



# Combinatorial links between quasisymmetric functions and tableaux for Coxeter groups.

Alina Mayorova

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# Liens combinatoires entre fonctions quasisymétriques et tableaux dans les groupes de Coxeter

Thèse de doctorat de l'Université Paris-Saclay  
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## *Abstract*

**Title:** Combinatorial links between quasisymmetric functions and tableaux for Coxeter groups.

**Keywords:** type B quasisymmetric functions, Schur functions, Cauchy identity, descent algebra, RSK-correspondence, hyperoctahedral group, domino tableaux, Schur positivity.

The algebra of symmetric functions is a major tool in algebraic combinatorics that plays a central rôle in the representation theory of the symmetric group. This thesis deals with quasisymmetric functions, a powerful generalisation introduced by Gessel in 1984, with significant applications in the enumeration of major combinatorial objects as permutations, Young tableaux and P-partitions and in the study of advanced symmetric functions like Schubert and Macdonald polynomials. More specifically we find a new connection between Chow's extension of quasisymmetric functions to Coxeter groups of type B and domino tableaux, i.e. Young diagrams tiled with  $2 \times 1$  and  $1 \times 2$  rectangles. It allows us to contribute new results to various fields including the structure constants of type B Solomon's descent algebra, the extension of the theory of Schur-positivity to signed permutations and the study a  $q$ -deformed type B Cauchy formula with important implications regarding statistics for domino tableaux.

Among the remarkable bases of the algebra of symmetric functions, Schur functions received particular attention as they are strongly related to the irreducible characters of the general linear group and Young diagrams. The Schur symmetric function indexed by an integer partition  $\lambda$  is the generating function for semistandard Young tableaux of shape  $\lambda$ . This result extends to skew shapes and allows to write any (skew-) Schur function as the sum of Gessel's fundamental quasisymmetric functions indexed by the descent set of all standard Young tableaux of a given shape. Furthermore the celebrated Cauchy formula for Schur functions gives an algebraic proof of the Robinson-Schensted-Knuth correspondence. Finally, the structure constants for the outer product and inner product of Schur polynomials are respectively the Littlewood-Richardson and Kronecker coefficients, two important families of coefficients with various combinatorial and algebraic applications. Using known results about Gessel's fundamental quasisymmetric functions we show that these properties imply directly and in a pure algebraic fashion, various results for the structure constants of the Solomon descent algebra of a finite Coxeter group of type A and the descent preserving property of the Robinson-Schensted correspondence, an essential tool to identify Schur-positive sets, i.e. sets of permutations whose associated quasisymmetric function is symmetric and can be written as a non-negative sum of Schur symmetric functions.

To extend these results to Coxeter groups of type B we introduce a family of modified generating functions for domino tableaux and relate it to Chow's type B fundamental quasisymmetric functions. Thanks to this relation we derive new formulas relating the structure constants of the type B Solomon's descent algebra with type B Kronecker and Littlewood-Richardson coefficients.

It further allows us to introduce a new type B extension of Schur-positivity based on a definition of descent for signed permutations that conforms to the abstract definition of Solomon for any Coxeter groups. We design descent preserving bijections between signed arc permutations and sets of domino tableaux to show that they are indeed type B Schur-positive. We also suggest an algebraical approach to prove the same fact.

Finally, we introduce a  $q$ -deformation of the modified generating functions for domino tableaux to extend a type B Cauchy identity by Lam and link it with Chow's quasisymmetric functions. We apply this result to a new framework of type B  $q$ -Schur positivity and to prove new equidistribution results for some sets of domino tableaux.

## *Résumé*

**Titre :** Liens combinatoires entre fonctions quasisymétriques et tableaux dans les groupes de Coxeter.

**Keywords :** fonctions quasisymétriques de type B, fonctions de Schur, identité de Cauchy, algèbre de descente, correspondance RSK, groupe hyperoctaédrique, tableaux de dominos, Schur positivité.

L'algèbre des fonctions symétriques est un outil majeur de la combinatoire algébrique qui joue un rôle central dans la théorie des représentations du groupe symétrique. Cette thèse traite des fonctions quasisymétriques, une puissante généralisation introduite par Gessel en 1984, avec des applications significatives dans l'énumération d'objets combinatoires majeurs tels que les permutations, les tableaux de Young et les P-partitions et dans l'étude de fonctions symétriques avancées telles que les polynômes de Schubert et Macdonald. Plus précisément, nous trouvons un nouveau lien entre l'extension des fonctions quasisymétriques de Chow à des groupes de Coxeter de type B et des tableaux de dominos, c'est-à-dire des diagrammes de Young pavés par des rectangles  $2 \times 1$  et  $1 \times 2$ . Ceci nous permet d'apporter de nouveaux résultats dans divers domaines, notamment les constantes de structure de l'algèbre de descente de Solomon de type B, l'extension de la théorie de la Schur-positivité aux permutations signées et l'étude d'une formule de Cauchy de type B  $q$ -déformée avec des implications importantes statistiques pour les tableaux dominos.

Parmi les bases remarquables de l'algèbre des fonctions symétriques, les fonctions de Schur ont fait l'objet d'une attention particulière car elles sont étroitement liées aux caractères irréductibles du groupe linéaire général et aux diagrammes de Young. La fonction symétrique de Schur indexée par une partition entière  $\lambda$  est la fonction génératrice des tableaux de Young semistandard de forme  $\lambda$ . Ce résultat s'étend aux formes gauches et permet d'écrire n'importe quelle fonction de Schur (gauche) comme la somme des fonctions quasisymétriques fondamentales de Gessel, indexées par l'ensemble de descente de tous les tableaux de Young standard d'une forme donnée. En outre, la célèbre formule de Cauchy pour les fonctions de Schur donne une preuve algébrique de la correspondance de Robinson-Schensted-Knuth. Enfin, les constantes de structure pour la multiplication et la comultiplication des polynômes de Schur sont respectivement les coefficients de Littlewood-Richardson et de Kronecker, deux familles importantes de coefficients ayant diverses applications combinatoires et algébriques. En utilisant des résultats connus sur les fonctions quasisymétriques fondamentales de Gessel, nous montrons que ces propriétés impliquent directement et de façon purement algébrique divers résultats pour les constantes de structure de l'algèbre de descente de Solomon d'un groupe de Coxeter fini de type A et la propriété de préservation de descente de la correspondance de Robinson-Schensted, un outil essentiel pour identifier les ensembles Schur-positifs, c'est-à-dire les ensembles de permutations dont la fonction quasisymétrique associée est symétrique et qui peut s'écrire sous la forme d'une somme non négative de fonctions symétriques de Schur.

Pour étendre ces résultats aux groupes de Coxeter de type B, nous avons introduit une famille de fonctions génératrices modifiées pour les tableaux de dominos et la relierons aux fonctions quasisymétriques fondamentales de type B de Chow. Grâce à cette relation, nous obtenons de nouvelles formules reliant les constantes de structure de l'algèbre de descente de Solomon de type B aux coefficients de Kronecker et de Littlewood-Richardson de type B.

Cela nous permet en outre d'introduire une nouvelle extension de type B de la Schur-positivité basée sur une définition de la descente pour les permutations signées, conforme à la définition abstraite de Solomon pour tous les groupes de Coxeter. Nous concevons des bijections préservant la descente entre des permutations d'arc signées et des ensembles de tableaux de dominos afin de montrer qu'ils sont bien type B Schur-positifs. Nous proposons également une approche algébrique pour prouver le même fait.

Enfin, nous introduisons une  $q$ -déformation des fonctions génératrices modifiées pour les tableaux de dominos afin d'étendre une identité de Cauchy de type B proposée par Lam et de la lier aux fonctions quasisymétriques de Chow. Nous appliquons ce résultat à un nouveau cadre de positivité de type B  $q$ -Schur et à la démonstration de nouveaux résultats d'équidistribution pour certains ensembles de tableaux de dominos.

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# Chapter 1

## Definitions and results for type A Coxeter groups

### 1.1 Permutations and Young tableaux

For any integer  $n$  write  $[n]$  for the set  $\{1, \dots, n\}$  of integers. Consider  $S_n$  the symmetric group on  $[n]$ , also known as a Coxeter group of type A. Let  $id_n$  be the identity permutation of  $S_n$ .

One important feature of a permutation to research is its descent set.

*Definition 1.* The **descent set** of a permutation  $\pi$  of  $S_n$  is the subset of  $[n - 1]$  defined as

$$\text{Des}(\pi) = \{1 \leq i \leq n - 1 \mid \pi(i) > \pi(i + 1)\}.$$

For instance, the descent set of the permutation  $\pi = (4, 1, 6, 2, 3, 5)$  is  $\{1, 3\}$ .

Various statistics are linked with the descent set, including the well known **major index** of a permutation

$$\text{maj}(\pi) = \sum_{i \in \text{Des}(\pi)} i.$$

The classical result of MacMahon [Mac60] concerning the equidistribution of major index and inverse number statistics over the symmetric group  $S_n$  opens a set of results regarding permutation statistics (see [FS78], [FZ90] for example). Several enumerative results were generalised to the hyperoctahedral group  $B_n$  (see [Rei93a], [Rei93b], [Ste94]). The usual tool for these results is a suggestion of an explicit bijection with certain properties. Further in this thesis we will show some equidistribution results using a purely algebraical approach.

*Definition 2.* A **composition**  $\alpha \models n$  is a sequence of positive integers  $\alpha = (\alpha_1, \dots, \alpha_p)$  such that  $\alpha_1 + \alpha_2 + \dots = n$ . A composition  $\lambda$  of  $n$  with its parts sorted in decreasing order is called a **partition** and denoted by  $\lambda \vdash n$  or  $|\lambda| = n$ . The length of a partition is the number of its parts  $l(\lambda) = p$ . We denote by  $p(n)$  the number of partitions of  $n$ .

Compositions of  $n$  are in natural bijection with subsets of  $[n-1]$ . Indeed, for  $\alpha \models n$  and  $I = \{i_1, i_2, \dots, i_m\}$  a subset of  $[n-1]$  such that  $i_1 < i_2 < \dots < i_m$ , we set  $set(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{p-1}\}$  and  $comp(I) = (i_1, i_2 - i_1, \dots, i_m - i_{m-1}, n - i_m)$  this bijection and its inverse. As a result the number of compositions of  $n$  is  $2^{n-1}$ .

A partition  $\lambda$  is usually represented as a Young diagram of  $n = |\lambda|$  boxes arranged in  $\ell(\lambda)$  left justified rows so that the  $i$ -th row from the top contains  $\lambda_i$  boxes. Other names for Young diagram appear in the literature, e.g. **Young frame** or **Ferrers diagram**.

*Definition 3.* The **conjugate**  $\lambda'$  of a partition  $\lambda$  is the partition corresponding to the transpose diagram of the partition  $\lambda$ .

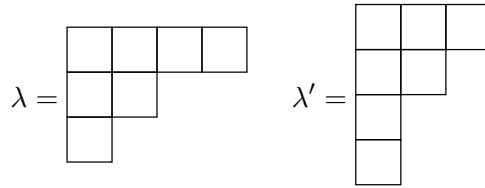


FIGURE 1.1: The Young diagram of the partition  $\lambda = (4, 2, 1)$  and the conjugate partition  $\lambda' = (3, 2, 1, 1)$ .

Partitions, or equivalently Young diagrams are used to parametrise a number of objects in algebra. First of all, they parametrise the conjugacy classes of the symmetric group  $S_n$  and therefore the irreducible characters of the symmetric group. Secondly, Young diagrams also parametrise the irreducible polynomial representations of the general linear group  $GL_n$ . They occur in representations of related groups like the special linear  $SL_n$  and special unitary groups  $SU_n$ .

Partitions also parametrise Schur functions, which are the Frobenius characteristics ([Fro00]) of the irreducible characters of the symmetric group, as well as the elements of all standard bases of the symmetric functions algebra. The question of *Schur-positivity* of certain functions and sets is therefore closely related to the representation theory of the symmetric group. See Section 1.10 for the details and Chapter 4 for a type B generalisation.

*Definition 4.* A **standard Young tableau**  $T$  is a Young diagram whose boxes are filled with the elements of  $[n]$  such that the entries are strictly increasing along the rows and down the columns. The partition given by the number of boxes in each row is its **shape**

and denoted  $\text{shape}(T)$ . We denote  $SYT(n)$  ( $SYT(\lambda)$ ) the set of standard Young tableaux of  $n$  boxes (of shape  $\lambda$ ).

**Example 1.1.** *The following diagrams are standard Young tableaux of shape  $\lambda = (6, 4, 2, 1, 1)$ .*

$T_1 =$ 

|    |    |   |    |   |    |
|----|----|---|----|---|----|
| 1  | 3  | 4 | 8  | 9 | 14 |
| 2  | 6  | 7 | 10 |   |    |
| 5  | 12 |   |    |   |    |
| 11 |    |   |    |   |    |
| 13 |    |   |    |   |    |

$T_2 =$ 

|    |    |   |   |    |    |
|----|----|---|---|----|----|
| 1  | 3  | 4 | 8 | 12 | 14 |
| 2  | 6  | 7 | 9 |    |    |
| 5  | 10 |   |   |    |    |
| 11 |    |   |   |    |    |
| 13 |    |   |   |    |    |

The dimension of a representation  $\pi_\lambda$  of the symmetric group  $S_n$  is equal to the number of standard Young tableaux of the corresponding shape. The well-known **hook-length** formula due to Frame, Robinson and Thrall [FdBRT54] allows us to compute the number of standard Young tableaux of shape  $\lambda$ .

$$\dim(\pi_\lambda) = f_\lambda = \frac{|\lambda|!}{\prod_b h_b},$$

where the product is over all boxes  $b$  and  $h_b$  is the number of boxes to the right and directly below  $b$  (including  $b$  itself).

*Example 1.*

$$f_{(4,2,1)} = \frac{7!}{6 \cdot 4 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 35.$$

**Definition 5.** Define the **descent set of a standard Young tableau**  $T$  as the subset of  $[n - 1]$  as

$$\text{Des}(T) = \{1 \leq i \leq n - 1 \mid i \text{ is in a strictly higher row than } i + 1\}.$$

For instance, the descent set of the tableaux  $T_1$  in Example 1.1 is  $\{1, 4, 9, 10, 12\}$ . We denote by  $d_{\lambda I}$  the number of standard Young tableaux of shape  $\lambda$  and descent set  $I$ .

**Definition 6.** A **semistandard Young tableau** is a Young diagram whose entries are strictly increasing down the columns and non-decreasing along the rows. A semistandard Young tableau  $T$  has weight  $w(T) = \mu = (\mu_1, \mu_2, \dots)$  if it has  $\mu_i$  entries equal to  $i$ . We denote by  $SSYT(n)$  ( $SSYT(\lambda)$ ) the set of semistandard Young tableaux of  $n$  boxes (of shape  $\lambda$ ).

**Example 1.2.** The following diagrams are semistandard Young tableaux of shape  $\lambda = (6, 4, 2, 1, 1)$  and weight  $\mu = (1, 2, 1, 0, 3, 1, 1, 1, 2, 2)$ .

$$T_1 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 7 & 10 \\ \hline 2 & 5 & 5 & 8 & & \\ \hline 5 & 9 & & & & \\ \hline 9 & & & & & \\ \hline 10 & & & & & \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 9 & 10 \\ \hline 2 & 5 & 5 & 7 & & \\ \hline 5 & 8 & & & & \\ \hline 9 & & & & & \\ \hline 10 & & & & & \\ \hline \end{array}$$

Given a partition  $\lambda$  and a composition  $\alpha$  denote the **Kostka number**  $K_{\lambda\alpha}$  the number of semistandard Young tableaux of shape  $\lambda$  and a weight composition is equal to  $\alpha$  (see [Kos82] for the details). An important property of  $K_{\lambda\alpha}$  is that it does not depend on the order of entries of  $\alpha$ , so given a partition rearrangement  $\mu$  of a composition  $\alpha$  one has  $K_{\lambda\alpha} = K_{\lambda\mu}$ . From a theoretical-representation point of view,  $K_{\lambda\mu}$  gives the dimension of the weight space corresponding to  $\mu$  in the irreducible representation  $W_\lambda$  of the general linear group  $GL_n$ , where  $l(\lambda)$  and  $l(\mu)$  are no more than  $n$ .

The coefficients  $d_{\lambda I}$  and Kostka numbers  $K_{\lambda\mu}$  may be used to compute the structure constants of the Solomon descent algebra, together with Kronecker coefficients, see Section 1.8 for the details. The type B generalisations of these formulas are given in Chapter 3.

There is an analogue of the hook-length formula for semistandard Young tableaux, called **hook-content** formula. It counts the number of semistandard Young tableaux of shape  $\lambda$  such that its entries are less than or equal to  $t \in \mathbb{Z}$ .

$$g_{\lambda,t} = \#\{SSYT, \max \leq t\} = \prod_{b \in \lambda} \frac{t + c(b)}{h_b},$$

where  $c(b) = i - j$  for the box  $b$  with coordinates  $(i, j)$ .

The dimension of the irreducible representation  $W(\lambda)$  of the general linear group  $GL_t$ , corresponding to the partition  $\lambda \vdash n$ ,  $l(\lambda) \leq t$ , may be computed using the formula above.

One may consider more general shapes for Young diagrams. Let  $n$  be a non-negative integer and let  $\lambda$  and  $\mu$  be two integer partitions such that the diagram of  $\mu$  is included in the diagram of  $\lambda$  and  $|\lambda| - |\mu| = n$ . We will write  $\mu \leq \lambda$  in this case. This is a partial order on the set of Young diagrams. The difference of the Young diagrams of shapes  $\lambda$  and  $\mu$  is called the **skew Young diagram** of skew shape  $\lambda/\mu$ . For instance,

$$\lambda/\mu = (4, 2, 1)/(2, 1) = \begin{array}{|c|c|c|c|} \hline \text{shaded} & \text{shaded} & & \\ \hline \text{shaded} & & & \\ \hline & \text{shaded} & & \\ \hline & & & \\ \hline \end{array}.$$

*Definition 7.* A **standard** (resp. **semistandard**) **skew Young tableau** of shape  $\lambda/\mu$  is a Young diagram of skew shape  $\lambda/\mu$  whose boxes are filled with the elements of  $[n]$  (resp. with positive integers) such that the entries are strictly increasing down the columns and along the rows (resp. strictly increasing down the columns and non decreasing along the rows). Denote by  $SYT(\lambda/\mu)$  (resp.  $SSYT(\lambda/\mu)$ ) the set of standard (resp. semistandard) skew Young tableaux of shape  $\lambda/\mu$ .

**Example 1.3.** The two tableaux  $T_1 \in SYT((6, 4, 2, 1, 1)/(3, 1))$  and  $T_2 \in SSYT((6, 4, 2, 1, 1)/(2, 1, 1))$  give an example of a standard and a semistandard tableaux.

$$T_1 = \begin{array}{|c|c|c|c|c|c|} \hline & & & 4 & 5 & 10 \\ \hline & 2 & 3 & 6 & & \\ \hline 1 & 8 & & & & \\ \hline 7 & & & & & \\ \hline 9 & & & & & \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|c|c|} \hline & & 3 & 6 & 9 & 10 \\ \hline & 5 & 5 & 7 & & \\ \hline & 8 & & & & \\ \hline 9 & & & & & \\ \hline 10 & & & & & \\ \hline \end{array}$$

The descent set of a standard skew Young tableau is defined similarly as in the case of usual shapes

$$\text{Des}(T) = \{1 \leq i \leq n-1 \mid i \text{ is in a strictly higher row than } i+1\}.$$

For  $I \subseteq [n-1]$ , denote by  $d_{\lambda/\mu I}$  the number of standard skew Young tableaux of shape  $\lambda/\mu$  and descent set  $I$ .

*Definition 8.* To any (standard or semistandard, skew or usual) Young tableau  $T$  we associate its **reading word**, i.e. the word obtained by concatenating the rows of  $T$  from bottom to top.

Skew and usual semistandard Young tableaux have a great importance in combinatorics, since they appear to be connected with numerous objects and questions.

Lascoux and Schützenberger in [LS81] study a structure called the *plactic monoid*, whose elements may be presented as semistandard Young tableaux. The monoid operation is the associative product on  $SSYT$ , introduced by Schensted [Sch61] in his study of the longest increasing subsequence of a permutation. This structure is also closely related to the notion of Knuth equivalence ([Knu70]) of two words.

The problem of counting Young tableaux leads to the definition and identities for Schur functions (see Section 1.4). They are involved in many algorithms like *jeu de taquin* and the Robinson-Schensted-Knuth correspondence (see Section 1.2).

All the topics mentioned above are closely related to each other.



The jeu de taquin algorithm ([dBR77]) defines an equivalence relation on the set of standard skew Young tableaux, called **jeu de taquin equivalence**. It is known that two semistandard skew tableaux are jeu de taquin equivalent if and only if the reading words are Knuth equivalent. As a consequence, both tableaux are jeu de taquin equivalent to the insertion tableau of the reading word of  $T$  under the Robinson-Schensted correspondence. The jeu-de-taquin algorithm is also linked with the Littlewood-Richardson coefficients and the Littlewood-Richardson rule ([LR34], [Tho74], [Tho78], [vL01], [Fom97]), that we use further in this thesis.

The Robinson-Schensted correspondence is closely related to the celebrated Cauchy identity for Schur functions, as shown in Section 1.9. Chapter 5 generalises domino functions with an additional parameter  $q$  and gives the analogue for Cauchy identity and for this link. In this thesis we also use the generalised Cauchy identity for  $q$ -domino functions to receive some equidistribution results and to generalise further the notion of type B Schur positivity.

## 1.2 RSK correspondence

The **Robinson-Schensted** (short RS) correspondence is a well-known bijection between permutations and pairs of standard Young tableaux of the same shape

$$RS : \pi \in S_n \rightarrow (P_\pi, Q_\pi) \in \bigcup_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda).$$

RS correspondence has many useful properties. In particular, it preserves the descent set, which is of interest in our research. This bijection has a lot of applications in combinatorics and representation theory (see Section 1.10).

The RS-correspondence was generalised by Knuth [Knu70] to the case of semistandard Young tableaux and generalised permutations. The resulting bijection is called the RSK-correspondence.

In this section we briefly describe the idea of the RS and RSK bijection. The shortest description of RS is due to Schensted (see [Sch61]) in 1961, but the correspondence was proved in a different form much earlier (1938) by Robinson (see [dBR38]). Further in this thesis we use an algebraical approach to reprove the existence of the correspondences with certain properties, including the RS-correspondence.

Consider the permutation  $\pi \in S_n$  in two-line notation

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}.$$

Starting from two empty standard Young tableaux  $(P_0, Q_0)$ , read the permutation from left to right. Construct the sequence of the pairs of standard Young tableaux of the same shape

$$(P_0, Q_0), (P_1, Q_1), \dots, (P_i, Q_i) \dots (P_n, Q_n) = (P_\pi, Q_\pi)$$

applying to the pair  $(P_{i-1}, Q_{i-1})$  the insertion procedure for each  $(i, \pi_i)$ . The insertion procedure for  $(i, \pi_i)$  follows.

Consider  $P_{i-1}$  and try to put  $\pi_i$  in the first row.

1. Find the minimal element  $a$  in a row, which is greater than  $\pi_i$ .
2. If there is no such element, we add a box with the new element in the end of the row to get  $P_i$  and finish the procedure. If not, put  $\pi_i$  instead of  $a$  in  $P_{i-1}$ .
3. Go to the next row with the new element  $a$  and repeat all the steps (try to put  $a$  in the next row).

This is the procedure to get  $P_i$ . In order to get  $Q_i$ , add a new box to  $Q_{i-1}$  with  $i$  as the entry such that  $\text{shape}(Q_i) = \text{shape}(P_i)$ .

*Example 2.* Let  $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix}$ . Figure 1.2 shows all the steps of the RS-correspondence for the permutation  $\pi$ .

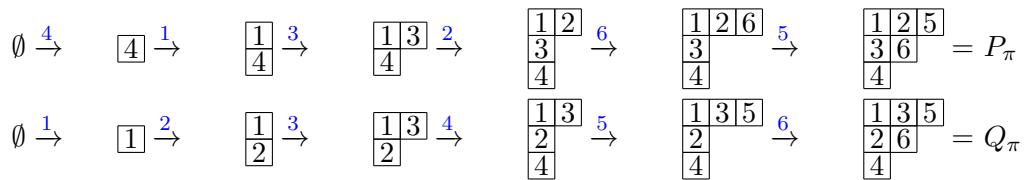


FIGURE 1.2: The sequences of pairs  $(P_i, Q_i)$  for the permutation  $\pi = (4, 1, 3, 2, 6, 5)$

The RS-correspondence features a number of important properties. In particular, in this thesis we will need the following ones:

- If  $\pi$  corresponds to  $(P_\pi, Q_\pi)$ , then  $\pi^{-1}$  will correspond to  $(Q_\pi, P_\pi)$ .
- $\text{Des}(\pi) = \text{Des}(Q_\pi)$  and  $\text{Des}(\pi^{-1}) = \text{Des}(P_\pi)$ , so it preserves the notion of descent.

To introduce the RSK-correspondence, we need to generalise the notion of permutation.

*Definition 9.* A **generalised permutation** is

- A two-line array  $w = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ b_1 & b_2 & \dots & b_{n-1} & b_n \end{pmatrix}$ ,  
such that  $a_1 \leq a_2 \leq \dots \leq a_{n-1} \leq a_n$ , for all  $i \in [n-1]$   $a_i = a_{i+1} \Rightarrow b_i \leq b_{i+1}$  and all  $a_s, b_s \in [m]$
- A matrix  $C \in \text{Mat}_{m \times m}(\mathbb{N})$  with  $\sum_{i,j} C_{i,j} = n$ .

*Example 3.* The two presentations of the same generalised permutation follow:

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}, \quad C_\pi = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Due to Knuth, there is a bijection between generalised permutations and pairs of semistandard Young tableaux  $(P_\pi, Q_\pi)$  of the same shape. The idea of the bijection is the same: we apply the inserting procedure to  $(P_{i-1}, Q_{i-1})$  for all  $(a_i, b_i)$  to get  $P_i$ . Then add a new box to  $Q_{i-1}$  with  $a_i$  as the entry such that  $\text{shape}(Q_i) = \text{shape}(P_i)$ .

### 1.3 Ring of symmetric functions

Let  $\Lambda$  be the ring of symmetric functions, i.e the formal power series in the infinitely many variables  $x_1, x_2, \dots$  that are invariant under permutation of indeterminates. Denote by  $\Lambda_n$  the vector space of symmetric functions that are homogeneous of degree  $n$ . Recall that the elements of the bases of  $\Lambda_n$  are indexed by partitions, in particular  $\dim(\Lambda_n) = p(n)$ . Here are the main examples of bases:

- Monomial symmetric functions

$$m_\lambda = \sum_{\beta} x_1^{\beta_1} x_2^{\beta_2} \dots x_r^{\beta_r},$$

where sum is over all distinct sequences  $\beta = (\beta_1, \beta_2, \dots, \beta_r)$  such that the partition rearrangement of  $\beta$  is equal to  $\lambda$ .

- Power sum symmetric functions

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_r},$$

where  $p_k = x_1^k + x_2^k + \dots$ .

- Elementary symmetric functions

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_r}, \quad e_k = \sum_{i_1 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}.$$

- Complete homogeneous symmetric functions

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_r}, \quad h_k = \sum_{i_1 \leq \dots \leq i_k} x_{i_1} x_{i_2} \dots x_{i_k}.$$

- Schur symmetric functions

$$s_\lambda = \sum_{T \in SSYT(n)} X^T,$$

where  $X^T = x_1^{\mu_1} x_2^{\mu_2} \dots$  for  $\mu = w(T)$ .

The set of symmetric functions could be completed with an inner product. Denote  $z_\lambda = 1^{k_1} k_1! 2^{k_2} k_2! \dots$ , where  $k_i$  is the multiplicity of  $i$  in  $\lambda$ . Let  $\langle \cdot, \cdot \rangle$  be the Hall inner product on  $\Lambda$  setting  $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ .

The bases  $m_\lambda$  and  $h_\lambda$  are dual to each other with respect to the Hall inner product, i.e.  $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$ . The algebra of symmetric functions  $\Lambda$  may be considered as a self-dual Hopf algebra and is well-studied from the Hopf algebraic point of view.

## 1.4 Schur functions

Schur polynomials in some way generalise both the elementary symmetric polynomials and the complete homogeneous symmetric polynomials. We call Schur functions the elements of ring  $\Lambda$  that correspond to Schur polynomials. They appear to be one of the important basis of the algebra of symmetric functions. Many of the combinatorial results involve Schur functions. As an example, Schur functions are related to representation theory. In particular, the characters of polynomial irreducible representations of  $GL(n, \mathbb{C})$  are given by Schur functions. They also correspond to the irreducible representations of the symmetric group via the Frobenius correspondence (see [Fro00]). Schur functions are also useful in mathematical physics, see [GH02].

There are various ways to define Schur polynomials. The original definition of Schur polynomials uses the Jacobi-Trudi formula ([Jac41], [Tru64])

*Definition 10.*

$$s_\lambda(x_1, \dots, x_m) = \frac{\det((x_j)^{\lambda_i + m - i})_{1 \leq i, j \leq m}}{\det((x_j)^{m - i})_{1 \leq i, j \leq m}}.$$

The denominator of the fraction is the Vandermonde determinant.

For the goals of this thesis, the tableaux-theoretic definition is more appropriate.

*Definition 11.* Given a semistandard tableau  $T$  of weight  $\mu$ , denote by  $X^T$  the monomial  $x_1^{\mu_1} x_2^{\mu_2} \dots$ . Schur polynomials are generating functions for semistandard Young tableaux

$$s_\lambda(X) = \sum_{T \in SSYT(\lambda)} X^T,$$

where sum is over all semistandard Young tableaux with shape  $\lambda$ .

*Example 4.* Let  $\lambda = (2, 1)$  and  $X = (x_1, x_2, x_3)$ , then

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3.$$

The list of semistandard Young tableaux of shape  $(2, 1)$  in  $X = (x_1, x_2, x_3)$  consists of the following tableaux

|   |   |
|---|---|
| 1 | 1 |
| 2 |   |

|   |   |
|---|---|
| 1 | 1 |
| 3 |   |

|   |   |
|---|---|
| 1 | 2 |
| 2 |   |

|   |   |
|---|---|
| 1 | 3 |
| 3 |   |

| |   |   | |---|---| | 2 | 2 | | 3 |   | | |   |   | |---|---| | 2 | 3 | | 3 |   | | |   |   | |---|---| | 1 | 2 | | 3 |   | | |   |   | |---|---| | 1 | 3 | | 2 |   | |

As already mentioned above, Schur polynomials form a  $\mathbb{Z}$ -basis for the ring of all symmetric polynomials  $\Lambda$ . For more details see [Mac99].

The (outer) product of Schur functions is a linear combination of Schur functions

$$s_\lambda s_\mu = \sum_{\nu} k_{\lambda\mu}^\nu s_\nu. \quad (1.1)$$

One can see that values of the coefficients depend only on the shapes, not on the entries of the tableaux and number of variables. The coefficients  $k_{\mu\nu}^\lambda$  are called **Littlewood-Richardson coefficients**.

*Definition 12.* A **lattice word** is a string composed of positive integers, such that any prefix of it contains at least as many positive integers  $i$  as integers  $i + 1$ . A **Yamanouchi word** is a sequence whose reversal is a lattice word.

**Theorem 1.1** (Littlewood-Richardson rule, [LR34]). *Littlewood-Richardson coefficients  $k_{\mu\nu}^\lambda$  may be computed as the number of semistandard skew tableaux of shape  $\nu/\lambda$  whose row reading is a Yamanouchi word and weight is equal to  $\mu$ .*

*Example 5.* Let  $\lambda, \mu = (2, 1)$  then the list of possible  $\nu$  consists of

$$(4, 2), (4, 1, 1), (3, 3), (3, 2, 1), (3, 1, 1, 1), (2, 2, 1), (2, 2, 1, 1).$$

In fact,

$$s_{(2,1)}s_{(2,1)} = s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(3,1,1,1)} + s_{(2,2,2)} + s_{(2,2,1)}.$$

Using the rule above one can get two special cases called the Pieri rule:

$$s_\lambda s_{(p)} = \sum_{\mu} s_{\mu},$$

where the sum is taken over all  $\mu$  that are obtained from  $\lambda$  by adding  $p$  boxes with no two in the same column;

$$s_\lambda s_{(1^p)} = \sum_{\mu} s_{\mu},$$

where the sum is taken over all  $\mu$  that are obtained from  $\lambda$  by adding  $p$  boxes with no two in the same row.

### 1.4.1 Skew Schur functions and Littlewood-Richardson coefficients

There also exists a skew analogue for the Schur function.

*Definition 13.* Let  $\lambda/\mu$  be a skew shape, i.e.  $\mu \leq \lambda$  as partitions or Young tableaux. The **skew Schur function**  $s_{\lambda/\mu}$  is the generating function for semistandard skew Young tableaux of shape  $\lambda/\mu$

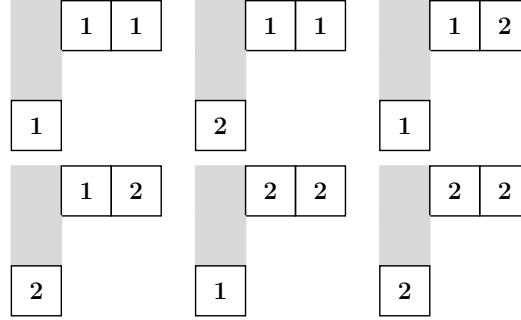
$$s_{\lambda/\mu}(X) = \sum_{\text{shape}(T)=\lambda/\mu} X^T,$$

where the sum is on all semistandard skew Young tableaux of shape  $\lambda/\mu$ .

*Example 6.* Let  $\lambda/\mu = (3, 1, 1)/(1, 1)$  and  $X = (x_1, x_2)$ , then

$$s_{(3,1,1)/(1,1)} = x_1^3 + 2x_1^2x_2 + 2x_1x_2^2 + x_2^3$$

The list of semistandard Young tableaux of skew shape  $(3, 1, 1)/(1, 1)$  in  $X = (x_1, x_2)$  consists of the following tableaux



Skew Schur functions are also symmetric functions and may be expressed using the usual ones with the same Littlewood-Richardson coefficients

$$s_{\lambda/\mu} = \sum_{\nu} k_{\mu\nu}^{\lambda} s_{\nu}. \quad (1.2)$$

The Littlewood-Richardson rule was first stated by Littlewood and Richardson in [LR34]. It has a long story of proving attempts, which became successful only after the works of Robinson, Schensted and Knuth on the RS and RSK correspondence. Afterwards, Zelevinsky in [Zel81] extended the Littlewood-Richardson rule to the case of skew Schur functions and obtained a decomposition of the product  $s_{\lambda}s_{\mu/\nu}$ .

Littlewood-Richardson coefficients have a strong combinatorial meaning. Besides the two relations concerning Schur functions, they also appear, for instance:

- as intersection numbers on a Grassmannian;
- in the decomposition of the tensor product of two Schur modules (irreducible representations of special linear groups), see [Ful97];
- as the number of standard Young tableaux of shape  $\lambda/\nu$  that are jeu de taquin equivalent to some fixed standard Young tableau of shape  $\mu$ ;
- as the number of Littlewood-Richardson tableaux of shape  $\lambda/\mu$  and weight  $\nu$ .

For the additional details concerning skew Schur functions we refer the reader to [BvW11], [Mcn14].

### 1.4.2 Cauchy identity

The Cauchy identity is one of the most useful relations concerning Schur functions. The combinatorial meaning of the relation deals with the celebrated RSK correspondence, see

Section 1.9 for details. This approach may be generalised in order to obtain bijections with certain properties, as in Section 5.2. Besides that, almost all the results included in this thesis more or less rely on this relation.

The **Cauchy identity** equates the sum over all integer partitions of the product of pairs of Schur functions and the Cauchy kernel. Namely,

$$\sum_{\lambda} s_{\lambda}(X)s_{\lambda}(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}, \quad (1.3)$$

where the sum runs over all integer partitions. By interpreting the right-hand side of Equation (1.3) as the generating function for *biwords* one recover that they are in weight preserving bijection with pairs of semistandard Young tableaux of the same shape, the main consequence of the *Robinson-Schensted-Knuth* correspondence.

More general, one may write

$$\sum_{\lambda} f_{\lambda}(X)g_{\lambda}(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}, \quad (1.4)$$

where  $\{f_{\lambda}\}_{\lambda}$  and  $\{g_{\lambda}\}_{\lambda}$  are dual bases with respect to the Hall inner product. Conversely, this equality is a criterion for bases to be dual. In particular, one get

$$\sum_{\lambda} h_{\lambda}(X)m_{\lambda}(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}, \quad (1.5)$$

In terms of Hall inner product the identity 1.3 means that  $\{s_{\lambda}\}_{\lambda \vdash n}$  is an orthonormal basis of  $\Lambda$ .

For any integer  $n$ , the Cauchy identity 1.3 can also be expressed as

$$\sum_{\lambda \vdash n} s_{\lambda}(X)s_{\lambda}(Y) = s_{(n)}(XY), \quad (1.6)$$

where  $X = \{x_1, x_2, \dots\}$  and  $Y = \{y_1, y_2, \dots\}$  are two commutative sets of variables,  $XY$  is the set of indeterminates  $\{x_i y_j; x_i \in X, y_j \in Y\}$ , ordered by the lexicographical order.

A similar decomposition may be given for any  $s_{\nu}(XY)$ :

$$s_{\nu}(XY) = \sum_{\lambda, \mu} g(\lambda, \mu, \nu) s_{\lambda}(X) s_{\mu}(Y). \quad (1.7)$$

It involves the **Kronecker coefficients**  $g(\lambda, \mu, \nu)$ , which may be computed as follows

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\omega \in S_n} \chi^{\lambda}(\omega) \chi^{\mu}(\omega) \chi^{\nu}(\omega),$$



where  $\chi^\lambda$  is the irreducible character of  $S_n$  indexed by partition  $\lambda$ .

## 1.5 Gessel's quasisymmetric functions

There are two natural generalisations of the symmetric functions algebra  $\Lambda$ , the algebra of noncommutative symmetric functions  $NSym$  and quasisymmetric functions algebra  $QSym$ , which are dual as Hopf algebras. While noncommutative symmetric functions are out of the scope of this thesis, quasisymmetric functions are of major interest. The basic theory of Hopf algebras, in particular the Hopf algebras of symmetric, quasisymmetric and noncommutative symmetric functions can be found in a book of Luoto, Mykytiuk and van Willigenburg [LMvW13].

*Definition 14.* Let  $X = \{x_1, x_2, \dots\}$  be a totally ordered set of commutative indeterminates and  $I \subseteq [n-1]$ . A quasisymmetric function is a bounded degree formal power series in  $\mathbb{C}[X]$  such that for any composition  $(\alpha_1, \dots, \alpha_p)$  and any strictly increasing sequence of distinct indices  $i_1 < i_2 < \dots < i_p$  the coefficient of  $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_p^{\alpha_p}$  is equal to the coefficient of  $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_p}^{\alpha_p}$ .

Quasisymmetric functions admit the monomial

$$M_I(X) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ k \in I \Leftrightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \dots x_{i_n}$$

and fundamental

$$F_I(X) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ k \in I \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \dots x_{i_n}$$

quasisymmetric functions as classical bases. These two bases are related through

$$F_I(X) = \sum_{I \subseteq J \subseteq [n-1]} M_J(X). \quad (1.8)$$

Note, that subsets of  $I \subset [n-1]$  are in bijection with compositions  $\alpha \Vdash n$ . Further in this thesis we will use both of them as indices.

As in the case of Schur functions, the  $F_\alpha$  have a representation-theoretic meaning as the characteristics of the irreducible characters of the (type A) 0-Hecke algebra, see [DKLT96], [KT97]. More details about this link can be found in [TvW14].

Obviously, any symmetric function is also quasisymmetric, so it may be decomposed in the bases above. For instance,

$$m_\lambda(X) = \sum_{\alpha} M_\alpha(X),$$

where the sum is over the compositions  $\alpha$  such that its partition rearrangement gives  $\lambda$ .

There is also an analogue of the Schur functions' basis in  $QSym$ . Haglund, Luoto, Mason and van Willigenburg [HLMvW11] introduced the basis of quasisymmetric Schur functions  $S_\alpha(X)$ , which refines Schur functions in a natural way

$$s_\lambda(X) = \sum_{\alpha} S_\alpha(X),$$

where sum is over the compositions  $\alpha$  such that its partition rearrangement gives  $\lambda$ . The authors also derive a Pieri rule for quasisymmetric Schur functions that refines the Pieri rule for Schur functions. These functions could be further generalised to the case of skew shapes, see [BLvW10], [BvW11] for the details and properties of (skew) quasisymmetric Schur functions.

In what follows, we may remove the reference to indeterminate  $X$  when there is no confusion.

## 1.6 Link between Schur functions and quasisymmetric functions

Using the tableaux-theoretic definition it is not hard to show that Schur polynomials are symmetric. Indeed, it is sufficient to prove that they are invariant under changing  $x_i$  and  $x_{i+1}$  for all  $i$ , see [BK72] or [vL01] for the details of the involution. From another point of view, any symmetric function is at the same time a quasisymmetric function, which admits a decomposition in terms of monomial and fundamental quasisymmetric functions as bases of  $QSym$ .

Recall  $d_{\lambda I}$ , the number of standard Young tableaux of shape  $\lambda$  and descent set  $I$  and the Kostka number  $K_{\lambda \text{comp}(I)}$ , i.e the number of semistandard Young tableaux of shape  $\lambda$  and weight  $\text{comp}(I)$ . We have

$$s_\lambda = \sum_{I \subseteq [n-1]} d_{\lambda I} F_I = \sum_{I \subseteq [n-1]} K_{\lambda \text{comp}(I)} M_I. \quad (1.9)$$

The main idea to prove these decompositions is to apply the following standardisation procedure to a semistandard Young tableau  $T$  in order to get a standard  $T_0$ .

Consider all boxes filled by 1 and relabel them with successive natural integers  $1, 2, \dots$  from bottom to top and from left to right. Then continue with all boxes filled by 2, by 3, etc. Finally, process all the entries and get a standard Young tableau.

**Example 1.4.** Here is a semistandard Young tableaux  $T$  and the result of the standardisation procedure  $T_0$ .

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 7 & 10 \\ \hline 2 & 5 & 5 & 8 & & \\ \hline 5 & 9 & & & & \\ \hline 9 & & & & & \\ \hline 10 & & & & & \\ \hline \end{array} \quad T_0 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 4 & 8 & 9 & 14 \\ \hline 2 & 6 & 7 & 10 & & \\ \hline 5 & 12 & & & & \\ \hline 11 & & & & & \\ \hline 13 & & & & & \\ \hline \end{array}$$

Add together all the terms  $X^T$  for the tableaux  $T$  with the same standardisation  $T_0$  to get  $F_{\text{Des}(T_0)}$ . As a result,

$$s_\lambda = \sum_{T \in SSYT(\lambda)} X^T = \sum_{T_0 \in SYT(\lambda)} F_{\text{Des}(T_0)} = \sum_{I \subseteq [n-1]} d_{\lambda I} F_I. \quad (1.10)$$

A similar decomposition may be written for the case of skew shapes:

$$s_{\lambda/\mu} = \sum_{I \subseteq [n-1]} d_{\lambda/\mu I} F_I = \sum_{I \subseteq [n-1]} K_{\lambda/\mu \text{ comp}(I)} M_I, \quad (1.11)$$

where  $d_{\lambda/\mu I}$  is the number of standard Young tableaux of skew shape  $\lambda/\mu$  and descent set  $I$ , and  $K_{\lambda/\mu \text{ comp}(I)}$  is the number of semistandard Young tableaux of skew shape  $\lambda/\mu$  and weight  $\text{comp}(I)$ .

## 1.7 Solomon's descent algebra

### 1.7.1 Descent algebra of a Coxeter group

Solomon in [Sol76] introduced a subalgebra of the integral group ring of the Coxeter group, called the Solomon descent algebra. Because of its important combinatorial and algebraic properties, the descent algebra received significant attention afterwards. In particular, Garsia and Reutenauer in [GR89] provide a decomposition of its multiplicative structure and a new combinatorial interpretation of the structure coefficients in terms of the number

of non-negative integer matrices with specified constraints. Bergeron and Bergeron give analogous results in [BB92] when the symmetric group is replaced by the hyperoctahedral group (Coxeter group of type B).

The Solomon descent algebra plays a rôle in many interesting situations. The descent structure of permutations is related to the shuffle and peak algebra. Besides, the result of Gessel [Ges84] shows that the Solomon descent algebra of the symmetric group is dual to the ring of quasisymmetric functions  $QSym$  as Hopf algebras. Further the duality was investigated by C. Malvenuto and C. Reutenauer in [MR95].

A good overview on Solomon descent algebra and its properties has been given by Schocker in [Sch04].

Let us start with the definition of the Solomon descent algebra. Recall the notion of a Coxeter group.

*Definition 15.* A group  $W$  with presentation

$$\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where  $m_{ii} = 1$  and  $m_{ij} \geq 2$  for  $i \neq j$  is called a Coxeter group. Given a Coxeter group  $W$  with the set of generators  $S = \{r_1, \dots, r_n\}$ , we call the pair  $(W, S)$  a Coxeter system.

Given  $w \in W$  denote

$$\text{Des}(w) = \{s \in S \mid l(ws) < l(w)\},$$

where  $l(w)$  is the Coxeter length of the element  $w$ , i.e. the length of a minimal expression for  $w$  as a product of elements of  $S$ . Note that for a Coxeter group of type A this definition turns into a notion of descent set of a permutation.

Consider the group algebra  $\mathbb{C}W$  of the Coxeter group  $W$ . Solomon showed in [Sol76] that the following elements of  $\mathbb{C}W$  span a subspace which is closed under the product of  $\mathbb{C}W$ :

$$Y_I = \sum_{\text{Des}(w)=I} w.$$

So, these elements generate a subalgebra of  $\mathbb{C}W$  of dimension  $2^{|S|}$  usually referred to as the **Solomon descent algebra** of  $W$ .

Denote

$$X_I = \sum_{J \subseteq I} Y_J = \sum_{\text{Des}(w) \subseteq I} w.$$

The two sets  $\{Y_I\}_{I \subseteq S}$  and  $\{X_I\}_{I \subseteq S}$  are bases of the Solomon descent algebra.

### 1.7.2 Descent algebra of the symmetric group

As defined above, the descent set of a permutation  $\pi$  of  $S_n$  is

$$\text{Des}(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}.$$

Denote by  $D_I$  (resp.  $B_I$ ) the generators of the Solomon descent algebra of the symmetric group, i.e. the elements of the algebra  $\mathbb{C}S_n$  defined as the formal sum of all the permutations  $\pi$  such that  $\text{Des}(\pi) = I$  (resp.  $\text{Des}(\pi) \subseteq I$ ). The dimension of the descent algebra is  $2^{n-1}$ .

More specifically, Solomon showed in [Sol76] that there exist (non negative integer valued) **structure constants**  $(a_{IJ}^K)_{I,J,K \subseteq [n-1]}$  and  $(b_{IJ}^K)_{I,J,K \subseteq [n-1]}$  that verify:

$$D_I D_J = \sum_{K \subseteq [n-1]} a_{IJ}^K D_K, \quad B_I B_J = \sum_{K \subseteq [n-1]} b_{IJ}^K B_K.$$

As a result the number of ways to write a fixed permutation  $\pi$  of  $D_K$  as the ordered product of two permutations  $\pi = \sigma\tau$  such that  $\sigma \in D_I$  and  $\tau \in D_J$  depends only on  $\text{Des}(\pi) = K$  and is equal to  $a_{IJ}^K$ . If we require instead  $\sigma$  and  $\tau$  to be respectively in  $B_I$  and  $B_J$  than the number of such products is  $\sum_{K' \supseteq K} b_{IJ}^{K'}$ .

*Remark 1.2.* The two families of structure constants are linked through the formula

$$\sum_{I' \subseteq I, J' \subseteq J} a_{I'J'}^K = \sum_{K' \supseteq K} b_{IJ}^{K'}. \quad (1.12)$$

Denote also

$$a_{IJK}^T = [D_T] D_I D_J D_K$$

the number of triples of permutations  $\sigma_1, \sigma_2, \sigma_3$  of  $S_n$  such that  $\text{Des}(\sigma_1) = I, \text{Des}(\sigma_2) = J, \text{Des}(\sigma_3) = K$  and  $\sigma_1 \sigma_2 \sigma_3 = \pi$  for some fixed permutation  $\pi$  such that  $\text{Des}(\pi) = T$ . Let

$$b_{IJK}^T = [B_T] B_I B_J B_K.$$

*Remark 1.3.* The two families of structure constants are also linked through the formula

$$\sum_{I' \subseteq I, J' \subseteq J, K' \subseteq K} a_{I'J'K'}^T = \sum_{T' \supseteq T} b_{IJK}^{T'}. \quad (1.13)$$

*Remark 1.4.* The structure constants of the Solomon descent algebra of a symmetric group verify the following identity

$$a_{IJK}^{\emptyset} = \sum_L a_{IL}^{\emptyset} a_{JK}^L.$$

*Proof.* Using the definition of  $a_{I^1 \dots I^p}^K$ ,  $p \in \{2, 3\}$  we get

$$\begin{aligned} a_{IJK}^{\emptyset} &= \#\{(\sigma_1, \sigma_2, \sigma_3): \sigma_1 \sigma_2 \sigma_3 = id, \sigma_1 \in D_I, \sigma_2 \in D_J, \sigma_3 \in D_K\} \\ &= \sum_{\sigma_3: \sigma_3^{-1} \in D_K} \#\{(\sigma_1, \sigma_2): \sigma_1 \sigma_2 = \sigma_3, \sigma_1 \in D_I, \sigma_2 \in D_J\} \\ &= \sum_{\sigma_3: \sigma_3^{-1} \in D_K} a_{IJ}^{\text{Des}(\sigma_3)} \\ &= \sum_L a_{IL}^{\emptyset} a_{JK}^L \end{aligned}$$

□

As shown by Norton in [Nor79], the descent algebra of a finite Coxeter group is also strongly related to its *0-Hecke algebra*. In particular, she proves that the dimension of each left principal indecomposable module of the 0-Hecke algebra is equal to the cardinality of the analogue of one the  $D_I$ 's. She further shows that the analogues of the  $a_{IJ}^{\emptyset}$  are the entries of the *Cartan matrix* giving the number of times each irreducible module is a composition factor of each indecomposable module. In the specific case of the symmetric group  $S_n$ , Carter in [Car86] uses the Robison-Schensted correspondence to explain the following relation obtained by computation of the Cartan matrix

$$a_{IJ}^{\emptyset} = \sum_{\lambda \vdash n} d_{\lambda I} d_{\lambda J}. \quad (1.14)$$

Recall that the Solomon's descent algebra is dual to the Hopf algebra of quasisymmetric functions. In particular, Gessel showed that the comultiplication table for their fundamental basis is given by the  $a_{IJ}^K$ 's. Further in the Section 1.8 we show that Equation (1.14) and its generalisations are a direct consequence of Gessel's result.

## 1.8 Computation of structure constants

### 1.8.1 Gessel's relation and its consequences

For two commutative sets of variables  $X = \{x_1, x_2, \dots, x_i, \dots\}$  and  $Y = \{y_1, y_2, \dots, y_i, \dots\}$ , we denote by  $XY$  ( $XYZ$ ) the set of indeterminates  $\{x_i y_j; x_i \in X, y_j \in Y\}$  ( $\{x_i y_j z_t; x_i \in$

$X, y_j \in Y, z_t \in Z\}$  ordered by the lexicographical order. Gessel shows in [Ges84] that for any subset  $K \subseteq [n-1]$

$$F_K(XY) = \sum_{I, J \subseteq [n-1]} a_{IJ}^K F_I(X) F_J(Y). \quad (1.15)$$

We can say that  $F_K(XY)$  is the generating series for the coefficients  $a_{IJ}^K$ . As stated above, Equation (1.14) and some of its generalisations are a direct consequence of Equation (1.15).

More precisely, denote by  $\chi^\lambda$  the irreducible character of  $S_n$  indexed by partition  $\lambda$  and for any  $\lambda, \mu, \nu \vdash n$  define the **Kronecker coefficients** the coefficients in the decomposition of the tensor product of two irreducible representations indexed by  $\lambda$  and  $\mu$  of a symmetric group into irreducible representations. They may be computed by

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\omega \in S_n} \chi^\lambda(\omega) \chi^\mu(\omega) \chi^\nu(\omega).$$

One of the main results of this chapter states that structure constants of the symmetric group, Kronecker coefficients and coefficients  $d_{\lambda I}$  are related by the following formulas.

**Theorem 1.5.** *Let  $I, J, K \subset [n-1]$ . Structure constants  $a_{IJ}^\emptyset$  and  $a_{IJK}^\emptyset$  can be computed using the following formulas:*

$$a_{IJ}^\emptyset = \sum_{\lambda \vdash n} d_{\lambda I} d_{\lambda J}, \quad (1.16)$$

$$a_{IJK}^\emptyset = \sum_{\lambda, \mu, \nu \vdash n} g(\lambda, \mu, \nu) d_{\lambda I} d_{\mu J} d_{\nu K}. \quad (1.17)$$

*Proof.* This theorem is a direct consequence of Equation (1.15). According to Equation (1.10)  $F_\emptyset = s_{(n)}$ . Then using the *Cauchy identity* for Schur functions

$$s_{(n)}(XY) = \sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) \quad (1.18)$$

and applying Equation (1.10) once again one gets:

$$\begin{aligned} \sum_{I, J \subseteq [n-1]} a_{IJ}^\emptyset F_I(X) F_J(Y) &= F_\emptyset(XY) = s_{(n)}(XY) \\ &= \sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) \\ &= \sum_{I, J \subseteq [n-1]} d_{\lambda I} d_{\lambda J} F_I(X) F_J(Y). \end{aligned}$$

Since the fundamental functions are the basis of the  $QSym$ , this proves the first statement of the theorem. This can be generalised to prove Equation (1.17) by noticing that the structure constants verify  $a_{IJK}^\emptyset = \sum_L a_{IL}^\emptyset a_{JK}^L$ . As a result, Equation (1.15) extends to

$$F_\emptyset(XYZ) = \sum_{I,J,K} a_{IJK}^\emptyset F_I(X) F_J(Y) F_K(Z).$$

Indeed,

$$\begin{aligned} F_\emptyset(XYZ) &= \sum_{I,L} a_{IL}^\emptyset F_I(X) F_L(YZ) \\ &= \sum_{I,J,K} \left( \sum_L a_{IL}^\emptyset a_{JK}^L \right) F_I(X) F_J(Y) F_K(Z) \\ &= \sum_{I,J,K} a_{IJK}^\emptyset F_I(X) F_J(Y) F_K(Z). \end{aligned}$$

Finally use the generalised version of Equation (1.18)

$$s_\nu(XY) = \sum_{\lambda, \mu} g(\lambda, \mu, \nu) s_\lambda(X) s_\mu(Y) \quad (1.19)$$

to decompose  $s_{(n)}(XYZ) = F_\emptyset(XYZ)$ . We get

$$\begin{aligned} \sum_{I,J,K} a_{IJK}^\emptyset F_I(X) F_J(Y) F_K(Z) &= F_\emptyset(XYZ) = s_{(n)}(XYZ) \\ &= \sum_{\nu \vdash n} s_\nu(XY) s_\nu(Z) \\ &= \sum_{\nu \vdash n} \sum_{\lambda, \mu} g(\lambda, \mu, \nu) s_\lambda(X) s_\mu(Y) s_\nu(Z) \\ &= \sum_{\substack{\lambda, \mu, \nu \vdash n \\ I, J, K \subseteq [n-1]}} g(\lambda, \mu, \nu) d_{\lambda I} d_{\mu J} d_{\nu K} F_I(X) F_J(Y) F_K(Z). \end{aligned}$$

This proves the second statement of the theorem.  $\square$

*Remark 1.6.* Denote by  $\lambda'$  the partition corresponding to the transposed Young diagram of shape  $\lambda$ . For  $I, J \subseteq [n-1]$ , the numbers  $a_{IJ}^{[n-1]}$  are given by

$$a_{IJ}^{[n-1]} = \sum_{\lambda \vdash n} d_{\lambda I} d_{\lambda' J}. \quad (1.20)$$



*Proof.* According to Equation (1.10)  $F_{[n-1]} = s_{(1^n)}$ . Then use the identity for Schur functions

$$s_{(1^n)}(XY) = \sum_{\lambda \vdash n} s_{\lambda}(X) s_{\lambda'}(Y)$$

and apply Equation (1.10) to  $s_{\lambda}$  and  $s_{\lambda'}$ .  $\square$

### 1.8.2 Extension to the RSK-correspondence

Generalising the RS-correspondence to matrices with non-negative integral entries, Knuth ([Knu70]) proved that for  $r, c \models n$  the number  $m_{r,c}$  of such matrices with row and column sums equal respectively to  $r$  and  $c$  is given by

$$m_{r,c} = \sum_{\lambda \vdash n} K_{\lambda r} K_{\lambda c}. \quad (1.21)$$

Equation (1.21) can be deduced algebraically from Equation (1.15) without the need of the bijective proof. Furthermore let  $p, q$  and  $r$  be three compositions of  $n$  and denote by  $m_{p,q,r}$  the number of three-dimensional arrays  $M = (M_{i,j,k})$  with non-negative integer entries such that  $p_k = \sum_{i,j} M_{i,j,k}$ ,  $q_j = \sum_{i,k} M_{i,j,k}$  and  $r_i = \sum_{j,k} M_{i,j,k}$ . Equation (1.15) implies

$$m_{p,q,r} = \sum_{\lambda, \mu, \nu \vdash n} g(\lambda, \mu, \nu) K_{\lambda p} K_{\mu q} K_{\nu r}. \quad (1.22)$$

So, we have the following corollary to Theorem 1.5.

**Corollary 1.7.** *Let  $p, q$  and  $r$  be three compositions of  $n$ , then*

$$\begin{aligned} m_{p,q} &= \sum_{\lambda \vdash n} K_{\lambda p} K_{\lambda q}, \\ m_{p,q,r} &= \sum_{\lambda, \mu, \nu \vdash n} g(\lambda, \mu, \nu) K_{\lambda p} K_{\mu q} K_{\nu r}. \end{aligned}$$

*Proof.* Let us start with another proof of Equation (1.21). Denote  $A_{I,J}^K = \sum_{I' \subseteq I, J' \subseteq J} a_{I',J'}^K$ . Then according to Equation (2.2)

$$\begin{aligned} \sum_{I', J' \subseteq [n-1]} a_{I',J'}^K F_{I'}(X) F_{J'}(Y) &= \sum_{I', J' \subseteq [n-1]} a_{I',J'}^K \sum_{I \supseteq I', J \supseteq J'} M_I(X) M_J(Y) \\ &= \sum_{I, J \subseteq [n-1]} M_I(X) M_J(Y) \sum_{I' \subseteq I, J' \subseteq J} a_{I',J'}^K \\ &= \sum_{I, J \subseteq [n-1]} A_{I,J}^K M_I(X) M_J(Y). \end{aligned}$$

As a result,  $A_{I,J}^\emptyset$  is the coefficient in  $M_I(X)M_J(Y)$  of  $F_\emptyset(XY)$  which we know is equal to  $\sum_{\lambda \vdash n} s_\lambda(X)s_\lambda(Y)$  (see above). Finally use the decomposition of a Schur function in terms of monomial quasisymmetric functions to get

$$A_{I,J}^\emptyset = \sum_{\lambda \vdash n} K_{\lambda \text{ comp}(I)} K_{\lambda \text{ comp}(J)}. \quad (1.23)$$

It remains to prove that  $A_{I,J}^\emptyset = m_{\text{comp}(I), \text{comp}(J)}$ . The combinatorial interpretation of [GR89] states that  $b_{IJ}^K$  is the number of non-negative integer matrices  $M$  with row and column sums equal to  $\text{comp}(I)$  and  $\text{comp}(J)$  respectively and with the word obtained by reading the entries of  $M$  row by row from top to bottom equal to  $K$  (zero entries being omitted). But according to Equation (1.12)  $A_{I,J}^\emptyset = \sum_{K \subseteq [n-1]} b_{IJ}^K$ . We get Equation (1.21).

In order to prove Equation (1.22) consider the relations from the proof of Theorem (1.5) in the monomial base. We get

$$\begin{aligned} \sum_{I,J,K} \sum_{I' \subseteq I, J' \subseteq J, K' \subseteq K} a_{I'J'K'}^\emptyset M_I(X)M_J(Y)M_K(Z) = \\ \sum_{I,J,K} \sum_{\lambda, \mu, \nu \vdash n} g(\lambda, \mu, \nu) K_{\lambda I} K_{\mu J} K_{\nu K} M_I(X)M_J(Y)M_K(Z). \end{aligned}$$

The result of Garsia and Reutenauer  $m_{\text{comp}(I), \text{comp}(J)} = \sum_T b_{IJ}^T$  may be extended to the case of three dimensional arrays as well as Gessel's relation (1.15). Together with Remark (1.12) it leads to formula

$$\begin{aligned} m_{\text{comp}(I), \text{comp}(J), \text{comp}(K)} &= \sum_T b_{IJK}^T \\ &= \sum_{I' \subseteq I, J' \subseteq J, K' \subseteq K} a_{I'J'K'}^\emptyset \\ &= \sum_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) K_{\lambda \text{ comp}(I)} K_{\mu \text{ comp}(J)} K_{\nu \text{ comp}(K)}. \end{aligned}$$

□

A bijective proof of Equation (1.22) is provided in [AAV12]. The proof involves semistandard Young tableaux and *Littlewood-Richardson tableaux*. Our approach allows us to recover these results immediately.

### 1.8.3 Skew shapes

Let  $I$  and  $J$  be two subsets of  $[n-1]$ . Gessel focuses in [Ges84] on the number  $a_{IJ}^{\lambda/\mu}$  of pairs of permutations  $\alpha, \beta$  such that  $\alpha$  has descent set  $I$ ,  $\beta$  has descent set  $J$  and  $\alpha\beta$  is

compatible with the skew shape  $\lambda/\mu$  (i.e.  $\alpha\beta$  is a correct reading word for the skew shape  $\lambda/\mu$ ). We need some more definitions introduced in [Ges84] to define this properly.

Given any partial order  $P$  on  $[n]$ , denote by  $L(P)$  the set of permutations  $\pi \in S_n$ , such that the corresponding linear order  $\pi(1) < \pi(2) < \dots < \pi(n)$  is a linearisation of  $P$ . To obtain a partial order  $P_{\lambda/\mu}$  from the skew shape  $|\lambda/\mu| = n$  we fill the corresponding skew Young diagram by integers  $[n]$  from bottom to top and from left to right and then consider relations  $i > k$  if  $k$  lay directly above or to the left of  $i$ . These relations generate the corresponding partial order  $P_{\lambda/\mu}$ .

**Example 1.5.** Let  $\lambda/\mu = (4, 4, 3)/(2, 1)$ . Figure 1.3 shows the corresponding partial order. Firstly fill the corresponding skew Young diagram by integers  $[7]$  from bottom to top and from left to right and then rotate it by 135 degrees counterclockwise and take its vertical mirror image to get the plane presentation of the ordering  $P_{(4,4,3)/(2,1)}$ .

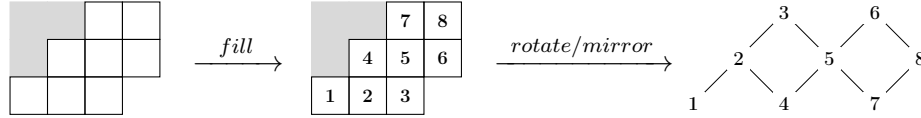
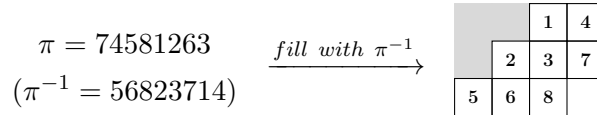


FIGURE 1.3: A partial order  $P_{(4,4,3)/(2,1)}$  obtained from the skew shape  $(4, 4, 3)/(2, 1)$ .

Due to Gessel [Ges84],  $L(P_{\lambda/\mu})$  may be identified with Young tableaux of shape  $\lambda/\mu$ . Given  $\pi \in L(P_{\lambda/\mu})$  construct the corresponding skew Young tableau by filling the skew shape  $\lambda/\mu$  with integers of  $\pi^{-1}$  from bottom to top and from left to right, as in Example 1.6. The permutations from  $L(P_{\lambda/\mu})$  are called **compatible** with the skew shape  $\lambda/\mu$ .

**Example 1.6.**



Using the bijection above, one can notice that

$$a_{IJ}^{\lambda/\mu} = \sum_{K \subseteq [|\lambda| - |\mu| - 1]} d_{\lambda/\mu K} a_{IJ}^K$$

where  $d_{\lambda/\mu K}$  is the number of standard Young tableaux of skew shape  $\lambda/\mu$  and descent set  $K$ .

**Proposition 1.8.** Let  $\lambda$  and  $\mu$  two integer partitions such that  $\lambda/\mu$  is a skew shape and  $I, J \subseteq [|\lambda| - |\mu| - 1]$ . The coefficients  $a_{IJ}^{\lambda/\mu}$  verify the following equality involving the

*Littlewood-Richardson coefficients, the Kronecker coefficients of the symmetric group and the numbers of standard Young tableaux of given shape and descent*

$$a_{IJ}^{\lambda/\mu} = \sum_{\nu, \rho, \varepsilon} k_{\mu\nu}^{\lambda} g(\nu, \rho, \varepsilon) d_{\rho I} d_{\varepsilon J}.$$

*Proof.* Gessel showed in [Ges84] that

$$s_{\lambda/\mu}(XY) = \sum_{I, J} a_{IJ}^{\lambda/\mu} F_I(X) F_J(Y). \quad (1.24)$$

Use Equation (1.1) to decompose skew Schur function  $s_{\lambda/\mu}(XY)$  using Littlewood-Richardson coefficients. Then use the generalised Cauchy identity (1.19) and decomposition (1.10) to prove the proposition.

$$\begin{aligned} s_{\lambda/\mu}(XY) &= \sum_{\nu} k_{\mu\nu}^{\lambda} s_{\nu}(XY) \\ &= \sum_{\nu, \rho, \varepsilon} k_{\mu\nu}^{\lambda} g(\nu, \rho, \varepsilon) s_{\rho}(X) s_{\varepsilon}(Y) \\ &= \sum_{I, J} \sum_{\nu, \rho, \varepsilon} k_{\mu\nu}^{\lambda} g(\nu, \rho, \varepsilon) d_{\rho I} d_{\varepsilon J} F_I(X) F_J(Y). \end{aligned} \quad \square$$

## 1.9 Descent preservation property of the Cauchy formula

The Cauchy identity may be used to show the existence of bijections with certain properties. The basic example of such an application is the Robinson-Schensted correspondence itself.

Recall that for  $\pi \in S_n$ ,

$$F_{\text{Des}(\pi)}(XY) = \sum_{\sigma, \rho \in S_n; \sigma\rho = \pi} F_{\text{Des}(\sigma)}(X) F_{\text{Des}(\rho)}(Y). \quad (1.25)$$

$$s_{\lambda}(X) = \sum_{T \in \text{SSYT}(\lambda)} X^T = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des}(T)}(X). \quad (1.26)$$

Given a Cauchy identity

$$\sum_{\lambda \vdash n} s_{\lambda}(X) s_{\lambda}(Y) = s_{(n)}(XY), \quad (1.27)$$

we use Equations (1.26) and (1.25) to rewrite Equation (1.27) as

$$\sum_{\substack{\lambda \vdash n \\ T, U \in SYT(\lambda)}} F_{\text{Des}(T)}(X) F_{\text{Des}(U)}(Y) = F_{\emptyset}(XY) = \sum_{\pi \in S_n} F_{\text{Des}(\pi)}(X) F_{\text{Des}(\pi^{-1})}(Y). \quad (1.28)$$

The fundamental functions are a basis of the quasisymmetric functions algebra, so permutations  $\pi$  in  $S_n$  are in bijection with pairs of standard Young tableaux  $T, U$  of the same shape such that  $\text{Des}(T) = \text{Des}(\pi)$  and  $\text{Des}(U) = \text{Des}(\pi^{-1})$ . This gives a direct analytical proof of one important property of the RS correspondence. More precisely, this proves the existence of a bijection that satisfies the corresponding properties of the RS correspondence.

## 1.10 Schur positivity

Many algebraic combinatorial problems are related to positivity questions, i.e. showing that some coefficients are positive integers, that may lead to a combinatorial interpretation of such coefficients. A good overview of open positivity questions was suggested by Richard Stanley in [Sta99].

In this thesis we focus on Schur-positivity questions.

*Definition 16.* A symmetric function is called **Schur-positive** if it may be decomposed in Schur functions with positive integer coefficients.

The algebra of symmetric functions  $\Lambda$  is linked with characters of the symmetric group  $S_n$  via the Frobenius characteristic map

$$ch(\chi) = \frac{1}{n!} \sum_{a \in S_n} \chi(a) p_{c(a)}(X),$$

where  $c(a)$  is the cycle type of a permutation  $a$ . There is a representation-theoretical interpretation of Schur functions as the Frobenius characteristic of an irreducible character  $ch(\chi_\lambda) = s_\lambda$ . The related notion of a **Kronecker product** of homogeneous symmetric functions  $f_1 = ch(\chi_1), f_2 = ch(\chi_2)$  is defined in terms of  $ch$  by  $f_1 \odot f_2 = ch(\chi_1 \chi_2)$ , where  $\chi_1 \chi_2(\sigma) = \chi_1(\sigma) \chi_2(\sigma)$  for any  $\sigma \in S_n$ . If  $\chi_1$  and  $\chi_2$  are characters of representations of  $S_n$ , then  $\chi_1 \chi_2$  is a character of the tensor product of these representations. The Kronecker coefficients in these terms are exactly coefficients  $\langle s_\lambda \odot s_\mu, s_\nu \rangle$ . At the same time, one can note that the Schur-positivity of a symmetric function means that it corresponds to some character of a symmetric group via the Frobenius characteristic map.

The Littlewood-Richardson rule (see Section 1.4) gives us the basic examples of Schur-positive functions. Skew Schur function  $s_{\lambda/\mu}$  and product of two Schur functions  $s_\lambda s_\mu$

are Schur-positive. For two skew shapes  $\lambda/\mu$  and  $\lambda'/\mu'$  the problem of determining the conditions whether or not  $s_{\lambda/\mu} - s_{\lambda'/\mu'}$  is Schur-positive has received much attention in recent years, see [BBR06], [FFLP05], [KWvW08], [Kir04], [LPP07].

Some papers also study the following situation: whether the expression  $s_{\lambda}s_{\mu} - s_{\nu}s_{\rho}$  is Schur-positive (see [BM04] for example).

*Definition 17.* Let  $X = \{x_1, x_2, \dots\}$  be a totally ordered set of commutative indeterminates. Given any subset  $\mathcal{A}$  of permutations in  $S_n$ , Gessel introduces in [Ges84] the following formal power series in  $\mathbb{C}[X]$ :

$$Q(\mathcal{A})(X) = \sum_{\pi \in \mathcal{A}} F_{\text{Des}(\pi)}(X).$$

In [GR93] Gessel and Reutenauer looked at the problem of characterising the sets  $\mathcal{A}$  for which  $Q(\mathcal{A})$  is symmetric. Further the question of determining Schur-positive sets, i.e. the sets  $\mathcal{A}$  for which  $Q(\mathcal{A})$  can be expanded with non-negative coefficients in the Schur basis received significant attention. Classical examples of Schur-positive sets include

- inverse descent classes,
- Knuth classes [Ges84],
- conjugacy classes [GR93].

As a more sophisticated example, Elizalde and Roichman proved [ER14] the Schur-positivity of **arc permutations**, i.e the set of permutations in  $S_n$  avoiding the patterns  $\sigma$  in  $S_4$  such that  $|\sigma(1) - \sigma(2)| = 2$ . Arc permutations are alternatively defined as the set of permutations  $\pi$  in  $S_n$  such that for any  $1 \leq j \leq n$ ,  $\{\pi(1), \pi(2), \dots, \pi(j)\}$  is an interval in  $\mathbb{Z}_n$ . The corresponding decomposition follows.

$$\sum_{\pi \in \mathcal{A}_n} F_{\text{Des}(\pi)} = s_{(n)} + s_{(1^n)} + 2 \sum_{1 \leq k \leq n-1} s_{(n-k, 1^k)} + \sum_{1 \leq k \leq n-2} s_{(n-k-2, 2, 1^k)}.$$

This result also leads to the problem of finding other Schur-positive pattern-avoiding sets. Some of them are given by Sagan in [Sag15]. Other advanced examples of Schur-positive sets can be found in [ER17].

Many of these results are the consequence of two main facts.

1. The Schur symmetric function indexed by  $\lambda \vdash n$ ,  $s_{\lambda}$  is the generating function for semistandard Young tableaux of shape  $\lambda$ .

2. There are various descent preserving bijections relating sets of permutations and standard Young tableaux, e.g. the Robinson-Schensted correspondence. The proof in [ER14] also uses such a bijection between arc permutations and standard Young tableaux of shapes  $(n - k, 1^k)$  and  $(n - k - 2, 2, 1^k)$ .

Chapter 4 is devoted to a new definition of type B Schur-positivity and the identification of type B Schur positive sets.

## 1.11 Material of the thesis

In this thesis we generalise the results of this chapter to the case of type B. The thesis relies on three main results, presented or accepted at the conferences:

- On the structure constants of the descent algebra of the hyperoctahedral group, Mayorova and Vassilieva, Eurocomb 2017, [MV17].
- A domino tableau-based view on type B Schur-positivity, Mayorova and Vassilieva, Permutation Patterns 2018, [MV18].
- A  $q$ -deformed type B Cauchy identity and Chow's quasisymmetric functions, Mayorova and Vassilieva, FPSAC 2019.

The following article is accepted as a full version of [MV17]:

[MV19] A new link between the descent algebra of type B, domino tableaux and Chow's quasisymmetric functions, Mayorova and Vassilieva, Discrete Mathematics, 342 (6):1658-1673, 2019.

The full versions of the two other results are in process.

## Chapter 2

# A new link between domino tableaux and Chow's type B quasisymmetric functions

This chapter plays a supporting rôle. We make an attempt to generalise all the main notions we used in Chapter 1 to the case of type B, trying to preserve the properties and links between them. We suggest a modified definition of domino tableaux and a definition of domino functions instead of Young tableaux and Schur functions. We show how to decompose the domino functions using type B Chow's fundamental and monomial functions and investigate the restrictions we needed to have such a decomposition.

### 2.1 Signed permutations and domino tableaux

#### 2.1.1 Hyperoctahedral group and descent set

Let  $B_n$  be the **hypercuboctahedral group** of order  $n$ , i.e. the Coxeter group of type B.  $B_n$  is composed of all permutations  $\pi$  on the set  $\{-n, \dots, -2, -1, 0, 1, 2, \dots, n\}$  such that for all  $i$  in  $\{0\} \cup [n]$ ,  $\pi(-i) = -\pi(i)$  (in particular  $\pi(0) = 0$ ). As a result, such permutations usually referred to as **signed permutations** are entirely described by their restriction to  $[n]$ .

Further we suggest an overview on the notions of descent set of a signed permutation. We start from the most essential definition.



*Definition 18.* The **descent set** of a signed permutation  $\pi$  of  $B_n$  is the subset of  $\{0\} \cup [n-1]$  defined by

$$\text{Des}(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}.$$

The main difference with respect to the case of the symmetric group is the possible descent in position 0 when  $\pi(1)$  is a negative integer. Further in the thesis we use bars instead the sign "-" for the convenience.

Another approach to define descent set in the hyperoctahedral group uses a different order on the elements of the set  $\{\bar{n}, \dots, \bar{1}, 1, \dots, n\}$ . Define:

$$\bar{1} <_r \bar{2} <_r \dots \bar{n} <_r 1 <_r 2 <_r \dots n.$$

One may define another notion of descent set  $\text{Des}_r$  using the less essential order  $>_r$ :

$$\text{Des}_r(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) >_r \pi(i+1)\}.$$

The descent set in this case is also a subset of  $0 \cup [n-1]$ .

A third option is the *signed descent*. Denote by  $\Sigma^B(n)$  the set consisting of all pairs  $(S, \varepsilon)$ , called **signed sets**, where  $S$  is subset of  $[n]$  and  $\varepsilon$  is map from  $S$  to  $\{-, +\}$ .

*Definition 19.* Denote by  $\text{sDes}(\pi)$  the **signed descent** of the signed permutation  $\pi$ , i.e. the signed set  $(S, \varepsilon)$ , such that

- $S$  contains all  $s \in [n-1]$  for which  $\pi(s) >_r \pi(s+1)$  or  $\pi(s)$  is barred and  $\pi(s+1)$  is not.
- $S$  contains  $n$ .
- For every  $s \in S$  we denote  $\varepsilon(s) = -$  if  $\pi(s)$  is barred and  $\varepsilon(s) = +$  otherwise.

Given a signed set  $\sigma = (S, \varepsilon) \in \Sigma^B(n)$  we will define two statistics which relies on it. Denote by  $\text{wDes}(\sigma)$  the set of elements  $s_i \neq s_n; s_i \in S$  such that  $\varepsilon(s_i) = \varepsilon(s_{i+1})$  or  $\varepsilon(s_i) = +, \varepsilon(s_{i+1}) = -$ . Denote also

$$\text{wDes}'(\sigma) = \begin{cases} \text{wDes}(\sigma), & \text{if } \varepsilon'(1) = +; \\ \text{wDes}(\sigma) \cup \{0\}, & \text{if } \varepsilon'(1) = -. \end{cases}$$

In case of signed permutations one can give a definition of  $\text{wDes}(\pi)$  and  $\text{wDes}'(\pi)$  without the notion of signed descent, as follows

$$\text{wDes}(\pi) = \text{wDes}(\text{sDes}(\pi)) = \{1 \leq i \leq n-1 \mid \pi(i) >_r \pi(i+1)\},$$

and

$$\text{wDes}'(\pi) = \text{wDes}'(\text{sDes}(\pi)) = \text{Des}_r(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) >_r \pi(i+1)\}.$$

So, the signed descent  $\text{sDes}$  refines  $\text{Des}_r$ .

One may note that the signed descent  $\text{sDes}$  refines also  $\text{Des}$ . Indeed, it is possible to continue the sign map  $\varepsilon$  from the set  $S$  to  $[n]$ . In particular, in the case of the descent set of a permutation it coincides with the map  $i \rightarrow \text{sign}(\pi(i))$ . Denote by  $\varepsilon'$  such a map. Further, any two permutations with the same signed descent set  $(S, \varepsilon)$  have also the same Solomon descent set, which may be obtained with the following rule:

- $0 \in \text{Des}(\pi)$  if and only if  $\varepsilon'(1) = -$ ,
- $k \in \text{Des}(\pi)$  if and only if
  - $k \in S$  and  $\varepsilon'(i) = \varepsilon'(i+1) = +$ , or
  - $k \notin S$  and  $\varepsilon'(i) = \varepsilon'(i+1) = -$ .

### 2.1.2 Domino tableaux

One way to generalise the notion of Young tableau to the case of type B is domino tableaux. Recall the definition from [Gar90].

*Definition 20.* For  $\lambda \vdash 2n$ , a **standard domino tableau**  $T$  of shape  $\lambda$  is a Young diagram of shape  $\lambda$  tiled by **dominoes**, i.e.  $2 \times 1$  or  $1 \times 2$  rectangles filled with the elements of  $[n]$  such that the entries are strictly increasing along the rows and down the columns. Denote by  $\text{SDT}(\lambda)$  ( $\text{SDT}(n)$ ) the set of standard domino tableaux of shape  $\lambda$  (of  $n$  dominoes).

We consider only the set  $\mathcal{P}^0(n)$  of *empty 2-core partitions*  $\lambda \vdash 2n$ , i.e. partitions that fit such a tiling.

Domino tableaux are related to Chow's quasisymmetric functions. Another advantage is the notion of the descent set, which is similar to the one in type A.

*Definition 21.* A standard domino tableau  $T$  has a descent in position  $i > 0$  if  $i+1$  lies strictly below  $i$  in  $T$  and has a descent in position 0 if the domino filled with 1 is vertical. We denote by  $\text{Des}(T)$  the set of all its descents.

For  $\lambda$  in  $\mathcal{P}^0(n)$  and  $I \subseteq \{0\} \cup [n-1]$ , denote by  $d_{\lambda I}^B$  the number of standard domino tableaux of shape  $\lambda$  and descent set  $I$ .

**Example 2.1.** The following standard domino tableaux have shape  $(5, 5, 4, 1, 1)$  and descent set  $\{0, 3, 5, 6\}$ .

|         |   |   |   |   |
|---------|---|---|---|---|
| $T_1 =$ | 1 | 2 | 3 | 5 |
|         |   |   |   | 6 |
|         | 4 |   | 8 |   |
|         | 7 |   |   |   |

|         |   |   |   |   |
|---------|---|---|---|---|
| $T_2 =$ | 1 | 2 | 3 |   |
|         |   |   | 4 | 5 |
|         | 6 |   | 8 |   |
|         | 7 |   |   |   |

*Definition 22.* A **semistandard domino tableau**  $T$  of shape  $\lambda \in \mathcal{P}^0(n)$  and weight  $w(T) = \mu = (\mu_0, \mu_1, \mu_2, \dots)$  with  $\mu_i \geq 0$  and  $\sum_i \mu_i = n$  is a tiling of the Young diagram of shape  $\lambda$  with horizontal and vertical dominoes labelled with integers of the set  $\{0, 1, 2, \dots\}$  such that labels are non decreasing along the rows, strictly increasing down the columns and exactly  $\mu_i$  dominoes are labelled with  $i$ . If the top leftmost domino is vertical, it cannot be labelled 0.

Our notion of semistandard domino tableau differs from the usual one (which is without zeroes). The reason is that we need zeroes in the tableau to make valid links to Chow's quasisymmetric functions.

*Remark 1.* The only possible sub-pattern of dominoes with label 0 in a semistandard domino tableau is a row composed of horizontal dominoes.

**Example 2.2.** The following semistandard tableau of shape  $(5, 5, 4, 3, 1)$  has weight  $\mu = (2, 0, 2, 0, 0, 4, 0, 1)$ .

|       |   |   |   |   |   |
|-------|---|---|---|---|---|
| $T =$ |   | 0 | 0 |   |   |
|       |   | 2 |   | 5 | 5 |
|       | 2 |   | 5 |   |   |
|       |   |   | 7 |   |   |
|       | 5 |   |   |   |   |

Denote by  $SSDT(\lambda)$  ( $SSDT(n)$ ) the set of semistandard domino tableaux of shape  $\lambda$  (of  $n$  dominoes) and  $K_{\lambda\mu}^B$  the number of semistandard domino tableaux of shape  $\lambda$  and weight  $\mu$ .

## 2.2 Stanton and White bijection

### 2.2.1 Bi-tableaux

A second way to generalise the notion of the Young tableau is to consider bi-tableaux. Bi-tableaux are closely related to Poirier's quasisymmetric functions. At the same time,

they are related to domino tableaux via the Stanton and White bijection.

*Definition 23.* Denote a pair of shapes  $(\lambda^-, \lambda^+)$  a **bi-shape** of  $n$  if  $|\lambda^-| + |\lambda^+| = n$ . A **standard Young bi-tableau** of bi-shape  $(\lambda^-, \lambda^+)$  is a pair of Young diagrams  $(T^-, T^+)$  with  $\text{shape}(T^-) = \lambda^-$  and  $\text{shape}(T^+) = \lambda^+$ , whose boxes are filled with the elements of  $[n]$  such that the entries are strictly increasing along the rows and down the columns for each of them.

*Remark 2.* The number of standard Young bi-tableaux of bi-shape  $(\lambda^-, \lambda^+)$  is exactly

$$f_{(\lambda^-, \lambda^+)} = \binom{|\lambda^-| + |\lambda^+|}{|\lambda^-|} f_{\lambda^-} f_{\lambda^+}.$$

For a standard Young bi-tableau it is also possible to denote a notion of a descent set. However, it differs from the usual one used in this thesis. In fact, one uses signed descent sets for the case of bi-tableaux. Recall,  $\Sigma^B(n)$  is the set consisting of all pairs  $(S, \varepsilon)$ , where  $S$  is subset of  $[n]$  and  $\varepsilon$  is map from  $S$  to  $\{-, +\}$ .

*Definition 24.* The **signed descent set**  $\text{sDes}((Q^+, Q^-))$  of a bi-tableau  $(Q^+, Q^-) \in \text{SYT}(\lambda, \mu)$  is the signed set  $(S, \varepsilon)$  defined as follows:

- $S$  contains all  $s \in [n-1]$  for which either both  $s$  and  $s+1$  appear in the same tableau and  $s+1$  is in a lower row than  $s$ , or  $s$  and  $s+1$  appear in different tableaux.
- $S$  contains  $n$ .
- For every  $s \in S$  we denote  $\varepsilon(s) = -$  if  $s$  appears in  $Q^-$  and  $\varepsilon(s) = +$  otherwise.

Recall, that in the case of permutations the notion of signed descent  $\text{sDes}$  generalises both the notions  $\text{Des}$  and  $\text{Des}_r$ . Given a signed descent  $\text{sDes}$  we can find the corresponding  $\text{Des}_r$ . In the case of bi-tableaux, this turns into the following definition for  $\text{Des}_r$ .

*Definition 25.* The **descent set**  $\text{Des}_r((Q^+, Q^-))$  of a bi-tableau  $(Q^+, Q^-) \in \text{SYT}(\lambda, \mu)$  is a subset of  $\{0\} \cup [n-1]$  containing:

- each  $s \in [n-1]$  such that both  $s$  and  $s+1$  appear in the same tableau and  $s+1$  is in a lower row than  $s$
- each  $s \in [n-1]$  such that  $s \in Q^+$  and  $s+1 \in Q^-$
- 0 if  $1 \in Q^-$ .

*Example 7.* The following standard bi-tableau has bi-shape  $((3), (2, 2, 2))$  and descent set  $\text{Des}_r$  equal to  $\{2, 4, 8\}$ .

$$\left( Q^- = \begin{array}{|c|c|c|} \hline 3 & 5 & 6 \\ \hline \end{array}, Q^+ = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 8 \\ \hline 7 & 9 \\ \hline \end{array} \right)$$

*Definition 26.* A **semistandard Young bi-tableau**  $(T^-, T^+)$  is a pair of Young diagrams of bi-shape  $(\lambda^-, \lambda^+)$  whose boxes are filled with nonnegative integers such that the entries are strictly increasing down the columns and non-decreasing along the rows for each of them. The additional constraint is that zeroes may appear only in  $T^+$ .

*Example 8.* The following standard bi-tableau has bi-shape  $((3), (2, 2, 2))$ .

$$\left( T^- = \begin{array}{|c|c|c|} \hline 2 & 4 & 4 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 4 & 6 \\ \hline 5 & 7 \\ \hline \end{array} \right)$$

*Remark 3.* The number of semistandard Young bi-tableaux of bi-shape  $(\lambda^-, \lambda^+)$  with its entries less than or equal to  $t \in \mathbb{Z}_+$  is exactly

$$g_{(\lambda^-, \lambda^+), t} = g_{\lambda^-, t} g_{\lambda^+, t+1}.$$

Recall,  $g_{\lambda, t}$  is the number of semistandard Young tableaux of shape  $\lambda$  with its entries less than or equal to  $t$ .

### 2.2.2 Description of the bijection

In [CL95] Carré and Leclerc studied a bijection due to Stanton and White [SW85]. They introduced an easier description of the bijection between semistandard domino tableaux (without zeroes) of  $n$  dominoes and pairs of semistandard Young tableaux with  $n$  cells in common. This bijection restricts to the sets of shapes, it does not depend on the tiling and numbers in cells. So,

$$|\mathcal{P}^0(n)| = \sum_{0 \leq k \leq n} p(k)p(n-k).$$

In this section, we further describe the idea of the algorithm and apply it to prove the similar statements for our definitions of a domino tableau and a bi-tableau.

Given a domino tableau  $T$  we consider  $d_l$  the diagonal line  $(i, j) : i = j - 2l$ . Such a line may intersect a domino in four different ways (see Figure 2.1).

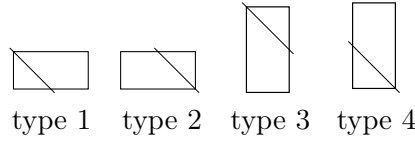


FIGURE 2.1: Four layouts of a domino on a diagonal.

Denote also  $d_l^+$  and  $d_l^-$  the diagonals  $(i, j) : i = j - l$  of the bi-tableau. Given a diagonal  $d_l$  we consider all the dominoes of types 1 and 4. They are positive and we put the cell with its entry in  $T^+$  on the diagonal  $d_l^+$ . The other dominoes are negative and go to  $T^-$  on the diagonal  $d_l^-$ . Due to [CL95] this correspondence is a bijection. The pair  $(T^-, T^+)$  is called the **2-quotient** of  $T$ .

**Example 2.3.** Figure 2.2 shows a domino tiling of a diagram and its 2-quotient.

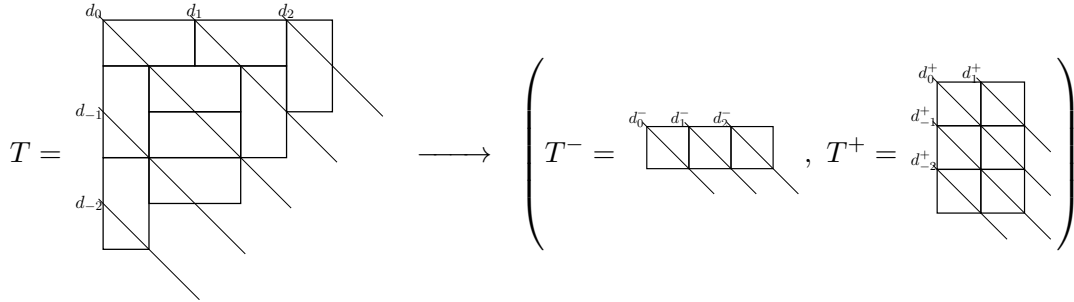


FIGURE 2.2: A domino tiling and its 2-quotient.

In fact, there is an easy way to determine if a domino is positive or negative. We colour the domino with "-" and "+" in the chess-board colouring. For a horizontal domino we look at the sign of the right cell, for vertical look at the sign of the top cell. It determines the sign of a domino itself. So, the bijection sends positive dominoes to  $T^+$  and negative dominoes to  $T^-$ , see Figure 2.3.

**Example 2.4.** Figure 2.3 shows a domino tableau of weight  $\mu = (2, 0, 2, 0, 0, 4, 0, 1)$  and its 2-quotient.

*Remark 2.1.* If the top leftmost domino of a semistandard domino tableau  $T$  is vertical (resp. horizontal), its label is used for  $T^-$  (resp.  $T^+$ ). Therefore not labelling the top leftmost domino with 0 if it is vertical is sufficient to ensure that only  $T^+$  may have entries equal to 0 and any bi-tableau with  $T^+$  containing entries equal to 0 gives a valid semistandard domino tableau according to our constraints. As a result, in our setup, the bijection is actually between semistandard domino tableaux indexed by labels in  $\{0, 1, 2, \dots\}$  and pairs of semistandard Young tableaux indexed respectively by  $\{1, 2, \dots\}$  and  $\{0, 1, 2, \dots\}$ .

$$T = \begin{array}{|c|c|c|c|} \hline - & 0 & + & - \\ \hline + & 2 & - & 2 & + & - \\ \hline - & + & 5 & - & + & - \\ \hline + & 5 & - & - & 7 & + \\ \hline - & & & & & - \\ \hline \end{array} \longrightarrow \left( T^- = \begin{array}{|c|c|c|} \hline 5 & 5 & 5 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 2 & 2 \\ \hline 5 & 7 \\ \hline \end{array} \right)$$

FIGURE 2.3: A semistandard domino tableau and its 2-quotient.

## 2.3 BV and Garfinkle bijections

*Definition 27.* Denote by  $\text{sp}(T)$  the **spin** of (semi-) standard domino tableau  $T$ , i.e. **half the number** of its vertical dominoes.

$$T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline & & & 6 \\ \hline 4 & & 8 & \\ \hline 7 & & & \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline & & 4 & \\ \hline 6 & & 8 & \\ \hline 7 & & & \\ \hline \end{array} \quad T_3 = \begin{array}{|c|c|c|c|} \hline 0 & 0 & & 5 \\ \hline 2 & 2 & & 5 \\ \hline & 5 & & \\ \hline 5 & 7 & & \\ \hline \end{array}$$

FIGURE 2.4: Two standard domino tableaux  $T_1$  and  $T_2$  of shape  $(5, 5, 4, 1, 1)$ , descent set  $\{0, 3, 5, 6\}$ , a semistandard tableau  $T_3$  of shape  $(5, 5, 4, 3, 1)$  and weight  $\mu = (2, 0, 2, 0, 0, 4, 0, 1)$ . All of the tableaux have a spin of 2.

We recall that there is a natural analogue of the RSK-correspondence for signed permutations involving **domino tableaux**. Barbash and Vogan ([BV82]) built a bijection between signed permutations of  $B_n$  and pairs of standard domino tableaux of equal shape in  $\mathcal{P}^0(n)$ . An independent development on the subject is due to Garfinkle [Gar92]. Taşkın ([Tas12, Prop. 26]) showed that the two standard domino tableaux  $T$  and  $U$  associated to a signed permutation  $\pi$  by the algorithm of Barbash and Vogan have respective descent sets  $\text{Des}(T) = \text{Des}(\pi^{-1})$  and  $\text{Des}(U) = \text{Des}(\pi)$  while Shimozono and White showed in [SW01] the **color-to-spin** property i.e. that

$$\text{tc}(\pi) = \text{sp}(T) + \text{sp}(U), \quad (2.1)$$

where  $\text{tc}(\pi)$  is the **total color** statistics, i.e. the number of negative values in  $\{\pi(i) \mid i \in [n]\}$ .

There is also a generalisation of the bijection to the case of colored permutations and ribbon tableaux of the same shape due to Shimozono and White ([SW02]) which satisfies the color-to-spin property.

## 2.4 Quasisymmetric functions of type B

Chow defines in [Cho01] an analogue of Gessel's algebra of quasisymmetric functions that is dual as Hopf algebras to the Solomon's descent algebra of type B.

This is not the only option of how to generalise the quasisymmetric functions. Another one is Poirier's quasisymmetric functions' algebra. However, Poirier's quasisymmetric functions are dual to the Mantaci-Reutenauer algebra and not to Solomon's descent algebra of type B. See [Pet05] for further details. Both Poirier's quasisymmetric functions and Mantaci-Reutenauer algebra are related to the notion of signed descent set.

In his thesis, Chow defines the algebra of type B quasisymmetric functions through its bases. He prove that the spanned vector space is an algebra.

*Definition 28.* Let  $X = \{\cdots, x_{-i}, \cdots, x_{-1}, x_0, x_1, \cdots, x_i, \cdots\}$  be a set of totally ordered commutative indeterminates with  $x_{-i} = x_i$  and  $I$  be a subset of  $\{0\} \cup [n-1]$ , define the monomial

$$M_I^B(X) = \sum_{\substack{0 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n},$$

and fundamental

$$F_I^B(X) = \sum_{\substack{0 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Chow's quasisymmetric functions.

Here we assume  $i_0 = 0$ . Note the particular rôle played by variable  $x_0$ .

These two bases are related through

$$F_I^B(X) = \sum_{I \subseteq J \subseteq \{0\} \cup [n-1]} M_J^B(X). \quad (2.2)$$

**Example 2.5.** Let  $n = 2$  and  $X = \{x_{-2}, x_{-1}, x_0, x_1, x_2\}$  then

$$\begin{aligned} F_{\emptyset}^B &= x_0^2 + x_1^2 + x_2^2 + x_0x_1 + x_0x_2 + x_1x_2, \\ F_{\{1\}}^B &= x_0x_1 + x_0x_2 + x_1x_2, \\ F_{\{0\}}^B &= x_1^2 + x_2^2 + x_1x_2, \\ F_{\{0,1\}}^B &= x_1x_2. \end{aligned}$$



*Remark 4.* Note that for any subset which include 0 Chow's quasisymmetric function is equal to the usual quasisymmetric function

$$F_{\{0\} \cup I}^B(X) = F_I(x_1, x_2, \dots), \quad I \subseteq [n-1].$$

In the case subset does not include 0 Chow's quasisymmetric function is equal to the usual quasisymmetric function with shifted alphabet

$$F_I^B(X) = F_I(x_0, x_1, \dots), \quad I \subseteq [n-1].$$

## 2.5 Modified domino functions

One of the central objects of this thesis is an analogue for Schur functions related to type B quasisymmetric functions. In this section, we use the tableaux-based way to define them and prove the basic formulas and decompositions.

We may consider generating functions for domino tableaux sometimes called *domino functions*. They are also well-studied objects (see e.g. [KLLT94]). We introduce a modified definition to get an analogue of Schur functions in Chow's quasisymmetric functions. Our development differs by the addition of "0" entries to the domino tableaux in some cases.

*Definition 29.* Given indeterminate  $X$  and a semistandard domino tableau  $T$  of weight  $\mu$ , denote by  $X^T$  the monomial  $x_0^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} \dots$ . For  $\lambda \in \mathcal{P}^0(n)$  we call the **domino function** indexed by  $\lambda$  the function

$$\mathcal{G}_\lambda(X) = \sum_{T \in SSDT(\lambda)} X^T. \quad (2.3)$$

*Example 9.* Let  $\lambda = (2, 2, 1, 1)$  and  $X = (x_0, x_1, x_2, x_3)$  then

$$\begin{aligned} \mathcal{G}_{(2,2,1,1)} = & x_0 x_1 x_2 + x_0 x_1 x_3 + x_0 x_2 x_3 + x_1^2 x_2 + x_1^2 x_3 + \\ & x_1 x_2^2 + 3x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2. \end{aligned}$$

The list of semistandard domino tableaux of shape  $(2, 2, 1, 1)$  in  $X = (x_0, x_1, x_2, x_3)$  consists of the following tableaux

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 |
| 1 | 1 | 2 |   |   |   |   |   |
| 2 | 3 | 3 | 2 | 3 | 2 |   |   |
|   |   |   |   |   |   |   |   |
| 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 2 | 2 | 3 | 3 |   |   |   |   |
| 3 | 3 | 2 | 3 | 3 | 3 | 3 |   |
|   |   |   |   |   |   |   |   |

## 2.6 Decomposition in Schur functions

Recall the bijection between semistandard domino tableaux and bi-tableaux from the Section 2.2. Given a semistandard domino tableau  $T$  denote by  $(T^-, T^+)$  its 2-quotient. Denote  $X^- = \{x_{-i}\}_{i>0}$  and  $X^+ = \{x_i\}_{i\geq 0}$  (note that  $X^- = X^+ \setminus \{x_0\}$  as  $x_{-i} = x_i$ ).

**Proposition 2.2.** *For  $\lambda \in \mathcal{P}^0(n)$ , the domino function  $\mathcal{G}_\lambda$  and the Schur symmetric functions are related through*

$$\mathcal{G}_\lambda(X) = s_{\lambda^-}(X^-)s_{\lambda^+}(X^+). \quad (2.4)$$

*Proof.* According to the definition of the 2-quotient above and Remark 2.1, one can get

$$\mathcal{G}_\lambda(X) = \sum_{\substack{\text{shape}(T^-)=\lambda^- \\ \text{shape}(T^+)=\lambda^+}} X^{-T^-} X^{T^+},$$

where the sum is on all pairs of semistandard Young tableau  $T^-$  and  $T^+$  of shape  $\lambda^-$  and  $\lambda^+$  such that no entry of  $T^-$  is equal to 0.  $\square$

## 2.7 Decomposition in Chow fundamental and monomial functions.

Recall that we denote by  $K_{\lambda I}^B$  the number of semistandard domino tableaux of shape  $\lambda$  and type  $\mu = \text{comp}(I)$  (note that  $\mu_0 = 0$  if and only if  $0 \in I$ ).  $d_{\lambda I}^B$  are the number of standard domino tableaux of shape  $\lambda$  and descent set  $I$ . The following proposition gives a decomposition of domino functions in two basic bases of type B quasisymmetric functions.

**Proposition 2.3.** *Domino functions admit the following expansion into Chow's fundamental and monomial bases.*

$$\mathcal{G}_\lambda = \sum_{I \subseteq \{0\} \cup [n-1]} d_{\lambda I}^B F_I^B = \sum_{I \subseteq \{0\} \cup [n-1]} K_{\lambda \text{comp}(I)}^B M_I^B. \quad (2.5)$$

The **standardisation**  $T^{st}$  of a semistandard domino tableau  $T$  of weight  $\mu$  is the standard domino tableau obtained by relabelling the dominoes of  $T$  with  $1, 2, \dots, n$  such that the dominoes labelled with  $i_m = \min\{i \mid \mu_i > 0\}$  are relabelled with  $1, 2, \dots, \mu_{i_m}$  from left to right and so on.

**Example 2.6.** Figure 2.5 shows a domino tableau of weight  $\mu = (2, 0, 2, 0, 0, 4, 0, 1)$  and its standardisation.

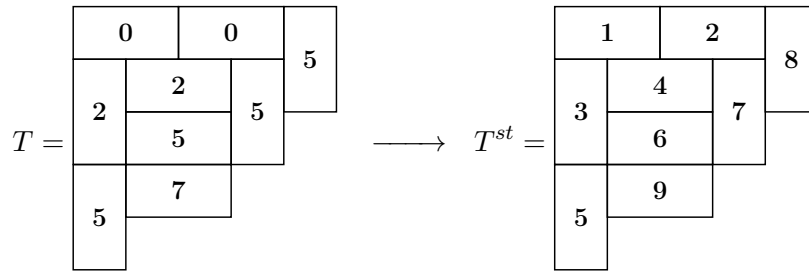


FIGURE 2.5: A semistandard domino tableau and its standardisation.

In order to prove Proposition 2.3 we need the following lemma.

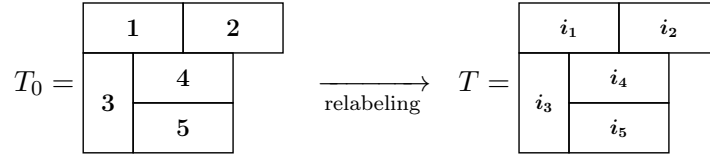
*Lemma 1.* Let  $T_0$  be a standard domino tableau. Then the fundamental quasisymmetric function  $F_{\text{Des}(T_0)}^B$  is the generating function for semistandard domino tableaux  $T$  such that  $T^{st} = T_0$ , i.e.

$$F_{\text{Des}(T_0)}^B = \sum_{T^{st}=T_0} X^T. \quad (2.6)$$

*Proof.* Given a fixed standard domino tableau  $T_0$ , we reverse the standardisation operation to obtain the set of semistandard domino tableaux with standardisation  $T_0$ . Since the standardisation operation is shape and domino tiling preserving, any such semistandard tableau  $T$  may be identified by the sequence of its entries  $i_1, i_2, \dots, i_n$ .

Relabel the entries of  $T_0$  with the sequence  $i_1, i_2, \dots, i_n$  such that  $k$  is mapped to  $i_k$ , as depicted on Figure 2.6. According to the standardisation procedure, we have

$0 \leq i_1 \leq i_2 \leq \dots \leq i_n$  and  $0 < i_1$  if the top leftmost domino is vertical. Furthermore such a sequence needs to verify additional conditions to give a valid semistandard domino tableau with standardisation equal to  $T_0$ .

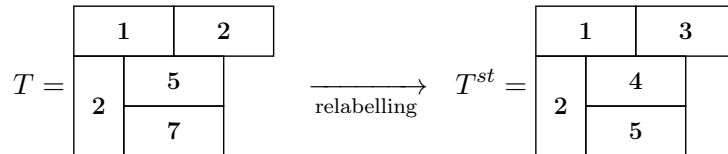
FIGURE 2.6: A standard domino tableau  $T_0$  and the template for its semistandard preimages.

Consider each constraint  $i_k \leq i_{k+1}$  separately. If  $i_k < i_{k+1}$  then locally the sequence gives a valid semistandard domino tableau (entries are increasing along the row and down the columns) and the standardisation process provides the expected result.

We look at the case  $i_k = i_{k+1}$  and show two properties.

- First, if  $i_k = i_{k+1}$  then  $k \notin \text{Des}(T_0)$ . Indeed assume  $k \in \text{Des}(T_0)$ . Domino  $k$  has to lay in a higher row than  $k + 1$  as in Figure 2.8.

But domino  $i_{k+1}$  may not be outside the green area defined in Figure 2.8 as the entries in the tableau are strictly increasing down the columns. As a consequence, domino  $i_{k+1}$  has to be on the left of and below domino  $i_k$ , as in Figure 2.8, picture (c). This is a contradiction since according to the definition of the standardisation operation, we should relabel the domino  $i_{k+1}$  before the domino  $i_k$  with such a pattern and  $T_0$  is not the standardisation of such a tableau. In fact, such a semistandard domino tableau corresponds to another standard domino tableau, without descent in the position  $k$  (see Figure 2.7 as an Example).

FIGURE 2.7:  $T^{st} \neq T_0$  (Wrong order of relabelling according the standardisation procedure.)

As a result  $k \in \text{Des}(T_0)$  implies  $i_k < i_{k+1}$ .

- Secondly, when  $k \notin \text{Des}(T_0)$  any sequence with  $i_k = i_{k+1}$  locally gives a valid semistandard tableau whose standardisation is equal to  $T_0$ . Indeed, if  $k \notin \text{Des}(T_0)$ , domino  $k$  does not lay in a higher row than  $k + 1$ .

As a result, domino  $i_{k+1}$  cannot be outside the green area defined in Figure 2.9 as entries are increasing along the rows and down the columns, i.e.  $i_{k+1}$  is to the right of and above domino  $i_k$ . This makes sure that  $i_k$  is relabelled before  $i_{k+1}$  in the standardisation procedure.

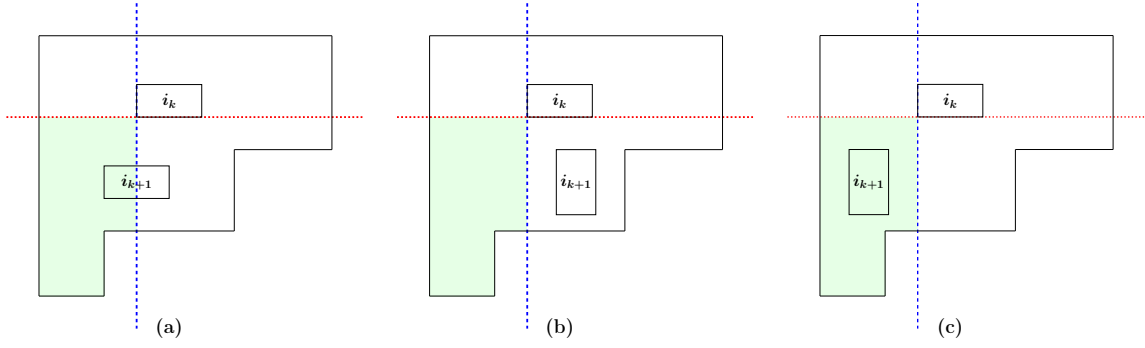


FIGURE 2.8: Layouts with a descent in position  $k$ . Green area corresponds to positions of domino  $i_{k+1}$  which do not violate the property of increasing entries along the rows and down the columns when  $i_k = i_{i+1}$ . In pictures (a) and (b) domino  $i_{k+1}$  lays outside the green area.

Picture (c) shows a valid layout of dominoes  $i_k$  and  $i_{k+1}$ .

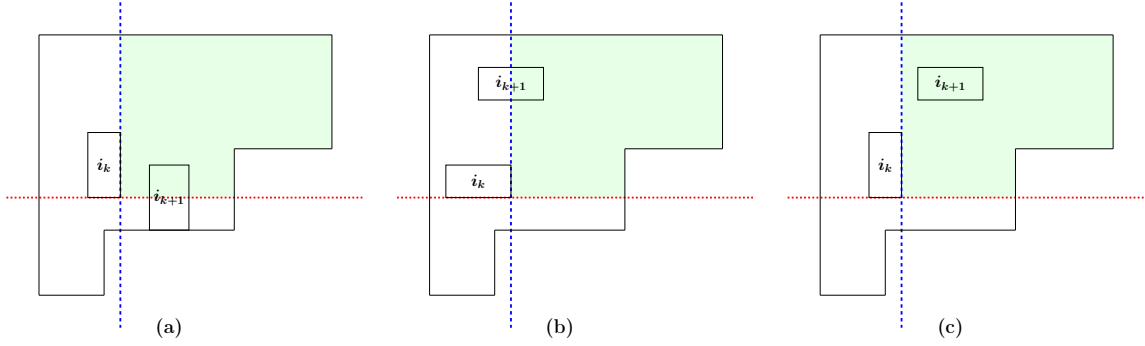


FIGURE 2.9: Layouts with no descent in position  $k$ . Green area corresponds to positions of domino  $i_{k+1}$  which do not violate the property of increasing entries along the rows and down the columns when  $i_k = i_{i+1}$ . In pictures (a) and (b) domino  $i_{k+1}$  lays outside the green area.

Picture (c) shows a valid layout of dominoes  $i_k$  and  $i_{k+1}$ .

As a consequence the preimages of  $T_0$  can be identified as the sequences  $0 \leq i_1 \leq i_2 \leq \dots \leq i_n$  such that  $k \in \text{Des}(T_0) \Rightarrow i_k < i_{k+1}$  for  $k \geq 0$  and  $i_0 = 0$ . Consider the constraints all together to ensure that such a sequence globally gives a valid semistandard tableau whose standardisation is equal to  $T_0$ .

Conversely, any such sequence gives a valid semistandard tableau  $T$  whose standardisation is  $T_0$ . Indeed, consider the entries of any row  $j_1 < \dots < j_r$  of  $T_0$ . Clearly,  $i_{j_1} \leq \dots \leq i_{j_r}$ . Consider any column  $j_1 < \dots < j_r$ , one also gets  $i_{j_1} < \dots < i_{j_r}$ , because otherwise the equality  $i_{j_t} = i_{j_{t+1}}$  leads to  $i_{j_t} = i_{j_{t+1}} = \dots = i_{j_{t+1}}$ . It implies  $j_t, j_{t+1}, \dots, j_{t+1} \notin \text{Des}(T_0)$ , so domino  $j_{t+1}$  could not lay right under domino  $j_t$ . Figure 2.9 picture (c) shows the only valid layout of dominoes  $i_k$  and  $i_{k+1}$  if  $i_k = i_{k+1}$ . Therefore, the standardisation of semistandard tableau  $T$  coincides with  $T_0$ . By definition, the fundamental function of Chow is exactly the generating function for the described set of semistandard domino

tableaux.

$$F_{\text{Des}(T_0)}^B(X) = \sum_{\substack{0 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ j \in \text{Des}(T_0) \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n} = \sum_{T^{st}=T_0} X^T.$$

□

We are now ready to prove Proposition 2.3.

*Proof of Proposition 2.3.* Classify the set of semistandard domino tableaux according to their standardisation and add the monomials corresponding to the tableaux in the same class. Using Lemma 1 we obtain the first part of the proposition.

$$\begin{aligned} \mathcal{G}_\lambda(X) &= \sum_{T \in SSDT(\lambda)} X^T \\ &= \sum_{T_0 \in SDT(\lambda)} \sum_{T \in SSDT(\lambda), T^{st}=T_0} X^T \\ &\stackrel{(2.6)}{=} \sum_{T_0 \in SDT(\lambda)} F_{\text{Des}(T_0)}^B(X) \\ &= \sum_{I \subseteq \{0\} \cup [n-1]} d_{\lambda I}^B F_I^B. \end{aligned}$$

To get the second part of the proposition, relabel the entries of a semistandard tableau  $T$  with successive integers  $0, 1, 2, \dots$  such that inequalities and equalities between entries in  $T$  are preserved after relabelling. If the top leftmost domino is vertical skip 0. This operation removes the zeros in the weight of the tableau except, possibly, the first one. Denote by  $\tilde{T}$  the resulting tableau and for  $\lambda \in \mathcal{P}^0(n)$ , by  $\widetilde{SSDT1}(\lambda)$  the set of semistandard domino tableaux of shape  $\lambda$  such that  $\tilde{T} = T$ . The second part of the statement is a consequence of the fact that the monomial quasisymmetric function  $M_{\text{set}(w(T_0))}^B$  is precisely the generating function for all semistandard domino tableaux  $T$  mapped to the same  $T_0 \in \widetilde{SSDT1}(\lambda)$  by this operation.

$$\begin{aligned} \mathcal{G}_\lambda(X) &= \sum_{T \in SSDT(\lambda)} X^T \\ &= \sum_{T_0 \in \widetilde{SSDT1}(\lambda)} \sum_{T \in SSDT(\lambda), \tilde{T}=T_0} X^T \\ &= \sum_{T_0 \in \widetilde{SSDT1}(\lambda)} M_{\text{set}(w(T_0))}^B \\ &= \sum_{I \subseteq \{0\} \cup [n-1]} K_{\lambda \text{comp}(I)}^B M_I^B. \end{aligned}$$

Equation (2.5) is fully proven.  $\square$

## 2.8 Skew shape decomposition

### 2.8.1 Skew domino tableaux

Let  $\lambda$  and  $\mu$  be two integer partitions such that  $\lambda/\mu$  is a skew shape,  $|\lambda| - |\mu| = 2n$  and such that the Young diagrams of shape  $\mu$  and  $\lambda/\mu$  (and by extension  $\lambda$ ) may be tiled by horizontal and vertical dominoes. We write  $\lambda/\mu \in \mathcal{P}^0(|\lambda|/2, |\mu|/2)$  when all these conditions are fulfilled. By abuse of notation, we also write  $\lambda/\mu \in \mathcal{P}^0(n)$ . A **standard skew domino tableau** is a tiling of the skew Young diagram of shape  $\lambda/\mu$  with horizontal and vertical dominoes labelled with integers of  $[n]$  such that labels are strictly increasing along the rows and down the columns.

Below we extend the bijection of Stanton and White (in the description of Carré and Leclerc [CL95]) to the case of skew shapes. As a result, there is a bijection between standard skew domino tableaux of shape  $\lambda/\mu$  and standard skew bi-tableaux of bi-shape  $(\lambda^-/\mu^-, \lambda^+/\mu^+)$ .

**Example 2.7.** A standard skew domino tableau  $T$  and its decomposition follows.

$$T = \begin{array}{ccccccc} & & & & & & 8 \\ & & & & & 3 & \\ & & & 2 & & & \\ & & 4 & & 9 & & \\ 1 & 6 & 7 & & & & \\ 5 & & & & & & \end{array} \longrightarrow \left( T^- = \begin{array}{ccc} & & \\ 1 & 7 & 9 \\ 5 & & \end{array}, T^+ = \begin{array}{ccc} 2 & 3 & 8 \\ 4 & & \\ 6 & & \end{array} \right)$$

Given a standard skew domino tableau  $T$ , we call in the sequel **negative (resp. positive) domino**, a domino that goes to  $T^-$  (resp.  $T^+$ ) according to the bijection above. We have the following definition.

*Definition 30* (Descent set of a standard skew domino tableau). A standard skew domino tableau  $T$  has a descent in position  $i > 0$  if  $i + 1$  lies strictly below  $i$  in  $T$  and has descent in position 0 if the domino filled with 1 is negative. Denote by  $\text{Des}(T)$  the set of all the descents of  $T$ . For  $\lambda/\mu \in \mathcal{P}^0(n)$  and  $I \subseteq \{0\} \cup [n-1]$  denote also  $d_{\lambda/\mu}^B I$  the number of standard skew domino tableaux of shape  $\lambda/\mu$  and descent set  $I$ .

**Example 2.8.** In Example 2.7 we have  $\text{Des}(T) = \{0, 3, 4, 8\}$ .

A **semistandard skew domino tableau** is a tiling of the skew Young diagram of shape  $\lambda/\mu$  with horizontal and vertical dominoes labelled with integers of the set  $\{0, 1, 2, \dots\}$  such that labels are non decreasing along the rows, strictly increasing down the columns, and negative dominoes cannot be labelled 0.

The bijection of Stanton and White also generalises to semistandard domino tableaux and pairs of semistandard Young tableaux such that dominoes labelled 0 may appear only in  $T^+$ .

**Proposition 2.4.** *Let  $\lambda/\mu$  be a skew shape, such that  $\mu \in \mathcal{P}^0(m)$  and  $\lambda/\mu \in \mathcal{P}^0(n-m)$ . There exists a bijection  $\varphi$  between semistandard domino tableaux (of shape  $\lambda/\mu$ ) and pairs of semistandard Young tableaux (of bi-shape  $(\lambda^-/\mu^-, \lambda^+/\mu^+)$ ), such that zeros may appear in the positive tableau of a pair. This bijection also restricts to the sets of shapes.*

*Proof.* Since  $\mu \in \mathcal{P}^0(m)$  one can fix some tiling of  $\mu$ . We apply the algorithm to get corresponding shapes  $\mu^-$  and  $\mu^+$ . Then we continue to proceed the shape  $\lambda/\mu$ : we look at dominoes on diagonals, check its types and add cells to the corresponding diagonals of  $T^-$  and  $T^+$ . At the end we got exactly shapes  $\lambda^-$  and  $\lambda^+$ . The row and column property is fulfilled since the construction of the Stanton and White bijection. The definition of semistandard domino tableaux states that zero dominoes go exactly to  $T^+$ .  $\square$

**Corollary 2.5.** *The same algorithm gives a bijection between standard domino tableaux and standard Young bi-tableaux.*

Consider two partial orderings. First one is a partial order on  $\mathcal{P}^0$ , such that  $\lambda \geq \mu$  if and only if shape  $\mu$  lay in shape  $\lambda$ . Second is partial order on bi-shapes,  $(\lambda^-, \lambda^+) \geq (\mu^-, \mu^+)$  if and only if  $\lambda^- \geq \mu^-$  and  $\lambda^+ \geq \mu^+$ .

*Remark 2.6.* Let  $\mu \in \mathcal{P}^0(m)$  and  $\lambda \in \mathcal{P}^0(n)$ . Their images under the bijection are  $(\mu^-, \mu^+)$  and  $(\lambda^-, \lambda^+)$ . If  $\lambda/\mu \in \mathcal{P}^0(n-m)$  then  $\lambda^- \geq \mu^-$  and  $\lambda^+ \geq \mu^+$ . And vice versa, if  $\lambda^- \geq \mu^-$  and  $\lambda^+ \geq \mu^+$  then  $\lambda/\mu \in \mathcal{P}^0(n-m)$  (in particular  $\lambda \geq \mu$ ).

So, if  $\mu \in \mathcal{P}^0(m)$  and  $\lambda \in \mathcal{P}^0(n)$  but  $\lambda/\mu \notin \mathcal{P}^0(n-m)$  then either  $\lambda^- \not\geq \mu^-$  or  $\lambda^+ \not\geq \mu^+$ .

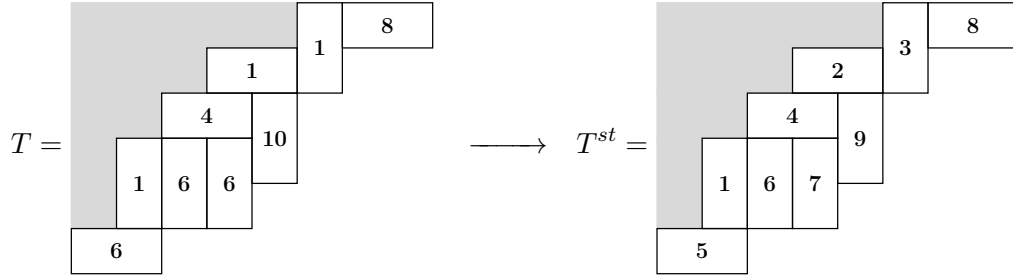
**Example 2.9.** *Decompose the following semistandard skew domino tableau.*

$$T = \begin{array}{ccccccc} & & & & & & 1 \\ & & & & & 0 & \\ & & & & 0 & & \\ & & & 3 & & & \\ & & 3 & 6 & 7 & 9 & \\ & 3 & 6 & 7 & & & \\ 6 & & & & & & \end{array} \longrightarrow \left( T^- = \begin{array}{ccc} & & \\ 3 & 7 & 9 \\ 6 & & \end{array}, T^+ = \begin{array}{ccc} & 0 & 0 & 1 \\ & 3 & & \\ & 6 & & \end{array} \right)$$



The standardisation of a semistandard skew domino tableau is defined in the same way as for non-skew shapes. We recall the procedure. Consider all boxes filled by 1 and relabel them with successive natural integers  $1, 2, \dots$  from bottom to top and from left to right. Then continue with all boxes filled by 2, by 3, etc.

**Example 2.10.** *The following picture shows a semistandard skew domino tableau  $T$  and its standardisation  $T^{st}$ .*



### 2.8.2 Skew domino functions

*Definition 31.* Given  $\lambda/\mu \in \mathcal{P}^0(n)$  define the **skew domino function**  $\mathcal{G}_{\lambda/\mu}$  as the generating function for the semistandard skew domino tableaux of shape  $\lambda/\mu$ .

$$\mathcal{G}_{\lambda/\mu}(X) = \sum_{T \in SSDT(\lambda/\mu)} X^T.$$

Proposition 2.2 generalises well to the case of skew shapes.

**Proposition 2.7.** *For  $\lambda/\mu \in \mathcal{P}^0(n)$ , the domino function  $\mathcal{G}_{\lambda/\mu}$  and the Schur symmetric functions are related through*

$$\mathcal{G}_{\lambda/\mu}(X) = s_{\lambda^-/\mu^-}(X^-) s_{\lambda^+/\mu^+}(X^+).$$

*Proof.* This identity is a direct consequence of the bijection between semistandard skew domino tableaux of shape  $\lambda/\mu$  and semistandard skew bi-tableaux of bi-shape  $(\lambda^-/\mu^-, \lambda^+/\mu^+)$ .  $\square$

Furthermore, we have the following proposition involving a type B analogue of the Littlewood-Richardson coefficients.

**Proposition 2.8.** *Given  $\lambda/\mu \in \mathcal{P}^0(n)$ ,  $\mu \in \mathcal{P}^0(m)$  and  $\nu \in \mathcal{P}^0(n)$  denote by  $l_{\mu\nu}^\lambda$  the **type B analogues of the Littlewood-Richardson coefficients** that we define as*

$$l_{\mu\nu}^\lambda = k_{\mu^-/\nu^-}^{\lambda^-} k_{\mu^+/\nu^+}^{\lambda^+}.$$

*Domino functions verify*

$$\mathcal{G}_\mu \mathcal{G}_\nu = \sum_{\lambda \in \mathcal{P}^0(n+m)} l_{\mu\nu}^\lambda \mathcal{G}_\lambda, \quad \mathcal{G}_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^0(n)} l_{\mu\nu}^\lambda \mathcal{G}_\nu.$$

*Proof.* Use Propositions 2.2 and 2.7 to show

$$\begin{aligned} \mathcal{G}_\mu(X) \mathcal{G}_\nu(X) &= s_{\mu^-}(X^-) s_{\mu^+}(X^+) s_{\nu^-}(X^-) s_{\nu^+}(X^+) \\ &= \sum_{\lambda^- \vdash |\mu^-| + |\nu^-|} k_{\mu^- \nu^-}^{\lambda^-} s_{\lambda^-}(X^-) \sum_{\lambda^+ \vdash |\mu^+| + |\nu^+|} k_{\mu^+ \nu^+}^{\lambda^+} s_{\lambda^+}(X^+) \\ &= \sum_{\lambda \in \mathcal{P}^0(n+m)} \left( k_{\mu^- \nu^-}^{\lambda^-} k_{\mu^+ \nu^+}^{\lambda^+} \right) \mathcal{G}_\lambda(X), \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_{\lambda/\mu}(X) &= s_{\lambda^-/\mu^-}(X^-) s_{\lambda^+/\mu^+}(X^+) \\ &= \left( \sum_{\nu^- \vdash |\lambda^-| - |\mu^-|} k_{\mu^- \nu^-}^{\lambda^-} s_{\nu^-}(X^-) \right) \left( \sum_{\nu^+ \vdash |\lambda^+| - |\mu^+|} k_{\mu^+ \nu^+}^{\lambda^+} s_{\nu^+}(X^+) \right) \\ &= \sum_{\nu \in \mathcal{P}^0(n)} \left( k_{\mu^- \nu^-}^{\lambda^-} k_{\mu^+ \nu^+}^{\lambda^+} \right) \mathcal{G}_\nu(X). \quad \square \end{aligned}$$

In order to get an analogue of Proposition 2.3, the skew shapes have to verify some additional constraints. As noticed in Remark 2.1 for non-skew shapes, the constraints of not labelling the vertical top leftmost domino of a semistandard domino tableau  $T$  with 0 and the constraint that no negative domino may be labelled with 0 are equivalent. With skew shapes, there is no analogue to this equivalence in the general case and we need to restrict the set of considered shapes. We proceed with the following definitions.

*Definition 32* (Top domino). Given a (semi-)standard domino tableau  $T$ , we say that domino  $d$  is a **top domino** of  $T$  if

1. there is no adjacent domino on the top of  $d$ ,
2. there is no non-top adjacent domino to the left of  $d$ .

One can note that the definition above is recursive, but it is not a problem. The definition is valid, because the status of the domino relies only on the status of the dominos which are to the left or to the top of it. So there will be no loops when we search the status of any domino.

*Remark 2.9.* The number of top dominos  $\nu$  for any shape  $\lambda/\mu \in \mathcal{P}^0(n, m)$ , such that  $\mu + \nu$  gives a valid shape is  $O(m)$ . (In each column there may be only two positions (vertical and horizontal) for such a top domino, plus two positions in the end of the first row. The maximal number of columns is  $2m$ , so the number of such positions is no more than  $4m + 2$ .)

We further say that a top domino  $d$  is **minimum positive** if  $d$  is the leftmost downmost positive top domino of  $T$  such that there is no adjacent (negative top) domino to the left of  $d$ .

There can be no minimum positive domino, for example, there are tableaux with only negative dominos.

**Example 2.11.** In Example 2.7, dominoes labelled with 1, 2, 3 and 8 are top dominoes and 2 is the minimum positive one.

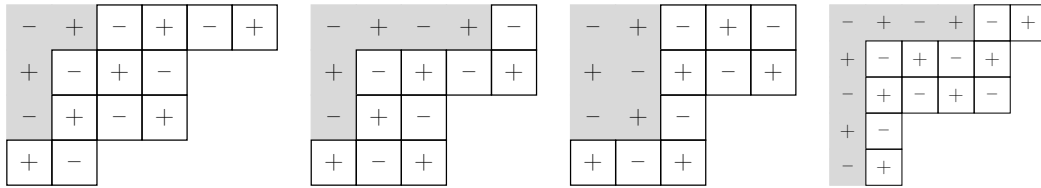
We look at the following subset of  $\mathcal{P}^0(n)$ .

*Definition 33* (Admissible skew shapes). An **admissible skew shape** is a skew shape  $\lambda/\mu \in \mathcal{P}^0(n)$  that cannot be tiled with horizontal and vertical dominos in a way that

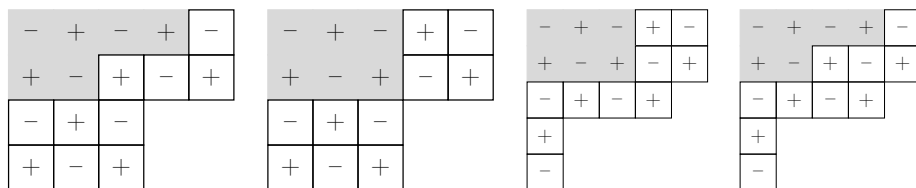
1. there is a minimum positive domino,
2. a negative top domino is placed to the right and above the minimum positive domino.

Denote by  $\tilde{\mathcal{P}}^0(n)$  ( $\tilde{\mathcal{P}}^0(n, m)$ ) the set of admissible skew shapes  $\lambda/\mu$  such that  $|\lambda| - |\mu| = 2n$  ( $|\lambda| = 2n, |\mu| = 2m$ ).

**Example 2.12.** The shape of the standard skew domino tableau in Example 2.7 is admissible as well as the following shapes.



These ones are not admissible.



**Proposition 2.10.** For  $\lambda/\mu \in \tilde{\mathcal{P}}^0(n)$ , the skew domino function  $\mathcal{G}_{\lambda/\mu}$  verify

$$\mathcal{G}_{\lambda/\mu} = \sum_{I \subseteq \{0\} \cup [n-1]} d_{\lambda/\mu I}^B F_I^B.$$

*Proof.* The proof is adapted from the one of Proposition 2.3. The first step is to show the analogue of Lemma 1, i.e. that for any standard skew domino tableau  $T_0$  of shape  $\lambda/\mu$ ,

$$F_{\text{Des}(T_0)}^B = \sum_{T^{st}=T_0} X^T.$$

To this end one needs to show that any sequence of relabelling of the dominoes  $1, 2, \dots, n$  in  $T_0$  by  $i_1 \leq i_2 \leq \dots \leq i_n$  such that  $k \in \text{Des}(T_0) \Rightarrow i_k < i_{k+1}$  with  $k \geq 0$  and  $i_0 = 0$  gives a valid semistandard skew domino tableau  $T$  such that  $T^{st} = T_0$ . If the domino labelled 1 in  $T_0$  is positive, i.e.  $\min(\text{Des}(T_0)) = k > 0$ , then any subsequence  $0 = i_1 = i_2 = \dots = i_l$  ( $l \leq k$ ) gives a valid semistandard skew domino tableau. Indeed, since  $\lambda/\mu$  is admissible, none of the dominoes  $2, 3, \dots, k$  are negative. As a result, no negative domino is labelled with 0 (a restriction in our definition of semistandard skew domino tableaux required to get Proposition 2.7). Other cases are similar to the proof of Lemma 1 and not detailed here.  $\square$

*Remark 5.* Actually the statement of Proposition 2.10 may be reversed. So, if

$$\mathcal{G}_{\lambda/\mu} = \sum_{I \subseteq \{0\} \cup [n-1]} d_{\lambda/\mu I}^B F_I^B.$$

then skew shape  $\lambda/\mu$  is admissible.

## 2.9 Admissible shapes

This section is devoted to the description of admissible shapes. We will show that asymptotically there is a few number of admissible shapes.

**Theorem 2.11.** Given  $n, m$ , such that  $m < n$  we consider numbers  $\frac{|\tilde{\mathcal{P}}^0(n, m)|}{|\mathcal{P}^0(n, m)|}$ . Then for any fixed  $m \in \mathbb{Z}_+$ .

$$\lim_{n \rightarrow \infty} \frac{|\tilde{\mathcal{P}}^0(n, m)|}{|\mathcal{P}^0(n, m)|} = 0.$$

### 2.9.1 Admissible and strictly admissible shapes

Let us consider the image of top dominoes under the Stanton and White's bijection. We define a top cell as follows.

*Definition 34* (Top cell). Given a (semi-)standard Young tableau  $T$ , we say that cell  $d$  is a **top cell** of  $T$  if there is no adjacent cell on the top of  $d$ .

*Remark 2.12.* The Stanton and White's bijection sends a top domino to a top cell.

Given a skew shape  $\lambda/\mu \in \mathcal{P}^0(n, m)$  denote by  $\delta(\lambda/\mu)$  its border. It consists of 4 parts:  $\mu$ -border,  $\lambda$ -border and two straight connecting lines. We consider  $\delta(\lambda/\mu)$  minus the  $\lambda$ -border. It may be presented as a sequence  $\hat{\lambda}/\hat{\mu}$  of angles (from bottom to top), as indicated at Figure 2.10. Obviously, two different skew shapes may have the same sequence of angles. The angles structure of a skew shape  $\lambda/\mu$  depends on the shape  $\mu$  and number of cells in the first row and in the first column in diagram  $\lambda$ .

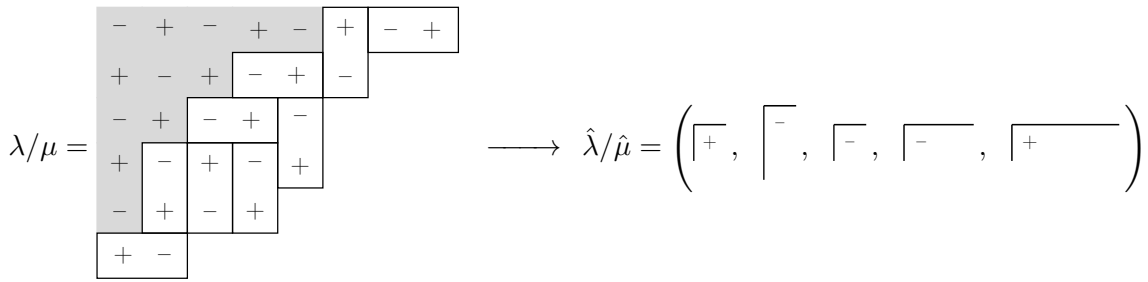










FIGURE 2.10: Skew shape and the corresponding sequence of angles.

All the angles are identified by two lengths  $a$  and  $b$  and the sign  $s$  in the corner. We divide all the angles into nine types.

| Types of angles |                               |   |
|-----------------|-------------------------------|---|
| type 0          | $a = 0$ or $b = 0$            |   |
| type 1          | $a \geq 2, b \geq 2, s = '+'$ |   |
| type 2          | $a = 1, b \geq 2, s = '-'$    |   |
| type 3          | $a \geq 2, b = 1, s = '+'$    |   |
| type 4          | $a = 1, b = 1, s = '+'$       |   |
| type 5          | $a = 1, b = 1, s = '-'$       |   |
| type 6          | $a \geq 2, b \geq 2, s = '-'$ |   |
| type 7          | $a = 1, b \geq 2, s = '+'$    |   |
| type 8          | $a \geq 2, b = 1, s = '-'$    |  |

Type 0 angles may appear only at the beginning or at the end of a sequence of angles  $\hat{\lambda}/\hat{\mu}$ .

*Definition 35.* We say that sequence  $\hat{\lambda}/\hat{\mu}$  is **open on the right side** if the corresponding skew shape  $\lambda/\mu$  have at least two cells in the first row, **open on the bottom** if they have at least two cells in the first column. Note that this definition does not depend on the choice of the skew shape  $\lambda/\mu$ , only on the sequence of angles  $\hat{\lambda}/\hat{\mu}$ . If  $\hat{\lambda}/\hat{\mu}$  is open on the right side and open on the bottom at the same time, we say it is **open**.

Note, if we have type 0 angles in  $\hat{\lambda}/\hat{\mu}$ , then it is closed (not open) on the corresponding side.

*Definition 36.* We say that sequence  $\hat{\lambda}/\hat{\mu}$  is **strictly admissible** ( $\hat{\lambda}/\hat{\mu} \in \widetilde{\mathcal{SP}}^0$ ) if for any  $\lambda/\mu \in \mathcal{P}^0(n, m)$  with angles structure  $\hat{\lambda}/\hat{\mu}$ , the skew shape  $\lambda/\mu$  is admissible.

*Remark 2.13.* If  $\hat{\lambda}/\hat{\mu}$  is strictly admissible then any  $\lambda/\mu \in \mathcal{P}^0(n, m)$  with such an angles structure is admissible. Conversely, consider any shape  $\lambda/\mu \in \widetilde{\mathcal{P}}^0(n, m)$ . There are two options:

- $\hat{\lambda}/\hat{\mu}$  is strictly admissible.
- $\hat{\lambda}/\hat{\mu}$  is not strictly admissible. One may consider an extension of shape  $\mu$  with two top dominoes (forming a skew shape  $\nu$ ) such that one domino is positive and the other is negative and the positive top domino is lower and to the left to the negative top domino. According to the definitions, the fact that  $\lambda/\mu$  is admissible means that for any  $\nu$ , satisfying the constraints above, shape  $\lambda$  could not be obtained from  $\mu + \nu$  with the domino tiling. In other words,  $\lambda/(\mu + \nu) \notin \mathcal{P}^0$ .

This remark allows us to study strictly admissible sequences and then move on to the admissible shapes. Now let us describe the structure of strictly admissible sequences.

**Proposition 2.14.** *Sequence  $\hat{\lambda}/\hat{\mu}$  of angles is strictly admissible if and only if it satisfies the following constraints:*

- *type 0 angles are allowed, they may appear without any constraint, at the beginning or at the end of the sequence.*
- *type 1 angles are strictly prohibited, they do not appear in the sequence.*
- *types 2 – 5 of angles are allowed, they may appear without any constraint,*
- *types 6 – 8 of angles are prohibited after types 1,2,3,6 in the angles' sequence.*

*Proof.* First of all, type 0 angles does not affect the status of the sequence of angles (whether it is strictly admissible or not). Note that each top domino lay in some angle. Type 1 angles can consist a negative top domino to the right of a positive top domino. It means that sequence with such an angle is not strictly admissible. Types 2 – 5 of angles either includes positive top dominoes or does not include top domino at all. Types 6 – 8 can include negative top domino. So, in the sequence, they can not be after angles of type 1,2,3,6 since they can consist positive top domino.  $\square$

*Lemma 2.* There are no open strictly admissible sequences of angles in  $\mathcal{P}^0$ .

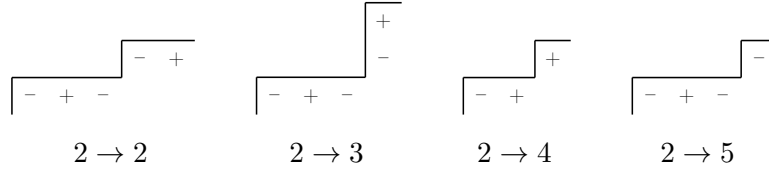
To prove this lemma we:

1. Figure out which transitions in the sequence are allowed and which are not.
2. Find the possible patterns for sequence goes from open on the bottom types of angles (3,6,8) to open on the right side types (2,6,7) such that the corresponding shape is strictly admissible due to constraints.
3. Prove that all the sequences of the patterns above do not give a  $\mathcal{P}^0$  shape.

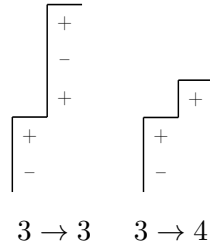
### 2.9.2 Allowed transitions

First of all, note that type 0 angles could appear at the beginning or at the end of the sequence, before or after any angle, without any restriction. We do not consider type 0 angles further in this subsection. The following lemmas concern allowed transitions in a sequence of angles.

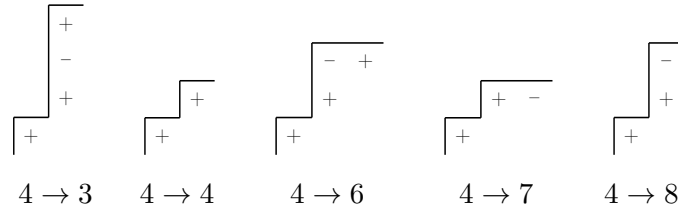
*Lemma 3.* After angles of type 2 there may be angles of types 2, 3, 4, 5. Here are some examples for each case.



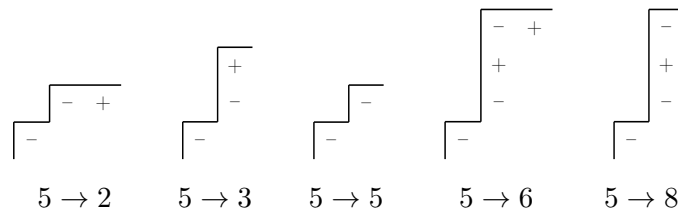
*Lemma 4.* After angles of type 3 there may be angles of types 3, 4. Here are some examples for each case.



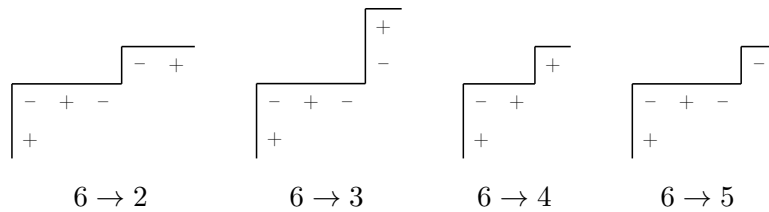
*Lemma 5.* After angles of type 4 there may be angles of types 3, 4, 6, 7, 8. Here are some examples for each case.



*Lemma 6.* After angles of type 5 there may be angles of types 2, 3, 5, 6, 8. Here are some examples for each case.

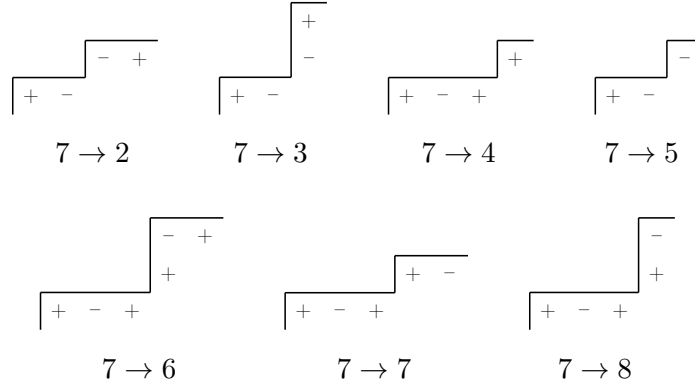


*Lemma 7.* After angles of type 6 there may be angles of types 2, 3, 4, 5. Here are some examples for each case.

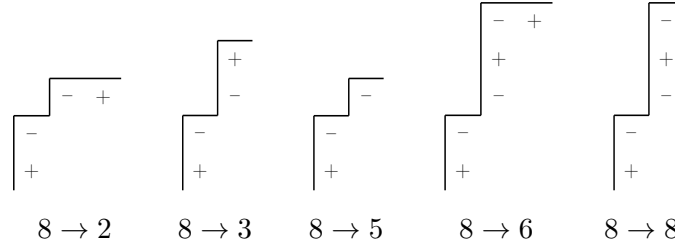




*Lemma 8.* After angles of type 7 there may be angles of types 2, 3, 4, 5, 6, 7, 8. Here are some examples for each case.



*Lemma 9.* After angles of type 8 there may be angles of types 2, 3, 5, 6, 8. Here are some examples for each case.



### 2.9.3 Patterns

In this subsection we present all possible patterns for sequence goes from open on the bottom types of angles (3, 6, 8) to open on the right side types (2, 6, 7) such that the constraints of Proposition 2.14 are fulfilled (we name them **valid** sequences). We will use brute force to write down all of them. Let us start with type 3 angle.

*Lemma 10.* There is no valid sequence  $\hat{\mu} = \hat{\lambda}/\hat{\mu} = (\hat{\mu}_1, \dots, \hat{\mu}_r)$  starting with type 3 and ending with open on the right side types (2, 6, 7).

*Proof.* Since the sequence  $\hat{\mu}$  begins with the angle of type 3, there could not appear types 6, 7, 8 as they are prohibited after the type 3 angle. Types of angles which are available at the second position are exactly 3 and 4. The set of valid types after 4 is  $\{3, 4, 6, 7, 8\}$ , three last types (6, 7 and 8) are prohibited. So, types 3 and 4 are the only types which may appear in  $\hat{\mu}$ , therefore it could not end with open on the right side types of angles.  $\square$

To describe a sequence of angles we will use the following notation. Round brackets  $(\dots)$  correspond to an optional part of a sequence. Square brackets  $[\dots]_i$  correspond to the part of a sequence which should be repeated  $i$  times.

*Lemma 11.* The pattern  $6(\rightarrow [2, 5]_{i \geq 0} \rightarrow 2)$  describes all valid sequences  $\hat{\mu} = \hat{\lambda}/\hat{\mu} = (\hat{\mu}_1, \dots, \hat{\mu}_r)$  starting with type 6 and ending with open on the right side types  $(2, 6, 7)$ .

*Proof.* Let us make the sequence starting from  $type(\hat{\mu}_1) = 6$ . We may finish here at the moment or continue the sequence.

Similarly to the Lemma 10, types 6, 7, 8 are prohibited after type 6, so we may end the sequence only with type 2 angle. In the second place, there may be 2, 3, 4 or 5. Lemma 10 and its proof show that  $\hat{\mu}$  can not consist type 3 or type 4 angles, because it is impossible to continue the sequence further to the end with an open on the right side type of angle. So, we may put only type 2 or type 5 angle in the second place.

After type 2 angle we may put either type 2 or type 5 (3 and 4 are prohibited). After type 5 we may put type 2 or type 5 also (3, 6, 8 are prohibited). Let us summarize, the pattern is type 6 angle, then some nonnegative number of type 2 or type 5 angles and then finish at some moment with type 2 angle:

$$6(\rightarrow [2, 5]_{i \geq 0} \rightarrow 2).$$

□

The element of a sequence labelled by  $*$  will further indicate the position when types 6, 7, 8 become prohibited.

*Lemma 12.* The two following patterns describe all valid sequences  $\hat{\mu} = \hat{\lambda}/\hat{\mu} = (\hat{\mu}_1, \dots, \hat{\mu}_r)$  starting with type 8 and ending with open on the right side types  $(2, 6, 7)$ .

$$\begin{aligned} 8 \rightarrow [8, 5]_{i \geq 0} \rightarrow 6_* (\rightarrow [2, 5]_{j \geq 0} \rightarrow 2) \\ 8 \rightarrow [8, 5]_{i \geq 0} \rightarrow 2_* (\rightarrow [2, 5]_{j \geq 0} \rightarrow 2) \end{aligned}$$

*Proof.* Note that types 6, 7, 8 are allowed until we get type 2, 3 or 6. First of all, let us consider the part of the sequence before these types.

Consider a sequence  $\hat{\mu}$  starting from  $type(\hat{\mu}_1) = 8$ . In the second place, there may be 2, 3, 5, 6 or 8. In Lemma 10 it is showed that  $\hat{\mu}$  can not consist type 3 angle. If  $type(\hat{\mu}_2) = 5$  we get the same set of allowed types 2, 3, 5, 6 and 8 at the third place. So, we may get any sequence of type 5 and type 8 angles before we get either type 2 or type 6. This part of the sequence corresponds to the pattern

$$8 \rightarrow [8, 5]_{i \geq 0}.$$

As soon as we got type 2 or type 6 we may either finish the sequence or continue. Since that moment types 6, 7, 8 are prohibited. Both types 2 and 6 have the same set of allowed types after them. So, Lemma 11 gives us all patterns starting with 2 and 6 and ending with open on the right side types.  $\square$

The following subsection gives a detailed construction of the patterns from this one.

### 2.9.4 Detailed information about possible patterns.

Let us start with the pattern from Lemma 11. Consider the sequence of angles  $\hat{\mu}$  of the pattern  $6 \rightarrow [2, 5]_{i \geq 0} \rightarrow 2$ .

If there is no optional part and  $\hat{\mu}$  consists of one angle of type 6 then  $\lambda/\mu$  is actually non-skew shape.

Let us show now that for any sequence of angles  $\hat{\mu} = \hat{\lambda}/\hat{\mu}$  of pattern  $6 \rightarrow [2, 5]_{i \geq 0} \rightarrow 2$  the corresponding shape  $\mu$  does not lay in  $\mathcal{P}^0(m)$ .

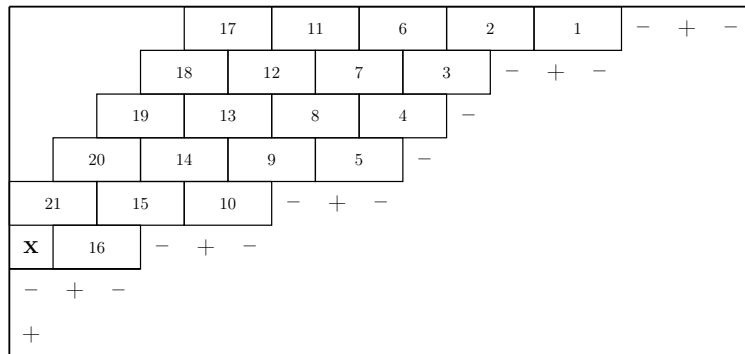


FIGURE 2.11: Sequence  $\hat{\mu} = 6 \rightarrow 2 \rightarrow 2 \rightarrow 5 \rightarrow 5 \rightarrow 2 \rightarrow 2$  and the only tiling of  $\mu$ . Numbers indicate the order of adding dominoes in the tiling.

*Lemma 13.* Pattern  $6 \rightarrow [2, 5]_{i \geq 0} \rightarrow 2$  correspond to shapes  $\mu$  which do not lay in  $\mathcal{P}^0(m)$ .

*Proof.* Consider the sequence of angles  $\hat{\mu}$  as sequence of right and up steps. Note, that it consists of steps to the right by 1 cell and steps up by odd number of cells. So, we may till the shape  $\mu$  from right to left with dominoes in the only possible way (with horizontal dominoes). Figure 2.11 shows an example of such tiling.

Note that we could not till the bottom left corner because of parity reasons. It means  $\mu$  does not lay in  $\mathcal{P}^0(m)$ .  $\square$

We proceed with pattern  $8 \rightarrow [8, 5]_{i \geq 0} \rightarrow 6_* (\rightarrow [2, 5]_{j \geq 0} \rightarrow 2)$ .

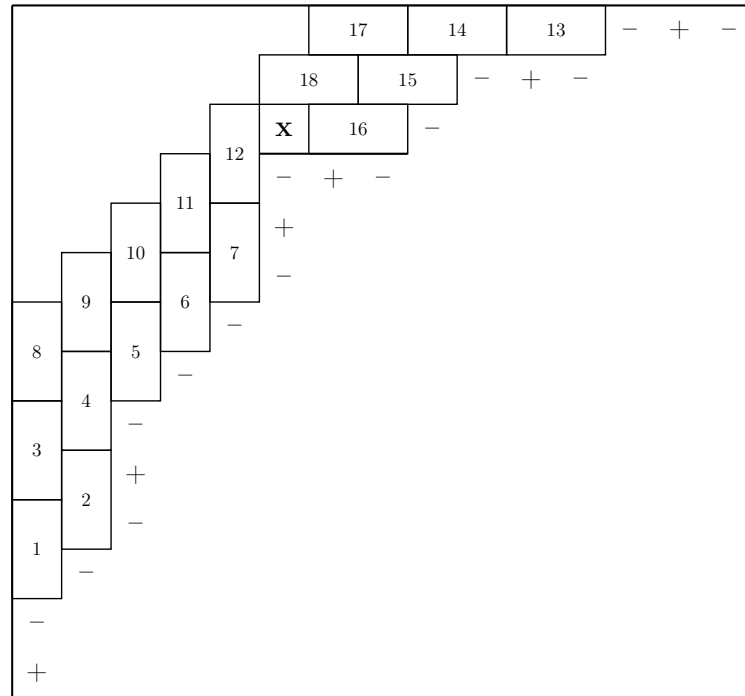


FIGURE 2.12: Sequence  $\hat{\mu} = 8 \rightarrow 5 \rightarrow 8 \rightarrow 5 \rightarrow 5 \rightarrow 6_* \rightarrow 5 \rightarrow 2 \rightarrow 2$  and the only tilling of  $\mu$ . Numbers indicate the order of adding dominoes in the tilling.

*Lemma 14.* Pattern  $8 \rightarrow [8, 5]_{i \geq 0} \rightarrow 6_* (\rightarrow [2, 5]_{j \geq 0} \rightarrow 2)$  correspond to shapes  $\mu$  which do not lay in  $\mathcal{P}^0(m)$ .

*Proof.* It is not hard to see that  $\hat{\mu}$  is divided into two parts (by corner of the type 6 angle). In the first part, we have steps up by 1 cell and steps to the right by an odd number of cells. In the second steps to the right by 1 cell and steps up by an odd number of cells. The second part is optional.

We will fill this shape starting with the first part from bottom to top with dominoes in the only possible way (with vertical dominoes). Then, if needed we fill the second part in the same way as in Lemma 13. Figure 2.13 shows an example of such tiling.

Note that we could not tile the top right corner if we do not have the optional part because of parity reasons. If we have the optional part, we get problems with the cell on the corner of type 6 angle. It means  $\mu$  does not lay in  $\mathcal{P}^0(m)$ .  $\square$

*Lemma 15.* Pattern  $8 \rightarrow [8, 5]_{i \geq 0} \rightarrow 2_*$  ( $\rightarrow [2, 5]_{j \geq 0} \rightarrow 2$ ) correspond to shapes  $\mu$  which do not lay in  $\mathcal{P}^0(m)$ .

*Proof.* The same as for Lemma 14. □

The three lemmas above together form the proof of Lemma 2.

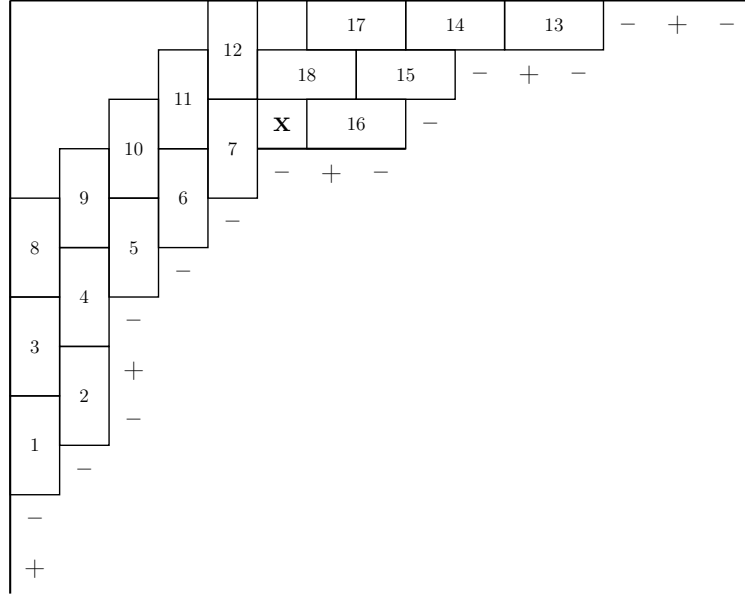


FIGURE 2.13: Sequence  $\hat{\mu} = 8 \rightarrow 5 \rightarrow 8 \rightarrow 5 \rightarrow 5 \rightarrow 2_* \rightarrow 5 \rightarrow 2 \rightarrow 2$  and the only tilling of  $\mu$ . Numbers indicate the order of adding dominoes in the tilling.

### 2.9.5 Estimations

In this subsection, we estimate the number of admissible shapes over all  $\mathcal{P}^0$  shapes. We use the structure of strictly admissible sequences and Remark 2.13 to move on to the admissible shapes. More precisely, it is already proved that strictly admissible sequences could not be open. It actually means that the width of such a shape (in its narrow part) is quite small. The percentage of such shapes is small too. Remark 2.13 allows us to estimate the number of all the admissible shapes which correspond to non strictly admissible sequences.

Let us precise what to mean by width of a skew shape  $\lambda/\mu$ . There is a number of ways how to define this notion. One way is to consider the minimal number of cells that needed to connect a cell of shape  $\mu$  to a cell out of shape  $\lambda$ . Proposition 2.15 below would be true for such a definition. However, further we need the width of a domino tableau to be related to the width of bi-tableau. So, it is better to use the less essential definition of the width for all types of skew diagram.

*Definition 37.* Given a skew shape  $\lambda/\mu \in \mathcal{P}^0(n, m)$  of a domino diagram, denote by  $wd(\lambda/\mu)$  the **width** of the shape as minimal of the number of cells on the diagonals  $d_l$ , such that  $d_l$  crosses both  $\lambda$  and  $\mu$  shapes. Recall,  $d_l$  denotes a diagonal line  $(i, j) : i = j - 2l$ . Denote by  $\mathcal{R}_k^0(n, m)$  the subset of  $\mathcal{P}^0(n, m)$  consists of skew shapes with width at least  $k$ .

*Definition 38.* Speaking about Young diagram  $\lambda/\mu$  we also count **width** as minimal of the number of cells on the diagonals  $d_l$ , such that  $d_l$  crosses both  $\lambda$  and  $\mu$ . This time diagonal  $d_l$  is the line  $(i, j) : i = j - l$ .

*Definition 39.* Given a bi-shape  $(\lambda^+/\mu^+, \lambda^-/\mu^-)$ , we count **width** of each part as an Young diagram

$$wd(\lambda^+/\mu^+, \lambda^-/\mu^-) = (wd(\lambda^+/\mu^+), wd(\lambda^-/\mu^-)).$$

In such definitions, if the sequence  $\hat{\lambda}/\hat{\mu}$  has type 0 angle, then  $wd(\lambda/\mu) = 0$ . The closed from some side sequence leads to  $wd(\lambda/\mu) \leq 1$ .

The following proposition states that the number of shapes which have a big width is much greater than number of shapes with a small width.

**Proposition 2.15.** *Fix  $m$ . Then, for any nonnegative integer  $k$ ,*

$$|\bar{\mathcal{R}}_k^0(n, m)| = |\mathcal{P}^0(n, m)| - |\mathcal{R}_k^0(n, m)| = o(|\mathcal{R}_k^0(n, m)|).$$

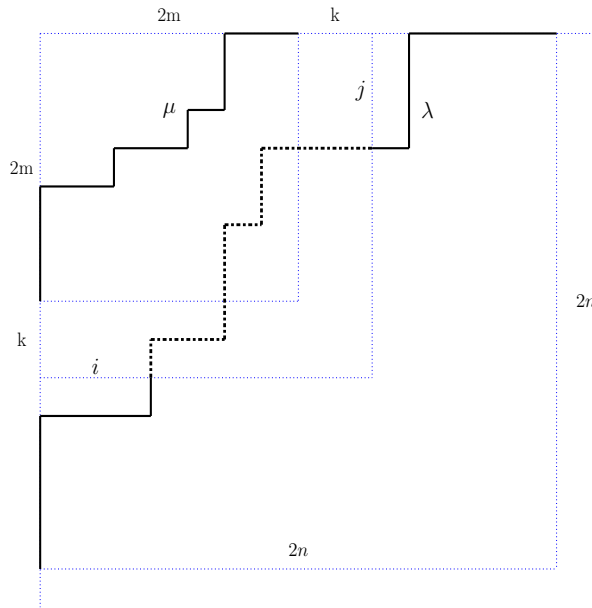


FIGURE 2.14: Shape of width less than  $k$ .

*Proof.* We start with evaluation of an upper bound for  $|\bar{\mathcal{R}}_k^0(n, m)|$ . Suppose  $2n$  is much greater than  $2m + k$ . Consider a skew shape  $\lambda/\mu \in \mathcal{P}^0(n, m)$ . The diagram  $\mu \vdash 2m$  lays in a square  $2m \times 2m$  while diagram  $\lambda \vdash 2n$  lays in a square  $2n \times 2n$ . In the case  $\lambda/\mu \in \bar{\mathcal{R}}_k^0(n, m)$  the diagram  $\lambda$  intersect with the square  $(2m+k) \times (2m+k)$ . We consider the diagram  $\lambda$  as a path from the bottom-left corner of the square  $2n \times 2n$  to its top-right corner. Counting the paths which have an intersection with the square  $(2m+k) \times (2m+k)$  gives an upper bound. There will be some extra skew shapes, which do not lay in  $\bar{\mathcal{R}}_k^0(n, m)$ , however, it does not matter for the proof. A path consists of three parts (see Figure 2.14):

- from the beginning to the first intersection with square  $(2m+k) \times (2m+k)$ ,
- part of paths which lay in the square  $(2m+k) \times (2m+k)$ ,
- from the last intersection with square  $(2m+k) \times (2m+k)$  to the top-right corner.

Let  $(i, 2m+k)$  and  $(2m+k, j)$  be the coordinates of the first and last intersection with the square  $(2m+k) \times (2m+k)$ . So, the first part of  $\lambda$  is a path from  $(0, 2n)$  to  $(i, 2m+k)$ . The number of paths from  $(0, 2n)$  to  $(i, 2m+k)$  is  $\binom{2n-(2m+k)+i}{i}$ . Similarly, the number of path from  $(2m+k, j)$  to  $(2n, 0)$  is  $\binom{2n-(2m+k)+j}{j}$ . The second part of the path consists of  $2m+k-i$  right steps and  $2m+k-j$  up steps, so it is bounded by  $\binom{4m+2k-i-j}{2m+k-j}$ .

The number of options for  $\mu$  is  $|\mathcal{P}^0(m)|$  which is a constant for any fixed  $m$ . So,

$$|\bar{\mathcal{R}}_k^0(n, m)| \leq |\mathcal{P}^0(m)| \cdot \sum_{i,j=0}^{2m+k} \binom{2n-(2m+k)+i}{i} \binom{4m+2k-i-j}{2m+k-j} \binom{2n-(2m+k)+j}{j}.$$

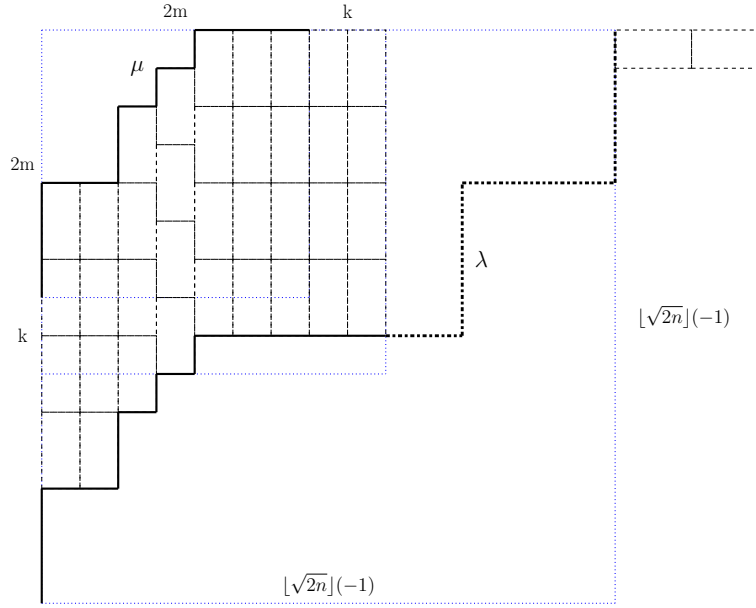
We estimate the binomial coefficients using that  $2n$  is much greater than  $2m+k$  and then use Stirling's approximation to finish estimations.

$$\begin{aligned} |\bar{\mathcal{R}}_k^0(n, m)| &\leq |\mathcal{P}^0(m)| \cdot (2m+k+1)^2 \binom{2n}{2m+k} \binom{4m+2k}{2m+k} \binom{2n}{2m+k} \sim \\ &\sim \text{Const}_1(m) \left[ \frac{(2n)^{2n} \sqrt{2\pi(2n)}}{(2m+k)^{2m+k} \sqrt{2\pi(2m+k)} (2n-(2m+k))^{2n-(2m+k)} \sqrt{2\pi(2n-(2m+k))}} \right]^2 \sim \\ &\sim \text{Const}_2(m) \left[ \frac{(2n)^{2n}}{(2n)^{2n-(2m+k)} \left(1 - \frac{(2m+k)}{2n}\right)^{2n-(2m+k)}} \right]^2 \\ &\sim \text{Const}_3(m) \cdot (2n)^{4m+2k}. \end{aligned}$$

Now let us evaluate the lower bound for  $|\mathcal{R}_k^0(n, m)|$ .

Given a shape  $\mu$  and suppose  $n$  is a positive integer such that  $\sqrt{2n}$  is much greater than  $2m+k$ . Construct shapes  $\lambda$  such that  $\lambda/\mu \in \mathcal{R}_k^0(n-m)$  with the following algorithm. This algorithm is valid for even  $k$ , but it is not hard to do the same for odd.

1. Consider a square  $\lfloor \sqrt{2n} \rfloor \times \lfloor \sqrt{2n} \rfloor$ . Note that all paths from the bottom-left corner to the top-right corner correspond to the diagram with no more than  $2n$  cells.

FIGURE 2.15: Construction of valid  $\lambda$ .

2. Either  $\lfloor \sqrt{2n} \rfloor$  or  $\lfloor \sqrt{2n} \rfloor - 1$  is even, denote this number by  $t$ . Add  $\frac{t}{2} - m$  vertical dominoes to shape  $\mu$  for  $2m + k$  first columns. That vertical dominoes gives the  $2m + k$  first columns of  $\lambda/\mu$ .
3. Proceed shape  $\lambda$  to the top-right corner using either 2 "up" steps or two "right" steps. So, columns of  $\lambda/\mu$  still have even number of cells and may be divided into dominoes. At the moment one got shape  $\lambda$  with a number of dominoes no more than  $n$ .
4. Finally, add additional cells to the first row in order to get exactly  $n$  dominoes in shape  $\lambda$ .

Suppose  $2k' = k$  and  $2t' = t$ . According to the point 3 of the algorithm, one makes  $t' - m$  up steps and  $t' - m - k'$  right steps, the length of each step is 2. Compute the number of shapes one got with this procedure.



$$\begin{aligned}
|\mathcal{R}_k^0(n, m)| &\geq |\mathcal{P}^0(m)| \cdot \binom{2t' - (2m + k')}{t' - m} \sim \\
&\sim |\mathcal{P}^0(m)| \frac{(2t' - (2m + k'))^{2t' - (2m + k')} \sqrt{2\pi(2t' - (2m + k'))}}{(t' - m)^{(t' - m)} \sqrt{2\pi(t' - m)} \cdot (t' - (m + k'))^{t' - (m + k')} \sqrt{2\pi(t' - (m + k'))}} \\
&\sim \text{Const}_4(m) \frac{(2\sqrt{\frac{n}{2}} - (2m + k'))^{2\sqrt{\frac{n}{2}} - (2m + k')}}{(\sqrt{\frac{n}{2}} - m)^{(\sqrt{\frac{n}{2}} - m)} \sqrt{2\pi(\sqrt{\frac{n}{2}} - m)} \cdot (\sqrt{\frac{n}{2}} - (m + k'))^{\sqrt{\frac{n}{2}} - (m + k')} \sqrt{2\pi(\sqrt{\frac{n}{2}} - (m + k'))}} \\
&\sim \frac{\text{Const}_4(m)}{\sqrt{2\pi(\sqrt{\frac{n}{2}} - m)} (\sqrt{\frac{n}{2}})^{(\sqrt{\frac{n}{2}} - m)} (1 - \frac{m}{\sqrt{\frac{n}{2}}})^{(\sqrt{\frac{n}{2}} - m)} (\sqrt{\frac{n}{2}})^{\sqrt{\frac{n}{2}} - (m + k')} (1 - \frac{m + k'}{\sqrt{\frac{n}{2}}})^{\sqrt{\frac{n}{2}} - (m + k')}} \frac{(2\sqrt{\frac{n}{2}})^{2\sqrt{\frac{n}{2}} - (2m + k')}}{2\sqrt{\frac{n}{2}}} \\
&\sim \frac{\text{Const}_5(m)}{\sqrt[4]{n}} \frac{(2\sqrt{\frac{n}{2}})^{2\sqrt{\frac{n}{2}} - (2m + k')}}{(\sqrt{\frac{n}{2}})^{(\sqrt{\frac{n}{2}} - m)} (\sqrt{\frac{n}{2}})^{\sqrt{\frac{n}{2}} - (m + k')}} \\
&\sim \frac{\text{Const}_6(m)}{\sqrt[4]{n}} \cdot 2^{\sqrt{2n}}.
\end{aligned}$$

Finally, it is not hard to see that  $|\bar{\mathcal{R}}_k^0(n, m)| = o(|\mathcal{R}_k^0(n, m)|)$ .

□

*Remark 2.16.* One can easily modify the proof above to the case of bi-shapes.

**Proposition 2.17.** *Fix any positive integer  $m$  and  $\mu \in \mathcal{P}^0(m)$ . Fix also  $\nu$  some 2-domino 'top' extension of  $\mu$  (i.e. extension with one positive and one negative top domino). Then for all  $k > 2$*

$$\#\{\lambda : \lambda/\mu \in \mathcal{R}_k^0(n, m), \lambda/(\mu + \nu) \notin \mathcal{P}^0(n, m + 2)\} = o(\#\{\lambda : \lambda/\mu \in \mathcal{R}_k^0(n, m)\}),$$

when  $n \rightarrow \infty$

*Proof.* Consider the bijection between skew shapes of  $\mathcal{P}^0(n, m)$  and bi-shapes. One can note that  $\lambda \geq \mu$  implies  $\lambda^- \geq \mu^-$  and  $\lambda^+ \geq \mu^+$ . Top dominoes goes to top cells in the corresponding bi-shape. The fact that  $\lambda/(\mu + \nu) \notin \mathcal{P}^0(n, m + 2)$  actually means that one of such top cells does not lay in  $\lambda$ . So, either  $wd(\lambda^-/\mu^-) = 0$  or  $wd(\lambda^+/\mu^+) = 0$ . Note that if  $\lambda/\mu \in \mathcal{P}^0(n, m)$  and

$$wd(\lambda^-/\mu^-) + wd(\lambda^+/\mu^+) \geq k.$$

then  $\lambda/\mu \in \mathcal{R}_k^0(n, m)$ . So,

$$\#\{\lambda : \lambda/\mu \in \mathcal{P}^0(n, m) \text{ and } wd(\lambda^-/\mu^-) + wd(\lambda^+/\mu^+) \geq k\} \leq \#\{\lambda : \lambda/\mu \in \mathcal{R}_k^0(n, m)\}.$$

Now it is not hard to show that

$$\begin{aligned}
& \#\{\lambda : \lambda/\mu \in \mathcal{R}_k^0(n, m), wd(\lambda^-/\mu^-) = 0 \text{ or } wd(\lambda^+/\mu^+) = 0\} \leq \\
& \quad \#\{\lambda : \lambda/\mu \in \mathcal{P}^0(n, m), wd(\lambda^-/\mu^-) = 0 \text{ or } wd(\lambda^+/\mu^+) = 0\} \leq \\
& \quad \#\{\lambda : \lambda^- \geq \mu^-, \lambda^+ \geq \mu^+, wd(\lambda^-/\mu^-) = 0\} + \#\{\lambda : \lambda^- \geq \mu^- \text{ and } \lambda^+ \geq \mu^+, wd(\lambda^+/\mu^+) = 0\} = \\
& \quad o(\#\{\lambda^- : \lambda^- \geq \mu^-, wd(\lambda^-/\mu^-) \geq k\}) \cdot (\#\{\lambda^+ : \lambda^+ \geq \mu^+\}) + \\
& \quad o(\#\{\lambda^+ : \lambda^+ \geq \mu^+, wd(\lambda^+/\mu^+) \geq k\}) \cdot (\#\{\lambda^- : \lambda^- \geq \mu^-\}) = \\
& \quad o(\#\{\lambda : \lambda/\mu \in \mathcal{P}^0(n, m) \text{ and } wd(\lambda^-/\mu^-) + wd(\lambda^+/\mu^+) \geq k\}) = \\
& \quad o(\#\{\lambda : \lambda/\mu \in \mathcal{R}_k^0(n, m)\}).
\end{aligned}$$

This is exactly what we needed.  $\square$

Now we are ready to prove Theorem 2.11.

*Proof.* Fix some  $k > 2$ . Restrict our set on shapes with big width using Lemma 2.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{|\tilde{\mathcal{P}}^0(n, m)|}{|\mathcal{P}^0(n, m)|} &= \lim_{n \rightarrow \infty} \frac{\left[ |\tilde{\mathcal{P}}^0(n, m)| - o(|\mathcal{R}_k^0(n, m)|) \right] + o(|\mathcal{R}_k^0(n, m)|)}{|\mathcal{R}_k^0(n, m)| + o(|\mathcal{R}_k^0(n, m)|)} = \\
&= \lim_{n \rightarrow \infty} \frac{\left[ |\tilde{\mathcal{P}}^0(n, m)| - |\tilde{\mathcal{P}}^0(n, m) \cap \overline{\mathcal{R}_k^0(n, m)}| \right]}{|\mathcal{R}_k^0(n, m)|} = \\
&= \lim_{n \rightarrow \infty} \frac{|\tilde{\mathcal{P}}^0(n, m) \cap \mathcal{R}_k^0(n, m)|}{|\mathcal{R}_k^0(n, m)|}.
\end{aligned}$$

Proposition 2.17 states that

$$\#\{\lambda : \lambda/\mu \in \mathcal{R}_k^0(n, m), \lambda/(\mu + \nu) \notin P^0(n, m + 2)\} = o(\#\{\lambda : \lambda/\mu \in \mathcal{R}_k^0(n, m)\}).$$

Take a sum over all  $\mu \in \mathcal{P}^0(m)$  and their 'top' extensions  $\nu$ . Further a pair  $(\mu, \lambda)$  is used only for skew shape  $\lambda/\mu$  with the constraint  $\mu \leq \lambda$  and a triple  $(\nu, \mu, \lambda)$  corresponds to skew shape and its 'top' extension. Since the number of 'top' extensions is no more than  $\binom{4(m+2)+2}{2}$  we get

$$\#\{(\nu, \mu, \lambda) : \lambda/\mu \in \mathcal{R}_k^0(n, m), \lambda/(\mu + \nu) \notin P^0(n, m + 2)\} = o(\#\{(\mu, \lambda) : \lambda/\mu \in \mathcal{R}_k^0(n, m)\}).$$

Consider the numerator of the fraction,  $|\tilde{\mathcal{P}}^0(n, m) \cap \mathcal{R}_k^0(n, m)|$ . It is the number of pairs  $(\mu, \lambda)$  such that the minimal distance between borders is not less  $k$  and  $\lambda/\mu \in \tilde{\mathcal{P}}^0(n, m)$ .

Remark 2.13 states that either  $\hat{\lambda}/\hat{\mu} \in \widetilde{\mathcal{SP}}^0(m)$  or for some 2-domino 'top' extension  $\lambda/(\mu + \nu) \notin \mathcal{P}^0$ . So,

$$\begin{aligned}
|\widetilde{\mathcal{P}}^0(n, m) \cap \mathcal{R}_k^0(n, m)| &= \\
&\sum_{\hat{\lambda}/\hat{\mu} \in \widetilde{\mathcal{SP}}^0(m)} \#\{(\mu, \lambda) : \lambda/\mu \in \mathcal{R}_k^0(n, m)\} + \sum_{\hat{\lambda}/\hat{\mu} \notin \widetilde{\mathcal{SP}}^0(m)} \#\{(\mu, \lambda) : \lambda/\mu \in \mathcal{R}_k^0(n, m)\} \\
&= 0 + \#\{(\mu, \lambda) : \hat{\lambda}/\hat{\mu} \notin \widetilde{\mathcal{SP}}^0(m), \lambda/\mu \in \mathcal{R}_k^0(n, m)\} \\
&= \#\{(\mu, \lambda) : \exists \nu, |\nu| = 2, \lambda/(\mu + \nu) \notin \mathcal{P}^0(n, m + 2), \lambda/\mu \in \mathcal{R}_k^0(n, m)\} \\
&\leq \#\{(\nu, \mu, \lambda) : \lambda/(\mu + \nu) \notin \mathcal{P}^0(n, m + 2), \lambda/\mu \in \mathcal{R}_k^0(n, m)\} \\
&= o(\#\{(\mu, \lambda) : \lambda/\mu \in \mathcal{R}_k^0(n, m)\}) = o(|\mathcal{R}_k^0(n, m)|).
\end{aligned}$$

This finishes the proof of Theorem 2.11

□

## Chapter 3

# Structure constants of the descent algebra of type B, type B Kronecker and Littlewood Richardson coefficients

This chapter is devoted to generalisation of the results from Section 1.8. First of all, we prove an analogue of Cauchy identity for domino functions. Further we use it to compute structure constants of the Solomon's descent algebra of the hyperoctahedral group  $B_n$ , i.e. the Coxeter group of type B.

More precisely, the idea is to take Cauchy formula for domino functions and, as in case of type A, decompose both sides of the equality using Chow's fundamental and monomial functions. Since we know they are bases of Chow's quasisymmetric functions, we derive the coincidence of coefficients in the decomposition. It links structure constants,  $d_{\lambda I}^B$  and Kronecker coefficients of  $B_n$ .

### 3.1 Type B Descent algebra

We denote by  $D_I^B$  (resp.  $B_I^B$ ) the formal sum in  $\mathbb{C}B_n$  of the signed permutations  $\pi$  with  $\text{Des}(\pi) = I$  (resp.  $\text{Des}(\pi) \subseteq I$ ). Let

$$c_{I_1 I_2 \dots I_p}^K = [D_K^B] \prod_{j=1}^p D_{I_j}^B; \quad e_{I_1 I_2 \dots I_p}^K = [B_K^B] \prod_{j=1}^p B_{I_j}^B$$

be the structure constants of Solomon's descent algebra of type B for  $I_1, \dots, I_p, K \subseteq \{0\} \cup [n-1]$ .

*Remark 3.1.* These constants verify

$$\sum_{I' \subseteq I, J' \subseteq J} c_{I'J'}^K = \sum_{K' \supseteq K} e_{IJ}^{K'}; \quad \sum_{I' \subseteq I, J' \subseteq J, K' \subseteq K} c_{I'J'K'}^T = \sum_{T' \supseteq T} e_{IJK}^{T'}. \quad (3.1)$$

One can also note the following analogue of Remark 1.4.

*Remark 3.2.* The structure constants of Solomon descent algebra of a symmetric group verify the following identity.

$$c_{IJK}^\emptyset = \sum_L c_{IL}^\emptyset c_{JK}^L.$$

Recall, there is a natural analogue of the RSK-correspondence for signed permutations involving **domino tableaux**. Barbash and Vogan ([BV82]) built a bijection between signed permutations of  $B_n$  and pairs of standard domino tableaux of equal shape in  $\mathcal{P}^0(n)$ . Taşkın ([Tas12]) showed that the two standard domino tableaux associated to a signed permutation  $\pi$  by the algorithm of Barbash and Vogan have respective descent sets  $\text{Des}(\pi^{-1})$  and  $\text{Des}(\pi)$ . As a result, we have the following analogue to Equation (1.14).

$$c_{IJ}^\emptyset = \sum_{\lambda \in \mathcal{P}^0(n)} d_{\lambda I}^B d_{\lambda J}^B. \quad (3.2)$$

Let  $Y$  be a second copy of  $X$ . Chow uses the theory of  $P$ -partitions of type B to show that for  $K \subseteq \{0\} \cup [n-1]$

$$F_K^B(XY) = \sum_{I, J \subseteq \{0\} \cup [n-1]} c_{IJ}^K F_I^B(X) F_J^B(Y). \quad (3.3)$$

Equation (3.2) is a consequence of Equation (3.3).<sup>4</sup>

**Theorem 3.3.** *Let  $I, J \subseteq \{0\} \cup [n-1]$ , then*

$$c_{IJ}^\emptyset = \sum_{\lambda \in \mathcal{P}^0(n)} d_{\lambda I}^B d_{\lambda J}^B. \quad (3.4)$$

## 3.2 Cauchy formula and Kronecker coefficients

In this section we show that domino functions satisfy the same relation as Schur functions.

The irreducible characters of  $B_n$  are naturally indexed by bi-partitions of  $n$ , thus by partitions of  $\mathcal{P}^0(n)$  also. Denote by  $\psi^\lambda$  the character indexed by  $\lambda \in \mathcal{P}^0(n)$  and define the **Kronecker coefficients of  $B_n$**  as  $g^B(\lambda, \mu, \nu) = 1/|B_n| \sum_{\omega \in B_n} \psi^\lambda(\omega) \psi^\mu(\omega) \psi^\nu(\omega)$ .

**Proposition 3.4.** *The domino functions  $\mathcal{G}_\lambda(XY)$  verify the following identities*

$$\begin{aligned} \mathcal{G}_{(2n)}(XY) &= \sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_\lambda(X) \mathcal{G}_\lambda(Y), \\ \mathcal{G}_\lambda(XY) &= \sum_{\mu, \nu \in \mathcal{P}^0(n)} g^B(\lambda, \mu, \nu) \mathcal{G}_\mu(X) \mathcal{G}_\nu(Y). \end{aligned}$$

*Proof.* Note that

$$(XY)^+ = \{x_i y_j\}_{(i,j) \geq (0,0)} = \{x_0 y_j\}_{j \geq 0} \cup \{x_i y_j\}_{i > 0, j \in \mathbb{Z}} = X^+ Y^+ \cup X^- Y^-.$$

Using Proposition 2.2 we get the first identity.

$$\begin{aligned} \mathcal{G}_{(2n)}(XY) &= s_{(n)}((XY)^+) \\ &= s_{(n)}(X^- Y^- \cup X^+ Y^+) \\ &= \sum_{k=0}^n s_{(k)}(X^- Y^-) s_{(n-k)}(X^+ Y^+) \\ &= \sum_{k=0}^n \sum_{\lambda^- \vdash k} s_{\lambda^-}(X^-) s_{\lambda^-}(Y^-) \sum_{\lambda^+ \vdash n-k} s_{\lambda^+}(X^+) s_{\lambda^+}(Y^+) \\ &= \sum_{\lambda \in \mathcal{P}^0(n)} s_{\lambda^-}(X^-) s_{\lambda^+}(X^+) s_{\lambda^-}(Y^-) s_{\lambda^+}(Y^+) \\ &= \sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_\lambda(X) \mathcal{G}_\lambda(Y). \end{aligned}$$

Using Proposition 2.2 and the theory of symmetric functions on wreath products we prove the second identity. Define as in [AAER17] for any set of indeterminates  $U$  and  $V$  and any partition  $\lambda$  of  $n$ :

$$p_\lambda^+(U, V) = \prod_i [p_{\lambda_i}(U) + p_{\lambda_i}(V)], \quad p_\lambda^-(U, V) = \prod_i [p_{\lambda_i}(U) - p_{\lambda_i}(V)].$$

Note that if  $U \cap V = \emptyset$ ,  $p_\lambda^+(U, V) = p_\lambda(U \cup V)$  and if  $V \subseteq U$ ,  $p_\lambda^-(U, V) = p_\lambda(U \setminus V)$ . Then for partitions  $\lambda \in \mathcal{P}^0(n)$  and  $\mu \in \mathcal{P}^0(n)$  of 2-quotient  $(\mu^-, \mu^+)$ , denote by  $\psi_\mu^\lambda$  the common value of character  $\psi^\lambda$  on all signed permutations of *cycle type*  $(\mu^-, \mu^+)$ . According to [Mac99, I, Appendix B],

$$p_{\mu^-}^-(U, V) p_{\mu^+}^+(U, V) = \sum_{\lambda \in \mathcal{P}^0(n)} \psi_\mu^\lambda s_{\lambda^-}(V) s_{\lambda^+}(U).$$

Set  $U = X^+$  and  $V = X^-$  and use Proposition 2.2 to get for any  $\mu \in \mathcal{P}^0(n)$

$$p_{\mu^+}(X)(x_0)^{n-|\mu^+|} = \sum_{\lambda \in \mathcal{P}^0(n)} \psi_\mu^\lambda \mathcal{G}_\lambda(X).$$

But  $p_{\mu^+}(XY)(x_0 y_0)^{n-|\mu^+|} = p_{\mu^+}(X)(x_0)^{n-|\mu^+|} p_{\mu^+}(Y)(y_0)^{n-|\mu^+|}$ . We get

$$\sum_{\lambda \in \mathcal{P}^0(n)} \psi^\lambda \mathcal{G}_\lambda(XY) = \sum_{\mu, \nu \in \mathcal{P}^0(n)} \psi^\mu \psi^\nu \mathcal{G}_\mu(X) \mathcal{G}_\nu(Y).$$

That yields the second identity.  $\square$

### 3.3 Structure constants of Solomon's descent algebra of the hyperoctahedral group.

A more general form of Theorem 3.3 can be stated as follows.

**Theorem 3.5.** *Let  $I, J$  and  $K \subseteq \{0\} \cup [n-1]$ . The following formula holds*

$$c_{IJK}^\emptyset = \sum_{\lambda, \mu, \nu \in \mathcal{P}^0(n)} g^B(\lambda, \mu, \nu) d_{\lambda I}^B d_{\mu J}^B d_{\nu K}^B. \quad (3.5)$$

Both Theorem 3.3 and Theorem 3.5 are consequences of Equation (3.3) and Proposition 3.4.

*Proof of Theorem 3.3.* Decompose the domino functions  $\mathcal{G}_{(2n)}(XY)$  in two ways.

$$\begin{aligned} \mathcal{G}_{(2n)}(XY) &= F_\emptyset^B(XY) = \sum_{I, J} c_{IJ}^\emptyset F_I(X) F_J(Y), \\ \mathcal{G}_{(2n)}(XY) &= \sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_\lambda(X) \mathcal{G}_\lambda(Y) = \sum_{I, J} \sum_{\lambda \vdash n} d_{\lambda I}^B d_{\lambda J}^B F_I^B(X) F_J^B(Y). \end{aligned} \quad \square$$

*Proof of Theorem 3.5.* Similarly,

$$\begin{aligned} \mathcal{G}_{(2n)}(XYZ) &= F_\emptyset^B(XYZ) = \sum_{I, J, K} c_{IJK}^\emptyset F_I^B(X) F_J^B(Y) F_K^B(Z), \\ \mathcal{G}_{(2n)}(XYZ) &= \sum_{\nu \in \mathcal{P}^0(n)} \mathcal{G}_\nu(XY) \mathcal{G}_\nu(Z) \\ &= \sum_{\lambda, \mu, \nu \in \mathcal{P}^0(n)} g^B(\lambda, \mu, \nu) \mathcal{G}_\lambda(X) \mathcal{G}_\mu(Y) \mathcal{G}_\nu(Z) \\ &= \sum_{I, J, K} \sum_{\lambda, \mu, \nu \in \mathcal{P}^0(n)} g^B(\lambda, \mu, \nu) d_{\lambda I}^B d_{\mu J}^B d_{\nu K}^B F_I^B(X) F_J^B(Y) F_K^B(Z). \end{aligned} \quad \square$$

*Remark 3.6.* Theorem 3.3 is a special case of Theorem 3.5. However, we felt that an independent proof that does not use Kronecker coefficients but only the elementary first identity of Proposition 3.4 was of interest.

### 3.3.1 A corollary to Theorem 3.5

The structure constants  $e_{IJ}^K$  also count templates  $M$  of the following type.

$$M = (m_{i,j}) = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,l(I)} \\ & b_{1,1} & \dots & b_{1,l(I)} \\ a_{1,0} & a_{1,1} & \dots & a_{1,l(I)} \\ \vdots & & & \vdots \\ & b_{l(J),1} & \dots & b_{l(J),l(I)} \\ a_{l(J),0} & a_{l(J),1} & \dots & a_{l(J),l(I)} \end{pmatrix}$$

*Definition 40.* Given  $q_s = a_{s,0} + \sum_k (a_{s,k} + b_{s,k})$  and  $q_0 = a_{0,0} + \sum_k a_{0,k}$  we denote  $q(M) = (q_0, q_1, \dots, q_{l(I)})$  the **row sums**. The **column sums**  $p(M) = (p_0, p_1, \dots, p_{l(J)})$  verify  $p_j = a_{0,j} + \sum_k (a_{k,s} + b_{k,s})$  and  $p_0 = a_{0,0} + \sum_k a_{k,0}$ . Successive reading of  $a$ -lines from left to right, and  $b$ -lines from right to left yields the **reading word**.

Denote by  $n_{IJ}^K$  the number of such templates with row sums  $I$ , column sums  $J$  and reading word  $K$  and  $n_{IJ} = \sum_K n_{IJ}^K$ . Bergeron and Bergeron in [BB92] prove that  $n_{IJ}^K = e_{IJ}^K$ . We get as a corollary to Theorem 3.5:

**Corollary 3.7.** *Let  $I, J$  and  $K \subseteq \{0\} \cup [n-1]$ . The numbers  $n_{I,J}$  and  $n_{I,J}^K$  verify*

$$n_{IJ} = \sum_{\lambda} K_{\lambda \text{ comp}(I)}^B K_{\lambda \text{ comp}(J)}^B,$$

$$\sum_{R, D \subseteq \{0\} \cup [n-1]} n_{IJ}^D n_{DK}^R = \sum_{\lambda, \mu, \nu \vdash n} g^B(\lambda, \mu, \nu) K_{\lambda \text{ comp}(I)}^B K_{\mu \text{ comp}(J)}^B K_{\nu \text{ comp}(K)}^B.$$

*Proof.* Considering relations, which give Equations (3.2) and (3.5), in the monomial basis instead of fundamental basis one gets

$$\sum_{I,J} \sum_{I' \subseteq I, J' \subseteq J} c_{I'J'}^{\emptyset} M_I^B(X) M_J^B(Y) = \sum_{I,J} \sum_{\lambda \vdash n} K_{\lambda \text{ comp}(I)}^B K_{\lambda \text{ comp}(J)}^B M_I^B(X) M_J^B(Y),$$

and

$$\sum_{I,J,K} \sum_{I' \subseteq I, J' \subseteq J, K' \subseteq K} c_{I'J'K'}^{\emptyset} M_I^B(X) M_J^B(Y) M_K^B(Z) =$$

$$\sum_{I,J,K} \sum_{\lambda, \mu, \nu \vdash n} g^B(\lambda, \mu, \nu) K_{\lambda \text{ comp}(I)}^B K_{\mu \text{ comp}(J)}^B K_{\nu \text{ comp}(K)}^B M_I^B(X) M_J^B(Y) M_K^B(Z).$$



The first equation of the statement is a consequence of the result of Bergeron and Bergeron and Equation (3.1).

$$n_{IJ} = \sum_K e_{IJ}^K = \sum_{I' \subseteq I, J' \subseteq J} c_{I'J'}^\emptyset = \sum_{\lambda \vdash n} K_{\lambda \text{ comp}(I)}^B K_{\lambda \text{ comp}(J)}^B.$$

To get the second identity compute  $e_{IJK}^R$  using structure constants  $\{e_{AB}^C\}_{A,B,C}$

$$\sum_R e_{IJK}^R B_R^B = B_I^B B_J^B B_K^B = \left( \sum_D e_{IJ}^D B_D^B \right) B_K^B = \sum_{R,D} e_{IJ}^D e_{DK}^R B_R^B.$$

Then one gets

$$\begin{aligned} \sum_{R,D} n_{IJ}^D n_{DK}^R &= \sum_{R,D} e_{IJ}^D e_{DK}^R \\ &= \sum_R e_{IJK}^R \\ &= \sum_{I' \subseteq I, J' \subseteq J, K' \subseteq K} c_{I'J'K'}^\emptyset \\ &= \sum_{\lambda, \mu, \nu \vdash n} g^B(\lambda, \mu, \nu) K_{\lambda \text{ comp}(I)}^B K_{\mu \text{ comp}(J)}^B K_{\nu \text{ comp}(K)}^B. \end{aligned} \quad \square$$

### 3.3.2 Skew shapes

Define for  $I, J \subseteq 0 \cup [n-1]$  the coefficients  $c_{IJ}^{\lambda/\mu} = \sum_{K \in 0 \cup [n-1]} d_{\lambda/\mu K}^B c_{IJ}^K$ . Formula (1.24) admits the following type B analogue.

**Proposition 3.8.** *For  $\lambda/\mu \in \tilde{\mathcal{P}}^0(n)$  and  $I, J \subseteq 0 \cup [n-1]$ , the coefficients  $c_{IJ}^{\lambda/\mu}$  verify*

$$\mathcal{G}_{\lambda/\mu}(XY) = \sum_{I, J \subseteq 0 \cup [n-1]} c_{IJ}^{\lambda/\mu} F_I^B(X) F_J^B(Y). \quad (3.6)$$

*Proof.* Apply Proposition 2.10 and then decompose Chow's fundamental quasisymmetric function  $F_K^B(XY)$  into  $F_I^B(X)$  and  $F_J^B(Y)$  to prove the statement.

$$\begin{aligned} \mathcal{G}_{\lambda/\mu}(XY) &= \sum_{K \subseteq \{0\} \cup [n-1]} d_{\lambda/\mu K}^B F_K^B(XY) \\ &= \sum_{I, J} \left( \sum_K d_{\lambda/\mu K}^B c_{IJ}^K \right) F_I^B(X) F_J^B(Y) \\ &= \sum_{I, J} c_{IJ}^{\lambda/\mu} F_I^B(X) F_J^B(Y). \end{aligned} \quad \square$$

We are now ready to state the main result of this subsection.

**Theorem 3.9.** For  $\lambda/\mu \in \tilde{\mathcal{P}}^0(n)$  and  $\nu \in \mathcal{P}^0(n)$  denote by  $l_{\mu\nu}^\lambda$  the type B analogue of the Littlewood-Richardson coefficients defined as in Proposition 2.8 by formula  $l_{\mu\nu}^\lambda = k_{\mu-\nu}^{\lambda-} - k_{\mu+\nu}^{\lambda+}$ . Then, for  $I, J \subseteq 0 \cup [n-1]$ , the coefficient  $c_{IJ}^{\lambda/\mu}$  is given by

$$c_{IJ}^{\lambda/\mu} = \sum_{\nu, \rho, \varepsilon \in \mathcal{P}^0(n)} l_{\mu\nu}^\lambda g^B(\nu, \rho, \varepsilon) d_{\rho I}^B d_{\varepsilon J}^B. \quad (3.7)$$

*Proof.* On the one hand expand  $\mathcal{G}_{\lambda/\mu}(XY)$  using Proposition 3.8. On the other hand expand the same skew domino function using Proposition 2.8, Proposition 3.4 and Proposition 2.3. Namely

$$\begin{aligned} \sum_{I, J \subseteq 0 \cup [n-1]} c_{I, J}^{\lambda/\mu} F_I^B(X) F_J^B(Y) &= \mathcal{G}_{\lambda/\mu}(XY) \\ &= \sum_{\nu \in \mathcal{P}^0(n)} l_{\mu\nu}^\lambda \mathcal{G}_\nu(XY) \\ &= \sum_{\nu \in \mathcal{P}^0(n)} l_{\mu\nu}^\lambda \sum_{\rho, \varepsilon \in \mathcal{P}^0(n)} g^B(\nu, \rho, \varepsilon) \mathcal{G}_\rho(X) \mathcal{G}_\varepsilon(Y) \\ &= \sum_{\substack{I, J \subseteq 0 \cup [n-1] \\ \nu, \rho, \varepsilon \in \mathcal{P}^0(n)}} l_{\mu\nu}^\lambda g^B(\nu, \rho, \varepsilon) d_{\rho I}^B d_{\varepsilon J}^B F_I^B(X) F_J^B(Y). \quad \square \end{aligned}$$



## Chapter 4

# Type B Schur positivity

This chapter is devoted to generalisation of a notion of Schur positivity to the case of type B, and to the question of determining type B Schur positive sets.

### 4.1 A new definition of type B Schur-positivity

Recall that many of the results concerning Schur positivity are the consequence of two main facts.

1. The Schur symmetric function indexed by  $\lambda \vdash n$ ,  $s_\lambda$  is the generating function for semistandard Young tableaux of shape  $\lambda$ . It follows (see e.g. [Sta01, 7.19.7]) that

$$s_\lambda(X) = \sum_{T \in SSYT(\lambda)} X^T = \sum_{T \in SYT(\lambda)} F_{\text{Des}(T)}(X). \quad (4.1)$$

2. There are various descent preserving bijections relating sets of permutations and standard Young tableaux, e.g. the celebrated *Robinson-Schensted* (RS) correspondence.

A **type B extension** of Schur-positivity deals with the hyperoctahedral group of order  $n$  instead of  $S_n$ . To extend items 1 and 2 above, two options are available and depend on the definition for the descent of signed permutations. In [AAER17], Adin et al. use the notion of **signed descent set**, i.e. the couple  $(S, \varepsilon)$  defined for  $\pi \in B_n$  as  $S = \{n\} \cup \{1 \leq i \leq n-1 \mid \pi(i) > 0 \text{ and } \pi(i) > \pi(i+1) \text{ or } \pi(i) < 0 \text{ and either } \pi(i+1) > 0 \text{ or } |\pi(i)| > |\pi(i+1)|\}$  and  $\varepsilon$  is the mapping from  $S$  to  $\{-, +\}$  defined as  $\varepsilon(s) = +$  if  $\pi(s) > 0$  and  $\varepsilon(s) = -$ , otherwise. There is a signed descent preserving analogue of the RS correspondence relating signed permutations and *bi-tableaux*, i.e. pairs of Young tableaux with specific constraints

and [AAER17] proves an analogue of Equality (4.1) between their generating function and *Poirier's signed quasisymmetric functions*. While the authors succeed in extending most of the results known in type A, providing another framework relying on a definition of descent conform to the theoretical definition of Solomon for signed permutations appears as a natural question. We use the notion of the descent set of  $\pi \in B_n$  as the subset of  $\{0\} \cup [n-1]$  equal to

$$\text{Des}(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}.$$

The Barbash and Vogan bijection provides descent preserving analogue of the RS correspondence in this case that relates signed permutations and domino tableaux. We use the generating function for domino tableaux and Chow's type B quasisymmetric functions to develop this alternative type B extension of Schur-positivity. Moreover, we introduce a new descent preserving bijection between *signed arc permutations* and domino tableaux to show that the former set is type B Schur-positive according to the definition of descent stated above.

*Definition 41.* Similarly to the type A case, we define for any subset  $\mathcal{B}$  of  $B_n$

$$Q(\mathcal{B}) = \sum_{\pi \in \mathcal{B}} F_{\text{Des}(\pi)}^B.$$

In order to extend the definition of Schur positivity, we need to answer two questions:

1. What is a Chow's type B **symmetric** function?
2. What is the analogue of Schur symmetric functions in this type B setup?

We proceed with the answer to the first question.

*Definition 42.* Let  $X = \{x_0, x_1, \dots\}$  and  $X^* = X \setminus \{x_0\}$ . Given a composition  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$  of  $n$  with  $\alpha_1 \geq 0$  and  $\alpha_i > 0$  for  $i > 1$  we define the type B monomial symmetric functions:

$$m_\alpha^B(X) = x_0^{\alpha_1} m_{\widetilde{\alpha \setminus \alpha_1}}(X^*),$$

where  $\widetilde{\gamma}$  is a partition obtained by reordering  $\gamma$ 's entries in decreasing order.

Given compositions  $\alpha$  and  $\beta$  with possible zeros in the beginning we see that  $m_\alpha^B(X)$  and  $m_\beta^B(X)$  coincide when  $\widetilde{\alpha \setminus \alpha_1} = \widetilde{\beta \setminus \beta_1}$ . So, such monomial functions are indexed by pairs  $\{(k, \lambda)\}_{k \in [n], \lambda \vdash n-k}$ .

*Definition 43.* Denote by  $\Lambda_n^B[X]$  the set of **type B symmetric functions** of degree  $n$  as the as the vector space spanned by the  $\{m_{(k, \lambda)}^B\}_{k \in [n], \lambda \vdash n-k}$ .

The natural bases of  $\Lambda_n^B[X]$  are  $\{x_0^{n-|\lambda|} f_\lambda(X^*)\}_{|\lambda| \leq n}$  where  $\{f_\lambda\}_{\lambda \vdash k}$  might be any usual basis of  $\Lambda_k[X^*]$ . We have

$$\Lambda_n^B[X] = \sum_{k=0}^n x_0^{n-k} \Lambda_k[X^*].$$

where  $\Lambda_k[X^*]$  the set of type A symmetric functions of degree  $k$ . Denote by  $p(n)$  the number of partitions  $\lambda \vdash n$ . The dimension of this space is

$$\dim(\Lambda_n^B[X]) = \sum_{i=0}^n p(i).$$

*Remark 4.1.* Type B monomial symmetric functions  $m_\lambda^B$  does not coincide with usual monomial symmetric functions  $m_\lambda$ :

$$m_\lambda(X) = \sum_{i=1}^{\ell(\lambda)+1} x_0^{\lambda_i} m_{\lambda \setminus \lambda_i}(X^*) \neq m_\lambda^B(X).$$

Regarding the type B Schur analogue, we may use either the family  $\{s_{(n-|\lambda|, \lambda)}^B(X) = x_0^{n-|\lambda|} s_\lambda(X^*)\}_{|\lambda| \leq n}$  or the family  $\{\mathcal{G}_\lambda\}_{\lambda \in \mathcal{P}^0(n)}$ .

*Definition 44.* We say that a set  $\mathcal{B} \subset B_n$  is  **$\mathcal{G}$ -positive** ( $s^B$ -positive) if the function  $Q(\mathcal{B})$  can be written as a non-negative sum of domino functions  $(\{s_{(n-|\lambda|, \lambda)}^B\})$ .

Both candidates have advantages and drawbacks. The family  $\{s_{(n-|\lambda|, \lambda)}^B\}$  is a natural basis of  $\Lambda_n^B[X]$  that looks like an equivalent of Schur functions in type A. However it doesn't seem to fulfil any nice combinatorial expressions. On the contrary, the functions  $\mathcal{G}_\lambda$  have a very strong combinatorial interpretation as shown in Equation (2.3) and (2.5). Furthermore, for any  $\lambda \in \mathcal{P}^0(n)$ ,  $\mathcal{G}_\lambda(X) \in \Lambda_n^B[X]$ . Indeed,

$$\mathcal{G}_\lambda(X) = \sum_{|\rho| \leq n} a_\rho^\lambda s_{(n-|\rho|, \rho)}^B(X), \quad (4.2)$$

where  $a_\rho^\lambda$  is a non-negative integer with

$$a_\rho^\lambda = \sum_{\nu \vdash |\rho| - |\lambda^-|; \lambda^+ \setminus \nu \text{ is a horizontal strip}} k_{\lambda^- \nu}^\rho,$$

where  $k$  is the Littlewood-Richardson coefficient and a **horizontal strip** is a skew shape composed by boxes, none of which are in the same column. But,  $\{\mathcal{G}_\lambda\}_{\lambda \in \mathcal{P}^0(n)}$  is not a basis of  $\Lambda_n^B[X]$  as:

$$|\{\mathcal{G}_\lambda\}_{\lambda \in \mathcal{P}^0(n)}| = |\mathcal{P}^0(n)| = \sum_{i=0}^n p(n-i)p(i) > \sum_{i=0}^n p(i) = \dim(\Lambda_n^B[X]).$$

The  $\mathcal{G}_\lambda$  are not linearly independent. One can show that the subfamily

$$\{\mathcal{G}_{\lambda^-,k}\}_{k \leq n, \lambda^- \vdash n-k}$$

is a basis of  $\Lambda_n^B[X]$ . However, if we use this subfamily even the most basic examples (see next subsection) are no longer  $\mathcal{G}$ -positive.

*Remark 4.2.* One interesting remark is that the functions  $\mathcal{G}_\lambda$  are all  $s^B$ -positive. As a result, if a function can be written (in a non unique way) as a positive combination of the  $\mathcal{G}_\lambda$ , then this function is proved to be  $s^B$ -positive.

#### 4.1.1 Basic examples of $\mathcal{G}$ -positivity

*Remark 4.3.* The inverse descent sets  $D_{n,J}^{B,-1} = \{\pi \in B_n \mid \text{Des}(\pi^{-1}) = J\}$  are  $\mathcal{G}$ -positive.

*Proof.* Thanks to the Barbash and Vogan correspondence, we have a descent preserving bijection between permutations  $\pi$  of hyperoctahedral group and pairs of standard domino tableaux  $(P_\pi, Q_\pi)$  that verifies  $\text{Des}(\pi) = \text{Des}(Q_\pi)$  and  $\text{Des}(\pi^{-1}) = \text{Des}(P_\pi)$ . We use this fact to compute  $Q(D_{n,J}^{B,-1})$ :

$$Q(D_{n,J}^{B,-1}) = \sum_{\substack{\pi \in B_n \\ \text{Des}(\pi^{-1})=J}} F_{\text{Des}(\pi)}^B = \sum_{\lambda \vdash n} \sum_{\substack{P \in \text{SDT}(\lambda); \\ \text{Des}(P)=J}} \sum_{Q \in \text{SDT}(\lambda)} F_{\text{Des}(Q)}^B.$$

Since  $\mathcal{G}_\lambda = \sum_{Q \in \text{SDT}(\lambda)} F_{\text{Des}(Q)}^B$  we get that  $Q(D_{n,J}^{B,-1})$  expands in domino functions with positive coefficients.  $\square$

Another essential example of a  $\mathcal{G}$ -positive set is type B Knuth classes. Given standard domino tableau  $T$  we denote the corresponding type B Knuth class

$$\mathcal{C}_T = \{\pi \in B_n \mid P_\pi = T\}.$$

*Remark 4.4.* The type B Knuth classes  $\mathcal{C}_T$  are  $\mathcal{G}$ -positive.

*Proof.* Similarly, we compute  $\mathcal{C}_T$  using the type B analogue for RS-correspondence:

$$Q(\mathcal{C}_T) = \sum_{\substack{\pi \in B_n \\ P_\pi=T}} F_{\text{Des}(\pi)}^B = \sum_{Q \in \text{SDT}(\text{shape}(T))} F_{\text{Des}(Q)}^B = \mathcal{G}_{\text{shape}(T)}.$$

$\square$

Next example is a consequence of the previous one.

*Definition 45.* A sequence  $(a_1, \dots, a_n)$  of distinct integers is called **unimodal** if there exists  $1 \leq m \leq n$  such that

$$a_1 > a_2 > \dots > a_m < a_{m+1} < \dots < a_n.$$

We say a permutation  $\pi \in B_n$  is **left-unimodal** if  $\pi^{-1}$  is unimodal as a sequence. So, permutation  $\pi$  is left-unimodal if and only if  $\text{Des}(\pi^{-1}) = \{1, 2, \dots, i\}$  or  $\text{Des}(\pi^{-1}) = \{0, 1, 2, \dots, i\}$  for some  $i \in [n-1]$ . Consider their images via the analog of RS-correspondence. Since this map preserves a notion of descent set, left-unimodal permutations are a union of Knuth classes, corresponding to standard domino tableaux with the constraint  $\text{Des}(T) = \{1, 2, \dots, i\}$  or  $\text{Des}(T) = \{0, 1, 2, \dots, i\}$ , so they are  $\mathcal{G}$ -positive.

All these examples are also  $s^B$ -positive as a consequence of Remark 4.2.

## 4.2 Application to signed arc permutations

*Definition 46.* A permutation  $\pi \in B_n$  is called a **signed arc permutation** if for  $1 \leq i \leq n$  the set  $\{|\pi(1)|, |\pi(2)|, \dots, |\pi(i-1)|\}$  is an interval in  $Z_n$  and  $\pi(i) > 0$  if  $|\pi(i)| - 1 \in \{|\pi(1)|, |\pi(2)|, \dots, |\pi(i-1)|\}$  and  $\pi(i) < 0$  otherwise. The set of signed arc permutations is denoted by  $\mathcal{A}_n^s$ .

One can prove (see [ER15]) that signed arc permutation are exactly those permutations of  $B_n$  that avoid the following 24 patterns:

$$[\pm 1, -2, \pm 3], [\pm 1, 3, \pm 2], [\pm 2, -3, \pm 1], [\pm 2, 1, \pm 3], [\pm 3, -1, \pm 2], [\pm 3, 2, \pm 1].$$

The main result of this section follows.

**Theorem 4.5.** *The set of signed arc permutations  $\mathcal{A}_n^s$  is  $\mathcal{G}$ -positive. Moreover,*

$$\begin{aligned} \sum_{\pi \in \mathcal{A}_n^s} F_{\text{Des}(\pi)}^B &= \mathcal{G}_{(2n)} + \mathcal{G}_{(2n-1,1)} + \mathcal{G}_{(2n-2,1,1)} + \mathcal{G}_{(2n-3,1,1,1)} + 2 \sum_{a \geq 2n-a \geq 2} \mathcal{G}_{(a,2n-a)} \\ &+ \sum_{a \geq 2n-a-2 \geq 2} \mathcal{G}_{(a,2n-a-2,2)} + \sum_{a \geq 2n-a-2 \geq 2} \mathcal{G}_{(a,2n-a-2,1,1)}. \end{aligned} \quad (4.3)$$

To prove Theorem 4.5 we introduce a new descent-preserving bijective map from  $\mathcal{A}_n^s$  to the sets of standard domino tableaux with shapes equal to the indices of the domino functions in formula 4.3.



### 4.2.1 Description of signed arc permutations

We start by giving a more precise description of signed arc permutations. Consider prefixes of the one-line notation of a signed arc permutation. Its absolute values form an interval in  $\mathbb{Z}_n$ , and if we go counterclockwise of the first value, the sign is "-", else we have "+", like at Figure 4.1.

*Definition 47.* Given two sequences of integers  $A$  and  $B$ , denote by  $A \text{ sh } B$  the set of **shuffles** of these sequences, i.e. the set of sequences  $\pi$  consisting of all elements of  $A$  and  $B$  such that each  $A$  and  $B$  forms an ordered subsequence of  $\pi$ .

We consider each signed permutation as a shuffle of its negative and positive subsequence.

The following table breaks the set of signed arc permutations into 6 non-overlapping types. Types 1 and 2 have only either positive or negative entries. The four remaining types have at least one negative and one positive integer and are characterised according to the sign of their entries with absolute value 1 and  $n$ .

| Type | Content  |
|------|--|
| 1    | $\bigcup_{k \in [2, n]} [k, \dots, n, 1, \dots, k-1] \bigcup [1, \dots, n]$  |
| 2    | $\bigcup_{k \in [2, n]} [-(k-1), \dots, -1, -n, \dots, -k] \bigcup [-n, \dots, -1]$  |
| 3    | For any $k, l$ such that $n > k > l \geq 1$ ,<br>$\bigcup_{k, l} (-k, -(k-1), \dots, -(l+1)) \text{ sh } (k+1, \dots, n, 1, \dots, l)$     |
| 4    | For any $k, l$ such that $n > l > k \geq 1$ ,<br>$\bigcup_{k, l} (-k, -(k-1), \dots, -1, -n, \dots, -(l+1)) \text{ sh } (k+1, \dots, l)\}$ |
| 5    | $\bigcup_{k \in [1, n-1]} (-k, -(k-1), \dots, -1) \text{ sh } (k+1, \dots, n)\}$   |
| 6    | $\bigcup_{k \in [1, n-1]} (-n, \dots, -(k+1)) \text{ sh } (1, \dots, k)$   |

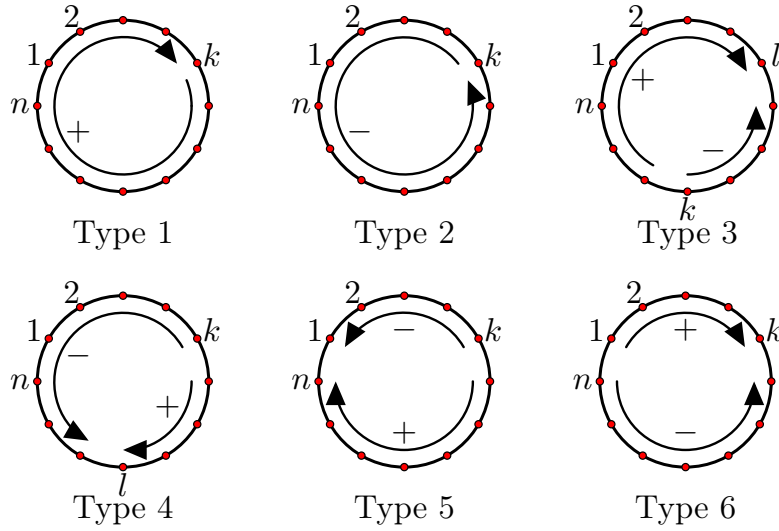


FIGURE 4.1: All types of signed arc permutations of  $n$ . One get an arc permutations taking a shuffle of positive and negative sequences with some order.

#### 4.2.2 Explicit bijections for all types

The next step is to build the bijections with standard domino tableaux for each type. Given a permutation, we build recursively a standard domino tableau. At step  $1 \leq i \leq n$  we add a domino with label  $i$ . For each step we will use one of the following rules:

- (Rule 1) Add a horizontal domino in the end of the first row.
- (Rule 2) Add either a horizontal domino in the end of the second row or a vertical domino across the two first rows. As the difference of lengths of the first and second row is always even, only one of these two positions is available.
- (Rule 3) Add either a horizontal domino in the end of the third row or a vertical domino across the rows number two and three. Only one of these two positions is available.
- (Rule 4) Add a vertical domino across the rows number three and four.

Types 5 and 6 shows the basic algorithm for these bijections.

**Proposition 4.6.** *Both type 5 and type 6 permutations are in descent-preserving bijection with the set of standard domino tableaux of shapes  $(a, 2n-a)$  for  $a$  such that  $a \geq 2n-a \geq 2$ .*

*Proof.* We give the proof for type 5 permutations but the same reasoning applies to type 6. Type 5 contains following permutations:

$$\bigcup_{k \in [1, n-1]} \{(-k, -(k-1), \dots, -1) \text{ sh } (k+1, \dots, n)\}.$$

Firstly, map  $\pi^0 = [-1, 2 \dots n]$  to the standard domino tableau  $T_0$  composed of  $n$  vertical dominoes. We have  $\text{Des}(\pi^0) = \text{Des}(T_0) = \{0\}$ . Secondly, let  $\pi = [\pi_1 \pi_2 \dots \pi_n] \neq \pi^0$ . We build recursively a two-row standard domino tableau. At step  $1 \leq i \leq n$  we add a domino with label  $i$ . The common rule is if  $\pi_i > 0$ , we apply Rule 1, otherwise, we apply Rule 2.

This mapping is clearly bijective. Indeed, given a tableau we get the sequence of signs  $\text{sign}(\pi_1), \text{sign}(\pi_2), \dots, \text{sign}(\pi_n)$ . The combined number of horizontal dominoes in the second row and vertical dominoes gives  $k$ . Finally, there is a descent in position  $i > 0$  in the tableau if and only if  $i$  is in the first row and  $i+1$  in the second row, i.e. if and only if  $\pi_i > 0$  and  $\pi_{i+1} < 0$ . At the same moment a type 5 signed arc permutation also has a descent in position  $i > 0$  if and only if  $\pi_i > 0 > \pi_{i+1}$ . There is a descent in 0 in the tableau if and only if  $\pi_1 < 0$ . As a result the bijection is descent preserving.  $\square$

The following bijections are modifications of the bijection above.

**Proposition 4.7.** *Type 1 and type 2 permutations together are in descent-preserving bijection with the set of standard domino tableaux of shapes  $(2n), (2n-1, 1), (2n-2, 1, 1)$  and  $(2n-3, 1, 1, 1)$ .*

*Proof.* Type 1 contains following permutations:

$$\bigcup_{k \in [2, n]} \{[k, \dots, n, 1, \dots, k-1]\} \bigcup \{[1, \dots, n]\}.$$

For type 1 permutation we use the same mapping as for type 5 signed arc permutations except for  $i$  such that  $\pi_i = 1$ ,  $i > 1$ , when we apply Rule 3. Type 1 signed arc permutations are in descent preserving bijection with standard domino tableaux of shapes  $(2n-2, 1, 1)$  and  $(2n)$ . If  $i$  is the index of the vertical domino across the second and the third row, then  $n-i+1 = k-1$ .

**Example 4.1.** *The signed arc permutation  $\pi = [4567123]$  is mapped to the following standard domino tableau.*

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 7 |
| 5 |   |   |   |   |   |

Type 2 contains following permutations:

$$\bigcup_{k \in [2, n]} \{[-(k-1), \dots, -1, -n, \dots, -k]\} \cup \{[-n, \dots, -1]\}.$$

For type 2 we use the following mapping. We map  $\pi^0 = [-n, \dots, -2, -1]$  to the standard domino tableau of shape  $(2n-1, 1)$ . Let  $\pi = [\pi_1 \pi_2 \dots \pi_n] \neq \pi^0$  with  $\pi_m = -n$ ,  $m > 1$ . We map  $\pi$  to the standard domino tableau with a vertical domino labelled by 1, all the other domino horizontal in the first row except the domino labelled by  $m$ , which is vertical across the rows 3 and 4. These permutations have two descents: 0 and  $m-1$ . Type 2 signed arc permutations are in descent preserving bijection with standard domino tableaux of shape  $(2n-3, 1, 1, 1)$  and  $(2n-1, 1)$ . If  $i$  is the index of the vertical domino across the second and the third row, then  $i-1 = k-1$ .

**Example 4.2.** The signed arc permutation  $\pi = [-3-2-1-7-6-5-4]$  is mapped to the following standard domino tableau.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 6 | 7 |
| 4 |   |   |   |   |   |

□

**Proposition 4.8.** Type 4 signed arc permutations are in descent preserving bijection with standard domino tableaux of shape  $(a, 2n-a-2, 1, 1)$  for  $a$  such that  $a \geq 2n-a-2 \geq 2$ .

*Proof.* Type 4 contains the following permutations:

$$\bigcup_{k,l} \{(-k, -(k-1), \dots, -1, -n, \dots, -(l+1)) \text{ sh } (k+1, \dots, l)\}.$$

We use the same mapping as for type 5 signed arc permutations except for step  $i$  such that  $\pi_i = -n$ ,  $i > 1$ , when we apply Rule 4. As  $-n$  is never the first negative number in  $\pi$ , the second row of the tableau is always of length greater or equal to 1. So, adding the domino labelled by  $i$  is possible. Type 4 signed arc permutations are in descent preserving bijection with standard domino tableaux of shape  $(2n-a, a, 1, 1)$  for all  $a > 0$ . Given  $i$  the index of the vertical domino across the third and the fourth row,  $k$  is equal to the common number of

- horizontal dominoes in the second row and
- vertical dominoes

with index strictly less than  $i$ . The number of such dominoes with index greater or equal to  $i$  is equal to  $n - l$ .

**Example 4.3.** The signed arc permutation  $\pi = [-3 \ 4 \ -2 \ -1 \ -7 \ 5 \ -6]$  is mapped to the following standard domino tableau.

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 4 | 6 |
|   | 3 |   | 7 |
| 5 |   |   |   |

□

The type 3 case is more complicated.

**Proposition 4.9.** Type 3 signed arc permutations are in descent preserving bijection with standard domino tableaux of shapes  $(a, 2n - a - 2, 2)$  for  $a$  such that  $a \geq 2n - a - 2 \geq 2$ .

*Proof.* Type 3 contains following permutations:

$$\bigcup_{k,l} (-k, -(k-1), \dots, -(l+1)) \text{ sh } (k+1, \dots, n, 1, \dots, l).$$

We consider seven different cases depending on the existence/absence of negative numbers before  $n$ , between  $n$  and 1 and after 1. We mostly use the common rule from type 5 for mapping, but for each case there are several exceptions, see table in Figure 4.2.

We suggest a reverse map to prove that this mapping is bijective. Given  $T$  a standard domino tableau of shape  $(a, 2n - a - 2, 2)$  we construct the permutation as follows.

First, distinguish the template of which case we need to use as follows.

1. If there are two dominoes across the second and third line, we are in case 5 or in case 6.
2. If entries of these dominoes are not adjacent integers then it is case 5. Otherwise, we consider the location of the next domino. If it lays in the first row, then it is also case 5. Otherwise, it is case 6. Thus, we can distinguish cases 5 and 6.
3. In the rest cases  $(1, 2, 3, 4, 7)$  there is exactly one horizontal domino in the third row. Denote by  $i$  the integer in this domino. Consider a subtableau  $T'$  of a tableau  $T$ , composed by the dominoes  $1, 2 \dots i$ .

| Case | Description | Exceptions  |
|------|-------------|---|
| 1    | $-n - 1 -$  | At the step $t + 1$ proceed positive $\pi_{t+1}$ with Rule 2. For the first negative $\pi_j$ after $n$ apply Rule 3.  |
| 2    | $-n - 1 +$  | Proceed positive $\pi_{t+1}$ and $\pi_i = 1$ with Rule 2. For the first negative $\pi_j$ after $n$ apply Rule 3.  |
| 3    | $-n + 1 -$  | Proceed positive $\pi_i = 1$ with Rule 2. For the first negative $\pi_j$ after 1 apply Rule 3.  |
| 4    | $-n + 1 +$  | Proceed positive $\pi_{t+1}$ with Rule 2. For $\pi_i = 1$ apply Rule 3.   |
| 5    | $+n - 1 -$  | So, we have at least two negative integers after $n$ . For two first negative integers after $n$ apply Rule 3.  |
| 6    | $+n - 1 +$  | We have at least one negative integer after $n$ . In case we have two negative integers between $n$ and 1, for two first negatives after $n$ apply Rule 3. Proceed positive $\pi_i = 1$ with Rule 2. If not, we apply Rule 3 for the first negative integer and for $\pi_i = 1$ . |
| 7    | $+n + 1 -$  | Proceed positive $\pi_i = 1$ with Rule 2. For the first negative $\pi_j$ after 1 apply Rule 3.  |

FIGURE 4.2: Table of exceptions for the case of type 3. We encode existence of negative number by "-" and absence by "+",  $t$  is an index of the last negative integer before  $n$  in the one-line notation of  $\pi$ .

4. We are in case 7 if and only if there is only one horizontal domino in the second row of  $T'$ .
5. In the cases 1,2,3,4 the last Rule 2 domino (horizontal in the second row or vertical across the first and second row) in  $T'$  correspond to the positive  $\pi_i$ , but was proceeded with Rule 2.
6. Denote by  $j$  the integer in this domino. If  $j - 1$  domino is positive, it may be only the case 3.
7. Among cases 1,2,4 only in case 4 there is no Rule 2 domino in  $T \setminus T'$ .
8. Cases 1 and 2 also may be distinguished by their templates of  $T \setminus T'$ . Indeed, in case 2 tableau  $T \setminus T'$  consists of some amount of Rule 2 dominoes and after that some amount of Rule 1 dominoes. In case 1 this template does not appear.

Secondly, when we know which case is appropriate for  $T$ , there is no problem further to construct the corresponding permutation using the template of the case. Indeed, for each case:

- For each step  $i$  we know the sign of  $\pi_i$  (including exceptional steps).

$$\begin{array}{l}
\text{Case 1} \quad \overbrace{*\dots*}^{n_1 \geq 0} - \overbrace{+\dots+n}^{n_2 \geq 1} \overbrace{-\dots-}^{n_3 \geq 1} 1 \overbrace{+\dots+}^{n_3 \geq 0} - \overbrace{*\dots*}^{n_3 \geq 0} \\
R \dots R \quad R \quad R2 \dots RR \quad R3 \dots R \quad R \quad R \dots R \quad R \quad R \dots R \\
\\
\text{Case 2} \quad \overbrace{*\dots*}^{n_1 \geq 0} - \overbrace{+\dots+n}^{n_2 \geq 1} \overbrace{-\dots-}^{n_3 \geq 1} 1 \overbrace{+\dots+}^{n_3 \geq 0} \\
R \dots R \quad R \quad R2 \dots RR \quad R3 \dots R \quad R2 \quad R \dots R \\
\\
\text{Case 3} \quad \overbrace{*\dots*}^{n_1 \geq 0} \quad n \quad 1 \quad \overbrace{+\dots+}^{n_3 \geq 0} - \overbrace{*\dots*}^{n_3 \geq 0} \\
R \dots R \quad R \quad R2 \quad R \dots R \quad R3 \quad R \dots R \\
\\
\text{Case 4} \quad \overbrace{*\dots*}^{n_1 \geq 0} - \overbrace{+\dots+n}^{n_2 \geq 1} 1 \overbrace{+\dots+}^{n_3 \geq 0} \\
R \dots R \quad R \quad R2 \dots RR \quad R3 \quad R \dots R \\
\\
\text{Case 5} \quad \overbrace{+\dots+}^{n_1 \geq 0} \quad n \quad \overbrace{-\dots-}^{n_1 \geq 2} 1 \overbrace{+\dots+}^{n_3 \geq 0} - \overbrace{*\dots*}^{n_3 \geq 0} \\
R \dots R \quad R \quad R3R3 \dots R \quad R \quad R \dots R \quad R \quad R \dots R \\
\\
\overbrace{+\dots+}^{n_1 \geq 0} \quad n \quad \overbrace{-}^{n_1=1} 1 \overbrace{+\dots+}^{n_3 \geq 0} - \overbrace{*\dots*}^{n_3 \geq 0} \\
R \dots R \quad R \quad R3 \quad R \quad R \dots R \quad R3 \quad R \dots R \\
\\
\text{Case 6} \quad \overbrace{+\dots+}^{n_1 \geq 0} \quad n \quad \overbrace{-\dots-}^{n_1 \geq 2} 1 \overbrace{+\dots+}^{n_3 \geq 0} \\
R \dots R \quad R \quad R3R3 \dots R \quad R2 \quad R \dots R \\
\\
\overbrace{+\dots+}^{n_1 \geq 0} \quad n \quad \overbrace{-}^{n_1=1} 1 \overbrace{+\dots+}^{n_3 \geq 0} \\
R \dots R \quad R \quad R3 \quad R3 \quad R \dots R \\
\\
\text{Case 7} \quad \overbrace{+\dots+n}^{n_1 \geq 1} 1 \overbrace{+\dots+}^{n_2 \geq 0} - \overbrace{*\dots*}^{n_3 \geq 0} \\
R \dots RR \quad R2 \quad R \dots R \quad R3 \quad R \dots R
\end{array}$$

FIGURE 4.3: All cases of signed arc permutations of type 3. First row describes a template for permutations of the case. We use  $+$  to indicate positive integers and  $-$  for negative ones,  $*$  means any sign of integer. Second row encodes a domino tableaux. We write  $R$  when step uses common rules from the case of type 5 and  $R2, R3$  to indicate an exception. Dots never replace non-exceptional steps.

- Rule 3 dominoes allow us to compute the position  $j$  for which  $\pi_j = 1$ . So, we can compute the number of positive  $\pi_i$ , such that  $i < j$ . From another point of view, it should be  $n - k$ . Thus, we get  $k$ .

As already noticed above, given  $k$  and signs of  $\pi_i$  we fix the signed arc permutation.  $\square$

One finishes the proof of Theorem 4.5 using the descent preserving bijections above to write:

$$\begin{aligned}
\sum_{\omega \in A_n^s} F_{\text{Des}(\omega)}^B &= \sum_{T \in \text{SDT}(2n)} F_{\text{Des}(T)}^B + \sum_{T \in \text{SDT}(2n-2,1,1)} F_{\text{Des}(T)}^B \\
&+ \sum_{T \in \text{SDT}(2n-1,1)} F_{\text{Des}(T)}^B + \sum_{T \in \text{SDT}(2n-3,1,1,1)} F_{\text{Des}(T)}^B \\
&+ \sum_{a, T \in \text{SDT}(a, 2n-a-2, 2)} F_{\text{Des}(T)}^B + \sum_{a, T \in \text{SDT}(a, 2n-a-2, 1, 1)} F_{\text{Des}(T)}^B \\
&+ 2 \sum_{a, T \in \text{SDT}(a, 2n-a)} F_{\text{Des}(T)}^B,
\end{aligned}$$

which gives Theorem 4.5 after application of Equation (2.3).

### 4.3 Another proof of Theorem 4.5

It is necessary to consider a big amount of cases to prove Propositions 4.6, 4.7, 4.8 and 4.9. We may avoid that and simplify the proofs, but the bijections will no longer be explicit. Such an approach uses bi-tableaux and Poirier's quasisymmetric functions. We will briefly introduce needed definitions (see Adin et al. [AAER17] for more details).

#### 4.3.1 Poirier's quasisymmetric functions

Recall the different notion of a descent set on  $B_n$ . A couple  $(S, \varepsilon)$  is called **signed descent set** of  $\pi \in B_n$  if

- $S = \{n\} \cup \{1 \leq i \leq n-1 \mid \begin{cases} \pi(i) > \pi(i+1), & \text{if } \pi(i) > 0 \\ \text{either } \pi(i+1) > 0 \text{ or } |\pi(i)| > |\pi(i+1)|, & \text{if } \pi(i) < 0 \end{cases}\}$
- $\varepsilon$  is the mapping from  $S$  to  $\{-, +\}$  defined as  $\varepsilon(s) = +$  if  $\pi(s) > 0$  and  $\varepsilon(s) = -$ , otherwise.

Given the signed set  $\sigma = (S, \varepsilon)$  we denote

$$\text{wDes}(\sigma) = \{s_k \in S \mid s_k \neq n, \varepsilon(s_k)\varepsilon(s_{k+1}) \in \{++, --, +- \}\}.$$

*Definition 48.* Poirier's quasisymmetric functions are defined as follows:

$$F_\sigma(X, Y) = \sum_{j \in \text{wDes}(\sigma): i_j < i_{j+1}} z_{i_1} \dots z_{i_n},$$

where  $i_1 \leq \dots \leq i_n$  and  $z_i = x_i$  if  $\varepsilon(i) = +$  and  $z_i = y_i$  else.



*Remark 4.10.* Consider an order  $<_r$  on  $[\bar{n}, \dots, \bar{1}, 1, \dots, n]$ . Recall,  $\bar{1} <_r \bar{2} <_r \dots \bar{n} <_r 1 <_r 2 <_r \dots <_r n$ . Given  $\pi \in B_n$  define

$$\text{Des}_r(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) >_r \pi(i+1)\}$$

the notion of descent set, associated with the order  $<_r$ . Note that the notion of signed descent set generalises both the notions  $\text{Des}$  and  $\text{Des}_r$ . So, given some  $\sigma \in \Sigma^B(n)$  we may determine  $\text{Des}(\sigma)$  and  $\text{Des}_r(\sigma)$ . Then Chow's quasisymmetric functions may be expressed in Poirier's quasisymmetric functions as follows:

$$F_{\text{Des}_r(\sigma)}^B(X) = F_\sigma(X^+, X^-).$$

*Proof.* Remind that

$$\text{Des}_r(\sigma) = \begin{cases} \text{wDes}(\sigma), & \text{if } \varepsilon'(1) = +; \\ \text{wDes}(\sigma) \cup \{0\}, & \text{if } \varepsilon'(1) = -. \end{cases}$$

□

*Lemma 16.* There is a bijection  $W_1 : B_n \rightarrow B_n$  satisfying

$$\text{Des}(\pi) = \text{Des}_r(W_1(\pi)); \quad \text{Des}_r(\pi) = \text{Des}(W_1(\pi)).$$

*Proof.* Indeed, fix some  $\pi \in B_n$ . For  $s \in [n-1]$  consider  $\varepsilon(s)$  and  $\varepsilon(s+1)$ . If at least one of them is not negative then

$$s \in \text{Des}_r(\pi) \leftrightarrow s \in \text{Des}(\pi).$$

If both  $\varepsilon(s)$ ,  $\varepsilon(s+1)$  are negative then  $s$  lies exactly in one of these sets. Using this property we suggest the following appropriate bijection  $W_1$ .

Consider the signed permutation  $\pi = [\pi_1, \pi_2, \dots, \pi_n]$ . It is a shuffle of its positive and negative subsequence. We let the positive elements stay on their places and reverse the negative subsequence.

*Example 10.* Given  $\pi = [-3 \ 8 \ 5 \ -2 \ 1 \ -9 \ -7 \ 4 \ 6]$  we get  $W_1(\pi) = [-7 \ 8 \ 5 \ -9 \ 1 \ -2 \ -3 \ 4 \ 6]$ .

□

### 4.3.2 Bi-tableaux

Section 2.2 tells us about the Stanton and White bijection between standard domino tableaux and standard bi-tableaux. However, this bijection does not preserve the descent set. The following lemma fixes this situation.

*Lemma 17.* There is an implicit bijection  $W_3$  between standard bi-tableaux and standard domino tableaux, such that

$$\text{Des}_r((Q^-, Q^+)) = \text{Des}(W_3(Q^-, Q^+)).$$

*Proof.* Proposition 4.2 in [AAER17] states that for all  $(\lambda, \mu)$ :

$$s_\lambda(X)s_\mu(Y) = \sum_{(Q^-, Q^+) \in SYT(\lambda, \mu)} F_{\text{sDes}((Q^-, Q^+))}(X, Y).$$

Recall, the Stanton and White bijection leads to the formula linked domino and Schur functions:

$$\mathcal{G}_\lambda(X) = s_{\lambda^+}(X^+)s_{\lambda^-}(X^-).$$

So, applying  $X := X^+$  and  $Y := X^-$  and using Remark 4.10 we further continue

$$\mathcal{G}_\lambda(X) = s_{\lambda^+}(X^+)s_{\lambda^-}(X^-) = \sum_{(Q^-, Q^+) \in SYT(\lambda^+, \lambda^-)} F_{\text{Des}_r(\text{sDes}(Q))}^B(X).$$

From the other point of view,

$$\mathcal{G}_\lambda(X) = \sum_{Q \in SDT(\lambda)} F_{\text{Des}(Q)}^B(X).$$

Note that  $F^B$  is a base of  $BQSym$ , so the following multisets coincide for any  $\lambda \in \mathcal{P}^0(n)$ :

$$\{\text{Des}_r((Q^-, Q^+))\}_{(Q^-, Q^+) \in SYT(\lambda^+, \lambda^-)} = \{\text{Des}(Q)\}_{Q \in SDT(\lambda)}.$$

The coincidence of the multisets means exactly the existence of the bijection with the needed property.  $\square$

### 4.3.3 Proof

We are going to present a bijection between signed arc permutations and a multiset of standard domino tableaux of several shapes. The bijection is composed of three mappings. Firstly, we apply bijection  $W_1$  from the Lemma 16. Further, we will denote a mapping  $W_2$  from  $W_1(A_n^s)$  to standard Young bi-tableaux. Restrictions of  $W_2$  on each of 6 types of

signed arc permutations are bijections with standard Young bi-tableaux of certain shapes. The last mapping is the bijection  $W_3$  from the Lemma 17.

*Lemma 18.* There is a mapping  $W_2$  from signed permutations of  $W_1(A_n^s)$  to standard Young bi-tableaux, such that it preserves a notion of signed descent. Moreover, the restriction  $W_2$  on any type of signed arc permutations is a bijection with standard Young bi-tableaux of some shapes.

*Proof.* The main idea is that in  $W_2(\pi)$  negative elements of  $\pi$  will correspond to the negative part of bi-tableau and positive elements to the positive part. This is to ensure that signed descents will coincide in positions with signs  $-+$  and  $+ -$ . Preserving signed descents in positions with signs  $--$  ( $++$ ) requires having the signed descents in the negative (positive) part of Young bi-tableau in the same positions as the signed descents in  $\pi$ .

Let  $\pi = \pi_1\pi_2\cdots\pi_n$  be a permutation of  $W_1(A_n^s)$ . Start from the empty bi-tableau ( $Q^- = \emptyset, Q^+ = \emptyset$ ). For  $1 \leq i \leq n$  build recursively a domino tableau with two rows according to the following procedure.

- if  $\pi_i > 1$  add a square with label  $i$  to the first row in  $Q^+$ .
- if  $\pi_i < -1$  add a square with label  $i$  to the first row in  $Q^-$ .
- if  $\pi_i = 1$  add a square with label  $i$  in  $Q^+$ . If  $n \in \pi$  then add it to the second row, otherwise add it to the first one.
- if  $\pi_i = -1$  add a square with label  $i$  in  $Q^-$ . If  $-n \in \pi$  then add it to the second row, otherwise add it to the first one.

We provide the result of  $W_1$  for each type and the template of corresponding standard Young bi-tableaux in the following table. Denote by  $p_a$  the position of  $\pm a$  in  $W_1(\pi)$ .

|                |   |
|----------------|---|
| type 1         | $W_1(\pi) \in \bigcup_{k \in [2, n]} \{[k, \dots, n, 1, \dots, k-1]\} \cup \{[1, \dots, n]\}$   |
| $W_2 W_1(\pi)$ | $(\emptyset, \begin{array}{ c c } \hline p_k & p_{k+1} \\ \hline p_1 & \end{array} \dots \dots \begin{array}{ c } \hline p_{k-1} \\ \hline \end{array}) \text{ or } (\emptyset, \begin{array}{ c c } \hline p_1 & p_2 \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_n \\ \hline \end{array})$ |
| type 2         | $W_1(\pi) \in \bigcup_{k \in [2, n]} \{[-k, \dots, -n, -1, \dots, -(k-1)]\} \cup \{[-1, \dots, -n]\}$   |
| $W_2 W_1(\pi)$ | $(\begin{array}{ c c } \hline p_k & p_{k+1} \\ \hline p_1 & \end{array} \dots \dots \begin{array}{ c } \hline p_{k-1} \\ \hline \end{array}, \emptyset) \text{ or } (\begin{array}{ c c } \hline p_1 & p_2 \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_n \\ \hline \end{array}, \emptyset)$ |
| type 3         | $W_1(\pi) \in \bigcup_{k, l} \{(-(l+1), \dots, -k) \text{ sh } (k+1, \dots, n, 1, \dots, l)\}$  |
| $W_2 W_1(\pi)$ | $(\begin{array}{ c c } \hline p_{l+1} & p_{l+2} \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_k \\ \hline \end{array}, \begin{array}{ c c } \hline p_{k+1} & p_{k+2} \\ \hline p_1 & \end{array} \dots \dots \begin{array}{ c } \hline p_l \\ \hline \end{array})$                            |
| type 4         | $W_1(\pi) \in \bigcup_{k, l} \{(-(l+1), \dots, -n, -1, \dots, -k) \text{ sh } (k+1, \dots, l)\}$  |
| $W_2 W_1(\pi)$ | $(\begin{array}{ c c } \hline p_{l+1} & p_{l+2} \\ \hline p_1 & \end{array} \dots \dots \begin{array}{ c } \hline p_k \\ \hline \end{array}, \begin{array}{ c c } \hline p_{k+1} & p_{k+2} \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_l \\ \hline \end{array})$                            |
| type 5         | $W_1(\pi) \in \bigcup_{k \in [1, n-1]} \{(-1, \dots, -k) \text{ sh } (k+1, \dots, n)\}$   |
| $W_2 W_1(\pi)$ | $(\begin{array}{ c c } \hline p_1 & p_2 \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_k \\ \hline \end{array}, \begin{array}{ c c } \hline p_{k+1} & p_{k+2} \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_n \\ \hline \end{array})$  |
| type 6         | $W_1(\pi) \in \bigcup_{k \in [1, n-1]} \{(-(k+1), \dots, -n) \text{ sh } (1, \dots, k)\}$   |
| $W_2 W_1(\pi)$ | $(\begin{array}{ c c } \hline p_{k+1} & p_{k+2} \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_n \\ \hline \end{array}, \begin{array}{ c c } \hline p_1 & p_2 \\ \hline \end{array} \dots \dots \begin{array}{ c } \hline p_k \\ \hline \end{array})$  |

Note, that restriction  $W_2$  to each type of signed arc permutations is exactly a descent-preserving bijection with all standard Young bi-tableaux of the corresponding shape template.  $\square$

Now we are ready to reprove Theorem 4.5. The bijection  $W_1$  satisfies

$$\text{Des}(\pi) = \text{Des}_r(W_1(\pi)).$$

For  $W_2$  we showed that it preserves signed descent set. This also means it preserves  $\text{Des}_r$ . And, finally,  $W_3$  satisfies

$$\text{Des}_r((Q^-, Q^+)) = \text{Des}(W_3(Q^-, Q^+)).$$

All together  $W_3 W_2 W_1$  preserve the notion of descent set:

$$\text{Des}(\pi) = \text{Des}_r(W_1(\pi)) = \text{Des}_r(W_2 W_1(\pi)) = \text{Des}(W_3 W_2 W_1(\pi)).$$

Applying bijection  $W_3$  to the bi-tableaux described in the table above, one get exactly the shapes from Propositions 4.6, 4.7, 4.8 and 4.9. Indeed, types 1 and 2 correspond to shapes  $\{(n), (n-1, 1), (n-2, 1, 1), (n-3, 1, 1, 1)\}$ , type 3 and 4 correspond to  $\{(a, n-a-2, 2)\}_{a \in [1 \dots n-2], a \geq n-a-2 \geq 2}$  and  $\{(a, n-a-2, 1, 1)\}_{a \in [1 \dots n-2], a \geq n-a-2 \geq 2}$  respectively. Finally, type 5 and 6 each gives  $\{(a, n-a)\}_{a \in [1 \dots n-2], a \geq n-a \geq 2}$ .

## Chapter 5

# Refined statistics with an additional parameter $q$

The goal of this chapter is to provide a **type B extension** of the following equation

$$\sum_{\substack{\lambda \vdash n \\ T, U \in SYT(\lambda)}} F_{\text{Des}(T)}(X) F_{\text{Des}(U)}(Y) = F_{\emptyset}(XY) = \sum_{\pi \in S_n} F_{\text{Des}(\pi)}(X) F_{\text{Des}(\pi^{-1})}(Y)$$

that involves an additional parameter  $q$  to keep track of some significant statistics. We then relate it to the question of type B ( $q$ -) Schur positivity and derive additional results.

### 5.1 Type B $q$ -Cauchy identity

To build our  $q$ -deformed framework, we introduce the following constraint of  **$q$ -symmetry** on alphabets.

*Definition 49* ( $q$ -symmetry). Let  $X = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$  be an alphabet of commutative indeterminates. We say that  $X$  is  **$q$ -symmetric** if for all  $i > 0$ ,

$$x_{-i} = qx_i. \tag{5.1}$$

In this section we let  $X = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$  and  $Y = \{\cdots, y_{-1}, y_0, y_1, \cdots\}$  be two  $q$ -symmetric alphabets. We proceed with the definition of the  $q$ -deformed modified domino functions.

*Definition 50* ( $q$ -domino functions). Given a semistandard domino tableau  $T$  of weight  $\mu$ ,  $X^T$  is the monomial  $x_0^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} \cdots$  and  $\text{sp}(T)$  is the spin of domino tableau  $T$ , i.e. half the

number of its vertical dominoes. For  $\lambda \in \mathcal{P}^0(n)$  define the  **$q$ -domino function** indexed by  $\lambda$  on alphabet  $X$  as

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SSDT(\lambda)} q^{\text{sp}(T)} X^T. \quad (5.2)$$

Setting  $q = 1$  gives our modified domino functions. We have the following lemma.

*Lemma 19.* For  $\lambda \in \mathcal{P}^0(n)$ , the  $q$ -domino function indexed by  $\lambda$  can be expanded in the basis of Chow's quasisymmetric functions as

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SDT(\lambda)} q^{\text{sp}(T)} F_{\text{Des}(T)}^B(X). \quad (5.3)$$

*Proof.* The proof is similar to the one in the case  $q = 1$ . □

*Remark 5.1.* Our domino functions resemble the *LLT-polynomials* introduced in [LLT96] but **they are not equal** because of the possible 0 labels in the semistandard domino tableaux of Remark 2.1. In particular our domino functions are **not symmetric** as the variable  $x_0$  has a particular rôle. More precisely let  $\lambda$  and  $\mu$  be two integer partitions such that  $\lambda/\mu$  is a skew shape,  $|\lambda| - |\mu| = 2n$  and such that the Young diagrams of shape  $\mu$  and  $\lambda/\mu$  (and by extension  $\lambda$ ) may be tiled by horizontal and vertical dominoes. By abuse of notation, we also write  $\lambda/\mu \in \mathcal{P}^0(n)$  when all these conditions are fulfilled. For  $\lambda/\mu \in \mathcal{P}^0(n)$  denote by  $SSDT^*(\lambda/\mu)$  the set of semistandard domino tableaux of shape  $\lambda/\mu$  that **do not have** 0 labels. The LLT-polynomial indexed by  $\lambda/\mu$  on alphabet  $X$  is

$$LLT_{\lambda/\mu}(X; q) = \sum_{T \in SSDT^*(\lambda/\mu)} q^{\text{sp}(T)} X^T. \quad (5.4)$$

For  $\lambda \in \mathcal{P}^0(n)$ , the two families of functions are related through

$$\mathcal{G}_\lambda(X; q) = \sum_{\substack{k \geq 0, \\ \lambda/2k \in \mathcal{P}^0(n-k)}} x_0^k LLT_{\lambda/2k}(X; q). \quad (5.5)$$

Let  $XY$  be the product alphabet  $\{x_i y_j\}_{i,j}$  ordered by the lexicographical order. Note that positive indices in  $XY$  may include a negative index for  $Y$  e.g.  $x_2 y_{-3}$  has a positive index within the alphabet  $XY$ . Our  $q$ -deformed type B Cauchy identity can be stated as follows.

**Theorem 5.2.** *Let  $X$  and  $Y$  be two  $q$ -symmetric alphabets, we have:*

$$\mathcal{G}_{(2n)}(XY; q) = \sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_\lambda(X; q) \mathcal{G}_\lambda(Y; q). \quad (5.6)$$

*Proof.* Lam shows in [Lam06] the following skew domino Cauchy formula for LLT polynomials

$$\sum_{\lambda} LLT_{\lambda/\alpha}(X; q) LLT_{\lambda/\beta}(Y; q) = \prod_{i,j>0} \frac{1}{(1-x_i y_j)(1-q x_i y_j)} \sum_{\mu} LLT_{\beta/\mu}(X; q) LLT_{\alpha/\mu}(Y; q). \quad (5.7)$$

Denote  $Z_+^* = (z_i)_{i>0}$ ,  $Z_+ = (z_i)_{i\geq 0}$ ,  $Z_- = (z_i)_{i<0}$  for  $Z = X, Y$  and  $z = x, y$ . As a consequence to Equation (5.7), one has:

$$\begin{aligned} \sum_{\lambda} \mathcal{G}_{\lambda}(X; q) \mathcal{G}_{\lambda}(Y; q) &= \sum_{k,l,\lambda} x_0^k y_0^l LLT_{\lambda/2k}(X; q) LLT_{\lambda/2l}(Y; q) \\ &= \prod_{i,j>0} \frac{1}{(1-x_i y_j)(1-q x_i y_j)} \sum_{k,l,m} x_0^k y_0^l s_{l-m}(X_+^*) s_{k-m}(Y_+^*) \\ &= \prod_{i,j>0} \frac{1}{(1-x_i y_j)(1-q x_i y_j)} \sum_m (x_0 y_0)^m \sum_{k'} x_0^{k'} s_{k'}(Y_+^*) \sum_{l'} y_0^{l'} s_{l'}(X_+^*) \\ &= \prod_{i,j>0} \frac{1}{(1-x_i y_j)(1-q x_i y_j)} \left( \frac{1}{1-x_0 y_0} \right) \prod_{i>0} \frac{1}{1-x_0 y_i} \prod_{i>0} \frac{1}{1-y_0 x_i} \\ &= \prod_{i,j\geq 0} \frac{1}{(1-x_i y_j)} \prod_{i,j>0} \frac{1}{(1-q x_i y_j)}. \end{aligned}$$

Looking at homogeneous parts of the same degree, we derive

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_{\lambda}(X; q) \mathcal{G}_{\lambda}(Y; q) &= s_n(X_+ Y_+ \cup q X_+^* Y_+^*) \\ &= s_n(X_+ Y_+ \cup X_+^* Y_-) \\ &= s_n((XY)_+) \\ &= \mathcal{G}_{(2n)}(XY; q) \end{aligned} \quad (5.8)$$

This is the desired formula.  $\square$

## 5.2 Analytical proof for the bijections

### 5.2.1 Barbash and Vogan correspondence

The following proposition gives an analytical proof of the properties of the Barbash and Vogan bijection. Let  $X$  and  $Y$  be two  $q$ -symmetric alphabets.



**Proposition 5.3.** *For any integer  $n$ , the following identity is a consequence to Theorem 5.2.*

$$\sum_{\pi \in B_n} q^{\text{tc}(\pi)} F_{\text{Des}(\pi)}^B(X) F_{\text{Des}(\pi^{-1})}^B(Y) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T, U \in \text{SDT}(\lambda)}} q^{\text{sp}(T) + \text{sp}(U)} F_{\text{Des}(T)}^B(X) F_{\text{Des}(U)}^B(Y). \quad (5.9)$$

*Proof.* To prove Equation (5.9), one should develop  $F_{\emptyset}^B(XY)$  in two different ways. Firstly, according to Lemma 19,  $F_{\emptyset}^B(XY) = \mathcal{G}_{(2n)}(XY; q)$ . Then, using Equation (5.8) and Lemma 19 yields the right-hand side. But using the definition of Chow's quasisymmetric functions one has:

$$F_{\emptyset}^B(XY) = \sum_{(0,0) \leq (i_1, j_1) \leq \dots \leq (i_n, j_n)} x_{i_1} y_{j_1} \dots x_{i_n} y_{j_n}.$$

The constraints on the indices  $(i_k, j_k)$  can be split in two disjoint cases:

- (i)  $i_k > i_{k+1}$  &  $j_k < j_{k+1}$ ,
- (ii)  $i_k \geq i_{k+1}$  &  $j_k \geq j_{k+1}$ .

Then using the theory of *Type B P-partitions* as in [Cho01, Thm. 2.3.4] and the fact that  $Y$  is  $q$ -symmetric yields the left-hand side of Equation (5.9).  $\square$

### 5.2.2 Two equidistributed statistics on pairs of domino tableaux

Recall the Carré and Leclerc bijection between semistandard domino tableaux of weight  $\mu$  and bi-tableaux. We need the following statistic on domino tableaux.

*Definition 51* (Number of negative dominoes). For a standard domino tableau  $T$  with 2-quotient  $(T^-, T^+)$  given by the algorithm of Carré and Leclerc, let  $\text{neg}(T)$  be the number of boxes in  $T^-$ . The statistic  $\text{neg}(T)$  depends only on the shape of  $T$ .

$$T = \begin{array}{|c|c|c|} \hline -0+ & -0+ & - \\ \hline +2 & -2+ & -5 \\ \hline - & +5- & + \\ \hline +5 & -7+ & \\ \hline - & & \\ \hline \end{array} \longrightarrow \left( T^- = \begin{array}{|c|c|c|} \hline 5 & 5 & 5 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 2 & 2 \\ \hline 5 & 7 \\ \hline \end{array} \right)$$

FIGURE 5.1: A semistandard domino tableau and its 2-quotient.

**Example 5.1.** Figure 5.1 shows a semistandard domino tableau  $T$  of weight  $\mu = (2, 0, 2, 0, 0, 4, 0, 1)$  and its 2-quotient. In this example  $\text{neg}(T) = 3$ .

Equation (5.8) actually implies a surprising result.

**Theorem 5.4.** *Let  $n$  be a positive integer. There is a one-to-one correspondence between pairs of standard domino tableaux  $T, U$  of the same shape  $\lambda \in \mathcal{P}^0(n)$  and pairs of standard domino tableaux  $R, S$  of the same shape  $\mu \in \mathcal{P}^0(n)$  (possibly  $\lambda \neq \mu$ ) such that*

$$\text{Des}(T) = \text{Des}(R), \text{Des}(U) = \text{Des}(S)$$

and

$$\text{sp}(T) + \text{sp}(U) = \text{neg}(R) = \text{neg}(S).$$

*Proof.* By successive application of Theorem 5.2 and Lemma 19, one may compute

$$\mathcal{G}_{(2n)}(XY; q) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T, U \in \text{SDT}(\lambda)}} q^{\text{sp}(T) + \text{sp}(U)} F_{\text{Des}(T)}^B(X) F_{\text{Des}(U)}^B(Y). \quad (5.10)$$

Then we use standard properties of Schur function and the fact that for  $\lambda \in \mathcal{P}^0(n)$ ,

$$\mathcal{G}_\lambda(X; 1) = s_{\lambda^+}(X_+) s_{\lambda^-}(X_+^*)$$

to expand  $\mathcal{G}_{(2n)}(XY; q)$  differently.

$$\begin{aligned} \mathcal{G}_{(2n)}(XY; q) &= s_{(n)}(X_+ Y_+ \cup q X_+^* Y_+^*) \\ &= \sum_k s_{(n-k)}(X_+ Y_+) s_{(k)}(q X_+^* Y_+^*) \\ &= \sum_k q^k s_{(n-k)}(X_+ Y_+) s_{(k)}(X_+^* Y_+^*) \\ &= \sum_{\substack{\lambda^-, \lambda^+ \\ |\lambda^-| + |\lambda^+| = n}} q^{|\lambda^-|} s_{\lambda^+}(X_+) s_{\lambda^-}(X_+^*) s_{\lambda^+}(Y_+) s_{\lambda^-}(Y_+^*) \\ &= \sum_{\lambda \in \mathcal{P}^0(n)} q^{|\lambda^-|} \mathcal{G}_\lambda(X; 1) \mathcal{G}_\lambda(Y; 1) \\ &= \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ R, S \in \text{SDT}(\lambda)}} q^{\text{neg}(R)} F_{\text{Des}(R)}^B(X) F_{\text{Des}(S)}^B(Y). \end{aligned} \quad (5.11)$$

Equating Equations (5.10) and (5.11) yields the desired result.  $\square$

*Remark 5.5.* Combining the two classical extensions of the RS correspondence to signed permutations, there is a natural one-to-one relation between pairs of bi-tableaux and pairs of domino tableaux such that the number of boxes in the first Young tableau of each bi-tableau and the sum of the spins of the domino tableaux are equal.

### 5.3 Type B $q$ -Schur positivity

Given any subset  $\mathcal{B}$  of  $B_n$  and a statistic  $stat$  defined on the set  $\mathcal{B}$  we look at the type B quasisymmetric function

$$Q(\mathcal{B}, stat)(X) = \sum_{\pi \in \mathcal{B}} q^{stat(\pi)} F_{Des(\pi)}^B(X).$$

*Definition 52* (Type B  $q$ -Schur positivity). We say that a pair composed of a set  $\mathcal{B} \subset B_n$  and a statistic  $stat$  defined on the set  $\mathcal{B}$  is **type B  $q$ -Schur positive** or  **$q$ - $\mathcal{G}$  positive** if the function  $Q(\mathcal{B}, stat)$  can be written as

$$Q(\mathcal{B}, stat)(X) = \sum_{\lambda} c_{\lambda}(q) \mathcal{G}_{\lambda}(X; q), \quad (5.12)$$

where the  $c_{\lambda}(q)$  are polynomials in  $q^{\frac{1}{2}}$  with non-negative integer coefficients.

*Remark 5.6.* The family  $\{\mathcal{G}_{\lambda}(X; q)\}_{\lambda \in \mathcal{P}^0(n)}$  is not linearly independent. As a result, there may be several ways to expand a type B quasisymmetric function as in Equation (5.12).

The previous chapter deals with the case  $q = 1$ . We extend part of our results for general  $q$ .

**Proposition 5.7** (Type B  $q$ -Schur positivity of Knuth and inverse descent classes). *Set  $stat = tc$ . Type B inverse descent classes and type B Knuth classes are type B  $q$ -Schur positive.*

*Proof.* Let  $J \subset \{0\} \cup [n-1]$ . Extracting the coefficient in  $F_J^B(Y)$  in Equation (5.9) yields:

$$\sum_{\pi \in D_J^n} q^{tc(\pi)} F_{Des(\pi)}^B(X) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T, U \in SDT(\lambda) \\ Des(U)=J}} q^{sp(T)+sp(U)} F_{Des(T)}^B(X) \quad (5.13)$$

$$\begin{aligned} &= \sum_{Des(U)=J} q^{sp(U)} \sum_{T \in SDT(shape(U))} q^{sp(T)} F_{Des(T)}^B(X) \\ &= \sum_{Des(U)=J} q^{sp(U)} \mathcal{G}_{shape(U)}(X; q). \end{aligned} \quad (5.14)$$

Furthermore, for a standard domino tableau  $T$ , we have

$$\sum_{\pi \in C_T^B} q^{tc(\pi)} F_{Des(\pi)}^B(X) = q^{sp(T)} \mathcal{G}_{shape(T)}(X; q).$$

This gives the desired result. □

*Remark 5.8.* As the proof for signed arc permutations is not based on the bijection of Barbash and Vogan, the extension of their type B Schur positivity to general  $q$  is not obvious and may not even be true for non-trivial choices of *stat*.

**Proposition 5.9.** *For a permutation  $\pi \in B_n$ , let  $l(\pi)$  be its type B Coxeter length. The pair  $(B_n, l)$  is type B  $q$ -Schur positive. Furthermore, given a standard domino tableau  $T$ , denote  $\text{maj}(T) = \sum_{i \in \text{Des}(T)} i$  (the same definition applies to signed permutations). We have*

$$\sum_{\pi \in B_n} q^{l(\pi)} F_{\text{Des}(\pi)}^B(X) = \sum_{\substack{\lambda \in \mathcal{P}^0(n), \\ Q \in \text{SDT}(\lambda)}} q^{2 \text{maj}(Q) + \text{sp}(Q)} \mathcal{G}_{\text{shape}(Q)}(X; q). \quad (5.15)$$

*Proof.* For  $\pi \in B_n$ , let  $\text{fmaj}(\pi) = 2 \text{maj}(\pi) + \text{tc}(\pi)$ . We use a result from Foata in [FH07] proving a bijection  $\varphi$  between signed permutations with a given Coxeter length  $k$  and signed permutation with  $\text{fmaj}$  statistic equal to  $k$  such that for  $\pi \in B_n$ ,  $\text{Des}(\varphi(\pi)) = \text{Des}(\pi^{-1})$ . As a result,

$$\begin{aligned} \sum_{\pi \in B_n} q^{l(\pi)} F_{\text{Des}(\pi)}^B(X) &= \sum_{\pi \in B_n} q^{l(\pi^{-1})} F_{\text{Des}(\pi)}^B(X) \\ &= \sum_{\sigma \in B_n} q^{2 \text{maj}(\sigma) + \text{tc}(\sigma)} F_{\text{Des}(\sigma)}^B(X). \end{aligned}$$

Then using the fact that the statistic  $\text{maj}$  depends only on the descent set and applying Equation (5.14) prove the property.  $\square$

*Remark 5.10.* Proposition 5.7 and Theorem 5.9 are seen as consequences of Theorem 5.2 but may be proved independently using the Barbash and Vogan algorithm and its descent preserving and color-to-spin properties.



# Appendix A

## Admissible shapes

In Section 2.9 we were interested in investigating the description of admissible shapes and counting the percentage of the admissible shapes over all  $\mathcal{P}^0(n, m)$ . One can note that there are some unsolved questions left in this topic. Below we discuss some of them.

So, let  $\mu \in \mathcal{P}^0(m)$ ,  $\lambda \in \mathcal{P}^0(n)$ ,  $\lambda/\mu \in \mathcal{P}^0(n, m)$ . We wrote a computer program to check whether a skew shape  $\lambda/\mu$  is admissible or not. Using this program we compute the fraction

$$\frac{\tilde{\mathcal{P}}^0(n, m)}{\mathcal{P}^0(n, m)} \cdot 100\%.$$

The following table contains the percentages of admissible shapes over all shapes of  $\mathcal{P}^0(n, m)$ .

| $n \setminus m$ | 0    | 1      | 2      | 3      | 4      | 5      | 6      | 7    |
|-----------------|------|--------|--------|--------|--------|--------|--------|------|
| 0               | —    | —      | —      | —      | —      | —      | —      | —    |
| 1               | 100% | —      | —      | —      | —      | —      | —      | —    |
| 2               | 100% | 100%   | —      | —      | —      | —      | —      | —    |
| 3               | 100% | 85.71% | 100%   | —      | —      | —      | —      | —    |
| 4               | 100% | 73.33% | 88.10% | 100%   | —      | —      | —      | —    |
| 5               | 100% | 62.07% | 75.56% | 87.13% | 100%   | —      | —      | —    |
| 6               | 100% | 53.70% | 66.12% | 75.44% | 85.96% | 100%   | —      | —    |
| 7               | 100% | 46.32% | 57.06% | 65.52% | 73.66% | 84.42% | 100%   | —    |
| 8               | 100% | 40.49% | 50.41% | 57.24% | 63.78% | 71.30% | 83.81% | 100% |

Looking at the table one can note that

- Given  $m = 0$  or  $m = n$ , one have  $\mathcal{P}^0(n, m) = \tilde{\mathcal{P}}^0(n, m)$ , so

$$\frac{\tilde{\mathcal{P}}^0(n, 0)}{\mathcal{P}^0(n, 0)} = 1 = \frac{\tilde{\mathcal{P}}^0(n, n)}{\mathcal{P}^0(n, n)}$$

- For any fixed  $m \in \mathbb{Z}_+$ , the entries of the  $m$ -th column tend to zero

$$\lim_{n \rightarrow \infty} \frac{\tilde{\mathcal{P}}^0(n, m)}{\mathcal{P}^0(n, m)} = 0$$

The first item is quite obvious, the second one was proved in Section 2.9. This two items leads us to the following question.

*Question 1.* Given a constant  $0 < \alpha < 1$  let  $m \sim \alpha n$ . Compute

$$\lim_{n \rightarrow \infty} \frac{\tilde{\mathcal{P}}^0(n, m)}{\mathcal{P}^0(n, m)}.$$

One can also note that for any fixed  $n$  the entries of the  $n$ -th row (except the first entry) compose an increasing sequence.

*Question 2.* Show (or find a counterexample to) the fact that sequence  $\{\frac{\tilde{\mathcal{P}}^0(n, m)}{\mathcal{P}^0(n, m)}\}_{m \in [1, n-1]}$  is increasing for any  $n \in \mathbb{Z}_+$ .

## Appendix B

# More type B Schur positive sets

Despite we have found and proved a number of examples of type B Schur positive sets, there are still some ones for which we think that they hopefully should be  $\mathcal{G}$ -positive.

One of the important examples of Schur-positive sets are conjugacy classes and therefore, any subsets of  $S_n$  closed under conjugation. Computer computations shows that at least for  $n = 2, 3, 4, 5$  all the conjugacy classes of  $B_n$  are  $\mathcal{G}$ -positive.

*Example 11.* Let  $n = 3$ . The set of domino functions have following decompositions in fundamental functions:

$$\begin{array}{ll} \mathcal{G}_6 = F_\emptyset, & \mathcal{G}_{3,1,1,1} = F_{0,1} + F_{0,2}, \\ \mathcal{G}_{5,1} = F_0, & \mathcal{G}_{2,2,2} = F_1 + F_{0,2} + F_{1,2}, \\ \mathcal{G}_{4,2} = F_0 + F_1 + F_2, & \mathcal{G}_{2,2,1,1} = F_{0,1} + F_{0,2} + F_{1,2}, \\ \mathcal{G}_{4,1,1} = F_1 + F_2, & \mathcal{G}_{2,1,1,1,1} = F_{1,2}, \\ \mathcal{G}_{3,3} = F_0 + F_1 + F_{0,2}, & \mathcal{G}_{1,1,1,1,1,1} = F_{0,1,2}. \end{array}$$

The table below shows all the conjugacy classes and one of the possible decompositions in domino functions.



| conjugacy class   | decomposition   |
|---|---|
| $\{(1, 2, 3)\}$   | $\mathcal{G}_6$   |
| $\{(1, 2, -3), (1, -2, 3), (-1, 2, 3)\}$  | $\mathcal{G}_{4,2}$   |
| $\{(1, -2, -3), (-1, 2, -3), (-1, -2, 3)\}$   | $\mathcal{G}_{3,1,1,1} + \mathcal{G}_{2,1,1,1,1}$                 |
| $\{(-1, -2, -3)\}$  | $\mathcal{G}_{1,1,1,1,1,1}$                                       |
| $\{(-1, -2, -3)\}$  | $\mathcal{G}_{1,1,1,1,1,1}$                                       |
| $\{(1, 3, 2), (3, 2, 1), (1, -3, -2),$<br>$(-3, 2, -1), (2, 1, 3), (-2, -1, 3)\}$                             | $\mathcal{G}_{4,2} + \mathcal{G}_{2,2,2}$                         |
| $\{(1, 3, -2), (1, -3, 2), (3, 2, -1),$<br>$(-3, 2, 1), (2, -1, 3), (-2, 1, 3)\}$                             | $\mathcal{G}_{4,2} + \mathcal{G}_{2,2,2}$                         |
| $\{(-1, 3, 2), (-1, -3, -2), (3, -2, 1),$<br>$(-3, -2, -1), (2, 1, -3), (-2, -1, -3)\}$                       | $\mathcal{G}_{3,1,1,1} + \mathcal{G}_{5,1} + \mathcal{G}_{2,2,2}$ |
| $\{(-1, 3, -2), (-1, -3, 2), (3, -2, -1),$<br>$(2, -1, -3), (-3, -2, 1), (-2, 1, -3)\}$                       | $\mathcal{G}_{3,1,1,1} + \mathcal{G}_{5,1} + \mathcal{G}_{2,2,2}$ |
| $\{(2, 3, 1), (2, -3, -1), (3, 1, 2), (-3, 1, -2),$<br>$(3, -1, -2), (-2, 3, -1), (-2, -3, 1), (-3, -1, 2)\}$ | $\mathcal{G}_{3,1,1,1} + \mathcal{G}_{4,2} + \mathcal{G}_{2,2,2}$ |
| $\{(2, 3, -1), (3, -1, 2), (2, -3, 1), (3, 1, -2),$<br>$(-3, -1, -2), (-2, 3, 1), (-3, 1, 2), (-2, -3, -1)\}$ | $\mathcal{G}_{3,1,1,1} + \mathcal{G}_{4,2} + \mathcal{G}_{2,2,2}$ |

*Question 3.* Show (or find a counterexample to) the fact that conjugacy classes of  $B_n$  are  $\mathcal{G}$ -positive for any  $n \in \mathbb{Z}_+$ .

In Chapter 4 we showed that type B analogue for arc permutations, called signed arc permutations, are  $\mathcal{G}$ -positive. However, there is another way to define a type B analogue of arc permutations.

*Definition 53.* A signed permutation  $\pi \in B_n$  is called  **$B$ -arc permutation** if for every  $i \in [n]$  the set

$$\{\pi(i), \pi(i+1), \dots, \pi(n)\}$$

is an interval in  $\mathbb{Z}_{2n}$ .

Computations for small  $n$  shows that these subsets are also  $\mathcal{G}$ -positive.

*Question 4.* Show (or find a counterexample to) the fact that  $B$ -arc permutations are  $\mathcal{G}$ -positive.

## Appendix C

# Spin and negative number statistics

Recall, given a standard domino tableaux  $T$  of shape  $\lambda$  one define

$$\text{Sp}(T) = 2\text{sp}(T) = \#\{\text{vertical dominoes in } T\}; \quad \text{neg}(T) = |\lambda^-|.$$

Let us consider the generating functions related to the statistics above

$$G_{\text{Sp}}(n) = \sum_{T \in \text{SDT}(2n)} q^{\text{Sp}(T)}; \quad G_{\text{neg}}(n) = \sum_{T \in \text{SDT}(2n)} q^{\text{neg}(T)}.$$

*Example 12.* Computations for small  $n$  shows that these two functions coincide.

| $n$ | $G_{\text{Sp}}(n) / G_{\text{neg}}(n)$                       |
|-----|--|
| 1   | $1q^0 + 1q^1$  |
| 2   | $2q^0 + 2q^1 + 2q^2$   |
| 3   | $4q^0 + 6q^1 + 6q^2 + 4q^3$                                  |
| 4   | $10q^0 + 16q^1 + 24q^2 + 16q^3 + 10q^4$                      |
| 5   | $26q^0 + 50q^1 + 80q^2 + 80q^3 + 50q^4 + 26q^5$              |
| 6   | $76q^0 + 156q^1 + 300q^2 + 320q^3 + 300q^4 + 156q^5 + 76q^6$ |

So, the interesting question to consider follows.

*Question 5.* Show the coincidence of two functions  $G_{\text{Sp}}(n)$  and  $G_{\text{neg}}(n)$  for any  $n \in \mathbb{Z}_+$ , i.e. equidistribution of two statistics or find a counterexample to this fact.

Secondly, one can note that sequences of coefficients of these two functions have several properties. The obvious property is that coefficient of  $q^k$  equals to the coefficient of  $q^{n-k}$ . Indeed, it is enough to conjugate the tableaux to prove that. Applying this operation

changes both the statistics from  $k$  to  $n - k$ . For negative number statistic, we may also change the positive and negative part of the corresponding bi-tableau.

Another one is not so obvious.

*Question 6.* Prove (or find a counterexample to) the fact that sequences of coefficients of two functions,  $G_{\text{Sp}}(n)$  and  $G_{\text{neg}}(n)$ , are unimodal, i.e. the coefficients grow up in the left half of the sequence and decrease in the right half.

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**Titre :** Liens combinatoires entre fonctions quasisymétriques et tableaux dans les groupes de Coxeter.

**Mots clés :** fonctions quasisymétriques de type B, fonctions de Schur, identité de Cauchy, algèbre de descente, groupe hyperoctaédrique, tableaux de dominos.

**Résumé :** L'algèbre des fonctions symétriques est un outil majeur de la combinatoire algébrique qui joue un rôle central dans la théorie des représentations du groupe symétrique. Cette thèse traite des fonctions quasisymétriques, une puissante généralisation introduite par Gessel en 1984, avec des applications significatives dans l'énumération d'objets combinatoires majeurs tels que les permutations, les tableaux de Young et les P-partitions et dans l'étude de fonctions symétriques avancées telles que les polynômes de Schubert et Macdonald. Plus précisément, nous trouvons un nouveau lien entre l'extension des fonctions

quasisymétriques de Chow à des groupes de Coxeter de type B et des tableaux de dominos, c'est-à-dire des diagrammes de Young pavés par des rectangles  $2 \times 1$  et  $1 \times 2$ . Ceci nous permet d'apporter de nouveaux résultats dans divers domaines, notamment les constantes de structure de l'algèbre de descente de Solomon de type B, l'extension de la théorie de la Schur-positivité aux permutations signées et l'étude d'une formule de Cauchy de type B  $q$ -déformée avec des implications importantes statistiques pour les tableaux dominos.

**Title :** Combinatorial links between quasisymmetric functions and tableaux for Coxeter groups.

**Keywords :** type B quasisymmetric functions, Schur functions, Cauchy identity, descent algebra, hyperoctahedral group, domino tableaux.

**Abstract :** The algebra of symmetric functions is a major tool in algebraic combinatorics that plays a central rôle in the representation theory of the symmetric group. This thesis deals with quasisymmetric functions, a powerful generalisation introduced by Gessel in 1984, with significant applications in the enumeration of major combinatorial objects as permutations, Young tableaux and P-partitions and in the study of advanced symmetric functions like Schubert and Macdonald polynomials. More specifically we find

a new connection between Chow's extension of quasisymmetric functions to Coxeter groups of type B and domino tableaux, i.e. Young diagrams tiled with  $2 \times 1$  and  $1 \times 2$  rectangles. It allows us to contribute new results to various fields including the structure constants of type B Solomon's descent algebra, the extension of the theory of Schur-positivity to signed permutations and the study a  $q$ -deformed type B Cauchy formula with important implications regarding statistics for domino tableaux.