A duality method for non-convex problems in Calculus of Variations

Tran Duc Minh Phan

UNIVERSITÉ DE TOULON

ÉCOLE DOCTORALE Mer et Sciences (ED548)

Institut de Mathématiques de Toulon

THÈSE

présentée par :

Tran Duc Minh PHAN

soutenue le : **28 Juin 2018**

pour obtenir le grade de Docteur en Mathématiques

Spécialité : Mathématiques appliquées

Une méthode de dualité pour des problèmes non convexes du Calcul des Variations

THÈSE dirigée par :

M. BOUCHITTÉ Guy	Professeur, Université de Toulon

JURY :

M. BOUCHITTÉ Guy	Professeur, Université de Toulon	Directeur
M. BOUSQUET Pierre	Professeur, Université de Toulouse	Rapporteur
M. BUTTAZZO Giuseppe	Professeur, Université de Pise	Président
M. CHAMPION Thierry	Maître de Conférences, Université de Toulon	Examinateur
Mme. FRAGALÀ Ilaria	Professeur, École polytechnique de Milan	Rapporteur
M. GALUSINSKI Cédric	Professeur, Université de Toulon	Examinateur
Mme. JIMENEZ Chloé	Maître de Conférences, Université de Brest	Examinateur

Remerciements

Je tiens à exprimer tout ma reconnaissance au Professeur Guy BOUCHITTÉ pour son aide précieuse tant morale que scientifique, notamment sa patience et tout ce qu'il m'a amicalement et gentiment donné tout au long de ces années de thèse. Je le remercie profondément d'avoir guidé avec enthousiasme mes premiers pas en recherche. Les orientations qui m'ont été suggérées sont très judicieuses et riches en enseignements.

J'exprime ma profonde gratitude au Professeur Cédric GALUSINSKI pour sa collaboration scientifique ainsi que ses nombreux conseils et encouragements. Grâce à son aide, j'ai pu réaliser la partie numérique de ma thèse en profitant de son expérience dans la mise au point d'algorithmes numériques et de leur implantation sur une interface de calcul performante.

Je souhaite remercier vivement la Professeure Ilaria FRAGALÀ, qui m'a beaucoup encouragé depuis le début de ma thèse à Toulon, pour sa disponibilité, ses conseils et les critiques qu'elle m'a adressées à l'occasion des comités de suivi de thèse. Elle a été pour moi un soutien incomparable. Je la remercie encore une fois d'avoir accepté la lourde tâche de rapporteur.

Je souhaite remercier sincèrement le Professeur Pierre BOUSQUET, qui m'a donné des conseils judicieux lors de nos discussions à Toulon et d'avoir accepté la lourde tâche d'être rapporteur.

Je voudrais particulièrement remercier le Professeur Giuseppe BUTTAZZO et Madame Chloé JIMENEZ d'avoir accepté de participer au jury de ma thèse. C'est un très grand honneur pour moi.

Je voudrais également remercier Thierry CHAMPION pour son aide et ses encouragements, et aussi d'avoir accepté de participer au jury de ma thèse.

Je tiens à remercier tous les membres de l'Institut de Mathématiques de Toulon (IMATH) pour leur accueil chaleureux et leur soutien très enthousiaste durant ces années de thèse.

Enfin, mes remerciements vont à ma famille et à tous mes amis qui ont toujours été à mes côtés pendant ces années de thèse.

Toulon, le 28 Juin 2018.

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Résumé de la thèse :

Une méthode de dualité pour des problèmes non convexes du Calcul des Variations

La recherche de minima globaux pour des fonctionnelles non convexes est un problème majeur. En raison du manque de convexité, un critère d'optimalité comme l'équation d'Euler ou les conditions d'optimalité du premier et second ordre ne sont pas suffisantes pour caractériser une solution globale parmi tous les points critiques et en particulier parmi les minimiseurs locaux. Une conséquence de cela est que les algorithmes classiques d'optimisation peuvent converger vers un minimiseur local sans aucune possibilité de vérifier s'il s'agit effectivement d'un minimiseur global. La difficulté est donc double : nous avons d'abord besoin d'un cadre théorique permettant d'obtenir des conditions nécessaires et suffisantes d'optimalité ; ensuite, nous devons proposer des nouvelles méthodes numériques qui permettent d'exclure tous les minimiseurs locaux (ou points critiques) et d'atteindre les minimiseurs globaux après un temps de calcul acceptable.

Motivée dans un tel contexte, cette thèse vise à étudier et développer un nouveau principe général de convexification permettant de traiter une classe spécifique mais importante de problèmes variationnels non convexes, notamment certains problèmes à frontière libre. Grâce à ce principe, nous avons pu mettre en œuvre les puissantes techniques de dualité en ramenant le problème non convexe étudé à des formulations de type primal–dual. Ceci nous a permis de proposer des algorithmes efficaces de recherche numérique de minima globaux.

La thèse est structurée en trois parties, chacune composée de plusieurs chapitres. Nous collectons dans la Partie A (Chapitre 1) des outils et techniques mathématiques de base se rapportant au sujet. La Partie B est composée des Chapitres 2, 3, 4. Cette partie théorique peut être considérée comme la « colonne vertébrale » de la thèse. Les contributions numériques sont présentées dans la Partie C constituée des Chapitres 5, 6 dans lesquels nous étudions la convergence des nouveaux algorithmes et présentons leur implémentation sur une plates-forme de calcul haute performance.

Un aperçu des chapitres 2, 3, 4, 5, 6 est donné ci-dessous.

Chapitre 2. Un principe général de dualité pour des problèmes variationnels non convexes

Dans ce chapitre, nous introduisons le principe général de convexification dans un cadre abstrait. Ensuite, nous l'appliquons à une classe de problèmes non convexes du calcul des variations pour lesquels un résultat de dualité est établi. Le problème dual consiste à maximimiser le flux d'un champ de vecteurs soumis à un système de contraintes ponctuelles convexes et à une contrainte de type divergence. Les résultats obtenus dans [20] dans le cas de fonctionnelles à croissance super linéaire (cadre $W^{1,p}$, avec $p > 1$) sont étendus au cas de fonctionnelles à croissance linéaire (définies dans $W^{1,1}$ et relaxées dans BV). Des conditions suffisantes d'optimalité sont reformulées dans ce cadre tenant compte des discontinuités des compétiteurs dans l'espace BV et de la relaxation de la condition de trace sur le bord.

Le principe général de convexification

On se donne une fonctionnelle $F : X \to [0, +\infty]$ de domaine non vide où X est un espace topologique muni d'une topologie τ. Pour fixer les idées, on peut mettre au départ l' hypothèse standard que les sous-niveaux $\{F \leq R\}$ sont des parties $\tau-$compactes de X. De cette façon, F atteint son minimum sur X.

On suppose que X s'injecte continuement dans un espace vectoriel topologique localement convexe Y. Dans la suite cette injection sera notée

$$\varphi : \ u \in X \ \mapsto \ \varphi_u \in Y.$$

L'idée est de construire une fonctionnelle *convexe* $G : X \to [0, +\infty]$ telle que :

$$G \circ \varphi = F \quad , \quad \inf_Y G = \inf_X F \ . \tag{1}$$

La construction d'une telle fonctionnelle convexe G s'avère en fait élémentaire lorsque l'on fait l'hypothèse suivante :

$$(H1) \qquad \begin{cases} \text{Il existe un sous-ensemble convexe métrisable compact } K \subset Y \\ \text{dont l'ensemble extrémal } \ddot{K} \text{ satisfait : } \ \varphi(X) \subset \ddot{K}. \end{cases}$$

Rappelons que v est un point extrémal de K si l'égalité $v = \theta v_1 + (1-\theta)v_2 \in K$ avec $v_1, v_2 \in K$ avec $\theta \in [0, 1]$ entraine que que $v = v_1 = v_2$. Un cas particulier important est celui où Y est le dual d'un espace de Banach séparable Z muni la topologie faible-étoile. Alors, Y^* peut être identifié avec Z lui-même et tout ensemble convexe, borné, $*$-faiblement fermé $K \subset Y$ est compact métrisable et coïncide avec l'enveloppe convexe de $\ddot{K}$.

Notant $< \cdot, \cdot >$ le crochet de dualité entre Y et son dual Y^*, introduisons la conjuguée de Fenchel de la fonctionnelle F_0 définie par $F_0(v) = F(u)$ si $v = \varphi(u)$ et $F_0(v) = +\infty$ si $v \notin \varphi(X)$. Par conséquent :

$$F_0^*(v) \ = \ \sup\{< v, g > -F(\varphi(u)) \ : \ u \in X\} \ .$$

On obtient alors une fonctionnelle G avec deux propriétés requises dans (1) en posant :

$$G(v) \ = \ F_0^{**}(v) \ = \ \sup\{< v, g > -F_0^*(g) \ : \ g \in Y^*\} \ .$$

Notons que la biconjuguée F_0^{**} coïncide avec la convexification séquentielle de F_0 donnée par :

$$(F_0)^{**}(v) = \inf\left\{ \liminf_h \sum_{i=1}^{n_h} t_i^h F(u_i^h) \ : \ \sum_{i=1}^{n_h} t_i^h \varphi_{u_i^h} \to v \right\},$$

où $\{t_i^h : i = 1, \cdots, n_h\}$ sont des scalaires dans $[0, 1]$ tels que $\sum_i t_i^h = 1$.

Le point clé pour établir l'égalié $F = F_0^{**} \circ \varphi$ est la propriété suivante des points extrémaux :

Lemme 1. *Soit $x \in \ddot{K}$, V un voisinage de x et notons $\mathbf{1}_V$ la fonction caractéristique de V. Alors*

$$(\mathbf{1}_V + \chi_K)^{**}(x) \;=\; 1 \;.$$

En fait, ayant en vue des applications en analyse asymptotique, on peut étendre facilement la construction de G au cas d'une suite de fonctionnelles $F^\varepsilon : X \to [0, +\infty]$ telle que F^ε $\Gamma-$ converge vers F pour la topologie τ. On définit alors $(F_0^\varepsilon)^* : Y^* \to [0, +\infty]$ puis $(F_0^\varepsilon)^{**} : Y \to [0, +\infty]$ comme ci-dessus et on suppose que la suite (F_ε) est uniformément coercive au sens suivant :

$$(H2) \qquad \left\{ \begin{array}{l} \text{Pour tout } R, \text{ il existe un sous-ensemble } \tau\text{-compact } C_R \text{ tel que :} \\ \forall \varepsilon > 0,\ \{u \in X \;:\; F^\varepsilon(u) \leq R\} \;\subset C_R \,. \end{array} \right.$$

On obtient :

Théorème 2. *Supposons que $G^\varepsilon = (F_0^\varepsilon)^{**}$ Γ-converge vers G pour la topologie faible de Y. Alors, sous les hypothèses (H1) et (H2), F^ε Γ-converge vers F sur (X, τ) où $F := G \circ \varphi$.*

Remarquons que, dans le cas particulier d'une suite constante $F^\varepsilon := F$ avec F non convexe et à priori non s.c.i, on obtient une formule de dualité pour la régularisée s.c.i. de F : $\overline{F}(u) = (F_0)^{**}(\varphi_u)$. Par ailleurs, le Théorème 2 permet d'offrir une alternative au calcul d'une Γ- limite. Pour cela, il est utile d'introduire une variante du résultat permettant de se ramener à des fonctionnelles homogènes G^ε. Il suffit d'ajouter l'hypothèse suivante :

$$(H3) \qquad \textit{Il existe une forme linéaire continue } l_0 \in Y^* \textit{ telle que } l_0 = 1 \textit{ sur } K \;.$$

qui revient à supposer que le compact convexe K est contenu dans la base d'un cône convexe fermé de Y. Pour tout $\varepsilon > 0$, nous introduisons l'ensemble convexe de Y^* :

$$D_\varepsilon \;:=\; \{g \in Y^* \;:\; (F_0^\varepsilon)^*(g) \;<\; 0\} \;=\; \{g \in Y^* \;: < \varphi_u, g > < \; F^\varepsilon(u)\ \forall u \in X\} \;,$$

Alors sous l'hypothèse supplémentaire $(H3)$, on obtient que $G^\varepsilon(v) = (F_0^\varepsilon)^{**}(v) = \chi^*_{D_\varepsilon}(v)$ pour tout $v \in K$. Le calcul de la Γ-limite G sur l'ensemble K (nécessaire pour calculer $G \circ \varphi$) se ramène alors à celui de la limite au sens de Kuratowski D de la suite (D_ε). Une fois que cet ensemble convexe D est identifié, on déduit la Γ-limite F à l'aide de la formule :

$$F(u) \;=\; \sup\{< \varphi(u), g > :\; g \in D\}.$$

Plus précisément, nous avons le résultat suivant :

Théorème 3. *Sous les hypothèses (H1), (H2), (H3) les trois conditions suivantes sont équivalentes :*

i) F^ε Γ–converge vers une limite F dans X,

ii) $(F_0^\varepsilon)^{**}$ Γ–converge vers une limite G dans Y,

iii) D_ε converge au sens de Kuratowski vers un ensemble D dans la topologie forte de Y^*.

De plus, si l'une de ces conditions est vraie, alors F, G et D satisfont les relations

$$D \;=\; \{g \in Y^* \;: < \varphi_u, g > \leq \; F(u)\ \forall u \in X\} \;, \tag{2}$$

$$G(v) \;=\; \sup_{g \in D} < v, g > \quad si < v, l_0 >= 1, \quad G(v) \;=\; +\infty \textit{ sinon,} \tag{3}$$

$$F(u) \;=\; G(\varphi_u) \;=\; \sup_{g \in D} < \varphi_u, g > \;. \tag{4}$$

Application aux problèmes variationnels non convexes

Le principe de convexification abstrait décrit ci-dessus va s'appliquer à une classe de problèmes variationnels non convexes où l'inconnue est une fonction scalaire $u \in L^1(\Omega)$ et Ω un domaine Lipschitzien borné de $\mathbb{R}^N$. Les espaces sont $X = L^1(\Omega)$, $Y = L^\infty(\Omega \times \mathbb{R})$ et l'injection de $\varphi : X \to Y$ est définie par :

$$\varphi : u \in X \mapsto \mathbf{1}_u \in Y \ , \qquad \mathbf{1}_u(x,t) := \begin{cases} 1 & \text{if } u(x) > t \\ 0 & \text{if } u(x) \leq t. \end{cases}$$

Il est facile de vérifier que l'application $u \mapsto \mathbf{1}_u$ est continue de $L^1(\Omega)$ à valeurs dans $L^\infty(\Omega \times \mathbb{R})$ muni de sa topologie $*$-faible (nous sommes dans le cas où $Y = Z^*$ si $Z = L^1(\Omega \times \mathbb{R})$). Par ailleurs l'image de φ est contenue dans l'ensemble convexe

$$K := \{v \in L^\infty(\Omega \times \mathbb{R}) \ : \ v(x,t) \in [0,1] \ \text{ pour p.p. } (x,t) \in \Omega \times \mathbb{R}\} \ ,$$

qui est un compact de $L^\infty(\Omega \times \mathbb{R})$ muni de sa topologie $*$-faible. Il est facile de vérifier que $\mathbf{1}_u$ est un point extrême de K car il prend ses valeurs dans $\{0,1\}$. On est donc bien dans la situation où $\varphi(X) \subset \ddot{K}$ conformément à l'hypothèse $(H1)$.

Soit $F : u \in L^1(\Omega) \to \mathbb{R} \cup +\infty$ une fonctionnelle s.c.i vérifiant l'hypothèse de coercivité ci-dessous :

$$\begin{cases} F(u) \ \geq \ k\,\|u\| - \frac{1}{k} \ , \ \text{ pour certaine constante adéquate } k > 0. \\ \text{Pour tout } R > 0, \text{ l'ensemble } \{u : F(u) \leq R\} \text{ est un compact de } L^1(\Omega). \end{cases} \tag{5}$$

Nous considérons le problème de minimisation

$$(\mathcal{P}) \qquad \inf\Big\{F(u) \ : \ u \in L^1(\Omega)\Big\}\,.$$

Sous l'hypothèse (5), le problème $(\mathcal{P})$ a au moins une solution et l'ensemble des solutions $\mathrm{Argmin}(\mathcal{P})$ est un compact non vide de $L^1(\Omega)$ (comme F est non convexe, nous nous attendons à priori à des solutions multiples). L'hypothèse $(H2)$ est vérifiée. Suivant la construction générale précédente, nous définissons pour tout couple $(v,g) \in L^\infty(\Omega \times \mathbb{R}) \times L^1(\Omega \times \mathbb{R})$

$$F_0^*(g) = \sup_{u \in L^1(\Omega)} \left\{ \int_{\Omega \times \mathbb{R}} g(x,t)\,\mathbf{1}_u\,dxdt - F(u) \right\}, \quad G(v) = \sup_{g \in L^\infty(\Omega \times \mathbb{R})} \left\{ \int_{\Omega \times \mathbb{R}} g\,v\,dxdt - F_0^*(g) \right\} \tag{6}$$

Appliquant le Théorème 2, on obtient donc

$$G(\mathbf{1}_u) \ = \ F(u) \quad \text{pour tout } u \in L^1(\Omega) \ . \tag{7}$$

Notre principe de convexification se conduit au problème d'optimisation convexe suivant

$$(\mathcal{Q}) \qquad \inf\{G(v) \ : \ v \in L^\infty(\Omega \times \mathbb{R}; [0,1])\} \ ,$$

dont l'ensemble des solutions $\mathrm{Argmin}(\mathcal{Q})$ est un faible-étoile compact non vide de $L^\infty(\Omega \times \mathbb{R}; [0,1])$. Du fait que $\inf_X F = \inf_Y F_0 = \inf_Y (F_0)^{**}$, les deux probèmes ont bien le même infimum (la seconde condition de (1)), d'où :

Lemme 4. *On a* $\inf(\mathcal{P}) = \inf(\mathcal{Q})$ *et l'équivalence :*

$$u \in \mathrm{Argmin}\,(\mathcal{P}) \iff \mathbf{1}_u \in \mathrm{Argmin}\,(\mathcal{Q}) \ .$$

Pour que ce résultat théorique soit exploitable, il s'agit maintenant d'identifier l'énergie convexifiée $G = (F_0)^{**}$ et d'établir un schéma de dualité pour le problème $(\mathcal{Q})$ permettant d'implanter des algorithmes numériques. D'autre part, s'il n'est pas réduit à un singleton, le convexe Argmin $(\mathcal{Q})$ contient des solutions v prenant des valeurs intermédiaires dans $(0,1)$ (i.e. v n'est pas de forme $\mathbf{1}_u$) ; il s'agit de comprendre la structure de ces solutions et d'en déduire les solutions du problème initial $(\mathcal{P})$.

Une réponse complète à ces questions est obtenue sous une hypothèse supplémentaire sur la fonctionnelle F permettant d'utiliser l' argument de « slicing » décrit ci-dessous. Notons

$$\mathcal{A} := \Big\{ v \in L^\infty(\Omega \times \mathbb{R}) \ : \ v(x,\cdot) \text{ non croissante (a.e. } x \in \Omega) \ , \ v(x,-\infty) = 1 \ , \ v(x,+\infty) = 0 \Big\} \ .$$

Pour tout $v \in \mathcal{A}$ et $s \in [0,1]$, nous définissons

$$u_s(x) := \inf \big\{ \tau \in \mathbb{R} \ : \ v(x,\tau) \le s \big\} \, . \tag{8}$$

Par construction, le sous-graphe de u_s coïncide à un ensemble Lebesgue-négligeable près avec l'ensemble de niveau $\{\tau \in \mathbb{R} \, : \, v(x,\tau) > s\}$. Plus précisément

$$\mathbf{1}_{u_s}(x,t) = \mathbf{1}_{\{v>s\}}(x,t) \qquad \text{pour p.p. } (x,t) \in \Omega \times \mathbb{R} \, . \tag{9}$$

Dans ce qui suit, nous noterons v_0 l'élément de $\mathcal{A}$ défini par

$$v_0(x,t) := \mathbf{1}_{\{t>0\}} \quad \text{(c'est-à-dire } v_0 = \mathbf{1}_{u_0} \text{ avec } u_0 \equiv 0) \ .$$

Définition 5. On dit qu'une fonctionnelle $J : L^\infty(\Omega \times \mathbb{R}) \to [0,+\infty]$ satisfait la *formule généralisée de co-aire* si pour tout $v \in L^\infty(\Omega \times \mathbb{R})$ la fonction $s \mapsto J(\mathbf{1}_{\{v>s\}})$ est Lebesgue-measurable dans $\mathbb{R}$ et

$$J(v) = \int_{-\infty}^{+\infty} J(\mathbf{1}_{\{v>s\}}) \, ds \qquad \forall v \in L^\infty(\Omega \times \mathbb{R}) \, . \tag{10}$$

Théorème 6. *Supposons que F satisfait l'hypothèse de coercivité (5) et qu'il existe une fonctionnelle faiblement-étoile s.c.i. et convexe $J : L^\infty(\Omega \times \mathbb{R}) \to [0,+\infty]$ satisfaisant la formule généralisée de co-aire et telle que*

$$J(\mathbf{1}_u) \ = \ F(u) \quad \textit{pour tout } u \in L^1(\Omega) \ . \tag{11}$$

Alors, si $\{u_s : s \in [0,1]\}$ est une famille paramétrisée associée à v par (8), on a

$$G(v) = \begin{cases} \displaystyle\int_0^1 F(u_s) \, ds & \textit{si } v \in \mathcal{A}, \\ +\infty & \textit{sinon.} \end{cases} \tag{12}$$

Donc, si $v \in$ Argmin G, alors $v \in \mathcal{A}$ et $u_s \in$ Argmin F for $\mathcal{L}^1$-p.p. $s \in (0,1)$. En particulier, si le problème initial $(\mathcal{P})$ admet un nombre fini de solutions $\{u^1, \dots, u^K\}$, alors

$$\text{Argmin } G \ = \ \left\{ \sum_{k=1}^K t_k \, \mathbf{1}_{u^k} \ : \ t_k \ge 0 \, , \ \sum_{k=1}^K t_k = 1 \right\} \, , \tag{13}$$

autrement dit, toute solution v du problème $(\mathcal{Q})$ est une fonction constante par morceaux.

C'est une conséquence remarquable du Théorème 6 qu'un minimiseur global pour le problème $(\mathcal{Q})$ choisi adéquatement (en prenant $t_k \in (0,1)$ dans (13)) peut encoder toutes les solutions multiples potentiellement du problème $(\mathcal{P})$. Nous référons au Chapitre 6 pour les simulations numériques de cette caractéristique intéressante.

Un schéma de dualité

Une classe assez large de fonctionnelles est couverte par le Théorème 6, en particulier

$$F_\lambda(u) = \begin{cases} \displaystyle\int_\Omega f(u, \nabla u)\, dx - \lambda \int_\Omega p(x) u\, dx\ , & \text{si } u \in W_0^{1,2}(\Omega) \\ +\infty & \text{sinon} \end{cases}$$

où $f = f(t,z)$ est une fonction $f : \mathbb{R} \times \mathbb{R}^N \to (-\infty, +\infty]$ satisfaisant :

$$f \text{ est s.c.i. dans } \mathbb{R}^N \times \mathbb{R}\ , \qquad \forall t \in \mathbb{R}\,,\ f(t,\cdot) \text{ est convexe}\ , \quad f(t,z) \geq k|z|^2 - \frac{1}{k}$$

où $k > 0$, λ est un paramètre positif et le terme source $p(x)$ (charge) appartient à $L^{r'}(\Omega)$ (r' est l'exposant conjugué de r) où r est compatible avec l'injection $W_0^{1,2}(\Omega) \subset L^r(\Omega)$ (c'est de dire $r \leq \frac{2N}{N+2}$ si $N \geq 3$, $r < +\infty$ si $N = 2$).

Notons que la non convexité de la densité d'énergie $f(u, \nabla u)$ ne se manifeste que par la dépendance par rapport à u. En fait, la convexité par rapport à la partie du gradient est nécessaire afin d'obtenir la semi-continuité inférieure de $F(u)$ et un problème primal bien posé . Il s'avère que la condition (11) va être obtenue en considérant la fonctionnelle convexe 1-homogène définie par :

$$J(v) := \int_{\Omega\times\mathbb{R}} h_f(t, Dv) \quad \text{où} \quad h_f(t, z^x, z^t) := \begin{cases} -z^t\, f\Big(t, \frac{z^x}{-z^t}\Big) & \text{si } z^t < 0 \\ +\infty & \text{si } z^t \geq 0. \end{cases} \tag{14}$$

Nous décrivons d'ici le problème dual dans le cas simple où f est de la forme $f(t,z) = g(t) + \varphi(z)$ étant donnés $\varphi : \mathbb{R}^N \to \mathbb{R}^+$ convexe continue avec $\varphi(0) = 0$ et $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ une fonction semi-continue inférieurement avec éventuellement de dénombrable discontinuités. Le problème primal s'écrit

$$(\mathcal{P}_\lambda) \qquad \inf \left\{ \int_\Omega \left(\varphi(\nabla u) + g(u)\right)\, dx - \lambda \int_\Omega p(x) u\, dx \ :\ u \in H_0^1(\Omega) \right\}$$

et sa version convexifiée est donnée par

$$(\mathcal{Q}_\lambda) \qquad \inf \left\{ \int_{\Omega\times\mathbb{R}} h_f(t, Dv) - \lambda \int_{\Omega\times\mathbb{R}} p(x)(v - v_0)\, dxdt \ :\ v \in \mathcal{A}\ ,\ v - v_0 \in BV_0(\Omega \times \mathbb{R}) \right\}$$

où $BV_0(\Omega \times \mathbb{R})$ désigne l'ensemble des fonctions intégrable à variations bornés dans $\Omega \times \mathbb{R}$ et dont la trace s'annule sur la frontière latérale $\partial\Omega \times \mathbb{R}$ (voir [103]).

Le problème dual (désigné par $(\mathcal{P}_\lambda^*)$) pour notre problème non convexe $(\mathcal{P}_\lambda)$ est alors recouvert en appliquant la dualité classique au problème $(\mathcal{Q}_\lambda)$. Les compétiteurs de ce problème dual sont des champs de vecteurs $\sigma = (\sigma^x, \sigma^t) : \Omega \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$ pris dans la classe

$$X_1(\Omega \times \mathbb{R}) = \left\{ \sigma \in L^\infty(\Omega \times \mathbb{R}; \mathbb{R}^{N+1}) \ :\ \operatorname{div} \sigma \in L^1_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}) \right\}\ ,$$

et $(\mathcal{P}_\lambda^*)$ consiste au problème du flux maximal ci-dessous :

$$(\mathcal{P}_\lambda^*) \qquad \sup \left\{ -\int_\Omega \sigma^t(x,0)\, dx \ :\ \sigma \in \mathcal{K}\ ,\ -\operatorname{div} \sigma = \lambda\, p \quad \text{dans } \Omega \times \mathbb{R} \right\}\ ,$$

où $\sigma \in \mathcal{K}$ signifie que le champs vectoriel $\sigma \in X_1(\Omega\times\mathbb{R})$ satisfait les contraintes locales (convexe) :

$$\begin{cases} \sigma^t(x,t) \geq \varphi^*(\sigma^x(x,t)) - g(t) \ \text{ pour } \mathcal{L}^{N+1}\text{-p.p. } (x,t) \in \Omega \times \mathbb{R} \\ \sigma^t(x,t) \geq -g(t) \quad \forall\, t \in S_g \text{ et pour } \mathcal{L}^N\text{-p.p. } x \in \Omega\,, \end{cases} \tag{15}$$

où S_g est l'ensemble de discontinuités de g.

Le résultat principal obtenu dans [20] est qu'il n'y a pas de saut de dualité.

Théorème 7. $\inf(\mathcal{P}_\lambda) \quad = \quad \sup(\mathcal{P}^*_\lambda)$.

Parmi les conséquences du Théorème 7, nous pouvons obtenir (voir [20]) des conditions nécessaires et suffisantes pour un minimiseur global de $(\mathcal{P}_\lambda)$, donc permettant d'exclure tous les minimiseurs locaux qui sont minimiseurs non globaux. Une deuxième conséquence est une caractérisation du point-selle qui permet la mise en œuvre de l'algorithme primal-dual efficace (voir Chapitre 5).

Le résultat ci-dessus peut être étendu aux problèmes aux conditions limites mixtes de type Dirichlet-Neumann. En particulier, si $u = u_0$ est prescrit sur un sous-ensemble $\Gamma_0 \subset \partial\Omega$ pour certain $u_0 \in W^{1,2}(\Omega)$, alors les compétiteurs σ de $(\mathcal{P}^*_\lambda)$ doivent satisfaire une condition de trace normale nulle sur $(\partial\Omega \setminus \Gamma_0) \times \mathbb{R}$, tandis que le terme linéaire à être maximisé devient $\int_{G_{u_0}} \sigma \cdot \nu_{u_0} d\mathcal{H}^N$, étant G_{u_0} le graphe de u_0 avec la normale unitaire ν_{u_0} orientée vers le bas. Pour le cas plus délicat avec les conditions aux limites de type Robin, nous référons à [20].

Lorsque la condition de trace u_0 est une fonction bornée, on peut en géréral établir par principe de comparaison que les minimiseurs du problème primal $(\mathcal{P}_\lambda)$ sont bornés. Dans ce cas, l'infimum est inchangé si l'on impose u à valeurs dans un intervalle fermé adéquat $\overline{I} := [m, M]$ de la droite réelle. Cela permet de réduire le problème primal $(\mathcal{P}_\lambda)$ à la classe de fonctions admissible $\Big\{u \in W^{1,2}(\Omega; I) \ : \ u = u_0 \text{ sur } \Gamma_0\Big\}$. Le résultat de dualité reste valable en prenant comme champs admissibles pour le problème dual $(\mathcal{P}^*_\lambda)$ la classe $\mathcal{K}(u_0, \Gamma_0, I)$ des éléments $\sigma \in X_1(\Omega \times I)$ satisfaisant les contraintes locales (15) sur $\Omega \times \overline{I}$ et les conditions d'équilibre

$$-\operatorname{div}\sigma = \lambda\, p(x) \ \text{ in } \Omega \times I \quad , \quad \sigma^x \cdot \nu_\Omega = 0 \ \text{ sur } \ (\partial\Omega \setminus \Gamma_0) \times I \ .$$

En conséquence, le problème convexifié $(\mathcal{Q}_\lambda)$ devient

$$(\mathcal{Q}_\lambda) \qquad \inf\left\{\int_{\Omega\times I} h_f(t, Dv) - \lambda \int_{\Omega\times I} p(x)(v - v_0)\, dxdt \ : \ v \in \mathcal{A}(u_0, \Gamma_0, I)\right\}$$

où l'ensemble des fonctions admissibles v est donné par

$$\mathcal{A}(u_0, \Gamma_0, I) := \Big\{v \in BV(\Omega \times I; [0,1]) : v = 1 \text{ sur } \Omega \times \{m\}, v = 0 \text{ sur } \Omega \times \{M\}, v = \mathbf{1}_{u_0} \text{ sur } \Gamma_0 \times I\Big\}$$

Si la solution de $(\mathcal{Q}_\lambda)$ est unique, alors elle est de la forme $\overline{v} = \mathbf{1}_{\overline{u}}$ où $\overline{u}$ est l'unique solution de $(\mathcal{P}_\lambda)$. Dans le cas de non unicité, la structure des solutions est donnée par le Théorème 6. Une caractérisation de type point-selle d'un couple optimal $(\overline{v}, \overline{\sigma})$ pour $(\mathcal{Q}_\lambda)$ et $(\mathcal{P}^*_\lambda)$ est obtenue en introduisant, pour toute couple (v, σ), avec $v \in BV(\Omega \times I; [0,1])$ et $\sigma \in X_1(\Omega \times I)$, le Lagrangien

$$L(v, \sigma) := \int_{\Omega\times I} (\sigma \cdot Dv) - \lambda \int_{\Omega\times I} p(x)\,(v - v_0)\, dxdt. \tag{16}$$

Théorème 8. *Posant $\widehat{\mathcal{A}} = \mathcal{A}(u_0, \Gamma_0, I)$ et $\widehat{\mathcal{K}} = \mathcal{K}(u_0, \Gamma_0, I)$, on a :*

$$\inf(\mathcal{P}_\lambda) \ = \ \inf_{v\in\widehat{\mathcal{A}}} \sup_{\sigma\in\widehat{\mathcal{K}}} L(v, \sigma) \ = \ \sup_{\sigma\in\widehat{\mathcal{K}}} \inf_{v\in\widehat{\mathcal{A}}} L(v, \sigma) \ = \ \sup(\mathcal{P}^*_\lambda)\,.$$

De plus, $(\overline{v}, \overline{\sigma})$ est optimal si et seulement s'il est un point-selle de L, i.e.

$$L(\overline{v}, \sigma) \ \le \ L(\overline{v}, \overline{\sigma}) \ \le L(v, \overline{\sigma}) \qquad \forall (v, \sigma) \in \widehat{\mathcal{A}} \times \widehat{\mathcal{K}}\,.$$

Γ-convergence de fonctionnelles non convexes

Le Théorème 6 peut être appliqué à une suite de fonctionnelles du type précédent i.e. :

$$F^\varepsilon(u) \ := \ \int_\Omega f_\varepsilon(u(x), \nabla u(x))\, dx,$$

Un cas particulier célèbre est celui étudié par Modica-Mortola où $f_e(t,z) = \varepsilon|z|^2 + \frac{1}{\varepsilon}W(t)$ où $W : \mathbb{R} \to \mathbb{R}^+$ est un potentiel non convexe qui s'annule uniquement en deux valeurs α, β (potentiel diphasique) et ε est un paramètre infinitésimal. Une variante un peu plus générale de cette fonctionnelle, étudiée dans [13] est de choisir $f_e(t,z) = \frac{1}{\varepsilon}f(t,\varepsilon\, z))$ où $f : \mathbb{R}\times\mathbb{R}^N \to [0,+\infty)$ satisfait les hypothèses ci-dessous :

i) f est continue par rapport à la première variable et convexe par rapport à la deuxième ;

ii) il existe deux nombres réels $0 < \alpha < \beta$ tels que $f(t,0) > 0$ si $t \neq \alpha, \beta$, $f(\alpha,0) = f(\beta,0) = 0$, et pour tout $z \neq 0$ et tout t, $f(t,z) > f(t,0)$;

iii) il existe $M > \beta$ tel que $f(t,\cdot)$ est borné localement, uniformément en $t \in [0,M]$;

iv) il existe une fonction ψ à croissance super-linéaire telle que $f(t,p) \geq \psi(p)$ pour tout $t \in \mathbb{R}$ et $p \in \mathbb{R}^N$.

L'approche par dualité proposée dans le Théorème 6 permet de calculer la $\Gamma-$ limite de F^ε dans $L^1(\Omega)$ par une méthode plus intuitive que celle utilisée dans [13]. Il est en effet facile de passer à la limite simple sur l'ensemble des contraintes associés au problème dual (convexe D_ε). Une fois connu la limite D de D_ε (au sens de Kuratowski), l'identification de la Γ-limite F à l'aide de la formule (4) permet alors d'obtenir :

Théorème 9. *Quand ε tend vers zero, F^ε Γ–converge dans $L^1(\Omega;[0,M])$ vers la fonctionnelle F donnée par*

$$F(u) \;=\; \begin{cases} \displaystyle\int_{S_u\cap\Omega} h(\nu_u)\, d\mathcal{H}^{N-1} & \text{si } u \in BV(\Omega;\{\alpha,\beta\}) \\ +\infty & \text{sinon .} \end{cases}$$

où $h(z)$ est la fonction convexe positivement homogène de degré un définie par :

$$h(z) = \int_\alpha^\beta f_c^{**}(t,z)\, dt \quad , \quad f_c(t,z) = \inf_{\varepsilon>0} f_\varepsilon(t,z) \ .$$

($f_c(t,\cdot)$ est l'enveloppe conique de $f(t,\cdot)$ et $f_c^{**}(t,)$ est son enveloppe convexe)

Extension du principe de dualité au cas des fonctionnelles à croissance linéaire

Jusqu'à présent nous avons considéré des fonctionnelles intégrales où l'intégrande $f(t,z)$ a un comportement super linéaire à l'infini en la variable $z = \nabla u$. Cette hypothèse est indispensable pour obtenir l'existence d'une solution du problème primal dans un espace de Sobolev de type $W^{1,p}(\Omega)$ avec $p > 1$. Motivés par les problèmes à frontière libre de type surfaces minimales ou variation totale qui seront abordés dans les chapitres suivant, il nous faut étendre les résultats de dualité au cas de fonctionnelles non convexes à croissance linéaire. Ceci est fait dans la dernière partie du Chapitre 2 où le problème primal se présente sous la forme

$$(\mathcal{P}) \qquad \inf\left\{\int_\Omega (\varphi(\nabla u) + g(u))\, dx - \int_\Omega p(x)u\, dx \;:\; u \in W^{1,1}(\Omega),\ u = u_0 \text{ sur } \partial\Omega\right\} \tag{17}$$

où $\varphi : \mathbb{R}^N \to \mathbb{R}^+$ est une fonction convexe continue telle que $k|z| - \frac{1}{k} \leq \varphi(z) \leq C(1+|z|)$ pour certains constants $k, C > 0$, $g : \mathbb{R} \to (-\infty,+\infty]$ est une fonction s.c.i. et u_0 un élément donné dans $W^{1,1}(\Omega)$. Pour que l'infimum de $(\mathcal{P})$ soit fini, nous imposons sur les fonctions g et p des conditions supplémentaires détaillées dans la Section 2.5. Le problème $(\mathcal{P})$ n'admet pas en général de solution dans $W^{1,1}(\Omega)$. Il faut le relaxer dans l'espace $BV(\Omega)$ ce qui est relativement classique (voir par exemple [71, 72]). Le problème relaxé $(\mathcal{P})$ s'écrit sous la forme

$$(\mathcal{PR}) \qquad \min\Big\{E(u) \;:\; u \in BV(\Omega)\Big\} \tag{18}$$

où l'énergie à minimiser est donnée par

$$E(u) := \int_\Omega \Big(\varphi(\nabla u) + g(u) - p(x)u\Big)\,dx + \int_{\partial\Omega} \varphi^\infty((u_0-u)\nu_\Omega)\,d\mathcal{H}^{N-1} + \int_\Omega \varphi^\infty\left(\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{S_u} [u]\varphi^\infty(\nu_u)\,d\mathcal{H}^{N-1} \tag{19}$$

(où $\varphi^\infty(z) := \lim_{s\to+\infty} \dfrac{\varphi(sz)}{s}$ est la fonction de récession de φ). L'infimum de ce problème relaxé $(\mathcal{PR})$ est atteint et que l'on a $\inf(\mathcal{P}) = \min(\mathcal{PR})$. Nous nous placerons dans le cas où le domaine de g est borné de sorte que les compétiteurs appartiennent à $BV(\Omega; I)$ où $I := [m, M]$ est un intervalle fermé adéquat.

$$(\mathcal{PR}) \qquad \min\Big\{E(u) \;:\; u \in BV(\Omega; I)\Big\} \tag{20}$$

Une théorie de dualité et des champs de calibration doit être réformulée pour ce problème $(\mathcal{PR})$ en tenant compte des discontinuités de compétiteurs dans l'espace BV. La fonctionnelle convexe 1-homogène J sur $v \in BV(\Omega \times I)$ à laquelle nous appliquerons la formule de coaire s'écrit :

$$J(v) := \int_{\Omega\times I} \widetilde{h}(t, Dv) \qquad \text{où} \qquad \widetilde{h}(t, z^x, z^t) := \begin{cases} -z^t\Big(\varphi(\frac{z^x}{-z^t}) + g(t)\Big) & \text{si } z^t < 0, \\ \varphi^\infty(z^x) & \text{si } z^t = 0, \\ +\infty & \text{si } z^t > 0. \end{cases} \tag{21}$$

Remarquons que la fonction 1-homogène $\widetilde{h}$ est maintenant bien définie pour $z^t = 0$ car φ^∞ est finie. Par ailleurs

$$J(\mathbf{1}_u) = F(u) := \int_\Omega \Big(\varphi(\nabla u) + g(u)\Big)dx + \int_\Omega \varphi^\infty\left(\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{S_u} [u]\varphi^\infty(\nu_u)\,d\mathcal{H}^{N-1}.$$

D'après Théorème 6, le problème convexifié de $(\mathcal{PR})$ s'écrit

$$(\mathcal{Q}) \qquad \inf\Big\{\widehat{E}(v) \;:\; v \in \mathcal{A}(I)\Big\} \tag{22}$$

où

$$\begin{aligned} \widehat{E}(v) &:= J(v) + \ell(v), \qquad \text{pour } v \in BV(\Omega\times I), \\ \ell(v) &:= -\int_{\Omega\times I} p(x)(v - v_0)\,dxdt + \int_{\partial\Omega\times I} \partial_t\gamma_{u_0}(x,t)(v-v_0)d\mathcal{H}^{N-1}dt + \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} \\ \gamma_{u_0}(x,t) &:= \varphi^\infty\Big((u_0(x) - t)\nu_\Omega(x)\Big) \quad \text{pour } (x,t) \in \partial\Omega \times I, \\ \mathcal{A}(I) &:= \Big\{v \in BV(\Omega\times I; [0,1]) \;:\; v = 1 \text{ sur } \Omega\times\{m\},\ v = 0 \text{ sur } \Omega\times\{M\}\Big\}. \end{aligned}$$

De manière similaire à (12), on a :

$$\widehat{E}(v) = \int_0^1 \widehat{E}(\mathbf{1}_{\{v>s\}}(x,t))ds = \int_0^1 \widehat{E}(\mathbf{1}_{u_s})ds = \int_0^1 E(u_s)ds.$$

pour tout $v \in \mathcal{A}(I)$ tel que $\widehat{E}(v) < +\infty$, et que $\inf(\mathcal{PR}) = \inf(\mathcal{Q})$ (voir Proposition 2.19).

De même façon d'établir le problème dual dans le cas où les fonctionnelles à croissance super linéaire, nous associons $(\mathcal{PR})$ à un problème dual de $(\mathcal{PR})$ écrit sous la forme

$$(\mathcal{P}^*) \qquad \sup\Big\{-\int_\Omega \sigma^t(x,0)dx + \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} \;:\; \sigma \in \mathcal{B}(I)\Big\} \tag{23}$$

où $\mathcal{B}(I)$ est une classe des champs de vecteurs définie dans le sens suivant :

- $\mathcal{K}(I)$ désigne la classe des champs de vecteurs $\sigma \in X_1(\Omega \times I)$ satisfaisant les contraintes convexe s

$$\sigma^t(x,t) \geq \varphi^*(\sigma^x(x,t)) - g(t) \quad \text{pour } \mathcal{L}^{N+1}\text{-p.p. } (x,t) \in \Omega \times I, \tag{24}$$
$$\sigma^t(x,t) \geq -\varphi(0) - g(t) \quad \forall\, t \in S_g \text{ et pour } \mathcal{L}^N\text{-p.p. } x \in \Omega. \tag{25}$$

  Ici, S_g désigne l'ensemble de discontinuités de g supposé Lebesgue négligeable.

- $\mathcal{B}(I)$ est la classe des champs $\sigma \in \mathcal{K}(I)$ tels que

$$-\operatorname{div}\sigma = p \quad \text{dans } \Omega \times I, \tag{26}$$
$$\sigma^x \cdot \nu_\Omega = -\partial_t \gamma_{u_0} \quad \text{sur } \partial\Omega \times I, \tag{27}$$

  où $\sigma^x \cdot \nu_\Omega$ est la trace normale faible de σ sur la frontière $\partial(\Omega \times I)$.

Nous utilisons de mêmes techniques dans [20] pour démontrer les résultats principaux de la théorie de dualité au cas de problèmes non convexes à croissance linéaire que : il n'y a pas de saut de dualité (voir Théorème 10) et que la condition d'optimalité (nécessaire et suffisante) est fourni par des champs de calibration sur les graphes de compétiteurs (notés par $\overline{G}_u$) du problème primal permettant de caractériser tous ses minimiseurs globaux (voir Théorème 11).

Théorème 10. *On a* $\inf(\mathcal{P}) = \min(\mathcal{PR}) = \sup(\mathcal{P}^*)$.

Théorème 11. *Soit u, σ admissible pour les problèmes $(\mathcal{PR})$ dans (20) et $(\mathcal{P}^*)$ dans (23), respectivement. Alors, u est solution de $(\mathcal{PR})$ et σ est solution de $(\mathcal{P}^*)$ si et seulement si*

$$\widetilde{h}(t, \widehat{\nu}_u) = \sigma \cdot \widehat{\nu}_u \qquad \mathcal{H}^N\text{-p.p. sur } \overline{G}_u. \tag{28}$$

ou de façon équivalente :

$$\sigma^x(x,u(x)) \in \partial\varphi(\nabla u(x)) \qquad \text{pour } \mathcal{L}^N\text{-p.p. } x \in u^{-1}(\mathbb{R} \setminus S_g), \tag{29}$$
$$\sigma^t(x,u(x)) = \varphi^*(\sigma^x(x,u(x))) - g(u(x)) \qquad \text{pour } \mathcal{L}^N\text{-p.p. } x \in u^{-1}(\mathbb{R} \setminus S_g), \tag{30}$$
$$\sigma^t(x,t) = -\varphi(0) - g(t) \qquad \forall t \in \mathbb{R} \text{ et pour } \mathcal{L}^N\text{-p.p. } x \in \{u = t\}, \tag{31}$$
$$\sigma^x(x,t) \cdot \nu_u = \varphi^\infty(\nu_u) \qquad \text{pour } \mathcal{H}^{N-1}\text{-p.p. } x \in S_u \text{ et } \forall t \in [u^-(x), u^+(x)], \tag{32}$$
$$\sigma^x(x,u^+(x)) \cdot \nu_u = \varphi^\infty(\nu_u) \qquad \text{pour } |D^c u|\text{-p.p. sur } \Omega. \tag{33}$$

où $u^\pm$ désignent les limites approximatives supérieures et inférieures de u, et $S_u = \{u^+ > u^-\}$. Nous appelons une tel σ une calibration pour la solution u.

À ce cas de problèmes non convexes à croissance linéaire, nous obtenons aussi une caractérisation de type point-selle d'un couple optimal $(\overline{v}, \overline{\sigma})$ pour $(\mathcal{Q})$ et $(\mathcal{P}^*)$. Pour cela nous introduisons, pour toute couple (v, σ), avec $v \in BV(\Omega \times I; [0,1])$ et $\sigma \in X_1(\Omega \times I)$, le Lagrangien

$$L(v,\sigma) := \ell(v) + \int_{\Omega \times I} \sigma \cdot Dv$$

où $\ell(v)$ est le terme affine de $\widehat{E}(v)$ (voir (22)).

Théorème 12. *On a*

$$\inf(\mathcal{P}) = \inf_{v \in \mathcal{A}(I)} \sup_{\sigma \in \mathcal{K}(I)} L(v,\sigma) = \sup_{\sigma \in \mathcal{K}(I)} \inf_{v \in \mathcal{A}(I)} L(v,\sigma) = \sup(\mathcal{P}^*).$$

De plus, $(\overline{v}, \overline{\sigma})$ est optimal si et seulement s'il est un point selle de L sur $\mathcal{A}(I) \times \mathcal{K}(I)$, i.e.

$$L(\overline{v}, \sigma) \leq L(\overline{v}, \overline{\sigma}) \leq L(v, \overline{\sigma}) \qquad \forall (v,\sigma) \in \mathcal{A}(I) \times \mathcal{K}(I).$$

Chapitre 3. Une méthode de relaxation convexe pour des problèmes aux frontières libres

Nous nous intéressons dans ce chapitre à une classe assez large de problèmes à frontière libre, notamment ceux que l'on rencontre en traitement d'images où la partie convexe de l'énergie est de type variation totale. Pour ce type de problèmes (en dimension 2), nous exploitons la théorie des champs de calibration 3d pour établir un principe d'exclusion qui généralise celui découvert par Visintin dans les années 1990. Grâce à ce résultat, nous pouvons ramener un problème bidimensionnel à frontière libre ou multi-phasique à la minimisation d'une fonctionnelle convexe en dimension deux. Les tests numériques basés sur cette minimisation conduisent à des interfaces d'une qualité surprenante.

Soit Ω un ouvert borné Lipschitzien de $\mathbb{R}^N$ et Γ un sous-ensemble Borélien de $\partial\Omega$. Nous désignons par ν_Ω la normale extérieure unitaire sur $\partial\Omega$. Étant donné $u_0 \in L^1(\partial\Omega)$, nous considérons le problème d'infimum

$$\inf\left\{\int_\Omega [h(\nabla u) + g(u) - p(x)u]dx \ : \ u \in W^{1,1}(\Omega)\ ,\ u = u_0 \text{ sur } \Gamma\right\} \tag{34}$$

Les hypothèses que nous imposons sur h, g, p sont précisées suivantes :

(H1) La fonction $h : \mathbb{R}^N \to \mathbb{R}$ est convexe continue, **positivement 1-homogène** et telle que

$$\forall z \in \mathbb{R}^N, \quad C_1|z| \le h(z) \le C_2(1+|z|)\ .$$

(H2) La fonction $g : \mathbb{R} \to (-\infty, +\infty]$ est une fonction semicontinue inférieurement avec éventuellement des discontinuités. Nous supposerons qu'il existe un ensemble *Lebesgue négligeable* $D \subset \mathbb{R}$ tel que g est semicontinue supérieurement sur $\mathbb{R} \setminus D$, i.e.

$$\limsup_{s\to t} g(s) \le g(t),\ \forall t \in \mathbb{R} \setminus D\,.$$

(H3) Le terme source p satisfait l'une des hypothèses suivantes :

a) $p \in L^1(\Omega)$ si $\operatorname{dom} g := \{t \in \mathbb{R} :\ g(t) < +\infty\}$ est borné,

b) $p \in L^{r'}(\Omega)$ si g satisfait $\liminf_{|t|\to\infty} \dfrac{g(t)}{|t|^r} > 0$ pour $r > 1$ $(r' = \frac{r}{r-1})$.

Sous ces hypothèses, l'infimum dans (34) est fini et les suites minimisantes sont bornées dans $W^{1,1}(\Omega)$. En utilisant la méthode directe au calcul des variations, nous obtenons l'existence de solutions pour le problème relaxé dans $BV(\Omega)$:

$$(\mathcal{P}_g) \qquad \inf\Big\{F_g(u) \ : \ u \in BV(\Omega)\Big\}$$

où F_g est défini dans $BV(\Omega)$ par

$$F_g(u) = \int_\Omega h(Du) + \int_\Omega (g(u) - p(x)\,u)\,dx + \int_\Gamma h((u_0 - u)\,\nu_\Omega)\,d\mathcal{H}^{N-1}\ .$$

Notons que dans la notation ci dessus, nous indiquons la dépendence relativement à g car nous aurons besoin de considérer plusieurs choix pour g.

Lemme 13. *Sous les hypothèses (H1)(H2)(H3), on a* $\inf(34) = \min(\mathcal{P}_g)$. *De plus, toutes les suites minimisantes pour* (34) *est faiblement relativement compacte dans* $BV(\Omega)$ *et admet une sous-suite convergeant dans* $L^1(\Omega)$ *vers une solution de* $(\mathcal{P}_g)$.

Nous soulignons que g est à priori une fonction *nonconvex* à valeurs dans $(-\infty, +\infty]$ et que les discontinuités sont autorisées. Ceci est crucial si nous souhaitons associer à (P_g) des problèmes à frontière libre ou multiphasique.

Le résultat principal de ce chapitre (Théorème 15) spécifie que, sous certaines conditions, le problème de minimisation (P_g) peut être relaxé sous la forme $(P_{g^{**}})$. Autrement dit, on peut légitimement remplacer g par son enveloppe convexe g^{**} sans changer la valeur de l'infimum. L'argument clé est le *principe d'exclusion* suivant :

Théorème 14. *Soit $-\infty < a < b < +\infty$ tels que $g_{a,b} < g$ dans (a,b) où $g_{a,b}$ est donné par*

$$g_{a,b}(t) := \begin{cases} g(t) & \text{si } t \notin (a,b) \\ m_{a,b}(t-a) + g(a) & \text{sinon} \end{cases} \qquad \text{et} \qquad m_{a,b} := \frac{g(b) - g(a)}{b - a}.$$

Supposons que $u_0 \notin (a,b)$ p.p. sur Γ. Alors, on a $\inf(\mathcal{P}_g) = \inf(\mathcal{P}_{g_{a,b}})$. En outre, si u est solution de $(\mathcal{P}_g)$ alors $u(x) \notin (a,b)$ pour p.p. $x \in \Omega$.

En particulier, le principe d'exclusion peut être utilisé sur une union dénombrable d'intervalles (a,b) sur laquelle $g > g_{a,b}$, c'est-à-dire l'ensemble suivant

$$Z := \{g > g^{**}\} \tag{35}$$

En fait, si Z est borné, nous pouvons appliquer le Lemme 3.5 (voir Chapitre 3) pour démontrer que Z est une union dénombrable d'intervalles ouverts. En conséquence, nous considerons les hypothèses supplémentaires suivantes :

(H4) $Z := \{g > g^{**}\}$ borné ;

(H5) $u_0 \in \mathbb{R} \setminus Z$ p.p. sur Γ.

Nous sommes prêt à énoncer notre résultat principal de relaxation convexe :

Théorème 15. *Sous les hypothèses (H1–H5), on a :*

i) $\inf(\mathcal{P}_g) = \inf(\mathcal{P}_{g^{**}})$

ii) Si u est solution de $(\mathcal{P}_g)$, alors $u(x) \in \mathbb{R} \setminus Z$ pour p.p. $x \in \Omega$.

En particulier si g est tel que $\mathrm{dom}(g) \setminus Z$ se compose d'un nombre fini de nombres réels $t_1 < \cdots < t_k$, alors solutions de $(\mathcal{P}_g)$ sont de la forme

$$u = \sum_{i=1}^{k} t_i \mathbf{1}_{A_i} \quad , \quad \text{avec } \{A_i, 1 \le i \le k\} \text{ partition Borelienne de } \Omega. \tag{36}$$

On a seulement $\operatorname{Argmin} \mathcal{P}_g \subset \operatorname{Argmin} \mathcal{P}_{g^{**}}$ mais l'égalité apparait au cas où $\mathcal{P}_{g^{**}}$ admet unique solution. Ce problème de unicité pour $(\mathcal{P}_{g^{**}})$ semble être une tâche très difficile car g^{**} n'est pas strictement convexe (du tout sur Z), nous référons au papier récent pour le resultat de unicité dans un contexte similaire [15]. Nous croyons que l'unicité pour $(\mathcal{P}_g)$ implique l'unicité pour $(\mathcal{P}_{g^{**}})$ même si nous ne sommes pas encore en mesure de le prouver. Actuellement, dans beaucoup des exemples des problèmes multiphasiques présentés dans Section 3.3, nous observons une très bonne convergence de l'approximation numérique de $(\mathcal{P}_{g^{**}})$ vers une solution de la forme (36). Par conséquent, nous croyons que l'unicité tient génériquement en ce qui concerne les données aux bords et les paramètres entrant dans la définition de g, Ω.

Dans ce chapitre, nous introduisons également des approches numériques basées sur des algorithmes primal-dual avec schémas explicite ou semi-implicite. La non différentiabilité de g^{**} est traitée par une méthode de projection sur l'épigraphe offrant une alternative très efficace à la méthode de régularisation classique. La conjonction de cette méthode de projection avec le caractère semi-implicite de l'algorithme permet d'obtenir une convergence rapide avec une excellente précision des interfaces obtenues (voir les Sections 3.3 et 3.4)

Chapitre 4. Une méthode de calibration pour des surfaces minimales avec frontière libre et un problème de type Cheeger

Nous appliquons la théorie des calibrations à un problème classique de surfaces minimales avec frontière libre et établissons de nouveaux résultats de comparaison avec sa variante où la fonctionnelle des surfaces minimales est remplacée par la variation totale (les deux fonctionnelles coïncident lorsque la surface est constituée exclusivement de plateaux). Nous montrons que les deux problèmes ne sont pas identiques en général notamment lorsque l'ouvert de référence n'est pas un ensemble de Cheeger. A cette occasion, nous généralisons la notion de calibrabilité introduite par Caselles-Chambolle et Al. et nous construisons explicitement une solution duale pour le problème associé à la seconde fonctionnelle. Aucune régularité n'est requise : le domaine est ouvert convexe quelconque de $\mathbb{R}^2$ et la construction utilise un potentiel localement Lipschitzien lié à la distance au *cut-locus*.

Nous nous intéressons aux deux problèmes à frontière libre suivants

$$\beta(\lambda) := \inf\left\{\int_D \sqrt{1+|\nabla u|^2}dx - \lambda|\{u \geq 1\}| \ : \ u \in W_0^{1,1}(D)\right\}, \tag{37}$$

$$\beta_0(\lambda) := \inf\left\{\int_D (1+|\nabla u|)dx - \lambda|\{u = 1\}| \ : \ u \in W_0^{1,1}(D),\ 0 \leq u \leq 1\right\}, \tag{38}$$

où D est un ouvert borné de $\mathbb{R}^N$. Le problème d'optimisation associé à $\beta_0(\lambda)$ est une variante de problème $\beta(\lambda)$, où l'on remplace la fonctionnelle $\int_D \sqrt{1+|\nabla u|^2}dx$ par la variation totale $\int_D(1+|\nabla u|)dx$. Par ailleurs, pour qu'une surface paramétrisée par $u \in W_0^{1,1}(D)$ soit minimale dans le contexte du problème $\beta(\lambda)$, il est nécessaire que u soit solution du problème aux limites

$$-\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{in } D \setminus \Omega(u), \tag{39}$$

$$u = 0 \quad \text{on } \partial D, \qquad u = 1 \quad \text{on } \partial\Omega(u), \tag{40}$$

où $\Omega(u) := \{u = 1\}$. La frontière libre correspond au bord de $\Omega(u)$.

La motivation de ce chapitre est de caractériser les solutions de $\beta(\lambda)$ et $\beta_0(\lambda)$ en utilisant la méthode duale des champs de calibration, puis de comparer la valeur des infima $\beta(\lambda)$ et $\beta_0(\lambda)$. Il est clair que $\beta(\lambda) \leq \beta_0(\lambda)$ pour tout $\lambda \geq 0$ (car on a $\sqrt{1+|z|^2} \leq 1+|z|$ pour tout $z \in \mathbb{R}^N$). La question se pose de déterminer les valeurs de λ pour lesquelles l'égalité $\beta(\lambda) = \beta_0(\lambda)$ a lieu.

Les énergies relaxées associées à $\beta(\lambda)$ et $\beta_0(\lambda)$ s'écrivent respectivement :

$$E_\lambda(u) := \int_D \sqrt{1+|\nabla u|^2}dx + \int_D d|D^s u| + \int_{\partial D} |u| d\mathcal{H}^{N-1} - \lambda|\{u \geq 1\}|,$$

$$E^0_\lambda(u) := |D| + \int_D d|Du| + \int_{\partial D} |u| d\mathcal{H}^{N-1} - \lambda|\{u = 1\}|.$$

Notons que $-\lambda|\{u = 1\}|$ peut être présenté comme une intégrale $\int_D g_0(u)dx$ où $g_0(t) = 0$ si $t \in [0,1)$, $g(1) = -\lambda$ et $g(t) = +\infty$ sinon. D'autre part E_λ et E^0_λ coïncident sur la classe des fonctions caractéristiques, i.e. $E_\lambda(\mathbf{1}_\Omega) = E^0_\lambda(\mathbf{1}_\Omega)$ pour $\Omega \subset D$. Du fait que la variation totale est homogène de degré un, le problème $\beta_0(\lambda)$ entre dans le cadre des résultats de relaxation convexe obtenus au Chapitre 3. En particulier une solution du problème $\beta_0(\lambda)$ ne peut pas prendre de valeurs intermédiaire dans $(0,1)$ et l'infimum $\beta_0(\lambda)$ reste inchangé si l'on remplace $g_0(t)$ par son enveloppe convexe $g_0^{**}(t)$. Par conséquent, on a :

$$\begin{aligned}\beta_0(\lambda) &= \min\left\{\int_D (1+|Du|-\lambda u)dx + \int_{\partial D}|u|d\mathcal{H}^{N-1} \ : \ u \in BV(D;\{0,1\})\right\} \\ &= \min\{|D| + P(\Omega) - \lambda|\Omega| \ : \ \Omega \subset D\} \\ &= |D| + m(\lambda, D)\end{aligned} \tag{41}$$

où $m(\lambda, D)$ désigne l'infimum du problème géométrique suivant :

$$m(\lambda, D) := \min \{P(\Omega) - \lambda|\Omega| \ : \ \Omega \subset D\}. \tag{42}$$

Le problème (42) est une variante intéressante du problème de Cheeger pour laquelle nous établissons les Propositions 4.4, 4.5, 4.6 et 4.7. Parmi celles ci, nous mentionnons deux résultats faisant intervenir l'ensemble de Cheeger maximal de D. Notant h_D la constante de Cheeger de D et C_D son ensemble de Cheeger maximal, on a :

Corollaire 16.

(i) Soit $\lambda > h_D$ et Ω une solution de $m(\lambda, D)$. Alors $\Omega \supset C_D$, $h_\Omega = h_D$, et Ω, D ont les mêmes ensembles de Cheeger.

(ii) Soit (λ_n) une suite telle que $\lambda_n > h_D$, $\lambda_n \downarrow h_D$ et Ω_n une solution de $m(\lambda_n, D)$. Alors :

$$C_D = \bigcap_{n\in\mathbb{N}} \Omega_n. \tag{43}$$

La structure particulière du problème non convexe $\beta_0(\lambda)$ sur $\mathbb{R}^N$ permet de réduire le problème dual initialement en dimension $(N+1)$ à un problème en dimension N (voir Remarque 4.2) : en effet le champ vectoriel $\sigma = (\sigma^x, \sigma^t)$ peut être cherché sous la forme $\sigma(x,t) = (-q(x), t\,\mathrm{div}_x\, q + a(x))$ et on est ramené à trouver $q : D \to \mathbb{R}^N$ solution de

$$\sup \left\{ \int_D (\mathrm{div}\, q - \lambda) dx \ : \ q \in L^\infty(D; \mathbb{R}^N),\ |q| \le 1,\ 0 \le \mathrm{div}\, q \le \lambda \right\}. \tag{44}$$

Ce problème dual (44) permet alors de caractériser les solutions du problème géométrique $m(\lambda, D)$, comme indiqué ci-dessous.

Théorème 17. *Le problème dual (44) admet une solution $\overline{q}$ dans $L^\infty(D; \mathbb{R}^N)$ et on a $m(\lambda, D) =$ sup (44). Un couple optimal $(\Omega, \overline{q})$ pour $m(\lambda, D)$ et (44) est caractérisé par les conditions suivantes*

$$|\overline{q}| \le 1 \quad p.p \text{ dans } D, \qquad 0 \le \mathrm{div}\, \overline{q} \le \lambda \quad p.p \text{ dans } D, \tag{45}$$

$$\overline{q} \cdot \nu_\Omega = 1 \quad \mathcal{H}^{N-1}\text{-}p.p. \text{ sur } \partial\Omega, \qquad \mathrm{div}\, \overline{q} = \lambda \quad p.p. \text{ dans } D \setminus \Omega. \tag{46}$$

Cette caractérisation suggère une extension de la notion de calibrabilité introduite par Caselles-Chambolle et Al. dans [4], qu'on appellera *θ-calibrabilité* :

Définition 18. Soit $\theta \ge 1$ et $\Omega \subset \mathbb{R}^N$ un ensemble borné à périmètre fini. On dit Ω *θ-calibrable* s'il existe un champ de vecteurs $q \in L^\infty(\Omega; \mathbb{R}^N)$ tel que

$$\|q\|_\infty \le 1, \qquad q \cdot \nu_\Omega = 1 \ \ \mathcal{H}^{N-1}\text{-p.p. sur } \ \partial\Omega, \qquad 0 \le \mathrm{div}\, q \le \theta\lambda_\Omega \ \text{ dans } \ \mathcal{D}'(\Omega).$$

à comparer avec la notion classique de calibrabilité [4] :

Définition 19. Soit Ω un sous-ensemble borné à périmètre fini dans $\mathbb{R}^N$. On dit Ω *calibrable* s'il existe un champ de vecteurs $q \in L^\infty(\Omega; \mathbb{R}^N)$ tel que

$$\|q\|_\infty \le 1, \qquad q \cdot \nu_\Omega = 1 \ \ \mathcal{H}^{N-1}\text{-p.p. sur } \ \partial\Omega, \qquad \mathrm{div}\, q = \lambda \ \text{ dans } \ \mathcal{D}'(\Omega),$$

pour certaine constante $\lambda \in \mathbb{R}$.

Une caractérisation des ensembles θ-calibrables est donné ci-dessous.

Proposition 20. *Soit $\Omega \subset \mathbb{R}^N$ un ensemble borné à périmetre fini. Les conditions suivantes sont équivalentes :*

(i) Ω *est solution du problème* $m(\theta\lambda_\Omega, \Omega)$ *pour certaine* $\theta \geq 1$.

(ii) $\mathbf{1}_\Omega$ *est solution de* (41) *pour* $\lambda = \theta\lambda_\Omega$.

(iii) Il existe un champ de vecteurs $q \in L^\infty(\Omega; \mathbb{R}^N)$ *tel que*

$$|q| \leq 1 \text{ p.p. dans } \Omega, \quad 0 \leq \operatorname{div} q \leq \theta\lambda_\Omega \text{ p.p. dans } \Omega, \quad q \cdot \nu_\Omega = 1 \ \mathcal{H}^{N-1}\text{-p.p. sur } \partial\Omega.$$

Au cas où Ω *est convexe et de classe* $C^{1,1}$*, ces trois conditions sont en effet équivalentes à*

(iv) On a

$$\max\left\{1, (N-1)\kappa_\infty(\partial\Omega)\lambda_\Omega^{-1}\right\} \leq \theta, \tag{47}$$

où $\kappa_\infty(\partial\Omega)$ *désigne la* L^∞*-norme de la courbure moyenne de* $\partial\Omega$.

Rappelons que si D est un convexe de $\mathbb{R}^2$, l'ensemble de Cheeger de D est unique et s'obtient en prenant la réunion des disques d'un certain rayon contenus dans D : posons pour tout $\delta > 0$ (δ doit être plus petit que le rayon intérieur R de D)

$$D^\delta := \{x \in D \ : \ d(x, D^c) > \delta\}$$

où $d(x, D^c)$ désigne la distance de x à $D^c := \mathbb{R}^2 \setminus D$. Alors il existe une valeur unique $\delta = \delta^*$ telle que $|D^\delta| = \pi\delta^2$. Alors, $h_D = 1/\delta^*$ et l'ensemble Cheeger de D est donné par

$$C_D = \cup\{B(x, \delta^*) : B(x, \delta^*) \subset D\} \ .$$

Cet ensemble C_D est calibrable au sens de la Définition 19. Nous allons construire des ensembles θ-calibrable de manière similaire. Pour cela nous introduisons l'ensemble Ω_λ suivant

$$\Omega_\lambda = \bigcup_{B(x,\lambda^{-1})\subset D} B(x, \lambda^{-1}). \tag{48}$$

Dans tout ce qui suit, on supposera que D **est un convexe borné de** $\mathbb{R}^2$**.**

Lemme 21. *Soit* $\lambda > 0$ *et* $\theta = \lambda\lambda_{\Omega_\lambda}^{-1}$*. Alors* Ω_λ *défini par* (48) *est un convexe* θ*-calibrable.*

Notons que dans la discussion de caractérisation de θ-calibrabilité, Ω_λ lui-même minimise le problème $m(\lambda, \Omega_\lambda)$. À l'esprit de la théorie des champs de calibration, nous arrivons à construire explicitement une calibration pour Ω_λ. Pour le faire, nous utiliserons un potentiel associé à la distance au cut-locus.

Potentiel associé au cut-locus

Nous introduisons la fonction $\rho : \overline{D} \to \mathbb{R}_+$ définie par

$$\rho(x) := \sup\{\delta \geq 0 \ : \ d(x, D^\delta) \leq \delta\}. \tag{49}$$

Nous appellerons ρ le *potentiel de cut-locus* associé à D. On peut vérifier que l'ensemble $\{\delta \ : \ d(x, D^\delta) - \delta \leq 0\}$ est exactement l' intervalle $[0, \rho(x)]$. Quelques propriétés intéressantes du potentiel ρ sont données ci-dessous :

Lemme 22. ρ *est continu, atteint son maximum* R *dans l'ensemble central de* D *et* $\rho(x) \geq d(x, D^c)$*. De plus, pour tout* $r > 0$*,*

$$\rho(x) = r \iff d(x, D^r) = r \iff x \in \partial\Omega_{\frac{1}{r}}. \tag{50}$$

Lemme 23. *ρ est continu sur $\overline{D}$ et localement Lipschitzien sur D (le gradient de ρ au point x peut exploser quand $d(x, D^c) \to 0$). Sur ∂D, on a :*

$$\rho(x) \leq \frac{1}{\kappa_{\partial D}(x)} \qquad et \qquad \rho(x) = \tau(x) \ ,$$

où $\kappa_{\partial D}(x)$ est la courbure au point x et $\tau(x)$ la distance normale au cut-locus :

$$\tau(x) = \sup\{t \geq 0 \ : \ x \text{ est l'unique projection de } (x - t\nu_D(x)) \text{ sur } \partial D\}.$$

Théorème 24. *Soit $D \subset \mathbb{R}^2$ un domaine convexe borné. Dans l'ouvert $\{0 < \rho < R\}$, le potentiel localement Lipschitzien ρ satisfait $\nabla\rho \neq 0$ p.p. et $\operatorname{div}\left(\frac{\nabla\rho}{|\nabla\rho|}\right) + \frac{1}{\rho} = 0$ (au sens des distributions).*

Corollaire 25. *Soit $\mu > 0$ et u_μ la solution unique de*

$$(\mathcal{Q}_\mu) \qquad \inf\left\{\int_{\mathbb{R}^2} |Du| + \frac{\mu}{2}\int_D (1-u)^2 \ : \ u \in BV_0(D)\right\}.$$

Alors, on a

$$u_\mu(x) \ = \ \begin{cases} \left(1 - \frac{1}{\mu\,\rho(x)}\right)_+ & si\ \rho(x) < \frac{1}{h_D} \\ \left(1 - \frac{h_D}{\mu}\right)_+ & si\ \rho(x) \geq \frac{1}{h_D} \quad (plateau). \end{cases} \tag{51}$$

Corollaire 26. *Soit $\mu > \lambda > h_D$. Alors, pour $s = 1 - \frac{\lambda}{\mu}$, l'ensemble niveau*

$$\{u_\mu > s\} = \{\rho > \lambda^{-1}\} = \Omega_\lambda$$

est solution du problème $m(\lambda, D)$.

Une construction explicite pour des champs de calibration sur D

Soit D un sous-ensemble convexe borné dans $\mathbb{R}^2$. On sait que pour $\lambda = h_D$ (la constante de Cheeger de D) le problème $m(\lambda, D)$ admet exactement deux solutions $\{\emptyset, C_D\}$ (où C_D est l'ensemble de Cheeger de D). Pour $\lambda > h_D$, l'unique minimiseur est l'ensemble Ω_λ défini dans (48). Nous retrouvons ce résultat en construisant une calibration $\overline{q}$ pour Ω_λ, qui satisfait les conditions suivantes :

$$|\overline{q}| \leq 1 \quad \text{p.p dans } D, \qquad 0 \leq \operatorname{div}\overline{q} \leq \lambda \quad \text{p.p dans } D, \tag{52}$$

$$\overline{q}\cdot\nu_\Omega = 1 \quad \mathcal{H}^{N-1}\text{-p.p. sur } \partial\Omega, \qquad \operatorname{div}\overline{q} = \lambda \quad \text{p.p. dans } D\setminus\Omega. \tag{53}$$

Pour cela, nous partons d'un champ de vecteurs $q_{h_D} \in L^\infty(\Omega_{h_D}; \mathbb{R}^2)$ tel que

$$|q_{h_D}| \leq 1, \quad \operatorname{div} q_{h_D} = h_D \quad \text{p.p. dans } \Omega_{h_D}, \qquad q_{h_D}\cdot\nu_{\Omega_{h_D}} = 1 \quad \mathcal{H}^1\text{-p.p. sur } \partial\Omega_{h_D}.$$

Comme Ω_{h_D} coïncide avec l'ensemble de Cheeger de D, l'existence d'un tel champ est assurée (mais malheureusement nous n'en avons pas une construction explicite). Ce champ q_{h_D} est compatible avec les conditions (52)-(53) sur l'ouvert Ω_{h_D} . La construction de $\overline{q}$ reste à réaliser en dehors de Ω_{h_D}.

La deuxième étape est de construire $\overline{q}$ dans $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$. Cette construction est donnée explicitement par $q_\rho := -\frac{\nabla\rho}{|\nabla\rho|}$ où ρ est le potentiel du cut-locus que nous venons de introduire. Ce champ q_ρ satisfait d'après le Théorème 24

$$|q_\rho| = 1 \quad \text{dans } \Omega_\lambda \setminus \overline{\Omega_{h_D}}, \qquad q_\rho\cdot\nu_{\Omega_\lambda} = 1 \quad \text{sur } \partial\Omega_\lambda,$$

$$\operatorname{div} q_\rho = \frac{1}{\rho}, \qquad \operatorname{div} q_\rho \in [h_D, \lambda] \quad \text{p.p. dans } \Omega_\lambda \setminus \overline{\Omega_{h_D}},$$

$$q_\rho\cdot\nu_{\Omega_{h_D}} = q_{h_D}\cdot\nu_{\Omega_{h_D}} = 1 \quad \text{sur } \partial\Omega_{h_D}.$$

Posant $\overline{q} = q_\rho$ sur $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$, nous voyons que $\overline{q}$ vérifie les conditions (52)-(53) sur tout l'ouvert Ω_λ (la divergence de $\overline{q}$ ne charge pas le bord de Ω_{h_D} car la trace normale $q_{h_D} \cdot \nu_{\Omega_{h_D}}$ est la même des deux cotés (égale à 1)).

La troisième étape est de donner une construction de $\overline{q}$ dans $D \setminus \overline{\Omega_\lambda}$. Cette construction est fournie par le résultat suivant :

Lemme 27. *Il existe un champ de vecteurs q_λ dans $D \setminus \overline{\Omega_\lambda}$ tel que*

$$|q_\lambda| \leq 1 \ \text{ dans } D \setminus \overline{\Omega_\lambda}, \qquad \operatorname{div} q_\lambda = \lambda \ \text{ dans } D \setminus \overline{\Omega_\lambda}, \qquad q_\lambda \cdot \nu_{\Omega_\lambda} = 1 \ \text{ sur } \partial\Omega_\lambda. \tag{54}$$

En fait le champ q_λ peut être construit explicitement dans un ensemble en général plus grand que D (voir la construction dans la preuve du lemme dans Section 4.3.4). Les conditions (54) sont compatibles avec les conditions (52)-(53) et la continuité de la trace normale est préservée à la traversée de $\partial\Omega_\lambda$.

En résumé, la construction de $\overline{q}$ est donnée par $\overline{q} = q_{h_D}$ dans Ω_{h_D}, $\overline{q} = q_\rho$ dans $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$ et $\overline{q} = q_\lambda$ dans $D \setminus \overline{\Omega_\lambda}$.

Résultats de comparaison

Discutons maintenant la relation entre $\beta(\lambda)$ et $\beta_0(\lambda)$ que nous avons évoquée au début du chapitre. Nous introduisons les valeurs critiques de λ suivantes

$$\lambda_0 = \sup\{\lambda \geq 0 \ : \ u \equiv 0 \text{ solution de } \beta(\lambda)\},$$
$$\lambda_1 = \inf\{\lambda \geq 0 \ : \ u \equiv 1 \text{ solution de } \beta(\lambda)\}.$$

On peut montrer facilemement le résultat suivant :

Théorème 28. *Si $\lambda_0 < \lambda_1$, alors $\beta(\lambda) < \beta_0(\lambda)$ pour tout $\lambda \in (\lambda_0, \lambda_1)$.*

Autrement dit , l'égalité $\beta(\lambda) = \beta_0(\lambda)$ n'est possible que si $\beta(\lambda)$ (et donc aussi $\beta_0(\lambda)$) admet pour solution $u \equiv 0$ ou $u \equiv 1$. Par conséquent, si $\beta_0(\lambda)$ admet une solution unique de la forme $u = \mathbf{1}_\Omega$ avec $0 < |\Omega| < |D|$, alors on aura l'inégalité stricte $\beta(\lambda) < \beta_0(\lambda)$.

Dans les deux propositions suivantes, nous avons obtenu quelques estimations de ces valeurs critiques λ_0, λ_1.

Proposition 29. *On a*

$$0 < \lambda_0 \ \leq h_D \ \leq \frac{P(D)}{|D|} \ \leq \ \lambda_1.$$

De plus, si D n'est pas un ensemble de Cheeger, on a l'inégalité stricte $\lambda_0 < h_D < \frac{P(D)}{|D|}$, donc $\beta(\lambda) < \beta_0(\lambda)$ pour tout $\lambda \in [h_D, \frac{P(D)}{|D|})$.

Proposition 30. *On a :*

(i) $\lambda_0 \geq \lambda_0^*$ où

$$\lambda_0^* := \begin{cases} I^{-1}(h_D) & \text{si } h_D \leq \frac{\pi}{2} \\ 1 + h_D - \frac{\pi}{2} & \text{si } h_D \geq \frac{\pi}{2} \end{cases} \quad , \quad I(y) = \int_0^y \frac{ds}{\sqrt{s(2-s)}} \, .$$

(ii) $\theta_D \lambda_D \leq \lambda_1 \leq \theta_D \lambda_D + 1$ où θ_D est la constante de calibration de D et $\lambda_D = P(D)/|D|$. En particulier, si D est un convexe de classe $C^{1,1}$, alors

$$\max\{\lambda_D, (N-1)\kappa_\infty(\partial D)\} \leq \lambda_1 \leq 1 + \max\{\lambda_D, (N-1)\kappa_\infty(\partial D)\}.$$

Quand ∂D admet un point singulier (cone normal non réduit à une seule direction), alors $\lambda_1 = +\infty$ (i.e. $u \equiv 1$ ne peut jamais être une solution de $\beta(\lambda)$).

Finalement nous présentons des simulations numériques dans le cas où $D = \{x \in \mathbb{R}^2 \ : \ |x| < R\}$ avec différentes valeurs de λ et R (voir Section 4.4). Il apparait que les fonctions $\lambda_0(R)$ et $\lambda_1(R)$ coïncident pour $R \leq 1$ alors que $\lambda_0(R) < \lambda_1(R)$ pour $R > 1$.

Chapitre 5. Un nouveau schéma semi-implicite basé sur la méthode d'Arrow-Hurwicz pour des problèmes de points selle

La dernière partie de la thèse est consacrée aux algorithmes d'optimisation utilisés pour la validation numérique de la méthode de convexification que nous avons proposée. Il s'agit notamment d'adapter les algorithmes existants de type primal-dual pour la recherche de points selle, en introduisant de nouvelles variantes plus efficaces en précision (finesse des interfaces) et temps calcul (notamment lorsque les champs de calibration du problème dual sont 3d). Nous avons en particulier introduit une variante semi-implicite de la méthode d'Arrow–Hurwicz (basée sur un Laplacien inverse) qui permet de réduire le nombre d'itérations nécessaires pour obtenir une qualité satisfaisante des interfaces. Par ailleurs nous avons traité la non différentiabilité structurelle des Lagrangiens utilisés à l'aide d'une méthode géométrique de projection sur l'épigraphe offrant ainsi une alternative aux méthodes classiques de régularisation (voir la Section 3.4 du Chapitre 3). De nombreux tests numériques sont présentés (2d ou 3d) avec un comparatif entre les différentes méthodes.

L'objectif de ce chapitre est de fournir des algorithmes efficaces pour chercher des points selle pour un certain Lagrangien $L(u,p)$ sur des convexes fermés C, K, qui sont caractérisés par les inégalités

$$L(\hat{u},p) \leq L(\hat{u},\hat{p}) \leq L(u,\hat{p}),\ \forall u \in C,\ \forall p \in K. \tag{55}$$

D'habitude, on pense immédiatement à la méthode de descente-montée suivant la direction du gradient. Pour un lagrangien général $L(u,p)$, l'approche la plus simple introduite par Arrow-Hurwicz a la forme

$$\begin{aligned} p_{n+1} &= \Pi_K\Big(p_n + \tau_n \frac{\partial L}{\partial p}(u_n,p_n)\Big) \\ u_{n+1} &= \Pi_C\Big(u_n - \tau_n \frac{\partial L}{\partial u}(u_n,p_n)\Big). \end{aligned}$$

où Π_K, Π_C sont les projecteurs orthogonaux sur des convexes fermés K et C respectivement. Il s'agit d'un algorithme d'optimisation itératif d'ordre un qui converge sous des conditions très strictes (notamment la stricte convexité-concavité) et pour un choix très précis des pas $\tau_n \to 0$, $\sum_{n=0}^{\infty} \tau_n = \infty$ (cf. M. Kallio et A. Ruszczynski). Pour s'affranchir de ces contraintes, L.D. Popov [97] a proposé une modification de la méthode d'Arrow-Hurwicz en introduisant un réajustement intermédiare noté $(\overline{u}_n, \overline{p}_n)$ suivant le schéma suivant :

$$\begin{aligned} p_{n+1} &= \Pi_K\Big(p_n + \tau \frac{\partial L}{\partial p}(\overline{u}_n,\overline{p}_n)\Big) \\ u_{n+1} &= \Pi_C\Big(u_n - \tau \frac{\partial L}{\partial u}(\overline{u}_n,\overline{p}_n)\Big) \\ \overline{p}_{n+1} &= \Pi_K\Big(p_{n+1} + \tau \frac{\partial L}{\partial p}(\overline{u}_n,\overline{p}_n)\Big) \\ \overline{u}_{n+1} &= \Pi_C\Big(u_{n+1} - \tau \frac{\partial L}{\partial u}(\overline{u}_n,\overline{p}_n)\Big). \end{aligned}$$

Il est démontré l'existence d'un scalaire positif τ_0 tel que l'algorithme modifié converge pour tout pas constant τ dans l'intervalle $0 < \tau < \tau_0$. Cette amélioration permet d'élargir la classe des problèmes où la méthode est applicable à tous les Lagrangiens convexes-concaves dont les dérivées sont Lipschitziennes. Par ailleurs l'introduction de points auxiliaires rend le processus itératif plus stable. Toutefois les projections supplémentaires introduites sont compliquées et rendent plus lourd le calcul numérique. Ensuite Chambolle-Pock et al. ont considéré des Lagrangien de type linéaires de la forme :

$$L(u,p) = \langle Au, p\rangle + \langle f, u\rangle - \langle g, p\rangle,$$

où A est un opérateur linéaire continu et $\langle \cdot, \cdot \rangle$ désigne le produit scalaire associé sur les espaces de Hilbert des variables u, p. Dans ce contexte, ils ont utilisé des points auxiliaires faciles à mettre en oeuvre : $\overline{p}_{n+1} = p_{n+1}$ et $\overline{u}_{n+1} = 2u_{n+1} - u_n$. Il est établi que l'algorithme itératif

$$\begin{aligned} p_{n+1} &= \Pi_K(p_n + \alpha(A\overline{u}_n - g)) \\ u_{n+1} &= \Pi_C(u_n - \beta(A^* p_{n+1} + f)) \\ \overline{u}_{n+1} &= 2u_{n+1} - u_n \end{aligned} \tag{56}$$

converge vers un point-selle de $L(u, p)$ si l'on choisit $\alpha, \beta > 0$ tels que $\alpha\beta\|A\|^2 < 1$. Ici A^* désigne l'opérateur adjoint de A. Il y a eu ensuite beaucoup d'efforts pour accélérer la convergence des algorithmes de type Arrow-Hurwicz par exemple dans [35] où une méthode de variation des pas de descente-montée est utilisée, ou en modifiant le choix de l'extrapolation des points auxiliaires, ou en implémentant des schémas implicites. Dans un papier plus récent [36], un changement de la métrique permet d'augmenter le pas de temps. Un tel résultat est obtenu pour un Lagrangien général du type

$$L(u, p) = \langle Au, p \rangle + F(u) - G(p). \tag{57}$$

où F et G sont différentiables. Dans ce cadre de nombreux résultats de l'ordre de convergence ont étés obtenus [35, 36]. L'idée principale y est de combiner la technique de type proximal avec des schémas implicites classiques. Mais, pour être efficaces, de telles méthodes demandent que l'opérateur proximal soit facile à calculer en pratique et pour ce faire il est nécessaire de pénaliser les contraintes convexes C et K avec des fonction lisses définies dans tout espace entier. Ces méthodes de pénalisation ne sont pas adaptées aux nos cas où les convexes C et K sont non lisses et comportent un grand nombre de contraintes. L'objet de ce chapitre est de remédier à ces difficultés en proposant des algorithmes nouveaux permettant de résoudre (57) dans le contexte d'un grand nombre de contraintes et ensuite dans le contexte de fonctions non différentiables F et G.

Nous considérons un Lagrangien $L(u, p)$ défini sur $V \times W$ où V, W sont des espaces de Hilbert. On se donne deux convexes fermés $C \subset V$ et $K \subset W$, puis deux fonctions convexes différentiables $F, G : V \to \mathbb{R}$. Les dérivées premières satisfont la condition de Lipschitz avec les constantes L_f, L_g, respectivement. Nous supposons en outre l'existence d'un point-selle pour le Lagrangien $L(u, p)$ (qui est acquise en pratique sous une hypothèse de coercivité ou de compacité).

Schéma explicite

Description de l'algorithme dans le schéma explicite
Initialisation : Soit $n \in \mathbb{N}$. Étant donné $(u_0, p_0) \in C \times K$, et $\overline{u}_0 = u_0$.

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha(A\overline{u}_n - G'(p_n))) \\ u_{n+1} = \Pi_C(u_n - \beta(A^* p_{n+1} + F'(u_n))) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n \end{cases} \tag{58}$$

où A^* désigne l'opérateur adjoint de A ; Π_K, Π_C désignent respectivement les projecteurs orthogonaux sur les fermés convexes K, C ; et les paramètres $\alpha, \beta > 0$ sont choisis de façon convenable (précisés après).

Nous obtenons le résultat de convergence suivant :

Théorème 31. *Pour tous α, β tels que*

$$\begin{gathered} 0 < \alpha < \frac{2}{L_g}, \qquad 0 < \beta < \frac{2}{L_f}, \\ \alpha\beta\left(\|A\|^2 - \frac{L_f L_g}{4}\right) + \frac{\alpha L_g}{2} + \frac{\beta L_f}{2} < 1, \end{gathered} \tag{59}$$

l'algorithme (58) *converge vers un point selle de $L(u,p)$ dans $C \times K$.*

Notons que, quand $F = G = 0$, le problème (57) réduit à

$$L(u,p) = \langle Au, p\rangle, \tag{60}$$

et l'algorithme (58) devient

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha A\overline{u}_n) \\ u_{n+1} = \Pi_C(u_n - \beta A^* p_{n+1}) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n. \end{cases} \tag{61}$$

Dans ce cas, les paramètres positifs α, β seront être choisis tels que $\alpha\beta\|A\|^2 < 1$, pour que (61) converge vers un point-selle de $L(u,p)$ sur $C \times K$. Le choix de α, β est maintenant plus flexible et ne fait pas intervenir les constantes de Lipschitz L_f, L_g. Noter que dans le cas où F, G sont affines, on retrouve le résultat de Chambolle-Pock et al. [96].

Schéma semi-implicite

Dans le schéma explicite lié à la méthode d'Arrow-Hurwicz, l'algorithme (58) est convergent sous une condition de bornitude de l'opérateur linéaire A (le choix de α, β dépendent de $\|A\|$). Dans la version semi-implicite présentée ci-dessous, cette dépendance n'existe plus.

Description de l'algorithme dans le schéma semi-implicite
Initialisation : Soit $n \in \mathbb{N}$. Étant donné $(u_0, p_0) \in C \times K$ et $\overline{u}_0 = u_0$.

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha(A\overline{u}_n - G'(p_n)) \\ u_{n+1} = \Pi_C(u_n - \beta(A^*A + I)^{-1}(A^* p_{n+1} + F'(u_n))) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n \end{cases} \tag{62}$$

où $\alpha, \beta > 0$ sont choisis convenablement (à préciser plus tard).

L'opérateur A est supposé satisfaire l'hypothèse suivante : Soit O est un fermé convexe de l'espace de Hilbert V tel que $0 \in O$ et Π_O le projecteur orthogonal sur O.

$$\forall O, \quad \forall u \in V, \quad \langle A(u - \Pi_O(u)), A\Pi_O(u)\rangle \geq 0. \tag{63}$$

Noter que A satisfait automatiquement l'hypothèse (63) quand A est une transformation orthogonale. En effet, comme $A^*A = I$, on a :

$$\langle A(u - \Pi_O(u)), A\Pi_O(u)\rangle = \langle A^*A(u - \Pi_O(u)), \Pi_O(u)\rangle = \langle u - \Pi_O(u), \Pi_O(u)\rangle.$$

La condition (63) sera également vérifiée dans les applications que nous traitons où A représente l'opérateur gradient ∇. Nous renvoyons pour cela à la Section 5.3.

Théorème 32. *Soit $A : V \to Z$ un opérateur linéaire, fermé de domaine dense, satisfaisant l'hypothèse* (63)*. Pour tous α, β tels que*

$$0 < \alpha < \frac{2}{L_g}, \qquad 0 < \beta < \frac{2}{L_f}, \qquad \alpha\beta + \frac{\alpha L_g}{2} < 1,$$

l'algorithme (62) *converge vers un point-selle de $L(u,p)$ sur $C \times K$.*

Lorsque le terme $F(u)$ est absent dans le Lagrangien $L(u,p)$ alors l'algorithme (62) se réduit à (voir Section 5.3 pour les détails)

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha(A\overline{u}_n - G'(p_n)) \\ u_{n+1} = \Pi_C(u_n - \beta(A^*A)^{-1}A^*p_{n+1}) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n \end{cases}$$

Dans cette situation, la condition (63) devra être remplacée par l'hypothèse que A^*A est défini positif (donc inversible) et si de plus G est linéaire, les paramètres positifs α, β devront satisfaire la contrainte $\alpha\beta < 1$.

En conclusion, le schéma explicite généralisé (58) est très bien adapté à la recherche de points selles de Lagrangiens du type (57). La contribution principale de ce chapitre est de proposer une nouvelle variante semi-implicite de cet algorithme qui converge sous des contraintes moins restrictives sur les paramètres numériques α, β qui ne dépend pas du pas de discrétisation. Le schéma semi-implicite couplé avec la méthode de splitting pour la partie explicite offre une diminution notoire du temps de calcul et du nombre d'itérations nécessaire pour atteindre une solution satisfaisante (voir Section 5.4 et Section 6.1 du Chapitre 6). Même si le temps dédié à une itération est long (en raison du solveur choisi pour le Laplacien), le coût global en temps calcul est réduit de facon particulièrement importante lorsqu'on traite une grille très fine afin d' obtenir des interfaces plus précises.

Pour terminer, quelques remarques sur le cas non différentiable : c'est une situation que nous rencontrons dans le traitement numérique des problème à frontière libre ou multiphasique abordés au Chapitre 3 (voir Section 3.4). La technique classique de régularisation du Lagrangien s'avère peu efficace en ce qui concerne la qualité de la frontière libre. Une alternative prometteuse est la mise en œuvre d'une méthode de projection géométrique sur les épigraphes de F et G. Cette méthode est complétement compatible avec le schéma implicite que nous proposons et s'avère spectaculairement efficace dans les exemples de problèmes à frontière libre traités dans la Section 3.4 du Chapitre 3.

Chapitre 6. Algorithmes primal-dual pour des problèmes à frontière libre

Ce chapitre est consacré aux simulations numériques de problèmes aux frontières libres. C'est l'occasion non seulement de rafraîchir le sujet avec une présentation illustrée par de nombreuses simulations, mais aussi de valider la théorie de la dualité introduite au Chapitre 2 pour les problèmes variationnels non-convexes. C'est ici en fait que sont testés les performances des algorithmes explicites, implicites basés sur la formulation primal-dual (discutés dans la Chapitre 5). Notons que les Lagrangiens que nous avons traités sont en dimension trois et qu'ils sont testés relativement à la convergence vers un point selle 3d. Une procédure est ensuite mise en place pour retrouvrer les solutions 2d des problèmes primals. Ces combinaisons de techniques fournissent une approche numérique intéressante pour une classe assez large des problèmes variationnels non convexes.

Perspectives et problèmes ouverts

En ce qui concerne la théorie de dualité étudiée et développée au Chapitre 2, nous sommes conduits à des perspectives et quelques problèmes ouverts qui peuvent motiver des travaux futurs.

Fonctionnelles définies sur une espace de fonctions à valeurs vectorielles

Les arguments dans la Section 2.2 ont été développés seulement dans le cas de fonctions scalaires. Précisément, l'espace X qui s'injecte dans les points extrêmaux d'un convexe compact a toujours été $L^1(\Omega)$ et l'injection l'application $\varphi : u \in X \mapsto \mathbf{1}_u \in Y$ où $Y = L^\infty(\Omega \times \mathbb{R}; [0,1])$. Une extension de la méthode à des fonctions à valeurs vectorielles requiert de construire une autre injection. Un choix simple serait d'associer au champ vectoriel $u \in L^1(\Omega; \mathbb{R}^N)$ la masse de Dirac en $u(x)$ de sorte que $\varphi(u)$ est la mesure de Young sur $\Omega \times \mathbb{R}^N$ définie par :

$$< \varphi(u), \psi >= \int_\Omega \psi(x, u(x))\, dx \ , \ \psi \in C^0(\Omega \times \mathbb{R}^N) \ .$$

Si $|\Omega| = 1$ et $u(x)$ est dans un compact convexe donné $K \subset \mathbb{R}^N$, alors $\varphi(u)$ est un point extrême de l'ensemnle de mesures de probabilité sur $\Omega \times K$ dont la première marginale coincide avec la mesure de Lebesgue sur Ω. Néanmoins, le calcul explicit de la fonctionnelle convexifiée semble difficile dans ce cadre général. Une alternative possible serait de faire appel à des outils de la théorie de la mesure géométrique tels que les courants Cartésiens ou les varifolds [90, 69, 70].

Fonctionnelles intégrales dépendant de la matrice Hessienne

Revenant au cas scalaire, il serait particulierement intéressant de développer une théorie de dualité pour la minimisation de fonctionnelles du type

$$F(u) \ = \ \int_\Omega f(\nabla u, \nabla^2 u)\, dx \ ,$$

où $f(z, M) : \mathbb{R}^N \times \mathbb{R}^{N\times N}_{\text{sym}}$ est une fonction convexe en M mais pas en z. La procédure générale de convexification décrite dans la Section 2.3 du Chapitre 2 pourrait être développée dans ce contexte. On s'attend à ce que le tenseur de courbure en chaque point $(x, u(x))$ du graphe de u joue un rôle clé dans le calcul de la fonctionnelle convexifiée.

Fonctionnelle de Mumford-Shah

Les problèmes à discontinuité libre ont été une motivation première pour le principe de convexification présenté dans le Chapitre 2. L'objectif du papier de référence [1] était de démontrer l'optimalité de certaines configurations pour le problème de segmentation d'images décrit ci-après. Soit Ω un domaine Lipschitzien borné de $\mathbb{R}^2$ et $g : \Omega \to [0,1]$ (représentant les données de niveaux de gris).

$$(\mathcal{P}) \qquad \inf_{u \in SBV(\Omega)} \left\{ \int_{\Omega \setminus S_u} \frac{1}{2} |\nabla u|^2 \, dx + \mathcal{H}^1(S_u) + \frac{1}{2} \int_\Omega |u - g(x)|^2 \, dx \right\}$$

Il s'agit d'un problème bien posé (existence au moins d'un minimiseur) dans l'espace $SBV(\Omega)$ des fonctions $u \in L^1(\Omega)$ dont le gradient au sens distributions Du se décompose en une partie régulière ∇u (coïncide avec le gradient défini p.p.) et une partie singulière concentrée sur l'ensemble de saut S_u qui est un sous-ensemble rectifiable de dimension 1 (inconnu) de Ω dont la longueur totale est notée $\mathcal{H}^1(S_u)$. Une formulation plus mécanique de $(\mathcal{P})$ (populaire en mécanique des fractures) s'écrit

$$\inf \left\{ \int_{\Omega \setminus K} \frac{1}{2} |\nabla u|^2 \, dx + \mathcal{H}^1(K) + \frac{1}{2} \int_\Omega |u - g(x)|^2 \, dx \ : \ K \text{ fermé} \subset \Omega \ , \ u \in C^1(\Omega \setminus K) \right\}$$

Comme le terme source g satisfait $0 \le g \le 1$, en utilisant un argument trivial de troncature, on vérifie facilement que l'infimum de $(\mathcal{P})$ est inchangé si restreignons aux compétiteurs u à valeurs dans $[0,1]$. Nous considérons donc l'espace métrique $X = L^1(\Omega;[0,1])$ sur lequel nous définissons la fonctionnelle

$$F(u) := \begin{cases} \displaystyle\int_{\Omega\setminus S_u} \frac{1}{2}|\nabla u|^2\,dx + \mathcal{H}^1(S_u) + \frac{1}{2}\int_\Omega |u-g(x)|^2\,dx & \text{si } u \in SBV(\Omega;[0,1]) \\ +\infty & \text{sinon.} \end{cases}$$

Alors, F est coercive et s.c.i. La fonctionnelle convexe $G : L^\infty(\Omega\times[0,1]) \to [0,+\infty]$ associée à F via la construction de la Section 2.3 du Chapitre 2 (voir aussi la Remarque 2.12), est définie formellement par :

$$G(v) = \sup_{g\in L^\infty(\Omega\times[0,1])} \left\{ \int_{\Omega\times[0,1]} g\,v\,dxdt - F_0^*(g) \right\},\ F_0^*(g) = \sup_{u\in X} \left\{ \int_{\Omega\times[0,1]} g(x,t)\,\mathbf{1}_u\,dxdt - F(u) \right\}.$$

On a bien $G(\mathbf{1}_u) = F(u)$ pour tout $u \in X$ et $\inf(\mathcal{P}) = \inf\{G(v) \ :\ v \in L^\infty(\Omega\times[0,1];[0,1])\}$.

Malheureusement, cette fonctionnelle G abstraite ne peut pas être identifiée en utilisant la formule de co-aire (12) et, à notre connaissance, aucune formule explicite pour G n'est disponible. Alternativement, dans [1], une autre fonctionnelle s.c.i. convexe J a été utilisée telle que $J \le G$ mais satisfaisant $J(\mathbf{1}_u) = G(\mathbf{1}_u) = F(u)$ pour tout $u \in X$. Bien que l'on ne sait pas si J a le même infimum que G, un schéma de dualité appliqué à J a été utile notamment pour vérifier l'optimalité de certains compétiteurs pour le problème $(\mathcal{P})$ (voir plusieurs exemples dans [1]). Dans le cadre de la minimisation de J, le problème dual s'écrit comme suit :

$$(\mathcal{Q}) \qquad \sup\left\{ -\int_\Omega \sigma^t(x,0)\,dx \ :\ \sigma \in \mathcal{K},\ \operatorname{div}\sigma = 0 \text{ dans } \Omega\times[0,1],\ \sigma^x\cdot\nu_\Omega = 0 \text{ sur } \partial\Omega\times[0,1] \right\}$$

où la contrainte convexe $\sigma \in \mathcal{K}$ se divise en deux conditions :

i) $\frac{1}{2}|\sigma^x|^2 \le \sigma^t + \frac{1}{2}|t-g(x)|^2$ p.p. dans $\Omega\times[0,1]$

ii) $\left|\int_{t_1}^{t_2} \sigma^x(x,s)\,ds\right| \le 1$, p.p. $x\in\Omega$ et pour tout $(t_1,t_2)\in[0,1]^2$

La deuxième condition prend en compte l'énergie de saut dans $F(u)$ et *est non locale*. La fonctionnelle J définie dans $BV(\Omega\times\mathbb{R};[0,1])$ vérifie la formule de coaire et satisfait

$$J(v) = \sup\left\{ \int_{\Omega\times\mathbb{R}} (Dv\cdot\sigma) \ :\ \sigma\in\mathcal{K}\ ,\ \sigma\in C^1(\Omega\times\mathbb{R}) \right\}\ .$$

On a bien $J(\mathbf{1}_u) = F(u)$ pour tout $u \in L^1(\Omega)$ tandis que $J(v) < +\infty$ implique que $v(x,\cdot)$ est non croissant. On obtient :

Proposition 33. *Soit $g \in L^\infty(\Omega;[0,1])$. Alors, $\inf(\mathcal{P}) \ge \sup(\mathcal{Q})$ avec l'égalité si, pour une paire (u,σ) admissible, on a*

$$\begin{aligned} &\sigma(x,u(x)) = (\nabla u(x), \tfrac{1}{2}(|\nabla u|^2 - |u-g|^2)) \quad \text{p.p. } x\in\Omega \\ &\textstyle\int_{u^-(x)}^{u^+(x)} \sigma^x(x,t)\cdot\nu_u = 1 \quad \mathcal{H}^1\text{-p.p. } x\in S_u\ , \end{aligned} \tag{64}$$

où $u^\pm$ désignent les limites approximatives supérieures et inférieures de u, et $S_u = \{u^+ > u^-\}$.

Notons que le dernier résultat n'est utile que lorsqu'il est possible de deviner des couples particuliers (u,σ) satisfaisant les conditions (64). Quand un compétiteur u est candidat à être un minimiseur global, chercher un σ fournit une condition suffisante d'optimalité. Notons que,

grâce aux relations (64), un tel champ de vecteurs σ (s'il existe) a ses valeurs prescrites sur le graphe de u (du moins sa trace normale). La difficulté est de l'étendre en dehors du graphe de u tout en préservant les contraintes i) et ii) et la condition de divergence nulle. Nous référons à [1] pour des constructions explicites dans le cas de conditions de type Dirichlet. Malheureusement, il n'a pas été possible de trouver un champ de calibration permettant de démontrer l'optimalité d'une fonction du type crack-tip. En fait, l'existence d'une telle calibration ne peut être garantie que si la conjecture suivante est vérifiée :

Conjecture : L'égalité suivante a lieu : $\inf(\mathcal{P}) = \sup(\mathcal{Q})$.

Postulant à priori la validité d'une telle conjecture, des schémas numérique liés à un algorithme primal-dual ont été utilisés récemment pour résoudre le problème $(\mathcal{P})$ (voir [96]). Au meilleur de nos connaissances, aucun saut numérique réfutant la conjecture n'a jamais été mis en évidence.

Question d'unicité de la solution pour un problème multiphasique

Dans le Chapitre 3, nous avons prouvé (voir Théorème 3.3) que tous minimiseurs du problème non convexe sont des minimiseurs du problème convexifié. Cependant, l'énoncé inverse pourrait être faux bien que nous n'ayons aucun contre-exemple. A ce stade, se pose une question ouverte profonde : Est-il possible de montrer l'unicité pour le problème convexifié malgré le fait que le potentiel convexifié g^{**} n'est pas strictement convexe ? Remarquez que dans le cas intéressant que nous avons traité, g^{**} est affine par morceaux.

Critère d'optimalité pour les interfaces d'un probème multiphasique anisotrope

Dans le Chapitre 3, les conditions d'optimalité géométriques que nous avons trouvées pour les interfaces ne sont valables que dans le cas isotrope de la fonctionnelle de la variation totale (la relation entre la courbure algébrique des interfaces et les sauts respectifs du minimiseur u et de la densité $g(u)$ est donnée par l'équation (3.11)). Au vu des résultats de simulation numérique de la Section 3.3.2, il serait intéressant de trouver l'équivalent de la formule (3.11) en termes de conditions nécessaire d'optimalité.

Perspectives pour les applications numériques

Notre algorithme semi-implicite (dans lequel un Laplacien inverse est utilisé) s'avère efficace non seulement en terme de précision (finesse des interfaces) mais aussi en temps de calcul. De plus quand ce schéma numérique est combiné avec la méthode de projection de l'épigraphe, il permet de traiter de nombreux Lagrangiens non différentiables. Dans des travaux futurs, nous envisageons une étude complète sur l'ordre de convergence de cette méthode, en particulier en comparaison avec le schéma implicite classique. D'autre part, lorsque le problème considéré nécessite un schéma de dualité en dimension trois, le nombre d'itérations nécessaires pour obtenir des interfaces nettes est considérablement plus grand et dans ce cas un temps de calcul acceptable requiert l'utilisation d'une plate-forme de calcul haute performance (parallèle). Un objectif futur est de réduire ce temps de calcul en affinant notre approche globale lorsqu'on est proche d'un optimum global par des outils spécifiques au voisinage des interfaces.

Introduction

The search for global minima of non convex variational problems is a delicate issue. Due to the lack of convexity, an optimality criterion like Euler's equation or second order conditions is not sufficient to characterize a global solution among all critical points and even in the subclass of local minimizers. As a dramatic consequence, all classical optimization algorithms may converge to a local minimizer without any possibility to check whether it is indeed a global one. The difficulty is twofold: firstly, we need a theoretical framework providing necessary and sufficient optimality conditions; secondly, we have to propose new numerical methods which allow to rule out all local minimizers (or critical points) and to reach global minimizers in an acceptable computation time.

Motivated in such a context, this thesis aims to study and develop a general new convexification principle allowing to treat a specific but large class of non convex variational problems. Thanks to this principle, we are able to enforce the powerful duality techniques and bring back such problems to primal-dual formulations, thus making the numerical search for global minima efficient.

The thesis is structured into three parts, each of them consisting of several chapters. We collect in Part A (Chapter 1) some useful mathematical background supporting essentially the research of the topic. Part B is composed of Chapters 2, 3, 4. This part can be considered as the "backbone" theoretical part of the thesis. The numerical contributions are presented in Part C along Chapters 5, 6 in which we study the convergence of new algorithms and present their implementation on high performance computing platforms.

A short summary of each of the five chapters is given below:

- In Chapter 2 the general convexification principle is presented. We then extend the results of the duality theory obtained in the case of super linear growth functionals (defined in $W^{1,p}$, with $p > 1$, see [1, 19, 20, 22]) in the case of linear growth functionals (defined in $W^{1,1}$ and relaxed in BV). A theory of duality and calibration fields is reformulated in this framework taking into account the discontinuities of the competitors in the BV space.

- In Chapter 3 we are interested in a fairly large class of free boundary problems, particularly in the context of image processing where the convex part of the energy is of the total variation type. For this kind of problem (in dimension 2), we exploit the theory of 3d calibration fields to establish an exclusion principle that generalizes the one discovered by Visintin in the 1990s [107]. Thanks to this result, we can reduce a two-dimensional free boundary or multiphase problem to the minimization of a convex functional in dimension two. Numerical tests based on this minimization lead to interfaces of unexpectedly high quality.

- In Chapter 4, we apply the theory of calibrations to a classical problem of minimal surfaces with free boundary and establish new results from the comparison with its variant where the functional of the minimal surfaces is replaced by the total variation (the two functionals coincide when the surface is exclusively constituted of plateaus). We show that the two

problems are not identical in general, especially when the integral domain is not a Cheeger set. On this occasion, we generalize the concept of calibrability introduced by Caselles-Chambolle and Al. [4] and we give an explicit construction of a dual solution for the problem associated with the second functional. No regularity is required: the domain is any convex open subset of $\mathbb{R}^2$. The construction uses a locally Lipschitzian potential related to the distance to the *cut-locus*.

- The two last Chapters 5 and 6 of the thesis are devoted to the optimization algorithms used for the numerical validation of the convexification method that we proposed. This involves adapting the existing primal-dual algorithms for the search of saddle points, by introducing new variants which are more efficient in precision (fineness of the interfaces) and computational time (notably when the calibration fields of the dual problem are 3d). In particular, we introduce a new semi-implicit variant of the Arrow-Hurwicz method (based on an inverse Laplacian) which permits to reduce the number of necessary iterations to obtain a high quality of the interfaces. Moreover, we treat the structural non-differentiability of Lagrangians used with a geometric projection method on the epigraph, thus offering an alternative to classical regularization methods [95]. Numerous numerical tests are presented (2d or 3d) with a comparison between the different methods.

Part A

Mathematical background and notations

Chapter 1

Preliminaries

This chapter is devoted to recall basic mathematical notions which provide important analysis tools for the thesis. We begin in the first section with the direct method in Calculus of Variations and the notion of Γ-convergence. In the next section we collect some useful tools of Convex Analysis. Then, we summarize the properties of lower semicontinuity of functionals defined in Sobolev spaces and the relaxation techniques for linear-growth functional in BV space (some notions in Geometric Measure Theory will be invoked). The last sections of this chapter are added to recall the methods of convex duality, classical algorithms based on primal-dual formulations.

1.1 Direct method in calculus of variations. Γ-convergence

The *direct method* in the calculus of variations is a general method for deriving the existence of a minimizer for a given functional. This method rests upon two important notions : *lower semicontinuity* and *compactness* conditions (called sometimes *coerciveness* conditions), in order to show that *minimizing sequences* admit a subsequence converging to a minimizer of the functional. Let us briefly present these two notions in a general topological space. For a very extensive survey we refer to [27, 72, 25].

In the following, we denote by X a normed vector space and τ a topology on X. By convention, if we do not specify the topology τ, it is the topology induced by norm. We denote by $f : X \to \mathbb{R} \cup \{+\infty\}$ an extended real-valued function, $\operatorname{dom} f = f^{-1}(\mathbb{R})$ the domain of f. An extended real-valued function f is said to be *proper* if $\operatorname{dom} f \neq \emptyset$, in other words it is not identically $+\infty$. We shall use the notation $\{f \leq \alpha\} := \{x \in X : f(x) \leq \alpha\}$, and similar. We define the epigraph of f (denoted by $\operatorname{epi}(f)$) as the subset of $X \times \mathbb{R}$:

$$\operatorname{epi}(f) := \{(x, \alpha) \in X \times \mathbb{R} \ : \ f(x) \leq \alpha\}.$$

1.1.1 Lower semicontinuity

Definition 1.1. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x \in X$.

(i) f is τ-lower semicontinuous (shortly seq. τ-l.s.c.) at x if

$$\forall \alpha \in \mathbb{R}, (f(x) > \alpha) \Rightarrow (\exists U \in \mathcal{N}_x \ : \ \inf_U f > \alpha) \tag{1.1}$$

where $\mathcal{N}_x$ denotes the family of all τ-neighborhoods of x.

(ii) f is τ-sequentially lower semicontinuous (τ-sequentially l.s.c.) at x if

$$\forall (x_j) \subset X, (x_j \overset{\tau}{\to} x) \Rightarrow (f(x) \leq \liminf_{j\to+\infty} f(x_j)) \tag{1.2}$$

or in other words

$$f(x) = \min\left\{\liminf_{j\to+\infty} f(x_j) \ : \ x_j \overset{\tau}{\to} x\right\}. \tag{1.3}$$

We see that if f is τ-l.s.c. at x, then f is τ-sequentially l.s.c. at x. The converse is true if x admits a countable basis of neighborhoods, thus the two notions are equivalent if for instance the topology τ is metrizable.

We say that f is τ-(sequentially) lower semicontinuous on X if it is τ-(sequentially) l.s.c. at all $x \in X$. And a function f is said to be τ-upper semicontinuous (τ-u.s.c.) if $-f$ is τ-l.s.c.

Remark 1.2. It can be checked that a function f is lower semicontinuous on X if and only if every lower level set $\{f \leq \alpha\}$ is closed or if and only if its epigraph is a closed subset of $X \times \mathbb{R}$.

By exploiting one of the equivalent properties in previous remark, it is easy to deduce the following stability property

Lemma 1.3. *If $(f_i)_{i\in I}$ is a family of τ-l.s.c. functions, then*

$$f(x) = \sup_{i\in I} f_i(x)$$

is τ-l.s.c.

Theorem 1.4. *Let τ be a topology on X such that (X, τ) is a compact topological space. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a τ-l.s.c function on X, then*

$$\operatorname{Argmin} f := \left\{x \in X \ : \ f(x) = \inf_X f\right\}$$

is a nonempty closed subset of X.

Proof. Let $C = f(X) \subseteq (-\infty, +\infty]$, and for $c \in C$ let $F_c = \{x \in X \ : \ f(x) \leq c\}$. Since f is τ-l.s.c., F_c is a τ-closed set. Suppose $\{c_1, c_2, ..., c_n\} \subset C$. By taking $c = \min\{c_k \ : \ 1 \leq k \leq n\} \in C$, we have

$$\bigcap_{k=1}^{n} F_{c_k} = F_c \neq \emptyset.$$

This shows that the collection $\{F_c \ : \ c \in C\}$ has the finite intersection property (the intersection of finitely many members of its is nonempty). But a topological space is compact if and only if for every collection of closed subsets with the finite intersection property the intersection of all the members of the collection is nonempty [91]. Applying this theorem, we get that

$$\bigcap_{c\in C} F_c \neq \emptyset.$$

and this intersection is closed because each member is closed.

Let $x \in \bigcap_{c\in C} F_c$. Then for all $c \in C$ we have $f(x) \leq c$, and because $C = f(X)$ this means that for all $y \in X$ we have $f(x) \leq f(y)$. Therefore, $x \in \operatorname{Argmin} f$. Now let $x \in \operatorname{Argmin} f$. Then for all $y \in X$ we have $f(x) \leq f(y)$, hence for all $c \in C$ we have $f(x) \leq c$, hence $x \in \bigcap_{c\in C} F_c$. Therefore $\operatorname{Argmin} f = \bigcap_{c\in C} F_c$, which we have shown is nonempty and closed, proving the claim. ■

Obviously the compactness assumption in Theorem 1.4 can be weakened by requiring the existence of a τ-compact subset K of X such that $\inf_K f = \inf_X f$. In practice, for applying the existence result, we need to find a topology τ for which this condition is fulfilled together with the lower semicontinuity of f.

1.1.2 Coerciveness conditions

We recall that a set $K \subset X$ is τ-precompact (or τ-relatively compact) in X if its closure is τ-compact. K is τ-sequentially precompact if

$$\forall (x_j) \subset K \; \exists (x_{j_k}) : x_{j_k} \; \tau\text{-converges in } X.$$

And K is said to be τ-sequentially compact if

$$\forall (x_j) \subset K \; \exists x \in K, \exists (x_{j_k}) : x_{j_k} \xrightarrow{\tau} x.$$

Definition 1.5. Let $f : X \to \mathbb{R} \cup \{+\infty\}$.

(i) f is τ-coercive if for all $\alpha \in \mathbb{R}$ the sublevel set $\{f \leq \alpha\}$ is τ-precompact;

(ii) f is τ-mildly coercive if there is a non-empty τ-compact set $K \subset X$ such that

$$\inf_X f = \inf_K f.$$

Remark 1.6. We observe that if f is τ-l.s.c. and τ-coercive, then for all $\alpha \in \mathbb{R}$ the set $\{f \leq \alpha\}$ is τ-compact. Besides, if f is τ-coercive then it is τ-mildly coercive. In fact, if f is not identically $+\infty$, then there exists $\alpha \in \mathbb{R}$ such that $\{f \leq \alpha\}$ is not empty, and K may be taken as the closure of $\{f \leq \alpha\}$ in X (when $f \equiv +\infty$, we can take K as any τ-compact subset of X). Notice that τ-mild coerciveness does not implies τ-coerciveness in general. For instance, any periodic function $f : \mathbb{R}^N \to \mathbb{R}$ is a mildly coercive function, but it is non-coercive. An intermediate condition between coerciveness and mild coerciveness is: there exists $\alpha \in \mathbb{R}$ such that $\{f \leq \alpha\}$ is non empty and τ-precompact.

Theorem 1.7. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function such that*

(i) f is τ-l.s.c.;

(ii) f is τ-mildly coercive.

Then there is $\overline{x} \in X$ such that $f(\overline{x}) = \inf_X f$ and the set Argmin f *is τ-compact.*

Theorem 1.7 is a direct consequence of Theorem 1.4. In fact, as f is τ-mildly coercive, there is a non-empty τ-compact subset K of X such that $\inf_X f = \inf_K f$. Then the restriction $f|_K$ is still τ-l.s.c. on the τ-compact K, hence f attains its minimum in K and the set $\text{Argmin}_X f = \text{Argmin}_K f$ is τ-compact.

As mentioned in Remark 1.6, the condition (ii) in Theorem 1.7 can be replaced by that f is τ-coercive. When τ is a sequential topology, we can use the sequential notions of lower semicontinuity and coerciveness instead. Indeed, in order to minimize a function on a sequential topological space, we need only the lower semicontinuity and the coerciveness of f along the so-called minimizing sequences.

Definition 1.8. A sequence $(x_j) \subset X$ is called a minimizing sequence for the function f on X if

$$\lim_{j \to +\infty} f(x_j) = \inf_X f.$$

Such a minimizing sequence always exists thanks to the definition of $\inf_X f$. We shall prove in the following Theorem 1.7 within a sequential topology τ (when τ is a general topology on X, the direct method might also be used for Theorem 1.7, just replacing sequences by nets and subsequence by cofinal subnets). Despite of repetition, the proof is short and very useful in practice, which provides the usual direct way to show the existence of minimizers for a minimum problem.

Proof of Theorem 1.7. Suppose that τ is a sequential topology. Let K be a τ-compact subset of X such that $\inf_K f = \inf_X f$. For every minimizing sequence for f on K, by possibly extracting a converging subsequence, we find a sequence $(x_j) \subset K$ such that $f(x_j) \to \inf_X f$ and $x_j \xrightarrow{\tau} \overline{x} \in K$. Then the lower semicontinuity of f induces

$$\inf_X f \leq f(\overline{x}) \leq \liminf_{j\to+\infty} f(x_j) = \inf_X f.$$

It completes the proof with $\overline{x} \in \operatorname{Argmin} f$. ■

It is figured out that the combination of lower semicontinuity and coerciveness conditions assures the existence of minimum points. This type of argument based on the use of minimizing sequences is referred to as the direct method of the calculus of variations (as we have already mentioned at the beginning of the section), which allows to construct a minimizing sequence converging to a minimum point of a given function. When dealing with a differentiable functional f defined on a topological vector space, the indirect method consists in studying the minimization problem via the Euler equation $f'(x) = 0$.

1.1.3 Relaxation

In many cases, f does not own the lower semicontinuity property. In such cases, minimum points of f may not exist. Nevertheless, we are interested in charcaterizing the limit points of minimizing sequences. To that aim, we introduce the so-called *l.s.c. envelop* of f whose existence follows from Lemma 1.3.

Definition 1.9. Let $f : X \to \mathbb{R} \cup \{+\infty\}$. We define the relaxed function $\overline{f}^{\tau}$ and the sequentially relaxed function $\overline{f}^{\tau\,\mathrm{seq}}$ by, for every $x \in X$:

$$\overline{f}^{\tau}(x) := \{g(x) \mid g : X \to \mathbb{R} \cup \{+\infty\},\ g \text{ is } \tau\text{-l.s.c. },\ g \leq f\},$$
$$\overline{f}^{\tau\,\mathrm{seq}}(x) := \inf\left\{\liminf_{j\to+\infty} f(x_j)\ :\ x_j \xrightarrow{\tau} x\right\}.$$

The relaxed function $\overline{f}^{\tau}$ is also called the τ-l.s.c. envelope of f (similarly, $\overline{f}^{\tau\,\mathrm{seq}}$ is the τ-sequentially l.s.c. envelope of f).

By Lemma 1.3, the function $\overline{f}^{\tau}$ is τ-l.s.c. and it is the greatest g τ-l.s.c. with $g \leq f$. Obviously, $\overline{f}^{\tau} = f$ if and only if f is τ-l.s.c. The function $\overline{f}^{\tau\,\mathrm{seq}}$ is τ-sequentially l.s.c. When τ is sequential, its holds for every $x \in X$,

$$\overline{f}^{\tau}(x) = \overline{f}^{\tau\,\mathrm{seq}}(x) = \liminf_{y\xrightarrow{\tau}x} f(y) = \min\left\{\liminf_{j\to+\infty} f(x_j)\ :\ x_j \xrightarrow{\tau} x\right\}.$$

Theorem 1.10. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, then*

(i) $\forall x \in X,\ \overline{f}^{\tau}(x) = \sup_{U\in\mathcal{N}_x} \inf_U f$,

(ii) $\operatorname{epi} \overline{f}^{\tau} = \operatorname{cl}_{X\times\mathbb{R}}(\operatorname{epi} f)$,

(iii) $\inf_X f = \inf_X \overline{f}^{\tau}$,

(iv) if f is τ-coercive then $\overline{f}^{\tau}$ attains its minimum and

$$\operatorname{Argmin} f = \operatorname{Argmin} \overline{f}^{\tau} \cap \left\{x \in X\ :\ f(x) = \overline{f}^{\tau}(x)\right\}.$$

We remark that the minimum points for $\overline{f}^{\tau}$ are exactly all the limits of converging sequences (x_j) such that $\lim_j f(x_j) = \inf_X f$.

1.1.4 Γ-Convergence

Following what has just been mentioned in Section 1.1.3, the minimum points of the relaxed function $\overline{f}^{\tau}$ are actually the cluster points of minimizing sequences of f. Thus, it is attracting to study the behavior of minimizing sequences of f. For certain reason arsing in concrete problems, one is led to consider the behavior of minimizing sequences of a family of functions. To that aim, the notion of Γ-*convergence* was introduced in the literature by De Giorgi in 1970s. In the very special case of Γ-convergence, the relaxed function $\overline{f}^{\tau}$ is indeed a $\Gamma(\tau)$-limit of the constant family (f_j) with $f_j = f$ for all $j \in \mathbb{N}$. We recall in the following the definition and some important properties of Γ-convergence. For more details on the proof of the results listed below and the complete view of this topic, we refer to the book of Dal Maso [45] or other references [7, 27, 25].

Definition 1.11. Given a sequence $f_j : X \to \mathbb{R} \cup \{+\infty\}$, we define its $\Gamma(\tau)$-liminf and $\Gamma(\tau)$-limsup as the following function from X to $\mathbb{R} \cup \{+\infty\}$:

$$\left(\Gamma(\tau)\text{-}\liminf_{j\to+\infty} f_j\right)(x) := \sup_{U\in\mathcal{N}_x} \liminf_{j\to+\infty} \inf_{y\in U} f_j(y)$$

$$\left(\Gamma(\tau)\text{-}\limsup_{j\to+\infty} f_j\right)(x) := \sup_{U\in\mathcal{N}_x} \limsup_{j\to+\infty} \inf_{y\in U} f_j(y).$$

If there is $F : X \to \mathbb{R} \cup \{+\infty\}$ such that

$$f = \Gamma(\tau)\text{-}\liminf_{j\to+\infty} f_j = \Gamma(\tau)\text{-}\limsup_{j\to+\infty} f_j$$

we say that the sequence f_j $\Gamma(\tau)$-converges to the $\Gamma(\tau)$-limit F, and we write $f_j \xrightarrow{\Gamma} f$.

It is clear that $\Gamma(\tau)\text{-}\liminf_j f_j \le \Gamma(\tau)\text{-}\limsup_j f_j$. The sequence f_j $\Gamma(\tau)$-converges to$f$ if and only if

$$\Gamma(\tau)\text{-}\limsup_j f_j \le f \le \Gamma(\tau)\text{-}\liminf_j f_j.$$

Remark 1.12. When (X, τ) is metrizable, f_j $\Gamma(\tau)$-converges to f on X if and only if the two assertions below hold

(i) for all $x \in X$, for every sequence x_j τ-converging to x, we have

$$f(x) \le \liminf_{j\to+\infty} f_j(x_j);$$

(ii) for all $x \in X$, there is a sequence x_j τ-converging to x such that

$$\limsup_{j\to+\infty} f_j(x_j) \le f(x).$$

Notice that the statement (ii) above can be replaced by:

(ii') for all $x \in X$, there exists a sequence x_j τ-converging to x such that

$$f(u) = \lim_{j\to+\infty} f_j(x_j).$$

We have the following compactness Theorem due to Kuratowski.

Theorem 1.13. *In a separable metric space every sequence f_j admits a Γ-converging subsequence.*

In the following we collect some important properties of Γ-limits.

Definition 1.14. We say that a sequence (f_j) is τ-equicoercive in X if for every $r \in \mathbb{R}$ there exists a τ-compact subset $K_r \subset X$ such that $\{f_j \leq r\} \subset K_r$ for all $j \in \mathbb{N}$.

In practice, we often use the following characterization for τ-equicoerciveness which is easier to handle (this characterization can be found in [45, Theorem 7.7]): A sequence (f_j) is τ-equicoercive if and only if there exists a τ-coercive and τ-l.s.c. function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ such that $f_j \geq \varphi$ on X for all $j \in \mathbb{N}$.

Proposition 1.15. *If f_j $\Gamma(\tau)$-converges to f, we have*

(i) for all $x \in X$, $f(x) = \inf\limits_{x_j \overset{\tau}{\to} x} \liminf\limits_{j\to+\infty} f_j(x_j)$;

(ii) the lower and upper $\Gamma(\tau)$-limits are τ-l.s.c. functions on X. In particular, if the $\Gamma(\tau)$-limit exists, then it is τ-l.s.c. on X;

(iii) (stability) if g is τ-continuous in X, then the sequence $(f_j + g)$ $\Gamma(\tau)$-converges to $f + g$ in X;

(iv) if a sequence of τ-equicoercive functions f_j satisfying

$$\lim_{j\to+\infty} \min_X (f_j + g) = \min_X (f + g)$$

for all g being τ-continuous and bounded from below, then $f_j \overset{\Gamma(\tau)}{\longrightarrow} f$;

(v) If (f_j) admits a minimizing sequence τ-relatively compact, then all limit points of this sequence minimizes f, in other words

$$\liminf_{j\to+\infty} \inf_X f_j = \min_X f.$$

Proposition 1.16. *The following facts hold true*

$$\Gamma(\tau)\text{-}\liminf_j f_j = \Gamma(\tau)\text{-}\liminf_j \overline{f}_j^{\tau}, \quad \Gamma(\tau)\text{-}\limsup_j f_j = \Gamma(\tau)\text{-}\limsup_j \overline{f}_j^{\tau}.$$

In particular, a sequence (f_j) $\Gamma(\tau)$-converges to f if and only if its relaxed sequence $(\overline{f}_j^{\tau})$ $\Gamma(\tau)$-converges to f.

The main result about the convergence of minima and minimizers of a Γ-convergent sequence of functions (or functionals) is summarized in the next theorem:

Theorem 1.17. *Let $f_j : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functions on (X, τ) such that:*

i) (f_j) is τ-equicoercive,

ii) $f_j \overset{\Gamma}{\to} f$ in (X, τ).

Then, for every minimizing sequence (x_j) (i.e. such that $f_j(x_j) - \inf_X f_j \to 0$), there exists a converging subsequence x_{j_k} such that

$$x_{j_k} \overset{\tau}{\to} x \quad , \quad x \in \operatorname{Argmin} f \quad , \quad f(x) = \lim_{k\to+\infty} f_{j_k}(x_{j_k}).$$

1.2 Convex analysis

We recall in this section some important properties of convex functions. As we know, convexity plays an important role in the search for minimum points of functions. One of the most powerful technical tools is the notion of Moreau-Fenchel conjugate. We quote [98, 100, 57, 14] for the details on this topic.

In the following, we denote by X a normed vector space, X^* the topological dual of X, and $\langle \cdot, \cdot \rangle$ the associated duality pairing. Let us first recall the definition of a convex function defined on X.

Definition 1.18. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex if for every $x, y \in X$ and for every $t \in [0, 1]$, there holds

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

f is called strictly convex if the inequality above is satisfied with the $<$ inequality.

A function f is convex if and only if its epigraph $\operatorname{epi} f$ is a convex set in the product space $X \times \mathbb{R}$. Convex functions enjoy nice continuity properties. We refer to [11] for a proof of

Theorem 1.19. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex and proper. If there is an open subset U of X such that $\sup_U f < +\infty$, then f is continuous and locally Lipschitzian on $\operatorname{int}(\operatorname{dom} f)$.*

The following important result is a well-known consequence of Hahn-Banach Theorem.

Theorem 1.20. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex and proper. Then f is l.s.c. with respect to the strong topology induced by norm in X if and only if f is l.s.c. with respect to the weak topology $\sigma(X, X^*)$.*

1.2.1 Moreau-Fenchel conjugate

Definition 1.21. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper. The *Moreau-Fenchel conjugate* $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}.$$

Analogously, if f^* is proper, we then introduce the *biconjugate* $f^{**} : X \to \mathbb{R} \cup \{+\infty\}$ as

$$f^{**}(x) := \sup_{x^* \in X^*} \{\langle x, x^* \rangle - f^*(x^*)\}.$$

Notice that in the definition above f does not need to be convex. A direct consequence is the so called Fenchel inequality

$$f(x) + f^*(x^*) \geq \langle x^*, x \rangle, \quad \forall x \in X, \forall x^* \in X^*.$$

Definition 1.22. Let $f : X \to \mathbb{R} \cup \{+\infty\}$. The *subdifferential* of f at $x \in X$ is the possibly empty subset $\partial f(x) \subset X^*$ defined by

$$\partial f(x) := \{x^* \in X^* \ : \ f(x) + f^*(x^*) = \langle x, x^* \rangle\}.$$

Clearly, $x^* \in \partial f(x)$ is equivalent to $x \in \partial f^*(x^*)$. It can be easily checked that $\partial f(x)$ is convex weak-* closed. If f is convex and continuous at $x \in X$ then $\partial f(x)$ is a nonempty and weakly-* compact subset of X^*. Additionally, whenever f is convex and has a differential (or Gâteaux derivative) $f'(x)$ at x, the subdifferential of f reduces to $\partial f(x) = \{f'(x)\}$. We summarize in the following basic properties of the Fenchel transform and then we give some examples.

Proposition 1.23. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper.*

(i) f^* is convex and $\sigma(X^*, X)$-l.s.c.,

(ii) $f^*(0) = -\inf_X f$,

(iv) if $f \geq g$ then $f^* \leq g^*$,

(v) $f^{**} \leq f$.

Example 1.24. Let $f : X \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{p}\|x\|_X^p$.

(a) If p is such that $1 < p < +\infty$ then

$$f^*(x^*) = \frac{1}{p'}\|x^*\|_{X^*}^{p'}$$

where p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

(b) If $p = 1$ the Moreau-Fenchel conjugate f^* coincides with the indicator function of the unit ball in X^*, that means

$$f^*(x^*) = \chi_{B^*}(x^*)$$

with $B^* = \{x^* \in X^* \ : \ \|x^*\|_{X^*} \leq 1\}$.

Example 1.25. Let χ_C be the indicator function of a nonempty closed convex subset $C \subset X$. The Moreau-Fenchel conjugate of χ_C is the support function of C, i.e.,

$$(\chi_C)^*(x^*) = h_C(x^*) = \sup_{x \in C} \langle x^*, x \rangle.$$

Example 1.26 (Integral functionals)**.** Let $1 < p < +\infty$ and $I_\varphi : (L^p_\mu)^d \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$u \mapsto I_\varphi(u) := \int_{\mathbb{R}^N} \varphi(x, u(x)) d\mu$$

where $\varphi : \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty]$ is a measurable integrand with respect to the product σ-algebra $\mathfrak{S}_\mu(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^d)$. Assume that I_φ is proper. Then, the Moreau-Fenchel conjugate of I_φ is given by

$$\begin{aligned} (I_\varphi)^* : (L^{p'}_\mu)^d &\to \mathbb{R} \cup \{+\infty\} \\ v &\mapsto \int_{\mathbb{R}^N} \varphi^*(x, v(x)) d\mu \end{aligned}$$

where $\varphi^*(x, z^*) = \sup \left\{ \langle z^*, z \rangle - \varphi(x, z) \ : \ z \in \mathbb{R}^d \right\}$ (notice that $\varphi^*(x, z^*)$ is a convex measurable integrand).

We recall here the two key results which are widely used in duality methods. For the proof of these results, we refer to [14].

Theorem 1.27. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex and proper. Then*

(i) f is l.s.c. if and only if $f^{**} = f$.

(ii) If f^* is proper (i.e., $\exists x_0^* \in X^*, \exists r \in \mathbb{R}, \forall x \in X, f(x) \geq \langle x_0^*, x \rangle - r$), then $f^{**} = \overline{f}$.

Theorem 1.28. *Let X be a normed space and let $f : X \to [0, +\infty]$ be a convex and proper function. Assume that f is continuous at 0, then*

(i) f^* *achieves its minimum on* X^*.

(ii) $f(0) = f^{**}(0) = -\inf_{X^*} f^*$

In this paragraph, we recall an important result of convex duality theory shown in Theorem 1.31. Before stating that result, let us invoke two properties of Fenchel conjugation with respect to summing and composing functions:

Proposition 1.29 (Conjugate of a sum)**.** *Let* $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *be convex such that*

$$\exists x_0 \in X \ : \ f \text{ is continuous at } x_0 \text{ and } g(x_0) < +\infty. \tag{1.4}$$

Then :

(i) $(f+g)^*(x^*) = \inf_{x_1^*+x_2^*=x^*} \{f^*(x_1^*) + g^*(x_2^*)\}$,

(ii) If both sides of the equality in (i) are finite, then the infimum in the right-hand side is achieved.

Proposition 1.30 (Composition)**.** *Let* X, Y *two Banach spaces and* $A : X \to Y$ *a linear operator with dense domain* $\mathrm{dom}(A)$*. Let* $\Psi : Y \to \mathbb{R} \cup \{+\infty\}$ *be a convex, l.s.c. function and let* $F : X \to \mathbb{R} \cup \{+\infty\}$ *be the convex functional defined by*

$$F(u) = \Psi(Au) \quad \text{if } u \in \mathrm{dom}(A), \qquad F(u) = +\infty \quad \text{otherwise}.$$

Assume that there exists $u_0 \in \mathrm{dom}\, A$ *such that* Ψ *is continuous at* Au_0*. Then :*

(i) The Fenchel conjugate of F *is given by*

$$\forall f \in X^*, \quad F^*(f) = \inf\{\Psi^*(\sigma) \ : \ \sigma \in Y^*, \ A^*\sigma = f\},$$

where, if both sides of the above equality are finite, the infimum in the right-hand side is achieved.

(ii) If in addition Y *is reflexive and* Ψ *is l.s.c., coercive, we have*

$$\overline{F}(u) = F^{**}(u) = \inf\{\Psi(p) \ : \ (u,p) \in \overline{G(A)}\},$$

where $G(A)$ *denote the graph of* A*.*

By combing two Propositions 1.29 and 1.30, one can prove the following duality result. It then can be applied to primal-dual formulations of convex variational problems. We can find in [14, 26] a proof of this powerful result.

Theorem 1.31. *Let* X *and* Y *be two Banach spaces and let be given the following functions:*

- *a linear operator* $A : X \to Y$ *with dense domain* $\mathrm{dom}\, A$*;*
- *a convex function* $\Phi : X \to \mathbb{R} \cup \{+\infty\}$*;*
- *a convex l.s.c function* $\Psi : Y \to \mathbb{R} \cup \{+\infty\}$ *which is continuous at some point* Au_0 *with* $u_0 \in \mathrm{dom}(A)$*;*

then

$$\inf_{u \in X} \{\Psi(Au) + \Phi(u)\} = \sup_{\sigma \in Y^*} \{-\Psi^*(\sigma) - \Phi^*(-A^*\sigma)\}$$

where the supremum on the right hand side is achieved. Furthermore, a pair $(\overline{u}, \overline{\sigma})$ *is optimal for the left hand side and the right hand side respectively if and only if it satisfies the relations*

$$\overline{\sigma} \in \partial\Psi(A\overline{u}) \quad \text{and} \quad -A^*\overline{\sigma} \in \partial\Phi(\overline{u}).$$

1.2.2 Convex optimization

Let X be a reflexive Banach space with norm denoted by $\|\cdot\|$, and $\mathcal{C}$ be a closed convex subset of X. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function on $\mathcal{C}$. We are interested in the constraint optimization problem

$$\inf_{x \in \mathcal{C}} f(x) \tag{1.5}$$

We call solution for problem (1.5) all element $\overline{x} \in \mathcal{C}$ such that $f(\overline{x}) = \inf_{x \in \mathcal{C}} f(x)$, and denote the set of these solutions by

$$\operatorname{Argmin}_{\mathcal{C}} f := \left\{ \overline{x} \in \mathcal{C} \ : \ f(\overline{x}) = \inf_{x \in \mathcal{C}} f(x) \right\}.$$

We will drop the subscript $\mathcal{C}$ whenever $\mathcal{C} \equiv X$, i.e. Argmin f instead. Notice that in many cases we prefer to replace the problem (1.5) by a minimum problem over the entire space X, to that aim, we associate f and the convex $\mathcal{C}$ with the functional $\widetilde{f} : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\widetilde{f}(x) = f(x) + \chi_{\mathcal{C}}(x).$$

Note that if f is convex l.s.c. function, then so is $\widetilde{f}$. Moreover, we have

$$\inf_{x \in \mathcal{C}} f(x) = \inf_{x \in X} \widetilde{f}(x) \quad \text{and} \quad \operatorname{Argmin}_{\mathcal{C}} f = \operatorname{Argmin} \widetilde{f}.$$

After summarizing two important characterizations of minimizers of Gâteaux differentiable functions in the two following propositions, we present an application to proximal approximations which is widely exploited in numerical methods.

Proposition 1.32. *Let f be convex l.s.c. function and be Gâteaux differentiable with f' continuous. Then, for $\overline{x} \in \mathcal{C}$, the three following conditions are equivalent*

(i) $\overline{x}$ is solution for (1.5);

(ii) $\langle f'(\overline{x}), x - \overline{x} \rangle \geq 0$ for all $x \in \mathcal{C}$;

(iii) $\langle f'(x), x - \overline{x} \rangle \geq 0$ for all $x \in \mathcal{C}$.

Proposition 1.33. *We suppose that $f = f_1 + f_2$ with $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ being convex l.s.c., and f_1 being Gâteaux differentiable (with Gâteaux derivative f_1'). Then, for $\overline{x} \in \mathcal{C}$, the three following conditions are equivalent*

(i) $\overline{x}$ is solution for (1.5);

(ii) $\langle f_1'(\overline{x}), x - \overline{x} \rangle + f_2(x) - f_2(\overline{x}) \geq 0$ for all $x \in \mathcal{C}$;

(iii) $\langle f_1'(x), x - \overline{x} \rangle + f_2(x) - f_2(\overline{x}) \geq 0$ for all $x \in \mathcal{C}$.

The inequalities in Proposition 1.32 (ii),(iii), Proposition 1.33 (ii),(iii) are called variational inequalities.

Example 1.34 (Application to proximal approximation)**.** We consider the function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \frac{1}{2}\|x - y\|^2 + \varphi(x)$$

where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is convex l.s.c. and proper, and y is given in X. Function $f_1 = \frac{1}{2}\|x - y\|^2$ is strictly convex and l.s.c. Hence, f is strictly convex l.s.c. On the other hand, f

is coercive. In fact, since φ admits a continuous affine minorant written as $\langle z, x\rangle + \alpha$ for some $z \in X$, $\alpha \in \mathbb{R}$, we have

$$f(x) \geq \frac{1}{2}\|x-y\|^2 + \langle z, x\rangle + \alpha$$
$$f(x) \geq \frac{1}{2}\|x+z-y\|^2 - \frac{1}{2}\|z-y\|^2 + \frac{1}{2}\|y\|^2 + \alpha,$$

and it shows that $f(x) \to +\infty$ as $\|x\| \to +\infty$. We may then apply the results of existence and uniqueness of solution to conclude that: there exists unique $\overline{x} \in X$ such that

$$f(\overline{x}) = \frac{1}{2}\|\overline{x}-y\|^2 + \varphi(\overline{x})$$

According to Proposition 1.33, $\overline{x}$ is characterized by one of the following conditions

$$\langle \overline{x}-y, z-\overline{x}\rangle + \varphi(z) - \varphi(\overline{x}) \geq 0, \quad \forall z \in X,$$
$$\langle z-y, z-\overline{x}\rangle + \varphi(z) - \varphi(\overline{x}) \geq 0, \quad \forall z \in X,$$

The mapping $y \to \overline{x} = \overline{x}(y)$ from X to itself, introduced by Moreau [88] is called *proximal operator* associated with the convex l.s.c. φ, and denoted by

$$\overline{x} = \operatorname{prox}_{\varphi} y. \tag{1.6}$$

In particular, when $\varphi = \chi_{\mathcal{C}}$, the indicator function of a closed convex subset $\mathcal{C}$ in X, we find the projection on $\mathcal{C}$ (we write $\Pi_{\mathcal{C}}$) and the equivalent conditions to (1.6) become

$$\overline{x} \in \mathcal{C} \text{ and } \langle \overline{x}-y, z-\overline{x}\rangle \geq 0, \ \forall z \in \mathcal{C}, \tag{1.7}$$
$$\overline{x} \in \mathcal{C} \text{ and } \langle z-y, z-\overline{x}\rangle \geq 0, \ \forall z \in \mathcal{C}, \tag{1.8}$$
$$\overline{x} = \Pi_{\mathcal{C}}(y) \tag{1.9}$$

We refer to [108] for more discussion on properties of orthogonal projection.

1.3 Integral functionals in Sobolev spaces

A central interest in the calculus of variations is the search for extrema of minimization problems of the type

$$\inf\{F(u) \ : \ u \in K\}$$

where the class K of admissible functions u is a subset of some Banach space. Usually, in concrete problems $F(u)$ is an integral functional defined on L^p spaces. The essence of the calculus of variations is the identification of necessary and sufficient conditions on the functional $F(u)$ so that the existence of minimizers is guaranteed. These relies on suitable growth conditions which provide coercivity for some topology. The key point is to verify the lower semicontinuity for this topology. If it is not the case we need to substitute F with its relaxed functional $\overline{F}$. In this section, we gather some useful results which can be used to check rapidly the lower semicontinuity for functionals defined in Sobolev spaces. We refer to the books [64, 27] for a complete study on these results. The proof of theorems listed below can be found in [64].

In what follows, we assume that Ω is an open subset of $\mathbb{R}^N$.

1.3.1 Integral functionals with integrand $f(x,z)$

We summarize in this subsection the lower semicontinuity properties of the integral functional I_f defined by

$$L^p(\Omega;\mathbb{R}^m) \ni z \mapsto I_f(z) := \int_\Omega f(x,z(x))dx$$

where $f : \Omega \times \mathbb{R}^m \to [-\infty,+\infty]$ is an $\mathcal{L}^N \times \mathcal{B}$-measurable function ($\mathcal{B}$ stands for the σ-algebra of Borel subsets of $\mathbb{R}^m$) and $1 \le p < +\infty$. We say that I_f is well-posed in $L^p(\Omega;\mathbb{R}^m)$ if for every $z \in L^p(\Omega;\mathbb{R}^m)$ it holds

$$\int_\Omega f^-(x,z(x))dx < +\infty$$

where f^- denotes the negative part of f.

Theorem 1.35 (Well-posedness). *Let $1 \le p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to [-\infty,+\infty]$ be a $\mathcal{L}^N \times \mathcal{B}$-measurable function. Then I_f is well-posed in $L^p(\Omega;\mathbb{R}^m)$ if and only if there is a nonnegative function $\gamma \in L^1(\Omega)$ and a constant $C > 0$ such that*

$$f(x,z) \ge -C|z|^p - \gamma(x)$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^m$.

Theorem 1.36 (strong lower semicontinuity). *Let $1 \le p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to [-\infty,+\infty]$ be a $\mathcal{L}^N \times \mathcal{B}$-measurable function. Asumme that the functional I_f is well-posed in $L^p(\Omega;\mathbb{R}^m)$. Then I_f is l.s.c. with respect to the strong topology in $L^p(\Omega;\mathbb{R}^m)$ if and only if $f(x,\cdot)$ is l.s.c. in $\mathbb{R}^m$ for a.e. $x \in \Omega$.*

Corollary 1.37 (Strong continuity). *Let $1 \le p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to [-\infty,+\infty]$ be a $\mathcal{L}^N \times \mathcal{B}$-measurable function. Assume that there is a nonnegative function $\gamma \in L^1(\Omega)$ and a constant $C > 0$ such that*

$$|f(x,z)| \le C|z|^p + \gamma(x)$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^m$. Then, the functional I_f is continuous with respect to the strong convergence in $L^p(\Omega;\mathbb{R}^m)$ if and only if $f(x,\cdot)$ is continuous in $\mathbb{R}^m$ for a.e. $x \in \Omega$.

Theorem 1.38 (Weak lower semicontinuity). *Let $1 \le p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to [-\infty,+\infty]$ be a $\mathcal{L}^N \times \mathcal{B}$-measurable function. Assume that $f(x,\cdot)$ is l.s.c. in $\mathbb{R}^m$ for a.e. $x \in \Omega$ and that the funtional I_f is well-defined in $L^p(\Omega;\mathbb{R}^m)$. Then, I_f is sequentially l.s.c. with respect to the weak convergence in $L^p(\Omega;\mathbb{R}^m)$ if and only if the two following conditions hold true:*

(i) $f(x,\cdot)$ is convex in $\mathbb{R}^m$ for a.e. $x \in \Omega$;

(ii) there are two functions $a \in L^1(\Omega)$ and $b \in L^{p'}(\Omega;\mathbb{R}^d)$ such that

$$f(x,z) \ge a(x) + b(x) \cdot z$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^m$.

1.3.2 Integral functionals with integrand $f(x,u,z)$

In this subsection, we summarize lower semicontinuity properties of the integral functional I_f defined by

$$L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m) \ni (u,z) \mapsto I_f(u,z) := \int_\Omega f(x,u(x),z(x))dx$$

where $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [-\infty, +\infty]$ is an $\mathcal{L}^N \times \mathcal{B}$-measurable function ($\mathcal{B}$ now stands for the σ-algebra of Borel subsets of $\mathbb{R}^d \times \mathbb{R}^m$) and $1 \leq p < +\infty$. We say that I_f is well-posed in $L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$ if for every $(u, z) \in L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$ it holds

$$\int_\Omega f^-(x, u(x), z(x))dx < +\infty$$

where f^- denotes the negative part of f.

Theorem 1.39 (Well-posednes). *Let $1 \leq p, q < +\infty$ and $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [-\infty, +\infty]$ be $\mathcal{L}^N \times \mathcal{B}$-measurable function. Then, the integral functional I_f is well-posed if and only if there exist a nonnegative function $\omega \in L^1(\Omega)$ and a constant $C > 0$ such that*

$$f(x, u, z) \geq -C(|u|^q + |z|^p) - \omega(x)$$

for a.e. $x \in \Omega$ and for evey $(u, z) \in \mathbb{R}^d \times \mathbb{R}^m$

Theorem 1.40 (Strong-strong lower semicontinuity). *Let $1 \leq p, q < +\infty$ and $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [-\infty, +\infty]$ be $\mathcal{L}^N \times \mathcal{B}$-measurable function. Assume that I_f is well-posed in $L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$. Then, I_f is sequentially l.s.c. with respect to the strong-strong convergence in $L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$ if and only if $f(x, \cdot, \cdot)$ is l.s.c. in $\mathbb{R}^d \times \mathbb{R}^m$ for a.e. $x \in \Omega$.*

Theorem 1.41 (Strong-weak lower semicontinuity). *Let $1 \leq p, q < +\infty$ and $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [-\infty, +\infty]$ be $\mathcal{L}^N \times \mathcal{B}$-measurable function. Assume that: $f(x, \cdot, \cdot)$ is l.s.c. in $\mathbb{R}^d \times \mathbb{R}^m$ for a.e. $x \in \Omega$, I_f is well defined in $L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$ and there is $z_0 \in L^p(\Omega;\mathbb{R}^m)$ such that*

$$\int_\Omega f(x, u(x), z_0(x))dx < +\infty$$

for every $u \in L^q(\Omega;\mathbb{R}^d)$. Then, the functional I_f is sequentially l.s.c. with respect to the strong-weak convergence in $L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$ if and only if the following conditions hold true

(i) $f(x, u, \cdot)$ is convex in $\mathbb{R}^m$ for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}^d$;

(ii) there are a constant $C > 0$ and two functions $\alpha \in L^1(\Omega)$, $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^m$ $\mathcal{L}^N \times \mathcal{B}$-measurable such that

$$f(x, u, z) \geq \alpha(x) + \beta(x, u) \cdot z - C|u|^q$$

for a.e. $x \in \Omega$ and for every $(u, z) \in \mathbb{R}^d \times \mathbb{R}^m$;

(iii) there are a constant $C_1 > 0$ and a function $b_1 \in L^1(\Omega)$ such that

$$|\beta(x, u)|^{p'} \leq C_1|u|^q + b_1(x)$$

for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}^d$.

1.4 Relaxation in BV

We recall in this section the relaxation in BV of linear-growth functionals defined in $W^{1,1}$. As consequence, the relaxed optimization problems are established in BV, particularly, the minimum is attained under appropriate coerciveness conditions. For the more detailed discussion on this topic, we refer to [17, 27, 6, 63].

Let us begin by recalling the space of functions of bounded variation and collecting some fine properties of functions in this space.

1.4.1 The space BV

Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset.

Definition 1.42. A function $u \in L^1(\Omega)$ is said to be a *function of bounded variation* in Ω if the distributional derivative of u is representable by a finite Radon measure in Ω, i.e. if

$$\int_\Omega u \nabla \phi dx = -\int_\Omega \phi dDu \quad \forall \phi \in C_0^\infty(\Omega)$$

for some $\mathbb{R}^N$-valued measure Du in Ω. The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.

Definition 1.43. Let $u \in L^1_{\text{loc}}(\Omega)$. The *variation* $V(u,\Omega)$ *of* u *in* Ω is defined by

$$V(u,\Omega) := \sup\left\{ \int_\Omega u \operatorname{div} \phi dx \ : \ \phi \in C_0^1(\Omega;\mathbb{R}^N),\ \|\phi\|_\infty \le 1 \right\}.$$

Proposition 1.44. *Let $u \in L^1(\Omega)$. Then, $u \in BV(\Omega)$ if and only if $V(u,\Omega) < +\infty$. In addition, $V(u,\Omega)$ coincides with $|Du|(\Omega)$ for any $u \in BV(\Omega)$ and the map $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega)$ with respect to the L^1_{loc}-topology.*

In view of Proposition 1.44, $|Du|(\Omega)$ is sometimes call the variation of u in Ω. We notice that the vector space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

A specific class of BV functions is characteristic functions of sets of finite perimeter. We recall that the *characteristic function* $\mathbf{1}_E$ of a subset $E \subset \mathbb{R}^N$ is given by

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in \mathbb{R}^N \setminus E. \end{cases}$$

In the following, we define the perimeter of an $\mathcal{L}^N$-measurable subset in $\mathbb{R}^N$ and then gather some remarkable properties of perimeter.

Definition 1.45 (Sets of finite perimeter)**.** Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For any open set $\Omega \subset \mathbb{R}^N$ the *perimeter of* E *in* Ω, denoted by $P(E,\Omega)$, is the variation of $\mathbf{1}_E$ in Ω, namely

$$P(E,\Omega) := \sup\left\{ \int_E u \operatorname{div} \phi dx \ : \ \phi \in C_0^1(\Omega;\mathbb{R}^N),\ \|\phi\|_\infty \le 1 \right\}.$$

We say that E is a *set of finite perimeter in* Ω if $P(E,\Omega) < +\infty$.

Theorem 1.46. *For any set E of finite perimeter in Ω, the distributional derivative $D\mathbf{1}_E$ is an $\mathbb{R}^N$-valued finite Radon measure in Ω. Moreover, $P(E,\Omega) = |D\mathbf{1}_E|(\Omega)$ and a generalized Gauss-Green formula holds*

$$\int_E \operatorname{div} \phi dx = -\int_\Omega (\nu_E \cdot \phi) d|D\mathbf{1}_E| \quad \forall \phi \in C_0^1(\Omega;\mathbb{R}^N)$$

where $D\mathbf{1}_E = \nu_E |D\mathbf{1}_E|$ is the polar decomposition of $D\mathbf{1}_E$.

Proposition 1.47 (Properties of perimeter)**.**

(i) The function $\Omega \mapsto P(E,\Omega)$ is the restriction to open sets of a Borel measure in $\mathbb{R}^N$.

(ii) $E \mapsto P(E,\Omega)$ is lower semicontinuous with respect to local convergence in measure in Ω.

(iii) $E \mapsto P(E,\Omega)$ *is local, i.e.* $P(E,\Omega) = P(F,\Omega)$ *whenever* $|\Omega \cap (E \Delta F)| = 0$.

(iv) $P(E,\Omega) = P(\mathbb{R}^N \setminus E, \Omega)$ *and*

$$P(E \cup F, \Omega) + P(E \cap F, \Omega) \leq P(E,\Omega) + P(F,\Omega).$$

Theorem 1.48 (Coarea formula in BV)**.** *For any open set* $\Omega \subset \mathbb{R}^N$ *and* $u \in L^1_{loc}(\Omega)$*, one has*

$$V(u,\Omega) = \int_{-\infty}^{+\infty} P(\{u > t\}, \Omega) dt.$$

In particular, if $u \in BV(\Omega)$ *the set* $\{u > t\}$ *has a finite perimeter in* Ω *for* $\mathcal{L}^1$*-a.e.* $t \in \mathbb{R}$ *and*

$$|Du|(B) = \int_{-\infty}^{+\infty} |D\mathbf{1}_{\{u>t\}}|(B) dt, \qquad Du(B) = \int_{-\infty}^{+\infty} D\mathbf{1}_{\{u>t\}}(B) dt$$

for any Borel set $B \subset \Omega$.

This paragraph is devoted to summarizing the notions of approximate continuity and differentiability of bounded variation.

Definition 1.49 (Approximate limits)**.** Let $u \in L^1_{\text{loc}}(\Omega)$. We say that u has an *approximate limit at* $x \in \Omega$ if there exist a $z \in \mathbb{R}$ such that

$$\lim_{\varepsilon \to 0^+} \fint_{B_\varepsilon(x)} |u(x) - z| dx = 0 \tag{1.10}$$

where $\fint_{B_\varepsilon(x)}$ stands for $\frac{1}{\mathcal{L}^N(B_\varepsilon(x))} \int_{B_\varepsilon(x)}$. The set D_u of points where this property does not hold is called the *approximate discontinuity set*. For any $x \in \Omega \setminus D_u$, z is uniquely determined by (1.10). We call z the approximate limit of u at x and denote by $\widetilde{u}(x)$. We say that u is approximately continuous at x if $x \notin D_u$ and $\widetilde{u}(x) = u(x)$, i.e. x is a Lebesgue point of u.

Notice that the set of points where the approximate limits exists does not depend on the representative in the equivalent class of u, but the property of being approximately continuous at x depends on the value of u at the point (this value could be different for functions in the same equivalent class).

Definition 1.50 (Approximate jump points)**.** Let $u \in L^1_{\text{loc}}(\Omega)$. We say that x is an *approximate jump point of* u if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{N-1}$ such that $a \neq b$ and

$$\lim_{\varepsilon \to 0^+} \fint_{B^+_\varepsilon(x,\nu)} |u(y) - a| dy = 0, \qquad \lim_{\varepsilon \to 0^+} \fint_{B^-_\varepsilon(x,\nu)} |u(y) - b| dy = 0, \tag{1.11}$$

where $B^\pm_\varepsilon(x,\nu) = x + \varepsilon\{y \in B_1(0) \ : \ \langle \nu, y \rangle \gtrless 0\}$. Up to a permutation of (a,b) and a change of sign of ν, the triple (a,b,ν) is uniquely determined by (1.11), we then denote by $(u^+(x), u^-(x), \nu_u(x))$ with a convention that $u^+(x) > u^-(x)$. The set of approximate jump points is denoted by S_u. The quantity $[u] := u^+ - u^-$ is the jump of u across the interface S_u and ν_u is the direction of the jump.

Definition 1.51 (Approximate differentiability)**.** Let $u \in L^1_{\text{loc}}(\Omega)$ and let $x \in \Omega \setminus D_u$. We say that u is *approximately differentiable at* x if there is a vector $P \in \mathbb{R}^N$ such that

$$\lim_{\varepsilon \to 0^+} \fint_{B_\varepsilon(x)} \frac{|u(y) - \widetilde{u}(x) - \langle P, y - x \rangle|}{\varepsilon} dx = 0. \tag{1.12}$$

If u is approximately differentiable at x, the vector P, uniquely determined by (1.12), is called the *approximate differential* (or *approximate gradient*) of u at x and denoted by $\nabla u(x)$.

Notice that S_u is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set contained in the approximate discontinuity set D_u. As recalled above, the space $BV(\Omega)$ consists of all functions of bounded variation in Ω, whose distributional derivative Du is a $\mathbb{R}^N$-valued Radon measure with total variation $|Du|$ bounded in Ω. The measure Du can be decompose into two parts: an absolutely continuous part (denoted by $\nabla u dx$) and a singular part (denoted by $D^s u$) with respect to the Lebesgue measure, namely

$$Du = \nabla u dx + D^s u.$$

In particular, denoting by D^c the *Cantor part* of the measure Du, one has

$$Du = \nabla u dx + D^c u + [u]\nu_u \mathcal{H}^{N-1} \llcorner S_u$$

where $[u] := u^+ - u^-$ is the jump of u. We say that a function $u \in BV(\Omega)$ is a *special function with bounded variation* if the Cantor part of its derivative $D^c u$ is zero. We denote by $SBV(\Omega)$ the space of all special functions with bounded variation. We obtain

$$Du = \nabla u dx + [u]\nu_u \mathcal{H}^{N-1} \llcorner S_u, \qquad \forall u \in SBV(\Omega).$$

The space $SBV(\Omega)$ was introduced by De Giorgi and Ambrosio in [49]. This space is to provide a weak formulation for some variational problems with free discontinuity or minimum problems characterized by a competition between volume energies, concentrated on N-dimensional sets, and surface energies concentrated on $(N-1)$-dimensional sets. The best-known example of a free discontinuity problem, proposed by D. Mumford and J. Shah in image segmentation, is the minimization of the functional

$$F(u) = \int_{\Omega \setminus S_u} \frac{1}{2}|\nabla u|^2 \, dx + \mathcal{H}^{N-1}(S_u) + \frac{1}{2}\int_\Omega |u - g(x)|^2 \, dx$$

where $g : \Omega \to [0,1]$ is a grey level data.

In the following, we recall two important boundary trace theorems of functions of bounded variation:

Theorem 1.52 (Boundary trace theorem)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set with bounded Lipschitz boundary and $u \in BV(\Omega)$. Then, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ there exists $\mathrm{Tr}(u)(x) \in \mathbb{R}$ such that*

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-N} \int_{\Omega \cap B_\varepsilon(x)} |u(y) - \mathrm{Tr}(u)(x)| dx = 0.$$

Moreover, $\|\mathrm{Tr}(u)\|_{L^1(\partial\Omega)} \le C\|u\|_{BV(\Omega)}$ for some constant $C > 0$ depending only on Ω. The extension $\overline{u}$ of u to 0 out of Ω belongs to $BV(\mathbb{R}^N)$ and, viewing Du as a measure on the whole of $\mathbb{R}^N$ and concentrated on Ω, $D\overline{u}$ is given by

$$D\overline{u} = Du + \mathrm{Tr}(u)\nu_\Omega \mathcal{H}^{N-1} \llcorner \partial\Omega.$$

Theorem 1.53 (Continuity of trace operator)**.** *Let Ω be an open subset of $\mathbb{R}^N$ with bounded Lipschitz boundary. Then, the trace operator $u \mapsto \mathrm{Tr}(u)$ is continuous between $BV(\Omega)$, endowed with the topology induced by strict convergence, and $L^1(\partial\Omega, \mathcal{H}^{N-1} \llcorner \partial\Omega)$.*

1.4.2 Integral representation of relaxed functionals

In this subsection, we gather some important properties of lower semicontinuity and relaxation for integral functionals of the type

$$F(u) := \begin{cases} \displaystyle\int_\Omega f(x,u,\nabla u)dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{if } u \in BV(\Omega) \setminus W^{1,1}(\Omega) \end{cases} \tag{1.13}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ is a Borel function, with $f(x,t,\cdot)$ convex on $\mathbb{R}^N$ for each $(x,t) \in \Omega \times \mathbb{R}$. First, let us recall a well-known result on the $L^1_{\rm loc}$-lower semicontinuity of F:

Theorem 1.54 (Serrin [101])**.** *Let $f = f(x,t,z)$ be continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, convex in the variable z, and satisfies one of the following conditions:*

(i) $f(x,t,z) \to +\infty$ as $z \to +\infty$ for each $(x,t) \in \Omega \times \mathbb{R}$.

(ii) $f(x,t,\cdot)$ is strictly convex in $\mathbb{R}^N$ for each $(x,t) \in \Omega \times \mathbb{R}$.

(iii) the derivatives $f_x(x,t,z)$, $f_z(x,t,z)$, $f_{xz}(x,t,z)$ exist and are continuous.

Then $F(u)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to L^1_{loc}-convergence.

The conditions given above may appear too stringent and many efforts have been done in order to find the minimal assumptions on f that ensure the $L^1_{\rm loc}$-lower semicontinuity of F on $W^{1,1}_{\rm loc}(\Omega)$. For instance, we quote [48, 50, 73]. In applications, it may happen that such a lower semicontinuity fails. Even if it is not the case, such a property is useless if we cannot ensure that sequences with uniformly bounded energy are weakly compact in $W^{1,1}_{\rm loc}$. This difficulty causes typically in optimization problems where the integrand f exhibits a linear growth condition. A fundamental issue consists then to identify the so called relaxed functional associated with F. Assuming that sequences with finite energy are relatively compact for the L^1-topology, this relaxed functional is defined by

$$\overline{F}(u) := \inf\left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n,\nabla u_n)dx \ : \ u_n \in W^{1,1}(\Omega), u_n \to u \text{ in } L^1(\Omega) \right\}.$$

Remark 1.55. It is easy to check that

$$\overline{F}(u) = \inf\left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n,\nabla u_n)dx \ : \ u_n \in W^{1,1}(\Omega), u_n \overset{\tau}{\to} u \right\}$$

for any topology τ such that $\{F \le R\}$ is τ-relatively compact for all $R > 0$.

A quite general result for the identification of $\overline{F}$ has been derived in [44]. Let us introduce:

Definition 1.56 (Recession function)**.** The *recession function* of f is given by

$$f^\infty(x,t,z) := \lim_{s\to+\infty} \frac{f(x,t,sz)}{s}.$$

It is easy to check that f^∞ is convex and positively homogenous of degree 1 with respect to the variable z. For example, $f^\infty(z) = |z|$ if $f(z) = \sqrt{1+|z|^2}$.

Assuming that Ω is an open bounded subset of $\mathbb{R}^N$, we set

$$\mathcal{F}(u) := \int_\Omega f(x,u,\nabla u)dx + \int_\Omega f^\infty\Big(x,\widetilde{u},\frac{dD^c u}{d|D^c u|}\Big)d|D^c u| + \int_{S_u}\int_{u^-(x)}^{u^+(x)} f^\infty\Big(x,s,\nu_u\Big)ds\, d\mathcal{H}^{N-1}$$

for every $u \in BV(\Omega)$. For simplicity, we sometimes use the following notation

$$\int_\Omega f\Big(x,u,Du\Big) := \int_\Omega f(x,u,\nabla u)dx + \int_\Omega f^\infty\Big(x,\widetilde{u},\frac{dD^c u}{d|D^c u|}\Big)d|D^c u| + \int_{S_u}\int_{u^-(x)}^{u^+(x)} f^\infty\Big(x,s,\nu_u\Big)ds\ d\mathcal{H}^{N-1}.$$

Then, we obtain :

Theorem 1.57 (Dal Maso [44]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,+\infty]$ be a Borel function such that:*

(i) for each $(x,t) \in \Omega\times\mathbb{R}$, the function $z \mapsto f(x,t,z)$ is convex in $\mathbb{R}^N$;

(ii) there exists a Borel subset B of $\Omega\times\mathbb{R}$, with $\mathcal{H}^N((\Omega\times\mathbb{R})\setminus B) = 0$, such that f is lower semicontinuous at each point of $B\times\mathbb{R}^N$;

(iii) there exists two constants $c > 0$, $d > 0$ such that

$$c|z| - d \le f(x,t,z)$$

for all $(x,t,z) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$;

(iv) the function $(x,t) \mapsto f(x,t,0)$ is locally bounded in $\Omega\times\mathbb{R}$.

Then the functional $\mathcal{F}(u)$ is lower semicontinuous on $BV_{loc}(\Omega)$ for the L^1_{loc}-topology.

1.4.3 Relaxed variational problems

Let us consider the following variational problem

$$(\mathcal{P}) \qquad \inf\{F(u,\Omega)\ :\ u \in W^{1,1}(\Omega),\ u = u_0 \text{ on } \partial\Omega\} \tag{1.14}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $u_0 \in L^1(\partial\Omega)$, and the functional F is given by (1.13). Assume that F satisfies all assumptions of Theorem 1.57 and such that

$$f(x,t,z) \le C(1+|z|), \qquad \forall (x,t,z) \in \Omega\times\mathbb{R}\times\mathbb{R}^N.$$

In order to facilitate our discussion, let us also consider the dependence of F on the integration domain Ω. The following minimization problem is called the *relaxed problem of* $(\mathcal{P})$ in $BV(\Omega)$:

$$(\mathcal{PR}) \qquad \inf\{\widetilde{F}(u,\Omega)\ :\ u \in BV(\Omega)\},$$

where

$$\widetilde{F}(u,\Omega) := \overline{F}(u,\Omega) + \int_{\partial\Omega}\int_{(u\wedge u_0)(x)}^{(u\vee u_0)(x)} f^\infty(x,s,\mathrm{sgn}(u_0-u)\nu_\Omega)ds\ d\mathcal{H}^{N-1} \tag{1.15}$$

and ν_Ω is the outer unit normal to Ω. Here $\overline{F}$ represents the relaxed functional associated with F. Notice that the boundary constraint $\{u = u_0 \text{ on } \partial\Omega\}$ has been dropped but it appears in a relaxed form in the definition of $\widetilde{F}$ through the boundary integral term in (1.15).

Let us check that $\inf(\mathcal{P}) = \min(\mathcal{PR})$. Let $\widetilde{\Omega}$ be a bounded open subset of $\mathbb{R}^N$ such that $\Omega \subset\subset \widetilde{\Omega}$. According to Gagliardo's Theorem in [67], for any $u_0 \in L(\partial\Omega)$, there exists a function $\widetilde{u}_0 \in W^{1,1}(\widetilde{\Omega})$ such that its trace on $\partial\Omega$ agrees with u_0. Then, for every $u \in BV(\Omega)$, we set

$$v = \begin{cases} u & \text{in } \Omega \\ \widetilde{u}_0 & \text{in } \widetilde{\Omega}\setminus\overline{\Omega}. \end{cases} \tag{1.16}$$

The function v defined by the cut-and-paste formulation (1.16) still belongs to $BV(\widetilde{\Omega})$, moreover $Dv = Du\llcorner\Omega + D\widetilde{u}_0\llcorner(\widetilde{\Omega}\setminus\overline{\Omega}) + (\widetilde{u}_0 - u)\nu_\Omega d\mathcal{H}^{N-1}\llcorner\partial\Omega$ (see for instance [6, Corollary 3.89]). This leads to

$$\overline{F}(v,\widetilde{\Omega}) = \overline{F}(u,\Omega) + \overline{F}(\widetilde{u}_0,\widetilde{\Omega}\setminus\overline{\Omega}) + \int_{\partial\Omega}\int_{(u\wedge u_0)(x)}^{(u\vee u_0)(x)} f^\infty(x,s,\mathrm{sgn}(u_0-u)\nu_\Omega)ds\; d\mathcal{H}^{N-1}. \quad (1.17)$$

We find that $\overline{F}(\widetilde{u}_0,\widetilde{\Omega}\setminus\overline{\Omega})$ does not depend on u so that minimizing $\overline{F}(v,\widetilde{\Omega})$ for $v\in BV(\widetilde{\Omega})$ with constraint $v=\widetilde{u}_0$ in $\widetilde{\Omega}\setminus\overline{\Omega}$ is equivalent to minimizing $\widetilde{F}(u,\Omega)$ on $BV(\Omega)$, namely

$$\min\{\overline{F}(v,\widetilde{\Omega})\ :\ v\in BV(\widetilde{\Omega}),\ v=\widetilde{u}_0 \text{ in } \widetilde{\Omega}\setminus\overline{\Omega}\} = \min\{\widetilde{F}(u,\Omega)\ :\ u\in BV(\Omega)\} + \overline{F}(\widetilde{u}_0,\widetilde{\Omega}\setminus\overline{\Omega}).$$

We conclude that $\min(\mathcal{PR}) = \inf(\mathcal{P})$ by noticing that

$$\inf\{F(v,\widetilde{\Omega})\ :\ v\in W^{1,1}(\widetilde{\Omega}),\ v=\widetilde{u}_0 \text{ in } \widetilde{\Omega}\setminus\overline{\Omega}\} = \min\{\overline{F}(v,\widetilde{\Omega})\ :\ v\in BV(\widetilde{\Omega}),\ v=\widetilde{u}_0 \text{ in } \widetilde{\Omega}\setminus\overline{\Omega}\}.$$

In the specific case where the integrand f is given by $f(t,z)=\varphi(z)+g(t)$, the relaxed problem $(\mathcal{PR})$ becomes

$$\min\left\{\int_\Omega \varphi(Du) + \int_\Omega g(u)dx + \int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}\ :\ u\in BV(\Omega)\right\}$$

where

$$\int_\Omega\varphi(Du) := \int_\Omega\varphi(\nabla u)dx + \int_\Omega\varphi^\infty\left(\frac{D^c u}{|D^c u|}\right)d|D^c u| + \int_{S_u\cap\Omega}[u]\varphi^\infty(\nu_u)d\mathcal{H}^{N-1}.$$

1.5 Convex duality. Primal-dual formulations

In this section, we briefly present: the two methods of the classical convex duality (duality by perturbation and duality by min-max), their coincidence and primal-dual formulations resulting therein. We refer to [57, 12] for the details on the topic.

1.5.1 Duality by perturbation

Let X, Y be normed vector spaces. We denote by X^*, Y^* the topological dual spaces of X and Y, respectively. Let $\Lambda : X\to Y$ be a continuous linear operator with its adjoint operator denoted by $\Lambda^* : Y^*\to X^*$. Let us consider the following problem

$$(\mathcal{P}) \qquad \inf_{u\in X} f(u,\Lambda u)$$

where f is a function from $X\times Y$ to $\mathbb{R}\cup\{+\infty\}$. Problem $(\mathcal{P})$ is called the *primal problem*. We introduce a perturbation $\Phi : X\times Y\to\mathbb{R}\cup\{+\infty\}$ given by

$$\Phi(u,p) = f(u,\Lambda u - p).$$

Then for each $p\in Y$ we obtain a perturbed problem

$$(\mathcal{P}_p) \qquad \inf_{u\in X}\Phi(u,p).$$

Obviously, $\Phi(u,0) = f(u,\Lambda u)$, thus $(\mathcal{P}_0)$ is identical to $(\mathcal{P})$. We say that $(\mathcal{P}_p)$ is the perturbed problem of $(\mathcal{P})$ associated with the perturbation Φ.

Let $\Phi^* : X^*\times Y^*\to\mathbb{R}\cup\{+\infty\}$ be the conjugate function of Φ. The following problem is called the *dual problem* of $(\mathcal{P})$ associated with the perturbation Φ:

$$(\mathcal{P}^*) \qquad \sup_{p^*\in Y^*}\left\{-\Phi^*(0,p^*)\right\}.$$

Then, it holds

Proposition 1.58. $-\infty \leq \sup(\mathcal{P}^*) \leq \inf(\mathcal{P}) \leq +\infty$.

If it holds the equality $\sup(\mathcal{P}^*) = \inf(\mathcal{P})$, we shall say there is no *duality gap*. Let us determine the dual problem $(\mathcal{P}^*)$. By using f^* (the Moreau-Fenchel conjugate of f), we obtain

$$\Phi^*(0,p^*) = \sup_{\substack{u\in X\\ p\in Y}} \{\langle p^*,p\rangle - f(u,\Lambda u - p)\} = \sup_{u\in X}\sup_{p\in Y} \{\langle p^*,p\rangle - f(u,\Lambda u - p)\}.$$

For fixed $u \in X$ and setting $q = \Lambda u - p$, we deduce that

$$\Phi^*(0,p^*) = \sup_{u\in X}\sup_{q\in Y} \{\langle p^*,\Lambda u\rangle - \langle p^*,q\rangle - f(u,q)\} = \sup_{\substack{u\in X\\ q\in Y}} \{\langle p^*,\Lambda u\rangle - \langle p^*,q\rangle - f(u,q)\},$$

in other words, $\Phi(0,p^*) = f^*(\Lambda^* p^*, -p^*)$. Hence, the dual problem $(\mathcal{P}^*)$ reads

$$\sup_{p^*\in Y^*} \{-f^*(\Lambda^* p^*, -p^*)\}.$$

In the following, we summarize the result of no duality gap and the optimality conditions.

Theorem 1.59. *Assume that f is convex,* $\inf \mathcal{P} < +\infty$, *and*

$$\begin{cases} \text{there exists } u_0 \in X \text{ such that } f(u_0,\Lambda u_0) < +\infty, \text{ the function} \\ p \mapsto f(u_0,p) \text{ being continuous at } \Lambda u_0. \end{cases} \tag{1.18}$$

Then, $(\mathcal{P}^*)$ attains its maximum and $\sup(\mathcal{P}^*) = \inf(\mathcal{P})$.

Proposition 1.60. *The two assertions below are equivalent:*

(i) $\overline{u}$ is a solution for $(\mathcal{P})$, $\overline{p}^*$ is a solution for $(\mathcal{P}^*)$ and $\inf(\mathcal{P}) = \sup(\mathcal{P}^*)$;

(ii) $\overline{u} \in X$, $\overline{p}^* \in Y^*$ satisfy the optimality condition:

$$f(\overline{u},\Lambda\overline{u}) + f^*(\Lambda^*\overline{p}^*, -\overline{p}^*) = 0 \tag{1.19}$$

or equivalently,

$$(\Lambda^*\overline{p}^*, -\overline{p}^*) \in \partial f(\overline{u},\Lambda\overline{u}). \tag{1.20}$$

Theorem 1.61. *Let X be a reflexive Banach space, $f : X \times Y \to \mathbb{R}\cup\{+\infty\}$ be convex l.s.c. function. Assume that (1.18) is satisfied, and that*

$$\lim_{\|u\|\to+\infty} f(u,\Lambda u) = +\infty.$$

Then, $(\mathcal{P})$ and $(\mathcal{P}^*)$ have solutions, $\inf \mathcal{P} = \sup \mathcal{P}^*$ *and the optimality condition (1.19)-(1.20) is verified.*

In particular, when f is of the form $f(u,\Lambda u) = \varphi(\Lambda u) + g(u)$ with $\varphi : Y \to \mathbb{R}\cup\{+\infty\}$ and $g : X \to \mathbb{R}\cup\{+\infty\}$, the primal problem $(\mathcal{P})$ reduces to

$$\inf_{u\in X} \{\varphi(\Lambda u) + g(u)\}.$$

It can be checked that $f^*(u^*,p^*) = \varphi^*(p^*) + g^*(u^*)$. Therefore, $(\mathcal{P}^*)$ is recast as

$$\sup_{p^*\in Y^*} \{-\varphi^*(-p^*) - g^*(\Lambda^* p^*)\}.$$

Notice that if φ, g are convex l.s.c., then f is convex l.s.c. Additionally, the condition (1.18) should be adapted with

$$\begin{cases} \text{there exists } u_0 \in X \text{ such that } g(u_0) < +\infty, \varphi(\Lambda u_0) < +\infty, \text{ and} \\ \varphi \text{ being continuous at } \Lambda u_0. \end{cases} \tag{1.21}$$

On the other hand, the optimality condition (1.19) reads

$$\begin{aligned} 0 &= f(\overline{u}, \Lambda\overline{u}) + f^*(\Lambda^*\overline{p}^*, -\overline{p}^*) \\ &= \{\varphi(\Lambda\overline{u}) + \varphi^*(-\overline{p}^*) - \langle -\overline{p}^*, \Lambda\overline{u}\rangle\} + \{g(\overline{u}) + g^*(\Lambda^*\overline{p}^*) - \langle \Lambda^*\overline{p}^*, \overline{u}\rangle\} \geq 0. \end{aligned}$$

The inequality above is obtained thanks to the Fenchel inequalities, thus these non negative quantities become null, which yield the following primal-dual optimality conditions

$$\begin{aligned} \varphi(\Lambda\overline{u}) + \varphi^*(-\overline{p}^*) - \langle -\overline{p}^*, \Lambda\overline{u}\rangle = 0, \\ g(\overline{u}) + g^*(\Lambda^*\overline{p}^*) - \langle \Lambda^*\overline{p}^*, \overline{u}\rangle = 0. \end{aligned}$$

These conditions can be equivalently rewritten as

$$\begin{aligned} \Lambda^*\overline{p}^* \in \partial g(\overline{u}), \\ -\overline{p}^* \in \partial\varphi(\Lambda\overline{u}). \end{aligned}$$

1.5.2 Duality by min-max

We recall in this subsection some very basic and important results of the saddle point theory. We refer to [57] for more details on the topic.

Saddle point problems

Let $L : X \times Y \to \mathbb{R}$ be a real-valued function defined on the product space $X \times Y$, that we call *Lagrangian function*. Let $\mathcal{A}$ be a subset of X and $\mathcal{B}$ be a subset of Y. Then, it holds:

Proposition 1.62.

$$\sup_{p\in\mathcal{B}} \inf_{u\in\mathcal{A}} L(u,p) \leq \inf_{u\in\mathcal{A}} \sup_{p\in\mathcal{B}} L(u,p). \tag{1.22}$$

Definition 1.63. A couple $(\overline{u}, \overline{p}) \in \mathcal{A} \times \mathcal{B}$ is called a *saddle-point* of L on $\mathcal{A} \times \mathcal{B}$ if

$$L(\overline{u}, p) \leq L(\overline{u}, \overline{p}) \leq L(u, \overline{p}), \qquad \forall u \in \mathcal{A}, \forall p \in \mathcal{B}. \tag{1.23}$$

Proposition 1.64. *$L : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ admits a saddle point $(\overline{u}, \overline{p})$ on $\mathcal{A} \times \mathcal{B}$ if and only if*

$$\max_{p\in\mathcal{B}} \inf_{u\in\mathcal{A}} L(u,p) = \min_{u\in\mathcal{A}} \sup_{p\in\mathcal{B}} L(u,p), \tag{1.24}$$

and this number is then equal to $L(\overline{u}, \overline{p})$.

Proof. Let $(\overline{u}, \overline{p})$ be a saddle point for L on $\mathcal{A} \times \mathcal{B}$. It holds

$$\inf_{u\in\mathcal{A}} \sup_{p\in\mathcal{B}} L(u,p) \leq \sup_{p\in\mathcal{B}} L(\overline{u},p) = L(\overline{u},\overline{p}) = \inf_{u\in\mathcal{A}} L(u,\overline{p}) \leq \sup_{p\in\mathcal{B}} \inf_{u\in\mathcal{A}} L(u,p). \tag{1.25}$$

The equalities in (1.25) is deduced from the inequalities (1.23). The combination of (1.22)-(1.25) then implies that

$$L(\overline{u},\overline{p}) = \sup_{p\in\mathcal{B}} L(\overline{u},p) = \min_{u\in\mathcal{A}} \sup_{p\in\mathcal{B}} L(u,p) = \inf_{u\in\mathcal{A}} L(u,\overline{p}) = \max_{p\in\mathcal{B}} \inf_{u\in\mathcal{A}} L(u,p).$$

Inversely, suppose that (1.24) holds, in which the minimum is attained at some $\overline{u} \in \mathcal{A}$ and the maximum is attained at some $\overline{p} \in \mathcal{B}$. We get that

$$\inf_{u\in\mathcal{A}} L(u,\overline{p}) \leq L(\overline{u},\overline{p}) \leq \sup_{p\in\mathcal{B}} L(\overline{u},p). \tag{1.26}$$

Taking into account the equality (1.24), the inequalities (1.26) become equalities from which $(\overline{u},\overline{p})$ is a saddle point for L on $\mathcal{A}\times\mathcal{B}$. ■

Remark 1.65. In particular, if the Lagrangian L has a saddle point on $\mathcal{A}\times\mathcal{B}$, then it holds

$$\sup_{p\in\mathcal{B}}\inf_{u\in\mathcal{A}} L(u,p) = \inf_{u\in\mathcal{A}}\sup_{p\in\mathcal{B}} L(u,p).$$

Characterization of saddle points

We collect in this paragraph some useful characterizations of saddle points.

Proposition 1.66. *Let $u_0 \in \mathcal{A}$, $p_0 \in \mathcal{B}$ and $\alpha \in \mathbb{R}$ such that*

$$L(u_0,p) \leq \alpha, \forall p \in \mathcal{B}, \qquad \text{and} \qquad \alpha \leq L(u,p_0), \forall u \in \mathcal{A}.$$

Then, (u_0,p_0) is a saddle point for L on $\mathcal{A}\times\mathcal{B}$ and

$$\alpha = \inf_{u\in\mathcal{A}}\sup_{p\in\mathcal{B}} L(u,p) = \sup_{p\in\mathcal{B}}\inf_{u\in\mathcal{A}} L(u,p).$$

One can prove that the set of all saddle points for L on $\mathcal{A}\times\mathcal{B}$ is of the form $\mathcal{A}_0\times\mathcal{B}_0$ where $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ (see for instance [57]). We remark in addition, the properties discussed above are established in rather flexible conditions that do not require the convex-concavity of L and that $\mathcal{A}$, $\mathcal{B}$ are arbitrary subsets. From now on, the Lagrangian L is supposed to satisfy the following assumptions:

Assumptions on Lagrangian L
Let X, Y be reflexive Banach spaces. We assume that

$$\mathcal{A} \subset X \text{ is nonempty closed convex,} \tag{1.27}$$
$$\mathcal{B} \subset Y \text{ is nonempty closed convex.} \tag{1.28}$$

The Lagrangian $L : \mathcal{A}\times\mathcal{B} \to \mathbb{R}$ is defined on $\mathcal{A}\times\mathcal{B}$ such that

$$\forall u \in \mathcal{A}, L(u,\cdot) \text{ is concave u.s.c.,} \tag{1.29}$$
$$\forall p \in \mathcal{B}, L(\cdot,p) \text{ is convex l.s.c..} \tag{1.30}$$

Proposition 1.67. *Let $L : \mathcal{A}\times\mathcal{B} \to \mathbb{R}$ satisfy the assumptions (1.27)–(1.30). The saddle point set $\mathcal{A}_0\times\mathcal{B}_0$ for L is convex. Moreover, if $L(u,p)$ is strictly convex in u (resp. strictly concave in p), then $\mathcal{A}_0$ (resp. $\mathcal{B}_0$) has at most one point.*

Proposition 1.68. *Let $L : \mathcal{A}\times\mathcal{B} \to \mathbb{R}$ satisfy the assumptions (1.27)–(1.30), and in addition, $L(u,\cdot)$ is Gâteaux differentiable for each $u \in \mathcal{A}$, $L(\cdot,p)$ is Gâteaux differentiable for each $p \in \mathcal{B}$. Then, $(\overline{u},\overline{p}) \in \mathcal{A}\times\mathcal{B}$ is a saddle point for L on $\mathcal{A}\times\mathcal{B}$ if and only if*

$$\left\langle \frac{\partial L}{\partial u}(\overline{u},\overline{p}), u-\overline{u} \right\rangle \geq 0 \quad \forall u \in \mathcal{A},$$
$$-\left\langle \frac{\partial L}{\partial p}(\overline{u},\overline{p}), p-\overline{p} \right\rangle \geq 0 \quad \forall p \in \mathcal{B}.$$

Proposition 1.69. *Let $L(u,p)=\varphi(u,p)+g(u,p)$ such that $g:\mathcal{A}\times\mathcal{B}\to\mathbb{R}$ is convex-concave, $\varphi:\mathcal{A}\times\mathcal{B}\to\mathbb{R}$ is convex-concave and Gâteaux differentiable. Then, $(\overline{u},\overline{p})\in\mathcal{A}\times\mathcal{B}$ is a saddle point for L on $\mathcal{A}\times\mathcal{B}$ if and only if*

$$\begin{aligned}\left\langle\frac{\partial\varphi}{\partial u}(\overline{u},\overline{p}),u-\overline{u}\right\rangle+g(u,\overline{p})-g(\overline{u},\overline{p})\geq 0\quad\forall u\in\mathcal{A},\\ -\left\langle\frac{\partial\varphi}{\partial p}(\overline{u},\overline{p}),p-\overline{p}\right\rangle-g(\overline{u},p)+g(\overline{u},\overline{p})\geq 0\quad\forall p\in\mathcal{B}.\end{aligned}$$

Existence of saddle points

In this paragraph, we summarize some compactness conditions ensuring the existence of saddle points of L on $\mathcal{A}\times\mathcal{B}$.

Proposition 1.70. *Let $L:\mathcal{A}\times\mathcal{B}\to\mathbb{R}$ satisfy the assumptions (1.27)–(1.30). Assume that*

$$\mathcal{A},\mathcal{B}\text{ are bounded.}\tag{1.31}$$

Then, L admits a saddle point $(\overline{u},\overline{p})$ on $\mathcal{A}\times\mathcal{B}$ and

$$L(\overline{u},\overline{p})=\min_{u\in\mathcal{A}}\max_{p\in\mathcal{B}}L(u,p)=\max_{p\in\mathcal{B}}\min_{u\in\mathcal{A}}L(u,p).$$

Proposition 1.71. *Let $L:\mathcal{A}\times\mathcal{B}\to\mathbb{R}$ satisfy the assumptions (1.27)–(1.30). Suppose in addition that*

$$\exists p_0\in\mathcal{B}\text{ such that }\lim_{\substack{u\in\mathcal{A}\\ \|u\|\to+\infty}}L(u,p_0)=+\infty,\tag{1.32}$$

$$\exists u_0\in\mathcal{A}\text{ such that }\lim_{\substack{p\in\mathcal{B}\\ \|p\|\to+\infty}}L(u_0,p)=-\infty.\tag{1.33}$$

Then, L admits a saddle point $(\overline{u},\overline{p})$ on $\mathcal{A}\times\mathcal{B}$, and

$$L(\overline{u},\overline{p})=\min_{u\in\mathcal{A}}\max_{p\in\mathcal{B}}L(u,p)=\max_{p\in\mathcal{B}}\min_{u\in\mathcal{A}}L(u,p).$$

Remark 1.72. The same results can be obtained if $\mathcal{A},\mathcal{B}$ are compact subsets of separable topological vector spaces. The conditions (1.31) and (1.32)-(1.33) can be combined to ensure the existence of saddle points, for instance: $\mathcal{A}$ is bounded and (1.33) holds, or $\mathcal{B}$ is bounded and (1.32) holds.

Proposition 1.73. *Let $L:\mathcal{A}\times\mathcal{B}\to\mathbb{R}$ satisfy the assumptions (1.27)-(1.30) and assume that $\mathcal{A}$ is bounded (or (1.32) holds). Then, we have*

$$\min_{u\in\mathcal{A}}\sup_{p\in\mathcal{B}}L(u,p)=\sup_{p\in\mathcal{B}}\inf_{u\in\mathcal{A}}L(u,p).$$

Proposition 1.74. *Let $L:\mathcal{A}\times\mathcal{B}\to\mathbb{R}$ satisfy the assumptions (1.27)-(1.30). Assume that the condition (1.32) holds, and*

$$\lim_{\substack{p\in\mathcal{B}\\ \|p\|\to+\infty}}\inf_{u\in\mathcal{A}}L(u,p)=-\infty.$$

Then, L admits a saddle point on $\mathcal{A}\times\mathcal{B}$.

The following is a generalized result where the coercive conditions on Lagrangian are simplified with compactness. We can find the proof of this result in [56, 57, 39].

Proposition 1.75. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty convex subsets of two locally convex topological vector spaces, and let $\mathcal{B}$ be compact. Assume that $L : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ is such that for every $p \in \mathcal{B}$, $L(\cdot, p)$ is convex, and for every $u \in \mathcal{A}$, $L(u, \cdot)$ is u.s.c. and concave. Then, if the quantity*

$$m := \inf_{u\in\mathcal{A}} \sup_{p\in\mathcal{B}} L(u,p)$$

is finite, we have $m = \sup_{p\in\mathcal{B}} \inf_{u\in\mathcal{A}} L(u,p)$, and there exists $\overline{p} \in \mathcal{B}$ such that $\inf_{u\in\mathcal{A}} L(u,\overline{p}) = m$. If in addition $\mathcal{A}$ is compact and for every $p \in \mathcal{B}$, $L(\cdot, p)$ is l.s.c., then there exists $\overline{u} \in \mathcal{A}$ such that $L(\overline{u},\overline{p}) = m$.

Application of saddle-point theorems to duality

We are now at point to make use of saddle point theorems for duality. This applications basically relies on Moreau-Fenchel conjugates which introduced in Section 1.2.1. We turn back to consider the infimum problem

$$(\mathcal{P}) \qquad \inf_{u\in X} f(u, \Lambda u)$$

where $f(u, \Lambda u) = \varphi(\Lambda u) + g(u)$ with $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ being convex l.s.c and proper on Y. Under this assumption, φ coincides with its biconjugate φ^{**}, thus we have

$$\varphi(\Lambda u) = \sup_{p^*\in Y^*} \{\langle \Lambda u, p^*\rangle - \varphi^*(p^*)\}. \tag{1.34}$$

Let us consider the Lagrangian L defined by

$$L(u, p^*) = g(u) + \langle \Lambda u, p^*\rangle - \varphi^*(p^*).$$

By substituting (1.34) into $(\mathcal{P})$, we obtain a saddle point problem

$$\inf_{u\in X} \sup_{p^*\in Y^*} L(u, p^*) \tag{1.35}$$

which is identical to $(\mathcal{P})$. Problem (1.35) is called a *primal-dual formulation* of the minimum problem $(\mathcal{P})$. Its dual problem $(\mathcal{P}^*)$ is then defined by

$$(\mathcal{P}^*) \qquad \sup_{p^*\in Y^*} \inf_{u\in X} L(u, p^*),$$

and there holds

$$-\infty \le \sup(\mathcal{P}^*) \le \inf(\mathcal{P}) \le +\infty.$$

Once $\inf\sup L = \sup\inf L$ happens, there is equivalence between the existence of saddle points for L and the existence of solutions for problems $(\mathcal{P})$, $(\mathcal{P}^*)$ with $\inf(\mathcal{P}) = \sup(\mathcal{P}^*)$. In this case, the saddle points are equal to the couples of solutions for $(\mathcal{P})$ and $(\mathcal{P}^*)$. In this approach, the optimality condition reads, for $(\overline{u},\overline{p}^*)$ being a saddle point,

$$\inf_{u\in X} L(u,\overline{p}^*) = L(\overline{u},\overline{p}^*) = \sup_{p^*\in Y^*} L(\overline{u}, p^*).$$

We remark that if f is non convex, the reformulation into a inf-sup problem is always feasible within a decomposition of f into convex part and non convex part.

The dual problem $(\mathcal{P}^*)$ is computed as

$$\sup_{p^*\in Y^*} \{-\varphi^*(p^*) - g^*(-\Lambda^* p^*)\}$$

by observing that

$$\inf_{u\in X} L(u, p^*) = -\varphi^*(p^*) - \sup_{u\in X} \{\langle \Lambda u, -p^*\rangle - g(u)\} = -\varphi^*(p^*) - g^*(-\Lambda^* p^*).$$

Additionally, we can derive the following optimality conditions

$$\begin{aligned} -\Lambda^*\overline{p}^* &\in \partial g(\overline{u}),\\ \overline{p}^* &\in \partial\varphi(\Lambda\overline{u}). \end{aligned}$$

1.5.3 Identification of duality methods

This subsection is devoted to the discussion of the coincidence of the two duality methods mentioned above. The proof of this result can be found in [57]. We also quote [10] for the detailed discussion on this topic. We shall interpret the following saddle point problem

$$\inf_{u\in\mathcal{A}}\sup_{p^*\in\mathcal{B}} L(u,p^*)$$

to the context of Section 1.5.1. Here, $L(u,p^*)$ denotes the Lagrangian defined on the subset $\mathcal{A}\times\mathcal{B}$ of the product space $X\times Y^*$. We introduce a function $\Phi : X\times Y\to\mathbb{R}$ determined by

$$\Phi(u,p)=\begin{cases}\sup\{\langle p,p^*\rangle+L(u,p^*)\ :\ p^*\in\mathcal{B}\} & \text{if } (u,p)\in\mathcal{A}\times Y\\ +\infty & \text{if } (u,p)\in(X\setminus\mathcal{A})\times Y.\end{cases}\tag{1.36}$$

It is evident that function $\Phi(u,p)$ is convex l.s.c. in p.

Lemma 1.76. *For every $u\in X$, function $p\mapsto\Phi(u,p)$ is convex l.s.c. on Y.*

In addition, if $\mathcal{A}$ is a closed convex subset of X, and for each $p^*\in\mathcal{B}$, $L(\cdot,p^*)$ is convex l.s.c., then Φ is convex l.s.c. on $X\times Y$. In fact, for $p^*\in\mathcal{B}$ fixed, the mapping $(u,p)\mapsto\langle p,p^*\rangle+L(u,p^*)$ is convex l.s.c. on $\mathcal{A}\times Y$. The extension by infinity outside a closed convex set still preserves the convexity and lower semicontinuity, thus Φ is convex l.s.c on the entire product space $X\times Y$. These are stated in the next lemma.

Lemma 1.77. *Assume that $\mathcal{A}$ is a closed convex subset of X and for each $p^*\in\mathcal{B}$, $L(\cdot,p^*)$ is convex l.s.c. Then, Φ is convex l.s.c. on $X\times Y$.*

Clearly, at $p=0$ it holds

$$\Phi(u,0)=\sup_{p^*\in\mathcal{B}} L(u,p^*).$$

Moreover, Φ actually defines a perturbation as mentioned in the circumstance of Section 1.5.1. Hence, the problem

$$\inf_{u\in X}\Phi(u,0)$$

is in duality with

$$\sup_{p^*\in Y^*}\{-\Phi^*(0,p^*)\}.\tag{1.37}$$

Proposition 1.78. *Let $\mathcal{B}\subset Y^*$ be a closed convex set. Assume that for each $u\in\mathcal{A}$, $L(u,\cdot)$ is concave u.s.c. on $\mathcal{B}$. Then, problem (1.37) coincides with*

$$\sup_{p^*\in\mathcal{B}}\inf_{u\in\mathcal{A}} L(u,p^*),$$

in other words, the two concepts of duality are identical.

1.6 Classical saddle-point algorithms

In this section, we recall the two important algorithms based on primal-dual formulations: Uzawa's algorithm and Arrow-Hurwicz' algorithm. These two algorithms was actually introduced to seek solutions of a constrained optimization problem. However, both of them basically relies on saddle point problems resulting from the primal-dual formulation of the constrained optimization problem. These classical algorithms have widely been used in numerical approximation and

since then there have been many improved variants of the algorithms. Let us shortly describe in the following the versions of the two algorithms that we can find in the book of Ekeland-Temam [57] the proof of convergence results.

V and W are Hilbert spaces. For simplicity, we denote by $\langle\cdot,\cdot\rangle$ the inner product and $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ the corresponding norm on both V and W. Let V^* and W^* respectively denote the associated topological dual spaces.

1.6.1 Uzawa's algorithm

Let $C \subset V$, $K \subset W$ be such that

$$C \text{ and } K \text{ are nonempty closed convex sets.} \tag{1.38}$$

We consider the following saddle point problem

$$\inf_{u\in C}\sup_{p\in K} L(u,p), \qquad L(u,p) = F(u) + \langle p, \Psi(u)\rangle\,, \tag{1.39}$$

where $F : C \to \mathbb{R}$ is a real-valued function, $\Psi : C \to W$ is a possibly nonlinear mapping (noticing that the scalar product $\langle\Psi(u), p\rangle$ in L is intended in the Hilbert space W). We remark that many optimization problems can be rewritten in this inf-sup form. In fact, the Lagrangian (1.39) can be derived from the method of Lagrange multipliers for the constrained infimum problem

$$\inf\{F(u) \ : \ u \in C, \Psi(u) = 0\}\,.$$

Standing assumptions
The Lagrangian L satisfies the following properties of coerciveness, differentiability, strong convexity, lower semicontinuity, and Lipschitz continuity:

$$K \text{ is bounded ;} \tag{1.40}$$
$$F : C \to \mathbb{R} \text{ is Gâteaux differentiable ;} \tag{1.41}$$
$$\alpha\|u-v\|^2 \le \langle F'(u) - F'(v), u - v\rangle\,, \quad \alpha > 0, \quad \forall u, v \in C; \tag{1.42}$$
$$\forall p \in K, \text{ the function } \langle p, \Psi(\cdot)\rangle \text{ is convex l.s.c. on } C; \tag{1.43}$$
$$\Psi : C \to W \text{ is Lipschitzian , i.e. } \|\Psi(u) - \Psi(v)\| \le c\|u - v\|, \ \forall u, v \in C. \tag{1.44}$$

We recall that a function $F : C \to \mathbb{R}$ being Gâteaux differentiable is strongly convex if it satisfies the condition (1.42), or equivalently

$$F(tu + (1-t)v) \le tF(u) + (1-t)F(v) - t(1-t)\frac{\alpha}{2}\|u-v\|^2$$

for all $u, v \in C$ and for all $t \in [0,1]$. Strong convexity plays a very important role in the convergence to the unique minimizer of F on C. Another equivalent condition for strong convexity is given by the inequality (1.45) in Lemma 1.79. We refer to [92] for more details on strongly convex functions. Before describing Uzawa's algorithm, let us summarize the well-posedness of the saddle point problem (1.39) in the two lemmas below.

Lemma 1.79. *$F : C \to \mathbb{R}$ is convex l.s.c. and such that*

$$F(u) \ge F(v) + \langle F'(v), u - v\rangle + \frac{\alpha}{2}\|u-v\|^2, \quad \forall u, v \in C, \tag{1.45}$$
$$\lim_{\substack{u\in C\\ \|u\|\to+\infty}} F(u) = +\infty. \tag{1.46}$$

Lemma 1.80. *Under the assumptions* (1.38) *and from* (1.40) *to* (1.44), *L admits a saddle point $(\hat{u}, \hat{p})$ on $C \times K$ where $\hat{u}$ is uniquely determined and minimize the function $u \mapsto \sup_{p\in K} L(u,p)$ on C.*

We give here a description of Uzawa's algorithm and the convergence result of this method. The proof of this result can be found in [57].

Description of Uzawa's algorithm
Initialization: Let $n \in \mathbb{N}$. Given $p_0 \in K$.

$$u_n = \operatorname{Argmin}_C \{F(u) + \langle p_n, \Psi(u)\rangle\}, \tag{1.47}$$
$$p_{n+1} = \Pi_K (p_n + \rho_n \Psi(u_n)), \tag{1.48}$$

where Π_K is an orthogonal projection on K and $\rho_n > 0$ is chosen appropriately.

Proposition 1.81. *Under the assumptions* (1.38) *and from* (1.40) *to* (1.44), *and let ρ_n be chosen such that*

$$0 < \rho_n < \frac{2\alpha}{c^2},$$

the Uzawa's algorithm defined by the iterative process (1.47)-(1.48) *is convergent, i.e. $u_n \to \hat{u}$ in V, where $\hat{u} = \operatorname{Argmin}_C \left\{\sup_{p\in K} L(\cdot, p)\right\}$.*

1.6.2 Arrow-Hurwicz' Algorithm

We observe that in the description of Uzawa's algorithm, the step (1.47) is not completely specified, that how to seek the minimizer is not discussed. This partly unfinishedness, however, permits us to arbitrarily choose a method for determining the current minimizer u_n. Arrow and Hurwicz introduced a variant in which the step (1.47) is computed explicitly.

For the sake of simplicity, the algorithm of Arrow and Hurwicz was proposed under the assumptions which are more specified than in the previous method: F is a quadratic form on V and Ψ is a linear operator.

Standing assumptions
V, W are Hilbert spaces, on which

$$K \subset W \text{ is a nonempty closed convex set.} \tag{1.49}$$

$\Psi : V \to W$, $A : V \to V^*$ are linear operators such that

$$A = A^*, \tag{1.50}$$
$$\langle Au, u\rangle \geq \alpha \|u\|^2, \quad \alpha > 0, \quad \forall u \in V, \tag{1.51}$$

(A^* is the adjoint of A), f is a element of V^* and

$$F(u) = \langle Au, u\rangle - 2\langle f, u\rangle. \tag{1.52}$$

We then deal with the saddle point problem

$$\inf_{u\in V} \sup_{p\in K} L(u,p), \qquad L(u,p) = F(u) + \langle p, \Psi u\rangle.$$

Under the assumptions from (1.49) to (1.52), the existence of saddle points are ensured since the hypotheses of Lemma 1.80 are all satisfied. That means L admits a saddle point $(\hat{u}, \hat{p})$ on

$V \times K$, and $\hat{u}$ is the unique minimizer of the function $u \mapsto \sup_{p \in K} L(u,p)$. We are ready to describe the Arrow-Hurwicz algorithm.

Description of Arrow-Hurwicz' algorithm
Initialization: Let $n \in \mathbb{N}$. Given $p_0 \in K$ and $u_0 \in V$.

$$u_{n+1} = u_n - \rho_1 S^{-1}(Au_n - f + \Psi^* p_n), \tag{1.53}$$
$$p_{n+1} = \Pi_K \left(p_n + \rho_2 \Psi u_{n+1}\right), \tag{1.54}$$

with a suitable choice of $\rho_1, \rho_2 > 0$, S being the canonical isomorphism from V to V^* and Π_K being an orthogonal projection on K.

We remark that the canonical isometrically isomorphism $S : V \to V^*$ is provided by F. Riesz' representation theorem, see for instance [108]. We observe that if the iterative process (1.53)-(1.54) converges to $(\hat{u}, \hat{p})$, then the couple $(\hat{u}, \hat{p})$ must satisfy

$$A\hat{u} - f + \Psi^* \hat{p} = 0,$$
$$\hat{p} = \Pi_K(\hat{p} + \rho \Psi \hat{u}), \quad \forall \rho > 0.$$

These conditions are obtained from the characterization of saddle points of L on $V \times K$. The following is a convergence result (see [57] for a proof of this result).

Proposition 1.82. *Under the assumptions from (1.49) to (1.52) and for any $\rho_1, \rho_2 > 0$ such that*

$$\|I - \rho_1 S^{-1} A\| \leq \beta < 1, \qquad \rho_2^2 \|\Psi\|^2 + \frac{2\rho_2(\beta - 1)}{\rho_1} < 0,$$

the Arrow-Hurwicz' algorithm defined by (1.53)-(1.54) converges, i.e. $u_n \to \hat{u}$ in V, being $\hat{u}$ the minimizer of $u \mapsto \sup_{p \in K} L(u,p)$.

Part B

Non convex variational problems

Chapter 2

A general duality principle for non convex variational problems

2.1 Introduction

The role of duality techniques is nowadays very well established in applied mathematics, mechanics and numerical analysis. In the context of infinite dimensional vector spaces, convex analysis has been a powerful mathematical tool taking a major part in this success. It is great to honor the pioneering contributions of J.J. Moreau which go back to the 60's [89] (lectures notes at collège de France) almost concomitantly with the work of T.Rockafellar [98] focused on the finite dimensional case. Such a mathematical step in functional analysis was crucial in order to make a rigorous existence theory in elasticity theory (existence of equilibrium strain/stress tensors, quasi-static evolution) and it could extended to non linear (but convex) situations, notably in plasticity theory [104, 105]. Let us emphasize that the impact of duality and convexity encompass a very broad area in optimization theory: in numerical analysis many efficient and stable algorithms are based on min-max (or saddle points) schemes and still recent progress in this area are very influential (for instance around proximal projection algorithms); in optimal mass transport [106] the existence and the characterization of an optimal map often goes through the existence of a solution for a dual problem; in asymptotic analysis (dimension reduction, homogenization) a huge number of results have been obtained by combining duality arguments and Γ-convergence techniques (for the latter notion we refer to Section 1.1.4 of Chapter 1).

Unfortunately, such a duality theory completely breaks down as soon as some nonconvexity appears in the optimization problem under study. In particular, this drawback is often met in Calculus of Variations, where even very classical problems involve non-convex energy costs. As no systematical tool is available to characterize a global optimum, a dramatic consequence is that all currently available numerical methods loose their efficiency, because they are not able to rule out local minimizers and detect the global ones.

The aim of this chapter is to present some new perspectives for exploiting duality in a context of *non convex* variational problems. We begin by presenting in Section 2.2 a general convexification recipe. It enlightens a new interpretation of the calibration field developed for the Munford-Shah segmentation problem [1][34] and suggests a new road for identifying the variational limit of non convex functionals. In Section 2.3, we show how the recipe can be applies to integral functionals satisfying a generalized coarea formula. Then we present a survey of the primal-dual framework obtained in [19][20] and we sketch a new proof for the Γ- convergence of Cahn-Hilliard models.

In the end of the thesis we discusss possible new developements and present a still unsolved conjecture.

2.2 General framework

We fix some preliminary notations. In this section (X,τ) denotes a topological space. We assume that there exists a continuous embedding

$$\varphi : u \in X \mapsto \varphi_u \in Y \ ,$$

where Y is a topological locally convex vector space. The symbol $< \cdot, \cdot >$ will denote the duality mapping between Y and its dual Y^*. Our convexification procedure is based on the following assumption:

$$(H1) \qquad \left\{ \begin{array}{l} \text{There exists a suitable compact metrizable convex subset } K \subset Y \\ \text{whose extremal set } \ddot{K} \text{ satisfies: } \quad \varphi(X) \subset \ddot{K}. \end{array} \right.$$

A typical situation is when Y is the dual of a separable Banach space Z equipped with the weak-star topology . Then Y^* can be identified with Z itself and every bounded and weakly-star closed convex subset of Y is compact metrizable. Recall that v is an extreme point of K if $v = \theta v_1 + (1-\theta)v_2 \in K$ with $v_1, v_2 \in K$ and $\theta \in (0,1)$ cannot occur unless $v = v_1 = v_2$. The set of all extreme points of K is called the extremal set of K.

We consider a sequence of proper functionals $F^\varepsilon : X \to [0,+\infty]$ which we assume to be uniformly coercive, that is:

$$(H2) \qquad \left\{ \begin{array}{l} \text{For every } R \text{ , there exists a } \tau\text{-compact subset } C_R \text{ such that:} \\ \forall \varepsilon > 0 \ , \ \{u \in X \ : \ F^\varepsilon(u) \leq R\} \ \subset C_R. \end{array} \right.$$

Define $F_0^\varepsilon : Y \to [0,+\infty]$ by setting

$$F_0^\varepsilon(\varphi_u) = F^\varepsilon(u) \quad \text{ for every } u \in X \quad , \quad F_0^\varepsilon(v) = +\infty \quad \text{if} \quad v \notin \varphi(X).$$

The Fenchel conjugate of F_0^ε is given on Y^* by:

$$(F_0^\varepsilon)^*(g) := \sup\{< v, g > - F_0^\varepsilon(v) \ : \ v \in Y\} = \sup\{< \varphi_u, g > - F^\varepsilon(u) \ : \ u \in X\}.$$

Then (being K metrizable) the biconjugate $(F_0^\varepsilon)^{**}$ will coincide with the sequential convexification of F_0^ε, that is, for every $v \in Y$:

$$(F_0^\varepsilon)^{**}(v) = \inf \left\{ \liminf_h \sum_{i=1}^{n_h} t_i^h F^\varepsilon(u_i^h) \ : \ \sum_{i=1}^{n_h} t_i^h \varphi_{u_i^h} \to v \right\},$$

where $\{t_i^h \ : \ i = 1, \cdots, n_h\}$ are real numbers in $[0,1]$ such that $\sum_i t_i^h = 1$.

The main result of this section states that the variational limit of F^ε at every $u \in X$ agrees with that of the convexification $(F_0^\varepsilon)^{**}$ at φ_u. To be more precise let us consider (see [45]) the (sequential) Γ-limits of F_ε defined on (X,τ) by

$$\Gamma - \liminf F^\varepsilon(u) = \inf \{ \liminf_{\varepsilon \to 0} F^\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u\}, \tag{2.1}$$

$$\Gamma - \limsup F^\varepsilon(u) = \inf \{ \limsup_{\varepsilon \to 0} F^\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u\}, \tag{2.2}$$

and in a similar way the Γ-limits of (F_0^ε) and $(F_0^\varepsilon)^{**}$ defined on Y. In order to simplify notations, in the following, we will denote by:

- F', F'' the $\Gamma - \liminf$ and $\Gamma - \limsup$ of F^ε (defined on X)
- F_0', F_0'', the $\Gamma - \liminf$ and $\Gamma - \limsup$ of F_0^ε (defined on Y)

- G', G'', the $\Gamma-\liminf$ and $\Gamma-\limsup$ of $(F_0^\varepsilon)^{**}$

Observe that due to (H2) by [27, Prop. 1.3.5], all sequential notions coincide with the topological ones. For instance it holds:

$$F'(u)(u) = \sup_{V\in\mathcal{V}(u)} \liminf_{\varepsilon} \inf_{V} F^\varepsilon \quad , \quad F''(u)(u) = \sup_{V\in\mathcal{V}(u)} \limsup_{\varepsilon} \inf_{V} F^\varepsilon \ . \tag{2.3}$$

On the other hand, in case of the constant sequence $F^\varepsilon = F$, the lower and upper Γ-limits F', F'' coincide and agree the usual notion of lower semicontinuous envelope:

$$\overline{F}(u) = \sup\{\Phi(u) \ : \ \Phi \text{ lower semicontinuous }, \ \Phi \le F \ \}$$

Theorem 2.1. *Under (H1) and (H2), there holds for every $u \in X$:*

$$\Gamma-\liminf F^\varepsilon(u) \le \Gamma-\liminf (F_0^\varepsilon)^{**}(\varphi_u) \le \Gamma-\limsup (F_0^\varepsilon)^{**}(\varphi_u) \le \Gamma - \limsup F^\varepsilon(u) \ .$$

In particular, if $F^\varepsilon := F$ for every ε, then

$$\overline{F}(u) = (F_0)^{**}(\varphi_u) \ .$$

The proof of Theorem 2.1 rests upon the following result. Let V an open subset Y and set

$$\theta_V(w) := \begin{cases} \inf\Big\{\nu(V) \ : \ [\nu] = w \ , \ \nu \in \mathcal{P}(K)\Big\} & \text{if } w \in K \\ +\infty & \text{otherwise} \end{cases} \tag{2.4}$$

where $\mathcal{P}(K)$ denotes the space of probability measures on Y supported on compact subset K and $[\nu]$ denotes the barycenter (i.e. $\int_K g d\nu = g([\nu])$ for every continuous linear form $g \in Y^*$, see [51]).

Lemma 2.2. *The function θ_V is convex, l.s.c and satisfies*

$$0 \le \theta_V \le 1 \text{ in } K \ , \ \theta_V = 0 \ \text{ in } K \setminus V \ , \ \theta_V = 1 \text{ in } \ddot{K} \cap V \ .$$

Moreover θ_V agrees with the convex l.s.c envelope of the function $\mathbf{1}_V + \chi_K$ (where $\chi_K = 0$ on K and $\chi_K = +\infty$ on $Y \setminus K$). It vanishes identically on K whenever $V \cap \ddot{K} = \emptyset$.

Proof. We recall that K being compact and metrizable, the set of probability measures $\mathcal{K}$ is a weakly-star compact subset on which the affine map $\nu \mapsto [\nu]$ is continuous and takes values in K. It is then straightforward to check that the function $\theta_V : Y \to [0, +\infty]$ is convex l.s.c. on Y. If $w \in K$, by taking ν to be the Dirac mass at w in (2.4), we infer that $\theta_V(w) \in [0, 1]$ whereas $\theta_V(w) = 0$ if $w \notin V$. If $w \in V$ is an extreme point of K, the latter choice $\nu = \delta_w$ turns out to be the unique one compatible with the condition $[\nu] = w$ and in this case we get $\theta_V(w) = \delta_w(V) = 1$. In fact it is a Choquet integral representation Theorem (see [51, Thm 25, p 283]) that every $w \in K$ is the barycenter of a suitable probability measure supported on $\ddot{K}$, thus θ_V vanishes identically on K whenever $V \cap \ddot{K} = \emptyset$.

On the other hand let us compute the Moreau-Fenchel conjugate of θ_V. For every $g \in Y^*$, we have

$$\begin{aligned} \theta_V^*(g) = \sup_{w\in K}\{< g, w > -\theta_V(w)\} &= \sup_{\nu\in\mathcal{P}(K)} \left\{\int_K (< g, w > -\mathbf{1}_V(w))\, \nu(dw)\right\} \\ &= \sup_{w\in K}\{< g, w > -\mathbf{1}_V(w)\} \ = \ (\mathbf{1}_V + \chi_K)^*(g), \end{aligned}$$

where, for the third equality, we used that the supremum over $\mathcal{P}(K)$ is reached by Dirac masses. As θ_V is convex l.s.c., we deduce that $\theta_V = (\theta_V)^{**} = (\mathbf{1}_V + \chi_K)^{**}$. Thus we have proved that θ_V coincides with the convex l.s.c. envelop of $\mathbf{1}_V + \chi_K$. ∎

Proof of Theorem 2.1. By the assumption (H2), the sequential caracterizations (2.1) for F', F'' can be used restricting ourselves to sequences $u_\varepsilon \to u$ where u_ε belong to a fixed compact subset $C(= C_R) \subset X$. Since the embedding $\varphi : C \mapsto \varphi(C)$ is bicontinuous, the convergences $u_\varepsilon \to u$ or $\varphi_{u_\varepsilon} \to \varphi_u$ are equivalent. Thus, by the indentity $F(u_\varepsilon) = F_0^\varepsilon(\varphi_{u_\varepsilon})$ (notice that $\Gamma-\liminf F_0^\varepsilon(v) = +\infty$ whenever $v \notin \varphi(X)$), we infer that

$$F_0' = F' \circ \varphi \quad , \quad F_0'' = F'' \circ \varphi \ .$$

We are therefore reduced to showing that for every $u \in X$:

i) $F_0'(\varphi_u) = \Gamma-\liminf F_0^\varepsilon(\varphi_u) \le \Gamma-\liminf (F_0^\varepsilon)^{**}(\varphi_u)$,

ii) $\Gamma-\limsup (F_0^\varepsilon)^{**}(\varphi_u) \le \Gamma - \limsup F_0^\varepsilon(\varphi_u) = F_0''(\varphi_u)$.

The inequality ii) is obvious since $(F_0^\varepsilon)^{**} \le F_0^\varepsilon$. Let us show i). Let v in $\ddot{K}$ and choose a real $t < F_0'(v)$. Then, by using the topological characterization of F_0' (see (2.3)), we may find a suitable open neighbourhood V of v such that $t < \inf_V F_\varepsilon^0$ holds for ε small enough. For such ε, we have $F_0^\varepsilon \ge t\,\theta_V$. Then, by using lemma 2.4 and by passing to the biconjugate, we obtain $(F_0^\varepsilon)^{**} \ \ge \ t\,(\theta_V)^{**} = t\,\theta_V$. We deduce

$$\Gamma-\liminf (F_0^\varepsilon)^{**}(v) \ \ge \ t\,\theta_V(v) \ = \ t \ .$$

The claim i) follows since by (H1) we have $\varphi_u \in \ddot{K}$ for every $u \in X$. ■

Homogeneous variant: In many cases the convex compact subset K appears to be the base of a closed convex cone. Namely we make the additional assumption

(*H*3) *There exists a continuous linear form* $l_0 \in Y^*$ *such that* $l_0 = 1$ *on* K .

This assumption allows us to simplify our duality scheme. For every $\varepsilon > 0$, we introduce the convex set of Y^*:

$$D_\varepsilon \ := \ \{g \in Y^* \ : \ (F_0^\varepsilon)^*(g) \ < \ 0\} \ = \ \{g \in Y^* \ : < \varphi_u, g > < \ F^\varepsilon(u) \ \forall u \in X\} \ ,$$

and in a similar way, we define D', $D'' \subset Y^*$ as

$$D' \ := \ \{g \in Y^* \ : \ (F_0')^*(g) \ \le \ 0\} \quad , \quad D'' \ := \ \{g \in Y^* \ : \ (F_0'')^*(g) \ \le \ 0\} \ . \tag{2.5}$$

Note that all results hereafter are unchanged if D_ε is defined alternatively with a large inequality. It turns out that functionals $(F_0^\varepsilon)^{**}, (F_0')^{**}, (F_0'')^{**}$ agree on K with the support functions of D_ε, D', D'' respectively (they are one homogeneous convex, l.s.c. functionals on Y).

Lemma 2.3. *For every* $v \in Y$ *with* $< v, l_0 >= 1$ *(in particular for* $v \in \varphi(X)$*), one has*

$$(F_0^\varepsilon)^{**}(v) \ = \ \sup_{g \in D_\varepsilon} < v, g > \quad , \quad (F_0')^{**}(v) \ = \ \sup_{g \in D'} < v, g > \quad , \quad (F_0'')^{**}(v) \ = \ \sup_{g \in D''} < v, g > \ .$$

Proof. Clearly, we have for every $v \in Y$,

$$(F_0^\varepsilon)^{**}(v) \ = \ \sup_{g \in Y^*} < v, g > -(F_0^\varepsilon)^*(g) \ge \ \sup_{g \in D_\varepsilon} < v, g > .$$

We need to prove that the converse inequality holds if v satisfies $< v, l_0 >= 1$. We notice that under (H3), we have for every $g \in Y^*$ and $\lambda \in \mathbb{R}$

$$(F_0^\varepsilon)^*(g - \lambda l_0) \ = \ \sup_{u \in X} \{< \varphi_u, g - \lambda l_0 > - F^\varepsilon(u)\} \ = \ (F_0^\varepsilon)^*(g) - \lambda \ . \tag{2.6}$$

In particular, by applying the identity above with $\lambda > (F_0^\varepsilon)^*(g)$, we obtain that $g_\lambda := g - \lambda l_0$ satisfies $(F_0^\varepsilon)^*(g_\lambda) < 0$ thus $g_\lambda \in D_\varepsilon$. Therefore, for every v such that $< v, l_0 >= 1$, one has

$$< v, g > -(F_0^\varepsilon)^*(g) \;= < v, g_\lambda > +\lambda - (F_0^\varepsilon)^*(g) \;\leq \sup_{h \in D_\varepsilon} < v, h > +\lambda - (F_0^\varepsilon)^*(g)$$

hence the desired inequality letting $\lambda \searrow (F_0^\varepsilon)^*(g)$. The proof is the same for $(F_0')^{**}$ and $(F_0'')^{**}$. ■

In the next lemma we establish a comparison between the sets D', D'' defined above and the lower and upper Kuratowski limits of the sets D_ε in Y^*, where Y^* is equipped with the strong topology (that is, the topology of the uniform convergence on the compact subsets of Y). Let us denote by $Li(D_\varepsilon)$ and $Ls(D_\varepsilon)$ these lower and upper Kuratowski limits. There are closed subset of Y^* whose indicator functions (see [45]) coincide respectively with $\Gamma-\liminf \chi_{D_\varepsilon}$ and $\Gamma-\limsup \chi_{D_\varepsilon}$. In other words:

i) $g \in Li(D_\varepsilon)$ iff there exist $g_\varepsilon \in D_\varepsilon$ such that $g_\varepsilon \to g$.

ii) $g \in Ls(D_\varepsilon)$ iff there exist a subsequenceg'_ε, with $g'_\varepsilon \in D'_\varepsilon$, such that $g'_\varepsilon \to g$.

Lemma 2.4. *With the notations above and D', D'' defined by (2.5), we have the following inclusions:*

i) $(F_0')^{**} \leq G' \leq G'' \leq (F_0'')^{**}$.

ii) $D' \subseteq Li(D_\varepsilon) \subseteq Ls(D_\varepsilon) \subseteq D''$.

Proof. Since K is compact and by exploiting the definition of F_0' on a minimizing sequence, it is easy to check that, for every $g \in Y^*$, it holds

$$-\limsup_\varepsilon (F_0^\varepsilon)^*(g) \;=\; \liminf_\varepsilon \inf_K \{F_0^\varepsilon - < \cdot, g >\} \;\geq\; \inf_K \{F_0' - < \cdot, g >\} \;=\; -(F_0')^*(g)\;. \quad (2.7)$$

Let us show i). Since $(F_0^\varepsilon)^{**} \leq F_0^\varepsilon$, we have $G'' \leq F_0''$. The inequality $G'' \leq (F_0'')^{**}$ is then a consequence of the fact that G'' is l.s.c and convex (as the $\Gamma-\limsup$-limit of sequence of convex functions). On the other hand, for every $g \in Y^*$ and every sequence $\{v_\varepsilon\}$ converging to v in Y, one has:

$$\liminf_\varepsilon (F_0^\varepsilon)^{**}(v_\varepsilon) \;\geq < v, g > - \limsup_\varepsilon (F_0^\varepsilon)^*(g) \geq < g, v > -(F_0')^*(g)$$

where we used Moreau-Fenchel inequality and (2.7). Thus $G'(v) \geq < g, v > -(F_0')^*(g)$. The inequality $(F_0')^{**} \leq G'$ follows by taking the supremum with repect to $g \in Y^*$.

Let us show ii): Let $g \in D'$ and assume first that $(F_0')^*(g) < 0$. Then, by (2.7), one has $(F_0^\varepsilon)^*(g) \leq 0$ for ε small enough (hence $g \in D_\varepsilon$) so that g belongs to $Li(D_\varepsilon)$. This conclusion can be extended to an element $g \in D'$ such that $(F_0')^*(g) = 0$. Indeed let $g_n := g - (1/n)\, l_0$. Then, by (2.6), $(F_0')^*(g_n) = -1/n < 0$. Therefore g_n belongs to the closed subset $Li(D_\varepsilon)$ while $g_n \to g$ as $n \to \infty$. Eventually we have proved that $D' \subseteq Li(D_\varepsilon)$.

It remains to show that $Ls(D_\varepsilon) \subseteq D''$. Let $g \in Ls(D_\varepsilon)$ and $v \in Y^*$. By the (sequential) definitions of $Ls(D_\varepsilon)$ and of F_0'', there exits a sequence $(v_\varepsilon, g_\varepsilon) \in Y \times Y^*$ such that

$$g_\varepsilon \in D_\varepsilon \text{ and } \; g_\varepsilon \to g \text{ strongly in } Y^* \quad , \quad v_\varepsilon \to v \;\text{ in } Y \text{ and } \limsup_\varepsilon F_0^\varepsilon(v_\varepsilon) \leq F_0''(v)\;.$$

Then, by applying Moreau Fenchel-inequality and (2.7), we are led to

$$< v, g > = \lim_\varepsilon < v_\varepsilon, g_\varepsilon > \leq \limsup_\varepsilon F_0^\varepsilon(v_\varepsilon) + \limsup_\varepsilon (F_0^\varepsilon)^*(g_\varepsilon) \;\leq\; F_0''(v) + 0\;,$$

holding for every $v \in Y$. Thus owing to the definition of D'' in (2.5), we get $g \in D''$. This proves that $Ls(D_\varepsilon) \subseteq D''$. ■

To summarize this section we give the following practical result, that will be useful in the applications, where we need to identify the Γ–limit of a sequence $\{F^\varepsilon\}$. Notice that by Kuratowski's compactness theorem [7] [45] and our assumptions (H1)(H2) (which allow to treat X as a separable metric space) such a Γ–limit exists, at least for a subsequence of $\{F^\varepsilon\}$.

Theorem 2.5. *Assume that (H1), (H2), (H3) hold. Then, the following three assertions are equivalent:*

i) F^ε *Γ–converges to a limit* F *in* X *(i.e.* $F' = F'' = F$*).*

ii) $(F_0^\varepsilon)^{**}$ *Γ–converges to a limit* G *in* Y *(i.e.* $G' = G'' = G$*).*

iii) D_ε *converges in the Kuratowski sense to a set* D *in the strong topology of* Y^* *(i.e.* $D' = D'' = D$*).*

In addition, if one of these assertions holds true, then F*,* G *and* D *satisfy the relations*

$$D = \{g \in Y^* : < \varphi_u, g > \leq F(u) \ \forall u \in X\}, \tag{2.8}$$

$$G(v) = \sup_{g \in D} < v, g > \quad \text{if } < v, l_0 >= 1, \quad G(v) = +\infty \text{ otherwise}, \tag{2.9}$$

$$F(u) = G(\varphi_u) = \sup_{g \in D} < \varphi_u, g > . \tag{2.10}$$

Proof. If $F^\varepsilon \xrightarrow{\Gamma} F$, then by definition $F = F' = F''$ so that $F_0' = F_0''$ and $D' = D''$. We conclude that ii) and iii) hold by invoking Lemma 2.4. We have $G = (F_0')^{**} = (F_0'')^{**}$ and $D = D' = D''$, showing (2.8), (2.9) and (2.10) as a consequence of Theorem2.1

Conversely, assume ii) or iii) holds. By compactness we consider F and a subsequence $\{F^{\varepsilon_k}\}$ that Γ–converges to F. Then, the reconstruction formula (2.10) shows that the limit F is uniquely determined. Hence the whole sequence $\{F^\varepsilon\}$ Γ–converges to F. ■

2.3 Application to non convex variational problems

We now apply the framework developed in Section 2.2 to the following situation. Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$. We consider the embedding of $X = L^1(\Omega)$ into $Y = L^\infty(\Omega \times \mathbb{R})$ defined by:

$$\varphi : u \in X \mapsto \mathbf{1}_u \in Y, \qquad \mathbf{1}_u(x,t) := \begin{cases} 1 & \text{if } u(x) > t \\ 0 & \text{if } u(x) \leq t. \end{cases}$$

Let us consider

$$K := \{v \in L^\infty(\Omega \times \mathbb{R}) \ : \ v(x,t) \in [0,1] \ a.e.(x,t) \in \Omega \times \mathbb{R}\} .$$

It is a compact subset of $L^\infty(\Omega \times \mathbb{R})$ equipped with its weak-star topology (we are in the case where $Y = Z^*$ if we set $Z = L^1(\Omega \times \mathbb{R})$). It is easy to check that $\mathbf{1}_u$ is an extreme point of K as it takes values in $\{0,1\}$. Moreover the map $u \mapsto \mathbf{1}_u$ is continuous from $L^1(\Omega)$ to $L^\infty(\Omega \times \mathbb{R})$ (embedded with its weak star topology).

Let $F : u \in L^1(\Omega) \to \mathbb{R} \cup +\infty$ be a possibly non convex functional. We simply assume that F is l.s.c. (with respect to the strong convergence in $L^1(\Omega)$) and that the following coercivity assumption holds:

$$\begin{cases} F(u) \geq k \|u\| - \frac{1}{k} \quad , \text{ for suitable constant } k > 0. \\ \text{For every } R > 0, \text{ the set } \{u : F(u) \leq R\} \text{ is a compact subset of } L^1(\Omega). \end{cases} \tag{2.11}$$

We consider the minimization problem

$$(\mathcal{P}) \qquad \inf\Big\{F(u) \ : \ u \in L^1(\Omega)\Big\}\,.$$

Under the assumption (2.11), this problem has at least one solution and the set of solutions $\mathrm{Argmin}\,\mathcal{P}$ is a non void compact subset of $L^1(\Omega)$ (since F is not convex, we expect à priori multiple solutions).

Following the construction developed in Section 2.2, we define for every pair $(v,g) \in L^\infty(\Omega\times\mathbb{R})\times L^1(\Omega\times\mathbb{R})$

$$F_0^*(g) = \sup_{u\in L^1(\Omega)}\left\{\int_{\Omega\times\mathbb{R}} g(x,t)\,\mathbf{1}_u\,dxdt - F(u)\right\}\ ,\ G(v) = \sup_{g\in L^\infty(\Omega\times\mathbb{R})}\left\{\int_{\Omega\times\mathbb{R}} g\,v\,dxdt - F_0^*(g)\right\} \tag{2.12}$$

Since F coincides with its l.s.c. envelope, it is a consequence of Theorem 2.1 that it holds

$$G(\mathbf{1}_u) \ = \ F(u) \quad \text{for all } u \in L^1(\Omega) \tag{2.13}$$

Our convexification recipe leads to the following convex optimization problem

$$(\mathcal{Q}) \qquad \inf\Big\{G(v) \ : \ v \in L^\infty(\Omega\times\mathbb{R};[0,1])\Big\},$$

whose set of solutions $\mathrm{Argmin}(\mathcal{Q})$ is a non empty weakly star compact subset of $L^\infty(\Omega\times\mathbb{R};[0,1])$.

Lemma 2.6. *It holds* $\inf(\mathcal{P}) = \inf(\mathcal{Q})$ *and the following equivalence holds:*

$$u \in \mathrm{Argmin}(\mathcal{P}) \iff \mathbf{1}_u \in \mathrm{Argmin}(\mathcal{Q})\ .$$

Proof. Applying (2.12) with $g = 0$, we get

$$\inf(\mathcal{Q}) = -G^*(0) = -(F_0)^*(0) = -\sup\{-F(u) : u \in X\} = \inf(\mathcal{P})\,.$$

The equivalence statement follows by using the identity (2.13). ∎

The next step is twofold: first we have to identify the convexified energy in practice in order to settle a duality scheme for $\mathcal{Q}$; then, as some solutions v for $(\mathcal{Q})$ may take intermediate values in $(0,1)$ (i.e. v is not of the form $\mathbf{1}_u$), we have to specify how solutions to $(\mathcal{P})$ can be recovered.

A complete answer to these two requirements will be obtained under an additional assumption on functional F. We will use the following slicing argument on the class

$$\mathcal{A} := \{v \in L^\infty(\Omega\times\mathbb{R}) \ : \ v(x,\cdot) \text{ non increasing (a.e. } x\in\Omega)\ ,\ v(x,-\infty) = 1\ ,\ v(x,+\infty) = 0\}\,.$$

For every $v \in \mathcal{A}$ and $s \in [0,1]$, let us define

$$u_s(x) := \inf\{\tau \in \mathbb{R} \ : \ v(x,\tau) \le s\}\,. \tag{2.14}$$

Notice that, by construction, the subgraph of u_s agrees up to a Lebesgue negligible set with the level set $\{\tau \in \mathbb{R} \ : \ v(x,\tau) > s\}$, namely

$$\mathbf{1}_{u_s}(x,t) = \mathbf{1}_{\{v>s\}}(x,t) \qquad \text{for a.e. } (x,t) \in \Omega\times\mathbb{R}\,. \tag{2.15}$$

In what follows we denote by v_0 the element of $\mathcal{A}$ defined by

$$v_0(x,t) := \mathbf{1}_{\{t>0\}} \quad (\text{that is } v_0 = \mathbf{1}_{u_0} \text{ with } u_0 \equiv 0)\ .$$

Definition 2.7. We say that a functional $J : L^\infty(\Omega \times \mathbb{R}) \to [0, +\infty]$ satisfies the *generalized coarea formula* if for every $v \in L^\infty(\Omega \times \mathbb{R})$ the function $s \mapsto J(\mathbf{1}_{\{v>s\}})$ is Lebesgue-measurable on $\mathbb{R}$ and there holds

$$J(v) = \int_{-\infty}^{+\infty} J(\mathbf{1}_{\{v>s\}})\, ds \qquad \forall v \in L^\infty(\Omega \times \mathbb{R})\,. \tag{2.16}$$

It is readily seen that a functional J satisfying the generalized coarea formula has to be positively 1-homogeneous (*i.e.* $J(\lambda v) = \lambda J(v)$ for all $\lambda \geq 0$) and that $J(v)$ vanishes for constant functions v.

Theorem 2.8. *Assume that F satisfies (2.11) and that there exists a convex and weakly-star l.s.c functional $J : L^\infty(\Omega \times \mathbb{R}) \to [0, +\infty]$ satisfying the generalized co-area formula and such that*

$$J(\mathbf{1}_u) \;=\; F(u) \qquad \textit{for every } u \in L^1(\Omega) \tag{2.17}$$

Then, if $\{u_s; s \in [0,1]\}$ is the parametrized family associated to v through (2.14), it holds

$$G(v) = \begin{cases} \displaystyle\int_0^1 F(u_s)\, ds & \textit{if } v \in \mathcal{A} \\ +\infty & \textit{otherwise} \end{cases} \tag{2.18}$$

Therefore, if $v \in \operatorname{Argmin} G$, then $v \in \mathcal{A}$ and $u_s \in \operatorname{Argmin} F$ for $\mathcal{L}^1$-a.e. $s \in (0,1)$. In particular, if the initial problem $\mathcal{P}$ admits a finite number of solutions $\{u^1, \dots, u^K\}$, then

$$\operatorname{Argmin} G \;=\; \left\{ \sum_{k=1}^K t_k\, \mathbf{1}_{u^k} \;:\; t_k \geq 0\,,\; \sum_{k=1}^K t_k = 1 \right\}, \tag{2.19}$$

meaning that a solution v of problem $\mathcal{Q}$ must be a piecewise constant function.

It is remarkable consequence of Theorem 2.8 that a global minimizer for problem $(\mathcal{Q})$ suitably chosen (taking $t_k \in (0,1)$ in (2.19)) can encode all the possibly mutiple solutions to problem $(\mathcal{P})$. We refer to [20] for the numerical illustration of this nice feature.

Before giving the proof let us notice first that the coarea condition (2.17) is used merely to minorize G. An upper bound for G is provided in the general case owing to the following result:

Lemma 2.9. *Let $v \in L^\infty(\Omega \times \mathbb{R})$ such that $G(v) < +\infty$. Then $v \in \mathcal{A}$ and it holds*

$$G(v) \;\leq\; \int_0^1 F(u_s)\, ds \qquad \textit{with } u_s \textit{ defined by (2.14)}\,.$$

Remark: By a slight modification of the proof, it is possible to show that the conclusions of Lemma 2.9 still hold if the first condition in (2.11) is replaced by: $F(u) \geq \int_\Omega \beta(|u|)$ where $\beta : \mathbb{R}^+ \to [0, +\infty]$ is non decreasing with $\beta(+\infty) = \infty$)

Proof. By using Fubini formula, one checks easily that, for every $u \in L^1(\Omega)$, one has

$$\int_\Omega |u|\, dx \;=\; \int_{\Omega\times\mathbb{R}} |\mathbf{1}_u - v_0|\; dx\, dt\,.$$

Therefore, by (2.11), for every v it holds:

$$F_0(v) \geq\; H(v) := \begin{cases} k \displaystyle\int_{\Omega\times\mathbb{R}} |v - v_0|\; dx\, dt - \frac{1}{k} & \text{if } v \in \mathcal{A} \\ +\infty & \text{otherwise} \end{cases}$$

It is easy to check that $H(v)$ is convex and weakly l.s.c. Indeed if $\liminf J(v_n) < +\infty$ holds for a sequence v_n in $\mathcal{A}$ such that $v_n \to v$ weakly-star, then $v(x,\cdot)$ is still non increasing and the inequality $\liminf \int_{\Omega\times\mathbb{R}} |v_n - v_0| \geq \int_{\Omega\times\mathbb{R}} |v - v_0|$ shows that $\int_{\mathbb{R}} |v-v_0|(x,t)\,dt < +\infty$ for a.e. $x \in \Omega$. Thus $v(x,+\infty) = 0$ and $v(x,-\infty) = 1$. It follows that $v \in \mathcal{A}$ and $\liminf H(v_h) \geq H(v)$. Now we may conclude by simply saying that $H = F_0^{**} \geq J$ so that $G(v) < +\infty$ implies that $H(v) < +\infty$ hence $v \in \mathcal{A}$ (in addition we get $(v - v_0) \in L^1(\Omega \times \mathbb{R})$).

For the second assertion, we apply Jensen's inequality to the convex functional G and to the family of functions $\{v_s, s \in [0,1]\}$ where $v_s(x,t) = \mathbf{1}_{u_s}(x,t)$ (see (2.15)). One checks easily that $\int_0^1 v_s(x,t)\,ds = v(x,t)$. Thus, recalling that $G(\mathbf{1}_{u_s}) = F(u_s)$ holds by Theorem 2.1, we conclude that

$$G(v) \;\leq\; \int_0^1 G(v_s)\,ds \;=\; \int_0^1 F(u_s)\,ds\;.$$

■

Proof of Theorem 2.8. By the definition of F_0, it holds $J \leq F_0$. Thus, as J is convex l.s.c., by taking the biconjugates we infer that $J \leq F_0^{**} = G$. Let $v \in \mathcal{A}$. By applying the assumption (2.17), we derive that $J(v) = \int_0^1 F(u_s)\,ds$ being u_s defined by by (2.14). Thus $G(v) \geq \int_0^1 F(u_s)\,ds$. By invoking Lemma 2.9, we are led to the identity (2.18). Assume now that $v \in \operatorname{Argmin} G$. By Lemma 2.6, F and G share the same infimum value which is a finite real α. Therefore, as $0 = G(v) - \alpha = \int_0^1 (F(u_s) - \alpha)\,ds$, we deduce that $u_s \in \operatorname{Argmin} F$ for a.e. $s \in [0,1]$. Therefore $(\mathcal{P})$ has infinitly many solutions unless v is piecewise constant. Conversely, if $(\mathcal{P})$ has a finite set of solutions $\{u^k \,:\, 1 \leq k \leq K\}$, then it is straightforward that the set of solutions to $(\mathcal{Q})$ coincides with the convex hull of $\{\mathbf{1}_{u^k} \,:\, 1 \leq k \leq K\}$. ■

2.4 Duality schemes and examples

A large class of functionals satisfying the assumptions required in Theorem 2.8 are of the kind

$$F_\lambda(u) = \begin{cases} \displaystyle\int_\Omega f(u, \nabla u)\,dx - \lambda \int_\Omega p(x)u\,dx\;, & \text{if } u \in W_0^{1,2}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

where the integrand $f = f(t,z)$ is a function $f : \mathbb{R} \times \mathbb{R}^N \to (-\infty, +\infty]$ sastisfying:

$$\forall t \in \mathbb{R}\,,\;\; f(t,\cdot) \text{ is convex} \;\;,\;\; f \text{ is l.s.c. on } \mathbb{R}^N \times \mathbb{R} \;\;,\;\; f(t,z) \geq k|z|^2 - \frac{1}{k} \tag{2.20}$$

where $k > 0$, λ is a non negative parameter and the source term $p(x)$ (load) belongs to $L^{r'}(\Omega)$ (r' conjugate exponent of r) where r is compatible with the embedding $W_0^{1,2}(\Omega) \subset L^r(\Omega)$ (that is $r \leq \frac{2N}{N+2}$ if $N \geq 3$, $r < +\infty$ if $N = 2$).

Notice that here the non convexity of the energy density $f(u, \nabla u)$ involves only the dependence with respect to u . In fact the convexity with respect to the gradient part is necessary to obtain lower semicontinuity for $F(u)$ and well posedness for the primal problem. It turns out that the condition (2.17) is satisfied by considering the convex 1-homogeneous functional defined by:

$$J(v) := \int_{\Omega\times\mathbb{R}} h_f(t, Dv) \quad \text{where} \quad h_f(t, z^x, z^t) := \begin{cases} -z^t\, f\Big(t, \frac{z^x}{-z^t}\Big) & \text{if } z^t < 0 \\ +\infty & \text{if } z^t \geq 0 \end{cases} \tag{2.21}$$

We refer to the recent paper [20] for further details namely for a proof of the coarea formula.

2.4.1 Dual problem in $\Omega \times \mathbb{R}$

Let us decribe the dual problem in the simpler case where f is of the form $f(t,z) = g(t) + \varphi(z)$ being $\varphi : \mathbb{R}^N \to \mathbb{R}_+$ convex continuous with $\varphi(0) = 0$ and $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function with possibly countably many discontinuities. The primal problem reads

$$(\mathcal{P}_\lambda) \qquad \inf \left\{ \int_\Omega (\varphi(\nabla u) + g(u))\, dx - \lambda \int_\Omega p(x) u\, dx \ : \ u \in H^1_0(\Omega) \right\}$$

and its convexified version

$$(\mathcal{Q}_\lambda) \qquad \inf \left\{ \int_{\Omega\times\mathbb{R}} h_f(t, Dv) - \lambda \int_{\Omega\times\mathbb{R}} p(x)(v - v_0)\, dxdt \ : \ v \in \mathcal{A}\ ,\ v - v_0 \in BV_0(\Omega \times \mathbb{R}) \right\}$$

where $BV_0(\Omega \times \mathbb{R})$ denotes the set of integrable functions with bounded variations on $\Omega \times \mathbb{R}$ and whose trace vanishes on the lateral boundary $\partial\Omega \times \mathbb{R}$ (see [103]).

The dual problem to our non convex problem $(\mathcal{P}_\lambda)$ is then recovered be applying classical duality to problem $(\mathcal{Q}_\lambda)$. The competitors of this dual problem $(\mathcal{P}^*_\lambda)$ are vector fields $\sigma = (\sigma^x, \sigma^t) : \Omega \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$ we take in the class

$$X_1(\Omega \times \mathbb{R}) = \left\{ \sigma \in L^\infty(\Omega \times \mathbb{R}; \mathbb{R}^{N+1}) \ : \ \operatorname{div} \sigma \in L^1_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}) \right\}\ ,$$

and $(\mathcal{P}^*_\lambda)$ consists in the following maximal flux problem:

$$(\mathcal{P}^*_\lambda) \qquad \sup \left\{ -\int_\Omega \sigma^t(x,0)\, dx \ : \ \sigma \in \mathcal{K}\ ,\ -\operatorname{div} \sigma = \lambda p \quad \text{in } \Omega \times \mathbb{R} \right\}\ ,$$

where $\sigma \in \mathcal{K}$ means that the vector field $\sigma \in X_1(\Omega \times \mathbb{R})$ satisfies the pointwise (convex) constraints:

$$\begin{cases} \sigma^t(x,t) \geq \varphi^*(\sigma^x(x,t)) - g(t) & \text{for } \mathcal{L}^{N+1}\text{-a.e. } (x,t) \in \Omega \times \mathbb{R} \\ \sigma^t(x,t) \geq -g(t) \quad \forall\, t \in S_g \text{ and for } \mathcal{L}^N\text{-a.e. } x \in \Omega\,, \end{cases} \tag{2.22}$$

where S_g is the set of discontinuities of g.

Notice here that the regularity condition $\sigma \in X_1(\Omega \times \mathbb{R})$ is required in order to be able to define the normal trace of σ on a N-dimensional rectifiable subset of $\mathbb{R}^{N+1}$ (in particular for every t, $\sigma^t(\cdot, t)$ is well defined for a.e. $x \in \Omega$). This allows also to compute the flux of σ through the graph of any competitor u for $(\mathcal{P}_\lambda)$. By applying a generalized Green formula for duality pairings (v, σ), it is possible to show (see [20] for details) that, for every admissible pair (u, σ) one has:

$$\int_\Omega f(u, \nabla u)\, dx - \lambda \int_\Omega p(x) u\, dx \quad \geq \quad -\int_\Omega \sigma^t(x,0)\, dx\ ,$$

thus it holds $\inf(\mathcal{P}_\lambda) \ \geq \ \sup(\mathcal{P}^*_\lambda)$. The core of our duality theory is the following no-gap result (see [20] for a complete proof in the case $p = 0$)

Theorem 2.10.

$$\inf(\mathcal{P}_\lambda) \quad = \quad \sup(\mathcal{P}^*_\lambda)\ .$$

Among consequences of of Theorem 2.10, we can derive (see [20]) necessary and sufficient conditions for a global optimum of $(\mathcal{P}_\lambda)$, thus allowing to rule out local minimizers which are non global ones. A second consequence is a saddle point characterization which fits to the implementation of efficient primal-dual algorithms ([95]). This is described in the next subsection.

Remark 2.11. The result above can be extended to mixed Dirichlet-Neumann conditions. In particular if $u = u_0$ is prescribed on a subset $\Gamma_0 \subset \partial\Omega$ for some $u_0 \in W^{1,2}(\Omega)$, then the competitors σ for $(\mathcal{P}^*_\lambda)$ have to satisfy a vanishing normal trace condition on $(\partial\Omega \setminus \Gamma_0) \times \mathbb{R}$, while the linear term to be maximized becomes $\int_{G_{u_0}} \sigma \cdot \nu_{u_0} d\mathcal{H}^N$, being G_{u_0} the graph of u_0 with unit normal ν_{u_0} pointing downwards. For the more delicate case of Robin type conditions, we refer to [20].

Remark 2.12. When the boundary datum on Γ_0 is a bounded function u_0, it exists in general a priori lower and upper bound for the minimizers of the primal problem $(\mathcal{P}_\lambda)$. In this case the infimum is unchanged if we impose u to take values in a suitable closed interval $\overline{I} := [m, M]$ of the real line. We are thus led to consider the variant of the primal problem $(\mathcal{P}_\lambda)$ where the class of admissible functions is restricted to $\Big\{u \in W^{1,2}(\Omega; I) \ : \ u = u_0 \text{ on } \Gamma_0\Big\}$. The duality result continues to hold (with a simpler proof), provided the admissible fields in the dual problem $(\mathcal{P}^*_\lambda)$ are taken in the class $\mathcal{K}(u_0, \Gamma_0, I)$ of elements $\sigma \in X_1(\Omega \times I)$ satisfying the pointwise constraints (2.22) on $\Omega \times \overline{I}$ and the equilibrium conditions

$$-\operatorname{div} \sigma = \lambda\, p(x) \ \text{ in } \Omega \times I \quad , \quad \sigma^x \cdot \nu_\Omega = 0 \ \text{ on } \ (\partial\Omega \setminus \Gamma_0) \times I \ .$$

Accordingly the convexified problem $(\mathcal{Q}_\lambda)$ becomes

$$(\mathcal{Q}_\lambda) \qquad \inf \left\{ \int_{\Omega\times I} h_f(t, Dv) - \lambda \int_{\Omega\times I} p(x)(v - v_0)\, dxdt \ : \ v \in \mathcal{A}(u_0, \Gamma_0, I) \right\}$$

where the set of admissible functions v is given by

$$\mathcal{A}(u_0, \Gamma_0, I) := \Big\{ v \in BV(\Omega \times I; [0,1]) \ : \ v = 1 \text{ on } \Omega \times \{m\},\ v = 0 \text{ on } \Omega \times \{M\},\ v = \mathbf{1}_{u_0} \text{ on } \Gamma_0 \times I \Big\}$$

(the condition $\int_{\Omega\times I} h_f(t, Dv) < +\infty$ implies implicitely that $v(x, \cdot)$ is monotone non increasing).

Remark 2.13. The growth condition (2.20) with exponent $r = 2$ can be considered with a different exponent $r \in (1, +\infty)$ and all the statements can be reformulated accordingly. The case $r = 1$ works pretty well for the dual problem see [19] but a lot of attention has to be devoted for the existence and compactness issue in the primal problem. Indeed the functional F is no more l.s.c. in $L^1(\Omega)$ and has to be relaxed in the space $BV(\Omega)$ whereas, in order that $\inf(\mathcal{P}_\lambda) > -\infty$, we need to chose λ in a finite interval $[0, \lambda^*)$ (this is in relation with the limit load problem is plasticity [24]).

To have in mind a prototype situation, let us mention for instance the free boundary problem studied in the seminal paper [3]:

$$\inf \left\{ \int_\Omega \frac{1}{2} |\nabla u|^2\, dx + \kappa |\{u > 0\}| \ : \ u \in W^{1,2}(\Omega)\,,\ u = 1 \text{ on } \partial\Omega \right\}, \tag{2.23}$$

the free boundary being the frontier of the positivity set $\{u > 0\}$ (see Figure 2.1, in which $\Omega = (0,1)^2 \subset \mathbb{R}^2$).

Clearly, problem (2.23) falls into this general framework, by taking $\Gamma_0 = \partial\Omega$, $u_0 \equiv 1$, $p \equiv 0$ and $\varphi(z) = \frac{1}{2}|z|^2$ and $g(t) = \kappa\, \mathbf{1}_{(0,+\infty)}(t)$ which jumps at $t = 0$. It is easy to check that solutions u exist and satify $0 \le u \le 1$ a.e. so that Remark 2.12 applies and the dual problem can be restricted to vector fields defined in $\Omega \times (0,1)$. Let $\mathcal{K}$ be the set of $\sigma \in X_1(\Omega \times (0,1))$ such that

$$\sigma^t(x,t) + \kappa \ge \frac{1}{2} |\sigma^x(x,t)|^2 \quad \text{a.e. on } \Omega \times \mathbb{R}\,, \qquad \sigma^t(x,0) \ge 0 \text{ a.e. on } \Omega\,.$$

The dual problem reads:

$$\sup\Big\{ -\int_\Omega \sigma^t(x,1)\,dx \;:\; \sigma\in\mathcal{K}\,,\ \operatorname{div}\sigma = 0 \quad \text{in } \Omega\times(0,1)\Big\}\,. \tag{2.24}$$

Notice that the integral on Ω represents the flux of σ across the graph of the boundary datum $u_0\equiv 1$. Thus problem (2.24) has a nice fluid mechanic interpretation: it consists in maximizing the downflow through the top face $\Omega\times\{1\}$ of an incompressible fluid constrained into the cylinder $\Omega\times\mathbb{R}$, whose speed σ satisfies the conditions above, preventing in particular the fluid to pass across the bottom face (see Figure 2.1 (b)).

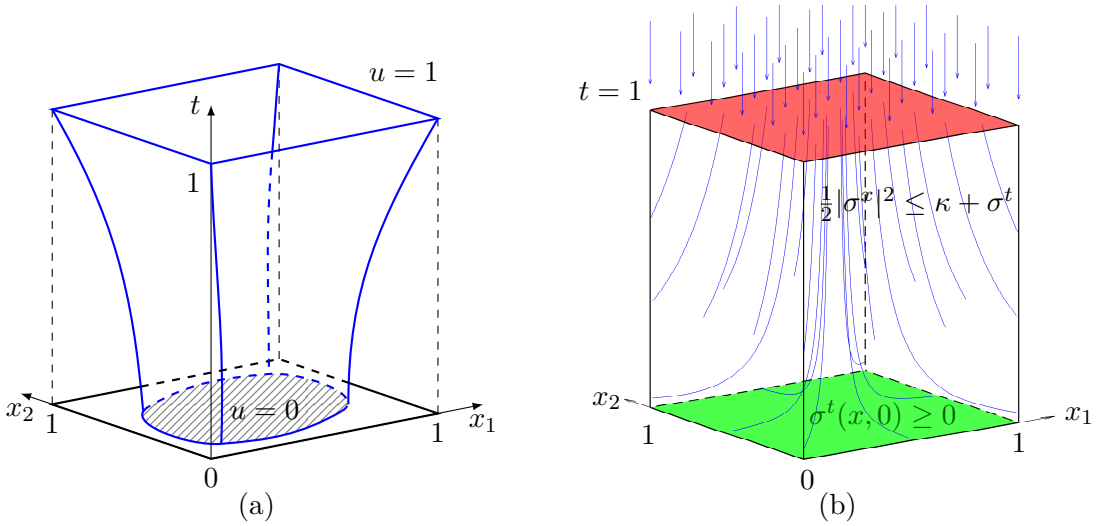


Figure 2.1: (a) The free boundary problem (2.23). (b) The optimal flow problem (2.24).

2.4.2 Saddle point characterization

We consider the variant described in Remark 2.12 where competitors for $(\mathcal{Q}_\lambda)$ are in the class $\widehat{\mathcal{A}} := \mathcal{A}(u_0,\Gamma_0,I)$ (here $I=(m,M)$) and competitors σ for $(\mathcal{P}^*_\lambda)$ are in the class $\widehat{\mathcal{K}} := \mathcal{K}(u_0,\Gamma_0,I)$. Let us introduce, for every pair (v,σ), with $v\in BV(\Omega\times I;[0,1])$ and $\sigma\in X_1(\Omega\times I)$, the following Lagrangian

$$L(v,\sigma) := \int_{\Omega\times I}(\sigma\cdot Dv) - \lambda\int_{\Omega\times I} p(x)\,(v-v_0)\,dxdt \tag{2.25}$$

Theorem 2.14. *There holds*

$$\inf(\mathcal{P}_\lambda) \;=\; \inf_{v\in\widehat{\mathcal{A}}}\sup_{\sigma\in\widehat{\mathcal{K}}} L(v,\sigma) \;=\; \sup_{\sigma\in\widehat{\mathcal{K}}}\inf_{v\in\widehat{\mathcal{A}}} L(v,\sigma) \;=\; \sup(\mathcal{P}^*_\lambda)\,.$$

Moreover, a pair $(\overline{v},\overline{\sigma})$ is optimal for $(\mathcal{Q}_\lambda)$ and for the dual problem $(\mathcal{P}^*_\lambda)$ if and only if it is a saddle point for L, namely

$$L(\overline{v},\sigma) \;\le\; L(\overline{v},\overline{\sigma}) \;\le L(v,\overline{\sigma}) \qquad \forall (v,\sigma)\in\widehat{\mathcal{A}}\times\widehat{\mathcal{K}}\,.$$

The proof is straightforward. Different numerical schemes (explicit and implicit) in order to solve the saddle point problem above are presented in [20] and [95]. In particular in the case of the 2d-example (2.23), some threshold value κ^* can be computed for which a numerical solution $v(x_1,x_2,t)$ is piecewise constant taking three values $0,\theta,1$ ($\theta\in(0,1)$). The upper level sets of v thus determine two global minimizers u_1,u_2 for the original free boundary problem ($u_1\equiv 1$ remains a solution for $\kappa<\kappa^*$ and is the unique one).

2.4.3 An example of Γ-convergence

We revisit here the celebrated asymptotic analysis of the Modica-Mortola functional which arises in the sharp interface model for Cahn-Hilliard fluids, showing how the duality approach developed in Section 2.2 can be used efficiently. In fact we can treat a sligthly more general model where we consider a family of functionals $(F^\varepsilon)_{\varepsilon>0}$, indexed with a (small) scale parameter $\varepsilon > 0$, of the following form (see [13])

$$F^\varepsilon(u) \;:=\; \frac{1}{\varepsilon}\int_\Omega f(u(x), \varepsilon\nabla u(x))\,dx,$$

where Ω is bounded domain of $\mathbb{R}^N$ with Lipschitz boundary and $f : \mathbb{R}\times\mathbb{R}^N \to [0,+\infty)$ the following assumptions on :

i) f is continuous in the first variable and convex in the second;

ii) there exist two real numbers $0 < \alpha < \beta$ such that $f(t,0) > 0$ if $t \neq \alpha, \beta$, $f(\alpha,0) = f(\beta,0) = 0$, and for every $z \neq 0$ and every t, $f(t,z) > f(t,0)$;

iii) there exists $M > \beta$ such that $f(t,\cdot)$ is locally bounded, uniformly in $t \in [0,M]$;

iv) there exists a function ψ with superlinear growth at ∞, such that $f(t,p) \geq \psi(p)$ for every $t \in \mathbb{R}$ and $p \in \mathbb{R}^N$.

Under such assumptions, it is not difficult to show that the family $\{F^\varepsilon, \varepsilon > 0\}$ is equicoercive in $X = L^1(\Omega;[0,M])$ (that is satisfies the condition (H2) in Section 2.2). Our aim is to compute the Γ-limit of F^ε as $\varepsilon \to 0$. For every $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$, we define

$$f_\varepsilon(t,z) = \frac{f(t,\varepsilon z)}{\varepsilon} \quad , \quad f_c(t,z) = \inf_{\varepsilon>0} f_\varepsilon(t,z) \quad , \quad h(z) = \int_\alpha^\beta f_c^{**}(t,z)\,dt\ .$$

By construction the conical enveloppe of f_c is one–homogeneous in z. It follows that h is a convex and one homogeneous function of z. An easy computation involving Moreau-Fenchel conjugates, for fixed t and with respect to the variable z, shows that

$$f_\varepsilon^*(t,z^*) = \frac{1}{\varepsilon}\,f^*(t,z^*) \quad , \quad f_c^{**}(z) = \sup_{z^*\in\mathbb{R}^N}\{z\cdot z^* \;:\; f^*(t,z^*) \leq 0\}\ . \tag{2.26}$$

Under these assumptions, we can show the following result:

Theorem 2.15. *As ε goes to zero, F^ε Γ–converges in $L^1(\Omega;[0,M])$ to the functional F given by*

$$F(u) \;=\; \begin{cases} \displaystyle\int_{S_u\cap\Omega} h(\nu_u)\,dH^{N-1} & \text{if } u \in BV(\Omega;\{\alpha,\beta\}) \\ +\infty & \text{otherwise .} \end{cases}$$

Here S_u denotes the discontinuity set of u given in the form $u = \alpha\mathbf{1}_A + \beta\mathbf{1}_{\Omega\setminus A}$, $\nu_u = \frac{Du}{|Du|}$ represents the inwards pointing normal to the interface $\partial A\cap\Omega$ and the integral on S_u is taken with respect to the $N-1$ dimensional measure.

The Modica-Mortola functional corresponds to taking $f(t,z) = \frac{1}{2}|z|^2 + W(t)$ where the double well potential $W : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$\lim_{t\to\infty}\frac{W(t)}{t} = +\infty \quad , \quad W(t) = 0 \iff t \in \{\alpha,\beta\}\ .$$

In that case we recover an isotropic interface energy $h(z) = c\,|z|$ where the surface tension coefficient is determined by $c = \int_\alpha^\beta \sqrt{2W(s)}ds$ (see [86]).

As an alternative to [13], we propose here a proof by duality exploiting Theorem 2.5 in Section 2.2. Let us sketch the different steps. We consider $X = L^1(\Omega;[0,M])$, $Y := L^\infty(\Omega\times(-1,M))$ (endowed with the weak star topology) and $\varphi : u \in X \mapsto \mathbf{1}_u(x,y)$. The assumptions $(H1)(H2)$ are fulfilled as well as $(H3)$ if we consider the (weakly star) continuous linear form $l_0(v) := |\Omega|^{-1}\int_{\Omega\times(-1,0)} v\,dxdt$ (as $u \in X$ is non negative, it holds $\mathbf{1}_u = 1$ on $\Omega\times(-1,0)$ so that $l_0(\mathbf{1}_u) = 1$).

Let us set $Q := \Omega\times(-1,M)$, $Q^- := \Omega\times(-1,0)$, $Q^+ := \Omega\times(0,M)$. First we compute:

$$D_\varepsilon := \left\{ p \in L^1(Q) : \int_Q \mathbf{1}_u\, p\, dxdt < F^\varepsilon(u) \quad \forall u \in X \right\}.$$

We observe that $p \in D_\varepsilon$ iff it holds $\inf\{F^\varepsilon(u) - \int_{Q^+} p\,\mathbf{1}_u\} > \int_{Q^-} p$. Thus, by applying the duality result of Theorem 2.10 (in the variants described in Remark 2.11 and Remark 2.12), we get

$$D_\varepsilon = \Big\{ p \in L^1(Q) \ : \ \exists \sigma \in \mathcal{K}_\varepsilon \ , \ -\operatorname{div}\sigma = p \quad \text{in } Q^+ \ , \ \sigma^x\cdot\nu_\Omega = 0 \quad \text{on } \partial\Omega\times(0,M) \\ \int_\Omega \sigma^t(x,0)\,dx + \int_{Q^-} p < 0 \Big\} .$$

where, in view of (2.26) and of the continuity of $f(\cdot,z)$,

$$\mathcal{K}_\varepsilon = \{\sigma \ \in L^\infty(Q^+;\mathbb{R}^{N+1}) \ : \ f^*(t,\sigma^x) \le \varepsilon\,\sigma^t \quad \text{a.e. in } Q^+\} .$$

Next we define D to be the closure in $L^1(Q)$ of the set D_0 given by

$$D_0 = \Big\{ p \in L^1(Q) \ : \ \exists \sigma \in \mathcal{K}_0 \ : , \ -\operatorname{div}\sigma = p \quad \text{in } Q^+ \ , \ \sigma^x\cdot\nu_\Omega = 0 \quad \text{on } \partial\Omega\times(0,M) \\ \int_\Omega \sigma^t(x,0)\,dx + \int_{Q^-} p < 0 \Big\} ,$$

where

$$\mathcal{K}_0 = \{\sigma \ \in C^1(\overline{Q^+}) \ : \ \sigma(x,t) \in \Gamma_0(t) \quad \forall (x,t) \in Q^+\} ,$$
$$C_0(t) = \Big\{ q = (z^*,\tau) \in \mathbb{R}^N\times\mathbb{R} : f^*(t,z^*) < 0\,,\ \tau > 0 \text{ if } t \in \{\alpha,\beta\} \Big\}$$

We introduce the following functional G defined on $L^1(Q)$:

$$G(v) \ := \ \sup_{p\in D} \int_Q v\cdot p\,dxdt \ = \ \sup_{p\in D_0} \int_Q v\cdot p\,dxdt \ .$$

By a straight forward computation, we observe that the support function of the convex constraint associated to subset $\mathcal{K}_0$ i.e. $C_0(t) = \Big\{ q = (z^*,\tau) \in \mathbb{R}^N\times\mathbb{R} : f^*(t,z^*) < 0\,,\ \tau > 0 \text{ if } t \in \{\alpha,\beta\} \Big\}$ is given by:

$$h(t,z^x,z^t) = f_c^{**}(t,z^x) \quad \text{if } z^t = 0 \ \text{ or } \ \Big[z^t \le 0 \text{ and } t \in \{\alpha,\beta\} \Big] , \quad h(t,z^x,z^t) = +\infty \quad \text{otherwise.}$$

As a consequence of a commutation argument between the symbols sup and $\int$ (see [17]), we infer that $G(v) = \int_Q h(t,Dv)$. In particular $G(v)$ is finite only for those functions $v(x,t)$ which are piecewise constant with respect to t and such that $v = 1$ for $t < \alpha$, $v = 0$ for $t > \beta$ and $v = \theta(x)$ for $t \in (\alpha,\beta)$ being θ an element of $BV(\Omega;[0,1])$. Then $G(v) = \int_\alpha^\beta \int_\Omega f_c^{**}(t,D\theta) = \int_\Omega h(D\theta)$. If $v = \mathbf{1}_u$, then we have $\theta = \mathbf{1}_A$ for a suitable subset $A \subset \Omega$ with finite perimeter so that $u = \alpha\mathbf{1}_A + \beta\mathbf{1}_{\Omega\setminus A}$ and $G(\mathbf{1}u) = F(u)$. Therefore $G(v)$ (which satisfy coarea formula) is nothing else but the convexified functional associated with the limit F given in Theorem 2.15. In other words we have showed that the set D_0 above satisfies

$$\overline{D_0} \ = \ D \ = \ \left\{ p \in L^1(Q) \ : \ \int_Q p\,\mathbf{1}_u\,dxdt \le F(u) \quad \forall u \in L^1(Q) \right\} . \tag{2.27}$$

Owing to Theorem 2.5, we deduce Theorem 2.15 by invoking the following result:

Lemma 2.16. *With the notations above, D_ε converges in the Kuratowski sense to D in $L^1(Q)$.*

Proof. First we prove that $D_0 \subset Li(D_\varepsilon)$ (thus $D \subset Li(D_\varepsilon)$). Let $p \in D_0$ and $\sigma \in \mathcal{K}_0$ associated. By continuity, σ^t is positive in a neighborhood of $t \in \{\alpha, \beta\}$ while in the complementary $f^*(t, \sigma^x)$ is majorized by a negative constant. Therefore $\sigma \in \mathcal{K}_\varepsilon$ and $p \in D_\varepsilon$ for ε small enough.

Let us show now that $Ls(D_\varepsilon) \subset D$. Let $p \in Ls(D_\varepsilon)$. Then it exists a sequence such that $p_\varepsilon \to p$ in $L^1(Q)$ and $p_\varepsilon = -\operatorname{div} \sigma_\varepsilon$ in Q^+, where σ_ε has a vanishing normal trace on $\partial\Omega \times (0, M)$ and satisfies

$$f^*(t, \sigma_\varepsilon^x) \le \varepsilon\, \sigma_\varepsilon^t \quad \text{in } Q^+ \quad , \quad \int_\Omega \sigma_\varepsilon^t(x, 0)\, dx + \int_{Q^-} p_\varepsilon < 0 \ . \tag{2.28}$$

By Gauss-Green formula applied on $\Omega \times [0, s]$ and by exploiting the vanishing normal trace condition on $\partial\Omega \times (0, M)$, we obtain that for every $s \in (0, M)$, it holds:

$$\int_\Omega \sigma_\varepsilon^t(x, s)\, dx - \int_\Omega \sigma_\varepsilon^t(x, 0)\, dx = \int_{\Omega \times [0,s]} p_\varepsilon dx dt \le \|p_\varepsilon\|_{L^1(Q)} \ .$$

Thus, by integrating in s over $(0, M)$ and taking into account (2.28), we deduce that

$$\int_{Q^+} f^*(t, \sigma_\varepsilon^x) \le \varepsilon \int_{Q^+} \sigma_\varepsilon^t(x, t)\, dx dt \ \le \ M\, \varepsilon\, \|p_\varepsilon\|_{L^1(Q)}$$

As $f^*(t, \sigma_\varepsilon^x) \ge -f(t, 0)$ is minorized, we infer that $\int_{Q^+} f^*(t, \sigma_\varepsilon^x) \to 0$ and that σ_ε^t is bounded in $L^1(Q^+)$. By the assumption iii) on f, this implies that $\{\sigma_\varepsilon^x\}$ is bounded and equi-integrable in $L^1(Q^+)$, hence up to a subsequence we may assume that $\sigma_\varepsilon^x \rightharpoonup \sigma^x$ in $L^1(Q^+; \mathbb{R}^N)$ for a suitable σ^x. By the convexity of $f^*(t, \cdot)$, it is easy to show that the weak limit σ^x satisfies $f^*(t, \sigma^x) \le 0$ a.e. in Q^+. Let $u \in L^1(\Omega)$ such that $F(u) < +\infty$. Then we have $u = \alpha \mathbf{1}_A + \beta \mathbf{1}_{\Omega \setminus A}$ where A is a subset of finite perimeter. We denote by ν_A the outward pointing normal to A which is well defined H^1 a.e. on ∂A (its essential boundary).

In view of (2.27), in order to show that $p \in D$, we are reduced to check that

$$\int_Q p\, \mathbf{1}_u\, dx dt \le F(u) = \int_{\partial A \cap \Omega} h(\nu_A) dH^1 \ .$$

Recalling that $p_\varepsilon \to p$ in $L^1(Q)$ while $p_\varepsilon = -\operatorname{div} \sigma_\varepsilon$ in Q^+ and $\mathbf{1}_u = 1$ in Q^-, we have

$$\begin{aligned}
\int_Q p\, \mathbf{1}_u\, dx dt \ &= \ \lim_{\varepsilon \to 0} \left(\int_{Q^-} p_\varepsilon\, dx dt - \int_{Q^+} \mathbf{1}_u \operatorname{div} \sigma_\varepsilon \right) \\
&\le \limsup_{\varepsilon \to 0} \left(-\int_A \sigma_\varepsilon^t(x, \beta)\, dx - \int_{\Omega \setminus A} \sigma_\varepsilon^t(x, \alpha)\, dx + \int_{\partial A \times (\alpha, \beta)} \sigma_\varepsilon^x(x, t) \cdot \nu_A\, H^1(dx) \otimes dt \right) \\
&\le \ \int_{\partial A \times (\alpha, \beta)} \sigma^x(x, t) \cdot \nu_A\, H^1(dx) \otimes dt,
\end{aligned}$$

where:

- in the second line we applied the generalized Gauss -Green formula on subset Q^+ taking into account the right hand side inequality in (2.28) and the fact that $\sigma_\varepsilon^x \cdot \nu_\Omega$ vanishes on $\partial\Omega \times (0, M)$,
- in the third line, we used the fact that $\sigma_\varepsilon^t(\cdot, t)$ is nonnegative for $t \in \{\alpha, \beta\}$ together with the weak convergence of the normale trace of σ_ε on $\partial A \times (\alpha, \beta)$.

Next we observe that, thanks to $f^*(t, \sigma^x) \le 0$, we have $f_c^*(t, \sigma^x) = 0$ and, by Moreau-Fenchel inequality, it holds: $\sigma_\varepsilon^x(x, t) \cdot \nu_A \le f_c^{**}(t, \nu_A)$. We can therefore conclude that

$$\int_Q p\, \mathbf{1}_u\, dx dt \ \le \ \int_{\partial A \times (\alpha, \beta)} f_c^{**}(t, \nu_A)\, H^1(dx) \otimes dt \ = \ \int_{\partial A} h(\nu_A) dH^1 \ = \ F(u) \ .$$

■

2.5 Extension of the duality principle to the case of linear-growth functionals

In this section, we extend the duality scheme developed in Section 2.4 and deeply studied in [20], to the case of linear-growth functionals. As we are motivated by specific examples which will be developed in Chapters 3 and 4, we shall focus on the case of variational problems of the particular form given by

$$(\mathcal{P}) \qquad \inf\left\{\int_\Omega (\varphi(\nabla u)+g(u))\,dx-\int_\Omega p(x)u\,dx \;:\; u\in W^{1,1}(\Omega),\ u=u_0 \text{ on } \partial\Omega\right\} \tag{2.29}$$

where $\varphi:\mathbb{R}^N\to\mathbb{R}^+$ is a convex continuous function such that

$$k|z|-\frac{1}{k}\ \le\ \varphi(z)\ \le\ C(1+|z|), \tag{2.30}$$

for suitable constants $k,C>0$, $g:\mathbb{R}\to(-\infty,+\infty]$ is a l.s.c function and u_0 is a fixed element in $L^1(\partial\Omega)$. In order that the infimum above is finite, we shall assume one of the following additional asumptions on functions g and p :

(A) $\quad \operatorname{dom} g:=\{t\in\mathbb{R}:\ g(t)<+\infty\}$ is bounded and $p\in L^1(\Omega)$;

(B) $\quad g$ satisfies $\liminf_{|t|\to+\infty}\dfrac{g(t)}{|t|^r}>0$ for some $r>1$ such that $p\in L^{r'}(\Omega)$ $(r'=\dfrac{r}{r-1})$.

The existence issue for $(\mathcal{P})$ is not trivial since minimizing sequences are merely bounded in $W^{1,1}(\Omega)$ and admit possibly discontinuous cluster points in the space $BV(\Omega)$. To obtain a well-posed primal problem, we need to relax it in the space $BV(\Omega)$. This is a well-known technical procedure (see [17]) that we describe shortly in the next subsection.

2.5.1 Relaxation in BV

We recall some preliminary notion of BV functions. For every function $u\in BV(\Omega)$, we recall $u^\pm(x)$ denote the *upper and lower approximate limits* of u, $S_u:=\{x\in\Omega\ :\ u^-(x)<u^+(x)\}$ is the set of all *approximate jump points* of u. Du is a bounded Radon measure, which can be decomposed into

$$Du=\nabla u dx+D^c u+(u^+-u^-)\nu_u d(\mathcal{H}^{N-1}\llcorner S_u)$$

where ν_u denotes the Radon-Nikodým density of Du with respect to its total variation $|Du|$, i.e. $\nu_u:=dDu/d|Du|$. Note that D^cu is the Cantor part of the measure Du. The quantity $[u]:=u^+-u^-$ is called the *jump* of u across the interface S_u and the direction of the jump is given by ν_u along S_u. Accordingly, the *complete graph* of function u, denoted by $\overline{G}_u$, is defined by

$$\overline{G}_u:=\bigcup_{x\in\Omega}\Big(\{x\}\times[u^-(x),u^+(x)]\Big).$$

It is a rectifiable subset of $\Omega\times\mathbb{R}$ with an oriented unit normal denoted by $\widehat{\nu}_u$. $\overline{G}_u$ is indeed the support of the bounded Radon vector measure $D\mathbf{1}_u$ in $\Omega\times\mathbb{R}$, namely

$$D\mathbf{1}_u=\widehat{\nu}_u d(\mathcal{H}^N\llcorner\overline{G}_u) \tag{2.31}$$

Actually, from writing the measure $D\mathbf{1}_u$ as the sum of

$$D\mathbf{1}_u\llcorner(S_u\times\mathbb{R}) \qquad \text{and} \qquad D\mathbf{1}_u\llcorner((\Omega\setminus S_u)\times\mathbb{R}) \tag{2.32}$$

one can derive a decomposition of the complete graph $\overline{G}_u$ into two parts: a *vertical part* and an *approximately continuous part* (denoted by G_u). Therefore, the density $\widehat{\nu}_u$ is determined by

$$\widehat{\nu}_u(x,t) = (\nu_u(x), 0) \quad \text{for } x \in S_u \text{ and } t \in [u^-(x), u^+(x)],$$

on the vertical part. On the approximately continuous part, it is identified as

$$\widehat{\nu}_u(x, u^+(x)) = \frac{(\nabla u(x), -1)}{\sqrt{1 + |\nabla u(x)|^2}} \tag{2.33}$$

if u is approximately differentiable at x (with its approximate gradient $\nabla u(x)$), and it is horizontal, i.e. $\widehat{\nu}_u(x, u^+(x)) = (\nu_u(x), 0)$, at points corresponding to the Cantor part of Du otherwise. Notice that $\nu_u = dD^c u/d|D^c u|$ $|D^c u|$-a.e in Ω. We remark also that the complete graph $\overline{G}_u$ of functions u belonging to $W^{1,p}(\Omega)$ agrees with the *usual graph* G_u, and $\widehat{\nu}_u(x, u(x))$ only has the form (2.33). But, it is not always the case when dealing with BV functions.

With the settings mentioned above, the relaxed functional $BV(\Omega)$ of the primal problem $(\mathcal{P})$ in (2.29), is given by

$$\begin{aligned} E(u) := &\int_\Omega \Big(\varphi(\nabla u) + g(u) - p(x)u\Big)\, dx + \int_{\partial\Omega} \varphi^\infty((u_0 - u)\nu_\Omega)\, d\mathcal{H}^{N-1} \\ &+ \int_\Omega \varphi^\infty\left(\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{S_u} [u]\varphi^\infty(\nu_u)\, d\mathcal{H}^{N-1} \end{aligned} \tag{2.34}$$

with φ^∞ being the *recession* function of φ, namely

$$\varphi^\infty(p) = \lim_{s\to+\infty} \frac{\varphi(sp)}{s}. \tag{2.35}$$

For the sake of simplification, we adopt the convention below

$$\int_\Omega \varphi(Du) := \int_\Omega \varphi(\nabla u)dx + \int_\Omega \varphi^\infty\left(\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{S_u} [u]\varphi^\infty(\nu_u)\, d\mathcal{H}^{N-1}.$$

Then, the relaxed problem of the problem $(\mathcal{P})$ in (2.29), is formulated as

$$(\mathcal{PR}) \qquad \min\Big\{E(u) \ : \ u \in BV(\Omega)\Big\}. \tag{2.36}$$

Problem $(\mathcal{PR})$ attains its minimum in $BV(\Omega)$ and if $\inf(\mathcal{P}) < +\infty$ then it holds

$$\inf(\mathcal{P}) = \inf(\mathcal{PR}).$$

Remark that, when the domain of g is bounded, there exist a priori lower and upper bounds for the minimizers of the primal problem $(\mathcal{PR})$. Accordingly, we can restrict $(\mathcal{PR})$ to the space $BV(\Omega; [a,b])$ where $-\infty < a < b < +\infty$ and $[a,b]$ is a suitable closed interval such that the minimum is unchanged.

2.5.2 Dual problem

From now on, we assume that $\inf(\mathcal{P}) < +\infty$. Additionally, we make the assumption (A), namely $\mathrm{dom}(g) \subset\subset I := [a,b]$ where $-\infty < a < b < +\infty$ so that the relaxed solutions of the problem $(\mathcal{PR})$ belongs to $BV(\Omega; I)$ and thus $(\mathcal{PR})$ reduces to

$$(\mathcal{PR}) \qquad \min\Big\{E(u) \ : \ u \in BV(\Omega; I)\Big\}. \tag{2.37}$$

Adopting the same strategy as in Section 2.3, we need to construct a functional $J : L^\infty(\Omega \times I) \to [0, +\infty]$ satisfying the co-area formula and such that $J(\mathbf{1}_u) = F(u)$ where

$$F(u) := \int_\Omega \Big(\varphi(Du) + g(u)\Big)dx$$

To that aim, we consider

$$J(v) := \int_{\Omega\times I} \widetilde{h}(t, Dv) \qquad \text{where} \qquad \widetilde{h}(t, z^x, z^t) := \begin{cases} -z^t\Big(\varphi(\frac{z^x}{-z^t}) + g(t)\Big) & \text{if } z^t < 0, \\ \varphi^\infty(z^x) & \text{if } z^t = 0, \\ +\infty & \text{if } z^t > 0. \end{cases} \tag{2.38}$$

Remark 2.17. We notice that, if $F(u) < +\infty$, then $u \in BV(\Omega; I)$ and as $g(u) < +\infty$ a.e., the function $v = \mathbf{1}_u$ (as an element of $BV(\Omega \times \mathbb{R}; [0, 1])$) is such that $v(x, \cdot)$ is non increasing with $v(x, a + 0) = 1$ and $v(x, b - 0) = 0$.

Remark 2.18. The important modification of $J(v)$ with respect to (2.21) lies on the fact that the positively 1-homogenous function $\widetilde{h}$ is now well defined for $z^t = 0$ since φ^∞ given in (2.35) is finite. It allows to handle the jump energy in our duality framework. This setting is compatible with functionals with linear growth, whose minimizers in BV spaces may occur discontinuity.

Let us check that $J(\mathbf{1}_u) = F(u)$ for every $u \in BV(\Omega; I)$. By the decomposition (2.32), there holds

$$\begin{aligned} \int_{\Omega\times I} \widetilde{h}(t, D\mathbf{1}_u) &= \int_{\Omega\times I} \widetilde{h}(t, \widehat{\nu}_u) d(\mathcal{H}^N \llcorner \overline{G}_u) \\ &= \int_{G_u} \widetilde{h}(t, \widehat{\nu}_u) d\mathcal{H}^N + \int_{S_u} \int_{u^-(x)}^{u^+(x)} \widetilde{h}(s, (\nu_u, 0)) ds \, d\mathcal{H}^{N-1}. \end{aligned}$$

The integral taken over the approximately continuous part (of the complete graph) G_u can be calculated as

$$\int_{G_u} \widetilde{h}(t, \widehat{\nu}_u) d\mathcal{H}^N = \int_\Omega \widetilde{h}\Big(u(x), (\nabla u(x), -1)\Big) dx + \int_\Omega \widetilde{h}\left(u^+(x), \Big(\frac{dD^c u}{d|D^c u|}, 0\Big)\right) d|D^c u|$$

by taking into account the positive 1-homogeneity of $\widetilde{h}$ and the disintegration

$$(H^N \llcorner G_u)(A \times B) = \int_A \delta_{u(x)}(B) \left(\sqrt{1 + |\nabla u|^2} dx + d|D^c u|\right) \tag{2.39}$$

for all Borel subsets $A \times B \subset (\Omega \setminus S_u) \times I$. Here, $\delta_{u(x)}$ denotes the Dirac measure. Notice in addition that the Jacobian of the mapping $\Omega \ni x \mapsto (x, u(x)) \in G_u$ is provided by $\sqrt{1 + |\nabla u|^2}$ that appears in the formula of disintegration above. Using the definition (2.38) of $\widetilde{h}$, we deduce that

$$\int_{\Omega\times I} \widetilde{h}(t, D\mathbf{1}_u) = \int_\Omega \Big(\varphi(\nabla u) + g(u)\Big) dx + \int_\Omega \varphi^\infty\left(\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{S_u} [u] \varphi^\infty(\nu_u) \, d\mathcal{H}^{N-1}$$

which yields $J(\mathbf{1}_u) = F(u)$.

Now, we are going to establish that $E(u)$ given in (2.34) satisfies

$$E(u) = \widehat{E}(\mathbf{1}_u) \tag{2.40}$$

where we define

$$\widehat{E}(v) := J(v) + \ell(v)$$

with $\ell(v)$ being a suitable affine functional on $BV(\Omega \times I)$. For that purpose, we have to define $\ell(v)$ so that

$$\ell(\mathbf{1}_u) = -\int_\Omega p(x) u(x) dx + \int_{\partial\Omega} \varphi^\infty((u_0 - u)\nu_\Omega) d\mathcal{H}^{N-1} \quad \forall u \in BV(\Omega; I). \tag{2.41}$$

Let us define for $v \in BV(\Omega \times I)$

$$\begin{aligned}\ell(v) \;:=\; &-\int_{\Omega\times I} p(x)(v-v_0)\,dxdt + \int_{\partial\Omega\times I} \partial_t\gamma_{u_0}(x,t)(v-v_0)d\mathcal{H}^{N-1}dt \\ &+\int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}\end{aligned} \tag{2.42}$$

where $v_0(x,t) = 1 - H(t)$ ($H(t)$ being the Heavyside function) and

$$\gamma_{u_0}(x,t) := \varphi^\infty\Big((u_0(x)-t)\nu_\Omega(x)\Big) \quad \text{for } (x,t) \in \partial\Omega \times I. \tag{2.43}$$

We observe that $\gamma_{u_0}(x,\cdot)$ is Lipschitz with

$$\partial_t\gamma_{u_0}(x,t) := \frac{\partial\gamma_{u_0}}{\partial t}(x,t) = \begin{cases}\varphi^\infty(-\nu_\Omega(x)) & \text{if } t > u_0(x),\\ -\varphi^\infty(\nu_\Omega(x)) & \text{if } t < u_0(x),\end{cases}$$

and that for every $u \in BV(\Omega)$, it holds true

$$(\mathbf{1}_u - v_0) = \begin{cases}\mathbf{1}_{[0,u(x)]} & \text{if } u(x) > 0,\\ -\mathbf{1}_{[u(x),0]} & \text{if } u(x) < 0.\end{cases} \tag{2.44}$$

We derive from the identity (2.44) that

$$u(x) = \int_{\mathbb{R}} (\mathbf{1}_u(x,t) - v_0(x,t))dt, \tag{2.45}$$

$$\int_{\mathbb{R}} \partial_t\gamma_{u_0}(x,t)(\mathbf{1}_u - v_0)dt = \varphi^\infty((u_0-u)\nu_\Omega) - \varphi^\infty(u_0\nu_\Omega). \tag{2.46}$$

Now, let $u \in BV(\Omega; I)$. Keeping in mind that as u and u_0 range in the interval I, the identity (2.44) remains true in I so that the integral representations (2.45)-(2.46) can be restricted to I. From these observations, we can verify that

$$\begin{aligned}&\int_{\partial\Omega\times I} \partial_t\gamma_{u_0}(x,t)(\mathbf{1}_u - v_0)d\mathcal{H}^{N-1}dt \\ &\qquad = \int_{\partial\Omega} \varphi^\infty((u_0-u)\nu_\Omega)\,dH^{N-1} - \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}.\end{aligned} \tag{2.47}$$

On the other hand, we have (by using the integral representation (2.45) above restricted to I) that

$$\int_{\Omega\times I} p(x)(\mathbf{1}_u - v_0)dxdt = \int_\Omega p(x)u(x)dx. \tag{2.48}$$

From (2.47) and (2.48), we find that $\ell(v)$ with its definition given in (2.42) satisfies (2.41). In combination with the fact that $J(\mathbf{1}_u) = F(u)$, it shows that the required identification $E(u) = \widehat{E}(\mathbf{1}_u)$ is supplied.

In virtue of Section 2.3, especially Lemma 2.6, the convexified problem of $(\mathcal{PR})$ in dimension $N+1$, reads

$$(\mathcal{Q}) \qquad \inf\Big\{\widehat{E}(v) \;:\; v \in \mathcal{A}(I)\Big\} \tag{2.49}$$

where, according to Remark 2.17, the admissible set is given by:

$$\mathcal{A}(I) := \Big\{v \in BV(\Omega\times I;[0,1]) \;:\; v = 1 \text{ on } \Omega\times\{a\},\; v = 0 \text{ on } \Omega\times\{b\}\Big\}. \tag{2.50}$$

Keep in mind that by definition of $\widetilde{h}$, the condition $\int_{\Omega\times I}\widetilde{h}(t,Dv) < +\infty$ implies implicitly that $v(x,\cdot)$ is monotone non increasing. We provide in the next proposition the so-called *slicing formula* which allows to establish the equality between the primal relaxed problem $(\mathcal{PR})$ and its convexified problem $(\mathcal{Q})$.

Proposition 2.19. *For every $v \in \mathcal{A}(I)$ such that $\widehat{E}(v) < +\infty$, it holds*

$$\widehat{E}(v) = \int_0^1 \widehat{E}(\mathbf{1}_{\{v>s\}}(x,t))ds = \int_0^1 \widehat{E}(\mathbf{1}_{u_s})ds = \int_0^1 E(u_s)ds. \tag{2.51}$$

As a consequence, we have $\inf(\mathcal{PR}) = \inf(\mathcal{Q})$.

Proof. Let $v \in \mathcal{A}(I)$ such that $\widehat{E}(v) < +\infty$. Since $v(x,\cdot)$ is non increasing (because of the finiteness of $J(v)$), v only takes values into $[0,1]$. We recall that the *layer cake representation formula* for any function w with values in $[0,1]$ is $w(x) = \int_0^1 \mathbf{1}_{\{w>s\}}(x)ds$. We get

$$\begin{aligned}\int_{\Omega\times I} p(x)(v-v_0)dxdt &= \int_{\Omega\times I} p(x)\left(\int_0^1 \mathbf{1}_{\{v>s\}}(x,t)ds - v_0\right)dxdt \\ &= \int_0^1 \int_\Omega p(x)\int_I (\mathbf{1}_{u_s} - v_0)dtdxds \\ &= \int_0^1 \int_\Omega p(x)u_s dxds\end{aligned} \tag{2.52}$$

where the second equality is obtained by using the identification (2.15) with the function u_s defined by (2.14), and the third equality is obtained by exploiting (2.45). Again, by the layer cake representation formula and then, by (2.15), we infer that

$$\begin{aligned}\int_{\partial\Omega\times I} \partial_t\gamma_{u_0}(x,t)(v-v_0)d\mathcal{H}^{N-1}dt &= \int_0^1 \int_{\partial\Omega\times I} \partial_t\gamma_{u_0}(x,t)(\mathbf{1}_{\{v>s\}} - v_0)d\mathcal{H}^{N-1}dtds \\ &= \int_0^1 \int_{\partial\Omega\times I} \partial_t\gamma_{u_0}(x,t)(\mathbf{1}_{u_s} - v_0)d\mathcal{H}^{N-1}dtds \\ &= \int_0^1 \int_{\partial\Omega} \varphi^\infty((u_0-u_s)\nu_\Omega)dH^{N-1}ds - \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}\end{aligned} \tag{2.53}$$

where third equality is obtained by the same computation of (2.47). As the functional $J(v)$ satisfies the coarea formula and the condition (2.17), it holds

$$J(v) = \int_{-\infty}^{+\infty} J(\mathbf{1}_{\{v>s\}}(x,t))ds = \int_0^1 J(\mathbf{1}_{u_s})ds = \int_0^1 \left(\int_\Omega \varphi(Du_s) + \int_\Omega g(u_s)dx\right)ds. \tag{2.54}$$

We deduce from the equalities (2.52)-(2.53)-(2.54) that

$$\widehat{E}(v) = \int_0^1 \widehat{E}(\mathbf{1}_{u_s})ds = \int_0^1 E(u_s)ds.$$

We observe that for $\mathcal{L}^1$-a.e. $s \in (0,1)$, the function u_s belongs to $BV(\Omega;I)$. This is to say $E(u_s) \geq \inf(\mathcal{P})$, and by the slicing formula (2.51), $\widehat{E}(v) \geq \inf(\mathcal{P})$. Therefore, because of the arbitrariness of $v \in \mathcal{A}(I)$, we conclude that $\inf(\mathcal{PR}) \geq \inf(\mathcal{P})$. Conversely, we have already known that $E(u) = \widehat{E}(\mathbf{1}_u)$ for $u \in BV(\Omega)$ (as shown in (2.40)). It follows that $\inf(\mathcal{P}) \geq \inf(\mathcal{PR})$. ■

In the next lemma, we give an extended integration by parts formula which intuitively describes the fluxes across the graphs of competitors u of the relaxed problem $(\mathcal{PR})$. Then, it permits formulating the dual problem of the problem $(\mathcal{PR})$ in terms of maximizing the fluxes.

Lemma 2.20. *Let $\widetilde{u} \in BV(\Omega;I)$. For every $\sigma \in X_1(\Omega\times I)$ satisfying*

$$-\operatorname{div}\sigma = p(x) \ \text{ in } \Omega\times I, \qquad \sigma^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0} \ \text{ on } \partial\Omega\times I, \tag{2.55}$$

and for every v of class $\mathcal{A}(I)$, it holds

$$\ell(v) + \int_{\Omega\times I}\sigma\cdot Dv = \int_{\widetilde{G}_{\widetilde{u}}} \sigma\cdot\widehat{\nu}_{\widetilde{u}}d\mathcal{H}^N - \int_\Omega p(x)\widetilde{u}dx + \int_{\partial\Omega}\varphi^\infty((u_0-\widetilde{u})\nu_\Omega)d\mathcal{H}^{N-1} \tag{2.56}$$

where $\ell(v)$ is defined in (2.42).
In particular, if $v = \mathbf{1}_u$ for some $u \in BV(\Omega; I)$, one has

$$\begin{aligned}\int_{\overline{G}_u} \sigma \cdot \widehat{\nu}_u d\mathcal{H}^N - \int_\Omega p(x) u dx + \int_{\partial\Omega} \varphi^\infty((u_0 - u)\nu_\Omega) d\mathcal{H}^{N-1} \\ = \int_{\overline{G}_{\widetilde{u}}} \sigma \cdot \widehat{\nu}_{\widetilde{u}} d\mathcal{H}^N - \int_\Omega p(x)\widetilde{u} dx + \int_{\partial\Omega} \varphi^\infty((u_0 - \widetilde{u})\nu_\Omega) d\mathcal{H}^{N-1}.\end{aligned} \tag{2.57}$$

Proof. For every $v \in \mathcal{A}(I)$ (see (2.50) for the definition of $\mathcal{A}(I)$), $v - \mathbf{1}_{\widetilde{u}} = 0$ on $\Omega \times \{a, b\}$. We apply the generalized Gauss-Green formula (keeping in mind the constraints of (2.55)) to obtain

$$\begin{aligned}\int_{\Omega\times I} \sigma \cdot D(v - \mathbf{1}_{\widetilde{u}}) &= -\int_{\Omega\times I} (v - \mathbf{1}_{\widetilde{u}}) \operatorname{div} \sigma + \int_{\partial\Omega\times I} \sigma^x \cdot \nu_\Omega (v - \mathbf{1}_{\widetilde{u}}) d\mathcal{H}^{N-1} dt \\ &= \int_{\Omega\times I} p(x)(v - \mathbf{1}_{\widetilde{u}}) - \int_{\partial\Omega\times I} \partial_t \gamma_{u_0}(x,t)(v - \mathbf{1}_{\widetilde{u}}) d\mathcal{H}^{N-1} dt \\ &= \int_{\Omega\times I} p(x)(v - v_0) - \int_\Omega p(x)\widetilde{u} dx - \int_{\partial\Omega\times I} \partial_t \gamma_{u_0}(x,t)(v - v_0) d\mathcal{H}^{N-1} dt \\ &\quad + \int_{\partial\Omega} \varphi^\infty((u_0 - \widetilde{u})\nu_\Omega) d\mathcal{H}^{N-1} - \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega) d\mathcal{H}^{N-1},\end{aligned} \tag{2.58}$$

wherein the third equality is done by decomposing $v - \mathbf{1}_{\widetilde{u}} = v - v_0 - (\mathbf{1}_{\widetilde{u}} - v_0)$ and invoking the equations (2.45)-(2.47). By means of (2.31), we have

$$\int_{\Omega\times I} \sigma \cdot D\mathbf{1}_{\widetilde{u}} = \int_{\overline{G}_{\widetilde{u}}} \sigma \cdot \widehat{\nu}_{\widetilde{u}} d\mathcal{H}^N$$

from which (2.58) yields (2.56). When v is of the kind $\mathbf{1}_u$ for some $u \in BV(\Omega; I)$, (2.57) is a direct consequence of (2.56) (again using (2.45)-(2.47) for the computation). ■

Notably, when one takes $\widetilde{u} = \widetilde{u}_0$ where $\widetilde{u}_0 \in BV(\Omega; I)$ and $\widetilde{u}_0 = u_0$ on $\partial\Omega$, the integration by parts formula (2.57) turns into

$$\begin{aligned}\int_{\overline{G}_{\widetilde{u}_0}} \sigma \cdot \widehat{\nu}_{\widetilde{u}_0} d\mathcal{H}^N - \int_\Omega p(x)\widetilde{u}_0 dx \\ = \int_{\overline{G}_u} \sigma \cdot \widehat{\nu}_u d\mathcal{H}^N - \int_\Omega p(x) u dx + \int_{\partial\Omega} \varphi^\infty((u_0 - u)\nu_\Omega) d\mathcal{H}^{N-1}.\end{aligned} \tag{2.59}$$

In particular, if $\widetilde{u} \equiv 0$, the subgraph function $\mathbf{1}_{\widetilde{u}}$ agrees with the function v_0, which belongs to $\mathcal{A}(I)$. Moreover, we obtain

$$\begin{aligned}-\int_\Omega \sigma^t(x,0) dx + \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega) d\mathcal{H}^{N-1} \\ = \int_{\overline{G}_u} \sigma \cdot \widehat{\nu}_u d\mathcal{H}^N - \int_\Omega p(x) u dx + \int_{\partial\Omega} \varphi^\infty((u_0 - u)\nu_\Omega) d\mathcal{H}^{N-1}.\end{aligned} \tag{2.60}$$

In our recent case, the primal problem $(\mathcal{P})$ in (2.29) that we are treating is a Dirichlet problem with the prescribed boundary value $u = u_0$. Then, $(\mathcal{P})$ is relaxed in $BV(\Omega; I)$ to become $(\mathcal{PR})$ in (2.37). By exploiting the relation (2.60), the dual problem of $(\mathcal{P})$ (in fact the dual of the relaxed problem $(\mathcal{PR})$) that we propose, is established as

$$(\mathcal{P}^*) \qquad \sup\left\{ -\int_\Omega \sigma^t(x,0) dx + \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega) d\mathcal{H}^{N-1} \ : \ \sigma \in \mathcal{B}(I) \right\} \tag{2.61}$$

where $\mathcal{B}(I)$ is a class of vector fields defined in the following sense:

- $\mathcal{K}(I)$ denotes the class of vector fields $\sigma \in X_1(\Omega \times I)$ satisfying the convex constraints

$$\sigma^t(x,t) \geq \varphi^*(\sigma^x(x,t)) - g(t) \quad \text{for } \mathcal{L}^{N+1}\text{-a.e. } (x,t) \in \Omega \times I, \tag{2.62}$$
$$\sigma^t(x,t) \geq -\varphi(0) - g(t) \quad \forall\, t \in S_g \text{ and for } \mathcal{L}^N\text{-a.e. } x \in \Omega. \tag{2.63}$$

  Here, S_g denotes the set of discontinuities of g. S_g is assumed to be Lebesgue negligible.

- $\mathcal{B}(I)$ is the class of fields $\sigma \in \mathcal{K}(I)$ such that

$$-\operatorname{div}\sigma = p \quad \text{in } \Omega \times I, \tag{2.64}$$
$$\sigma^x \cdot \nu_\Omega = -\partial_t \gamma_{u_0} \quad \text{on } \partial\Omega \times I, \tag{2.65}$$

  where $\sigma^x \cdot \nu_\Omega$ is the weak normal trace of σ on the boundary $\partial(\Omega \times I)$.

Let us remark that the affine form in σ appearing in the dual problem $(\mathcal{P}^*)$, namely

$$\mathsf{L}(\sigma) := -\int_\Omega \sigma^t(x,0)dx + \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}$$

can be rewritten as

$$\mathsf{L}(\sigma) = \int_{\overline{G}_{\widetilde{u}_0}} \sigma \cdot \widehat{\nu}_{\widetilde{u}_0} d\mathcal{H}^N - \int_\Omega p(x)\widetilde{u}_0 dx$$

being $\widetilde{u}_0$ any element of $BV(\Omega;I)$ such that $\widetilde{u}_0 = u_0$ on $\partial\Omega$. This can be checked easily by using the extended Green formula in Lemma 2.20.

In the next lemma, we show that the infimum of $(\mathcal{P})$ is bounded from below by the extremum of its dual problem $(\mathcal{P}^*)$.

Lemma 2.21. *It holds* $\sup(\mathcal{P}^*) \leq \inf(\mathcal{P})$.

Proof. It is easy to verify the inequality $\sup(\mathcal{P}^*) \leq \inf(\mathcal{P})$ $(= \inf(\mathcal{PR}))$. Indeed, taking u, σ the competitors of $(\mathcal{PR})$ and $(\mathcal{P}^*)$ respectively, we have

$$\begin{aligned}
\int_{\Omega\times I} \sigma \cdot D\mathbf{1}_u &= \int_{\overline{G}_u} \sigma \cdot \widehat{\nu}_u d\mathcal{H}^N \\
&= \int_\Omega [\sigma^x(x,u(x)) \cdot \nabla u(x) - \sigma^t(x,u(x))]dx + \int_\Omega \sigma^x(x,u^+(x)) \cdot \nu_u \, d|D^c u| \\
&\quad + \int_{S_u} \int_{u^-(x)}^{u^+(x)} \sigma^x(x,s) \cdot \nu_u \, ds \, d\mathcal{H}^{N-1} \\
&\leq \int_\Omega \Big[\varphi^*(\sigma^x(x,u(x))) + \varphi(\nabla u(x)) - \sigma^t(x,u(x))\Big]dx \\
&\quad + \int_\Omega \sigma^x(x,u^+(x)) \cdot \nu_u \, d|D^c u| + \int_{S_u} \int_{u^-(x)}^{u^+(x)} \sigma^x(x,s) \cdot \nu_u \, ds \, d\mathcal{H}^{N-1} \\
&\leq \int_\Omega \Big[\varphi(\nabla u(x)) + g(u(x))\Big]dx + \int_\Omega \sigma^x(x,u^+(x)) \cdot \nu_u \, d|D^c u| \\
&\quad + \int_{S_u} \int_{u^-(x)}^{u^+(x)} \sigma^x(x,s) \cdot \nu_u \, ds \, d\mathcal{H}^{N-1}.
\end{aligned}$$

In the calculation above, we derive the first and second equalities by using (2.31), the decomposition (2.32) and the disintegration (2.39); the first inequality by making use of the Moreau-Fenchel inequality; and the second inequality by involving the constraint (2.62).

On the other hand, we recall that the recession function of a closed proper convex function is characterized as the support function of the effective domain of its convex conjugate function,

see [98, Theorem 13.3]. Keeping in mind that the integrand $\varphi(z)$ is continuous and convex, it happens $\sigma^x(x,t)\cdot\nu_u \le \varphi^\infty(\nu_u)$. Then, we deduce that

$$\int_{\Omega\times\mathbb{R}} \sigma\cdot D\mathbf{1}_u \le \int_\Omega \Big[\varphi(\nabla u)+g(u)\Big]dx + \int_\Omega \varphi^\infty(\nu_u)\, d|D^c u| + \int_{S_u}[u]\varphi^\infty(\nu_u)\, d\mathcal{H}^{N-1}.$$

By adding $\int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1} - \int_\Omega p(x)u dx$ in both sides of the inequality above and using the relation (2.60), the definition (2.34), the flux of any $\sigma\in\mathcal{B}(I)$ passing through the interface $t=0$ is bounded from above

$$-\int_\Omega \sigma^t(x,0)dx + \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} \le E(u),$$

where $E(u)$ is the energy of the relaxed problem $(\mathcal{PR})$. This entails that

$$\sup(\mathcal{P}^*) \le \inf(\mathcal{PR}) = \inf(\mathcal{P}).$$

■

Theorem 2.22. *There is no duality gap between the primal and dual problems defined respectively in (2.29) and (2.61), i.e.*

$$\inf(\mathcal{P}) = \sup(\mathcal{P}^*).$$

Proof. By Proposition 2.19, we have $\inf(\mathcal{Q}) = \inf(\mathcal{PR})$ $(= \inf(\mathcal{P}))$. To prove the equality $\inf(\mathcal{P}) = \sup(\mathcal{P}^*)$, we can alternatively demonstrate that $\inf(\mathcal{Q}) = \sup(\mathcal{P}^*)$. For this purpose, we introduce a perturbation of the dual problem

$$\Phi(\eta) := \inf\Big\{\int_\Omega \sigma^t(x,0)dx - \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} \;:\; \sigma\in X_1(\Omega\times I),$$
$$-\operatorname{div}\sigma = p,\ \sigma^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0} \text{ on } \partial\Omega\times I,\ \sigma+\eta\in\mathcal{K}(I)\Big\}$$

for $\eta\in C_0(\Omega\times I;\mathbb{R}^{N+1})$. For the sake of simplicity, by convention, the constraint in the abbreviated form $-\operatorname{div}\sigma = p$ appearing in the definition of Φ is meant to be satisfied $\mathcal{L}^{N+1}$-a.e. in $\Omega\times I$. We can easily verify the convexity of the map $\eta\mapsto\Phi(\eta)$, and evidently, for $\eta=0$, $-\Phi(0)$ is nothing else but $(\mathcal{P}^*)$, i.e.

$$\sup(\mathcal{P}^*) = -\Phi(0). \tag{2.66}$$

Moreover, Φ is continuous at 0, namely, for any η with $\|\eta\|_\infty\le\delta$, there is $\sigma_0\in\mathcal{K}(I)$ such that

$$\Phi(\eta) \le \int_\Omega \sigma^t(x,0)dx - \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}.$$

In fact, we can take σ_0 of the kind

$$\sigma_0(x,t) := (-\psi(x,t), (c_\Omega - p(x))t + \mu)$$

where $\mu\in\mathbb{R}$ is determined later, $c_\Omega := |\partial\Omega|/|\Omega|$, $\psi(x,t) := \nabla w^+$ if $t > u_0(x)$ and $\psi(x,t) := \nabla w^-$ if $t < u_0(x)$, with $w^\pm$ being the unique solution to the boundary value problems $\Delta w^+ = c_\Omega$ in Ω, $w^+_\nu = \varphi^\infty(-\nu_\Omega)$ on $\partial\Omega$ and $\Delta w^- = c_\Omega$ in Ω, $w^-_\nu = -\varphi^\infty(\nu_\Omega)$ on $\partial\Omega$. We see that, by construction, $\sigma_0\in L^\infty(\Omega\times I;\mathbb{R}^{N+1})$, $\operatorname{div}\sigma_0 = -p$ in $\Omega\times I$ and $(\sigma_0)^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0}$ on $\partial\Omega\times I$. Let us show that μ can be chosen so that $\sigma_0+\eta\in\mathcal{K}(I)$ for every η with $\|\eta\|_\infty\le\delta$. In other words, $\sigma_0+\eta$ must satisfy the conditions (2.62)-(2.63). To that aim, we have to choose μ so that the following implication is ensured

$$|q|<\delta \Rightarrow \begin{cases}(c_\Omega - p)t + \mu + q^t \ge \varphi^*(\psi(x,t)) - g(t)\\ (c_\Omega - p)t + \mu + q^t \ge -\varphi(0) - g(t).\end{cases}$$

Clearly, it is possible to choose μ since $t \in I$, and $\varphi^*(\psi(x,t)) \leq k\|\psi(x,t)\|_\infty + 1/k$ (see the growth condition of φ in (2.30)).

Therefore, we have

$$-\Phi(0) = -\Phi^{**}(0) = \min(\Phi^*). \tag{2.67}$$

where Φ^* denotes the Moreau-Fenchel conjugate of Φ. Notice that the conjugate is taken in the duality between continuous functions and bounded measures. We shall show that

$$\Phi^*(\lambda) = \begin{cases} \widehat{E}(v) & \text{if } \lambda = Dv \text{ with } v \in \mathcal{A}(I), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.68}$$

Then, as a consequence, there holds $\min(\Phi^*) = \min\left\{\widehat{E}(v) \ : \ v \in \mathcal{A}(I)\right\}$. Thus, by (2.66)-(2.67), we conclude that $\inf(\mathcal{Q}) = \sup(\mathcal{P}^*)$, and the proof is completed.

Let us sketch the proof of the claim (2.68). The proof is delicate, very technical, and can be achieved in the similar way of the results of G. Bouchitté and I. Fragàla in [20]. We refer to this paper for further details. Let λ be a bounded vector measure such that $\Phi^*(\lambda) < +\infty$. Basically, the proof of the claim (2.68) is supported by the three statements :

(i) For every $\sigma \in X_1(\Omega \times I)$ and every $v \in BV_\infty(\Omega \times I)$, it holds

$$\int_{\Omega\times I} \widetilde{h}(t, Dv) = \sup\left\{\int_{\Omega\times I} (\sigma+\eta)\cdot Dv \ : \ \eta \in \mathcal{D}(\Omega\times I;\mathbb{R}^{N+1}),\ \sigma+\eta \in \mathcal{K}(I)\right\};$$

(ii) For every compact neighborhood U of the boundary $\partial\Omega$ and every bounded continuous $\psi : \overline{\Omega} \times I \to \mathbb{R}^{N+1}$, there hold

$$\langle \lambda - Dv_0, \psi\rangle = 0 \text{ whenever } \operatorname{div}\psi = 0 \text{ in } \Omega\times I \text{ and } \psi = 0 \text{ on } U\times I,$$
$$\int_{(\Omega\setminus U)\times I} \widetilde{h}(t,\lambda) < +\infty;$$

(iii) There exists a scalar function $v \in L^1_{\text{loc}}(\Omega\times I)$ with $v(x,\cdot)$ monotone non-increasing such that $\lambda = Dv$. Furthermore, up to adding a constant to v, we have $v \in \mathcal{A}(I)$, i.e.

$$v \in BV_\infty(\Omega\times I;[0,1]), \quad v(x,a) = 1, \quad v(x,b) = 0.$$

We are going to compute $\Phi^*(\lambda)$. Let σ be an element of $X_1(\Omega\times I)$. By the assertion (iii) above, the duality pair $(\sigma\cdot\lambda) = (\sigma\cdot Dv)$ is well defined. On the other hand, by (i), it holds

$$\int_{\Omega\times I} \widetilde{h}(t,Dv) = \sup\left\{\int_{\Omega\times I} (\sigma+\eta)\cdot Dv \ : \ \eta\in\mathcal{D}(\Omega\times I;\mathbb{R}^{N+1}),\ \sigma+\eta\in\mathcal{K}(I)\right\}.$$

The Moreau-Fenchel conjugate of Φ is computed as followings

$$\begin{aligned}
\Phi^*(\lambda) &= \sup\left\{\int_{\Omega\times I} \eta\cdot Dv - \Phi(\eta) \ : \ \eta\in C_0(\Omega\times I;\mathbb{R}^{N+1}\right\} \\
&= \sup\left\{\int_{\Omega\times I} \eta\cdot Dv - \Phi(\eta) \ : \ \eta\in \mathcal{D}(\Omega\times I;\mathbb{R}^{N+1}\right\} \\
&= \sup\Big\{ \int_{\Omega\times I} (\eta+\sigma)\cdot Dv - \int_\Omega \sigma^t(x,0)dx + \int_{\partial\Omega} \varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} - \langle Dv,\sigma\rangle \ : \\
&\qquad \eta\in\mathcal{D}(\Omega\times I;\mathbb{R}^{N+1}),\ \sigma\in X_1(\Omega\times\mathbb{R}), \\
&\qquad\qquad -\operatorname{div}\sigma = p,\ \sigma^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0} \text{ on } \partial\Omega\times I,\ \sigma+\eta\in\mathcal{K}(I)\Big\} \\
&= \int_{\Omega\times I}\widetilde{h}(t,Dv) + \sup\Big\{ \langle Dv_0 - Dv,\sigma\rangle + \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} \ : \\
&\qquad \sigma\in X_1(\Omega\times\mathbb{R}),\ -\operatorname{div}\sigma = p,\ \sigma^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0} \text{ on } \partial\Omega\times I,\ \sigma\in\mathcal{K}(I)\Big\}.
\end{aligned}$$

In the calculation above, we used : the definition of Φ^* in the first equality; the density of $\mathcal{D}(\Omega\times I;\mathbb{R}^{N+1})$ in $C_0(\Omega\times I;\mathbb{R}^{N+1})$ and the continuity of Φ at 0 in the second equality; the definition of Φ in the third equality; and the assertion (i) in the fourth equality. Finally, to finish the calculation, by invoking (2.56) (with $\widetilde{u}\equiv 0$), we find that the supremum in the fourth equality of $\Phi^*(\lambda)$ above agrees with $\ell(v)$, hence $\Phi^*(\lambda)$ is equal to the energy of the convexified problem $(\mathcal{PR})$, namely

$$\Phi^*(\lambda)=\int_{\Omega\times I}\widetilde{h}(t,Dv)+\ell(v)=\widehat{E}(v).$$

As shown by the assertion (iii), v belongs to $\mathcal{A}(I)$, the demonstration of (2.68) is done. ■

Remark 2.23. The dual problem $(\mathcal{P}^*)$ can be interpreted as a problem accompanying with the constraint of free divergence. To that aim, we set $\widetilde{\sigma}=(\widetilde{\sigma}^x,\widetilde{\sigma}^t)$ such that

$$\widetilde{\sigma}^x=\sigma^x,\qquad \widetilde{\sigma}^t=\sigma^t+p(x)t\qquad \forall\sigma\in\mathcal{K}(I).$$

Clearly, $\operatorname{div}\widetilde{\sigma}=0$, and $\widetilde{\sigma}$ satisfies

$$\begin{cases}\widetilde{\sigma}^t(x,t)\geq\varphi^*(\widetilde{\sigma}^x(x,t))-g(t)+p(x)t\ \text{ for } \mathcal{L}^{N+1}\text{-a.e. } (x,t)\in\Omega\times I,\\ \widetilde{\sigma}^t(x,t)\geq-\varphi(0)-g(t)+p(x)t\quad \forall\, t\in S_g \text{ and for } \mathcal{L}^N\text{-a.e. } x\in\Omega.\end{cases}\tag{2.69}$$

At that moment, the flux formula (2.56) reduces to

$$\int_{\Omega\times I}\widetilde{\sigma}\cdot Dv+\int_{\partial\Omega\times I}\partial_t\gamma_{u_0}(x,t)(v-v_0)d\mathcal{H}^{N-1}dt+\int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}$$
$$=\int_{\overline{G}_{\widetilde{u}}}\widetilde{\sigma}\cdot\widehat{\nu}_{\widetilde{u}}d\mathcal{H}^N+\int_{\partial\Omega}\varphi^\infty((u_0-\widetilde{u})\nu_\Omega)d\mathcal{H}^{N-1}.$$

When $v=\mathbf{1}_u$ for some $u\in BV(\Omega;I)$, one has

$$\int_{\overline{G}_u}\widetilde{\sigma}\cdot\widehat{\nu}_u d\mathcal{H}^N+\int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}$$
$$=\int_{\overline{G}_{\widetilde{u}}}\widetilde{\sigma}\cdot\widehat{\nu}_{\widetilde{u}}d\mathcal{H}^N+\int_{\partial\Omega}\varphi^\infty((u_0-\widetilde{u})\nu_\Omega)d\mathcal{H}^{N-1}.\tag{2.70}$$

In addition, if $\widetilde{u}=\widetilde{u}_0$ where $\widetilde{u}_0\in BV(\Omega;I)$ and $\widetilde{u}_0=u_0$ on $\partial\Omega$, it holds

$$\int_{\overline{G}_{\widetilde{u}_0}}\widetilde{\sigma}\cdot\widehat{\nu}_{\widetilde{u}_0}d\mathcal{H}^N=\int_{\overline{G}_u}\widetilde{\sigma}\cdot\widehat{\nu}_u d\mathcal{H}^N+\int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}.$$

In the specific case where $\widetilde{u}\equiv 0$, the flux formula (2.70) becomes

$$-\int_\Omega\widetilde{\sigma}^t(x,0)dx+\int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}=\int_{\overline{G}_u}\widetilde{\sigma}\cdot\widehat{\nu}_u d\mathcal{H}^N+\int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}.$$

On the other hand, the two fluxes of $\widetilde{\sigma}$ and σ passing through the interface $t=0$ are identical, precisely as $\widetilde{\sigma}-\sigma=(0,p(x)t)$ we have

$$\int_\Omega(\widetilde{\sigma}^t(x,0)-\sigma^t(x,0))dx=0.$$

Following these observations, it allows to establish another equivalent dual problem. Let us denote by $\mathcal{K}_0(I)$ the class of vector fields in $X_1(\Omega\times\mathbb{R})$ verifying the convex constraints (2.69). Then, there holds

$$\sup(\mathcal{P}^*)=\sup\Big\{-\int_\Omega\widetilde{\sigma}^t(x,0)dx+\int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}\ :\ \widetilde{\sigma}\in\mathcal{K}_0(I),$$
$$\operatorname{div}\widetilde{\sigma}=0\text{ in }\Omega\times I,\ (\widetilde{\sigma})^x\cdot\nu_\Omega=-\partial_t\gamma_{u_0}\text{ on }\partial\Omega\times I\Big\}.$$

2.5.3 Optimality conditions and saddle point characterization

We provide in the following the necessary and sufficient conditions for optimality.

Theorem 2.24. *Let u, σ be admissible to problems $(\mathcal{PR})$ in (2.37) and $(\mathcal{P}^*)$ in (2.61), respectively. Then, u is a solution to $(\mathcal{PR})$ and σ is a solution to $(\mathcal{P}^*)$ if and only if*

$$\widetilde{h}(t,\widehat{\nu}_u) = \sigma\cdot\widehat{\nu}_u \qquad \mathcal{H}^N\text{-a.e. on } \overline{G}_u. \tag{2.71}$$

We call such σ a calibration for u.

Proof. Assume that u, σ are respectively admissible to problems $(\mathcal{PR})$ in (2.37) and $(\mathcal{P}^*)$ in (2.61), and they satisfy the condition (2.71).

$$\begin{aligned}
\sup(\mathcal{P}^*) &\geq -\int_\Omega \sigma^t(x,0)dx + \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}\\
&= \int_{\overline{G}_u}\sigma\cdot\widehat{\nu}_u d\mathcal{H}^N - \int_\Omega p(x)u dx + \int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}\\
&= \int_{\overline{G}_u}\widetilde{h}(t,\widehat{\nu}_u)d\mathcal{H}^N - \int_\Omega p(x)u dx + \int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}\\
&= E(u) \geq \inf(\mathcal{PR}) = \inf(\mathcal{P}).
\end{aligned}$$

In the comparison above, we used in the order the definition of $\sup(\mathcal{P}^*)$, Lemma 2.20 (see also (2.60)), the condition (2.71), the equality $J(\mathbf{1}_u) = F(u)$, and the definition of $\inf(\mathcal{PR})$ (note that $\inf(\mathcal{PR}) = \inf(\mathcal{P})$). As we concluded in Lemma 2.21, we infer that all the equalities become equalities. It means u, σ are optimal correspondingly for the (relaxed) primal and dual problems.

Suppose now that u, σ are optimal correspondingly for the (relaxed) primal and dual problems. Again, we apply in the order Lemma 2.20 (see also (2.60)), the optimality of σ, the no duality gap, the optimality of u, and the equality $J(\mathbf{1}_u) = F(u)$ to the sequence of calculation below

$$\begin{aligned}
&\int_{\overline{G}_u}\sigma\cdot\widehat{\nu}_u d\mathcal{H}^N - \int_\Omega p(x)u dx + \int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}\\
&= -\int_\Omega \sigma^t(x,0)dx + \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1}\\
&= \sup(\mathcal{P}^*) = \inf(\mathcal{P}) = \inf(\mathcal{PR}) = E(u)\\
&= \int_{\overline{G}_u}\widetilde{h}(t,\widehat{\nu}_u)d\mathcal{H}^N - \int_\Omega p(x)u dx + \int_{\partial\Omega}\varphi^\infty((u_0-u)\nu_\Omega)d\mathcal{H}^{N-1}.
\end{aligned}$$

We deduce that $\int_{\overline{G}_u}\sigma\cdot\widehat{\nu}_u d\mathcal{H}^N = \int_{\overline{G}_u}\widetilde{h}(t,\widehat{\nu}_u)d\mathcal{H}^N$, which implies (2.71). ∎

Corollary 2.25. *Assume u, σ are optimal correspondingly for the (relaxed) primal and dual problems. The condition (2.71) is satisfied if and only if it holds*

$$\sigma^x(x,u(x)) \in \partial\varphi(\nabla u(x)) \qquad \text{for } \mathcal{L}^N\text{-a.e. } x\in u^{-1}(\mathbb{R}\setminus S_g), \tag{2.72}$$

$$\sigma^t(x,u(x)) = \varphi^*(\sigma^x(x,u(x))) - g(u(x)) \qquad \text{for } \mathcal{L}^N\text{-a.e. } x\in u^{-1}(\mathbb{R}\setminus S_g), \tag{2.73}$$

$$\sigma^t(x,t) = -\varphi(0) - g(t) \qquad \forall t\in\mathbb{R} \text{ and for } \mathcal{L}^N\text{-a.e. } x\in\{u=t\}, \tag{2.74}$$

$$\sigma^x(x,t)\cdot\nu_u = \varphi^\infty(\nu_u) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x\in S_u \text{ and } \forall t\in[u^-(x),u^+(x)], \tag{2.75}$$

$$\sigma^x(x,u^+(x))\cdot\nu_u = \varphi^\infty(\nu_u) \qquad \text{for } |D^c u|\text{-a.e. on } \Omega. \tag{2.76}$$

Proof. On the complete graph $\overline{G}_u$, at the points corresponding to singular parts $D^j u$ and $D^c u$, the conditions (2.75)-(2.76) agree with (2.71).

On the usual graph G_u except the points corresponding the Cantor part of Du where $\widehat{\nu}_u^t \neq 0$ (see the definition (2.33) of $\widehat{\nu}_u$ on G_u), the following computation is done $\mathcal{H}^N$-a.e. on

$G_u \cap [\Omega \times (\mathbb{R} \setminus S_g)]$ with making use of the conditions (2.72)-(2.73) and noticing the definition of $\widetilde{h}$,

$$\begin{aligned}\sigma \cdot \widehat{\nu}_u &= \widehat{\nu}_u^x \cdot \sigma^x + \widehat{\nu}_u^t\Big(\varphi^*(\sigma^x) - g(u)\Big) = -\widehat{\nu}_u^t\left(-\frac{\widehat{\nu}_u^x}{\widehat{\nu}_u^t} \cdot \sigma^x - \varphi^*(\sigma^x) + g(u)\right)\\ &= -\widehat{\nu}_u^t\left(\nabla u \cdot \sigma^x - \varphi^*(\sigma^x) + g(u)\right) = -\widehat{\nu}_u^t\Big(\varphi(\nabla u) + g(u)\Big)\\ &= -\widehat{\nu}_u^t\Big(\varphi(-\frac{\widehat{\nu}_u^x}{\widehat{\nu}_u^t}) + g(u)\Big) = \widetilde{h}(t, \widehat{\nu}_u).\end{aligned}$$

Besides, because of (2.74),

$$\sigma \cdot (-e_{N+1}) = -\sigma^t(x,t) = \varphi(0) + g(t) = \widetilde{h}(t, -e_{N+1})$$

$\mathcal{H}^N$-a.e. on $G_u \cap [\Omega \times S_g]$ (here the vector e_{N+1} is the $(N+1)$-th element of the standard basis of $\mathbb{R}^{N+1}$).

Conversely, suppose that (2.71) holds true. As σ verifies the constraint (2.22), there holds $\mathcal{H}^N$-a.e. on $G_u \cap [\Omega \times (\mathbb{R} \setminus S_g)]$ (except the points corresponding the Cantor part of Du):

$$\begin{aligned}\widetilde{h}(t, \widehat{\nu}_u) = \sigma \cdot \widehat{\nu}_u &\le \widehat{\nu}_u^x \cdot \sigma^x + \widehat{\nu}_u^t\Big(\varphi^*(\sigma^x) - g(u)\Big) = -\widehat{\nu}_u^t\left(-\frac{\widehat{\nu}_u^x}{\widehat{\nu}_u^t} \cdot \sigma^x - \varphi^*(\sigma^x) + g(u)\right)\\ &= -\widehat{\nu}_u^t\left(\nabla u \cdot \sigma^x - \varphi^*(\sigma^x) + g(u)\right) \le -\widehat{\nu}_u^t\Big(\varphi(\nabla u) + g(u)\Big) = -\widehat{\nu}_u^t\Big(\varphi(-\frac{\widehat{\nu}_u^x}{\widehat{\nu}_u^t}) + g(u)\Big)\\ &= \widetilde{h}(t, \widehat{\nu}_u).\end{aligned}$$

As we see, these inequalities above is indeed equalities, which give (2.72)-(2.73). By using the second constraint in (2.22), we infer that

$$\widetilde{h}(t, -e_{N+1}) = \sigma \cdot (-e_{N+1}) = -\sigma^t(x,t) \le \varphi(0) + g(t) = \widetilde{h}(t, -e_{N+1})$$

$\mathcal{H}^N$-a.e. on $G_u \cap (\Omega \times \mathbb{R})$. Hence, this inequality actually holds with equality, which yields the condition (2.74). ∎

Our duality result in the extended case of linear-growth functionals also provide a min-max formulation. In an analogous way, let us establish this saddle point characterization. Let $v \in BV(\Omega \times I)$ and $\sigma \in X_1(\Omega \times I)$, we introduce the Lagrangian

$$L(v, \sigma) := \ell(v) + \int_{\Omega \times I} \sigma \cdot Dv \tag{2.77}$$

where $\ell(v)$ is the affine form defined in (2.42).

Theorem 2.26. *We have*

$$\inf(\mathcal{P}) = \inf_{v \in \mathcal{A}(I)} \sup_{\sigma \in \mathcal{K}(I)} L(v, \sigma) = \sup_{\sigma \in \mathcal{K}(I)} \inf_{v \in \mathcal{A}(I)} L(v, \sigma) = \sup(\mathcal{P}^*).$$

Furthermore, a pair $(\overline{v}, \overline{\sigma})$ is optimal for the convexified problem $(\mathcal{Q})$ (in (2.49)) and for the dual problem $(\mathcal{P}^*)$ (in (2.61)) if and only if it is a saddle point for $L(v, \sigma)$, i.e.

$$L(\overline{v}, \sigma) \le L(\overline{v}, \overline{\sigma}) \le L(v, \overline{\sigma}) \qquad \forall (v, \sigma) \in \mathcal{A}(I) \times \mathcal{K}(I).$$

Proof. We observe that

$$\begin{aligned} L(v,\sigma) =& \ell(v) + \int_{\Omega\times I} \sigma\cdot D(v-v_0) - \int_\Omega \sigma^t(x,0)dx \\ =& -\int_{\Omega\times I} p(x)(v-v_0)\,dxdt + \int_{\partial\Omega\times I} \partial_t\gamma_{u_0}(x,t)(v-v_0)d\mathcal{H}^{N-1}dt + \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} \\ & - \int_{\Omega\times I}(v-v_0)\,\mathrm{div}\,\sigma + \int_{\partial\Omega\times I}(v-v_0)(\sigma^x\cdot\nu_\Omega)d\mathcal{H}^{N-1}dt - \int_\Omega \sigma^t(x,0)dx \end{aligned}$$

where we used the generalized Gauss-Green formula. It easy to see that for $v = v_0 + \psi$ with $\psi \in \mathcal{D}(\Omega\times I)$, the infimum $\inf_{v\in\mathcal{A}(I)} L(v,\sigma)$ is finite only if $\mathrm{div}\,\sigma = 0$, and when we take $\psi\in\mathcal{D}(\overline{\Omega}\times I)$ it must hold $\sigma^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0}(x,t)$. This is to say that

$$\inf_{v\in\mathcal{A}(I)} L(v,\sigma) = \begin{cases} -\displaystyle\int_\Omega \sigma^t(x,0)dx + \int_{\partial\Omega}\varphi^\infty(u_0\nu_\Omega)d\mathcal{H}^{N-1} & \text{if } \sigma\in\mathcal{B}(I) \\ -\infty & \text{if } \sigma\notin\mathcal{B}(I). \end{cases}$$

So it turns out that

$$\sup(\mathcal{P}^*) = \sup_{\sigma\in\mathcal{K}(I)}\inf_{v\in\mathcal{A}(I)} L(v,\sigma).$$

Besides, we have

$$\begin{aligned} \inf(\mathcal{P}) &= \inf\{\widehat{E}(v)\ :\ v\in\mathcal{A}(I)\} = \inf\{J(v)+\ell(v)\ :\ v\in\mathcal{A}(I)\} \\ &= \inf_{v\in\mathcal{A}(I)}\sup_{\sigma\in\mathcal{K}(I)}\left\{\int_{\Omega\times I}\sigma\cdot Dv + \ell(v)\right\} = \inf_{v\in\mathcal{A}(I)}\sup_{\sigma\in\mathcal{K}(I)} L(v,\sigma) \end{aligned}$$

where we used the definition of $\widehat{E}(v)$ and the identification

$$J(v) = \int_{\Omega\times I}\widetilde{h}(t,Dv) = \sup_{\sigma\in\mathcal{K}(I)}\int_{\Omega\times I}\sigma\cdot Dv. \tag{2.78}$$

Notice that for every $t\in\mathbb{R}$, the function $\widetilde{h}(t,\cdot)$ (defined in (2.38)) is the support function of the convex subset of $\mathbb{R}^{N+1}$

$$K(t) := \{q=(q^x,q^t)\ :\ q^t\geq\varphi^*(q^x)-g(t)\}$$

(we can find the proof of this fact in [98, Corollary 13.5.1]). This result can be extended to the integral functional $\int_{\Omega\times I}\widetilde{h}(t,\cdot)$ on the space of bounded vector-valued measure [20], namely

$$\int_{\Omega\times I}\widetilde{h}(t,\lambda) = \sup\left\{\langle\lambda,\psi\rangle\ :\ \psi\in C_0(\Omega\times I;\mathbb{R}^{N+1}),\ \psi(x,t)\in K(t)\text{ in }\Omega\times I\right\}$$

for any bounded vector-valued measure $\lambda\in\mathcal{M}(\Omega\times I;\mathbb{R}^{N+1})$. Moreover, this equality remains true if we restrict the supremum above on smooth functions $\psi\in\mathcal{D}(\Omega\times I;\mathbb{R}^{N+1})$. And then the identification (2.78) can be proved. We refer to [20] for its delicate proof.

The last part of the theorem is the classical result in min-max theory (see [57]). ■

Chapter 3

A convex relaxation method for free boundary problems

We introduce in this chapter a convex relaxation method for a large class of non convex variational problems where the dependence of the functional with respect to gradient satisfies the assumption of positive one-homogeneity. This method is directly applied to free boundary or multiphase problems in which we minimize the classical total variation or anisotropic norms. One of the main results of the convexification procedure is the so-called exclusion principle, which states that optimal solutions for the relaxed problem avoid taking values in the non convex regions of the functional. We also present in this chapter various numerical experiments based on a primal-dual algorithm where the non differentiability of Lagrangians is treated by using a new epigraph projection.

3.1 Introduction

In this chapter we focus on a particular case of the convexification method developed in Chapter 2 dealing with non convex variational problems of the kind

$$\inf\left\{\int_\Omega (h(\nabla u)+g(u)-p(x)\,u)\,dx \;:\; u\in W^{1,1}(\Omega),\ u=u_0 \text{ on } \Gamma\right\}$$

where Γ be a Borel subset of $\partial\Omega$. We will still assume that the convex function $h:\mathbb{R}^N\to[0,+\infty)$ is convex continuous, but we assume in addition that it satisfies the 1-homogeneity condition:

$$h(tz) \;=\; t\,h(z) \qquad \text{for all } z\in\mathbb{R}^N \text{ and } t\geq 0.$$

It turns out in this case that, under mild conditions on the possibly discontinuous non convex function g and a suitable compatibility condition on the prescribed Dirichlet datum u_0, the duality results obtained in Chapter 2 holding in the higher dimension space $\mathbb{R}^{N+1}$ can be reduced to the initial dimension N. As a consequence, we deduce that optimizers u do not take values in the non convexity subset of g defined by $Z:=\{g>g^{**}\}$; it is what we call the *exclusion principle*. As a consequence the infimum of the initial non convex problem coincides with that of its direct convexified version:

$$\inf\left\{\int_\Omega (h(\nabla u)+g^{**}(u)-p(x)\,u)\,dx : u\in W^{1,1}(\Omega),\ u=u_0 \text{ on } \Gamma\right\}\ .$$

In fact the exclusion principle mentioned above is not new: it has been discovered by A. Visintin in the 1990's (see [107]) in the different context of Cahn-Hilliard type models where g is a double well potential and u represents the density of a two-phase fluid. In this case the boundary condition is replaced by a prescribed total mass condition $\int_\Omega u = m$ and the arguments of the proof relying on rearrangements tricks allow to replace the term $\int_\Omega h(\nabla u)dx$ by a general convex 1-homogeneous functional satisfying the co-area formula.

The method we use here, based on the calibration theory developed in Chapter 2, gives a very interesting alternative. The main benefit is that we can extend the exclusion principle to situations where the potential g is discontinuous, possibly infinite. In particular, as we are motivated by more general free boundary problems or multiphase problems, we will apply the result when g is chosen so that the coincidence set $\{t \in \mathbb{R} : g(t) = g^{**}(t) < +\infty\}$ is a finite subset $\{t_1, ..., t_n\}$ (sorted in increasing order) while the boundary data u_0 avoids all intermediate values. Doing so we can reduce some n-multiple phases problem to a classical convex optimization problem for which many efficient algorithms are available. Note however that, by construction, g^{**} will not be differentiable at the values t_i corresponding to the searched phases. To overcome this difficulty we propose two numerical methods: the first one combines an explicit primal-dual scheme with a geometric (polyhedral) projection on the epigraph of g^{**} whereas the second one combines the semi-implicit algorithms studied in Chapter 5 with a projection on the epigraph of g^*. The efficiency of the method is evidenced in many numerical simulations including the case where the homogeneous integrand h is associated with a crystalline norm. An optimality criterion for the interfaces is derived in the case where h is the Euclidean norm.

3.2 Exclusion principle and main results

Let Ω be an open bounded domain of $\mathbb{R}^N$ with Lipschitz boundary and let Γ be a Borel subset of $\partial\Omega$. We denote by ν_Ω the unit exterior normal on $\partial\Omega$. For a given $u_0 \in L^1(\partial\Omega)$, we consider the infimum problem

$$\inf\left\{\int_\Omega [h(\nabla u) + g(u) - p(x)u]dx \ : \ u \in W^{1,1}(\Omega)\ ,\ u = u_0 \text{ on } \Gamma\right\} \tag{3.1}$$

We recall that, by Gagliardo's Theorem, the trace map: $u \in W^{1,1}(\Omega) \to \mathrm{Tr}(u) \in L^1(\partial\Omega)$ is continuous and *surjective*.

The standing hypotheses on h, g, p are listed below:

(H1) The function $h : \mathbb{R}^N \to (-\infty, +\infty]$ is lower semicontinuous, positively 1-homogeneous, convex satisfying the standard growth condition

$$\forall z \in \mathbb{R}^N, \quad C_1|z| \le h(z) \le C_2(1 + |z|)\ .$$

(H2) The function $g : \mathbb{R} \to (-\infty, +\infty]$ is a lower semicontinuous function with possibly many jumps; more precisely we assume that there exists a *Lebesgue negligible* set $D \subset \mathbb{R}$ such that g is upper semicontinuous on $\mathbb{R} \setminus D$, namely

$$\limsup_{s\to t} g(s) \le g(t),\ \forall t \in \mathbb{R} \setminus D\,.$$

(H3) The source term p satisfies one of the hypothesis listed below:

a) $p \in L^1(\Omega)$ if $\mathrm{dom}\, g := \{t \in \mathbb{R} :\ g(t) < +\infty\}$ is bounded

b) $p \in L^{r'}(\Omega)$ if g satisfies $\liminf_{|t|\to\infty} \dfrac{g(t)}{|t|^r} > 0$ for $r > 1$ ($r' = \frac{r}{r-1}$).

Under these assumptions, the infimum in (3.1) is finite and minimizing sequences are bounded in $W^{1,1}(\Omega)$. By using the classical direct method of Calculus of Variations, we obtain existence of solutions for the relaxed problem in $BV(\Omega)$ namely:

$$(\mathcal{P}_g) \qquad \inf\Big\{F_g(u) \ : \ u \in BV(\Omega)\Big\}$$

being F_g defined on $BV(\Omega)$ by

$$F_g(u) = \int_\Omega h(Du) + \int_\Omega (g(u) - p(x)\,u)\,dx + \int_\Gamma h((u_0 - u)\,\nu_\Omega)\,d\mathcal{H}^{N-1}\ .$$

Note that in the notation above, we underlined the dependence with respect to g since we will need to consider different choices for g. The next result is classical.

Lemma 3.1. *Under (H1)(H2)(H3), it holds* $\inf(3.1) = \min(\mathcal{P}_g)$. *Furthermore any minimizing sequence for* (3.1) *is weakly relatively compact in* $BV(\Omega)$ *and admits a subsequence converging in* $L^1(\Omega)$ *to a solution of* $(\mathcal{P}_g)$.

We emphasize that g is a priori a *nonconvex* function ranging in $(-\infty, +\infty]$ and that discontinuities are allowed. This is crucial in order to associate (P_g) with some free boundary or multi-phase problems.

The main result of this section (Theorem 3.3) specifies under which conditions the minimum problem (P_g) can be relaxed in $(P_{g^{**}})$, being g^{**} the convex enveloppe of g. The key argument is the following fundamental *exclusion principle*:

Theorem 3.2. *Let* $-\infty < a < b < +\infty$. *Assume that* $g_{a,b} < g$ *in* (a,b) *where* $g_{a,b}$ *is the affine interpolant given by:*

$$g_{a,b}(t) := \begin{cases} g(t) & \text{if } t \notin (a,b) \\ m_{a,b}(t-a) + g(a) & \text{otherwise}\,, \end{cases} \qquad \text{and} \qquad m_{a,b} := \frac{g(b) - g(a)}{b-a}.$$

Then if $u_0 \notin (a,b)$ *a.e. on* Γ, *it holds* $\inf(\mathcal{P}_g) = \inf(\mathcal{P}_{g_{a,b}})$. *Moreover any solution* u *to* $(\mathcal{P}_g)$ *is such that* $u(x) \notin (a,b)$ *for a.e.* $x \in \Omega$.

Before giving the proof of Theorem 3.2, we notice that the exclusion principle can be used on a countable union of intervals (a,b) on which $g > g_{a,b}$, namely to the following set

$$Z \ := \ \{g > g^{**}\} \tag{3.2}$$

Indeed, if Z is bounded, we may apply Lemma 3.5 to show that Z is a countable union of open intervals. Accordingly we consider the following additional assumptions:

(H4) $Z := \{g > g^{**}\}$ bounded ;

(H5) $u_0 \in \mathbb{R} \setminus Z$ a.e. on Γ.

We are now in position to state our main "convex relaxation result":

Theorem 3.3. *Under (H1–H5), it holds:*

i) $\inf(\mathcal{P}_g) = \inf(\mathcal{P}_{g^{**}})$

ii) If u *is a solution to* $(\mathcal{P}_g)$, *then* $u(x) \in \mathbb{R} \setminus Z$ *for a.e.* $x \in \Omega$.

In particular if g is such that $\mathrm{dom}(g) \setminus Z$ consists of finitely many reals numbers $t_1 < \cdots < t_k$, then solutions to $(\mathcal{P}_g)$ are of the form

$$u = \sum_{i=1}^{k} t_i \mathbf{1}_{A_i} \quad , \quad \text{with } \{A_i, 1 \le i \le k\} \text{ Borel partition of } \Omega. \tag{3.3}$$

Remark 3.4. We have merely $\mathrm{Argmin}\,\mathcal{P}_g \subset \mathrm{Argmin}\,\mathcal{P}_{g^{**}}$ but equality holds in the case $\mathcal{P}_{g^{**}}$ admits a unique solution. This uniqueness issue for $(\mathcal{P}_{g^{**}})$ seems to be a very difficult task since g^{**} is not stricly convex (at all on Z) and we refer for that to the recent paper for a uniqueness result in a similar context [15]. We believe that uniqueness for $(\mathcal{P}_g)$ implies uniqueness for $(\mathcal{P}_{g^{**}})$ although we are not yet able to prove it. Actually in many examples of mutiphase problems as presented in Section 3.3, we observed a very good convergence of the numerical approximation of $(\mathcal{P}_{g^{**}})$ to a solution of the form (3.3). Therefore we believe that uniqueness holds generically with respect to the boundary data and the parameters entering in the definition of g, Ω.

Lemma 3.5. *Let $g : \mathbb{R} \to (-\infty, +\infty]$ be a l.s.c. non convex function whose Fenchel biconjugate g^{**} is proper. Let Z be the non empty set $Z := \{t \in \mathbb{R} \ : \ g(t) > g^{**}(t)\}$ and let $[m, M]$ be the closure of the interval $\{t \in \mathbb{R} : g^{**}(t) < +\infty\}$ $(-\infty \le m < M \le +\infty)$. Then, we have:*

(i) Z is an open set and $Z \subset (m, M)$;

(ii) Assume that Z is bounded. Then there exists an at most countably disjoint family of finite intervals $\{(a_i, b_i)\}_{i \in I}$ so that $Z = \cup_{i \in I}(a_i, b_i)$. Furthermore g^{**} is affine in each interval (a_i, b_i).

Proof. (i) Obviously one has $Z \subset [m, M]$ since $g(t) = g^{**}(t) = +\infty$ for $t \notin [m, M]$. On the other hand, as we are on the real line, the convex envelope $\mathrm{co}\, g$ of g is given by:

$$g^{**}(t) = (\mathrm{co}\, g)(t) = \inf\{\theta g(t') + (1-\theta) g(t'') \ : \ \theta \in [0,1],\ t', t'' \in [m, M],\ \theta t' + (1-\theta) t'' = t\}$$

Keeping in mind that if $g^{**}(t) < +\infty$, we have to choose t', t" $\in \mathrm{dom}\, g$, we infer that if $m \in \mathbb{R}$ then $g^{**}(m) = g(m)$, i.e. $m \notin Z$. Thus g^{**} agrees with g on the extreme points of $[m, M]$, that is $m, M \notin Z$ and $Z \subset (m, M)$

We show now that $Z^c = \mathbb{R} \setminus Z$ is a closed subset of $\mathbb{R}$. Let $t_n \in Z^c$ be such tha $t_n \to t$. We need to check that $t \in Z^c$ which is already known if $t \notin (m, M)$. If $t \in (m, M)$, we exploit the continuity of the convex function g^{**} on the open intervall (m, M) together with the lower semicontinuity of g:

$$g^{**}(t) = \limsup_n g^{**}(t_n) \ge \liminf_n g(t_n) \ge g(t).$$

Thus $g^{**}(t) = g(t)$ and $t \in Z^c$.

(ii) As a bounded open subset of (m, M), Z can be written as $Z = \cup_{i \in I}(a_i, b_i)$ where I is finite or countable and $(a_i, b_i) \subset (m, M)$ (thus $g(a_i), g(b_i)$ are finite). It remains to show that g^{**} is affine in each interval (a_i, b_i). Let us denote by $\hat{g}$ the convex function which coincides with g^{**} on Z^c and with the affine interpolant of g^{**} on each interval $[a_i, b_i]$. Clearly one has:

$$g^{**} \le \hat{g} \quad , \quad g^{**} = \hat{g} = g \text{ in } Z^c \quad , \quad \partial \hat{g}(t) = \{(\hat{g})'(t)\} = \{m_i\} \quad \forall t \in (a_i, b_i)\ ,$$

where $m_i := \frac{g(b_i) - g(a_i)}{b_i - a_i}$ denotes the slope of $\hat{g}$ on $[a_i, b_i]$. We are done if we can show the inequality $g \ge \hat{g}$ since then $g^{**} \ge \hat{g}$ by taking the convex envelope, thus $g^{**} = \hat{g}$. We observe that the function $\varphi := g - \hat{g}$ vanishes on Z^c and is l.s.c. (since g is l.s.c. and $\hat{g}$ is continuous on (m, M)). Asssume that $\inf \varphi < 0$. Then, recalling that Z is bounded, φ achieves its global minimum at some $\bar{t} \in Z$. Let $i \in I$ such that $\bar{t} \in (a_i, b_i)$. Then $\hat{g}(t) \ge \hat{g}(\bar{t}) + m_i(t - \bar{t})$ whereas $\varphi(t) \ge \varphi(\bar{t})$ for all $t \in \mathbb{R}$. So far we obtain that $g(t) \ge g(\bar{t}) + m_i(t - \bar{t})$ thus $m_i \in \partial g(\bar{t})$. This is inpossible since $g^{**}(\bar{t}) < g(\bar{t})$. ∎

Lemma 3.6. *Let $u \in BV(\Omega)$. If u does not take value in the interval (a,b) almost everywhere in Ω then neither does its boundary trace $\mathcal{H}^{N-1}$-almost everywhere in $\partial\Omega$.*

Proof. For $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, the boundary trace of u (denoted by $\mathrm{Tr}(u)$) is defined in a way such that

$$\lim_{r\to 0^+} r^{-N} \int_{\Omega\cap B(0,r)} |u(y) - \mathrm{Tr}(u)(x)|\, dy = 0. \tag{3.4}$$

We shall prove the lemma by contradiction. Suppose that $\mathrm{Tr}(u)(x) := \alpha \in (a,b)$. By setting $\eta := \min\{|a-\alpha|, |b-\alpha|\}$, we find that for a.e. $y \in \Omega$, since $u(y) \notin (a,b)$,

$$|u(y) - \alpha| > \eta > 0.$$

Then we infer that

$$\int_{\Omega\cap B(0,r)} |u(y) - \mathrm{Tr}(u)(x)|\, dy > \eta|\Omega \cap B(0,1)| = O(r^{-N}).$$

This contradicts to the definition of trace (3.4). ∎

Remark 3.7. Since the set of non-convexity $Z := \{g > g^{**}\}$ is a union of open intervals (as shown in Lemma 3.5), it is the case to apply Lemma 3.6 so that if u does not take value in Z then neither does its boundary trace. This explains the necessity of condition $u_0 \in Z^c$ appearing in (H5). Moreover, there is a counterexample figuring out that it does not hold $\inf(\mathcal{P}_g) = \inf(\mathcal{P}_{g^{**}})$ if $u_0 \in Z$. We refer to Example 3.9.

Remark 3.8. The positive 1-homogeneity property of the integrand h in the hypothesis (H1) must also be guaranteed in order to validate Theorems 3.2 and 3.3. We describe in Example 3.10 a counterexample where h lacks its homogeneity property.

Proof of Theorem 3.2

In order to prove the Theorem, we exploit the duality framework developed in Chapter 2, especially its extension discussed in details in Section 2.5 of this chapter (see also Remark 2.23). Accordingly let us first write the dual problems associated with $(\mathcal{P}_g)$ and $(\mathcal{P}_{g_{a,b}})$ respectively. To that aim, we use the part a) of the assumption (H3), namely $\mathrm{dom}(g) \subset\subset I := [\alpha,\beta]$ where $-\infty < \alpha < \beta < +\infty$. We set $K := \mathrm{dom}(h^*)$. By the linear growth condition (H1), K is a convex compact subset of $\mathbb{R}^N$. Let $\mathcal{K}_g$ denote the class of fields $\sigma = (\sigma^x, \sigma^t)$ in the space

$$X_1(\Omega\times I) := \Big\{\sigma \in L^\infty(\Omega\times I;\mathbb{R}^{N+1}) \ :\ \mathrm{div}\,\sigma \in L^1(\Omega\times I)\Big\},$$

satisfying the following convex pointwise constraints

$$\begin{aligned} &\sigma^x(x,t) \in K, \quad \forall (x,t) \in \Omega\times I,\\ &\sigma^t(x,t) + g(t) - p(x)t \geq 0 \quad \text{for } \mathcal{L}^{N+1}\text{-a.e. } (x,t) \in \Omega\times I. \end{aligned}$$

Then, taking into account Remark 2.23 for including the source term $p(x)$ in the constraint, the dual problem of $(\mathcal{P}_g)$ can be written as

$$(P_g^*) \qquad \sup\Big\{ -\int_\Omega \sigma^t(x,a)dx + \int_\Gamma h((u_0-a)\nu_\Omega)d\mathcal{H}^{N-1} \ :\ \sigma \in \mathcal{K}_g,$$
$$\mathrm{div}\,\sigma = 0,\ \sigma^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0} \text{ on } \Gamma\times I\Big\}$$

where γ_{u_0} is defined on $\partial\Omega \times I$ by

$$\gamma_{u_0}(x,t) := h((u_0(x)-t)\nu_\Omega(x)), \qquad \partial_t\gamma_{u_0} := \frac{\partial\gamma_{u_0}}{\partial t}(x,t) = \begin{cases} h(-\nu_\Omega(x)) & \text{if } t > u_0(x) \\ -h(\nu_\Omega(x)) & \text{if } t < u_0(x). \end{cases} \tag{3.5}$$

Recall that the divergence-free constraint $\operatorname{div}\sigma = 0$ is taken in the distribution sense in the open subset $\Omega \times I$. Similarly, the dual problem of $(\mathcal{P}_{g_{a,b}})$ can be written as

$$(\mathcal{P}^*_{g_{a,b}}) \quad \sup\Big\{ -\int_\Omega \sigma^t(x,a)dx + \int_\Gamma h((u_0-a)\nu_\Omega)d\mathcal{H}^{N-1} \ : \ \sigma \in \mathcal{K}_{g_{a,b}}, \\ \operatorname{div}\sigma = 0, \ \sigma^x\cdot\nu_\Omega = -\partial_t\gamma_{u_0} \text{ on } \Gamma\times I\Big\}$$

where the convex constraint $\mathcal{K}_{g_{a,b}}$ is the counterpart of $\mathcal{K}_g$, precisely the fields σ must satisfy

$$\sigma^x(x,t) \in K, \quad \forall (x,t) \in \Omega\times I,$$
$$\sigma^t(x,t) + g_{a,b}(t) - p(x)t \geq 0 \quad \text{for } \mathcal{L}^{N+1}\text{-a.e. } (x,t)\in\Omega\times I.$$

As $g \geq g_{a,b}$, by exploiting the non gap result of Theorem 2.22, we get:

$$\sup(\mathcal{P}^*_g) = \inf(\mathcal{P}_g) \geq \inf(\mathcal{P}_{g_{a,b}}) = \sup(\mathcal{P}^*_{g_{a,b}}).$$

As a consequence we will deduce the desired equality $\inf(\mathcal{P}_g) = \inf(\mathcal{P}_{g_{a,b}})$ if we can prove that

$$\sup(\mathcal{P}^*_g) \ \leq \ \sup(\mathcal{P}^*_{g_{a,b}}). \tag{3.6}$$

The idea to demonstrate this claim consists in changing any admissible σ for $(\mathcal{P}^*_g)$ to an admissible $\sigma_{a,b}$ for $(\mathcal{P}^*_{g_{a,b}})$ keeping the flux across the interface $t=a$ unchanged. To that aim, setting $\theta(t) := \Big(\frac{t-a}{b-a}\wedge 1\Big)_+$ and $q(x) := \frac{1}{b-a}\int_a^b \sigma^x(x,s)ds$, we associate to each $\sigma\in\mathcal{K}_g$ a laminated field $\sigma_{a,b}$ defined by

$$\sigma_{a,b}(x,t) = \begin{cases} \sigma(x,t) & \text{if } t \notin (a,b) \\ \Big(q(x), (1-\theta(t))\sigma^t(x,a) + \theta(t)\sigma^t(x,b)\Big) & \text{if } t\in[a,b]. \end{cases} \tag{3.7}$$

We can show that $\sigma_{a,b}$ belongs to $\mathcal{K}_{a,b}$. Actually, as $g\geq g_{a,b}$ and $-\operatorname{div}_x q$ agrees with the slope of $\sigma^t_{a,b}(x,\cdot)$, we infer that

$$\sigma\in\mathcal{K}_g, \quad \operatorname{div}\sigma = 0 \qquad \Longrightarrow \qquad \sigma_{a,b}\in\mathcal{K}_{a,b}, \quad \operatorname{div}\sigma_{a,b} = 0.$$

This construction of σ^b_a is described in Figure 3.1.

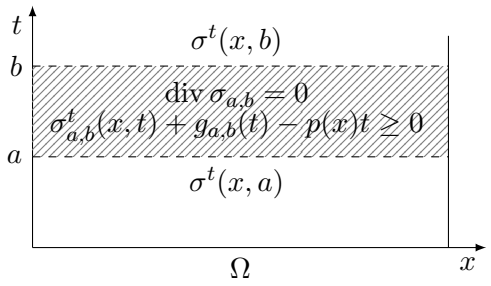


Figure 3.1: The field $\sigma_{a,b}$ defined by (3.7).

Let us check now that $\sigma_{a,b}$ still satisfies the normal trace condition on $\Gamma\times I$. This is clearly true on $\Gamma\times(I\setminus(a,b))$. On the other hand, since the boundary value u_0 is outside the interval (a,b),

the function $\partial_t \gamma_{u_0}(x,t)$ given by (3.5) is constant with respect to t on $[a,b]$. Thus, we get for $\mathcal{H}^N$-a.e. $(x,t) \in \Gamma \times (a,b)$,

$$(\sigma_{a,b})^x \cdot \nu_\Omega = q \cdot \nu_\Omega = \frac{1}{b-a} \int_a^b \sigma^x \cdot \nu_\Omega \, dt = -\partial_t \gamma_{u_0}.$$

Therefore, as the flux across the interface $t = a$ is clearly unchanged after we change σ into $\sigma_{a,b}$, we obtain:

$$\begin{aligned} -\int_\Omega \sigma^t(x,a)dx + \int_\Gamma h((u_0 - a)\nu_\Omega)d\mathcal{H}^{N-1} &= -\int_\Omega \sigma^t_{a,b}(x,a)dx + \int_\Gamma h((u_0 - a)\nu_\Omega)d\mathcal{H}^{N-1} \\ &\leq \sup(\mathcal{P}^*_{g_{a,b}}) = \inf(\mathcal{P}_{g_{a,b}}). \end{aligned}$$

Then, by taking the supremum on the left hand side of the inequality above over all admissible $\sigma \in \mathcal{K}_g$, we obtain claim (3.6).

Eventually, let $u \in \mathrm{Argmin}(\mathcal{P}_g)$. Then

$$F_{g_{a,b}}(u) \leq F_g(u) = \inf(\mathcal{P}_g) = \inf(\mathcal{P}_{g_{a,b}}).$$

This implies that $u \in \mathrm{Argmin}(\mathcal{P}_{g_{a,b}})$ and also that $F_{g_{a,b}}(u) = F_g(u)$. Thus, $g_{a,b}(u(x)) = g(u(x))$ holds for a.e. $x \in \Omega$. As $g > g_{a,b}$ on (a,b), this means that the minimizer u does not take values in the interval (a,b). The proof of Theorem 3.2 is finished.

3.3 Application to free boundary and multiphase problems

In this Section we consider several multi-phase problems on the unit square $\Omega = (0,1)^2$ of $\mathbb{R}^2$. The boundary data u_0 will take two values $u_0 = 0$ on Γ_0 and $u_0 = 1$ on Γ_1, with $\{\Gamma_0, \Gamma_1\}$ being a partition of $\Gamma := \partial\Omega$. We treat first a three phase problem (see subsection 3.3.1) and then a four phase problem (see subsection 3.3.2).

3.3.1 A three phases problem

We consider the variational problem

$$\inf \left\{ \int_\Omega |\nabla u| dx + \lambda \left| \{u \neq \frac{1}{2}\} \right| \ : \ u \in W^{1,1}(\Omega), \ u = 0 \text{ on } \Gamma_0, \ u = 1 \text{ on } \Gamma_1 \right\} \tag{3.8}$$

where the non convex function $g(t)$ is given by

$$g(t) = \begin{cases} 0 & \text{if} \quad t = \frac{1}{2} \\ \lambda & \text{if} \quad t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ +\infty & \text{otherwise .} \end{cases} \tag{3.9}$$

The convex envelope of $g(t)$ reads

$$g^{**}(t) = \begin{cases} 2\lambda |t - \frac{1}{2}| & \text{if } t \in (0,1) \\ +\infty & \text{otherwise} \end{cases}$$

In Figure 3.2 below, we represent g, g^{**} and $\mathrm{epi}\, g := \{(t,\alpha) \in \mathbb{R}^2 \ : \ g(t) \leq \alpha\}$ the epigraph of g.

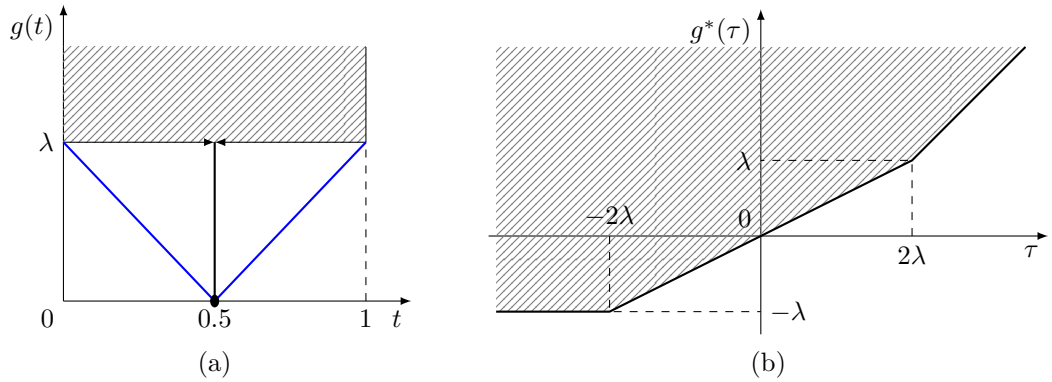


Figure 3.2: (a) Representation of g and of g^{**} (in blue). (b) Representation of g^*. The epigraphs $\operatorname{epi} g$ and $\operatorname{epi} g^*$ are filled in dashed.

Notice that in this case the set $Z = \{g^{**} < g\}$ is the complement of $\{0, \frac{1}{2}, 1\}$. In view of Theorem 3.3, a solution to the relaxed problem in $BV(\Omega)$ associated with (3.8) takes only three values represented by the phases $A_0 := \{u = 0\}$, $A_{\frac{1}{2}} := \{u = \frac{1}{2}\}$, $A_1 := \{u = 1\}$. The convexified problem that we are going to solve in order to recover these three phases solutions reads

$$\min\left\{\int_\Omega |Du| + \int_{\Gamma_0} |u| + \int_{\Gamma_1} |1-u| + 2\lambda \int_\Omega \left|u - \frac{1}{2}\right| dx \ : \ u \in BV(\Omega; [0,1])\right\} \tag{3.10}$$

If the solution u to (3.10) is unique, then (see Theorem 3.3) it is of the form $u = \frac{1}{2}\mathbf{1}_{A_{\frac{1}{2}}} + \mathbf{1}_{A_1}$ which allows to determine the searched optimal partition $\{A_0, A_{\frac{1}{2}}, A_1\}$ of Ω. This scenario is confirmed by numerical simulations, namely in the particular case described in Figure 3.4 (see also the 4 phases variant of (3.10) in Figures 3.9.

Dual problem and interface conditions

The dual problem to $(\mathcal{P}_{g^{**}})$ reads:

$$(\mathcal{Q}) \qquad \sup\left\{\int_{\Gamma_1} q \cdot \nu_\Omega - \int_\Omega g^*(\operatorname{div} q) \ : \ q \in L^\infty(\Omega; \mathbb{R}^N),\ |q| \le 1\right\}$$

Let $\bar{q}$ solve $(\mathcal{Q})$. Then a function $\overline{u}(x)$ taking values in $\{0, 1/2, 1\}$ for a.e. $x \in \Omega$ solves problem $(\mathcal{P}_g)$ (then also $(\mathcal{P}_{g^{**}})$) if and only if

$$\bar{q} = \nu_{\overline{u}} \ \text{ on } S_{\overline{u}}\,, \qquad \operatorname{div} \bar{q} \in \partial g^{**}(\overline{u}) \text{ a.e. in } \Omega$$

where $S_{\overline{u}}$ denotes the jump set of $\overline{u}$.

We now give optimality conditions for minimal partitions, called recovering interface conditions. Euler equation for shape variations of problem $(\mathcal{P}_{g^{**}})$ is written in the form of conservation law (see [21])

$$\operatorname{div} \mathcal{A} = 0 \quad \text{ in } \Omega$$

where $\mathcal{A}$ is a tensor defined on the product space $\operatorname{Argmin}(\mathcal{P}_{g^{**}}) \times \operatorname{Argmax}(\mathcal{Q})$:

$$\mathcal{A}(u, q) := (|Du| + g^{**}(u))I_2 - Du \otimes q.$$

Let $(\overline{u}, \overline{q})$ be an element of $\operatorname{Argmin}(\mathcal{P}_{g^{**}}) \times \operatorname{Argmax}(\mathcal{Q})$. The vector field $\bar{q}$ identifies to $D\overline{u}/|D\overline{u}|$, and the function $\overline{u}$ only takes values in $\{0, 1/2, 1\}$ so that $g^{**}(\overline{u}) = g(\overline{u})$. Thus, we have

$$\mathcal{A}(\overline{u}, \overline{q}) = (I_2 - \nu_{\overline{u}} \otimes \nu_{\overline{u}})\,[\overline{u}]\delta_{S_{\overline{u}}} + g(\overline{u})\mathbf{1}_\Omega\, I_2.$$

In particular, on the interface $S_{\overline{u}} \cap \Omega$ the conservation law reduces to

$$[\overline{u}]\kappa_{S_{\overline{u}}} + [g(\overline{u})] = 0 \tag{3.11}$$

provided $(I_2 - \nu_{\overline{u}} \otimes \nu_{\overline{u}})\delta_{S_{\overline{u}}}$ being the tangential projection of the curve vector measure $\nu_{\overline{u}}\delta_{S_{\overline{u}}}$ and the distribution $\kappa_{S_{\overline{u}}}\nu_{\overline{u}} = \operatorname{div}\Big((I_2 - \nu_{\overline{u}} \otimes \nu_{\overline{u}})\delta_{S_{\overline{u}}}\Big)$ being the algebraic curvature vector of $S_{\overline{u}}$, see [16, 18]. Notice that the curvature form (3.11) is achieved thanks to the fact that $[\overline{u}]$ is constant along the interface $S_{\overline{u}} \cap \Omega$. In our case, $[\overline{u}] = \frac{1}{2}$, $[g(\overline{u})] = \lambda$, and hence $|\kappa_{S_{\overline{u}}}| = 2\lambda$. It is to say that the interfaces of partitions are curves with constant curvature (arcs of radius $\frac{1}{2\lambda}$), see the configuration in Figure 3.3.

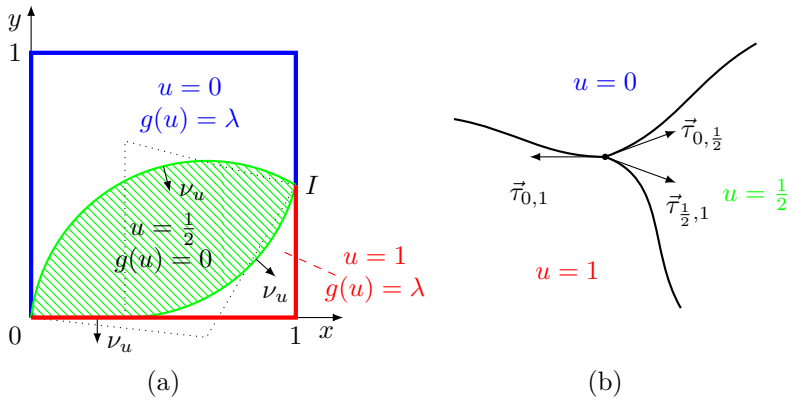


Figure 3.3: (a) Configuration of minimal partitions. (b) Equilibrium of junction.

On the other hand, the vector measure $F := \operatorname{div}\mathcal{A}$ defines a force field on $\overline{\Omega}$ which, by the stabilization of shape variations, belongs to the normal cone to the domain Ω. From this point of view, if a junction J of three phases occurs then it must be balanced, therefore the force must vanish at that point, i.e. $F(J) = 0$. Inversely, F is singular at the triple junction and $F(J) = \vec{\tau}_{0,1} + \frac{1}{2}(\vec{\tau}_{0,\frac{1}{2}} + \vec{\tau}_{\frac{1}{2},1}) \neq \vec{0}$, see Figure 3.3.

Numerical results

In this subsection we present numerical simulations in $\mathbb{R}^2$ where we solve the convexified problem (3.10) by using the algorithm described in Section 3.4. They are done over the unit square $\Omega = (0,1)^2$ and $\lambda = 3/4$. The Dirichlet conditions are varied on the boundary:

$$u = 0 \text{ on } \Gamma_0 \quad \text{and} \quad u = 1 \text{ on } \Gamma_1^\alpha \tag{3.12}$$

where $\{\Gamma_0, \Gamma_1^\alpha\}$ is a partition of $\partial\Omega$ with $\Gamma_1^\alpha = \Big((0,1] \times \{0\}\Big) \bigcup \Big(\{1\} \times [0,\alpha)\Big)$ for $\alpha \in \Big\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\Big\}$.

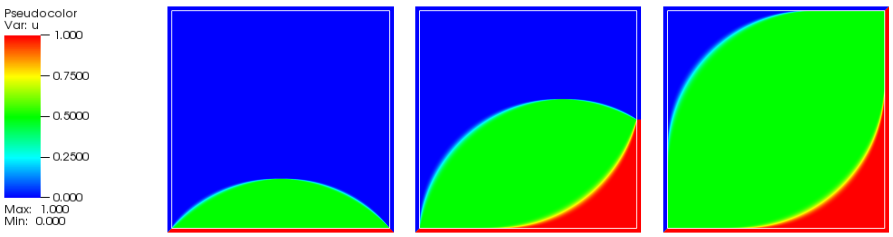


Figure 3.4: Problem of three phases or minimal partitions (in (3.10)). Respectively, $\{u = 0\}$ is blue, $\{u = 1/2\}$ is green, $\{u = 1\}$ is red.

Anisotropic variant

According to hypothesis (H1), we may substitute the classical total variation $\int |Du|$ with $\int h(Du)$ being h any crystalline norm, that is

$$h(x) = \|x\|_B := \inf\{r \geq 0 \ : \ x \in rB\} \ ,$$

where B is a convex compact polyhedron containing 0 in its interior. The related anisotropic perimeter P_B in $\mathbb{R}^N$ is defined as

$$P_B(A) := \int_{\partial A} \|\nu_A\|_B d\mathcal{H}^{N-1} \qquad \text{for } A \subset \mathbb{R}^N,$$

where ν_A denotes the outward unit normal to the boundary ∂A of A. Let K be the polar of B that is

$$K = \{z^* \in \mathbb{R}^N : z \cdot z^* \leq 1 \ , \ \forall z \in B\}.$$

As the crystalline norm $h(z)$ coincides with the support function of K, optimal multiphase configurations will favour interfaces whose normal are orthogal to the faces of K. This is confirmed by the numerical simulations below (Figures 3.6 and 3.7) which have been performed in $\mathbb{R}^2$ in the case where K is a square or a regular hexagon that we rotate with different angles θ, see Figure 3.5.

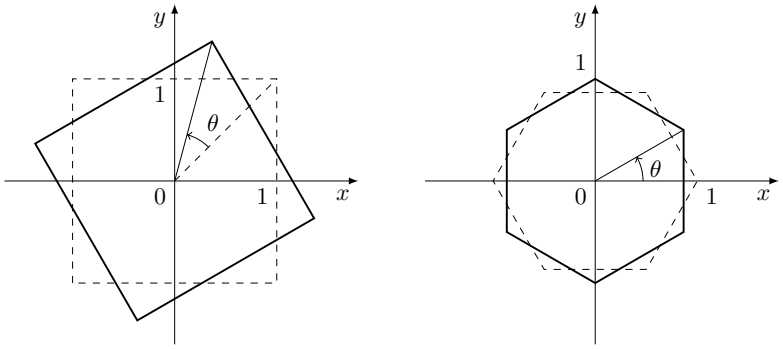


Figure 3.5: Rotation is applied to convex sets in oder to generate variants of orientation on which we define anisotropic seminorms.

The numerical simulations are enforced with the primal-dual algorithms stated in Section 3.4.

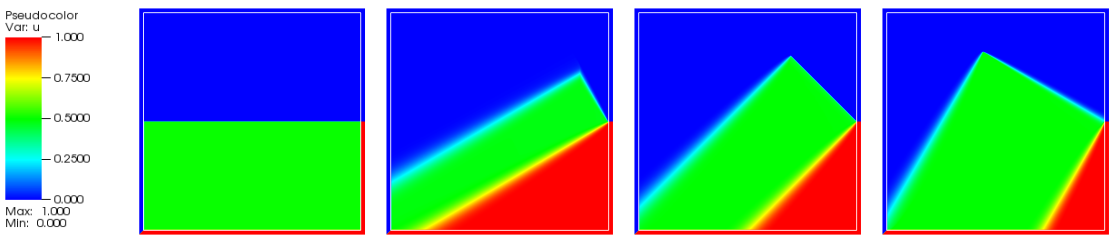


Figure 3.6: Here h is the support function of K_θ deduced from $(-1,1)^2$ after a rotation of angle θ. From the left to the right the solution for $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, all with $\lambda = 1$.

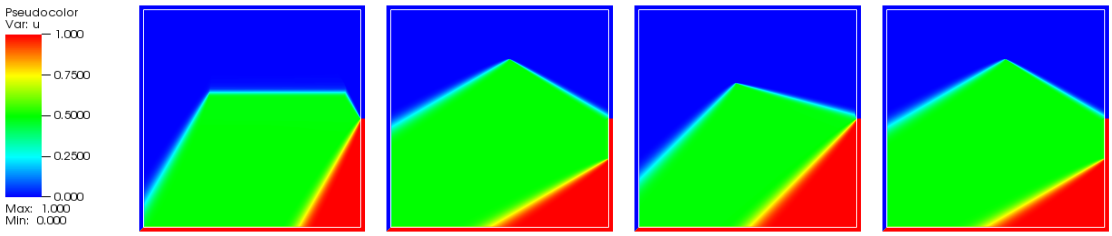


Figure 3.7: Here h is the support of $K = H_\theta$ where H_θ is a regular hexagon deduced form H_0 by a rotaion of angle θ. From the left to the right, the solution for $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$, all with $\lambda = 4/5$.

3.3.2 A 4 phases problem

We exhibit in this section a 4-phase problem with many nice numerical results which highlight the validity of Theorem 3.3. The non convex term g and its convex envelop are set up as below

$$g(t) = \begin{cases} \lambda & \text{if } t \in [0,a) \\ -\frac{\kappa}{b-a}(t-a) & \text{if } t \in [a,b] \\ \mu & \text{if } t \in (b,1] \\ +\infty & \text{otherwise} \end{cases} \qquad g^{**}(t) = \begin{cases} -\frac{\lambda}{a}t + \lambda & \text{if } t \in [0,a) \\ -\frac{\kappa}{b-a}(t-a) & \text{if } t \in [a,b] \\ \frac{\mu+\kappa}{1-b}t - \frac{b\mu+\kappa}{1-b} & \text{if } t \in (b,1] \\ +\infty & \text{otherwise} \end{cases}$$

We shall put some conditions on parameters so that function g is well-defined (see Figure 3.8 for this configuration):

$$0 < a < b, \qquad 0 < \lambda, \eta, \qquad 0 < \kappa < \frac{\lambda(b-a)}{a}.$$

The third condition assures the convex envelop of g to link to value a in order to have 4-phase solution. If not g^{**} is affine in the interval $[0,b]$. At that moment, by the exclusion principle (Theorem 3.2), optimal solutions will be kept away from value a. And hence, the problem degenerates and reduces to a 3-phase problem.

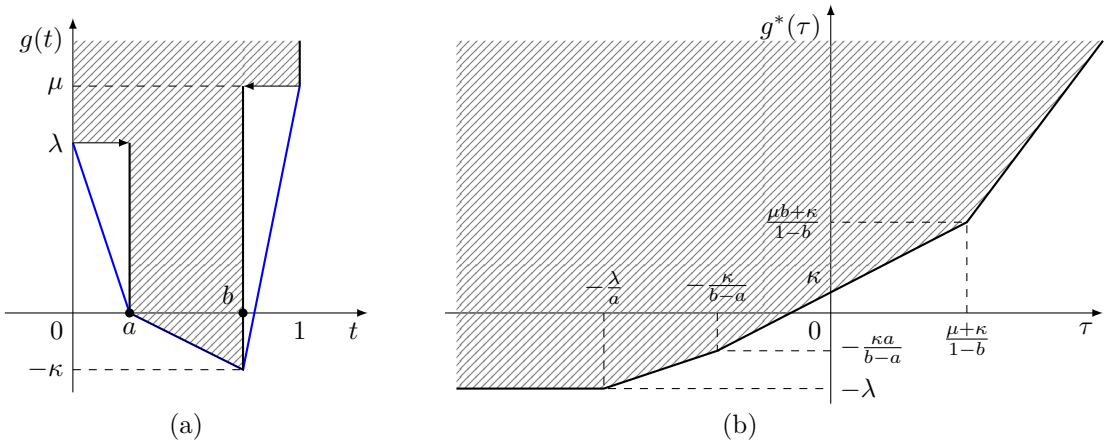


Figure 3.8: (a) Graphs of functions g and g^{**} (in blue). (b) Illustration of the Fenchel conjugate g^*. Epigraphs epi g, epi g^* are displayed in dashed.

We shall use the algorithms proposed in Section 3.4 for the simulation. In our numerical experiment, we maintain the prescribed boundary conditions on the partition $\left\{\Gamma_0, \Gamma_1^{1/2}\right\}$ of $\partial\Omega$ as the settings (3.12) for the 3-phase problem studied in Section 3.3.1. This arrangement permits having boundary junctions which are one singular point and one regular point of the domain's

boundary $\partial\Omega$. It will serve on further discussions about interface conditions and phase-junctions. In the following (see Figure 3.9), we present numerical results for some variants where we combine anisotropic effects due to the use of a crystalline norm (with different θ-rotations as discussed in the section 3.3.1) and a tune of the weight parameters λ, κ, μ. For the numerical simulations shown in Figure 3.9, the parameters are fixed as below:

$$a = 0.25, \quad b = 0.75, \quad \mu = 0.3, \quad \kappa = 0.7,$$

the weight parameter λ and the angle of rotation θ will be varied. With this configuration, an optimal solution (by Theorem 3.3) would take only four values in $\left\{0, \frac{1}{4}, \frac{3}{4}, 1\right\}$ which are called 4 phases. Such a 4-phase solution clearly gives a partition of the domain Ω.

Notice that in the case of the total variation, the geometric optimality condition (3.11) still holds on $S_{\overline{u}} \cap \Omega$ (which consists of all interfaces appearing in Figure 3.9). The determination of the counterpart of condition (3.11) in case of crystalline norms is a delicate issue worth of further investigations.

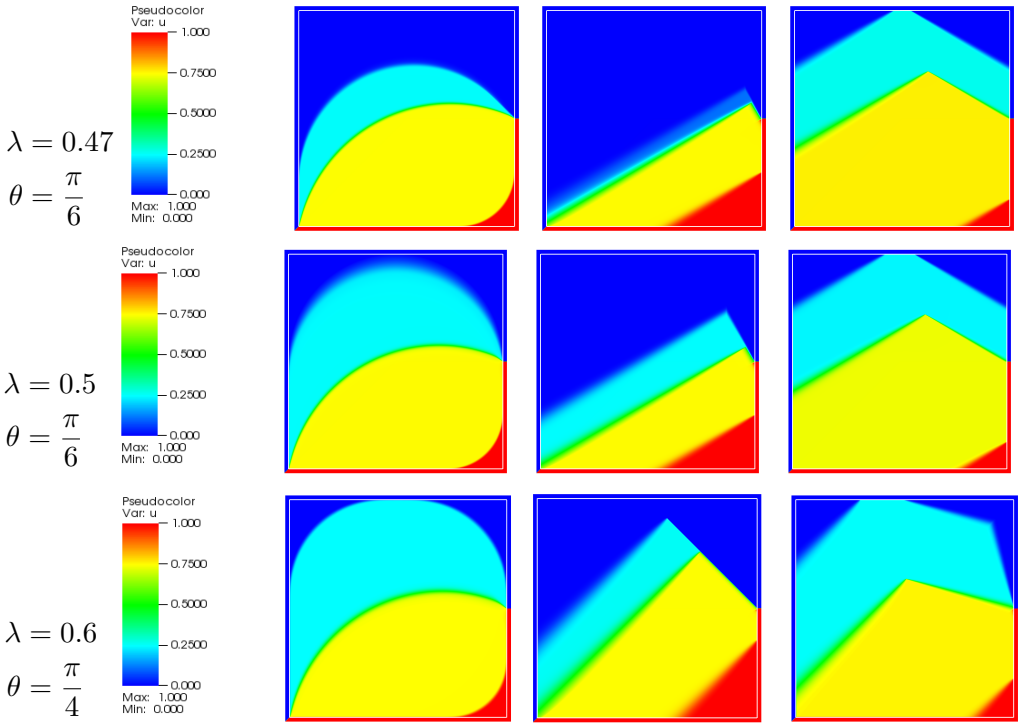


Figure 3.9: Problem of 4-phases. Each row corresponds to a choice of certain (λ, θ). In a left-to-right order, the norm h is generated by a ball (Euclidian norm), by a square and by a regular hexagon (crystalline norms).

3.4 Min-max approach. A primal-dual algorithm for non differentiable Lagrangians

In this section, we shall deal with multiphase problems introduced in previous sections by means of numerical approach. The convexification principle fundamentally based on Theorem 3.3 is applied to bring us a convex problem under the form

$$\inf_{u \in C} \int_\Omega [h(\nabla u) + g^{**}(u) - p(x)u] dx$$

where C, which is comprised the boundary condition $u = u_0$ on Γ, is a closed convex subset in $W^{1,1}(\Omega)$. By a duality argument, we can rewrite this infimum problem as a saddle point problem

$$\inf_{u\in C}\sup_{q\in K}\int_\Omega [\nabla u\cdot q + g^{**}(u) - p(x)u]dx \tag{3.13}$$

with $K = \mathrm{dom}(h^*)$. Remark that under the assumption on positive homogeneity of h (see (H1)), one has $h^*(q) = 0$ for all $|q| \le 1$, $h^*(q) = +\infty$ otherwise. So, if we replace the density h by a gauge (Minkowski functional) of a closed convex set then the constraint K is indeed the polar of that closed convex set. We denote by $L(u, q)$ the Lagrangian associated to the inf-sup problem (3.13)

$$L(u,q) = \int_\Omega [\nabla u\cdot q + g^{**}(u) - p(x)u]dx.$$

We can use classical primal-dual algorithms based on Arrow-Hurwicz' method to seek a saddle point of $L(u, q)$ in $C \times K$. But, to that aim, we shall need a regularization of function g^{**} since this function is usually piecewise affine and hence non differentiable. Take for instance the 3-phase problem introduced in the previous section, g^{**} is not differentiable at $1/2$. A regularization for g^{**} in that case can be done with

$$g_\varepsilon(t) = \begin{cases} 2\lambda\sqrt{(t-\frac{1}{2})^2+\varepsilon^2} & \text{if } t\in(0,1)\\ +\infty & \text{otherwise .}\end{cases}$$

Correspondingly, we denote L_ε the regularized Lagrangian which is obtained by substituting g^{**} with its regularization g_ε. Then by using the explicit algorithm described in Chapter 5, we produce an iterative process with initialization $\eta, \theta > 0$, $(u_0^h, q_0^h) \in C \times K$, $\overline{u}_0^h = u_0^h$,

$$\begin{aligned} q_{n+1}^h &= \Pi_K^h\Big(q_n^h + \eta\frac{\partial L_\varepsilon}{\partial q}(\overline{u}_n^h, q_n^h)\Big)\\ u_{n+1}^h &= \Pi_C^h\Big(u_n^h - \theta\frac{\partial L_\varepsilon}{\partial u}(u_n^h, q_{n+1}^h)\Big)\\ \overline{u}_{n+1}^h &= 2u_{n+1}^h - u_n^h\end{aligned}$$

where Π_K^h, Π_C^h are respectively the orthogonal projections on closed convex sets K, C. The superscript h indicates that a discretization with mesh size h was done. This algorithm converges to a saddle point (proved in Chapter 5) under the following constraint on the step sizes

$$0 < \eta, \qquad 0 < \theta < \frac{2}{L_{g'_\varepsilon}}, \qquad \eta\theta\|\nabla^h\|^2 + \frac{\theta L_{g'_\varepsilon}}{2} < 1$$

where $L_{g'_\varepsilon}$ denotes the Lipschitz constant of g'_ε. In Chapter 5, we also proposed a semi-implicit scheme which considerably accelerates the convergence of the iterative algorithm. This implicit scheme is realized with inspiration from the splitting method for Navier-Stokes system (in this circumstance, the couple (u, q) stands for the pressure and the speed of the fluid):

$$\begin{aligned} q_{n+1}^h &= \Pi_K^h\Big(q_n^h + \eta\nabla^h\overline{u}_n^h\Big)\\ u_{n+1}^h &= u_n^h - \theta(I-\Delta^h)^{-1}(-\operatorname{div}^h q_{n+1} + g'_\varepsilon(u_n) - p^h)\\ \overline{u}_{n+1}^h &= 2u_{n+1}^h - u_n^h\end{aligned}$$

with Δ^h being a discretized Laplacian and I standing for the identity operator. Note that the projection on C is just to maintain the boundary condition $u = u_0$ on Γ and this can be simply

implemented within Laplacian operator, thus it is hidden. This scheme gives an iterative process converging under conditions which are more flexible

$$0 < \eta, \qquad 0 < \theta < \frac{2}{L_{g'_\varepsilon}}, \qquad \eta\theta < 1.$$

However, the step size θ is still constrained by the Lipschitz constant of the derivative of the regularization g_ε which is of order $O(1/\varepsilon)$. This is a limited aspect of regularization method since handling the inverse of Laplacian within an iterative algorithm, as we know, is costly. To overcome this challenge, we introduce a new approach using a geometric projection on epigraph. And we shall demonstrate that the stepsizes are free from Lipschitz constants.

3.4.1 Explicit scheme with projection on $\mathrm{epi}(g^{**})$

Let us rewrite problem (3.13) by introducing a new variable $\alpha \in L^1(\Omega)$:

$$\inf_{\substack{u\in C\\ g^{**}(u)\le\alpha}} \sup_{q\in K} \int_\Omega \Big(\nabla u \cdot q - p(x)u + \alpha\Big)dx \tag{3.14}$$

and denote by $\widetilde{L}(u,q)$ the corresponding Lagrangian

$$\widetilde{L}(u,q) := \int_\Omega \Big(\nabla u \cdot q - p(x)u + \alpha\Big)dx.$$

We introduce some notations

$$\begin{gathered} \widetilde{u} := (u,\alpha), \quad \widetilde{q} := (q,-p,\beta), \\ \widetilde{C} := \Big\{(u,\alpha) \ : \ u\in C,\ g^{**}(u)\le\alpha\Big\}, \quad \widetilde{K} := \Big\{(q,-p,1) \ : \ q\in K\Big\}, \\ A\widetilde{u} := \Big(\nabla u, u, \alpha\Big), \quad A^*\widetilde{q} = \Big(-\operatorname{div} q - p, \beta\Big). \end{gathered} \tag{3.15}$$

Then, A defines a linear operator from $W^{1,1}(\Omega)\times L^1(\Omega)$ to $L^1(\Omega;\mathbb{R}^N)\times L^1(\Omega)\times L^1(\Omega)$, and its adjoint is A^*. Problem (3.14) now falls into the form

$$\inf_{\widetilde{u}\in\widetilde{C}} \sup_{\widetilde{q}\in\widetilde{K}} \langle\langle A\widetilde{u}, \widetilde{q}\rangle\rangle \tag{3.16}$$

where $\langle\langle\cdot\rangle\rangle$ is a scalar product defined by

$$\langle\langle A\widetilde{u}, \widetilde{q}\rangle\rangle := \langle\nabla u, q\rangle - \langle p, u\rangle + \langle\alpha,\beta\rangle.$$

We remark that the presence of β is just to define the scalar product $\langle\langle\cdot\rangle\rangle$, it will be kept equal to 1 by the convex constraint $\widetilde{K}$. And the convex $\widetilde{C}$ is indeed (a part of) epigraph of function g^{**} ($u\in C$ is the boundary condition). Problem (3.16) is a standard problem which has the simplest form. We can easily apply the explicit algorithm based on Arrow-Hurwicz' method described in Chapter 5 to obtain

$$\begin{aligned} \widetilde{q}^h_{n+1} &= \Pi^h_{\widetilde{K}}(\widetilde{q}^h_n + \eta A\overline{u}^h_n) \\ \widetilde{u}^h_{n+1} &= \Pi^h_{\widetilde{C}}(\widetilde{u}^h_n - \theta A^*\widetilde{q}^h_{n+1}) \\ \overline{u}^h_{n+1} &= 2\widetilde{u}^h_{n+1} - \widetilde{u}^h_n \end{aligned}$$

The choice of step sizes is now of order $O(h)$ and it is no longer squeezed by Lipschitz constants

$$0 < \eta, \qquad 0 < \theta, \qquad \eta\theta\|\nabla^h\|^2 < 1.$$

It is evident that transforming problem (3.13) to the standard one (3.16) with the projection on epigraph allows facing difficulties resulting from the non-differentiability of g^{**}. At the moment, a semi-implicit scheme should be generated to help speed up the convergence of the algorithm. The splitting technique may be applied to the Lagrangian $\widetilde{L}(u,q)$ to create such a semi-implicit scheme. More precisely, we rewrite the inf-sup problem (3.14) into the form

$$\inf_{(\phi,u,\alpha)\in\ddot{C}} \sup_{q\in K} \int_\Omega \Big(\phi\cdot q - p(x)u + \alpha\Big)dx \tag{3.17}$$

with $\ddot{C}$ given by

$$\ddot{C} := \Big\{(\nabla u, u, \alpha) \ : \ u\in C,\ g^{**}(u)\le\alpha\Big\}.$$

And the implicitness of the scheme in variable u is effectively enforced thanks to the following projection

$$\inf_{\substack{v=u_0 \text{ on } \Gamma\\ g^{**}(v)\le a}} \frac{1}{2}\int_\Omega \Big(|\nabla v-\phi|^2 + |v-u|^2 + |a-\alpha|^2\Big)dx.$$

This is a slightly heavy projection. But pay attention that if we remove the epigraph component $(g^{**}(v)\le a)$ then that projection just becomes a proximal operator of a quadratic form with easily solvable resolvent which appears as $(I-\Delta^h)^{-1}$ in discretized schemes. For more details in this topic, we refer to the discussion in Chapter 5. Fortunately, removing the epigraph part in variable u is doable and even simple by exploiting the conjugate g^* instead of g^{**}. In the next subsection, we shall see that the semi-implicit scheme is still well adapted for the epigraph projection method. This will be a quite complete setting in order to reach a better acceleration in convergence of our algorithms.

3.4.2 Semi-implicit scheme with projection on $\mathrm{epi}(g^*)$

We start with recasting the saddle point problem (3.13) as

$$\inf_{u\in C} \sup_{\substack{q\in K\\ \tau\in L^\infty(\Omega)}} \int_\Omega [\nabla u\cdot q + \tau u - g^*(\tau) - p(x)u]dx = \inf_{u\in C} \sup_{\substack{q\in K\\ g^*(\tau)\le\beta}} \int_\Omega [\nabla u\cdot q + \tau u - \beta - p(x)u]dx \tag{3.18}$$

We use here the epigraph of the conjugate g^* (instead of g^{**}). Let us set some notations to be suitable for the current context

$$\begin{aligned} \widehat{u} &:= (u,\alpha), \quad \widehat{q} := (q,-p,\tau,\beta),\\ \widehat{C} &:= \Big\{(u,-1) \ : \ u\in C\Big\}, \quad \widehat{K} := \Big\{(q,-p,\tau,\beta) \ : \ q\in K,\ g^*(\tau)\le\beta\Big\},\\ \widehat{A}\widehat{u} &:= \Big(\nabla u, u, u, \alpha\Big), \quad (\widehat{A})^*\widehat{q} = \Big(-\operatorname{div} q - p + \tau, \beta\Big). \end{aligned} \tag{3.19}$$

We observe that the convex set $\widehat{C}$ is merely the boundary condition whilst $\widehat{K}$ is independently combined by the convex constraint K and the epigraph of g^*. With these configuration, problem (3.18) becomes

$$\inf_{\widehat{u}\in\widehat{C}} \sup_{\widehat{q}\in\widehat{K}} \langle\langle \widehat{A}\widehat{u}, \widehat{q}\rangle\rangle \tag{3.20}$$

where the corresponding scalar product $\langle\langle\cdot\rangle\rangle$ is given by

$$\langle\langle \widehat{A}\widehat{u}, \widehat{q}\rangle\rangle := \langle\nabla u, q\rangle - \langle p, u\rangle + \langle\tau, u\rangle + \langle\alpha,\beta\rangle.$$

The semi-implicit scheme (described in Chapter 5) is available and perfectly fits the standard problem (3.20). It provides a convergent iterative algorithm

$$\begin{aligned}
\widehat{q}^h_{n+1} &= \Pi^h_{\widehat{K}}(\widehat{q}^h_n + \eta \widehat{A}\overline{u}^h_n) \\
\widehat{u}^h_{n+1} &= \Pi^h_{\widehat{C}}\Big(\widehat{u}^h_n - \theta(\widehat{A}^*\widehat{A})^{-1}(\widehat{A}^*(\widehat{q}^h_{n+1}))\Big) \\
\overline{u}^h_{n+1} &= 2\widehat{u}^h_{n+1} - \widehat{u}^h_n
\end{aligned} \tag{3.21}$$

under a rather comfortable choice of step sizes

$$0 < \eta, \qquad 0 < \theta, \qquad \eta\theta < 1.$$

For conveniences in practice, let us unfold the algorithm (3.21) in the expression of primal variables u, q and extra variables τ, β:

$$\begin{aligned}
q^h_{n+1} &= \Pi^h_K(q^h_n + \eta\nabla^h\overline{u}^h_n) \\
(\tau^h_{n+1}, \beta^h_{n+1}) &= \Pi^h_{\mathrm{epi}(g^*)}\Big((\tau^h_n, \beta^h_n) + \eta(\overline{u}^h_n, -1)\Big) \\
u^h_{n+1} &= u^h_n - \theta(2I - \Delta^h)^{-1}\Big(-\mathrm{div}^h(q^h_{n+1}) + \tau^h_{n+1} - p^h\Big) \\
\overline{u}^h_{n+1} &= 2u^h_{n+1} - u^h_n
\end{aligned} \tag{3.22}$$

In summary, the algorithm (3.22) is an effective combination between the epigraph projection method and the semi-implicit scheme. This coupling provides a quite powerful numerical approach to treat non-differentiable Lagrangians of saddle point problems of the same type, and contemporaneously brings considerable profit on acceleration of convergence of the algorithm.

To simplify our discussion, we denote in the following

- EReg: explicit scheme combined with regularization,
- EEpi: explicit scheme combined with epigraph projection,
- IEpi: semi-implicit scheme combined with epigraph projection.

As shown in Figures 3.10, 3.11 and Table 3.1, the combination of the semi-implicit scheme and the epigraph projection method in IEpi is more efficient in term of precision of the interfaces and reduces many necessary iterations.

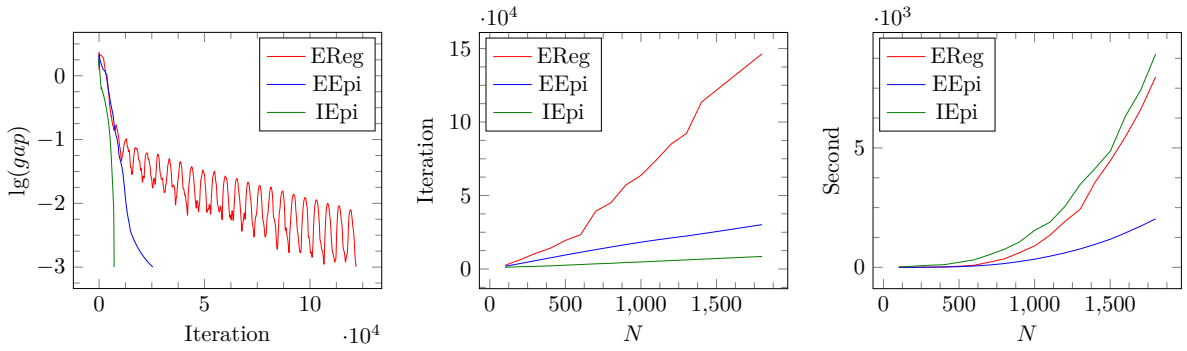


Figure 3.10: Convergence of primal-dual gap (in $N = 1500$) and comparison in terms of iteration and computational time for the 3-phase problem with the classical total variation and $\lambda = 0.75$.

primal-dual gap $< 10^{-3}$ with MPI in 6 processes						
N	iteration			time (second)		
	EReg	**EEpi**	**IEpi**	**EReg**	**EEpi**	**IEpi**
201	6329	3706	1566	2.361	1.599	49.047
401	14164	7600	2127	20.414	13.269	110.211
601	23393	11379	3045	87.592	55.751	316.240
801	45089	14939	3854	356.529	158.577	760.536
1001	63759	18264	4790	897.927	342.014	1551.827

Table 3.1: Comparison of numerical methods in terms of iteration and computational time for the 3-phase problem with the classical total variation and $\lambda = 0.75$.

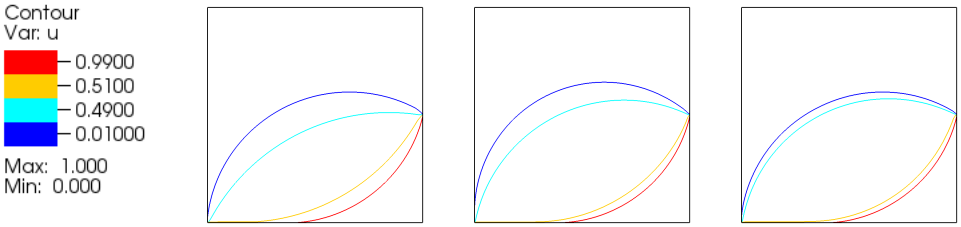


Figure 3.11: Approximation of interfaces of discontinuity. Here are contour lines of numerical solution u for 3-phase problem ($N = 1500, \lambda = 0.75$, primal-dual gap $< 10^{-3}$).

3.5 Counterexamples

In this section, we discuss about the necessary of the assumptions on positive homogeneity of h in (H1) and on the boundary constraint (H5). We shall show that if one of the two assumptions mentioned above is violated, the convexification procedure fails, meaning in particular that $\inf(\mathcal{P}_{g_\lambda^{**}}) < \inf(\mathcal{P}_{g_\lambda})$. In the two following examples, we choose the non convex g to be $g = g_\lambda$ where

$$g_\lambda(t) = \begin{cases} 0 & \text{if } t \in [0,1) \\ -\lambda & \text{if } t = 1 \\ +\infty & \text{if } t \notin [0,1] \end{cases}$$

and λ is a positive real parameter.

In that case, the subset Z defined in (3.5) is given by $Z = (0,1)$ and the convex envelop reads

$$g_\lambda^{**}(t) = \begin{cases} -\lambda t & \text{if } t \in [0,1] \\ +\infty & \text{if } t \notin [0,1] \end{cases} \tag{3.23}$$

Since $\mathrm{dom}(g_\lambda) \setminus Z := \{0,1\}$, the application of Theorem 3.3 should lead to solutions u ranging in $\{0,1\}$.

Example 3.9. In this example the assumption (H5) is not satisfied. Let Ω be the unit cube in $\mathbb{R}^2$, i.e. $\Omega = (0,1)^2$. We take the boundary data $u_0 = \frac{1}{2}$ and consider the variational problem

$$\inf\left\{\int_\Omega |\nabla u| dx + \int_\Omega g_\lambda(u) dx \ : \ u \in W^{1,1}(\Omega), u = \frac{1}{2} \text{ on } \partial\Omega\right\}$$

that we rewrite in its relaxed form

$$(\mathcal{P}_{g_\lambda}) \qquad \inf\left\{\int_\Omega |Du| + \int_{\partial\Omega} |u-\frac{1}{2}|d\mathcal{H}^1 + \int_\Omega g_\lambda(u)dx \ : \ u \in BV(\Omega)\right\}.$$

Let u be optimal for problem $(\mathcal{P}_{g_\lambda})$. Then it satisfies $\frac{1}{2} \le u \le 1$. Indeed, if we set $\tilde{u} = u \vee \frac{1}{2}$, then

$$\int_\Omega |\nabla u|dx - \lambda|u=1| \ge \int_\Omega |\nabla \tilde{u}|dx - \lambda|\tilde{u}=1|$$

so that equality holds and $\tilde{u} = u$ a.e. We can then apply to u the following identity

$$\int_\Omega |Dw| + \int_{\partial\Omega} |w-\frac{1}{2}|d\mathcal{H}^1 \ = \ \int_{\frac{1}{2}}^1 P(w>t)\,dt \quad \text{for all } w \in BV\left(\Omega;\left[\frac{1}{2},1\right]\right) \tag{3.24}$$

(which is derived from the coarea formula applied to the function v defined by $v = w - \frac{1}{2}$ in Ω and $v = 0$ in $\mathbb{R}^2 \setminus \Omega$).

Now by taking into account $\{u=1\} \subset \{u>t\}$ for all $t \in [\frac{1}{2},1]$, we infer that

$$\min(\mathcal{P}_{g_\lambda}) \ = \ \int_\Omega |Du| + \int_{\partial\Omega} |u-\frac{1}{2}|d\mathcal{H}^1 - \lambda|u=1| \ \ge \ \int_{\frac{1}{2}}^1 \Big(P(u>t) - 2\lambda|u>t|\Big)dt. \tag{3.25}$$

Let h_Ω denote the Cheeger constant of Ω, that is

$$h_\Omega := \inf_{A\subset\Omega} \frac{P(A)}{|A|}.$$

Then, if $\lambda \le \frac{h_\Omega}{2}$, it holds

$$\int_{\frac{1}{2}}^1 \Big(P(u>t) - 2\lambda|u>t|\Big)dt \ge \int_{\frac{1}{2}}^1 \Big(P(u>t) - h_\Omega|u>t|\Big)dt \ge 0\ .$$

Therefore, from (3.25), we conclude that

$$\inf(\mathcal{P}_{g_\lambda}) \ge 0 \quad \text{whenever } \lambda \le \frac{h_\Omega}{2}\ .$$

Now if we consider the convex problem $(P_{g_\lambda^{**}})$ obtained by substituting g_λ with g_λ^{**}, by taking $u \equiv \frac{1}{2}$ as a competitor, we get

$$\inf(P_{g_\lambda^{**}}) \le -\frac{\lambda}{2}|\Omega| = -\frac{\lambda}{2}$$

This shows that for all $\lambda \le \frac{h_\Omega}{2}$

$$\inf(\mathcal{P}_{g_\lambda^{**}}) < \inf(\mathcal{P}_{g_\lambda}).$$

Remark In fact it can be readily checked that $u \equiv \frac{1}{2}$ is the unique solution to problem $(\mathcal{P}_{g_\lambda^{**}})$ for all $\lambda < h_\Omega$. Indeed , as for $(\mathcal{P}_g)$, any solution u to $(\mathcal{P}_{g_\lambda^{**}})$ belongs to $BV\left(\Omega;\left[\frac{1}{2},1\right]\right)$ so that by applying (3.24), we get

$$\int_\Omega |Du| + \int_{\partial\Omega} |u-\frac{1}{2}|d\mathcal{H}^1 - \lambda\int_\Omega u \ \ge \ \int_{\frac{1}{2}}^1 \Big(P(u>t) - \lambda|u>t|\Big)dt - \frac{\lambda}{2}|\Omega|\ ,$$

where, since $\lambda < h_\Omega$, the integral in the right hand side is strictly positive unless $u = \frac{1}{2}$ a.e.

Example 3.10. In the following, taking the same g_λ as before, we consider now the case where h is not homogeneous. More precisely we consider two different integrands

$$h(z) = \sqrt{1+|z|^2}, \qquad h_0(z) = 1 + |z|.$$

Correspondingly, we set

$$\beta(\lambda) := \inf_{u \in W_0^{1,1}(\Omega)} \int_\Omega h(\nabla u)dx - \lambda\Big|\{u=1\}\Big|\,, \quad \beta_0(\lambda) := \inf_{u \in W_0^{1,1}(\Omega)} \int_\Omega h_0(\nabla u)dx - \lambda\Big|\{u=1\}\Big|,$$

and we look at the associated relaxed problems in $BV(\Omega)$:

$$\beta(\lambda) = \inf\left\{\int_\Omega h(Du) + \int_{\partial\Omega} |u| d\mathcal{H}^1 - \lambda\Big|\{u=1\}\Big| \;:\; u \in BV(\Omega)\right\},$$
$$\beta_0(\lambda) = \inf\left\{\int_\Omega h_0(Du) + \int_{\partial\Omega} |u| d\mathcal{H}^1 - \lambda\Big|\{u=1\}\Big| \;:\; u \in BV(\Omega)\right\}.$$

Let us observe that $\beta(\lambda)$ is the infimum of a minimal surface problem with free boundary which is quite challenging (see for instance [46, 29, 6, 71, 62, 77, 85]). The infimum problem corresponding to $\beta_0(\lambda)$ is much simpler. In particular as it well fits the assumptions (H1)-(H5) of our convexification result (Theorem 3.3), its solutions only take value in $\{0,1\}$ and by substituting u with $\mathbf{1}_A$, we obtain:

$$\beta_0(\lambda) = |\Omega| + \inf\left\{P(A) - \lambda|A| \;:\; A \subset \Omega\right\}\,.$$

Suppose that the convexification principle holds also for problem $\beta(\lambda)$, then observing that $\int_\Omega h(Du) = \int_\Omega h_0(Du)$ for $u = \mathbf{1}_A$, we infer that the equality $\beta(\lambda) = \beta_0(\lambda)$ would hold for every $\lambda \geq 0$. We are going to show that such an equality does not hold in general even in a very standard situation where explicit solutions are known.

To that aim we consider the radial case where $\Omega = \{x \in \mathbb{R}^2 : |x| < R\}$.

In this case, we obtain that the unique solution of problem $\beta_0(\lambda))$ is $u_0 \equiv 0$ for $\lambda < h_\Omega$ where $h_\Omega = \frac{2}{R}$ is the Cheeger constant of Ω. In contrast the unique solution is $u_1 \equiv 1$ for $\lambda > h_\Omega$. For the precise value $\lambda = h_\Omega$ we obtain exactly two solutions u_0, u_1. Accordingly,

$$\beta_0(\lambda) = \begin{cases} |\Omega| & \text{if } \lambda \leq h_\Omega \\ P(\Omega) + (1-\lambda)|\Omega| & \text{if } \lambda > h_\Omega. \end{cases}$$

Let us now turn to the determination of the infimum $\beta(\lambda)$. By a rearrangement argument, we can prove that solutions are all radial of the form $u(x) = \varphi(\frac{|x|}{R})$ being $\varphi(t) : [0,1] \to [0,1]$ monotone non increasing. The plateau $\{u=1\}$ is associated with an interval $t \in [0,\rho]$ for a suitable value of $\rho \in [0,1]$ to be determined. For such a plateau, the minimal surface problem reads

$$J(\rho) := \inf_{\substack{\varphi(\rho)=1 \\ \varphi(1)=0}} I(\varphi)\,, \qquad I(\varphi) := R\int_\rho^1 \sqrt{R^2+\varphi'^2}\, t dt\,. \tag{3.26}$$

The first integral of Euler equation for this minimization problem reads

$$\frac{t\varphi'}{\sqrt{R^2+\varphi'^2}} = \mu \tag{3.27}$$

for some constant μ. As $\varphi(1) = 0$, we are led to the explicit form

$$\varphi(t) = K(\mu, t)\,, \qquad K(\mu,t) := \mu\, R \log\left(\frac{1+\sqrt{1-\mu^2}}{t+\sqrt{t^2-\mu^2}}\right)\,, \tag{3.28}$$

provided we can find $\mu \in [0,\rho]$ such that $\varphi(\rho) = K(\mu,\rho) = 1$. In fact, the function $\mu \mapsto K(\mu,\rho)$ is stricly increasing on $[0,\rho]$ so that such a μ is unique and exists if and only if

$$1 \;\leq\; K(\rho,\rho) = \rho\, R\, \log\left(\frac{1+\sqrt{1-\rho^2}}{\rho}\right). \tag{3.29}$$

If the inequality above is strict, then we obtain that the solution to (3.26) is unique and smooth. In the limit case where (3.29) is an equality, one has $\mu = \rho$ and $\varphi'(\rho) = +\infty$. If $K(\rho,\rho) < 1$, then no solution to (3.26) exists but merely a (unique) relaxed solution which exhibits a jump at $t = \rho$ of amplitude $1 - K(\rho,\rho)$. In all cases, we have determined, in term of parameter $\rho \in [0,1]$, an optimized radial configuration whose plateau $\{u = 1\}$ agrees with the disk $B(0,\rho)$. Its total energy is given by

$$E(\rho) := 2\pi\, J(\rho) + (1-\lambda)\pi\rho^2 R^2\ .$$

In order to minimize $E(\rho)$ on interval $[0,1]$, we introduce

$$\overline{\mu}(\rho) := \sup_{0\leq\mu\leq\rho} \{\mu : K(\mu,\rho) \leq 1\}.$$

It is easy to check that for every $\rho \in [0,1]$ such that if $K(\rho,\rho) < 1$ then $\overline{\mu}(\rho) = \rho$. Otherwise, $\overline{\mu}(\rho)$ is the unique solution of equation $K(\mu,\rho) = 1$. After a straightforward computation and exploiting (3.27), we obtain

$$J(\rho) = R^2 \int_\rho^1 \frac{t^2}{\sqrt{t^2-\overline{\mu}(\rho)^2}} dt + \rho R\Big(1 - K(\overline{\mu}(\rho),\rho)\Big)\,.$$

Thus, noticing that $K(\overline{\mu}(\rho),\rho) \leq 1$, we are led to:

$$E(\rho) = \pi R^2 \left(\sqrt{1-\overline{\mu}(\rho)^2} - \rho\sqrt{\rho^2-\overline{\mu}(\rho)^2} + \overline{\mu}(\rho)^2 \log\frac{1+\sqrt{1-\overline{\mu}(\rho)^2}}{\rho+\sqrt{\rho-\overline{\mu}(\rho)^2}} + \frac{2\rho\Big(1-K(\overline{\mu}(\rho),\rho)\Big)}{R} + (1-\lambda)\rho^2 \right) \tag{3.30}$$

Summarizing, an optimal $\overline{\rho}$ for $E(\rho)$ will give a radial function $\overline{u}(x) = \overline{\varphi}(\frac{|x|}{R}) = K(\overline{\mu}(\overline{\rho}), \frac{|x|}{R})$ (defined in (3.28)) which minimizes the relaxed problem associated with $\beta(\lambda)$. This solution is continuous if $K(\overline{\rho},\overline{\rho}) = 1$ and otherwise exhibits a jump of amplitude $1 - K(\overline{\rho},\overline{\rho})$ before reaching the value 1 on the plateau. This is illustrated in Figure 3.12.

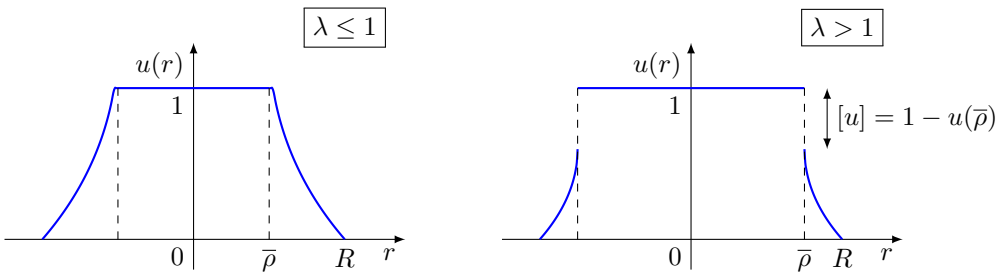


Figure 3.12: Illustration for an optimal u and optimality conditions.

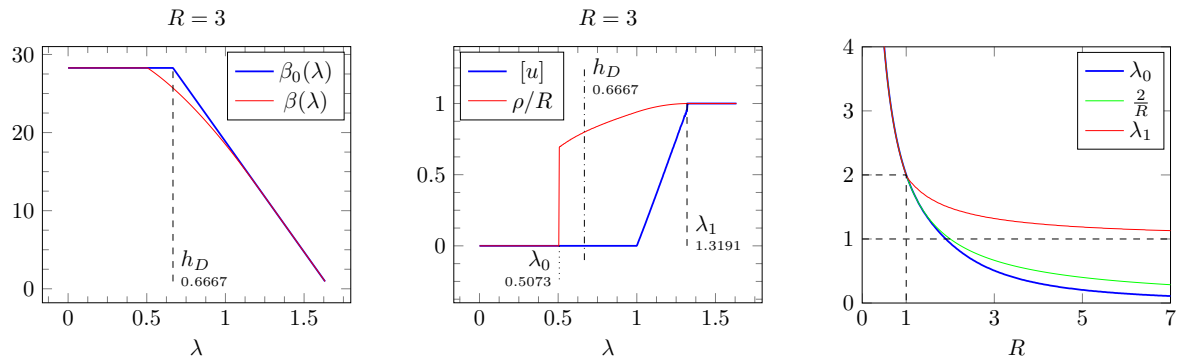


Figure 3.13: Numerical computation of functions $\beta_0(\lambda)$, $\beta(\lambda)$, critical values of λ: $\lambda_0(R), \lambda_1(R)$, and Cheeger constant $h_\Omega = \frac{2}{R}$.

The minimization of $E(\rho)$ is performed by using Matlab for different values of R and λ. It turns out that:

- For $R > 1$, the exists constants $\lambda_0(R) < h_\Omega < \lambda_1(R)$ where $h_\Omega = \frac{2}{R}$ (Cheeger constant of Ω) and such that $\beta(\lambda) < \beta_0(\lambda)$ for every $\lambda \in (\lambda_0, \lambda_1)$. Solutions to $\beta(\lambda)$ do or do not have jump depending on the position of λ with respect to 1: $[u] = 0$ if $\lambda \leq 1$ and $[u] > 0$ if $\lambda > 1$ (see Figure 3.12 and the middle subfigure of Figure 3.13). In contrast it holds $\beta(\lambda) = \beta_0(\lambda)$ for $\lambda \leq \lambda_0$ (in this case $u \equiv 0$ solves both problems) or for $\lambda \geq \lambda_1$ (in that case $u \equiv 1$ is a solution).
- For small balls $R \leq 1$, we have the equality $\beta(\lambda) = \beta_0(\lambda)$ (and $\lambda_0 = \lambda_1 = \frac{2}{R}$ which is shown in the last subfigure of Figure 3.13).

Chapter 4

Calibrating fields for minimal surfaces with free boundary and Cheeger-type problem

In this chapter, we apply the duality theory introduced in Chapter 2 to study a problem of minimal surfaces with free boundary. The aim is to characterize optimal surfaces and their free boundaries by finding a calibration field solving an associated dual problem. An interesting upper bound of this problem is given by a variant of Cheeger problem, which we revisit with using an explicit 2-dimensional calibration method based on the construction of a cut-locus potential. The comparison with the original problem is discussed in detail.

Let D be an open bounded nonempty subset of $\mathbb{R}^N$ with $N \geq 2$. Given $\lambda \geq 0$. A subset $\Omega \subset D$ have its area as N-dimensional Lebesgue measure denoted by $|\Omega| := \mathcal{L}^N(\Omega)$ and its perimeter as $(N-1)$-dimensional Hausdorff measure of the boundary $\partial\Omega$ denoted by $P(\Omega) := \mathcal{H}^{N-1}(\partial\Omega)$.

4.1 Presentation of two free boundary problems

We consider in the following the variational problem of minimal surfaces with free boundary

$$\beta(\lambda) := \inf\left\{ \int_D \sqrt{1+|\nabla u|^2}dx - \lambda|\{u \geq 1\}| \ : \ u \in W_0^{1,1}(D) \right\}.$$

We denote by E_λ the relaxed functional of problem $\beta(\lambda)$ in $BV(D)$, namely

$$E_\lambda(u) := \int_D \sqrt{1+|\nabla u|^2}dx + \int_D d|D^s u| + \int_{\partial D} |u| d\mathcal{H}^{N-1} - \lambda|\{u \geq 1\}|. \tag{4.1}$$

By convention, when we say $\overline{u}$ is a solution of $\beta(\lambda)$, we intend to say $\overline{u}$ minimizes the relaxed problem $\inf\{E_\lambda(u) \ : \ u \in BV(D)\}$. The competitive term $-\lambda|\{u \geq 1\}|$ can be written as an integral $\int_D g(u)dx$ with

$$g(t) = \begin{cases} 0 & \text{if } t < 1, \\ -\lambda & \text{if } t \geq 1. \end{cases}$$

We see that the integrand $g(t)$ is non convex and discontinuous at $t = 1$.

By a comparison argument, one can figure out that, if u is optimal for $\beta(\lambda)$ then $0 \leq u \leq 1$. Let $\Omega(u) := \{u = 1\}$. The variational problem $\beta(\lambda)$ is actually to minimize the area of surfaces

on $D \setminus \Omega(u)$ in competition with the λ-scaled Lebesgue measure of $\Omega(u)$ (i.e. $\lambda|\Omega(u)|$). These surfaces are given by graphs of functions u in $\mathbb{R}^{N+1}$. Minimal surfaces have a prescribed boundary realized by the constraint $u = 0$ on ∂D and are free on the boundary $\partial\Omega(u)$. Because of this reason, $\partial\Omega(u)$ is called *free boundary*. See Figure 4.1 for illustration. A surface is minimal in this context, with a free boundary, must satisfy the following partial differential equation

$$-\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{in } D \setminus \Omega(u), \tag{4.2}$$

$$u = 0 \quad \text{on } \partial D, \qquad u = 1 \quad \text{on } \partial\Omega(u), \tag{4.3}$$

which is indeed derived from the Euler-Lagrange equation of the functional $E_\lambda(u)$. Note that the left-hand side of the equation (4.2) is well-known as a representation of the scalar mean curvature of the graph of function u, up to a division by $N-1$. In this sense, minimal surfaces have zero mean curvature.

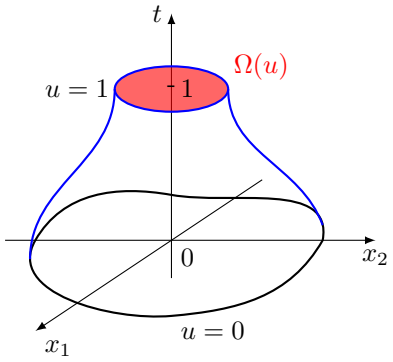


Figure 4.1: A surface with prescribed boundary $u = 0$ on ∂D and free boundary $\partial\{u = 1\}$.

By replacing the surface-area integral in problem $\beta(\lambda)$ by the total variation of u, we obtain a variant with minimizing total variation with free boundary

$$\beta_0(\lambda) := \inf\left\{ \int_D (1+|\nabla u|)dx - \lambda|\{u=1\}| \ : \ u \in W_0^{1,1}(D),\ 0 \le u \le 1 \right\}. \tag{4.4}$$

It leads to another free boundary problem whose partial differential equation is given by

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \quad \text{in } D \setminus \Omega(u),$$

$$u = 0 \quad \text{on } \partial D, \qquad u = 1 \quad \text{on } \partial\Omega(u).$$

In $BV(D)$, the relaxed functional of problem $\beta_0(\lambda)$ is

$$E_\lambda^0(u) := |D| + \int_D d|Du| + \int_{\partial D} |u| d\mathcal{H}^{N-1} - \lambda|\{u=1\}|. \tag{4.5}$$

Similarly, when we say $\overline{u}$ is a solution to problem $\beta_0(\lambda)$, we means, by convention, $\overline{u}$ minimizes its relaxed problem, i.e. $\inf\{E_\lambda^0(u) \ : \ u \in BV(D)\}$. We remark that $\overline{u}$ only takes values in $[0,1]$. In fact, by truncating $\overline{u}$ to obtain $\tilde{u} = \max\{\min\{\overline{u}, 1\}, 0\}$, we decrease the energy E_λ^0. Since $\{\overline{u}=1\} \subset \{\tilde{u}=1\}$ and $\int_D d|D\overline{u}|dx \ge \int_D d|D\tilde{u}|dx$, we have $E_\lambda^0(\overline{u}) \ge E_\lambda^0(\tilde{u})$. It turns out that the relaxed problem of $\beta_0(\lambda)$ can be restricted to $BV(D;[0,1])$. Besides, the competitive term $-\lambda|\{u=1\}|$ in (4.4) can also be recast as $\int_D g_0(u)dx$ where

$$g_0(t) = 0 \text{ in } [0,1), \qquad g_0(1) = -\lambda, \qquad \text{and} \qquad g_0(t) = +\infty \text{ for } t \notin [0,1].$$

g_0 is a non convex function with discontinuity at $t = 1$.

Here two important observations are in order:

Firstly, if u is the characteristic function of some finite perimeter subset $\Omega \subset D$, we find from (4.1) and (4.5) that

$$E_\lambda(u) = E_\lambda(\mathbf{1}_\Omega) = |D| + P(\Omega) - \lambda|\Omega| = E^0_\lambda(\mathbf{1}_\Omega) = E^0_\lambda(u). \tag{4.6}$$

Therefore, E_λ and E^0_λ coincide together on the class of characteristic functions.

Secondly, the non convex variational problem $\beta_0(\lambda)$ in (4.4) satisfies the positive one–homogeneity assumption used in Chapter 3 and we may apply the convexification argument (see Theorem 3.3). Therefore,

- solutions u do not take intermediate value in $(0,1)$, that is $u = \mathbf{1}_\Omega$ for some Ω being a finite perimeter subset of D;
- the infimum $\beta_0(\lambda)$ is unchanged if we substitute g_0 with its convex envelop

$$g_0^{**}(t) = -\lambda t \text{ in } [0,1], \quad \text{and} \quad g_0^{**}(t) = +\infty \text{ for } t \notin [0,1]. \tag{4.7}$$

As a consequence, we may rewrite $\beta_0(\lambda)$ as the infimum of a convex problem

$$\begin{aligned}\beta_0(\lambda) &= \inf\left\{\int_D (1 + |\nabla u| + g_0^{**}(u))dx \ : \ u \in W^{1,1}_0(D;[0,1])\right\} \\ &= \min\left\{\int_D (1 + |Du| - \lambda u)dx + \int_{\partial D} |u| d\mathcal{H}^{N-1} \ : \ u \in BV(D;[0,1])\right\} \\ &= \min\left\{\int_D (1 + |Du| - \lambda u)dx + \int_{\partial D} |u| d\mathcal{H}^{N-1} \ : \ u \in BV(D;\{0,1\})\right\}.\end{aligned} \tag{4.8}$$

By substituting u with $\mathbf{1}_\Omega$ (while $\Omega = \{u = 1\}$) in the last equality, we obtain

$$\begin{aligned}\beta_0(\lambda) &= \min\left\{|D| + P(\Omega) - \lambda|\Omega| \ : \ \Omega \subset D\right\} \\ &= |D| + m(\lambda, D)\end{aligned} \tag{4.9}$$

where we denote by $m(\lambda, D)$ the infimum of the geometric problem

$$m(\lambda, D) := \min\left\{P(\Omega) - \lambda|\Omega| \ : \ \Omega \subset D\right\}. \tag{4.10}$$

Remark 4.1. The equality (4.9) can be obtained directly by using the co-area formula for BV-functions. In fact, we take $u \in BV(\mathbb{R}^N;[0,1])$ such that $u = 0$ in $\mathbb{R}^N \backslash D$. Since $\{u = 1\} \subset \{u > s\}$ for all $s \in (0,1)$, we have

$$E^0_\lambda(u) = |D| + \int_0^1 P(\{u > s\})ds - \lambda|\{u = 1\}| \geq |D| + \int_0^1 P(\{u > s\})ds - \lambda|\{u > s\}|. \tag{4.11}$$

As u is a BV-function in D, its level sets are finite perimeter subsets of D. We deduce from (4.11) that $\beta_0(\lambda) \geq |D| + m(\lambda, D)$. The inequality $\beta_0(\lambda) = |D| + m(\lambda, D)$ occurs when we evaluate E^0_λ at $\mathbf{1}_\Omega$ with Ω being a solution to problem $m(\lambda, D)$.

Obviously as $|z| \leq \sqrt{1 + |z|^2} \leq 1 + |z|$ for all $z \in \mathbb{R}^N$ we have the following inequalities

$$m(\lambda, D) = \beta_0(\lambda) - |D| \leq \beta(\lambda) \leq \beta_0(\lambda). \tag{4.12}$$

A natural question is to know in which cases $\beta(\lambda) < \beta_0(\lambda)$. This is the main objective of Section 4.4 where we demonstrate that the inequality $\beta(\lambda) < \beta_0(\lambda)$ will happen for arranging in some interval of $\mathbb{R}_+$ (which is non empty as soon as D is non smooth) and the solutions to these two problems do not always coincide. We also provide numerical illustrations of this non coincidence issue.

For clarity, we give here the synopsis of the next sections of this chapter:

- Section 4.2: We present in Subsection 4.2.1 the dual problems associated with $\beta(\lambda)$ and $\beta_0(\lambda)$ in dimension $N+1$. After studying some general properties of the geometric problem $m(\lambda, D)$ in Subsection 4.2.2, we consider a N-dimensional dual problem associated with problem $m(\lambda, D)$, which allows to characterize solutions of $m(\lambda, D)$ (in Subsection 4.2.3). We then introduce in Subsection 4.2.4 the notion of θ-calibrability which generalizes the classical notion of calibrability.

- Section 4.3: We recall the construction of the Cheeger set of convex sets in $\mathbb{R}^2$ in Subsection 4.3.1 and then explicitly construct θ-calibrable sets of convex sets in $\mathbb{R}^2$ (the set Ω_λ) in Subsection 4.3.2. We introduce in Subsection 4.3.3 a function called cut-locus potential and use this function to construct explicitly a vector field on D which calibrates the set Ω_λ.

- Section 4.4: We discuss about the occurrence of the inequality $\beta(\lambda) < \beta_0(\lambda)$ according to the value of λ and to the geometric properties of D. Some numerical simulations are given in the case of a disk.

4.2 Dual problems and calibration method

The variational problems associated respectively with $\beta(\lambda)$ and $\beta_0(\lambda)$ are non convex with linear growth. They enter the duality framework developed in Section 2.5 of Chapter 2, where a characterization of global minimizers is provided (cf. Theorem 2.24).

4.2.1 Dual problems in dimension $N+1$

This subsection is devoted to state the dual problems associated with $\beta(\lambda)$ and $\beta_0(\lambda)$ in $N+1$ dimensions. Let us first set $Q = D \times [0,1]$.

Dual problem of $\beta(\lambda)$

The dual variational problem of $\beta(\lambda)$ is given by

$$\sup\left\{-\int_D \sigma^t(x,0) \ : \ \operatorname{div}\sigma = 0 \text{ in } Q,\ \sigma^t + \sqrt{1-|\sigma^x|^2} \geq 0 \text{ in } Q,\ \sigma^t(x,1) \geq \lambda - 1 \text{ on } D\right\},$$

where $\sigma = (\sigma^x, \sigma^t) \in L^\infty(Q;\mathbb{R}^N)$. Note that the constraint $\sigma^t(x,1) \geq \lambda - 1$ on D is associated with the discontinuity of the non convex part $g(t)$ at $t = 1$.

The dual problem can be interpreted in the point of view of fluid dynamics as an optimal flow problem in the cube Q wherein we maximize the downward flow $\sigma = (\sigma^x, \sigma^t)$ of an incompressible fluid ($\operatorname{div}\sigma = 0$ in Q) through the bottom interface $D \times \{0\}$ subject to the pointwise non linear constraint $\sigma^t + \sqrt{1-|\sigma^x|^2} \geq 0$ in Q and the pointwise linear constraint $\sigma^t(x,1) \geq \lambda - 1$ on the upper interface $D \times \{1\}$.

By means of calibration fields, we can characterize minimal surfaces (with free boundary) thanks to the following optimality conditions (see Theorem 2.24, Corollary 2.25): u is an optimal solution for problem $\beta(\lambda)$ if and only if there exists a calibration field $\sigma \in L^\infty(Q;\mathbb{R}^N)$, $\sigma = (\sigma^x(x,t), \sigma^t(x,t))$ being admissible to the dual problem and satisfying on the complete

graph $\overline{G}_u$ of u that

$$\begin{aligned}
\sigma^x(x,u(x)) &= \frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}} && \text{for } \mathcal{L}^N\text{-a.e. } x \in u^{-1}([0,1)),\\
\sigma^t(x,u(x)) &= -\sqrt{1-|\nabla u(x)|^2} && \text{for } \mathcal{L}^N\text{-a.e. } x \in u^{-1}([0,1)),\\
\sigma^t(x,1) &= \lambda - 1 && \text{for } \mathcal{L}^N\text{-a.e. } x \in \{u=1\},\\
\sigma(x,t)\cdot\nu_u &= 1 && \text{for } \mathcal{H}^N\text{-a.e. } x \in S_u \text{ and } \forall t \in [u^-(x),u^+(x)],\\
\sigma^x(x,u^+(x)) &= 1 && \text{for } |D^c u|\text{-a.e. } x \in \Omega,
\end{aligned}$$

where S_u denotes the jump set of u. In particular, $\overline{u} \equiv 0$ is an optimal solution to $\beta(\lambda)$ if and only if

$$\sigma^x(x,0) = 0, \quad \sigma^t(x,0) = -1, \quad \sigma^t(x,1) \geq \lambda - 1 \quad \text{on } D, \tag{4.13}$$

$$\sigma^t + \sqrt{1-|\sigma^x|^2} \geq 0, \quad \operatorname{div}\sigma = 0 \quad \text{in } Q. \tag{4.14}$$

And, $\overline{u} \equiv 1$ is optimal to $\beta(\lambda)$ if and only if there hold

$$\sigma^x(x,1) = 0, \quad \sigma^t(x,1) = \lambda - 1 \quad \text{on } D, \quad \sigma^x \cdot \nu_D = 1 \quad \text{on } \partial D,$$
$$\sigma^t + \sqrt{1-|\sigma^x|^2} \geq 0, \quad \operatorname{div}\sigma = 0 \quad \text{in } Q.$$

With these optimality conditions, it is easy to check that if $u \equiv 0$ solves $\beta(\lambda)$, then it solves $\beta(\lambda')$ for all $0 \leq \lambda' < \lambda$, and similarly, if $u \equiv 1$ minimizes $\beta(\lambda)$, then it minimizes $\beta(\lambda'')$ for all $\lambda'' > \lambda$. We shall say $u \equiv 0$ and $u \equiv 1$ are trivial solution of $\beta(\lambda)$.

Let us introduce two following critical values of λ which provide the dependence on λ of the appearance of the trivial solutions for problem $\beta(\lambda)$:

$$\lambda_0 = \sup\{\lambda \geq 0 \ : \ u \equiv 0 \text{ solves } \beta(\lambda)\}, \tag{4.15}$$

$$\lambda_1 = \inf\{\lambda \geq 0 \ : \ u \equiv 1 \text{ solves } \beta(\lambda)\}. \tag{4.16}$$

According to this definition, if $\lambda_0 < \lambda_1$ and $\lambda \in (\lambda_0, \lambda_1)$ then a solution to problem $\beta(\lambda)$ of the kind $\mathbf{1}_\Omega$ is such that $0 < |\Omega| < |D|$.

Dual problem of $\beta_0(\lambda)$

We apply the same duality framework to obtain the dual problem of $\beta_0(\lambda)$. There holds

$$\beta_0(\lambda) = \sup\left\{\int_D -\sigma^t(x,0)dx \ : \ \operatorname{div}\sigma = 0,\ |\sigma^x| \leq 1,\ \sigma^t + 1 \geq 0 \text{ in } Q,\ \sigma^t(x,1) \geq \lambda - 1 \text{ on } D\right\}$$

where $\sigma = (\sigma^x, \sigma^t) \in L^\infty(Q;\mathbb{R}^N)$. The pointwise constraint $\sigma^t(x,1) \geq \lambda - 1$ is involved with the discontinuity of g_0 at $t = 1$.

A characteristic function $\mathbf{1}_\Omega$ for some $\Omega \subset D$ is optimal for problem $\beta_0(\lambda)$ if and only if there is a calibration field $\sigma \in L^\infty(Q;\mathbb{R}^N)$, $\sigma = (\sigma^x(x,t), \sigma^t(x,t))$ verifying

$$\operatorname{div}\sigma = 0, \ \ |\sigma^x| \leq 1, \ \ \sigma^t + 1 \geq 0 \ \ \mathcal{L}^{N+1}\text{-a.e. in } Q, \qquad \sigma^t(x,1) \geq \lambda - 1 \ \ \mathcal{L}^N\text{-a.e. in } D, \tag{4.17}$$

$$\sigma^t(x,1) = \lambda - 1 \ \ \mathcal{L}^N\text{-a.e. in } \Omega, \qquad \sigma^t(x,0) = -1 \ \ \mathcal{L}^N\text{-a.e. in } D \setminus \Omega, \tag{4.18}$$

$$\sigma^x \cdot \nu_\Omega = 1 \ \ \mathcal{H}^N\text{-a.e. on } \partial\Omega \times (0,1). \tag{4.19}$$

Note that constraints (4.17) make the vector field σ admissible to the dual problem while conditions (4.18)-(4.19) are interpreted from the optimality conditions in Theorem 2.24, Corollary 2.25.

Remark 4.2. In the particular case of the variational problem associated to $\beta_0(\lambda)$, due to the homogeneity of the total variation, there exists a closed relation between the $(N+1)$-dimensional calibration fields above and the solutions of a more classical N-dimensional dual problem, namely we have the following assertions:

(i) If $(\mathbf{1}_\Omega, \sigma)$ is a pair of calibration then there exists a vector field $q \in L^\infty(D;\mathbb{R}^N)$, which is given by $q(x) := -\int_0^1 \sigma^x(x,t)dt$, verifying

$$|q| \leq 1 \text{ in } D, \qquad \operatorname{div} q \leq \lambda \text{ in } \Omega, \qquad \operatorname{div} q \geq \lambda \text{ in } D \setminus \Omega, \qquad q \cdot \nu_\Omega = -1 \text{ on } \partial\Omega. \quad (4.20)$$

(ii) If $q \in L^\infty(D;\mathbb{R}^N)$ is a vector field satisfying (4.20), and if in addition, $\operatorname{div} q \geq 0$ in D and $\operatorname{div} q = \lambda$ in $D \setminus \Omega$ then $\sigma_q(x,t)$ calibrates $\mathbf{1}_\Omega$ where

$$\sigma_q(x,t) := (-q(x), t \operatorname{div} q(x) - 1 + (\lambda - \operatorname{div} q)_+). \quad (4.21)$$

Let us sketch a proof of these two statements. We set here $u = \mathbf{1}_\Omega$ for the sake of simplicity. (i) Assume that (u,σ) is a pair of calibration and $q(x) = -\int_0^1 \sigma^x(x,t)dt$. We have

$$|q(x)| = |\int_0^1 \sigma^x(x,t)dt| \leq \int_0^1 |\sigma^x(x,t)|dt \leq 1.$$

and on the other hand,

$$\operatorname{div} q(x) = -\operatorname{div}_x \int_0^1 \sigma^x(x,t)dt = -\int_0^1 \operatorname{div}_x \sigma^x(x,t)dt.$$

Since σ is a divergence-free vector field, i.e. $\operatorname{div}\sigma = \operatorname{div}_x \sigma^x + \frac{\partial \sigma^t}{\partial t} = 0$, it holds

$$\operatorname{div} q(x) = \int_0^1 \frac{\partial \sigma^t}{\partial t} dt = \sigma^t(x,1) - \sigma^t(x,0).$$

Under conditions (4.17)-(4.18), we derive that $\operatorname{div} q \leq \lambda$ in Ω and $\operatorname{div} q \geq \lambda$ in $D \setminus \Omega$. In fact, we have $\sigma^t(x,0) \geq -1$, $\sigma^t(x,1) = \lambda - 1$ in Ω, and in $D \setminus \Omega$, $\sigma^t(x,0) = -1$, $\sigma^t(x,1) \geq \lambda - 1$. For verifying $q \cdot \nu_\Omega = 1$ on $\partial\Omega$, it suffice to use (4.19), that is for every $x \in \partial\Omega$,

$$q(x) \cdot \nu_\Omega(x) = \left(-\int_0^1 \sigma^x(x,t)dt\right) \cdot \nu_\Omega(x) = -\int_0^1 \sigma^x(x,t) \cdot \nu_\Omega(x)dt = -1.$$

(ii) The vector field σ_q defined by (4.21) is a calibration for $\mathbf{1}_\Omega$ if and only if it satisfies the optimality conditions (4.17)-(4.19). It is easy to verify these conditions.

Remark 4.2 is to say that we probably obtain among $(N+1)$-dimensional calibration fields a calibration σ whose first component σ^x does not depend on t and whose second component σ^t is linear in t. At the moment, the $(N+1)$-dimensional calibration is completely determined by the N-dimensional field $q(x) := -\int_0^1 \sigma^x(x,t)dt$ and vice versa.

In Subsection 4.2.3 we shall come back to the determination of this N-dimensional vector field $q(x)$ as a solution of a N-dimensional variational problem.

4.2.2 Geometric problem $m(\lambda, D)$

This subsection is devoted to the properties of the geometric problem $m(\lambda, D)$ in (4.10). We minimize the shape functional

$$J_\lambda(\Omega) := P(\Omega) - \lambda|\Omega|$$

among all open subset of D with finite perimeter.

Theorem 4.3 (existence)**.** *The infimum $m(\lambda, D)$ is finite and attained.*

Proof. We prove by using the direct method of Calculus of Variations. For every $\Omega \subset D$, we observe that

$$P(\Omega) - \lambda|\Omega| \geq P(\Omega) - \lambda|D| \geq -\lambda|D|.$$

This is to say that $m(\lambda, D)$ is finite, i.e. $\inf\{P(\Omega) - \lambda|\Omega| : \Omega \subset D\} \geq -\lambda|D|$. We now take a minimizing sequence (Ω_n), i.e. $P(\Omega_n) - \lambda|\Omega_n| \to m(\lambda, D)$. Since $0 \leq |\Omega_n| \leq |D|$, we have $\sup_n P(\Omega_n) < +\infty$. Therefore, the sequence of functions $u_n := \mathbf{1}_{\Omega_n}$ is bounded in $BV(D)$. As consequence, there exists a subsequence (u_{n_k}) converging to some u strongly in $L^1(D)$. Thus, there is a subset $\Omega \subset D$ such that $u = \mathbf{1}_\Omega$ a.e. and $D\mathbf{1}_{\Omega_{n_k}}$ converges weakly-* to $D\mathbf{1}_\Omega$. Since $|\Omega_{n_k}| = \int_D u_{n_k} \longrightarrow \int_D u = |\Omega|$ (furthermore, Ω_{n_k} converges to Ω in measure in D, i.e. $|D \cap (\Omega_{n_k} \Delta \Omega)| \to 0$) and because of lower semicontinuity of perimeter, i.e. $\liminf_k P(\Omega_{n_k}) \geq P(\Omega)$, we obtain

$$m(\lambda, D) = \liminf_k (P(\Omega_{n_k}) - \lambda|\Omega_{n_k}|) \geq P(\Omega) - \lambda|\Omega| \geq m(\lambda, D).$$

This show that Ω is a minimizer. ∎

We recall that the Cheeger constant of D is given by

$$h_D := \inf\{\lambda_\Omega : \Omega \subset D, |\Omega| > 0\}, \qquad \text{with } \lambda_\Omega := \frac{P(\Omega)}{|\Omega|}, \tag{4.22}$$

An optimal shape of this geometric problem is called Cheeger set of D. In case D is convex, it turns out that such a Cheeger set is unique (see for instance [33], [82], [94]) and we shall denote it by C_D. In general, it can be proved (see [82], [28]) that a union of Cheeger sets is a Cheeger set. Thus, we may extend the definition of C_D by setting

$$C_D := \bigcup \{\Omega \ : \ \Omega \text{ is a Cheeger set of } D\}.$$

Then, C_D is called the *maximal Cheeger set.*

Proposition 4.4. *The following equality holds: $h_D = \sup\{\lambda \geq 0 \ : \ m(\lambda, D) = 0\}$. More precisely, one has*

(i) If $\lambda \in (0, h_D)$ then $m(\lambda, D) = 0$ and the empty set $\emptyset$ is the unique solution.

(ii) If $\lambda = h_D$ then $m(\lambda, D) = 0$ and $\operatorname{Argmin} J_\lambda(\Omega) = \{\emptyset,$ Cheeger sets of $D\}$.

(iii) If $\lambda > h_D$ then $m(\lambda, D) < 0$ and solutions of $m(\lambda, D)$ have strictly positive Lebesgue measure. (In other words, $\emptyset$ does not solve $m(\lambda, D)$ anymore).

Proof. Assume that $\lambda \leq h_D$. Then, for all measurable subset $\Omega \subset D$, one has

$$J_\lambda(\Omega) = P(\Omega) - \lambda|\Omega| \geq (h_D - \lambda)|\Omega| \geq 0 \tag{4.23}$$

Thus, $m(\lambda, D) = 0$ where $\emptyset$ is a solution. Observing that by (4.23), $J_\lambda(\Omega) > 0$ whenever $|\Omega| > 0$ and $\lambda < h_D$, we deduce that $\Omega = \emptyset$ is the unique solution (up to a Lebesgue negligible set) in the case $\lambda < h_D$. If $\lambda = h_D$, the non empty solutions to $m(\lambda, D)$ are by definition the Cheeger sets of D. In particular, if D is convex, we derive that $m(\lambda, D)$ has exactly two solutions $\{\emptyset, C_D\}$.

Assume now that $\lambda > h_D$. Then, if Ω is a Cheeger set of D, one has

$$J_\lambda(\Omega) = |\Omega| \left(\frac{P(\Omega)}{|\Omega|} - \lambda \right) = |\Omega|(h_D - \lambda) < 0.$$

and therefore, $m(\lambda, D) < 0$. It follows that any solution Ω to $m(\lambda, D)$ has strictly positive Lebesgue, i.e. $|\Omega| > 0$. (Indeed, $|\Omega| = 0$ implies that $J_\lambda(\Omega) \geq 0$). ∎

Proposition 4.5. *Let D be a Cheeger set of itself, i.e. $h_D = P(D)/|D|$. For all $\lambda > h_D$, it holds $m(\lambda, D) = (h_D - \lambda)|D|$ and D is the unique solution for $m(\lambda, D)$.*

Proof. As, by Proposition 4.4, $m(\lambda, D) < 0$, any solution Ω is such that $|\Omega| > 0$. On the other hand, for such a solution Ω it holds

$$J_\lambda(\Omega) = P(\Omega) - h_D|\Omega| + (h_D - \lambda)|\Omega| \geq (h_D - \lambda)|\Omega|$$

whereas since $P(D) = h_D|D|$, we have

$$J_\lambda(D) = (h_D - \lambda)|D| \leq (h_D - \lambda)|\Omega|.$$

It follows that D is also a solution and that $m(\lambda, D) = (h_D - \lambda)|D| = (h_D - \lambda)|\Omega|$. Thus, $\Omega = D$ is the unique solution (up to a Lebesgue negligible set) to $m(\lambda, D)$. ■

Proposition 4.6 (monotonicity)**.**

(i) If $C \subset D$ then $m(\lambda, C) \geq m(\lambda, D)$. Consequently, if Σ is a solution to $m(\lambda, D)$ then it is also a solution to $m(\lambda, \Sigma)$ and hence, $m(\lambda, D) = m(\lambda, \Sigma)$.

(ii) Let $\Omega(\lambda)$, $\Omega(\mu)$ be solutions of $m(\lambda, D)$ and $m(\mu, D)$ respectively. Then, if $0 < \lambda < \mu$, there hold

$$\Omega(\lambda) \subset \Omega(\mu) \quad \text{and} \quad m(\mu, D) \leq m(\lambda, D) \leq 0.$$

Particularly, one has $m(\mu, D) < m(\lambda, D) < 0$ if $h_D < \lambda < \mu$.

(iii) If D is a solution to $m(\lambda, D)$ then D is also a solution to $m(\mu, D)$ for any $\mu \geq \lambda$.

Proof. (i) Let $C \subset D$. We assume that Ω and Σ are solutions to $m(\lambda, C)$ and $m(\lambda, D)$ respectively. Since $\Omega \subset C \subset D$, it holds

$$m(\lambda, C) = P(\Omega) - \lambda|\Omega| \geq m(\lambda, D).$$

By applying this result to $C = \Sigma$, we obtain $m(\lambda, \Sigma) \geq m(\lambda, D)$. On the other hand, since Σ is a minimizer of $m(\lambda, D)$, it turns out that for all $A \subset \Sigma (\subset D)$,

$$P(A) - \lambda|A| \geq m(\lambda, D) = P(\Sigma) - \lambda|\Sigma|.$$

Thus, by taking the infimum of the left hand side of the inequality above over all subsets $A \subset \Sigma$, we have $m(\lambda, \Sigma) \geq P(\Sigma) - \lambda|\Sigma|$. This show that Σ is a minimizer of $m(\lambda, \Sigma)$. So, the equality holds $m(\lambda, \Sigma) = m(\lambda, D)$.
(ii) It is a fact [6, Proposition 3.38] that for any two finite perimeter sets A, B in $\mathbb{R}^N$ we have

$$P(A) + P(B) \geq P(A \cap B) + P(A \cup B). \tag{4.24}$$

Let $0 < \lambda < \mu$ and let $\Omega(\lambda)$, $\Omega(\mu)$ be solutions of $m(\lambda, D)$ and $m(\mu, D)$ respectively. Then, it holds

$$\begin{aligned} P(\Omega(\lambda)) - \lambda|\Omega(\lambda)| &\leq P(\Omega(\lambda) \cap \Omega(\mu)) - \lambda|\Omega(\lambda) \cap \Omega(\mu)|, \\ P(\Omega(\mu)) - \mu|\Omega(\mu)| &\leq P(\Omega(\lambda) \cup \Omega(\mu)) - \mu|\Omega(\lambda) \cup \Omega(\mu)|. \end{aligned}$$

By adding both inequalities above and using (4.24), we obtain

$$0 \leq \lambda\Big(|\Omega(\lambda)| - |\Omega(\lambda) \cap \Omega(\mu)|\Big) - \mu\Big(|\Omega(\lambda) \cup \Omega(\mu)| - |\Omega(\mu)|\Big) = (\lambda - \mu)|\Omega(\lambda) \backslash \Omega(\mu)| \leq 0.$$

This means $(\lambda-\mu)|\Omega(\lambda)\backslash\Omega(\mu)|=0$. Since $\lambda<\mu$, we get $|\Omega(\lambda)\backslash\Omega(\mu)|=0$, hence $\Omega(\lambda)\subset\Omega(\mu)$. By using again $\lambda<\mu$, $\Omega(\lambda)\subset\Omega(\mu)$ (that we have already known) and (i), we deduce that

$$m(\lambda,D)=P(\Omega(\lambda))-\lambda|\Omega(\lambda)|\geq P(\Omega(\lambda))-\mu|\Omega(\lambda)|\geq m(\mu,\Omega(\mu))=m(\mu,D). \tag{4.25}$$

As proved in Proposition 4.4, both $m(\lambda,D)$ and $m(\mu,D)$ are non positive. Notice that the first inequality in (4.25) becomes "$>$" if $\Omega(\lambda)\neq\emptyset$. In the case where $h_D<\lambda<\mu$, by invoking Proposition 4.4 (iii), we infer that $m(\mu,D)<m(\lambda,D)<0$.
(iii) Let $0<\lambda\leq\mu$. If D is solution to $m(\lambda,D)$ then for every $\Omega\subset D$,

$$\begin{aligned}P(\Omega)-\mu|\Omega| &= P(\Omega)-\lambda|\Omega|+(\lambda-\mu)|\Omega|\\ &\geq P(D)-\lambda|D|+(\lambda-\mu)|D|\\ &= P(D)-\mu|D|\end{aligned}$$

Thus, D is also optimal for $m(\mu,D)$. ■

Proposition 4.7. *Let (λ_n) be an strictly increasing sequence converging to λ and Ω_n be a solution of $m(\lambda_n,D)$. Then, $\Omega(\lambda)^+:=\bigcup_n\Omega_n$ is a minimizer of $m(\lambda,D)$. Furthermore, $P(\Omega_n)\to P(\Omega(\lambda)^+)$. Similarly, assume that $\lambda_n\downarrow\lambda$, then $\Omega(\lambda)^-:=\bigcap_n\Omega_n$ is a minimizer of $m(\lambda,D)$ and $P(\Omega_n)\to P(\Omega(\lambda)^-)$.*

Proof. Assume that $\lambda_n\uparrow\lambda$, and for each $n\in\mathbb{N}$, Ω_n is a solution of $m(\lambda_n,D)$. Then, for every subset $\Omega\subset D$, one has

$$P(\Omega_n)-\lambda_n|\Omega_n|\leq P(\Omega)-\lambda_n|\Omega|. \tag{4.26}$$

Since the family (Ω_n) is monotone (see Proposition 4.6 (ii)) and the perimeter is lower semi-continuous (see [6, Proposition 3.38]), i.e. $P(\Omega(\lambda)^+)\leq\liminf_n P(\Omega_n)$, we derive from (4.26) by passing to the limit inferior that

$$P(\Omega(\lambda)^+)-\lambda|\Omega(\lambda)^+|\leq P(\Omega)-\lambda|\Omega|.$$

This shows that $\Omega(\lambda)^+$ is a solution of $m(\lambda,D)$. Besides, as Ω_n is a minimizer of $m(\lambda_n,D)$, one has

$$P(\Omega_n)-\lambda_n|\Omega_n|\leq P(\Omega(\lambda)^+)-\lambda_n|\Omega(\lambda)^+|.$$

By taking the limit superior on both sides of the inequality above, we obtain

$$\limsup_n P(\Omega_n)\leq P(\Omega(\lambda)^+).$$

Once again with the lower semicontinuity of the perimeter, we conclude $\lim_n P(\Omega_n)=P(\Omega(\lambda)^+)$. The remained assertion is proved analogously. ■

Corollary 4.8.

(i) Let $\lambda>h_D$ and Ω be a solution to $m(\lambda,D)$. Then $\Omega\supset C_D$, $h_\Omega=h_D$, and Ω, D share the same Cheeger sets.

(ii) Let (λ_n) be a sequence such that $\lambda_n>h_D$, $\lambda_n\downarrow h_D$, and let Ω_n be a solution to $m(\lambda_n,D)$. Then it holds

$$C_D=\bigcap_{n\in\mathbb{N}}\Omega_n. \tag{4.27}$$

Proof. (i) Let $\lambda > h_D$. Assume that Ω is a solution to $m(\lambda, D)$. Since Ω solves $m(\lambda, D)$, it follows Proposition 4.6 (ii) that Ω contains all Cheeger sets of D, hence, $\Omega \supset C_D$.

On the other hand, if A is a Cheeger set of D then by (i), we infer that $A \subset C_D \subset \Omega \subset D$. Thus, A is also a Cheeger set of Ω. Conversely, supposed that A is now a Cheeger set of Ω, then as $C_D \subset \Omega$, it holds

$$\frac{P(A)}{|A|} \leq \frac{P(C_D)}{|C_D|} = h_D.$$

It shows by definition, that A is a Cheeger set of D. We conclude that A is a Cheeger set of Ω if and only if it is a Cheeger set of D. Consequently, $h_\Omega = h_D$.

(ii) We set $\Omega_\infty := \bigcap_{n\in\mathbb{N}} \Omega_n$. It is obvious (by (i)) that $C_D \subset \Omega_\infty$. By Proposition 4.7, Ω_∞ is a Cheeger set of D. Hence, $\Omega_\infty = C_D$ (note that C_D is the maximal Cheeger set of D). ■

Remark 4.9. The assertion iii) in Corollary 4.8 gives an alternative to the construction of the maximal Cheeger set proposed in [28]. Actually, we may extend the assertion iii) to any sequence (λ_n) satisfying $\lambda_n > h_D$ and $\lambda_n \to h_D$. And then the maximal Cheeger set of D is determined by

$$C_D = \bigcap_{\lambda > h_D} \{\Omega \ : \ \Omega \text{ solves } m(\lambda, D)\}.$$

4.2.3 Dual problem associated with $m(\lambda, D)$

In view of Remark 4.2, our goal is to evidence a dual problem where the unknown q is a vector field $q : D \to \mathbb{R}^N$. In order that this problem is easy to handle, we slightly change the convex integrand g_0^{**} which appears in (4.8). We consider the new convex function

$$\widetilde{g}_0(t) = -\lambda \min\{t, 1\} \qquad \forall t \in \mathbb{R}.$$

Then, it is easy to check that

$$m(\lambda, D) = \inf \left\{ \int_D (|\nabla u| + \widetilde{g}_0(u))\, dx \ : \ u \in W_0^{1,1}(D) \right\}. \tag{4.28}$$

Indeed, by a trivial truncation argument, we observe that the infimum in the right hand side is achieved for functions u taking values in the interval $[0, 1]$ on which $\widetilde{g}_0 = g_0$.

We now compute in the classical convex framework the dual problem of (4.28). As the usual qualification assumption is satisfied (see Theorem 1.31, Chapter 1), this dual problem reads

$$\sup \left\{ -\int_D (\widetilde{g}_0)^*(\operatorname{div} q) dx \ : \ q \in L^\infty(D; \mathbb{R}^N),\ |q| \leq 1 \text{ a.e. in } D \right\}. \tag{4.29}$$

As $(\widetilde{g}_0)^*(t^*) = t^* + \lambda$ for $t^* \in [-\lambda, 0]$, and $(\widetilde{g}_0)^*(t^*) = +\infty$ otherwise, we obtain

$$m(\lambda, D) = \sup \left\{ \int_D (\operatorname{div} q - \lambda) dx \ : \ q \in L^\infty(D; \mathbb{R}^N),\ |q| \leq 1,\ 0 \leq \operatorname{div} q \leq \lambda \right\}. \tag{4.30}$$

Theorem 4.10. *The dual problem* (4.29) *admits a solution* $\overline{q}$ *in* $L^\infty(D; \mathbb{R}^N)$ *and it holds* $m(\lambda, D) = \sup$ (4.29). *An optimal pair* $(\Omega, \overline{q})$ *for* $m(\lambda, D)$ *and* (4.29) *is characterized by the following conditions*

$$|\overline{q}| \leq 1 \quad \text{a.e. in } D, \qquad 0 \leq \operatorname{div} \overline{q} \leq \lambda \quad \text{a.e. in } D, \tag{4.31}$$

$$\overline{q} \cdot \nu_\Omega = 1 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \qquad \operatorname{div} \overline{q} = \lambda \quad \text{a.e. in } D \setminus \Omega. \tag{4.32}$$

Proof. Since our convex function $\widetilde{g}_0$ satisfies the qualification assumption ($\widetilde{g}_0$ is continuous), we deduce from the classical duality theory that the dual problem (4.29) attains its maximum at some $\overline{q} \in L^\infty(D;\mathbb{R}^N)$ and $m(\lambda, D) = \sup(4.29)$. We refer to [26], [57] for more details on these duality results. Notice that Ω is a solution to $m(\lambda, D)$ if and only if $\mathbf{1}_\Omega$ is a solution of the variational problem (4.28), which is identical to $\beta_0(\lambda)$. Then, the optimal pair $(\Omega, \overline{q})$ for $m(\lambda, D)$ and (4.29) is characterized by applying the classical duality results to the dual pair $(\mathbf{1}_\Omega, \overline{q})$ which gives the conditions (4.31)-(4.31). ■

Theorem 4.10 provides optimality conditions (4.31)-(4.32) from which minimizers of the geometric problem $m(\lambda, D)$ will be characterized once a calibrating field $\overline{q}$ exists and is determined. In particular, in light of Remark 4.2, the N-dimensional vector field $\overline{q}$ may be used to create an $(N+1)$-calibration for the problem $\beta_0(\lambda)$ defined in (4.4). In Section 4.3, we shall describe an explicit construction of a 2-dimensional field $\overline{q}$ in support of a potential associated with the distance to the cut-locus. But, before doing that, we present in the subsection below a generalized notion of classical calibrable sets, the so-called θ-calibrable sets. These sets are actually optimal shapes of the geometric problem $m(\lambda, D)$.

4.2.4 θ-calibrable bodies

The shape optimization problem $m(\lambda, D)$ maintains pretty many interesting properties of the Cheeger problem. For instance, Cheeger sets are calibrable sets in the sense introduced by Caselles-Chambolle et Al. in [4] while solutions of the problem $m(\lambda, D)$ are calibrable but in a more general sense as stated in the following definition.

Definition 4.11. Let $\theta \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded set of finite perimeter. Ω is called *θ-calibrable* if there exists a vector field $q \in L^\infty(\Omega;\mathbb{R}^N)$ such that

$$\|q\|_\infty \leq 1, \qquad q \cdot \nu_\Omega = 1 \ \ \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \qquad 0 \leq \operatorname{div} q \leq \theta\lambda_\Omega \ \text{ in } \ \mathcal{D}'(\Omega).$$

Notice that the condition $q \cdot \nu_\Omega = 1$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ can be understood as $q \cdot D\mathbf{1}_\Omega = -|D\mathbf{1}_\Omega|$ in the sense of measure in $\overline{\Omega}$. It is easy to see that if Ω is θ-calibrable then it is also γ-calibrable for every $\gamma \geq \theta$. Thus, we define the *calibration constant* of a subset $\Omega \subset \mathbb{R}^N$ by the minimum value of $\theta \geq 1$ such that Ω is θ-calibrable, namely

$$\theta_\Omega := \inf\{\theta \ : \ \theta \geq 1, \ \Omega \text{ is } \theta\text{-calibrable}\}. \tag{4.33}$$

Naturally, we have $\theta_\Omega \geq 1$. In the special case where $\theta_\Omega = 1$, we find that Ω is actually a Cheeger set, in other words, *calibrable* in the sense introduced in [4]:

Definition 4.12. Let Ω is a bounded set of finite perimeter in $\mathbb{R}^N$. Ω is said to be *calibrable* if there exists a vector field $q \in L^\infty(\Omega;\mathbb{R}^N)$ such that

$$\|q\|_\infty \leq 1, \qquad q \cdot \nu_\Omega = 1 \ \ \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \qquad \operatorname{div} q = \lambda \ \text{ in } \ \mathcal{D}'(\Omega),$$

for some constant $\lambda \in \mathbb{R}$.

We remark that the constant λ appearing in Definition 4.12 is not an arbitrary real value but determined by $\lambda = \lambda_\Omega := P(\Omega)/|\Omega|$. In fact, once Ω is calibrable, it turn out that

$$\lambda|\Omega| = \int_{\mathbb{R}^N} (\operatorname{div} q)\mathbf{1}_\Omega dx = \int_{\partial\Omega} q \cdot \nu_\Omega d\mathcal{H}^{N-1} = P(\Omega)$$

which yields $\lambda = \lambda_\Omega$.

Proposition 4.13. *Let $\theta \geq 1$, and $\Omega \subset \mathbb{R}^N$ be a bounded set of finite perimeter. The following assertions are equivalent:*

(i) Ω *is* θ*-calibrable,*

(ii) Ω *is a minimizer of* $m(\theta\lambda_\Omega, \Omega)$.

Proof. $(i) \Rightarrow (ii)$. Assume that Ω is θ-calibrable. By definition of θ-calibrability, for every $A \subset \Omega$ being a subset of finite perimeter, we have $\operatorname{div} q - \theta\lambda_\Omega \leq 0$ a.e. in Ω. This implies that

$$\int_\Omega (\operatorname{div} q - \theta\lambda_\Omega)\mathbf{1}_A dx \geq \int_\Omega \operatorname{div} q \, dx - \theta\lambda_\Omega|\Omega| = P(\Omega) - \theta\lambda_\Omega|\Omega|.$$

Since $\int_\Omega \operatorname{div} q\mathbf{1}_A dx = \int_{\partial A} q \cdot \nu_A d\mathcal{H}^{N-1} \leq P(A)$, we obtain immediately

$$P(A) - \theta\lambda_\Omega|A| \geq P(\Omega) - \theta\lambda_\Omega|\Omega|.$$

It is to say that Ω minimizes $m(\theta\lambda_\Omega, \Omega)$.

$(ii) \Rightarrow (i)$. We now assume that Ω is a solution of $m(\theta\lambda_\Omega, \Omega)$. According to Theorem 4.10, there exists $q \in X_1(\Omega; \mathbb{R}^N)$ with $\|q\|_\infty \leq 1$ such that $0 \leq \operatorname{div} q \leq \theta\lambda_\Omega$ a.e. in Ω, and

$$P(\Omega) - \theta\lambda_\Omega|\Omega| = \int_\Omega (\operatorname{div} q - \theta\lambda_\Omega) dx.$$

This implies that $q \cdot \nu_\Omega = 1$ $D\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$. We showed that Ω is a θ-calbrable set. ■

Corollary 4.14. *There hold*

(i) A bounded finite-perimeter set Ω *in* $\mathbb{R}^N$ *is 1-calibrable if and only if it is a self-Cheeger set.*

(ii) 1-calibrability coincides with the clasical calibrability in the sense of Definition 4.12.

Proof. (i) It is easy to figure out that Ω is a minimizer of the problem $m(\lambda_\Omega, \Omega)$ is equivalent to that it is a self-Cheeger set. Indeed, we observe that for every $A \subset \Omega$ with finite perimeter and $|A| > 0$,

$$P(\Omega) - \lambda_\Omega|\Omega| \leq P(A) - \lambda_\Omega|A| \Longleftrightarrow \frac{P(\Omega)}{|\Omega|} \leq \frac{P(A)}{|A|},$$

which means $\lambda_\Omega = \inf\{P(A)/|A| \ : \ A \subset \Omega\} = h_\Omega$.

(ii) When Ω is a 1-calibrable set, we have $0 \leq \operatorname{div} q \leq \lambda_\Omega$ a.e. in Ω and $q \cdot \nu_\Omega = 1$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ so that

$$\lambda_\Omega = \frac{P(\Omega)}{|\Omega|} = \frac{1}{|\Omega|}\int_\Omega \operatorname{div} q.$$

We see that $(\lambda_\Omega - \operatorname{div} q)$ is a positive Radon measure and $\int_\Omega(\lambda_\Omega - \operatorname{div} q) = 0$. This means $\operatorname{div} q = \lambda_\Omega$ in $\mathcal{D}'(\Omega)$. ■

Characterization of θ-calibrable sets

We provide here a characterization of θ-calibrable sets among the class of finite perimeter subsets in $\mathbb{R}^N$.

Proposition 4.15. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded subset of finite perimeter. The following conditions are equivalent:*

(i) Ω *is a solution for problem* $m(\theta\lambda_\Omega, \Omega)$ *for some* $\theta \geq 1$.

(ii) $\mathbf{1}_\Omega$ *is a relaxed solution for the variational problem defined by (4.28) for* $\lambda = \theta\lambda_\Omega$.

(iii) There exists a vector field $q \in L^\infty(\Omega;\mathbb{R}^N)$ *such that*

$$|q| \le 1 \text{ a.e. in } \Omega, \quad 0 \le \operatorname{div} q \le \theta\lambda_\Omega \text{ a.e. in } \Omega, \quad q\cdot\nu_\Omega = 1 \ \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{4.34}$$

In the case Ω *being convex and of class* $C^{1,1}$*, these three conditions are indeed equivalent to*

(iv) It holds

$$\max\left\{1, (N-1)\kappa_\infty(\partial\Omega)\lambda_\Omega^{-1}\right\} \le \theta, \tag{4.35}$$

where $\kappa_\infty(\partial\Omega)$ *denotes the* L^∞*-norm of the mean curvature of* $\partial\Omega$*.*

The equivalences between the triple conditions (i)-(ii)-(iii) in Proposition 4.15 mean significantly that all level sets $\{\overline{u} > t\}$, $t \in (0,1)$, of a relaxed solution $\overline{u}$ of the variational problem (4.28) (which is identical to $\beta_0(\lambda)$) are $\lambda\lambda_\Omega^{-1}$-calibrable sets if $\lambda \ge h_D$. The geometric characterization (4.35) is derived from the result below.

Proposition 4.16 ([4])**.** *Let* Ω *be a bounded convex subset in* $\mathbb{R}^N$ *of class* $C^{1,1}$*. Let* $\widetilde{\Omega}$ *be the solution of* $m(\lambda,\Omega)$ *with* $\lambda > 0$*. Then,* $\widetilde{\Omega} = \Omega$ *if and only if* $\lambda \ge \max\{\lambda_\Omega, (N-1)\kappa_\infty(\partial\Omega)\}$*.*

For more details on this geometric characterization, we refer to the study of F. Alter, V. Caselles, and A. Chambolle in [4]. In what follows, we show that solutions of the geometric problem $m(\lambda, D)$ are θ-calibrable sets.

Remark 4.17. The condition (iv) in Proposition 4.15 is actually to say that the calibration constant of a nonempty bounded open subset Ω in $\mathbb{R}^N$ of class $C^{1,1}$ is determined by

$$\theta_\Omega = \max\left\{1, (N-1)\kappa_\infty(\partial\Omega)\lambda_\Omega^{-1}\right\}.$$

Proposition 4.18. *Let* $\Omega \subset \mathbb{R}^N$ *be a finite perimeter subset with* $|\Omega| > 0$*. Then, if* Ω *solves* $m(\lambda, D)$ *for some* $\lambda \ge 0$*, it holds* $\lambda_\Omega \le \lambda$*. As a consequence,* Ω *is a* θ*-calibrable set with* $\theta = \lambda\lambda_\Omega^{-1}$*.*

Proof. Suppose that Ω solves problem $m(\lambda, D)$. Then, Ω automatically solves $m(\lambda,\Omega)$, see Proposition 4.6. If $\lambda \ge \lambda_\Omega$, we have Ω is θ-calibrable with $\theta = \lambda\lambda_\Omega^{-1}$. If $\lambda < \lambda_\Omega$, Ω will solve $m(\lambda_\Omega,\Omega)$, thus Ω is 1-calibrable, or equivalently, a Cheeger set of D. It turns out that $\lambda = \lambda_\Omega = h_D$. ∎

Remark 4.19. We shall see later that solutions Ω of problem $m(\lambda, D)$ in 2 dimensions are indeed θ-calibrable sets with their calibration constants determined by $\theta_\Omega = \lambda\lambda_\Omega^{-1}$.

Eventually, let us give some interesting comments. The notion of θ-calibrability as we have seen, covers the classical conception of calibrability while the calibration constant θ_Ω intuitively represents how far Ω stretches out from its Cheeger set. In this context, when θ_Ω is near the critical value 1, we say Ω concentrates on its Cheeger set. Otherwise, as θ_Ω is near $+\infty$, we say that it diffuses. For instance, we consider the family of non void solutions of the geometric problem $m(\lambda, D)$ for a convex nonsmooth body D in $\mathbb{R}^N$ with it curvature $\kappa_\infty(\partial D) = +\infty$ (it means D has sharp corners on its boundary). We denote this family by $\{\Omega(\theta)\}_\theta$ with $\theta = \lambda\lambda_\Omega^{-1}$, $\lambda \in [h_D, +\infty]$. Then, the family is a increasing sequence of sets which continuously fills the space between the maximal Cheeger set of D and D itself, namely

$$C_D = \Omega(1) \subset \ldots \subset \Omega(\theta) \subset \ldots \subset \Omega(+\infty) = D.$$

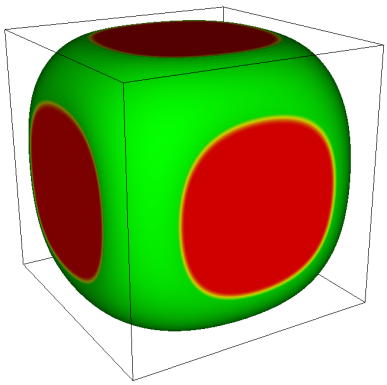


Figure 4.2: A θ-calibrable set of the unit cube in three dimensions.

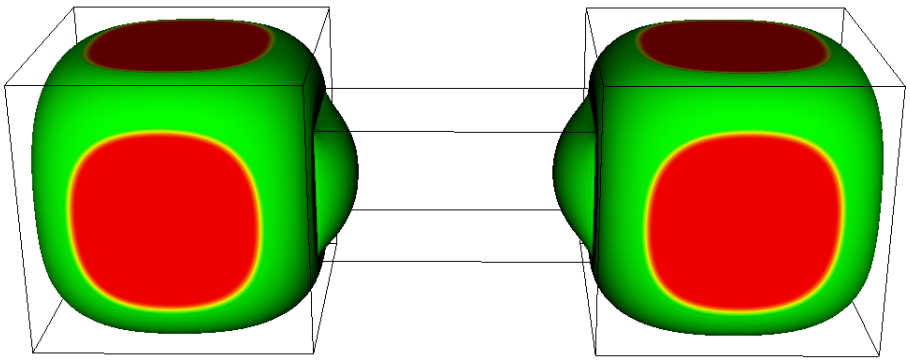


Figure 4.3: A θ-calibrable set of nonconvex domain in three dimensions.

We illustrate in Figures 4.2 and 4.3 θ-calibrable sets of convex and non convex domains in $\mathbb{R}^3$. They are obtained by using the primal-dual algorithm discussed in Chapter 5 to solve the variational problem (4.28). Figure 4.2 is done with D being the unit cube and $\lambda = 5.5$. In Figure 4.3, the non convex domain D is generated by a pair of unit cube joining with a small rectangular solid tube, and (4.28) is solved with $\lambda = 5.8$. In both figures, the green surfaces present the boundary of θ-calibrable sets inside D, and the red regions indicate their common boundaries with ∂D.

4.3 Cut-locus potential and an explicit construction of 2-dimensional calibrating fields

The following metric notions will be used. For x, y being two points in $\mathbb{R}^2$, the segment joining them denoted by $[x, y] := \{(1-t)x + ty \ : \ t \in [0,1]\}$. The distance between them is measured as $|x - y|$. For every nonempty subset A of $\mathbb{R}^2$, we denote by $d(x, A)$ the distance from x to A, namely

$$d(x, A) := \inf \{|x - y| \ : \ y \in A\}.$$

If $x \in A$, then $d(x, A) = 0$. The distance between two nonempty subsets A, B of $\mathbb{R}^2$ is given by $d(A, B) := \inf \{d(x, B) \ : \ x \in A\}$.

In this section we assume that D is a bounded convex open subset of $\mathbb{R}^2$.

The distance function to the complement of D will be given by $d(\cdot, D^c)$. Let R be the inradius of D, given by $R = \max_{x\in D} d(x, D^c)$. For every δ such that $0 \le \delta < R$, we set

$$D^\delta := \{x \in D \ : \ d(x, D^c) > \delta\}.$$

We have $D^0 = D$. It is easy to see that D^δ is a subset of D, whose boundary is the inner parallel set to ∂D at distance δ. Thus, D^δ is convex as D (a formal proof can be found in the proof of Lemma 4.21). We denote by D^R the so-called *core* of D,

$$D^R := \bigcap_{0<\delta<R} D^\delta = \{x : B(x, R) \subset \overline{D}\}.$$

Hereafter, $B(x, r)$ denotes the open ball in $\mathbb{R}^2$ centered at x with radius $r > 0$.

For all $\lambda \ge R^{-1}$, we define a specified subset of D, denoted by Ω_λ, which is the union of all balls of radius λ^{-1} contained in D, namely

$$\Omega_\lambda = \bigcup_{B(x,\lambda^{-1})\subset D} B(x, \lambda^{-1}). \tag{4.36}$$

Evidently, Ω_λ coincides with the union of the family of balls of radius $\delta = 1/\lambda$ centered in D^δ. We call $\Omega_{\frac{1}{R}}$ *central set* of D. We give in the following results on the convexity and θ-calibrability of Ω_λ. Let us begin with recalling the characterization of calibrable sets (Cheeger sets) in two dimensions since they are special cases of θ-calibrability.

4.3.1 Cheeger sets in $\mathbb{R}^2$

In 2-dimensional case, Cheeger set of a convex set is convex and uniquely defined as the union of a family of disks of suitable radius, for more details we refer to [102, Theorem 3.32 i)]. The radius was then identified as $1/h_D$.

Following is a well-known result we can find in the work of by Bernd Kawohl and Thomas Lachand-Robert in [79]. We also recall here its proof for convenience.

Theorem 4.20. *There exists a unique value $\delta = \delta^*$ such that $|D^\delta| = \pi\delta^2$. Then, $h_D = 1/\delta^*$ and the Cheeger set of D is $C_D = \cup\{B(x, \delta^*) : B(x, \delta^*) \subset D\}$.*

Proof. Let R be the radius of D. It is easy to see that $|D^\delta|$ and $\pi\delta^2$ are monotone functions in δ, which are strictly decreasing and strictly increasing respectively in $(0, R)$. So, there is a unique $\delta^* \in (0, R)$ such that $|D^\delta| = \pi\delta^2$.
Let C is the Cheeger set of D. Then, C is the union of disks containing in D of radius $\delta = 1/h_D$, [102, Theorem 3.32 i)]. Let $C^\delta := \{x \in C : d(x, C^C) > \delta\}$. By Steiner's formulae, we typically have

$$|C| = |C^\delta| + \delta P(C^\delta) + \pi\delta^2 \tag{4.37}$$

$$P(C) = P(C^\delta) + 2\pi\delta. \tag{4.38}$$

We will show that $1/h_D = \delta^*$. Since C is Cheeger set of D, by definition, $1/\delta = h_D = P(C)/|C|$ or $\delta P(C) = |C|$. Use (4.37) and (4.38), we can derive that $|C^\delta| = \pi\delta^2$. Let us figure out that $C^\delta = D^\delta$, and then, this completes the proof by the uniqueness of δ^*. For any $x \in C^\delta$, as $C \subset D$, $d(x, D^C) \ge d(x, C^C) > \delta$. This shows $C^\delta \subset D^\delta$. Inversely, if $x \in D^\delta$ then $B(x, \delta) \subset D$ would be contained in C. Hence, x is in C^δ. ∎

4.3.2 Characterization of θ-calibrable sets in $\mathbb{R}^2$. The set Ω_λ

Inspired from Theorem 4.20, θ-calibrable sets in $\mathbb{R}^2$ are also formed by a cover of balls with certain radius.

Lemma 4.21. *Let $\lambda > 0$ and $\theta = \lambda\lambda_{\Omega_\lambda}^{-1}$. Then Ω_λ defined by (4.36) is convex and θ-calibrable.*

Notice that by the characterization of θ-calibrability discussed in Section 4.2.4, Ω_λ itself minimizes problem $m(\lambda, \Omega_\lambda)$.

Proof. First of all, we will prove that Ω_λ is convex. We observe that D^δ is a convex set. In fact, D^δ coincides with $\{x \in D : B(x,\delta) \subset D\}$ and this is a convex set. For any $x_1, x_2 \in D^\delta$, there exist $B(x_1,\delta), B(x_2,\delta) \subset D$. It is easy to see that

$$B(\frac{x_1+x_2}{2}, \delta) \subset \operatorname{conv}\left(B(x_1,\delta) \cup B(x_2,\delta)\right) \subset \operatorname{conv}(D) = D.$$

Thus, $(x_1+x_2)/2$ is in D^δ. Now, for every $x, y \in \Omega_\lambda$, there exist $B(x_0,\lambda^{-1}), B(y_0,\lambda^{-1}) \subset D$ such that $x \in B(x_0,\lambda^{-1}), y \in B(y_0,\lambda^{-1})$. Then, we get $\dfrac{x+y}{2} \in B(\dfrac{x_0+y_0}{2}, \dfrac{1}{\lambda})$ since

$$\left|\frac{x+y}{2} - \frac{x_0+y_0}{2}\right| \le \left|\frac{x-x_0}{2}\right| + \left|\frac{y-y_0}{2}\right| \le \frac{1}{\lambda}.$$

As D^δ is convex, we have $B(\dfrac{x_0+y_0}{2}, \dfrac{1}{\lambda}) \subset \Omega_\lambda$. So, $\dfrac{x+y}{2}$ is in Ω_λ.

Secondly, by Proposition 4.15 (iv), we shall prove that $\kappa_\infty(\partial\Omega_\lambda) \ge \lambda_{\Omega_\lambda}$ so that Ω_λ is $\lambda\lambda_{\Omega_\lambda}^{-1}$-calibrable. For the sake of simplicity, let us set, only in this proof, $\delta = \lambda^{-1}$ and $\delta^* = h_D^{-1}$. Since Ω_λ is a union of disks of radius δ containing in D, we obtain by Steiner's formulae [74],

$$|\Omega_\lambda| = |D^\delta| + \delta P(D^\delta) + \pi\delta^2, \tag{4.39}$$

$$P(\Omega_\lambda) = P(D^\delta) + 2\pi\delta, \tag{4.40}$$

on the other hand, the curvature of $\partial\Omega_\lambda$ is finite and $\kappa_\infty(\partial\Omega_\lambda) = \lambda$. Hence, it suffice to show that

$$\lambda_{\Omega_\lambda} = \frac{P(\Omega_\lambda)}{|\Omega_\lambda|} \le \lambda. \tag{4.41}$$

By substituting (4.39)-(4.40) in (4.41), we get inequalities which are equivalent to (4.41),

$$\frac{|D^\delta| + \delta P(D^\delta) + \pi\delta^2}{P(D^\delta) + 2\pi\delta} \ge \delta \iff \frac{|D^\delta| - \pi\delta^2}{\delta P(D^\delta) + 2\pi\delta^2} \ge 0 \iff |D^\delta| - \pi\delta^2 \ge 0$$

The last inequality is true as $\delta \le \delta^*$. In fact, since $D^{\delta^*} \subset D^\delta$, we have, by Theorem 4.20,

$$|D^\delta| \ge |D^{\delta^*}| = \pi(\delta^*)^2 \ge \pi\delta^2.$$

We close the proof with repeating the conclusion that Ω_λ is $\lambda\lambda_{\Omega_\lambda}^{-1}$-calibrable, in other words, Ω_λ itself solves problem $m(\lambda, \Omega_\lambda)$. ∎

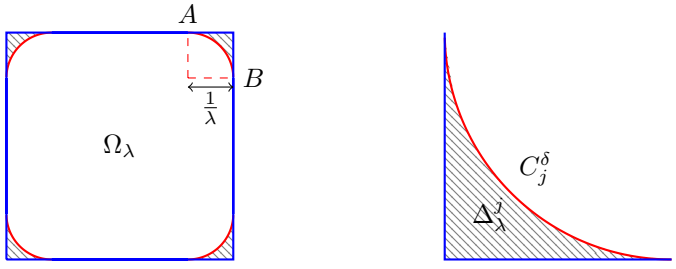


Figure 4.4: Configuration of Ω_λ inside a square domain D.

Figure 4.4 gives a configuration for the $\lambda\lambda_{\Omega_\lambda}^{-1}$-calibrable subset Ω_λ of a square domain D in $\mathbb{R}^2$. Note that $\kappa_\infty(\partial D) = +\infty$. The infinite curvature occurs at corners of the square. Ω_λ is constructed by four circular arcs which tangentially cover the four corners in the way to remove the infinity of curvature. These circular arcs are uniquely determined by their positions (meaning being tangent to ∂D) and their radii of λ^{-1}. Thanks to the geometric characterization (4.35), it is clear to say that the calibration constant of Ω_λ is given by $\theta_{\Omega_\lambda} = \lambda\lambda_{\Omega_\lambda}^{-1}$. When $\lambda = h_D$, we get the Cheeger set of D and $\theta_{\Omega_\lambda} = 1$. As discussed in Section 4.2.4, the family $\{\Omega_\lambda\}_\lambda$ is a increasing sequence of θ_{Ω_λ}-calibrable sets which continuously spread out from the Cheeger kernel C_D to reach D. This evolution corresponds to the slide of calibration constant θ_{Ω_λ} in the rail from 1 to $+\infty$.

In Figure 4.5, we give a numerical θ- calibrable set of a non convex subset in $\mathbb{R}^2$. The non convex subset comprises two unit square linking together by a small rectangle. This θ-calibrable set is obtained by solving numerically the variational problem $\widetilde{m}(\lambda, D)$ with $\lambda = 3.94$ by the primal-dual algorithms described in Chapter 5.

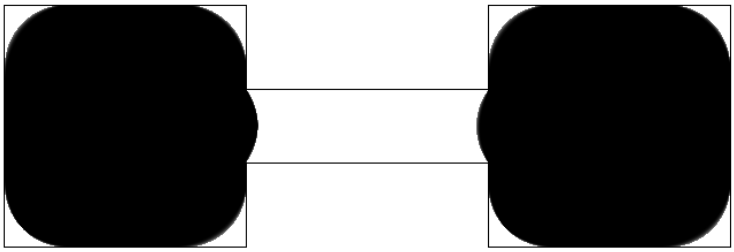


Figure 4.5: A θ-calibrable set of nonconvex domain in two dimensions.

Is an ellipse domain θ-calibrable ?

Let us observe an ellipse in its standard form

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

And, D is defined as $D = \{x \in \mathbb{R}^2 : x_1^2/a^2 + x_2^2/b^2 < 1\}$. By the geometric characterization (4.35), as an ellipse always have finite curvature, it is certainly θ-calibrable for some $\theta \geq 1$. The question to investigate is when an ellipse is θ-calibrable ?

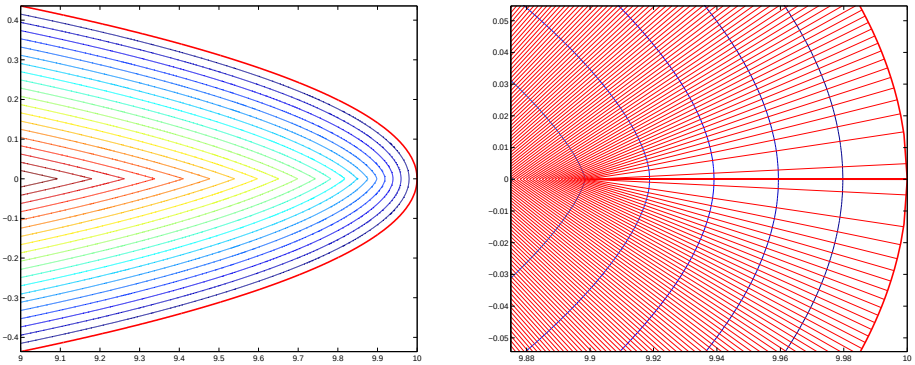


Figure 4.6: Ellipse domain $D = \{x \in \mathbb{R}^2 : 0.1x_1^2 + x_2^2 < 1\}$ and its D^δ subsets. Red lines in the right subfigure are geodesic distance to ∂D, which are orthogonal to ∂D^δ.

Take for instance, an ellipse stretched out so that the maximum curvature is large enough and $\kappa_\infty(\partial D) > h_D$. Then, it is not a self-Cheeger set. By construction of Ω_λ, we shall obtain a family of θ_{Ω_λ}-calibrable sets with their calibration constants θ_{Ω_λ} being bounded, i.e.

$1 \le \theta_{\Omega_\lambda} \le \theta_D < +\infty$. In Figure 4.6, we present $D = \{x \in \mathbb{R}^2 : 0.1x_1^2 + x_2^2 < 1\}$ and D^δ subsets. As we can see, for some δ' large enough, $D^{\delta'}$ occurs singular points on its boundary. And, if h_D^{-1} is less than all such δ', D is self-Cheeger set. Otherwise, D must be θ_D-calibrable for some $\theta_D > 1$.

Let us turn back to answer precisely the question when a general ellipse domain D is θ-calibrable. We restrict our case in the first quadrant of the 2-dimensional Cartesian coordinate system so that we have a parameterization of the ellipse,

$$x_2 = \frac{b}{a}\sqrt{a^2 - x_1^2}\,.$$

Following calculations are first derivatives and then, the mean curvature of the elliptic curve,

$$x_2' = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x_1^2}}\,, \quad x_2'' = -\frac{ba}{(a^2 - x_1^2)^{\frac{3}{2}}}\,, \quad k(x_1) = \frac{x_2''}{(1 + x_2'^2)^{\frac{3}{2}}} = -\frac{ba^4}{[a^4 - x_1^2(a^2 - b^2)]^{\frac{3}{2}}}\,.$$

Clearly, its maximal mean curvature is attained at $x = a$. Thus, $\kappa_\infty(D) = |k(a)| = a/b^2$. Besides, the area of the ellipse is formulated with

$$|D| = \frac{b}{a}\int_{-a}^{a} 2\sqrt{a^2 - s^2}ds = \pi ab.$$

According to Proposition 4.15, it is now reasonably said that the ellipse D is a θ-calibrable set if and only if

$$P(D) \ge \frac{\pi a^2}{b\theta}\,. \tag{4.42}$$

On the other hand, the circumference of the ellipse is given by $P(D) = 4aE(e)$, where e is the eccentricity $\sqrt{1 - b^2/a^2}$, and $E(e)$ is a complete elliptic integral of the second kind,

$$E(e) := \int_0^{\pi/2} \sqrt{1 - e^2\sin^2\varphi}d\varphi. \tag{4.43}$$

It turns out that inequality (4.42) is equivalent to

$$\frac{bE(e)}{a} \ge \frac{\pi}{4\theta}\,. \tag{4.44}$$

By setting $t := b/a$, we can verify that function $G(t) := tE(\sqrt{1 - t^2})$ is monotonically increasing in the unit interval $(0, 1)$. Or, $G(\sqrt{1 - e^2})$ monotonically decreases in e in $(0, 1)$. Therefore, it is easy to see that for each θ given in $[1, +\infty)$, the inequality (4.44) is satisfied for any ellipse with eccentricity smaller than the critical value e_θ where the equality of (4.44) happens, see Figure 4.7 for visualization.

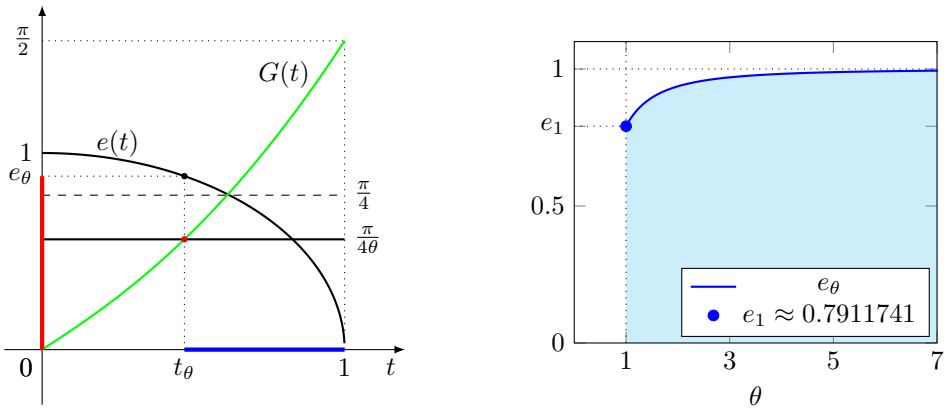


Figure 4.7: Is an ellipse domain θ-calibrable?

In Figure 4.7, t_θ is the solution of $G(t) = \pi/(4\theta)$. The blue interval $(t_\theta, 1)$ with

$$(t_\theta, 1) := \{t \in (0,1) : G(t) \geq \pi/(4\theta)\}$$

defines the red interval $(0, e_\theta)$ which represents all of ellipses with eccentricity less than e_θ. Of course, the quantity e_θ can be easily approximated by available numerical methods. As a direct result, every ellipse with eccentricity less than e_1 (in this case, $\theta = 1$) is a self-Cheeger set. And the inequality (4.44) reduces to be $\sqrt{1-e^2}E(e) \geq \pi/4$. At that moment, e_1 can be numerically estimated by 0.7911741. If we carefully regard in Figure 4.7, as θ tends to infinity, e_θ approaches 1. The interval $(0, e_\theta)$ is then enlarged to fit the unit interval $(0,1)$. In detail, e_θ belongs to $(0, e_\mu)$ for all $\mu \geq \theta$. It is to say that if D is θ-calibrable, it is immediately μ-calibrable. This emphasizes again an attractive illustration of Proposition 4.6 (iii).

4.3.3 Cut-locus potential

Let $\mathbf{n} : \partial D^\delta \to S^1$ be the Gauss map assigning each point x on the boundary of D^δ with its normal $\nu_{D^\delta}(x)$ which is in the unit sphere S^1 ($\mathbf{n}$ is universal, thus, we prefer ignoring its dependence on D^δ, i.e. using $\mathbf{n}$ instead of $\mathbf{n}_{\partial D^\delta}$ for simplicity). We call $\varphi^\delta(x)$ the angle of $\nu_{D^\delta}(x)$ in S^1. As D^δ is a convex set, the singular set of $\mathbf{n}$ is at most countable. We denote it by $\partial_s D^\delta := \{x_j : j \in J\}$. Then, the regular part of ∂D^δ is given by $\partial_r D^\delta := \partial D^\delta \setminus \partial_s D^\delta$. For each singular point x in $\partial_s D^\delta$, the Gauss map $\mathbf{n}$ has its limits inferior and superior which will be respectively denoted as $\nu^-_{D^\delta}(x)$ and $\nu^+_{D^\delta}(x)$. Accordingly, $\varphi^\delta_-(x)$ and $\varphi^\delta_+(x)$ are defined. For each $x \in \overline{D}$, we denote by $N_{D^\delta}(x)$ the *normal cone* of D^δ at x, namely

$$N_{D^\delta}(x) := \left\{x^* \in \mathbb{R}^2 \ : \ \langle x^*, y - x\rangle \leq 0, \ \forall y \in D^\delta\right\}.$$

The normal cone $N_{D^\delta}(x)$ will reduce to $\{0\}$ if x is inside D^δ. When x is on the boundary of D^δ, the normal cone is generated by the unit vector $\nu^-_{D^\delta}(x)$ and $\nu^+_{D^\delta}(x)$, that is

$$N_{D^\delta}(x) := \left\{a\nu^+_{D^\delta}(x) + b\nu^-_{D^\delta}(x) \ : \ a, b \in \mathbb{R}_+\right\}.$$

If $x \in \partial_r D^\delta$, it happens $\mathbf{n}(x) = \nu^+_{D^\delta}(x) = \nu^-_{D^\delta}(x)$ and $N_{D^\delta}(x)$ is the positive ray spanned by $\nu_{D^\delta}(x)$, i.e. $N_{D^\delta}(x) = \mathbb{R}_+\nu_{D^\delta}(x)$. We then define

$$N^\delta(x) := x + \{p \in N_{D^\delta}(x) \ : \ |p| \leq \delta\}, \text{ for } x \in \overline{D^\delta}, \tag{4.45}$$

$$C^\delta(x) := x + \{p \in N_{D^\delta}(x) \ : \ |p| = \delta\}, \text{ for } x \in \partial D^\delta. \tag{4.46}$$

For every δ, δ' such that $0 \leq \delta' < \delta < R$, we define

$$M^\delta_{\delta'}(x) := x + \left\{p \in N_{D^\delta}(x) \ : \ \left\langle p, \nu^\pm_{D^\delta}(x)\right\rangle \leq \delta - \delta'\right\}, \text{ for } x \in \overline{D^\delta}. \tag{4.47}$$

For all $x \in \overline{D}$ and for A being a closed subset of $\mathbb{R}^2$, $\Pi_A(x)$ denotes the orthogonal projection in $\mathbb{R}^2$ of x on A. We note by $\Pi_\delta(x)$ the projection in $\mathbb{R}^2$ of x on the closure $\overline{D^\delta}$.

All notations above can be defined for every convex nonempty subset of D, but for convenience, we just recall them on the family of special subsets $\{D^\delta\}_\delta$. We begin in the followings with some preparatory lemmas before discussing about cut-locus potential.

Some preparatory lemmas

The lemma below provides the projection on the boundary of a closed convex set from inside.

Lemma 4.22. *Let Ω be a closed convex subset of $\mathbb{R}^2$. Let $y \in \Omega$ and $y^* \in \partial\Omega$ be a projection of y on $\partial\Omega$, i.e.:*

$$|y - z| \geq |y - y^*| \quad \forall z \in \partial\Omega. \tag{4.48}$$

Then, it holds $\langle x - y^*, y - y^*\rangle \geq 0$ for every $x \in \Omega$.

Proof. We may assume that $y \in \operatorname{int}\Omega$ so that $y \neq y^*$. Under (4.48), it is clear that Ω contains the ball $B(y, |y-y^*|)$. Therefore the normal cone $N_\Omega(y^*)$ (which is non reduced to $\{0\}$) must be a subset of the normal cone to $B(y, |y-y^*|)$ i.e. of $\mathbb{R}_+(y^*-y)$. ∎

Let Λ be the singular set of $d(\cdot, D^c)$ (i.e. the set of point $x \in D$ such that $d(\cdot, D^c)$ is not differentiable at x, or equivalently such that $\Pi_{D^c}(x)$ is not a singleton). The closure of Λ in $\overline{D}$ is called the *cut-locus of* D and denoted by $\overline{\Lambda}$. Let $\gamma : \overline{D} \to [0, R]$ be geodesic distance passing x from ∂D to cut-locus $\overline{\Lambda}$, namely

$$\gamma : x \in \overline{D} \mapsto \zeta(x) + d(x, D^c),$$

where $\zeta(x)$ is the *normal distance* to cut-locus $\overline{\Lambda}$ from x, i.e.

$$\zeta(x) := \begin{cases} \min\{t \geq 0 : x + t\nabla d(x, D^c) \in \overline{\Lambda}\} & \forall x \in \overline{D} \setminus \overline{\Lambda}, \\ 0 & \forall x \in \overline{\Lambda}. \end{cases} \tag{4.49}$$

Here, $\nabla d(x, D^c)$ denotes the gradient of the distance function, and hence, it is a unit vector providing the direction of the geodesic distance starting from x to cut-locus $\overline{\Lambda}$. We refer to [42, 41, 43, 40] for more details on this topic.

Lemma 4.23. *If $x \in \overline{D}$ and $\delta = \gamma(x)$ then*

$$P_{D^\delta}(x) = \begin{cases} x & \text{if } x \in \overline{\Lambda}, \\ x + \zeta(x)\nabla d(x, D^c) & \text{if } x \in \overline{D} \setminus \overline{\Lambda}. \end{cases}$$

Proof. If $x \in \overline{\Lambda}$, x is in $\overline{D^\delta}$. Assume now $x \in \overline{D} \setminus \overline{\Lambda}$. Let $x_* \in \overline{\Lambda}$ such that

$$x_* = x + \zeta(x)\nabla d(x, D^c).$$

It is evidently that $|x - x_*| = \zeta(x)|\nabla d(x, D^c)| = \zeta(x)$. Besides, we observe

$$\gamma(x) = d(\Pi_\delta(x), D^c) \leq d(\Pi_\delta(x), x) + d(x, D^c).$$

This implies that

$$d(\Pi_\delta(x), x) \geq \zeta(x) = |x - x_*|$$

Thus, by uniqueness of projection, $\Pi_\delta(x)$ must coincide with x_*. ∎

Lemma 4.24. *Let D be a bounded convex set in $\mathbb{R}^N$. Given $x \in D$ such that $d(x, D^\delta) \leq \delta$ for some $\delta > 0$. Then, for every δ' such that $0 < \delta' < \delta$, we have $d(x, D^{\delta'}) < \delta'$.*

Proof. It is trivial if $x \in D^\delta$, since $D^\delta \subset D^{\delta'}$ for any $\delta' < \delta$. We now assume that $x \notin D^\delta$. Let x_δ and $x_{\delta'}$ be the projections of x on $\overline{D^\delta}$, $\overline{D^{\delta'}}$, respectively. As $d(x, D^\delta) \leq \delta$, x is contained in $\overline{B(x_\delta, \delta)}$. Let us show that we can always find out a ball $B(z, \delta') \subset B(x_\delta, \delta)$ such that $x \in \overline{B(z, \delta')}$ and z is an interior point of $D^{\delta'}$. And hence, by the fact that x has a unique projection on the boundary of the closed convex set $\overline{D^{\delta'}}$, the lemme is proved thanks to

$$d(x, D^{\delta'}) = |x - x_{\delta'}| < |x - z| \leq \delta'.$$

If $d(x, D^\delta) \leq \delta'$. Take $z = x_\delta$, we have $x \in \overline{B(z, \delta')} \subset B(x_\delta, \delta) \subset D$. Then, $d(z, D^c) > \delta'$, it means $z \in D^\delta$. Otherwise, $\delta' < d(x, D^\delta) \leq \delta$. We set

$$z = (1-t)x + tx_\delta, \text{ for some } t \in [0, 1].$$

We have $|x-z| = t|x-x_\delta| = t\delta$. For every $t \in [0,1]$, $y \in B(z, t\delta)$,

$$|y - x_\delta| \le |y-z| + |z - x_\delta| = t\delta + (1-t)|x - x_\delta| \le \delta.$$

Thus, $B(z,t\delta) \subset B(x_\delta, \delta)$ and $x \in \overline{B(z,t\delta)}$. Besides, $d(z, D^c) \ge d(z, \partial B(z,t\delta)) = t\delta$. The inequality becomes equality if and only if $B(z,t\delta)$ touches the boundary of D, or equivalently, $B(z,t\delta) \equiv B(x_\delta, \delta)$. We observe that $B(z,t\delta) \equiv B(x_\delta,\delta)$ if and only if $d(x, D^\delta) = \delta$ and $t = 1$. Therefore, for $t = \delta'/\delta$, z is an interior point of $D^{\delta'}$ since $d(z, D^c) > d(z, \partial B(z,\delta')) = \delta'$. We then obtain a ball $B(z,\delta')$ such that $B(z,\delta') \subset B(x_\delta,\delta)$ and $x \in \overline{B(z,\delta')}$. ■

Lemma 4.25. *We have :*

(i) For all $\delta' < \delta$, it holds

$$D^\delta = \left\{x \in D^{\delta'} \ : \ d(x, (D^{\delta'})^c) > \delta - \delta'\right\} = \left\{x \in D^{\delta'} \ : \ B(x, \delta - \delta') \subset D^{\delta'}\right\}.$$

(ii) Given $x \in \partial_r D^\delta$, $z \in \partial D \cap M_0^\delta(x)$ (see (4.47) for the definition of $M_0^\delta(x)$). Then, for every $y \in M_0^\delta(x)$, we have $|y-z| = d(y, D^c)$. In particular, $|x-z| = d(x, D^c) = \delta$.

(iii) For each $\delta' < \delta$, it holds

$$\partial_s D^{\delta'} \subset \bigcup_{x \in \partial_s D^\delta} M_0^\delta(x).$$

As consequence, for every $x \in \overline{D}$, if $\Pi_\delta(x)$ is in $\partial_r D^\delta$ then $\Pi_{\delta'}(x)$ belongs to $\partial_r D^{\delta'}$ for all $\delta' < \delta$.

Proof. (i) Let $E := \left\{x \in D^{\delta'} \ : \ d(x, (D^{\delta'})^c) > \delta - \delta'\right\}$. It is easy to verify that

$$E = \left\{x \in D^{\delta'} \ : \ B(x, \delta - \delta') \subset D^{\delta'}\right\}.$$

We will prove that $D^\delta = E$.

For every $x \in D^\delta$, $B(x,\delta) \subset D$ implies that

$$\delta < d(x, D^c) \le d(x, \partial D^{\delta'}) + d(\partial D^{\delta'}, D^c) = d(x, \partial D^{\delta'}) + \delta'.$$

That means $d(x, (D^{\delta'})^c) = d(x, \partial D^{\delta'}) > \delta - \delta'$. It is to say that x is in E. So, we get $D^\delta \subset E$.

For every $x \in E$, $B(x, \delta - \delta') \subset D^{\delta'}$ implies that for all $y \in B(x, \delta - \delta')$, y is in $D^{\delta'}$ and $d(y, D^c) > \delta'$. Thus, we obtain

$$\begin{aligned} d(x, D^c) &= d(x, \partial B(x, \delta - \delta')) + d(\partial B(x, \delta - \delta'), D^c) \\ &> (\delta - \delta') + \delta' = \delta. \end{aligned}$$

It shows that $x \in D^\delta$. The inverse inclusion $D^\delta \supset E$ is true.

(ii) Given $x \in \partial_r D^\delta$ and $z \in \partial D \cap M_0^\delta(x)$, we have $z = x + p$ with $|p| \le \delta$. It follows that

$$\delta = d(x, D^c) \le |x - z| \le \delta.$$

Or, $d(x,z) = d(x, D^c) = \delta$.

For every $y \in M_0^\delta(x)$, we will prove that $|y-z| = d(y, D^c)$. Suppose that $d(y, D^c) < |y-z|$ and $d(y, D^c) = |y - \overline{z}|$ for some $\overline{z} \in \partial D$, $\overline{z} \ne z$. Then, $|y - \overline{z}| < |y - z|$. Recall that as $x \in \partial_r D^\delta$, $M_0^\delta(x)$ is a segment joining x and z. We have

$$\begin{aligned} |x - \overline{z}| &\le |x - y| + |y - \overline{z}| \\ &< |x - y| + |y - z| = \delta \end{aligned}$$

while $|x-\overline{z}| \geq d(x, D^c) = \delta$. This gives a contradiction. So, $d(y, D^c) = |y-z|$.

(iii) It is equivalent to prove that for every $\delta' < \delta$, $y \in \partial_s D^{\delta'}$ implies that $\Pi_\delta(y) \in \partial_s D^\delta$. Suppose that $y \in \partial_s D^{\delta'}$ and $\Pi_\delta(y) \in \partial_r D^\delta$. From (i) and (ii), we derive that $|y - \Pi_\delta(y)| = \delta - \delta'$. As $\Pi_\delta(y) \in \partial_r D^\delta$, the ball $B(\Pi_\delta(y), \delta)$ then touches the boundary of D at a unique point called z. Let w be the intersection of segment $M_0^\delta(\Pi_\delta(y)) = [\Pi_\delta(y), z]$ and $\partial D^{\delta'}$. By using (ii), $d(w, D^c) = \delta'$. The ball $B(w, \delta')$ contained in $B(\Pi_\delta(y), \delta)$ touches ∂D at and only at z. In other words, w is in $\partial_r D^{\delta'}$. Besides, both y and w are in $M_0^\delta(\Pi_\delta(y))$. It is easy to see that they coincide. We conclude that y belongs to $\partial_r D^{\delta'}$, a contradiction. The proof is completed. ■

Lemma 4.26. *The followings hold true:*

(i) Given $x \in \partial D^\delta$ and $x^* \in \partial D$ such that $|x - x^*| = d(x, D^c)$. For every $\delta' \leq \delta$, let z^* be the intersection of the segment $[x, x^*]$ and $\partial D^{\delta'}$. Then, we have

$$\Pi_{\delta'}(x^*) = z^*.$$

(ii) For every $y \in M_0^\delta(x)$, with $x = \Pi_\delta(y)$, we always have $\Pi_{\delta'}(y) \in M_0^\delta(x)$ for every $\delta' < \delta$.

Proof. (i) For each $z \in [x, x^*]$, z can be parametrized as

$$z(t) := (1-t)x^* + tx.$$

If we take $z^* = z(\delta'/\delta)$ then $z^* \in \partial D^{\delta'}$. In fact, it holds $d(z^*, D^c) = \delta'$ since

$$\delta' = |z^* - x^*| \geq d(z^*, D^c) \geq d(z^*, \partial B(z^*, |z^* - x^*|)) = \delta'.$$

Besides, we have

$$\delta' = |x^* - z^*| \geq \inf_{y \in D^{\delta'}} |x^* - y| \geq \delta'.$$

It turns out that $\Pi_{\delta'}(x^*) = z^*$.

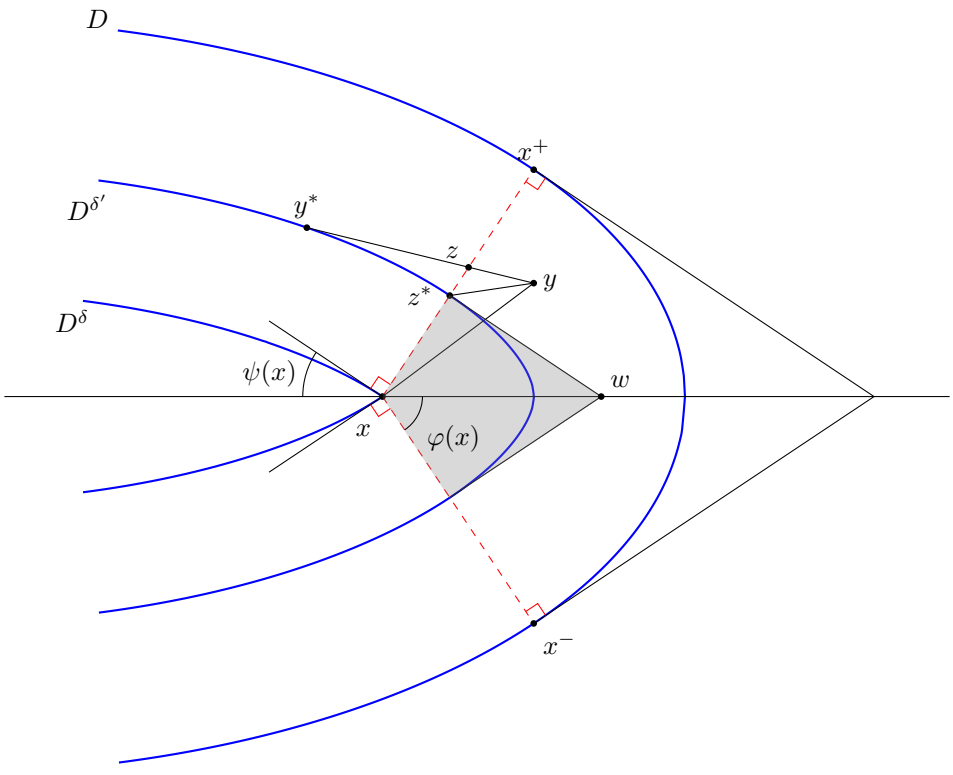


Figure 4.8: Proof. The kite in gray color represents the set $M_{\delta'}^\delta$.

(ii) See Figure 4.8 for illustration. Suppose that $y^* := \Pi_{\delta'}(y) \notin M_0^{\delta}(x)$. Without loss of generality, we can assume that the segment $[y, y^*]$ intersects $[x, x^+]$ at z, where $x^+ \in \partial D$ such that $|x - x^+| = d(x, D^c)$. Let z^* be the intersection of the segment $[x, x^+]$ and $\partial D^{\delta'}$. By (i), we get that $\Pi_{\delta'}(x^+) = z^*$. So, the segment $[z^*, x^+]$ is contained in $M_0^{\delta'}(z^*)$. Therefore, for every $z \in [z^*, x^+]$, z admits z^* as its unique projection on $D^{\delta'}$. It is then clearly that

$$|y - z^*| \leq |y - z| + |z - z^*| \leq |y - z| + |z - y^*| = |y - y^*|$$

gives a contradiction to the fact that y^* is the nearest point on $D^{\delta'}$ to y. ■

Lemma 4.27. *Let D be a convex subset of $\mathbb{R}^2$ with positive measure, i.e. $|D| > 0$. For every $x \in \partial D$, we define*

$$k_{\partial D}(x) := \frac{1 + \nu_D^+(x) \cdot \nu_D^-(x)}{2}.$$

Then, we have

(i) for every $x \in \partial D$, $0 < k_{\partial D}(x) \leq 1$;

(ii) $k_{\partial D}(x) = 1$ for every $x \in \partial_r D$;

(iii) $\forall \varepsilon \in (0, 1)$, $\exists n \in \mathbb{N}$ such that $\#\{x \,:\, k_{\partial D}(x) < \varepsilon\} \leq n$ (# measures the cardinality of sets).

Proof. We recall that the normal cone of D at x is given by

$$N_D(x) := \left\{ a\nu_D^+(x) + b\nu_D^-(x) \;:\; a, b \in \mathbb{R}_+ \right\}.$$

For every $x \in \partial D$, we denote by $\varphi(x)$ the angle

$$\varphi(x) := \frac{1}{2} \angle(\nu_D^-(x), \nu_D^+(x)),$$

by $T_D(x)$ the tangent cone of D at x

$$\begin{aligned} T_D(x) &:= \mathrm{cl}\,\{s(y - x) \;:\; y \in D,\ s \geq 0\} \\ &= \left\{ aT_D^+(x) + bT_D^-(x) \;:\; a, b \in \mathbb{R}_+ \right\}, \end{aligned}$$

where $T_D^{\pm}(x)$ are the left and right tangent unit vectors of D at x. We denote by $\psi(x)$ the angle

$$\psi(x) := \frac{1}{2} \angle(T_D^-(x), T_D^+(x)).$$

Since tangent and normal cones are polar each other, i.e. $N_D(x) = (T_D(x))^o$, for every $x \in \partial D$,

$$\varphi(x) + \psi(x) = \frac{\pi}{2}.$$

(i) Since $|D| > 0$, $\mathrm{int}\, D \neq \emptyset$, in other words, $\mathrm{int}\, T_D(x) \neq \emptyset$. It is to say $\psi(x) > 0$ (and $\psi(x) \leq \pi/2$ as D convex). This implies that $\varphi(x) = \frac{\pi}{2} - \psi(x) < \frac{\pi}{2}$. Then, we have

$$0 \leq \varphi(x) < \frac{\pi}{2}.$$

For all $x \in \partial D$, it holds

$$k_{\partial D}(x) = \frac{1 + \cos 2\varphi(x)}{2} = \cos^2 \varphi(x) = \sin^2 \psi(x).$$

Hence, $0 < k_{\partial D}(x) \leq 1$.

(ii) D is convex, its boundary ∂D admits at most countably many singular points. On the regular part $\partial_r D = \partial D \setminus \partial_s D$, we get $\nu_D^-(x) = \nu_D^+(x)$, i.e. $\varphi(x) = 0$. Then, $k_{\partial D}(x) = 1$.

(iii) Suppose there exists $0 < \varepsilon < 1$ such that for every $n \in \mathbb{N}$, $\#\{x \ : \ k_{\partial D}(x) < \varepsilon\} > n$. Consider the function $h(\varphi)$ defined by

$$h(\varphi) := \frac{1 + \cos 2\varphi}{2}, \qquad \text{for } \varphi \in [0, \frac{\pi}{2}).$$

h is a decreasing function in $[0, \frac{\pi}{2})$, namely

$$\varphi < \varphi' \Rightarrow h(\varphi) \geq h(\varphi').$$

Moreover, for every $x \in \partial D$, it holds $h(\varphi(x)) = k_{\partial D}(x)$. There always exists $\overline{\varphi} \in (0, \pi/2)$ such that $h(\overline{\varphi}) = \varepsilon$. We set

$$E := \{x \in \partial D \ : \ k_{\partial D}(x) < \varepsilon = h(\overline{\varphi})\}.$$

Then, for all $x \in E$, by the monotonicity of h, it must be satisfied that $\varphi(x) > \overline{\varphi}$. Hence, for every $n \in \mathbb{N}$,

$$n\overline{\varphi} = \sum_{i=1}^{n} \overline{\varphi} < \sum_{x \in E} \varphi(x) \leq \sum_{x \in \partial D} \varphi(x) \leq 2\pi.$$

That means $\overline{\varphi} \leq 2\pi/n$ for all $n \in \mathbb{N}$. In other words, we deduce that $\overline{\varphi} = 0$ (or $\varepsilon = h(\overline{\varphi}) = 1$). This gives a contradiction. ∎

Lemma 4.28. *Let $\alpha(x, \delta)$ be a function defined by*

$$\alpha(x, \delta) := d(x, D^\delta) - \delta \tag{4.50}$$

for $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, and $\delta \in [0, R]$. Then, for each $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, function $\alpha(x, \cdot)$ satisfies

$$\alpha(x, \delta) \begin{cases} = -\delta & \text{if } 0 < \delta < d(x, D^c) \\ = -d(x, D^c) & \text{if } d(x, D^c) \leq \delta \leq \gamma(x) \\ \text{strictly increasing} & \text{if } \gamma(x) < \delta < R. \end{cases}$$

Furthermore, there is unique $\delta = \delta(x) \in (\gamma(x), R)$ such that $\alpha(x, \delta) = 0$.

Proof. For $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, $d(x, D^c) > 0$. For all δ such that $0 < \delta \leq d(x, D^c)$, x is in $\overline{D^\delta}$. So, $\alpha(x, \delta) = -\delta$.

If $\delta = \gamma(x)$, then by Lemma 4.23, $\Pi_\delta(x) = x + \zeta(x)\nabla d(x, D^c)$. We have

$$\alpha(x, \gamma(x)) = \zeta(x) - \gamma(x) = -d(x, D^c).$$

For $d(x, D^c) < \delta < \gamma(x)$, let $\overline{x} \in \overline{\Lambda}$ such that

$$\overline{x} = x + \zeta(x)\nabla d(x, D^c),$$

$x^* \in \partial D$ such that $|\overline{x} - x^*| = d(\overline{x}, D^c)$, and w be the intersection of the segment $[\overline{x}, x^*]$ and ∂D^δ. It is clearly that $x \in [\overline{x}, x^*]$ and $w \in \partial_r D^\delta$. By Lemma 4.26 (i), we obtain $w = \Pi_\delta(x^*)$. As consequence, $\Pi_\delta(x) = w$ and hence, keeping in mind that $|x - x^*| = d(x, D^c)$ (by Lemma 4.25 (ii)), we have

$$d(x, D^\delta) = |x - w| = |x^* - w| - |x - x^*| = \delta - d(x, D^c).$$

So, we conclude that $\alpha(x,\delta) = d(x,D^\delta) - \delta = -d(x,D^c)$ for every δ such that $d(x,D^c) \le \delta \le \gamma(x)$.

In the interval $(\gamma(x), R)$, function $\alpha(x,\delta)$ is strictly increasing in δ. In fact, for every δ, δ' such that $\gamma(x) < \delta' < \delta < R$, x is outside $D^{\delta'}$. By property of projection, the ball $B(x, |x - \Pi_{\delta'}(x)|)$ is contained in $(D^{\delta'})^c$. We have

$$\begin{aligned} d(x,D^\delta) - d(x,D^{\delta'}) &= |x - \Pi_\delta(x)| - |x - \Pi_{\delta'}(x)| \\ &= d(\Pi_\delta(x), B(x, |x - \Pi_{\delta'}(x)|)) \\ &\ge d(\Pi_\delta(x), \partial D^{\delta'}) \\ &= \delta - \delta'. \end{aligned}$$

The inequality becomes equality, i.e. $d(\Pi_\delta(x), \partial D^{\delta'}) = d(\Pi_\delta(x), B(x, |x - \Pi_{\delta'}(x)|))$, if and only if

$$\Pi_{\delta'}(\Pi_\delta(x)) = \Pi_{\delta'}(x) = \Pi_{\overline{B(x,|x-\Pi_{\delta'}(x)|)}}(\Pi_\delta(x)) \in [\Pi_\delta(x), x], \tag{4.51}$$

where $[\Pi_\delta(x), x]$ denotes the segment joining $\Pi_\delta(x)$ and x. The expression (4.51) also shows that $\Pi_\delta(x) \in \partial_r D^\delta$. It follows Lemma 4.25 by (iii) that $\Pi_{\delta'}(x) \in \partial_r D^{\delta'}$ and moreover, by (ii) that $M_0^\delta(\Pi_\delta(x))$ is a segment comprising regular points of $d(\cdot, D^c)$ in the direction orthogonal to ∂D, which includes x and $\Pi_{\delta'}(x)$. Therefore, $\Pi_\delta(x)$ can be rewritten as

$$\Pi_\delta(x) = x + t_\delta \nabla d(x, D^c),$$

for some $t_\delta > 0$. Since $\gamma(x) < \delta$, we get $\zeta(x) < t_\delta$, or equivalently, $x + \zeta(x)\nabla d(x,D^c)$ contained in $M_0^\delta(\Pi_\delta(x))$. This gives a contradiction to the fact that $x + \zeta(x)\nabla d(x,D^c)$ is a singular point of $d(\cdot, D^c)$. So, we conclude that the relation (4.51) never occurs and, then $\alpha(x,\cdot)$ is strictly increasing.

Finally, it is evidently that

$$\begin{aligned} \alpha(x,\gamma(x)) &= d(x,D^{\gamma(x)}) - \gamma(x) = -d(x,D^c) < 0, \\ \alpha(x,R) &= d(x,D^R) - R > 0. \end{aligned}$$

So, by continuity of $\alpha(x,\cdot)$, there exists unique $\delta = \delta(x)$ in $(\gamma(x), R)$ so that $\alpha(x,\delta) = 0$. ■

Remark 4.29. We notice that if $\alpha(x,\delta) \le 0$ for some $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$ and $\delta > 0$ then for every δ' satisfying $0 < \delta' < \delta$, it holds $\alpha(x,\delta') < 0$. This is described in Lemma 4.24. By using Lemma 4.25 (i), Ω_λ can be recast as

$$\Omega_\lambda = \{x \in D \ : \ \alpha(x, \lambda^{-1}) < 0\}.$$

As a consequence of the monotonicity of function $\alpha(x,\cdot)$, the family $\{\Omega_\lambda\}_{\lambda \ge R^{-1}}$ is clearly strictly monotone with respect to λ. In other words, it holds $\Omega_\lambda \subsetneq \Omega_{\lambda'}$ for all $\lambda' > \lambda \ge R^{-1}$.

Cut-locus potential

We introduce the function $\rho : \overline{D} \to \mathbb{R}_+$ defined by

$$\rho(x) := \sup\{\delta \ge 0 \ : \ d(x,D^\delta) \le \delta\}. \tag{4.52}$$

We call ρ *cut-locus potential*.

It is clear that function ρ can be rewritten as

$$\rho(x) = \sup\{\delta \ge 0 \ : \ \alpha(x,\delta) \le 0\}.$$

This supremum is attained at a unique δ_x such that $\alpha(x,\delta_x) = 0$. In fact, $\{\delta \ : \ d(x,D^\delta) - \delta \le 0\}$ is the full interval $[0, \rho(x)]$. See Figure 4.9.

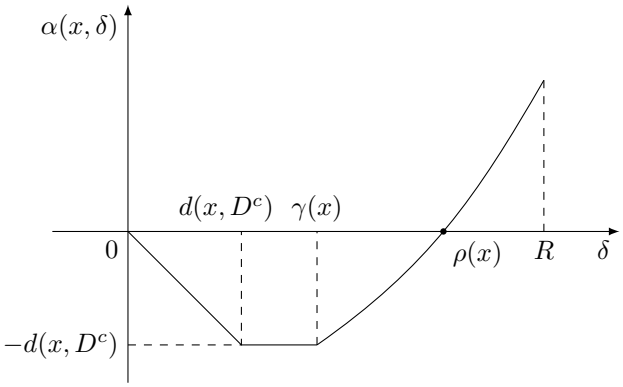


Figure 4.9: Function $\alpha(x,\cdot)$, for $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$.

Lemma 4.30. *ρ is continuous, achieves its maximum R on the central set of D and $\rho(x) \geq d(x,D^c)$. Moreover, for every $r>0$,*

$$\rho(x) = r \iff d(x,D^r) = r \iff x \in \partial\Omega_{\frac{1}{r}}. \tag{4.53}$$

Proof. The continuity of ρ can be deduced from the continuity of the map $\delta \to d(x,D^\delta)$ (notice that the map $\delta \to d(x,D^\delta)$ is continuous only if D is convex). Since we have the inclusions $(0,d(x,D^c)] \subset \{\delta>0 : d(x,D^\delta) \leq \delta\} \subset (0,R]$, it is clear that ρ is bounded, namely

$$d(x,D^c) \leq \rho(x) \leq R.$$

Moreover, $\rho(x) = R$ for every $x \in \Omega_{\frac{1}{R}}$.

We now prove that $\Omega_{\frac{1}{r}} = \{\rho > r\}$ for every $r>0$, then, the equivalences (4.53) are consequent. In fact, if $x \in \Omega_{\frac{1}{r}}$ then $d(x,D^r) < r$. This implies that $\rho(x) \geq r$. Since $\Omega_{\frac{1}{r}}$ is an open set, we have $\Omega_{\frac{1}{r}} \subset \{\rho > r\}$. For the inverse inclusion, for any $x \in \{\rho > r\}$, there exists $\delta > r$ such that $\delta > d(x,D^\delta)$. This shows that $x \in \Omega_{\frac{1}{\delta}} \subset \Omega_{\frac{1}{r}}$. Thus, $\{\rho > r\} \subset \Omega_{\frac{1}{r}}$. ∎

Remark 4.31. By Lemma 4.30, we consider Ω_λ as a superlevel set of ρ,

$$\Omega_\lambda = \left\{x \in D \ : \ \rho(x) > \lambda^{-1}\right\}.$$

Then, level sets of ρ being $C^\delta := \{x \in D \ : \ \rho(x) = \delta\} = \partial\Omega_{\frac{1}{\delta}}$ will give a partition to D, namely

$$D = \bigcup_{0<\delta\leq R} C^\delta,$$

where $C^R = \Omega_{\frac{1}{R}}$. The family $\{C^\delta\}_{0<\delta<R}$ being the boundaries of $\Omega_{\frac{1}{\delta}}$ in D are indeed arcs of radius δ.

Lemma 4.32. *ρ is locally Lipschitz in D (the Lipschitz constant blows-up as $d(x,D^c) \to 0$). On ∂D, $\rho(x) \leq \frac{1}{\kappa_{\partial D}(x)}$ and $\rho(x) = \tau(x)$ the normal distance to the cut locus, namely*

$$\tau(x) = \sup\{t \geq 0 \ : \ x \text{ is the unique projection of } x - t\nu_D(x) \text{ on } \partial D\}.$$

Notice that $\tau : \partial D \to [0,R]$ is a function defined on ∂D (it measures the normal distance from boundary to cut-locus) and $\tau(x) = \zeta(x)$ for all $x \in \partial D$ where $\zeta : \overline{D} \to [0,R]$ given by (4.49) (ζ is defined in the entire $\overline{D}$).

Proof. Firstly, we prove that ρ is locally Lipschitz in D. Given $\delta > 0$, for every $x \in \Omega_{\delta^{-1}} \cap D$, we have

$$d(x, D^c) > 0 \qquad \text{and} \qquad \delta = |x - \Pi_\delta(x)|.$$

We set $r := d(x, D^c)$. Let z be the point lying outside the disk $B(\Pi_\delta(x), \delta)$, on the line passing x, $\Pi_\delta(x)$ such that $|z - x| = r$. So, z is in $\overline{D}$. For every $\delta' < \delta$, we take y as the point inside D, on the same latter line such that the disks $B(y, \delta')$ and $B(\Pi_\delta(x), \delta)$ have the same tangents passing z. See Figure 4.10 for our settings.

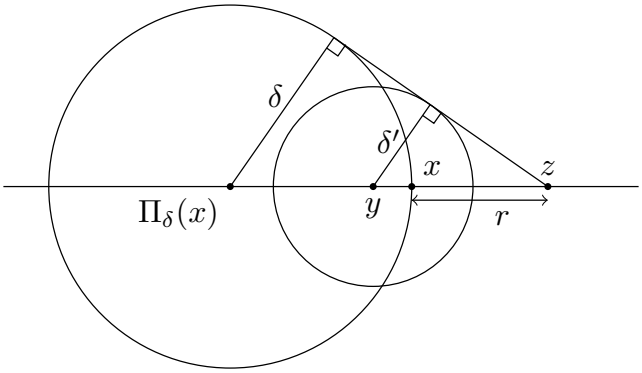


Figure 4.10: To prove that ρ is locally Lipschitzian.

Thales' Theorem is applied,

$$\frac{|z - y|}{|z - \Pi_\delta(x)|} = \frac{\delta'}{\delta},$$

then, we get

$$|z - y| = \frac{\delta'}{\delta}(\delta + r). \tag{4.54}$$

We notice that $B(\Pi_\delta(x), \delta) \subset \overline{D}$ and, $z \in \overline{B(x, r)} \subset \overline{D}$. Since D convex, we have

$$\text{conv}\left[\overline{B(\Pi_\delta(x), \delta)} \cup \{z\}\right] \subset \overline{D}.$$

This implies $B(y, \delta') \subset \overline{D}$. Thus, $y \in \overline{D^{\delta'}}$ and, by using the equality (4.54), it holds

$$d(x, D^{\delta'}) \leq |x - y| = |z - y| - r = \delta' - r(1 - \frac{\delta'}{\delta}).$$

We now can summarize that

$$\forall x \in \partial\Omega_{\delta^{-1}} \cap D,\ \forall \delta' < \delta, \quad d(x, D^{\delta'}) \leq \delta' - d(x, D^c)(1 - \frac{\delta'}{\delta}). \tag{4.55}$$

We observe that for each $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, by using Remark 4.31, $x \in \partial\Omega_{\delta^{-1}} \cap D$ with $\delta = \rho(x)$. As a consequence of statement (4.55), for every $x, x' \in D \setminus \overline{\Omega_{\frac{1}{R}}}$ and, for every $\delta' < \delta = \rho(x)$, we have

$$\begin{aligned} d(x', D^{\delta'}) - \delta' &\leq |x - x'| + d(x, D^{\delta'}) - \delta' \\ &\leq |x - x'| - d(x, D^c)\left[1 - \frac{\delta'}{\delta}\right]. \end{aligned}$$

If $\rho(x') < \rho(x)$, we can choose $\delta' = \rho(x')$ and, then $d(x', D^{\delta'}) - \delta' = 0$. It turns out that

$$d(x, D^c)\left(1 - \frac{\rho(x')}{\rho(x)}\right) \leq |x - x'|.$$

Therefore, for every $x, x' \in D \setminus \overline{\Omega_{\frac{1}{R}}}$ such that $\rho(x') < \rho(x)$, we have

$$|\rho(x') - \rho(x)| \leq \frac{\rho(x)}{d(x, D^c)}|x' - x|.$$

Finally, we conclude that for every $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, $x' \in B(x, \varepsilon) \subset D$,

$$|\rho(x') - \rho(x)| \leq \frac{R}{d(B(x, \varepsilon), D^c)}|x' - x|. \tag{4.56}$$

In other words, ρ is locally Lipschitz in $D \setminus \overline{\Omega_{\frac{1}{R}}}$, thus, it is locally Lipschitz in D.

Secondly, we show that on ∂D, it holds $\rho(x) \leq \frac{1}{\kappa_{\partial D}(x)}$. As shown in Example 4.37, the function ρ is at risk of exploding if it touches ∂D. However, near the boundary, this positive function has an upper bound characterized by geometry of D: For $x \in \partial D$, we get an estimation

$$0 \leq \overline{\rho}(x) \leq \frac{1}{\kappa_{\partial D}(x)}$$

where $\overline{\rho}$ denotes the limit inferior of ρ at x

$$\overline{\rho}(x) := \sup_{\varepsilon > 0} \inf_{B(x,\varepsilon) \cap D} \rho$$

(As ρ is continuous, ρ coincides with its limit inferior $\overline{\rho}$ on ∂D). In fact, for every $\varepsilon > 0$, we denote

$$\rho_\varepsilon := \inf_{B(x,\varepsilon) \cap D} \rho.$$

Then, for each $y \in B(x, \varepsilon) \cap D$, $\rho(y) \geq \rho_\varepsilon$ and $d(y, D^{\rho(y)}) \leq \rho(y)$. By Lemma 4.24, we have $d(y, D^{\rho_\varepsilon}) \leq \rho_\varepsilon$. Thus, $d(x, D^{\rho_\varepsilon}) \leq \rho_\varepsilon$. It follows

$$\overline{\rho}(x) \leq d(x, D^{\overline{\rho}}) = \sup_{\varepsilon > 0} d(x, D^{\rho_\varepsilon}) \leq \sup_{\varepsilon > 0} \rho_\varepsilon = \overline{\rho}(x).$$

This implies $B(\Pi_{\overline{\rho}(x)}(x), \overline{\rho}(x)) \subset \overline{D}$ and $x \in \partial B(\Pi_{\overline{\rho}(x)}(x), \overline{\rho}(x))$, i.e. this ball touches D at x. It is to say that

$$\overline{\rho}(x) \leq \frac{1}{\kappa_{\partial D}(x)}.$$

Finally, $\rho(x) = \tau(x)$ on ∂D is a direct consequence of the fact that $d(x, D^c) = 0$ and $\{\delta \,:\, \alpha(x, \delta) \leq 0\} = [0, \gamma(x)]$. ∎

Remark 4.33. Y. Li and L. Nirenberg proved in [83] that τ is Lipschitz if D is $C^{2,1}$ but it is untrue for a general convex domain (even $C^{2,\alpha}$ with $\alpha < 1$ is not enough).

Theorem 4.34. *Let $D \subset \mathbb{R}^2$ be a bounded convex domain. On the open subset $\{0 < \rho < R\}$, the locally Lipschitz potential ρ satisfies $\nabla\rho \neq 0$ a.e. and $\operatorname{div}\left(\frac{\nabla\rho}{|\nabla\rho|}\right) + \frac{1}{\rho} = 0$ in the sense of distributions.*

Proof. Note that the open subset $\{0 < \rho < R\}$ is indeed $D \setminus \overline{\Omega_{\frac{1}{R}}}$ where ρ is not constant. On this subset, the vector field $\eta := -\nabla\rho/|\nabla\rho|$ can be rewritten as

$$\eta(x) = \frac{x - \Pi_{\rho(x)}(x)}{\rho(x)}.$$

We are going to prove that $\Pi_{\rho(x)}(x)$ is locally Lipschitz in x and hence, so is η.

We firstly claim that for every δ, δ' satisfying $0 < \delta' < \delta$, for each $y \in D$, there is a constant K_δ such that

$$|\Pi_\delta(y) - \Pi_{\delta'}(y)| \leq K_\delta |\delta - \delta'|. \tag{4.57}$$

It follows immediately that for every $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, $x' \in B(x, \varepsilon) \subset D$, keeping in mind (4.56),

$$\begin{aligned} |\Pi_{\rho(x)}(x) - \Pi_{\rho(x')}(x')| &\leq |\Pi_{\rho(x)}(x) - \Pi_{\rho(x')}(x)| + |x - x'| \\ &\leq K_{\rho(x)} |\rho(x) - \rho(x')| + |x - x'| \\ &\leq \left(K_{\rho(x)} + \frac{R}{d(B(x,\varepsilon), D^c)} \right) |x - x'|. \end{aligned}$$

Hence, η is locally Lipschitz.

To complete the proof of the first part, we now make the assertion (4.57) evident. Given $0 < \delta' < \delta$, for every $y \in D$, by Lemma 4.26 (ii), $\Pi_{\delta'}(y)$ is always in $M^\delta_{\delta'}(\Pi_\delta(y))$, see Figure 4.8 for illustration. We obtain

$$|\Pi_\delta(y) - \Pi_{\delta'}(y)| \leq |w - x| = \frac{|\delta - \delta'|}{\cos\varphi(x)}, \tag{4.58}$$

where $x = \Pi_\delta(y)$ and w is the extreme point of $M^\delta_{\delta'}(x)$ in the complement of $D^{\delta'}$ (see Figure 4.8). By passing Lemma 4.27 (iii), there exists $K_{\partial D^\delta} > 0$ such that

$$K_{\partial D^\delta} = \min\left\{ k_{\partial D}(s) \ : \ s \in \partial D^\delta \right\} = \min\left\{ \cos^2\varphi(s) \ : \ s \in \partial D^\delta \right\}.$$

Since $x \in \partial D^\delta$, we have

$$\frac{1}{\cos\varphi(x)} \leq \frac{1}{\sqrt{K_{\partial D^\delta}}}. \tag{4.59}$$

We then use the inequalities (4.58) and (4.59) to derive that

$$|\Pi_\delta(y) - \Pi_{\delta'}(y)| \leq \frac{1}{\sqrt{K_{\partial D^\delta}}} |\delta - \delta'|.$$

So, the inequality (4.57) is proved with $K_\delta = (K_{\partial D^\delta})^{-1/2}$.

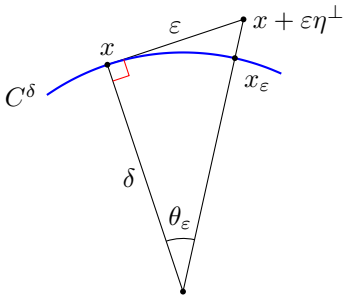


Figure 4.11: Divergence of η along C^δ.

The next step is to prove that $\operatorname{div}\eta = \dfrac{1}{\rho}$ in $D \setminus \overline{\Omega_{\frac{1}{R}}}$. The vector field η is indeed the unit normal to the level sets $C^\delta = \{x \in D \ : \ \rho(x) = \delta\}$ which are arcs of radius δ. Given x and x_ε on C^δ illustrated by Figure 4.11. Let us evaluate locally divergence of η along directions η and $\eta^\perp$. We recall that

$$(D\eta)h \cdot h := \lim_{\varepsilon \to 0} \left\langle \frac{\eta(x+\varepsilon h) - \eta(x)}{\varepsilon}, h \right\rangle$$

for some non null direction h. Since $|\eta| = 1$, we have

$$2(D\eta)h \cdot \eta = \left\langle \frac{\eta(x+\varepsilon h) - \eta(x)}{\varepsilon}, \eta(x+\varepsilon h) + \eta(x) \right\rangle = \frac{1}{\varepsilon}\left[|\eta|^2(x+\varepsilon h) - |\eta|^2(x)\right] = 0.$$

Thus, for $h = \eta$, we get

$$(D\eta)\eta \cdot \eta = 0. \tag{4.60}$$

As η locally Lipschitz, there is some constant M such that

$$\left|\frac{\eta(x+\varepsilon h) - \eta(x_\varepsilon)}{\varepsilon}\right| \leq \frac{M}{\varepsilon}|x + \varepsilon h - x_\varepsilon| = \frac{M\delta}{\varepsilon}\left(\frac{1}{\cos\theta_\varepsilon} - 1\right) = \frac{M\delta}{\varepsilon}\left(\sqrt{1 + \frac{\varepsilon^2}{\delta^2}} - 1\right) \sim \frac{M\varepsilon}{2\delta}.$$

Hence, for $h = \eta^\perp$,

$$\begin{aligned}\left\langle \frac{\eta(x+\varepsilon h) - \eta(x)}{\varepsilon}, h \right\rangle &= \left\langle \frac{\eta(x+\varepsilon h) - \eta(x_\varepsilon)}{\varepsilon}, h \right\rangle + \left\langle \frac{\eta(x_\varepsilon) - \eta(x)}{\varepsilon}, h \right\rangle \\ &\sim \frac{M\varepsilon}{2\delta} + \frac{\sin\theta_\varepsilon}{\varepsilon} \sim \frac{M\varepsilon}{2\delta} + \frac{\tan\theta_\varepsilon}{\varepsilon} \\ &= \frac{M\varepsilon}{2\delta} + \frac{1}{\delta}.\end{aligned}$$

We obtain

$$(D\eta)\eta^\perp \cdot \eta^\perp = \frac{1}{\delta}. \tag{4.61}$$

From the equations (4.60) and (4.61), we can derive that

$$\operatorname{div}\eta = (D\eta)^T : Id = (D\eta)^T : (\eta \otimes \eta + \eta^\perp \otimes \eta^\perp) = (D\eta)\eta \cdot \eta + (D\eta)\eta^\perp \cdot \eta^\perp = \frac{1}{\delta} = \frac{1}{\rho}.$$

As η is locally Lipschitz on the open set $\{0 < \rho < R\}$, the equality above holds not only a.e. on this set but also in the distributional sense. ■

Corollary 4.35. *Let $\mu > 0$ and u_μ the unique solution to*

$$(\mathcal{Q}_\mu) \qquad \inf\left\{\int_{\mathbb{R}^2} |Du| + \frac{\mu}{2}\int_D (1-u)^2 \ : \ u \in BV_0(D)\right\}.$$

Then, we have

$$u_\mu(x) \ = \ \begin{cases} \left(1 - \frac{1}{\mu\,\rho(x)}\right)_+ & \text{if } \rho(x) < \frac{1}{h_D} \\ \left(1 - \frac{h_D}{\mu}\right)_+ & \text{if } \rho(x) \geq \frac{1}{h_D} \quad \text{(plateau).} \end{cases} \tag{4.62}$$

Proof. It is easy to check that the function $\overline{u}$ defined by

$$\overline{u}(x) := \begin{cases} \left(1 - \frac{1}{\mu\,\rho(x)}\right)_+ & \text{if } \rho(x) < \frac{1}{h_D} \\ \left(1 - \frac{h_D}{\mu}\right)_+ & \text{if } \rho(x) \geq \frac{1}{h_D} \end{cases}$$

satisfied the Euler-Lagrange equation of the problem $(\mathcal{Q}_\mu)$:

$$\begin{aligned} -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) &= \mu(1-u) \qquad \text{in } D, \\ \frac{\nabla u}{|\nabla u|} \cdot \nu_D &= -1 \qquad \text{on } \partial D. \end{aligned}$$

Since $(\mathcal{Q}_\mu)$ is strictly convex, by the uniqueness of solution, one has $u_\mu = \overline{u}$. ■

Corollary 4.36. *Let $\mu > \lambda > h_D$. Then, for $s = 1 - \frac{\lambda}{\mu}$, the upper level set*

$$\{u_\mu > s\} = \{\rho > \lambda^{-1}\} = \Omega_\lambda$$

solves the problem $m(\lambda, D)$.

Proof. From the explicit expression of the unique solution u_μ in (4.62), we can easily check that $\{u_\mu > s\} = \{\rho > \lambda^{-1}\}$, then by Lemma 4.30 these upper level sets agree with Ω_λ. The optimality of $\{u_\mu > s\}$ for the problem $m(\lambda, D)$ follows the results of F. Alter, V. Caselles, and A. Chambolle in [4, Proposition 4] ■

We remark that according to Theorem 4.10 the optimality of Ω_λ for problem $m(\lambda, D)$ can be achieved by constructing an explicit calibrating field for Ω_λ which will be done in the next subsection.

In Figure 4.12, we present the explicit solution u_μ of problem $(\mathcal{Q}_\mu)$ given by Corollary 4.35 in a quarter of the square $(0,1)^2$. The white curve determines the maximum-valued plateau of the solution u_μ. As we observe, the maximum-valued plateau corresponds to the Cheeger set of the domain, it means $\{x \in D \ : \ u_\mu(x) = \max_D u_\mu\} = \Omega_{h_D}$.

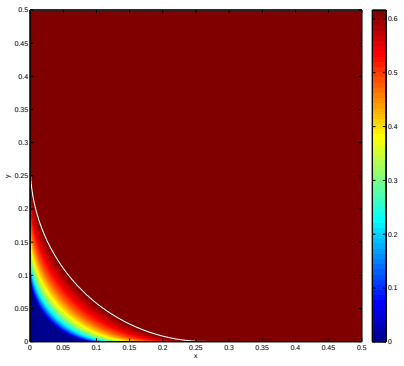


Figure 4.12: The unique solution u_μ of problem $(\mathcal{Q}_\mu)$.

Example 4.37 (Explicit ρ in case of a square domain)**.**

We take the unit square centered at the origin $D = (-1/2, 1/2)^2$. We shall find ρ is a $C^{1/2}$-function.

The inradius of D is $R = 1/2$. So, we denote by Ω_2 the central set of D. We recall that

$$C^\delta = \{x \in D \ : \ d(x, D^\delta) = \delta\},$$

and for each $x \in D \setminus \Omega_2$, by Lemma 4.24, there exists a unique C^δ such that $x \in C^\delta$. Hence, ρ is determined by its level sets, i.e. C^δ. It is easy to check that $\rho(x) = 1/2$ for all $x \in \overline{\Omega}_2$. Let us formulate ρ in $D \setminus \overline{\Omega}_2$. We set $x = (x_1, x_2)$, and $\delta = 1/2 - t$ for $t \in (0, 1/2)$. Then, $x \in C^\delta$ if and only if it holds

$$\begin{cases} (x_1 - t)^2 + (x_2 - t)^2 = (\frac{1}{2} - t)^2 \\ x_1^2 + x_2^2 \geq \frac{1}{4}. \end{cases}$$

Thanks to these relations, we can get t as a function in x

$$t = x_1 + x_2 - \frac{1}{2} - \sqrt{2}\sqrt{\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)}.$$

Thus, we have

$$\rho(x) = \frac{1}{2} - t = 1 - (x_1 + x_2) + \sqrt{2}\sqrt{\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)}.$$

For some $x = (x_1, x_2)$, $y = (1/2, x_2)$, we have $|x - y| = |1/2 - x_1|$, then

$$\begin{aligned} |\rho(x) - \rho(y)| &\leq |\frac{1}{2} - x_1| + \sqrt{2}\sqrt{\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)} \\ &\leq |x - y| + \sqrt{|x - y|}. \end{aligned}$$

We notice that ρ is locally Lipschitzian but not Lipschitzian in $\overline{D}$. Moreover, ρ is a $C^{1/2}$-function. We give in the followings some calculations for the normalized gradient of ρ:

$$\nabla\rho(x) = \begin{pmatrix} -1 - \frac{\sqrt{2}}{2}a(x) \\ -1 - \frac{\sqrt{2}}{2}\frac{1}{a(x)} \end{pmatrix}, \qquad a(x) := \frac{\sqrt{\frac{1}{2} - x_2}}{\sqrt{\frac{1}{2} - x_1}};$$

$$|\nabla\rho(x)| = 1 + \frac{1}{\sqrt{2}}\left(a(x) + \frac{1}{a(x)}\right) = \frac{\rho(x)}{\sqrt{2}\sqrt{\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)}}.$$

In Figure 4.13, we present the cut-locus potential ρ and its normalized gradient on a square domain $D = (-1/2, 1/2)^2$. The simulation is done on a first quarter of D. We plot in the left subfigure the contours of ρ while in the right one with a magnifying glass the normalized gradient of ρ, i.e. $\nabla\rho/|\nabla\rho|$, in streamlines (in black) which start from $\partial\Omega_2$ and orthogonally across the geodesic levels of ρ (in varied colors).

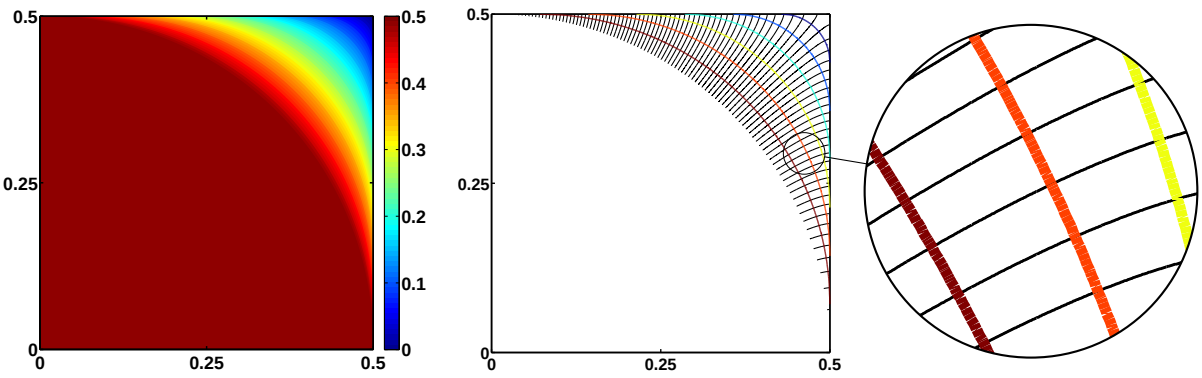


Figure 4.13: Illustration of ρ and its noramlized gradient $\nabla\rho/|\nabla\rho|$ in case $D = (-1/2, 1/2)^2$.

4.3.4 An explicit construction for calibrating fields on D

As we have already known, for each bounded convex open subset D of $\mathbb{R}^2$, for $\lambda = h_D$, problem $m(\lambda, D)$ has exactly two solution $\{\emptyset, C_D\}$ (see Proposition 4.4). The nonempty solution, i.e. C_D, is the Cheeger set of D, which actually is the union of balls contained in D of radius h_D^{-1}. Naturally, we expect that the unique solution to problem $m(\lambda, D)$ with $\lambda > h_D$ has the similar form which can be built of (4.36), namely the set Ω_λ. Evidently, C_D is Ω_{h_D}. According to Theorem 4.10, the optimality of Ω_λ is equivalent to finding an optimal solution $\overline{q}$ to the problem (4.30) satisfying

$$|\overline{q}| \leq 1 \quad \text{a.e. in } D, \qquad 0 \leq \operatorname{div} \overline{q} \leq \lambda \quad \text{a.e. in } D, \tag{4.63}$$

$$\overline{q} \cdot \nu_{\Omega_\lambda} = 1 \quad \mathcal{H}^1\text{-a.e. on } \partial\Omega_\lambda, \qquad \operatorname{div} \overline{q} = \lambda \quad \text{a.e. in } D \setminus \Omega_\lambda. \tag{4.64}$$

In the point of view of calibrability, $\overline{q}$ is a *calibrating field* for Ω_λ so that Ω_λ is a $\lambda\lambda_{\Omega_\lambda}^{-1}$-calibrable subset of D. Such a vector field can be explicitly constructed on D by starting from a calibrating field of the Cheeger set of D. This semi-explicit construction is provided with the existence of a calibrating field for the Cheeger set of D, that means $\overline{q} = q_{h_D}$ in Ω_{h_D} where $q_{h_D} \in L^\infty(\Omega_{h_D}; \mathbb{R}^2)$ satisfies

$$|q_{h_D}| \leq 1, \quad \operatorname{div} q_{h_D} = h_D \quad \text{a.e. in } \Omega_{h_D}, \qquad q_{h_D} \cdot \nu_{\Omega_{h_D}} = 1 \quad \mathcal{H}^1\text{-a.e. on } \partial\Omega_{h_D}.$$

It remains the explicit part of the semi-explicit construction which will be devised in two different modules: a construction of $\overline{q}$ on $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$ by using cut-locus potential ρ and a construction of $\overline{q}$ by a unit vector field with constant divergence on $D \setminus \overline{\Omega_\lambda}$.

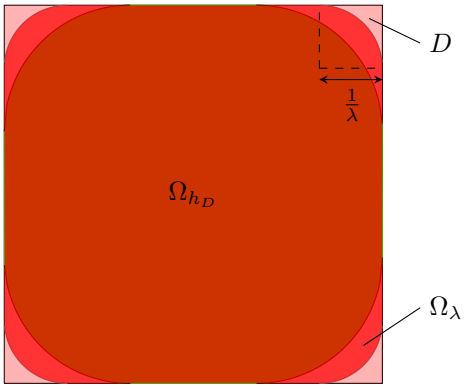


Figure 4.14: The inclusion $\Omega_{h_D} \subset \Omega_\lambda \subset D$.

Figure 4.14 illustrates the strict inclusion $\Omega_{h_D} \subset \Omega_\lambda \subset D$ for $\lambda > h_D$ and D being a square. We start the construction of $\overline{q}$ with the calibrating field for the Cheeger set Ω_{h_D} and build it outside continuously: in $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$, then in $D \setminus \Omega_\lambda$. This procedure will produce a calibrating field for Ω_λ on D.

A construction of $\overline{q}$ on $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$ by using cut-locus potential ρ

The construction for the calibrating field $\overline{q}$ only has meaning when D is not a self-Cheeger set. We assume D is not a Cheeger set so that $\Omega_{h_D} \neq \Omega_\lambda$. On the nonempty components of $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$, we define $\overline{q} = q_\rho$ with

$$q_\rho := -\frac{\nabla \rho}{|\nabla \rho|}$$

where ρ is the cut-locus potential discussed in the previous paragraph. Immediately, we have

$$|q_\rho| = 1 \quad \text{in } \Omega_\lambda \setminus \overline{\Omega_{h_D}}, \qquad q_\rho \cdot \nu_{\Omega_\lambda} = 1 \quad \text{on } \partial\Omega_\lambda. \tag{4.65}$$

The normalized field q_ρ inherits from the cut-locus potential ρ the locally Lipschitzian property. By Lemma 4.30 and Theorem 4.34, it holds $h_D < \rho^{-1} < \lambda$ in $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$, hence,

$$\operatorname{div} q_\rho = \frac{1}{\rho}, \qquad \operatorname{div} q_\rho \in [h_D, \lambda] \quad \text{a.e. in } \Omega_\lambda \setminus \overline{\Omega_{h_D}}. \tag{4.66}$$

Additionally, this is a continuous construction of $\overline{q}$ in $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$ since it maintains the orthogonal trace of the field, i.e.

$$q_\rho \cdot \nu_{\Omega_{h_D}} = q_{h_D} \cdot \nu_{\Omega_{h_D}} = 1 \quad \text{on } \partial\Omega_{h_D}. \tag{4.67}$$

It is clear that the properties (4.65), (4.66), (4.67) are compatible with the conditions (4.63)-(4.64).

A construction of $\overline{q}$ by a unit vector field with constant divergence on $D \setminus \overline{\Omega_\lambda}$

We shall complete the construction of $\overline{q}$ on the entire D with designing a unit vector field q_λ in $D \setminus \overline{\Omega_\lambda}$ such that

$$q_\lambda \cdot \nu_{\Omega_\lambda} = q_\rho \cdot \nu_{\Omega_\lambda} = 1 \quad \text{on } \partial\Omega_\lambda,$$

and q_λ fits in with the conditions (4.63)-(4.64).

Lemma 4.38. *There exists a vector field q_λ in $D \setminus \overline{\Omega_\lambda}$ satisfying*

$$|q_\lambda| \le 1 \ \text{ in } D \setminus \overline{\Omega_\lambda}, \qquad \operatorname{div} q_\lambda = \lambda \ \text{ in } D \setminus \overline{\Omega_\lambda}, \qquad q_\lambda \cdot \nu_{\Omega_\lambda} = 1 \ \text{ on } \partial\Omega_\lambda. \tag{4.68}$$

Proof. Only here, we set $\delta = 1/\lambda$ for the simplification of notations. We reuse the notations introduced at the beginning of Section 4.3.3. Since D^δ is a convex set, the singular part of its boundary $\partial_s D^\delta$ has many at most countable points, denoted by $\{x_j : j \in J\} := \partial_s D^\delta$. For each point $x \in \partial_s D^\delta$, the normal cone of D^δ at the point, denoted by $N_{D^\delta}(x)$, is generated by the two limit vectors $\nu^-_{D^\delta}(x)$ and $\nu^+_{D^\delta}(x)$. $\varphi^\delta_-(x)$ and $\varphi^\delta_+(x)$ are the corresponding angles of the two vectors in S^1. We recall the definition of the sets $N^\delta(x)$, $C^\delta(x)$, and $M^\delta_{\delta'}(x)$:

$$\begin{aligned} N^\delta(x) &:= x + \{p \in N_{D^\delta}(x) \ : \ |p| \le \delta\}, \ \text{for } x \in \overline{D^\delta}, \\ C^\delta(x) &:= x + \{p \in N_{D^\delta}(x) \ : \ |p| = \delta\}, \ \text{for } x \in \partial D^\delta, \\ M^\delta_{\delta'}(x) &:= x + \left\{p \in N_{D^\delta}(x) \ : \ \left\langle p, \nu^\pm_{D^\delta}(x)\right\rangle \le \delta - \delta'\right\}, \ \text{for } x \in \overline{D^\delta},\ 0 \le \delta' < \delta < R. \end{aligned}$$

Then, Ω_λ can be characterized by using $N^\delta(x)$ and $C^\delta(x)$,

$$\overline{\Omega_\lambda} = \bigcup_{x \in D^\delta} N^\delta(x), \qquad \partial\Omega_\lambda = \bigcup_{x \in \partial D^\delta} C^\delta(x). \tag{4.69}$$

It is remarkable that $\{N_{D^\delta}(x) \ : \ x \in D^\delta\}$ is a family of disjoint sets. In fact, if it isn't true, there exists a point that have at least two distinguished projections on $\overline{D^\delta}$, a contradiction to the fact that $\overline{D^\delta}$ is a closed convex set. So, $\{N^\delta(x) : x \in D^\delta\}$ will give a partition to $\overline{\Omega_\lambda}$. We also observe that $C^\delta(x)$ describing the boundary of $\cup_{x \in D^\delta} N^\delta(x)$ is a point or an arc of a circle centered at x of radius δ. By definition of Ω_λ, $C^\delta(x)$ is clearly contained in $\overline{\Omega}_\lambda$. When x is a regular point of ∂D^δ, i.e. $x \in \partial_r D^\delta$, the normal cone $N_{D^\delta}(x)$ will reduce to only one direction and $C^\delta(x)$ will be a singleton, let's say $C^\delta(x) = \{y_x\}$. In that case, y_x must lie on $\partial D \cap \partial\Omega_\lambda$. Otherwise, if $x \in \partial_s D^\delta$, $N^\delta(x)$ is indeed the intersection between the normal cone $x + N_{D^\delta}(x)$ and the closed ball $\overline{B(x,\delta)}$ and, $C^\delta(x)$ is the arc of $B(x,\delta)$ in $N^\delta(x)$. Since ρ is increasing map, $N^\delta(x)$ is characterized by

$$N^\delta(x) = x + \left(\operatorname{cone}\{\nu^-_{D^\delta}(x), \nu^+_{D^\delta}(x)\} \cap \overline{B(0,\delta)}\right).$$

And, $C^\delta(x)$ is determined by the interval $(\varphi^\delta_-(x), \varphi^\delta_+(x))$. In addition, these angle intervals must be less than π. Because $\operatorname{int} D^\delta \neq \emptyset$, for very $x \in \partial D^\delta$, the tangent cone at x is non trivial. As consequence, $N_{D^\delta}(x)$ being the polar of tangent cone always has its angle less than π. Inversely, for every $y \in \partial\Omega_\lambda$, y can be decomposed as $y = \Pi_\delta(y) + (y - \Pi_\delta(y))$ with $|y - \Pi_\delta(y)| = \delta$. Or, $y \in C^\delta(\Pi_\delta(y))$. Hence, (4.69) is verified. Moreover, $\{C^\delta(x) \ : \ x \in \partial D^\delta\}$ is a partition of $\partial\Omega_\lambda$. For abbreviation, for each $x_j \in \partial_s D^\delta$, we set

$$N^\delta_j := N^\delta(x_j), \qquad C^\delta_j := C^\delta(x_j), \tag{4.70}$$

$$\nu_j := \frac{\nu^+_{D^\delta}(x_j) + \nu^-_{D^\delta}(x_j)}{2}, \qquad \varphi_j := \frac{\varphi^\delta_+(x_j) - \varphi^\delta_-(x_j)}{2}. \tag{4.71}$$

We remark that $\partial\Omega_\lambda \cap D \subset \bigcup_{j\in J} C^\delta_j$, keeping in mind C^δ_j are arcs of radius δ given by a triple of center, angle, and oriented unit vector, respectively $(x_j, \varphi_j, \nu_j) \in \partial_s D^\delta \times (0, \pi/2) \times S^1$. For instance, see Figures 4.15.

Now, let us introduce the regions where we want to construct the vector field q_λ,

$$\text{for } x \in D^\delta, \quad M^\delta_0(x) = x + \left\{p \in N_{D^\delta}(x) \ : \ \left\langle p, \nu^+_{D^\delta}(x)\right\rangle \leq \delta, \ \left\langle p, \nu^-_{D^\delta}(x)\right\rangle \leq \delta\right\}, \tag{4.72}$$

$$\Sigma_\lambda := \bigcup_{x\in D^\delta} M^\delta_0(x), \qquad \Delta_\lambda := \Sigma_\lambda \setminus \overline{\Omega_\lambda}. \tag{4.73}$$

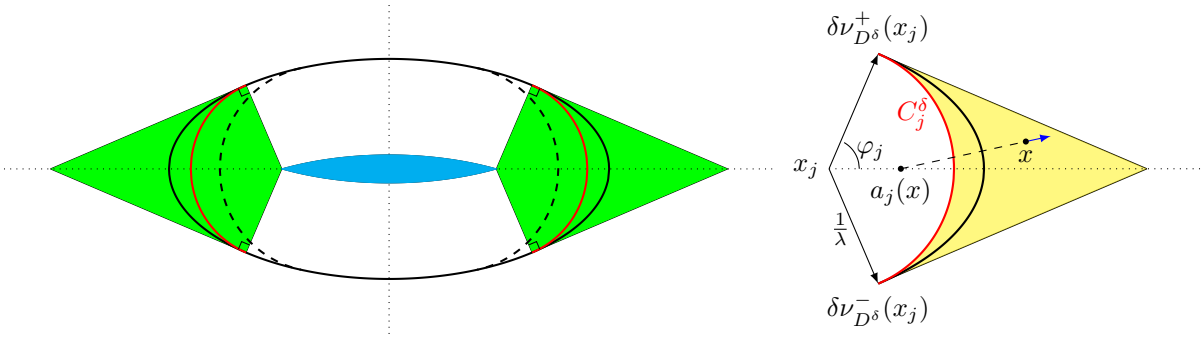


Figure 4.15: A configuration for the extension of a calibrating field on an ellipse domain.

Figure 4.15 presents the settings of the construction when D is an ellipse domain. The thick black curve is the boundary of D. The dashed curve is the boundary of the Cheeger set Ω_{h_D} of D. The cyan region is the set D^δ which generates Ω_λ. Ω_λ is illustrated in the figure with its boundary in red color. The sets $M^\delta_0(x_j)$ are in green background. The region in light yellow background is a component of Δ_λ. C^δ_j are red arcs of radius δ, centered at x_j, determined by angle φ_j and the oriented unit vector ν_j. $\{C^\delta_j\}$ are the boundaries of Ω_λ inside D, i.e. $\partial\Omega_\lambda \cap D$.

$\overline{D}$ should be contained in Σ_λ. In fact, for every $y \in \overline{D}$, let $x = \Pi_\delta(y)$ and $p = y - x$. Since $|x - (x + \delta\nu^\pm_{D^\delta}(x))| = \delta = d(x, D^c)$ and $x + \delta\nu^\pm_{D^\delta}(x) \in \partial D$, we have that $x + \delta\nu^\pm_{D^\delta}(x)$ are projections of x on D^c. As $\overline{D}$ is convex, these projections implies that for all $z \in \overline{D}$,

$$\left\langle x - (x + \delta\nu^-_{D^\delta}(x)), z - (x + \delta\nu^-_{D^\delta}(x))\right\rangle \geq 0,$$
$$\left\langle x - (x + \delta\nu^+_{D^\delta}(x)), z - (x + \delta\nu^+_{D^\delta}(x))\right\rangle \geq 0.$$

Therefore, for $z = y$, we obtain $\langle p, \nu^\pm_{D^\delta}(x)\rangle \leq \delta$. Hence, by definition (4.72), $y = x + p \in M^\delta_0(x)$. In particular, when $x \in \partial_r D^\delta$, the left and right limits are the same, i.e. $\nu^-_{D^\delta}(x) = \nu^+_{D^\delta}(x) =: \nu_{D^\delta}(x)$, while $N^\delta(x)$ coincides with $M^\delta_0(x)$ and they are folded up to be a segment. At that moment, p and $\nu_{D^\delta}(x)$ are co-linear, then, $\langle p, \nu_{D^\delta}(x)\rangle = |p| \leq \delta$. This is to say that $y \in N^\delta(x) = M^\delta_0(x)$.

Of course, when $y \in D^\delta$ or $y = \Pi_\delta(y)$, the associated cones degenerate and shrink to a point. We get $y = \Pi_\delta(y) = N^\delta(y) = M_0^\delta(y)$.

We notice that since the angle of $N_{D^\delta}(x)$ is always less than π, $M_0^\delta(x)$ is bounded for every $x \in D^\delta$. Σ_λ is then bounded. It is clear that $\{M_0^\delta(x) \,:\, x \in D^\delta\}$ is a partition of Σ_λ and hence, Δ_λ admits a decomposition,

$$\Delta_\lambda = \bigcup_{x\in D^\delta} M_0^\delta(x) \setminus \overline{\Omega}_\lambda = \bigcup_{x\in \partial_s D^\delta} M_0^\delta(x) \setminus \overline{\Omega_\lambda}. \tag{4.74}$$

For short, we set

$$\Delta_\lambda^j = M_0^\delta(x_j) \setminus \overline{\Omega}_\lambda \text{ for some } x_j \in \partial_s D^\delta, \qquad \text{and} \qquad \Delta_\lambda = \bigcup_{j\in J} \Delta_\lambda^j. \tag{4.75}$$

Δ_λ has many at most countable disjoint components. Figures 4.15 and 4.4 are examples performing this configuration. In Figure 4.15, Σ_λ strictly contains D whereas they coincide in case of Figure 4.4.

We now explicitly construct the vector field q_λ in Δ_λ satisfying conditions (4.68). In each component Δ_λ^j, we set $x = (s,t)$ and

$$\Delta_\lambda^j \ni (s,t) \mapsto q_1^j(s,t) := (s - a_j(s,t)\nu_j^s, t - a_j(s,t)\nu_j^t), \tag{4.76}$$

where $\nu_j = (\nu_j^s, \nu_j^t)$ is the oriented unit vector defined C_j^δ and $a_j(s,t) \geq 0$ such that $\operatorname{div} q_1^j = 1$. In fact, q_1^j is the unit normal of the ball of radius 1 centered at point $a_j\nu_j$,

$$[s - a_j(s,t)\nu_j^s]^2 + [t - a_j(s,t)\nu_j^t]^2 = 1. \tag{4.77}$$

We observe also that

$$\operatorname{div} q_1^j(s,t) = 1 \Longleftrightarrow \partial_s a_j(s,t)\nu_j^s + \partial_t a_j(s,t)\nu_j^t = 1. \tag{4.78}$$

From equation (4.77), we can find out explicitly a_j in function of (s,t), and in such a way, (4.78) is fulfilled,

$$a_j(s,t) = s\nu_j^s + t\nu_j^t - \sqrt{1 - (s\nu_j^t - t\nu_j^s)^2}. \tag{4.79}$$

Therefore, $q_\lambda(s,t) = q_\lambda^j(s,t) := q_1^j(\lambda s, \lambda t)$ in Δ_λ^j is a construction that we expected. This completes the proof. ∎

Example 4.39 (Construction of a calibration)**.**

(a) In case $D = (-\frac{1}{2}, \frac{1}{2})^2$, Σ_λ coincides with D, see Figure 4.4. The boundary of D^δ has 4 singular points and $\partial\Omega_\lambda \cap D = \cup_{j=1}^4 C_j^\delta$. The oriented vectors of C_j^δ are $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Take $\nu_1 = (1/\sqrt{2}, 1/\sqrt{2})$ for example, it is easy to give an explicit construction of the calibrating field q_λ in Δ_λ^1. Thanks to (4.79) and (4.76), we have, for $(s,t) \in \Delta_\lambda^1$,

$$a_1(s,t) = \frac{s+t}{\sqrt{2}} - \sqrt{1 - \frac{(s-t)^2}{2}}, \qquad q_1^1(s,t) = \left(s - \frac{a_1(s,t)}{\sqrt{2}}, t - \frac{a_1(s,t)}{\sqrt{2}}\right).$$

Then, the expected construction of q_λ in Δ_λ^1 is given by $q_\lambda(s,t) = q_\lambda^1(s,t) = q_1^1(\lambda s, \lambda t)$. By the same way, we obtain the construction of q_λ in other components of Δ_λ.

(b) In case D given by an ellipse of standard form, see Figure 4.15, D is strictly contained in Σ_λ. Δ_λ now has two components and the boundary of Ω_λ inside D is the union of arcs C_1^δ and C_2^δ whose oriented vectors are $(\pm 1, 0)$. Take $\nu_1 = (1, 0)$ for example to construct q_λ in Δ_λ^1, we get, for $(s, t) \in \Delta_\lambda^1$,

$$a_1(s,t) = s - \sqrt{1-t^2}, \qquad q_1^1(s,t) = \left(\sqrt{1-t^2}, t\right).$$

Therefore, we obtain $q_\lambda(s,t)$ by scaling $q_1^1(s,t)$, i.e. $q_\lambda(s,t) = q_\lambda^1(s,t) = q_1^1(\lambda s, \lambda t)$.

In summary, the vector field $\overline{q}$ can be built with q_{h_D} on Ω_{h_D}, with q_ρ on $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$ by means of cut-locus potential ρ, then glued with q_λ on $D \setminus \overline{\Omega_\lambda}$ so that we can obtain a calibrating field $\overline{q}$ which calibrate Ω_λ in the sense of θ-calibrability. We remark also that the construction of the calibrating field $\overline{q}$ can be done in a region which is more large than D, i.e. in Σ_λ.

4.4 Comparison results

We now go back to the initial question about the comparison between free boundary problems $\beta(\lambda)$ and $\beta_0(\lambda)$ that we have mentioned at the beginning of this chapter.

As observed from the optimality conditions given in Sections 4.2.1, if $u \equiv 0$ solves $\beta(\lambda)$, then it solves $\beta(\lambda')$ for all $0 \le \lambda' < \lambda$, and similarly, if $u \equiv 1$ solves $\beta(\lambda)$, then it solves $\beta(\lambda'')$ for all $\lambda'' > \lambda$. Accordingly, we have introduced the critical values $\lambda_0, \lambda_1 \in [0, +\infty]$ defined by

$$\lambda_0 = \sup\{\lambda \ge 0 \ : \ u \equiv 0 \text{ solves } \beta(\lambda)\}, \tag{4.80}$$

$$\lambda_1 = \inf\{\lambda \ge 0 \ : \ u \equiv 1 \text{ solves } \beta(\lambda)\}. \tag{4.81}$$

Following this definition, if $\lambda_0 < \lambda_1$ and $\lambda \in (\lambda_0, \lambda_1)$ then a solution to problem $\beta(\lambda)$ of the kind $\mathbf{1}_\Omega$ is such that $0 < |\Omega| < |D|$. The key argument in order to show that $\beta(\lambda) < \beta_0(\lambda)$ for all λ in (λ_0, λ_1) (see Theorem 4.40) will be showing that such a characteristic function $\mathbf{1}_\Omega$ cannot be optimal for $\beta(\lambda)$. On the other hand, as the relaxed funcionals E_λ and E_λ^0 coincide on characteristic functions (see (4.6)), it is clear that $\beta(\lambda) = \beta_0(\lambda)$ for all $\lambda \notin (\lambda_0, \lambda_1)$.

Theorem 4.40. *Assume that $\lambda_0 < \lambda_1$. Then it holds $\beta(\lambda) < \beta_0(\lambda)$ for every $\lambda \in (\lambda_0, \lambda_1)$.*

Proof. Let $\lambda \in (\lambda_0, \lambda_1)$. We shall prove the theorem by contradiction. Suppose that $\beta(\lambda) = \beta_0(\lambda)$. Then, $\overline{u} = \mathbf{1}_\Omega$ is a solution for $\beta(\lambda)$ where $\Omega \subset D$ is such that $0 < |\Omega| < |D|$ (since $\lambda \in (\lambda_0, \lambda_1)$) and $\mathbf{1}_\Omega$ solves $\beta_0(\lambda)$. For such Ω, let us define

$$\alpha_\varepsilon = \inf\left\{\int_{D\setminus\Omega}\left(\sqrt{1+|\nabla u|^2}-1\right)dx + \int_{\partial D\setminus\overline{\Omega}} |u| d\mathcal{H}^{N-1} : u \in W^{1,1}(D\setminus\Omega), u = \varepsilon \text{ on } \partial\Omega \cap D\right\}.$$

We firstly claim that

$$\lim_{\varepsilon\to 0^+} \frac{\alpha_\varepsilon}{\varepsilon} = 0. \tag{4.82}$$

By definition of α_ε, for every $\varepsilon > 0$, there is $\widetilde{u}_\varepsilon \in W^{1,1}(D\setminus\Omega)$ such that $\widetilde{u}_\varepsilon = \varepsilon$ on $\partial\Omega \cap D$, and

$$\int_{D\setminus\Omega}\left(\sqrt{1+|\nabla\widetilde{u}_\varepsilon|^2}-1\right)dx + \int_{\partial D\setminus\overline{\Omega}} |u_\varepsilon| d\mathcal{H}^{N-1} < \alpha_\varepsilon + \varepsilon^2.$$

We then set

$$u_\varepsilon = \begin{cases} 1 & \text{in } \Omega, \\ \widetilde{u}_\varepsilon & \text{in } D\setminus\Omega. \end{cases}$$

Clearly, $u_\varepsilon \to \overline{u}$ as $\varepsilon \to 0$. On the other hand, we observe that

$$\begin{aligned} E_\lambda(u_\varepsilon) - E_\lambda(\overline{u}) &= \int_{D\setminus\Omega} \left(\sqrt{1+|\nabla \widetilde{u}_\varepsilon|^2} - 1\right) dx + \int_{\partial D\setminus\overline{\Omega}} |u_\varepsilon| d\mathcal{H}^{N-1} - \varepsilon P(\partial\Omega \cap D) \\ &\leq \alpha_\varepsilon + \varepsilon^2 - \varepsilon P(\partial\Omega \cap D). \end{aligned}$$

This leads to the contradiction

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_\lambda(u_\varepsilon) \leq \lim_{\varepsilon\to 0^+} \left(\frac{\alpha_\varepsilon}{\varepsilon} + \varepsilon\right) - P(\partial\Omega \cap D) < 0.$$

To complete the proof, let us now prove the claim (4.82). For each $\eta > 0$, there exists $v \in W^{1,1}(D \setminus \Omega)$ such that $v = 1$ on $\partial\Omega \cap D$ and

$$\int_{\partial D\setminus\overline{\Omega}} |v| d\mathcal{H}^{N-1} < \eta.$$

Let $u_\varepsilon = \varepsilon v$. Then, $u_\varepsilon = \varepsilon$ on $\partial\Omega \cap D$, and hence,

$$\begin{aligned} \alpha_\varepsilon &\leq \int_{D\setminus\Omega} \left(\sqrt{1+|\nabla u_\varepsilon|^2} - 1\right) dx + \int_{\partial D\setminus\overline{\Omega}} |u_\varepsilon| d\mathcal{H}^{N-1} \\ &\leq \int_{D\setminus\Omega} \left(\sqrt{1+\varepsilon^2|\nabla v|^2} - 1\right) dx + \varepsilon\eta. \end{aligned}$$

We derive that for all $\eta > 0$, $\limsup\limits_{\varepsilon\to 0^+} \dfrac{\alpha_\varepsilon}{\varepsilon} \leq \eta$. So, it holds true (4.82). ■

In the one dimensional case, it is possible to compute explicitly the values of λ_0, λ_1. It turns out that these values coincide if and only if the length of the interval D is less than 2. Without loss of generality, lets us consider $\beta(\lambda)$ and $\beta_0(\lambda)$ associated with the domain $D_h = (-h, h)$ (whose Cheeger constant is $h_D = \frac{1}{h}$):

$$\begin{aligned} \beta(\lambda) &= \inf\left\{\int_{-h}^{h} \sqrt{1+u'^2}\, dx - \lambda|\{u \geq 1\}| \ : \ u \in W^{1,1}(-h,h),\ u(\pm h) = 0\right\}, \\ \beta_0(\lambda) &= 2h + \inf\left\{\int_{-h}^{h} |u'|\, dx - \lambda|\{u = 1\}| \ : \ u \in W^{1,1}(-h,h),\ u(\pm h) = 0\right\}. \end{aligned}$$

Since D_h is a Cheeger set, the solution to $\beta_0(\lambda)$ is either $u \equiv 0$ or $u \equiv 1$ (by invoking the identity (4.9) and Propositions 4.4, 4.5). Thus:

$$\beta_0(\lambda) = \min\{2h, 2(1 + h(1-\lambda))\}\ .$$

For problem $\beta(\lambda)$, it is easy to check that the optimal solution is either $u \equiv 0$ or of the radial form $u_\alpha(x) = \min\{1, (h - |x|)/\alpha\}$ for a suitable value of parameter $\alpha \in [0, h]$. Plugging in the expression of E_λ given above, we are then reduced to minimize the convex function

$$r_h(\alpha) := 2\left((1-\lambda)(h-\alpha) + \sqrt{1+\alpha^2}\right), \quad \alpha \in [0, h]$$

and to compare its minimum value with $E_\lambda(0) = 2h$. Note that the derivative $r_h'(\alpha) = 2\left(\frac{\alpha}{\sqrt{1+\alpha^2}} - (1-\lambda)\right)$ vanishes at a unique point α_λ which satisfies $\sqrt{1+\alpha_\lambda^2} = 1 - \lambda$. We have to consider three cases depending on whether or not $\alpha_\lambda \in [0, h]$ and $h \leq 1$.

- If $\lambda \geq 1$, then the minimum of r_h is reached at $\alpha = 0$ and therefore $\beta(\lambda) = \beta_0(\lambda) = \min\{2h, 2(1+h(1-\lambda))\}$. The solution u_λ is then $u_\lambda \equiv 0$ if $\lambda h < 1$ and $u_\lambda \equiv 1$ if $\lambda h > 1$, whereas the two solutions coexist for $\lambda h = 1$.

- If $0 \le \lambda < 1$ and $h \le 1$, the for every $\alpha \in [0,h]$, we have $r_h(\alpha) \ge 2\sqrt{1+\alpha^2} \ge 2\,h$. Thus, noticing that $\lambda h < 1$, we are led to $\beta(\lambda) = \beta_0(\lambda) = 2h$ and the unique solution is $u_\lambda \equiv 0$. Accordingly, combining with the first case above, we infer that $\lambda_0 = \lambda_1 = \frac{1}{h}$ if $h \le 1$.

- If $0 \le \lambda < 1$ and $h > 1$, the situation is a litle more complicated as we may find a solution which differs from 0 and 1. The value α_λ belongs to $[0,h]$ if and only if $\lambda_* := 1 - \frac{h}{\sqrt{1+h^2}} \le \lambda < 1$. If $\lambda < \lambda_*$, then the minimum value of $r_h(\alpha)$ is reached for $\alpha = h$. Since $r_h(h) = 2\sqrt{1+h^2} > 2h$, we obtain a unique solution $u_\lambda \equiv 0$ and $\beta(\lambda) = \beta_0(\lambda) = 2h$. If $\lambda \in [\lambda_*, 1)$, then we have to check the sign of $r_h(\alpha_\lambda) - 2h$. After a tedious but straightforward computation, we obtain that this quantity is positive unless $\lambda^* := \frac{2}{1+h^2} \le \lambda < 1$ (this latter situation is possible since, for $h > 1$, it holds $\lambda_* < \lambda^* < 1$). All in all, we deduce that still $u_\lambda \equiv 0$ is the unique solution and $\beta(\lambda) = \beta_0(\lambda) = 2h$ if $\lambda \in [0, \lambda^*)$. In contrast, for $\lambda \in (\lambda^*, 1)$, the solution becomes $u_\lambda = u_{\alpha_\lambda}$ and we obtain a strict inequality $\beta(\lambda) < \beta_0(\lambda)$.

Summarizing we have shown that the critical values λ_0, λ_1 differ if and only if $h > 1$. More precisely these values are given in terms of h as follows:

$$\lambda_0(h) \;=\; \begin{cases} \frac{1}{h} & \text{if } h \le 1 \\ \frac{2}{1+h^2} & \text{if } h \ge 1 \end{cases} \quad , \quad \lambda_1(h) \;=\; \begin{cases} \frac{1}{h} & \text{if } h \le 1 \\ 1 & \text{if } h \ge 1 \end{cases} \tag{4.83}$$

In higher dimension $N \ge 2$, explicit computations are not available (except in the radial case). However we are able to derive some estimates where the role of the Cheeger constant of D is enlightened.

Proposition 4.41. *It holds*

$$0 < \lambda_0 \;\le h_D \;\le \frac{P(D)}{|D|} \;\le\; \lambda_1.$$

Furthermore, if D is not a Cheeger set, one has the strict inequalities $\lambda_0 < h_D < \frac{P(D)}{|D|}$, thus $\beta(\lambda) < \beta_0(\lambda)$ for every $\lambda \in [h_D, \frac{P(D)}{|D|})$.

Proof. By definition of h_D, we have $h_D \le \lambda_D := P(D)/|D|$. If $u \equiv 0$ is a solution to $\beta(\lambda)$ then it holds

$$\beta(\lambda) = \beta_0(\lambda) = |D|.$$

Since $\beta_0(\lambda) = |D| + m(\lambda, D)$ (see (4.9)), the second equality above implies that $m(\lambda, D) = 0$. Thus, by invoking Proposition 4.4, we deduce that $\lambda_0 \le h_D \le \lambda_D$. In addition, we observe that

$$E_\lambda(1) - E_\lambda(0) = P(D) - \lambda|D| > 0$$

for all $\lambda < \lambda_D$. Therefore, whenever $u \equiv 1$ is a solution to the problem $\beta(\lambda)$, it must hold $\lambda \ge \lambda_D$ (since $E_\lambda(1) \le E_\lambda(0)$). Hence, by definition of λ_1, we get $\lambda_D \le \lambda_1$.

Let us now prove that $\lambda_0 > 0$. To this end, we shall exploit calibration fields of the dual problem associated to $\beta(\lambda)$ in order to show that there exists a small strictly positive value λ so that $u \equiv 0$ minimizes $\beta(\lambda)$. Then, by definition of λ_0, we infer that $\lambda_0 > 0$. Notice that if $u \equiv 0$ minimizes $\beta(\lambda)$ then it minimizes $\beta_0(\lambda)$, too.

Assume that λ is small enough such that $0 < \lambda < \lambda_0$, namely $0 < \lambda \ll 1$. Let $q : D \to \mathbb{R}^N$ be a vector field satisfying

$$|q| \le \frac{\lambda}{h_D} \quad \text{and} \quad \operatorname{div} q = \lambda \quad \text{in } D.$$

We notice that such a vector field exists, for instance $q = \frac{\lambda}{h_D}\widetilde{q}$ where $\widetilde{q}$ is the calibrating field for the Cheeger set of D. Let $Q := D \times I$ with $I := [0,1]$. Now we construct a vector field $\sigma = (\sigma^x(x,t), \sigma^t(x,t))$ in Q, of the form below

$$\sigma^x(x,t) = -a(t)q(x), \quad \sigma^t(x,t) = \lambda A(t), \quad \text{for } (x,t) \in Q, \tag{4.84}$$

where the real valued functions $a : I \to \mathbb{R}$ and $A : I \to \mathbb{R}$ satisfy

$$a(0) = 0, \qquad A(t) = \int_0^t a(s)ds - \frac{1}{\lambda}. \tag{4.85}$$

In order to obtain a calibration field σ of the form (4.84)-(4.85) which calibrates $u \equiv 0$ for problem $\beta(\lambda)$, the field σ must satisfy the optimality conditions (4.13)-(4.14), namely

$$\sigma^x(x,0) = 0, \quad \sigma^t(x,0) = -1, \quad \sigma^t(x,1) \geq \lambda - 1 \quad \text{on } D, \tag{4.86}$$

$$\sigma^t + \sqrt{1 - |\sigma^x|^2} \geq 0, \quad \operatorname{div}\sigma = 0 \quad \text{in } Q. \tag{4.87}$$

It is easy to see that for any function $a(t)$ such that

$$|a(t)| \leq \frac{h_D}{\lambda} \quad \forall t \in I, \qquad \int_0^1 a(s)ds \geq 1, \qquad \sqrt{1 - a(t)^2|q(x)|^2} + \lambda A(t) \geq 0 \quad \forall (x,t) \in Q,$$

the vector field σ defined by (4.84)-(4.85) fulfills the conditions (4.86)-(4.87). Fortunately, such a function $a(t)$ is available. For example, we can choose $a(t) = 2t$ whereas λ should be taken small enough but strictly positive (i.e. $0 < \lambda \ll 1$) and such that

$$\lambda \leq \min\left\{\frac{h_D}{2}, \frac{h_D^2}{4}\right\}. \tag{4.88}$$

We have finished the proof of $\lambda_0 > 0$. In summary, we have $0 < \lambda_0 \leq h_D \leq \frac{P(D)}{|D|} \leq \lambda_1$. When D is not a Cheeger set, it is clear that $h_D < \frac{P(D)}{|D|}$. In the case $\lambda = h_D$, let Ω be the Cheeger set of D. Then Ω must be strictly contained in D so that $0 < |\Omega| < |D|$. And in this case, $\mathbf{1}_\Omega$ is a solution to $\beta_0(\lambda)$. We repeat the argument that the characteristic function $\mathbf{1}_\Omega$ with $0 < |\Omega| < |D|$ cannot solve the problem $\beta(\lambda)$ (which is evidenced in the proof of Theorem 4.40), so as to show $\beta(h_D) < \beta_0(h_D)$. In other words, $\lambda_0 < h_D$, and hence, we get $\beta(\lambda) < \beta_0(\lambda)$ for every $\lambda \in [h_D, \frac{P(D)}{|D|})$. ■

Remark 4.42. In the case where D is a θ-calibrable set but not a Cheeger set, the calibration constant of D is strictly larger than 1, namely $\theta_D > 1$ (see (4.33) for the definition of θ_D). Then we have $[h_D, \theta_D\lambda_D) \subset (\lambda_0, \lambda_1)$ (where $\lambda_D := P(D)/|D|$). In fact, by Proposition 4.41, it holds $\lambda_0 < h_D$. On the other hand, for all $\lambda \in (h_D, \theta_D\lambda_D)$, any solution Ω of the geometric problem $m(\lambda, D)$ (see (4.10)) is strictly contained in D, thus $0 < |\Omega| < |D|$. In view of (4.8)-(4.9), the characteristic function $\mathbf{1}_\Omega$ minimizes problem $\beta_0(\lambda)$. But, $\mathbf{1}_\Omega$ (with $0 < |\Omega| < |D|$) cannot be a solution for problem $\beta(\lambda)$ (see the proof of Theorem 4.40). This is to say that $\beta(\lambda) < \beta_0(\lambda)$ for all $\lambda \in (h_D, \theta_D\lambda_D)$.

Proposition 4.43. *We have:*

(i) $\lambda_0 \geq \lambda_0^*$ *where*

$$\lambda_0^* := \begin{cases} I^{-1}(h_D) & \text{if } h_D \leq \frac{\pi}{2} \\ 1 + h_D - \frac{\pi}{2} & \text{if } h_D \geq \frac{\pi}{2} \end{cases}, \quad I(y) = \int_0^y \frac{ds}{\sqrt{s(2-s)}}. \tag{4.89}$$

(ii) $\theta_D\lambda_D \le \lambda_1 \le \theta_D\lambda_D + 1$ *where* θ_D *is the calibration constant of* D *and* $\lambda_D := P(D)/|D|$. *Particularly, if* D *is a convex set of class* $C^{1,1}$, *then*

$$\max\{\lambda_D, (N-1)\kappa_\infty(\partial D)\} \le \lambda_1 \le 1 + \max\{\lambda_D, (N-1)\kappa_\infty(\partial D)\}. \tag{4.90}$$

When ∂D *exhibits a corner,* $\lambda_1 = +\infty$ *(i.e.* $u \equiv 1$ *can't neither be a solution to* $\beta(\lambda)$*).*

Proof. Let $Q := D \times [0,1]$.
i) We are seeking a vector field of the type

$$\sigma(x,t) = \Big(-a(t)q(x),\ A(t)\operatorname{div} q(x) - 1\Big) \quad \text{for } (x,t) \in Q$$

with

$$A'(t) = a(t), \quad a(0) = A(0) = 0, \quad \text{and} \quad |q| \le 1, \quad \operatorname{div} q = h_D \quad \text{in } D\,, \tag{4.91}$$

so that σ is a calibration of $u \equiv 0$ for the problem $\beta(\lambda)$ (Note that q exists as the calibrating field of Cheeger set of D). This happens if and only if σ satisfies the optimality conditions (4.13)-(4.14), namely

$$\sigma^x(x,0) = 0, \quad \sigma^t(x,0) = -1, \quad \sigma^t(x,1) \ge \lambda - 1 \quad \text{on } D,$$
$$\sigma^t + \sqrt{1 - |\sigma^x|^2} \ge 0, \quad \operatorname{div}\sigma = 0 \quad \text{in } Q.$$

We deduce from these conditions that the function $a(t)$ and its primitive function $A(t)$ should be chosen, taking into account (4.91), such that

$$A(1)h_D \ge \lambda, \quad \text{and} \quad \sqrt{1 - |a(t)|^2} + A(t)h_D \ge 1, \quad \forall t \in [0,1].$$

We then set

$$M(h) := \sup\Big\{A(1)\ :\ A(0) = 0,\ \sqrt{1 - |A'(t)|^2} + A(t)h \ge 1\ , \forall t \in [0,1]\Big\}.$$

It follows immediately that $\lambda \le M(h_D)h_D$. If we set $\lambda_0^* := M(h_D)h_D$, then by definition, $\lambda_0 \ge \lambda_0^*$. Let $\psi(t) := h_D A(t)$ in $[0,1]$, we have

$$h_D M(h_D) = \sup\left\{\psi(1)\ :\ \psi(0) = 0,\ \sqrt{1 - \frac{|\psi'|^2}{h_D^2}} + \psi \ge 1\right\}. \tag{4.92}$$

Observe that the inequality constraint on ψ in (4.92) can be rewritten equivalently as

$$1 - \frac{|\psi'|^2}{h_D^2} \ge ((1-\psi)_+)^2,$$

or in other words,

$$\frac{|\psi'|^2}{h_D^2} \le \begin{cases} \psi(2-\psi) & \text{if } \psi \le 1 \\ 1 & \text{if } \psi > 1. \end{cases}$$

For simplicity, let us introduce

$$\gamma(s) := \begin{cases} \sqrt{s(2-s)} & \text{if } s \le 1 \\ 1 & \text{if } s > 1 \end{cases}, \quad I(y) = \int_0^y \frac{1}{\gamma(s)} ds\,.$$

We deduce that $\frac{\psi'}{\gamma(\psi)} \leq h_D$ and $(I \circ \psi)'(t) \leq h_D$. Remarkably, the integral $I(y)$ is strictly increasing, then the maximum (4.92) attains when $I \circ \psi(t) = h_D t$ for all $t \in [0,1]$. At that moment, since $I(y)$ is invertible, $\psi(t)$ is completely determined by

$$\psi(t) = I^{-1}(h_D t).$$

Moreover, the integral $I(x)$ can be computed explicitly

$$I(x) = \begin{cases} 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{2}}\right) & \text{if } x \leq 1, \\ \frac{\pi}{2} + x - 1 & \text{if } x > 1. \end{cases}$$

Hence, we obtain

$$\psi(1) = \begin{cases} 2\sin^2\left(\frac{h_D}{2}\right) & \text{if } h_D \leq \frac{\pi}{2}, \\ 1 + h_D - \frac{\pi}{2} & \text{if } h_D > \frac{\pi}{2}, \end{cases}$$

which yields (4.89).

ii) For the inequality $\theta_D \lambda_D \leq \lambda_1$, we refer to Remark 4.42.

We now prove that $\lambda > 1 + \theta_D \lambda_D$ implies $\lambda \geq \lambda_1$. And hence, we get $\lambda_1 \leq 1 + \theta_D \lambda_D$. To that aim, let $\lambda > 1 + \theta_D \lambda_D$. Then, D is a θ-calibrable set and itself minimizes problem $m(\lambda - 1, D)$. Thus, there exists a calibrating field q satisfying

$$|q| \leq 1, \quad 0 \leq \operatorname{div} q \leq \lambda - 1 \text{ in } D, \quad \text{and} \quad q \cdot n = 1 \text{ on } \partial D.$$

We now construct a vector field σ from q under the form

$$\sigma(x,t) = (-q(x), (t-1)\operatorname{div} q(x) + \lambda - 1).$$

It is easy to verify that σ is admissible for the dual problem of $\beta(\lambda)$, that is

$$\sigma^t(x,1) = \lambda - 1 \text{ in } D, \quad \operatorname{div} \sigma = 0 \text{ in } Q,$$

and for all $(x,t) \in Q$,

$$\sqrt{1 - |\sigma^x(x,t)|^2} + \sigma^t(x,t) \geq (t-1)\operatorname{div} q(x) + \lambda - 1 \geq -\operatorname{div} q(x) + \lambda - 1 \geq 0.$$

Therefore, it holds

$$\beta(\lambda) \geq -\int_D \sigma^t(x,0) = (1-\lambda)|D| + P(D) = \beta_0(\lambda).$$

We deduce that $\beta(\lambda) = \beta_0(\lambda)$, that means $u \equiv 1$ solves $\beta(\lambda)$. Hence, $\lambda \geq \lambda_1$.

We have finished the proof of the inequalities

$$\theta_D \lambda_D \leq \lambda_1 \leq \theta_D \lambda_D + 1. \tag{4.93}$$

Now, for the specific case where D is convex and of class $C^{1,1}$, we recall (Remark 4.17) that the calibration constant of D is given by

$$\theta_D = \max\left\{1, (N-1)\kappa_\infty(\partial D)\lambda_D^{-1}\right\}.$$

Equivalently, we have $\theta_D \lambda_D = \max\{\lambda_D, (N-1)\kappa_\infty(\partial D)\}$ which can be substituted in (4.93) to obtain (4.90). We observe that if $(N-1)\kappa_\infty(\partial D) \leq \lambda_D$, then D is a Cheeger set. Thus, $h_D = \lambda_D$ and $\theta_D = 1$. The estimation (4.93) reduces to $h_D \leq \lambda_1 \leq h_D + 1$. In contrast, when $\lambda_D < (N-1)\kappa_\infty(\partial D)$, D is a θ_D-calibrable set and $(N-1)\kappa_\infty(\partial D) \leq \lambda_1 \leq (N-1)\kappa_\infty(\partial D) + 1$. ■

Example 4.44. Let $D = \{x \in \mathbb{R}^2 : \|x\| < R\}$ be a disk of radius R in $\mathbb{R}^2$. Recall that disks are self-Cheeger sets and that the Cheeger constant of a disk is completely determined by its radius, i.e. $h_D = 2/R$.

Here we shall reuse the computation of Example 3.10 and observe the behaviors of $\beta(\lambda)$, $\beta_0(\lambda)$ with respect to λ and R. Recall that according to Example 3.10, the relaxed problem of $\beta(\lambda)$ can be interpreted as $\inf\{E(r),\, r \in [0,1]\}$ where

$$E(r) = \pi R^2 \left(\sqrt{1-\overline{\mu}(r)^2} - r\sqrt{r^2-\overline{\mu}(r)^2} + \overline{\mu}(r)^2 \log \frac{1+\sqrt{1-\overline{\mu}(r)^2}}{r+\sqrt{r-\overline{\mu}(r)^2}} + \frac{2r\Big(1-K(\overline{\mu}(r),r)\Big)}{R} + (1-\lambda)r^2 \right).$$

Then, the solution $\overline{u}$ of problem $\beta(\lambda)$ has its plateau $\{\overline{u} = 1\}$ determined by the disk centered at 0 of radius $\overline{r}R$, where $\overline{r}$ is the minimizer of $E(r)$. The radical function $\overline{u}$ is evaluated explicitly with $\overline{u}(x) = \overline{\varphi}(\frac{|x|}{R}) = K(\overline{\mu}(\overline{r}), \frac{|x|}{R})$ (given by (3.28)). The graph of $\overline{u}$ except the plateau $\{\overline{u} = 1\}$ is a minimal surface with a prescribed boundary fixed on the boundary of D. The minimal surface has a free boundary corresponding to the boundary of the plateau $\{\overline{u} = 1\}$ (See Figure 4.16).

Figure 4.16: Continuous and discontinuous minimizers of $\beta(\lambda)$ in case D is a disk.

As predicted by Theorem 4.40, $\beta_0(\lambda)$ coincides with $\beta(\lambda)$ outside the interval (λ_0, λ_1) where their common minimizers are trivial characteristic functions either $\overline{u} \equiv 0$ $(\lambda \le \lambda_0)$ or $\overline{u} \equiv 1$ $(\lambda \ge \lambda_1)$. When $\lambda_0 < \lambda_1$, the strict inequality $\beta(\lambda) < \beta_0(\lambda)$ happens. At the moment, the minimizers $\overline{u}$ for problem $\beta(\lambda)$ are not a characteristic functions any more.

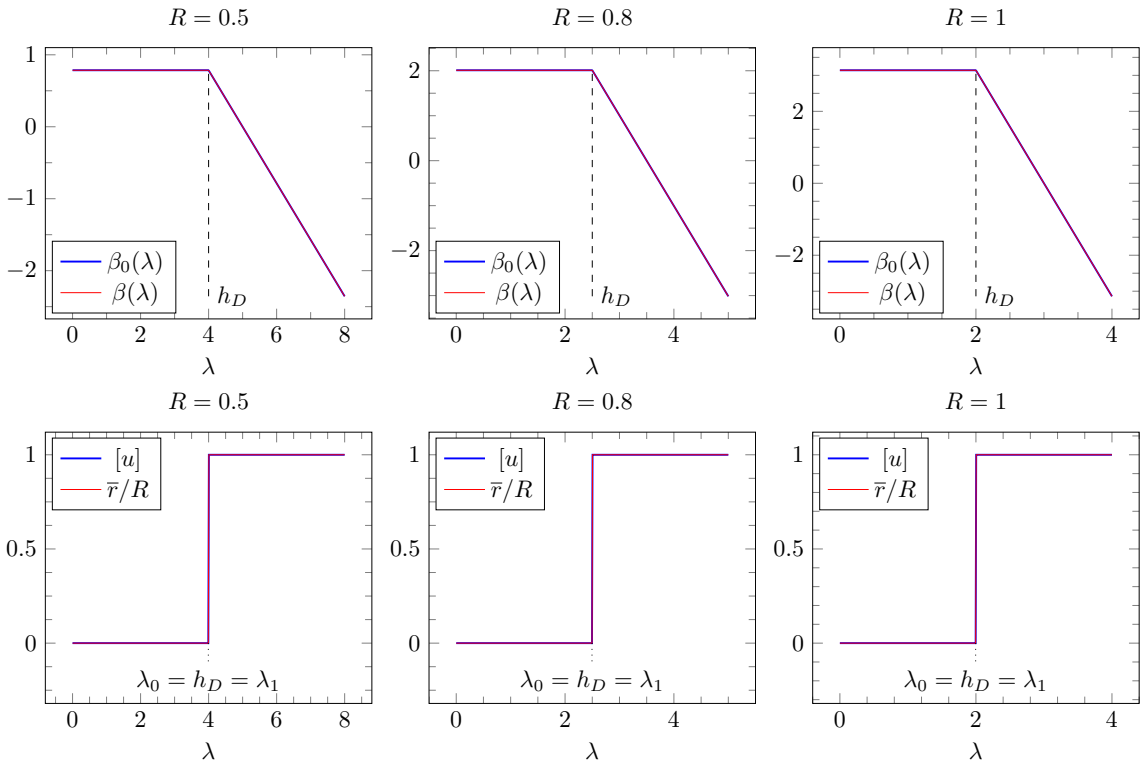


Figure 4.17: Case $R \le 1$. Behaviors of $\beta(\lambda)$, and features of their solutions.

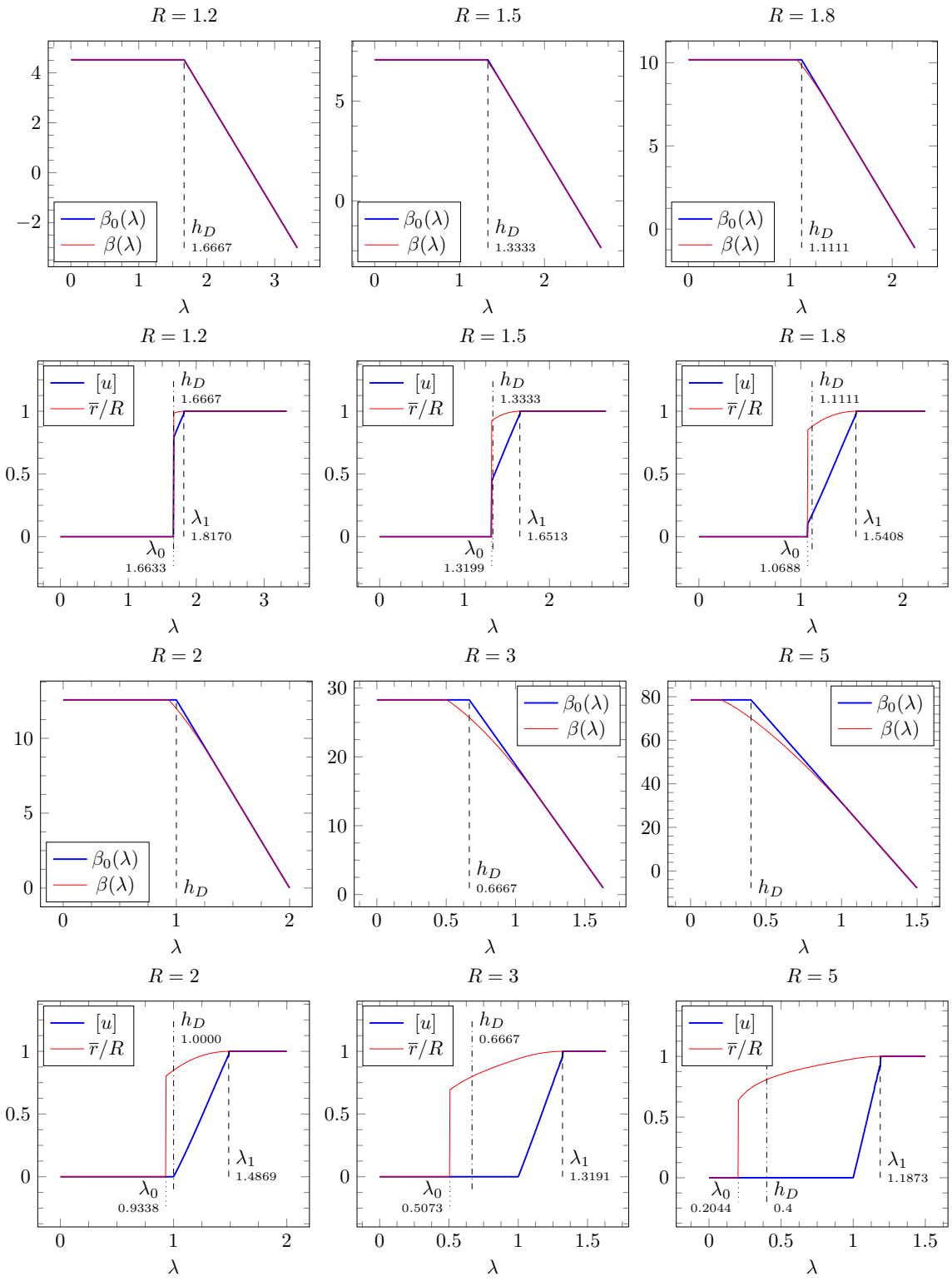


Figure 4.18: Case $R > 1$. Behaviors of $\beta(\lambda)$ and features of its solutions.

By numerical experiments, we observe that in case $R \leq 1$, $\beta_0(\lambda)$ is identical to $\beta(\lambda)$ for all $\lambda \geq 0$, see Figure 4.17. In case $R > 1$, the strict inequality $\beta(\lambda) < \beta_0(\lambda)$ appears for certain interval (λ_0, λ_1), see Figure 4.18. As shown in these figures, the critical values λ_0, λ_1 detaches the Cheeger constant h_D in the opposite sides (i.e. $\lambda_0 < h_D < \lambda_1$) as soon as $R > 1$. Another attractive phenomena is made evident that the discontinuity of minimizers of problem $\beta(\lambda)$ presents when λ passes 1 (as shown in Figure 4.18 by the jump of u denoted by $[u]$).

Since D is a disk, $\theta_D \lambda_D = h_D = 2/R$. Proposition 4.43 provides bounds on λ_0 and λ_1:

$$\lambda_0^* \leq \lambda_0 \leq \frac{2}{R} \leq \lambda_1 \leq 1 + \frac{2}{R}$$

where λ_0^* is determined as

$$\lambda_0^* = \begin{cases} 2\sin^2\left(\frac{1}{R}\right) & \text{if } R \geq \frac{4}{\pi}, \\ 1 + \frac{2}{R} - \frac{\pi}{2} & \text{if } R < \frac{4}{\pi}. \end{cases}$$

In Figure 4.19 we visualize these estimations. The critical values λ_0, λ_1 are numerically computed. As we can see, a conjecture to determine λ_1 can be derived from numerical results, that

$$\lambda_1 = \max\left\{\frac{2}{R}, 1 + \frac{1}{R}\right\}.$$

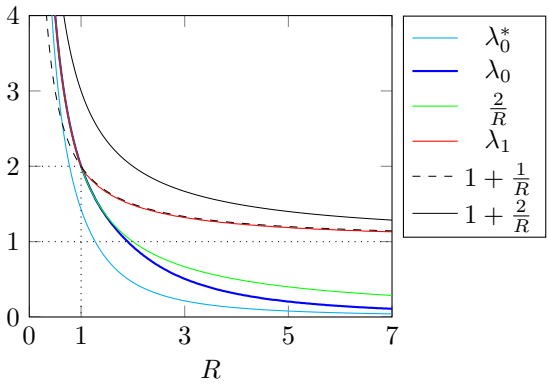


Figure 4.19: Critical values of λ in the case where D is a disk in $\mathbb{R}^2$.

Part C

Numerical methods

Chapter 5

A new semi-implicit scheme based on Arrow-Hurwicz method for saddle point problems

This chapter provides numerical algorithms for searching saddle points for a large class of convex-concave Lagrangians which covers many nice examples treated in this thesis. A generalized explicit iterative scheme based on Arrow-Hurwicz method is proved to be convergent to a saddle point of the problem. We also propose in this chapter, a convergent semi-implicit scheme in order to accelerate the convergence of the iterative process. Numerical experiments are provided for a nontrivial numerical problem modeling an optimal shape problem of thin torsion rods [2]. The semi-implicit scheme is figured out in practice robustly efficient in comparison with the explicit one. This chapter is the reproduction of our submitted paper [95].

5.1 Introduction

The initial motivation of this chapter arises from the numerical approximation of problems of the calculus of variations of the kind

$$(\mathcal{P}) \qquad \inf\left\{\int_\Omega [\varphi(\nabla u) + g(u)]\,dx \;:\; u \in W^{1,p}(\Omega), u = u_0 \text{ on } \partial\Omega\right\} \tag{5.1}$$

where $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a convex, *non differentiable* function, and $g : \mathbb{R} \to (-\infty, +\infty]$ is a probably non convex function (with possible discontinuities).

In the case where g is convex and smooth, many numerical methods based on duality or minimax algorithms exist in the literature. However, in specific cases, the non differentiability of φ creates difficulties for the convergence in particular at the interfaces where the gradient of u is close to the singular set of φ.

We shall focus on a model example where the interface represents boundary of an optimal shape for thin torsion rods studied in [2]. For any value of parameter s ($s \geq 0$ represents the mass of material to be placed in subset Ω) we have to solve

$$m(s) := \inf\left\{\int_\Omega \varphi(\nabla u) : u \in H_0^1(\Omega), \int_\Omega u = s\right\}.$$

Here $g = 0$ and the convex function

$$\varphi(z) := \begin{cases} \frac{1}{2}(|z|^2 + 1) & \text{if } |z| \geq 1 \\ |z| & \text{if } |z| \leq 1 \end{cases}$$

is non differentiable at 0. Our goal is to determine the subsets $\{|\nabla u| > 1\}$ (full region), $\{0 < |\nabla u| \leq 1\}$ (homogenized region) and $\{\nabla u = 0\}$ (empty region).

By the classical duality argument (see for instance [57]), we may reformulate $(\mathcal{P})$ as an inf-sup problem

$$(\mathcal{M}) \qquad \inf_{u \in C} \sup_{p \in K} L(u,p)\ , \qquad L(u,p) = \int_\Omega [\nabla u \cdot p + g(u) - \varphi^*(p)]dx$$

where φ^* is the Fenchel conjugate of φ, $C = \{u \in W^{1,p}(\Omega), u = u_0 \text{ on } \partial\Omega\}$, and $K = L^{p'}(\Omega; \mathbb{R}^N)$. When g is convex, the Lagrangian $L(u,p)$ is convex, concave, and solving $(\mathcal{P})$ amounts to searching saddle points of $(\mathcal{M})$, that is pairs $(\hat{u}, \hat{p})$ in $C \times K$ such that

$$L(\hat{u}, p) \leq L(\hat{u}, \hat{p}) \leq L(u, \hat{p}),\ \forall u \in C,\ \forall p \in K. \tag{5.2}$$

When g is non convex, the duality theory studied in Chapter 2 or in [19, 20, 22] provides a convexified version of $(\mathcal{P})$ in dimension $(N+1)$ from which an equivalent saddle point formulation (like (5.2)) can be obtained (but in higher dimension, except if φ is one-homogeneous (see Chapter 3)).

The core of this chapter is to develop efficient methods for solving problems of the type (5.2). Usually, we immediately think of the gradient descent-ascent so as to seek a saddle point. For a general Lagrangian $L(u,p)$, the simplest approach introduced by Arrow and Hurwicz has the form

$$\begin{aligned} p_{n+1} &= \Pi_K\Big(p_n + \tau_n \frac{\partial L}{\partial p}(u_n, p_n)\Big) \\ u_{n+1} &= \Pi_C\Big(u_n - \tau_n \frac{\partial L}{\partial u}(u_n, p_n)\Big). \end{aligned}$$

where Π_K, Π_C are orthogonal projections on the closed convex sets K and C, respectively. However, this first-order iterative optimization algorithm converges under very stringent conditions (like strict convexity-concavity) and special choosing of stepsizes $\tau_n \to 0$, $\sum_{n=0}^{\infty} \tau_n = \infty$ [78]. To overcome these difficulties, L. D. Popov [97] gave a modification of the Arrow-Hurwicz method by introducing the so-called "leading" point, denoted by $(\overline{u}_n, \overline{p}_n)$, which is an auxiliary point in order to jump to the next approximation with the help of the gradient direction,

$$\begin{aligned} p_{n+1} &= \Pi_K\Big(p_n + \tau \frac{\partial L}{\partial p}(\overline{u}_n, \overline{p}_n)\Big) \\ u_{n+1} &= \Pi_C\Big(u_n - \tau \frac{\partial L}{\partial u}(\overline{u}_n, \overline{p}_n)\Big) \\ \overline{p}_{n+1} &= \Pi_K\Big(p_{n+1} + \tau \frac{\partial L}{\partial p}(\overline{u}_n, \overline{p}_n)\Big) \\ \overline{u}_{n+1} &= \Pi_C\Big(u_{n+1} - \tau \frac{\partial L}{\partial u}(\overline{u}_n, \overline{p}_n)\Big). \end{aligned}$$

In his paper, he proved that there exists a positive scalar τ_0 such that the modified algorithm converges for all constant stepsize τ taken in the interval $0 < \tau < \tau_0$. This improvement enlarges the class of applicable problems whose convex-concave Lagrangians have derivatives satisfying Lipschitz conditions. It is clear that leading points makes the iterative processes more stable. But, because of extra projections, the more complicated the projections are, the heavier

computation is. So, in some cases, replacing the extra projections as well as leading points is necessary. Chambolle-Pock et al. [96, 35] started dealing with a typical Lagrangian which is a linear form

$$L(u,p) = \langle Au, p\rangle + \langle f, u\rangle - \langle g, p\rangle,$$

where A is a bounded linear operator and $\langle\cdot,\cdot\rangle$ denotes the associated scalar product on Hilbert spaces of variables u, p. With these settings, they used simply computed leading points: $\overline{p}_{n+1} = p_{n+1}$ and $\overline{u}_{n+1} = 2u_{n+1} - u_n$. The replaced leading points are just a linear extrapolation based on the current and previous iterates. And, it is proved that the iterative process

$$\begin{aligned} p_{n+1} &= \Pi_K(p_n + \alpha(A\overline{u}_n - g)) \\ u_{n+1} &= \Pi_C(u_n - \beta(A^* p_{n+1} + f)) \\ \overline{u}_{n+1} &= 2u_{n+1} - u_n \end{aligned} \tag{5.3}$$

converges to a saddle point of $L(u,p)$ if we choose $\alpha, \beta > 0$ such that $\alpha\beta\|A\|^2 < 1$. Here, A^* denotes the adjoint of operator A. We remark that the algorithm (5.3) without the projectors Π_K and Π_C, can be interpreted as the Arrow-Hurwicz algorithm for the augmented Lagrangian

$$L_\beta(u,p) = L(u,p) - \beta\langle A^* p, A^* p\rangle$$

and the augmented parameter β is optimal for the convergence.

After that, many efforts has been made to accelerate the convergence of algorithms of Arrow-Hurwicz type, as for instance in [35], where moving stepsizes, modified extrapolation of leading points and implicit schemes are used. In a more recent paper [36], metric changes allow to enlarge the stepsize. Such results are obtained for a general Lagrangian of the kind

$$L(u,p) = \langle Au, p\rangle + F(u) - G(p). \tag{5.4}$$

In addition, many results on the convergence rates are achieved in [35, 36]. The main ideas therein are to combine proximal techniques with implicit schemes. But, to be efficient, such methods need that the proximal map is easy to compute in practice and for that it is necessary to penalize the convex constraint C and K by smooth functions defined on the whole space. These penalization methods are not well adapted to our case where convex sets C and K are non smooth and involve to many constraints. Therefore, we shall proceed with non trivial projections and we shall adapt algorithms solving (5.4) in this context. Let us emphasize that the convergence results obtained in this chapter are completely new.

The explicit scheme is discussed in Section 5.2 with a proof of the convergence under the classical CFL like conditions. The new semi-implicit schemes is introduced in Section 5.3 and the proof of the convergence is obtained under conditions on α, β which do not depend on the size of the mesh (and of the norm of operator A). Section 5.4 deals with numerical results and evidences the advantage of the semi-implicit algorithm, namely of its variant where a splitting technique allows to reduce the iteration number of implicit solvers. We remark that because of the presence of the adjoint operator A^* in the (explicit, implicit) algorithms, we have to choose discretization methods ensuring the condition $\langle Au, p\rangle = \langle u, A^* p\rangle$. Among such discretization methods, we favored the Marker and Cell (MAC) grids which well fit for implementing an effective parallel computation. The computational costs of the different algorithms are then compared, showing again the interest of our semi-implicit algorithm.

5.2 Saddle point problem and explicit scheme

Let C and K be closed convex non empty subsets of Hilbert spaces V and W, respectively. We denote by $\langle\cdot,\cdot\rangle$ the inner product and by $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ the corresponding norm on both Hilbert

spaces without ambiguity. Given $A : V \to W$ a continuous linear operator with its induced norm

$$\|A\| := \sup\{\|Av\| : v \in V, \|v\| \leq 1\}.$$

We consider an inf-sup optimization problem in the very generic form

$$\inf_{u\in C}\sup_{p\in K} L(u,p) \quad \text{with} \quad L(u,p) = \langle Au, p\rangle + F(u) - G(p), \tag{5.5}$$

where F, G are convex functions and supposed to be differentiable. Their derivatives satisfy the Lipschitz condition with constants L_f, L_g, respectively. We assume that the set of saddle points $\hat{C} \times \hat{K}$ of Lagrangian $L(u,p)$ is not empty.

The aim is to find a saddle point $(\hat{u}, \hat{p})$ of $L(u,p)$ in $C \times K$. Now, let us generalize the explicit scheme introduced in (5.3) for the general saddle point problem (5.5). Basically, we keep the main idea of the convergence proof by Chambolle-Pock et al. [96] with additional technical difficulties due to additional convex function F and G. We propose an iterative algorithm as follows.

Description of the algorithm in explicit scheme
Initialization: Let $n \in \mathbb{N}$. Given $(u_0, p_0) \in C \times K$, and $\overline{u}_0 = u_0$.

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha(A\overline{u}_n - G'(p_n))) \\ u_{n+1} = \Pi_C(u_n - \beta(A^*p_{n+1} + F'(u_n))) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n \end{cases} \tag{5.6}$$

where A^* stands for the adjoint of operator A; Π_K, Π_C respectively denote the orthogonal projectors on closed convex sets K, C; and the stepsizes $\alpha, \beta > 0$ are chosen suitably.

We get below the convergence result:

Theorem 5.1. *Under the standing assumption, for all α, β such that*

$$\begin{aligned} 0 < \alpha < \frac{2}{L_g}, \qquad 0 < \beta < \frac{2}{L_f}, \\ \alpha\beta\left(\|A\|^2 - \frac{L_f L_g}{4}\right) + \frac{\alpha L_g}{2} + \frac{\beta L_f}{2} < 1, \end{aligned} \tag{5.7}$$

the proposed algorithm (5.6) converges to a saddle point of $L(u,p)$ in the set $C \times K$.

Before proving the theorem, let us recall some important properties of orthogonal projection on a closed convex set.

Proposition 5.2. *Let Π_D be an orthogonal projection on a closed convex subset D in a Hilbert space V. Followings hold true:*

(i) For every $u, v \in V$,

$$\langle \Pi_D(u) - \Pi_D(v), u - v\rangle \geq \|\Pi_D(u) - \Pi_D(v)\|^2. \tag{5.8}$$

As a consequence, the projection is a monotone 1-Lipschitz operator.

(ii) For every $z \in D$,

$$\|u - \Pi_D(u)\|^2 + \|z - \Pi_D(u)\|^2 \leq \|u - z\|^2.$$

In particular, if $0 \in D$ then

$$\|u - \Pi_D(u)\|^2 + \|\Pi_D(u)\|^2 \leq \|u\|^2. \tag{5.9}$$

(iii) For every $v \in V$,

$$\Pi_D(u) - v = \Pi_{D-v}(u - v). \tag{5.10}$$

where $D - v$ is the Minkowski addition, i.e. $D - v := \{d - v : d \in D\}$.

Proof. (i) For every $u \in V$, the characterization of $\Pi_D(u)$ is given by

$$\langle u - \Pi_D(u), z - \Pi_D(u)\rangle \le 0 \quad \forall z \in D \tag{5.11}$$

We choose $z = \Pi_D(v)$, then

$$\langle u - \Pi_D(u), \Pi_D(v) - \Pi_D(u)\rangle \le 0. \tag{5.12}$$

Similarly, we have

$$\langle v - \Pi_D(v), \Pi_D(u) - \Pi_D(v)\rangle \le 0. \tag{5.13}$$

Summing inequalities (5.12) and (5.13), we obtain the inequality (5.8). The monotonicity and Lipschitz continuity are direct consequences.

(ii) For every $z \in D$, by using (5.11), we have

$$\begin{aligned}\langle u - z, u - z\rangle =& \|u - \Pi_D(u)\|^2 + \|z - \Pi_D\|^2 + 2\langle u - \Pi_D(u), \Pi_D(u) - z\rangle \\ \ge& \|u - \Pi_D(u)\|^2 + \|z - \Pi_D\|^2.\end{aligned}$$

(iii) For every $z \in D$, we recall the characterization (5.11),

$$\langle u - \Pi_D(u), z - \Pi_D(u)\rangle \le 0.$$

This is equivalent to

$$\langle u - v - (\Pi_D(u) - v), z - v - (\Pi_D(u) - v)\rangle \le 0, \quad \forall z \in D.$$

It is to say that $\Pi_D(u) - v$ is the projection of $(u - v)$ on the set $D - v$. ∎

Proof of Theorem 5.1. Let $(\hat{u}, \hat{p}) \in \hat{C} \times \hat{K}$ be a saddle point of $L(u, p)$. Beside the characterization given by the inequalities (5.2), saddle points of the problem (5.5) can be characterized by, see [57] in detail:

$(\hat{u}, \hat{p})$ is a saddle point of $L(u, p)$ in $C \times K$ if and only if

$$\langle A^*\hat{p} + F'(\hat{u}), u - \hat{u}\rangle \ge 0 \ \text{ for all } \ u \in C \tag{5.14}$$

$$\langle A\hat{u} - G'(\hat{p}), p - \hat{p}\rangle \le 0 \ \text{ for all } \ p \in K. \tag{5.15}$$

Under the standing assumption, F and G have second derivatives defined almost everywhere. Then, we obtain

$$F'(u) - F'(v) = \int_0^1 F''((1-\theta)v + \theta u))(u - v)\, d\theta \quad \forall u, v \in V, \tag{5.16}$$

$$G'(p) - G'(q) = \int_0^1 G''((1-\theta)q + \theta p))(p - q)\, d\theta \quad \forall p, q \in W. \tag{5.17}$$

We define

$$f_{v,u} := \int_0^1 F''((1-\theta)v + \theta u))d\theta, \qquad g_{q,p} := \int_0^1 G''((1-\theta)q + \theta p))d\theta.$$

We observe that $f_{v,u}$, $g_{q,p}$ are symmetric bilinear forms on product spaces $V \times V$ and $W \times W$, respectively. Since F, G are convex functions, $f_{v,u}$ and $g_{q,p}$ are positive semi-definite. Such bilinear forms admit factorizations by operators, see [99] Theorem 12.33 p.331, as

$$\langle f_{v,u}V, V\rangle = \langle M_{f_{v,u}}V, M_{f_{v,u}}V\rangle, \qquad \langle g_{q,p}W, W\rangle = \langle M_{g_{q,p}}W, M_{g_{q,p}}W\rangle.$$

Furthermore, by Lipschitz conditions, we have

$$\langle f_{v,u}v', u'\rangle \leq L_f\|v'\|\|u'\| \quad \forall u', v' \in V \quad \text{and} \quad \langle g_{q,p}q', p'\rangle \leq L_g\|q'\|\|p'\| \quad \forall p', q' \in W.$$

Hence, $f_{v,u}$, $g_{q,p}$ are continuous bilinear forms, i.e. $f_{v,u} \in \mathcal{L}(V \times V; \mathbb{R})$, $g_{q,p} \in \mathcal{L}(W \times W; \mathbb{R})$. In what follows, the role of symmetric positive semi-definite bilinear forms $f_{v,u}$, $g_{q,p}$ is exploited.

We firstly introduce some useful notation:

$$\begin{aligned} &P_n = p_n - \hat{p}, \quad U_n = u_n - \hat{u}, \quad \overline{U}_n = \overline{u}_n - \hat{u}, \quad f_n = f_{\hat{u},u_n}\,, \quad g_n = g_{\hat{p},p_n}\,, \\ &\widetilde{K} = K - \hat{p}, \qquad \widetilde{C} = C - \hat{u}. \end{aligned} \tag{5.18}$$

By using (5.10) with the settings (5.18) above, we can rewrite the iterative process (5.6) as

$$\begin{cases} P_{n+1} = \Pi_{\widetilde{K}}(P_n + \alpha(A\overline{u}_n - G'(p_n))) \\ U_{n+1} = \Pi_{\widetilde{C}}(U_n - \beta(A^*p_{n+1} + F'(u_n))) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n. \end{cases}$$

We notice that the couple of equations (5.16) - (5.17) give us the following representation

$$F'(u_n) - F'(\hat{u}) = f_nU_n \tag{5.19}$$
$$G'(p_n) - G'(\hat{p}) = g_nP_n. \tag{5.20}$$

Since $0 \in \widetilde{K}$, we can handle the inequality (5.9) in order to deduce that

$$\begin{aligned} \|P_{n+1}\|^2 &\leq \|P_n + \alpha(A\overline{u}_n - G'(p_n))\|^2 - \|P_{n+1} - P_n - \alpha(A\overline{u}_n - G'(p_n))\|^2 \\ &= \|P_n\|^2 - \|P_{n+1} - P_n\|^2 + 2\alpha\langle A\overline{u}_n - G'(p_n), P_{n+1}\rangle \\ &\leq \|P_n\|^2 - \|P_{n+1} - P_n\|^2 + 2\alpha\langle A\overline{U}_n - g_nP_n), P_{n+1}\rangle. \end{aligned}$$

The second inequality is obtained by adding a non negative amount $-\langle A\hat{u} - G'(\hat{p}), p - \hat{p}\rangle$, see (5.15) for evidence and passing the equality (5.20). Similarly, we just repeat the same procedure for the variable u

$$\begin{aligned} \|U_{n+1}\|^2 &\leq \|U_n - \beta(A^*p_{n+1} + F'(u_n))\|^2 - \|U_{n+1} - U_n + \beta(A^*p_{n+1} + F'(u_n))\|^2 \\ &\leq \|U_n\|^2 - \|U_{n+1} - U_n\|^2 - 2\beta\langle A^*P_{n+1} + f_nU_n, U_{n+1}\rangle. \end{aligned}$$

For short, let us denote

$$\begin{aligned} S_n &= \alpha\|U_n\|^2 + \beta\|P_n\|^2, \\ R_n &= \alpha\|U_{n+1} - U_n\|^2 + \beta\|P_{n+1} - P_n\|^2. \end{aligned}$$

It then follows that

$$S_{n+1} \leq S_n - R_n - 2\alpha\beta\left(\langle A^*P_{n+1} + f_nU_n, U_{n+1}\rangle - \langle A\overline{U}_n - g_nP_n, P_{n+1}\rangle\right). \tag{5.21}$$

By definition of adjoint operator, $\langle A^*P_{n+1}, U_{n+1}\rangle = \langle AU_{n+1}, P_{n+1}\rangle$, we observe that

$$\begin{aligned} &-2\alpha\beta\left[\langle A^*P_{n+1} + f_nU_n, U_{n+1}\rangle - \langle A\overline{U}_n - g_nP_n, P_{n+1}\rangle\right] \\ =&-2\alpha\beta\left[\langle f_nU_n, U_{n+1}\rangle + \langle g_nP_n, P_{n+1}\rangle - \langle A(\overline{U}_n - U_{n+1}), P_{n+1}\rangle\right]. \end{aligned}$$

In the followings, let us do some necessary estimations. By using definition of leading point $\overline{U}_n$ and continuity of operator A, we deduce that

$$\begin{aligned}\langle A(U_{n+1}-\overline{U}_n),P_{n+1}\rangle =&\langle A(U_{n+1}-U_n),P_{n+1}\rangle-\langle A(U_n-U_{n-1}),P_{n+1}\rangle\\ =&\langle A(U_{n+1}-U_n),P_{n+1}\rangle-\langle A(U_n-U_{n-1}),P_n\rangle\\ &-\langle A(U_n-U_{n-1}),P_{n+1}-P_n\rangle\\ \geq&\langle A(U_{n+1}-U_n),P_{n+1}\rangle-\langle A(U_n-U_{n-1}),P_n\rangle\\ &-\|A\|\|U_n-U_{n-1}\|\|P_{n+1}-P_n\|.\end{aligned}$$

We have already mentioned that the bilinear form f_n have positive square root M_{f_n}. By taking into account this factorization, we get

$$\begin{aligned}-\langle f_nU_n,U_{n+1}\rangle &= -\langle M_{f_n}U_{n+1},M_{f_n}U_{n+1}\rangle-\langle M_{f_n}(U_n-U_{n+1}),M_{f_n}U_{n+1}\rangle\\ &\leq(-1+1)\|M_{f_n}U_{n+1}\|^2+\frac{1}{4}\|M_{f_n}(U_{n+1}-U_n)\|^2\\ &\leq\frac{L_f}{4}\|U_{n+1}-U_n\|^2,\end{aligned}\tag{5.22}$$

and analogously,

$$-\langle g_nP_n,P_{n+1}\rangle\leq\frac{L_g}{4}\|P_{n+1}-P_n\|^2.\tag{5.23}$$

We recall that the inequality $2ab\leq(\delta a^2+b^2/\delta)$ holds true for any a,b and any $\delta>0$. We now make use of the previous estimations. For any $\delta>0$, it holds

$$\begin{aligned}S_{n+1}\leq S_n&-\alpha\left(1-\frac{\beta L_f}{2}\right)\|U_{n+1}-U_n\|^2-\beta\left(1-\frac{\alpha L_g}{2}\right)\|P_{n+1}-P_n\|^2\\ &-2\alpha\beta\left[\langle A(U_{n+1}-U_n),P_{n+1}\rangle-\langle A(U_n-U_{n-1}),P_n\rangle\right]\\ &+2\alpha\beta\|A\|\|U_n-U_{n-1}\|\|P_{n+1}-P_n\|\\ \leq S_n&-\alpha\left(1-\frac{\beta L_f}{2}\right)\|U_{n+1}-U_n\|^2-\beta\left(1-\frac{\alpha L_g}{2}\right)\|P_{n+1}-P_n\|^2\\ &-2\alpha\beta\left[\langle A(U_{n+1}-U_n),P_{n+1}\rangle-\langle A(U_n-U_{n-1}),P_n\rangle\right]\\ &+\alpha\beta\|A\|\left(\delta\|U_n-U_{n-1}\|^2+\frac{1}{\delta}\|P_{n+1}-P_n\|^2\right).\end{aligned}$$

Given $M\geq N\geq 1$ and $\gamma>0$, let us take the sum of the inequality (5.21) from N up to M:

$$\begin{aligned}S_{M+1}\leq S_N&-\alpha\left(1-\frac{\beta L_f}{2}\right)\sum_{n=N}^{M}\|U_{n+1}-U_n\|^2-\beta\left(1-\frac{\alpha L_g}{2}\right)\sum_{n=N}^{M}\|P_{n+1}-P_n\|^2\\ &-2\alpha\beta\left[\langle A(U_{M+1}-U_M),P_{M+1}\rangle-\langle A(U_N-U_{N-1}),P_N\rangle\right]\\ &+\alpha\beta\|A\|\delta\sum_{n=N-1}^{M-1}\|U_{n+1}-U_n\|^2+\frac{\alpha\beta\|A\|}{\delta}\sum_{n=N}^{M}\|P_{n+1}-P_n\|^2\\ \leq S_N&-\alpha\left(1-\frac{\beta L_f}{2}-\beta\|A\|\delta\right)\sum_{n=N}^{M-1}\|U_{n+1}-U_n\|^2\\ &-\beta\left(1-\frac{\alpha L_g}{2}-\frac{\alpha\|A\|}{\delta}\right)\sum_{n=N}^{M}\|P_{n+1}-P_n\|^2\\ &-\alpha\left(1-\frac{\beta L_f}{2}\right)\|U_{M+1}-U_M\|^2+\alpha\beta\|A\|\delta\|U_N-U_{N-1}\|^2\\ &+2\alpha\beta\langle A(U_N-U_{N-1}),P_N\rangle\\ &+\alpha\beta\|A\|\left(\gamma\|U_{M+1}-U_M\|^2+\frac{1}{\gamma}\|P_{M+1}\|^2\right).\end{aligned}$$

We develop S_{M+1} and rearrange the previous calculation to obtain

$$\begin{aligned}
&\alpha\|U_{M+1}\|^2+\beta\left(1-\frac{\alpha\|A\|}{\gamma}\right)\|P_{M+1}\|^2\\
&\quad+\alpha\left(1-\frac{\beta L_f}{2}-\beta\|A\|\delta\right)\sum_{n=N}^{M-1}\|U_{n+1}-U_n\|^2\\
&\quad+\beta\left(1-\frac{\alpha L_g}{2}-\frac{\alpha\|A\|}{\delta}\right)\sum_{n=N}^{M}\|P_{n+1}-P_n\|^2\\
&\quad+\alpha\left(1-\frac{\beta L_f}{2}-\beta\|A\|\gamma\right)\|U_{M+1}-U_M\|^2\\
&\leq S_N+\alpha\beta\|A\|\delta\|U_N-U_{N-1}\|^2+2\alpha\beta\langle A(U_N-U_{N-1}),P_N\rangle.
\end{aligned}\tag{5.24}$$

We observe that for $\alpha,\beta,\delta,\gamma$ being positive numbers, it holds

$$\begin{cases}
\beta\left(1-\frac{\alpha\|A\|}{\gamma}\right)>0\\
\alpha\left(1-\frac{\beta L_f}{2}-\beta\|A\|\delta\right)>0\\
\beta\left(1-\frac{\alpha L_g}{2}-\frac{\alpha\|A\|}{\delta}\right)>0\\
\alpha\left(1-\frac{\beta L_f}{2}-\beta\|A\|\gamma\right)>0
\end{cases}\quad\Longleftrightarrow\quad
\begin{cases}
\alpha\|A\|<\gamma<\frac{2-\beta L_f}{2\beta\|A\|}\\
\frac{2\beta\|A\|}{2-\beta L_f}<\frac{1}{\delta}<\frac{2-\alpha L_g}{2\alpha\|A\|}.
\end{cases}\tag{5.25}$$

We then derive from (5.25) that

$$\begin{aligned}
0<\alpha<\frac{2}{L_g},\qquad 0<\beta<\frac{2}{L_f},\\
\alpha\beta\left(\|A\|^2-\frac{L_fL_g}{4}\right)+\frac{\alpha L_g}{2}+\frac{\beta L_f}{2}<1.
\end{aligned}\tag{5.26}$$

Therefore, if we choose (α,β) satisfying (5.26) and (δ,γ) given in the intervals defined by (5.25), the left-hand side of inequality (5.24) is positive. We see that two sequences $\|U_{M+1}\|$ and $\|P_{M+1}\|$ are bounded while both $\|U_{M+1}-U_M\|$ and $\|P_{M+1}-P_M\|$ converge to 0 as $M\to+\infty$. Or, equivalently, $\{u_M\}$ and $\{p_M\}$ are bounded and the sequences $\|u_{M+1}-u_M\|$, $\|p_{M+1}-p_M\|$ go to 0 as $M\to+\infty$. So, there exists a subsequence (u_{M_k},p_{M_k}) weakly converging to some $(u_*,p_*)\in C\times K$, furthermore, u_{M_k+1} and $\overline{u}_{M_k}$ converge to u_* whilst p_{M_k+1} converges to p_*. By passing in the limit in (5.6), we have

$$\begin{aligned}
p_*&=\Pi_K(p_*+\alpha(Au_*-G'(p_*)),\\
u_*&=\Pi_C(u_*-\beta(A^*p_*+F'(u_*)).
\end{aligned}$$

This shows that (u_*,p_*) is solution to problem (5.5). We now can replace $\hat{u}=u_*$, $\hat{p}=p_*$ and $N=M_k$ in (5.24). Then, as k is large enough, the right-hand side of (5.24) will arbitrarily small. Thus, for every $M>M_k$, $\|U_M\|$ and $\|P_M\|$ are as small as we want. We conclude that (U_M,P_M) converges to $(0,0)$ as $M\to+\infty$. ■

Remark 5.3. If $F=G=0$, the problem (5.5) reduces to

$$\inf_{u\in C}\sup_{p\in K}\langle Au,p\rangle,\tag{5.27}$$

and the algorithm (5.6) becomes

$$\begin{cases}
p_{n+1}=\Pi_K(p_n+\alpha A\overline{u}_n)\\
u_{n+1}=\Pi_C(u_n-\beta A^*p_{n+1})\\
\overline{u}_{n+1}=2u_{n+1}-u_n.
\end{cases}\tag{5.28}$$

In that case, positive parameters α, β will be chosen such that $\alpha\beta\|A\|^2 < 1$, so that the process (5.28) converges to a saddle point of $L(u,p)$ in $C \times K$. The choice of α, β is now more flexible, without upper bounds of Lipschitz constants.

5.3 Semi-implicit scheme

In the explicit scheme based on Arrow - Hurwicz method, the algorithm (5.6) is convergent under the permanent appearance of the boundedness of the linear operator A. We wonder whether we find out an process which converges to a saddle point of $L(u,p)$ whose numerical parameters do not depend on the boundedness of A. In the next works, we shall show up such a process, namely semi-implicit algorithm.

The idea of the following algorithm comes from the ascertainment that the steps α and β of the algorithms (5.6) and (5.28)are limited by large eigenvalues of the operator A whereas this steps could be increased for the part of the iterate associated to low eigenvalues. We then aim for progress with optimal steps α and β whatever the considered eigenmode of the iterate.

We suppose that $C \subset \mathrm{dom}(A)$. Problem (5.5) can be written as

$$\min_{(x,q)\in\ddot{C}} \max_{y\in K} \langle q, p\rangle + F(u) - G(p) \tag{5.29}$$

with $\ddot{C} := \{(u, Au) : u \in C\}$. Now, if we apply the process (5.6) to the problem in this form, we obtain

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha(\overline{q}_n - G'(p_n))) \\ (u_{n+1}, q_{n+1}) = \Pi_{\ddot{C}}(u_n - \beta F'(u_n), q_n - \beta p_{n+1}) \\ \overline{q}_{n+1} = 2q_{n+1} - q_n \end{cases} \tag{5.30}$$

It is evident that $\ddot{C} \subset V \times \mathrm{Im}\, A \subset V \times W$. As $V \times \mathrm{Im}\, A$ is a linear subspace of $V \times W$, we deduce that

$$\begin{aligned} \Pi_{\ddot{C}}(u_n - \beta F'(u_n), q_n - \beta p_{n+1}) &= \Pi_{\ddot{C}}(\Pi_{V\times \mathrm{Im}\, A}(u_n - \beta F'(u_n), q_n - \beta p_{n+1})) \\ &= \Pi_{\ddot{C}}((u_n, q_n) - \beta\, \Pi_{V\times \mathrm{Im}\, A}(F'(u_n), p_{n+1})). \end{aligned}$$

The projection $\Pi_{V\times \mathrm{Im}\, A}(u^0, q^0)$ is indeed to search an optimizer for the problem

$$\text{minimize} \quad u \mapsto \frac{1}{2}\left(\|Au - q^0\|^2 + \|u - u^0\|^2\right).$$

This is in fact a proximal operator of a quadratic form. Its resolvent is easily determined,

$$u^* = \mathrm{prox}_{A,q^0}(u^0) = (A^*A + I)^{-1}(A^*q^0 + u^0).$$

Besides, whenever operator A is bounded, it holds

$$\|Au - Au^*\|^2 + \|u - u^*\|^2 \le (\|A\|^2 + 1)\|u - u^*\|^2.$$

Then, the proximal operator assures the couple $(\Pi_C(u^*), A\Pi_C(u^*))$ isn't too far from (u^*, Au^*). This leads to the idea of replacing the projection $\Pi_{\ddot{C}}$ in the process (5.30) by a simpler approximation

$$\begin{cases} u_{n+1} = \Pi_C(u_n - \beta\widetilde{u}_n) \\ q_{n+1} = Au_{n+1} \end{cases} \tag{5.31}$$

with $\widetilde{u}_n$ being the proximal point determined by

$$\widetilde{u}_n = \mathrm{prox}_{A,p_{n+1}}(F'(u_n)) = (A^*A + I)^{-1}(A^*p_{n+1} + F'(u_n)). \tag{5.32}$$

Therefore, we introduce a semi-implicit scheme:

Description of the algorithm in semi-implicit scheme
Initialization: Let $n \in \mathbb{N}$. Given $(u_0, p_0) \in C \times K$ and $\overline{u}_0 = u_0$.

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha(A\overline{u}_n - G'(p_n)) \\ u_{n+1} = \Pi_C(u_n - \beta(A^*A + I)^{-1}(A^*p_{n+1} + F'(u_n))) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n \end{cases} \tag{5.33}$$

where $\alpha, \beta > 0$ is taken appropriately.

Remark 5.4. If $C = V$, replacing the projection $\Pi_{\ddot{C}}$ in the process (5.30) by (5.31) is straightforward. Ortherwise, we notice that once A is an isometric operator, i.e. $\|Au\| = \|u\|$, $\forall u \in \mathrm{dom}(A)$, this replacement is clearly equivalent. Then, in other words, the expression (5.31) defines a projector on $\ddot{C}$ which naturally coincides with the projector $\Pi_{\ddot{C}}$. In addition, if A is an orthogonal operator, in this case, Au_{n+1} is indeed the projection of $A(u_n - \beta\widetilde{u}_n)$ on the image AC, which will be shown in Lemma 5.5.

Lemma 5.5. *Let O be a closed convex subset of Hilbert space V such that $0 \in O$ and Π_O be the orthogonal projector on O. For every densely defined, closed, linear operator $A : V \to W$ satisfying*

$$\forall O, \quad \forall u \in V, \quad \langle A(u - \Pi_O(u)), A\Pi_O(u)\rangle \geq 0, \tag{5.34}$$

the process (5.31) is identified with

$$\begin{cases} u_{n+1} = \Pi_C(u_n - \beta\widetilde{u}_n) \\ q_{n+1} = \Pi_{AC}(Au_n - \beta A\widetilde{u}_n). \end{cases} \tag{5.35}$$

Proof. By making use of notation (5.10), it is easy to verify that

$$u_{n+1} = \Pi_C(u_n - \beta\widetilde{u}_n) \quad \Longleftrightarrow \quad u_{n+1} - u_0 = \Pi_{C-u_0}(u_n - u_0 - \beta\widetilde{u}_n), \ \forall u_0 \in C.$$

Since 0 is always in $C - u_0$, and by dealing with the characterization of the projection Π_{C-u_0}, we derive that for all $u_0 \in C$,

$$\langle u_n - u_0 - \beta\widetilde{u}_n - (u_{n+1} - u_0), u_0 - u_{n+1}\rangle \leq 0.$$

We deduce from the hypothesis (5.34) on the operator A that

$$\forall u_0 \in C, \quad \langle Au_n - \beta A\widetilde{u}_n - Au_{n+1}, Au_0 - Au_{n+1}\rangle \leq 0. \tag{5.36}$$

AC is closed convex. Convexity is preserved by linearity and closedness is preserved by closedness of A. The inequality (5.36) well defines a projection on AC,

$$Au_{n+1} = \Pi_{AC}(Au_n - \beta A\widetilde{u}_n).$$

If q_{n+1} is a projection of $Au_n - \beta A\widetilde{u}_n$ on AC then by the uniqueness of projection, q_{n+1} must coincide with Au_{n+1}. The proof is completed. ∎

Remark 5.6. When A is an orthogonal operator, it evidently satisfies the hypothesis (5.34). At the moment, keeping in mind that $A^*A = I$, it gives an identification

$$\langle A(u - \Pi_O(u)), A\Pi_O(u)\rangle = \langle A^*A(u - \Pi_O(u)), \Pi_O(u)\rangle = \langle u - \Pi_O(u), \Pi_O(u)\rangle.$$

In practice, A usually stands for gradient operator ∇. Let us show that, in this case, gradient operator satisfies the hypothesis (5.34). In the following lemma, we regard u as a function of variable x in a suitable space, for example $u \in V = L^2(\Omega)$. We say that u has local constraints in the convex C if $u(x) \in C(x)$ for all $x \in \Omega$.

Lemma 5.7. *Let O be a closed convex subset of Hilbert space V such that $0 \in O$ and Π_O be the orthogonal projector on O. We suppose, in addition, that the projection Π_O is local in the sense that $\mathring{u} = \Pi_O(u)$ is equivalent to $\mathring{u}(x) = \Pi_{O(x)}(u(x)), \forall x \in \Omega$. Then, the gradient operator ∇ satisfies the hypothesis (5.34).*

Proof. For every $u, v \in V$, let $\mathring{u}$, $\mathring{v}$ be the projections of u, v on O, respectively. By using the inequality (5.8), we have that

$$\langle u - v - (\mathring{u} - \mathring{v}), \mathring{u} - \mathring{v}\rangle \geq 0.$$

Since the projection Π_O is realized locally, choosing $v \in V$ such that $v(x) = u(x + th)$ for $t > 0$ and some given $h \in \mathbb{R}^N$, we have

$$\frac{1}{t^2}\langle u(x) - u(x + th) - [\mathring{u}(x) - \mathring{u}(x + th)], \mathring{u}(x) - \mathring{u}(x + th)\rangle \geq 0.$$

We recall that the Gateaux derivative of u is defined by

$$\langle \nabla u(x), h\rangle := \lim_{t \to 0^+} \frac{u(x + th) - u(x)}{t}.$$

By passing to limit as $t \to 0^+$, we obtain

$$\langle \nabla u(x) - \nabla \mathring{u}(x), \nabla \mathring{u}(x)\rangle \geq 0.$$

In other words, we get

$$\langle \nabla u - \nabla \Pi_O(u), \nabla \Pi_O(u)\rangle \geq 0.$$

This completes the proof of lemma. ■

It is ready to prove the convergence of semi-implicit scheme proposed in (5.33). Here are the main result:

Theorem 5.8. *Let $A : V \to W$ be a densely defined, closed linear operator satisfied the hypothesis (5.34). For all α, β such that*

$$0 < \alpha < \frac{2}{L_g}, \qquad 0 < \beta < \frac{2}{L_f}, \qquad \alpha\beta + \frac{\alpha L_g}{2} < 1,$$

the iterative process defined by (5.33) converges to a saddle point of $L(u, p)$ in the set $C \times K$.

Proof. We maintain using the notation (5.18) and introduce some more notation:

$$\widetilde{q}_n = A\widetilde{u}_n, \quad q_n = Au_n, \quad Q_n = AU_n, \quad \overline{Q}_n = A\overline{U}_n.$$

Under the above settings and passing the translation of projection (5.10), we can rewrite the process (5.33) as

$$\begin{cases} P_{n+1} = \Pi_{\widetilde{K}}(P_n + \alpha(\overline{q}_n - G'(p_n))) \\ U_{n+1} = \Pi_{\widetilde{C}}(U_n - \beta\widetilde{u}_n) \\ q_{n+1} = Au_{n+1} \\ \overline{q}_{n+1} = 2q_{n+1} - q_n. \end{cases}$$

We manipulate again the property of projection (5.9), the characterization (5.15) and the representation (5.20) to obtain that

$$\begin{aligned} \|P_{n+1}\|^2 &\leq \|P_n + \alpha(\overline{q}_n - G'(p_n))\|^2 - \|P_{n+1} - P_n - \alpha(\overline{q}_n - G'(p_n))\|^2 \\ &\leq \|P_n\|^2 - \|P_{n+1} - P_n\|^2 + 2\alpha\langle \overline{Q}_n - g_n P_n, P_{n+1}\rangle \end{aligned}$$

and similarly,

$$\begin{aligned} \|U_{n+1}\|^2 &\leq \|U_n - \beta\widetilde{u}_n\|^2 - \|U_{n+1} - U_n + \beta\widetilde{u}_n\|^2 \\ &\leq \|U_n\|^2 - \|U_{n+1} - U_n\|^2 - 2\beta\langle \widetilde{u}_n, U_{n+1}\rangle. \end{aligned}$$

Within the spirit of Lemma (5.5), Q_{n+1} is indeed the projection of $(Q_n - \beta\widetilde{q}_n)$ on $A\widetilde{C}$. We then have

$$\begin{aligned} \|Q_{n+1}\|^2 &\leq \|Q_n - \beta\widetilde{q}_n\|^2 - \|Q_{n+1} - Q_n + \beta\widetilde{q}_n\|^2 \\ &\leq \|Q_n\|^2 - \|Q_{n+1} - Q_n\|^2 - 2\beta\langle \widetilde{q}_n, Q_{n+1}\rangle \end{aligned}$$

We denote that

$$\begin{aligned} S_n &= \alpha(\|U_n\|^2 + \|Q_n\|^2) + \beta\|P_n\|^2 \\ R_n &= \alpha(\|U_{n+1} - U_n\|^2 + \|Q_{n+1} - Q_n\|^2) + \beta\|P_{n+1} - P_n\|^2 \end{aligned}$$

Then, it holds

$$S_{n+1} \leq S_n - R_n - 2\alpha\beta\left(\langle \widetilde{u}_n, (A^*A + I)U_{n+1}\rangle - \langle \overline{Q}_n - g_n P_n, P_{n+1}\rangle\right). \tag{5.37}$$

Keeping in mind that $\widetilde{u}_n = (A^*A + I)^{-1}(A^*p_{n+1} + F'(u_n))$, $(A^*A + I)$ is self-adjoint, and use the inequality (5.14) and the representation (5.19), we get

$$\begin{aligned} \langle \widetilde{u}, (A^*A + I)U_{n+1}\rangle =& \langle (A^*A + I)^{-1}(A^*p_{n+1} + F'(u_n)), (A^*A + I)U_{n+1}\rangle \\ =& \langle A^*p_{n+1} + F'(u_n), U_{n+1}\rangle \\ \geq& \langle A^*P_{n+1} + f_n U_n, U_{n+1}\rangle. \end{aligned}$$

Besides, we derive that

$$\begin{aligned} &\langle A^*P_{n+1} + f_n U_n, U_{n+1}\rangle - \langle \overline{Q}_n - g_n P_n, P_{n+1}\rangle \\ =& \langle f_n U_n, U_{n+1}\rangle + \langle g_n P_n, P_{n+1}\rangle - \langle \overline{Q}_n - Q_{n+1}, P_{n+1}\rangle \\ =& \langle M_{f_n} U_n, M_{f_n} U_{n+1}\rangle + \langle M_{g_n} P_n, M_{g_n} P_{n+1}\rangle \\ &+ \langle Q_{n+1} - Q_n, P_{n+1}\rangle - \langle Q_n - Q_{n-1}, P_n\rangle - \langle Q_n - Q_{n-1}, P_{n+1} - P_n\rangle \\ \geq& -\frac{L_f}{4}\|U_{n+1} - U_n\|^2 - \frac{L_g}{4}\|P_{n+1} - P_n\|^2 \\ &+ \langle Q_{n+1} - Q_n, P_{n+1}\rangle - \langle Q_n - Q_{n-1}, P_n\rangle - \|Q_n - Q_{n-1}\|\|P_{n+1} - P_n\| \end{aligned}$$

In the inequality above, we reused the estimates (5.22)-(5.23) in the proof of congvergence of explicit scheme. We recall that $2ab \leq (\delta a^2 + b^2/\delta)$ for any a, b and any $\delta > 0$. Therefore, for every $\delta > 0$, it follows

$$\begin{aligned} S_{n+1} \leq S_n &- \alpha\|Q_{n+1} - Q_n\|^2 - \alpha\left(1 - \frac{\beta L_f}{2}\right)\|U_{n+1} - U_n\|^2 \\ &- \beta\left(1 - \frac{\alpha L_g}{2}\right)\|P_{n+1} - P_n\|^2 \\ &- 2\alpha\beta\left[\langle Q_{n+1} - Q_n, P_{n+1}\rangle - \langle Q_n - Q_{n-1}, P_n\rangle\right] \\ &+ 2\alpha\beta\|Q_n - Q_{n-1}\|\|P_{n+1} - P_n\|, \end{aligned}$$

and then

$$\begin{aligned} S_{n+1} \leq S_n &- \alpha\|Q_{n+1} - Q_n\|^2 - \alpha\left(1 - \frac{\beta L_f}{2}\right)\|U_{n+1} - U_n\|^2 \\ &- \beta\left(1 - \frac{\alpha L_g}{2}\right)\|P_{n+1} - P_n\|^2 \\ &- 2\alpha\beta\left[\langle Q_{n+1} - Q_n, P_{n+1}\rangle - \langle Q_n - Q_{n-1}, P_n\rangle\right] \\ &+ \alpha\beta\left(\delta\|Q_n - Q_{n-1}\|^2 + \frac{1}{\delta}\|P_{n+1} - P_n\|^2\right). \end{aligned}$$

Let $M \geq N \geq 1$ and $\gamma > 0$. We take the sum of the inequality (5.37) from N up to M in order to obtain

$$\begin{aligned} S_{M+1} \leq S_N &- \alpha\sum_{n=N}^{M}\|Q_{n+1} - Q_n\|^2 \\ &- \alpha\left(1 - \frac{\beta L_f}{2}\right)\sum_{n=N}^{M}\|U_{n+1} - U_n\|^2 - \beta\left(1 - \frac{\alpha L_g}{2}\right)\sum_{n=N}^{M}\|P_{n+1} - P_n\|^2 \\ &- 2\alpha\beta\left[\langle Q_{M+1} - Q_M, P_{M+1}\rangle - \langle Q_N - Q_{N-1}, P_N\rangle\right] \\ &+ \alpha\beta\delta\sum_{n=N-1}^{M-1}\|Q_{n+1} - Q_n\|^2 + \frac{\alpha\beta}{\delta}\sum_{n=N}^{M}\|P_{n+1} - P_n\|^2, \end{aligned}$$

which leads to

$$\begin{aligned} S_{M+1} \leq S_N &- \alpha(1 - \beta\delta)\sum_{n=N}^{M-1}\|Q_{n+1} - Q_n\|^2 - \alpha\left(1 - \frac{\beta L_f}{2}\right)\sum_{n=N}^{M}\|U_{n+1} - U_n\|^2 \\ &- \beta\left(1 - \frac{\alpha L_g}{2} - \frac{\alpha}{\delta}\right)\sum_{n=N}^{M}\|P_{n+1} - P_n\|^2 \\ &- \alpha\|Q_{M+1} - Q_M\|^2 + \alpha\beta\delta\|Q_N - Q_{N-1}\|^2 \\ &+ 2\alpha\beta\langle Q_N - Q_{N-1}, P_N\rangle \\ &+ \alpha\beta\left(\gamma\|Q_{M+1} - Q_M\|^2 + \frac{1}{\gamma}\|P_{M+1}\|^2\right). \end{aligned}$$

After rearranging, it turns out immediately that

$$\begin{aligned}
&\alpha\|Q_{M+1}\|^2 + \alpha\|U_{M+1}\|^2 + \beta\left(1-\frac{\alpha}{\gamma}\right)\|P_{M+1}\|^2 \\
&\quad + \alpha(1-\beta\delta)\sum_{n=N}^{M-1}\|Q_{n+1}-Q_n\|^2 \\
&\quad + \alpha\left(1-\frac{\beta L_f}{2}\right)\sum_{n=N}^{M}\|U_{n+1}-U_n\|^2 \\
&\quad + \beta\left(1-\frac{\alpha L_g}{2}-\frac{\alpha}{\delta}\right)\sum_{n=N}^{M}\|P_{n+1}-P_n\|^2 \\
&\quad + \alpha(1-\beta\gamma)\|Q_{M+1}-Q_M\|^2 \\
&\leq S_N + \alpha\beta\delta\|Q_N - Q_{N-1}\|^2 + 2\alpha\beta\langle Q_N - Q_{N-1}, P_N\rangle.
\end{aligned} \tag{5.38}$$

We extract all the coefficients in the left-hand side of the inequality (5.38) and regard that for any $\alpha, \beta, \delta, \gamma$ being positive parameters, the following holds true

$$\begin{cases}
\beta\left(1-\frac{\alpha}{\gamma}\right) > 0 \\
\alpha(1-\beta\delta) > 0 \\
\alpha\left(1-\frac{\beta L_f}{2}\right) > 0 \\
\beta\left(1-\frac{\alpha L_g}{2}-\frac{\alpha}{\delta}\right) > 0 \\
\alpha(1-\beta\gamma) > 0
\end{cases} \iff \begin{cases}
\beta < \frac{2}{L_f} \\
\alpha < \gamma < \frac{1}{\beta} \\
\beta < \frac{1}{\delta} < \frac{2-\alpha L_g}{2\alpha}.
\end{cases} \tag{5.39}$$

We then derive from the right-hand side of (5.39) that

$$0 < \alpha < \frac{2}{L_g}, \qquad 0 < \beta < \frac{2}{L_f}, \qquad \alpha\beta + \frac{\alpha L_g}{2} < 1. \tag{5.40}$$

So, if we choose the couple (α, β) as in (5.40) and (γ, δ) in the intervals defined by the right-hand side of (5.39) then the left-hand side of (5.38) is positive. We deduce that the sequences $\|U_{M+1}\|$, $\|Q_{M+1}\|$, $\|P_{M+1}\|$ are bounded while $\|U_{M+1} - U_M\|$, $\|Q_{M+1} - Q_M\|$ and $\|P_{M+1} - P_M\|$ must converge to 0 as $M \to +\infty$. We then have the same conclusion for the sequences $\|u_{M+1}\|$, $\|q_{M+1}\|$, $\|p_{M+1}\|$, $\|u_{M+1} - u_M\|$, $\|q_{M+1} - q_M\|$ and $\|p_{M+1} - p_M\|$, respectively. Therefore, the sequences $\{u_{M+1}\}$, $\{q_{M+1}\}$, $\{p_{M+1}\}$ have subsequences converging in weak topology. Let say u_*, q_*, and p_* are corresponding limits. By substituting $\hat{u} = u_*$, $\hat{q} = q_*$, $\hat{p} = p_*$, and handling again the inequality (5.38) we can derive that sequences $\{u_M\}$, $\{q_M\}$, $\{p_M\}$ are indeed Cauchy sequences, hence converge to the limits u_*, q_*, and p_*, respectively. By passing to the limit in (5.33) and (5.35), we have

$$\begin{aligned}
p_* &= \Pi_K(p_* + \alpha(Au_* - G'(p_*)) \\
u_* &= \Pi_C(u_* - \beta\widetilde{u}_*) \\
Au_* &= \Pi_{AC}(Au_* - \beta A\widetilde{u}_*)
\end{aligned}$$

where $\widetilde{u}_*$ is the proximal point $\operatorname{prox}_{A,p_*}(F'(u_*))$ (see (5.32)), that is

$$\widetilde{u}_* = (A^*A + I)^{-1}(A^*p_* + F'(u_*)).$$

It is easy to see that for every $u \in C$

$$0 \leq \langle \widetilde{u}_*, u - u_*\rangle + \langle A\widetilde{u}_*, A(u - u_*)\rangle = \langle (A^*A + I)\widetilde{u}_*, u - u_*\rangle.$$

We deduce that

$$\begin{aligned}\langle Au_* - G'(p_*), p - p_*\rangle \le 0 \quad \forall p \in K,\\ \langle A^*p_* + F'(u_*), u - u_*\rangle \ge 0 \quad \forall u \in C.\end{aligned}$$

This is to say that (u_*, p_*) is a saddle point of $L(u,p)$ in $C \times K$. ■

Remark 5.9. We emphasize that if the term $F(u)$ is absent in the Lagrangian $L(u,p)$, the projection on $\ddot{C}$ can be restricted only on AC. Then, the algorithm (5.33) reads

$$\begin{cases} p_{n+1} = \Pi_K(p_n + \alpha(\overline{q}_n - G'(p_n)) \\ q_{n+1} = \Pi_{AC}(q_n - \beta A(A^*A)^{-1}A^*p_{n+1}) \\ \overline{q}_{n+1} = 2q_{n+1} - q_n \end{cases}$$

where $q_n = Au_n$. In this situation, the hypothesis (5.34) should be replaced by A^*A being positive definite so that A^*A is invertible. And when the term $G(p)$ is not also present, the positive parameters α, β just have to satisfy the constraint $\alpha\beta < 1$ to ensure the convergence of the algorithm.

5.4 Application to the shape optimization of thin torsion rods

Let D be a bounded connected domain in $\mathbb{R}^2$ and s be a real parameter. We are interested in considering the variational problem, studied in [2]

$$m(s) := \inf\left\{\int_D \varphi(\nabla u) : u \in H_0^1(D), \int_D u = s\right\}$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a convex function given by

$$\varphi(z) := \begin{cases} \frac{1}{2}(|z|^2 + 1) & \text{if } |z| \ge 1 \\ |z| & \text{if } |z| \le 1. \end{cases}$$

We see that the integrand φ is not strictly convex and not differentiable at 0. Its Fenchel conjugate is the positive part of a quadratic form

$$\varphi^*(p) = \frac{1}{2}\left(|p|^2 - 1\right)_+.$$

It is clear that φ^* is convex but non strictly convex, too. See Figure 5.1 for illustration.

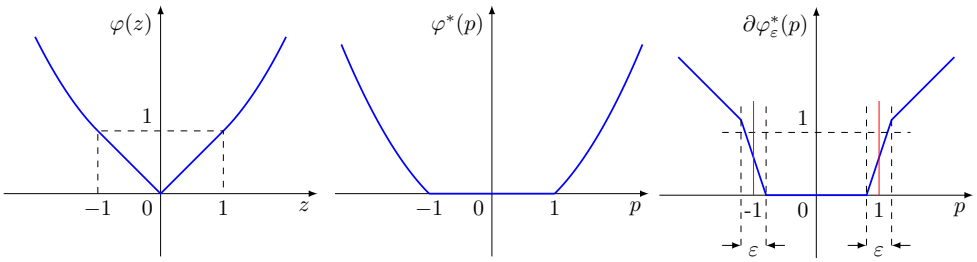


Figure 5.1: From left to right, the description of function $\varphi(z)$, its Fenchel conjugate $\varphi^*(p)$ and the regularization of $\varphi^*(p)$.

We recall that the Fenchel conjugate of an integral functional is an integral of the Fenchel conjugate of the corresponded integrand, i.e.

$$\left(\int_D f(p(x))dx\right)^* = \int_D f^*(p^*(x))dx.$$

For more details on this topic, we refer to [57]. We are going to apply this fact to calculate the conjugate of $m(s)$. For every $\lambda \in \mathbb{R}$,

$$\begin{aligned} m^*(\lambda) &= \sup_{s\in\mathbb{R}} \{\lambda s - m(s)\} = - \inf_{u\in H_0^1(D)} \left\{\int_D \varphi(\nabla u) - \lambda \int_D u\right\} \\ &= - \inf_{u\in H_0^1(D)} \sup_{p\in L^2(D;\mathbb{R}^2)} \left\{\int_D p\cdot\nabla u - \int_D \varphi^*(p) - \lambda \int_D u\right\} \end{aligned}$$

The second equality occurs with a replacement of the functional $\int_D \varphi(\nabla u)$ by a Fenchel conjugate. We regard that the functional

$$p \mapsto \inf_{u\in H_0^1(D)} \left\{\int_D p\cdot\nabla u - \int_D \varphi^*(p) - \lambda \int_D u\right\}$$

is finite if and only if $-\operatorname{div} p = \lambda$. So, by passing the inf-sup permutation argument, $m^*(\lambda)$ can be rewritten as

$$m^*(\lambda) = \inf\left\{\int_D \varphi^*(p) : p \in L^2(D;\mathbb{R}^2), -\operatorname{div} p = \lambda\right\}. \tag{5.41}$$

Furthermore, optimal solutions of $m(s)$ and $m^*(\lambda)$ are characterized by certain optimality conditions

$$\begin{cases} u \text{ solution to } m(s) \\ p \text{ solution to } m^*(\lambda) \\ \lambda \in \partial m(s) \end{cases} \iff \begin{cases} \int_D u = s \\ -\operatorname{div} p = \lambda \\ p \in \partial\varphi(\nabla u) \text{ a.e.} \end{cases}$$

Many interesting properties of functions $m(s)$ and $m^*(\lambda)$ were studied in [2]. One of them which is obviously seen is that the Fenchel equality is satisfied

$$m(s) + m^*(\lambda) = s\lambda,$$

since $\lambda \in \partial m(s)$.

We shall focus on the inf-sup formulation of $m^*(\lambda)$ which is adapted within our numerical schemes. But we must notice that φ^* is neither strictly convex and nor differentiable on the unit circle $\{p \in \mathbb{R}^2 : |p| = 1\}$. A regularization for φ^* should be done before enforcing the algorithms. See Figure 5.1 for visualization. Instead of taking the subgradient of φ^*, we regularize it by removing the discontinuities with an ε-affine symmetric connection

$$\partial\varphi_\varepsilon^*(p) = \begin{cases} \frac{2+\varepsilon}{2\varepsilon}\left(|p| + \frac{\varepsilon}{2} - 1\right)\frac{p}{|p|} & \text{if } ||p| - 1| \le \frac{\varepsilon}{2} \\ \partial\varphi^*(p) & \text{otherwise.} \end{cases}$$

It is ready to find a saddle point of the problem with ε-regularization.

The solution of such a problem in the context of shape optimization of thin torsion rods exhibits regions where u is constant corresponding to regions without material, regions where $|\nabla u| > 1$ corresponds to the optimal region for the material in order to struggle torsion. Regions where $0 < |\nabla u| \le 1$ describes the regions of homogenized material for which the convexity of the Lagrangian is not strict. This makes the problem nontrivial. Depending on the mass constraint, such a homogenization region can appear (low mass constraint leading to the so-called

"homogeneous solution") or not ("special solution"). To answer the question whether an optimal design contains some homogenization region is equivalent to investigate when the special solution exists. And naturally, we wonder in which domain D special solutions present. These are still open issue. In Figure 5.2, the magnitude of the gradient of the solution is plotted, a special solution can be found in the left figure, a homogenized solution is in the middle with weak gradient (lower than 1) on regions limited by thick lines corresponding to $|\nabla u| = 1$. In the right figure, the value λ is close to the critical Cheeger constant while as we see, the optimal shape becomes thinner and tends to the boundary of the Cheeger set of the domain D [79].

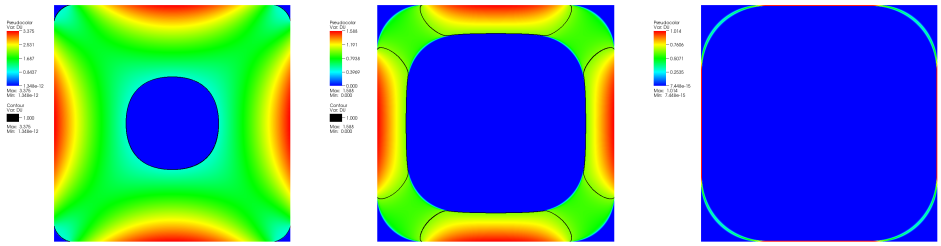


Figure 5.2: Special (left) and homogenized (middle and right) solutions depending on the mass constraint.

It is easy to see that if D is a symmetric simple connected domain then solutions of $m(s)$ are symmetric. When D is a square, the symmetrization allows dividing D in four parts. Without lost of generality, let us describe the discretization settings with $D = [0, 0.5]^2$ instead of unit square. Let $x = (x_1, x_2)$, $p = (p_1, p_2)$. The subdivision leads to the appearance of extra boundary conditions

$$u(\cdot, 0.5) = u(0.5, \cdot) = 0$$
$$p_1(0, \cdot) = p_2(\cdot, 0) = 0.$$

Because of the presence of the adjoint operator A^* in the algorithms (5.6) and (5.33), we must choose discretization methods which ensure the condition $\langle Au, p\rangle = \langle u, A^*p\rangle$. Among such methods, we prefer to implement our algorithms with staggered MAC (Marker and Cell) grids [75]. We shall adopt the superscript h for indicating the discretization with respect to a mesh size h. For instance, ∇^h and div^h stand for the discrete gradient and divergence operators, respectively. The discretization with MAC grids not only adapts to the condition $\mathrm{div}^h = -(\nabla^h)^*$, but also facilitates a parallel computation by a simple decomposition of the domain D. In Figure 5.3, we briefly describe the configuration of the MAC grids in our implementation, wherein the third subfigure presents a strip partitioning of the domain while the gray dashed patterned region indicates the communication interface used within a parallel computation.

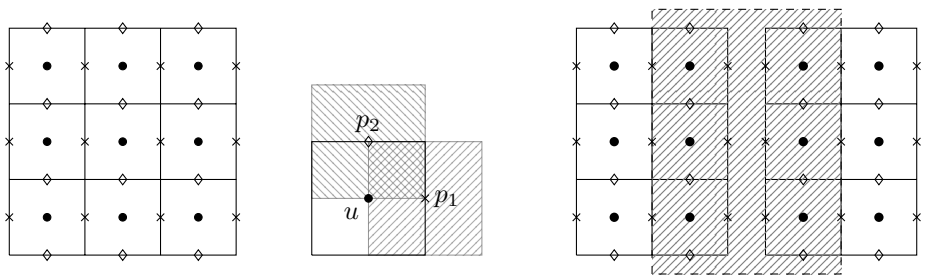


Figure 5.3: Staggered MAC grids.

We provide an explicit iterative process in discrete scheme with

$$\begin{cases} p_{n+1}^h = p_n^h + \alpha(\nabla^h \overline{u}_n^h - \partial\varphi_\varepsilon^*(p_n^h)) \\ u_{n+1}^h = \Pi_C^h(u_n^h + \beta(\mathrm{div}^h(p_{n+1}^h) + \lambda)) \\ \overline{u}_{n+1}^h = 2u_{n+1}^h - u_n^h. \end{cases} \tag{5.42}$$

where Π_C^h is the discretized projection. In this case, the projection is just simple to keep the boundary condition $u = 0$ on ∂D. Since Lipschitz constants $L_f = 0$, $L_g = (2+\varepsilon)/2\varepsilon$, the positive parameters α, β should be chosen such that

$$\alpha < \frac{4\varepsilon}{2+\varepsilon}, \qquad \alpha\beta c_h^2 + \frac{\alpha(2+\varepsilon)}{4\varepsilon} \leq 1 \qquad \text{with } c_h := ||\nabla^h|| = \frac{2\sqrt{2}}{h}.$$

We remark that $\alpha < 4$ and tends to 0 as $\varepsilon \to 0$. But with the second condition, the product $\alpha\beta$ should be of order $O(h^2)$.

In the point of view of implicit scheme, the proposed convergent process reads

$$\begin{cases} p_{n+1}^h = p_n^h + \alpha(\nabla^h \overline{u}_n^h - \partial\varphi_\varepsilon^*(p_n^h)) \\ u_{n+1}^h = u_n^h + \beta(I - \Delta^h)^{-1}(\mathrm{div}^h(p_{n+1}^h) + \lambda) \\ \overline{u}_{n+1}^h = 2u_{n+1}^h - u_n^h. \end{cases} \tag{5.43}$$

We replaced $\mathrm{div}^h(p_{n+1}^h) + \lambda$ in the explicit process by $\widetilde{u}_n = (I - \Delta^h)^{-1}(\mathrm{div}^h(p_{n+1}^h) + \lambda)$ which is solution of the equations

$$\begin{cases} (I - \Delta^h)v = \mathrm{div}^h(p_{n+1}^h) + \lambda \\ v = 0 \quad \text{on} \quad \partial D \end{cases}$$

The projection Π_C^h then disappears since the boundary condition on u is added within resolving $\widetilde{u}_n$. Moreover, the positive parameters are simplified

$$\alpha < \frac{4\varepsilon}{2+\varepsilon}, \qquad \alpha\beta + \frac{\alpha(2+\varepsilon)}{4\varepsilon} < 1.$$

We see that the choices of α, β now do not depend on c_h and thus, the product $\alpha\beta$ is of order $O(1)$ with respect to h. Nevertheless, the stepsize α is still restricted by small ε.

The regularizing parameter ε is linked to the grid size, in practice we take for instance $\varepsilon = 3h$. This relation makes the stepsize α be of order of h for the implicit algorithm (5.43). It leads to introduce a new algorithm with κ sub-iterations on the explicit part of (5.43), in implementation $\kappa = 50$:

$$\begin{cases} p_{n+1,0}^h = p_{n,\kappa}^h \\ p_{n+1,k+1}^h = p_{n+1,k}^h + \frac{\alpha}{\kappa}(\nabla^h \overline{u}_n^h - \partial\varphi_\varepsilon^*(p_{n+1,k}^h)), \qquad k = 0, 1, ..., \kappa - 1 \\ u_{n+1}^h = u_n^h + \beta(I - \Delta^h)^{-1}(\mathrm{div}^h(p_{n+1,\kappa}^h) + \lambda) \\ \overline{u}_{n+1}^h = 2u_{n+1}^h - u_n^h. \end{cases} \tag{5.44}$$

The algorithm (5.44) allows α to be κ times bigger and then reduces the number of iterations in n.

From now on, the algorithm (5.44) is called Implicit Sub-iteration Scheme (ISS), the algorithm (5.43) is called Implicit Scheme (IS) and the algorithm (5.42) is called Explicit Scheme (ES).

In Figure 5.4, we present the Euclidean norm of the gradient of a solution, i.e. $|\nabla u|$. The left figure shows an expected solution with the explicit scheme. The centered and the right ones are in a quarter domain which are done with implicit methods (IS) and (ISS), respectively. We see

that in implicit scheme (IS), the contact zones are still not tangent to the boundary of domain (the region on which it is difficult to converge the algorithm). And, the (IS) algorithm, expected to be slower than (ISS), introduces an other drawback with numerical artefacts.

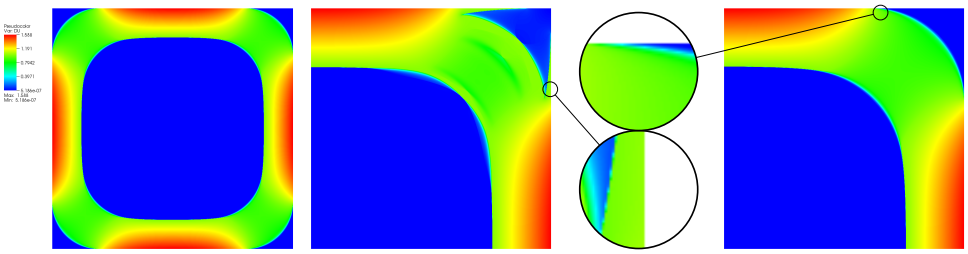


Figure 5.4: Case $\lambda = 5$ for (ES) on the left, (IS) in the middle, (ISS) on the right, for the same grid size.

In the processes (5.43)-(5.44), the inverse Laplacian computation is the most costly. The computational cost thus depends highly on the solver used for the inverse Laplacian operator. If one uses a multigrid or a FFT solver, it can be of order $\dfrac{1}{h^2 \log h}$. Besides, handling a multigrid solver, for instance AGMGPAR (A parallel version of Algebraic Multigrid method), see [93], and Marker and Cell (MAC) grids, it easy to implement our algorithms with MPI (Massage Passing Interface) library which provides an effective environment for parallel computation. In the following computations, comparisons between the different schemes are done with a fixed number of process to 6, leading to good scalability for all methods.

Before comparing the computational cost, we ensure that the algorithms converge to the exact solution as the grid size h goes to zero. We then consider a case in which the unique exact solution is known, that is when D is a disk. In such a case, the solution is a special solution and it is radial with mass concentrated on the periphery and with an internal radius of $\overline{R} = \frac{2}{\lambda}$ (see Figure 5.5), see [2] for the expression of the exact solution.

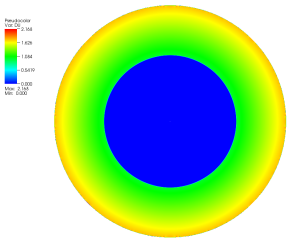


Figure 5.5: Exact solution when the constraint domain D is a disk.

We are then able to compute the numerical internal radius r_N, R_N defined respectively as the maximal value of radius where $|\nabla u|$ is smaller than a half and the minimal value of radius where $|\nabla u|$ is bigger than one. The error on the internal radius is measured as a number of cells for different cell sizes. The radius error is of order of a half of grid size whatever the grid, as shown on Figure 5.6.

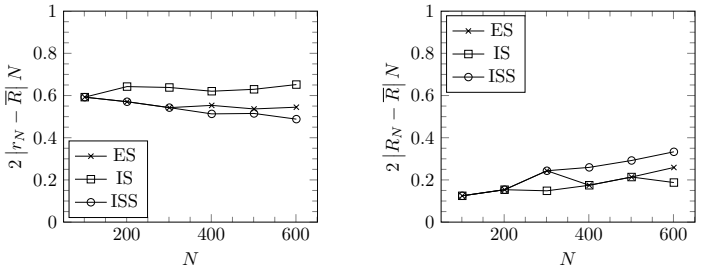


Figure 5.6: Graph of $2|r_N - \overline{R}|N$ and $2|R_N - \overline{R}|N$ with respect to the grid size N.

We are now concerned with the comparison of computational cost between schemes (ES), (IS), (ISS) in the case where $\lambda = 5$, for a unitary square D, so that an homogeneous solution occurs.

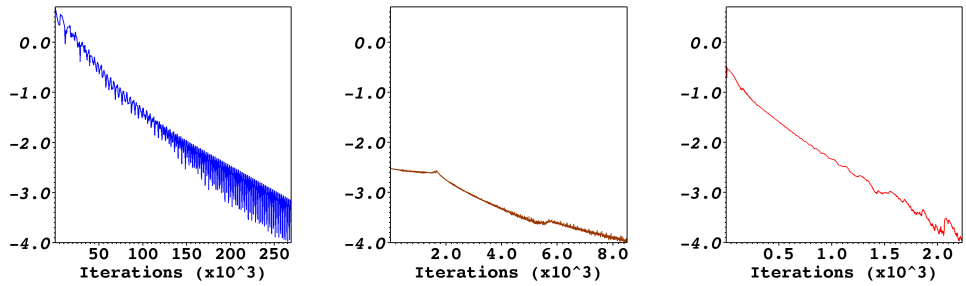


Figure 5.7: Convergence of algorithms with criterion $\|\operatorname{div} p + \lambda\|_{L^2} < 10^{-4}$ in case $\lambda = 5$. From left to right, the first figure is in (ES), the second with (IS), and the last one is done with (ISS)

We remark that we have to well pay for the cost of inverse Laplacian operator. So, the choice of solvers should be carefully considered. But, the positive side of semi-implicit scheme is to reduce globally many iterations of the iterative process. When the projections on convex sets become more expensive, this reduction of iteration will evidently save the computational time. At the moment, semi-implicit scheme is the most efficient, see Table 5.1, Figure 5.7 and Figure 5.8.

$\|\operatorname{div} p + \lambda\|_{L^2} < 10^{-4}$ with MPI in 6 processes						
N	iterations			time (seconds)		
	ES	**IS**	**ISS**	**ES**	**IS**	**ISS**
101	9451	763	360	0.59	12.75	5.84
201	21647	1340	545	4.92	47.63	20.18
301	34719	1857	733	17.96	92.01	39.93
501	59438	2822	856	113.29	311.92	121.36
801	98484	4072	1251	848.15	819.10	467.12
1201	154107	5777	1596	3590.69	2625.47	1460.76
1701	232793	7629	2038	11520.75	7642.35	3876.28
1921	268999	8507	2230	17183.18	11174.52	5717.89

Table 5.1: Comparison of three methods in iterations and computational time in case $\lambda = 5$. The stop criterion is $\|\operatorname{div} p + \lambda\|_{L^2} < 10^{-4}$. They are implemented with MPI in 6 processes.

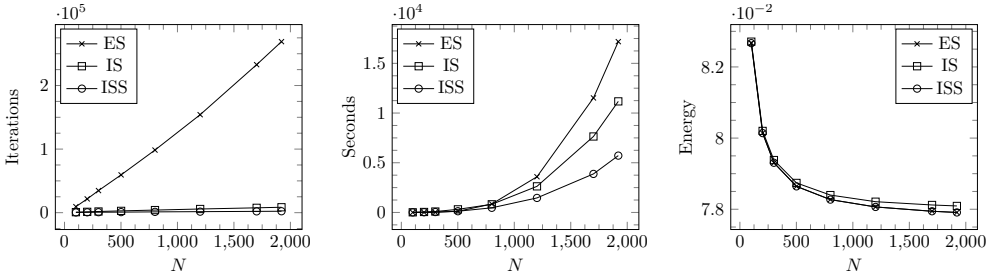


Figure 5.8: Comparison of three methods in term of iterations, computational time, convergence of energy function in case $\lambda = 5$.

Remark 5.10. When dealing with non differentiable Larangians, we propose a more efficient alternative for the regularization method that we used for simulations in this chapter a new approach by using a geometric epigraph projection. We refer to the numerical part of Chapter 3 for the description of the epigraph projection method.

5.5 Conclusion

The generalized explicit scheme for searching saddle points of Lagrangians of type (5.4) is convergent and widely applicable. The main contribution of this chapter is to propose a semi-implicit extension of such an algorithm. It remains convergent under less restrictive constraint on numerical parameters. Non differentiability can possibly occur in Lagrangians, and in this case, to fix it, we regularize derivatives of Lagrangians, with a smoothing parameter linked to the space discretization parameter. The semi-implicit scheme, coupled to a splitting method for the rapidly computed explicit part, provides a robust acceleration of the computational cost in comparison with the fully explicit one. The number of iterations in order to reach a precise convergence is widely reduced as shown in Section 5.4. Even if an iteration is more costly for the semi-implicit algorithm, the global computational cost is clearly reduced, specially for fine grids and provides accurate solutions. Furthermore, such an algorithm can reveal even more performing than explicit scheme when Lipschitz constants functions F and G are lower and also if the Lagrangian contains large quadratic terms with respect to the p variable. This algorithm has then been willingly tested on a stiff problem. In any case, the solver of a Laplacian type problem must be carefully chosen since it mainly contributes to the computational cost. Once set up, the heavier projections are, the more efficiency the semi-implicit scheme shows.

Chapter 6

Application of primal-dual algorithms to free boundary problems

This chapter is devoted to numerical simulations of free boundary problems. It is an occasion not only to refresh the topic with an exhibition of numerical solutions but also to validate the worthy fruits from the theory of duality and calibrations for non-convex variational problems presented in Chapter 2. This is actually the place where explicit, implicit algorithms based on primal-dual formulation shows their surprising efficiency, particularly in three dimensions.

6.1 An elliptic-type free boundary problem

In this section we present our numerical approximation for an elliptic-type free boundary problem which appeared in [20]. We consider a free boundary problem given by the elliptic partial differential equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \{u = 0\}, \\ u = 1 & \text{on } \partial\Omega, \end{cases}$$

where Ω is assumed to be a bounded open subset of $\mathbb{R}^N$. The frontier of the zero level set $E := \{u = 0\}$ is called free boundary. This elliptic-type free boundary problem has been firstly considered in the pioneering paper [3]. (See also [29] for this elliptic case). A variational formulation for this problem is given by

$$\mathcal{I}(\Omega, \lambda) := \inf \left\{ \int_\Omega \frac{1}{2}|\nabla u|^2 + \lambda |\{u > 0\}| \ : \ u \in W^{1,2}(\Omega)\,,\ u = 1 \text{ on } \partial\Omega \right\}. \tag{6.1}$$

Actually, the infimum problem $\mathcal{I}(\Omega, \lambda)$ can be viewed as a shape optimization problem:

$$\inf_E \left\{ \int_\Omega \frac{1}{2}|\nabla u_E|^2 + \lambda |\Omega \setminus E| \right\},$$

with u_E being the solution to the partial differential equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus E, \\ u = 0 & \text{in } E, \\ u = 1 & \text{on } \partial\Omega. \end{cases}$$

Problem (6.1) is a non convex variational problem which is easily seen through our framework by setting its integrand with

$$f(t,z) = \frac{1}{2}|z|^2 + \lambda\chi_{(0,+\infty)}(t) \tag{6.2}$$

and the prescribed boundary term $u_0 \equiv 1$. Notice that the function f admits a discontinuity at $\{t = 0\}$.

According to the duality theory presented in Chapter 2, there is no gap between $\mathcal{I}(\Omega,\lambda)$ and its dual problem, denoted by $\mathcal{I}^*(\Omega,\lambda)$,

$$\mathcal{I}^*(\Omega,\lambda) := \sup\left\{ -\int_\Omega \sigma^t(x,1)\,dx \;:\; \sigma \in \mathcal{B} \right\}, \tag{6.3}$$

where $\mathcal{B}$ is the class of bounded divergence free vector field on $\Omega\times\mathbb{R}$ satisfying the pointwise constraints

$$\sigma^t(x,t) + \lambda \geq \frac{1}{2}|\sigma^x(x,t)|^2 \quad \text{a.e. on } \Omega\times\mathbb{R}\,, \qquad \sigma^t(x,0) \geq 0 \text{ a.e. on } \Omega\,. \tag{6.4}$$

The dual problem is stated in higher dimension, however, it is just a linear programming on a convex set.

It is easy to check that any solution $u \in W^{1,2}(\Omega)$ to problem (6.1) takes values in $[0,1]$. Therefore, we can treat the problem $\mathcal{I}^*(\Omega,\lambda)$ on the bounded subset $\Omega\times[0,1]$ instead of $\Omega\times\mathbb{R}$. Then, an optimal pair $(\overline{u},\overline{\sigma})$ for the couple of problems in duality $\mathcal{I}(\Omega,\lambda)$ and $\mathcal{I}^*(\Omega,\lambda)$, in view of Theorem 2.14, is actually a saddle point for the Lagrangian

$$\inf_{v\in\widehat{\mathcal{A}}}\ \sup_{\sigma\in\mathcal{K}} \int_{\Omega\times[0,1]} \sigma\cdot Dv\,, \tag{6.5}$$

on closed convex subsets

$$\widehat{\mathcal{A}} = \Big\{ v\in BV_\infty(\Omega\times\mathbb{R}) \;:\; v=1 \text{ for } t<0\,,\ v=0 \text{ for } t>1\,,\ v=1 \text{ on } \partial\Omega\times[0,1] \Big\},$$

$$\mathcal{K} = \Big\{ \sigma\in X_1(\Omega\times(0,1)) \;:\; \sigma^t + \lambda \geq \tfrac{1}{2}|\sigma^x|^2 \text{ a.e. on } \Omega\times(0,1)\,,\ \sigma^t(\cdot,0)\geq 0 \text{ a.e. on } \Omega \Big\}.$$

Notice carefully that the integration domain in (6.5) is the product of Ω times the *closed* interval $[0,1]$. In fact, minimizing over $\widehat{\mathcal{A}}$ the functional $v\mapsto\int_{\Omega\times[0,1]}\sigma\cdot Dv$ appearing in (6.5) is equivalent to minimizing over the space of functions $v\in BV(\Omega\times(0,1))$ satisfying the boundary condition $v=1$ on $\partial\Omega\times[0,1]$ the functional

$$v\mapsto \int_{\Omega\times(0,1)}\sigma\cdot Dv + \int_\Omega \left[\sigma^t(x,0)(v(x,0^+)-1) - \sigma^t(x,1)v(x,1^-)\right]dx$$

where $v(x,0^+)$ and $v(x,1^-)$ are respectively the traces of v on $\Omega\times\{0\}$ and $\Omega\times\{1\}$.

We recall that, if $(\overline{v},\overline{\sigma})$ is an optimal pair for the inf-sup problem (6.5), the function $\overline{v}$ should be a step function. Indeed, the primal problem (6.1) is expected to admit only one or at most a finite number of solutions. Then, by Theorem 2.8, the step function $\overline{v}$ will take only the values 0 and 1 in case of a unique solution, or a finite number of values in $[0,1]$ in case of multiple solutions.

6.1.1 Discretization

We use two different numerical schemes proposed in the Chapter 5 to solve the saddle point problem (6.5).

In the explicit scheme, we choose an initial point $(v_0, \sigma_0) \in \widehat{\mathcal{A}} \times \mathcal{K}$ and two positive stepsizes α, β. Then, for each $n \in \mathbb{N}$, denoting by h the size parameter of a cartesian grid in $\mathbb{R}^{N+1}$, we let

$$\begin{cases} \sigma^h_{n+1} = \Pi^h_K(\sigma^h_n + \alpha \nabla^h \overline{v}^h_n) \\ v^h_{n+1} = v^h_n + \beta \, \mathrm{div}^h(\sigma^h_{n+1}) \\ \overline{v}^h_{n+1} = 2v^h_{n+1} - v^h_n \,, \end{cases} \tag{6.6}$$

where Π^h_K is a suitable projection operator associated with the convex constraint $K(t)$. The convergence for system (6.6) requires that the stringent condition $\alpha\beta\, c_h^2 \le 1$ is satisfied, where c_h equals $2\sqrt{N+1}/h$ (namely the norm of the discretized gradient operator).

The second scheme is the following semi-implicit algorithm

$$\begin{cases} \sigma^h_{n+1} = \Pi^h_K(\sigma^h_n + \alpha \nabla^h \overline{v}^h_n) \\ v^h_{n+1} = v^h_n - \beta(\Delta^h)^{-1}(\mathrm{div}^h(\sigma^h_{n+1})) \\ \overline{v}^h_{n+1} = 2v^h_{n+1} - v^h_n \,. \end{cases} \tag{6.7}$$

Notice that (6.7) differs from (6.6) just in the term $-(\Delta^h)^{-1}(\mathrm{div}^h(\sigma^h_{n+1}))$ which replaces $\mathrm{div}^h(\sigma^h_{n+1})$. The theoretical convergence of this second algorithm is proved under the condition $\alpha\beta \le 1$, which is independent from both the mesh side and the space dimension (see Chapter 5). Moreover, in this case the convergence occurs after a relatively small number of iterations (see Figure 6.4 and Table 6.1).

6.1.2 Some simulations in case $N = 1$

When the open set Ω is an interval $(0, a)$ of the real line, we can solve explicitly the primal problem, which reads

$$\mathcal{I}(a, \lambda) := \inf \Big\{ \int_0^a \frac{|u'|^2}{2} + \lambda |\{u \neq 0\}| \, dt \ : \ u \in W^{1,2}(0, a) \,, \ u(0) = u(a) = 1 \Big\} . \tag{6.8}$$

The Euler-Lagrange equation written in the integrated conservation law form reads

$$\frac{1}{2}|u'|^2 - \lambda \chi_{\{u \neq 0\}} = C \,. \tag{6.9}$$

Two cases may occur, according to whether the measure of the level set $\{u = 0\}$ is null or strictly positive. In the first case, the solution is the constant function equal to 1 on $(0, a)$, with cost equal to λa. In the second case, the constant C in (6.9) equals zero, so that $u' \in \{0, \pm\sqrt{2\lambda}\}$. Setting $E^{\pm} = \{x \in (0, a) \, : \, u' = \pm\sqrt{2\lambda}\}$, since $\int_0^a u' = 0$, there holds $|E^+| = |E^-|$, and the cost is $4\lambda|E^-|$. On the other hand, since $u(0) = 1$ and u reaches the level zero, we have the lower bound $|E^-| \ge 1/\sqrt{2\lambda}$. Therefore, such a function u can be a minimizer only if $a \ge 2\sqrt{2/\lambda}$, and in this case the minimal cost is larger than or equal to $2\sqrt{2\lambda}$, with equality if $E^- = (0, 1/\sqrt{2\lambda})$, $E^+ = (h - 1/\sqrt{2\lambda}, a)$. To summarize, we have $\mathcal{I}(\lambda, a) = \min\{\lambda a, 2\sqrt{2\lambda}\}$, and

(i) for $a \in (0, 2\sqrt{\frac{2}{\lambda}}]$, the unique solution is $\overline{u}_1 \equiv 1$;

(ii) for $a > 2\sqrt{\frac{2}{\lambda}}$, the unique solution is

$$\overline{u}_2(x) = \begin{cases} -\sqrt{2\lambda}x + 1 & \text{if } x \in [0, \frac{1}{\sqrt{2\lambda}}] \\ 0 & \text{if } x \in [\frac{1}{\sqrt{2\lambda}}, a - \frac{1}{\sqrt{2\lambda}}] \\ \sqrt{2\lambda}x + 1 - \sqrt{2}a & \text{if } x \in [a - \frac{1}{\sqrt{2\lambda}}, a] \,. \end{cases}$$

(iii) for $a = 2\sqrt{\frac{2}{\lambda}}$ there are two solutions, given by the two functions $\overline{u}_1$ and $\overline{u}_2$.

Below we give some numerical results obtained, for $a = 2$, by using the algorithm (6.7).

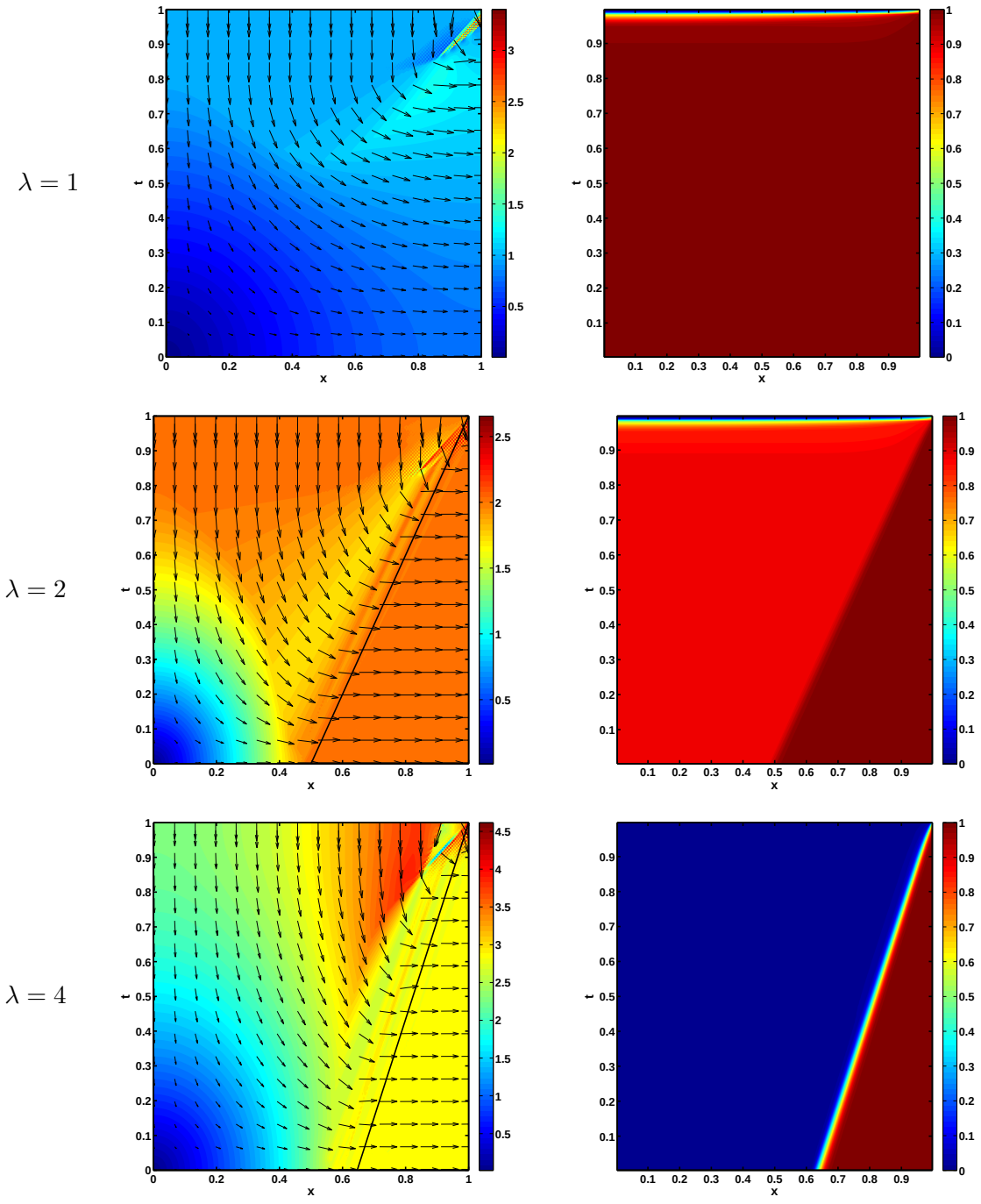


Figure 6.1: Streamlines of $\overline{\sigma}$ and level sets of $\overline{v}$.

In Figure 6.1 we present three cases $\lambda = 1, 2, 4$. They represent the behaviour of the optimal $\overline{\sigma}$ and $\overline{v}$ in each case. Up to a translation of the interval $\Omega = (0, 2)$ into $(-1, 1)$, we can work on the cylinder $(-1, 1) \times (0, 1)$; then, for symmetry reasons, we limit ourselves to plot our functions on the right part $(0, 1) \times (0, 1)$ of the cylinder. Notice that the most important issue is the location of the discontinuity set of $\overline{v}$, as the free boundary is given by the intersection of this set with the horizontal axis.

For $\lambda = 2$, we recover the two solutions $\overline{u}_1$ and $\overline{u}_2$ since the optimal function $\overline{v}$ exhibits three values (see Figure 6.1, were the regions in blue, red, and brown correspond respectively to the

level sets $\{\overline{v} = 0\}$, $\{\overline{v} = 0.8886\}$, and $\{\overline{v} = 1\}$).

In constrast, for $\lambda = 1$ or $\lambda = 4$, when the primal problem admits a unique solution, the function $\overline{v}$ exhibits only two values (see the regions in blue and brown in Figure 6.1).

6.1.3 Some simulations in case $N = 2$

By using the concavity of the map $\lambda \mapsto \mathcal{I}(\Omega, \lambda)$ one can check that, similarly to the one dimensional case, there exists a critical value $\lambda^* = \lambda^*(\Omega)$ below which the unique solution of the primal problem is $\overline{u}_1 \equiv 1$, corresponding to the function $\overline{v}_1 \in \widehat{\mathcal{A}}$ which vanishes identically in $\Omega \times (0, 1)$. For $\lambda = \lambda^*(\Omega)$ this solution may coexists with a non constant solution $\overline{u}_2$, exhibiting a free boundary E. Moreover, the function $\Omega \mapsto \lambda^*(\Omega)$ turns out to be monotone decreasing with respect to domain inclusions. In the special case when $\Omega = B_R := \{|x| < R\}$, we find the explicit value $\lambda^*(B_R) = \frac{2e}{R^2}$.

We now present some numerical simulations obtained for $\Omega = (-1, 1)^2$. Noticing that $B_1 \subset \Omega \subset B_{\sqrt{2}}$, we can predict a critical value $\lambda^*(\Omega)$ in the interval $(e, 2e)$. In fact, by using the second algorithm described above with a mesh size 10^{-2} and by tuning the value of λ, we obtained the estimate $\lambda^*(\Omega) \sim 4.7$.

In Figures 6.2 and 6.3 we represent respectively the behaviour of the optimal field $\overline{\sigma}$ and of the optimal function $\overline{u}$ for $\lambda = 2e$ (for symmetry reasons, Figure 6.2 is referred just to a quarter of Ω, namely to the set $(0, 1)^2$). Notice that the free boundary is given by the frontier of the region in dark blue.

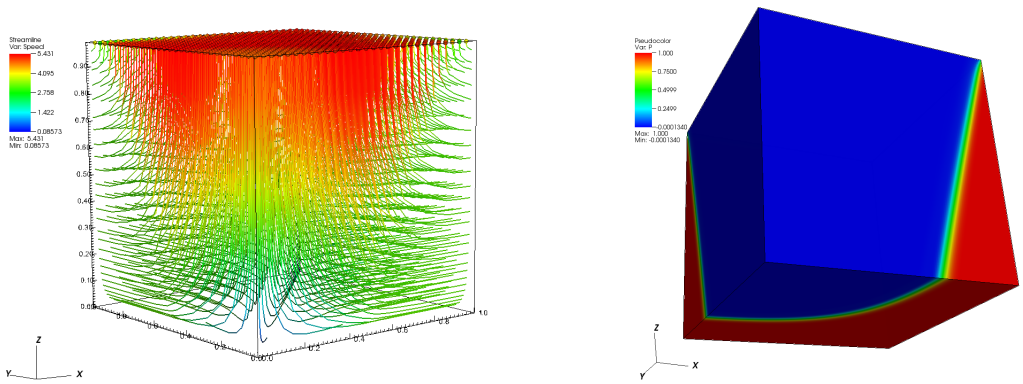


Figure 6.2: Streamlines of $\overline{\sigma}$ and plots of $\overline{v}$ in the case $\lambda = 2e$.

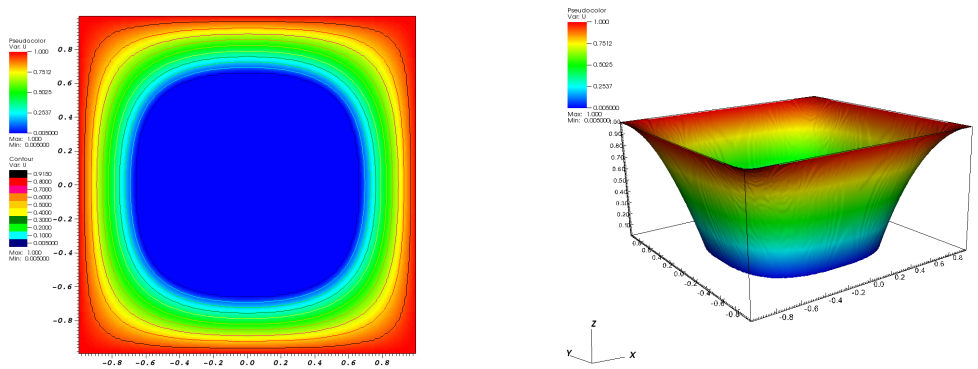


Figure 6.3: Level sets and plots of $\overline{u}$ in the case $\lambda = 2e$.

$\|\operatorname{div}\sigma\|_{L^2} < 10^{-3}$ with MPI in 6 processes				
$\sqrt[3]{N}$	iterations		time (seconds)	
	explicit	**implicit**	**explicit**	**implicit**
63	10662	621	62.54	39.65
123	31147	926	1810.78	501.15
183	59501	1222	11775.74	2333.48
243	93526	1521	44432.96	7430.99

Table 6.1: Free boundary problem. $\lambda = 2e$. Comparison of two methods in iterations and computational time. The stop criterion is $\|\operatorname{div}\sigma\|_{L^2} < 10^{-3}$. They are implemented with MPI in 6 processes.

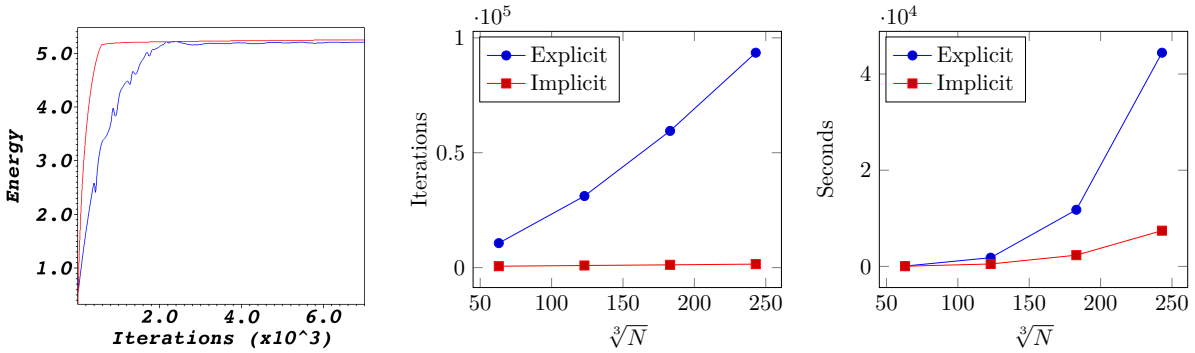
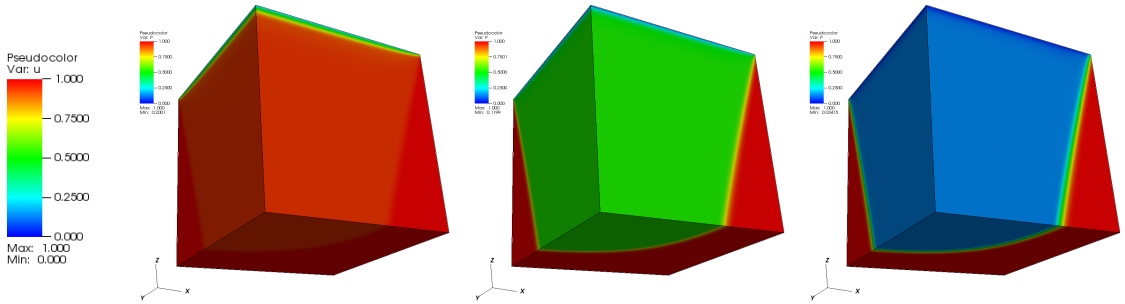


Figure 6.4: Free boundary problem. $\lambda = 2e$. Comparison of two methods in term of iterations, computational time.



$\lambda = 4.6$ $\lambda = 4.7$ $\lambda = 4.8$

Figure 6.5: Plots of optimal $\overline{v}$ in case λ approaching the Cheeger constant h_D.

6.2 Free boundary problems with linear growth

We switch to the integrand

$$f(t,z) = \varphi(z) + \lambda\chi_{(0,+\infty)}(t) \tag{6.10}$$

where φ is a function with linear growth. Replacing the quadratic term $|z|^2/2$ in (6.2) by $\varphi(z)$ still reproduce variants of free boundary in the analogical context. Nevertheless, solutions of the related variational problems are not obligatory to be continuous. Thus, these problems should be relaxed in BV space so that their minimum are attained and their minimizers have possibly discontinuities. When the solutions accept discontinuities, solving numerically the

variational problems is required to approach the discontinuity interfaces well or at least agreeably. One of the simplest ways is to augment the nodes of the discretized grid. Yet it results in a very big computation, especially, 3-dimensional grids. The difficulty is then multiplied since the primal-dual algorithms are iterative processes. We meet the challenge of reducing the computational time because the more the iteration is, the more it takes time. We have to balance these purposes to obtain a pertinent approximation of the discontinuity interfaces in an acceptable amount of time. The semi-implicit scheme is expectantly designed to do this task.

In case where we take $\varphi(z) = \sqrt{1+|z|^2}$, we get a variational formulation of minimal surfaces having free boundary

$$\mathcal{J}(\Omega,\lambda) := \inf\left\{\int_\Omega \sqrt{1+|\nabla u|^2} + \lambda|\{u>0\}| \ : \ u\in W^{1,2}(\Omega)\,,\ u=1 \text{ on } \partial\Omega\right\}. \tag{6.11}$$

Up to a constant, the problem $\mathcal{J}(\Omega,\lambda)$ is actually a reformulation of the problem $\beta(\lambda)$ in Chapter 4. The prescribed boundary data is now $u_0 \equiv 1$ instead of $u_0 \equiv 0$. The dual problem of $\mathcal{J}(\Omega,\lambda)$ reads

$$\mathcal{J}^*(\Omega,\lambda) := \sup\left\{-\int_\Omega \sigma^t(x,1)\,dx \ : \ \sigma\in\mathcal{B}\right\},$$

where the class $\mathcal{B}$ of bounded divergence-free vector field on $\Omega\times\mathbb{R}$ satisfies the pointwise constraints

$$|\sigma^x(x,t)|\le 1 \ \text{ a.e. on } \Omega\times\mathbb{R}, \quad \sigma^t(x,t)+\sqrt{1-|\sigma^x(x,t)|^2}+\lambda\ge 0 \ \text{ a.e. on } \Omega\times\mathbb{R}, \tag{6.12}$$

$$\sigma^t(x,0)\ge -1 \ \text{ a.e. on } \Omega. \tag{6.13}$$

With the same approach in Section 6.1, we are interesting in solving the primal and dual problems, $\mathcal{J}(\Omega,\lambda)$ and $\mathcal{J}^*(\Omega,\lambda)$, by means of the saddle point problem

$$\inf_{v\in\widehat{\mathcal{A}}}\ \sup_{\sigma\in\mathcal{K}} \int_{\Omega\times[0,1]} \sigma\cdot Dv\,, \tag{6.14}$$

where closed convex subsets are now adapted with

$$\widehat{\mathcal{A}} = \Big\{v\in BV_\infty(\Omega\times\mathbb{R}) \ : \ v=1 \text{ for } t<0\,,\ v=0 \text{ for } t>1\,,\ v=1 \text{ on } \partial\Omega\times[0,1]\Big\},$$

$$\mathcal{K} = \Big\{\sigma\in X_1(\Omega\times(0,1)) \ : \ |\sigma^x|\le 1,\ \sigma^t+\sqrt{1-|\sigma^x|^2}+\lambda\ge 0 \text{ a.e. in } \Omega\times(0,1),$$
$$\sigma^t(\cdot,0)\ge -1 \text{ a.e. on } \Omega\Big\}.$$

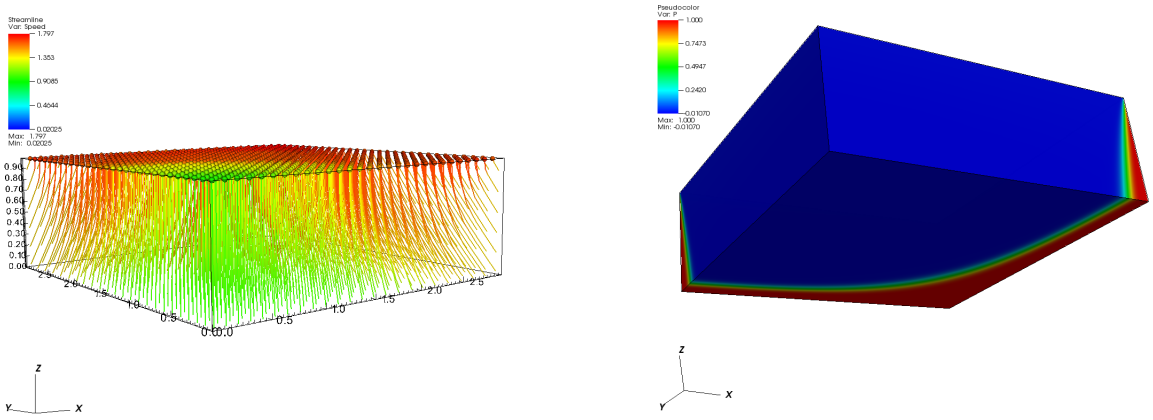


Figure 6.6: Streamlines of $\overline{\sigma}$ and plots of $\overline{v}$ in the case $\lambda = 0.8$.

In view of Theorem 2.14, we can recover the optimal pair $(\overline{u}, \overline{\sigma})$ for the couple of problems in duality $\mathcal{J}(\Omega, \lambda)$ and $\mathcal{J}^*(\Omega, \lambda)$ from any saddle point of (6.14). We then apply the semi-implicit algorithm (6.7) to solve (6.14). The only thing changed in this scheme is the projection on the closed convex set $\mathcal{K}$. We present in Figure 6.6 a saddle point of the problem (6.14) for $\Omega = [-3, 3]$ and $\lambda = 0.8$. Because of symmetry reason, we plot them in a quarter of the domain Ω. As we can see, the optimal $\overline{v}$ almost takes values in $\{0, 1\}$ and has a thin interface of intermediate values in $(0, 1)$. This interface presents the theoretical discontinuity of $\overline{v}$ which allows recovering the solution to the primal problem $\mathcal{J}(\Omega, \lambda)$. In Figure 6.7, we plot the optimal $\overline{u}$ of $\mathcal{J}(\Omega, \lambda)$ which is obtained from $\overline{v}$. We chose in this example the square Ω large enough (of sizes 6×6) and Ω small enough ($\lambda = 0.8$) so that we can find an optimal $\overline{u}$ that does not have any discontinuity. Note that the free boundary is given by the frontier of the dark blue region.

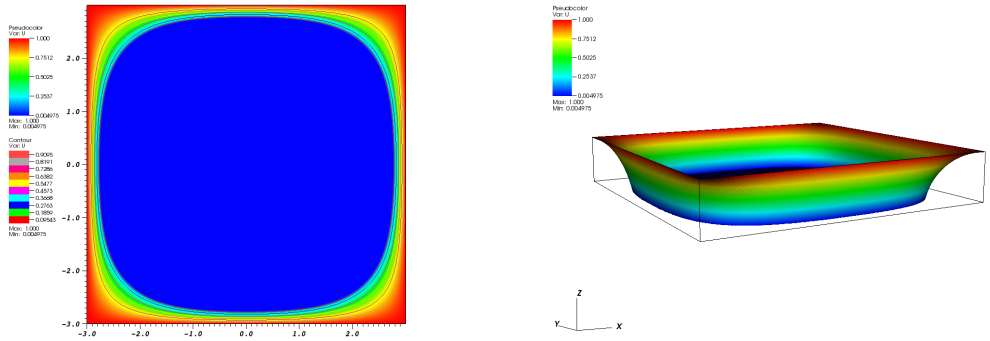


Figure 6.7: Level sets and plots of $\overline{u}$ in the case $\lambda = 0.8$.

As discussed widely in Section 4.4 of Chapter 4, when we replace $\varphi(z)$ in (6.10) by $\varphi_0(z) = 1 + |z|$, the problem $\mathcal{J}(\Omega, \lambda)$ coincide with the problem minimizing total variation

$$\mathcal{J}_0(\Omega, \lambda) := |\Omega| + \inf \left\{ \int_\Omega |\nabla u| dx + \lambda |\{u > 0\}| \ : \ u \in W^{1,2}(\Omega)\,,\ u = 1 \text{ on } \partial\Omega \right\}$$

if and only if their common solutions are trivial, i.e. either $\overline{u} \equiv 0$ or $\overline{u} \equiv 1$. And in dimension 2, solutions of the problem $\mathcal{J}_0(\Omega, \lambda)$ are explicit, and of the form $\mathbf{1}_{\Omega \setminus \Omega_\lambda}$ where Ω_λ is the union of all ball of radius $1/\lambda$ contained in Ω. Nonetheless, we also have some simulation in dimension 2. In our implementation, we took $\Omega = [-1/2, 1/2]$. The Cheeger constant of Ω is explicitly computed $h_\Omega = 2 + \sqrt{\pi}$. We show in Figure 6.8 the solution of the primal problem $\mathcal{J}_0(\Omega, \lambda)$ and the saddle point for the problem (6.14) for $\lambda = h_\Omega = 2 + \sqrt{\pi}$. Figure 6.9 displays the approximation of the Cheeger set C_Ω of Ω and its calibrating field (in the first quarter of the domain). It is easy to see that the optimal $\overline{v}$ is actually the characteristic function of the complement of C_Ω in Ω, i.e. $\overline{v} = \mathbf{1}_{\Omega \setminus C_\Omega}$. Then, $\overline{u}$ is determined by $u_s(x) = \inf\{t \in \mathbb{R} \ : \ \overline{v}(x, t) \leq s\}$ for any $s \in (0, 1)$.

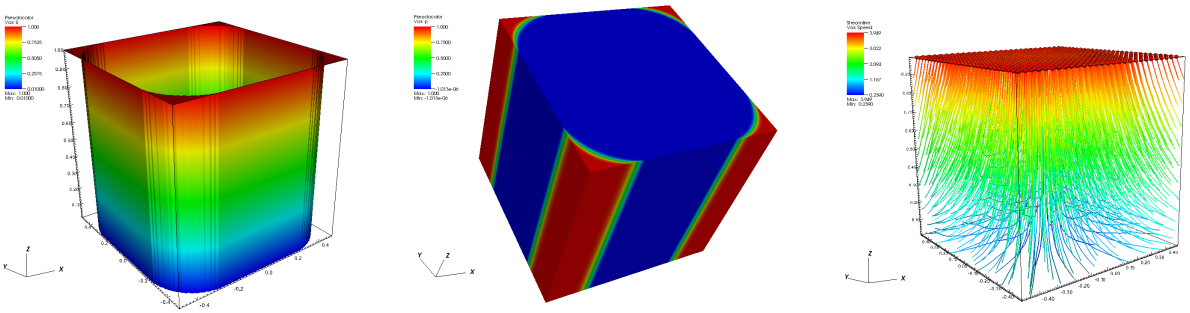


Figure 6.8: Plots of $\overline{u}$, $\overline{v}$ and streamlines of $\overline{\sigma}$ in the case $\lambda = h_\Omega = 2 + \sqrt{\pi}$.

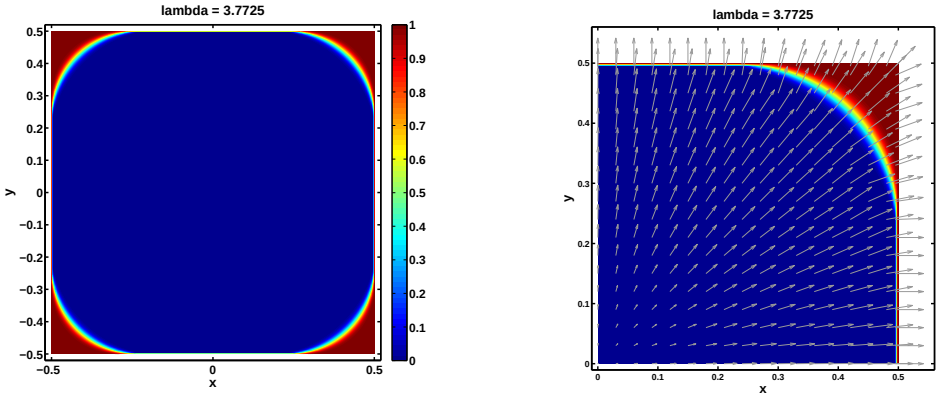


Figure 6.9: Approximation of Cheeger set and calibrating field.

Perspectives and open problems

Concerning the duality theory studied and developed in Chapter 2, we are led to perspectives and open problems which motivates many interests for future works. Let us describe them in the following:

Functionals involving vector valued functions

The arguments in Section 2.2 have been developped merely in the case of scalar functions. Namely the space X for which we construct an embedding in the extreme points of some convex compact subset has always been $L^1(\Omega)$. An extension of the method working for vector valued functions requires to construct another embedding. A very simple choice would be to associate to a vector field $u \in L^1(\Omega;\mathbb{R}^N)$ the Dirac mass at $u(x)$ so that $\varphi(u)$ becomes the Young measure on $\Omega \times \mathbb{R}^N$ defined by:

$$< \varphi(u), \psi >= \int_\Omega \psi(x, u(x))\, dx \ , \ \psi \in C^0(\Omega \times \mathbb{R}^N) \ .$$

If $|\Omega| = 1$ and $u(x)$ is assigned to stay in a given convex compact subset $K \subset \mathbb{R}^N$, then $\varphi(u)$ is an extreme point in the set of probabilities measures on $\Omega \times K$ whose first marginal agrees with the Lebesgue measure on Ω. However the explicit computation of the convexified functional seems difficult in this framework. A possible issue would be to consider more involved tools of geometric measure theory as Cartesian currents or varifolds [90, 69, 70].

Functionals involving second order gradients

Going back to the scalar case, many problems involve functionals of the kind

$$F(u) \ = \ \int_\Omega f(\nabla u, \nabla^2 u)\, dx \ ,$$

being $f(z, M) : \mathbb{R}^N \times \mathbb{R}^{N\times N}_{\text{sym}}$ a function convex in M but not in z. Applying the convexification procedure like in Section 2.3 seems to be a nice perspective in this context; It will involve the curvature tensor at each point $(x, u(x))$ of the graph of u.

Mumford -Shah functional

The free discontinuity problems have been the first motivation for the convexification recipe presented in Chapter 2. The goal was to prove the optimality of some specific configurations for the image segmentation problem described hereafter. Let Ω be a bounded and Lipschitz domain of $\mathbb{R}^2$ and $g : \Omega \to [0, 1]$ (grey level data).

$$(\mathcal{P}) \qquad \inf_{u\in SBV(\Omega)} \left\{ \int_{\Omega\setminus S_u} \frac{1}{2}|\nabla u|^2\, dx + \mathcal{H}^1(S_u) + \frac{1}{2}\int_\Omega |u - g(x)|^2\, dx \right\}$$

This setting turns out to be well posed (existence of at least one minimizer) in the space $SBV(\Omega)$ of functions $u \in L^1(\Omega)$ whose distributional gradient Du consists in a regular part ∇u (coincides

with the a.e. defined gradient) and a singular part concentrated on the jump set S_u which is a rectifiable one dimensional (unknown) subset of Ω whose total length is denoted $\mathcal{H}^1(S_u)$. A more mechanical formulation of $(\mathcal{P})$ (popular in fracture mechanics) reads

$$\inf\left\{\int_{\Omega\setminus K}\frac{1}{2}|\nabla u|^2\,dx+\mathcal{H}^1(K)+\frac{1}{2}\int_\Omega|u-g(x)|^2\,dx\ :\ K\text{ closed subset}\subset\Omega\ ,\ u\in C^1(\Omega\setminus K)\right\}$$

As the source term g satisfies $0\le g\le 1$, by using a trivial truncation argument, one checks easily that the infimum of $(\mathcal{P})$ is unchanged if restricted to competitors u taking values in $[0,1]$. Accordingly we consider the metric space $X=L^1(\Omega;[0,1])$ on which we define the functional

$$F(u):=\begin{cases}\displaystyle\int_{\Omega\setminus S_u}\frac{1}{2}|\nabla u|^2\,dx+\mathcal{H}^1(S_u)+\frac{1}{2}\int_\Omega|u-g(x)|^2\,dx & \text{if } u\in SBV(\Omega;[0,1])\\ +\infty & \text{otherwise}\end{cases}.$$

Then F turns out to be coercive and l.s.c. Recalling the construction in Section 2.3 (see also Remark 2.12), we can define a convex functional G on $L^\infty(\Omega\times[0,1])$ by setting:

$$G(v)=\sup_{g\in L^\infty(\Omega\times[0,1])}\left\{\int_{\Omega\times[0,1]}gv\,dxdt-F_0^*(g)\right\},\ F_0^*(g)=\sup_{u\in X}\left\{\int_{\Omega\times[0,1]}g(x,t)\,\mathbf{1}_u\,dxdt-F(u)\right\}.$$

so that $G(\mathbf{1}_u)=F(u)$ for every $u\in X$ and $\inf(\mathcal{P})=\inf\{G(v)\ :\ v\in L^\infty(\Omega\times[0,1];[0,1])\}$.

Unfortunately this functional G cannot be recovered by using the coarea formula (2.18) and, to our knowledge, no explicit formula for G is available. Alternatively, in [1], another convex l.s.c. functional J was used such that $J\le G$ but satisfying $J(\mathbf{1}_u)=G(\mathbf{1}_u)=F(u)$ for every $u\in X$. Although it is not known whether or not J shares the same infimum as G, a duality scheme applied to J has been unexpectedly useful for checking the optimality of some competitors for problem $(\mathcal{P})$ (see many examples in [1]). In this framework the dual problem reads as follows:

$$(\mathcal{Q})\quad\sup\left\{-\int_\Omega\sigma^t(x,0)\,dx\ ,\ \sigma\in\mathcal{K}\ ,\ \mathrm{div}\sigma=0\ \text{ on }\Omega\times[0,1]\ ,\ \sigma^x\cdot\nu_\Omega=0\ \text{ on }\partial\Omega\times[0,1]\right\},$$

where the convex constraint $\sigma\in\mathcal{K}$ splits into the two conditions:

i) $\dfrac{1}{2}|\sigma^x|^2\le\sigma^t+\dfrac{1}{2}|t-g(x)|^2$ a.e. in $\Omega\times[0,1]$

ii) $\left|\displaystyle\int_{t_1}^{t_2}\sigma^x(x,s)\,ds\right|\ \le\ 1$, a.e $x\in\Omega$ and for every $(t_1,t_2)\in[0,1]^2$

The second condition takes into account the jump energy in $F(u)$ and *is non local*. The functional J defined in $BV(\Omega\times\mathbb{R};[0,1])$ can be recovered by duality:

$$J(v)\ =\ \sup\left\{\int_{\Omega\times\mathbb{R}}(Dv\cdot\sigma)\ :\ \sigma\in\mathcal{K}\ ,\ \sigma\in C^1(\Omega\times\mathbb{R})\right\}\ .$$

As claimed before this convex functional satisfies $J(\mathbf{1}_u)=F(u)$ whereas $J(v)<+\infty$ implies that $v(x,\cdot)$ is non increasing. It is then possible to prove:

Proposition I. *Let $g\in L^\infty(\Omega;[0,1])$. Then it holds $\inf(\mathcal{P})\ge\sup(\mathcal{Q})$ with equality if, for an admissible pair (u,σ), one has*

$$\begin{array}{l}\sigma(x,u(x))\ =\ (\nabla u(x),\frac{1}{2}(|\nabla u|^2-|u-g|^2))\quad a.e.\ x\in\Omega\\ \int_{u^-(x)}^{u^+(x)}\sigma^x(x,t)\cdot\nu_u\ =\ 1\quad\mathcal{H}^1\ a.e.\ x\in S_u\ ,\end{array}\qquad(*)$$

where $u^\pm$ denote the upper and lower approximate limits of u and $S_u=\{u^+>u^-\}$.

Let us notice that the latter result is useful merely when it is possible to guess particular pairs (u, σ) satisfying conditions $(*)$. When a competitor u is candidate to be a global minimizer, finding a σ provides a sufficient condition of optimality. This calibrating vector field σ if it exists is determined on the graph of u by relations $(*)$. The difficulty is to extend it outside the graph of u while preserving the constraints $i)$ and $ii)$ and the divergence free condition. We refer to [1] for explicit constructions in case of particular Dirichlet boundary data. Unfortunately a calibration field for proving the optimality of a function of type crack-tip could not yet be found. A very challenging issue which will be worth for further investigations is the following:

Conjecture: The following equality holds: $\inf(\mathcal{P}) = \sup(\mathcal{Q})$.

Postulating a priori the validity of such a conjecture, numerical schemes based on a primal-dual algorithm are actually used to solve problem $(\mathcal{P})$ (see [96]). To our knowledge, it has neither been evidenced any numerical gap disproving the conjecture.

Uniqueness of solution for a multiphase problem

In Chapter 3, we proved (see Theorem 3.3) that any minimizer of the non convex problem is a minimizer of the convexified. However the converse statement could be untrue although we have no counterexample. At this stage, arises a deep open issue: is it possible to show uniqueness for the convexified problem despite the fact that the convexified potential g^{**} is not strictly convex ? Notice that in the interesting case we have treated, g^{**} is piecewise affine.

Optimality criterion for phase interfaces of an anisotropic multiphase problem

In Chapter 3, the geometric optimality conditions that we found for the phase interfaces are valid only in the isotropic case of the total variation functional (relation between algebraic curvature of the interfaces and the respective jumps of the minimizer u and of the density $g(u)$ (see (3.11)). We are interested in finding the counterpart of these optimality conditions in the anisotropic variant where crystalline norms are considered.

Perspectives of numerical approximation

Our semi-implicit algorithm (wherein an inverse Laplacian is utilized) is proved to be efficient not only in term of precision (fineness of interfaces) but also in computational time. Especially when this numerical scheme is combined with the epigraph projection method, it allows to treat many non differentiable Lagrangians. In future works, we expect a complete study on the convergence rate of this method, in particular in comparison with the classical implicit scheme. On the other hand, when the problem considered requires a duality scheme in dimension 3, the number of iterations necessary to obtain sharp interfaces is considerably increased and an acceptable computation time requires using a high performance (parallel) computing platform. A further objective is to reduce the computation time by making our global approach accurate with additional tools in the vicinity of the interfaces.

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Tran Duc Minh PHAN

Institut de Mathématiques de Toulon, Université de Toulon

Une méthode de dualité pour des problèmes non convexes du Calcul des Variations

Dans cette thèse, nous étudions un principe général de convexification permettant de traiter certains problèmes variationnels non convexes sur $\mathbb{R}^d$. Grâce à ce principe nous pouvons mettre en œuvre les puissantes techniques de dualité et ramener de tels problèmes à des formulations de type primal–dual dans $\mathbb{R}^{d+1}$, rendant ainsi efficace la recherche numérique de minima globaux. Une théorie de la dualité et des champs de calibration est reformulée dans le cas de fonctionnelles à croissance linéaire. Sous certaines hypothèses, cela nous permet de généraliser un principe d'exclusion découvert par Visintin dans les années 1990 et de réduire le problème initial à la minimisation d'une fonctionnelle convexe sur $\mathbb{R}^d$. Ce résultat s'applique notamment à une classe de problèmes à frontière libre ou multi-phasique donnant lieu à des tests numériques très convaincants au vu de la qualité des interfaces obtenues. Ensuite nous appliquons la théorie des calibrations à un problème classique de surfaces minimales avec frontière libre et établissons de nouveaux résultats de comparaison avec sa variante où la fonctionnelle des surfaces minimales est remplacée par la variation totale. Nous généralisons la notion de calibrabilité introduite par Caselles-Chambolle et Al. et construisons explicitement une solution duale pour le problème associé à la seconde fonctionnelle en utilisant un potentiel localement Lipschitzien lié à la distance au cut-locus. La dernière partie de la thèse est consacrée aux algorithmes d'optimisation de type primal-dual pour la recherche de points selle, en introduisant de nouvelles variantes plus efficaces en précision et temps calcul. Nous avons en particulier introduit une variante semi-implicite de la méthode d'Arrow-Hurwicz qui permet de réduire le nombre d'itérations nécessaires pour obtenir une qualité satisfaisante des interfaces. Enfin nous avons traité la non différentiabilité structurelle des Lagrangiens utilisés à l'aide d'une méthode géométrique de projection sur l'épigraphe offrant ainsi une alternative aux méthodes classiques de régularisation.

Mot clés : Optimisation non convexe, Principe de min-max, Frontière libre, Discontinuités libres, Gamma-convergence, Algorithme primal-dual, Projection épigraphique.

A duality method for non-convex problems in Calculus of Variations

In this thesis, we study a general principle of convexification to treat certain non convex variational problems in $\mathbb{R}^d$. Thanks to this principle we are able to enforce the powerful duality techniques and bring back such problems to primal-dual formulations in $\mathbb{R}^{d+1}$, thus making efficient the numerical search of a global minimizer. A theory of duality and calibration fields is reformulated in the case of linear-growth functionals. Under suitable assumptions, this allows us to revisit and extend an exclusion principle discovered by Visintin in the 1990s and to reduce the original problem to the minimization of a convex functional in $\mathbb{R}^d$. This result is then applied successfully to a class of free boundary or multiphase problems that we treat numerically obtaining very accurate interfaces. On the other hand we apply the theory of calibrations to a classical problem of minimal surfaces with free boundary and establish new results related to the comparison with its variant where the minimal surfaces functional is replaced by the total variation. We generalize the notion of calibrability introduced by Caselles-Chambolle and Al. and construct explicitly a dual solution for the problem associated with the second functional by using a locally Lipschitzian potential related to the distance to the cut-locus. The last part of the thesis is devoted to primal-dual optimization algorithms for the search of saddle points, introducing new more efficient variants in precision and computation time. In particular, we experiment a semi-implicit variant of the Arrow-Hurwicz method which allows to reduce drastically the number of iterations necessary to obtain a sharp accuracy of the interfaces. Eventually we tackle the structural non-differentiability of the Lagrangian arising from our method by means of a geometric projection method on the epigraph, thus offering an alternative to all classical regularization methods.

Keywords : Non-convex optimization, Min-Max principle, Free boundary, Free discontinuities, Gamma-convergence, Primal-dual algorithm, Epigraph projection.