Stabilisation de systèmes hyperboliques non-linéaires en dimension un d'espace

Amaury Hayat

Sorbonne Université
Paris 6

Laboratoire Jacques-Louis Lions
UMR 7598

Thèse de Doctorat de Sorbonne Université

Présentée et soutenue publiquement le jeudi 16 mai 2019

pour l'obtention du grade de
Docteur de Sorbonne Université

Spécialité : Mathématiques Appliquées

par

Amaury HAYAT

sous la direction de
Jean-Michel CORON
et
Sébastien BOYAVAL

Sujet de la thèse :

Stabilisation de systèmes hyperboliques non-linéaires en dimension un d'espace

devant le jury composé de :

Mme	Karine Beauchard	Examinatrice
M.	Sébastien Boyaval	Directeur de Thèse
M.	Eric Cancès	Examinateur
M.	Yacine Chitour	Examinateur
M.	Jean-Michel Coron	Directeur de Thèse
M.	Antoine Girard	Rapporteur
M.	Günter Leugering	Rapporteur
M.	Benoit Perthame	Examinateur

École Doctorale de Sciences Mathématiques
de Paris Centre ED 386

Faculté de Mathématiques
UFR 929

Amaury HAYAT :

Sorbonne Université, Université Paris-Diderot SPC, CNRS, INRIA, Laboratoire Jacques-Louis Lions, LJLL, équipe CAGE, F-75005 Paris.

Adresse électronique : hayat@ljll.math.upmc.fr

Remerciements

Je souhaite commencer par adresser mes plus chaleureux remerciements à Jean-Michel Coron. Je savais en arrivant en thèse que j'allais être encadré par un grand mathématicien, au niveau scientifique plus qu'impressionnant, le dire est presque un lieu commun. Ce que je ne savais pas, c'est que j'allais avoir un directeur de thèse extrêmement humble, toujours très bienveillant, très optimiste, et qui se rend disponible en cas de besoin malgré tous ses engagements. C'était un véritable plaisir de travailler avec toi pendant ces presque trois ans. Merci Jean-Michel. Je souhaite aussi remercier Sébastien Boyaval qui s'est toujours rendu disponible pour discuter et m'aiguiller, et dont les remarques ont toujours été très pertinentes. Je remercie aussi Nicole Goutal et Eric Demay qui m'ont soutenu, écouté et questionné, et j'ai beaucoup appris de nos discussions pendant ces trois ans.
J'aimerais remercier Antoine Girard et Günter Leugering pour avoir accepté de rapporter ma thèse et pour leurs commentaires, ainsi que Karine Beauchard, Eric Cancès, Yacine Chitour et Benoit Perthame pour le temps qu'ils prennent pour assister à ma soutenance et évaluer ma thèse. C'est un honneur que vous me faites.
Une organisation tourne quand ceux qui la font tourner au jour le jour le font bien, c'est aussi valable pour le Laboratoire Jacques-Louis Lions. J'aimerais remercier Catherine, Malika et Salima qui m'ont toujours aidé avec une grande ouverture d'esprit même quand ma situation administrative semblait désespérée, et Khashayar toujours très réactif et prompt à résoudre tous les problèmes avec une efficacité impressionnante. Le laboratoire a de la chance de vous avoir.
Pendant cette thèse plusieurs personnes se sont intéressées à mes travaux, m'ont soutenu et m'ont aidé à poursuivre vers une carrière de recherche. Je leur en suis très reconnaissant, en particulier Eric Cancès, Alexandre Ern, Tony Lelièvre, Gabriel Stoltz, Julien Reygner, Emmanuel Trélat, Federico Capasso et Henri-Jean Drouhin. J'aimerais aussi remercier tous les scientifiques qui m'ont conseillé dans mon parcours, et m'ont partagé leur goût des sciences, en commençant par mes professeurs de prépa, de l'X et de lycée. Je remercie aussi tout particulièrement l'ensemble du Capasso Group et en premier lieu Federico Capasso pour m'avoir fait découvrir concrètement la recherche dans un cadre exceptionnel ainsi que Balthasar Müller qui m'a encadré pour travailler ensemble sur un sujet fascinant.

Cette thèse n'a pas débuté par hasard, et je remercie Karine Beauchard à plusieurs titres : d'abord pour m'avoir convaincu de faire des mathématiques quand j'hésitais avec la physique, pour m'avoir conseillé dans ma recherche d'une thèse et pour m'avoir recommandé auprès de Jean-Michel. Il est au moins une autre personne sans qui cette thèse n'aurait peut-être jamais eu lieu : je remercie Françoise Prêteux pour son soutien en amont et tout au long de cette thèse. Je remercie aussi le Corps des IPEF et le ministère chargé de l'environnement qui ont permis cette thèse, ainsi que François Moreau pour ses conseils. Enfin, je remercie également Frédéric Marbach pour ses conseils.

Loin d'avoir travaillé seul ces dernières années, je souhaite remercier grandement Peipei Shang, pour sa bonne humeur permanente, pour sa patience, pour m'avoir supporté pendant nos longues séances de travail, et pour m'avoir invité à Tongji. Je remercie aussi Georges Bastin pour m'avoir initié à des problèmes très intéressants, pour son enthousiasme, pour tout ce qu'il m'a appris et pour ses remarques toujours très pertinentes. Je remercie Zhiqiang Wang et Tatsien Li, pour leur accueil chaleureux et les discussions que nous avons eues à Fudan. Enfin je conclus ce paragraphe en adressant un immense remerciement à Tristan Rivière et à tout le personnel du FIM pour le très bel accueil que j'ai eu à l'ETH Zürich et les formidables mois que j'y ai passés.

A ce point vous avez peut-être peur que ces remerciements s'éternisent. Peur justifiée. Au moment d'aborder le paragraphe des doctorants, il y a deux personnes que je me dois de mentionner en premier : Christophe et Shengquan. La Team. Je ne compte plus les expériences passées ensemble ces dernières années. D'abord à Zurich avec les visites de Google (merci Manu), les discussions mathématiques (merci Alexis M.), les dîners, le Zeughauskeller, les escapades au lac (merci Ulrike), les fins d'après-midi d'été, les sorties du DMAT, le lac des quatre Cantons, l'Uetliberg, etc. Pèle-mèle je citerais aussi l'ICM, Rio, la SMAI, Naples, les tarots (merci Audrey et Aurore), et bien sûr le bac d'eau. Merci pour nos discussions, pour votre sympathie, et pour tout ce qu'on a partagé ces dernières années.
Je remercie aussi le groupe des irréductibles : Ludie Ludie [1] j'espère que tu continueras à voir la vie en orange, Federica à qui je dois encore une demie Saint-Valentin, Antoine paradoxalement aussi raisonnable que délirant et armé d'une sympathie sans faille, Camille dont la gentillesse n'égale que l'originalité de son humour et Alex D. dont zénitude semble à toute épreuve.
En thèse, les gens avec qui on passe le plus de temps sont ceux de son bureau, et j'ai eu la chance d'avoir un bureau exceptionnel : d'abord Chaoyu et sa gentillesse, Carlo et son flegme cool, Shuyang et son humour, puis Gabriela avec son expressivité et nos longues discussions, Alex R. et son enthousiasme, Elise sa bonne humeur et nos séances de thé [2], Rim, Léa et tous ceux qui ont été de passage. Enfin je remercie l'ensemble des doctorantes et doctorants que j'ai croisés au LJLL et avec qui j'ai partagé ces dernières années au laboratoire. En particulier Jean et Idriss [3] (Team GTT rpz), Anouk, Olivier, Allen, Fatima, Martin, Gontran, Eu et tous les autres que ne je peux pas mentionner ici par manque de place, mais ce n'est pas l'envie qui m'en manque.

J'aimerais aussi remercier mes amis et mes proches de tous bords qui n'ont parfois pas contribué directement à cette thèse mais qui ont largement contribué à m'amener ici. Je commence par un groupe de personnes qui me sont très chères et que j'ai rencontrées sur le terrain de rugby d'un lycée préparationnaire [4] il y a bientôt dix ans. William et Mathilde, Sylvain et Sophie C., Romain et Sophie M., Quentin et Mathieu, Maxence, Léa et bien sûr Anrong. Je ne ferai pas de commentaires, tant il y aurait de choses à dire. On dit souvent qu'on est la somme des gens avec qui on passe le plus de temps, j'ai de la chance que vous en soyez. Je souhaite aussi remercier Alexis, le vieux loup, Coralie, Chloé, Thomas, Etienne, Soso, Louis, Aglaé, Paul-Henri avec qui on se suit depuis longtemps, Benjamin D., Tanguy et Adrien, la fine équipe de Cambridge, Jérome, Moh, Pierre et Guillaume, qui me supportent au quotidien, Jean-Vincent, Adélie et Romuald, et tant d'autres avec qui j'ai passé tant de bons moments et que j'ai la chance de recroiser plus ou moins fréquemment. Je remercie aussi Valentine, Fanny, Cédric, Quentin L., Benjamin C., Lalau, Hélène, Mathias, Arnaud, Alban, Clémentine, et toute la promotion IPEF 2016.

Enfin, je remercie aussi (et bien sûr !) toute ma famille, mes grands-parents Roger, Michelle, Hélène et Noël, les cousins, tout le clan des Hayat, mes sœurs formidables Lauriane et Amandine, qui se sont habituées à avoir un grand frère spécial et m'auront pardonné, j'espère, d'être parti de la maison pour aller en internat à 17 ans. Je finirai sur ceux qui ont toujours été là pour moi sans limite, et à qui je dois le plus : mes parents Serge et Martine.

1. qui m'a fait promettre d'apparaitre deux fois dans ces remerciements, c'est chose faite.
2. content de t'avoir comme élève.
3. dont les efforts pour avoir une note de bas de page ont payé, merci pour les discussions sur les références historiques au passage.
4. rendons à Ginette ce qui est à Ginette, j'y ai passé deux années fondatrices.

à mes grands-parents
Roger, Michelle, Hélène et Noël,
qui ont toujours encouragé ma curiosité
et mon goût pour les sciences.

Table des matières

Chapitre 1

Introduction

1.1 Equations différentielles

Comprendre et décrire le monde qui nous entoure a toujours été une motivation des mathématiques. Dans cette démarche, l'apparition de la notion de dérivée a été un tournant majeur et a permis l'arrivée d'un outil fondamental permettant de décrire la dynamique du monde : les équations différentielles. L'équation différentielle la plus célèbre est sans doute le principe fondamental de la dynamique énoncé par Newton en 1687 qui s'écrit ainsi :

$$m\frac{d^2x}{dt^2} = \sum F_{ext}, \tag{1.1.1}$$

où m est la masse de l'objet considéré, x son vecteur position et F_{ext} les forces extérieures vectorielles qui s'appliquent dessus. Cette équation ouvrit la porte à la description des systèmes physiques ponctuels et fut utilisée massivement dès lors et ce jusqu'à aujourd'hui.

En réalité la notion de dérivée n'a été formalisée dans sa forme la plus rigoureuse qu'au 19ème siècle, ce qui n'a pas empêché l'essor des équations différentielles dès la fin du 17ème siècle et l'établissement d'un nombre exceptionnel d'études et de méthodes de résolutions. Depuis, les équations différentielles ont fédéré de nombreux domaines des mathématiques, de l'algèbre aux probabilités en passant par la géométrie et la théorie des nombres.

1.2 Equations aux dérivées partielles

Les équations différentielles ordinaires modélisent très bien les systèmes ponctuels mais représentent mal les systèmes continus. Comment décrire par exemple le mouvement d'une vague ? Ou la propagation d'une vibration dans une corde ? Pour ces systèmes on aimerait introduire un nombre infini de points, chacun obéissant à une équation différentielle. C'est ce qui motiva l'apparition des équations aux dérivées partielles ou EDP en abrégé. On peut alors définir les dérivées par rapport à une variable pour des fonctions de plusieurs variables, $\partial_t f(t,x)$ représente par exemple la dérivée par rapport au temps d'une quantité qui dépend à la fois du temps t et de la position x. La conservation de la masse d'un fluide avec une densité ϱ qui se déplace avec une vitesse V peut s'écrire par exemple :

$$\partial_t \varrho(t,x) = -\partial_x(\varrho(t,x)V(t,x)). \tag{1.2.1}$$

Les EDP ne sont en fait apparues que vers le milieu du 18ème siècle. A l'époque les mathématiciens cherchaient des méthodes de résolution, souvent géométriques, pour ramener les problèmes qui auraient dû mener à des EDP à des problèmes plus simples. On doit leur apparition à Euler et d'Alembert vers 1750 [69, 73, 87]. Les premières méthodes générales de résolution sont dues à Charpit en 1784 [92], dont les travaux n'ont été connus que plus tard et ont été complétés par Lagrange et Monge. Ces travaux ont permis l'arrivée de la méthode des caractéristiques pour les équations de premier ordre. Depuis, les EDP ont nourri de nombreux problèmes mathématiques. Elles sont omniprésentes en physique, en mécanique, et ont mené à de nombreuses théorie mathématiques (distributions, solutions de viscosité, etc.).

1.3 Théorie du contrôle

Quand on sait comment fonctionne un système et qu'on peut agir dessus, que peut-on lui faire faire ? Cette question, que Kalman s'est posée en 1960, n'est pas nouvelle et n'aurait pas été reniée par le *Prince* de Machiavel. Le mérite de Kalman est d'avoir amené cette question dans un domaine quantitatif et connu pour ses outils extrêmement puissants : les mathématiques. Le contrôle peut se décomposer en trois branches :

— La contrôlabilité : quand mon système part d'un état x_0, est-il possible de l'amener à un état x_1 voulu ?

— Contrôle optimal : s'il existe des trajectoires amenant mon système à l'état voulu, y en a t-il une optimale pour un critère donné, et si oui laquelle ?

— Stabilisation : si mon état ou ma trajectoire est fixée, est-il possible de la rendre stable, c'est-à-dire robuste aux perturbations qui pourraient se produire ?

Dans les deux premiers problèmes, le contrôle dépend de l'état initial du système [1] x_0, c'est ce qu'on appelle une boucle ouverte. Dans le troisième, par contre, le contrôle est une fonction de l'état du système au temps t, c'est ce qu'on appelle un feedback ou une rétroaction. On dit alors que le système est en boucle fermée. D'un point de vue mathématique cette différence fondamentale signifie que plusieurs questions doivent être repensées. Par exemple le caractère bien posé de l'équation $\dot{x} = f(t, x, u(t))$ avec une condition initiale x_0 donnée est bien connu quand u n'est qu'une fonction de x_0 et t. Celui de $\dot{x} = f(t, x, u(t, x))$ en revanche nécessite d'être un peu plus attentif aux propriétés du contrôle u et sa dépendance en x. D'un point de vue pratique, cette différence signifie qu'on agit sur le système en fonction de ce qu'on mesure. Et, bien évidemment, dans un système de dimension infinie où l'état du système x est une fonction, se pose la question de savoir ce qu'on s'autorise à mesurer.

En dimension infinie, c'est-à-dire pour les problèmes modélisés par des équations aux dérivées partielles, il existe deux grands types de contrôles [2] :

— **Le contrôle interne.** Dans ce cas le contrôle est directement inclus à la dynamique du système et agit sur un ensemble de points à l'intérieur du domaine, souvent de mesure non-nulle, par exemple

$$\begin{aligned} &\partial_t u + \partial_x u = f(t, x), \ \text{sur } [0, 1] \\ &u(t, 0) = u(t, 1) \end{aligned} \tag{1.3.1}$$

où f est un contrôle avec un support spatial inclus dans $[0, 1]$. C'est le cas par exemple d'un champ magnétique qu'on applique à un matériau conducteur.

— **Le contrôle aux bords.** Dans ce cas le contrôle agit aux limites du domaine (et donc sur un sous-ensemble de mesure nulle). La dynamique du système reste inchangée mais est couplée avec des conditions aux bords sur lesquelles on agit. Par exemple

$$\begin{aligned} &\partial_t u + \partial_x u = 0, \ \text{sur } [0, 1] \\ &u(t, 0) = f(t) \end{aligned} \tag{1.3.2}$$

où f est un contrôle. C'est le cas par exemple d'un fleuve sur lequel on peut agir via des barrages situés à ses extrémités, ou encore d'une autoroute dont on contrôle le flux entrant à l'aide de feux rouges.

Pour de nombreux systèmes il est difficile d'avoir accès directement à l'intérieur du système pour le contrôler. Pour les fleuves par exemple, on peut uniquement agir sur les barrages en amont et en aval. Les contrôles aux bords sont donc souvent un bon choix physique, c'est pourquoi ce sont les contrôles que nous considérerons

1. En réalité les solutions des problèmes de contrôle optimal peuvent parfois aussi prendre la forme de feedback (cf. équations de Riccati [167, 190])

2. Cette distinction n'existe pas en dimension finie puisqu'il n'y a pas de condition aux bords à prescrire, tous les contrôles sont donc des contrôles internes

dans la suite. Par ailleurs on s'intéresse au problème de stabilisation, les contrôles que nous utiliserons seront donc des feedbacks, c'est-à-dire $f(t) = f(u(t,\cdot))$. Enfin, on s'imposera une contrainte supplémentaire : on ne considérera que des contrôles "simples", c'est-à-dire des contrôles qui ne nécessitent de mesurer que l'état du système aux bords du domaine. En d'autres termes

$$f(u(t,\cdot)) = f(u(t,\cdot)_{|\partial\Omega}), \tag{1.3.3}$$

où Ω est le domaine. L'idée derrière une telle contrainte est double : d'une part ces contrôles sont historiquement les plus utilisés dans les études mathématiques ce qui nous permet à la fois de comparer nos résultats et nos méthodes. D'autre part cette contrainte permet de faire en sorte que l'implémentation des contrôles dans les systèmes physiques soit simple, sans avoir à mesurer de quantité à l'intérieur du domaine (par exemple à l'intérieur d'un fleuve), quitte à ce que les mathématiques derrière soient plus compliquées.

1.4 Stabilisation des systèmes d'équations aux dérivées partielles

Contrairement aux équations différentielles ordinaires, il est difficile d'étudier toutes les équations aux dérivées partielles comme une seule classe d'objet. Néanmoins on peut essayer de les classifier en trois types :

- Les équations elliptiques.
- Les équations paraboliques.
- Les équations hyperboliques.

Certaines équations, plus compliquées, sont un mélange entre ces trois types. On peut alors essayer d'identifier, au moins localement, un comportement prédominant correspondant à une de ces classes. Les solutions des équations elliptiques sont entièrement déterminées par leurs conditions aux bords, le problème de leur stabilisation par des contrôles aux bords n'a donc pas vraiment de sens. Concernant les équations paraboliques, de nombreux résultats existent (par exemple [9, 32, 63, 89, 149, 191], voir aussi [177, Section 7]), même s'il reste encore beaucoup de questions ouvertes. On peut encore citer [67, 174, 175, 189] pour des exemples concernant des équations des fluides avec des comportements paraboliques. Enfin, la stabilisation des équations et des systèmes d'équations hyperboliques est encore mal connue, même en dimension 1 d'espace. C'est à ces systèmes que nous nous intéresserons dans cette thèse.

1.4.1 Généralités sur les systèmes hyperboliques 1D

Un système d'EDP hyperbolique quasilinéaire 1D a la forme suivante :

$$\partial_t \mathbf{Y} + F(\mathbf{Y})\partial_x \mathbf{Y} + G(\mathbf{Y}, x) = 0. \tag{1.4.1}$$

où $\mathbf{Y} \ : [0, +\infty[\times[0, L] \to \mathbb{R}^n$ est l'état du système, F est une fonction de U dans $\mathcal{M}_n(\mathbb{R})$, l'espace des matrices carrées de dimension n, telle que pour tout $\mathbf{Y} \in U$, $F(\mathbf{Y})$ est diagonalisable avec des valeurs propres distinctes et réelles, U est un ouvert connexe non vide de $\mathbb{R}^n$, et G est une fonction de $U \times [0, L]$ dans $\mathbb{R}^n$. On suppose par ailleurs que pour tout $\mathbf{Y} \in U$, les valeurs propres de $F(\mathbf{Y})$ ne s'annulent pas, et que F et G sont de classe C^1.

Un changement de coordonnées Par définition, un système hyperbolique est un système qui possède des quantités qui se propagent. Néanmoins ces quantités peuvent être différentes des composantes de $\mathbf{Y}$ quand la matrice $F(\mathbf{Y})$ n'est pas diagonale. Pour les retrouver localement autour d'un état stationnaire $\mathbf{Y}^*$, on peut introduire le changement de variables suivant : $\mathbf{u} = S(\mathbf{Y} - \mathbf{Y}^*)$, avec S une matrice qui diagonalise $F(\mathbf{Y}^*)$. Les composantes de $\mathbf{u}$ représentent alors localement les quantités qui se propagent avec, pour vitesse de propagation, les valeurs propres de $F(\mathbf{Y}^*)$. Le système prend alors la forme

$$\partial_t \mathbf{u} + A(\mathbf{u}, x)\partial_x \mathbf{u} + B(\mathbf{u}, x) = 0. \tag{1.4.2}$$

où $A(\mathbf{0}, x) = \Lambda(x)$ est la matrice diagonale des valeurs propres de $F(\mathbf{Y}^*)$, et $B(\mathbf{0}, x) = 0$. A et B sont alors à nouveau de classe C^1.

Un phénomène caractéristique : les chocs. Les systèmes non-linéaires hyperboliques peuvent présenter un phénomène étonnant : même en partant d'une condition initiale très régulière à $t = 0$, la solution peut spontanément évoluer vers une solution qui "casse" en temps fini et présente des discontinuités. Ces discontinuités sont appelées des chocs. On peut illustrer cela avec l'équation de Burgers qu'on présentera plus en détail en Section 1.7.1,

$$\partial_t u + u\partial_x u = 0. \tag{1.4.3}$$

Pour simplifier on étudie cette équation sur $\mathbb{R}$ tout entier avec pour condition initiale $u_0 \in C^1(\mathbb{R})$. Supposons que la solution est régulière pour tout temps, on définit la courbe caractéristique $x(t)$ démarrant en x_0 comme l'unique solution de

$$\begin{aligned} x'(t) &= u(t, x(t)), \\ x(0) &= x_0. \end{aligned} \tag{1.4.4}$$

Alors, le long d'une caractéristique $x(t)$, la solution u de (1.4.3) vérifie

$$\frac{d}{dt}(u(t, x(t))) = 0. \tag{1.4.5}$$

Fixons deux caractéristiques $(x_1(t), t)$ et $(x_2(t), t)$ démarrant en $x_1(0) = x_{1,0} < x_{2,0} = x_2(0)$. D'après (1.4.3) et (1.4.5) ces caractéristiques vérifient :

$$x_1(t) = u_0(x_{1,0})t + x_{1,0} \quad \text{et} \quad x_2(t) = u_0(x_{2,0})t + x_{2,0}. \tag{1.4.6}$$

Supposons maintenant que $u_0(x_{1,0}) > u_0(x_{2,0})$, alors pour $t = \frac{x_1 - x_2}{u_0(x_2) - u_0(x_1)} > 0$, $x_1(t) = x_2(t)$. Mais comme ce sont des caractéristiques, d'après (1.4.5),

$$u(t, x_1) = u_0(x_{1,0}) \quad \text{et} \quad u(t, x_2) = u_0(x_{2,0}), \tag{1.4.7}$$

si bien que $u_0(x_{1,0}) = u_0(x_{2,0})$, ce qui amène à une contradiction. Définir une solution pour tout temps dans cette situation impose alors nécessairement qu'elle soit discontinue. En intégrant (1.4.3), on peut trouver la vitesse à laquelle ces discontinuités se propagent et, pour une discontinuité localisée en $x_s(t)$,

$$\dot{x}_s(t) = \frac{(u(x_s^+(t)))^2 - (u(x_s^-(t)))^2}{2(u(x_s^+(t)) - u(x_s^-(t)))}. \tag{1.4.8}$$

Cela amène un nouveau problème : les solutions discontinues ne sont pas uniques. Cela signifie non seulement que le système n'est plus bien posé mais aussi que certaines des solutions considérées ne sont pas physiques. Pour régler cela, Lax introduit une condition supplémentaire [134] : pour être acceptables les deux caractéristiques arrivant à une discontinuité, si elles étaient prolongés positivement en temps, devraient croiser la courbe de discontinuité qui avance à la vitesse (1.4.8). En d'autres termes dans notre cas,

$$u(x_s^+) < \dot{x}_s < u(x_s^-). \tag{1.4.9}$$

Cette condition assure l'unicité de la solution et donc un principe de causalité c'est-à-dire que à chaque temps $t > 0$ toute l'information de $u(t, \cdot)$ était déjà contenue dans la condition initiale u_0, ce qui est, somme toute, assez physique. Muni de cette condition le système est de nouveau bien posé, on dit que cette solution est entropique. Dans les cas plus généraux on peut aussi définir des solutions discontinues et entropiques. L'étude de ces solutions, entre autres pour les études de stabilité, est souvent un problème compliqué.

1.4.2 Définition de la stabilité

On couple maintenant le système (1.4.1) avec le contrôle aux bords, pour l'instant formel,

$$B(\mathbf{Y}(t, 0), \mathbf{Y}(t, L)) = u(t). \tag{1.4.10}$$

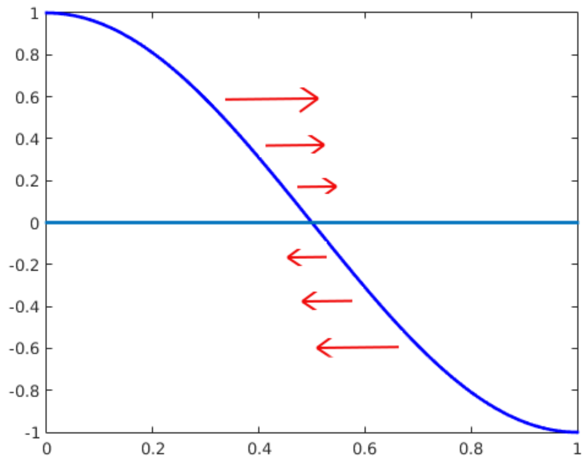

FIGURE 1.1 – Exemple de condition initiale u_0 amenant à une discontinuité : la partie gauche de la solution de (1.4.3) est positive et se déplace à vitesse positive, tandis que la partie droite est négative et se déplace à vitesse négative.

où $u(t)$ est le contrôle qui dépend de l'état $\mathbf{Y}$. Comme mentionné précédemment, on dit alors que $u(t)$ est une rétroaction ou un feedback. La stabilisation consiste à trouver des rétroactions $u(t)$ telles que, quelle que soit la condition initiale, toute solution de (1.4.1), (1.4.10) converge vers un état stationnaire fixé $\mathbf{Y}^*$, qui joue le rôle de cible à atteindre. On peut en donner une définition mathématiques plus précise : pour un espace de Banach X muni d'une norme $\|\cdot\|_X$, qu'on appellera norme X dans la suite, on définit la stabilité exponentielle de la façon suivante :

Définition 1.4.1. L'état stationnaire $\mathbf{Y}^*$ du système (1.4.1), (1.4.10) est exponentiellement stable pour la norme X s'il existe $\gamma > 0$, $\eta > 0$, et $C > 0$ tels que pour toute condition initiale $\mathbf{Y}^0 \in X$ compatible avec (1.4.10) [3] et telle que $\|\mathbf{Y}^0 - \mathbf{Y}^*\|_X \leq \eta$, le problème de Cauchy (1.4.1), (1.4.10),($\mathbf{Y}(0,x) = \mathbf{Y}^0$) a une unique solution sur $[0,+\infty[$ dans X et

$$\|\mathbf{Y}(t,\cdot) - \mathbf{Y}^*\|_X \leq Ce^{-\gamma t}\|\mathbf{Y}^0 - \mathbf{Y}^*\|_X, \ \forall t \in [0,+\infty[. \tag{1.4.11}$$

On peut faire trois remarques :

— Cette définition dépend de la norme X considérée. Une question qu'on pourrait légitimement se poser est : est-ce qu'il est vraiment nécessaire de préciser la norme X, du moins tant que le système est bien posé ? La réponse est oui. En général, pour les systèmes non-linéaires hyperboliques, la stabilité dans les différentes normes n'est pas équivalente [61]. Nous y reviendrons dans la partie I (voir la Section 1.6).

— On voit que cette définition nécessite que le système soit bien posé, ce qui peut parfois poser un problème. En effet, comme on l'a déjà un peu mentionné, même le caractère bien-posé des systèmes usuels doit être ré-étudié quand le contrôle est un feedback. Dans le Chapitre 6 par exemple, on est obligé de montrer ce caractère bien posé pour l'équation de Burgers munie de nos feedbacks avant d'étudier la stabilité en tant que telle. Une définition moins exigeante consisterait à simplement exiger que des solutions existent sans demander l'unicité et qu'elles vérifient toutes l'estimée de stabilité (1.4.11). Dans la plupart des cas néanmoins, il n'est pas dit que cela simplifie beaucoup le problème.

— La stabilité exponentielle dont il est question est une stabilité locale. En effet, lorsqu'on stabilise des systèmes hyperboliques non-linéaires par des contrôles aux bords, il est en général impossible

3. Une définition précise de ce qu'est une solution compatible est donnée dans le Chapitre 2

d'obtenir une stabilisation globale, c'est-à-dire pour des perturbations aussi large qu'on le souhaite. Cela est dû aux vitesses de propagation finies du système qui font que si l'état du système commence suffisamment près d'un choc à l'intérieur du domaine, alors le choc se formera avant que l'influence des contrôles aux bords ne l'atteigne.

Enfin, si on considère le changement de coordonnées $\mathbf{u}(t,x) = S(x)(\mathbf{Y}(t,x) - \mathbf{Y}^*(x))$, l'état stationnaire à stabiliser est maintenant $\mathbf{u}^* = 0$. Bien évidemment la stabilité exponentielle de $\mathbf{Y}^*$ pour le système d'origine (1.4.1), (1.4.10) est équivalente à la stabilité exponentielle de l'état stationnaire $\mathbf{u}^* = 0$ pour le nouveau système (1.4.2) muni du contrôle aux bords correspondant. C'est pourquoi dans la suite nous considérerons la plupart du temps ce dernier système.

Un contrôle aux bords célèbre : le contrôle proportionnel La stabilisation existait déjà bien avant la première définition de contrôlabillité de Kalman [115, 116]. Dès l'antiquité on cherchait déjà stabiliser des systèmes, et le contrôle proportionnel est peut-être le premier contrôle qui a été conçu sur des systèmes de dimension finie. Sans vraiment le mathématiser, Ctésibios l'utilisait déjà au IIIème siècle avant J.C. pour stabiliser le débit d'une clepsydre [136]. Bien plus tard, en 1788 c'est le régulateur à boules de Watt [4] utilisé pour stabiliser les machines à vapeurs qui reprit ce même principe. Comme son nom l'indique, ce contrôle consiste à agir sur le système proportionnellement à ce qu'on mesure. Si on prend par exemple l'équation $\dot{x} = x + u(t)$ où $u(t)$ est le contrôle, on voit que choisir $u(t) = -kx$ est un moyen simple de stabiliser exponentiellement le système si $k > 1$, la solution étant alors $x(t) = x_0 e^{-(k-1)t}$. Pour les systèmes de dimension infinie le principe est le même et on attribue à notre contrôle aux bords $u(t)$ une valeur proportionnelle aux valeurs de $\mathbf{Y}$ aux bords du domaine, c'est-à-dire en 0 et en L. Le contrôle aux bords (1.4.10) devient alors

$$\begin{pmatrix} (S(0)(\mathbf{Y}(t,0) - \mathbf{Y}^*(0))_+ \\ (S(L)(\mathbf{Y}(t,L) - \mathbf{Y}^*(L))_- \end{pmatrix} = K \begin{pmatrix} (S(L)(\mathbf{Y}(t,L) - \mathbf{Y}^*(L))_+ \\ (S(0)(\mathbf{Y}(t,0) - \mathbf{Y}^*(0))_- \end{pmatrix}, \tag{1.4.12}$$

où $K \in \mathcal{M}_n(\mathbb{R})$ est une matrice qu'on peut choisir, S diagonalise $F(\mathbf{Y}^*)$ de telle sorte que $S(\mathbf{Y} - \mathbf{Y}^*)$ représente localement les composantes qui se propagent avec pour vitesse les valeurs propres de $F(\mathbf{Y}^*)$. La notation $S(\mathbf{Y} - \mathbf{Y}^*)_+$ désigne les composantes qui se propagent avec une vitesse positive et $S(\mathbf{Y} - \mathbf{Y}^*)_-$ celle qui se propagent avec une vitesse négative. Cette loi de contrôle aux bords signifie donc simplement que l'information entrante dans le système est une fonction de l'information sortante, ce qui est la notion la plus simple de rétroaction. Par extension on peut considérer un contrôle non-linéaire :

$$\begin{pmatrix} (S(0)(\mathbf{Y}(t,0) - \mathbf{Y}^*(0))_+ \\ (S(L)(\mathbf{Y}(t,L) - \mathbf{Y}^*(L))_- \end{pmatrix} = G \begin{pmatrix} (S(L)(\mathbf{Y}(t,L) - \mathbf{Y}^*(L))_+ \\ (S(0)(\mathbf{Y}(t,0) - \mathbf{Y}^*(0))_- \end{pmatrix}, \tag{1.4.13}$$

où G est maintenant une fonction C^1 (et non plus une matrice) telle que $G(0) = 0$. Puisque G est C^1, l'expression (1.4.13) se linéarise localement sous la forme (1.4.12) avec $K = G'(0)$. C'est pourquoi ces contrôles sont la généralisation naturelles des contrôles proportionnels, que nous appellerons dans la suite feedback de sortie ou contrôles de sortie. Ces contrôles sont les plus célèbres et probablement les plus étudiés en mathématiques (voir la Section 1.6.1, [13, 142]). Ce sont ceux que nous étudierons dans les deux premières parties. Néanmoins, dans l'industrie, une variante permettant de gagner en robustesse existe : le contrôle proportionnel-intégral, ou contrôle PI [4, 6, 157]. On en donnera une définition précise dans la partie III (voir la Section 1.8) dédiée à l'étude plus spécifique de ces contrôles.

Une question naturelle se pose alors : comment trouver un contrôle qui stabilise notre système ? Autrement dit, comment choisir G ?

4. Même si c'est le nom de Watt qui est resté, ce régulateur était probablement déjà inventé avant. La paternité reste difficile à établir mais Mead utilisait un régulateur similaire en 1787 et un schéma de di Giorgio Martini du XVème siècle semble représenter un régulateur similaire.

1.4.3 Un outil tout-puissant pour les systèmes linéaires : le Spectral Mapping Theorem

Pour les systèmes linéaires 1D il existe un outil tout puissant : le Spectral Mapping Theorem qui s'énonce ainsi [5] [143, 161, 176]

Théorème 1.4.1. *Soit un système hyperbolique linéaire de la forme* (1.4.2)*, où A ne dépend pas de $\mathbf{u}$ et $B = M(x)\mathbf{u}$, muni de conditions aux bords de la forme* (1.4.12)*. On définit $\mathcal{L}$ par*

$$\mathcal{L}(\mathbf{u}) = -A\partial_x\mathbf{u} - M\mathbf{u} \tag{1.4.14}$$

sur le domaine $D(\mathcal{L}) := \{(\mathbf{u}) \in W^{1,2}((0,L),\mathbb{C}^n) | ((\mathbf{u}(0))_+, (\mathbf{u}(L))_-)^T = K((\mathbf{u}(L))_+, (\mathbf{u}(0))_-)^T\}$*. Alors*

$$\sigma(e^{\mathcal{L}t}) \setminus \{0\} = \overline{e^{\sigma(\mathcal{L})t}} \setminus \{0\}, \ \text{pour } t \geq 0, \tag{1.4.15}$$

où $\sigma(C)$ désigne le spectre de l'opérateur C, et $\overline{e^{\sigma(\mathcal{L})t}}$ l'adhérence de $e^{\sigma(\mathcal{L})t}$.

Ce théorème, s'il peut sembler aride à première vue, signifie en résumé qu'il suffit de connaître les valeurs propres de l'opérateur pour connaître les limites de la stabilité du système linéaire. Il rend ainsi relativement simple la résolution d'un problème compliqué comme la stabilité d'un système de dimension infinie.

Malheureusement, dès que le système est non-linéaire, ce théorème ne s'applique plus, même s'il n'est que très faiblement non-linéaire [6] ou qu'on ne l'étudie que localement. Pour s'en convaincre on peut regarder le système suivant tiré de [61] :

$$\begin{aligned} \partial_t u + \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}+10^{-5}u_2} \end{pmatrix} \partial_x u = 0, \\ u(t,0) = a \begin{pmatrix} 1 & \xi \\ -1 & \eta \end{pmatrix} u(t,1). \end{aligned} \tag{1.4.16}$$

Dans [61], les auteurs montrent qu'on peut trouver $a \in \mathbb{Q}$, $\xi > 1$ et $\eta > 1$, tels que le système linéarisé associé à (1.4.16) est exponentiellement stable, tandis que le système (1.4.16) en lui-même n'est pas exponentiellement stable pour la norme C^1, et ce même quand $\|\mathbf{u}\|_{C^1}$ est aussi proche de 0 qu'on veut. Ajouter même une très faible non-linéarité peut ainsi changer radicalement la stabilité. Il faut donc trouver d'autres méthodes pour stabiliser les systèmes non-linéaires.

1.4.4 Une nouvelle façon de formuler le problème : l'approche Lyapunov

De nombreuses stratégies ont été établies pour stabiliser les systèmes non-linéaires, parmi elles :

— La méthode des caractéristiques. C'est la méthode historique et la plus naturelle. Pour les systèmes hyperboliques 1D, il est possible de remonter les caractéristiques du système pour avoir des estimées explicites. Néanmoins cette méthode devient vite complexe et peu utilisable quand le système se complique, en particulier quand il présente des termes sources, c'est-à-dire $B \neq 0$.

— La méthode du Grammien. Introduite d'abord en dimension finie puis pour des systèmes linéaires de dimension infinie par [179] et [122, 123], elle consiste à utiliser les propriétés des semi-groupes pour définir une rétroaction et en déduire la stabilité. Cette approche semi-groupe peut se généraliser aux systèmes non-linéaire dans certains cas.

— La méthode du Backstepping. Cette méthode, très puissante, introduite dans [39, 121, 195] pour les systèmes de dimension finie a été adaptée dans [51] puis modifiée dans [9, 32] (voir aussi [128, 181]) pour les systèmes de dimension infinie. Pour les systèmes de dimension infinie cette méthode modifiée

5. Techniquement le résultat énoncé par [143] ne concerne que des conditions aux bords locales au sens où le contrôle en 0 ne dépend que des valeurs mesurées en 0 et le contrôle en L ne dépend que des valeurs mesurées en L. Néanmoins on peut toujours s'y ramener via un doublement des variables approprié comme présenté dans [70].

6. La définition de faiblement non-linéaire n'a pas l'air d'être clairement établie mais on comprend l'idée : le Spectral Mapping Theorem ne pourra pas s'appliquer stricto sensu à un système de la forme $u_t + (1 + 0.00001u)u_x = 0$.

revient à transformer, via une transformation linéaire inversible, le système en un système simple à stabiliser, puis à effectuer ensuite la transformation inverse une fois qu'on a trouvé un feedback pour ce système plus simple. Malheureusement, ces transformations successives aboutissent à des contrôles compliqués dépendant en général de tout l'état du système, et non plus seulement de l'état aux bords.
— L'approche Lyapunov, que nous détaillons ici.

L'esprit de l'approche Lyapunov consiste à trouver une fonctionnelle V qui décroît le long des trajectoires $\mathbf{u}$, tend vers 0, et telle que $V(\mathbf{u}) = 0$ implique $\mathbf{u} = 0$. Intuitivement, V peut être vue comme une énergie des perturbations $\mathbf{u}$ qu'on cherche à ramener à 0. L'idée derrière est qu'on dispose de plus de marge de manoeuvre en étudiant $V(\mathbf{u})$ que $\mathbf{u}$. En effet, même si la norme de $\mathbf{u}$ tend vers 0 il est possible que ce soit difficile à montrer si elle ne décroît pas tout le temps, si elle fait des oscillations amorties par exemple. Par contre si elle tend vers 0, il est possible de fabriquer une enveloppe de la solution $V(\mathbf{u})$ qui, elle, décroît tout le temps. La convergence de $V(\mathbf{u})$ est alors plus facile à montrer puisqu'il "suffit" de calculer sa dérivée par rapport au temps. On illustre cela dans la figure suivante (Fig. 1.2).

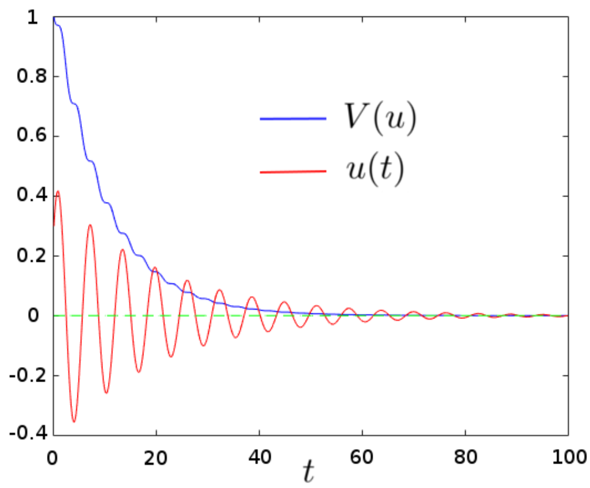

FIGURE 1.2 – Exemple de fonction de Lyapunov $V(u(t))$, décroissante avec le temps t tandis que la solution $u(t)$ ne décroît pas avec t et a des oscillations amorties. Ici $u(t)$ correspond à l'angle d'un pendule simple comme décrit après en Section 1.4.4 et $V(u(t))$ à l'énergie mécanique associée.

Cette méthode a d'abord été introduite pour les systèmes de dimension finie par Lyapunov en 1892 [150, 151].

Théorème 1.4.2 (Lyapunov)**.** *Soit un système $\dot{x} = f(x)$ avec f localement Lipschitz et $f(0) = 0$. S'il existe un voisinage U de l'origine et une fonction $V \in C^1(U, \mathbb{R})$ telle que*

$$\begin{aligned} &V(x) > 0, \ \textit{pour tout } x \in U \setminus \{0\}, \quad V(0) = 0, \\ &\nabla V(x).f(x) < 0 \ \textit{pour tout } x \in U \setminus \{0\}, \end{aligned} \tag{1.4.17}$$

alors on dit que V est une fonction de Lyapunov, et le système est (localement) asymptotiquement stable.

Dans certains cas il est difficile d'obtenir une inégalité stricte dans (1.4.17). Ce théorème peut alors être complété par le principe d'invariance de LaSalle [7] :

Théorème 1.4.3 (LaSalle)**.** *Soit le même système que précédemment, s'il existe un voisinage U de l'origine et une fonction $V \in C^1(U, \mathbb{R})$ telle que*

$$\begin{aligned} &V(x) > 0, \ \textit{pour tout } x \in U \setminus \{0\}, \quad V(0) = 0, \\ &\nabla V(x).f(x) \leq 0 \ \textit{pour tout } x \in U \setminus \{0\}, \end{aligned} \tag{1.4.18}$$

7. En réalité le principe de LaSalle est un peu plus général que 1.4.3 et le théorème qu'on donne est son application directe à la stabilité d'un point, voir [190, Théorème 13.3.3] pour plus de détails.

et telle que l'ensemble $\{x \in U | \nabla V(x).f(x) = 0\}$ ne contient aucune trajectoire non identiquement nulle, alors le système est (localement) asymptotiquement stable.

En dimension infinie cette méthode s'applique toujours, avec quelques précisions. En effet, pour fonctionner, le principe d'invariance de LaSalle nécessite de montrer la pré-compacité des trajectoires du système de dimension infinie [40], ce qui est parfois technique et compliqué. Dans la suite on s'intéresse à la stabilisation exponentielle (voir la Définition 1.4.1), c'est pourquoi on définit les fonctions de Lyapunov de la façon suivante, légèrement plus stricte que précédemment.

Définition 1.4.2. Soit un système quasilinéaire hyperbolique de la forme (1.4.2) muni de conditions aux bords telles que le système soit bien posé dans [8] X. On appelle fonction de Lyapunov pour la norme X une fonctionnelle $V \in C^0(X, \mathbb{R})$ telle qu'il existe $\gamma > 0$, $c_1 > 0$ et $c_2 > 0$ et un voisinage de l'origine $W \subset X$ tels que

$$c_1 \|\mathbf{U}\|_X < V(\mathbf{U}) < c_2 \|\mathbf{U}\|_X, \text{ pour tout } \mathbf{U} \in W, \tag{1.4.19}$$

et telle que pour tout $T > 0$ et toute solution du système $\mathbf{u}$ sur $[0, T]$ dont la trajectoire est contenue dans W,

$$\frac{dV(\mathbf{u}(t, \cdot))}{dt} \leq -\gamma V(\mathbf{u}(t, \cdot)), \tag{1.4.20}$$

au sens des distributions sur $[0, T]$ [9].

On peut remarquer qu'on n'exige plus que la fonction V soit C^1, ce qui se traduit par une inégalité (1.4.20) au sens des distributions. Cela sera utile pour la suite lorsqu'on définira des fonctions de Lyapunov équivalente à la norme L^∞ ou $W^{l,\infty}$ et pas forcément différentiable [10]. Par ailleurs cette définition est un peu plus stricte que celle qu'on utilisait précédemment parce qu'on exige une décroissance de la fonction de Lyapunov exponentielle et non plus seulement asymptotique. Cette légère contrainte permet sans surprise d'obtenir la stabilité exponentielle du système :

Proposition 1.4.4. *Soit un quasilinéaire hyperbolique de la forme* (1.4.2) *muni de conditions aux bords telles que le système soit bien posé dans X. S'il existe une fonction de Lyapunov pour la norme X, alors le système est exponentiellement stable pour la norme X.*

Cette méthode est très utile et très générale, mais le prix à payer est qu'elle est très abstraite et ne donne pas beaucoup d'indications sur la façon de construire la fonction V, surtout en dimension infinie quand V appartient $C^0(X, \mathbb{R}_+)$, un espace très large.

Un cas important : les fonctions de Lyapunov basiques.

Parmi l'ensemble des fonctions de Lyapunov possibles, il existe une classe naturelle et importante : les fonctions de Lyapunov basiques.

Définition 1.4.3. Pour un espace de Sobolev $W^{l,p}(]0, L[)$, où $(l, p) \in \mathbb{N} \times \mathbb{N}^* \cup \{+\infty\}$, on appelle fonction de Lyapunov basique pour la norme $W^{l,p}$ une fonction $V \in C^0(W^{l,p}(]0, L[), \mathbb{R})$ définie par :

$$V(\mathbf{U}) = \sum_{n=0}^{l} \|F(\cdot)E(\mathbf{U}, \cdot)D_n\mathbf{U}\|_{L^p(]0,L[)}, \quad \forall\ \mathbf{U} \in W^{l,p}(]0, L[), \tag{1.4.21}$$

où $E(\mathbf{U}, x)$ est une matrice qui diagonalise $A(\mathbf{U}, x)$, D_n est l'opérateur défini itérativement par $D_1\mathbf{U} = -A(\mathbf{U}, x)\partial_x\mathbf{U} + B(\mathbf{U}, x)$, et pour $n \geq 2$, $D_n\mathbf{U} = \partial_{\mathbf{U}}(D_{n-1}\mathbf{U})D_1\mathbf{U}$, avec $\partial_{\mathbf{U}}$ l'opérateur différentiation par rapport à $\mathbf{U}$, et $F = (f_1, ..., f_n)$ sont des fonctions C^1 à valeurs strictement positives sur $[0, L]$, telles que V est une fonction de Lyapunov pour la norme $W^{l,p}$.

8. Précisément, on exige que pour tout $T > 0$ le système est bien posé sur $[0, T]$ dans X, des définitions plus précises et un peu moins abstraites se trouvent au début de chaque chapitre concerné.

9. Pour une définition précise d'une inégalité au sens des distributions, voir la Remarque 2.3.3. Dans le cas présent (1.4.20) est équivalente à $V(\mathbf{u}(t, \cdot)) \leq V(\mathbf{u}(t', \cdot))e^{-\gamma(t-t')}$ pour tout $0 < t \leq t' < T$, ce qui explique qu'on parle de décroissance exponentielle.

10. Le problème ne se pose pas vraiment en dimension finie : toutes les normes étant équivalentes on peut toujours choisir une norme plus régulière que la norme infinie.

Définies ainsi, ces fonctions pourraient ne pas sembler si naturelles que cela. En réalité, appliquées à une solution de (1.4.2), l'expression (1.4.21) devient

$$V(\mathbf{u}) = \sum_{n=0}^{l} \| F(\cdot)E(\mathbf{u}(t,\cdot),\cdot)\partial_t^n \mathbf{u}(t,\cdot)\|_{L^p}, \tag{1.4.22}$$

ce qui justifie la forme (1.4.21). Ces fonctions de Lyapunov sont donc des sortes de normes à poids pour les espaces de Sobolev $W^{l,p}$ en considérant les dérivées temporelles au lieu des dérivées spatiales. Si ces fonctions sont utilisées depuis longtemps pour la norme L^2, pour leurs liens avec les entropies dissipatives que nous verrons dans le paragraphe suivant, la dénomination *fonction de Lyapunov basiques* apparaît dans [11], tandis que nous introduisons la version C^1 dans [106], même si des fonctions similaires étaient déjà utilisées dans [48].

Lien entropie dissipative - fonction de Lyapunov Les systèmes physiques présentent des quantités physiques, comme l'énergie ou l'entropie, qui peuvent aider à étudier leur stabilité. La stabilité d'un pendule simple soumis à des frottements, par exemple, s'étudie facilement à l'aide de l'énergie mécanique. La dynamique du pendule est donnée par

$$\ddot{\theta} = -\frac{g}{mR}\sin(\theta) - \frac{f}{m}\dot{\theta}, \tag{1.4.23}$$

où R est la longueur du pendule, m sa masse, θ son angle avec la verticale, $f > 0$ le coefficient de frottements et g l'accélération de la pesanteur à la surface de la Terre. En introduisant l'énergie mécanique $V(\theta) = (1/2)m(R^2(\dot{\theta})^2 + 2gR(1-\cos(\theta)))$ et en utilisant (1.4.23) on obtient l'équation d'énergie le long des trajectoires

$$\frac{dV(\theta)}{dt} = \frac{1}{2}(-2gR\sin(\theta)(\dot{\theta}) - 2R^2 f\dot{\theta}^2 + 2gR\sin(\theta)\dot{\theta}) = -\frac{R^2 f}{m}\dot{\theta}^2 \leq 0. \tag{1.4.24}$$

On applique alors le Théorème 1.4.3. En effet $\dot{V}(\theta) = 0$ implique $\dot{\theta} = 0$ et, d'après (1.4.23), la seule trajectoire contenue dans un voisinage de l'origine telle que $\dot{\theta} = 0$ pour tout temps est la trajectoire identiquement nulle $\theta = 0$. Le système est donc asymptotiquement stable et, accessoirement, V est une fonction de Lyapunov faible, c'est-à-dire qu'elle vérifie (1.4.18). De la même façon si on considère un solide isolé avec une température inhomogène à $t = 0$, le second principe de la thermodynamique impose que l'entropie augmente, ce qui impose le retour à un équilibre uniforme pour la température.

Tout cela a motivé la définition d'une notion mathématique, appelée entropie, qui, par abus de langage, englobe les quantités physiques précédentes mais aussi d'autres quantités auxquelles on peine parfois à donner un sens physique, comme on le verra (voir Chapitre 6 et 7).

Définition 1.4.4. Soit un système de la forme (1.4.2), on appelle entropie une fonction $\eta \in C^\infty(W \times [0,L];\mathbb{R})$, où $W \subset \mathbb{R}^n$ est un voisinage de l'origine, telle que pour tout $\mathbf{U} \in W$ et tout $x \in [0,L]$,

$$\partial_{\mathbf{U}}^2\eta(\mathbf{U},x)A(\mathbf{U},x) = A(\mathbf{U},x)^T\partial_{\mathbf{U}}^2\eta(\mathbf{U},x). \tag{1.4.25}$$

Clairement, si la dimension n est trop grande, cette condition impose plus qu'une seule équation tandis que η n'a qu'une composante. Le problème n'admet donc pas forcément de solution. Néanmoins, ce qui est admirable, c'est que les systèmes physiques ont souvent une structure spéciale qui leur en confère une, comme on le verra avec l'exemple des systèmes densité-vélocité dans la partie II. Si une entropie existe on peut définir le flux associé à l'entropie par

$$\partial_{\mathbf{U}}q = \partial_{\mathbf{U}}\eta A \tag{1.4.26}$$

Evidemment, η n'est définie qu'à une fonction affine près et q n'est défini qu'à une constante près. C'est pourquoi, pour une entropie convexe donnée η, on peut définir une nouvelle entropie par

$$\eta_r(\mathbf{U}) = \eta(\mathbf{U}) - \eta(\mathbf{U}^*) - \partial_{\mathbf{U}}\eta(\mathbf{U}^*)(\mathbf{U}-\mathbf{U}^*). \tag{1.4.27}$$

de telle sorte que $\eta_r(\mathbf{U}^*) = \partial_{\mathbf{U}}\eta_r(\mathbf{U}^*) = 0$ et que η_r soit localement quadratique au voisinage de $\mathbf{U}^*$. Cette entropie est appelée entropie relative par rapport à $\mathbf{U}^*$. Enfin, ces entropies sont dites dissipatives si elles vérifient la condition de production d'entropie suivante :

$$\partial_{\mathbf{U}}\eta(\mathbf{U},x)(B(\mathbf{U},x)) - \partial_x q(\mathbf{U},x) \geq 0 \tag{1.4.28}$$

L'existence d'une entropie dissipative convexe garantit la stabilité “interne” du système. En d'autres termes si le système est étudié sur $\mathbb{R}$ tout entier (et non pas sur un domaine borné) et qu'il existe une entropie dissipative convexe, alors le système sera asymptotiquement stable pour la norme L^2. Dans notre problème de stabilisation, on peut choisir les conditions aux bords, elles ne posent donc pas de problème. C'est pourquoi, quand il existe une telle entropie, il existe toujours un contrôle aux bords tel que le système est stable. Par ailleurs la forme de η permet de trouver des conditions explicites sur ce contrôle.

Les entropies dissipatives et les fonctions de Lyapunov sont très liées. On peut remarquer que l'intégrande des fonctions de Lyapunov basiques pour la norme L^2 est directement une entropie dissipative convexe, localement au moins. C'est aussi le cas des fonctions de Lyapunov basiques pour la norme H^p en voyant la norme H^p comme la norme L^2 du système augmenté qui prend comme variable $(\mathbf{u}, \partial_t\mathbf{u}, ..., \partial_t^p\mathbf{u})$ et qui est obtenu en dérivant obtenu successivement p fois le système (1.4.2) par rapport au temps (voir l'Annexe 2.8.4 par exemple). En effet, au premier ordre $\partial_{\mathbf{u}}^2\eta$ est alors diagonale tout comme A, ce qui règle le problème de la condition (1.4.25), à un reste d'ordre supérieur près, tandis que la condition (1.4.28) correspond exactement à la condition intérieure qu'on retrouve dans [13, Théorème 6.10] (voir la Section 1.6.1) pour les fonctions de Lyapunov basiques pour la norme H^p. Par ailleurs, on peut noter que (1.4.25) est exactement la condition demandée par les auteurs dans [16] pour définir une fonction de Lyapunov basique pour la norme L^2 pour un système densité-vélocité linéarisé. Enfin, les entropies convexes dissipatives sont des bonnes fonctions de Lyapunov et prennent localement la forme quadratique $\eta(\mathbf{U}) = \frac{1}{2}\mathbf{U}^T\partial_{\mathbf{U}}^2\eta(\mathbf{0},x).\mathbf{U} + O(|\mathbf{U}|^2)$, au voisinage de 0, qui se rapproche [11] de l'intégrande des fonctions de Lyapunov basiques pour la norme L^2.

1.4.5 Principales difficultés

La stabilisation des systèmes hyperboliques 1D et homogènes, c'est-à-dire de la forme (1.4.2) avec $B \equiv 0$, est un problème bien connu et assez bien circonscrit quand il est étudié dans des espaces réguliers et par des contrôles aux bords proportionnels où des feebacks de sortie (voir par exemple [140] et [61]). La stabilisation des systèmes généraux, en revanche, est encore peu connue parce que plus compliquée. Deux phénomènes rendent intrinsèquement difficile l'étude de la stabilité des systèmes généraux :

— **Les termes sources (ou inhomogénéités).** Ces termes sources induisent plusieurs problèmes, entre autres, un couplage intrinsèque entre les différentes équations du système qui ne peut pas être résolu en utilisant simplement un changement de variables (voir Section 1.6 pour plus de détails). Ce sera l'objet des Chapitres 2 et 3 et en partie des Chapitres 4, 5 et 9.

— **Les chocs.** L'apparition de discontinuités complique les méthodes de stabilisation, en particulier lorsqu'elles changent le signe d'une des vitesses de propagation. C'est pourquoi les études se restreignent généralement aux solutions régulières. L'objet des Chapitres 6 et 7 sera d'étendre ces résultats à la stabilisation d'états stationnaires présentant un choc.

Par ailleurs une difficulté supplémentaire peut survenir dans le choix du contrôle. C'est le cas quand on veut utiliser un contrôle qui n'est plus seulement proportionnel, mais proportionnel-intégral. Comme on le verra dans la partie III, les contrôles PI sont encore mal compris mathématiquement quand ils sont appliqués à des systèmes non-linéaires de dimension infinie. Un exemple l'illustre bien : c'est celui de l'équation de transport traité dans le Chapitre 8. Si trouver des conditions optimales pour stabiliser ce système avec un contrôle proportionnel ou un feedback de sortie est aisé et peut se traiter en un paragraphe, le faire avec un contrôle PI nécessite une nouvelle méthode, qu'on introduit dans le Chapitre 8, et prend une vingtaine de pages.

11. En réalité on peut montrer que quand A est diagonale, puisqu'elle a des valeurs propres distinctes, $\partial_{\mathbf{U}}^2\eta(\mathbf{0},x)$ est diagonale, et $E(\mathbf{U},x) = Id$, ce qui correspond donc exactement à l'intégrande des fonctions de Lyapunov basiques.

1.5 Objectifs de la thèse

Les systèmes hyperboliques sont étudiés depuis de nombreuses années pour leur large intérêt mathématique et pratique. Au vu de la multitude de leurs applications, la question de leur stabilité et leur stabilisation est fondamentale et a nourri de nombreuses recherches ces quarante dernières années. Cette thèse se place dans leur suite et s'articule autour de trois parties.
Dans la première partie, on cherche des résultats généraux pour garantir la stabilité en norme C^1 des systèmes inhomogènes.
La deuxième partie s'intéresse aux équations de la mécanique des fluides et on montre en particulier l'existence d'une entropie locale dissipative pour les équations de type "densité-vélocité" 1-D, dont font partie les équations de Saint-Venant et les équations d'Euler isentropiques sous leurs formes les plus générales. Cette entropie trivialise l'étude de leurs stabilités quel que soit le cadre. Enfin on introduit une méthode pour traiter les chocs et stabiliser un état stationnaire possédant un choc.
Dans la troisième partie on s'intéresse aux contrôles proportionnels-intégraux (PI), qui sont très utilisés dans l'industrie pour leur robustesse mais très peu compris mathématiquement quand ils sont appliqués aux systèmes non-linéaires de dimension infinie. On introduit une méthode dite "d'extraction" pour trouver des conditions de stabilité optimales pour une équation scalaire. On s'intéresse ensuite aux équations de Saint-Venant stabilisées par un contrôle PI.

Résumé de la thèse :

1.6 Partie 1 : Stabilité des systèmes inhomogènes quasilinéaires pour la norme C^1.

Comme mentionné dans la Section 1.4.2, les estimées de stabilité du type (1.4.11) pour différentes normes ne sont pas équivalentes quand le système est non-linéaire. Pour un système différentiel non-linéaire de premier ordre, les solutions classiques naturelles sont les solutions C^1. C'est pourquoi l'étude générale de la stabilité de ces systèmes avec des contrôles aux bords a historiquement été conduite pour cette norme et pour des systèmes homogènes. Ces systèmes homogènes, c'est-à-dire sans terme source, s'écrivent

$$\partial_t \mathbf{Y} + F(\mathbf{Y})\partial_x \mathbf{Y} = 0, \tag{1.6.1}$$

où $F(\mathbf{Y})$ est diagonalisable autour d'un état stationnaire $\mathbf{Y}^*$ avec des valeurs propres distinctes et réelles, comme dans la Section 1.4.1. Les systèmes inhomogènes, eux, s'écrivent

$$\partial_t \mathbf{Y} + F(\mathbf{Y})\partial_x \mathbf{Y} + G(\mathbf{Y}, x) = 0. \tag{1.6.2}$$

Ces systèmes apportent de nombreuses difficultés par rapport au cas homogène :

— **Etats-stationnaires non-uniformes**. Quand il y a un terme source les états stationnaires sont en général non-uniformes avec des variations qui peuvent être amples, comme on peut le voir en cherchant une solution stationnaire à (1.6.2). La stabilisation d'un de ces états stationnaires peut alors toujours se ramener à la stabilisation de l'état stationnaire constant égal à 0 via un changement de variables comme vu en Section 1.4.1. Mais il y a un prix à payer : le système prend la forme (1.4.2) et la matrice de transport A et le terme source B dépendent alors explicitement de x, que les matrices d'origine en dépendent ou non.

— **Couplage interne entre les équations**. Lorsque le système est homogène et qu'on utilise le changement de variables $\mathbf{u} = S(\mathbf{Y} - \mathbf{Y}^*)$ pour le mettre sous la forme (1.4.2), $B \equiv 0$ et la matrice A est diagonale pour $\mathbf{u} = 0$. Le système prend donc une forme localement diagonale autour de $\mathbf{u}^* = 0$. Les composantes de $\mathbf{u}$ ne sont donc localement que faiblement couplées entre elles à cause de la non-linéarité. Quand le système est inhomogène, a contrario, $B \neq 0$ et les composantes sont intrinsèquement couplées entre elles par le terme source, même localement.

Ces différences rendent l'étude de la stabilité des systèmes inhomogènes significativement plus difficile que celle des systèmes homogènes. A ma connaissance, les seuls résultats généraux se trouvent dans [13, Théorème

6.10] et [114] qui étudient la stabilité dans la norme H^2 pour surmonter ces difficultés. Dans [13, Théorème 6.10], Bastin et Coron utilisent des contrôles aux bords de sortie, donc de la forme (1.4.13). Dans [114] les auteurs ont une approche complètement différente et utilisent la méthode du backstepping, déjà mentionnée en Section 1.4.4, qui donne un résultat qui fonctionne tout le temps mais qui donne lieu à des contrôles aux bords souvent beaucoup plus compliqués et difficiles à mettre en oeuvre en pratique.

Néanmoins, dans les deux cas on perd ainsi la norme naturelle C^1 au profit une norme plus régulière et donc plus restrictive : la norme H^2 (on rappelle que, grâce à l'injection de Sobolev, une fonction H^2 est automatiquement aussi C^1 en 1D). Dans la suite nous ne considérerons que les contrôles aux bords proportionnels et les feedbacks de sortie de la forme (1.4.13) qui sont à la fois parmi les plus étudiés, mais aussi parmi les plus pratiques car ils ne nécessitent de connaître l'état du système qu'aux bords, comme mentionné précédemment. Le but de cette partie est de retrouver la norme C^1 en utilisant une approche similaire à celle de Bastin et Coron mais dont on verra qu'elle apporte des conditions significativement différentes. Enfin, on étudiera plus en détail les systèmes de deux équations et on verra qu'on peut établir des liens entre la stabilité en norme H^2 et la stabilité en norme C^1 pour ces systèmes.

1.6.1 Résultats existants

Systèmes homogènes Le premier résultat pour les systèmes quasilinéaires hyperboliques est probablement celui de Li et Greenberg en 1984 [93]. Dans cet article, Li et Greenberg étudient un système homogène de deux équations, donc $B \equiv 0$ dans sa forme diagonalisée (1.4.2), avec des contrôles aux bords locaux de la forme :

$$\begin{aligned} \mathbf{u}_+(0) &= f(\mathbf{u}_-(0)), \\ \mathbf{u}_-(L) &= g(\mathbf{u}_+(L)), \end{aligned} \tag{1.6.3}$$

et montrent en étudiant précisément les caractéristiques que le système est exponentiellement stable en norme C^1 si

$$|f'(0)g'(0)| < 1. \tag{1.6.4}$$

On remarque que les contrôles (1.6.3) sont un cas particulier des feedback de sortie (1.4.13). En effet quand le système est sous forme diagonale avec $\mathbf{u} = S(\mathbf{Y} - \mathbf{Y}^*)$, les contrôles (1.4.13) s'écrivent

$$\begin{pmatrix} \mathbf{u}_+(t,0) \\ \mathbf{u}_-(t,L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}_+(t,L) \\ \mathbf{u}_-(t,0) \end{pmatrix}, \tag{1.6.5}$$

ce qui couvre bien le cas de (1.6.3). Ce résultat a ensuite été généralisé aux systèmes de dimension $n \in \mathbb{N}^*$ par Qin [172] puis Zhao [204], puis avec des conditions aux bords générales de la forme (1.6.5) par de Halleux et al. [71] puis récemment par [48]. La preuve de ce dernier article utilise des fonctions de Lyapunov d'une manière dont nous nous inspirerons tout en les modifiant pour pouvoir gérer la nouvelle difficulté liée au terme source. Avec des conditions aux bords générales de la forme (1.6.5), le système est alors exponentiellement stable en norme C^1 si

$$\rho_\infty(G'(0)) < 1 \tag{1.6.6}$$

où ρ_∞ est défini par

$$\rho_\infty(M) = \inf(\|\Delta M \Delta^{-1}\|_\infty, \Delta \in D_n^+), \tag{1.6.7}$$

avec D_n^* l'ensemble des matrices diagonales à coefficients strictement positifs.

Plus tard, Coron et Bastin ont montré que la stabilité est plus facile à obtenir dans l'espace plus régulier (et donc plus restreint) H^2. La condition dans cet espace devient :

$$\rho_2(G'(0)) < 1 \tag{1.6.8}$$

où ρ_2 est défini par

$$\rho_2(M) = \inf(\|\Delta M \Delta^{-1}\|_2, \Delta \in D_n^+), \tag{1.6.9}$$

et on peut vérifier que $\rho_2 \leq \rho_\infty$ [50]. Au vu de la similarité des conditions (1.6.7) et (1.6.8) on peut se demander s'il est possible de trouver des conditions de stabilité pour d'autres espaces de Sobolev en utilisant ρ_p défini par

$$\rho_p(M) = \inf(\|\Delta M \Delta^{-1}\|_p, \Delta \in D_n^+). \tag{1.6.10}$$

Coron et Nguyên ont montré que c'était le cas [61], plus précisément le système homogène de la forme (1.4.2) avec $B \equiv 0$ muni du contrôle (1.6.5) est exponentiellement stable pour la norme $W^{2,p}$ si

$$\rho_p(G'(0)) < 1. \tag{1.6.11}$$

Reste à savoir laquelle de ces conditions est la moins restrictive. Si on peut montrer que $\rho_\infty \geq \rho_p$ pour tout $p \in \mathbb{N}^*$, et donc que la stabilité en norme C^1 est plus difficile à obtenir que la stabilité en norme $W^{2,p}$ pour tout $p \in \mathbb{N}^*$, la comparaison des ρ_p entre eux n'est pas claire. Puisque moins la solution est régulière, plus elle est générale, on pourrait s'attendre à ce que la stabilité dans les espaces de solutions moins réguliers (et donc plus larges) soit plus difficile à obtenir et donc à ce que $\rho_{p_2} \geq \rho_{p_1}$ pour $p_1 > p_2$. Cette intuition est en fait trompeuse, car $\rho_1 = \rho_\infty$ et donc $\rho_1 \leq \rho_p$ pour tout $p \in \mathbb{N}^* \cup \{\infty\}$ [56, Remarque 1.4]. Mais dans tous ces cas, on s'éloigne de la norme la moins restrictive et la plus naturelle : la norme C^1 (ou $W^{1,\infty}$).

Système inhomogènes Si des systèmes hyperboliques inhomogènes particuliers ont intéressé les mathématiciens depuis des années, l'étude de la stabilité des systèmes hyperboliques inhomogènes en général n'a commencé que très récemment. On considère maintenant un système de la forme (1.4.1) et on veut stabiliser un état stationnaire $\mathbf{Y}^*$, potentiellement non-uniforme, avec des contrôles aux bords de la forme (1.4.13). Pour simplifier l'étude, on utilise à nouveau le changement de variables $\mathbf{u} = S(\mathbf{Y} - \mathbf{Y}^*)$ présenté en Section 1.4.1 pour que le système se ramène à

$$\partial_t \mathbf{u} + A(\mathbf{u}, x)\partial_x \mathbf{u} + B(\mathbf{u}, x) = 0 \tag{1.6.12}$$

où $A(\mathbf{0}, \cdot) = \Lambda$, et $B(\mathbf{0}, x) = 0$, comme décrit en introduction, et dont on cherche à stabiliser l'état stationnaire 0. Les contrôles aux bords (1.4.13) ont à nouveau la forme (1.6.5), et on introduit par ailleurs la notation $M(\mathbf{0}, x) = \frac{\partial B}{\partial \mathbf{u}}(\mathbf{0}, x)$.
A nouveau, le premier résultat vraiment général pour ces systèmes concernait les systèmes 2×2, c'est-à-dire composés de deux équations potentiellement couplées, avec des vitesses de propagation de signes différents [11]. En toute rigueur le résultat de [11] concerne des systèmes linéaires mais il utilise une approche qui s'étend presque immédiatement aux systèmes non-linéaires pour la norme H^2 et se formule ainsi :

Théorème 1.6.1 ([11])**.** *Soit un système quasilinéaire hyperbolique de dimension 2 de la forme* (1.4.2) *où* A, B *et* G *sont de classe* C^2, $\Lambda_1 > 0$ *et* $\Lambda_2 < 0$. *Il existe un contrôle de la forme* (1.6.5) *tel qu'il existe une fonction de Lyapunov basique pour la norme* H^2 *pour ce système si et seulement s'il existe une fonction* η *définie sur* $[0, L]$ *et solution de :*

$$\begin{aligned} &\eta'(x) = \left| \frac{M_{21}(\mathbf{0}, \cdot)}{\varphi|\Lambda_2|}\eta^2 + \varphi \frac{M_{12}(\mathbf{0}, \cdot)}{\Lambda_1} \right|, \\ &\eta(0) = 0, \end{aligned} \tag{1.6.13}$$

où φ *est une notation qui désigne :*

$$\varphi = \exp\left(\int_0^x \frac{M_{11}(\mathbf{0}, s)}{\Lambda_1} + \frac{M_{22}(\mathbf{0}, s)}{|\Lambda_2|} ds \right). \tag{1.6.14}$$

Par ailleurs pour tout $\sigma > 0$ *tel que*

$$\begin{aligned} &\eta_\sigma' \geq \left| \frac{M_{21}(\mathbf{0}, \cdot)}{\varphi|\Lambda_2|}\eta^2 + \varphi \frac{M_{12}(\mathbf{0}, \cdot)}{\Lambda_1} \right|, \\ &\eta_\sigma(0) = \sigma, \end{aligned} \tag{1.6.15}$$

a une solution η_σ sur $[0,L]$ si

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \textit{ avec } l_1^2 < \eta_\sigma^2(0) \textit{ et } l_2^2 < \frac{1}{\eta_\sigma^2(L)}, \tag{1.6.16}$$

il existe une fonction de Lyapunov basique pour la norme H^2 et le système (1.4.2), (1.6.16) est exponentiellement stable en la norme H^2.

On voit apparaître un phénomène intéressant qui n'avait pas lieu avant : comme avant il y a une condition sur le contrôle aux bords, mais il y a aussi maintenant une condition intérieure (1.6.15), indépendante du contrôle et donc intrinsèque au système. Puisque l'équation différentielle (1.6.15) est non-linéaire, ses solutions peuvent exploser et cesser d'exister si L est trop grand, c'est par exemple le cas si $M_{11}(0,\cdot) = M_{22}(0,\cdot) = 0$ et $M_{12}(0,\cdot) = M_{21}(0,\cdot) = 1$. Dans ce cas η est simplement la fonction tangente et explose en $L = \pi/2$. Cette condition signifie qu'on n'est pas capable de garantir la stabilité au dessus d'une certaine distance $L_{\max}$. C'est une condition inhérente au système qui lie la forme et la taille des termes sources avec la distance.
Ce résultat a ensuite été généralisé dans [13, Chapitre 6] au cas général $n \times n$ et les auteurs ont montré :

Théorème 1.6.2 ([13]). *Soit un système hyperbolique quasi-linéaire* (1.4.2) *avec un contrôle aux bords* (1.6.5) *tels que A, B et G sont de classe C^2. Le système est exponentiellement stable pour la norme H^2 si*

— *(Condition intérieure) la matrice*

$$-(Q\Lambda)'(x) + Q(x)M(\mathbf{0},x) + M(\mathbf{0},x)^T Q(x)^T \tag{1.6.17}$$

est définie positive pour tout $x \in [0,L]$,

— *(Condition aux bords) la matrice*

$$\begin{pmatrix} \Lambda_+(L)Q_+(L) & 0 \\ 0 & -\Lambda_-(0)Q_-(0) \end{pmatrix} - K^T \begin{pmatrix} \Lambda_+(0)Q_+(0) & 0 \\ 0 & -\Lambda_-(L)Q_-(L) \end{pmatrix} K \tag{1.6.18}$$

est positive.

On retrouve bien les deux conditions : la condition (1.6.18) sur le contrôle aux bords et la condition intérieure (1.6.17) inhérente au système. A nouveau, ce résultat utilise des fonctions de Lyapunov basiques pour la norme H^2.

1.6.2 Nouveaux résultats

Cas général Dans le Chapitre 2 on s'intéresse à la norme C^1 et on montre le résultat suivant :

Théorème 1.6.3. *Soit un système quasilinéaire hyperbolique de la forme* (1.4.2), (1.6.5) *avec A et B de classe C^1. Si les deux propriétés suivantes sont vérifiées,*

1. *(condition intérieure) le système*

$$\Lambda_i f_i' \leq -2\left(-M_{ii}(\mathbf{0},x)f_i + \sum_{k=1,k\neq i}^{n} |M_{ik}(\mathbf{0},x)| \frac{f_i^{3/2}}{\sqrt{f_k}}\right), \tag{1.6.19}$$

   a une solution $(f_1,...,f_n)$ sur $[0,L]$ telle que pour tout $i \in [1,n]$, $f_i > 0$,

2. *(Conditions aux bords) il existe une matrice diagonale Δ avec des coefficients strictement positifs tels que*

$$\|\Delta G'(0)\Delta^{-1}\|_\infty < \frac{\inf_i \left(\frac{f_i(d_i)}{\Delta_i^2}\right)}{\sup_i \left(\frac{f_i(L-d_i)}{\Delta_i^2}\right)}, \tag{1.6.20}$$

   où $d_i = L$ si $\Lambda_i > 0$ et $d_i = 0$ sinon,

alors il existe une fonction de Lyapunov basique pour la norme C^1 et le système (1.4.2), (1.6.5) *est exponentiellement stable pour la norme C^1.*

On retrouve à nouveau une condition intérieure et une condition aux bords. Notons que l'existence d'une solution $(f_1, ...f_n)$ sur $[0, L]$ au système

$$f_i' = -\frac{2}{\Lambda_i}\left(-M_{ii}(\mathbf{0}, x)f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(\mathbf{0}, x)|\frac{f_i^{3/2}}{\sqrt{f_k}}\right) \tag{1.6.21}$$

avec $f_i > 0$ pour tout $i \in \{1, ..., n\}$, est aussi une condition intérieure suffisante, quoique plus stricte. D'autre part, lorsque $M \equiv 0$, on retrouve bien le résultat de Li, Greenberg, Qin, Zhao et de Halleux et al. pour les systèmes homogènes : la condition intérieure est satisfaite par n'importe quelles fonctions constantes $(f_1, ..., f_n)$ et en choisissant $f_i = \Delta_i^2$, la condition aux bords se ramène à l'existence de $\Delta \in D_+^n$ telle que $\|\Delta G'(0)\Delta^{-1}\|_\infty < 1$, ce qui est équivalent à $\rho_\infty(G'(0)) < 1$.

Pour montrer ce résultat on introduit les fonctions de Lyapunov basiques pour la norme C^1, qui sont l'analogue de celle pour la norme H^2 utilisée dans [11, 13], et on les approche par des fonctions équivalentes à la norme $W^{1,p}$ où on fait tendre p vers l'infini, dans la même veine que ce qui est fait dans [48]. La principale différence vient du fait que, quand on dérive par rapport au temps les fonctions équivalentes à la norme $W^{1,p}$, apparaît à cause de l'inhomogénéité un polynôme dont on aimerait qu'il soit positif. Dans [13] qui traite le cas de la norme H^2 ce polynôme est une forme quadratique. Ici, il s'agit d'un polynôme de degré p et garantir son signe nécessite donc un peu plus de travail (voir (2.5.31)–(2.5.38) et le Lemme 2.5.1 pour plus de détails). Par ailleurs on va un peu plus loin en montrant le résultat suivant qui illustre la précision de la condition intérieure du Théorème 1.6.3 :

Théorème 1.6.4. *Soit un système quasilinéaire hyperbolique de la forme* (1.4.2) *avec A et B de classe C^3. Il existe un contrôle aux bords de la forme* (1.6.5) *tel qu'il existe une fonction de Lyapunov basique pour la norme C^1 si et seulement si*

$$\Lambda_i f_i' \leq -2\left(-M_{ii}(\mathbf{0}, x)f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(\mathbf{0}, x)|\frac{f_i^{3/2}}{\sqrt{f_k}}\right), \tag{1.6.22}$$

admet une solution $(f_1, ..., f_n)$ sur $[0, L]$ telle que pour tout $i \in [1, n]$, $f_i > 0$.

Ce résultat est démontré par l'absurde, en supposant qu'il existe une fonction de Lyapunov basique pour la norme C^1 quand cette condition n'est pas vérifiée et en montrant qu'on peut trouver une perturbation initiale qui la rend croissante au moins sur un court intervalle de temps.

Remarque 1.6.1. *La régularité des coefficients n'est pas exactement la même dans le Théorème 1.6.3 et 1.6.4. L'explication est dans la perturbation qu'on construit pour prouver le Théorème 1.6.4 : on cherche à la rendre plus régulière que seulement C^1 pour pouvoir estimer la dérivée de la fonction de Lyapunov dans sens classique. Il est très probable que la régularité puisse être abaissée à celle du Théorème 1.6.3. Néanmoins, dans les cas physiques A et B sont souvent non seulement C^∞ mais même analytiques. Il faut bien voir que c'est la régularité de A et B comme fonctionnelle qu'on impose et pas celle $A(\mathbf{u}(t, \cdot), \cdot)$ qui, elle, pourrait être inférieure si $\mathbf{u}$ est moins régulière.*

Ces résultats se généralisent en fait à la stabilisation en norme C^q pour tout $q \geq 1$, sous les mêmes conditions. En d'autres termes si A, B et G sont de classe C^q (resp. C^{q+2}) alors le Théorème 1.6.3 (resp. Théorème 1.6.4) s'applique toujours en remplaçant la stabilité en norme C^1 par la stabilité en norme C^q. Néanmoins, on cherche à avoir la stabilisation dans la norme la moins régulière possible pour couvrir l'espace de solutions le plus grand possible. C'est pourquoi c'est la norme C^1 qui nous intéresse le plus en général. Enfin, dans le cas où le système est semi-linéaire, c'est-à-dire quand A ne dépend pas de $\mathbf{u}$, ces résultats se généralisent à la norme C^0.

Cas 2×2 et comparaison avec la norme H^2 Dans le Chapitre 3 on s'intéresse au cas où le système est composé de seulement deux équations. Ce cas est intéressant à regarder, pas seulement parce que c'est le cas le plus simple de systèmes présentant un couplage entre deux équations, mais aussi parce qu'il correspond à de très nombreux systèmes physiques (transmission de signal le long d'une ligne électrique, dynamique d'un fleuve, comportement d'un gaz isentropique, trafic routier, etc.) et qu'il présente des comportements particuliers. Dans la suite on considère le cas $n = 2$ et on introduit les notations :

$$\begin{aligned}
\varphi_1 &= \exp\left(\int_0^x \frac{M_{11}(\mathbf{0}, s)}{\Lambda_1} ds\right), \\
\varphi_2 &= \exp\left(\int_0^x \frac{M_{22}(\mathbf{0}, s)}{\Lambda_2} ds\right), \\
\varphi &= \frac{\varphi_1}{\varphi_2},
\end{aligned} \tag{1.6.23}$$

$$\begin{aligned}
a &= \varphi M_{12}(\mathbf{0}, \cdot), \\
b &= \varphi^{-1} M_{21}(\mathbf{0}, \cdot).
\end{aligned} \tag{1.6.24}$$

Avant de continuer, prenons un moment pour expliquer ces notations. Tout d'abord on peut remarquer que φ correspond exactement au φ introduit dans le Théorème 1.6.1 de Bastin et Coron. En fait, les fonctions φ_1 et φ_2 représentent l'influence des termes diagonaux de $M(\mathbf{0}, \cdot)$ qui mèneraient à une variation exponentielle de l'amplitude s'il n'y avait pas de couplage entre u_1 et u_2. Les fonctions a et b représentent, elles, ce couplage induit par $M(\mathbf{0}, \cdot)$ après un changement de variable pour enlever les termes diagonaux de M (voir (3.4.1) et (3.4.4) pour plus de détails). Ce changement de variable visant à supprimer les termes diagonaux de M, qu'on obtient en posant $z_i = \varphi_i u_i$, est en apparence anodin mais il permet de faire en sorte que l'opérateur linéarisé $\mathcal{L}(\mathbf{U}) = -A(\mathbf{0}, x)\partial_x U - M(\mathbf{0}, x)\mathbf{U}$ et l'opérateur linéarisé homogène $\mathcal{L}^*(\mathbf{U}) = -A(\mathbf{0}, x)\partial_x U$ soient munis respectivement de semigroupes S et S^* dont la différence est compacte. Ce qui n'est pas le cas quand M a des termes diagonaux non nuls. Cette remarque formulée par Russell en 1978 [177] et prouvée par Hu et Olive dans [113] peut s'avérer utile pour obtenir des propriétés de contrôlabilité (voir par exemple [113]), bien que ce ne soit pas explicitement utilisé ici.

On montre alors les résultats suivants :

— **Cas où les vitesses de propagation ont le même signe :**
Théorème 1.6.5. *Soit un système quasilinéaire hyperbolique de la forme* (1.4.2), *avec* $n = 2$, *où* A, B *et* G *sont de classe* C^2 *et tel que* $\Lambda_1 \Lambda_2 > 0$. *Si*

$$\begin{aligned}
G'(0) &= \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \text{, où} \\
k_1^2 &< \exp\left(\int_0^L 2\frac{M_{11}(0, s)}{|\Lambda_1|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right), \\
k_2^2 &< \exp\left(\int_0^L 2\frac{M_{22}(0, s)}{|\Lambda_2|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right).
\end{aligned} \tag{1.6.25}$$

Alors il existe une fonction de Lyapunov basique pour la norme C^1 *et une fonction de Lyapunov basique pour la norme* H^2. *En particulier le système est exponentiellement stable pour la norme* C^1 *et la norme* H^2.

— **Cas où les vitesses de propagation ont des signes différents :**
Théorème 1.6.6. *Soit un système quasilinéaire hyperbolique de la forme* (1.4.2) *avec* A *et* B *de classe* C^3 *avec* $\Lambda_1 > 0$ *et* $\Lambda_2 < 0$. *Il existe un contrôle de la forme* (1.6.5) *tel qu'il existe une fonction de Lyapunov basique pour la norme* C^1 *si et seulement si*

$$d_1' = \frac{|a(x)|}{\Lambda_1} d_2, \tag{1.6.26}$$

$$d_2' = -\frac{|b(x)|}{|\Lambda_2|} d_1, \tag{1.6.27}$$

admet une solution positive d_1, d_2 sur $[0,L]$*, ou de façon équivalente*

$$\begin{aligned}\eta' &= \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right|\eta^2,\\ \eta(0) &= 0,\end{aligned} \tag{1.6.28}$$

admet une solution sur $[0,L]$*, où a et b sont définies par* (1.6.24)*.*
Par ailleurs si une des conditions précédentes est vérifiée et

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ avec } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ et } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{1.6.29}$$

où d_1 et d_2 forment une solution de (1.6.26)–(1.6.27), alors le système (1.4.2), (1.6.5) *est exponentiellement stable pour la norme C^1.*

Remarque 1.6.2. *On peut faire une remarque intéressante : dans le Théorème 1.6.3 la condition intérieure nécessaire et suffisante concerne l'existence de solutions à un système d'inéquations différentielles. Ce théorème montre avec* (1.6.26)–(1.6.27) *que, dans le cas des systèmes 2×2, c'est équivalent à l'existence d'une solution à un système d'équations différentielles, linéaires qui plus est, ce qui est pratique. Ce résultat est faux en général en dimension $n \geq 3$.*

Ce n'est pas par hasard si on appelle à nouveau η la solution de l'ODE (1.6.28) : ces conditions sont différentes de celles obtenues précédemment pour la norme H^2, et rappelées dans le Théorème 1.6.1, mais ont la même forme. Il est donc naturel de vouloir les comparer, et c'est ce qu'on fait dans le corollaire suivant :

Corollaire 1. *Soit un système quasilinéaire hyperbolique de la forme* (1.4.2), (1.6.5) *avec $n = 2$ et A et B de classe C^2, tel que $\Lambda_1 > 0$ et $\Lambda_2 < 0$.*

1. *S'il existe une fonction de Lyapunov basique pour la norme C^1 alors il existe un contrôle aux bords de la forme* (1.6.5) *tel qu'il existe une fonction de Lyapunov basique pour la norme H^2.*
   De plus, si $M_{12}(0,\cdot)M_{21}(0,\cdot) \geq 0$, alors la réciproque est vraie.

2. *En particulier si le système* (1.4.2), (1.6.5) *admet une fonction de Lyapunov basique pour la norme C^1 et*

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ avec } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ et } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{1.6.30}$$

   où d_1 et d_2 sont des solutions positives de (1.6.26)–(1.6.27), alors avec le même contrôle aux bords il existe une fonction de Lyapunov basique pour la norme H^2.
   Réciproquement, si le système admet une fonction de Lyapunov basique pour la norme H^2 et si $M_{12}(0,\cdot)M_{21}(0,\cdot) \geq 0$ et

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ avec } k_2^2 < \left(\frac{\varphi(L)}{\eta(L)}\right)^2 \text{ et } k_1^2 < \eta(0)^2, \tag{1.6.31}$$

   où η est une solution positive de

$$\eta' = \left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta^2\right|, \tag{1.6.32}$$

   alors il existe une fonction de Lyapunov basique pour la norme C^1 et le système est stable pour la norme C^1.

Remarque 1.6.3. *L'existence d'une solution à* (1.6.32) *quand il existe une fonction de Lyapunov basique pour la norme H^2 est garantie par le Théorème 1.6.1.*

A ma connaissance, le seul lien de ce genre qui existait jusqu'ici concernait le cas trivial $M \equiv 0$ où il existe toujours à la fois une fonction de Lyapunov basique pour la norme H^2 et une fonction de Lyapunov basique pour la norme C^1. Ce lien peut en fait être étendu à la stabilité en norme H^p et C^q pour $p \geq 2$ et $q \geq 1$ sous les même conditions.

Puisque le Théorème 1.6.1 et le Théorème 1.6.6 utilisent tous les deux des fonctions de Lyapunov basiques pour la norme H^2 et la norme C^1 respectivement, on peut vouloir construire l'une en fonction de l'autre quand on sait, grâce au Corollaire 1, que les deux existent. C'est ce qu'on fait dans ce théorème :

Théorème 1.6.7. *S'il existe un contrôle aux bords de la forme* (1.6.5) *tel qu'il existe une fonction de Lyapunov basique pour la norme* C^1 *avec pour coefficient* g_1 *et* g_2*, alors pour tout* $0 < \varepsilon < \min_{[0,L]}((\varphi_2/\varphi_1)\sqrt{g_1/g_2})/L$ *il existe un contrôle aux bords de la forme* (1.6.5) *tel que*

$$\frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}}\varphi_1\varphi_2 - \varphi_1^2\varepsilon Id\right) \quad et \quad \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}}\varphi_1\varphi_2 \tag{1.6.33}$$

sont les coefficients d'une fonction de Lyapunov basique pour la norme H^2*, où* Id *désigne la fonction identité.*

S'il existe une fonction de Lyapunov basique pour la norme H^2 *avec pour coefficients* (q_1, q_2) *et si* $M_{12}(0,\cdot)M_{21}(0,\cdot) \geq 0$*, alors pour tout* $A \geq 0$ *et* $\varepsilon > 0$ *il existe un contrôle aux bords de la forme* (1.6.5) *tel que* g_1 *et* g_2 *définies par :*

$$g_1(x) = A\exp\left(2\int_0^x \frac{M_{11}(0,\cdot)}{\Lambda_1} - \frac{|M_{12}(0,\cdot)|}{\Lambda_1}\sqrt{\frac{|\Lambda_1|q_1}{|\Lambda_2|q_2}}ds - \varepsilon x\right), \tag{1.6.34}$$

$$g_2 = \frac{|\Lambda_2|q_2}{\Lambda_1 q_1}g_1, \tag{1.6.35}$$

sont les coefficients d'une fonction de Lyapunov basique pour la norme C^1*.*

Le Corollaire 1 donne une condition suffisante pour que l'existence d'une fonction de Lyapunov basique pour la norme H^2 soit équivalente à l'existence d'une fonction de Lyapunov basique pour la norme C^1. En regardant les conditions nécessaires (1.6.28) et (1.6.15), on voit rapidement qu'il existe des contre-exemples quand la condition suffisante du Corollaire 1 n'est pas vérifiée. On pourrait croire que ces contre-exemples sont artificiels mais en fait, et de façon un peu surprenante, il y a même des contre-exemples qui correspondent à des systèmes physiques bien connus. C'est par exemple le cas des équations de Saint-Venant qui servent à modéliser l'atmosphère et les voies navigables (plus de détails sont donnés dans la Section 1.7.3). On verra dans la partie II une forme générale des équations de Saint-Venant, mais ici pour simplifier on introduit un cas particulier dans lequel la pente est constante, la section rectangulaire uniforme, et le modèle de frottement bien déterminé. Les équations de Saint-Venant s'écrivent alors :

$$\begin{aligned} &\partial_t H + \partial_x(HV) = 0, \\ &\partial_t V + \partial_x\left(\frac{V^2}{2} + gH\right) + \left(\frac{kV^2}{H} - C\right) = 0, \end{aligned} \tag{1.6.36}$$

où C est l'influence de la pente, k le coefficient de frottement, g l'accélération de la pesanteur à la surface de la Terre et les variables H et V désignent la hauteur d'eau et sa vitesse horizontale moyenne, par exemple dans un fleuve. On note (H^*, V^*) l'état stationnaire qu'on cherche à stabiliser et on introduit les contrôles aux bords

$$\begin{aligned} &(H(t,0) - H^*(0)) = b_1(V(t,0) - V^*(0)), \\ &(H(t,L) - H^*(L)) = b_2(V(t,L) - V^*(L)). \end{aligned} \tag{1.6.37}$$

Puisque le système n'est pas sous forme diagonalisée, H et V ne sont pas les quantités qui se propagent. Néanmoins on peut montrer que ce contrôle aux bords peut quand même se mettre sous la forme (1.6.5) (voir la Section 3.5 en particulier (3.3.13), (3.5.14)–(3.5.17)). Quand l'influence des frottements est plus forte que celle de la pente, on peut toujours trouver b_1 et b_2 tels qu'il existe une fonction de Lyapunov basique pour la norme C^1, plus précisément

Proposition 1.6.8. *Soit les équations de Saint-Venant non-linéaires (1.6.36) avec le contrôle aux bords* (1.6.37)*. Si* $kV^{*2}(0)/H^*(0) > C$ *et*

$$b_1 \in \left]-\frac{H^*(0)}{V^*(0)}, -\frac{V^*(0)}{g}\right[\quad et\ b_2 \in \mathbb{R} \setminus \left[-\frac{H^*(L)}{V^*(L)}, -\frac{V^*(0)}{g}\right], \tag{1.6.38}$$

alors le système admet une fonction de Lyapunov basique pour la norme C^1*.*

Quand l'influence de la pente est plus forte que celle des frottements, en revanche, il n'existe pas toujours de fonction de Lyapunov basique pour la norme C^1. Ce résultat est d'autant plus intéressant qu'il existe toujours une fonction de Lyapunov basique pour la norme H^2 (voir la partie II, Théorème 1.7.1).

Théorème 1.6.9. *Soit les équations de Saint-Venant non-linéaires (1.6.36) définies sur un domaine* $[0, L]$*, avec un état stationnaire* (H^*, V^*)*. Si* $kV^{*2}(0)/H^*(0) < C$ *alors :*

1. *Il existe* $L_1 > 0$ *tel que si* $L < L_1$*, il existe un contrôle aux bords de la forme* (1.6.37) *tel que le système admet une fonction de Lyapunov basique pour la norme* C^1*.*

2. *Il existe* $L_2 > 0$ *indépendante du contrôle aux bords telle que, si* $L > L_2$*, le système n'admet pas de fonction de Lyapunov basique pour la norme* C^1*.*

Généralisation aux normes H^p et C^q. Comme précédemment, tous ces résultats se généralisent aux normes H^p et C^q pour tous entiers $p \geq 2$ et $q \geq 1$, sous les mêmes conditions, pourvu que A, B, et G soient suffisamment régulières. De plus, quand le système est semi-linéaire, ces résultats se généralisent aussi aux normes H^1 et C^0.

1.7 Partie 2 : Equations physiques et systèmes densité-vélocité.

Dans cette partie on étudie des équations issues de la mécanique des fluides, et en particulier trois exemples célèbres : l'équation de Burgers, les équations d'Euler, et les équations de Saint-Venant.

1.7.1 L'équation de Burgers

Bien qu'en apparence la plus simple des équations de cette partie, l'équation de Burgers est en fait la plus récente. Elle s'écrit ainsi :

$$\partial_t y + \partial_x \left(\frac{y^2}{2}\right) = 0. \tag{1.7.1}$$

Au vu de sa simplicité, il est difficile dater les premiers travaux qui la concerne. Forsyth la mentionnait déjà en 1906 dans [88] avec un terme de viscosité supplémentaire, tout comme Bateman en 1915 [23] qui la déduisait d'un problème physique et se demandait déjà quel serait le comportement des solutions si la viscosité tendait vers 0. En 1939, c'est Burgers qui s'y intéresse et supprime le terme de viscosité pour ne conserver que la forme (1.7.1) qui possède déjà les comportements qui l'intéressent. Ses multiples travaux sur le sujet jusqu'en 1948, résumés dans [35] et [36], la popularisent et lui donnent son nom actuel. Derrière cette approche, Burgers s'intéressait en fait à la turbulence qu'il étudiait déjà depuis 1923 [37], et voyait en l'équation de Burgers un modèle simple pour l'étudier. Malheureusement, les travaux Hopf en 1950 [111] et de Cole en 1951 [45] montrèrent qu'il est possible de résoudre explicitement cette équation, du moins quand elle est étudiée sur $\mathbb{R}$ tout entier.

Néanmoins, si l'équation de Burgers est aussi célèbre, c'est qu'elle possède des comportements génériques qu'on retrouve dans les équations de la mécanique des fluides. Parmi ces comportements, le plus célèbre est peut-être le phénomène de chocs : même en partant d'une solution régulière à $t = 0$ le système peut naturellement "casser" la solution et évoluer vers une solution qui présente des discontinuités (voir la Section 1.4.1 pour plus de détails sur les chocs).
Sa simplicité en fait donc un bon moyen d'étudier les grandes lignes de ces comportements, et un bon banc d'essai pour éprouver de nouvelles techniques mathématiques en vue d'une utilisation ultérieure sur des

équations plus compliquées, comme par exemple les équations d'Euler qu'on verra juste après. La première étude des chocs, par exemple, formulée par Hopf en 1950 dans [111] (voir aussi [133, 134]), commence par l'équation de Burgers.
L'équation de Burgers est aussi parfois utilisée directement pour la modélisation de systèmes physiques, soit parce qu'elle représente déjà une bonne approximation du phénomène, soit parce qu'on cherche à modéliser un réseau qui couple de nombreuses équations identiques entre elles et il est alors utile d'avoir une équation de base suffisamment simple pour réussir à tirer des conclusions sur le réseau tout entier. C'est le cas par exemple du réseau routier [103, Chapitre 3] dont les embouteillages sont bien modélisés par des chocs [28]. Loin d'être un sujet épuisé, ces modélisations ont encore beaucoup de marge de progrès et donnent lieu à des problèmes ouverts intéressants.
Il y a beaucoup à dire sur cette équation et on pourrait probablement consacrer plusieurs thèses aux travaux qui ont été menés sur l'équation de Burgers. On s'arrêtera là, mais pour une description un peu plus détaillée on peut se référer à [153, 1.2.1].

1.7.2 Les équations d'Euler

L'histoire des équations d'Euler est directement liée à celle des équations aux dérivées partielles. La majeure partie de la première équation est en fait proposée par d'Alembert en 1749 dans un mémoire pour un concours de l'Académie des Sciences de Berlin, publié en 1752 [69]. Son mémoire est aujourd'hui reconnu comme une pierre angulaire de la mécanique des fluides et introduit plusieurs notions nouvelles [186] : les dérivées partielles, la notion de pression interne d'un fluide, le champs des vitesses, et la première équation décrivant les fluides comme une fonction de plusieurs variables. Sur décision du jury, dont fait partie Euler, le concours sera en fait décalé à 1752 puis remporté par Adami, un protégé d'Euler. D'Alembert [12] en tient Euler pour directement responsable et il faudra plus de dix ans pour qu'ils se réconcilient. Euler reste néanmoins très intéressé par les travaux de d'Alembert et les complète en 1751 [13] puis 1755 [87] en y ajoutant un élément fondamental : le gradient de pression. Ses équations s'écrivent alors :

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{V}) = 0 \\ &\partial_t(\varrho \mathbf{V}) + \operatorname{div}(\varrho \mathbf{V}\mathbf{V}) = -\nabla p - \varrho \mathbf{g}. \end{aligned} \tag{1.7.2}$$

où ϱ est la densité du fluide, $\mathbf{V}$ son vecteur vitesse, p la pression et $\mathbf{g}$ est le vecteur gravité, complétant ainsi ce qui sera l'ingrédient principal de la mécanique des fluides jusqu'à nos jours.

Ces équations seront à leur tour complétées plus tard par les travaux de Navier en 1823 et Stokes en 1845 (avec des contributions de Cauchy et Poisson en 1829 et de Saint-Venant en 1843) pour tenir compte de la viscosité et donner les célèbres équations de Navier-Stokes. Enfin, si ϱ est une constante, ces équations deviennent alors les équations d'Euler incompressibles.

Les équations d'Euler sont liées à de nombreux problèmes compliqués. En 3D, même dans le cas incompressible, leur caractère bien posé, c'est-à-dire l'existence et l'unicité d'une solution, est encore un des problèmes mathématiques les plus difficiles à l'heure actuelle qui, selon Villani [14], est encore plus difficile à résoudre que le caractère bien posé de l'équation de Navier-Stokes, pourtant un des problèmes du millénaire.

En 1D leur caractère bien posé est beaucoup mieux maîtrisé. En regardant (1.7.2), on voit qu'il faut une équation supplémentaire pour la pression pour décrire le système. Une simplification consiste à considérer que la pression est une fonction (croissante) de la densité du fluide ρ, ce qui donne les équations d'Euler

12. Il faut néanmoins rappeler qu'à cette date d'Alembert est un membre étranger de l'Académie de Berlin et avait remporté le concours pour l'année 1746, dans lequel Euler était déjà membre du jury. Les relations entre d'Alembert et Euler sont complexes et sont décrites plus en détail dans [185, 186]

13. Au même moment, dans un autre domaine et un tout autre contexte sort la première édition d'une certaine *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, dirigée par Diderot et d'Alembert, mais ceci est une autre histoire.

14. dans *Théorème vivant*

isentropiques que nous étudierons dans la suite :

$$
\begin{aligned}
&\partial_t \varrho + \partial_x(\varrho V) = 0, \\
&\partial_t V + V\partial_x V = -\frac{1}{\varrho}\partial_x(P(\varrho)) - \frac{1}{2}\theta V|V| - g\sin\alpha(x),
\end{aligned}
\tag{1.7.3}
$$

où α est l'angle de la pente, P la pression, et θ est un terme de frottements qui décrit la friction entre le fluide et l'extérieur et qui apparaît à partir des conditions aux bords quand on intègre l'équation 3D pour passer à l'équation 1D. Enfin, d'autres simplifications existent :

— Approximation des gaz polytropiques : $P(\rho) = C\rho^\gamma$ où C est une constante et $\gamma > 1$.

— Approximation isotherme : $P(\rho) = C\rho$ où C est une constante.

La première approximation est souvent directement considérée avec les équations d'Euler isentropiques, pour les rendre explicites, même si elle ne leurs sont pas rigoureusement liées. La deuxième approximation donne lieu aux équations d'Euler isothermes. Nous n'utiliserons aucune de ces deux simplifications dans la suite.

1.7.3 Les équations de Saint-Venant

Découvertes de manière heuristique par Saint-Venant en 1871 pour décrire l'écoulement dans un canal unidimensionnel [10], les équations de Saint-Venant modélisent le mouvement des fluides quand la profondeur est faible comparée à la longueur du système et que le système évolue à surface libre. Elles sont donc très adaptées à la description des fleuves en 1D, tandis que leur généralisation naturelle en 2D sert à la modélisation de l'atmosphère. Elles s'écrivent de la façon suivante :

$$
\begin{aligned}
&\partial_t A + \partial_x(AV) = 0 \\
&\partial_t V + V\partial_x V + g\partial_x H - S_b(x) + S_f(A, V, x) = 0
\end{aligned}
\tag{1.7.4}
$$

où A est la section immergée, V la vitesse du fluide, H la hauteur du fluide, $S_b = -\partial_x \mathcal{Z}$ l'influence de la pente avec $\mathcal{Z}$ la bathymétrie, et S_f l'influence des frottements. On peut remarquer que plus la hauteur d'eau augmente, plus la section immergée augmente, tant qu'on n'atteint pas le haut de la paroi au moins. En d'autres termes, il existe une fonction F qui dépend du profil du fleuve telle que $A = F(H, x)$ et $\partial_H F(H, x) > 0$. De manière surprenante, les équations énoncées par Saint-Venant en 1871 sont déjà les plus générales en 1D et n'ont donc pas été considérablement complétées par la suite, même si des variantes avec viscosité et de nombreux autres modèles moins célèbres mais plus adaptés à certaines situations ont été et sont toujours développés (ex. [29–31, 155]). Comme on le verra plus tard, ces équations ont même été simplifiées dans un premier temps dans les études de stabilité, les équations générales s'avérant trop compliquées.

Si Saint-Venant les a obtenues heuristiquement, ces équations sont cohérentes avec les équations déjà existantes en mécanique des fluides et peuvent en fait se déduire des équations d'Euler sous l'hypothèse de faible profondeur. Une variante plus récente incluant une viscosité a été introduite en 2001 par Gerbeau et Perthame dans [91] comme conséquence des équations de Navier-Stokes sous cette même hypothèse de faible profondeur.

Leur relative précision et leur simplicité apparente, ont fait de ces équations à la fois un objet mathématique très étudié, analytiquement et numériquement [90], et un outil technique très utilisé en ingénierie, en particulier dans la régulation des voies navigables. Cette dernière application est loin d'être anecdotique quand on sait qu'il y a près de 700 000 km de voies naviguables dans le monde. Par ailleurs l'hydroélectricité représente 70% de l'énergie renouvelable produite à l'échelle mondiale, et l'hydroélectricité fluviale est celle dont le développement local est le plus facile et provoque le moins de dégâts environnementaux [15].

Pour les équations de Saint-Venant, il existe un phénomène de choc célèbre : le ressaut hydraulique. Afin de comprendre ce qu'est un ressaut hydraulique il faut se demander quelle la différence entre un fleuve et un torrent ? La première réponse qui vient à l'esprit c'est la vitesse de l'écoulement, et c'est à peu près correct. Dans un écoulement en régime torrentiel $V > \sqrt{gH}$ et on peut montrer que les deux vitesses de

15. Un point de vue défendu dans [163]

propagation de (1.7.4), données par $V+\sqrt{gH}$ et $V-\sqrt{gH}$ (voir Chapitre 4), sont strictement positives. Dans un écoulement en régime fluvial, par contre, $V<\sqrt{gH}$ et on peut montrer que les vitesses de propagation ont des signes opposés. La transition du premier régime au deuxième s'accompagne d'un choc entropique et s'appelle un ressaut hydraulique. Ces ressauts hydrauliques apparaissent fréquemment dans les fleuves. Ils sont à l'origine des mascarets, par exemple dans la Gironde. Ils apparaissent aussi parfois dans l'atmosphère et sont à l'origine des Morning Glory Clouds [44], nuages qui peuvent être observés de temps à autre en Australie. Ils sont aussi soupçonnés d'avoir été à l'origine de certains crash de planeurs [129]. L'importance de ces ressauts ne tient pas seulement à leurs apparitions spontanées mais aussi à leur utilité en ingénierie où on provoque parfois volontairement un ressaut dans des installations hydrauliques pour dissiper de l'énergie et protéger le bassin naturel ou les installations.

1.7.4 Un cadre plus général : les systèmes densité-vélocité

Gagnons un peu en abstraction pour introduire un cadre un peu plus général qui nous sera utile pour la suite. Comme leur nom l'indique, les systèmes "densité-vélocité" sont les systèmes décrit par la densité d'une certaine quantité (matière, probabilité, etc.) et la vélocité de cette même quantité pour laquelle le flux est conservé. Ils sont constitués d'une équation de continuité (ou encore conservation du flux) couplée avec une équation d'énergie ou une équation de moment, et s'écrivent sous la forme

$$\begin{aligned}&\partial_t \varrho + \partial_x(\varrho V) = 0,\\ &\partial_t V + V\partial_x V + \partial_x(P(\varrho,x)) + S(\varrho,V,x) = 0,\end{aligned} \tag{1.7.5}$$

où ϱ est la densité, V la vitesse, P est une fonction croissante de ϱ et représente par exemple l'influence des forces de pression ou de forces dérivant d'un potentiel et S est un terme source. Ce cadre couvre de nombreux systèmes en mécanique des fluides, en particulier les équations d'Euler isentropiques et les équations de Saint-Venant décrites juste avant, mais aussi d'autres exemples comme les équations décrivant le mouvement d'un fluide dans une canalisation rigide, qu'on peut trouver dans [13, Chapitre 1],

$$\begin{aligned}\partial_t\left(\exp\left(\frac{gP}{c^2}\right)\right)+\partial_x\left(V\exp\left(\frac{gP}{c^2}\right)\right)=0,\\ \partial_t V + V\partial_x V + \partial_x(gP) + S_f(V,x) - gC(x) = 0\end{aligned} \tag{1.7.6}$$

où P est la tête piezométrique, $V>0$ la vitesse du fluide, c la vélocité du son dans le fluide, g l'accélération de la pesanteur à la surface de la Terre, S_f le terme de frottements et C l'influence de la pente. Un autre exemple encore sont les équations régissant le phénomène d'osmose tel que modélisé dans [154] et qui sont constituées des équations d'Euler isentropiques avec une barrière de potentiel supplémentaire qui agit sur le soluté :

$$\begin{aligned}&\partial_t\varrho+\partial_x(\varrho V) = 0,\\ &\partial_t V+V\partial_x V + \frac{\partial_x(P(\varrho))}{\varrho} + \frac{1}{2}\theta V|V| + g\sin\alpha(x) - c(x)\partial_x\mathcal{U} = 0,\end{aligned} \tag{1.7.7}$$

où $\mathcal{U}$ est le profil de la barrière de potentiel, à support compact, et c la concentration du soluté.

Dans cette partie, on cherche à garantir la stabilité en norme H^2 de ces systèmes par des feedbacks de sortie. Pour cela, on doit régler les deux problèmes mentionnés en Section 1.4.5 : les termes sources et les chocs.

Avant de continuer on peut se poser la question suivante : pourquoi considérer la norme H^2, alors même qu'on a passé la Section 1.6 à expliquer que la norme C^1 est plus naturelle mathématiquement ? La réponse est à chercher du coté de la physique. Dans cette partie on cherche à garantir la stabilité en utilisant des quantités physiques cachées dans la structure du système, comme des énergies ou des entropies. Or ces quantités sont souvent localement des intégrales de formes quadratiques et ont donc la forme d'une norme H^p, avec $p\geq 0$, comme mentionné dans la Section 1.4.4. Mais, en général, la régularité H^p minimale pour laquelle on sait étudier la stabilité des systèmes quasilinéaires est la norme H^2. C'est d'ailleurs la raison pour laquelle c'est cette norme qui est considérée dans [13, Chapitre 6].

1.7.5 Une nouvelle entropie pour les systèmes de densité-vélocité inhomogènes.

L'étude de la stabilisation des systèmes densité-vélocité par des contrôles aux bords n'est pas nouvelle. Pour les mêmes raisons que dans la Section 1.6, on ne parlera ici que des résultats avec des rétroactions qui ne nécessitent de mesurer que l'état du système aux bords. D'autres résultats utilisant la méthode du backstepping et des rétroactions nécessitant la connaissance de l'état du système sur tout le domaine sont décrits en introduction du Chapitre 4. Pour les rétroactions qui ne nécessitent que la connaissance du système aux bords, les trois approches mentionnés en Section 1.4.4 ont été utilisées par le passé avec une particularité : les fonctions de Lyapunov utilisées sont souvent des entropies ou des énergies physiques pour le système, telle l'énergie mécanique.

Pour les équations de Saint-Venant, les premiers résultats sur le système non-linéaire datent de [52], en 1999, où les auteurs étudient les équations non-linéaires homogènes à l'aide d'une fonction de Lyapunov. Cette fonction de Lyapunov n'est pas stricte et nécessite l'utilisation du principe d'invariance de LaSalle pour conclure à la stabilité asymptotique. Plus tard, dans [53], est introduite une fonction de Lyapunov stricte valable pour tous les systèmes homogènes qui se mettent sous la forme d'invariants de Riemann et couvre ainsi l'ensemble des systèmes densité-vélocité en l'absence de terme source. Plus tard encore, dans [83, 171] les auteurs montrent en utilisant la méthode des caractéristiques qu'on peut stabiliser les équations de Saint-Venant avec terme source quand les termes sources sont suffisamment faibles en norme C^1, avec une borne qui dépend de la longueur du domaine L. Cette borne peut aussi être vue comme une borne sur la longueur L pour un terme source fixé. Dans [47, Section 13.4] et [50] les auteurs montrent le même type de résultat, mais en utilisant une fonction de Lyapunov. Puis, dans [18], les auteurs étudient les équations linéaires inhomogènes sans borne sur le terme source, et montrent que la fonction de Lyapunov utilisée dans [53] peut aussi servir à stabiliser ce système mais dans le cas très particulier où l'état stationnaire est uniforme. Enfin, récemment dans [16] les auteurs arrivent à stabiliser les équations non-linéaires inhomogènes pour la norme H^2 sans aucune borne sur le terme source ou la longueur du domaine, mais en l'absence de pente, c'est-à-dire dans le cas où le terme source est dissipatif et ne comporte que des frottements. Ce dernier résultat est, à ma connaissance, le plus abouti en la matière avant les travaux présentés dans cette partie. Malheureusement dans la pratique les états stationnaires les plus fréquents sont justement ceux où l'influence de la pente est plus forte que celle des frottements, les autres cessent d'exister après une distance finie parfois courte [110], [26, Chapitre 5]. Cette étude, néanmoins, donne un cadre et une condition générale concernant la stabilisation des systèmes densité-vélocité en norme H^2.

Concernant les équations d'Euler isentropiques, plusieurs résultats ont été obtenus en rajoutant l'hypothèse isothermale présentée avant : les auteurs de [82] montrent qu'il est possible de stabiliser le système par des feedbacks de sortie pour la norme H^1 à l'aide d'une fonction de Lyapunov. Notons que la stabilité en norme H^1, potentiellement meilleure que la stabilité en norme H^2, est alors rendue possible par l'hypothèse isothermale. La stabilité avec une incertitude sur les conditions aux bords et sans terme source a été étudiée dans [101]. Toujours avec la même hypothèse, le cas d'un réseau a été étudié dans [81, 94, 95]. Dans [99] les auteurs arrivent à la fois à stabiliser exponentiellement les équations d'Euler isothermales pour la norme H^2 et à donner une estimation du taux de décroissance, mais avec une limite sur la longueur du domaine. Enfin dans [16], sans utiliser l'hypothèse isothermale mais seulement l'hypothèse, moins restrictive, des gaz polytropiques, les auteurs arrivent à démontrer la stabilité des équations nonlinéaires pour la norme H^2 sans aucune limite sur la taille du domaine mais en l'absence de pente.

Ces études traitent donc à chaque fois de cas particuliers, soit parce que la longueur du domaine ou les termes sources doivent respecter une certaine borne, soit parce qu'ils ont une forme particulière. Jusqu'ici aucun résultat n'existe dans le cas général, qui présente au moins trois différences importantes : l'absence de borne sur le terme source et la longueur du domaine, la présence d'une pente qui peut rendre le terme source non-dissipatif, c'est-à-dire qu'il fait gagner de l'énergie au système plutôt que lui en faire perdre, et la présence d'une section variable.

Avant de continuer, on peut remarquer que pour un état stationnaire régulier (ϱ^*, V^*) de (1.7.5), trois cas sont possibles :

— Régime torrentiel : $\partial_H P(\varrho^*(x), x)\varrho < V^{*2}(x)$ pour tout $x \in [0, L]$ et les vitesses de propagation du système sont strictement positives. Ce cas est traité par le Théorème 1.6.5 de la partie I.

— Régime critique : $\partial_H P(\varrho^*(x), x)\varrho = V^{*2}(x)$ pour tout $x \in [0, L]$, d'après (1.7.5) ce cas est très particulier et n'existe pas toujours. Par ailleurs une des vitesses de propagation s'annule.

— Régime fluvial : $\partial_H P(\varrho^*(x), x)\varrho > V^{*2}(x)$ pour tout $x \in [0, L]$ et les vitesses de propagation ont des signes opposés. Ce sont les cas compliqués et ce sont ceux qu'on regardera.

A partir de maintenant on supposera donc qu'on se place en régime fluvial. Comme mentionné dans la Section 1.6, un système inhomogène stabilisé par des feedback de sortie a *a priori* une longueur intrinsèque $L_{\max}$ au-delà de laquelle on échoue à démontrer la stabilité quel que soit le choix de notre contrôle aux bords. Et ce même en norme H^2. Dans le Chapitre 4 on montre qu'en réalité ce phénomène de longueur maximale $L_{\max}$ ne se produit jamais pour les équations de Saint-Venant avec section rectangulaire. Ce résultat est vrai même quand le terme source n'est pas dissipatif, c'est-à-dire qu'il ne fait pas perdre d'énergie au système mais qu'au contraire il lui en fait gagner. Ce dernier cas de figure se présente quand l'influence de la pente est plus forte que celle des frottements, ce qui correspond à une grande partie des cas pratiques. Plus précisément, on montre le résultat suivant :

Théorème 1.7.1. *Considérons les équations de Saint-Venant non-linéaires* (1.7.4) *avec* $A = H$ *et* $S(A, V, \cdot) = kV^2/H - gC$*, où* $C \in C^2([0, L])$*, munies de contrôles aux bords de la forme*

$$V(t, 0) = \mathcal{B}(H(t, 0)), \quad V(t, L) = \mathcal{B}(H(t, L)). \tag{1.7.8}$$

Si les conditions suivantes sont vérifiées

$$\begin{aligned} &\mathcal{B}'(H^*(0)) \in \Big(-\frac{g}{V^*(0)}, -\frac{V^*(0)}{H^*(0)} \Big), \\ \textit{et} \quad &\mathcal{B}'(H^*(L)) \in \mathbb{R} \setminus \Big[-\frac{g}{V^*(L)}, -\frac{V^*(L)}{H^*(L)} \Big], \end{aligned} \tag{1.7.9}$$

alors le système est exponentiellement stable pour la norme H^2*.*

Ce résultat signifie qu'on peut toujours stabiliser les équations de Saint-Venant à l'aide de feedbacks de sortie, quels que soient le terme source et la longueur du domaine. En fait ce résultat est plus fort que cela : de façon très surprenante, les lois de commandes ne dépendent pas directement ni des frottements, ni de la pente. [16] On montre même dans le Chapitre 5 que ce résultat se généralise aux équations de Saint-Venant générales et que les lois de commandes ne dépendent pas non plus du profil de la section ou du modèle utilisé pour les frottements. On peut donc stabiliser le système en ne connaissant que la hauteur et la vitesse aux bords et celles de l'état stationnaire visé, en ignorant tout le reste y compris une partie du modèle ainsi que toutes les données physiques du fleuve parfois difficiles à récupérer (bathymétrie, coefficients de frottements, profil de la section le long du fleuve).

Une entropie locale dissipative Ce résultat découle en fait de l'existence d'une entropie locale dissipative pour les équations de Saint-Venant inhomogènes. Pour les équations de Saint-Venant homogènes, l'entropie naturelle est bien connue : il s'agit de l'énergie mécanique $gH^2/2 + HV^2/2$, largement utilisée [52, 160]. Pour les équations de Saint-Venant inhomogènes, en revanche, cette énergie mécanique ne correspond pas à une entropie dissipative, quand bien même le terme inhomogène ne contiendrait que des frottements et donc serait purement dissipatif [16, Remarque 1]. Dans le Chapitre 4, on montre qu'on peut tout de même former localement une entropie dissipative autour de n'importe quel état stationnaire (H^*, V^*), et ce quel que soit le terme source. Cette entropie s'écrit

$$\begin{aligned} \mathcal{E} = &\exp\left(\int_0^x \frac{-3gC}{gH^* - V^{*2}} ds\right) \frac{H^{*7/2}\sqrt{gH^*}}{gH^* - V^{*2}} \\ &\times \left(g(H - H^*)^2 + 2V^*(H - H^*)(V - V^*) + H^*(V - V^*)^2\right). \end{aligned} \tag{1.7.10}$$

16. Evidemment elles en dépendent de manière indirecte à travers les états stationnaires. Mais quand on veut stabiliser une hauteur d'eau et une vitesse, le minimum c'est de savoir à quelle hauteur et à quelle vitesse on veut les stabiliser.

Elle correspond à un cas limite puisque l'inégalité de production d'entropie (1.4.28) introduite dans la Section 1.4.4, qui donne le caractère dissipatif, devient en fait une égalité pour tout une classe de perturbations : si $HV = H^*V^*$,

$$\partial_{(H,V)}\mathcal{E}(H,V,x).P(H,V,x) = 0, \tag{1.7.11}$$

où $P(H,V,x) = \begin{pmatrix} 0 \\ S(H,V,x) \end{pmatrix}$ est le terme source de l'équation de Saint-Venant.

On a vu en Section 1.4.4 le lien entre les entropies dissipatives et les fonctions de Lyapunov. Pour comprendre d'où vient cette entropie, on peut donc regarder le point de vue Lyapunov. D'après le résultat de Bastin et Coron rappelé dans le Théorème 1.6.1, il existe une fonction de Lyapunov basique pour la norme H^2 si et seulement si [17].

$$\eta' = \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta^2 \right|, \tag{1.7.12}$$

admet une solution η sur $[0,L]$ telle que $\eta(0) > 0$. Ici $\Lambda_1 > 0$, $\Lambda_2 < 0$ sont les vitesses de propagation et a et b sont les quantités du systèmes définies par (1.6.24) dans le cas des équations de Saint-Venant. Une expression explicite est donnée dans le Chapitre 4 (voir (4.3.17)–(4.3.20), (4.3.21)–(4.3.23), (4.3.26)). En général, trouver des solutions explicites à (1.7.12) est vain quand les fonctions $a/|\Lambda_1|$ et $b/|\Lambda_2|$ ne sont pas triviales. Une méthode consiste donc à trouver à la place une fonction η explicite et définie sur $[0,L]$, telle que $\eta(0) > 0$ et

$$\eta' > \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta^2 \right|, \tag{1.7.13}$$

en essayant de se rapprocher au plus possible du cas d'égalité. C'est ce qui est utilisé par exemple dans [16] pour les équations de Saint-Venant. On montre que les équations de Saint-Venant ont en fait la remarquable propriété suivante :

Lemme 1.7.1. *La fonction*

$$\eta = \frac{|\Lambda_2|}{\Lambda_1}\varphi \tag{1.7.14}$$

est une solution de l'équation (1.7.12)*.*

La fonction η qu'on trouve ici est donc optimale au sens où on est capable d'atteindre le cas d'égalité. Par ailleurs, plus la valeur de $\eta(0)$ est élevée, moins la condition sur le contrôle en 0 est restrictive (plus de détails dans le Chapitre 4, en particulier (4.3.40) et (4.3.43), ou [11]). Il est donc légitime de se demander s'il existe une autre fonction η_2 encore meilleure, qui serait aussi une solution de (1.7.12) pour toute amplitude du terme source et toute longueur $L > 0$ et telle que $\eta_2(0) > |\Lambda_2|(0)\phi(0)/\Lambda_1(0)$. La réponse, négative, est donnée par la proposition suivante

Proposition 1.7.2. *Soit un état stationnaire* (H^*,V^*) *tel que* $S(H^*,V^*,x) < 0$*. Pour tout* $\eta_{2,0} > |\Lambda_2(0)|/\Lambda_1(0)$*, il existe* $L > 0$ *telle que* $(H^*,V^*) \in C^1([0,L];\mathbb{R}^2)$ *et la solution* η_2 *de* (1.7.12) *avec* $\eta_2(0) = \eta_{2,0}$ *vérifie*

$$\lim_{x\to L} \eta_2(x) = +\infty. \tag{1.7.15}$$

Cette fonction $|\Lambda_2|\varphi/\Lambda_1$ et le Lemme 1.7.1 sont la clé de voûte du Théorème 1.7.1.

Dans le Chapitre 5 on montre que cette entropie locale dissipative et ce résultat sont en fait généralisables non seulement aux équations de Saint-Venant générales mais aussi à tous les systèmes de densité-vélocité, dont on a donné quelques exemples plus haut. Plus précisément si on considère le système

$$\begin{aligned} &\partial_t \varrho + \partial_x(\varrho V) = 0, \\ &\partial_t V + V\partial_x V + \partial_x(P(\varrho,x)) + S(\varrho,V,x) = 0, \end{aligned} \tag{1.7.16}$$

17. Cette forme est légèrement différente de celle donnée dans la Section 1.6.2, mais parfaitement équivalente comme on le montre dans [107] et dans le Chapitre 3 et à l'avantage de libérer la condition initiale.

introduit dans la Section 1.7.4, où $S \in C^2(]0,+\infty[^3;\mathbb{R})$ et $P \in C^2(]0,+\infty[^2;\mathbb{R})$ et muni du contrôle aux bords

$$\begin{aligned} V(t,0) &= \mathcal{B}(A(t,0)), \\ V(t,L) &= \mathcal{B}(A(t,L)), \end{aligned} \tag{1.7.17}$$

on montre :

Théorème 1.7.3. *Soit* (ϱ^*, V^*) *un état stationnaire du système* (1.7.16)–(1.7.17) *tel que*

$$\frac{\partial_V S(\varrho^*, V^*(x), x)}{V^*(x)} - \frac{\partial_\varrho S(\varrho^*, V^*(x), x)}{\partial_\varrho P(\varrho^*, x)} \geq 0, \ \forall\, x \in [0, L]. \tag{1.7.18}$$

Si les conditions suivantes sont vérifiées

$$\begin{aligned} \mathcal{B}'(\varrho^*(0)) &\in \left]-\frac{\partial_\varrho P(\varrho^*(0),0)}{V^*(0)}, -\frac{V^*(0)}{\varrho^*(0)}\right[, \\ \mathcal{B}'(\varrho^*(L)) &\in \mathbb{R} \setminus \left[-\frac{\partial_\varrho P(\varrho^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{\varrho^*(L)}\right], \end{aligned} \tag{1.7.19}$$

alors l'état stationnaire (ϱ^*, V^*) *est exponentiellement stable pour la norme* H^2.

Remarque 1.7.1. *La condition* (1.7.19) *qui apparaît dans ce théorème est en fait une condition très physique. Notons qu'elle est vérifiée pour tous les termes sources issus de forces extérieures et qui ne dépendent ni de* A *ni de* V*. Pour les équations de Saint-Venant elle est évidemment vérifiée puisque la pente ne dépend ni de* A *ni de* V *tandis que les frottements croissent avec la vitesse et décroissent (au sens large) avec la section, donc* $\partial_V S(A^*, V^*, \cdot) \leq 0$ *et* $\partial_A S(A^*, H^*, \cdot) \geq 0$*. Elle est aussi vérifiée pour tous les exemples donnés dans la Section 1.7.4.*

1.7.6 Stabilisation d'états stationnaires avec chocs

Comme mentionné en introduction, les équations aux dérivées partielles hyperboliques non-linéaires ont ceci de particulier qu'elle peuvent faire apparaître des discontinuités naturellement. Ces discontinuités, appelées chocs, entraînent des difficultés techniques dans l'étude des équations, si bien que la grande majorité des études de stabilité se concentrent sur des solutions et des états stationnaires réguliers, comme nous l'avons fait jusqu'ici. Néanmoins les chocs correspondent à des phénomènes physiques concrets, tels le ressaut hydraulique pour les équations de Saint-Venant, et leur étude est loin d'être dénuée d'intérêt.

Par le passé plusieurs travaux ont considéré ce problème. Dans [125, 182], par exemple, sont considérés des solutions régulières et des états stationnaires réguliers mais dont les profils se rapprochent d'un choc. D'autres travaux se sont attaqués à des systèmes pouvant avoir des solutions discontinues. Dans [33] par exemple est étudiée la contrôlabilité d'un système hyperbolique homogène pour des solutions de classe BV, qui est généralement la classe la plus générale de fonctions avec chocs pour les systèmes hyperboliques [18]. La stabilisation d'une équation scalaire est traitée dans [27, 164] tandis que celle d'un système hyperbolique homogène est étudiée dans [33, 57]. Mais dans tous ces articles l'état stationnaire à stabiliser est régulier (et même constant), or dans l'idéal on aimerait pouvoir stabiliser un état stationnaire présentant un choc, comme par exemple un ressaut hydraulique. C'est le but des Chapitres 6 et 7 qui traitent respectivement des équations de conservation scalaires et des équations de Saint-Venant. Plus précisément dans le Chapitre 6 on considère le système suivant constitué de l'équation de Burgers sur un domaine borné $[0, L]$ avec deux contrôles aux bords $u_0(t)$ et $u_L(t)$:

$$\begin{aligned} &\partial_t y + \partial_x \left(\frac{y^2}{2}\right) = 0 \\ &y(t, 0^+) = u_0(t) \\ &y(t, L^-) = u_L(t). \end{aligned} \tag{1.7.20}$$

18. Etonnamment cet article montre avec un exemple que la contrôlabilité exacte ne peut pas être obtenue en général, alors que ce phénomène ne se produit pas quand les solutions sont régulières.

On s'intéresse à des états y qui comportent un choc localisé en $x_s \in]0,L[$ et qui sont réguliers ailleurs. La dynamique de ce choc peut être obtenue à partir de (1.7.20) et des relations de Rankine-Hugoniot qui imposent

$$\dot{x}_s(t) = \frac{y(t,x_s(t)^+) + y(t,x_s(t)^-)}{2} \tag{1.7.21}$$

On cherche à stabiliser les états stationnaires suivants :

$$\begin{aligned} y^*(x) &= \begin{cases} 1, & x \in [0,x_0[, \\ -1, & x \in]x_0,L], \end{cases} \\ x_s^* &= x_0. \end{aligned} \tag{1.7.22}$$

où $x_0 \in]0,L[$. On vérifie aisément que cet état stationnaire est entropique, et en fait tout état stationnaire entropique de (1.7.20) avec un seul choc peut se ramener à celui-ci sans perte de généralité, moyennant un changement d'échelle. Il est clair que si on ne cherchait pas à stabiliser la position du choc en x_0 mais seulement l'état du système y, on pourrait choisir des contrôles constants $u_0(t) = -u_L(t) = 1$ et se placer en boucle ouverte, c'est-à-dire avec un contrôle sans rétroaction. Notre but, et c'est la principale difficulté, est de stabiliser complètement l'état stationnaire : c'est-à-dire à la fois l'amplitude y avant et après le choc mais aussi la position du choc x_s. On optera donc pour des feedbacks de la forme :

$$\begin{aligned} u_0(t) &= 1 + k_1(y(t,x_s(t)^-) - 1) + b_1(x_0 - x_s(t)), \\ u_L(t) &= -1 + k_2(y(t,x_s(t)^+) + 1) + b_2(x_0 - x_s(t)). \end{aligned} \tag{1.7.23}$$

Ces feedbacks sont sans doute les plus simples que l'on puisse prendre au sens où ils nécessitent le minimum d'informations possible : admettons que l'état du système soit proche de l'état stationnaire, alors d'après (1.7.20) la vitesse de propagation est positive avant le choc et négative après le choc, donc aucune information ne transite d'un coté à l'autre du choc. Cela signifie qu'il est nécessaire de mesurer l'état du système au moins en un point de chaque coté, et c'est ce qui est utilisé par le premier terme de rétroaction de (1.7.23). Par ailleurs, puisqu'on veut stabiliser la position du choc, le minimum est de connaître la position du choc x_s, et c'est ce qui est utilisé par le second terme de rétroaction de (1.7.23).

On montre d'une part que ce système est bien posé :

Théorème 1.7.4. *Pour tout $T > 0$, il existe $\delta(T) > 0$ tel que, pour tout $x_{s0} \in]0,L[$ et $y_0 \in H^2(]0,x_{s0}[;\mathbb{R}) \cap H^2(]x_{s0},L[;\mathbb{R})$ satisfaisant les conditions de compatibilité d'ordre 1 associées à* (1.7.23) *et*

$$\begin{aligned} &|y_0 - 1|_{H^2(]0,x_{s0}[;\mathbb{R})} + |y_0 + 1|_{H^2(]x_{s0},L[;\mathbb{R})} \leq \delta(T), \\ &|x_{s0} - x_0| \leq \delta(T), \end{aligned} \tag{1.7.24}$$

alors le système (1.7.20), (1.7.21), $y(0,\cdot) = y_0$, $x_s(0) = x_{s0}$, (1.7.23) *a une unique solution entropique C^1 par morceaux $y \in C^0([0,T]; H^2(]0,x_s(t)[;\mathbb{R}) \cap H^2(]x_s(t),L[;\mathbb{R}))$ avec $x_s \in C^1([0,T];]0,L[)$ son seul choc. Par ailleurs, il existe $C(T)$ telle que l'estimée suivante est vérifiée pour tout $t \in [0,T]$*

$$\begin{aligned} &|y(t,\cdot) - 1|_{H^2(]0,x_s(t)[;\mathbb{R})} + |y(t,\cdot) + 1|_{H^2(]x_s(t),L[;\mathbb{R})} + |x_s(t) - x_0| \\ &\qquad \leq C(T)\left(|y_0 - 1|_{H^2(]0,x_{s0}[;\mathbb{R})} + |y_0 + 1|_{H^2(]x_{s0},L[;\mathbb{R})} + |x_{s0} - x_0|\right). \end{aligned} \tag{1.7.25}$$

En fait on peut même prouver de manière très similaire à ce qui est fait dans [134, Preuve du Théorème 3.4] que cette solution est l'unique solution entropique pour ce système et ces conditions initiales. En résumé, ce résultat nous apprend donc que si l'état commence avec un seul choc alors il garde un seul choc durant son évolution. Plus de précisions sur les définitions, en particulier sur les conditions de compatiblité et la définition d'une solution C^1 par morceaux entropique, sont à trouver dans le Chapitre 6.

D'autre part, puisque la solution n'est pas régulière et qu'on cherche à stabiliser aussi la position du choc, on doit légèrement changer la définition de la stabilité exponentielle de la façon suivante :

Définition 1.7.1. L'état stationnaire $(y^*, x_0) \in H^2(]0, x_0[;\mathbb{R}) \cap H^2(]x_0, L[;\mathbb{R}) \times]0, L[$ du système (1.7.20), (1.7.21), $y(0, \cdot) = y_0$, $x_s(0) = x_{s0}$, (1.7.23) est exponentiellement stable pour la norme H^2 avec un taux de décroissance γ, s'il existe $\delta^* > 0$ et $C > 0$ tels que si $y_0 \in H^2(]0, x_{s0}[;\mathbb{R}) \cap H^2(]x_{s0}, L[;\mathbb{R})$ et $x_{s0} \in]0, L[$ sont compatibles avec (1.7.23) et vérifient

$$\begin{aligned} &|y_0 - y_1^*(0,\cdot)|_{H^2(]0,x_{s0}[;\mathbb{R})} + |y_0 - y_2^*(0,\cdot)|_{H^2(]x_{s0},L[;\mathbb{R})} \leqslant \delta^*, \\ &|x_{s0} - x_0| \leqslant \delta^* \end{aligned} \tag{1.7.26}$$

le système a une unique solution $(y, x_s) \in C^0([0, +\infty[; H^2(]0, x_s(t)[;\mathbb{R}) \cap H^2(]x_s(t), L[;\mathbb{R})) \times C^1([0, +\infty[;\mathbb{R})$ et

$$\begin{aligned} &|y(t,\cdot) - y_1^*(t,\cdot)|_{H^2(]0,x_s(t)[;\mathbb{R})} + |y(t,\cdot) - y_2^*(t,\cdot)|_{H^2(]x_s(t),L[;\mathbb{R})} + |x_s(t) - x_0| \\ &\quad \leqslant Ce^{-\gamma t}\left(|y_0 - y_1^*(0,\cdot)|_{H^2(]0,x_{s0}[;\mathbb{R})} + |y_0 - y_2^*(0,\cdot)|_{H^2(]x_{s0},L[;\mathbb{R})} + |x_{s0} - x_0|\right), \ \forall t \in [0, +\infty[. \end{aligned} \tag{1.7.27}$$

On montre alors le résultat de stabilité exponentielle suivant :

Théorème 1.7.5. *Soit $\gamma > 0$. Si les conditions suivantes sont vérifiées :*

$$b_1 \in \left]\gamma e^{-\gamma x_0}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}}\right[, \ b_2 \in \left]\gamma e^{-\gamma(L-x_0)}, \frac{\gamma e^{-\gamma(L-x_0)}}{1 - e^{-\gamma(L-x_0)}}\right[, \tag{1.7.28a}$$

$$k_1^2 < e^{-\gamma x_0}\left(1 - \frac{b_1}{\gamma}\left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{1.7.28b}$$

$$k_2^2 < e^{-\gamma(L-x_0)}\left(1 - \frac{b_2}{\gamma}\left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{1.7.28c}$$

alors l'état stationnaire (y^*, x_0) du système (1.7.20), (1.7.21), $y(0, \cdot) = y_0$, $x_s(0) = x_{s0}$, (1.7.23) *est exponentiellement stable pour la norme H^2 avec un taux de décroissance $\gamma/4$.*

Remarque 1.7.2. *On montre aussi que pour tout $\gamma > 0$ il existe bien des paramètres b_1, b_2, k_1 et k_2 vérifiant* (1.7.28)*. Ce résultat est donc en fait un résultat de stabilisation dite "rapide", puisque le taux de décroissance γ peut être choisi aussi grand qu'on le souhaite.*

Pour le montrer, la difficulté est double :
- D'une part les solutions ne sont pas régulières.
- D'autre part on a bien un contrôle direct sur l'amplitude de la solution de part et d'autre du choc via les conditions aux bords de (1.7.20), mais on n'a pas de contrôle direct sur la position du choc dont la dynamique est donnée par les relations de Rankine-Hugoniot (1.7.21).

On vient à bout de la première difficulté en utilisant le Théorème 1.7.4 qui dit que si la solution commence avec un seul choc elle reste avec un seul choc. On peut donc diviser le problème en deux : avant et après le choc. Cela donne deux solutions régulières z_1 et z_2 définies sur un domaine variable mais on peut redresser le domaine avec un changement de coordonnées pour se ramener à un domaine fixe. Cela revient à définir la variable $\mathbf{z}$ suivante (voir (6.3.1)–(6.3.3) pour plus de détails).

$$\mathbf{z}(t,x) = \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} = \begin{pmatrix} y(t, x\frac{x_s(t)}{x_0}) - 1 \\ y(t, L + x\frac{x_s(t)-L}{x_0}) + 1 \end{pmatrix}, \ \ x \in]0, x_0[, \tag{1.7.29}$$

où z_1 représente l'état du système avant le choc et z_2 représente l'état du système après le choc.

La deuxième difficulté signifie en substance qu'on ne peut s'en sortir qu'en utilisant le couplage entre l'état du système et la position du choc donnée par (1.7.21). Pour cela on définit une nouvelle forme de fonction de Lyapunov dont la forme générale est donnée dans le Chapitre 6 à (6.4.3)–(6.4.15) mais qui s'écrit pour une solution de (1.7.20)–(1.7.21), avec $\mathbf{z}$ défini juste avant par (1.7.29) :

$$V(\mathbf{z}, x_s) = V_1(\mathbf{z}) + V_2(\mathbf{z}, x_s) + V_3(\mathbf{z}, x_s) + V_4(\mathbf{z}, x_s) + V_5(\mathbf{z}, x_s) + V_6(\mathbf{z}, x_s) \tag{1.7.30}$$

avec

$$V_1(\mathbf{z}) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_1^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_2^2 dx, \tag{1.7.31}$$

$$V_2(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_{1t}^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_{2t}^2 dx, \tag{1.7.32}$$

$$V_3(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt}^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt}^2 dx, \tag{1.7.33}$$

$$V_4(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_1 (x_s - x_0)\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2 (x_s - x_0)\, dx + \kappa (x_s - x_0)^2, \tag{1.7.34}$$

$$V_5(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1t} \dot{x}_s\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2t} \dot{x}_s\, dx + \kappa (\dot{x}_s)^2, \tag{1.7.35}$$

$$V_6(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt} \ddot{x}_s\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt} \ddot{x}_s\, dx + \kappa (\ddot{x}_s)^2. \tag{1.7.36}$$

où, μ, p_1, p_2, $\bar{p}_1$, $\bar{p}_2$ sont des constantes strictement positives, $\kappa > 1$ et

$$\eta_1 = 1, \quad \eta_2 = \frac{x_0}{L - x_0}. \tag{1.7.37}$$

On va ensuite un peu plus loin et on montre que le Théorème 1.7.5 peut s'étendre à toute loi de conservation scalaire de la forme

$$\partial_t y + \partial_x (f(y)) = 0 \tag{1.7.38}$$

avec f de classe C^3, convexe, telle que [19] $f(1) = f(-1)$ et vérifiant :

$$\min(f'(1), |f'(-1)|) \geq 1. \tag{1.7.39}$$

Remarque 1.7.3. *On peut faire une remarque intéressante sur ce feedback : si on prend $k_1 = k_2 = b_1 = b_2 = 0$, alors, d'après* (1.7.23)*, $u_0(t) \equiv 1$ et $u_L(t) \equiv -1$ et le système n'est pas exponentiellement stable car on ne peut pas stabiliser la position du choc. Regardons maintenant les conditions* (1.7.28)*. S'il semble logique que plus γ est grand, plus les gains k_1 et k_2 doivent être faibles, il peut sembler par contre contre-intuitif que b_1 et b_2 doivent tendre vers 0 quand γ tend vers $+\infty$, puisque si on pose $b_1 = 0$ et $b_2 = 0$, on ne peut pas stabiliser la position du choc. En d'autres termes, pour tout $\gamma > 0$ le feedback qu'on définit fonctionne, alors que le feedback limite qu'on obtiendrait en faisant tendre $\gamma \to +\infty$ ne peut même pas assurer la stabilité asymptotique du système. L'explication derrière ce paradoxe apparent est que si γ tend vers l'infini, la fonction de Lyapunov utilisée n'est plus équivalente à la norme de la solution et ne peut plus garantir sa décroissance exponentielle. Plus de détails sont donnés dans le Chapitre 6 (en particulier la Remarque 6.2.6).*

Il est tout de même utile de mentionner qu'un préprint [165], sorti sur Hal peu après la soumission de nos travaux, vise aussi à stabiliser un état stationnaire avec un choc à l'aide de contrôles aux bords pour la norme L^1 dans l'espace des solutions de classe BV. La norme est donc plus générale, la stabilisation par contre est simplement exponentielle avec un taux de décroissance qui ne peut pas être aussi grand que voulu contrairement à notre étude. La méthode quant à elle est très intéressante et totalement différente.

19. On peut en fait faire cette hypothèse sans perte de généralité, les deux hypothèses importantes sont la convexité et (1.7.39)

Stabilisation d'un état ressaut hydraulique Dans le Chapitre 7 on montre que cette méthode peut-être étendue aux équations de Saint-Venant et au phénomène du ressaut hydraulique, avec une difficulté supplémentaire néanmoins, comme on va le voir.

On s'intéresse ici aux équations de Saint-Venant homogènes dans le cas particulier où la section est rectangulaire,

$$\begin{aligned} &\partial_t H + \partial_x Q = 0, \\ &\partial_t Q + \partial_x \left(\frac{gH^2}{2} + \frac{Q^2}{H} \right) = 0. \end{aligned} \tag{1.7.40}$$

L'intérêt de considérer un tel cas particulier est d'étudier le phénomène de ressaut hydraulique tout en gardant le cadre le plus simple possible. Néanmoins, au vu de la Section 1.7.5, il est probable que le résultat puisse s'étendre aux équations de Saint-Venant générales. Par ailleurs nous considérons ici la formulation des équations de Saint-Venant avec pour variables la hauteur H et le débit $Q = HV$ et une deuxième équation qui représente un bilan des forces, plutôt que les variables H et V et une deuxième équation qui représente un bilan d'énergie, comme nous le faisions précédemment avec (1.6.36) et (1.7.4). En fait, quand les solutions sont régulières ces deux formulations sont équivalentes. Quand les solutions comportent des discontinuités, par contre, cette équivalence est perdue et cela peut se voir à travers les état stationnaires : la formulation (1.7.40) impose une continuité du débit Q^* et des forces appliqués $(gH^{*2} + Q^{*2}/H^*)$, ce qui est compatible avec une discontinuité en hauteur d'eau pour l'état stationnaire. A contrario une formulation comme (1.7.4) impliquerait une continuité du débit H^*V^* et de l'énergie $(gH^* + V^{*2}/2)$ ce qui impliquerait directement la continuité de H^* et V^* et donc empêcherait tout ressaut (voir Chapitre 7 Section 7.1, paragraphe Physical remarks). C'est en fait assez logique : quand un ressaut hydraulique se produit il y a une perte locale d'énergie, dissipée dans le ressaut, ce qui empêche la conservation de l'énergie.

Puisqu'on cherche à étudier un système comportant un ressaut hydraulique, dont on note la position x_s^*, les états stationnaires (H^*, Q^*) de (1.7.40) doivent être en régime torrentiel avant le ressaut et en régime fluvial après le ressaut. On peut montrer que cela signifie que les états stationnaires sont les fonctions qui vérifient les conditions suivantes :

1. Q^* est une constante strictement positive, $x_s^* \in]0, L[$ et

$$H^* = \begin{cases} H_1^* > 0, & x \in [0, x_s^*[, \\ H_2^* > 0, & x \in]x_s^*, L], \end{cases} \tag{1.7.41}$$

2. Condition de régime torrentiel avant le ressaut :

$$\lambda_1 = \frac{Q^*}{H_1^*} - \sqrt{gH_1^*} > 0, \quad \lambda_2 = \frac{Q^*}{H_1^*} + \sqrt{gH_1^*} > 0, \qquad \text{pour } x \in [0, x_s^*), \tag{1.7.42}$$

   Condition de régime fluvial après le ressaut :

$$-\lambda_3 = \frac{Q^*}{H_2^*} - \sqrt{gH_2^*} < 0, \quad \lambda_4 = \frac{Q^*}{H_2^*} + \sqrt{gH_2^*} > 0, \qquad \text{pour } x \in]x_s^*, L]. \tag{1.7.43}$$

   Ce qui signifie en particulier que $H_1^* < H_2^*$.

3. Condition à l'interface

$$\frac{H_2^*}{H_1^*} = \frac{-1 + \sqrt{1 + 8\frac{(Q^*)^2}{g(H_1^*)^3}}}{2}. \tag{1.7.44}$$

La dernière condition est appelée équation de Bélanger [42] et se déduit de (1.7.40) et des relations de Rankine-Hugoniot.

Par rapport à l'équation de Burgers, une difficulté supplémentaire apparaît : dans le cas de l'équation de Burgers il n'y a qu'une vitesse de propagation, et elle passe d'un signe positif avant le choc à un signe négatif

après le choc. Cela permet d'avoir une information entrante en $x = 0$ et en $x = L$ et d'avoir ainsi un contrôle sur chacune des deux moitiés de la solution (cf. Fig 1.3).
Dans le cas des équations de Saint-Venant par contre, on a deux composantes et deux vitesses de propagation. Les deux sont discontinues mais une seule change de signe. Ainsi, en divisant le problème en deux entre avant le choc et après le choc, on a en tout quatre quantités (deux avant, deux après), et on aimerait avoir le nombre correspondant de contrôles aux bords. Avant le choc, les deux vitesses de propagation sont positives, on peut donc bien avoir deux contrôles aux bords. Après le choc, par contre, une des vitesses de propagation a changé de signe et permet d'avoir une information entrante en $x = L$ et donc un contrôle, mais l'autre n'a pas changé de signe, l'information correspondante en $x = L$ est toujours sortante et on ne peut donc pas avoir de contrôle dessus (cf. Fig 1.3). On a donc quatre quantités avec seulement trois contrôles aux bords et on doit donc exploiter le couplage non-seulement pour stabiliser la position du choc mais aussi pour stabiliser cette quatrième composante.

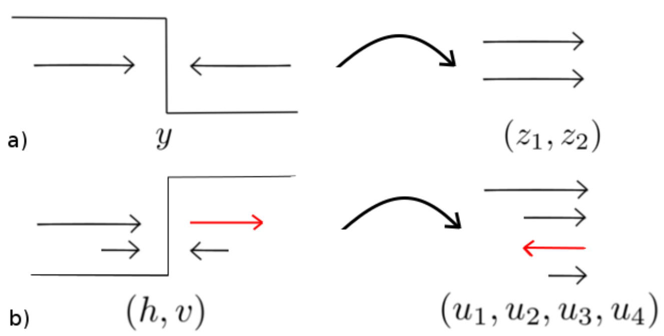

FIGURE 1.3 – Changement de variables pour ramener le problème à un système régulier. **a)** Equation de Burgers. **b)** Equations de Saint-Venant, en rouge la composante correspondant à la vitesse de propagation qui ne change pas de signe et sur laquelle on ne peut pas appliquer de contrôle aux bords.

Cette difficulté nécessite d'être un peu plus précis dans les estimées que pour une équation scalaire, mais la méthode fonctionne toujours en choisissant des contrôles similaires au cas précédent :

$$\begin{pmatrix} H(t,0) - H_1^* \\ Q(t,0) - Q^* \\ Q(t,L) - Q^* \end{pmatrix} = G \begin{pmatrix} Q(t,x_s^-) - Q^* \\ Q(t,x_s^+) - Q^* \\ H(t,x_s^-) - H_1^* \\ x_s - x_s^* \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ G_4(H(t,L) - H_2^*) \end{pmatrix} \tag{1.7.45}$$

où $G = (G_1, G_2, G_3)^\intercal : \mathbb{R}^4 \to \mathbb{R}^3$ et $G_4 : \mathbb{R} \to \mathbb{R}$ sont de classe C^2 et vérifient

$$G(\mathbf{0}) = \mathbf{0}, \quad G_4(0) = 0, \quad G_4'(0) = -\lambda_4. \tag{1.7.46}$$

Et, en introduisant les matrices D, $\tilde{D}$, K et les b_i définis par

$$D(x,\gamma) = \mathrm{diag}\left(\frac{s_i(1-s_i\frac{\lambda_i}{\lambda_4})}{b_i}e^{\frac{\gamma}{x_i\lambda_i}(x_s^*-x)}, i\in\{1,2,3\}\right),$$

$$\widetilde{D}(\gamma) = \mathrm{diag}\left(\left(\sum_{j=1}^{3} e^{\frac{\gamma x_s^*}{x_i\lambda_i}-\frac{\gamma x_s^*}{x_j\lambda_j}}\right)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)^2, i\in\{1,2,3\}\right),$$

$$K = \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} & -\frac{\lambda_1}{\lambda_2-\lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1-\lambda_2} & -\frac{\lambda_2}{\lambda_1-\lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3+\lambda_4} \end{pmatrix} G'(\mathbf{0}) \begin{pmatrix} 1 & 1 & 0 \\ \frac{\lambda_1}{\lambda_4} & \frac{\lambda_2}{\lambda_4} & 1+\frac{\lambda_3}{\lambda_4} \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.7.47}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} & -\frac{\lambda_1}{\lambda_2-\lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1-\lambda_2} & -\frac{\lambda_2}{\lambda_1-\lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3+\lambda_4} \end{pmatrix} G'(\mathbf{0}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

et en notant $s_1 = s_2 = 1$, $s_3 = -1$, $x_1 = x_2 = 1$, $x_3 = x_s^*/(L-x_s^*)$ et $x_4 = x_s^*/(x_s^*-L)$, on obtient le résultat suivant :

Théorème 1.7.6. *Pour tout état stationnaire* $((H^*,Q^*)^\intercal, x_s^*)$ *du système* (1.7.40) *vérifiant* (1.7.41)-(1.7.44) *et les conditions aux bords* (1.7.45), *pour tout* $\gamma > 0$, *si pour* $i = 1, 2, 3$

$$\begin{aligned} b_i &\in \left]\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^*-H_2^*)}{3s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)(1-e^{-\frac{\gamma}{x_i\lambda_i}x_s^*})}, \frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^*-H_2^*)}{3s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)}\right[, \quad si \quad s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right) < 0, \\ b_i &\in \left]\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^*-H_2^*)}{3s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)}, \frac{-\gamma x_i e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}(H_1^*-H_2^*)}{3s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)(1-e^{-\frac{\gamma}{x_i\lambda_i}x_s^*})}\right[, \quad si \quad s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right) > 0, \end{aligned} \tag{1.7.48}$$

et si la matrice

$$D(x_s^*,\gamma) - K^\intercal D(0,\gamma)K - \left(\sum_{k=1}^{3}\frac{2}{\gamma^2(H_1^*-H_2^*)^2}b_k s_k(1-s_k\frac{\lambda_k}{\lambda_4})(e^{\frac{\gamma x_s^*}{x_k\lambda_k}}-1)\right)\widetilde{D}(\gamma) \tag{1.7.49}$$

est définie positive, avec $(b_1,b_2,b_3)^\intercal$, D, $\widetilde{D}$ *et* K *définis par* (1.7.47), *alors les états stationnaire* $((H^*,Q^*)^\intercal, x_s^*)$ *sont exponentiellement stable pour la norme* H^2 *avec taux de décroissance* $\gamma/4$.

Remarque 1.7.4.
- *La fonction de contrôle G est en fait très similaire aux contrôles utilisés pour l'équation de Burgers : son linéarisé s'écrit de façon analogue à $u_0(t)$ et $u_L(t)$ définis par* (1.7.23), *où K joue le rôle des k_i et où les b_i jouent le même rôle qu'avant. On a gardé ici G non-linéaire pour montrer, comme on pouvait s'y attendre, que la linéarité du contrôle ne joue pas de rôle dans cette méthode.*
- *Ici aussi, pour tout $\gamma > 0$ on peut trouver G telle que K et $(b_1,b_2,b_3)^\intercal$ définis par* (1.7.47) *vérifient* (1.7.48)-(1.7.49). *Ce résultat est donc à nouveau un résultat de stabilisation rapide.*

1.8 Partie 3 : Contrôles PI

Backstepping mis à part, la plupart des études de stabilisation utilisent des feedbacks de sortie, donc de la forme (1.4.13). Puisque le but est d'appliquer ces contrôles à des systèmes pratiques, il est naturel de se demander ce qui se passe quand on fait de petites erreurs, par exemple une petite erreur constante sur le contrôle qu'on applique. Que se passe t-il si, dans le Théorème 1.7.1, au lieu d'appliquer $\mathcal{B}(H(t,L))$ on applique $\mathcal{B}(H(t,L))+\varepsilon$? La réponse est simple : plus rien ne fonctionne. [20] Ces lois de contrôles ne sont donc pas robustes aux erreurs dites d'"off-set"[6, Chapitre 11.3]. Heureusement il existe une solution simple connue depuis longtemps à ce problème : rajouter un terme intégral qui utilise aussi l'information des erreurs précédentes et absorbe cette erreur de mesure[157]. Le contrôle devient alors :

$$V(t,L) = k_p(H(t,L) - H^*(L)) - k_I Z \tag{1.8.1}$$

où Z est le terme intégral et vérifie $\dot{Z} = H(t,L)$. Cette solution est connue depuis longtemps, sa première apparition remonte aux travaux des frères Périer vers la fin du XVIIIème siècle qui utilisaient un contrôle intégral pour stabiliser leur "pompe à feu" [72, Pages 50-51 et figure 231, Plate 26], [25, Chapitre 2]. Puis le régulateur de Jenkins, qui fonctionne sur le même principe, a ensuite été étudié par Maxwell qui mentionne ce type de régulateur dans son article célèbre :"On governors" présenté à la Royal Society of London [21] en 1868. Evidemment ces régulateurs n'étaient pas encore appelés "contrôles PI" et leur théorie n'était pas encore aussi mathématisée, mais ils fonctionnaient sur le même principe. C'est Minorsky au début du XXème siècle qui commença à théoriser cette classe de contrôles pour des systèmes de dimension finie. A l'époque Minorsky les étudiait pour les appliquer aux pilotes automatiques des navires de la US Navy. Depuis, motivées par le très grand nombre d'applications [74], de nombreuses études ont été publiées dans le cas de systèmes de dimension finie [4–6]. Puis, en analysant les valeurs propres et en utilisant le Spectral Mapping Theorem décrit en Section 1.4.3, d'autres travaux ont étudié des systèmes linéaires de dimension infinie [22], en particulier [21, 84, 130, 169, 194, 199, 200].
Les systèmes non-linéaires de dimension infinie, eux, et en particulier les systèmes non-linéaires hyperboliques, sont restés très peu étudiés puisque très difficiles à manipuler avec ces contrôles. La difficulté supplémentaire rajoutée par la dimension infinie n'est plus compensée par la possible utilisation du Spectral Mapping Theorem et assez peu de techniques sont connues. Le but de cette partie est de contribuer à leur étude.

1.8.1 Equation de transport

A notre connaissance un seul résultat général existe pour un système non-linéaire de dimension infinie : c'est le cas de l'équation de transport avec un contrôle intégral :

$$\partial_t z + \lambda(z)\partial_x z = 0, \tag{1.8.2}$$
$$z(0,t) = -k_I I(t), \tag{1.8.3}$$
$$\dot{I} = z(L,t), \tag{1.8.4}$$

où $\lambda(0) > 0$ et k_I est le coefficient de contrôle à choisir. L'équation de transport (1.8.2) est le système hyperbolique non-linéaire le plus simple mais est pourtant déjà assez riche. Elle est étudiée soit comme modèle-jouet, comme étape préparatoire à l'étude de systèmes plus compliqués, soit comme un modèle basique de transport pour modéliser par exemple la dynamique d'une population [166], le trafic routier, ou le transport du gaz en première approximation. Elle couvre aussi par exemple l'équation de Burgers, dont on a déjà parlé à la Section 1.7.1, quand elle est étudiée autour d'un état stationnaire constant non nul.

Pour ce système, le système linéarisé associé est facile à étudier en utilisant le Spectral Mapping Theorem, et les valeurs propres ϱ vérifient l'équation suivante :

$$k_I + \varrho e^{\frac{\varrho L}{\lambda_0}} = 0. \tag{1.8.5}$$

20. Notons que dans ce cas il est parfois quand même possible d'atteindre un état stationnaire proche : c'est le cas par exemple des systèmes stabilisés par un régulateur à boules. Plus d'informations sur ce régulateur peuvent être trouvées dans [157, Section X]. Suivant les applications, les contrôles proportionnels et les feedbacks de sortie ont donc quand même un intérêt en eux-mêmes, en plus d'être une étape préparatoire naturelle aux contrôles PI.

21. Son papier fut en fait ignoré pendant plusieurs années, parce que visiblement jugé en partie incompréhensible [158]

22. Le Spectral Mapping Theorem tel que défini dans la Section 1.4.3 n'est valable que pour des feedbacks de sortie. Néanmoins d'autres versions existent avec un contrôle PI en considérant l'intégrateur I comme une variable supplémentaire.

où $\lambda_0 := \lambda(0)$. On peut montrer les propriétés suivantes :

— Si ϱ est solution de (1.8.5), alors $\bar{\varrho}$ est aussi solution.

— Si $k_I \leq 0$, il existe toujours une solution réelle ϱ de partie réelle positive ou nulle à (1.8.5).

— Si $k_I \in \left]0, \frac{2k\pi\lambda_0}{L} + \frac{\pi\lambda_0}{2L}\right[$, alors il existe exactement $2k$ solutions $\varrho \in \mathbb{C}$ à (1.8.5) de parties réelles positives qui sont conjuguées [192].

On obtient alors le résultat suivant : le système linéarisé est exponentiellement stable si et seulement si

$$k_I \in \left]0, \frac{\pi\lambda_0}{2L}\right[. \tag{1.8.6}$$

Le système non-linéaire, lui, a été étudié par Trinh, Andrieu et Xu [194] qui ont montré le résultat suivant à l'aide d'une approche Lyapunov :

Théorème 1.8.1 ([194]). *Il existe $k_I > 0$ tel que le système* (1.8.2)–(1.8.4) *est exponentiellement stable pour la norme H^2*

En réalité leur preuve est un peu plus forte et montre que le système est exponentiellement stable pour $k_I \in]0, \lambda(0)\Pi(2-\sqrt{2})/2L[$, où $\Pi(x) = \sqrt{x(2-x)}e^{-x/2}$ et donc $\Pi(2-\sqrt{2})/2 \approx 0.34$. Ce résultat est une condition suffisante et donc sans doute trop stricte. Naturellement, on a envie de savoir quelle serait la condition optimale sur k_I, c'est à dire sa limite maximale, et on aimerait savoir si c'est $\pi\lambda(0)/2L$ comme pour le système linéaire, ou non. C'est ce problème que nous étudions dans le Chapitre 8.

Notre approche commence par trouver une fonction de Lyapunov un peu différente de celle utilisée par Trinh, Andrieu et Xu qui exploite le couplage entre l'intégrateur et l'état du système et qui nous permet d'obtenir la proposition suivante :

Proposition 1.8.2. *Le système non-linéaire* (1.8.2)–(1.8.4) *est exponentiellement stable pour la norme H^2 si*

$$k_I \in \left]0, \frac{\lambda(0)}{L}\right[. \tag{1.8.7}$$

Malheureusement cette condition est encore trop stricte, et tous nos essais pour trouver une fonction de Lyapunov relativement simple permettant d'aller au-delà ont échoué. Ce problème s'inscrit dans un problématique plus générale déjà un peu mentionnée : les fonctions de Lyapunov sont de très bons outils pour étudier la stabilité des systèmes mais, à moins de trouver une fonction parfaite, elles conduisent souvent à des conditions trop strictes quand les systèmes deviennent un peu compliqués. Nous avons donc élaboré une nouvelle méthode pour atteindre les conditions optimales : la méthode d'extraction.

La méthode d'extraction L'idée est la suivante : si on regarde le système linéarisé, il y a deux valeurs propres qui limitent la stabilité qu'on note ϱ_1 et $\bar{\varrho}_1$ et qui correspondent [23] aux valeurs propres dont la partie réelle est positive quand $\pi\lambda_0/2L < k_I < 2\pi\lambda_0/L + \pi\lambda_0/2L$. Si ces valeurs propres n'existaient pas alors, d'après les propriétés de (1.8.5) qu'on a énoncées, le système serait aussi stable pour des k_I plus grands et en fait pour $0 < k_I < 5\pi\lambda_0/2L$. Donc si on arrivait à extraire de la solution la partie correspondant à ces deux valeurs propres à l'aide d'un projecteur adéquat, on se retrouverait avec deux parties :

— Une partie (φ_1, X_1) correspondant aux valeurs propres ϱ_1 et $\bar{\varrho}_1$, qui appartient à un espace de dimension 2 et qui est le facteur limitant de stabilité (n'est plus exponentiellement stable à partir de $k_I = \pi\lambda_0/2L$ car alors ϱ_1 a une partie réelle positive ou nulle).

— Une partie (φ_2, X_2) qui est le "reste", qui appartient à un espace de dimension infinie mais qui est stable jusqu'à des valeurs de k_I plus grandes (au moins $5\pi\lambda_0/2L$).

Notre but est alors de réussir à prouver la stabilité à l'aide de fonctions de Lyapunov pour pouvoir transposer le raisonnement au non-linéaire. Etudions les deux parties séparément : puisque (φ_1, X_1) appartient à un espace de dimension finie relativement simple, trouver une fonction de Lyapunov optimale qui prouve la

23. On voit que cette méthode part déjà du principe que les valeurs propres qui limitent la stabilité sont continues, ce qui motive la Conjecture 1

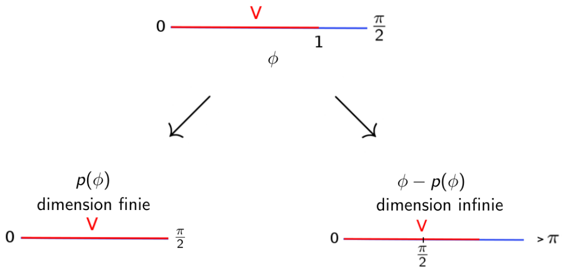

FIGURE 1.4 – Illustration de la stratégie de la méthode d'extraction. En rouge le domaine de stabilité qu'on arrive à obtenir directement avec une fonction de Lyapunov et en bleu le domaine de stabilité total. Les valeurs sont normalisées de façon à ce que $\lambda_0/L = 1$.

stabilité de (φ_1, X_1) jusqu'à $k_I = \pi\lambda_0/2L$ est relativement aisé. La deuxième partie (φ_2, X_2) appartient à un espace de dimension infinie, mais est stable pour des k_I qui peuvent aller bien plus loin que $\pi\lambda_0/2L$, donc si on utilise notre fonction de Lyapunov pour la deuxième partie, même si elle est trop stricte dans l'absolu, il y a un espoir pour que la limite sur k_I soit au-delà de $\pi\lambda_0/2L$, et ainsi la somme des deux fonctions aurait pour limite de stabilité totale $\pi\lambda_0/2L$. Tout cela est fait jusqu'ici pour le système linéaire, mais puisqu'on utilise des fonctions de Lyapunov qui ont une forme basique on peut espérer que cela fonctionne avec le non-linéaire. C'est en fait le cas, avec une différence principale : pour le système linéarisé les équations sur (φ_1, X_1) et (φ_2, X_2) ne sont pas couplées et donc (φ_1, X_1) et (φ_2, X_2) peuvent être étudiées séparément, tandis que pour le système non-linéaire φ_1 dépend faiblement de (φ_2, X_2), et inversement, ce qui impose quand même de les étudier ensemble.

Les principales difficultés sont les suivantes :
- Trouver le projecteur, et prouver que c'est bien un projecteur. Trouver la forme potentielle du projecteur n'est en fait pas très compliqué, mais, à notre surprise, réussir à montrer que c'est bien un projecteur s'avère plus ardu. Pour s'en sortir il faut utiliser précisément la relation (1.8.5).
- Comme on l'a vu plus haut avec la Proposition 1.8.2, utiliser directement la fonction de Lyapunov sur la solution totale c'est à dire $(\varphi_1, X_1) + (\varphi_2, X_2)$ n'est pas satisfaisant. Dans cette méthode on utilise notre fonction de Lyapunov sur (φ_2, X_2) à la place et on veut utiliser l'information supplémentaire qu'on a sur (φ_2, X_2) pour améliorer la condition qu'on obtient. Le problème est que cette information s'avère assez difficile à exploiter. Pour y arriver on utilise le projecteur pour trouver une fonction θ telle que $\int_0^L \varphi_2(t,x).\theta(x)dx = 0$, on peut donc ensuite rajouter $\kappa \int_0^L \varphi_2(t,x).\theta(x)dx = 0$ à l'expression de la fonction de Lyapunov dérivée par rapport au temps le long des trajectoires et optimiser κ (voir (8.5.14)–(8.5.21) pour plus de détails).

En résumé, voici schématiquement les différentes étapes :
1. Trouver une bonne fonction de Lyapunov candidate V.
2. Isoler les valeurs propres qui posent problème, éventuellement via une relation implicite.
3. Trouver un projecteur sur l'espace engendré par ces valeurs propres.
4. Trouver une fonction θ telle que $\int_0^L \varphi_2(t,x).\theta(x)dx = 0$ à l'aide du projecteur.
5. Trouver une fonction de Lyapunov optimale pour la partie de dimension finie (φ_1, X_1).
6. Utiliser θ pour améliorer la dérivée par rapport au temps de V prise le long des trajectoires (φ_2, X_2) pour atteindre la limite donnée par les valeurs propres qui posent problème.

Si la condition qu'on obtient sur (φ_2, X_2) n'est pas satisfaisante, on peut itérer l'extraction pour enlever autant de valeurs propres que nécessaire. Evidemment la dernière étape devient plus précise au fur et à mesure qu'on extrait des valeurs propres mais d'autant plus compliquée à réaliser car on a plus d'information à exploiter et les espaces propres ne sont pas nécessairement orthogonaux entre eux.

En utilisant cette méthode, on montre alors le résultat final :

Théorème 1.8.3. *Le système non-linéaire* (1.8.2)–(1.8.4) *est exponentiellement stable pour la norme H^2 si*

$$k_I \in \left]0, \frac{\pi\lambda(0)}{2L}\right[. \tag{1.8.8}$$

On retrouve donc bien la même condition que pour le système linéarisé. L'optimalité est illustrée par la proposition suivante :

Proposition 1.8.4. *Il existe $k_1 > \pi\lambda(0)/2L$, tel que pour tout $\pi\lambda(0)/2L < k_I < k_1$ le système non-linéaire* (1.8.2)–(1.8.4) *est instable pour la norme H^2.*

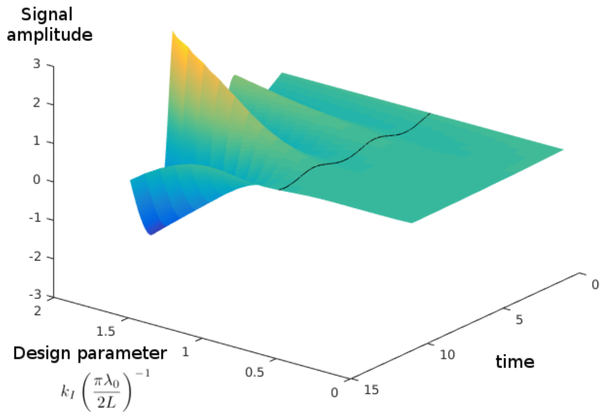

FIGURE 1.5 – Exemple de simulation numérique de $z(t,0)$ en fonction du temps t variant entre 0 et 10 pour différentes valeurs de k_I entre $0.1k_{I,c}$ et $2k_{I,c}$, où $k_{I,c} = \pi\lambda_0/2L$ est la valeur critique du Théorème 1.8.3 et de la Proposition 1.8.4. La ligne noire représente la trajectoire pour $k_I = k_{I,c}$. Sur la gauche k_I est grand et le système (non-linéaire) est instable. Sur la droite k_I est plus faible et le système est stable. Les paramètres du système sont choisis tels que $\lambda(x) = 1 + x$, $\lambda_0 = L = 1$, et $\varphi_0(x) = 0.1$ sur $[0, L/2]$ et $\varphi_0(L) = 0$ de telle sorte que φ_0 soit compatible pour tout $k_I \in [0.1k_{I,c}, 2k_{I,c}]$.

Remarque 1.8.1. *On remarque que cette méthode vise à montrer que le système non-linéaire est exponentiellement stable avec la même limite que le système linéarisé. Or on a vu en introduction que c'est faux pour certains systèmes. Il serait donc intéressant de savoir quand est-ce que c'est le cas et quand ça ne l'est pas. C'est l'objet de la conjecture qu'on donnera plus loin en Section 1.9.*

En fait ce résultat est un peu plus général qu'il n'y parait : on peut ramener la stabilisation de tout état stationnaire régulier d'une équation quasilinéaire hyperbolique à ce problème. Plus précisément, si on considère le système

$$\begin{aligned} &\partial_t y + \lambda(y,x)\partial_x y + g(y,x) = 0, \\ &y(t,0) = y^*(0) - k_I Z, \\ &\dot{Z} = (y(t,L) - y^*(L)), \end{aligned} \tag{1.8.9}$$

où y^* est l'état stationnaire régulier qu'on cherche à stabiliser avec $\lambda(y^*,\cdot) > 0$, alors on montre le résultat suivant :

Theorem 1.8.5. *L'état stationnaire y^* du système* (1.8.9) *est exponentiellement stable pour la norme H^2 si*

$$k_I \in \left]0, \frac{\pi \exp\left(\int_0^L \frac{\partial_y g(y^*,s)+y_x^*\partial_y\lambda(y^*,s)}{\lambda(y^*,s)}ds\right)}{2l^{-1}(L)}\right[, \tag{1.8.10}$$

où l est la primitive de $\lambda(y^*,\cdot)$ s'annulant en 0.

Et la proposition,

Proposition 1.8.6. *Il existe $k_1 > k_c := \pi \exp\left(\int_0^L (\partial_y g(y^*,s) + y_x^* \partial_y \lambda(y^*,s))/\lambda(y^*,s)ds\right)/2l^{-1}(L)$, tel que pour tout $k_I \in]k_c, k_1[$ le système non-linéaire* (1.8.9) *est instable pour la norme H^2.*

1.8.2 Equations de Saint-Venant

Dans la pratique, les voies navigables, qui sont modélisées par les équations de Saint-Venant, sont régulées par des contrôleurs PI [148], [13, Chapitre 8]. Pour ce faire les lois de contrôle sont obtenues soit en linéarisant le système [22, 146, 147, 200, 201], [145, Chapitre 1] et en utilisant le Spectral Mapping Theorem décrit en Section 1.4.3, soit en approximant le modèle par un système de dimension finie fixée [148]. Ces deux approches ont un désavantage majeur : rien ne garantit *a priori* que le "vrai" système non-linéaire sera stable si ces systèmes approchés le sont. Le but de cette sous-partie est d'y remédier en traitant le cas non-linéaire.

On considère pour simplifier les équations de Saint-Venant avec section rectangulaire et modèle de frottements défini,

$$\begin{aligned} &\partial_t H + \partial_x (HV) = 0, \\ &\partial_t V + \partial_x \left(\frac{V^2}{2} + gH\right) + \left(\frac{kV^2}{H} - C(x)\right) = 0, \end{aligned} \tag{1.8.11}$$

où k est à nouveau le coefficient de frottement et C l'influence de la pente. Comme dans la Section 1.7, nos résultats pourront se généraliser aux équations de Saint-Venant générales ou aux systèmes densité-vélocité. Pour un état stationnaire (H^*, V^*), le contrôleur PI est donné par

$$\begin{aligned} H(t,L)V(t,L) - H(L)^*V(L)^* =& k_p(H(t,L) - H^*(L)) - k_I Z, \\ \dot{Z} =& H(t,L) - H^*(L), \end{aligned} \tag{1.8.12}$$

tandis que le débit amont est une constante Q_0 inconnue,

$$H(0)V(0) = Q_0. \tag{1.8.13}$$

On peut remarquer trois choses par rapport aux contrôles utilisés précédemment :
— D'une part on n'utilise qu'un seul contrôle en aval, la condition en amont est imposée.
— D'autre part le contrôle PI donné par (1.8.12) agit sur le débit et non plus directement sur la vitesse. Cela ne change pas l'analyse, mais c'est plus logique d'un point de vue pratique : la plupart du temps les installations hydrauliques contrôlent en réalité le débit.
— Enfin, puisque le contrôle PI est défini à une constante près grâce à l'intégrateur Z, il n'est pas nécessaire de connaître $V^*(L) = Q_0/H^*(L)$, il suffit de connaître la valeur de $H^*(L)$ qu'on veut atteindre. On peut alors ré-écrire le contrôle sous la forme, plus simple,

$$\begin{aligned} H(t,L)V(t,L) =& k_p(H(t,L) - H^*(L)) - k_I Z, \\ \dot{Z} =& H(t,L) - H^*(L). \end{aligned} \tag{1.8.14}$$

Pour ce système, de nombreuses études linéaires reposant sur le Spectral Mapping Theorem existent. Dans [200, 201], par exemple, les auteurs donnent des conditions suffisantes pour assurer la stabilité du système inhomogène linéarisé, tandis que dans [13, Sections 2.2.4.1, 3.4.4], on peut trouver une condition nécessaire et suffisante de stabilité pour les équations homogènes linéarisées. Pour le système non-linéaire, quelques études de stabilité ont été réalisées, dans [13, Section 2.2.4.2] les auteurs trouvent une condition nécessaire et suffisante pour le système homogène et une condition suffisante pour le système inhomogène dans [13, Sections 5.4.4,5.5] mais dans le cas particulier où les états stationnaires sont constants. Le résultat le plus abouti, à ma connaissance, est le suivant, tirée de [17], qui traite du cas où il n'y a pas de pente :

Théorème 1.8.7 ([17])**.** *Si $C \equiv 0$ alors soit un état stationnaire $(H^*, V^*) \in H^2([0, L]; \mathbb{R}^2)$ de* (1.8.11)–(1.8.13)*, si les conditions suivantes sont vérifiées*

$$k_p > 1, \quad k_I > 0, \tag{1.8.15}$$

alors le système est exponentiellement stable pour la norme H^2.

Malheureusement, comme mentionné dans la deuxième partie, le cas sans pente correspond à des états stationnaires particuliers qui n'existent que sur des distances généralement faibles et ne sont pas les cas les plus fréquents dans la pratique. Notre but est donc d'enlever cette hypothèse sur la pente, comme dans la deuxième partie. En particulier on montre :

Théorème 1.8.8. *Soit un état stationnaire $(H^*, V^*) \in H^2([0, L]; \mathbb{R}^2)$ de* (1.8.11)–(1.8.13)*, si les conditions suivantes sont vérifiées*

$$\begin{aligned} &k_p > 0 \ \text{ et } k_I > 0, \\ \text{ou } \ &k_p < -\frac{gH^*(L) - V^{*2}(L)}{V^*(L)} \ \text{ et } k_I < 0, \end{aligned} \tag{1.8.16}$$

alors le système est exponentiellement stable pour la norme H^2.

Notons ici que les conditions sur k_I ne dépendent pas des paramètres du système. De façon plus étonnante, si on se restreint à la première condition elles ne dépendent pas non plus de l'état stationnaire considéré. Elles sont par ailleurs moins restrictives que les conditions obtenues dans le résultat précédent (Théorème 1.8.7). Il serait intéressant de savoir si ces conditions sont optimales ou non. Malheureusement, au contraire de l'équation de transport, même dans le cas linéarisé on ne connaît pas les conditions optimales de stabilité. On pourra néanmoins noter que quand le système est homogène, les conditions (1.8.16) sont nécessaires et suffisantes [22].

En fait le résultat qu'on obtient dans le Chapitre 9 est même un peu plus précis que le Théorème 1.8.8 car on fait cette étude dans le cas, plus général, où ce qu'on cherche à stabiliser ne sont pas des états stationnaires mais des états pouvant varier faiblement avec le temps tout en gardant une hauteur fixée H_c en aval, ce qui correspond souvent à l'objectif industriel. Ce choix obéit à une certaine logique : dans la réalité le débit entrant en amont du cours d'eau n'est pas constant et peut varier beaucoup, mais lentement. Ainsi il n'existe plus d'états stationnaires et les états qu'on cherche à stabiliser sont des états nécessairement variables, mais qui varient lentement. Pour un débit entrant $Q_0 \in L^\infty$ fixé et appartenant à $C^3([0, +\infty[)$, on aimerait donc stabiliser les états cibles (H_1, V_1) suivants :

$$\begin{aligned} &\partial_t H_1 + \partial_x (H_1 V_1) = 0, \\ &\partial_t V_1 + V_1 \partial_x V_1 + g \partial_x H_1 + \left(\frac{k V_1^2}{H_1} - C(x) \right) = 0, \\ &H_1(t, 0) V_1(t, 0) = Q_0(t), \\ &H_1(t, L) = H_c, \end{aligned} \tag{1.8.17}$$

avec pour conditions initiales $H_1(0, \cdot) = H^*$, $V_1(0, \cdot) = V^*$, où (H^*, V^*) serait l'état stationnaire si Q_0 était constant égal à $Q_0(0)$. Pour prouver l'existence de ces états cibles on va définir une famille de fonctions intermédiaire (H_0, V_0). Pour chaque $t^* \in [0, +\infty[$ fixé, on définit $(H^*_{t^*}, V^*_{t^*})$ l'unique solution de

$$\begin{aligned}
&\partial_x(HV) = 0,\\
&V\partial_x V + g\partial_x H + \left(\frac{kV^2}{H} - C(x)\right) = 0,\\
&H(L) = H_c,
\end{aligned} \tag{1.8.18}$$

avec la condition initiale $H^*_{t^*}(0)V^*_{t^*}(0) = Q_0(t^*)$. Pour H_c fixé on peut trouver une borne Q_∞ telle que si $Q_0(t^*) \leq Q_\infty$ la solution existe sur $[0, L]$ et le problème est bien posé (voir [110] ou le Chapitre 4 pour plus de détails). Cette borne n'est pas forcément faible et correspond au fait qu'on doit garder $\sqrt{gH^*_{t^*}} > V^*_{t^*}$ sur $[0, L]$ pour que la solution soit en régime fluvial. La fonction $(H^*_{t^*}, V^*_{t^*})$ correspond alors à ce qui serait l'état stationnaire du problème si Q_0 était constant égal à $Q_0(t^*)$. Si $\|Q_0\|_{L^\infty} \leq Q_\infty$, on peut ainsi définir $(H_0, V_0)\ : (t, x) \to (H^*_t(x), V^*_t(x))$ qui peut être vue comme une famille de fonctions indexées par t ou une fonction de deux variables. On peut montrer que cette fonction de deux variables est aussi régulière en temps que Q_0 ([105][Chapitre 5, Th 3.1], voir Chapitre 9 pour plus de détails). On montre alors la proposition suivante :

Proposition 1.8.9. *Il existe des constantes $C > 0$, $\nu > 0$, $\mu > 0$ et $\delta > 0$ telle que si $\|\partial_t Q_0\|_{C^2([0,+\infty[)} \leq \delta$, alors il existe une unique solution $(H_1, V_1) \in C^0([0, +\infty[, H^2(]0, L[))$ et*

$$\begin{aligned}
&\|H_1(t, \cdot) - H_0(t, \cdot)\|_{H^2(]0,L[)} + \|V_1(t, \cdot) - V_0(t, \cdot)\|_{H^2(]0,L[)}\\
&\leq C\left(\int_0^t |\partial_t Q_0(s) + \partial^2_{tt} Q_0(s) + \partial^3_{ttt} Q_0(s)| e^{\frac{\mu s}{2}} ds\right) e^{-\frac{\mu t}{2}}.
\end{aligned} \tag{1.8.19}$$

On cherche maintenant à stabiliser l'état cible (H_1, V_1). On voit vite qu'on ne peut plus utiliser le contrôle PI donné par (1.8.14), tout simplement parce que $V_1(t, L)$ n'est plus une constante et (H_1, V_1) ne vérifie donc plus les conditions (1.8.14). On définit donc le nouveau contrôle naturel :

$$\begin{aligned}
H(t, L)V(t, L) - H_1 V_1(t, L) &= k_p(H(t, L) - H_c) + k_I Z,\\
\dot{Z} &= H(t, L) - H_c.
\end{aligned} \tag{1.8.20}$$

On montre alors le résultat attendu :

Théorème 1.8.10. *Il existe $\delta > 0$ tel que si $\|\partial_t Q_0\|_{C^3([0,+\infty[)} \leq \delta$, et si*

$$\begin{aligned}
&k_p > 0 \ \text{et}\ k_I > 0,\\
\text{ou}\ \ &k_p < -\frac{gH_1(t, L) - V_1^2(t, L)}{V_1(t, L)}\ \text{et}\ k_I < 0,
\end{aligned} \tag{1.8.21}$$

alors l'état-cible (H_1, V_1) du système (1.8.11), (1.8.13), (1.8.20) *est exponentiellement stable*[24] *pour la norme H^2.*

Au vu de la loi de contrôle (1.8.20), on pourrait se demander ce qui se passe quand on évolue dans l'inconnu et qu'on ne connaît pas le débit entrant Q_0. Dans ce cas impossible de connaître H_1 et V_1 et donc d'imposer la loi de contrôle (1.8.20). Puisqu'on ne connaît pas la cible, il n'est pas possible d'obtenir la stabilité exponentielle ne serait-ce que parce qu'on ne sait pas mettre une condition aux bords compatible avec la cible. Néanmoins on peut tout de même donner une borne sur l'erreur qu'on commet en utilisant la loi de contrôle (1.8.14) avec $H^*(L) = H_c$, et on prouve le résultat d'*input-to-state stability* (ISS) suivant :

Théorème 1.8.11. *Il existe des constantes $\nu > 0$, $\delta > 0$, $\gamma > 0$ et C, telle que si $\|\partial_t Q_0\|_{C^2([0,+\infty[)} \leq \delta$, alors pour tout $T > 0$ et $(H^0, V^0) \in (H^2(]0, L[))^2$ telle que*

$$\|H^0 - H^*\|_{H^2(]0,L[)} + \|V^0 - V^*\|_{H^2(]0,L[)} \leq \nu,$$

24. Lorsque la trajectoire cible qui dépend du temps, la définition de stabilité exponentielle doit être légèrement changée pour s'assurer qu'elle ne dépende pas du temps initial. Pour une définition rigoureuse, voir la Définition 9.2.1.

le système (1.8.11)*,* (1.8.13)*,* (1.8.14) *avec condition initiale* (H^0, V^0) *a une unique solution* $(H, V) \in C^0([0,T], H^2(]0,L[))$ *qui satisfait l'inégalité d'ISS suivante*

$$\begin{aligned} &\|H(t,\cdot) - H_0(t,\cdot)\|_{H^2(]0,L[)} + \|V(t,\cdot) - V_0(t,\cdot)\|_{H^2(]0,L[)} \\ &\leq Ce^{-\gamma t}\left(\|H^0 - H^*, V^0 - V^*\|_{H^2(]0,L[)} + \int_0^t |\partial_t Q_0(s) + \partial_{tt}^2 Q_0(s) + \partial_{ttt}^3 Q_0(s)|e^{\gamma s} ds\right). \end{aligned} \tag{1.8.22}$$

1.9 Question intéressantes concernant les systèmes hyperboliques 1D

Avant de terminer cette introduction, on peut revenir sur une question à laquelle nous sommes encore loin de donner une réponse parfaite :

Si le système linéarisé est exponentiellement stable, est-ce que le système non-linéaire est exponentiellement stable localement ? C'est-à-dire si la condition initiale est arbitrairement proche de l'état stationnaire à stabiliser, est-ce que la connaissance de la stabilité du système linéarisé associé suffit ? La réponse est non, on l'a vu dans la Section 1.4.3. La question plus approfondie consisterait alors à se demander : à quelle condition la stabilité exponentielle du système linéarisé garantit-elle la stabilité du système non-linéaire ? Si nous n'avons pas la réponse, on peut essayer de formuler la conjecture suivante.

Conjecture 1. *Soit un système hyperbolique de la forme* (1.4.2)*, muni de conditions aux bords de la forme* (1.4.13)*. On définit* $\mathcal{L}$ *l'opérateur du système linéarisé associé. Si*

— *le système linéarisé associé est exponentiellement stable,*

— *toutes les valeurs propres de* $\mathcal{L}$ *sont continues avec* $A(\mathbf{0},\cdot)$ *et* $B(\mathbf{0},\cdot)$*,*

alors le système est exponentiellement stable pour la norme C^1 *et* H^2*.*

Dans cette conjecture, la condition supplémentaire déterminante est donc la continuité des valeurs propres du système linéarisé par rapport aux paramètres du système.

1.10 Perspectives

Ces résultats et ces travaux apportent plusieurs perspectives :

— Dans la première partie on utilise des fonctions de Lyapunov basiques pour trouver une condition intérieure qui garantit qu'on puisse stabiliser un système hyperbolique pour la norme C^1 par des feedbacks de sortie. Cette condition est plus restrictive que celle trouvée pour la norme H^2 par la même méthode. Il serait intéressant d'étendre la méthode à la norme $W^{2,p}$ pour savoir comment se comportent les conditions intérieures qu'on obtiendrait. Seraient-elles plus restrictives que celle pour la norme H^2 et moins restrictives que celle pour la norme C^1 ? Sinon, quelles seraient les plus et les moins restrictives en fonction de p ? Est-ce que les liens qu'on obtient dans le cas 2×2 seraient toujours valables avec ces normes ?

— Dans la deuxième partie on montre qu'on peut toujours stabiliser les états stationnaires des systèmes densité-vélocité par des contrôles aux bords quelque soit le terme source. Il serait intéressant de savoir si on peut utiliser la même approche pour stabiliser des systèmes avec plus d'équations, par exemple les équations d'Euler sans l'hypothèse isentropique.

— Ces travaux ont été réalisés en dimension 1 d'espace. Est-il possible de les étendre en dimension supérieure ? C'est probablement une des principales perspectives. Loin d'être une simple généralisation technique, il est à prévoir que de nouveaux comportements très différents peuvent apparaître. Cette question, très intéressante, est d'autant plus pertinente que l'analyse de la stabilité par le Spectral Mapping Theorem n'est plus valable même pour les systèmes linéaires pour une dimension d'espace

$n \geq 2$.

- Quand on stabilise un état stationnaire avec un choc, on considère des solutions discontinues qui n'ont qu'un seul choc. Serait-il possible de considérer un espace de solution plus large comme l'espace des solutions de classe BV entropique ? Tant qu'il n'y a qu'un seul choc qui change une des vitesses de propagation, on peut probablement utiliser la même méthode pour traiter le cas d'une solution qui a un nombre fini de chocs. Le cas des solutions de classe BV est sans doute nettement plus ambitieux.

- La stabilisation exponentielle du ressaut hydraulique est-elle toujours possible lorsqu'il y a un terme source ? Au vu de la partie II il est probable que la méthode fonctionne toujours en changeant la fonction de Lyapunov pour tenir compte des termes sources. Néanmoins il est aussi probable que le résultat ne permette pas une stabilisation rapide. A contrario il est aussi probable que le problème de stabilisation de la position du choc soit simplifié quand les états stationnaires ne sont plus uniformes.

- Dans la troisième partie, on stabilise les équations de Saint-Venant avec un contrôle PI et on obtient des conditions qui sont optimales lorsqu'il n'y a pas de terme source ou quand les états stationnaires qu'on stabilise sont constants, mais on n'en sait pas plus. Sont-elles optimales en général ? Sinon, peut-on les rendre optimales ? La réponse à cette question serait très intéressante car cela clôturerait la série des études de stabilisation des équations de Saint-Venant avec un contrôle PI.

- En pratique, de nombreux systèmes de stabilisation en dimension finie utilisent un régulateur PID qui se déduit du PI en ajoutant un terme dérivé. De façon très étonnante, ce régulateur ne permet pas d'obtenir la stabilité dans le cas d'une équation de dimension infinie [64]. Est-il possible, en appliquant un filtrage sur le terme dérivé, de retrouver la stabilité exponentielle et d'avoir un retour à l'équilibre plus rapide qu'avec un régulateur PI ?

- Enfin, il serait intéressant d'essayer d'utiliser la méthode d'extraction sur des systèmes 2×2 voire $n \times n$. Et, si on a montré que cette méthode fonctionne bien avec un contrôle PI, est-il possible de l'étendre avec succès à d'autre type de contrôle ?

Part I

Stability of quasilinear inhomogeneous systems for the C^1-norm.

Chapter 2

Exponential stability of general 1-D quasilinear systems with source terms for the C^1 norm under boundary conditions

This chapter is taken from the following article (also referred to as [106]):

Amaury Hayat. Exponential stability of general 1-D quasilinear systems with source terms for the C^1 norm under boundary conditions. Preprint, 2017.

Abstract. We address the question of the exponential stability for the C^1 norm of general 1-D quasilinear systems with source terms under boundary conditions. To reach this aim, we introduce the notion of basic C^1 Lyapunov functions, a generic kind of exponentially decreasing function whose existence ensures the exponential stability of the system for the C^1 norm. We show that the existence of a basic C^1 Lyapunov function is subject to two conditions: an interior condition, intrinsic to the system, and a condition on the boundary controls. We give explicit sufficient interior and boundary conditions such that the system is exponentially stable for the C^1 norm and we show that the interior condition is also necessary to the existence of a basic C^1 Lyapunov function. Finally, we show that the results conducted in this chapter are also true under the same conditions for the exponential stability in the C^p norm, for any $p \geq 1$.

2.1 Introduction

Hyperbolic systems have been studied for several centuries, as their importance in representing physical phenomena is undeniable. From gaz dynamics to population evolution through wave equations and fluid dynamics they are found in many areas. As they represent the propagation phenomena of numerous physical or industrial systems [1, 94, 137], the issue of their controllability and stability is a major concern, with both theoretical and practical interest. If the question of controllability has been well-studied [140], the problem of stabilization under boundary control, however, is only well known in the particular case of an absence of source term. However, in many case neglecting the source term is a crude approximation and reduces greatly the analysis, in particular because it implies that the system can be reduced to decoupled equations or slightly coupled equations (see [71] for instance). For most physical equations the source term cannot therefore be neglected and the steady-states we aim at stabilizing can be non-uniform with potentially large variations of amplitude (e.g. Saint-Venant equations, see [42] Chapter 5 or [110], Euler equations, see [81] or [100], Telegrapher equations, etc.). Taking into account these nonuniform steady-states and stabilizing them is impossible when not taking the source term into account, although it is an important issue in many applications. In presence of a source term some results exist for the H^2 norm (and actually H^p, $p \geq 2$), however, few results exist for the more natural C^1 norm (and consequent C^p norms, $p \geq 1$). It has to be underlined that for nonlinear systems the stability in these two main topologies are not equivalent as shown in [61]. In this article we deal with the stability in C^1 norm of such hyperbolic systems of quasilinear partial differential equations with source term under boundary conditions.

Several methods are usually used to study the stability of systems. The Lyapunov approach, one of the most famous, is the one we opted for in this article. This approach has the advantage, among others, of guaranteeing some robustness and of being convenient to deal with non-linear problems [47, 119]. We first introduce the *basic C^1 Lyapunov functions*, a kind of natural Lyapunov functions for the C^1 norm and we then find a sufficient condition such that the system admits a basic C^1 Lyapunov function. We show that this sufficient condition is twofold: a first intrinsic condition on the system and a second condition on the boundary controls. We show then that this sufficient condition on the system is in fact necessary in the general case for the existence of a basic C^1 Lyapunov function.

The organisation of this chapter is as follows: In Section 1, we recall some preliminary properties about 1-D quasilinear hyperbolic system. Section 2 presents an overview of the context and previous results. Section 3 states the main results, which are proven in Section 4. Section 5 presents several remarks and further detail to the results.

2.2 Preliminary properties of 1-D quasilinear hyperbolic systems

A general quasilinear hyperbolic system can be written as:

$$\mathbf{Y}_t + F(\mathbf{Y})\mathbf{Y}_x + D(\mathbf{Y}) = 0, \tag{2.2.1}$$

$$\mathcal{B}(\mathbf{Y}(t,0), \mathbf{Y}(t,L)) = 0, \tag{2.2.2}$$

with $\mathbf{Y} : [0, +\infty) \times [0, L] \to \mathbb{R}^n$ and $F : U \to \mathcal{M}_n(\mathbb{R})$ and $D : U \to \mathbb{R}^n$ where U is a non empty connected open set of $\mathbb{R}^n$ and F is strictly hyperbolic, i.e. for all $\mathbf{Y} \in U$, $F(\mathbf{Y})$ has real, distinct eigenvalues. We suppose in addition that these eigenvalues are non-vanishing. $\mathcal{B}$ is a map from $U \times U$ to $\mathbb{R}$ whose form will be precised later on, such that the system (2.2.1)–(2.2.2) is well-posed.

We call $\mathbf{Y}^*$ a steady-state of the previous system that we aim at stabilizing. Note that, due to the source term, $\mathbf{Y}^*$ is not necessarily uniform and the problem cannot be directly treated as a null stabilization. We therefore use the following transformation:

$$\mathbf{u}(x,t) = N(x)(\mathbf{Y}(x,t) - \mathbf{Y}^*(x)), \tag{2.2.3}$$

where N is such that:

$$NF(\mathbf{Y}^*)N^{-1} = \Lambda, \tag{2.2.4}$$

where Λ is diagonal and corresponds to the eigenvalues of $F(\mathbf{Y}^*)$. Note that such N exists as the system is strictly hyperbolic. Therefore, the system (2.2.1)–(2.2.2) is equivalent to

$$\mathbf{u}_t + A(\mathbf{u}, x)\mathbf{u}_x + B(\mathbf{u}, x) = 0, \tag{2.2.5}$$

$$\mathcal{B}(N(0)^{-1}\mathbf{u}(0,t) + \mathbf{Y}^*(0), N(L)^{-1}\mathbf{u}(L,t) + \mathbf{Y}^*(L)) = 0, \tag{2.2.6}$$

with

$$A(\mathbf{u}, x) = N(x)F(\mathbf{Y})N^{-1}(x) = N(x)F(N^{-1}(x)\mathbf{u} + \mathbf{Y}^*(x))N^{-1}(x), \tag{2.2.7}$$

$$B(\mathbf{u}, x) = N(F(\mathbf{Y})(\mathbf{Y}^*_x + (N^{-1})'\mathbf{u}) + D(\mathbf{Y})). \tag{2.2.8}$$

The difficulty when there is a source term is twofold, and its first aspect can be seen in (2.2.7): we cannot assume that the steady state $\mathbf{Y}^*$ we aim at stabilizing is uniform. Therefore A depends not only on $\mathbf{u}$ but also directly on x, and having $A(\mathbf{u}(t,x))$ is different from having $A(\mathbf{u}(t,x),x)$ especially when $\mathbf{u}$ is a perturbation: if $\mathbf{u}$ can still be seen as a perturbation, the dependency on x can no longer be seen itself as a perturbation.

Its second aspect is that the source term creates a coupling between the two quantities which is a zero order term that can disturb the Lyapunov function and we will see in Section 2, 3 and 4 that this implies that there does not always exist a simple quadratic Lyapunov function ensuring exponential stability even when the boundary conditions can be chosen arbitrarly, while this phenomenon cannot appear in the absence of source term.

From the strict hyperbolicity we can denote by m the integer such that

$$\Lambda_i > 0,\ \forall i \leq m,\ \text{and } \Lambda_i < 0,\ \forall i \in [m+1, n]. \tag{2.2.9}$$

We now denote by $\mathbf{u}_+$ the vector of components associated to positive eigenvalues $(u_1, ..., u_m)^T$ and similarly $\mathbf{u}_-$ refers to $(u_{m+1}, ..., u_n)^T$. In the special cases where $m = 0$ or $m = n$ $\mathbf{u}$ is equal to $\mathbf{u}_-$ or $\mathbf{u}_+$ respectively. From now on we will focus on boundary conditions of the form

$$\begin{pmatrix} \mathbf{u}_+(t,0) \\ \mathbf{u}_-(t,L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}_+(t,L) \\ \mathbf{u}_-(t,0) \end{pmatrix}. \tag{2.2.10}$$

Note that with these boundary conditions the incoming signal is a function of the outgoing signal, which is what is typically expected from a feedback control law and enables the well-posedness of the system (see Theorem 2.2.1 later on). However the method presented in this article could also be applied to any other boundary conditions of the form (2.2.2) that also ensure well-posedness.
We also introduce the consequent first order compatibility conditions for an initial condition $\mathbf{u}^0$:

$$\begin{pmatrix} \mathbf{u}^0_+(0) \\ \mathbf{u}^0_-(L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}^0_+(L) \\ \mathbf{u}^0_-(0) \end{pmatrix}, \tag{2.2.11}$$

$$\begin{gathered} \begin{pmatrix} \left(A(\mathbf{u}^0(0),0)\partial_x\mathbf{u}^0(0) + B(\mathbf{u}^0(0),0)\right)_+ \\ \left(A(\mathbf{u}^0(L),L)\partial_x\mathbf{u}^0(L) + B(\mathbf{u}^0(L),L)\right)_- \end{pmatrix} = \\ G' \begin{pmatrix} \mathbf{u}^0_+(L) \\ \mathbf{u}^0_-(0) \end{pmatrix} \begin{pmatrix} \left(A(\mathbf{u}^0(L),L)\partial_x\mathbf{u}^0(L) + B(\mathbf{u}^0(L),L)\right)_+ \\ \left(A(\mathbf{u}^0(0),0)\partial_x\mathbf{u}^0(0) + B(\mathbf{u}^0(0),0)\right)_- \end{pmatrix}. \end{gathered} \tag{2.2.12}$$

Well-posedness of the system (2.2.5),(2.2.10) for any initial condition $\mathbf{u}^0$ that satisfies the compatibility conditions (2.2.11),(2.2.12) is given by Li [142] (see also [172]), one has the following theorem:

Theorem 2.2.1. *For all $T > 0$ there exist $C_1(T) > 0$ and $\eta(T) > 0$ such that, for every $\mathbf{u_0} \in C^1([0,L], \mathbb{R}^n)$ satisfying the compatibility conditions (2.2.11), (2.2.12) and such that $|\mathbf{u}_0|_1 \leq \eta(T)$, the system (2.2.5)-(2.2.10), with A and B of class C^1, has a unique solution on $[0,T] \times [0,L]$ with initial condition $\mathbf{u_0}$. Moreover one has:*

$$|\mathbf{u}(t,\cdot)|_1 \leq C_1(T)|\mathbf{u}(0,\cdot)|_1,\ \forall t \in [0,T]. \tag{2.2.13}$$

2.3 Context and previous results

General hyperbolic system without source term The exponential stability of general strictly hyperbolic systems of the form (2.2.5) without source term, i.e. $B \equiv 0$, has been mainly studied in the linear or non-linear case (see for instance [21, 48, 50, 57, 93, 141, 180]) under various boundary conditions or boundary controls (e.g. Proportional-integral control, dead beat control, single boundary control, etc.). A large part of these studies has been conducted using boundary conditions of the form (2.2.10). For such boundary conditions in non-linear systems the exponential stability depends on the topology [61] and in particular that the stability in H^2 norm does not imply the stability in C^1 norm. In [61] the authors also gave a sufficient condition for stability in the $W^{2,p}$ norm for $p \in [1, +\infty]$:

$$\rho_p(G'(0)) < 1, \tag{2.3.1}$$

where G is given in (2.2.10) and the definition of ρ_p is

$$\rho_p(M) = \inf(\|\Delta M \Delta^{-1}\|_p, \Delta \in D_n^+),\ 1 \leq p \leq +\infty \tag{2.3.2}$$

where $\|\cdot\|_p$ is the usual p norm for matrices and D_n^+ are the diagonal $n \times n$ matrices with positive eigenvalues.

The case of the C^1 norm for systems with no source term has also been treated in [48] by Jean-Michel Coron and Georges Bastin by a Lyapunov approach that inspired the first part of this chapter. There, they proved the following sufficient condition for exponential stability through a Lyapunov approach:

$$\rho_\infty(G'(0)) < 1. \tag{2.3.3}$$

However the general case with a non-zero source term changes several things. As mentioned previously it implies that the steady-states $\mathbf{Y}^*$ are no longer necessarily uniform and as a direct consequence the matrix A defined in (2.2.7) depends explicitly not only on $\mathbf{u}$ but also on x. In addition, there are some cases where, for any G, no basic quadratic H^2 Lyapunov function can be found (see for instance [13] and in particular Proposition 5.12) or no basic C^1 Lyapunov function can be found, as shown later on.

General hyperbolic system with non-zero source term in the H^p norm For general quasilinear hyperbolic systems with source term, also called inhomogeneous quasilinear hyperbolic systems, the analysis of the exponential stability is much less advanced and actual knowledge in the matter is still partial. To our knowledge the exponential stability of such systems with non zero and non negligible source term was only treated in the framework of the H^p norm for $p \in \mathbb{N} \setminus \{0, 1\}$ and in [13] (in Chapter 6) the authors find a sufficient (but *a priori* non-necessary) condition: exponential stability of the system (2.2.5)–(2.2.13) for the H^p norm where $p \geq 2$ is achieved if there exists $Q \in C^1([0, L], D_n^+)$ such that the two following conditions hold:

- (Interior condition) the matrix

$$-(Q\Lambda)'(x) + Q(x)M(\mathbf{0}, x) + M(\mathbf{0}, x)^T Q(x)^T \tag{2.3.4}$$

  is positive definite for all $x \in [0, L]$,
- (Boundary conditions) the matrix

$$\begin{pmatrix} \Lambda_+(L)Q_+(L) & 0 \\ 0 & -\Lambda_-(0)Q_-(0) \end{pmatrix} - K^T \begin{pmatrix} \Lambda_+(0)Q_+(0) & 0 \\ 0 & -\Lambda_-(L)Q_-(L) \end{pmatrix} K \tag{2.3.5}$$

  is positive semi-definite

where $M(\mathbf{0}, \cdot) = \frac{\partial B}{\partial \mathbf{u}}(\mathbf{0}, \cdot)$ and $K = G'(\mathbf{0})$.

It has to be underlined that with a non-zero source term there does not always exist a simple quadratic Lyapunov function ensuring exponential stability for the H^p norm whatever the boundary conditions are.

Thus appears not only a boundary condition (2.3.5) as in the previous paragraph but also an interior condition (2.3.4).
This phenomenon is not specific to non-linear systems but also appears in linear systems: In [11] for instance, the authors study a linear 2×2 system and found a necessary and sufficient condition for the existence of Q such that (2.3.4) hold. In general for linear hyperbolic systems the condition (2.3.4) also appears although it is only sufficient when $n > 2$. This is the consequence of the non-uniformity of the steady-states combined with non-identically vanishing zero order term even close to the steady states. If this phenomenon is not new, we will see however that the interior condition that appears for the C^1 norm is different from the condition that typically appears when studying Lyapunov functions for H^p norms.

Our contribution in this article is to deal with the exponential stability for the C^1 norm of such general hyperbolic systems with source term. This article intends to give a necessary and sufficient interior condition to the existence of a simple quadratic Lyapunov function ensuring exponential stability in the C^1 (and actually C^p) norm of the system and a sufficient condition on the boundary conditions.

Useful observations and notations Before going any further let us note that by definition of B and as $\mathbf{Y}^*$ is a steady-state

$$B(\mathbf{0}, x) = N(0)(F(\mathbf{Y}^*)(\mathbf{Y}^*_x) + D(\mathbf{Y}^*)) = 0. \tag{2.3.6}$$

Thus if we assume that F and Y^* are C^3 functions, then, from (2.2.8), B is C^2 and there exists $\eta_0 > 0$ and $M \in C^1(\mathcal{B}_{\eta_0} \times [0, L], \mathcal{M}_n(\mathbb{R}))$, where $\mathcal{B}_{\eta_0}$ is the ball of radius η_0 in the space of continuous function endowed with the L^∞ topology, such that,

$$\begin{aligned} B(\mathbf{u}, x) &= M(\mathbf{u}, x)\mathbf{u}, \\ \text{and therefore, } \frac{\partial B}{\partial \mathbf{u}}(\mathbf{0}, x) &= M(\mathbf{0}, x). \end{aligned} \tag{2.3.7}$$

Besides, A is also a C^2 function and $\eta_0 > 0$ can be chosen small enough such that there exists $E \in C^2(\mathcal{B}_{\eta_0} \times [0, L], \mathcal{M}_n(\mathbb{R}))$, satisfying (see [13] in particular Lemma 6.7),

$$E(\mathbf{u}, x)A(\mathbf{u}, x) = \lambda(\mathbf{u}, x)E(\mathbf{u}, x) \ \forall \ (\mathbf{u}, x) \in \mathcal{B}_{\eta_0} \times [0, L], \tag{2.3.8}$$

$$\text{and } E(\mathbf{0}, x) = Id, \tag{2.3.9}$$

where λ is a diagonal matrix, whose diagonal entries are the eigenvalues of $A(\mathbf{u}, x)$.

Also we introduce the following notations:

Definition 2.3.1. For a C^0 function $\mathbf{U} = (U_1, ..., U_n)^T$ on $[0, L]$ we define the C^0 norm $|\mathbf{U}|_0$ by

$$|\mathbf{U}|_0 := \sup_i \left(\sup_{[0,L]} (|U_i|) \right). \tag{2.3.10}$$

For a C^1 function $\mathbf{U} = (U_1, ..., U_n)^T$ on $[0, L]$, we denote similarly the C^1 norm $|\mathbf{U}|_1$ by

$$|\mathbf{U}|_1 := |\mathbf{U}|_0 + |\partial_x \mathbf{U}|_0. \tag{2.3.11}$$

In the following for a C^1 function $\mathbf{u}$ on $[0, T] \times [0, L]$, we will sometimes note for simplicity $|\mathbf{u}|_0$ instead of $|\mathbf{u}(t, \cdot)|_0$ and $|\mathbf{u}|_1$ instead of $|\mathbf{u}(t, \cdot)|_1$.

We recall the definition of the exponential stability for the C^1 norm:

Definition 2.3.2. The steady state $\mathbf{u}^* = 0$ of the system (2.2.5),(2.2.10) is exponentially stable for the C^1 norm if there exist $\gamma > 0$, $\eta > 0$, and $C > 0$ such that for every $\mathbf{u}^0 \in C^1([0, L])$ satisfying the compatibility conditions (2.2.11),(2.2.12) and $|\mathbf{u}^0|_1 \leq \eta$, the Cauchy problem (2.2.5),(2.2.10),($\mathbf{u}(0, x) = \mathbf{u}^0$) has a unique C^1 solution and

$$|\mathbf{u}(t, \cdot)|_1 \leq Ce^{-\gamma t}|\mathbf{u}^0|_1, \ \forall t \in [0, +\infty[. \tag{2.3.12}$$

Remark 2.3.1. *Given our change of variable* $\mathbf{Y} \to \mathbf{u}$*, proving the exponential stability for the* C^1 *norm of the steady state* 0 *of the system (2.2.5),(2.2.10) is equivalent to proving the exponential stability for the* C^1 *norm of the steady state* $\mathbf{Y}^*$ *of the system (2.2.1) and the associated boundary condition.*

Definition 2.3.3. We call *basic* C^1 *Lyapunov function* a function V defined by

$$
\begin{aligned}
V(\mathbf{U}) = &\left|(\sqrt{f_1}U_1, ..., \sqrt{f_n}U_n)^T\right|_0 \\
&+ \left|\left((E(\mathbf{U},x)(A(\mathbf{U},x)\mathbf{U}_x + B(\mathbf{U},x)))_1\sqrt{f_1}, ..., (E(\mathbf{U},x)(A(\mathbf{U},x)\mathbf{U}_x + B(\mathbf{U},x)))_n\sqrt{f_n}\right)^T\right|_0,
\end{aligned}
\tag{2.3.13}
$$

for some $(f_1, ...f_n) \in C^1\left([0,L];\mathbb{R}_+^*\right)^n$, such that there exist $\gamma > 0$ and $\eta > 0$ such that for any $T > 0$ and any solution $\mathbf{u}$ of the system (2.2.5)–(2.2.10) with $|\mathbf{u}^0|_1 \leq \eta$,

$$
V(t) \leq V(t')e^{-\gamma(t-t')}, \ \forall\ 0 \leq t' \leq t \leq T. \tag{2.3.14}
$$

Also, in that case, $(f_1, ..., f_n)$ are called coefficients inducing a basic C^1 Lyapunov function.

Remark 2.3.2. *Note from (2.2.5), that when* $\mathbf{u}$ *is a solution of the system (2.2.5), (2.2.10),* $V(\mathbf{u}(t,\cdot))$ *becomes*

$$
V(\mathbf{u}(t,\cdot)) = \left|(\sqrt{f_1}u_1, ..., \sqrt{f_n}u_n)^T\right|_0 + \left|(E\mathbf{u}_t)_1\sqrt{f_1}, ..., (E\mathbf{u}_t)_n\sqrt{f_n})^T\right|_0, \tag{2.3.15}
$$

where we denoted $E = E(\mathbf{u}(t,x),x)$ *to lighten the notations. The previous definition (2.3.13) is used so that* V *is actually defined as function on* $C^1([0,L])$ *only and to underline that therefore, the function* $V(\mathbf{u}) : t \to V(\mathbf{u}(t,\cdot))$ *does only depend on the state of the system at time* t*. Looking at* (2.3.15)*, one could wonder why we consider the components of* $\mathbf{u}$ *while we consider the components of* $E\mathbf{u}_t$ *for the derivative. The interest of considering* $E\mathbf{u}_t$ *instead of* $\mathbf{u}_t$ *is that* E *diagonalizes* A *and therefore when differentiating the Lyapunov function appears* $2(E\mathbf{u}_t)_n(E(\mathbf{u})_{tt})_n = -\lambda_n(\mathbf{u},x)((E\mathbf{u}_{tx})_n^2)$ *and first order derivative terms, and there is no crossed term of second order derivative which would be impossible to bound with the* C^1 *norm (the full computation is done in Appendix 2.8.1). Differentiating* u_n^2*, though, gives* $-\lambda_n(u_n^2)_x - u_n((A-\lambda).\mathbf{u}_x)_n$ *and zero order derivative terms, and the second term is a cubic perturbation that can be bounded by the cube of the* C^1 *norm. Nevertheless, the proof would work as well with* $E\mathbf{u}$ *instead of* $\mathbf{u}$*, but we consider* $\mathbf{u}$ *to keep the computations as simple as we can in the main proof (Section 2.5). Finally, we use in the definition* (2.3.13) *the weights* $\sqrt{f_i}$ *instead of using directly the weights* f_i *to be coherent with the existing definition of basic quadratic Lyapunov function for the* L^2 *norm introduced in [11] (see in particular (34)) for linear systems and to facilitate a potential comparison.*

Remark 2.3.3. *Note also that, in Definition 2.3.3, the condition* (2.3.14) *is actually equivalent to the condition*

$$
\frac{dV(\mathbf{u})}{dt} \leq -\gamma V(\mathbf{u}), \tag{2.3.16}
$$

in a distributional sense on $(0,T)$*, where we say that* $d \geq 0$ *in a distributional sense on* $(0,T)$ *with* $d \in \mathcal{D}'(0,T)$ *when, for any* $\phi \in C_c^\infty((0,T),\mathbb{R}^+)$*,*

$$
< d, \phi > \ \geq \ 0. \tag{2.3.17}
$$

Note that the existence of such basic C^1 Lyapunov function for a system guaranties the exponential stability of the system for the C^1 norm. More precisely we have the following proposition:

Proposition 2.3.1. *Let a quasilinear hyperbolic system be of the form (2.2.5),(2.2.10), with* A *and* B *of class* C^1 *such that there exists a basic* C^1 *Lyapunov function, then the system is exponentially stable for the* C^1 *norm.*

Proof of proposition 2.3.1. From Theorem 2.2.1, let $T > 0$ and $\mathbf{u}_0 \in C^1([0,L],\mathbb{R}^n)$ satisfying the compatibility conditions (2.2.11) and such that $|\mathbf{u}_0|_1 \leq \min(\eta(T), \eta_0/C_1(T))$, where $\eta(T)$ and $C_1(T)$ are given by Theorem 2.2.1 and η_0 is given by (2.3.7)–(2.3.9). From Theorem 2.2.1 there exists a unique solution $\mathbf{u} \in C^1([0,T]\times[0,L])$. Suppose that V is a basic C^1 Lyapunov function, induced by $(f_1,...f_n)$ and γ and η_1 are the constants associated. From its definition $V(\mathbf{u}(t,\cdot))$ is closely related to $|\mathbf{u}(t,\cdot)|_1$, indeed, using that for all $i \in \{1,n\}$, f_i are positive and bounded on $[0,L]$, it is easy to see that there exists a constant $c_2 > 0$ such that

$$\frac{1}{c_2}(|\mathbf{u}(t,\cdot)|_0 + |E\partial_t\mathbf{u}(t,\cdot)|_0) \leq V(\mathbf{u}(t,\cdot)) \leq c_2(|\mathbf{u}(t,\cdot)|_0 + |E\partial_t\mathbf{u}(t,\cdot)|_0). \tag{2.3.18}$$

But as, from (2.2.13) and the assumption on $|\mathbf{u}_0|_1$, $|\mathbf{u}(t,\cdot)|_1 \leq \eta_0$ for any $t\in[0,T]$. Thus from (2.3.8)–(2.3.9) there exists a constant c_1 depending only on η_0 and the system such that

$$\frac{1}{c_1}|\partial_t\mathbf{u}(t,\cdot)|_0 \leq |E\partial_t\mathbf{u}(t,\cdot)|_0 \leq c_1|\partial_t\mathbf{u}(t,\cdot)|_0, \tag{2.3.19}$$

thus, there exists $c_0 > 0$ such that

$$\frac{1}{c_0}(|\mathbf{u}(t,\cdot)|_0 + |\partial_t\mathbf{u}(t,\cdot)|_0) \leq V(\mathbf{u}(t,\cdot)) \leq c_0(|\mathbf{u}(t,\cdot)|_0 + |\partial_t\mathbf{u}(t,\cdot)|_0). \tag{2.3.20}$$

But observe that, as $\mathbf{u}$ is a solution of (2.2.5), there exists $\eta_a > 0$ such that for $|\mathbf{u}(t,\cdot)|_0 < \eta_a$

$$|\partial_t\mathbf{u}(t,\cdot)|_0 \leq 2\sup_i\left(|\Lambda_i|_0\right)|\partial_x\mathbf{u}(t,\cdot)|_0 + 2\sup_{i,j}\left(|M_{ij}(\mathbf{0},\cdot)|_0\right)|\mathbf{u}(t,\cdot)|_0, \tag{2.3.21}$$

and similarly

$$|\partial_x\mathbf{u}(t,\cdot)|_0 \leq \frac{2}{\inf_{i,x\in[0,L]}\left(\Lambda_i(x)\right)}\left(|\partial_t\mathbf{u}(t,\cdot)|_0 + \sup_{i,j}\left(|M_{ij}(\mathbf{0},\cdot)|_0\right)|\mathbf{u}(t,\cdot)|_0\right), \tag{2.3.22}$$

which implies that there exists $c > 0$ constant such that for $|\mathbf{u}(t,\cdot)|_0 < \eta_a$

$$\frac{1}{c}|\mathbf{u}(t,\cdot)|_1 \leq V(\mathbf{u}) \leq c|\mathbf{u}(t,\cdot)|_1. \tag{2.3.23}$$

Let $T \in \mathbb{R}_+^*$, with $T > 0$ and T large enough such that $c^2e^{-\gamma T} < \frac{1}{2}$. From (2.3.14), for all solution $\mathbf{u}$ such that $|\mathbf{u}^0|_1 < \min(\eta(T), \eta_1, \eta_a/C(T))$ where $C(T)$ is defined in (2.2.13),

$$V(\mathbf{u},T) \leq V(\mathbf{u},0)e^{-\gamma T}. \tag{2.3.24}$$

Now, using (2.3.23) we get

$$|\mathbf{u}(T,\cdot)|_1 \leq |\mathbf{u}(0,\cdot)|_1 c^2 e^{-\gamma T}, \tag{2.3.25}$$

And from the hypothesis on T

$$|\mathbf{u}(T,\cdot)|_1 \leq \frac{1}{2}|\mathbf{u}(0,\cdot)|_1, \tag{2.3.26}$$

and this imply that $\mathbf{u}$ is defined on $[0,+\infty)$ and that we can find C and γ_1 such that

$$|\mathbf{u}(t,0) - \mathbf{u}^*|_1 \leq Ce^{-\gamma t}|\mathbf{u^0} - \mathbf{u}^*|_1,\ \forall t \in [0,+\infty[, \tag{2.3.27}$$

which gives the exponential stability and concludes the proof. □

2.4 Main results

The aim of this article is to show the following results:

Theorem 2.4.1. *Let a quasilinear hyperbolic system be of the form (2.2.5), (2.2.10), with A and B of class C^1, Λ defined as in (2.2.4) and M as in (2.3.7). Let assume that the two following properties hold*

1. *(Interior condition) the system*

$$\Lambda_i f_i' \leq -2 \left(-M_{ii}(0,x) f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{f_k}} \right), \tag{2.4.1}$$

   admits a solution $(f_1, ..., f_n)$ *on* $[0, L]$ *such that for all* $i \in [1, n]$, $f_i > 0$,

2. *(Boundary conditions) there exists a diagonal matrix* Δ *with positive coefficients such that*

$$\|\Delta G'(0)\Delta^{-1}\|_\infty < \frac{\inf_i \left(\frac{f_i(d_i)}{\Delta_i^2} \right)}{\sup_i \left(\frac{f_i(L-d_i)}{\Delta_i^2} \right)}, \tag{2.4.2}$$

   where $d_i = L$ *if* $\Lambda_i > 0$ *and* $d_i = 0$ *otherwise.*

Then there exists a basic C^1 *Lyapunov function for the system (2.2.5), (2.2.10).*

Remark 2.4.1. *Note that when* $M \equiv 0$ *we recover the result found in [48] in the absence of source term: the interior condition is always verified by any positive constant functions* $(f_1, ..., f_n)$ *and when choosing* $f_i = \Delta_i^2$ *the boundary condition reduces to the existence of* $\Delta \in D_+^n$ *such that* $\|\Delta G'(0)\Delta^{-1}\|_\infty < 1$ *which is equivalent to* $\rho_\infty(G'(0)) < 1$.

Note also that the existence of a solution $(f_1, ...f_n)$ *with* $f_i > 0$ *on* $[0, L]$ *for all* $i \in \{1, ..., n\}$ *for the system*

$$f_i' = -\frac{2}{\Lambda_i} \left(-M_{ii}(0,x) f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{f_k}} \right) \tag{2.4.3}$$

is also a sufficient interior condition as it obviously implies the existence of a solution with positive components for (2.4.1).

Moreover, we show in the following Theorem that condition (2.4.1) is also necessary in order to ensure the existence of a basic C^1 Lyapunov function.

Theorem 2.4.2. *Let a quasilinear hyperbolic system be of the form (2.2.5) with* A *and* B *of class* C^3, *there exists a control of the form (2.2.10) such that there exists a basic* C^1 *Lyapunov function for the system (2.2.5),(2.2.10) if and only if*

$$\Lambda_i f_i' \leq -2 \left(\sum_{k=1, k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{f_k}} - M_{ii}(0,x) f_i \right), \tag{2.4.4}$$

admits a solution $(f_1, ..., f_n)$ *on* $[0, L]$ *such that for all* $i \in [1, n]$, $f_i > 0$.

Remark 2.4.2. *Note that Theorem 2.4.2 illustrates the sharpness of (2.4.1) by showing that it is a necessary condition. This is not trivial as, to our knowledge, there is no similar condition for the* H^p *norm when* $n > 2$ *yet. Note also that we have not imposed anything on the initial values of the* $(f_1, ..., f_n)$ *but we see from Theorem 2.4.1 and (2.4.2) that the more liberty we give them, the more restrictive the condition on the boundary (2.4.2) might become.*

The proof of these two results is given in the next section.

2.5 C^1 Lyapunov stability of $n \times n$ quasilinear hyperbolic system

In this Section we shall prove Theorem 2.4.1 and Theorem 2.4.2. We will first start by proving the following Lemma which will be useful for finding the interior condition in the proof of Theorem 2.4.1 and for proving Theorem 2.4.2:

Lemme 2.5.1. *Let* $(a_i, b_{ij})_{(i,j)\in[\![1,n]\!]^2} \in C([0,L],\mathbb{R})^n \times C([0,L],\mathbb{R})^{n^2}$,

If

$$(i)\ \exists p_1 \in \mathbb{N}^* : \sum_{i=1}^{n}\left(a_i(x)y_i^{2p} + \sum_{j=1}^{n} b_{ij}(x)y_i^{2p-1}y_j\right) > 0,\ \forall p > p_1, \forall y \in \mathbb{R}^n \setminus \{0\}, \forall x \in [0,L], \tag{2.5.1}$$

then

$$(ii)\ a_i(x) \geq \sum_{j=1, j\neq i}^{n} |b_{ij}(x)| - b_{ii}(x),\ \forall i \in [1,n], \forall x \in [0,L]. \tag{2.5.2}$$

And if

$$(iii)\ a_i(x) > \sum_{j=1, j\neq i}^{n} |b_{ij}(x)| - b_{ii}(x), \forall i \in [1,n], \forall x \in [0,L], \tag{2.5.3}$$

then (i) *holds.*

Proof of Lemma 2.5.1. We start with $(i) \Rightarrow (ii)$. Let $x \in [0,L]$, let $i_1 \in [1,n]$, assuming (i) is true for all $y \in \mathbb{R}^n \setminus \{0\}$, we take $m \in \mathbb{N}^*$, and define $y_{i_1} := 1$, $y_j := -\operatorname{sgn}(b_{i_1 j})m/(m+1)$ for $j \neq i_1$. Then as (2.5.1) is true there exists $p_1 \in \mathbb{N}^*$ such that

$$\begin{aligned}&\sum_{i=1, i\neq i_1}^{n}\left(a_i(x)y_i^{2p} + \sum_{j=1}^{n} b_{ij}(x)y_i^{2p-1}y_j\right)\\ &+ a_{i_1}(x) + b_{i_1 i_1} + \sum_{j=1, j\neq i_1}^{n} b_{i_1 j}(x)y_j > 0,\ \forall p > p_1, \forall x \in [0,L].\end{aligned} \tag{2.5.4}$$

Note that for any $i \neq i_1$, $\lim_{p\to+\infty} |y_i|^{2p} = 0$. Thus, by letting $p \to +\infty$ one gets

$$a_{i_1}(x) + b_{i_1 i_1}(x) \geq \frac{m}{m+1} \sum_{j=1, j\neq i_1}^{n} |b_{i_1 j}(x)|,\ \forall x \in [0,L]. \tag{2.5.5}$$

Hence, as it is true for all $m \in \mathbb{N}^*$, letting $m \to +\infty$

$$a_{i_1}(x) + b_{i_1 i_1}(x) \geq \sum_{j=1, j\neq i_1}^{n} |b_{i_1 j}(x)|,\ \forall x \in [0,L]. \tag{2.5.6}$$

This can be done for any $i_1 \in [1,n]$, which concludes $(i) \Rightarrow (ii)$.

Now let us prove that $(iii) \Rightarrow (i)$. First of all observe that we can suppose without loss of generality that $\forall i \in [1,n]$, $b_{ii} := 0$: one just has to redefine $a_i := a_i + b_{ii}$. Then by (2.5.3), $a_i > \sum_{j=1}^{n} |b_{ij}|$, $\forall i \in [1,n]$, then let us define:

$$d_i(x) := a_i(x) - \sum_{k=1}^{n} |b_{ik}(x)|, \tag{2.5.7}$$

then d_i is C^0 and positive on $[0,L]$. We denote by

$$d_i^{(0)} := \inf_{[0,L]}(d_i) = \min_{[0,L]}(d_i) > 0. \tag{2.5.8}$$

Now, let $y \in \mathbb{R}^n \setminus \{0\}$, we can select i_1 such that

$$|y_{i_1}| = \max_{i\in[1,n]}(|y_i|), \tag{2.5.9}$$

thus $y_{i_1} \neq 0$ and proving (2.5.1) is equivalent to proving that there exists $p_1 \in \mathbb{N}^*$ such that for all $p > p_1$,

$$\sum_{i=1}^{n}\left(a_i(x)\left|\frac{y_i}{y_{i_1}}\right|^{2p} + \sum_{k=1}^{n} b_{ik}(x)\left(\frac{y_i}{y_{i_1}}\right)^{2p-1}\frac{y_k}{y_{i_1}}\right) > 0,\ \forall x \in [0, L]. \tag{2.5.10}$$

Denoting $z_i = y_i/y_{i_1}$, (2.5.10) becomes

$$I := \sum_{i=1}^{n}\left(a_i z_i^{2p} + \sum_{k=1}^{n} b_{ik} z_i^{2p-1} z_k\right) > 0,\ \text{on } [0, L]. \tag{2.5.11}$$

Using (2.5.7) we know that

$$I = \sum_{i=1}^{n} d_i z_i^{2p} + \sum_{k=1}^{n} |b_{ik}| z_i^{2p} + \sum_{k=1}^{n} b_{ik} z_i^{2p-1} z_k. \tag{2.5.12}$$

By definition for $i = i_1$, $|z_{i_1}| = 1$, and for $i \neq i_1$, $|z_k| \leq 1$, therefore

$$d_{i_1} z_{i_1}^{2p} + \sum_{k=1}^{n} |b_{i_1 k}| z_{i_1}^{2p} + \sum_{k=1}^{n} b_{i_1 k} z_{i_1}^{2p-1} z_k \geq d_{i_1} \geq d_{i_1}^{(0)}. \tag{2.5.13}$$

Therefore

$$\begin{aligned} I &\geq d_{i_1}^{(0)} + \sum_{i=1, i\neq i_1}^{n}\left(d_i z_i^{2p} + \sum_{k=1}^{n} |b_{ik}||z_i|^{2p} - \sum_{k=1}^{n} |b_{ik}||z_i|^{2p-1}\right), \\ &= d_{i_1}^{(0)} + \sum_{i=1, i\neq i_1}^{n}\left(d_i z_i^{2p} - \sum_{k=1}^{n} |b_{ik}|(1 - |z_i|)|z_i|^{2p-1}\right). \end{aligned} \tag{2.5.14}$$

We introduce

$$g : z \mapsto g(z) = -(1 - z) z^{2p-1}, \tag{2.5.15}$$

We know that g is C^1 on [0,1] and admits a minimum on $[0, 1]$ at $z = 1 - \frac{1}{2p}$, as one can check that

$$g'(z) = (2pz - (2p - 1)) z^{2p-2}. \tag{2.5.16}$$

Therefore

$$I \geq d_{i_1}^{(0)} - \frac{1}{2p} \sum_{i=1, i\neq i_1}^{n} \sum_{k=1}^{n} |b_{ik}(x)|, \tag{2.5.17}$$

and this is true for all $x \in [0, L]$. Let us point out that there exists $p_1 > 0$ such that

$$\frac{1}{2p} \sum_{i=1, i\neq i_1}^{n} \sum_{k=1}^{n} |b_{ik}|_0 < d_{i_1}^{(0)},\ \forall p > p_1. \tag{2.5.18}$$

Here p_1 is a constant and does not depend on x. Hence we can conclude that $I > 0,\ \forall p > p_1, \forall x \in [0, L], \forall y \in \mathbb{R}^n$. Therefore (2.5.1) holds. □

Now let us prove Theorem 2.4.1.

Proof of Theorem 2.4.1. Let $T \in \mathbb{R}_+^*$. Let assume that A and B are of class C^2, and let $\mathbf{u}$ be a C^2 solution of system (2.2.5),(2.2.10) such that $|\mathbf{u}^0|_1 \leq \varepsilon$. Such solution exists for ε small enough and $\mathbf{u}_0 \in C^2([0,L],\mathbb{R}^n)$ which verifies the compatiblity conditions (2.2.11) (see [13] in particular Theorem 4.21). We suppose here a C^2 regularity for technical reason but the final estimate will not depend on the C^2 norm and will be also true by density for A and B of class C^1 and for $\mathbf{u}$ a C^1 solution. Recall that λ_i are the eigenvalues of A as defined in (2.2.7). We denote $s_i := \mathrm{sgn}(\lambda_i(\mathbf{u},x))$ which only depends on i from the hypothesis of non-vanishing eigenvalues and the continuity of A. We define:

$$W_{1,p} := \left(\int_0^L \sum_{i=1}^n f_i(x)^p u_i^{2p} e^{-2p\mu s_i x} dx\right)^{1/2p}, \tag{2.5.19}$$

with $p \in \mathbb{N}^*$, and $f_i > 0$ on $[0,L]$ to be determined. Clearly $W_{1,p} > 0$ for $\mathbf{u} \neq 0$, and $W_{1,p} = 0$ when $\mathbf{u} \equiv 0$. If we differentiate $W_{1,p}$ with respect to time along the C^2 trajectories, we have

$$\begin{aligned}\frac{dW_{1,p}}{dt} =& W_{1,p}^{1-2p} \int_0^L \sum_{i=1}^n f_i(x)^p u_i^{2p-1} \left[-\sum_{k=1}^n a_{ik}(\mathbf{u},x) u_{kx}\right.\\ &\left.-\sum_{k=1}^n M_{ik}(\mathbf{u},x) u_k\right] e^{-2p\mu s_i x} dx,\end{aligned} \tag{2.5.20}$$

where $(a_{ij})_{(i,j)\in[1,n]^2} = A$ and M is defined in (2.3.7). We know that the a_{ij} are C^2 and from (2.3.7) that $a_{ij}(0,\cdot) = \delta_{i,j}\Lambda_i(\cdot)$. Here $\delta_{i,j}$ stands for the Kronecker delta. Hence

$$a_{ij}(\mathbf{u},\cdot) = \delta_{i,j}\Lambda_i(\cdot) + V_{ij}.\mathbf{u}, \tag{2.5.21}$$

where V_{ij} are C^1. Therefore using integration by parts

$$\begin{aligned}\frac{dW_{1,p}}{dt} =& -\frac{W_{1,p}^{1-2p}}{2p}\left[\sum_{i=1}^n \lambda_i f_i(x)^p u_i^{2p} e^{-2p\mu s_i x}\right]_0^L\\ &- W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^n f_i(x)^p u_i^{2p-1}\left[\left(\sum_{k=1}^n M_{ik}u_k\right) + \sum_{k=1}^n (V_{ik}(\mathbf{u},x).\mathbf{u})u_{kx}\right] e^{-2p\mu s_i x}dx\\ &+ \frac{W_{1,p}^{1-2p}}{2}\int_0^L \sum_{i=1}^n \left(\lambda_i(\mathbf{u},x) f_i(x)^{p-1} f_i' u_i^{2p} + \frac{d}{dx}\frac{(\lambda_i(\mathbf{u},x))}{p} f_i(x)^p u_i^{2p}\right) e^{-2p\mu s_i x}dx\\ &- \mu W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^n |\lambda_i| f_i^p u_i^{2p} e^{-2p\mu s_i x}dx.\end{aligned} \tag{2.5.22}$$

We denote

$$\mathrm{I}_2 := \frac{W_{1,p}^{1-2p}}{2p}\left[\sum_{i=1}^n \lambda_i f_i(x)^p u_i^{2p} e^{-2p\mu s_i x}\right]_0^L, \tag{2.5.23}$$

and

$$\begin{aligned}\mathrm{I}_3 :=& W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^n f_i(x)^p u_i^{2p-1}\left(\sum_{k=1}^n M_{ik}u_k\right) e^{-2p\mu s_i x}dx\\ &- \frac{W_{1,p}^{1-2p}}{2}\int_0^L \sum_{i=1}^n \lambda_i(\mathbf{u},x) f_i(x)^{p-1} f_i' u_i^{2p} e^{-2p\mu s_i x}dx.\end{aligned} \tag{2.5.24}$$

We supposed that $|\mathbf{u}^0|_1 \leq \varepsilon$, where $\varepsilon > 0$ can be chosen arbitrarily small but, of course, independent of p. From (2.2.13) and denoting $\eta = C_1(T)\varepsilon$ we have: $|\mathbf{u}|_0 \leq \eta$. Choosing ε sufficiently small is thus equivalent to choosing η sufficiently small, so we will rather choose η in the following and this choice of η

will always be independent of p. Besides, observe that there exists $\eta_1 > 0$ sufficiently small such that for all $\mathbf{u}_0 \in C^0([0,L],\mathbb{R}^n)$ such that $|\mathbf{u}|_0 \leq \eta_1$

$$\min_{x\in[0,L]}\left(\min_{i\in[1,n]}\left(|\lambda_i\left(\mathbf{u},x\right)|\right)\right) \geq \min_{x\in[0,L]}\left(\min_{i\in[1,n]}\left(\frac{|\Lambda_i(x)|}{2}\right)\right). \tag{2.5.25}$$

Recall that $\Lambda = \lambda(0,\cdot)$ and is defined in (2.2.4). As $[0,L]$ is a closed segment, and the $|\Lambda_i|$ are strictly positive continuous functions we can define the positive constant $\alpha_0 := \min_{x\in[0,L]}\left(\min_{i\in[1,n]}\left(|\Lambda_i(x)|/2\right)\right) > 0$. We suppose from now on that $\eta < \eta_1$. Therefore from (2.5.22), (2.5.23), (2.5.24) and (2.5.25)

$$\begin{aligned}\frac{dW_{1,p}}{dt} \leq & - I_2 - \mu\alpha_0 W_{1,p} - I_3 \\ & - W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^n f_i^p u_i^{2p-1}\left(\sum_{j=1}^n (V_{ik}(\mathbf{u},x).\mathbf{u})u_{kx}\right)e^{-2p\mu s_i x} \\ & + \frac{W_{1,p}^{1-2p}}{2p}\int_0^L \sum_{i=1}^n \left(\frac{\partial\lambda_i}{\partial\mathbf{u}}.\mathbf{u}_x + \partial_x\lambda_i\right) f_i(x)^p u_i^{2p} e^{-2p\mu s_i x}dx.\end{aligned} \tag{2.5.26}$$

We now estimate the two last terms, starting by the last one. The λ_i are C^2 and in particular C^1 in $\mathbf{u}$ therefore

$$\begin{aligned}\frac{W_{1,p}^{1-2p}}{2p}&\int_0^L \sum_{i=1}^n \left(\frac{\partial\lambda}{\partial\mathbf{u}}.\mathbf{u}_x + \partial_x\lambda_i\right) f_i(x)^p u_i^{2p} e^{-2p\mu s_i x}dx \\ &\leq \frac{C_1}{2p}W_{1,p} + \frac{C_2}{2p}W_{1,p}|\mathbf{u}|_1,\end{aligned} \tag{2.5.27}$$

where C_1 and C_2 are constants that depend on η and the system but are independent from p and $\mathbf{u}$ provided that $|\mathbf{u}|_1 < \eta$. Besides we have

$$W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^n f_i^p u_i^{2p-1}\left(\sum_{j=1}^n (V_{ik}(\mathbf{u},x).\mathbf{u})u_{kx}\right)e^{-2p\mu x s_i}dx \leq C_3 W_{1,p}|\mathbf{u}|_1. \tag{2.5.28}$$

where C_3 is a constant that does not depend on on p and $\mathbf{u}$. Therefore (2.5.26) can be written as

$$\frac{dW_{1,p}}{dt} \leq -I_2 - I_3 - \left(\mu\alpha_0 - \frac{C_1}{2p}\right)W_{1,p} + \left(\frac{C_2}{2p} + C_3\right)W_{1,p}|\mathbf{u}|_1. \tag{2.5.29}$$

As $\alpha_0 > 0$, it is easy to see that there exists $p_1 \in \mathbb{N}^*$ such that $\forall p \geq p_1$

$$\frac{dW_{1,p}}{dt} \leq -I_2 - I_3 - \frac{\mu\alpha_0}{2}W_{1,p} + C_4 W_{1,p}|\mathbf{u}|_1. \tag{2.5.30}$$

Here p_1 depends only on α_0 and η, while C_4 does not depend on p and $\mathbf{u}$. Before going any further, we see here that if we can manage to prove that $I_2 > 0$ and $I_3 \geq 0$ we may be able to conclude to the existence of a Lyapunov function that looks like a L^{2p} norm where p can be as large as we want and therefore we start to see the forecoming basic C^1 Lyapunov function. We are now left with studying I_2 and I_3 which will correspond respectively to the boundary condition and the interior condition we mentioned in Section 2 and in Theorem 2.4.1.

Let us first deal with I_3:

$$\mathrm{I}_3 = W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^n \left(f_i^p u_i^{2p-1}\left(\sum_{k=1}^n M_{ik}u_k\right) - \frac{\lambda_i f_i'}{2}f_i^{p-1}u_i^{2p}\right)e^{-2p\mu s_i x}dx. \tag{2.5.31}$$

Let suppose that the system (2.4.1) admits a positive solution $(g_1, ...g_n)$ on $[0, L]$, which is the interior condition. Then we can write this as

$$
-\Lambda_i g_i' = 2\left(\sum_{k=1,k\neq i}^{n} |M_{ik}(0,x)|\frac{g_i^{3/2}}{\sqrt{g_k}} - M_{ii}(0,x)g_i\right) + h_i, \tag{2.5.32}
$$

where h_i are non-negative functions. By continuity (see for instance [105], in particular Theorem 2.1 in Chapter 5) there exists $\sigma_1 > 0$ such that for all $\sigma \in [0, \sigma_1]$ there exists a unique solution to

$$
\begin{aligned}
-\Lambda_i f_i' =& 2\left(\sum_{k=1,k\neq i}^{n} |M_{ik}(0,x)|\frac{f_i^{3/2}}{\sqrt{f_k}} - M_{ii}(0,x)f_i\right) + h_i + \sigma,\\
f_i(0) =& g_i(0).
\end{aligned} \tag{2.5.33}
$$

We denote $(f_{1,\sigma}, ...f_{n,\sigma})$ this solution, which is continuous with σ. Therefore there exists $\sigma_2 \in (0, \sigma_1]$ such that for all $i \in [1, n]$, and all $\sigma \in (0, \sigma_2]$, $f_{i,\sigma} > 0$, on $[0, L]$ and

$$
-\Lambda_i f_{i,\sigma}' > 2\left(\sum_{k=1,k\neq i}^{n} |M_{ik}(0,x)|\frac{f_{i,\sigma}^{3/2}}{\sqrt{f_{k,\sigma}}} - M_{ii}(0,x)f_{i,\sigma}\right). \tag{2.5.34}
$$

We choose now $f_i := f_{i,\sigma}$ where $\sigma \in (0, \sigma_2]$. As M and λ are continuous in $\mathbf{u}$, there exists $\eta_2 > 0$ such that for $|\mathbf{u}|_0 < \eta_2$

$$
-\lambda_i(\mathbf{u}, x)\frac{f_i'}{f_i} > 2\sum_{k=1,k\neq i}^{n} |M_{ik}(\mathbf{u}, x)|\sqrt{\frac{f_i}{f_k}} - 2M_{ii}(\mathbf{u}, x). \tag{2.5.35}
$$

Therefore from Lemma 2.5.1

$$
\sum_{i=1}^{n}\left(-\frac{\lambda_i f_i'}{2f_i}y_i^{2p} + \sum_{j=1}^{n} M_{ik}\frac{\sqrt{f_i}}{\sqrt{f_k}}y_k y_i^{2p-1}\right) > 0,\ \forall\ \mathbf{y} = (y_i)_{i\in[1,n]} \in \mathbb{R}^n \setminus \{0\}, \tag{2.5.36}
$$

applying this for $(y_i)_{i\in[1,n]} = (\sqrt{f_i}u_i)_{i\in[1,n]}$, it implies that

$$
W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^{n}\left(-\frac{\lambda_i f_i'}{2}f_i^{p-1}u_i^{2p} + \sum_{k=1}^{n} M_{ik}u_k f_i^p u_i^{2p-1}\right)dx \geq 0. \tag{2.5.37}
$$

Therefore by continuity, there exists a $\mu_1 > 0$ such that $\forall \mu \in [0, \mu_1]$

$$
\mathrm{I}_3 = W_{1,p}^{1-2p}\int_0^L \sum_{i=1}^{n}\left(-\frac{\lambda_i f_i'}{2}f_i^{p-1}u_i^{2p} + f_i^p u_i^{2p-1}\left(\sum_{k=1}^{n} M_{ik}u_k\right)\right)e^{-2p\mu s_i x}dx > 0. \tag{2.5.38}
$$

Now let us deal with I_2, which will lead to the boundary condition. Recall that

$$
\begin{aligned}
\mathrm{I}_2 =& \frac{W_{1,p}^{1-2p}}{2p}\left[\sum_{i=1}^{n}\lambda_i(\mathbf{u}(t,L),L)f_i(L)^p u_i^{2p}(t,L)e^{-2p\mu s_i L}\right.\\
&\left.-\sum_{i=1}^{n}\lambda_i(\mathbf{u}(t,0),0)f_i(0)^p u_i^{2p}(t,0)\right].
\end{aligned} \tag{2.5.39}
$$

Recall that m is the integer such that $\Lambda_i > 0$, for all $i \leq m$ and $\Lambda_i < 0$, for all $i > m$, we have

$$\begin{aligned}
\mathrm{I}_2 =& \frac{W_{1,p}^{1-2p}}{2p}\left(\sum_{i=1}^{m}|\lambda_i(\mathbf{u}(t,L),L)|f_i(L)^p u_i^{2p}(t,L)e^{-2p\mu L}\right.\\
&-\sum_{i=1}^{m}|\lambda_i(\mathbf{u}(t,0),0)|f_i(0)^p u_i^{2p}(t,0)\\
&-\sum_{i=m+1}^{n}|\lambda_i(\mathbf{u}(t,L),L)|f_i(L)^p u_i^{2p}(t,L)e^{2p\mu L}\\
&\left.+\sum_{i=m+1}^{n}|\lambda_i(\mathbf{u}(t,0),0)|f_i(0)^p u_i^{2p}(t,0)\right),
\end{aligned} \tag{2.5.40}$$

We denote $K := G'(0)$ and we know that under assumption (2.4.2) there exists $\Delta = (\Delta_1, ..., \Delta_n)^T \in (\mathbb{R}_+^*)^n$ such that

$$\theta := \sup_{\|\xi\|_\infty \leq 1}(\sup_i(|\sum_{j=1}^{n}(\Delta_i K_{ij}\Delta_j^{-1})\xi_j|)) < \frac{\inf_i\left(\frac{g_i(d_i)}{\Delta_i^2}\right)}{\sup_i\left(\frac{g_i(L-d_i)}{\Delta_i^2}\right)}. \tag{2.5.41}$$

where $(g_i)_{i\in[1,n]}$ denote the positive solution of (2.4.1) introduced previously in (2.5.32). Note that we have in fact $\theta = \sup_i(\sum_{i=0}^{n}|K_{ij}|\frac{\Delta_i}{\Delta_j})$. Let:

$$\xi_i = \Delta_i u_i(t,L) \text{ for } i \in [1,m], \tag{2.5.42}$$

$$\xi_i = \Delta_i u_i(t,0) \text{ for } i \in [m+1,n]. \tag{2.5.43}$$

From (2.2.10) and using the fact that G is C^1, we have

$$\begin{pmatrix} u_+(t,0) \\ u_-(t,L) \end{pmatrix} = K\begin{pmatrix} u_+(t,L) \\ u_-(t,0) \end{pmatrix} + o\left(\left|\begin{pmatrix} u_+(t,L) \\ u_-(t,0) \end{pmatrix}\right|\right), \tag{2.5.44}$$

where $o(x)$ refers to a function such that $o(x)/|x|$ tends to 0 when $|\mathbf{u}|_0$ tends to 0. Thus we get

$$\begin{aligned}
\mathrm{I}_2 =& \frac{W_{1,p}^{1-2p}}{2p}\left(\sum_{i=1}^{m}\lambda_i(\mathbf{u}(t,L),L)\frac{f_i(L)^p}{\Delta_i^{2p}}(u_i(t,L)\Delta_i)^{2p}e^{-2p\mu L}\right.\\
&+\sum_{i=m+1}^{n}|\lambda_i(\mathbf{u}(t,0),0)|\frac{f_i(0)^p}{\Delta_i^{2p}}(u_i(t,0)\Delta_i)^{2p}\\
&-\sum_{i=1}^{m}\lambda_i(\mathbf{u}(t,0),0)\frac{f_i(0)^p}{\Delta_i^{2p}}(\sum_{k=1}^{n}K_{ik}\xi_k(t)\frac{\Delta_i}{\Delta_k}+o(\xi))^{2p}\\
&\left.-\sum_{i=m+1}^{n}|\lambda_i(\mathbf{u}(t,L),L)|\frac{f_i(L)^p}{\Delta_i^{2p}}(\sum_{k=1}^{n}K_{ik}\xi_k(t)\frac{\Delta_i}{\Delta_k}+o(\xi))^{2p}e^{2p\mu L}\right)
\end{aligned} \tag{2.5.45}$$

As the λ_i are C^1 in $\mathbf{u}$ we have

$$\begin{aligned}
\mathrm{I}_2 =& \frac{W_{1,p}^{1-2p}}{2p}\left(\sum_{i=1}^{m}(\Lambda_i(L)+O(\xi))\frac{f_i(L)^p}{\Delta_i^{2p}}(u_i(t,L)\Delta_i)^{2p}e^{-2p\mu L}\right.\\
&+\sum_{i=m+1}^{n}|(\Lambda_i(0)+O(\xi))|\frac{f_i(0)^p}{\Delta_i^{2p}}(u_i(t,0)\Delta_i)^{2p}\\
&-\sum_{i=1}^{m}(\Lambda_i(0)+O(\xi))\frac{f_i(0)^p}{\Delta_i^{2p}}(\sum_{k=1}^{n}K_{ik}\xi_k(t)\frac{\Delta_i}{\Delta_k}+o(\xi))^{2p}\\
&\left.-\sum_{i=m+1}^{n}|(\Lambda_i(L)+O(\xi))|\frac{f_i(L)^p}{\Delta_i^{2p}}(\sum_{k=1}^{n}K_{ik}\xi_k(t)\frac{\Delta_i}{\Delta_k}+o(\xi))^{2p}e^{2p\mu L}\right)
\end{aligned} \tag{2.5.46}$$

where $O(x)$ refers to a function such that $O(x)/|x|$ is bounded when $|\mathbf{u}|_0$ tends to 0. Now let $t \in [0,T]$, there exists i_0 such that $\max_i(\xi_i^2(t)) = \xi_{i_0}^2$, to simplify the notations we introduce d_i such that $d_i = L$ for $i \leq m$ and $d_i = 0$ for $i \geq m+1$. Then there exists a constant $C > 0$ independant of $\mathbf{u}$ and p such that

$$\begin{aligned} \mathrm{I}_2 \geq & \frac{W_{1,p}^{1-2p}}{2p}((|\Lambda_{i_0}(d_{i_0})| - C|\xi_{i_0}|)\frac{f_{i_0}^p(d_{i_0})}{\Delta_{i_0}^{2p}}\xi_{i_0}^{2p}(t)e^{-2p\mu d_{i_0}} \\ & - \sum_{i=1}^{n}(|\Lambda_i(L-d_i)| + C|\xi_{i_0}|)\frac{f_i^p(L-d_i)}{\Delta_i^{2p}}(\theta + l(\xi_{i_0}))^{2p}\xi_{i_0}^{2p}e^{2p\mu(L-d_i)}) \end{aligned} \tag{2.5.47}$$

where l is a continuous and positive function which satisfies $l(0) = 0$. thus

$$\begin{aligned} \mathrm{I}_2 \geq & \frac{W_{1,p}^{1-2p}}{2p}((|\Lambda_{i_0}(d_{i_0})| - C|\xi_{i_0}|)\frac{f_{i_0}^p(d_{i_0})}{\Delta_{i_0}^{2p}}\xi_{i_0}^{2p}(t)e^{-2p\mu d_{i_0}} \\ & - n\sup_{i\in[1,n]}\left((|\Lambda_i(L-d_i)| + C|\xi_{i_0}|)\frac{f_i^p(L-d_i)}{\Delta_i^{2p}}e^{2p\mu(L-d_i)}\right)(\theta + l(\xi_{i_0}))^{2p}\xi_{i_0}^{2p}) \end{aligned} \tag{2.5.48}$$

Now, from (2.4.2) we have

$$\theta^2 < \frac{\inf_i\left(\frac{g_i(d_i)}{\Delta_i^2}\right)}{\sup_i\left(\frac{g_i(L-d_i)}{\Delta_i^2}\right)}, \tag{2.5.49}$$

where $(g_i)_{i\in[1,n]}$ still denote the positive solution of (2.4.1). Remark that we set earlier $f_i := f_{i,\sigma}$ where $\sigma \in (0,\sigma_2]$ and can be chosen arbitrary small, and recall that the functions $f_{i,\sigma}$ are continuous in σ on this neighbourhood of 0. Therefore there exists $\sigma \in (0,\sigma_2]$ such that

$$\theta^2 < \frac{\inf_i\left(\frac{f_i(d_i)}{\Delta_i^2}\right)}{\sup_i\left(\frac{f_i(L-d_i)}{\Delta_i^2}\right)}. \tag{2.5.50}$$

But as the inequality is strict, there exist by continuity $\eta_3 \in (0,\eta_2)$, $p_3 > 0$ and μ_3 such that for all $|\mathbf{u}|_0 < \eta_3$ and $p > p_3$

$$(\theta + l(\xi_{i0}))^2 < \left(\frac{\inf_i|\Lambda_i(d_i)| - C|\xi_{i_0}|}{n\left(\sup_i|\Lambda_i(L-d_i)| + C|\xi_{i_0}|\right)}\right)^{1/p}\frac{\inf_i\left(\frac{f_i(d_i)}{\Delta_i^2}\right)}{\sup_i\left(\frac{f_i(L-d_i)}{\Delta_i^2}\right)}e^{-4\mu L},\ \forall\mu\in[0,\mu_3], \forall p \geq p_3. \tag{2.5.51}$$

Therefore from (2.5.51) and (2.5.48) $\mathrm{I}_2 > 0$. We can conclude that there exist p_4 and $\mu > 0$

$$\frac{dW_{1,p}}{dt} \leq -\frac{\mu\alpha_0}{2}W_{1,p} + C_4 W_{1,p}|\mathbf{u}|_1,\ \forall p \geq p_4. \tag{2.5.52}$$

We now have our first estimate and we have seen appear both an interior condition and a boundary condition that explains the conditions that appear in Theorem 2.4.1. Yet there remains a potentially non-negative term in $|\mathbf{u}|_1$ and the function we considered in (2.5.19) does not have the form of a basic C^1 Lyapunov function. The last step is now to convert $W_{1,p}$ in a basic C^1 Lyapunov function. Defining

$$W_{2,p} = \left(\int_0^L\sum_{i=1}^n f_i(x)^p(E\mathbf{u_t})_i^{2p}e^{-2p\mu s_i x}dx\right)^{1/2p}, \tag{2.5.53}$$

where $E = E(\mathbf{u}(t,x),x)$ is given by (2.3.8), and proceeding the same way and observing that, for C^2 solutions,

$$\mathbf{u}_{tt} + A(\mathbf{u},x)\mathbf{u}_{tx} + \left[\frac{\partial A}{\partial\mathbf{u}}(\mathbf{u},x).\mathbf{u}_t\right]\mathbf{u}_x + \frac{\partial B}{\partial\mathbf{u}}(\mathbf{u},x)\mathbf{u}_t = 0, \tag{2.5.54}$$

where $\partial A/\partial \mathbf{u}.\mathbf{u}_t$ refers to the matrix with coefficients $\sum_{k=1}^{n} \partial A_{ij}/\partial \mathbf{u}_k(\mathbf{u}, x).\partial_t \mathbf{u}_k(t, x)$, we can obtain similarly

$$\frac{dW_{2,p}}{dt} \leq -\frac{\mu\alpha_0}{2} W_{2,p} + C_5 W_{2,p} |\mathbf{u}|_1. \tag{2.5.55}$$

In order to avoid overloading this article, the proof -which is very similar to the proof of (2.5.52)- is given in the Appendix (see 2.8.1).
Now let us define $W_p := W_{1,p} + W_{2,p}$, there exists $\eta_4 > 0$ (independent of p), $\mu > 0$, C (independent of p and $\mathbf{u}$), and p_5 such that, with $|\mathbf{u}|_1 < \eta_4$,

$$\frac{dW_p}{dt} \leq -\frac{\mu\alpha_0}{2} W_p + C W_p |\mathbf{u}|_1, \ \forall p \geq p_5. \tag{2.5.56}$$

Here we see that this estimate does not depend on the C^2 norm of the solution $\mathbf{u}$ and of the C^2 norms of A and B and is therefore also true by density for solutions that are only of class C^1 and for A and B also only C^1. To be fully rigourous, this statement assumes the well-posedness of the system (2.2.5), (2.2.10), ($\mathbf{u} = \mathbf{u}^0$) in $W^{1,\infty}$ when $u_0 \in W^{1,\infty}([0, L])$, but such well posedness is true (see [142]). We choose such η, μ, p_5, and we define our basic C^1 Lyapunov function candidate

$$\begin{aligned} V := &|\sqrt{f_1} u_1 e^{-\mu x \frac{\lambda_1}{|\lambda_1|}}, ..., \sqrt{f_n} u_n e^{-\mu x \frac{\lambda_n}{|\lambda_n|}}|_0 \\ &+ |\sqrt{f_1} (E\mathbf{u_t})_1 e^{-\mu x \frac{\lambda_1}{|\lambda_1|}}, ..., \sqrt{f_n} (E\mathbf{u_t})_n e^{-\mu x \frac{\lambda_n}{|\lambda_n|}}|_0. \end{aligned} \tag{2.5.57}$$

Similarly to the method used in [48] we can first choose $\eta_5 < \min(\eta_1, \eta_2, \eta_3, \eta_4)$ such that for all $\eta < \eta_5$

$$|\mathbf{u}|_1 < \frac{\mu\alpha_0}{4C}. \tag{2.5.58}$$

Remark 2.5.1. *Recall that $|\mathbf{u}|_1 \leq \eta$ and that for convenience we are choosing η the bound on $|\mathbf{u}|_1$ instead of choosing ε, the bound on $|\mathbf{u}^0|_1$, but from (2.2.13) it is equivalent. Hence the previous only means choosing $\varepsilon_2 > 0$ small enough, and such that for all $\varepsilon < \varepsilon_2$*

$$|\mathbf{u}(0, \cdot)|_1 < \frac{\mu\alpha_0}{4C_1(T)C}, \tag{2.5.59}$$

where $C_1(T)$ is the constant defined in (2.2.13).

Therefore from (2.5.56) and (2.5.58)

$$\frac{dW_p}{dt} \leq -\frac{\mu\alpha_0}{4} W_p(t), \ \forall p \geq p_5. \tag{2.5.60}$$

Thus, using Gronwall Lemma, one has, for any $p \geq p_5$ and any $0 \leq t' \leq t \leq T$,

$$W_p(t) \leq W_p(t') e^{-\frac{\mu\alpha_0}{4}(t-t')}. \tag{2.5.61}$$

Then, by definitions of W_p and V

$$\lim_{p \to +\infty} W_p(t) = V^2(t), \ \forall t \in [0, T], \tag{2.5.62}$$

Therefore

$$V(t) \leq V(t') e^{-\frac{\mu\alpha_0}{8}(t-t')}, \ \forall \ 0 \leq t' \leq t \leq T. \tag{2.5.63}$$

Therefore V is a basic C^1 Lyapunov function with the associated constants $\gamma = \frac{\mu\alpha_0}{8}$ and $\eta = \eta_5$. □

Proof of Theorem 2.4.2

Proof. The sufficient way is simply proven by using Theorem 2.4.1 with $G \equiv 0$ for instance. We are left with proving the necessary way. Let us suppose that there exists a basic C^1 Lyapunov function V induced by coefficients $(f_1, ...f_n)$ and γ and η_1 the constants associated such that V is a Lyapunov function for all $\mathbf{u}$ smooth solution that satisfies the compatibility conditions and such that $|\mathbf{u}|_0 < \eta_1$. Suppose now by contradiction that the system (2.4.4) does not admit a solution $(g_1, ..., g_n)$ on $[0, L]$ such that for all $i \in [1, n]$, $g_i > 0$. Then there exist $x_0 \in [0, L]$ and $i_0 \in [1, n]$ such that

$$-\Lambda_{i_0}(x_0)f'_{i_0}(x_0) < 2\sum_{k=1,k\neq i_0}^{n} |M_{i_0k}(0,x_0)|\frac{f_{i_0}^{3/2}(x_0)}{\sqrt{f_k(x_0)}} - 2M_{i_0i_0}(0,x_0)f_{i_0}(x_0), \tag{2.5.64}$$

as, if not, $(f_1, ...f_n)$ would be a solution on $[0, L]$ to (2.4.4) with $f_i > 0$, for all $i \in [1, n]$. We can rewrite (2.5.64) simply as

$$-\sum_{k=1,k\neq i_0}^{n} |M_{i_0k}(0,x_0)|\frac{\sqrt{f_{i_0}(x_0)}}{\sqrt{f_k(x_0)}} - \frac{\Lambda_{i_0}(x_0)f'_{i_0}(x_0)}{2f_{i_0}(x_0)} + M_{i_0i_0}(0,x_0) < 0. \tag{2.5.65}$$

For simplicity we can assume without losing any generality that $i_0 = 1$. By continuity there exists $\varepsilon > 0$ such that (2.5.65) is true on $[x_0 - \varepsilon, x_0 + \varepsilon] \cap [0, L]$. We actually can suppose without loss of generality that $x_0 \in (0, L)$ and that $[x_0 - \varepsilon, x_0 + \varepsilon] \subset (0, L)$.

Then we take $u_1^0 \in (-\eta_2, \eta_2)$ positive, where η_2 is a positive constant arbitrary so far, and define the vector $\mathbf{u}^0$ by

$$u_i^0 := -u_1^0\left(1 - \frac{1}{k}\right)\operatorname{sgn}(M_{1i}(0,x_0)),\ \forall i \neq 1, \tag{2.5.66}$$

where $k \in \mathbb{N}^*$ is arbitrary and $\operatorname{sgn}(0) = 0$. As the system is strictly hyperbolic, $\min(|\lambda_i(x_0)|)$ is achieved at most for two $i \in [1, n]$. If so, we denote i_0 and i_1 the corresponding index, and if $i_0 \neq 1$ and $i_1 \neq 1$ we can redefine $u_{i_1}^0$ by

$$u_{i_0}^0 := -u_1^0\left(1 - \frac{1}{k_2}\right)\operatorname{sgn}(M_{1i_0}(0,x_0)), \tag{2.5.67}$$

where $k_2 \in \mathbb{N}^*$ with $k_2 > k$. The goal of this redefinition is that in both cases we can choose k large enough so that

$$(i \neq i_0) \Rightarrow \left|\frac{u_i^0}{\lambda_i(x_0)}\right| < \left|\frac{u_{i_0}^0}{\lambda_{i_0}(x_0)}\right|. \tag{2.5.68}$$

We now define the initial condition by

$$u_i(0,x) := \frac{u_i^0}{m}\chi(x)\frac{e^{-m(x-x_0)-c}}{\lambda_i(x)\sqrt{f_i(x)}}, \tag{2.5.69}$$

where $\chi : [0, L] \to \mathbb{R}$ is a C^∞ function with compact support in $(0, L)$ to be determined, such that $|\chi|_0$ is independent of $m \in \mathbb{N}^*$ which will be set large enough and c is a constant independent from m, also to be determined. In order to simplify the notations we will suppose here that $\lambda_1 > 0$, otherwise one only needs to replace $e^{-m(x-x_0)-c}$ by $e^{-\operatorname{sgn}(\lambda_1)(m(x-x_0)+c)}$ to obtain the same result. Note here that the compatibility conditions are satisfied for this initial condition as the function and its derivatives vanish on the boundaries. From (2.5.66) and (2.5.69), we can choose η_2 small enough and independent of m such that $|\mathbf{u}(0,\cdot)|_1 < \eta_1$. Well-posedness of the system guaranties the existence and uniqueness of a solution y to the system (2.2.5),(2.2.10) with such initial condition (see Theorem 2.2.1). For simplicity we will conduct the proof assuming that the system is linear, (i.e. $\lambda_i(\mathbf{u},\cdot) = \Lambda_i$, $a_{ij}(\mathbf{u},\cdot) = \delta_{ij}\Lambda_i(\cdot)$, $E(\mathbf{u},\cdot) = Id$, and $M(\mathbf{u},\cdot) = M(0,\cdot)$) although it is also not needed and is only to simplify the computations. A way to transform the proof for non-linear system is given in the Appendix (see 2.8.3).

Before going any further and selecting χ, we shall first give the idea and explain our strategy. We want to select χ such that $|\sqrt{f_1}\partial_t u_1(0,\cdot), ..., \sqrt{f_n}\partial_t u_n(0,\cdot)|_0$ is achieved for $i = 1$ and $x = x_1$ close to x_0 and only for such i and x_1. We also want $d/dt|\sqrt{f_1}u_1(0,\cdot), ..., \sqrt{f_n}u_n(0,\cdot)|_0(0)$ to exist and to be $O\left(d/dt\left(\sqrt{f_1(x_0)}\partial_t u_1(0,x_0)/m\right)\right)$ such that $dV/dt(0)$ will exist and its sign will be given by the sign of $\sqrt{f_1(x_0)}\partial_{tt}^2 u_1(0,x_0)$. Then we will show that this sign is positive.

Now let us select χ in order to achieve these goals. Rephrasing our first objective, we want that for all $i \neq 1$

$$\sqrt{f_1(x_1)}\left|\lambda_1(x_1)\partial_x u_1(0,x_1) + \sum_{j=1}^{n} M_{1j}u_j(0,x_1)\right| > \sup_{x\in[0,L]}\left(\sqrt{f_i(x)}\left|\lambda_i(x)\partial_x u_i(0,x) + \sum_{j=1}^{n} M_{ij}u_j(0,x)\right|\right), \tag{2.5.70}$$

while the maximum of $\sqrt{f_1}|\lambda_1\partial_x u_1(0,\cdot) + \sum_{j=1}^{n} M_{1j}u_j(0,\cdot)|$ is achieved only in x_1, close to x_0.
We search χ under the form

$$\chi = \phi(m(x - x_0)), \tag{2.5.71}$$

where ϕ is a positive C^∞ function with compact support. And we search χ such that all the $\left|\sqrt{f_i}\partial_t u_i(0,\cdot)\right|$ admit their maximum at a single point in a small neighbourhood of x_0. In that case note that from (2.5.66) we would indeed get that for m large enough $|\sqrt{f_1}\partial_t u_1(0,\cdot), ..., \sqrt{f_n}\partial_t u_n(0,\cdot)|_0$ is attained for $i = 1$ only and at a single point close to x_0. This will be shown rigorously later (see (2.5.79)). Now let us look at $\sqrt{f_i}\partial_t u_i(0,\cdot)$

$$\begin{aligned}\sqrt{f_i}\partial_t u_i(0,x) = &- u_i^0 e^{-m(x-x_0)-c}\Bigg[-\chi(x) + \frac{\chi'(x)}{m} + \frac{\chi(x)\lambda_i\sqrt{f_i}}{m}\left(\frac{1}{\lambda_i\sqrt{f_i}}\right)' \\ &+\frac{1}{m}\sum_{j=1}^{n} M_{ij}\left(\frac{u_j^0}{u_i^0}\right)\left(\sqrt{\frac{f_i}{f_j}}\right)\frac{1}{\lambda_j}\chi(x)\Bigg].\end{aligned} \tag{2.5.72}$$

Using (2.5.71) and a change of variable $y = m(x - x_0)$, (2.5.72) becomes

$$\begin{aligned}\sqrt{f_i}\partial_t u_i(0,x) = &- u_i^0 e^{-y-c}\Bigg[- \phi(y) + \phi'(y) \\ &+\left(\frac{g_i(\frac{y}{m} + x_0)}{m} + \sum_{j=0}^{n}\frac{f_{ij}(\frac{y}{m} + x_0)}{m}\right)\phi(y)\Bigg],\end{aligned} \tag{2.5.73}$$

where g_i and f_{ij} are C^2 bounded functions on $[0, L]$ independent of m. This comes from the fact that A and B are of class C^3. This hypothesis, that does not appear in Theorem 2.4.1, is used to apply the implicit function theorem later on (see (2.5.77) and (2.5.82)). Theorem 2.4.2 might also be proven with lower hypothesis on the regularity A and B, however in most physical case A and B are C^3 even when the solutions of the system are much less regular. We can see that the coefficients of the equation (2.5.73) in ϕ and ϕ' depend on m and are close to be constant for large m. One can show that there exists a function ψ_0 such that $\psi_0 \in C_c^3((-1,1))$, such that $|(\psi_0(y) - \psi_0'(y))e^{-y}|$ has a unique maximum on $[-1,1]$ which is 1, and such that the second derivative of $|(\psi_0(y) - \psi_0'(y))e^{-y}|$ does not vanish in this point, i.e. there exists a unique $y_1 \in (-1,1)$ such that

$$|\psi_0(y) - \psi_0'(y)|\,e^{-y} < 1 = |\psi_0(y_1) - \psi_0'(y_1)|\,e^{-y_1},\ \forall y \in [-1,1]\setminus\{y_1\}, \tag{2.5.74}$$

$$\left(|\psi_0 - \psi_0'|\,e^{-Id}\right)''(y_1) \neq 0. \tag{2.5.75}$$

The existence of this function ψ_0 is shown in the Appendix (see 2.8.2). We set $\phi : y \to \psi_0(y+y_1)$ and $c = y_1$. Therefore

$$e^{-y-c}\left[-\phi(y) + \phi'(y)\right] = (-\psi_0(y + y_1) + \psi_0'(y + y_1))e^{-(y+y_1)}, \tag{2.5.76}$$

which has a maximum absolute value for $y = 0$ with value equal to 1. Hence, there exists $m_1 > 0$ such that for all $m > m_1$ and all $i \in [1, n]$

$$\exists! x_i \in [x_0 - \varepsilon, x_0 + \varepsilon] : |\sqrt{f_i(x_i)}\partial_t u_i(0, x_i)| = \sup_{[0,L]}(|\sqrt{f_i}\partial_t u_i(0, \cdot)|), \tag{2.5.77}$$

$$-u_i^0 - \frac{C_i}{m}|u_i^0| \leq \sqrt{f_i(x_i)}\partial_t u_i(0, x_i) \leq -u_i^0 + \frac{C_i}{m}|u_i^0|, \tag{2.5.78}$$

where C_i are constants that do not depend on m. The unicity in (2.5.77) comes from the condition (2.5.75) which ensures that the maximum stays unique when the function is slightly perturbated. We can actually replace C_i by $C = \max_i(C_i) > 0$. Therefore, there exists $m_2 > m_1$ such that for all $m > m_2$ and $i \in [2, n]$

$$\sup_{[0,L]}(|\sqrt{f_i}\partial_t u_i(0, \cdot)|) \leq (1 - \frac{1}{k})\left(1 + \frac{C}{m}\right) u_1^0 < u_1^0 \left(1 - \frac{C}{m}\right) \leq \sup_{[0,L]}(|\sqrt{f_1}\partial_t u_1(0, \cdot)|). \tag{2.5.79}$$

Hence, as we announced earlier,

$$|\sqrt{f_1}\partial_t u_1(0, \cdot), ..., \sqrt{f_n}\partial_t u_n(0, \cdot)|_0 = |\sqrt{f_i}\partial_t u_i(0, x)| \Longleftrightarrow i = 1, x = x_1. \tag{2.5.80}$$

Hence, as $u_1^0 > 0$ and from (2.5.77) and (2.5.78),

$$|\sqrt{f_1}\partial_t u_1(0, \cdot), ..., \sqrt{f_n}\partial_t u_n(0, \cdot)|_0 = -\sqrt{f_1(x_1)}\partial_t u_1(0, x_1). \tag{2.5.81}$$

Therefore, as the maximum is unique and the inequality of (2.5.79) is strict, and from (2.5.75) and the implicit function theorem, provided that m is large enough there exist $t_1 > 0$ and $x_a \in C^1([0, t_1]; [0, L])$ such that

$$\begin{aligned} &|\sqrt{f_1}\partial_t u_1(t, \cdot), ..., \sqrt{f_n}\partial_t u_n(t, \cdot)|_0 = -\sqrt{f_1(x_a(t))}\partial_t u_1(t, x_a(t)),\ \forall t \in [0, t_1], \\ &x_a(0) = x_1. \end{aligned} \tag{2.5.82}$$

We seek now to obtain a similar relation for $|\sqrt{f_1}u_1(t, \cdot), ..., \sqrt{f_n}u_n(t, \cdot)|_0$. One can show that it is possible to find ψ_0 that satisfies the previous hypothesis (2.5.74) and (2.5.75) and such that in addition, there exists $y_2 \in [-1, 1]$ such that

$$|\psi_0(y)|\, e^{-y} < |\psi_0(y_2)|\, e^{-y_2},\ \forall y \in [-1, 1] \setminus \{y_2\}, \tag{2.5.83}$$

$$|\psi_0(y_2) - \psi_0''(y_2)| > 0, \tag{2.5.84}$$

and such that there exists $m_3 > 0$ such that for all $m > m_3$, if $\sup_{y\in[-1,1]}(\psi_0(y + y_1)\frac{e^{-(y+y_1)}}{\lambda_i(\frac{y}{m}+x_0)})$ is achieved in $y_m \in [-1, 1]$, then

$$|\psi_0(y_m + y_1) - \psi_0''(y_m + y_1)| > c_1, \tag{2.5.85}$$

where c_1 is a positive constant that does not depend on m. The example of ψ_0 provided in the Appendix is suitable. Thus with $h_i(l, y) = \frac{u_i^0}{\lambda_i(yl+x_0)}\phi(y)e^{-y-y_1}$ one has:

$$\partial_y h_i(0, y_2 - y_1) = 0. \tag{2.5.86}$$

Note that from (2.5.83), $\psi_0(y_2) = \psi_0'(y_2)$, thus from (2.5.84)

$$|\partial_{yy} h_i(0, y_2 - y_1)| > 0. \tag{2.5.87}$$

Therefore from the implicit function theorem, there exists $m_4 > m_3$ such that for all $m > m_4$ and each $i \in [1, n]$ there exists a unique $y_i \in [-1 - y_1, 1 - y_1]$ such that

$$\partial_y h_i\left(\frac{1}{m}, y_i\right) = 0, \tag{2.5.88}$$

$$|y_i - (y_2 - y_1)| \leq \frac{C_a}{m}, \tag{2.5.89}$$

where C_a is a constant independent of m. From (2.5.68) there exists $m_5 > m_4$ such that for all $m > m_5$,

$$\left|\frac{u_i^0}{\lambda_i\left(\frac{y_i}{m}+x_0\right)}\right| C_b < \left|\frac{u_{i_0}^0}{\lambda_{i_0}\left(\frac{y_{i_0}}{m}+x_0\right)}\right|,\ \forall\, i \neq i_0, \tag{2.5.90}$$

where $C_b > 1$ is a constant independent of m. From (2.5.89), we have for any $i \in [1, n]$

$$\left|\frac{\phi(y_{i_0})e^{-y_{i_0}}}{\phi(y_i)e^{-y_i}}\right| \geq 1 - \frac{C_r}{m}, \tag{2.5.91}$$

where C_r is a constant independent of m. Therefore there exists $m_6 > m_5$ such that for all $m > m_6$

$$\left|\frac{u_i^0}{\lambda_i\left(\frac{y_i}{m}+x_0\right)}\phi(y_i)e^{-y_i}\right| \frac{(1+C_b)}{2} < \left|\frac{u_{i_0}^0}{\lambda_{i_0}\left(\frac{y_{i_0}}{m}+x_0\right)}\phi(y_{i_0})e^{-y_{i_0}}\right|,\ \forall\, i \neq i_0. \tag{2.5.92}$$

This means that for all $m > m_6$ there exists a unique $i_0 \in [1, n]$ and a unique $x_{a_0} \in [x_0 - \varepsilon, x_0 + \varepsilon]$ such that

$$|\sqrt{f_{i_0}(x_{a_0})}u_{i_0}(0, x_{a_0})| = \sup_{i\in[1,n],x\in[0,L]} |\sqrt{f_i}u_i(0, \cdot)|. \tag{2.5.93}$$

Now if we denote $g(t, x) := \partial_x(\sqrt{f_{i_0}(x)}u_{i_0}(t, x)\operatorname{sgn}(u_{i_0}(0, x_{a_0})))$, one has that

$$g(0, x_{a_0}) = 0, \tag{2.5.94}$$

hence

$$\frac{-\lambda_{i_0}'(x_{a_0})}{m\lambda_{i_0}(x_{a_0})}\chi(x_{a_0}) + \frac{\chi'(x_{a_0})}{m} = \chi(x_{a_0}). \tag{2.5.95}$$

Therefore

$$\begin{aligned}\partial_x g(0, x_{a_0}) = &-\operatorname{sgn}(\lambda_{i_0})\frac{|u_{i_0}^0|}{m}e^{-m(x_{a_0}-x_0)-y_1}\left(\left(\frac{1}{\lambda_{i_0}}\right)''(x_{a_0})\chi(x_{a_0}) + \chi''(x_{a_0})\frac{1}{\lambda_{i_0}(x_{a_0})}\right.\\ &+2\chi'(x_{a_0})\left(\frac{1}{\lambda_{i_0}}\right)'(x_{a_0}) - m\left(\frac{\chi}{\lambda_{i_0}}\right)'(x_{a_0})\\ &\left.-m\left(\left(\frac{1}{\lambda_{i_0}}\right)'(x_{a_0})\chi(x_{a_0}) - m\frac{\chi(x_{a_0})}{\lambda_{i_0}(x_{a_0})} + \chi'(x_{a_0})\frac{1}{\lambda_{i_0}(x_{a_0})}\right)\right).\end{aligned} \tag{2.5.96}$$

Defining $c_{i0} := -\operatorname{sgn}(\lambda_{i_0})|u_{i_0}^0|$ which is a non-zero constant, we have from (2.5.71) and the definition of ϕ

$$\partial_x g(0, x_{a_0}) = c_{i0}m\frac{e^{-y_{i_0}-y_1}}{\lambda_{i_0}\left(\frac{y_{i_0}}{m}+x_0\right)}\left(\psi_0''(y_{i_0}+y_1) - 2\psi_0'(y_{i_0}+y_1) + \psi_0(y_{i_0}+y_1) + O\left(\frac{1}{m^2}\right) + O\left(\frac{1}{m}\right)\right). \tag{2.5.97}$$

Observe that, by definition, y_{i_0} maximises $\left|\psi_0(y+y_1)\frac{e^{-y-y_1}}{\lambda_{i_0}(\frac{y}{m}+x_0)}\right|$, therefore we have from (2.5.85) and (2.5.95)

$$\begin{aligned}|\partial_x g(0, x_{a_0})| =&|c_{a_0}|m\left|\frac{e^{-y_{i_0}-y_1}}{\lambda_{i_0}\left(\frac{y_{i_0}}{m}+x_0\right)}\left(\psi_0''(y_{i_0}+y_1) - \psi_0(y_{i_0}+y_1) + O\left(\frac{1}{m}\right)\right)\right|,\\ =&|c_{i_0}|m\left|\frac{e^{-y_{i_0}-y_1}}{\lambda_{i_0}(\frac{y_{i_0}}{m}+x_0)}\right|\left(c_1 + O\left(\frac{1}{m}\right)\right).\end{aligned} \tag{2.5.98}$$

Hence, as the inequality (2.5.92) is strict and from the implicit function theorem, there exists $m_7 > m_6$ such that for all $m > m_7$, $x_b \in C^1([0, t_2]; [0, L])$ and $i_0 \in [1, n]$ such that

$$|\sqrt{f_1}u_1(t, \cdot), ..., \sqrt{f_n}u_n(t, \cdot)|_0 = \sqrt{f_{i_0}(x_b(t))}u_{i_0}(t, x_b(t))\operatorname{sgn}(u_{i_0}(0, x_{a_0})),\ \forall t \in [0, t_2],\ x_b(0) = x_{a_0}. \tag{2.5.99}$$

Hence V is C^1 on $[0,t_3)$ where $t_5 = \min(t_1,t_2) > 0$ and, denoting $s_{a_0} := \operatorname{sgn}(u_{i_0}(0,x_{a_0}))$, we have from the definition of V, (2.5.82) and (2.5.99)

$$\begin{aligned}\frac{dV}{dt}(0) = &-\sqrt{f_1(x_1)}\partial_{tt}u_1(0,x_1) - \frac{\partial}{\partial x}(\sqrt{f_1}\partial_t u_1(0,\cdot))(x_1)\frac{dx_a}{dt}(0)\\ &+ s_{a_0}\left(\sqrt{f_{i_0}(x_{a_0})}\partial_t u_{i_0}(t,x_{a_0}) + \frac{\partial}{\partial x}\left(\sqrt{f_{i_0}}u_{i_0}(0,\cdot)\right)(x_{a_0})\frac{dx_b}{dt}(0)\right).\end{aligned} \tag{2.5.100}$$

But now observe that for a fixed m, x_{a_0} is an interior maximum thus

$$\frac{d}{dx}(\sqrt{f_{i_0}}u_{i_0}(0,\cdot))(x_{a_0}) = 0. \tag{2.5.101}$$

Also as $\frac{d}{dx}(\sqrt{f_1}\partial_t u_1(0,\cdot))(x_1) = 0$, we have

$$\frac{dV}{dt}(0) = -\sqrt{f_1(x_1)}\partial^2_{tt}u_1(0,x_1) + s_{a_0}\sqrt{f_{i_0}(x_a)}\partial_t u_{i_0}(t,x_{a_0}). \tag{2.5.102}$$

Besides as ϕ has compact support in $[-1-y_1, 1-y_1]$, we have

$$\left|e^{m(x-x_0)+y_1}\chi(x)\right| \leq e^1\|\chi\|_\infty, \tag{2.5.103}$$

and the right-hand side does not depend on m, thus

$$\lim_{m\to+\infty}\left|\frac{e^{m(x-x_0)+y_1}}{m}\chi(x)\right| = 0, \tag{2.5.104}$$

uniformally on $[0,L]$ and therefore in particular for x_{a_0} (even though x_{a_0} might depend on m). We denote

$$V_2 := -\sqrt{f_1(x_a(t))}\partial_t u_1(t,x_a(t)). \tag{2.5.105}$$

Using (2.5.69) and $\frac{d}{dx}(\sqrt{f_1}\partial_t u_1(0,\cdot))(x_1) = 0$, we have

$$\begin{aligned}\frac{dV_2}{dt}(0) &= -\sqrt{f_1(x_1)}\partial^2_{tt}u_1(0,x_1)\\ &= -\sqrt{f_1(x_1)}\partial_t(-\lambda_1\partial_x u_1(\cdot,x_1) - \sum_{j=1}^n M_{1j}u_j(\cdot,x_1))(0)\\ &= -\sqrt{f_1(x_1)}(-\lambda_1\partial_x(\partial_t u_1(0,x_1)) - \sum_{j=1}^n M_{1j}\partial_t u_j(0,x_1))\\ &= -\sqrt{f_1(x_1)}(\lambda_1\frac{(\sqrt{f_1})'}{\sqrt{f_1}}\partial_t u_1(0,x_1) - \sum_{j=1}^n M_{1j}\partial_t u_j(0,x_1))\\ &= -\sqrt{f_1(x_1)}(\frac{\lambda_1 f_1'}{2f_1}\partial_t u_1(0,x_1) - \sum_{j=1}^n M_{1j}\partial_t u_j(0,x_1)).\end{aligned} \tag{2.5.106}$$

And from (2.5.72) and (2.5.78)

$$\begin{aligned}\frac{dV_2}{dt}(0) =& u_1^0\left(\frac{\lambda_1 f_1'}{2f_1}\left(1+O\left(\frac{1}{m}\right)\right)\right.\\ &\left.-\sum_{j=1}^n M_{1j}(0,x_1)\frac{u_j^0}{u_1^0}\frac{\sqrt{f_1(x_1)}}{\sqrt{f_j(x_1)}}\left(1+O\left(\frac{1}{m}\right)+\frac{\sqrt{f_j}(x_j)\partial_t u_j(x_j)-\sqrt{f_j}(x_1)\partial_t u_j(x_1)}{u_j^0}\right)\right).\end{aligned} \tag{2.5.107}$$

We know that if $M_{1j}(0,x_0) \neq 0$, then there exists $m_8 \in \mathbb{N}^*$ such that for all $m > m_8$, $\text{sgn}(M_{1j}(0,x_0)) = \text{sgn}(M_{1j}(0,x_1))$. We denote by $\mathcal{N}$ the subset of $j \in \{1,...,n\}$ such that $M_{1j}(0,x_0) = 0$. Therefore from (2.5.107) and (2.5.66)

$$\begin{aligned}\frac{dV_2}{dt}(0) =&u_1^0\left(\left[\frac{\lambda_1 f_1'}{2f_1} - M_{11}(0,x_1) + \sum_{j=2, j\in\mathcal{N}^c}^{n} |M_{1j}(0,x_1)|\left(1-\frac{1}{k}\right)\frac{\sqrt{f_1}}{\sqrt{f_j}}\right]\right.\\ &\left.+O\left(\frac{1}{m}\right) + \sum_{j=0}^{n} C_j\left(\frac{\sqrt{f_j}(x_j)\partial_t u_j(x_j) - \sqrt{f_j}(x_1)\partial_t u_j(x_1)}{u_j^0}\right)\right),\end{aligned} \tag{2.5.108}$$

where C_j are constants that do not depend on m. Now, keeping in mind (2.5.102), we are going to add $s_{a_0}\sqrt{f_{i_0}(x_{a_0})}\partial_t u_{i_0}(0,x_{a_0})$ to obtain dV/dt at $t=0$. But first observe that using (2.5.101) and (2.5.103)

$$\begin{aligned}\sqrt{f_{i_0}(x_{a_0})}\partial_t u_{i_0}(0,x_{a_0}) &= \sqrt{f_{i_0}(x_{a_0})}(-\lambda_i\partial_x u_{i_0}(0,x_{a_0}) - \sum_{j=1}^{n} M_{i_0 j}u_j(0,x_{a_0}))\\ &= \sqrt{f_{i_0}(x_{a_0})}\left(\lambda_i\frac{(\sqrt{f_{i_0}})'(x_{a_0})}{\sqrt{f_{i_0}(x_{a_0})}}\frac{u_{i_0}^0}{m}\chi(x_{a_0})\frac{e^{-m(x_{a_0}-x_0)-y_1}}{\lambda_i\sqrt{f_{i_0}(x_{a_0})}} - \sum_{j=1}^{n} M_{i_0 j}\frac{u_j^0}{m}\chi(x_{a_0})\frac{e^{-m(x_{a_0}-x_0)-y_1}}{\lambda_i\sqrt{f_i(x_{a_0})}}\right)\\ &= O\left(\frac{1}{m}\right).\end{aligned} \tag{2.5.109}$$

Therefore

$$\begin{aligned}\frac{dV}{dt}(0) =&\frac{dV_2}{dt}(0) + O\left(\frac{1}{m}\right)\\ =&\, u_1^0\left(\left[\frac{\lambda_1 f_1'}{2f_1} - M_{11}(0,x_1) + \sum_{j=2, j\in\mathcal{N}^c}^{n} |M_{1j}(0,x_1)|\left(1-\frac{1}{k}\right)\frac{\sqrt{f_1}}{\sqrt{f_j}}\right]\right.\\ &\left.+O\left(\frac{1}{m}\right) + \sum_{j=0}^{n} C_j\left(\frac{\sqrt{f_j}(x_j)\partial_t u_j(x_j) - \sqrt{f_j}(x_1)\partial_t u_j(x_1)}{u_j^0}\right)\right) + O\left(\frac{1}{m}\right).\end{aligned} \tag{2.5.110}$$

And from (2.5.72) and the definition of x_j

$$\lim_{m\to+\infty}\left(\frac{\sqrt{f_j}(x_j)\partial_t u_j(x_j) - \sqrt{f_j}(x_1)\partial_t u_j(x_1)}{u_j^0}\right) = 0. \tag{2.5.111}$$

Note that x_1 and x_j both depend on m and tend to x_0 when m goes to infinity. Also we know that for all $m > m_2$, we have $x_1 \in [x_0-\varepsilon, x_0+\varepsilon]$. Thus from (2.5.65),

$$\lim_{m\to+\infty}\left[\frac{\lambda_1(x_1)f_1'(x_1)}{2f_1(x_1)} - M_{11}(0,x_1) + \sum_{j=2, j\in\mathcal{N}^c}^{n}|M_{1j}(0,x_1)|\left(1-\frac{1}{k}\right)\frac{\sqrt{f_1(x_1)}}{\sqrt{f_j(x_1)}}\right] > 0. \tag{2.5.112}$$

Therefore there exists $m_9 > 0$ such that for all $m > m_9$

$$\frac{dV}{dt}(0) > 0. \tag{2.5.113}$$

But we know from (2.3.14) that

$$\frac{dV}{dt}(0) \leq -\gamma V(0) < 0. \tag{2.5.114}$$

Note that (2.5.114) is true as V is C^1 in $[0,t_1)$ and from (2.3.14), for any $t\in[0,t_1)$,

$$\frac{V(t)-V(0)}{t}\leq V(0)\frac{e^{-\gamma t}-1}{t} \tag{2.5.115}$$

which, letting $t\to 0$, gives (2.5.114) and a contradiction. This ends the proof of Theorem 2.4.2. □

2.6 Further details

The previous results were derived for the C^1 norm but actually they can be extended to the C^p norm, for $p\in\mathbb{N}^*$, with the same conditions. Namely we can extend the definition of *basic C^p Lyapunov function* for $p\in\mathbb{N}^*$ by replacing V in Definition 2.3.3 by

$$V(\mathbf{u}(t,\cdot))=\sum_{k=0}^{p}\left|\sqrt{f_1}(E\partial_t^k\mathbf{u}(t,\cdot))_1,...,\sqrt{f_n}(E\partial_t^k\mathbf{u}(t,\cdot))_n\right|_0. \tag{2.6.1}$$

Defining the $p-1$ compatibility conditions as in [13] at (4.136) (see also (4.137)-(4.142)), the well-posedness still holds [13] and we can state:

Theorem 2.6.1. *Let a quasilinear hyperbolic system be of the form (2.2.5),(2.2.10), with A and B of class C^p, Λ defined as in (2.2.4) and M as in (2.3.7), if*

1. *(Interior condition) the system*

$$\Lambda_i f_i'\leq -2\left(-M_{ii}(0,x)f_i+\sum_{k=1,k\neq i}^{n}|M_{ik}(0,x)|\frac{f_i^{3/2}}{\sqrt{f_k}}\right), \tag{2.6.2}$$

admits a solution $(f_1,...,f_n)$ on $[0,L]$ such that for all $i\in[1,n]$, $f_i>0$,

2. *(Boundary condition) there exists a diagonal matrix Δ with positive coefficients such that*

$$\|\Delta G'(0)\Delta^{-1}\|_\infty<\frac{\inf_i\left(\frac{f_i(d_i)}{\Delta_i^2}\right)}{\sup_i\left(\frac{f_i(L-d_i)}{\Delta_i^2}\right)}, \tag{2.6.3}$$

where $d_i=L$ if $\Lambda_i>0$, and $d_i=0$ otherwise.

Then there exists a basic C^p Lyapunov function for the system (2.2.5),(2.2.10).

Theorem 2.6.2. *Let a quasilinear hyperbolic system be of the form (2.2.5) with A and B of class C^{p+2}, there exists a control of the form (2.2.10) such that there exists a basic C^p Lyapunov function if and only if*

$$\Lambda_i f_i'\leq -2\left(-M_{ii}(0,x)f_i+\sum_{k=1,k\neq i}^{n}|M_{ik}(0,x)|\frac{f_i^{3/2}}{\sqrt{f_k}}\right), \tag{2.6.4}$$

admits a solution $(f_1,...,f_n)$ on $[0,L]$ such that for all $i\in[1,n]$, $f_i>0$.

A proof of this is included in the Appendix (see 2.8.4).

This article therefore fills the blank about the exponential stability for the C^p norm for quasilinear hyperbolic systems with non-zero source term using a Lyapunov approach, for any $p\in\mathbb{N}^*$.
We introduced the notion of *basic C^1 Lyapunov function* that can be seen as natural Lyapunov function for the C^1 norm. For general quasilinear hyperbolic systems we gave a sufficient interior condition on the system and a sufficient boundary condition such that there exists a basic C^1 Lyapunov function that ensure exponential stability of the system for the C^1 norm. We also showed that the interior condition is necessary for the existence of such basic C^1 Lyapunov function. Therefore in some cases, there cannot exist such basic C^1 Lyapunov function whatever the boundary conditions are.

2.7 Acknowledgement

The author would like to thank his advisor Jean-Michel Coron for suggesting this problem, for his constant support, and for many fruitful discussions. The author would also like to thank Georges Bastin, Sébastien Boyaval, Nicole Goutal, Peipei Shang, Shengquan Xiang and Christophe Zhang for their valuable remarks, and many interesting discussions. The author would also like to thank the ETH - FIM and the ETH - ITS for their support and their warm welcome. Finally the author would like to thank the ANR project Finite 4SoS ANR 15-CE23-0007.

2.8 Appendix

2.8.1 Bound on the derivative of $W_{2,p}$

2.8.1.1 Derivative of $W_{2,p}$

Recall that we have from (2.5.53)

$$W_{2,p} = \left(\int_0^L \sum_{i=1}^n f_i(x)^p (E\mathbf{u}_t)_i^{2p} e^{-2p\mu s_i x} dx\right)^{1/2p},$$

where $E = E(\mathbf{u}(t,x),x)$ given by (2.3.8)–(2.3.9) and that $\mathbf{u}_t$ satisfies the following equation

$$\mathbf{u}_{tt} + A(\mathbf{u},x)u_{tx} + \left[\frac{\partial A}{\partial \mathbf{u}}(\mathbf{u},x).\mathbf{u}_t\right]\mathbf{u}_x + \frac{\partial B}{\partial \mathbf{u}}(\mathbf{u},x)\mathbf{u}_t = 0, \tag{2.8.1}$$

where $\partial A/\partial \mathbf{u}.u_t$ is the matrix with coefficients $\sum_{k=1}^n \partial A_{ij}/\partial \mathbf{u}_k(\mathbf{u},x).\partial_t \mathbf{u}_k(t,x)$. We can again differentiate $W_{2,p}$ with respect to time along the trajectories which are of class C^2 (recall that we are proving the estimate (2.5.56) for C^2 solutions first). Using integration by parts as previously:

$$\begin{aligned}
\frac{dW_{2,p}}{dt} =& -\frac{W_{2,p}^{1-2p}}{2p}\left[\sum_{i=1}^n \lambda_i f_i(x)^p (E\mathbf{u}_t)_i^{2p} e^{-2p\mu s_i x}\right]_0^L \\
&- W_{2,p}^{1-2p}\int_0^L \sum_{i=1}^n f_i(x)^p (E\mathbf{u}_t)_i^{2p-1}\left[\left(E\left(D_a + \frac{\partial B}{\partial \mathbf{u}}(\mathbf{u},x)\right).\mathbf{u}_t\right)_i\right. \\
&\left.-\left(\left(\frac{\partial E}{\partial \mathbf{u}}.\mathbf{u}_t\right)\mathbf{u}_t + \lambda\left(\frac{\partial E}{\partial \mathbf{u}}.\mathbf{u}_x\right)\mathbf{u}_t + \lambda(\partial_x E)\mathbf{u}_t\right)_i\right] e^{-2p\mu s_i x} dx \\
&+ \frac{W_{2,p}^{1-2p}}{2}\int_0^L \sum_{i=1}^n \Big(\lambda_i(\mathbf{u},x) f_i(x)^{p-1} f_i'(x)(E\mathbf{u}_t)_i^{2p} \\
&+ \frac{d}{dx}\frac{(\lambda_i(\mathbf{u},x))}{p} f_i(x)^p (E\mathbf{u}_t)_i^{2p}\Big) e^{-2p\mu s_i x} dx \\
&- \mu W_{2,p}^{1-2p}\int_0^L \sum_{i=1}^n |\lambda_i| f_i^p(x)(E\mathbf{u}_t)_i^{2p} e^{-2p\mu s_i x} dx,
\end{aligned} \tag{2.8.2}$$

where D_a is the matrix with coefficient $\sum_{k=1}^n (\partial A_{ik}/\partial u_j)(\mathbf{u}_x)_k$, so that $D_a.\mathbf{u}_t = \left[\frac{\partial A}{\partial \mathbf{u}}(\mathbf{u},x).\mathbf{u}_t\right]\mathbf{u}_x$. Observe that E is C^2 and invertible by definition (given by (2.3.8)–(2.3.9)), thus $\mathbf{u}_t = E^{-1}(E\mathbf{u}_t)$. We can therefore denote, similarly as previously

$$\mathrm{I}_{21} := \frac{W_{2,p}^{1-2p}}{2p}\left[\sum_{i=1}^n \lambda_i f_i(x)^p (E\mathbf{u}_t)_i^{2p} e^{-2p\mu s_i x}\right]_0^L, \tag{2.8.3}$$

and

$$
\begin{aligned}
\mathrm{I}_{31} =& W_{2,p}^{1-2p}\left(\int_0^L \sum_{i=1}^n f_i(x)^p (E\mathbf{u_t})_i^{2p-1}\left(\sum_{k=1}^n R_{ik}(\mathbf{u},x)(E\mathbf{u}_t)_k\right)e^{-2p\mu s_i x}dx\right)\\
&-\frac{W_{2,p}^{1-2p}}{2}\int_0^L\sum_{i=1}^n \lambda_i(\mathbf{u},x)f_i(x)^{p-1}f_i'(x)(E\mathbf{u}_t)_i^{2p}e^{-2p\mu s_i x}dx.
\end{aligned}
\tag{2.8.4}
$$

where $R=(R_{ij})_{(i,j)\in[1,n]^2}$ is defined as $R := E\left(D_a+\frac{\partial B}{\partial \mathbf{u}}\right)E^{-1}$. As E is C^1 and its inverse is continuous, and from (2.3.9), there exists a constant C_0 independant of $\mathbf{u}$ (and p) such that

$$
\max_{(i,j)\in[1,n]^2}\left|\left(\left(\frac{\partial E}{\partial \mathbf{u}}.\mathbf{u}_t\right)E^{-1}+\left(\frac{\partial E}{\partial \mathbf{u}}.\mathbf{u}_x\right)E^{-1}+(\partial_x E)E^{-1}\right)_{ij}\right|\leq C_0|\mathbf{u}|_1. \tag{2.8.5}
$$

Note that we used (2.3.9) and the fact that $\partial_x(E(\mathbf{0},x))=0$. Thus, similarly as for (2.5.30), we have

$$
\frac{dW_{2,p}}{dt}\leq -I_{21}-I_{31}-(\mu\alpha_0-\frac{C_6}{2p})W_{2,p}+C_5W_{2,p}|\mathbf{u}|_1, \tag{2.8.6}
$$

where C_5 and C_6 are constants that does not depend on p or $\mathbf{u}$ provided that $|\mathbf{u}|_1<\eta$ for η small enough but independent of p. Recall that α_0 is defined in Section 2.5 right before (2.5.26). Just as previously, a sufficient condition such that there exist $p_1\in\mathbb{N}^*$, $\eta_1>0$ and μ_1 such that $I_{31}>0$ for $\mu<\mu_1$, $p>p_1$ and $|\mathbf{u}|_1<\eta_1$ is

$$
-\lambda_i\frac{f_i'}{f_i}>2\sum_{k=1,k\neq i}^n|R_{ik}(\mathbf{u},x)|\sqrt{\frac{f_i}{f_k}}-2R_{ii}, \tag{2.8.7}
$$

But we have from the definition of D_a, (2.3.9) and (2.3.7):

$$
E\left(D_a+\frac{\partial B}{\partial \mathbf{u}}\right)E^{-1}=\frac{\partial B}{\partial \mathbf{u}}(0,x)+O(|\mathbf{u}|_1)=M(0,x)+O(|\mathbf{u}|_1), \tag{2.8.8}
$$

and recall that in the proof $(f_1,...,f_n)$ have been selected such that

$$
-\Lambda_i\frac{f_i'}{f_i}>2\sum_{k=1,k\neq i}^n|M_{ik}(0,x)|\sqrt{\frac{f_i}{f_k}}-2M_{ii}(0,x). \tag{2.8.9}
$$

Thus from (2.8.8) and (2.8.9) there exist $\eta_2>0$, $p_1\in\mathbb{N}^*$ and μ_1 such that if $\mu<\mu_1$, $p>p_1$ and $|\mathbf{u}|_1<\eta_2$, then $I_{31}>0$. It remains to deal with I_{21}. As E is C^1, and from (2.3.9),

$$
(E\mathbf{u}_t)=\mathbf{u}_t+(\mathbf{u}.\mathcal{V})\mathbf{u}_t \tag{2.8.10}
$$

where $\mathcal{V}=\mathcal{V}(\mathbf{u}(t,x),x)$ is continuous on $\mathcal{B}_{\eta_0}\times[0,L]$. Using (2.8.10) together with (2.8.3) and proceeding exactly as previously for I_2, we get

$$
\begin{aligned}
\mathrm{I}_{21} =& \frac{W_{2,p}^{1-2p}}{2p}\left(\sum_{i=1}^m\lambda_i(\mathbf{u}(t,L),L)f_i(L)^p((\mathbf{u}_t)_i(t,L)+((\mathbf{u}(t,L).\mathcal{V})\mathbf{u}_t(t,L))_i)^{2p}e^{-2p\mu L}\right.\\
&-\sum_{i=1}^m\lambda_i(\mathbf{u}(t,0),0)f_i(0)^p((\mathbf{u}_t)_i(t,0)+((\mathbf{u}(t,0).\mathcal{V})\mathbf{u}_t(t,0))_i)^{2p}\\
&-\sum_{i=m+1}^n|\lambda_i(\mathbf{u}(t,L),L)|f_i(L)^p((\mathbf{u}_t)_i(t,L)+((\mathbf{u}(t,L).\mathcal{V})\mathbf{u}_t(t,L))_i)^{2p}e^{2p\mu L}\\
&\left.+\sum_{i=m+1}^n|\lambda_i(\mathbf{u}(t,0),0)|f_i(0)^p((\mathbf{u}_t)_i(t,0)+((\mathbf{u}(t,0).\mathcal{V})\mathbf{u}_t(t,0))_i)^{2p}\right).
\end{aligned}
\tag{2.8.11}
$$

Recall that $K = G'(0)$ and $\Delta = (\Delta_1, ..., \Delta_n)^T \in (\mathbb{R}_+^*)^n$ are chosen such that

$$\theta := \sup_{\|\xi\|_\infty \leq 1} (\sup_i (|\sum_{j=1}^n (\Delta_i K_{ij} \Delta_j^{-1}) \xi_j|)) < \frac{\inf_i \left(\frac{f_i(d_i)}{\Delta_i^2}\right)}{\sup_i \left(\frac{f_i(L-d_i)}{\Delta_i^2}\right)}. \tag{2.8.12}$$

We denote again

$$\xi_i := \Delta_i (\mathbf{u}_t)_i (t, L) \text{ for } i \in [1, m], \tag{2.8.13}$$
$$\xi_i := \Delta_i (\mathbf{u}_t)_i (t, 0) \text{ for } i \in [m+1, n]. \tag{2.8.14}$$

From the fact that G and $\mathbf{u}$ are C^1, we can differentiate (2.2.10) with respect to time, and we have

$$\begin{pmatrix} (\mathbf{u}_t)_+(t,0) \\ (\mathbf{u}_t)_-(t,L) \end{pmatrix} = K \begin{pmatrix} (\mathbf{u}_t)_+(t,L) \\ (\mathbf{u}_t)_-(t,0) \end{pmatrix} + o\left(\left| \begin{pmatrix} (\mathbf{u}_t)_+(t,L) \\ (\mathbf{u}_t)_-(t,0) \end{pmatrix} \right| \right), \tag{2.8.15}$$

where $o(x)$ refers to a function such that $o(x)/|x|$ tends to 0 when $|\mathbf{u}|_1$ tends to 0. Thus

$$\begin{aligned}
\mathrm{I}_{21} = & \frac{W_{2,p}^{1-2p}}{2p} \left(\sum_{i=1}^m \lambda_i(\mathbf{u}(t,L),L) \frac{f_i(L)^p}{\Delta_i^{2p}} ((\mathbf{u}_t)_i(t,L)\Delta_i + o(|\xi|))^{2p} e^{-2p\mu L} \right. \\
& + \sum_{i=m+1}^n |\lambda_i(\mathbf{u}(t,0),0)| \frac{f_i(0)^p}{\Delta_i^{2p}} ((\mathbf{u}_t)_i(t,0)\Delta_i + o(|\xi|))^{2p} \\
& - \sum_{i=1}^m \lambda_i(\mathbf{u}(t,0),0) \frac{f_i(0)^p}{\Delta_i^{2p}} (\sum_{k=1}^n K_{ik} \xi_k(t) \frac{\Delta_i}{\Delta_k} + o(|\xi|))^{2p} \\
& \left. - \sum_{i=m+1}^n |\lambda_i(\mathbf{u}(t,L),L)| \frac{f_i(L)^p}{\Delta_i^{2p}} (\sum_{k=1}^n K_{ik} \xi_k(t) \frac{\Delta_i}{\Delta_k} + o(|\xi|))^{2p} e^{2p\mu L} \right)
\end{aligned} \tag{2.8.16}$$

We end by proceeding exactly as for I_2. Therefore under assumption (2.4.2), there exist p_3, μ_3 and $\eta_3 > 0$ such that for $\mu < \mu_3$ and $|\mathbf{u}|_1 < \eta_3$, $I_{21} < 0$. Therefore, as stated in the main text, there exist η_4, p_5 and μ such that for all $p > p_5$ and $|\mathbf{u}|_1 < \eta_4$

$$\frac{dW_{2,p}}{dt} \leq -\frac{\mu\alpha_0}{2} W_{2,p} + C_7 W_{2,p} |\mathbf{u}|_1. \tag{2.8.17}$$

2.8.2 Existence of ψ_0

We want to find a function ψ_0 that is C^1 with compact support in $[-1, 1]$ such that there exists a unique $y_1 \in (-1, 1)$ such that

$$|\psi_1(y) - \psi_1'(y)| e^{-y} < |\psi_1(y_1) - \psi_1'(y_1)| e^{-y_1}, \ \forall y \in [-1, 1] \setminus \{y_1\}. \tag{2.8.18}$$

Let χ be a positive C_c^1 with compact support in in $[-1, 1]$ such that

$$\begin{aligned}
&\chi \equiv 1 \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right], \\
&|\chi| \leq 1 \text{ on } [-1, 1], \\
&|\chi'| \leq 3 \text{ on } \left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right],
\end{aligned} \tag{2.8.19}$$

and let us define $f : y \to e^{-n_1 y^2}$ where $n_1 \in \mathbb{N}^*$ will be chosen later on. We have

$$(f(y) - f'(y)) e^{-y} = e^{-n_1 y^2 - y} (1 + 2n_1 y). \tag{2.8.20}$$

Therefore

$$|f(y) - f'(y)|e^{-y} \leq e^{-\frac{n_1}{4}+1}(1+2n_1) \text{ on } \left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]. \tag{2.8.21}$$

As $\lim_{n\to+\infty} e^{-\frac{n}{4}+1}(1+2n) = 0$ we can choose $n_1 \geq 1$ large enough such that

$$e^{-\frac{n_1}{4}+1}(1+2n_1) \leq \frac{1}{3}. \tag{2.8.22}$$

Now let us consider $\psi_1 = \chi f$, one has

$$|\psi_1(y) - \psi_1'(y)|e^{-y} = |\chi(y)(f(y) - f'(y))e^{-y} - \chi'(y)f(y)e^{-y}|. \tag{2.8.23}$$

Therefore from (2.8.19), (2.8.21) and (2.8.22), we have on $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$

$$|\psi_1(y) - \psi_1'(y)|e^{-y} \leq \frac{1}{3} + \frac{3}{9} < 1. \tag{2.8.24}$$

As $g : y \to |\psi_1(y) - \psi_1'(y)|e^{-y}$ has compact support on $[-1, 1]$ we can define d as

$$d := \sup_{y\in[-1,1]} (|\psi_1(y) - \psi_1'(y)|e^{-y}), \tag{2.8.25}$$

and d is attained in at least one point. But as $g(0) = 1$ and as from (2.8.24) $|g| < 1$ on $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$, d is attained only on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, and on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ we have

$$|\psi_1(y) - \psi_1'(y)|e^{-y} = |f(y) - f'(y)|e^{-y}. \tag{2.8.26}$$

Let us show now that $|f(y)-f'(y)|e^{-y}$ admits a unique maximum on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We know that $|f(y)-f'(y)|e^{-y}$ attains a maximum $d \geq 1$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and when it attains this maximum $((f(y) - f'(y))e^{-y})'$ vanishes, therefore

$$e^{-n_1y^2-y}(2n_1 - 4n_1^2y^2 - 1 - 4n_1y) = 0, \tag{2.8.27}$$

hence

$$4n_1^2y^2 + 4n_1y + (1 - 2n_1) = 0. \tag{2.8.28}$$

This equation has only two solutions: $y_\pm = \frac{-1\pm\sqrt{2n_1}}{2n_1}$ but

$$|f(y_-) - f'(y_-)|e^{-y_-} = \sqrt{2n_1}e^{\frac{1-2n_1+4\sqrt{2n_1}}{4n_1}} > \sqrt{2n_1}e^{\frac{1-2n_1-4\sqrt{2n_1}}{4n_1}} = |f(y_+) - f'(y_+)|e^{-y_+}. \tag{2.8.29}$$

Therefore $|f(y) - f'(y)|e^{-y}$ admits its maximum on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ at most one time. But we also know that it does admit a maximum on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, hence and from (2.8.24)

$$\exists! y_1 \in (-1, 1) : \ |\psi_1(y) - \psi_1'(y)|e^{-y} < |\psi_1(y_1) - \psi_1'(y_1)|e^{-y_1}, \ \forall y \in [-1, 1] \setminus \{y_1\}. \tag{2.8.30}$$

Now we just need to normalize the function and define $\psi_0 := \frac{1}{d}\psi_1$ where d is given in (2.8.25) to obtain the desired function ψ_0.

Observe that this function also satisfies (2.5.83), (2.5.84) and (2.5.85): Let $y \in (-1/2, 1/2)$, then $\psi_0(y)e^{-y}$ is positive and one has

$$(\psi_0(\cdot)e^{-Id})'(y) = \frac{1}{d}(-1 - 2n_1y)e^{-y-n_1y^2}, \tag{2.8.31}$$

thus on $(-1/2, 1/2)$, $|\psi_0|e^{-Id}$ has a unique maximum achieved in $y_2 = -1/2n_1$. Now let $y \in [-1, 1] \setminus (-1/2, 1/2)$, we have from (2.8.22)

$$|\psi_0(y)|e^{-y} \leq \frac{e^{1-\frac{n_1}{4}}}{d} \leq \frac{e^{-\frac{3}{4n_1}}}{d} = \psi_0(y_2)e^{-y_2}. \tag{2.8.32}$$

Hence the function admit a unique maximum on $[-1,1]$ and (2.5.83) is verified. And from (2.8.22) we have

$$\psi_0(y_2) - \psi_0''(y_2) = (-2n_1 + 2) > 0. \tag{2.8.33}$$

This implies (2.5.84) and we are left with proving (2.5.85). Let again $y + y_1 \in [-1,1] \setminus (-1/2, 1/2)$, for $i \in [1,n]$ and m large enough, from (2.8.22)

$$\left|\psi_0(y+y_1)\frac{e^{-y-y_1}}{\lambda_i(\frac{y}{m}+x_0)}\right| \leq e^{-\frac{n_1}{4}}\frac{e^{y_1-y_1}}{d\inf_{[x_0-\frac{1+y_1}{m},x_0+\frac{(1-y_1)}{m}]}|\lambda_1|} < \left|\psi_0(0)\frac{e^{y_1-y_1}}{\lambda_1(x_0)}\right|, \tag{2.8.34}$$

which means that $\sup\limits_{[-1,1]}\left|\psi_0(y+y_1)\frac{e^{-y-y_1}}{\lambda_i(\frac{y}{m}+x_0)}\right|$ can only be achieved on $(-1/2-y_1, 1/2-y_1)$. But we also know that on $[-1/2,1/2]$, $\psi_0 = d^{-1}f$. Therefore let be a y_m maximizing $\sup_{[-1,1]}\left|\psi_0(y+y_1)\frac{e^{-y-y_1}}{\lambda_i(\frac{y}{m}+x_0)}\right|$, we know that y_m exists as [-1,1] is a compact, that y_m is an interior maximum and we have

$$\partial_y\Big(e^{-n_1(y+y_1)^2}\frac{e^{-y-y_1}}{d\lambda_i(\frac{y}{m}+x_0)}\Big)(y_m) = 0. \tag{2.8.35}$$

Hence

$$2n_1\left((y_m+y_1)+1+\frac{\lambda_i'(\frac{y_m}{m}+x_0)}{m\lambda_i(\frac{y_m}{m}+x_0)}\right)\frac{e^{-n_1^2(y_m+y_1)-y_m-y_1}}{d\lambda_i(\frac{y_m}{m}+x_0)} = 0, \tag{2.8.36}$$

thus

$$(y_m+y_1) = -\frac{1}{2n_1} - \frac{\lambda_i'(\frac{y_m}{m}+x_0)}{(2n_1)m\lambda_i(\frac{y_m}{m}+x_0)}. \tag{2.8.37}$$

All it remains to show is that for m large enough we have (2.5.85). Let us compute $\psi_0''(y_m+y_1)$

$$\begin{aligned}\psi_0''(y_m+y_1) &= d^{-1}f''(y_m+y_1) = f(y_m+y_1)(-2n_1+4n_1^2(y_m+y_1)^2) \\ &= f(y_m+y_1)\Big(-2n_1+1+\left(\frac{\lambda_i'(\frac{y_m}{m}+x_0)}{m\lambda_i(\frac{y_m}{m}+x_0)}\right)^2+\left(\frac{\lambda_i'(\frac{y_m}{m}+x_0)}{m\lambda_i(\frac{y_m}{m}+x_0)}\right)\Big).\end{aligned} \tag{2.8.38}$$

Therefore there exists $m_3 > 0$ such that for all $m > m_3$,

$$|\psi_0''(y_m+y_1) - \psi_0(y_m+y_1)| > e^{-\frac{1}{n_1}}(2n_1-3), \tag{2.8.39}$$

and as we chose n_1 large enough, $C := e^{-\frac{1}{n_1}}(2n_1-3) > 0$. This ends the proof of the existence of ψ_0.

2.8.3 Adapting proof of Theorem 2.4.2 in the nonlinear case

For all $\mathbf{u}(0,\cdot) \in \mathcal{B}_{\eta_1}$, we can still define

$$u_i(0,x) = \frac{u_i^0}{m}\chi(x)\frac{e^{-m(x-x_0)-y_1}}{\Lambda_i(x)\sqrt{f_i(x)}}, \tag{2.8.40}$$

which is the analogous of (2.5.69) in the proof of Theorem 2.4.2. If there are two index i_0 and i_1 such that $\min\limits_i(|\Lambda_i(x_0)|)$ is achieved we can still redefine $u_{i_1}^0$ as in (2.5.67). Observe then that if (2.5.64) is satisfied, then there exists $\eta_3 > 0$ such that if $|\mathbf{u}|_0 < \eta_3$ then

$$-\lambda_{i_0}(\mathbf{u},x_0)f_{i_0}'(x_0) < 2\sum_{k=1,k\neq i_0}^{n}|M_{i_0k}(\mathbf{u},x_0)|\frac{f_{i_0}^{3/2}(x_0)}{\sqrt{f_k(x_0)}} - 2M_{i_0i_0}(\mathbf{u},x_0)f_{i_0}(x_0). \tag{2.8.41}$$

From (2.8.40), (2.5.72) becomes

$$
\begin{aligned}
\sqrt{f_i}\partial_t u_i(0,x) = &- u_i^0 e^{-m(x-x_0)-y_1}\left[-\chi(x) + \frac{\chi'(x)}{m} + \frac{\chi(x)\lambda_i\sqrt{f_i}}{m}\left(\frac{1}{\lambda_i\sqrt{f_i}}\right)'\right.\\
&+\frac{1}{m}\sum_{j=1}^{n} M_{ij}(\mathbf{u},x)\left(\frac{u_j^0}{u_i^0}\right)\left(\sqrt{\frac{f_i}{f_j}}\right)\frac{1}{\lambda_j}\chi(x)\\
&\left.+\sum_{j=1}^{n}(V_{ij}(\mathbf{u},x).\mathbf{u}(0,x))\left(\frac{u_j^0}{u_i^0}\right)\left(\sqrt{\frac{f_i}{f_j}}\right)\frac{1}{\lambda_j}\left(-\chi(x) + \frac{\chi'(x)}{m} + \frac{\chi(x)\lambda_j\sqrt{f_j}}{m}\left(\frac{1}{\lambda_j\sqrt{f_j}}\right)'\right)\right].
\end{aligned}
\tag{2.8.42}
$$

where V_{ij} are C^2 functions as we assume that A is of class C^3. Therefore

$$
\begin{aligned}
\sqrt{f_i}\partial_t u_i(0,x) = &- u_i^0 e^{-m(x-x_0)-y_1}\left[\left(-\chi(x) + \frac{\chi'(x)}{m}\right)\left(1+\sum_{j=1}^{n}(V_{ij}(\mathbf{u},x).\mathbf{u}(0,x))\left(\frac{u_j^0}{u_i^0}\right)\left(\sqrt{\frac{f_i}{f_j}}\right)\frac{1}{\lambda_j}\right)\right.\\
&+\chi(x)\left(\frac{\lambda_i\sqrt{f_i}}{m}\left(\frac{1}{\lambda_i\sqrt{f_i}}\right)' + \sum_{j=1}^{n}(V_{ij}(\mathbf{u},x).\mathbf{u}(0,x))\left(\frac{u_j^0}{u_i^0}\right)\left(\sqrt{\frac{f_i}{f_j}}\right)\frac{\sqrt{f_j}}{m}\left(\frac{1}{\lambda_j\sqrt{f_j}}\right)'\right)\\
&\left.+\frac{1}{m}\sum_{j=1}^{n} M_{ij}(\mathbf{u},x)\left(\frac{u_j^0}{u_i^0}\right)\left(\sqrt{\frac{f_i}{f_j}}\right)\frac{1}{\lambda_j}\chi(x)\right].
\end{aligned}
\tag{2.8.43}
$$

Now, after the change of variable (2.5.71), one has

$$
\begin{aligned}
\sqrt{f_i}\partial_t u_i(0,x) = &- u_i^0 e^{-y-y_1}\left[-\phi(y)\left(1+\frac{g_i(\mathbf{u}(0,y/m+x_0),y/m+x_0)}{m}\right)\right.\\
&\left.+\phi'(y)\left(1+\frac{h_i(\mathbf{u}(0,y/m+x_0),y/m+x_0)}{m}\right)\right].
\end{aligned}
\tag{2.8.44}
$$

where h_i and g_i are bounded functions in the C^2 norm and are independent of m. Thus

$$
\begin{aligned}
\sqrt{f_i}(E\partial_t\mathbf{u})_i(0,x) = &-\left(u_i^0 + \sum_{j=1}^{n}\sqrt{\frac{f_i}{f_j}}Z_{ij}(\mathbf{u}(0,y/m+x_0),y/m+x_0)u_j^0\right)e^{-y-y_1}\left[-\phi(y)+\phi'(y)\right]\\
&- u_i^0 e^{-y-y_1}\left[-\phi(y)\left(\frac{g_i(\mathbf{u}(0,y/m+x_0),y/m+x_0)}{m}\right) + \phi'(y)\left(\frac{h_i(\mathbf{u}(0,y/m+x_0),y/m+x_0)}{m}\right)\right]\\
&-\sum_{j=1}^{n}\sqrt{\frac{f_i}{f_j}}Z_{ij}(\mathbf{u}(0,y/m+x_0),y/m+x_0)u_j^0 e^{-y-y_1}\left[-\phi(y)\left(\frac{g_j(\mathbf{u}(0,y/m+x_0),y/m+x_0)}{m}\right)\right.\\
&\left.+\phi'(y)\left(\frac{h_j(\mathbf{u}(0,y/m+x_0),y/m+x_0)}{m}\right)\right].
\end{aligned}
\tag{2.8.45}
$$

where $Z(\mathbf{u},x) = (Z_{ij})_{(i,j)\in[1,n]^2} := \mathbf{u}.V(\mathbf{u},x)$, with V given by (2.8.10). In addition one also has

$$
u_i(0,\frac{y}{m}+x_0) = \frac{u_i^0}{m}\phi(y)\frac{e^{-y-y_1}}{\Lambda_i(\frac{y}{m}+x_0)\sqrt{f_i(\frac{y}{m}+x_0)}},\ \forall i\in[1,n]. \tag{2.8.46}
$$

Thus, the function $y \to \mathbf{u}(0,y/m+x_0)$ is $O(1/m)$ in the C^2 norm, which means that $g_i(\mathbf{u}(0,y/m+x_0),y/m+x_0)$ and $g_i(\mathbf{u}(0,y/m+x_0),y/m+x_0)$ are $O(1)$ in the C^2 norm when m tends to $+\infty$. Similarly, Z is a C^2

function as E is a C^3 function (recall that A is C^3 for Theorem 2.4.2), and there exists a constant C independant of $\mathbf{u}$ and m such that $\max_{(i,j)\in[1,n]^2}|Z_{ij}(\mathbf{u}(0,y/m+x_0),(0,y/m+x_0))|\leq C|\mathbf{u}(0,y/m+x_0)|$. This, with (2.8.46), implies that the terms which involves Z in (2.8.45) are all $O(1/m)$ in the C^2 norm. Therefore we can process similarly as previously for the existence of $(x_i)_{i\in[1,n]}$, t_1 and $x_a\in C^1([0,t_1))$ such that

$$V_2(t)=|\sqrt{f_1}E\partial_t u_1(t,\cdot),...,\sqrt{f_n}E\partial_t u_n(t,\cdot)|_0=-\sqrt{f_1(x_a(t))}(E\partial_t\mathbf{u})_1(t,x_a(t)),\ \forall t\in[0,t_1), \tag{2.8.47}$$

The only thing that remains to be checked is whether we still have the existence of $x_b\in C^1([0,t_2))$ for some t_2 positive and independent of m. Existence of a unique i_0 and $x_{a_0}\in[x_0-\varepsilon,x_0+\varepsilon]$ such that

$$|\sqrt{f_{i_0}(x_{a_0})}u_{i_0}(0,x_{a_0})|=\sup_{i\in[1,n],x\in[0,L]}|\sqrt{f_i}u_i(0,\cdot)| \tag{2.8.48}$$

is granted by the same argument as previously. As $\mathbf{u}(0,x)$ is defined exactly as in the linear case, we still have for our choice of χ

$$\partial_x g(0,x_{a_0})\neq 0. \tag{2.8.49}$$

This implies the existence of $x_b\in C^1([0,t_2))$ for some t_2 positive and independent of m.

If we look now at the computation of $dV_2/dt(0)$ and $dV_1/dt(0)$, one has, proceeding as in Section 2.5 and using (2.3.8)

$$\begin{aligned}\frac{dV_2}{dt}(0)&=-\sqrt{f_1(x_1)}\left(E\partial^2_{tt}\mathbf{u}+\left(\frac{\partial E}{\partial\mathbf{u}}.\partial_t\mathbf{u}\right)\partial_t\mathbf{u}\right)_1(0,x_1)\\&=+\sqrt{f_1(x_1)}\left(\lambda_1(E\partial^2_{tx}\mathbf{u})_1(0,x_1)+(E\left(D_a+\frac{\partial B}{\partial\mathbf{u}}\right)E^{-1}E\partial_t\mathbf{u})_1(0,x_1)\right)\\&\quad-\sqrt{f_1(x_1)}\left(\left(\frac{\partial E}{\partial\mathbf{u}}.\partial_t\mathbf{u}\right)E^{-1}E\partial_t\mathbf{u}\right)_1(0,x_1)\\&=\sqrt{f_1(x_1)}\left(\lambda_1(\partial_x(E\partial_t\mathbf{u}))_1(0,x_1)+(R(\mathbf{u},x)E\partial_t\mathbf{u})_1(0,x_1)\right)\\&\quad-\sqrt{f_1(x_1)}\left((\lambda\left(\frac{\partial E}{\partial\mathbf{u}}\partial_x\mathbf{u}+\partial_x E\right)\partial_t\mathbf{u})_1(0,x_1)+\left(\left(\frac{\partial E}{\partial\mathbf{u}}.\partial_t\mathbf{u}\right)E^{-1}E\partial_t\mathbf{u}\right)_1(0,x_1)\right)\end{aligned} \tag{2.8.50}$$

where $R=E(D_a+\frac{\partial B}{\partial\mathbf{u}})E^{-1}$. Therefore from the definition of $\mathbf{u}(0,x)$ given by (2.8.40), one has

$$\begin{aligned}\frac{dV_2}{dt}(0)&=-\sqrt{f_1(x_1)}\left(-\left(\Lambda_1+\frac{l(\mathbf{u}(0,x_1),x_1)}{m}\right)(\partial_x(E\partial_t\mathbf{u}))_1(0,x_1)+\left(\left(R(\mathbf{0},x)+\frac{v(\mathbf{u}(0,x_1),x_1)}{m}\right)E\partial_t\mathbf{u}\right)_1(0,x_1)\right)\\&\quad-\sqrt{f_1(x_1)}\left((\lambda\left(\frac{\partial E}{\partial\mathbf{u}}\partial_x\mathbf{u}+\partial_x E\right)\partial_t\mathbf{u})_1(0,x_1)+\left(\left(\frac{\partial E}{\partial\mathbf{u}}.\partial_t\mathbf{u}\right)E^{-1}E\partial_t\mathbf{u}\right)_1(0,x_1)\right),\end{aligned} \tag{2.8.51}$$

where l and v are bounded functions on $\mathcal{B}_{\eta_3}\times[0,L]$ with a bound independent of m from (2.5.103). Hence, using this together with (2.3.9) and noting that $R(\mathbf{0},x)=M(\mathbf{0},x)$,

$$\begin{aligned}\frac{dV_2}{dt}(0)&=-\sqrt{f_1(x_1)}\left((\Lambda_1+\frac{l(\mathbf{u}(0,x_1),x_1)}{m})\frac{(\sqrt{f_1})'}{\sqrt{f_1}}(E\partial_t\mathbf{u})_1(0,x_1)\right.\\&\quad\left.-\sum_{j=1}^n(M_{1j}(0,x_1)+\frac{v_{1j}(\mathbf{u}(0,x_1),x_1)}{m})(E\partial_t\mathbf{u})_j(0,x_1)\right)+O\left(|\mathbf{u}|_1^2\right)\\&=-\sqrt{f_1(x_1)}\left(\frac{\Lambda_1 f_1'}{2f_1}(E\partial_t\mathbf{u})_1(0,x_1)\left(1+O\left(\frac{1}{m}\right)\right)-\sum_{j=1}^n\left(M_{1j}(0,x_1)+O\left(\frac{1}{m}\right)\right)(E\partial_t\mathbf{u})_j(0,x_1)\right)\\&\quad+O\left(|\mathbf{u}|_1^2\right).\end{aligned} \tag{2.8.52}$$

where the O does not depends on u_1^0 but only on an upper bound of u_1^0 (we can choose η_0 for instance). Observe that, from (2.8.40), (2.8.45), and the fact that $Z = O(1/m)$ for the C^2 norm, we can proceed as previously and we obtain (2.5.108) with an additional $O(|\mathbf{u}|_1^2)$. Similarly as previously we can obtain

$$\sqrt{f_{i_0}(x_{a_0})}\partial_t u_{i_0}(0, x_{a_0}) = O\left(\frac{1}{m}\right). \tag{2.8.53}$$

The rest of the proof to get (2.5.110)–(2.5.112) can then be done identically as all the relations used in the proof still hold in the nonlinear case. But actually looking at (2.5.110)–(2.5.112), together with (2.8.43), (2.8.46), (2.8.52)–(2.8.53), there exists $a > 0$ independant of m and C independant of m and u_1^0 such that for any $m > m_9$,

$$\frac{dV_1}{dt} + \frac{dV_2}{dt} \geq au_1^0 - C(|u_1^0|^2). \tag{2.8.54}$$

Thus, there exists $\eta_2 > 0$ independant of m such that, for any $m > m_9$,

$$\frac{dV}{dt} > 0 \tag{2.8.55}$$

which ends the proof in the nonlinear case.

2.8.4 Extension of the proof to the C^q norm

To be able to extend the proof for the C^q norm one should first define the corresponding compatibility conditions of order $q-1$ that are given for instance in [13] at (4.136) and see also (4.137)-(4.142). Then one only needs to realize that if we now consider the state $\mathbf{y} = (\mathbf{u}, \partial_t \mathbf{u}, ..., \partial_t^{q-1}\mathbf{u})$, $\mathbf{y}$ is still the solution of a quasilinear hyperbolic system of the form

$$\mathbf{y}_t + A_1(\mathbf{y}, x)\mathbf{y}_x + M_1(\mathbf{y}, x).\mathbf{y} + C = 0 \tag{2.8.56}$$

where $|C_i|_0 = O\left(|u, ..., \partial_t^{i-1}u|_0^2\right)$, and where the principal matrix A_1 verifies

$$A_1 = \begin{pmatrix} A(\mathbf{u}, x) & (0) & ... & \\ (0) & A(\mathbf{u}, x) & (0) & ... \\ (0) & (0) & A(\mathbf{u}, x) & ... \\ ... & ... & ... & ... \end{pmatrix} \tag{2.8.57}$$

and is therefore block diagonal with blocks that are all A as previously. Similarly $M_1(0, x)$ is also block diagonal with blocks that are all $M(0, x)$. Therefore if we consider the following functions

$$W_{k+1,p} = \left(\int_0^L \sum_{i=1}^n f_i(x)^p (E\partial_t^k \mathbf{u})_j^{2p} e^{-2p\mu s_i x} dx\right)^{1/2p}, \tag{2.8.58}$$

for all $k \in [0, q]$, where f_i are chosen as previously, and if we perform as previously (Section 2.5 and Appendix 2.8.1), we have existence of $C_k > 0$ constants, $\eta_k > 0$ and $p_k \in \mathbb{N}^*$ such that for all $p > p_k$ and $|\mathbf{y}|_1 < \eta_k$ we have relations of the type

$$\frac{dW_{k+1,p}}{dt} \leq -\frac{\mu\alpha_0}{2} W_{k+1,p} + C_{k+1} \sum_1^k W_{r+1,p} |\mathbf{y}|_1. \tag{2.8.59}$$

for all $k \in [0, q]$. Thus denoting $W_p = \sum_{k=0}^q W_{k+1,p}$, there exists $C > 0$ constant, $p_l \in \mathbb{N}^*$ and $\eta_l > 0$ such that for all $p > p_l$ and $|\mathbf{y}|_1 < \eta_l$,

$$\frac{dW_p}{dt} \leq -\frac{\mu\alpha_0}{2} W_p + CW_p |\mathbf{y}|_1. \tag{2.8.60}$$

and we could perform as previously to obtain the exponential decay of $V = \sum_{k=0}^q V_{k+1}$ where $V_{k+1} = |\sqrt{f_1}(\partial_t^k \mathbf{u})_1, ..., \sqrt{f_n}(\partial_t^k \mathbf{u})_n|_0$ and therefore stability for the C^q norm.

Chapter 3

On boundary stability of inhomogeneous 2×2 1-D hyperbolic systems for the C^1 norm.

This chapter is taken from the following article (also referred to as [107]):

Amaury Hayat. On boundary stability of inhomogeneous 2×2 1-D hyperbolic systems for the C^1 norm. Preprint, 2017.

Abstract. We study the exponential stability for the C^1 norm of general 2×2 1-D quasilinear hyperbolic systems with source terms and boundary controls. When the propagation speeds of the system have the same sign, any nonuniform steady-state can be stabilized using boundary feedbacks that only depend on measurements at the boundaries and we give explicit conditions on the gain of the feedback. In other cases, we exhibit a simple numerical criterion for the existence of basic C^1 Lyapunov function, a natural candidate for a Lyapunov function to ensure exponential stability for the C^1 norm. We show that, under a simple condition on the source term, the existence of a basic C^1 (or C^p , for any $p \geq 1$) Lyapunov function is equivalent to the existence of a basic H^2 (or H^q , for any $q \geq 2$) Lyapunov function, its analogue for the H^2 norm. Finally, we apply these results to the nonlinear Saint-Venant equations. We show in particular that in the subcritical regime, when the slope is larger than the friction, the system can always be stabilized in the C^1 norm using static boundary feedbacks depending only on measurements at the boundaries, which has a large practical interest in hydraulic and engineering applications.

3.1 Introduction

Hyperbolic systems are widely studied, as their ability to model physical phenomena gives rise to numerous applications. The 2×2 hyperbolic systems, in particular, are very interesting at two extends: on the one hand they are the simplest systems that present a coupling, and on the other hand, by modeling the systems of two balance laws, they represent a huge number of physical systems from fluid dynamics in rivers and shallow waters [42], to road traffic [8], signal transmission, laser amplification [86], etc. In order to use these models in industrial or practical applications, the question of their stability or their possible stabilization is fundamental. While for linear 1-D systems, or nonlinear 1-D systems without source term, many results exist (see in particular [14, Section 4.5] [48, 138]) the question of the stabilization in general for 2×2 1-D nonlinear systems has often been treated for the H^p norm and only few results exist for the more natural C^1 (or C^p) norm when a source term occurs. In [106], however, were presented some results for the C^p stability ($p \geq 1$) of general $n \times n$ quasilinear hyperbolic system using basic Lyapunov functions for the C^p norm.
In this article we consider the stability for the C^1 norm of 2×2 general quasilinear 1-d hyperbolic systems. We show several results and we use them to study the exponential stability of the general nonlinear Saint-

Venant equations for the C^1 norm. Firstly introduced in 1871 by Barré de Saint-Venant and used to model flows under shallow water approximation, the Saint-Venant equations can be derived from the Navier-Stokes equations and have been widely used in the last centuries in many areas such as agriculture, river regulation, and hydraulic electricity production. For instance they are used in Belgium for the control of the Meuse and Sambre river (see [54], [71]). Their indisputable usefulness in the field of fluid mechanics or in engineering applications makes them a well-studied example in stability theory ([14], [110], [71]) although their stability for the C^1 norm by means of boundary controls seems to be only known so far in the particular case when when both the slope and the friction are sufficiently small (or equivalently the size of the river is sufficiently small) [171].

We first show that the results presented in [106] can be simplified for 2×2 systems in conditions that are easier to check in practice. In particular, any 2×2 quasilinear hyperbolic system with propagation speeds of the same sign can be stabilized by means of static boundary feedback and we give here explicit conditions on the gain of the feedbacks to achieve such result. In the general case we also give a simple linear numerical criterion to design good boundary controls and estimate the limit length above which stability is not guaranteed anymore.

Then we deduce a link between the H^p stability and C^q stability under appropriate boundary control for any $p \geq 2$ and $q \geq 1$. In particular we give a practical way to construct a basic Lyapunov function for the C^1 norm from a basic quadratic Lyapunov function for the H^2 norm and reciprocally.

Finally, we use these results to study the C^1 stability of the general nonlinear Saint-Venant equations taking into account the slope and the friction. We show that when the friction is stronger than the slope the system can always be made stable for the C^1 norm by applying appropriate boundary controls that are given explicitly. When the slope is higher than the friction, however, there always exists a length above which the system do not admit a basic C^1 Lyapunov function that would ensure the stability, whatever the boundary controls are. This results is all the more interesting that it has been shown that there always exists a basic quadratic H^2 Lyapunov function ensuring the stability for the H^2 norm under suitable boundary controls (see [110]).

Nevertheless in that last case the results given in this article allow to find good Lyapunov function numerically and estimate the limit length under which the stability can be guaranteed. We provide at the end of this chapter numerical computations of this limit for the Saint-Venant equations that illustrate that for most applications the stability can be guaranteed by means of explicit static boundary feedback. This article is organised as follows: in Section 3.2 we present several properties of 2×2 quasilinear hyperbolic system, as well as some useful definitions and we review some existing results. Section 3.3 present the main results for the general case and for the particular case of the Saint-Venant equations. Section 3.4 is devoted to the proof of the results in the general case and to the link between the H^p and C^q stability, while the proofs of the results about the Saint-Venant equations are given in Section 3.5. Finally, we provide some numerical computations in Section 3.6 and some comments in Section 3.7.

3.2 General considerations and previous results

3.2.1 General considerations

A 2×2 quasilinear hyperbolic system can be written in the form:

$$\mathbf{Y}_t + F(\mathbf{Y})\mathbf{Y}_x + D(\mathbf{Y}) = 0, \tag{3.2.1}$$

$$\mathcal{B}(\mathbf{Y}(t,0), \mathbf{Y}(t,L)) = 0. \tag{3.2.2}$$

As the goal of this study is to deal with the exponential stability of the system around a steady-state we assume that there exists $\mathbf{Y}^*$ a steady-state that we aim at stabilizing. Note that this steady-state is not necessarily uniform and can potentially have large variations of amplitude. As we are looking at the local stability around this steady-state, we study F and D on $\mathcal{U} = \mathcal{B}_{\mathbf{Y}^*,\eta_0}$, the ball of radius η_0 centered in $\mathbf{Y}^*$ in the space of the continuous functions endowed with the L^∞ norm, for some η_0 small enough to be precised. We assume that the system is strictly hyperbolic around $\mathbf{Y}^*$ with non vanishing propagation speeds, i.e. non vanishing eigenvalues of $F(\mathbf{Y})$, then $F(\mathbf{Y}^*)$ is diagonalisable and denoting by N a matrix of eigenvector we introduce the following change of variables:

$$\mathbf{u} = N(x)(\mathbf{Y} - \mathbf{Y}^*) \tag{3.2.3}$$

and the system (3.2.1)-(3.2.2) is equivalent to

$$\begin{gathered} \mathbf{u}_t + A(\mathbf{u}, x)\mathbf{u}_x + B(\mathbf{u}, x) = 0, \\ \mathcal{B}(N^{-1}(0)\mathbf{u}(t,0) + Y^*(0), N^{-1}(L)\mathbf{u}(t,L) + Y^*(L)) = 0, \end{gathered} \tag{3.2.4}$$

where

$$A(\mathbf{u}, x) = N(x)F(\mathbf{Y}^* + N^{-1}\mathbf{u})N^{-1}(x), \tag{3.2.5}$$

$$A(0,x) = \begin{pmatrix} \Lambda_1(x) & 0 \\ 0 & \Lambda_2(x) \end{pmatrix}, \tag{3.2.6}$$

and B is given in Appendix 3.8.1. Let us assume that F and D are C^1 on $\mathcal{B}_{\mathbf{Y}^*,\eta_0}$, then A and B are C^1 on $\mathcal{B}_{\mathbf{0},\eta_0} \times [0, L]$ (see Appendix 3.8.1). As $\mathbf{Y}^*$ is a stationary state, one has $B(\mathbf{0}, \cdot) \equiv 0$ and B can be written:

$$B(\mathbf{u}, x) = M(\mathbf{u}, x).\mathbf{u}. \tag{3.2.7}$$

Therefore the system (3.2.1)-(3.2.2) is now equivalent to

$$\begin{gathered} u_t + A(\mathbf{u}, x)u_x + M(\mathbf{u}, x).\mathbf{u} = 0, \\ \mathcal{B}(N^{-1}(0)\mathbf{u}(t,0) + Y^*(0), N^{-1}(L)\mathbf{u}(t,L) + Y^*(L)) = 0. \end{gathered} \tag{3.2.8}$$

We can suppose without loss of generality that $\Lambda_1 \geq \Lambda_2$. As the system is strictly hyperbolic with non-vanishing eigenvalues we can denote by $\mathbf{u}_+$ the components associated with positive eigenvalues, i.e. $\Lambda_i > 0$, and $\mathbf{u}_-$ the component associated with negative eigenvalues. We focus now on boundary conditions of the form:

$$\begin{pmatrix} \mathbf{u}_+(0) \\ \mathbf{u}_-(L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}_+(L) \\ \mathbf{u}_-(0) \end{pmatrix}. \tag{3.2.9}$$

For the rest of the article, unless otherwise stated, we will assume that F, D and G are C^1 when dealing with the C^1 norm and that F,D and G are C^2 when dealing with the H^2 norm. We also introduce the associated first order compatibility condition on an initial condition $\mathbf{u}^0$:

$$\begin{aligned} &\begin{pmatrix} \mathbf{u}^0_+(0) \\ \mathbf{u}^0_-(L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}^0_+(L) \\ \mathbf{u}^0_-(0) \end{pmatrix}, \\ &\begin{pmatrix} \left(A(\mathbf{u}^0(0),0)\partial_x \mathbf{u}^0(0) + B(\mathbf{u}^0(0),0)\right)_+ \\ \left(A(\mathbf{u}^0(L),L)\partial_x \mathbf{u}^0(L) + B(\mathbf{u}^0(L),L)\right)_- \end{pmatrix} = G' \begin{pmatrix} \mathbf{u}^0_+(L) \\ \mathbf{u}^0_-(0) \end{pmatrix} \\ &\times \begin{pmatrix} \left(A(\mathbf{u}^0(L),L)\partial_x \mathbf{u}^0(L) + B(\mathbf{u}^0(L),L)\right)_+ \\ \left(A(\mathbf{u}^0(0),0)\partial_x \mathbf{u}^0(0) + B(\mathbf{u}^0(0),0)\right)_- \end{pmatrix}. \end{aligned} \tag{3.2.10}$$

With these boundary conditions the incoming information is a function of the outgoing information which enables the system to be well-posed (see [142], [172] or [14] in particular Theorem 6.4).

Theorem 3.2.1. *Let $T > 0$, there exists $\delta(T) > 0$ and $C(T) > 0$ such that for any $\mathbf{u}_0 \in C^1([0, L])$ satisfying the compatibility conditions (3.2.10) and*

$$|\mathbf{u}_0|_1 \leq \delta, \tag{3.2.11}$$

the system (3.2.8)–(3.2.9) with initial condition $\mathbf{u}_0$ has a unique maximal solution $\mathbf{u} \in C^1([0,T] \times [0,L])$ and we have the estimate:

$$|\mathbf{u}(t,\cdot)|_1 \leq C_1(T)|\mathbf{u}(0,\cdot)|_1, \ \forall\, t \in [0,T]. \tag{3.2.12}$$

where $|\cdot|_1$ is the C^1 norm that is recalled later on in Definition 3.2.1. Moreover if $\mathbf{u}_0 \in H^2([0,L])$ and

$$\|\mathbf{u}_0\|_{H^2((0,L))} \leq \delta, \tag{3.2.13}$$

then the solution $\mathbf{u}$ belongs to $C^0([0,T], H^2(0,L))$.

3.2.2 Context and previous results

Exponential stability of 2×2 hyperbolic systems.

- In [14] (see Theorem 4.3) and [48] respectively it has been shown that when there is no source term, i.e. $M \equiv 0$, it is always possible to guarantee the exponential stability of the system (3.2.8) with boundary controls of the form (3.2.9), both for the H^p and the C^q norm (with $p \geq 2$ and $q \geq 1$). Moreover, when the system is linear, this is also true for the L^2 and C^0 norm.

- In [11] the authors study a linear 2×2 system and found a necessary and sufficient interior condition to have existence of quadratic Lyapunov function for the L^2 norm with a boundary control of the form (3.2.9) when the system (3.2.8) is linear (Theorem 3.4.1). However it is straightforward to extend this results to the existence of a basic quadratic Lyapunov function for the H^p norm with $p \geq 2$ when the system (3.2.8) is nonlinear (see Theorem 3.4.2). At it is mentioned in [11] the existence of a basic quadratic Lyapunov function for the H^p norm implies the exponential stability of the system in the H^p norm.

- In [106] the author gives a necessary and sufficient condition on (3.2.8) such that there exists a basic C^1 Lyapunov function with a boundary control of the form (3.2.9), guaranteeing therefore the stability for the C^1 norm of the system (3.2.8)-(3.2.9). The results can be extended with the same condition to the C^p norm, with $p \geq 1$.

Exponential stability of the Saint-Venant Equations. The Saint Venant equations correspond to a system of the form (3.2.8) where the eigenvalues of A satisfy $\Lambda_1\Lambda_2 < 0$ when the flow is in the fluvial regime and $\Lambda_1\Lambda_2 > 0$ when in the torrential regime. The stability of the Saint-Venant equations has been well-studied in the past twenty years and, to our knowledge, the most advanced contribution in the area would refer, but not exclusively, to the following:

- In [16] the authors show that when there is no slope, i.e. $C \equiv 0$, there always exists a Lyapunov function in the fluvial regime (i.e. the eigenvalues satisfy $\Lambda_1\Lambda_2 < 0$) for the H^p norm for the nonlinear system under boundary controls of the form (3.2.9) and they give an explicit example. In [110] the authors show that this is true even when the slope is arbitrary.

- In [21] it is found, through a time delay approach, a necessary and sufficient condition for the stability of the linearized system under proportional integral control.

- In [197] and [65, 196] the authors use a backstepping method to stabilize respectively a linear 2×2 1-d hyperbolic systems and a nonlinear 2×2 1-d hyperbolic systems. These results cover in particular the linearized Saint-Venant equations and the nonlinear Saint-Venant equations. However, in both cases this method gives rise to full-state feedback laws that are harder to implement in practice than static feedback laws depending only on the measurements at the boundaries.

In this article we intend to show that there always exists a Lyapunov function ensuring exponential stability in the C^1 (and actually C^p) norm under boundary controls of the form (3.2.9) when the system is in the fluvial regime and the slope is smaller than the friction. However, in the fluvial regime when the slope is larger than the friction, there exists a maximal length L_{max} beyond which there never exists a basic C^1 Lyapunov function whatever the boundary controls are. Nevertheless, this maximal length L_{max} can be estimated numerically and can be shown to be large enough to ensure the feasibility of nearly all hydraulic applications.

Notations and definitions. We recall the definition of the C^1 norm:

Definition 3.2.1. Let $\mathbf{U} \in C^0([0, L], \mathbb{R}^2)$, its C^0 norm $|\mathbf{U}|_0$ is defined by:

$$|\mathbf{U}|_0 = \max(\|U_1\|_\infty, \|U_2\|_\infty), \tag{3.2.14}$$

and if $\mathbf{U} \in C^1([0,L],\mathbb{R}^2)$, its C^1 norm $|\mathbf{U}|_1$ is defined by

$$|\mathbf{U}|_1 = |\mathbf{U}|_0 + |\partial_x \mathbf{U}|_0. \tag{3.2.15}$$

We recall the definition of exponential stability for the C^1 (resp. H^2) norm:

Definition 3.2.2. The null steady-state $\mathbf{u}^* \equiv 0$ of the system (3.2.8)–(3.2.9) is said exponentially stable for the C^1 (resp. H^2) norm if there exists $\gamma > 0$, $\delta > 0$ and $C > 0$ such that for any $\mathbf{u}_0 \in C^1([0,L])$ (resp. $H^2([0,L])$) satisfying the compatibility conditions (3.2.10) and such that $\|u_0\|_{C^1([0,L])} \leq \delta$ (resp.$\|u_0\|_{H^2((0,L))} \leq \delta$), the system (3.2.8)–(3.2.9) has a unique solution $\mathbf{u} \in C^1([0,+\infty) \times [0,L])$ (resp. $\mathbf{u} \in C^1([0,+\infty) \times [0,L]) \cap C^0([0,+\infty), H^2((0,L))))$ and

$$\begin{gathered}\|\mathbf{u}(t,\cdot)\|_{C^1([0,L])} \leq Ce^{-\gamma t}\|\mathbf{u}_0\|_{C^1([0,L])},\ \forall\, t \in [0,+\infty)\\ (\text{resp. } \|\mathbf{u}(t,\cdot)\|_{H^2((0,L))} \leq Ce^{-\gamma t}\|\mathbf{u}_0\|_{H^2((0,L))},\ \forall\, t \in [0,+\infty)),\end{gathered} \tag{3.2.16}$$

Remark 3.2.1. *The exponential stability of the steady state $\mathbf{u}^* \equiv 0$ of the system (3.2.8)–(3.2.9) is equivalent to the exponential stability of the steady-states $\mathbf{Y}^*$ (*(3.2.16)* could in fact even be seen as a definition of the exponential stability of $\mathbf{Y}^*$). We see here one of the interests of the change of variables given by (3.2.3): from the stabilization of a potentially nonuniform steady-state the problem is reduced to the stabilization of a null steady-state.*

We now recall the definition of two useful tools. The first one deals with the basic C^1 Lyapunov functions described in [106]:

Definition 3.2.3. We call *basic C^1 Lyapunov function* for the system (3.2.8),(3.2.9) the function $V : C^1([0,L]) \to \mathbb{R}_+$ defined by:

$$\begin{aligned}V(\mathbf{U}) =& |\sqrt{f_1}U_1, \sqrt{f_2}U_2|_0\\ &+ |(A(\mathbf{U},\cdot).\mathbf{U}_x + B(\mathbf{U},\cdot))_1\sqrt{f_1}, (A(\mathbf{U},\cdot).\mathbf{U}_x + B(\mathbf{U},\cdot))_2\sqrt{f_2}|_0,\end{aligned} \tag{3.2.17}$$

where f_1 and f_2 belong to $C^1([0,L],\mathbb{R}_+^*)$, and such that there exists $\gamma > 0$ and $\eta > 0$ such that for any $\mathbf{u} \in C^1([0,L])$ solution of the system (3.2.8),(3.2.9) with $|\mathbf{u}^0|_1 \leq \eta$ and for any $T > 0$:

$$\frac{dV(\mathbf{u})}{dt} \leq -\gamma V(\mathbf{u}), \tag{3.2.18}$$

in a distributional sense on $(0,T)$. In that case f_1 and f_2 are called coefficients of the basic C^1 Lyapunov function.

Remark 3.2.2. *Note that for any $u \in C^1([0,L] \times [0,T])$ solution of (3.2.8), one has*

$$\begin{aligned}V(\mathbf{u}(t,\cdot)) =& |\sqrt{f_1}u_1(t,\cdot), \sqrt{f_2}u_2(t,\cdot)|_0,\\ &+ |(u_1(t,\cdot))_t\sqrt{f_1}, (u_2(t,\cdot))_t\sqrt{f_2}|_0.\end{aligned} \tag{3.2.19}$$

The previous definition (3.2.17) of V is only stated to show that V is in fact a function on $C^1([0,L])$ and therefore only depends on t through $\mathbf{u}$. Besides, one could wonder why using a weight $\sqrt{f_i}$ instead of f_i in the definition. The goal is to facilitate the comparison with the existing definition of basic quadratic Lyapunov functions for the L^2 (resp. H^2) norm introduced by Jean-Michel Coron and Georges Bastin in [11] and recalled below.

Definition 3.2.4. We call *basic quadratic Lyapunov function* for the L^2 norm (resp. for the H^2 norm) and for the system (3.2.8), (3.2.9) the function V defined on $L^2(0,L)$ (resp. $H^2(0,L)$) by:

$$
\begin{aligned}
V(\mathbf{U}) &= \int_0^L q_1 U_1^2 + q_2 U_2^2 dx \\
\Bigg(\text{resp. } V(\mathbf{U}) &= \int_0^L q_1 U_1^2 + q_2 U_2^2 dx \\
&+ \int_0^L (A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x))_1^2 q_1 + (A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x))_1^2 q_2 dx \\
&+ \int_0^L q_1 (\partial_{\mathbf{U}} A.[A.\mathbf{U}_x + B].\mathbf{U}_x + A\frac{d}{dx}\left(A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x)\right) + \partial_{\mathbf{U}} B.[A.\mathbf{U}_x + B])_1^2 \\
&+ q_2 (\partial_{\mathbf{U}} A.[A.\mathbf{U}_x + B].\mathbf{U}_x + A\frac{d}{dx}\left(A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x)\right) + \partial_{\mathbf{U}} B.[A.\mathbf{U}_x + B])_2^2 dx \Bigg),
\end{aligned}
\tag{3.2.20}
$$

where q_1 and q_2 belong to $C^1([0,L],\mathbb{R}_+^*)$ and such that there exists $\gamma > 0$ and $\eta > 0$ such that for any $\mathbf{u} \in L^2(0,L)$ (resp. $H^2(0,L)$) solution of the system (3.2.8),(3.2.9) with $|\mathbf{u}^0|_{L^2(0,L)} \leq \eta$ (resp. $|\mathbf{u}^0|_{H^2(0,L)} \leq \eta$) and any $T > 0$

$$
\frac{dV(\mathbf{u}(t))}{dt} \leq -\gamma V(\mathbf{u}(t)), \tag{3.2.21}
$$

in a distributional sense on $(0,T)$. The function q_1 and q_2 are called *coefficients* of the *basic quadratic Lyapunov function.*

Remark 3.2.3. *As for the basic C^1 Lyapunov functions, note that for any $\mathbf{u} \in C^0([0,T],H^2(0,L))$ solution to* (3.2.8) *the expression* (3.2.20) *of a basic quadratic Lyapunov function for the H^2 norm becomes*

$$
\begin{aligned}
V(\mathbf{U}) = &\int_0^L q_1 u_1^2 + q_2 u_2^2 dx \\
&+ \int_0^L (u_1)_t^2 q_1 + (u_2)_t^2 q_2 dx \\
&+ \int_0^L (u_1)_{tt}^2 q_1 + (u_2)_{tt}^2 q_2 dx,
\end{aligned}
\tag{3.2.22}
$$

which justifies the expression (3.2.20).

Remark 3.2.4. (*Lyapunov functions and stability*)

— *The existence of a basic C^1 Lyapunov function for a quasilinear hyperbolic system implies the exponential stability for the C^1 norm of this system. A proof for the general case is given in [106].*

— *Similarly the existence a basic quadratic Lyapunov function for the L^2 (resp. H^2) norm implies the exponential stability of the system for the L^2 (resp. H^2) norm (see for instance the proof in [14] and in particular (4.50)).*

Finally we introduce the following notations, useful for the rest of the article,

$$
\begin{aligned}
\varphi_1 &= \exp\left(\int_0^x \frac{M_{11}(0,s)}{\Lambda_1} ds\right), \\
\varphi_2 &= \exp\left(\int_0^x \frac{M_{22}(0,s)}{\Lambda_2} ds\right), \\
\varphi &= \frac{\varphi_1}{\varphi_2},
\end{aligned}
\tag{3.2.23}
$$

$$
\begin{aligned}
a &= \varphi M_{12}(0,\cdot), \\
b &= M_{21}(0,\cdot)/\varphi.
\end{aligned}
\tag{3.2.24}
$$

While the function φ_1 and φ_2 represent the influence of the diagonal terms of $M(0,\cdot)$ that would lead to an exponential variation of the amplitude on $[0,L]$ in the absence of coupling between u_1 and u_2, the function a and b represent the coupling term of $M(0,\cdot)$ after a change of variables on the system to remove the diagonal coefficients of M (see (3.4.1) and (3.4.4)).

We can now state the main results.

3.3 Main results

3.3.1 Stability of a general 2×2 hyperbolic system for the C^1 norm

Theorem 3.3.1. *Let a 2×2 quasilinear hyperbolic system of the form (3.2.8) be such that $\Lambda_1\Lambda_2 > 0$. Assume that*

$$G'(0) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \text{, where}$$
$$k_1^2 < \exp\left(\int_0^L 2\frac{M_{11}(0,s)}{|\Lambda_1|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right), \tag{3.3.1}$$
$$k_2^2 < \exp\left(\int_0^L 2\frac{M_{22}(0,s)}{|\Lambda_2|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right).$$

Then there exists a basic C^1 Lyapunov function and a basic quadratic H^2 Lyapunov function. In particular, the null steady-state $\mathbf{u}^* \equiv 0$ of the system (3.2.8)–(3.2.9) is exponentially stable for the C^1 and the H^2 norms.

This theorem is a direct consequence of Theorem 3.1 in [106] and will be proven in Appendix 3.8.3. From this theorem, when the eigenvalues of the hyperbolic system have the same sign, the coupling between the two equations does not raise any obstruction to the stability in the H^2 and in C^1 norm, so this case poses no challenge. We will therefore focus on the case where the eigenvalues have opposite signs, and without loss of generality we can assume that $\Lambda_1 > 0$ and $\Lambda_2 < 0$.

Theorem 3.3.2. *Let a 2×2 quasilinear hyperbolic system be of the form (3.2.8), where A and B are C^3 functions with $\Lambda_1 > 0$ and $\Lambda_2 < 0$. There exists a control of the form (3.2.9) such that there exists a basic C^1 Lyapunov function, if and only if*

$$d_1' = \frac{|a(x)|}{\Lambda_1} d_2, \tag{3.3.2}$$
$$d_2' = -\frac{|b(x)|}{|\Lambda_2|} d_1, \tag{3.3.3}$$

admit a positive solution d_1, d_2 on $[0,L]$ or equivalently

$$\begin{aligned} \eta' &= \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right| \eta^2, \\ \eta(0) &= 0, \end{aligned} \tag{3.3.4}$$

admits a solution on $[0,L]$, where a and b are defined in (3.2.24).
Moreover if one of the previous condition is verified and

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2\left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{3.3.5}$$

where d_1 and d_2 are any positive solution of (3.3.2)–(3.3.3), then the system (3.2.8)–(3.2.9) is exponentially stable for the C^1 norm.

Remark 3.3.1. *This result can be used in general to find good Lyapunov functions numerically and to estimate the limit length under which the stability is guaranteed by solving linear ODEs which are quite simple to handle.*

The third equivalence together with the criterion given in [11] (recalled in Section 3.4) can be used to show a link between the H^2 and C^1 stability. This link is given in the following corollary

Corollary 1. *Let a 2×2 quasilinear hyperbolic system be of the form (3.2.8),(3.2.9), where A and B are C^3 functions and such that $\Lambda_1 > 0$ and $\Lambda_2 < 0$.*

1. *If there exists a basic C^1 Lyapunov function then there exists a boundary control of the form (3.2.9) such that there exists a basic quadratic Lyapunov function for the H^2 norm.*
Moreover, if $M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0$, then the converse is true.

2. *In particular if the system (3.2.8),(3.2.9) admits a basic C^1 Lyapunov function and*

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{3.3.6}$$

where d_1 and d_2 are positive solutions of (3.3.2)–(3.3.3), then under the same boundary control there exists a basic quadratic Lyapunov function for the H^2 norm.
Conversly if the system admits a basic quadratic H^2 Lyapunov function and $M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0$ and

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \left(\frac{\varphi(L)}{\eta(L)}\right)^2 \text{ and } k_1^2 < \eta(0)^2, \tag{3.3.7}$$

where η is a positive solution of

$$\eta' = \left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta^2\right|, \tag{3.3.8}$$

then there exists a basic C^1 Lyapunov function.

Remark 3.3.2. — *The existence of a positive solution to (3.3.8) is guaranteed by [11] when there exists a basic quadratic Lyapunov function for the H^2 norm. This result is recalled in Theorem 3.4.2.*

— *The converse of 1. is wrong in general. An example where the system admits a basic quadratic H^2 Lyapunov function but no basic C^1 Lyapunov function, whatever are the boundary controls, is provided in Appendix 3.8.2.*

— *To our knowledge the only such link that existed so far consists in the trivial case where $B \equiv 0$ and where there consequently always exists both a basic quadratic H^2 Lyapunov function and a basic C^1 Lyapunov function. This link can be in fact extended to the H^p and C^q stability with $p \geq 2$ and $q \geq 1$ with the same condition (see Section 6).*

This theoretical link can be complemented by the following practical theorem that enables to construct basic quadratic H^2 Lyapunov functions from basic C^1 Lyapunov functions and conversely when possible.

Theorem 3.3.3. *If there exists a boundary control of the form (3.2.9) such that there exists a basic C^1 Lyapunov function with coefficients g_1 and g_2, then for any $0 < \varepsilon < \min_{[0,L]}((\varphi_2/\varphi_1)\sqrt{g_1/g_2})/L$ there exists a boundary control of the form (3.2.9) such that*

$$\frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}}\varphi_1\varphi_2 - \varphi_1^2\varepsilon Id\right) \quad \text{and} \quad \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}}\varphi_1\varphi_2 \tag{3.3.9}$$

are coefficients of a basic quadratic Lyapunov functions for the H^2 norm, where Id refers to the identity function.

If there exists a basic quadratic H^2 Lyapunov function with coefficients (q_1, q_2) and if $M_{12}(0,\cdot)M_{21}(0,\cdot) \geq 0$, then for all $A \geq 0$ and $\varepsilon > 0$ there exists a boundary control of the form (3.2.9) such that g_1 and g_2 defined by:

$$g_1(x) = A\exp\left(2\int_0^x \frac{M_{11}(0,\cdot)}{\Lambda_1} - \frac{|M_{12}(0,\cdot)|}{\Lambda_1}\sqrt{\frac{|\Lambda_1|q_1}{|\Lambda_2|q_2}}ds - \varepsilon x\right), \tag{3.3.10}$$

$$g_2 = \frac{|\Lambda_2|q_2}{\Lambda_1 q_1} g_1, \tag{3.3.11}$$

induce a basic C^1 Lyapunov function.

3.3.2 Stability of the general Saint-Venant equations for the C^1 norm

We introduce the nonlinear Saint-Venant equations with a slope and a dissipative source term resulting from the friction:

$$\begin{aligned}
&\partial_t H + \partial_x(HV) = 0,\\
&\partial_t V + \partial_x\left(\frac{V^2}{2} + gH\right) + \left(\frac{kV^2}{H} - C\right) = 0,
\end{aligned} \tag{3.3.12}$$

where $k > 0$ is the constant friction coefficient, g is the acceleration of gravity, and C is the constant slope coefficient. We denote by (H^*, V^*) the steady-state around which we want to stabilize the system, and we assume $gH^* - V^{*2} > 0$ such that the propagation speeds have opposite signs, i.e. the system is in fluvial regime (see [2] in particular (63)). The case where the propagation speeds have same sign raises no difficulty and is treated by Theorem 3.1. We show two results depending on whether the slope or the friction is the most influent.

Theorem 3.3.4. *Consider the nonlinear Saint-Venant equations (3.3.12) with the boundary control:*

$$\begin{aligned}
h(t,0) &= b_1 v(t,0),\\
h(t,L) &= b_2 v(t,L),
\end{aligned} \tag{3.3.13}$$

such that

$$b_1 \in \left(-\frac{H^*(0)}{V^*(0)}, -\frac{V^*(0)}{g}\right) \text{ and } b_2 \in \mathbb{R} \setminus \left[-\frac{H^*(L)}{V^*(L)}, -\frac{V^*(0)}{g}\right]. \tag{3.3.14}$$

If $kV^{*2}(0)/H^*(0) > C$, this system admits a basic C^1 Lyapunov function and the steady-state (H^*, V^*) is exponentially stable for the C^1 norm.

Remark 3.3.3. *It could seem surprising at first that the condition (3.3.14) that appears is the same as the condition that appears for the existence of a basic quadratic H^2 Lyapunov function (see [110]). This is an illustration of the second part of Corollary 1.*

Theorem 3.3.5. *Consider the nonlinear Saint-Venant equations (3.3.12) on a domain $[0, L]$. If $kV^{*2}(0)/H^*(0) < C$ then:*

1. *There exists $L_1 > 0$ such that if $L < L_1$, there exists boundary controls of the form (3.2.9) such that the system admits a basic C^1 Lyapunov function and (H^*, V^*) is exponentially stable for the C^1 norm.*

2. *There exists $L_2 > 0$ independent from the boundary control such that, if $L > L_2$, the system does not admit a basic C^1 Lyapunov function.*

Remark 3.3.4. *This last result is all the more interesting since it has been shown that for any $L > 0$ the system always admits a basic quadratic H^2 Lyapunov function (see [110]).*

3.4 $\mathbf{C^1}$ stability of a 2×2 quasilinear hyperbolic system and link with basic quadratic $\mathbf{H^2}$ Lyapunov functions

In this section we prove Theorem 3.3.2, Corollary 1 and Theorem 3.3.3. For convenience in the computations, let us first introduce the following change of variables to remove the diagonal coefficients of the source term:

$$\begin{aligned} z_1(t,x) &= \varphi_1(x)u_1(t,x), \\ z_2(t,x) &= \varphi_2(x)u_2(t,x), \end{aligned} \tag{3.4.1}$$

where φ_1 and φ_2 are given by (3.2.23). This change of variables can be found in [11] and is inspired from [127, Chapter 9]. Then the system (3.2.8) becomes

$$\mathbf{z}_t + A_2(\mathbf{z},x)\mathbf{z}_x + M_2(\mathbf{z},x)\mathbf{z} = 0, \tag{3.4.2}$$

where

$$A_2(0,x) = A(0,x), \tag{3.4.3}$$

$$M_2(0,x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}, \tag{3.4.4}$$

with a and b given by (3.2.24), and (3.2.9) becomes:

$$\begin{pmatrix} \mathbf{z}_+(0) \\ \mathbf{z}_-(L) \end{pmatrix} = G_1 \begin{pmatrix} \mathbf{z}_+(L) \\ \mathbf{z}_-(0) \end{pmatrix}. \tag{3.4.5}$$

where G_1 as the same regularity than G. Showing the existence of a basic C^1 Lyapunov function (resp. a basic quadratic H^2 Lyapunov function) for the system (3.2.8)–(3.2.9) is obviously equivalent to showing the existence of a basic C^1 Lyapunov function (resp. a basic quadratic H^2 Lyapunov function) for the system (3.4.2), (3.4.5), and the stability of the steady-state $\mathbf{u}^* \equiv 0$ in (3.2.8)–(3.2.9) is equivalent to the stability of the steady-state $\mathbf{z}^* \equiv 0$ in (3.4.2), (3.4.5). We now state two useful Lemma that can be found for instance in [105]:

Lemme 3.4.1. *Let $n \in \mathbb{N}^*$. Consider the ODE problem*

$$\begin{aligned} y' &= f(x,y,s), \\ y(0) &= y_0, \end{aligned} \tag{3.4.6}$$

where $y_0 \in \mathbb{R}^n$. If $f \in C^0(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and is locally Lipschitz in y for any $s \in \mathbb{R}$, then for all $s \in \mathbb{R}$ (3.4.6) has a maximum solution y_s defined on an interval I_s, and the function $(x,s) \to y_s(x)$ is continuous on $\{(x,s) \in \mathbb{R}^2 : s \in \mathbb{R}, x \in I_s\}$.

Lemme 3.4.2. *Let $L > 0$ and let g and f be continuous functions on $[0,L] \times \mathbb{R}_+$ and locally Lipshitz with respect to their second variable such that*

$$g(x,y) \geq f(x,y) \geq 0, \ \forall \ (x,y) \in [0,L] \times \mathbb{R}_+. \tag{3.4.7}$$

If there exists a solution y_1 on $[0,L]$ to

$$\begin{aligned} y_1' &= g(x,y_1), \\ y_1(0) &= y_0, \end{aligned} \tag{3.4.8}$$

with $y_0 \in \mathbb{R}_+$, then there exists a solution y on $[0,L]$ to

$$\begin{aligned} y' &= f(x,y), \\ y(0) &= y_0, \end{aligned} \tag{3.4.9}$$

and in addition $0 \leq y \leq y_1$ on $[0,L]$.

Let us now prove Theorem 3.3.2, which is mainly based on the results in [106].

Proof of Theorem 3.3.2. Let a 2×2 quasilinear hyperbolic system be of the form (3.4.2). Using Theorem 3.2 in [106] on (3.4.2) we know that there exists a boundary control of the form of (3.2.9) such that there exists a basic C^1 Lyapunov function if and only if:

$$\begin{aligned} f_1' &\leq -\frac{2|a(x)|}{\Lambda_1}\frac{f_1^{3/2}}{\sqrt{f_2}}, \\ f_2' &\geq \frac{2|b(x)|}{|\Lambda_2|}\frac{f_2^{3/2}}{\sqrt{f_1}}, \end{aligned} \tag{3.4.10}$$

admit a solution on [0,L] with $f_1 > 0$ and $f_2 > 0$ on $[0, L]$. But as f_1 and f_2 are positive this is equivalent to say that:

$$\left(\frac{1}{\sqrt{f_1}}\right)' \geq \frac{|a(x)|}{\Lambda_1}\frac{1}{\sqrt{f_2}}, \tag{3.4.11}$$

$$\left(\frac{1}{\sqrt{f_2}}\right)' \leq -\frac{|b(x)|}{|\Lambda_2|}\frac{1}{\sqrt{f_1}}. \tag{3.4.12}$$

Denoting $d_1 = 1/\sqrt{f_1}$ and $d_2 = 1/\sqrt{f_2}$ and checking that $(f_1, f_2) \in \mathbb{R}_+^*$ is equivalent to $(d_1, d_2) \in \mathbb{R}_+^*$, the existence of a solution with positive components to (3.4.10) is equivalent to having a solution with positive components on $[0, L]$ to the system:

$$\begin{aligned} d_1' &\geq \frac{|a(x)|}{\Lambda_1}d_2, \\ d_2' &\leq -\frac{|b(x)|}{|\Lambda_2|}d_1. \end{aligned} \tag{3.4.13}$$

Let us show that this is equivalent to the existence of a solution with positive components on $[0, L]$ to the system (3.3.2)-(3.3.3). One way is obvious: if there exists a solution with positive components to (3.3.2)-(3.3.3) then it is also a solution with positive components to (3.4.13). Let us show the other way: suppose that there exists a solution (d_1, d_2) to (3.4.13) with positive components on $[0, L]$. Then:

$$\left(\frac{d_1}{d_2}\right)' \geq \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\left(\frac{d_1}{d_2}\right)^2. \tag{3.4.14}$$

Hence from Lemma 3.4.2 the system:

$$\eta' = \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \tag{3.4.15}$$

$$\eta(0) = \frac{d_1(0)}{d_2(0)}, \tag{3.4.16}$$

admits a solution on $[0, L]$. We can now define g_2 as the unique solution of:

$$\begin{aligned} g_2' &= -\eta\left|\frac{b(x)}{\Lambda_2}\right|g_2, \\ g_2(0) &= d_2(0) > 0, \end{aligned} \tag{3.4.17}$$

and $g_1 = \eta g_2$. Thus g_1 and g_2 exist on $[0, L]$, and take only positive values and

$$g_1' = \frac{|a(x)|}{\Lambda_1}g_2, \tag{3.4.18}$$

$$g_2' = -\left|\frac{b(x)}{\Lambda_2}\right|g_1. \tag{3.4.19}$$

Therefore this system admits a solution (g_1, g_2) with positive components on $[0, L]$. This ends the proof of the first equivalence.

To prove the second equivalence, note from the previous that if there exists a solution to (3.4.13) with positive components on $[0, L]$ then there exists a function η on $[0, L]$ such that:

$$\begin{aligned} \eta' &= \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \\ \eta(0) &> 0. \end{aligned} \tag{3.4.20}$$

Therefore by comparison the system :

$$\begin{aligned} \eta' &= \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \\ \eta(0) &= 0, \end{aligned} \tag{3.4.21}$$

admits a solution on $[0, L]$.

Conversely, if (3.4.21) admits a solution on $[0, L]$ then there exists $\varepsilon > 0$ such that:

$$\begin{aligned} \eta' &= \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \\ \eta(0) &= \varepsilon, \end{aligned} \tag{3.4.22}$$

admits a solution η_ε on $[0, L]$. Defining as previously g_2 the unique solution of:

$$\begin{aligned} g_2' &= -\eta_\varepsilon \left| \frac{b(x)}{\Lambda_2} \right| g_2, \\ g_2(0) &= \eta_\varepsilon(0) > 0, \end{aligned} \tag{3.4.23}$$

and $g_1 = \eta_\varepsilon g_2$, then g_1 and g_2 and (g_1, g_2) is solution on $[0, L]$ of the system (3.3.2)-(3.3.3). This ends the proof of the second equivalence.

It remains now only to prove that if one of the previous conditions is verified, and if the boundary conditions (3.2.9) satisfy (3.3.5), then the system (3.2.8)–(3.2.9) is exponentially stable for the C^1 norm. Suppose that the system (3.3.2)-(3.3.3) admits a solution (d_1, d_2) on $[0, L]$ where d_1 and d_2 are positive, then from the previous, (3.4.10) admits a solution (f_1, f_2) on $[0, L]$ where $f_1 = d_1^{-2}$ and $f_2 = d_2^{-2}$ are positive. Therefore, as $y \to y_1^{3/2}/\sqrt{y_2}$ is C^1 and hence locally Lipshitz on $\mathbb{R}_+^*$, and from Lemma 3.4.1, there exists $\sigma_1 > 0$ such that for all $0 \le \sigma < \sigma_1$ there exists on $[0, L]$ a solution $(f_{1,\sigma}, f_{2,\sigma})$ of the system

$$\begin{aligned} f_{1,\sigma}' &= -\frac{2|a(x)|}{\Lambda_1} \frac{f_{1,\sigma}^{3/2}}{\sqrt{f_{2,\sigma}}} - \sigma, \\ f_{2,\sigma}' &= \frac{2|b(x)|}{|\Lambda_2|} \frac{f_{2,\sigma}^{3/2}}{\sqrt{f_{1,\sigma}}} + \sigma, \end{aligned} \tag{3.4.24}$$

with $f_{1,\sigma} > 0$ and $f_{2,\sigma} > 0$. Note that from the proof of Theorem 3.1 in [106] (see in particular (4.29),(4.37),(4.45) and note that $K := G'(0)$), when these $f_{1,\sigma}$ and $f_{2,\sigma}$ exist, one only needs to show the following condition to have a basic C^1 Lyapunov function:

$$\begin{aligned} &\exists \alpha > 0,\ \exists \mu > 0,\ \exists p_1 \ge 0 : \forall p \ge p_1, \\ &\Lambda_1(L) f_{1,\sigma}(L)^p (z_1(t, L))^{2p} e^{-2p\mu L} - |\Lambda_2(L)| f_{2,\sigma}(L)^p \left(G_1'(0). \begin{pmatrix} z_1(t, L) \\ z_2(t, 0) \end{pmatrix} \right)_2^{2p} e^{2p\mu L} \\ &+ |\Lambda_2(0)| f_{2,\sigma}(0)^p z_2^{2p}(t, 0) - \Lambda_1(0) f_{1,\sigma}(0)^p \left(G_1'(0). \begin{pmatrix} z_1(t, L) \\ z_2(t, 0) \end{pmatrix} \right)_1^{2p} > \alpha(z_1^{2p} + z_2^{2p}) \end{aligned} \tag{3.4.25}$$

where G_1 is given by (3.4.5). As (3.4.25) only needs to be true for one particular $\sigma > 0$ and using that $f_{1,\sigma}$ and $f_{2,\sigma}$ are continuous with σ and $f_{i,0} = f_i$ for $i \in \{1, 2\}$, by continuity one only needs to show that:

$$\begin{aligned}
&\exists \alpha > 0,\ \exists \mu > 0,\ \exists p_1 \geq 0 : \forall p \geq p_1,\\
&\Lambda_1(L) f_1(L)^p (z_1(t,L))^{2p} e^{-2p\mu L} - |\Lambda_2(L)| f_2(L)^p \left(G_1'(0).\begin{pmatrix} z_1(t,L) \\ z_2(t,0) \end{pmatrix}\right)_2^{2p} e^{2p\mu L}\\
&+ |\Lambda_2(0)| f_2(0)^p z_2^{2p}(t,0) - \Lambda_1(0) f_1(0)^p \left(G_1'(0).\begin{pmatrix} z_1(t,L) \\ z_2(t,0) \end{pmatrix}\right)_1^{2p} > \alpha(z_1^{2p} + z_2^{2p}).
\end{aligned} \tag{3.4.26}$$

Now under hypothesis (3.3.5) and with the change of variables (3.4.1) we have

$$G_1'(0) = \begin{pmatrix} 0 & k_1 \\ \frac{k_2}{\varphi(L)} & 0 \end{pmatrix}. \tag{3.4.27}$$

Therefore the condition (3.4.26) becomes:

$$\begin{aligned}
&\exists \alpha > 0,\ \exists \mu > 0,\ \exists p_1 \geq 0 : \forall p \geq p_1,\\
&(\Lambda_1(L) f_1(L)^p e^{-2p\mu L} - k_2^{2p} \varphi^{-2p}(L) |\Lambda_2|(L) f_2(L)^p e^{2p\mu L})(z_1(t,L))^{2p}\\
&+ (|\Lambda_2(0)| f_2(0)^p - k_1^{2p} \Lambda_1(0) f_1(0)^p) z_2^{2p}(t,0) > \alpha(z_1^{2p} + z_2^{2p}).
\end{aligned} \tag{3.4.28}$$

But as $f_1 = d_1^{-2}$ and $f_2 = d_2^{-2}$, from (3.3.5)

$$\begin{aligned}
\varphi^{-2}(L) f_2(L) k_2^2 &< f_1(L),\\
f_1(0) k_1^2 &< f_2(0).
\end{aligned} \tag{3.4.29}$$

Therefore by continuity there exists $\alpha > 0$, $\mu > 0$ and $p_1 \geq 0$ such that

$$\begin{aligned}
\forall p \geq p_1,\ e^{2\mu L} (|\Lambda_2|(L))^{1/p} \varphi^{-2}(L) f_2(L) k_2^2 &< f_1(L) (\Lambda_1(L))^{1/p} e^{-2\mu L},\\
\text{and} \quad (\Lambda_1(0))^{1/p} f_1(0) k_1^2 &< f_2(0) (|\Lambda_2|(0))^{1/p},
\end{aligned} \tag{3.4.30}$$

and therefore (3.4.28) is verified, hence the system admits a basic C^1 Lyapunov function and is exponentially stable for the C^1 norm. This ends the proof of Theorem 3.3.2. □

Remark 3.4.1. *This theorem has a theoretical interest as it gives a simple criterion to ensure the stability of the system, but it has also a numerical interest. By computing numerically d_2 and seeking the first point where it vanishes, one can find the limit length L_{max} above which there cannot exist a basic C^1 Lyapunov function and under which the stability is guaranteed. Then the coefficients of the boundary feedback control can also be designed numerically using d_1 and d_2 thus computed. Moreover, finding d_1 and d_2 only consists in solving two linear ODEs and is therefore computationally very easy to achieve. An example is given with the Saint-Venant equations in Section 3.5 to illustrate this statement. Finally the second equivalence is useful to show Corollary 1.*

Before proving Corollary 1, let us first state the following theorem dealing with the stability in the L^2 norm of linear hyperbolic systems:

Theorem 3.4.1 ([11]). *Let a linear hyperbolic system be of the form:*

$$z_t + \begin{pmatrix} \Lambda_1(x) & 0 \\ 0 & \Lambda_2(x) \end{pmatrix} z_x + \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix} z = 0, \tag{3.4.31}$$

with $\Lambda_1 > 0$ and $\Lambda_2 < 0$. There exists a boundary control of the form (3.2.9) such that there exists a basic quadratic Lyapunov function for the L^2 norm and for this sytem if and only if there exists a function η on $[0, L]$ solution of:

$$\begin{aligned}
\eta' &= \left| \frac{b}{|\Lambda_2|} \eta^2 + \frac{a}{\Lambda_1} \right|,\\
\eta(0) &= 0.
\end{aligned} \tag{3.4.32}$$

Besides for any $\sigma > 0$ such that

$$\begin{aligned} \eta_\sigma' &= \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta_\sigma^2 \right|, \\ \eta_\sigma(0) &= \sigma, \end{aligned} \tag{3.4.33}$$

has a solution η_σ on $[0, L]$ then

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \text{ with } l_1^2 < \eta_\sigma^2(0) \text{ and } l_2^2 < \frac{1}{\eta_\sigma^2(L)}, \tag{3.4.34}$$

are suitable boundary conditions such that there exists a quadratic L^2 Lyapunov function for the system (3.4.31), (3.4.34).

Such Lyapunov function guarantees the global exponential stability in the L^2 norm for a linear system under suitable boundary controls of the form (3.2.9). This result can be extended to the stability in the H^2 norm when the system is nonlinear, namely we have:

Theorem 3.4.2. *Let a quasilinear hyperbolic system be of the form (3.4.2) with $\Lambda_1 > 0$ and $\Lambda_2 < 0$, where the Λ_i are defined in (3.2.6). There exists a boundary control of the form (3.4.5) such that there exists a basic quadratic Lyapunov function for the H^2 norm for this system if and only if there exists a function η on $[0, L]$ solution of:*

$$\begin{aligned} \eta' &= \left| \frac{b}{|\Lambda_2|}\eta^2 + \frac{a}{\Lambda_1} \right|, \\ \eta(0) &= 0. \end{aligned} \tag{3.4.35}$$

Besides for any $\sigma > 0$ such that

$$\begin{aligned} \eta_\sigma' &= \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta_\sigma^2 \right|, \\ \eta_\sigma(0) &= \sigma, \end{aligned} \tag{3.4.36}$$

has a solution η_σ on $[0, L]$ then

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \text{ with } l_1^2 < \eta_\sigma^2(0) \text{ and } l_2^2 < \frac{1}{\eta_\sigma^2(L)}, \tag{3.4.37}$$

are suitable boundary conditions such that there exists a quadratic H^2 Lyapunov function for the system (3.4.2), (3.4.37).

The proof of this theorem is straightforward and is given in Appendix 3.8.3. Knowing Theorem 3.4.2, we can prove Corollary 1.

Proof of Corollary 1. Let a 2×2 quasilinear hyperbolic system be of the form (3.4.2).
Let us suppose that there exists a boundary control such that there exists a basic C^1 Lyapunov function. Then from Theorem 3.3.2 there exists η_1 solution on $[0, L]$ of:

$$\begin{aligned} \eta_1' &= \left| \frac{a}{\Lambda_1} \right| + \left| \frac{b}{\Lambda_2} \right| \eta_1^2, \\ \eta_1(0) &= 0, \end{aligned} \tag{3.4.38}$$

and from Lemma 3.4.2 there also exists η solution on $[0, L]$ of

$$\begin{aligned} \eta' &= \left| \frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|}\eta^2 \right|, \\ \eta(0) &= 0. \end{aligned} \tag{3.4.39}$$

Therefore, from Theorem 3.4.2 there exists a boundary control of the form (3.2.9) such that there exists a basic quadratic Lyapunov function for the H^2 norm for this system.
Let us suppose now that $M_{12}(0,\cdot).M_{22}(0,\cdot) \geq 0$, then from (3.2.24) $ab \geq 0$. Thus (3.4.39) and (3.4.38) are the same equations and therefore, from Theorems 3.3.2 and Theorem 3.4.2 if there exists a boundary control of the form (3.4.5) such that there exists a basic quadratic Lyapunov function for the H^2 norm, then there also exists a boundary control of the form (3.4.5) such that the system admits a basic C^1 Lyapunov function. This ends the proof of the first part of Corollary 1.

Let us now show the second part of Corollary 1. As previously, from (3.4.1) we only need to show the result for the equivalent system (3.4.2), (3.4.5). Observe first that from Theorem 3.4.2, for any $\sigma > 0$ such that

$$\begin{aligned} \eta_2' &= \left| \frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|}\eta_2^2 \right|, \\ \eta_2(0) &= \sigma, \end{aligned} \tag{3.4.40}$$

has a solution η_2 on $[0, L]$, then

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \quad \text{with} \quad l_1^2 < \eta_2^2(0) \quad \text{and} \quad l_2^2 < \frac{1}{\eta_2^2(L)}, \tag{3.4.41}$$

are suitable boundary conditions such that there exists a basic quadratic H^2 Lyapunov function for the system (3.4.2), (3.4.5), where G_1 is given by (3.4.5). Now, let us suppose that there exists a basic C^1 Lyapunov function for the system (3.4.2), (3.4.5). From the first part of Corollary 1, there exists a boundary control of the form (3.4.5) such that there exists a basic quadratic Lyapunov function for the H^2 norm. Let us suppose that

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)} \right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)} \right)^2, \tag{3.4.42}$$

where d_1 and d_2 are positive solutions of (3.3.2)–(3.3.3). Note that defining $\eta_3 = d_1/d_2$ and $\sigma = \eta_3(0) = d_1(0)/d_2(0) > 0$ the condition (3.4.42) is equivalent to

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \quad \text{with} \quad l_1^2 < \eta_3^2(0) \quad \text{and} \quad l_2^2 < \frac{1}{\eta_3^2(L)}. \tag{3.4.43}$$

As from (3.3.2)-(3.3.3),

$$\begin{aligned} \eta_3' &= \frac{d_1'}{d_2} - \frac{d_1 d_2'}{d_2^2} = \left| \frac{a}{\Lambda_1} \right| + \left| \frac{b}{\Lambda_2} \right| \eta_3^2, \\ \text{and } \eta_3(0) &= \sigma, \end{aligned} \tag{3.4.44}$$

then from Lemma 3.4.2 the problem (3.4.40) has a solution η_2 on $[0, L]$ and $\eta_3(L) \geq \eta_2(L)$. Therefore $G'(0)$ also satisfies (3.4.41). Hence there exists a basic quadratic Lyapunov function for the H^2 norm and the system is exponentially stable for the H^2 norm.

Let us now show the other way. Suppose that $M_{12}(0,\cdot)M_{22}(0,\cdot) \geq 0$ and that the system admits a basic quadratic Lyapunov function for the H^2 norm. Then from Theorem 3.4.2 and by continuity of the solutions with respect to the initial conditions there exists $\sigma > 0$ such that:

$$\begin{aligned} \eta_2' &= \left| \frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|}\eta_2^2 \right|, \\ \eta_2(0) &= \sigma. \end{aligned} \tag{3.4.45}$$

has a solution on $[0, L]$, that we denote η_2. From hypothesis (3.3.7) there exists such $\sigma > 0$ such that the condition (3.3.7) is still satisfied with η_2. We can define

$$d_2 = \exp\left(-\int_0^x \eta_2(s) \left| \frac{b(s)}{\Lambda_2(s)} \right| ds \right), \tag{3.4.46}$$

$$d_1 = \eta_2 d_2. \tag{3.4.47}$$

As $|ab| = ab$, then (d_1, d_2) is a solution of (3.3.2)–(3.3.3) and $d_1 > 0$ and $d_2 > 0$, and from (3.3.7) and (3.4.46)–(3.4.47).

$$k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2. \tag{3.4.48}$$

Hence from Theorem 3.3.2 there exists a basic C^1 Lyapunov function. This ends the proof of Corollary 1.

□

Proof of Theorem 3.3.3. Let us first note from (3.4.1) that (g_1, g_2) are the coefficients of a basic C^1 Lyapunov function for the system (3.2.8) if and only if (f_1, f_2) are the coefficients of a basic C^1 Lyapunov for the system (3.4.2) with

$$f_i = \frac{g_i}{\varphi_i^2},\ i \in \{1,2\}, \tag{3.4.49}$$

Therefore, we will first prove the result for (3.4.2) and then use the change of coordinates (3.4.1) and the transformation (3.4.49) to come back to the system (3.2.8). Let a 2×2 quasilinear hyperbolic system be of the form (3.4.2). Suppose that there exist boundary controls of the form (3.2.9) such that there exists a basic C^1 Lyapunov function with coefficients f_1 and f_2. Then from [106] (see in particular Theorem 3.2), one has:

$$\begin{aligned} f_1' &\leq -\frac{2|a(x)|}{\Lambda_1}\frac{f_1^{3/2}}{\sqrt{f_2}}, \\ f_2' &\geq \frac{2|b(x)|}{|\Lambda_2|}\frac{f_2^{3/2}}{\sqrt{f_1}}, \end{aligned} \tag{3.4.50}$$

Now let us denote $d_1 = f_1^{-1/2}$ and $d_2 = f_2^{-1/2}$, then

$$d_1' \geq \frac{|a(x)|}{\Lambda_1} d_2, \tag{3.4.51}$$

$$d_2' \leq -\frac{|b(x)|}{|\Lambda_2|} d_1. \tag{3.4.52}$$

Therefore:

$$-\left(\frac{d_1}{d_2}\right)'\left(\frac{d_2}{d_1}\right)' \geq \left(\left|\frac{b}{\Lambda_2}\right|\left(\frac{d_1}{d_2}\right)^2 + \left|\frac{a}{\Lambda_1}\right|\right)\left(\left|\frac{b}{\Lambda_2}\right| + \left|\frac{a}{\Lambda_1}\right|\left(\frac{d_2}{d_1}\right)^2\right). \tag{3.4.53}$$

Hence:

$$-\left(\frac{d_1}{d_2}\right)'\left(\frac{d_2}{d_1}\right)' \geq \left(\left|\frac{b}{\Lambda_2}\right|\left(\frac{d_1}{d_2}\right) + \left|\frac{a}{\Lambda_1}\right|\left(\frac{d_2}{d_1}\right)\right)^2. \tag{3.4.54}$$

Now let us take $\varepsilon > 0$ such that $\varepsilon L < \min_{[0,L]} (d_2/d_1)$. Note that from (3.4.49) this is equivalent to $\varepsilon L < \min_{[0,L]} \left(\varphi_2/\varphi_1\sqrt{g_1/g_2}\right)$, where g_1 and g_2 are the coefficients of the basic C^1 Lyapunov function for the original system (3.2.8). We have:

$$\left(\frac{d_1(x)}{d_2(x)}\right)' \geq 0 \tag{3.4.55}$$

$$\text{and } -\left(\frac{d_1(x)}{d_2(x)}\right)'\left(\frac{d_2(x)}{d_1(x)} - \varepsilon x\right)' > \left(\left|\frac{b(x)}{\Lambda_2(x)}\right|\left(\frac{d_1(x)}{d_2(x)}\right) + \left|\frac{a(x)}{\Lambda_1(x)}\right|\left(\frac{d_2(x)}{d_1(x)} - \varepsilon x\right)\right)^2. \tag{3.4.56}$$

It can be shown (see [11] or [14] in particular Theorem 6.10 for more details) that this condition implies that

$$\frac{1}{\Lambda_1}\left(\frac{d_2}{d_1} - \varepsilon Id\right) \text{ and } \frac{d_1}{|\Lambda_2| d_2}$$

are the coefficients of a basic quadratic Lyapunov function for the H^2 norm for the system (3.4.2) for some boundary controls of the form (3.4.5). Equivalently this means that, for some boundary controls of the form (3.4.5),

$$\begin{aligned} V(t) = & \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{f_1(x)}{f_2(x)}} - \varepsilon x\right) z_1^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{f_2}{f_1}} z_2^2(t,x)dx \\ & + \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{f_1(x)}{f_2(x)}} - \varepsilon x\right) (\partial_t z_1)^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{f_2}{f_1}} (\partial_t z_2)^2(t,x)dx \\ & + \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{f_1(x)}{f_2(x)}} - \varepsilon x\right) (\partial_{tt}^2 z_1)^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{f_2}{f_1}} (\partial_{tt}^2 z_2)^2(t,x)dx \end{aligned} \tag{3.4.57}$$

is a Lyapunov function for the H^2 norm. Therefore using (3.4.49) and performing the inverse change of coordinates to go from (3.4.2) to (3.2.8), V can also be written as

$$\begin{aligned} V(t) = & \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}} - \frac{\varphi_1}{\varphi_2}\varepsilon x\right) \varphi_1\varphi_2 u_1^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}} \varphi_1\varphi_2 u_2^2(t,x)dx \\ & + \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}} - \frac{\varphi_1}{\varphi_2}\varepsilon x\right) \varphi_1\varphi_2 (\partial_t u_1(t,x))^2 + \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}} \varphi_1\varphi_2 (\partial_t u_2(t,x))^2dx \\ & + \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}} - \frac{\varphi_1}{\varphi_2}\varepsilon x\right) \varphi_1\varphi_2 (\partial_{tt}^2 u_1(t,x))^2 + \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}} \varphi_1\varphi_2 (\partial_{tt}^2 u_2(t,x))^2dx, \end{aligned} \tag{3.4.58}$$

where g_1 and g_2 are the coefficients of the basic C^1 Lyapunov function of the system (3.2.8), and this concludes the proof of the first part of Theorem 3.3.3.

To show the second part of Theorem 3.3.3 suppose that $M_{12}M_{21} \geq 0$. Therefore from (3.2.24), $ab \geq 0$. Suppose also that (l_1, l_2) are the coefficients of a basic quadratic Lyapunov functions for the H^2 norm for the system (3.4.2), (3.4.5). Define $h_i = |\Lambda_i| l_i$ and

$$f_1(x) = A \exp\left(-\int_0^x 2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_1}{h_2}}ds - \varepsilon x\right), \tag{3.4.59}$$

$$f_2 = \frac{h_2}{h_1} f_1, \tag{3.4.60}$$

where $A > 0$ and $\varepsilon > 0$ are taken arbitrarly. We have:

$$f_1' = -2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_1}{h_2}} f_1 - \varepsilon f_1 < -2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_1}{h_2}} f_1 = -2\frac{|a|}{\Lambda_1}\frac{f_1^{3/2}}{\sqrt{f_2}}, \tag{3.4.61}$$

and

$$\left(\left(\frac{h_2}{h_1}\right)'\right)^2 = \left(h_2'\frac{1}{h_1} + \left(\frac{1}{h_1}\right)' h_2\right)^2 \geq 4h_2'\left(\frac{1}{h_1}\right)'\frac{h_2}{h_1}. \tag{3.4.62}$$

Besides l_i are the coefficients of a basic quadratic Lyapunov function for the H^2 norm for (3.4.2) therefore (see [11], in particular (41)-(43)) $h_1' < 0$, $h_2' > 0$ and

$$h_2'\left(\frac{1}{h_1}\right)' > \left(\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right)^2. \tag{3.4.63}$$

Let us denote

$$I_1 := \left(h_2'\left(\frac{1}{h_1}\right)'\right)^{1/2} - \left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right| > 0. \tag{3.4.64}$$

Thus from (3.4.61), (3.4.62) and (3.4.63):

$$\begin{aligned} f_2' &\geq 2\sqrt{\frac{h_2}{h_1}}\left|\frac{a}{\Lambda_1}+\frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right| f_1 + 2\sqrt{\frac{h_2}{h_1}} I_1 f_1 + f_1'\frac{h_2}{h_1} \\ &= 2\sqrt{\frac{h_2}{h_1}}\left|\frac{a}{\Lambda_1}+\frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right| f_1 + 2\sqrt{\frac{h_2}{h_1}} I_1 f_1 - 2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_2}{h_1}} f_1 - \varepsilon f_1 \frac{h_2}{h_1}. \end{aligned} \tag{3.4.65}$$

Assuming now that $\varepsilon < 2\ \min_{[0,L]}(I_1\sqrt{h_1/h_2})$, we have, as $|ab| = ab$ and h_1 and h_2 are positive,

$$\begin{aligned} f_2' &> 2\sqrt{\frac{h_2}{h_1}}\frac{|b|}{|\Lambda_2|}\frac{h_2}{h_1} f_1 \\ &= 2\frac{|b|}{|\Lambda_2|}\frac{f_2^{3/2}}{\sqrt{f_1}}. \end{aligned} \tag{3.4.66}$$

Therefore from (3.4.65) and (3.4.66) and Theorem 3.1 in [106], there exists a boundary control of the form (3.4.5) such that (f_1, f_2) induce a basic C^1 Lyapunov function for the system (3.4.2). Performing the inverse change of coordinates and using (3.4.1) there exists a boundary control of the form (3.2.9) such that (f_1, f_2) induce a C^1 basic Lyapunov function, where

$$g_1(x) = A\exp\left(2\int_0^x \frac{M_{11}(0,\cdot)}{\Lambda_1} - \frac{|M_{12}(0,\cdot)|}{\Lambda_1}\sqrt{\frac{|\Lambda_1|q_1}{|\Lambda_2|q_2}} ds - \varepsilon x\right), \tag{3.4.67}$$

$$g_2 = \frac{|\Lambda_2|q_2}{|\Lambda_1|q_1} g_1, \tag{3.4.68}$$

and (q_1, q_2) are the coefficients inducing a basic quadratic Lyapunov function for the H^2 norm for the system (3.2.8)–(3.2.9). This ends the proof of Theorem 3.3.3.

□

3.5 An application to the Saint-Venant equations

In this section we will show Theorem 3.3.4 and Theorem 3.3.5. Before proving these results, we recall some properties of the Saint-Venant equations. The steady-states (H^*, V^*) of (3.3.12) are the solutions of:

$$\begin{aligned} \partial_x(H^* V^*) &= 0, \\ \partial_x\left(\frac{V^{*2}}{2} + gH^*\right) &= \left(C - \frac{kV^{*2}}{H^*}\right). \end{aligned} \tag{3.5.1}$$

Under the assumption of physical fluvial (also called subcritical) regime, i.e. $0 < V^* < \sqrt{gH^*}$, these equations reduce to:

$$\begin{aligned} H_x^* &= -H^*\frac{V_x^*}{V^*}, \\ V_x^* &= V^*\frac{\frac{kV^{*2}}{H^*} - C}{gH^* - V^{*2}}, \end{aligned} \tag{3.5.2}$$

and have a unique maximal solution for a given $H^*(0)$ and $V^*(0)$ verifying $V^*(0) < \sqrt{gH^*(0)}$. Observe now that there are three different cases:

— $\frac{kV^{*2}(0)}{H^*(0)} > C$, i.e. the friction is larger than the slope. In this case V^* is an increasing function, H^* is a decreasing function and therefore the friction stays larger than the slope on the whole domain, i.e. $kV^{*2}/H^* > C$.

— $\frac{kV^{*2}(0)}{H^*(0)} = C$, in this case the steady-states are uniform and thus defined on $[0, +\infty)$ and in particular they are defined on $[0, L]$ for any $L > 0$.

— $\frac{kV^{*2}(0)}{H^*(0)} < C$, in this case V^* is an decreasing function, H^* is an increasing function, therefore the slope stays larger than the friction on the whole domain. Note that the system moves away from the critical regime, therefore the solution (H^*, V^*) is defined on $[0, +\infty)$ and in particular it is defined on $[0, L]$ for any $L > 0$. A more rigorous proof will be given later on (see the the beginning of the proof of Theorem 3.3.5).

Therefore, it is enough to look at the difference between the friction and the slope at the initial point $x = 0$ to know whether the slope or the friction is larger on the whole domain.
We will consider a steady-state (H^*, V^*) with $H(0)^* = H_0^*$ and $V(0)^* = V_0^*$ the associated initial conditions, and we define now the perturbations:

$$h = H - H^* \text{ and } v = V - V^*. \tag{3.5.3}$$

Assuming subcritical regime, i.e. $V^* < \sqrt{gH^*}$, the Saint-Venant equations (3.3.12) can be transformed using the transformation described by (3.2.1)–(3.2.2)→(3.2.8)–(3.2.9) in

$$\partial_t \mathbf{u} + A(\mathbf{u}, x)\partial_x \mathbf{u} + M(\mathbf{u}, x)\mathbf{u} = 0, \tag{3.5.4}$$

where

$$\mathbf{u} = \begin{pmatrix} v + h\sqrt{\frac{g}{H^*}} \\ v - h\sqrt{\frac{g}{H^*}} \end{pmatrix}, \tag{3.5.5}$$

$$\Lambda_1 = V^* + \sqrt{gH^*}, \tag{3.5.6}$$

$$\Lambda_2 = V^* - \sqrt{gH^*}, \tag{3.5.7}$$

$$\begin{aligned} M(0,\cdot) = & \frac{kV^{*2}}{H^*}\begin{pmatrix} -\frac{3}{4(\sqrt{gH^*}+V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}} & -\frac{1}{4(\sqrt{gH^*}+V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}} \\ \frac{1}{4(\sqrt{gH^*}-V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}} & \frac{3}{4(\sqrt{gH^*}-V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}} \end{pmatrix} \\ & - C\begin{pmatrix} -\frac{3}{4(\sqrt{gH^*}+V^*)} & -\frac{1}{4(\sqrt{gH^*}+V^*)} \\ \frac{1}{4(\sqrt{gH^*}-V^*)} & \frac{3}{4(\sqrt{gH^*}-V^*)} \end{pmatrix}. \end{aligned} \tag{3.5.8}$$

Observe that the system is indeed strictly hyperbolic under small perturbations as $\Lambda_1 > 0$ and $\Lambda_2 < 0$. The derivation of Λ_1, Λ_2 and $M(0,\cdot)$ will not be detailed here but is quite straightforward and the expression (3.5.6)–(3.5.8) can be found for instance in [14, Section 1.4.2].

Proof of Theorem 3.3.4. Let us suppose that the flow is in the fluvial regime on $[0, L]$, therefore

$$V^*(x) < \sqrt{gH^*(x)}, \ \forall\, x \in [0, L], \tag{3.5.9}$$

and suppose that $kV_0^{*2}/H_0^* > C$. From the previous we have $kV^{*2}/H^* > C$ on $[0, L]$. Thus from (3.5.8) $M_{12}(0,\cdot) \geq 0$ and $M_{21}(0,\cdot) \geq 0$. Before going any further let us note that we know from [110] that there exists a basic quadratic H^2 function for the system (3.5.4) for some boundary controls of the form (3.4.5), therefore Corollary 1 applies and there exists a boundary control of the form (3.4.5) such that there exists a basic C^1 Lyapunov function for this system.

Moreover from [110] (see Lemma 3.1), we know that

$$\eta = \frac{\varphi_1 |\Lambda_2|}{\varphi_2 \Lambda_1}, \tag{3.5.10}$$

is a positive solution of:

$$\eta' = \frac{b}{|\Lambda_2|}\eta^2 + \frac{a}{\Lambda_1}. \tag{3.5.11}$$

Therefore from the second part of Corollary 1 one has that if

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \tag{3.5.12}$$

$$\text{with } k_2^2 < \frac{\varphi^2(L)}{\eta^2(L)} \text{ and } k_1^2 < \eta^2(0), \tag{3.5.13}$$

where G is given by (3.2.9) and η by (3.5.10), then the system (3.5.4), (3.2.9) admits a basic C^1 Lyapunov function and is exponentially stable for the C^1 norm.

It is therefore enough to show that the boundary conditions (3.3.13) under hypothesis (3.3.14) are equivalent to boundary conditions of the form (3.2.9) satisfying the previous condition (3.5.12)–(3.5.13) in the new system (3.5.4), obtained by the change of variables (3.5.5). Observe that from (3.3.13) we have

$$\begin{aligned}
v(t,0) + h(t,0)\sqrt{\frac{g}{H^*(0)}} &= k_1\left(v(t,0) - h(t,0)\sqrt{\frac{g}{H^*(0)}}\right),\\
v(t,L) - h(t,L)\sqrt{\frac{g}{H^*(L)}} &= k_2\left(v(t,L) + h(t,L)\sqrt{\frac{g}{H^*(0)}}\right),
\end{aligned} \tag{3.5.14}$$

where

$$k_1 := \left(\frac{1+\sqrt{\frac{g}{H^*(0)}}b_1}{1-\sqrt{\frac{g}{H^*(0)}}b_1}\right), \tag{3.5.15}$$

$$\frac{1}{k_2} := \left(\frac{1+\sqrt{\frac{g}{H^*(L)}}b_2}{1-\sqrt{\frac{g}{H^*(L)}}b_2}\right), \tag{3.5.16}$$

and therefore:

$$\begin{aligned}
u_1(t,0) &= k_1 u_2(t,0),\\
u_2(t,L) &= k_2 u_1(t,L).
\end{aligned} \tag{3.5.17}$$

Therefore after the change of variables given by (3.5.5), the boundary conditions (3.3.13) are equivalent to boundary conditions of the form (3.2.9) satisfying (3.5.12). All it remains to do is to prove that under the hypothesis (3.3.14), the boundary conditions (3.5.17) also satisfy the condition (3.5.13). Now observe that from (3.5.10), (3.5.15) and (3.5.16), the condition (3.5.13) becomes:

$$\left(\frac{1+\sqrt{\frac{g}{H^*(0)}}b_1}{1-\sqrt{\frac{g}{H^*(0)}}b_1}\right)^2 < \left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2, \tag{3.5.18}$$

$$\left(\frac{1+\sqrt{\frac{g}{H^*(L)}}b_2}{1-\sqrt{\frac{g}{H^*(L)}}b_2}\right)^2 > \left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2, \tag{3.5.19}$$

which is equivalent to

$$\left(1-\left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2\right) + \left(1-\left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2\right)\left(\sqrt{\frac{g}{H^*(0)}}b_1\right)^2 + 2\left(1+\left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2\right)\sqrt{\frac{g}{H^*(0)}}b_1 < 0, \tag{3.5.20}$$

$$\left(1-\left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2\right) + \left(1-\left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2\right)\left(\sqrt{\frac{g}{H^*(L)}}b_2\right)^2 + 2\left(1+\left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2\right)\sqrt{\frac{g}{H^*(L)}}b_2 > 0, \tag{3.5.21}$$

and from (3.5.6) and (3.5.7) this is equivalent to having

$$b_1 \in \left(-\frac{H^*(0)}{V^*(0)}, -\frac{V^*(0)}{g}\right) \text{ and } b_2 \in \mathbb{R}\setminus\left[-\frac{H^*(L)}{V^*(L)}, -\frac{V^*(0)}{g}\right]. \tag{3.5.22}$$

which is exactly (3.3.14). Therefore under boundary conditions (3.3.13) and hypothesis (3.3.14) the system admits a basic C^1 Lyapunov function and is therefore stable for the C^1 norm, this ends the proof. □

Proof of Theorem 3.3.5. Let us suppose that $C - kV_0^{*2}/H_0^* > 0$ and $gH_0^* > V_0^{*2}$. Then there exists a unique maximal solution (H^*, V^*) to the equations (3.5.2). Let us prove that this solution is defined on $[0, +\infty)$. Denoting $L_0 \in (0, \infty]$ the limit such that the maximal solution is defined on $[0, L_0)$, we have from the beginning of this section, in particular (3.5.9), that for all $x \in [0, L_0)$, H^* and V^* are continuous, positive, and:

$$\frac{kV^{*2}}{H^*} > 0, \tag{3.5.23}$$
$$gH^* > V^{*2}. \tag{3.5.24}$$

Therefore from (3.5.2), H^* is an increasing function and V^* is a decreasing function. Besides, as H^*V^* remains constant, both H^* and V^* remain positive. From (3.5.2) we can get an estimate on the growth of H^*:

$$H_x^*(1 - \frac{V^{*2}}{gH^*}) = \frac{C}{g} - \frac{kV^{*2}}{gH^*}, \tag{3.5.25}$$

therefore

$$\frac{C}{g} - \frac{kV_0^{*2}}{gH_0^*} \leq H_x^*(1 - \frac{V^{*2}}{gH^*}) \leq \frac{C}{g}, \tag{3.5.26}$$

hence

$$0 < \frac{C}{g} - \frac{kV_0^{*2}}{gH_0^*} \leq H_x^* \leq \frac{C}{g(1 - \frac{V_0^{*2}}{gH_0^*})}. \tag{3.5.27}$$

Thus H_x^* is bounded, Hence $L_0 = +\infty$, as H^* is an increasing function and cannot explode in finite length and as $V^* = H_0^* V_0^*/H^*$ from (3.5.2).

Consider now the Saint-Venant equations transformed into the system (3.5.4) with (3.5.5)–(3.5.8). The first part of the theorem is straightforward from Theorem 3.3.2 as there exists $L_1 > 0$ such that (3.3.4) admits a solution on $[0, L_1]$. Let us now suppose by contradiction that for any $L > 0$ there exists a basic C^1 Lyapunov function for the system. Then from Theorem 3.3.2, for any $L > 0$ there exists a solution η on $[0, L]$ of

$$\begin{aligned} \eta' &= \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right|\eta^2, \\ \eta(0) &= 0, \end{aligned} \tag{3.5.28}$$

where a, b, Λ_1 and Λ_2 are given by (3.2.24), (3.5.6) and (3.5.7). From (3.5.27), H goes to $+\infty$ when x goes to $+\infty$, and from (3.5.8) we have

$$\begin{aligned}
\frac{M_{11}(0,\cdot)}{\Lambda_1} + \frac{M_{22}(0,\cdot)}{|\Lambda_2|} =& \frac{3C}{4}\left(\frac{1}{\Lambda_1^2} - \frac{1}{\Lambda_2^2}\right) + \frac{kV^*}{H^*}\left(\frac{1}{\Lambda_1} + \frac{1}{|\Lambda_2|}\right) \\
&+ \frac{kV^{*2}}{H^*}\left[\frac{3}{4}\left(\frac{1}{|\Lambda_2|^2} - \frac{1}{\Lambda_1^2}\right) + (\frac{1}{|\Lambda_2|} - \frac{1}{\Lambda_1})\frac{1}{2\sqrt{gH^*}}\right], \\
=& \frac{-3C\sqrt{gH^*}V^*}{(gH^* - V^{*2})^2} + \sqrt{\frac{g}{H^*}}\frac{2kV^*}{gH^* - V^{*2}} + \frac{kV^{*2}}{H^*}\left[\frac{3\sqrt{gH^*}V^*}{(gH^* - V^{*2})^2} + \frac{V^*}{(gH^* - V^{*2})\sqrt{gH^*}}\right], \\
=& \sqrt{\frac{g}{H^*}}\frac{1}{(gH^* - V^{*2})}\left[\frac{-3CQ}{gH^* - V^{*2}} + \frac{2kQ}{H^*}\right] \\
&+ \frac{kQ^2}{H^{*3}}\left[\sqrt{\frac{g}{H^*}}\frac{3Q}{(gH^* - V^{*2})^2} + \frac{2Q}{H^*(gH^* - V^{*2})\sqrt{gH^*}}\right], \\
=& \sqrt{\frac{g}{H^*}}\frac{1}{(gH^* - V^{*2})}\left[\frac{-3CQ + 2kQg - 2kQ\frac{Q^2}{H^{*3}}}{gH^* - V^{*2}}\right] + O\left(\frac{1}{H^{*5}}\right), \\
=& O\left(\frac{1}{H^{*5/2}}\right).
\end{aligned} \tag{3.5.29}$$

We used here that H^*V^* is constant and therefore $1/(gH^* - V^{*2}) = O(1/H^*)$ when H^* (or equivalently x) goes to infinity. But from (3.5.27) we know that $H^* \geq \left(C/g - kV_0^{*2}/gH_0^*\right) x + H_0^*$. Therefore

$$\frac{M_{11}(0,x)}{\Lambda_1} + \frac{M_{22}(0,x)}{|\Lambda_2|} = O\left(\frac{1}{x^{5/2}}\right) \text{ for } x \to +\infty, \tag{3.5.30}$$

and thus is integrable. Hence

$$\lim_{x\to+\infty} \frac{\varphi_1(x)}{\varphi_2(x)} = C_2 > 0. \tag{3.5.31}$$

Let us look at a/Λ_1 and $b/|\Lambda_2|$. We have

$$\begin{aligned}\frac{a}{\Lambda_1} =& \varphi\frac{M_{12}(0,\cdot)}{\Lambda_1} \\ =& \varphi\left(\frac{C - \frac{kV^{*2}}{H^*}}{4(\sqrt{gH^*}+V^*)} + \frac{kV^{*2}}{H^*\Lambda_1}\left[\frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right]\right) \\ =& \frac{\varphi}{gH^*}\left(\frac{C - \frac{kV^{*2}}{H^*}}{4(1+\frac{V^*}{\sqrt{gH^*}})^2} + \frac{kQ^2 g}{H^{*2}(\sqrt{gH^*}+V^*)}\left[\frac{H^*}{Q} + \frac{1}{2\sqrt{gH^*}}\right]\right).\end{aligned} \tag{3.5.32}$$

Therefore

$$\lim_{x\to+\infty} \frac{agH^*}{\Lambda_1} = \frac{C_2 C}{4}. \tag{3.5.33}$$

Similarly we can obtain:

$$\lim_{x\to+\infty} \frac{bgH^*}{|\Lambda_2|} = -\frac{C}{4C_2}. \tag{3.5.34}$$

Therefore there exists $x_1 \in (0,\infty)$ such that for all $x > x_1$

$$\left|\frac{a}{\Lambda_1}\right| \geq \frac{C_2 C}{5gH^*}, \tag{3.5.35}$$

$$\left|\frac{b}{\Lambda_2}\right| \geq \frac{C}{5C_2 gH^*}. \tag{3.5.36}$$

Let $L > x_1$, by assumption equation (3.5.28) has a solution η defined on $[0,L]$ and from (3.5.35) and (3.5.36), for all $x > x_1$

$$\eta' \geq \frac{C}{5gH^*}(C_2 + \frac{\eta^2}{C_2}) \geq \frac{C_3 C}{5gH^*}(1+\eta^2), \tag{3.5.37}$$

where $C_3 = \min\left(C_2, \frac{1}{C_2}\right)$. From (3.5.27) right-hand side and (3.5.37) we have

$$\frac{\eta'}{(1+\eta^2)} \geq \frac{C_3 C}{5gH^*} \geq \frac{C_3\left(1 - \frac{V_0^{*2}}{gH_0^*}\right)}{5(x + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*)}, \tag{3.5.38}$$

hence

$$\int_{x_1}^{x} \frac{\eta'}{(1+\eta^2)}dx \geq \int_{x_1}^{x} \frac{C_3\left(1 - \frac{V_0^{*2}}{gH_0^*}\right)}{5\left(x + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*\right)}dx. \tag{3.5.39}$$

Thus

$$\arctan(\eta(x)) - \arctan(\eta(x_1)) \geq \frac{C_3}{5}\left(1 - \frac{V_0^{*2}}{gH_0^*}\right) \ln\left(\frac{x + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}{x_1 + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}\right). \tag{3.5.40}$$

Note that the right-hand side does not depend on η and L and that

$$\lim_{x\to+\infty}\frac{C_3}{5}\left(1-\frac{V_0^{*2}}{gH_0^*}\right)\ln\left(\frac{x+\frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C}gH_0^*}{x_1+\frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C}gH_0^*}\right)=+\infty. \tag{3.5.41}$$

Therefore, as this is true for any $L>0$ we can choose L such that

$$\frac{C_3}{5}\left(1-\frac{V_0^{*2}}{gH_0^*}\right)\ln\left(\frac{L+\frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C}gH_0^*}{x_1+\frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C}gH_0^*}\right)\geq\frac{\pi}{2}. \tag{3.5.42}$$

By hypothesis, there still exist a function η that verifies (3.5.40), is positive and defined on $[0,L]$ with this choice of L. Hence, $\arctan(\eta(L))<\pi/2$. But, as η is positive, $\arctan(\eta(x_1))>0$ so we have from (3.5.40) and (3.5.42)

$$\arctan(\eta(L))>\frac{\pi}{2} \tag{3.5.43}$$

Hence we have a contradiction. Therefore there exists $L_2>0$ such that for any $L>L_2$ there do not exist a basic C^1 Lyapunov function whatever the boundary conditions are. This ends the proof. □

3.6 Numerical estimation

From Theorem 3.3.4 when $kV_0^{*2}/H_0^*\geq C$ there exists explicit static boundary controls under which the general Saint-Venant equations are exponentially stable for the C^1 norm, whatever the length of the channel. When $kV_0^{*2}/H_0^*<C$ no such explicit result exists but in practice we can however use Theorem 3.3.2 to find the limit length under which stability can be guaranteed. We provide here some numerical estimations under reasonable conditions ($Q^*=1\ m^2.s^{-1}$, $V_0^*=0.5\ m.s^{-1}$, $k=0.002$). On Figure 3.1, one can see that for a 50km channel with a constant slope such that $C=2kV_0^{*2}/H_0^*$, η exists and there is no problem. In Figure 3.2, we extended the channel until the limit length L_{max} for this system and it appears that $L_{max}>10^4km$ and that $H^*(L_{max})>100m$ which is quite unrealistic in current hydraulic applications. This suggest that for nearly all hydraulic applications it will be possible to design boundary conditions such that there exists a basic C^1 Lyapunov function that ensures the stability of the system for the C^1 norm.

3.7 Further details

The previous results were derived for the C^1 and the H^2 norm but they can actually be extended to the C^q and the H^p norm with the same conditions, for any $p\in\mathbb{N}^*\setminus\{1\}$ and $q\in\mathbb{N}^*$. To show that, one only needs to realize that Theorem 3.1 and 3.2 in [106] and Theorem 3.4.1 (and therefore Theorem 3.4.2) are true for the C^q and the H^p norm with the same conditions, for any $p\in\mathbb{N}^*\setminus\{1\}$ and $q\in\mathbb{N}^*$.
In conclusion we gave explicit conditions on the gain of the feedbacks to get exponential stability for the C^1 norm for 2×2 quasilinear hyperbolic systems with propagation speeds of the same sign. In the general case we derived a simple criterion for the existence of basic C^1 Lyapunov functions and a practical way to derive admissible static feedback gains when this criterion is satisfied, simply by solving an ODE. We showed that under some conditions on the coefficients of the source term the existence of a H^p and C^q basic Lyapunov function for any $p\in\mathbb{N}^*\setminus\{1\}$ and $q\in\mathbb{N}^*$ are equivalent and that, in the general case, the existence of a C^q Lyapunov function for any $q\in\mathbb{N}^*$ for some appropriate boundary controls implies the existence of a basic quadratic H^p Lyapunov function for any $p\in\mathbb{N}^*\setminus\{1\}$ for some appropriate boundary controls. Finally we showed that when the friction is larger than the slope the general nonlinear Saint-Venant equations can be

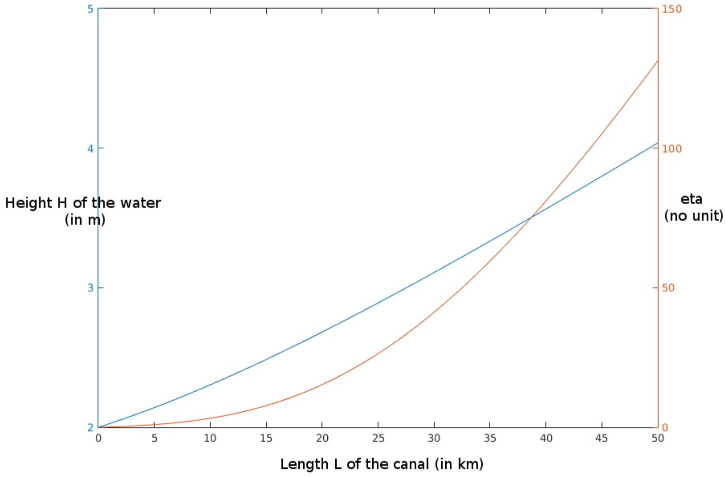

Figure 3.1 – In the x-axis is represented the length of the water channel, in blue the height of the water for the stationary state and in red the value of η.

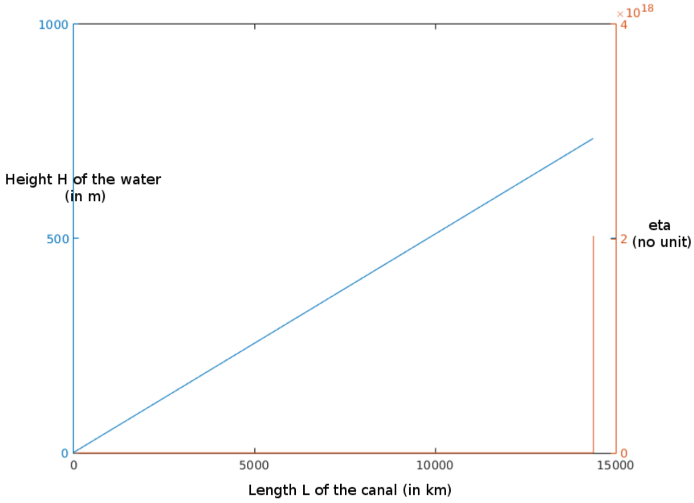

Figure 3.2 – In the x-axis is represented the length of the water channel, in blue the height of the water in stationary state and in red the value of η, one can see the limit length L_{max} at which η explodes.

stabilized for the C^1 norm by means of simple pointwise feedback, and we gave explicit conditions on the feedbacks. When the the slope is larger than the friction no such general result can be shown. However, we showed that for nearly all applications the Saint-Venant equations can be stabilized by means of such simple feedbacks.

3.8 Appendix

3.8.1 Explicit form of B and regularity of A and B

Applying the transformation (3.2.3) on the system (3.2.8), B is given by:

$$B(\mathbf{u}, x) = N(x)(F(\mathbf{Y})(\mathbf{Y}_x^* + (N^{-1}(x))'\mathbf{u}) + D(\mathbf{Y})). \tag{3.8.1}$$

As $\mathbf{Y}^*$ is a steady-state, it verifies the equation:

$$F(\mathbf{Y}^*)\partial_x \mathbf{Y}^* = -G(\mathbf{Y}^*). \tag{3.8.2}$$

Thus if we suppose that F and G are C^p on $\mathcal{B}_{\mathbf{Y}^*,\eta_0}$, where $p \in \mathbb{N}^*$, as F is strictly hyperbolic with non-vanishing eigenvalues, $\mathbf{Y}^*$ is C^{p+1} on $[0, L]$. Therefore, using (3.2.5) and (3.8.1), A and B are also C^p on $\mathcal{B}_{\mathbf{0},\eta_0} \times [0, L]$.

3.8.2 Counter exemple of the converse of Corollary 1 in general

As mentionned earlier, from [110] we know that for any $L > 0$ the system (3.5.4) corresponding to the Saint-Venant equations with boundary conditions (3.4.5) admits a basic quadratic H^2 Lyapunov function. However from Theorem 3.3.5 we know that there exists L_{max} such that for $L > L_{max}$ the system does not admit a C^1 Lyapunov function whatever the boundary control is. This is a counter exemple of the Corollary 1 when one cannot ensure that M_{12} and M_{21} have the same sign.

3.8.3 Proof of Theorem 3.3.1

In this Theorem we rely mainly on Theorem 3.1 of [106]. Let a quasilinear 2×2 hyperbolic system of the form (3.2.8)–(3.2.9) be with $\Lambda_1\Lambda_2 > 0$. Without loss of generality we can assume that $\Lambda_1 > 0$ and $\Lambda_2 > 0$. As previously this system is equivalent to the system (3.4.2), (3.4.5) (see Section 3.4) and the existence of a basic C^1 (resp. basic quadratic H^2) Lyapunov function for this system is equivalent to the existence of a basic C^1 (resp. basic quadratic H^2) Lyapunov function for the system (3.2.8)–(3.2.9). Let us suppose that

$$\begin{aligned} G'(0) &= \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \text{ such that} \\ k_1^2 &< \exp\left(\int_0^L 2\frac{M_{11}(0,s)}{\Lambda_1} - 2\max\left(\frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2}\right) ds\right), \\ k_2^2 &< \exp\left(\int_0^L 2\frac{M_{22}(0,s)}{\Lambda_2} - 2\max\left(\frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2}\right) ds\right), \end{aligned} \tag{3.8.3}$$

then from (3.4.1) we have

$$\begin{aligned} G_1'(0) &= \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \text{ such that} \\ l_1^2 &< \exp\left(-2\int_0^L \max\left(\frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2}\right) ds\right), \\ l_2^2 &< \exp\left(-2\int_0^L \max\left(\frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2}\right) ds\right), \end{aligned} \tag{3.8.4}$$

where G_1 is defined in (3.4.5). Let us define $f_1 = f_2 = f$ by

$$f(x) = \exp\left(-2\int_0^x \max\left(\frac{|a(s)|}{\Lambda_1(s)}, \frac{|b(s)|}{\Lambda_2(s)}\right) ds\right), \forall x \in [0, L]. \tag{3.8.5}$$

Then f is positive and C^1 on $[0, L]$ and

$$\begin{aligned} f' &\leq -2\frac{|a(x)|}{\Lambda_1(x)}\frac{f^{3/2}}{f^{1/2}}, \\ \text{and } f' &\leq -2\frac{|b(x)|}{\Lambda_2(x)}\frac{f^{3/2}}{f^{1/2}}. \end{aligned} \tag{3.8.6}$$

Besides, as $G_1'(0)$ is diagonal, if we define $\rho_\infty : M \to \min(\|\Delta M \Delta^{-1}\|_\infty : \Delta \in D_2^+)$ where D_2^+ is the space of diagonal 2×2 matrix with positive coefficients we have from (3.8.4):

$$\rho_\infty(G_1'(0)) = \max(l_1, l_2) < \sqrt{\frac{f(L)}{f(0)}}. \tag{3.8.7}$$

Therefore, from (3.8.6) and (3.8.7) and Theorem 3.1 in [106], there exists a basic C^1 Lyapunov function and therefore the system (3.2.8)–(3.2.9) is stable for the C^1 norm.
Let us now show the stability for the H^2 norm by showing that there exists a basic quadratic H^2 Lyapunov function for the system (3.4.2), (3.4.5). From (3.8.7) and by continuity we know that there exists $\sigma > 0$ such that

$$\max(l_1, l_2) < \sqrt{\frac{g(L)}{g(0)}}, \tag{3.8.8}$$

where g is defined by

$$g(x) = \exp\left(-2\int_0^x \max\left(\frac{|a(s)|}{\Lambda_1(s)}, \frac{|b(s)|}{\Lambda_2(s)}\right) ds\right) - \sigma, \forall x \in [0, L]. \tag{3.8.9}$$

We want now to be able to apply Theorem 6.6 in [14] which would give the result. Note that if we now define $q_1 = g/\Lambda_1$ and $q_2 = g/\Lambda_2$ we have

$$\begin{aligned}&-\begin{pmatrix}(\Lambda_1 q_1)' & 0\\ 0 & (\Lambda_2 q_2)'\end{pmatrix} + \begin{pmatrix}q_1 & 0\\ 0 & q_2\end{pmatrix}\begin{pmatrix}0 & a\\ b & 0\end{pmatrix} + \begin{pmatrix}0 & a\\ b & 0\end{pmatrix}^T\begin{pmatrix}q_1 & 0\\ 0 & q_2\end{pmatrix}\\ &= \begin{pmatrix}-g' & g\left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right)\\ g\left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right) & -g'\end{pmatrix},\end{aligned} \tag{3.8.10}$$

and this matrix is positive definite as $g' < 0$ and:

$$g'^2 - g^2\left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right)^2 > \left(4\max\left(\frac{|a|}{\Lambda_1}, \frac{|b|}{\Lambda_2}\right)^2 - \left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right)^2\right) g^2 \geq 0. \tag{3.8.11}$$

Besides

$$\begin{aligned}&\begin{pmatrix}q_1(L)\Lambda_1(L) & 0\\ 0 & q_2(L)\Lambda_2(L)\end{pmatrix} - G_1'(0)^T\begin{pmatrix}q_1(0)\Lambda_1(0) & 0\\ 0 & q_2(0)\Lambda_2(0)\end{pmatrix}G_1'(0)\\ &= \begin{pmatrix}g(L) - (1-\sigma)l_1^2 & 0\\ 0 & g(L) - (1-\sigma)l_2^2\end{pmatrix}\end{aligned} \tag{3.8.12}$$

is positive semi-definite from (3.8.8). Therefore from Theorem 6.6 in [14] we know that the system admits a basic quadratic H^2 Lyapunov function and is stable for the H^2 norm. This ends the proof of Theorem 3.3.1.

Remark 3.8.1. *Although the existence of a basic quadratic H^2 Lyapunov function is not stated directly in Theorem 6.6 in [14] one can easily check that the theorem actually proves the H^2 stability by showing that there exists a basic quadratic H^2 Lyapunov function as defined in Definition 3.2.4 (see in particular Lemma 6.8 in [14]).*

Acknowledgments

The author would like to thank Jean-Michel Coron for his advice and his constant support. The author would like to thank Sébastien Boyaval, Shengquan Xiang and Christophe Zhang for their remarks and fruitful discussions. Finally the author would also like to thank the ETH-FIM, for their warm welcome and their support, as well as the ANR project Finite4SoS (No.ANR 15-CE23-0007) and the French Corps des IPEF for their financial support.

Part II

Physical equations and density-velocity systems

Chapter 4

A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope

This chapter is taken from the following article (also referred to as [110]):

Amaury Hayat and Peipei Shang. A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope. Automatica J. IFAC, 100:52-60, 2019.

Abstract. The exponential stability problem of the nonlinear Saint-Venant equations is addressed in this chapter. We consider the general case where an arbitrary friction and space-varying slope are both included in the system, which lead to non-uniform steady-states. An explicit quadratic Lyapunov function as a weighted function of a small perturbation of the steady-states is constructed. Then we show that by a suitable choice of boundary feedback controls, that we give explicitly, the local exponential stability of the nonlinear Saint-Venant equations for the H^2-norm is guaranteed.

4.1 Introduction

Since discovered in 1871 by Barré de Saint-Venant [10], the shallow water equations (or Saint-Venant equations in unidimensional form) have been frequently used by hydraulic engineers in their practice. Their apparent simplicity and their ability to describe fairly well the behaviour of rivers and water channel make them a useful tool for many applications as for instance the regulation of navigable rivers and irrigation networks in agriculture. Among which, the problem of designing control tools to regularize the water level and the flow rate in the open hydraulic systems has been studied for a long time [71, 97, 137, 139, 141].

The Saint-Venant equations constitute a nonlinear 2×2 1-D hyperbolic system. In the last decades, the boundary feedback stabilization problem for 1-D hyperbolic systems has been widely investigated, and many tools have been developed. To our knowledge, the first result for nonlinear 2×2 homogeneous systems was obtained by Greenberg and Li [93] in the framework of C^1 solutions by using the characteristic method. Later on, this result was generalized by Qin [172] to $n \times n$ homogeneous systems. In 1999, Coron et al. introduced another method: the quadratic Lyapunov function, firstly used to analyze the asymptotic behavior of linear hyperbolic equations in the L^2 norm but then generalized for nonlinear hyperbolic equations in the framework of C^1 and H^2 solutions [48, 50, 52, 53]. Both of these two methods guarantee the exponential stability of the nonlinear homogeneous hyperbolic systems when the boundary conditions satisfy an appropriate sufficient dissipativity property. Such boundary conditions are the so-called static boundary feedback control and lead to feedbacks that only depend on the measures at the boundaries. However, when inhomogenous systems are considered, it is usually difficult (or even impossible) to construct a quadratic Lyapunov function with static boundary feedback [13, Chapter 5.6] or in [12]. The backstepping method introduced by Krstic et al. in [128] is a powerful tool to deal with the exponential stabilization of inhomogenous hyperbolic systems. Initially developed for parabolic equations [181], this method has been firstly applied to first order hyperbolic equations in [126], and then generalized to $(n+1) \times (n+1)$ linear hyperbolic systems with n positive and one negative characteristic speed in [76, 197]. The case of general bidirectional linear systems was recently treated in [112]. For the nonlinear case, one can refer to [65], where the authors designed a full-state feedback control actuated on only one boundary and achieved exponential stability for the closed-loop 2×2 quasilinear hyperbolic systems in H^2-norm. Later on, this result was generalized in [114] to $n \times n$ quasilinear hyperbolic systems. In particular, in the context of the Saint-Venant equations, the backstepping method has been used to achieve exponential stabilization of the linearized Saint-Venant Exner equations with arbitrary slope or friction in [78], both in subcritical and supercritical regime. It has also been used in [80] to stabilize a linearized bilayered Saint-Venant model, a 4×4 system of two Saint-Venant systems interacting with each other. With the backstepping method, one can realize rapid decay (i.e., exponential decay with arbitrary rate) or even finite time stabilization for some linear case [7, 60, 112]. However, one requires a full-state feedback control rather than static boundary feedback control depending only on the values at the boundaries. Nevertheless, in some cases, it is possible to design an observer to tackle this issue [75, 77, 78, 197]. In this chapter, we will use a direct Lyapunov approach to study the exponential stability for nonhomogeneous Saint-Venant equations with arbitrary friction and space-varying slope. The advantage is that using this method, we only need to measure the value at the boundary, which is much easier for practical implementations.

The first result concerning this method applied to Saint-Venant equations was obtained by Coron et al. in 1999 for the homogeneous case, i.e., without any friction or slope [52]. There, they use an entropy of the system as a Lyapunov function. But this Lyapunov function has only a semi-negative definite time derivative. One has to conclude the stability result using LaSalle's invariance principle which is usually difficult to apply due to the problem of precompactness of the trajectories. Later on, the authors introduce in [53] a strict Lyapunov function for conservation laws that can be diagonalized with Riemann invariants. The time derivative of the Lyapunov function can be made strictly negative definite by an appropriate choice of the boundary conditions. They apply the result to regulate the level and flow in an open channel without friction or slope. Under the assumption that the friction and the slope are sufficiently small in C^1-norm, with a bound depending on the length of the channel, the stability may be proved using the method of characteristics as in [83, 171]. Note that in [83], the results are applied to real data obtained from the river Sambre in Belgium. Semigroup approaches have also been developed in [85] but using proportional integral control rather than of purely static feedback control. The stability with sufficiently small coefficients may also be proved using a Lyapunov approach as in [47, 50]. In the special case where the slope is a constant and "compensate" with the influence of the friction, thus resulting the uniform steady-states, the stability of the linearized system was considered in [18]. There, the authors use the same Lyapunov function as in [53]. More recently, in [16], the authors managed to study the stability in H^2-norm of the nonlinear and nonhomogeneous system when the friction is arbitrary but in the absence of the slope. It should be noted that this last result was proved using a basic quadratic Lyapunov function (see [12] for a proper definition) independent of the length of the water channel which is not trivial as, usually, the existence of a basic quadratic Lyapunov function for a nonhomogenous system depends on the size of the domain [99]. Of course for realistic description of the behaviour of rivers one can easily understand that adding a slope is essential not only because it is the prime mover of the flow but also because in some common cases the effect of the slope can be much larger than the effect of the friction, both being non-negligible: it is the steep-slope regime (see for instance [43, Chapter 5-3]). To our knowledge, no study so far takes into account arbitrary non-negligible friction, slope, and river length through static feedback control, except in the special case mentioned previously where the slope compensate exactly the friction and cancels the source term [18].

Our contribution in this chapter is that we managed to construct an explicit Lyapunov control function to analyze the local exponential stability in H^2-norm of the nonlinear Saint-Venant equations with static boundary feedbacks in the case where both the friction and the slope are arbitrary (not necessarily small), and where the river length can be arbitrary as well. This enables us to design robust static feedback controllers to ensure the exponential stability of the steady-states of the system. Especially we deal with the case where the slope may vary with respect to the space variable. This is all the more important that the slope is likely to vary in a river, even sometimes on short distances. We first describe three regimes depending whether the influence of the slope is smaller, equal or greater to the influence of the friction and we show that the dynamics in two opposite regimes are inverted. Then we construct a quadratic Lyapunov function for the H^2-norm whatever the friction and the slope are.

The organization of the chapter is as follows. In Section 4.2, we give a description of the non-linear Saint-Venant equations together with some definitions and we state our main result (Theorem 4.2.2). In Section 4.3, the exponential stability of the linearized system is firstly studied by constructing a quadratic Lyapunov function. Based on the results of the linearized system, we then show that a similar expanded Lyapunov function enables us to get the exponential stability of the nonlinear system by properly choosing the boundary feedback controls. In Section 4.4, some numerical illustrations are given to support our theoretical result.

4.2 Description of the Saint-Venant equations and the main result

The non-linear Saint-Venant equations with slope and friction are given by the following system:

$$\partial_t H + \partial_x (HV) = 0, \tag{4.2.1}$$

$$\partial_t V + \partial_x \left(\frac{V^2}{2} + gH \right) + \left(\frac{kV^2}{H} - gC \right) = 0, \tag{4.2.2}$$

where $H(t,x)$ is the water depth, $V(t,x)$ is the horizontal water velocity. The slope $C(\cdot) \in C^2([0,L])$ is defined by $C(x) = -\dfrac{dB}{dx}$ with $B(x)$ the elevation of the bottom (bathymetry) which is therefore supposed to be C^3, g is the constant gravity acceleration and k is a constant friction coefficient. Note from (4.2.1) and (4.2.2) that the equilibrium H^* and V^* verifies:

$$H^*(x)V^*(x) = Q^*, \tag{4.2.3}$$

$$V^* V^*_x + gH^*_x + \left(\frac{kV^{*2}}{H^*} - gC \right) = 0. \tag{4.2.4}$$

As we are interested in physical stationary states, we suppose that $H^* > 0$ and $V^* > 0$. Therefore $Q^* > 0$ is any given constant set point and corresponds to the flow rate. Substituting (4.2.3) to (4.2.4), we get that V^* satisfies

$$V^*_x = \frac{V^{*2}(\frac{kV^{*3}}{Q^*} - gC)}{gQ^* - V^{*3}}. \tag{4.2.5}$$

Observe that the steady-states are therefore non necessary uniform. As we are interested in navigable rivers we also suppose that the flow is in the fluvial regime, i.e.,

$$gH^* > V^{*2} \tag{4.2.6}$$

or equivalently $gQ^* - V^{*3} > 0$. Then the system (4.2.1) and (4.2.2) has a positive and a negative eigenvalue and for any flow rate Q^*, equation (4.2.5) has a unique C^3 solution on $[0,L]$ with any given boundary data $V^*(0) = V^*_0$. Moreover, the steady-states have three possible dynamics depending on the slope as the following.

1. When $gC < \dfrac{kV^{*2}}{H^*}$, also known in hydraulic engineering as "mild slope regime". This covers also the case without slope. Note from (4.2.3) and (4.2.5) that in this case, H^* decreases while V^* increases and consequently the system becomes closer to the critical point where $gH^* = V^{*2}$ which is the limit of the fluvial regime.
2. When $gC = \dfrac{kV^{*2}}{H^*}$, which means that the friction and the slope "compensate" each other. When the slope C is in additionally constant, the steady-states are uniform. This special case has been studied in [18] and only for the linearized system.

3. When $gC > \frac{kV^{*2}}{H^*}$, also known in hydraulic engineering as "steep slope regime". Then the dynamics of the steady-states are inverted: H^* tends to increase while V^* decreases and consequently the system moves away from the limit of the fluvial regime defined by the critical point where $gH^* = V^{*2}$.

Our goal is to ensure the exponential stability of the steady-states of the nonlinear system (4.2.1) and (4.2.2) for all the above three cases under some boundary conditions of the form:

$$V(t,0) = \mathcal{B}(H(t,0)), \quad V(t,L) = \mathcal{B}(H(t,L)), \tag{4.2.7}$$

where $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ is of class C^2. These kind of boundary conditions are imposed by physical devices located at the ends of the channel where the controls are implemented, as for instance mobile spillways or tunable hydraulic gates as in irrigation canals and navigable rivers.

Taking for instance a spillway outflow gate at $x = L$ modelled by a simple linear model and imposing the inflow with a tunable hydraulic gate at $x = 0$, the boundary conditions are

$$H(t,0)V(t,0) = U_0(t),\ H(t,L)V(t,L) = \kappa(H(t,L) - U_1(t)),$$

where κ is a constant depending on the gate, and where U_0 the inflow and U_L the spillway gate elevation are our control functions with $H(t,0)$ and $H(t,L)$ the observations as shown in (4.2.7). Some other explicit expressions of the boundary conditions are given in [13, Section 1.4].

We will first prove the exponential stability for the linearized system for the L^2-norm. Note that for nonlinear systems, the stability depends on the topology considered as shown in [61]. In this chapter, we will consider the exponential stability in H^2-norm.

For any given initial condition

$$H(0,x) = H_0(x), \quad V(0,x) = V_0(x), \quad x \in [0,L], \tag{4.2.8}$$

we suppose that the following compatibility conditions hold

$$\begin{aligned} &V_0(0) = \mathcal{B}(H_0(0)), \quad V_0(L) = \mathcal{B}(H_0(L)), &(4.2.9)\\ &\partial_x(\frac{V_0^2}{2} + gH_0)(0) + \frac{kV_0^2}{H_0}(0) - gC(0) \\ &= \mathcal{B}'(H_0(0))\partial_x(H_0V_0)(0), &(4.2.10)\\ &\partial_x(\frac{V_0^2}{2} + gH_0)(L) + \frac{kV_0^2}{H_0}(L) - gC(L) \\ &= \mathcal{B}'(H_0(L))\partial_x(H_0V_0)(L). &(4.2.11) \end{aligned}$$

These compatibility conditions guarantee the well-posedness of the system (4.2.1), (4.2.2), (4.2.7) and (4.2.8) for sufficiently small initial data. More precisely, we have (see [13, Appendix B])

Theorem 4.2.1. *There exists $\delta_0 > 0$ such that for every $(H_0, V_0)^T \in H^2((0,L);\mathbb{R}^2)$ satisfying*

$$\|(H_0 - H^*, V_0 - V^*)^T\|_{H^2((0,L);\mathbb{R}^2)} \le \delta_0$$

and compatibility conditions (4.2.9) *to* (4.2.11)*. The Cauchy problem* (4.2.1)*,* (4.2.2)*,* (4.2.7) *and* (4.2.8) *has a unique maximal classical solution*

$$(H,V)^T \in C^0([0,T); H^2((0,L);\mathbb{R}^2))$$

with $T \in (0,+\infty)$.

We recall the definition of the exponential stability in H^2-norm:

Definition 4.2.1. The steady-state $(H^*, V^*)^T$ of the system (4.2.1), (4.2.2) and (4.2.7) is exponentially stable for the H^2-norm if there exist $\gamma > 0$, $\delta > 0$ and $C > 0$ such that for every $(H_0, V_0)^T \in H^2((0,L);\mathbb{R}^2)$ satisfying $\|(H_0 - H^*, V_0 - V^*)^T\|_{H^2((0,L);\mathbb{R}^2)} \le \delta$ and the compatibility conditions (4.2.9) to (4.2.11), the

Cauchy problem (4.2.1), (4.2.2), (4.2.7) and (4.2.8) has a unique classical solution on $[0,+\infty)\times[0,L]$ and satisfies

$$\begin{aligned}&\|(H(t,\cdot)-H^*,V(t,\cdot)-V^*)^T\|_{H^2((0,L);\mathbb{R}^2)}\\ \leq& Ce^{-\gamma t}\|(H_0-H^*,V_0-V^*)^T\|_{H^2((0,L);\mathbb{R}^2)},\end{aligned} \tag{4.2.12}$$

for all $t\in[0,+\infty)$.

Based on this definition, our main result is the following:

Theorem 4.2.2. *The nonlinear Saint-Venant system* (4.2.1), (4.2.2) *and* (4.2.7) *is exponentially stable for the H^2-norm provided that the boundary conditions satisfy*

$$\begin{aligned}&\mathcal{B}'(H^*(0))\in\Big(-\frac{g}{V^*(0)},-\frac{V^*(0)}{H^*(0)}\Big),\\ \textit{and}\quad&\mathcal{B}'(H^*(L))\in\mathbb{R}\setminus\Big[-\frac{g}{V^*(L)},-\frac{V^*(L)}{H^*(L)}\Big].\end{aligned} \tag{4.2.13}$$

The proof of this theorem is given in Section 3. To that end, we will first prove the exponential stability result (Proposition 4.3.1) for the linearized system for the L^2-norm by finding a suitable Lyapunov function. Then we show that this Lyapunov function enables us to obtain the exponential stability for the H^2-norm for the nonlinear system under boundary control conditions (4.2.7) with properties (4.2.13).

4.3 Exponential stability for the H^2-norm with arbitrary friction and space-varying slope

4.3.1 Exponential stability of the linearized system

In this section, we study the exponential stability of the linearized system about a steady-state $(H^*,V^*)^T$ for the L^2-norm. We define the perturbation functions h and v as

$$h(t,x)=H(t,x)-H^*(x), \tag{4.3.1}$$
$$v(t,x)=V(t,x)-V^*(x). \tag{4.3.2}$$

The linearization of the system (4.2.1) and (4.2.2) about the steady-state is

$$\begin{pmatrix}h\\v\end{pmatrix}_t+\begin{pmatrix}V^* & H^*\\ g & V^*\end{pmatrix}\begin{pmatrix}h\\v\end{pmatrix}_x+\begin{pmatrix}V^*_x & H^*_x\\ f^*_H & V^*_x+f^*_V\end{pmatrix}\begin{pmatrix}h\\v\end{pmatrix}=0, \tag{4.3.3}$$

where f^*_H and f^*_V are defined by

$$f^*_V=\frac{2kV^*}{H^*},\quad f^*_H=-\frac{kV^{*2}}{H^{*2}}. \tag{4.3.4}$$

The corresponding linearization of the boundary conditions (4.2.7) are given by

$$v(t,0)=b_0h(t,0),\quad v(t,L)=b_1h(t,L), \tag{4.3.5}$$

where

$$b_0=\mathcal{B}'(H^*(0)),\quad b_1=\mathcal{B}'(H^*(L)). \tag{4.3.6}$$

The initial condition is given as follows

$$h(0,x)=h_0(x),\quad v(0,x)=v_0(x), \tag{4.3.7}$$

where $(h_0,v_0)^T\in L^2((0,L);\mathbb{R}^2)$. The Cauchy problem (4.3.3), (4.3.5) and (4.3.7) is well-posed (see [13, Appendix A]). Note that the exponential stability of the linearized system is now a problem of null-stabilization for h and v. We have the following result:

Proposition 4.3.1. *For the linearized Saint-Venant system* (4.3.3)*,* (4.3.5) *and* (4.3.7)*, if the boundary conditions satisfy*

$$b_0 \in \Big(-\frac{g}{V^*(0)}, -\frac{V^*(0)}{H^*(0)} \Big),\ b_1 \in \mathbb{R} \setminus \Big[-\frac{g}{V^*(L)}, -\frac{V^*(L)}{H^*(L)} \Big]. \tag{4.3.8}$$

Then there exists a constant $\mu > 0$, $q_1 \in C^1([0,L];(0,+\infty))$, $q_2 \in C^1([0,L];(0,+\infty))$ and $\delta > 0$ such that the following control Lyapunov function candidate

$$V(h,v) = \int_0^L \frac{q_1+q_2}{H^*}\Big(gh^2 + 2\frac{q_1-q_2}{q_1+q_2}\sqrt{gH^*}hv + H^*v^2 \Big) dx \tag{4.3.9}$$

verifies:

$$V(h,v) \geq \delta(\|h\|^2_{L^2(0,L)} + \|v\|^2_{L^2(0,L)}) \tag{4.3.10}$$

for any $(h,v) \in L^2((0,L);\mathbb{R}^2)$, where $L^2(0,L)$ denotes $L^2((0,L);\mathbb{R})$. If in addition, $(h,v)^T$ is a solution of the system (4.3.3)*,* (4.3.5) *and* (4.3.7)*, we have*

$$\frac{d}{dt}(V(h(t,\cdot),v(t,\cdot))) \leq -\mu V(h(t,\cdot),v(t,\cdot)) \tag{4.3.11}$$

in the distribution sense which implies the exponential stability of the linearized system (4.3.3)*,* (4.3.5) *and* (4.3.7) *for the L^2-norm.*

In order to prove Proposition 4.3.1, we introduce the following lemma, the proof of which is given in the Appendix.

Lemme 4.3.1. *The function η defined by*

$$\eta = \frac{\lambda_2}{\lambda_1}\varphi \tag{4.3.12}$$

is a solution to the equation

$$\eta' = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2 \right|, \tag{4.3.13}$$

where λ_1 and λ_2 are defined in (4.3.14)*, φ is given by* (4.3.23)*, a and b are given by* (4.3.26) *below.*

Proof of Proposition 4.3.1. Let us denote

$$A(x) = \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix}.$$

Under the subcritical condition (4.2.6), the matrix $A(x)$ has two real distinct eigenvalues λ_1 and $-\lambda_2$ with

$$\lambda_1(x) = \sqrt{gH^*} + V^* > 0,\ \lambda_2(x) = \sqrt{gH^*} - V^* > 0. \tag{4.3.14}$$

We define the characteristic coordinates as follows

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{g}{H^*}} & 1 \\ -\sqrt{\frac{g}{H^*}} & 1 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}. \tag{4.3.15}$$

With these definitions and notations, the linearized Saint-Venant equations (4.3.3) are written in characteristic form:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x + \begin{pmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0. \tag{4.3.16}$$

In (4.3.16),

$$\gamma_1(x) = -\frac{3f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)} + \frac{kV^*}{H^*} - \frac{kV^{*2}}{2H^*\sqrt{gH^*}}, \tag{4.3.17}$$

$$\delta_1(x) = -\frac{f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)} + \frac{kV^*}{H^*} + \frac{kV^{*2}}{2H^*\sqrt{gH^*}}, \tag{4.3.18}$$

$$\gamma_2(x) = \frac{f(H^*,V^*)}{4(\sqrt{gH^*}-V^*)} + \frac{kV^*}{H^*} - \frac{kV^{*2}}{2H^*\sqrt{gH^*}}, \tag{4.3.19}$$

$$\delta_2(x) = \frac{3f(H^*,V^*)}{4(\sqrt{gH^*}-V^*)} + \frac{kV^*}{H^*} + \frac{kV^{*2}}{2H^*\sqrt{gH^*}}, \tag{4.3.20}$$

where $f(H^*,V^*) = \dfrac{kV^{*2}}{H^*} - gC$.

As the diagonal coefficients of the source term in (4.3.16) may bring complexity on the analysis of the stability, we then make a coordinate transformation inspired by [128] (see also [16]) to remove the diagonal coefficients. We introduce the notations

$$\varphi_1(x) = \exp\Big(\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\,ds\Big), \tag{4.3.21}$$

$$\varphi_2(x) = \exp\Big(-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\,ds\Big), \tag{4.3.22}$$

$$\varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)}, \tag{4.3.23}$$

and the new coordinates

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{4.3.24}$$

Then system (4.3.16) is transformed into the following system expressed in the new coordinates

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_x + \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{4.3.25}$$

with

$$a(x) = \varphi(x)\delta_1(x), \quad b(x) = \varphi^{-1}(x)\gamma_2(x). \tag{4.3.26}$$

From (4.3.5), (4.3.15) and (4.3.24), we obtain the following boundary conditions for system (4.3.25)

$$\begin{aligned} y_1(t,0) &= k_0 \frac{\varphi_1(0)}{\varphi_2(0)}\, y_2(t,0), \\ y_2(t,L) &= k_1 \frac{\varphi_2(L)}{\varphi_1(L)}\, y_1(t,L), \end{aligned} \tag{4.3.27}$$

where

$$k_0 = \frac{b_0 H^*(0) + \sqrt{gH^*(0)}}{b_0 H^*(0) - \sqrt{gH^*(0)}}, \; k_1 = \frac{b_1 H^*(L) - \sqrt{gH^*(L)}}{b_1 H^*(L) + \sqrt{gH^*(L)}}. \tag{4.3.28}$$

Note that from (4.3.28), it is easy to check that condition (4.3.8), using our notation (4.3.14), is equivalent to

$$k_0^2 < \Big(\frac{\lambda_2(0)}{\lambda_1(0)}\Big)^2, \quad k_1^2 < \Big(\frac{\lambda_1(L)}{\lambda_2(L)}\Big)^2. \tag{4.3.29}$$

Let us define

$$\begin{aligned} &V: \quad L^2(0,L) \times L^2(0,L) \to \mathbb{R}^+ \\ &V(\psi_1,\psi_2) = \int_0^L \Big(f_1(x)\psi_1^2(x) e^{-\frac{\mu}{\lambda_1}x} + f_2(x)\psi_2^2(x) e^{\frac{\mu}{\lambda_2}x} \Big)\, dx \end{aligned} \tag{4.3.30}$$

where the parameter $\mu > 0$ and two functions $f_1 \in C^1([0,L];(0,+\infty))$ and $f_2 \in C^1([0,L];(0,+\infty))$ are to be determined. Obviously, there exists $\bar{\delta} > 0$ such that for any $(\psi_1, \psi_2) \in L^2((0,L);\mathbb{R}^2)$

$$V(\psi_1,\psi_2) \geq \bar{\delta}(\|\psi_1\|^2_{L^2(0,L)} + \|\psi_2\|^2_{L^2(0,L)}). \tag{4.3.31}$$

For any arbitrary C^1-solution y_1 and y_2 to system (4.3.25) and (4.3.27), we denote $V(t)$ by

$$V(t) = V(y_1(t,\cdot), y_2(t,\cdot)). \tag{4.3.32}$$

From (4.3.30) and differentiating V with respect to time t we get

$$\begin{aligned}\frac{dV}{dt} =& -\mu V - \left[\lambda_1 f_1 e^{-\frac{\mu}{\lambda_1}x} y_1^2 - \lambda_2 f_2 e^{\frac{\mu}{\lambda_2}x} y_2^2\right]_0^L \\ &- \int_0^L \Bigg[\left(-(\lambda_1 f_1)_x - \frac{\mu x \lambda_{1x}}{\lambda_1} f_1\right) e^{-\frac{\mu}{\lambda_1}x} y_1^2 \\ &+ \left((\lambda_2 f_2)_x - \frac{\mu x \lambda_{2x}}{\lambda_2} f_2\right) e^{\frac{\mu}{\lambda_2}x} y_2^2 \\ &+ 2(f_1 e^{-\frac{\mu}{\lambda_1}x} a(x) + f_2 e^{\frac{\mu}{\lambda_2}x} b(x)) y_1 y_2 \Bigg] dx.\end{aligned} \tag{4.3.33}$$

We observe that in (4.3.33), there is a term relying on the boundary controls that will be chosen to make this term negative along the system trajectories. Moreover, there also appears to have an interior term which is intrinsic to the system. Let us deal firstly with the interior term, we denote by

$$\begin{aligned}I_1 := \int_0^L \Bigg[& \left(-(\lambda_1 f_1)_x - \frac{\mu x \lambda_{1x}}{\lambda_1} f_1\right) e^{-\frac{\mu}{\lambda_1}x} y_1^2 \\ &+ \left((\lambda_2 f_2)_x - \frac{\mu x \lambda_{2x}}{\lambda_2} f_2\right) e^{\frac{\mu}{\lambda_2}x} y_2^2 \\ &+ 2(f_1 e^{-\frac{\mu}{\lambda_1}x} a(x) + f_2 e^{\frac{\mu}{\lambda_2}x} b(x)) y_1 y_2 \Bigg] dx.\end{aligned} \tag{4.3.34}$$

To ensure that for $\mu > 0$ sufficiently small, I_1 is positive for any $t > 0$ and any solution (y_1, y_2), one only needs to construct f_1 and f_2 to guarantee that for any $x \in [0,L]$

$$-(\lambda_1 f_1)_x > 0, \quad (\lambda_2 f_2)_x > 0, \tag{4.3.35}$$

$$-(\lambda_1 f_1)_x(\lambda_2 f_2)_x - (f_1(x)a(x) + f_2(x)b(x))^2 > 0. \tag{4.3.36}$$

Indeed in this case from the strict inequality in (4.3.35) and (4.3.36), by continuity, there exists $\mu_1 > 0$ such that for all $\mu \in (0, \mu_1]$, I_1 is positive.
Let us point out that there exist f_1 and f_2 such that (4.3.35) and (4.3.36) hold as soon as there exists a positive function η well defined on $[0,L]$ and satisfying the following equation (see [12])

$$\eta' = \left|\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2\right|. \tag{4.3.37}$$

Therefore, one of the key points to prove Proposition 4.3.1 is to find a positive solution to (4.3.37). And from Lemma 4.3.1 we know that such solution does exist. Hence, we can define a map

$$f : (\eta, \varepsilon) \to \left|\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2\right| + \varepsilon,$$

which is locally Lipschitz (and even C^1) in ε around 0. From Lemma 4.3.1 we know that

$$\begin{cases}\eta' = f(\eta, 0), \\ \eta(0) = \dfrac{\lambda_2(0)}{\lambda_1(0)}\varphi(0)\end{cases} \tag{4.3.38}$$

admits a unique solution on $[0, L]$ which is given by (4.3.12). Therefore, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, the Cauchy problem

$$\begin{cases} \eta_\varepsilon' = \left| \dfrac{a}{\lambda_1} + \dfrac{b}{\lambda_2}\eta_\varepsilon^2 \right| + \varepsilon, \\ \eta_\varepsilon(0) = \dfrac{\lambda_2(0)}{\lambda_1(0)}\varphi(0) \end{cases} \tag{4.3.39}$$

admits a unique solution η_ε on $[0, L]$. Moreover as $\eta_\varepsilon(0) > 0$, we have $\eta_\varepsilon(x) > 0$ for all $x \in [0, L]$. Now proceeding as in [12], we choose f_1 and f_2 as

$$f_1 = f_{1,\varepsilon} := \frac{1}{\lambda_1 \eta_\varepsilon}, \quad f_2 = f_{2,\varepsilon} := \frac{\eta_\varepsilon}{\lambda_2}, \tag{4.3.40}$$

then we have for any $\varepsilon \in (0, \varepsilon_0]$ that

$$-(\lambda_1 f_1)_x > 0, \quad (\lambda_2 f_2)_x > 0, \tag{4.3.41}$$

$$\begin{aligned} -(\lambda_1 f_1)_x(\lambda_2 f_2)_x &- (f_1 a + f_2 b)^2 \\ &= \frac{\varepsilon^2 + 2\varepsilon|\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta_\varepsilon^2|}{\eta_\varepsilon^2} > 0. \end{aligned} \tag{4.3.42}$$

Thus, (4.3.35) and (4.3.36) hold. Now, let us consider the boundary term in (4.3.33), we denote by

$$I_2 := -\left[\lambda_1 f_1 e^{-\frac{\mu}{\lambda_1}x} y_1^2 - \lambda_2 f_2 e^{\frac{\mu}{\lambda_2}x} y_2^2\right]_0^L. \tag{4.3.43}$$

Suppose that (4.3.8) is satisfied, from (4.3.12), (4.3.29), (4.3.39) and (4.3.40), we have

$$k_0^2 < \Big(\frac{\lambda_2(0)}{\lambda_1(0)}\Big)^2 = \frac{\lambda_2(0) f_{2,0}(0)}{\lambda_1(0) f_{1,0}(0)}\varphi^{-2}(0) = \frac{\lambda_2(0) f_{2,\varepsilon}(0)}{\lambda_1(0) f_{1,\varepsilon}(0)}\varphi^{-2}(0), \tag{4.3.44}$$

$$k_1^2 < \Big(\frac{\lambda_1(L)}{\lambda_2(L)}\Big)^2 = \frac{\lambda_1(L) f_{1,0}(L)}{\lambda_2(L) f_{2,0}(L)}\varphi^2(L). \tag{4.3.45}$$

By the continuity of $f_{1,\varepsilon}(L)$ and $f_{2,\varepsilon}(L)$ with ε, there exists $0 < \varepsilon_1 < \varepsilon_0$ such that for any $\varepsilon \in (0, \varepsilon_1]$

$$k_1^2 < \frac{\lambda_1(L) f_{1,\varepsilon}(L)}{\lambda_2(L) f_{2,\varepsilon}(L)}\varphi^2(L), \tag{4.3.46}$$

thus, there exists $0 < \mu_2 < \mu_1$ such that for any $\mu \in (0, \mu_2]$

$$k_1^2 < \frac{\lambda_1(L) f_{1,\varepsilon}(L) e^{-\frac{\mu}{\lambda_1(L)}L}}{\lambda_2(L) f_{2,\varepsilon}(L) e^{\frac{\mu}{\lambda_2(L)}L}}\varphi^2(L). \tag{4.3.47}$$

Combining (4.3.44) and (4.3.47), we get

$$\begin{aligned} I_2 = &- \left[\lambda_1 f_1 e^{-\frac{\mu}{\lambda_1}x} y_1^2 - \lambda_2 f_2 e^{\frac{\mu}{\lambda_2}x} y_2^2\right]_0^L \\ = &\Big(k_1^2 \lambda_2(L) f_{2,\varepsilon}(L)\varphi^{-2}(L) e^{\frac{\mu}{\lambda_2(L)}L} \\ &- \lambda_1(L) f_{1,\varepsilon}(L) e^{-\frac{\mu}{\lambda_1(L)}L}\Big) y_1^2(t, L) \\ &+ \big(k_0^2 \lambda_1(0) f_{1,\varepsilon}(0)\varphi^2(0) - \lambda_2(0) f_{2,\varepsilon}(0)\big) y_2^2(t, 0) < 0. \end{aligned} \tag{4.3.48}$$

From (4.3.33), (4.3.41), (4.3.42) and (4.3.48), we obtain

$$\frac{dV}{dt} < -\mu V \tag{4.3.49}$$

along the C^1-solutions of the system (4.3.25) and (4.3.27) for any $\mu \in (0, \mu_2]$. Since the C^1-solutions are dense in the set of L^2-solutions, inequality (4.3.49) also holds in the sense of distributions for the L^2-solutions (see [13, Section 2.1] for the details).
Let us define

$$q_1 := f_1 \varphi_1^2 e^{-\frac{\mu}{\lambda_1} x} \quad \text{and} \quad q_2 := f_2 \varphi_2^2 e^{\frac{\mu}{\lambda_2} x}. \tag{4.3.50}$$

For any $(h, v) \in L^2((0, L); \mathbb{R}^2)$, let (ψ_1, ψ_2) be the result of the change of variable as in (4.3.15) and (4.3.24), we get immediately from (4.3.30) and (4.3.50) the expression of Lyapunov function candidate as in (4.3.9). Moreover, from the positivity of f_1 and f_2, we had (4.3.31), which implies that there exists $\delta > 0$ such that (4.3.10) holds using the change of variable (4.3.15) and (4.3.24). From (4.3.49), we get (4.3.11) as well. The proof of Proposition 4.3.1 is completed. □

4.3.2 Exponential stability of the steady-state of the nonlinear system in H^2-norm

We will now prove our main result, Theorem 4.2.2. Firstly, we recall the following theorem which gives sufficient conditions for the exponential stability of the steady-state of the nonlinear system (4.2.1), (4.2.2) and (4.2.7).

Theorem 4.3.2. *The steady-state $(H^*, V^*)^T$ of the system* (4.2.1), (4.2.2) *and* (4.2.7) *is exponentially stable for the H^2-norm if*

— *There exists two functions $f_1, f_2 \in C^1([0, L]; (0, +\infty))$ such that*

$$-(\lambda_1 f_1)_x > 0, \quad (\lambda_2 f_2)_x > 0 \tag{4.3.51}$$

and

$$-(\lambda_1 f_1)_x (\lambda_2 f_2)_x > \left(\frac{a(x)}{\lambda_1(x)} f_1(x) + \frac{b(x)}{\lambda_2(x)} f_2(x) \right)^2 \tag{4.3.52}$$

for any $x \in [0, L]$, where a and b are given by (4.3.26).

— *The following inequalities are satisfied:*

$$\begin{aligned} &\Big(\frac{b_0 H^*(0) + \sqrt{g H^*(0)}}{b_0 H^*(0) - \sqrt{g H^*(0)}} \Big)^2 < \frac{\lambda_2(0) f_2(0)}{\lambda_1(0) f_1(0)} \varphi^{-2}(0), \\ &\Big(\frac{b_1 H^*(L) - \sqrt{g H^*(L)}}{b_1 H^*(L) + \sqrt{g H^*(L)}} \Big)^2 < \frac{\lambda_1(L) f_1(L)}{\lambda_2(L) f_2(L)} \varphi^2(L), \end{aligned} \tag{4.3.53}$$

where b_0, b_1 and φ are given by (4.3.6) and (4.3.23) respectively.

Remark 4.3.1. *This theorem comes directly from [13, Theorem 6.6 and 6.10]. Note that finding such f_1 and f_2 corresponds to finding a quadratic Lyapunov function V for the H^2-norm of the perturbations* (4.3.1) *and* (4.3.2) *such that:*

$$\frac{1}{\beta} \| (h, v)^T \|_{H^2} \le V \le \beta \| (h, v)^T \|_{H^2} \quad \text{and} \quad \frac{dV}{dt} \le -\alpha V \tag{4.3.54}$$

for some $\alpha > 0$ and $\beta > 0$. In particular, such Lyapunov function has some robustness with respect to small perturbations of the system dynamics. More details about the construction of such Lyapunov function as well as the proof of this theorem can be found in the Appendix.

Using Theorem 4.3.2, we shall finally prove Theorem 4.2.2 that is now straightforward.

Proof of Theorem 4.2.2. Note that the condition (4.3.51) and (4.3.52) are exactly the same with conditions (4.3.35) and (4.3.36). Therefore from the proof of Proposition 4.3.1 for all $\varepsilon \in (0, \varepsilon_0]$, there exist $f_1 = f_{1,\varepsilon}$

and $f_2 = f_{2,\varepsilon}$ defined by (4.3.40), continuous with respect to ε, such that (4.3.51) and (4.3.52) are verified and

$$\frac{f_{1,0}\lambda_1}{f_{2,0}\lambda_2}\varphi^2 = \left(\frac{\lambda_1}{\lambda_2}\right)^2, \tag{4.3.55}$$

where φ is given by (4.3.23). Under hypothesis (4.2.13) of Theorem 4.2.2, we have (4.3.29), which together with (4.3.55) gives (4.3.44). Recall that by the continuity of $f_{1,\varepsilon}$ and $f_{2,\varepsilon}$ with respect to ε, (4.3.46) holds for any $\varepsilon \in (0, \varepsilon_1]$. Combining (4.3.44), (4.3.46) and noticing (4.3.28), we obtain that

$$\begin{aligned} &\left(\frac{b_0 H^*(0) + gH^*(0)}{b_0 H^*(0) - gH^*(0)}\right)^2 < \frac{\lambda_2(0) f_{2,\varepsilon}(0)}{\lambda_1(0) f_{1,\varepsilon}(0)}\varphi^{-2}(0), \\ &\left(\frac{b_1 H^*(L) - gH^*(L)}{b_1 H^*(L) + gH^*(L)}\right)^2 < \frac{\lambda_1(L) f_{1,\varepsilon}(L)}{\lambda_2(L) f_{2,\varepsilon}(L)}\varphi^{2}(L). \end{aligned} \tag{4.3.56}$$

Thus, we get from Theorem 4.3.2 that the steady-state $(H^*, V^*)^T$ of the system (4.2.1), (4.2.2) and (4.2.7) is exponentially stable for the H^2-norm. This ends the proof of Theorem 4.2.2. □

Remark 4.3.2. *We emphasize that the exponential stability in H^p-norm holds in fact for any $p \in \mathbb{N} \setminus \{0, 1\}$ under the same condition (4.2.13) given in Theorem 4.2.2 when the map $\mathcal{B}$ is of class C^p and the definition of the exponential stability involves an appropriate extension of the compatibility conditions of order $p-1$ (see [13, Page.153] for the definition). This is a consequence of [13, Theorem 6.10]. Roughly speaking, this can be obtained by considering the augmented systems and then using the same method as in the proof of Theorem 4.3.2.*

4.4 Numerical simulations

In this section, we illustrate Theorem 4.2.2 by providing numerical simulations of the H^2-norm of the solutions of the nonlinear system (4.2.1), (4.2.2) and (4.2.7). We focus on the steep slope regime (i.e. $gC > kV^2/H$) as it is the most challenging situation to stabilize. The steady-state is chosen with initial condition prescribed as $H^*(0) = 2\,m$, $V^*(0) = 0.5\,m/s$. And the parameters are chosen as follows: $k = 0.002$, $L = 3.10^3 m$, $gC = 2kV^{*2}(0)/H^*(0)$, the boundary controls are chosen such that $\mathcal{B}'(H^*(0)) = 1.1 b_{0,lim}$, $\mathcal{B}'(H^*(L)) = 0.9 b_{1,lim}$ where $b_{0,lim} = -V^*(0)/H^*(0)$ and $b_{1,lim} = -V^*(L)/H^*(L)$ are the critical upper bounds given in Theorem 4.2.2. The initial perturbations are chosen to be compatible at the boundaries and sinusoidal in the domain with an amplitude of 0.2 m in height and $0.05\,m/s$ in velocity. In Fig.4.1, the blue curve represents the variation of the H^2-norm of the perturbations for the water level with time, while in Fig.4.2, the red curve represents the behavior of the downstream controller $V(t, L)$, which converges to the value $V^*(L)$ (in green).

4.5 Conclusion

In this chapter we addressed the problem of the exponential stability of the Saint-Venant equations with arbitrary friction and space-varying slope. An explicit boundary condition was given which guarantees the exponential stability of the nonlinear system in H^2-norm. To that end, we first studied a corresponding linearized system and proved the exponential stability result in L^2-norm by constructing a quadratic Lyapunov function. Then by expanding the Lyapunov function, we obtained the exponential stability of the nonlinear system in H^2-norm by requiring proper conditions on the boundaries. These boundary conditions are related to physical devices located at the ends of the channel where the controls acting as feedback are implemented. Finally, some numerical simulations are given to support our main result, Theorem 4.2.2.

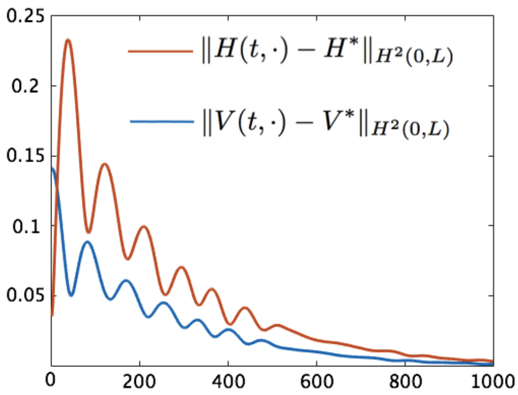

Figure 4.1 – Convergence of the height and velocity.

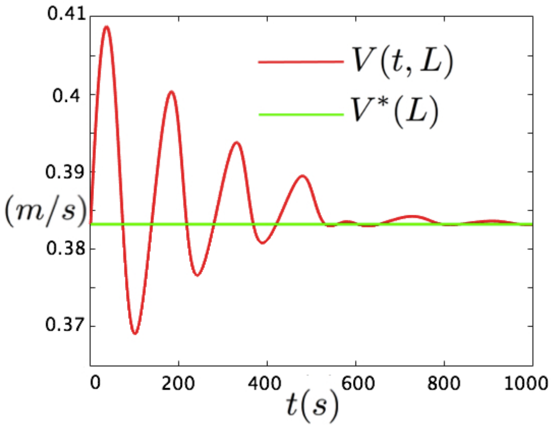

Figure 4.2 – Convergence of the controller at $x = L$ to the value $V^*(L)$.

4.6 Appendix

4.6.1 Proof of Lemma 4.3.1

Proof. From (4.2.3), (4.2.5), (4.3.17) to (4.3.23) and (4.3.26), we get that

$$\begin{aligned}
\left(\frac{\lambda_2}{\lambda_1}\varphi\right)' &= \frac{\lambda_2'\lambda_1 - \lambda_1'\lambda_2}{\lambda_1^2}\varphi + \frac{\lambda_2}{\lambda_1}\left(\frac{\gamma_1}{\lambda_1} + \frac{\delta_2}{\lambda_2}\right)\varphi \\
&= \left(\frac{3\sqrt{gH^*}V^*\left(gC - \frac{kV^{*2}}{H^*}\right)}{\lambda_1^2(gH^* - V^{*2})} + \frac{\lambda_2\gamma_1 + \delta_2\lambda_1}{\lambda_1^2}\right)\varphi \\
&= \left(\frac{3\sqrt{gH^*}V^*\left(gC - \frac{kV^{*2}}{H^*}\right)}{\lambda_1^2(gH^* - V^{*2})}\right. \\
&\quad + \frac{1}{\lambda_1^2}\left[\frac{3}{4}\left(gC - \frac{kV^{*2}}{H^*}\right)\left[\frac{\sqrt{gH^*} - V}{\sqrt{gH^*} + V} - \frac{\sqrt{gH^*} + V}{\sqrt{gH^*} - V}\right]\right. \\
&\quad \left.\left. + \frac{kV^{*2}}{H^*}\left(\frac{2\sqrt{gH^*}}{V^*} + \frac{V^*}{\sqrt{gH^*}}\right)\right]\right)\varphi
\end{aligned}$$

$$= \frac{kV^{*2}}{H^*}\left(\frac{2\sqrt{gH^*}}{V^*} + \frac{V^*}{\sqrt{gH^*}}\right)\frac{\varphi}{\lambda_1^2}.$$

Besides, we have

$$\begin{aligned}\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2 &= \left(\frac{\delta_1\lambda_1 + \gamma_2\lambda_2}{\lambda_1^2}\right)\varphi \\ &= \frac{kV^{*2}}{H^*}\left(\frac{2\sqrt{gH^*}}{V^*} + \frac{V^*}{\sqrt{gH^*}}\right)\frac{\varphi}{\lambda_1^2} > 0.\end{aligned}$$

Therefore

$$\eta' = \frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2 = \left|\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2\right|.$$

This ends the proof of Lemma 4.3.1. □

4.6.2 Proof of Theorem 4.3.2

Proof. This theorem is a particular case of [13, Theorem 6.10]. One just need to check that the system (4.2.1), (4.2.2) and (4.2.8) with boundary conditions (4.2.7) satisfying the dissipative conditions (4.2.13) can be written in the form of [13, (6.54)-(6.57)]. Note that this also implies the well-posedness of the system as well. Indeed, we perform the change of variable

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \varphi_1\sqrt{\frac{g}{H^*}} & \varphi_1 \\ -\varphi_2\sqrt{\frac{g}{H^*}} & \varphi_2 \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix}, \tag{4.6.1}$$

where h and v are the perturbations given by (4.3.1) and (4.3.2) and φ_1 and φ_2 are given by (4.3.21) and (4.3.22) respectively. If we denote by $\mathbf{z} = (z_1, z_2)^T$, the nonlinear system (4.2.1), (4.2.2) and (4.2.7) is equivalent to:

$$\mathbf{z}_t + A(\mathbf{z}, x)\mathbf{z}_x + B(\mathbf{z}, x) = 0 \tag{4.6.2}$$

$$\begin{pmatrix} z_1(t,0) \\ z_2(t,L) \end{pmatrix} = G\begin{pmatrix} z_1(t,L) \\ z_2(t,0) \end{pmatrix}, \tag{4.6.3}$$

where

$$A(\mathbf{0}, x) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix},\ B(\mathbf{0}, x) = 0$$

$$\text{and}\ \ \frac{\partial B}{\partial \mathbf{z}}(\mathbf{0}, x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}$$

and

$$G(\mathbf{0}) = \mathbf{0}, \quad G'(\mathbf{0}) = \begin{pmatrix} 0 & k_0\varphi(0) \\ k_1\varphi^{-1}(L) & 0 \end{pmatrix} \tag{4.6.4}$$

with k_0 and k_1 defined in (4.3.28). Note that the boundary condition (4.6.3) obtained from (4.2.7) is true at least locally, thus is in the form used in [13, Theorem 6.10]. To be more precise, noticing $\varphi_1(0) = 1$, we get from (4.6.1) that

$$\begin{aligned} z_1(t,0) &= v(t,0) + \sqrt{\frac{g}{H^*(0)}}h(t,0) \\ &= V(t,0) - V^*(0) + \sqrt{\frac{g}{H^*(0)}}h(t,0) \\ &= \mathcal{B}(H(t,0)) - V^*(0) + \sqrt{\frac{g}{H^*(0)}}h(t,0) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{B}(h(t,0)+H^*(0)) - V^*(0) + \sqrt{\frac{g}{H^*(0)}}h(t,0) \\
&:= l_1(h(t,0)).
\end{aligned}
\tag{4.6.5}
$$

Similarly note that $\varphi_2(0)=1$, we obtain

$$
\begin{aligned}
z_2(t,0) &= v(t,0) - \sqrt{\frac{g}{H^*(0)}}h(t,0) \\
&= \mathcal{B}(h(t,0)+H^*(0)) - V^*(0) - \sqrt{\frac{g}{H^*(0)}}h(t,0) \\
&:= l_2(h(t,0)).
\end{aligned}
\tag{4.6.6}
$$

From (4.3.6) and (4.6.6), we have

$$
l_2'(0) = \mathcal{B}'(H^*(0)) - \sqrt{\frac{g}{H^*(0)}} = b_0 - \sqrt{\frac{g}{H^*(0)}},
\tag{4.6.7}
$$

which together with the definition of b_0 in (4.3.8) gives that

$$
l_2'(0) < 0.
\tag{4.6.8}
$$

Thanks to the implicit function theorem, we get from (4.6.5), (4.6.6) and (4.6.8) that in a neighborhood of 0

$$
z_1(t,0) = m_1(z_2(t,0)).
\tag{4.6.9}
$$

Similarly, we can obtain in a neighborhood of 0 that

$$
z_2(t,L) = m_2(z_1(t,L)).
\tag{4.6.10}
$$

Note that (4.6.9) and (4.6.10) are indeed in the form of (4.6.3). Then [13, Theorem 6.6 and 6.10] can be directly applied to this particular case and gives the sufficient conditions (4.3.51), (4.3.52) and (4.3.53). We remark here that the essential element of the proof for [13, Theorem 6.6 and 6.10] is that finding such f_1 and f_2 corresponds to finding a quadratic Lyapunov function for the H^2-norm of the form:

$$
\begin{aligned}
V = &\int_0^L \left(f_1(x)z_1^2(t,x) + f_2(x)z_2^2(t,x)\right)dx \\
&+ \int_0^L \left(f_1(x)z_{1t}^2(t,x) + f_2(x)z_{2t}^2(t,x)\right)dx \\
&+ \int_0^L \left(f_1(x)z_{1tt}^2(t,x) + f_2(x)z_{2tt}^2(t,x)\right)dx.
\end{aligned}
$$

One can look at in particular Lemma 6.8 and (6.19) to (6.22) in [13]. This completes the statement of Remark 4.3.1.

□

Acknowledgment

The authors would like to thank Jean-Michel Coron, Tatsien Li and Sébastien Boyaval for their constant support. The authors would like also to thank Georges Bastin, Shengquan Xiang and Christophe Zhang for many fruitful disscussions. Finally, the authors would like to thank National Natural Science Foundation of China (No. 11771336), ETH-ITS, ETH-FIM, ANR project Finite4SoS (No.ANR 15-CE23-0007) and the French Corps des IPEF for their financial support.

Chapter 5

Exponential stability of density-velocity systems with boundary conditions and source term for the H^2 norm

This chapter takes most of the following article (also referred to as [109]):

Amaury Hayat and Peipei Shang. Exponential stability of density-velocity systems with boundary conditions and source term for the H^2 norm. Preprint, 2019.

Abstract.
In this chapter, we address the problem of the exponential stability of density-velocity systems with boundary conditions. Density-velocity systems are omnipresent in physics as they encompass all systems that consist in a flux conservation and a momentum equation. In this chapter we show that any such system can be stabilized exponentially quickly in the H^2 norm using simple local feedbacks, provided a condition on the source term which holds for most physical systems. This is true even when the source term is not dissipative. Besides, the feedback laws obtained only depends on the target values at the boundaries, which implies that they do not depend on the expression of the source term or the force applied on the system and makes them very easy to implement in practice and robust to model errors. For instance, for a river modelled by the Saint-Venant equations this means that the feedback laws do not require any information on the friction model, the slope or the shape of the channel considered. This feat is obtained by showing the existence of a basic H^2 Lyapunov functions and we apply it to numerous systems: the general Saint-Venant equations, the isentropic Euler equations, the motion of water in rigid-pipe, the osmosis phenomenon, etc.

5.1 Introduction

Density-velocity systems are important hyperbolic systems as they represent the physical phenomena where the flux is conserved, while the energy can be either increased or decreased. In physics they are found in fluid mechanics, electromegnetism, etc. The increase or decrease of the energy leads to nonuniform steady-states with sometimes large variations in space. In this chapter, we address the exponential stability of such nonlinear systems for the H^2 norm, although the result is also true for the H^p norm for any $p \geq 2$. Mentioning the norm is not superfluous as, for nonlinear systems, the stability for different norms are not equivalent [62]. In particular it has been shown in [12] that the basic quadratic Lyapunov functions fail to ensure the stabilization in the L^2 norm for nonlinear hyperbolic systems systems and that one has to study the H^2 norm instead. Other attempt of basic Lyapunov functions have been constructed to ensure the stability of hyperbolic systems in the C^1 norm, for instance [49, 106, 107].

Physical density-velocity systems often have well-known conservative or dissipative energy or entropy functions when no source term occurs [68]. These dissipative energy or entropy functions are quite useful for the analysis of such system and enable to prove stability results (see for instance [52, 55] for the use of entropy as control Lyapunov function for Saint-Venant equations and [46] for the Euler equations). When source terms appear, however, no such function is usually known, especially when the source term is not dissipative. In the previous contribution [16], the authors also studied the stabilization of hyperbolic density-velocity equations, but with dissipative source terms only depending on the unknown functions. This is the case for Saint-Venant equations with no slope and with a constant friction, or for the isentropic Euler equations when the gas pressure is simply assumed to be a function of a gas density and a friction proportional to the square of the velocity. However, the source terms may also depend on the space variable in practice and may not be dissipative. This is the case for example for Saint-Venant equations with both slope and arbitrary friction, or Euler equations with arbitrary friction slope, and general gas pressure, which are more realistic.

For general density-velocity systems, we find that for any H^2 steady-state, there always exists a basic quadratic Lyapunov function for the H^2 norm (or basic H^2 Lyapunov function) that guarantees the exponential stability of the steady-state for the H^2 norm provided suitable boundary conditions and physical condition on the source term. For the link between dissipative entropies and basic quadratic Lyapunov functions, one can look at Section 1.4.4 in the Introduction. Our result in this chapter is quite generic and can be widely used in applications, we illustrate it by applying it to several physical systems: the general nonlinear Saint-Venant equations, the general isentropic Euler equations, the motion of water in a rigid pipe, and a flow model under osmosis phenomenon. Moreover, our method has many advantages when applying it in the real world. For example, to stabilize the Saint-Venant equations, we require some knowledge on the section and the velocities only at the boundaries. No information on the internal section profile, on the slope or on the friction model is required. This is very convenient in practice, as this feedback law can be applied without a clear information of the inner state of the channel (bathymetry, material, profile, etc.) since there may be no way to know properly the precise shape or material of the channel. Besides, while many friction models exist (see e.g. [42, Section 4.5]), it also completes the debate about which friction model to use as this feedback law works for any of them, without requiring to know it.

The organization of the chapter is as follows. In Section 5.2, we present the main results: the exponential stabilization of general density-velocity systems with two boundary controls. Moreover in Theorem 5.2.2, the exponential stabilization result with single boundary control is presented. Then we apply the result to several physical models. In Section 5.3, we give the proof of our main results, namely Theorem 5.2.1 and Theorem 5.2.2. Finally, some detailed computations are provided in the appendix.

5.2 Model considered and main result

A nonlinear hyperbolic density-velocity system is composed of a mass conservation law and a balance of momentum [16] and is thus given by

$$\partial_t H + \partial_x (HV) = 0, \tag{5.2.1}$$

$$\partial_t V + V\partial_x V + \partial_x (P(H, x)) + S(H, V, x) = 0, \tag{5.2.2}$$

where $t \in [0, +\infty)$, $x \in [0, L]$ with $L > 0$ any arbitrary constant. In many applications, $H : [0, +\infty) \times [0, L] \to (0, +\infty)$ denotes the density, $V : [0, +\infty) \times [0, L] \to (0, +\infty)$ denotes the propagation velocity, HV is the flow density and and $S(H, V, x)$ is a source term resulting of non-conservative forces acting on the system, such as slope or friction. The first equation expresses the flux conservation and is often known as continuity equation, while the second equation is usually referred as dynamical or momentum equation. In this second equation, $V\partial_x V$ represents the variation of the kinetic energy, while $\partial_x (P(H, x))$ represents the variation of the potential energy and corresponds to a conservative force (e.g. pressure, gravitation, etc.). As we are interested in physical systems, we assume that $S \in C^2((0, +\infty)^3; \mathbb{R})$, $P \in C^2((0, +\infty)^2; \mathbb{R})$ and here and hereafter, we also assume that

$$H>0, \quad V>0, \quad \partial_H P(H,x)>0. \tag{5.2.3}$$

The steady-states (H^*,V^*) of (5.2.1)–(5.2.2) are the solutions of

$$H_x^* = V_x^* \frac{H^*}{V^*}, \tag{5.2.4}$$

$$V_x^* = V^* \frac{S(H^*,V^*,\cdot)+\partial_x P(H^*,\cdot)}{\partial_H P(H^*,\cdot)H^* - V^{*2}}. \tag{5.2.5}$$

For each initial condition $(H^*(0),V^*(0)) \in (0,+\infty)^2$ satisfying $\partial_H P(H^*(0),0)H^*(0) - V(0)^{*2} > 0$, there exists a unique maximal solution to (5.2.4)-(5.2.5), and this maximal solution exists as soon as the condition $\partial_H P(H^*,\cdot)H^* > V^{*2}$ is satisfied. Besides, as hyperbolic systems with propagation velocities of the same sign can always be stabilized by the means of proportional boundary feedback (see e.g. [107]), we assume in the following that the propagation velocities of this system have opposite signs, which, from (5.2.1)–(5.2.2), means that $\partial_H P(H^*,\cdot)H^* > V^{*2}$. This holds for example in the case of the fluvial regime for Saint-Venant equations.

In the following, we give two strategies of boundary controls. As a first strategy, Theorem 5.2.1 relies on two boundary controls, i.e. the number of controls are equal to the number of the unknown functions. While in practice, one may control only one boundary. In the regulation of navigable rivers, for instance, one usually apply only one control at the downstream of the channel. Theorem 5.2.2 is thus concerned with the stabilization of general density-velocity systems with a single boundary control.

Two boundary controls We aim at stabilizing the steady-states of (5.2.1)–(5.2.2) with boundary feedback controls. We suppose that the boundary conditions have the form

$$\begin{aligned} V(t,0) &= \mathcal{B}_1(H(t,0)),\\ V(t,L) &= \mathcal{B}_2(H(t,L)), \end{aligned} \tag{5.2.6}$$

where the control function $\mathcal{B}=(\mathcal{B}_1,\mathcal{B}_2):\mathbb{R}^2\to\mathbb{R}^2$ is of class C^2. These kind of boundary conditions are imposed by physical devices in engineering system (e.g. sluice gates, feeding valves, pumps, etc.). This control function is one of the most simple possible feedback law as one does not need to know the full state of the system. Moreover, this control is local, which means that one only needs to measure the value at the same end where the control acts.

As we study the stabilization in the H^2 norm, for any given initial condition

$$H(0,x)=H_0(x), \quad V(0,x)=V_0(x), \quad x\in[0,L], \tag{5.2.7}$$

with $(H_0,V_0)\in H^2((0,L);\mathbb{R}^2)$, we need to add the following first-order compatibility condition for any given initial condition sufficiently regular

$$\begin{aligned} V_0(0) =& \mathcal{B}_1(H_0(0)),\\ V_0(L) =& \mathcal{B}_2(H_0(L)),\\ \left(V_0\partial_x V_0 + \partial_H P(H_0,\cdot)\partial_x H_0 + \partial_x P(H_0,\cdot) + S(H_0,V_0,\cdot)\right)(0) =& \mathcal{B}_1'(H_0(0))\partial_x(H_0V_0)(0),\\ \left(V_0\partial_x V_0 + \partial_H P(H_0,\cdot)\partial_x H_0 + \partial_x P(H_0,\cdot) + S(H_0,V_0,\cdot)\right)(L) =& \mathcal{B}_2'(H_0(L))\partial_x(H_0V_0)(L). \end{aligned} \tag{5.2.8}$$

We recall now the definition of the exponential stability for the H^2 norm.

Definition 5.2.1. A steady-state (H^*,V^*) of the system (5.2.1), (5.2.2), (5.2.6) is exponentially stable for the H^2 norm if there exist $\delta>0$, $\gamma>0$ and $C>0$ such that, for any $(H_0,V_0)\in H^2((0,L);\mathbb{R}^2)$ satisfying

$$|H_0-H^*|_{H^2} + |V_0-V^*|_{H^2} < \delta, \tag{5.2.9}$$

and the compatibility conditions (5.2.8), and for any $T>0$, the Cauchy problem (5.2.1), (5.2.2), (5.2.6) and (5.2.7) has a unique solution $(H(t,\cdot),V(t,\cdot))\in H^2((0,L);\mathbb{R}^2)$ satisfying

$$|H(t,\cdot)-H^*|_{H^2} + |V(t,\cdot)-V^*|_{H^2} \le Ce^{-\gamma t}\left(|H_0-H^*|_{H^2} + |V_0-V^*|_{H^2}\right),\ \forall\, t\in[0,T). \tag{5.2.10}$$

Our main result is the following

Theorem 5.2.1. *Assume that* $S \in C^2((0,+\infty)^3;\mathbb{R})$ *and* $P \in C^2((0,+\infty)^2;\mathbb{R})$*, let* (H^*,V^*) *be a steady-state of the nonlinear hyperbolic density-velocity system* (5.2.1), (5.2.2), (5.2.6) *satisfying*

$$\frac{\partial_V S(H^*,V^*,\cdot)}{V^*} - \frac{\partial_H S(H^*,V^*,\cdot)}{\partial_H P(H^*,\cdot)} \geq 0. \tag{5.2.11}$$

If the boundary conditions satisfy:

$$\begin{aligned} \mathcal{B}_1'(H^*(0)) &\in \left[-\frac{\partial_H P(H^*(0),0)}{V^*(0)}, -\frac{V^*(0)}{H^*(0)}\right], \\ \mathcal{B}_2'(H^*(L)) &\in \mathbb{R}\setminus\left[-\frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right], \end{aligned} \tag{5.2.12}$$

then the steady-state (H^*,V^*) *is exponentially stable for the* H^2 *norm.*

Remark 5.2.1. *Note that condition* (5.2.11) *holds naturally for most physical systems with source terms, (e.g. friction, slope, electric field, external forces, etc.), as illustrated in Section 5.2.1. Besides, note also that the source term does not have to be dissipative, as* S *could be negative.*

The proof of this result is given in Section 5.3. As announced in the introduction, this is done by showing the existence of a basic quadratic Lyapunov function for the H^2 norm (or basic H^2 Lyapunov function).

Single boundary control Suppose now that we have only a single feedback control, the other boundary condition being imposed, for instance by a constant but unknown upstream flow rate on which we cannot act. The boundary conditions are now

$$\begin{aligned} &H(t,0)V(t,0) = Q_0, \\ &V(t,L) = \mathcal{B}_2(H(t,L)), \end{aligned} \tag{5.2.13}$$

where Q_0 is the unknown constant inflow upstream, while $\mathcal{B}_2 : \mathbb{R} \to \mathbb{R}$ of class C^2 is still the control function. Using the same basic quadratic Lyapunov function for the H^2 norm we can still achieve the exponential stability which is a direct application of Theorem 5.2.1 by noticing now that $\mathcal{B}_1(H(t,0)) = Q_0/H(t,0)$ and that the steady-state satisfies $H^*(x)V^*(x) = Q_0$.

Theorem 5.2.2. *Assume that* $S \in C^2((0,+\infty)^3;\mathbb{R})$ *and* $P \in C^2((0,+\infty)^2;\mathbb{R})$*, let* (H^*,V^*) *be a steady-state of the nonlinear hyperbolic density-velocity system* (5.2.1), (5.2.2), (5.2.13) *satisfying* (5.2.11)*. If the boundary control satisfies:*

$$\mathcal{B}_2'(H^*(L)) \in \mathbb{R}\setminus\left[-\frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right], \tag{5.2.14}$$

then the steady-state (H^*,V^*) *is exponentially stable for the* H^2 *norm.*

Remark 5.2.2. *Note that* Q_0 *is assumed to be constant otherwise no steady-state* (H^*,V^*) *exists. However, the stabilization of slowly-varying target-state when* Q_0 *can vary, possibly a lot, but slowly would hold under the same condition, adapting the control as in [108].*

Remark 5.2.3. *Theorem 5.2.2 still holds if the control is located in* $x = 0$ *instead, while the imposed flow is located in* $x = L$.

5.2.1 Applications to physical systems

The system (5.2.1), (5.2.2) with boundary conditions (5.2.6) covers many well-known systems and we give now a few examples.

General Saint-Venant equations The Saint-Venant equations are the basis model for the regulation of navigable rivers and irrigation networks in agriculture. The stabilization of the Saint-Venant equations by means of local boundary feedbacks has been widely studied [16, 18, 52, 53, 83, 171]. Recently in [110], the authors obtained the stabilization of the Saint-Venant equations with non-negligible friction and arbitrary slope. However, this result is obtained under the assumption of a rectangular cross section with a constant width and a known friction model. In the following, we show that our result applies to the most general 1D Saint-Venant equations with arbitrary varying slope, section profile and friction model [10]:

$$\partial_t A + \partial_x (AV) = 0, \tag{5.2.15}$$

$$\partial_t (AV) + \partial_x (AV^2) + gA(\partial_x H - S_b(x) + S_f(A, V, x)) = 0, \tag{5.2.16}$$

where A is the cross-sectional area of the section, V is the velocity, AV is consequently the flux, H is the height of the water, S_b is the slope, S_f is the friction and g is the gravity acceleration. Note that the friction logically depends on H and V but can also depend on x for external reasons, for instance if the material of the channel changes. Whatever is the section profile, A is strictly increasing with H, thus there exists a function G strictly increasing with A such that $H = G(A, x)$ and consequently, using (5.2.15), (5.2.16) can be written as

$$\partial_t V + V \partial_x V + g\partial_A G(A, x)\partial_x A + g\partial_x G(A, x) + g(S_f(A, V, x) - S_b(x)) = 0. \tag{5.2.17}$$

Thus, system (5.2.15)-(5.2.16) has the form (5.2.1)–(5.2.2) with $P = gG(A, x)$ and $S = g(S_f - S_b)$. Besides, to be physically acceptable, the friction term has to be increasing with V and decreasing with A. Hence $\partial_V S = g\partial_V S_f > 0$ and $\partial_A S = g\partial_A S_f < 0$. Noticing that $\partial_A P = g\partial_A G(A, x) > 0$, condition (5.2.11) is satisfied and we have the following theorem

Theorem 5.2.3. *Any steady-state* (A^*, V^*) *of the general Saint-Venant equations* (5.2.15), (5.2.17) *with boundary conditions* (5.2.6) *with A instead of H, is exponentially stable for the H^2 norm provided that*

$$\begin{aligned}
\mathcal{B}'(A^*(0)) &\in \left[-\frac{g\partial_A G(A^*(0), 0)}{V^*(0)}, -\frac{V^*(0)}{A^*(0)}\right], \\
\mathcal{B}'(A^*(L)) &\in \mathbb{R} \setminus \left[-\frac{g\partial_A G(A^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{A^*(L)}\right].
\end{aligned} \tag{5.2.18}$$

Water motion in a rigid pipe The water motion in a rigid pipe is a common example for engineering system, whose equations are given in [15] as follows

$$\begin{aligned}
\partial_t \left(\exp\left(\frac{gP}{c^2}\right)\right) + \partial_x \left(V \exp\left(\frac{gP}{c^2}\right)\right) &= 0, \\
\partial_t V + V\partial_x V + \partial_x (gP) + S_f(V, x) &= 0,
\end{aligned} \tag{5.2.19}$$

where P is the piezometric head, $V > 0$ is the speed of the water, c is the sound velocity in water, g is the gravity acceleration, and S_f is the friction term. As previously, to be physically acceptable, the friction term has to be nondecreasing with V, thus (5.2.11) holds. Denoting $H = \exp\left(gP/c^2\right)$, this system has the form of (5.2.1)–(5.2.2) with $P = gH$. And obviously $\partial_H P > 0$ thus Theorem 5.2.1 applies again.

The isentropic Euler equations The isentropic Euler equations are used to model the gas transportation in pipelines. There are many literatures on the stabilization of the isentropic Euler equations [16, 81, 82, 94, 95, 99, 101]. But all those results are obtained without considering the pipeline slope and using the polytropic gas assumpation or the isothermal assumpation. The isentropic Euler equations with slope and friction have exactly the form (5.2.1)–(5.2.2) as (see e.g. [15][1.8.1] or [102])

$$\begin{aligned}
&\partial_t \varrho + \partial_x (\varrho V) = 0, \\
&\partial_t V + V\partial_x V + \frac{\partial_x (\mathcal{P}(\varrho))}{\varrho} + \frac{1}{2}\theta V|V| + g\sin\alpha(x) = 0,
\end{aligned} \tag{5.2.20}$$

where ϱ is the gas density, V is the velocity, $\alpha \in C^2(\mathbb{R})$ is the slope of the pipe, g is the acceleration gravity, $\mathcal{P}(\varrho)$ is the pressure (increasing with ϱ) with $\sqrt{\mathcal{P}'(\varrho)} > 0$ being the sound speed in the gas, $\theta = \lambda/D$ with $\lambda > 0$ is the friction coefficient and $D > 0$ the diameter of the pipe. In this case, $P := \int_0^{\varrho} \mathcal{P}'(s)/s\, ds$ and $S = 1/2\theta V|V| + g \sin\alpha$. Thus, $\partial_\varrho P(\varrho) > 0$, $\partial_\varrho S = 0$, $\partial_V S > 0$ as long as $V > 0$, which implies that (5.2.11) holds and that Theorem 5.2.1 applies. Note that this holds in particular in the case where the gas is polytropic, i.e. $\mathcal{P} = a^2\varrho^\gamma$ with $\gamma > 1$ (as in [16]), and in the case of the isothermal Euler equation, i.e. $\mathcal{P} = a^2\varrho$ (as in [102]).

Flow under osmosis Osmosis is a spontaneous movement of solvant or solute through a semipermeable membrane in a solute/solvant mix. This phenomenon is extremely important in chemistry and biology as it is the main way by which water is transported out of cells in living organisms. Besides, biological membranes allow much faster filtration than any artificial mechanical membrane, thus attempts have been recently made to design active membranes that would mimic this behavior and a mechanical model for this phenomenon can be found in [154].
Osmosis phenomenon through a membrane permeable to the solute but not to the solvant can be modeled by a potential barrier which acts on the solute. This creates, from Newton's law, a volume force on the fluid $-c(x)\partial_x\mathcal{U}$, where $\mathcal{U}$ is the profile of the potential barrier, compactly supported, c is the concentration and x is the space variable [154]. In an inviscid fluid modeled by the isentropic Euler equations (5.2.20), this reduces to adding a pressure term. Therefore, we still have $\partial_\varrho P(\varrho) > 0$, $\partial_\varrho S = 0$, $\partial_V S > 0$ as long as $V > 0$, and Theorem 5.2.1 applies. Note that any external potential acting on a fluid modeled by the isentropic Euler equations would fit in our framework, osmosis is only an example.

5.2.2 Optimality of the conditions on the control

In this section, we will show the optimality of the conditions on the control in the sense that no basic quadratic Lyapunov function that would always exist for density-velocity systems satisfying (5.2.11) can give strictly less restrictive boundary conditions than (5.2.12) and (5.2.14), making these conditions quite sharp. These results are given by Theorem 5.2.4 and Theorem 5.2.5.

Theorem 5.2.4. *Assume that $S \in C^2((0,+\infty)^3;\mathbb{R})$ and $P \in C^2((0,+\infty)^2;\mathbb{R})$. Let a steady-state $(H^*, V^*) \in C^1([0,L])$ of the nonlinear hyperbolic density-velocity system (5.2.1), (5.2.2), (5.2.6) satisfying (5.2.11) with $S(H^*, V^*, \cdot) \leq 0$. For any $\varepsilon > 0$ there exists $L > 0$ such that, if there exists a basic quadratic Lyapunov function for the the H^2 norm, then*

$$\begin{aligned} \mathcal{B}'(H^*(0)) &\in \left(-\varepsilon - \frac{\partial_H P(H^*(0),0)}{V^*(0)}, \varepsilon - \frac{V^*(0)}{H^*(0)}\right), \\ \mathcal{B}'(H^*(L)) &\in \mathbb{R} \setminus \left[\varepsilon - \frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\varepsilon - \frac{V^*(L)}{H^*(L)}\right]. \end{aligned} \tag{5.2.21}$$

This theorem shows therefore that the condition (5.2.12) given in Theorem 5.2.1 is quite sharp and cannot be significantly improved. The situation is even clearer in the case of a single boundary control.

Theorem 5.2.5. *Assume that $S \in C^2((0,+\infty)^3;\mathbb{R})$ and $P \in C^2((0,+\infty)^2;\mathbb{R})$. Let a steady-state $(H^*, V^*) \in C^1([0,L])$ of the nonlinear hyperbolic density-velocity system (5.2.1), (5.2.2), (5.2.13) satisfying (5.2.11). There exists a basic quadratic Lyapunov function for the the H^2 norm if and only if*

$$\mathcal{B}'(H^*(L)) \in \mathbb{R} \setminus \left[-\frac{\partial_H P(H^*(L),L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right]. \tag{5.2.22}$$

The proofs of Theorems 5.2.4 and 5.2.5 are given in Section 5.3.

5.3 Exponential stability of density-velocity hyperbolic systems

In this section we prove Theorem 5.2.1. Let (H^*, V^*) be a steady-state of (5.2.1)–(5.2.2). We start by proving the exponential stability of the linearized system around this steady-state for the L^2 norm to give an idea of how the proof works and then, we show that the same type of Lyapunov function can be applied to ensure the exponential stability of the nonlinear system for the H^2 norm.

5.3.1 Exponential stability of the linearized system

Around the steady-state (H^*, V^*), the linearized system of (5.2.1)-(5.2.2) and (5.2.6) is given by:

$$\begin{aligned}
&\partial_t h + V^*\partial_x h + H^*\partial_x v + V^*_x h + H^*_x v = 0,\\
&\partial_t v + V^*\partial_x v + V^*_x v + \partial_H P(H^*, x)\partial_x h + \partial^2_{HH} P(H^*, x)H^*_x h + \partial^2_{xH} P(H^*, x)h\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \partial_H S(H^*, V^*, x)h + \partial_V S(H^*, V^*, x)v = 0.
\end{aligned} \tag{5.3.1}$$

and

$$\begin{aligned}
v(t, 0) &= c_1 h(t, 0),\\
v(t, L) &= c_2 h(t, L),
\end{aligned} \tag{5.3.2}$$

where $h = H - H^*$ and $v = V - V^*$ are the perturbations and $c_1 = \mathcal{B}'(H^*(0))$ and $c_2 = \mathcal{B}'(H^*(L))$. To simplify the notations, we denote from now on

$$\begin{gathered}
\partial_H P(H^*, x) := f(H^*, x), \quad S_{H^*} := \partial_H S(H^*, V^*, x), \quad S_{V^*} := \partial_V S(H^*, V^*, x),\\
\mathscr{S}_{H^*} := \partial_x f(H^*, x) + S_{H^*}.
\end{gathered} \tag{5.3.3}$$

Thus, the linearized system of (5.2.1)-(5.2.2) and (5.2.6) around the steady-state (H^*, V^*) given by (5.3.1) becomes

$$\begin{pmatrix} h \\ v \end{pmatrix}_t + \begin{pmatrix} V^* & H^* \\ f(H^*, x) & V^* \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix}_x + \begin{pmatrix} V^*_x & H^*_x \\ \mathscr{S}_{H^*} + \partial_H f(H^*, x)H^*_x & S_{V^*} + V^*_x \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.3.4}$$

We prove the following proposition

Proposition 5.3.1. *Let (H^*, V^*) be any given steady-state such that (5.2.11) holds, if the boundary conditions satisfy:*

$$\begin{aligned}
c_1 &\in \left[-\frac{f(H^*(0), 0)}{V^*(0)}, -\frac{V^*(0)}{H^*(0)}\right],\\
c_2 &\in \mathbb{R} \setminus \left[-\frac{f(H^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right],
\end{aligned} \tag{5.3.5}$$

then the null steady-state $h = 0$, $v = 0$ of the system (5.3.4)–(5.3.2) is exponentially stable for the L^2 norm.

Proof. Observe that the matrix $\begin{pmatrix} V^* & H^* \\ f(H^*, \cdot) & V^* \end{pmatrix}$ can be diagonalized, therefore the system can be put under the Riemann invariant form by the following change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{f(H^*, x)}{H^*}} & 1 \\ -\sqrt{\frac{f(H^*, x)}{H^*}} & 1 \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix}. \tag{5.3.6}$$

Then (5.3.4) becomes (see Appendix 5.4.1)

$$\begin{aligned}
\partial_t z_1 + \lambda_1\partial_x z_1 + \gamma_1 z_1 + \delta_1 z_2 = 0,\\
\partial_t z_2 - \lambda_2\partial_x z_2 + \gamma_2 z_1 + \delta_2 z_2 = 0,
\end{aligned} \tag{5.3.7}$$

where

$$\lambda_1 = V^* + \sqrt{f(H^*,x)H^*} > 0, \quad \lambda_2 = \sqrt{f(H^*,x)H^*} - V^* > 0 \tag{5.3.8}$$

and

$$\begin{aligned}
\gamma_1 =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - 3\lambda_2\frac{V_x^*}{V^*} - \lambda_1\frac{\partial_x f(H^*,x)}{f(H^*,x)} + \lambda_2\frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right),\\
\gamma_2 =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + \lambda_1\frac{V_x^*}{V^*} - \lambda_2\frac{\partial_x f(H^*,x)}{f(H^*,x)} + \lambda_1\frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right),\\
\delta_1 =& \frac{1}{4}\left(2S_{V^*} - 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - \lambda_2\frac{V_x^*}{V^*} + \lambda_1\frac{\partial_x f(H^*,x)}{f(H^*,x)} - \lambda_2\frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right),\\
\delta_2 =& \frac{1}{4}\left(2S_{V^*} - 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + 3\lambda_1\frac{V_x^*}{V^*} + \lambda_2\frac{\partial_x f(H^*,x)}{f(H^*,x)} - \lambda_1\frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right).
\end{aligned} \tag{5.3.9}$$

The boundary conditions (5.3.2) become

$$z_1(t,0) = k_1 z_2(t,0), \quad z_2(t,L) = k_2 z_1(t,L) \tag{5.3.10}$$

with

$$k_1 = \frac{c_1 + \sqrt{\frac{f(H^*(0),0)}{H^*(0)}}}{c_1 - \sqrt{\frac{f(H^*(0),0)}{H^*(0)}}}, \quad k_2 = \frac{c_2 - \sqrt{\frac{f(H^*(L),L)}{H^*(L)}}}{c_2 + \sqrt{\frac{f(H^*(L),L)}{H^*(L)}}}. \tag{5.3.11}$$

With these conditions, the Cauchy problem (5.3.7), (5.3.10) with any given initial condition $\mathbf{z}(0,x) = (z_{10}, z_{20}) \in L^2((0,L);\mathbb{R}^2)$ is well-posed (see [15, Appendix A]), which implies that the original system in physical coordinates is also well-posed.
We define the function ϕ by

$$\phi(x) = \exp\left(\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)} + \frac{\delta_2(s)}{\lambda_2(s)}ds\right) \tag{5.3.12}$$

and we introduce the following lemma, that can also be found in [110] in the particular case of the Saint-Venant equations with constant rectangular section and friction given by $S_f = kV^2A^{-1}$ where $k > 0$ is a constant friction coefficient.

Lemme 5.3.1. *For any $x \in [0,L]$,*

$$\left(\frac{\lambda_2}{\lambda_1}\phi\right)'(x) = \frac{\phi\delta_1}{\lambda_1} + \frac{\phi^{-1}\gamma_2}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}\phi\right)^2. \tag{5.3.13}$$

The proof of this lemma is given in the Appendix 5.4.2. We introduce now the following Lyapunov function candidate for the L^2 norm

$$V = \int_0^L \left(f_1(x)e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds}e^{-\frac{\mu}{\lambda_1}x}z_1^2(t,x) + f_2(x)e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds}e^{\frac{\mu}{\lambda_2}x}z_2^2(t,x)\right)dx, \tag{5.3.14}$$

where $\mu > 0$ is a constant and f_1, f_2 are positive C^1 functions to be chosen later on. From the positivity of f_1 and f_2, there exist a_1 and a_2 positive constants such that

$$a_2\|(z_1,z_2)\|_{L^2((0,L);\mathbb{R}^2)} \leq V \leq a_1\|(z_1,z_2)\|_{L^2((0,L);\mathbb{R}^2)} \tag{5.3.15}$$

which means that V is equivalent to the L^2 norm of (z_1,z_2), thus is equivalent to the L^2 norm of (h,v) from the linear change of variables (5.3.6). Therefore, it suffices to show the exponential decay of V to obtain the exponential stability of (5.3.4) and (5.3.2) for the L^2 norm. Differentiating (5.3.14) with time along the C^1 solutions, one has

$$
\begin{aligned}
\frac{dV}{dt} =& -\left[\lambda_1 f_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x} z_1^2 - \lambda_2 f_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x} z_2^2\right]_0^L \\
&- \int_0^L \Big[(2f_1\gamma_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x} - (\lambda_1 f_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x})')z_1^2 \\
&\quad + (2f_2\delta_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x} + (\lambda_2 f_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x})')z_2^2 \\
&\quad + 2(f_1\delta_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x} + f_2\gamma_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x})z_1 z_2\Big]dx. \\
=& -\mu V - \left[\lambda_1 f_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x} z_1^2 - \lambda_2 f_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x} z_2^2\right]_0^L \\
&- \int_0^L \Big[-(\lambda_1 f_1)' e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x} z_1^2 + (\lambda_2 f_2)' e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x} z_2^2 \\
&\quad - \frac{\mu x \lambda_1'}{\lambda_1} f_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x} z_1^2 - \frac{\mu x \lambda_2'}{\lambda_2} f_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x} z_2^2 \\
&\quad + 2(f_1\delta_1 e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}x} + f_2\gamma_2 e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}x})z_1 z_2\Big]dx.
\end{aligned} \tag{5.3.16}
$$

Using the boundary conditions (5.3.10), we get

$$
\begin{aligned}
\frac{dV}{dt} =& -\left(\lambda_1(L)f_1(L)e^{2\int_0^L \frac{\gamma_1(s)}{\lambda_1(s)}ds} e^{-\frac{\mu}{\lambda_1}L} - k_2^2\lambda_2(L)f_2(L)e^{-2\int_0^L \frac{\delta_2(s)}{\lambda_2(s)}ds} e^{\frac{\mu}{\lambda_2}L}\right) z_1^2(t,L) \\
&- \left(\lambda_2(0)f_2(0) - k_1^2\lambda_1(0)f_1(0)\right) z_2^2(t,0) \\
&- \int_0^L \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} z_1 \\ e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} z_2 \end{pmatrix}^T I \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} z_1 \\ e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} z_2 \end{pmatrix} dx,
\end{aligned} \tag{5.3.17}
$$

where

$$
I = \begin{pmatrix} \left(-(\lambda_1 f_1)' - \frac{\mu x \lambda_1'}{\lambda_1} f_1\right) e^{-\frac{\mu}{\lambda_1}x} & f_1\delta_1\phi(x)e^{-\frac{\mu}{\lambda_1}x} + f_2\gamma_2\phi^{-1}(x)e^{\frac{\mu}{\lambda_2}x} \\ f_1\delta_1\phi(x)e^{-\frac{\mu}{\lambda_1}x} + f_2\gamma_2\phi^{-1}(x)e^{\frac{\mu}{\lambda_2}x} & \left((\lambda_2 f_2)' - \frac{\mu x \lambda_2'}{\lambda_2} f_2\right) e^{\frac{\mu}{\lambda_2}x} \end{pmatrix}. \tag{5.3.18}
$$

Therefore, by continuity and using the definition of ϕ given in (5.3.12), it suffices to show that there exist f_1 and f_2, such that the following matrix

$$
I_0 = \begin{pmatrix} -(\lambda_1 f_1)' & f_1\delta_1\phi(x) + f_2\gamma_2\phi^{-1}(x) \\ f_1\delta_1\phi(x) + f_2\gamma_2\phi^{-1}(x) & (\lambda_2 f_2)' \end{pmatrix} \tag{5.3.19}
$$

is positive definite and that

$$
\begin{aligned}
&\lambda_1(L)f_1(L)\phi^2(L) - k_2^2\lambda_2(L)f_2(L) > 0, \\
&\lambda_2(0)f_2(0) - k_1^2\lambda_1(0)f_1(0) > 0
\end{aligned} \tag{5.3.20}
$$

to prove the exponential decay of V.

Before going any further, observe that under the assumption (5.2.11), from (5.3.9), (5.3.8) and noticing the notations (5.3.3), one has

$$
\begin{aligned}
\frac{\phi\delta_1}{\lambda_1} + \frac{\phi^{-1}\gamma_2}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}\phi\right)^2 &= \frac{\phi}{\lambda_1^2}(\lambda_1\delta_1 + \lambda_2\gamma_2) \\
&= \frac{\phi}{\lambda_1^2}\left(\frac{(\lambda_1+\lambda_2)}{2}S_{V^*} + \frac{(\lambda_2-\lambda_1)}{2}\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + \frac{(\lambda_1^2-\lambda_2^2)}{4}\frac{\partial_x f(H^*,x)}{f(H^*,x)}\right)
\end{aligned}
$$

$$
= \frac{\phi}{\lambda_1^2}\left(\sqrt{f(H^*,x)H^*}S_{V^*} - V^*\sqrt{\frac{H^*}{f(H^*,x)}}S_{H^*}\right) \geq 0, \tag{5.3.21}
$$

which together with Lemma 5.3.1 implies that $\lambda_2\phi/\lambda_1$ is a solution to the differential equation

$$
\eta' = \left|\frac{\delta_1}{\lambda_1}\phi + \frac{\gamma_2}{\lambda_2}\phi^{-1}\eta^2\right|, \quad \eta(0) = \frac{\lambda_2(0)}{\lambda_1(0)} \tag{5.3.22}
$$

on $[0, L]$. Thus, there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in [0, \varepsilon_1)$, there exists a solution η_ε on $[0, L]$ to

$$
\eta_\varepsilon' = \left|\frac{\delta_1}{\lambda_1}\phi + \frac{\gamma_2}{\lambda_2}\phi^{-1}\eta_\varepsilon^2\right| + \varepsilon, \quad \eta_\varepsilon(0) = \frac{\lambda_2(0)}{\lambda_1(0)} \tag{5.3.23}
$$

and such that we can define a map $\varepsilon \to \eta_\varepsilon$ which is C^0 on $[0, \varepsilon_1)$ [105]. Let us define

$$
f_1 = (\lambda_1\eta_\varepsilon)^{-1} \text{ and } f_2 = \lambda_2^{-1}\eta_\varepsilon, \tag{5.3.24}
$$

where $\varepsilon \in (0, \varepsilon_1)$ can be chosen later on. One has from (5.3.5) and (5.3.11) that

$$
k_1^2 \leq \left(\frac{\lambda_2(0)}{\lambda_1(0)}\right)^2, \quad k_2^2 < \left(\frac{\lambda_1(L)}{\lambda_2(L)}\right)^2. \tag{5.3.25}
$$

Therefore from the continuity of $\varepsilon \to \eta_\varepsilon$, there exists $0 < \varepsilon_2 < \varepsilon_1$ such that for all $\varepsilon \in (0, \varepsilon_2)$

$$
k_1^2 < \frac{\lambda_2(0)f_2(0)}{\lambda_1(0)f_1(0)}, \quad k_2^2 < \frac{\lambda_1(L)f_1(L)}{\lambda_2(L)f_2(L)}\phi^2(L), \tag{5.3.26}
$$

which is exactly the same as condition (5.3.20) from the definition of ϕ in (5.3.12). We choose such $\varepsilon \in (0, \varepsilon_2)$, and we are left to prove that I_0 defined by (5.3.19) is positive definite. We have from (5.3.19), (5.3.24) and (5.3.23) that

$$
\begin{aligned}
\det(I_0) = &- (\lambda_1 f_1)'(\lambda_2 f_2)' - \left(f_1\delta_1\phi + f_2\gamma_2\phi^{-1}\right)^2 \\
= &\frac{1}{\eta_\varepsilon^2}\left((\eta_\varepsilon')^2 - \left(\frac{\delta_1}{\lambda_1}\phi + \frac{\gamma_2}{\lambda_2}\phi^{-1}\eta_\varepsilon^2\right)^2\right) > 0.
\end{aligned} \tag{5.3.27}
$$

Besides, from (5.3.23) and (5.3.24), one has $-(\lambda_1 f_1)' > 0$ and $(\lambda_2 f_2)' > 0$, hence I is positive definite. By continuity, there exists $\mu > 0$ such that

$$
\frac{dV}{dt} \leq -\mu V \tag{5.3.28}
$$

along the C^1-solutions of the system (5.3.7) and (5.3.10) for any $\mu \in (0, \mu_1)$. Since the C^1-solutions are dense in the set of L^2-solutions, inequality (5.3.28) also holds in the sense of distributions for the L^2-solutions (see [15, Section 2.1]) for the details). Thus, the exponential stability of (5.3.4)–(5.3.2) in the L^2 norm is also guaranteed thanks to the linear change of variables (5.3.6). This ends the proof of Proposition 5.3.1. □

5.3.2 Exponential stability of the nonlinear system

For the exponential stability of nonlinear system, the proof will be similar to the linearized case. For a given steady-state (H^*, V^*) defined on $[0, L]$, we can still define $h = H - H^*$ and $v = V - V^*$ as previously and (z_1, z_2) using the same change of variables (5.3.6). Then, for (z_1, z_2) small enough, the system (5.2.1)–(5.2.2), (5.2.6) is equivalent to

$$
\mathbf{z}_t + A(\mathbf{z}, x)\mathbf{z}_x + M(\mathbf{z}, x)\mathbf{z} = 0, \tag{5.3.29}
$$

where

$$A(\mathbf{0},x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & -\lambda_2(x) \end{pmatrix}, \quad M(\mathbf{0},x) = \begin{pmatrix} \gamma_1(x) & \delta_1(x) \\ \gamma_2(x) & \delta_2(x) \end{pmatrix}, \tag{5.3.30}$$

and

$$\begin{aligned} z_1(t,0) =& m_1(z_2(t,0)), \\ z_2(t,L) =& m_2(z_1(t,L)), \end{aligned} \tag{5.3.31}$$

with

$$m_1'(0) = k_1, \quad m_2'(0) = k_2, \tag{5.3.32}$$

here, k_1 and k_2 are defined as (5.3.11). In (5.3.31), m_1 and m_2 are found by the implicit function theorem around $\mathbf{0}$, for z_1 and z_2 small enough (see [110, A.2] for more details in a similar case). Noticing that the exponential stability of the steady-state (H^*, V^*) of system (5.2.1)–(5.2.2) and (5.2.6) is therefore equivalent to the exponential stability of the null steady-state ($z_1 = 0$, $z_2 = 0$) of system (5.3.29)–(5.3.32), we use the following theorem, which is a direct application of [15, Theorem 6.10].

Theorem 5.3.2. *If there exists C^1 functions $g_1(x) > 0$ and $g_2(x) > 0$ such that, with $Q = diag(g_1(x), g_2(x))$, one has*

$$-(QA(\mathbf{0},\cdot))' + QM(\mathbf{0},x) + M^T(\mathbf{0},x)Q \tag{5.3.33}$$

is positive definite on $[0,L]$ and the following inequalities hold

$$k_1^2 \leq \frac{\lambda_2(0)g_2(0)}{\lambda_1(0)g_1(0)}, \quad k_2^2 < \frac{\lambda_1(L)g_1(L)}{\lambda_2(L)g_2(L)}, \tag{5.3.34}$$

then the null steady-state of the system (5.3.29)–(5.3.32) is exponentially stable for the H^2 norm.

Remark 5.3.1. *This theorem actually shows the existence of a Lyapunov function for the H^2 norm of the form*

$$\begin{aligned} V =& \int_0^L (f_1(E(\mathbf{z},x)\mathbf{z})_1^2 + f_2(E(\mathbf{z},x)\mathbf{z})_2^2)dx + \int_0^L (f_1(E(\mathbf{z},x)\mathbf{z}_x)_1^2 + f_2(E(\mathbf{z},x)\mathbf{z}_x)_2^2)dx \\ &+ \int_0^L (f_1(E(\mathbf{z},x)\mathbf{z}_{xx})_1^2 + f_2(E(\mathbf{z},x)\mathbf{z}_{xx})_2^2)dx, \end{aligned} \tag{5.3.35}$$

where $E(\mathbf{0},\cdot) = Id$ (see [15, Chapter 6] for more details). This is the reason why we claim that this proof is actually the same as the proof of the exponential stability in the linearized case, and we will now see that we can use a similar Lyapunov function but for the H^2 norm.

Proof of Theorem 5.2.1. Let

$$g_1 := e^{2\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} f_1, \quad g_2 := e^{-2\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} f_2,$$

where f_1 and f_2 are defined in (5.3.24). One can directly check that

$$-(QA(\mathbf{0},\cdot))' + QM(\mathbf{0},x) + M^T(\mathbf{0},x)Q = \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} & 0 \\ 0 & e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} \end{pmatrix} I_0 \begin{pmatrix} e^{\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds} & 0 \\ 0 & e^{-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds} \end{pmatrix}$$

with I_0 defined as (5.3.19), as I_0 is positive definite from (5.3.27), condition (5.3.33) is thus satisfied. Condition (5.3.34) is satisfied from (5.3.26) by noticing the definition of ϕ given in (5.3.12). Thus, Theorem 5.3.2 applies and Theorem 5.2.1 holds. □

5.3.3 Optimality of the control

In this subsection we show Theorem 5.2.4 and 5.2.5.

Proof of Theorem 5.2.4. Let assume that along H^*, V^*, $S(H^*, V^*, x) + \partial_x P(H^*, V^*, x) \leq 0$. Then from (5.2.1)–(5.2.2), the steady-state (H^*, V^*) exists and is C^1 for any length $L > 0$. Suppose that there exists $\varepsilon_1 > 0$ such that for any length $L > 0$, there exists a basic quadratic Lyapunov function for the H^2 norm with

$$\begin{aligned} \mathcal{B}'(H^*(0)) &\in \mathbb{R} \setminus \left(-\varepsilon - \frac{\partial_H P(H^*(0), 0)}{V^*(0)}, \varepsilon - \frac{V^*(0)}{H^*(0)} \right), \\ \mathcal{B}'(H^*(L)) &\in \left[\varepsilon - \frac{\partial_H P(H^*(L), L)}{V^*(L)}, -\varepsilon - \frac{V^*(L)}{H^*(L)} \right]. \end{aligned} \tag{5.3.36}$$

We can then use the same change of variables (5.3.6), as in Section 5.3. The system (5.2.1)–(5.2.2), (5.2.6) becomes (5.3.29) with boundary conditions (5.3.31). From (5.3.36), we have

$$\begin{aligned} k_1^2 &> \frac{\phi^2(L)}{\eta^2(L)} \\ \text{or } k_2^2 &> \eta^2(0), \end{aligned} \tag{5.3.37}$$

where k_1, k_2 are defined by (5.3.32) and $\eta = \lambda_2 \phi / \lambda_1$. We define now

$$a = \delta_1 \phi, \qquad b = \gamma_2 \phi^{-1}. \tag{5.3.38}$$

As there exists a basic quadratic Lyapunov function for the H^2 norm, thus from [12] (see also [107, Theorem 3.5], and [107, (24),(40)–(43)]), there exists a function $\eta_2 \in C^1([0, L])$ such that

$$\eta' = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right| \tag{5.3.39}$$

on $[0, L]$ and there exists $\varepsilon_1 > 0$ depending only on ε such that

$$\begin{aligned} &\eta_2(L) \leq \eta(L) - \varepsilon_1, \\ &\text{or } \eta_2(0) \geq \eta(0) + \varepsilon_1. \end{aligned} \tag{5.3.40}$$

Now, as L can be taken arbitrarily, η_2 exists for any L, and thus on $[0, +\infty)$. We claim now that

$$\lim_{x \to +\infty} \eta_2(x) \in \mathbb{R}_+^*. \tag{5.3.41}$$

Indeed, let assume that $\lim\limits_{x \to +\infty} \eta_2(x) = +\infty$, then when x is large enough we have (see [107, Section 4])

$$\eta_2' = \frac{|b|}{\lambda_2} \eta_2^2 - \frac{}{\lambda_1}, \tag{5.3.42}$$

thus using the estimates of [107, Section 4], there would exists $C > 0$ and $x_1 > 0$ such that for any $x \geq x_1$,

$$\eta_2' \geq \frac{C}{x} \eta_2^2, \tag{5.3.43}$$

hence

$$\eta_2 \geq \frac{1}{\frac{1}{\eta_2(x_1)} - C \ln(x/x_1)}. \tag{5.3.44}$$

And η_2 exist and is positive and $[x_1, +\infty)$, hence the contradiction. Thus η_2 converges when x goes to $+\infty$ to a limit $\eta_{2,\infty}$. Note that ϕ converges to $\phi_\infty > 0$ [107, Section 4]. Besides,

$$\eta_2' = \frac{|\gamma_2|}{\lambda_1 \phi(x)} \left| \phi^2(x) - \frac{\lambda_1 |\delta_1|}{\lambda_2 \gamma_2} \eta^2 \right|. \tag{5.3.45}$$

As $\frac{\lambda_1|\delta_1|}{\lambda_2\gamma_2}$ goes to 1 when x goes to infinity, assume by contradiction that $\eta_{2,\infty} \neq \phi_\infty$, there exists C_3 and x_3 such that for all $x > x_3$,

$$\eta_2' \geq \frac{C_3}{x}, \tag{5.3.46}$$

which implies that $\lim\limits_{x\to+\infty} \eta_2(x) = +\infty$, hence contradiction. Thus η_2 converges to ϕ_∞, just as $\eta(L)$, which implies that in any cases the condition at $x = L$ become arbitrarily close to the one we obtain with η when L goes to infinity and prove that the first inequality of (5.3.40) is impossible.
Now let assume by contradiction that the second inequality of (5.3.40) is satisfied. Then $\eta_2(0) > \eta(0)$ and from (5.3.39),

$$\eta_2' \geq \left(\frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta_2^2 \right), \tag{5.3.47}$$

which implies that

$$(\eta_2 - \eta)' \geq -2\frac{|b|}{\lambda_2} (\eta_2 - \eta)\phi_\infty. \tag{5.3.48}$$

Thus

$$(\eta_{2,\infty} - \eta_\infty) \geq (\eta_2(0) - \eta(0)) \exp\left(-\phi_\infty \int_0^{+\infty} 2\frac{|b|}{\lambda_2} \right). \tag{5.3.49}$$

But, as seen in [107, Section 4], $\int_0^{+\infty} 2\frac{|b|}{\lambda_2} < +\infty$, which implies, using that $\eta_2(0) > \eta(0)$,

$$\eta_{2,\infty} > \eta_\infty, \tag{5.3.50}$$

while we know that $\eta_{2,\infty} = \eta_\infty$, hence the contradiction. This ends the proof of Theorem 5.2.4. □

We can now prove Theorem 5.2.5 in a very similar fashion.

Proof of Theorem 5.2.5. Let $(H^*, V^*) \in C^1([0, L])$ be a steady-state of (5.2.1)–(5.2.2). Let assume by contradiction that there exists a basic quadratic Lyapunov function for the H^2 norm and that

$$\mathcal{B}'(H^*(L)) \in \left[-\frac{\partial_H P(H^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)} \right]. \tag{5.3.51}$$

Then using again the change of variables (5.3.6) the system (5.2.1)–(5.2.2), (5.2.6) is again equivalent to (5.3.29) with boundary conditions (5.3.31) From (5.3.51), one has

$$k_1^2 := (\mathcal{D}_1'(0))^2 = \eta^2(0). \tag{5.3.52}$$

where k_1 is again given by (5.3.32) and $\eta = \lambda_2 \phi / \lambda_1$. As previously, as there exists a basic quadratic Lyapunov function for the H^2 norm, from [12] (see also [107, Theorem 3.5]) there exists a function $\eta_2 \in C^1([0, L])$ such that

$$\eta' = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right| \tag{5.3.53}$$

on $[0, L]$, where a and b are defined by (5.3.38), and there exists $\varepsilon_1 > 0$such that

$$\begin{aligned} &\eta_2(L) \leq \eta(L) - \varepsilon_1, \ \forall \ L > 0, \\ &\eta_2(0) \geq \eta(0). \end{aligned} \tag{5.3.54}$$

Using (5.3.54) and the same argument as (5.3.47)–(5.3.50), we get that $\eta_2(L) \geq \eta(L)$ thus

$$\eta_2(L) \geq \lambda_2(L)\phi(L)/\lambda_1(L) \tag{5.3.55}$$

which is in contradiction with (5.3.54). This ends the proof of Theorem 5.2.5. □

5.4 Appendix

5.4.1 Derivation of γ_1, γ_2, δ_1 and δ_2

Looking at 5.3.6, we denote by

$$\Delta = \begin{pmatrix} \sqrt{\frac{f(H^*,x)}{H^*}} & 1 \\ -\sqrt{\frac{f(H^*,x)}{H^*}} & 1 \end{pmatrix}$$

$$\Delta^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{H^*}{f(H^*,x)}} & -\sqrt{\frac{H^*}{f(H^*,x)}} \\ 1 & 1 \end{pmatrix}$$

Then, using the notations (5.3.3), (5.3.4) becomes

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\Delta\Delta_x^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \Delta\begin{pmatrix} V_x^* & H_x^* \\ \mathscr{S}_{H^*} + \partial_H f(H^*,x)H_x^* & S_{V^*} + V_x^* \end{pmatrix}\Delta^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.4.1}$$

where λ_1 and λ_2 are given by (5.3.8). Let us compute the coefficient of the first part of the source term,

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\Delta\Delta_x^{-1} = \frac{1}{4}\begin{pmatrix} \lambda_1\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) & -\lambda_1\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) \\ \lambda_2\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) & -\lambda_2\left(\frac{H_x^*}{H^*} - \frac{\frac{d}{dx}(f(H^*,x))}{f}\right) \end{pmatrix}. \tag{5.4.2}$$

The coefficient of the second part of the source term is

$$\begin{aligned} &\Delta\begin{pmatrix} V_x^* & H_x^* \\ \mathscr{S}_{H^*} + \partial_H f(H^*,x)H_x^* & S_{V^*} + V_x^* \end{pmatrix}\Delta^{-1} \\ &= \frac{1}{2}\begin{pmatrix} \partial_H f H_x^*\sqrt{\frac{H^*}{f}} + \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} + H_x^*\sqrt{\frac{f}{H^*}} + 2V_x^* + S_{V^*} & -\partial_H f H_x^*\sqrt{\frac{H^*}{f}} - \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} + H_x^*\sqrt{\frac{f}{H^*}} + S_{V^*} \\ \partial_H f H_x^*\sqrt{\frac{H^*}{f}} + \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} - H_x^*\sqrt{\frac{f}{H^*}} + S_{V^*} & -\partial_H f H_x^*\sqrt{\frac{H^*}{f}} - \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} - H_x^*\sqrt{\frac{f}{H^*}} + 2V_x^* + S_{V^*} \end{pmatrix}. \end{aligned} \tag{5.4.3}$$

Thus,

$$\begin{aligned} \gamma_1 =& \frac{1}{4}\left[\lambda_1\left(\frac{H_x^*}{H^*} - \frac{\partial_H f(H^*,x)H_x^* + \partial_x f(H^*,x)}{f}\right) + 2\left(\partial_H f H_x^*\sqrt{\frac{H^*}{f}} + \mathscr{S}_{H^*}\sqrt{\frac{H^*}{f}} + H_x^*\sqrt{\frac{f}{H^*}} + 2V_x^* + S_{V^*}\right)\right] \\ =& \frac{1}{4}\Big[2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - \lambda_1\frac{\partial_x f(H^*,x)}{f(H^*,x)} \\ &+ (V^* + \sqrt{fH^*})\left(\frac{H_x^*}{H^*} - \frac{\partial_H f(H^*,x)H_x^*}{f}\right) + 2\partial_H f H_x^*\sqrt{\frac{H^*}{f}} + 2H_x^*\sqrt{\frac{f}{H^*}} + 4V_x^*\Big] \\ =& \frac{1}{4}\left(2S_{V^*} + 2\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} - 3\lambda_2\frac{V_x^*}{V^*} - \lambda_1\frac{\partial_x f(H^*,x)}{f(H^*,x)} + \lambda_2\frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right), \end{aligned}$$

and γ_2, δ_1 and δ_2 can be found similarly.

5.4.2 Proof of Lemma 5.3.1

Differentiating $\lambda_2\phi/\lambda_1$ using (5.3.8), (5.3.9) and (5.3.12), we have

$$
\begin{aligned}
\left(\frac{\lambda_2}{\lambda_1}\phi\right)' &= \frac{\phi}{\lambda_1^2}\left(\lambda_1\lambda_2' - \lambda_1'\lambda_2 + (\lambda_2\gamma_1 + \lambda_1\delta_2)\right)\\
&= \frac{\phi}{\lambda_1^2}\Bigg\{(V^* + \sqrt{f(H^*,x)H^*})\left(-V_x^* + \frac{(f(H^*,x)H^*)'}{2\sqrt{f(H^*,x)H^*}}\right)\\
&\qquad -(\sqrt{f(H^*,x)H^*} - V^*)\left(V_x^* + \frac{(f(H^*,x)H^*)'}{2\sqrt{f(H^*,x)H^*}}\right)\\
&+\frac{1}{4}\left[(\lambda_1^2-\lambda_2^2)\left(3\frac{V_x^*}{V^*} - \frac{\partial_H f(H^*,x)H_x^*}{f(H^*,x)}\right) + 2(\lambda_2-\lambda_1)\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + 2(\lambda_2+\lambda_1)S_{V^*}\right]\Bigg\}\\
&= \frac{\phi}{\lambda_1^2}\Bigg(V^*\sqrt{\frac{H^*}{f(H^*,x)}}\partial_x f(H^*,x) + H_x^*V^*\sqrt{\frac{f(H^*,x)}{H^*}} + V_x^*\sqrt{f(H^*,x)H^*}\\
&\qquad\qquad -V^*\mathscr{S}_{H^*}\sqrt{\frac{H^*}{f(H^*,x)}} + \sqrt{f(H^*,x)H^*}S_{V^*}\Bigg).
\end{aligned}
\tag{5.4.4}
$$

Noticing the notations (5.3.3) and from (5.2.4), (5.4.4) becomes

$$
\left(\frac{\lambda_2}{\lambda_1}\phi\right)' = \frac{\phi}{\lambda_1^2}\left(\sqrt{f(H^*,x)H^*}S_{V^*} - V\sqrt{\frac{H^*}{f(H^*,x)}}S_{H^*}\right), \tag{5.4.5}
$$

which, together with (5.3.21) gives

$$
\left(\frac{\lambda_2}{\lambda_1}\phi\right)' = \frac{\phi\delta_1}{\lambda_1} + \frac{\phi^{-1}\gamma_2}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}\phi\right)^2. \tag{5.4.6}
$$

Chapter 6

Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation

This chapter is taken from the following article (also referred to as [20]):

Georges Bastin, Jean-Michel Coron, Amaury Hayat, and Peipei Shang. Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation. Math. Models Methods Appl. Sci., 29(2):271-316, 2019.

Abstract. In this chapter, we study the exponential stabilization of a shock steady state for the inviscid Burgers equation on a bounded interval. Our analysis relies on the construction of an explicit strict control Lyapunov function. We prove that by appropriately choosing the feedback boundary conditions, we can stabilize the state as well as the shock location to the desired steady state in H^2-norm, with an arbitrary decay rate.

6.1 Introduction

The problem of asymptotic stabilization for hyperbolic systems using boundary feedback control has been studied for a long time. We refer to the pioneer work due to Rauch and Taylor [173] and Russell [177] for linear coupled hyperbolic systems. The first important result of asymptotic stability concerning quasilinear hyperbolic equations was obtained by Slemrod [180] and Greenberg and Li [93]. These two works dealt with local dissipative boundary conditions. The result was established by using the method of characteristics, which allows to estimate the related bounds along the characteristic curves in the framework of C^1 solutions. Another approach to analyze the dissipative boundary conditions is based on the use of Lyapunov functions. Especially, Coron, Bastin and Andrea-Novel [53] used this method to study the asymptotic behavior of the nonlinear hyperbolic equations in the framework of H^2 solutions. In particular, the Lyapunov function they constructed is an extension of the entropy and can be made strictly negative definite by properly choosing the boundary conditions. This method has been later on widely used for hyperbolic conservation laws in the framework of C^1 solutions [48, 106, 107] or H^2 solutions [12, 16, 18, 47, 50, 66, 110] (see [13] for an overview of this method).
But all of these results concerning the asymptotic stability of nonlinear hyperbolic equations focus on the convergence to regular solutions, i.e., on the stabilization of regular solutions to a desired regular steady state. It is well known, however, that for quasilinear hyperbolic partial differential equations, solutions may

break down in finite time when their first derivatives break up even if the initial condition is smooth [134]. They give rise to the phenomena of shock waves with numerous important applications in physics and fluid mechanics. Compared to classical case, very few results exist on the stabilization of less regular solutions, which requires new techniques. This is also true for related fields, as the optimal control problem [41, 168]. For the problem of control and asymptotic stabilization of less regular solutions, we refer to [33] for the controllability of a general hyperbolic system of conservation laws, [27, 164] for the stabilization in the scalar case and [33, 57] for the stabilization of a hyperbolic system of conservation laws. In [27, 57, 164], by using suitable feedback laws on both side of the interval, one can steer asymptotically any initial data with sufficiently small total variations to any close constant steady states. All those results concern the boundary stabilization of constant steady states. In particular, as the target state is regular there is no need to stabilize any shock location. In this work, we will study the boundary stabilization of steady states with jump discontinuities for a scalar equation. We believe that our method can be applied to nonlinear hyperbolic systems as well. While preparing the revised version, our attention was drawn to a very recent work [165] studying a similar problem in the BV norm. The method and the results are quite different and complementary to this work.

Hyperbolic systems have a wide application in fluid dynamics, and hydraulic jump is one of the best known examples of shock waves as it is frequently observed in open channel flow such as rivers and spillways. Other physical examples of shock waves can be found in road traffic or in gas transportation, with the water hammer phenomenon. In the literature, Burgers equation often appears as a simplification of the dynamical model of flows, as well as the most studied scalar model for transportation. Burgers turbulence has been investigated both analytically and numerically by many authors either as a preliminary approach to turbulence prior to an occurrence of the Navier-Stokes turbulence or for its own sake since the Burgers equation describes the formation and decay of weak shock waves in a compressible fluid [120, 144, 187]. From a mathematical point of view, it turns out that the study of Burgers equation leads to many of the ideas that arise in the field of nonlinear hyperbolic equations. It is therefore a natural first step to develop methods for the control of this equation. For the boundary stabilization problem of viscous Burgers equation, we refer to works by Krstic et al. [125, 182] for the stabilization of regular shock-like profile steady states and [38, 124] for the stabilization of null-steady-state. In [182], the authors proved that the shock-like profile steady states of the linearized unit viscous Burgers equation is exponentially stable when using high-gain “radiation” boundary feedback (i.e. static boundary feedback only depending on output measurements). However, they showed that there is a limitation in the decay rate achievable by radiation feedback, i.e., the decay rate goes to zero exponentially as the shock becomes sharper. Thus, they have to use another strategy (namely backstepping method) to achieve arbitrarily fast local convergence to arbitrarily sharp shock profiles. However, this strategy requires a kind of full-state feedback control, rather than measuring only the boundary data.

In this chapter, we study the exponential asymptotic stability of a shock steady state of the Burgers equation in H^2-norm, which has been commonly used as a proper norm for studying the stability of hyperbolic systems (see e.g. [65, 114, 184]), as it enables to deal with Lyapunov functions that are integrals on the domain of quadratic quantities, which is relatively easy to handle. To that end, we construct an explicit Lyapunov function with a strict negative definite time derivative by properly choosing the boundary conditions. Though it has been shown in [61] that exponential stability in H^2-norm is not equivalent to C^1-norm, our result could probably be generalized to the C^1-norm for conservation laws by transforming the Lyapunov functions as in [48, 106].

The first problem is to deal with the well-posedness of the corresponding initial boundary value problem (IBVP) on a bounded domain. The existence of the weak solution to the initial value problem (IVP) of Burgers equation was first studied by Hopf by using vanishing viscosity [111]. The uniqueness of the entropy solution was then studied by Oleinik [162]. One can refer to [134] for a comprehensive study of the well-posedness of hyperbolic conservation laws in piecewise continuous entropy solution case and also to [68] in the class of entropy BV functions. Although there are many results for the well-posedness of the (IVP) for hyperbolic conservation laws, the problem of (IBVP) is less studied due to the difficulty of handling the boundary condition. In [3], the authors studied (IBVP) but in the quarter plane, i.e., $x > 0, t > 0$. By requiring that the boundary condition at $x = 0$ is satisfied in a weak sense, they can apply the method introduced by LeFloch [135] and obtain the explicit formula of the solution. However, our case is more complicated since we consider the Burgers equation defined on a bounded interval.

The organization of the chapter is the following. In Section 6.2, we formulate the problem and state our main results. In Section 6.3, we prove the well-posedness of the Burgers equation in the framework of piecewise continuously differentiable entropy solutions, which is one of the main results in this chapter. Based on this well-posedness result, we then prove in Section 6.4 by a Lyapunov approach that for appropriately chosen boundary conditions, we can achieve the exponential stability in H^2-norm of a shock steady state with any given arbitrary decay rate and with an exact exponential stabilization of the desired shock location. This result also holds for the H^k-norm for any $k \geqslant 2$. In Section 6.5, we extend the result to a more general convex flux by requiring some additional conditions on the flux. Conclusion and some open problems are provided in Section 6.6. Finally, some technical proof are given in the Appendix.

6.2 Problem statement and main result

We consider the following nonlinear inviscid Burgers equation on a bounded domain

$$y_t(t,x) + \left(\frac{y^2}{2}\right)_x (t,x) = 0 \tag{6.2.1}$$

with initial condition

$$y(0,x) = y_0(x), \quad x \in (0,L), \tag{6.2.2}$$

where $L > 0$ and boundary controls

$$y(t,0^+) = u_0(t), \quad y(t,L^-) = u_L(t). \tag{6.2.3}$$

In this article, we will be exclusively concerned with the case where the controls $u_0(t) > 0$, $u_L(t) < 0$ have opposite signs and the state $y(t,.)$ at each time t has a jump discontinuity as illustrated in Figure 6.1. The

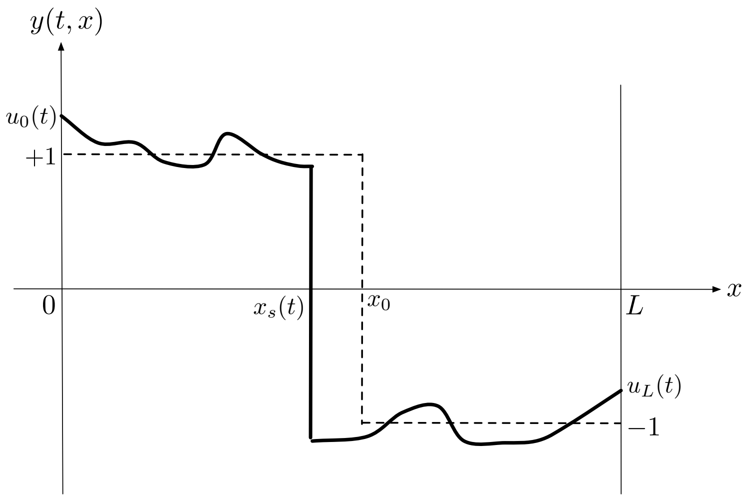

Figure 6.1 – Entropy solution to the Burgers equation with a shock wave.

discontinuity is a shock wave that occurs at position $x_s(t) \in (0,L)$. According to the Rankine-Hugoniot condition, the shock wave moves with the speed

$$\dot{x}_s(t) = \frac{y(t,x_s(t)^+) + y(t,x_s(t)^-)}{2} \tag{6.2.4}$$

which satisfies the Lax entropy condition [134]

$$y(t,x_s(t)^+) < \dot{x}_s(t) < y(t,x_s(t)^-), \tag{6.2.5}$$

together with the initial condition

$$x_s(0) = x_{s0}. \tag{6.2.6}$$

Under a constant control $u_0(t) = -u_L(t) = 1$ for all t, for any $x_0 \in (0, L)$, the system (6.2.1), (6.2.3), (6.2.4) has a steady state (y^*, x_s^*) defined as follows:

$$\begin{aligned} y^*(x) &= \begin{cases} 1, & x \in [0, x_0), \\ -1, & x \in (x_0, L], \end{cases} \\ x_s^* &= x_0. \end{aligned} \tag{6.2.7}$$

These equilibria are clearly not isolated and, consequently, not asymptotically stable. Indeed, one can see that for any given equilibrium y^* satisfying (6.2.7), we can find initial data arbitrarily close to y^* which is also an equilibrium of the form (6.2.7). As the solution cannot be approaching the given equilibrium when t tends to infinity as long as the initial data is another equilibrium, this feature prevents any stability no matter how close the initial data is around y^*. With such open-loop constant control another problem could appear: any small mistake on the boundary control could result in a non-stationary shock moving far away from x_0. It is therefore relevant to study the boundary feedback stabilization of the control system (6.2.1), (6.2.3), (6.2.4).

In this chapter, our main contribution is precisely to show how we can exponentially stabilize any of the steady states defined by (6.2.7) with boundary feedback controls of the following form:

$$\begin{aligned} u_0(t) &= k_1 y(t, x_s(t)^-) + (1 - k_1) + b_1(x_0 - x_s(t)), \\ u_L(t) &= k_2 y(t, x_s(t)^+) - (1 - k_2) + b_2(x_0 - x_s(t)). \end{aligned} \tag{6.2.8}$$

Here, it is important to emphasize that, with these controls, we are able not only to guarantee the exponential convergence of the solution $y(t, x)$ to the steady state y^* but also to exponentially stabilize the location of the shock discontinuity at the exact desired position x_0. In practice, if the system was used for instance to model gas transportation, the measures of the state around the shock could be obtained using sensors in the pipe. Note that if the control is applied properly, sensors would be only needed on a small region as the shock would remain located in a small region.

Before addressing the exponential stability issue, we first show that there exists a unique piecewise continuously differentiable entropy solution with $x_s(t)$ as its single shock for system (6.2.1)–(6.2.4), (6.2.6), (6.2.8) provided that y_0 and x_{s0} are in a small neighborhood of y^* and x_0 respectively.

For any given initial condition (6.2.2) and (6.2.6), we define the following zero order compatibility conditions

$$\begin{aligned} y_0(0^+) &= k_1 y_0(x_{s0}^-) + (1 - k_1) + b_1(x_0 - x_{s0}), \\ y_0(L^-) &= k_2 y_0(x_{s0}^+) - (1 - k_2) + b_2(x_0 - x_{s0}). \end{aligned} \tag{6.2.9}$$

Differentiating (6.2.9) with respect to time t and using (6.2.4), we get the following first order compatibility conditions

$$\begin{aligned} y_0(0^+) y_{0x}(0^+) =& k_1 y_0(x_{s0}^-) y_{0x}(x_{s0}^-) - k_1 y_{0x}(x_{s0}^-) \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2} + b_1 \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2}, \\ y_0(L^-) y_{0x}(L^-) =& k_2 y_0(x_{s0}^+) y_{0x}(x_{s0}^+) - k_2 y_{0x}(x_{s0}^-) \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2} + b_2 \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2}. \end{aligned} \tag{6.2.10}$$

The first result of this chapter deals with the well-posedness of system (6.2.1)–(6.2.4), (6.2.6), (6.2.8) and is stated in the following theorem.

Theorem 6.2.1. *For all $T > 0$, there exists $\delta(T) > 0$ such that, for every $x_{s0} \in (0, L)$ and $y_0 \in H^2((0, x_{s0}); \mathbb{R}) \cap H^2((x_{s0}, L); \mathbb{R})$ satisfying the compatibility conditions* (6.2.9)–(6.2.10) *and*

$$\begin{aligned} &|y_0 - 1|_{H^2((0,x_{s0});\mathbb{R})} + |y_0 + 1|_{H^2((x_{s0},L);\mathbb{R})} \leqslant \delta(T), \\ &|x_{s0} - x_0| \leqslant \delta(T), \end{aligned} \tag{6.2.11}$$

the system (6.2.1)–(6.2.4), (6.2.6), (6.2.8) *has a unique piecewise continuously differentiable entropy solution* $y \in C^0([0,T]; H^2((0,x_s(t));\mathbb{R})) \cap H^2((x_s(t),L);\mathbb{R}))$ *with* $x_s \in C^1([0,T];(0,L))$ *as its single shock. Moreover, there exists* $C(T)$ *such that the following estimate holds for all* $t \in [0,T]$

$$\begin{aligned}|y(t,\cdot)-1|_{H^2((0,x_s(t));\mathbb{R})} &+ |y(t,\cdot)+1|_{H^2((x_s(t),L);\mathbb{R})} + |x_s(t)-x_0| \\ &\leqslant C(T)\left(|y_0-1|_{H^2((0,x_{s0});\mathbb{R})} + |y_0+1|_{H^2((x_{s0},L);\mathbb{R})} + |x_{s0}-x_0|\right). \quad (6.2.12)\end{aligned}$$

The proof of this result is given in Section 6.3.

Our next result deals with the exponential stability of the steady state (6.2.7) for the H^2-norm according to the following definition.

Definition 6.2.1. The steady state $(y^*, x_0) \in (H^2((0,x_0);\mathbb{R}) \cap H^2((x_0,L);\mathbb{R})) \times (0,L)$ of the system (6.2.1), (6.2.3), (6.2.4), (6.2.8) is exponentially stable for the H^2-norm with decay rate γ, if there exists $\delta^* > 0$ and $C > 0$ such that for any $y_0 \in H^2((0,x_{s0});\mathbb{R}) \cap H^2((x_{s0},L);\mathbb{R})$ and $x_{s0} \in (0,L)$ satisfying

$$\begin{aligned}&|y_0 - y_1^*(0,\cdot)|_{H^2((0,x_{s0});\mathbb{R})} + |y_0 - y_2^*(0,\cdot)|_{H^2((x_{s0},L);\mathbb{R})} \leqslant \delta^*, \\ &|x_{s0}-x_0| \leqslant \delta^*\end{aligned} \quad (6.2.13)$$

and the compatibility conditions (6.2.9)–(6.2.10), and for any $T > 0$ the system (6.2.1)–(6.2.4), (6.2.6), (6.2.8) has a unique solution $(y, x_s) \in C^0([0,T]; H^2((0,x_s(t));\mathbb{R}) \cap H^2((x_s(t),L);\mathbb{R})) \times C^1([0,T];\mathbb{R})$ and

$$\begin{aligned}|y(t,\cdot)-y_1^*(t,\cdot)|_{H^2((0,x_s(t));\mathbb{R})} &+ |y(t,\cdot)-y_2^*(t,\cdot)|_{H^2((x_s(t),L);\mathbb{R})} + |x_s(t)-x_0| \\ \leqslant Ce^{-\gamma t}\left(|y_0-y_1^*(0,\cdot)|_{H^2((0,x_{s0});\mathbb{R})} \right.&\left.+ |y_0-y_2^*(0,\cdot)|_{H^2((x_{s0},L);\mathbb{R})} + |x_{s0}-x_0|\right), \ \forall t \in [0,T). \quad (6.2.14)\end{aligned}$$

In (6.2.13) and (6.2.14),

$$\begin{aligned}y_1^*(t,x) &= y^*\left(x\frac{x_0}{x_s(t)}\right), \\ y_2^*(t,x) &= y^*\left(\frac{(x-L)x_0}{x_s(t)-L}\right).\end{aligned} \quad (6.2.15)$$

Remark 6.2.1. *At first glance it could seem peculiar to define* y_1^* *and* y_2^* *and to compare* $y(t,\cdot)$ *with these functions. However the steady state* y^* *is piecewise* H^2 *with discontinuity at* x_0*, while the solution* $y(t,x)$ *is piecewise* H^2 *with discontinuity at the shock* $x_s(t)$*, which may be moving around* x_0*. Thus, to compare the solution* y *with the steady state* y^* *on the same space interval, it is necessary to define such functions* y_1^* *and* y_2^*.

Remark 6.2.2. *We emphasize here that the "exponential stability for the* H^2*-norm" is not the usual convergence of the* H^2*-norm of* $y - y^*$ *taken on* $(0,L)$ *as* y *and* y^* *do not belong to* $H^2(0,L)$*. This definition enables to define an exponential stability in* H^2*-norm for a function that has a discontinuity at some point and is regular elsewhere. Note that, the convergence to* 0 *of the* H^2*-norm in the usual sense does not ensure the convergence of the shock location* x_s *to* x_0*. Thus, to guarantee that the state converges to the shock steady state, we have to take account of the shock location, which is explained in Definition 2.1.*

Remark 6.2.3. *Note that this definition of exponential stability only deals a priori with* $t \in [0,T)$ *for any* $T > 0$*. However this, together with Theorem 6.2.1 implies the global existence in time of the solution* (y, x_s) *and the exponential stability on* $[0,+\infty)$*. This is shown at the end of the proof of Theorem 6.4.1.*

We can now state the main result of this chapter

Theorem 6.2.2. *Let $\gamma > 0$. If the following conditions hold:*

$$b_1 \in \left(\gamma e^{-\gamma x_0}, \frac{\gamma e^{-\gamma x_0}}{1-e^{-\gamma x_0}}\right),\ b_2 \in \left(\gamma e^{-\gamma(L-x_0)}, \frac{\gamma e^{-\gamma(L-x_0)}}{1-e^{-\gamma(L-x_0)}}\right), \tag{6.2.16a}$$

$$k_1^2 < e^{-\gamma x_0}\left(1 - \frac{b_1}{\gamma}\left(b_1 \frac{1-e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1-e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{6.2.16b}$$

$$k_2^2 < e^{-\gamma(L-x_0)}\left(1 - \frac{b_2}{\gamma}\left(b_1 \frac{1-e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1-e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right), \tag{6.2.16c}$$

then the steady state (y^*, x_0) of the system (6.2.1), (6.2.3), (6.2.4), (6.2.8) *is exponentially stable for the H^2-norm with decay rate $\gamma/4$.*

The proof of this theorem is given in Section 6.4.

Remark 6.2.4. *One can actually check that for any $\gamma > 0$ there exist parameters b_1, b_2 and k_1, k_2 satisfying* (6.2.16) *as, for $b_1 = \gamma e^{-\gamma x_0}$ and $b_2 = \gamma e^{-\gamma(L-x_0)}$, one has*

$$\begin{aligned} &1 - \frac{b_1}{\gamma}\left(b_1 \frac{1-e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1-e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right) \\ =& 1 - e^{-\gamma x_0}(2 - e^{-\gamma x_0} - e^{-\gamma(L-x_0)}) \\ =& e^{-2\gamma x_0}(e^{\gamma x_0} - 1)^2 + e^{-\gamma L} > 0. \end{aligned} \tag{6.2.17}$$

Similarly, we get

$$\begin{aligned} &1 - \frac{b_2}{\gamma}\left(b_1 \frac{1-e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1-e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right) \\ =& e^{-2\gamma(L-x_0)}(e^{\gamma(L-x_0)} - 1)^2 + e^{-\gamma L} > 0. \end{aligned} \tag{6.2.18}$$

Therefore, by continuity, there exist b_1 and b_2, satisfying condition (6.2.16a) *such that there exist k_1 and k_2 satisfying* (6.2.16b) *and* (6.2.16c)*. This implies that γ can be made arbitrarily large. And, from* (6.2.16a)–(6.2.16c)*, we can note that for large γ the conditions on the k_i tend to*

$$k_1^2 < e^{-\gamma x_0}, \quad k_2^2 < e^{-\gamma(L-x_0)}.$$

Remark 6.2.5. *The result can also be generalized to H^k-norm for any integer $k \geqslant 2$ in the sense of Definition 6.2.1 by replacing H^2 with H^k. This can be easily done by just adapting the Lyapunov function defined below by* (6.4.3)-(6.4.9) *as was done in [13, Sections 4.5 and 6.2].*

Remark 6.2.6. *If we set $k_1 = k_2 = b_1 = b_2 = 0$, then from* (6.2.8)*, $u_0(t) \equiv 1$ and $u_L(t) \equiv -1$. Thus it seems logical that the larger γ is, the smaller k_1 and k_2 are. However, it could seem counter-intuitive that b_1 and b_2 have to tend to 0 when γ tends to $+\infty$, as if one sets $b_1 = 0$ and $b_2 = 0$, one cannot stabilize the location of the system just like in the constant open-loop control case. In other words for any $\gamma > 0$ the prescribed feedback works while the limit feedback we obtain by letting $\gamma \to +\infty$ cannot even ensure the asymptotic stability of the system. The explanation behind this apparent paradox is that when γ tends to infinity, the Lyapunov function candidate used to prove Theorem 6.4.1 is not equivalent to the norm of the solution and cannot guarantee anymore the exponential decay of the solution in the H^2-norm. More precisely, one can see, looking at* (6.4.69) *and* (6.4.71)*, that the hypothesis* (6.4.16) *of Lemma 6.4.1 does not hold anymore.*

6.3 An equivalent system with shock-free solutions

Our strategy to analyze the existence and the exponential stability of the shock wave solutions to the scalar Burgers equation (6.2.1) is to use an equivalent 2×2 quasilinear hyperbolic system having shock-free solutions. In order to set up this equivalent system, we define the two following functions

$$y_1(t,x) = y(t, x\frac{x_s(t)}{x_0}), \quad y_2(t,x) = y(t, L + x\frac{x_s(t)-L}{x_0}) \tag{6.3.1}$$

and the new state variables as follows:

$$\mathbf{z}(t,x) = \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} = \begin{pmatrix} y_1(t,x) - 1 \\ y_2(t,x) + 1 \end{pmatrix}, \quad x \in (0, x_0). \tag{6.3.2}$$

The idea behind the definition of y_1, y_2 is to describe the behavior of the solution $y(t,x)$ before and after the moving shock, while studying functions on a time invariant interval. Observe indeed that the functions y_1 and y_2 in (6.3.1) correspond to the solution $y(t,x)$ on the time varying intervals $(0, x_s(t))$ and $(x_s(t), L)$ respectively, albeit with a time varying scaling of the space coordinate x which is driven by $x_s(t)$ and allows to define the new state variables (z_1, z_2) on the fixed time invariant interval $(0, x_0)$. The reason to rescale y_2 on $(0, x_0)$ instead of (x_0, L) is to simplify the analysis by defining state variables on the same space interval with the same direction of propagation.
Besides, from (6.3.2), the former steady state (y^*, x_0) corresponds now to the steady state $(\mathbf{z} = \mathbf{0}, x_s = x_0)$ in the new variables. With these new variables, the dynamics of (y, x_s) can now be expressed as follows:

$$\begin{aligned} &z_{1t} + \left(1 + z_1 - x\frac{\dot{x}_s}{x_0}\right) z_{1x} \frac{x_0}{x_s} = 0, \\ &z_{2t} + \left(1 - z_2 + x\frac{\dot{x}_s}{x_0}\right) z_{2x} \frac{x_0}{L - x_s} = 0, \\ &\dot{x}_s(t) = \frac{z_1(t,x_0) + z_2(t,x_0)}{2}, \end{aligned} \tag{6.3.3}$$

with the boundary conditions:

$$\begin{aligned} z_1(t,0) &= k_1 z_1(t,x_0) + b_1(x_0 - x_s(t)), \\ z_2(t,0) &= k_2 z_2(t,x_0) + b_2(x_0 - x_s(t)), \end{aligned} \tag{6.3.4}$$

and initial condition

$$\mathbf{z}(0,x) = \mathbf{z}^0(x), \quad x_s(0) = x_{s0}, \tag{6.3.5}$$

where $\mathbf{z}^0 = (z_1^0, z_2^0)^T$ and

$$\begin{aligned} z_1^0(x) &= y_0\left(x\frac{x_{s0}}{x_0}\right) - 1, \\ z_2^0(x) &= y_0\left(L + x\frac{x_{s0} - L}{x_0}\right) + 1. \end{aligned} \tag{6.3.6}$$

Furthermore, in the new variables, the compatibility conditions (6.2.9)–(6.2.10) are expressed as follows:

$$\begin{aligned} z_1^0(0) &= k_1 z_1^0(x_0) + b_1(x_0 - x_{s0}), \\ z_2^0(0) &= k_2 z_2^0(x_0) + b_2(x_0 - x_{s0}), \end{aligned} \tag{6.3.7}$$

and

$$\begin{aligned} (1 + z_1^0(0)) z_{1x}^0(0)\frac{x_0}{x_{s0}} &= k_1\left(1 + z_1^0(x_0) - \frac{z_1^0(x_0) + z_2^0(x_0)}{2}\right) z_{1x}^0(x_0)\frac{x_0}{x_{s0}} + b_1\frac{z_1^0(x_0) + z_2^0(x_0)}{2}, \\ (1 - z_2^0(0)) z_{2x}^0(0)\frac{x_0}{L - x_{s0}} &= k_2\left(1 - z_2^0(x_0) + \frac{z_1^0(x_0) + z_2^0(x_0)}{2}\right) z_{2x}^0(x_0)\frac{x_0}{L - x_{s0}} + b_2\frac{z_1^0(x_0) + z_2^0(x_0)}{2}. \end{aligned} \tag{6.3.8}$$

Concerning the existence and uniqueness of the solution to the system (6.3.3)–(6.3.5), we have the following lemma.

Lemme 6.3.1. *For all $T > 0$, there exists $\delta(T) > 0$ such that, for every $x_{s0} \in (0, L)$ and $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$ satisfying the compatibility conditions* (6.3.7)–(6.3.8) *and*

$$|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \delta(T), \quad |x_{s0} - x_0| \leqslant \delta(T), \tag{6.3.9}$$

the system (6.3.3)–(6.3.5) *has a unique classical solution* $(\mathbf{z}, x_s) \in C^0([0,T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0,T]; (0, L))$. *Moreover, there exists $C(T)$ such that the following estimate holds for all $t \in [0, T]$*

$$|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s(t) - x_0| \leqslant C(T) \left(|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} + |x_{s0} - x_0|\right). \tag{6.3.10}$$

Proof. The proof of Lemma 6.3.1 is given in Appendix 6.7.1. □

From this lemma, it is then clear that the proof of Theorem 6.2.1 follows immediately.

Proof of Theorem 6.2.1. The change of variables (6.3.1), (6.3.2) induces an equivalence between the classical solutions $(\mathbf{z}, x_s)$ of the system (6.3.3)–(6.3.5) and the entropy solutions with a single shock (y, x_s) of the system (6.2.1)–(6.2.4), (6.2.6), (6.2.8). Consequently, from (6.3.2) and provided $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)}$ and $|x_{s0} - x_0|$ are sufficiently small, the existence and uniqueness of a solution with a single shock (y, x_s) to the system (6.2.1)–(6.2.4), (6.2.6), (6.2.8) satisfying the entropy condition (6.2.5) when (y_0, x_{s0}) is in a sufficiently small neighborhood of (y^*, x_0), follows directly from the existence and uniqueness of the classical solution $(\mathbf{z}, x_s)$ to the system (6.3.3)–(6.3.5) which is guaranteed by Lemma 6.3.1. □

Remark 6.3.1. *Under the assumption in Lemma 6.3.1, if we assume furthermore that $\mathbf{z}^0 \in H^k((0, x_0); \mathbb{R}^2)$ with $k \geqslant 2$ satisfying the k-th order compatibility conditions (see the definition in [13, p.143]), then $(\mathbf{z}, x_s) \in C^0([0,T]; H^k((0, x_0); \mathbb{R}^2)) \times C^k([0,T]; \mathbb{R})$ and* (6.3.10) *still holds. This is a straightforward extension of the proof in Appendix 6.7.1, thus we will not give the details of this proof here.*

6.4 Exponential stability for the H^2-norm

This section is devoted to the proof of Theorem 6.2.2 concerning the exponential stability of the steady state of system (6.2.1), (6.2.3), (6.2.4), (6.2.8). Actually, on the basis of the change of variables introduced in the previous section, we know that we only have to prove the exponential stability of the steady state of the auxiliary system (6.3.3)–(6.3.4) according to the following theorem which is equivalent to Theorem 6.2.2.

Theorem 6.4.1. *For any $\gamma > 0$, if condition* (6.2.16) *on the parameters of the feedback holds, then there exist $\delta^* > 0$ and $C > 0$ such that for any $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$ and $x_{s0} \in (0, L)$ satisfying*

$$|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \delta^*, \quad |x_{s0} - x_0| \leqslant \delta^* \tag{6.4.1}$$

and the compatibility conditions (6.3.7)–(6.3.8)*, and for any $T > 0$ the system* (6.3.3)–(6.3.5) *has a unique classical solution $(\mathbf{z}, x_s) \in C^0([0,T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0,T]; \mathbb{R})$ such that*

$$|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s(t) - x_0| \leqslant Ce^{-\gamma t/4} \left(|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} + |x_{s0} - x_0|\right), \quad \forall t \in [0, T). \tag{6.4.2}$$

When this theorem holds, we say that the steady state $(\mathbf{z} = \mathbf{0}, x_s = x_0)$ of the system (6.3.3)–(6.3.4) is exponentially stable for the H^2-norm with convergence rate $\gamma/4$. Recall that, from Remark 6.2.4, there always exist parameters such that (6.2.16) holds.
Before proving Theorem 6.4.1, let us give an overview of our strategy. We first introduce a Lyapunov function candidate V with parameters to be chosen. Then, in Lemma 6.4.1, we give a condition on the parameters such that V is equivalent to the square of the H^2-norm of $\mathbf{z}$ plus the absolute value of $x_s - x_0$, which implies that proving the exponential decay of V with rate $\gamma/2$ is enough to show the exponential stability of the system with decay rate $\gamma/4$ for the H^2-norm. In Lemma 6.4.2, we show that in order to obtain Theorem 6.4.1,

it is enough to prove that V decays along any solutions $(\mathbf{z}, x_s) \in C^3([0,T]\times[0,x_0];\mathbb{R}^2)\times C^3([0,T];\mathbb{R})$ with a density argument. Then in Lemma 6.4.3, we compute the time derivative of V along any C^3 solutions of the system and we give a sufficient condition on the parameters such that V satisfies a useful estimate along these solutions. Finally, we show that there exist parameters satisfying the sufficient condition of Lemma 4.3. This, together with Lemma 4.2, ends the proof of Theorem 4.1.
We now introduce the following candidate Lyapunov function which is defined for all $\mathbf{z} = (z_1, z_2)^T \in H^2((0,x_0);\mathbb{R}^2)$ and $x_s \in (0, L)$:

$$V(\mathbf{z}, x_s) = V_1(\mathbf{z}) + V_2(\mathbf{z}, x_s) + V_3(\mathbf{z}, x_s) + V_4(\mathbf{z}, x_s) + V_5(\mathbf{z}, x_s) + V_6(\mathbf{z}, x_s) \tag{6.4.3}$$

with

$$V_1(\mathbf{z}) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_1^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_2^2 dx, \tag{6.4.4}$$

$$V_2(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_{1t}^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_{2t}^2 dx, \tag{6.4.5}$$

$$V_3(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt}^2 + p_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt}^2 dx, \tag{6.4.6}$$

$$V_4(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_1 (x_s - x_0)\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_2 (x_s - x_0)\, dx + \kappa (x_s - x_0)^2, \tag{6.4.7}$$

$$V_5(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1t}\dot{x}_s\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2t}\dot{x}_s\, dx + \kappa (\dot{x}_s)^2, \tag{6.4.8}$$

$$V_6(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt}\ddot{x}_s\, dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt}\ddot{x}_s\, dx + \kappa (\ddot{x}_s)^2. \tag{6.4.9}$$

In (6.4.4)–(6.4.9), μ, p_1, p_2, $\bar{p}_1$, $\bar{p}_2$ are positive constants. Moreover

$$\eta_1 = 1, \quad \eta_2 = \frac{x_0}{L - x_0} \tag{6.4.10}$$

and

$$\kappa > 1. \tag{6.4.11}$$

Actually, in this section, we will need to evaluate $V(\mathbf{z}, x_s)$ only along the system solutions for which the variables $\mathbf{z}_t = (z_{1t}, z_{2t})$, $\mathbf{z}_{tt} = (z_{1tt}, z_{2tt})$, $\dot{x}_s$ and $\ddot{x}_s$ that appear in the definition of V can be well defined as functions of $(\mathbf{z}, x_s) \in H^2((0,x_0);\mathbb{R}^2)\times(0,L)$ from the system (6.3.3)–(6.3.4) and their space derivatives. For example, z_{1t} and z_{2t} are defined as functions of $(\mathbf{z}, x_s)$ by

$$z_{1t} := -\left(1 + z_1 - x\frac{z_1(x_0) + z_2(x_0)}{2x_0}\right) z_{1x}\frac{x_0}{x_s}, \tag{6.4.12}$$

$$z_{2t} := -\left(1 - z_2 + x\frac{z_1(x_0) + z_2(x_0)}{2x_0}\right) z_{2x}\frac{x_0}{L - x_s}, \tag{6.4.13}$$

and z_{1tt} and z_{2tt} as functions of $(\mathbf{z}, x_s)$ by

$$z_{1tt} := -\left(1 + z_1 - x\frac{z_1(x_0) + z_2(x_0)}{2x_0}\right)(z_{1t})_x\frac{x_0}{x_s} - \left(z_{1t} - x\frac{z_{1t}(x_0) + z_{2t}(x_0)}{2x_0}\right) z_{1x}\frac{x_0}{x_s} - z_{1t}\frac{z_1(x_0) + z_2(x_0)}{2x_s}, \tag{6.4.14}$$

$$z_{2tt} := -\left(1 - z_2 + x\frac{z_1(x_0) + z_2(x_0)}{2x_0}\right)(z_{2t})_x\frac{x_0}{L - x_s}$$

$$+\left(z_{2t}-x\frac{z_{1t}(x_0)+z_{2t}(x_0)}{2x_0}\right)z_{2x}\frac{x_0}{L-x_s}+z_{2t}\frac{z_1(x_0)+z_2(x_0)}{2(L-x_s)}. \quad (6.4.15)$$

The functions z_{1t} and z_{2t} which appear in (6.4.14) and (6.4.15) are supposed to be defined by (6.4.12) and (6.4.13) respectively.

Remark 6.4.1. *When looking for a Lyapunov function to stabilize the state (z_1, z_2) in H^2-norm, the component $(V_1+V_2+V_3)$ can be seen as the most natural and easiest choice, as it is equivalent to a weighted H^2-norm by properly choosing the parameters. This kind of Lyapunov function, sometimes called basic quadratic Lyapunov function, is used for instance in [12] or [13, Section 4.4]. However, in the present case one needs to stabilize both the state $\mathbf{z}$ and the shock location x_s, which requires to add additional terms to the Lyapunov function in order to deal with x_s. Besides, as we have no direct control on x_s (observe that none of the terms of the right-hand side of (6.2.4), or equivalently of the third equation of (6.3.3), is a control), we need to add some coupling terms between the state $\mathbf{z}$ on which we have a control and the shock location x_s in the Lyapunov function. Thus, V_4 is designed to provide such coupling with the product of the component of $\mathbf{z}$ and x_s, while V_5 and V_6 are its analogous for the time derivatives terms (as V_2 and V_3 are the analogous of V_1 respectively for the first and second time derivative of $\mathbf{z}$).*

We now state the following lemma, providing a condition on μ, p_1, p_2, $\bar{p}_1$ and $\bar{p}_2$ such that $V(\mathbf{z}, x_s)$ is equivalent to $(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)}+|x_s-x_0|^2)$.

Lemme 6.4.1. *If*

$$\max(\Theta_1,\Theta_2)<2, \tag{6.4.16}$$

where

$$\Theta_1 := \frac{\bar{p}_1^2}{p_1}\frac{\eta_1}{\mu}\left(1-e^{\frac{-\mu x_0}{\eta_1}}\right),\quad \Theta_2 := \frac{\bar{p}_2^2}{p_2}\frac{\eta_2}{\mu}\left(1-e^{-\frac{\mu x_0}{\eta_2}}\right), \tag{6.4.17}$$

there exists $\beta>0$ such that

$$\beta\left(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)}+|x_s-x_0|^2\right)\leqslant V\leqslant\frac{1}{\beta}\left(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)}+|x_s-x_0|^2\right) \tag{6.4.18}$$

for any $(\mathbf{z},x_s)\in H^2((0,x_0);\mathbb{R}^2)\times(0,L)$ satisfying

$$|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)}+|x_s-x_0|^2<\beta^2. \tag{6.4.19}$$

Proof of Lemma 6.4.1. Let us start with

$$V_4=\int_0^{x_0}\bar{p}_1e^{\frac{-\mu x}{\eta_1}}z_1(x_s-x_0)\,dx+\int_0^{x_0}\bar{p}_2e^{\frac{-\mu x}{\eta_2}}z_2(x_s-x_0)\,dx+\kappa(x_s-x_0)^2. \tag{6.4.20}$$

Using Young's inequality we get

$$\begin{aligned}
&-\frac{1}{2}\left(\int_0^{x_0}\bar{p}_1e^{\frac{-\mu x}{\eta_1}}z_1\,dx\right)^2-\frac{(x_s-x_0)^2}{2}-\frac{1}{2}\left(\int_0^{x_0}\bar{p}_2e^{\frac{-\mu x}{\eta_2}}z_2\,dx\right)^2-\frac{(x_s-x_0)^2}{2}\\
&+\kappa(x_s-x_0)^2\leqslant V_4\leqslant\frac{1}{2}\left(\int_0^{x_0}\bar{p}_1e^{\frac{-\mu x}{\eta_1}}z_1\,dx\right)^2\\
&+\frac{(x_s-x_0)^2}{2}+\frac{1}{2}\left(\int_0^{x_0}\bar{p}_2e^{\frac{-\mu x}{\eta_2}}z_2\,dx\right)^2+\frac{(x_s-x_0)^2}{2}+\kappa(x_s-x_0)^2.
\end{aligned} \tag{6.4.21}$$

Hence, using the Cauchy-Schwarz inequality and the expression of V_1 given in (6.4.4),

$$\begin{aligned}
&p_1(1-\frac{1}{2}\Theta_1)\int_0^{x_0}e^{\frac{-\mu x}{\eta_1}}z_1^2dx+p_2(1-\frac{1}{2}\Theta_2)\int_0^{x_0}e^{\frac{-\mu x}{\eta_2}}z_2^2\,dx\\
&+(x_s-x_0)^2(\kappa-1)\leqslant V_1+V_4\leqslant p_1(1+\frac{1}{2}\Theta_1)\int_0^{x_0}e^{\frac{-\mu x}{\eta_1}}z_1^2dx\\
&+p_2(1+\frac{1}{2}\Theta_2)\int_0^{x_0}e^{\frac{-\mu x}{\eta_2}}z_2^2\,dx+(x_s-x_0)^2(\kappa+1),
\end{aligned} \tag{6.4.22}$$

and similarly

$$
\begin{aligned}
& p_1(1-\frac{1}{2}\Theta_1)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1t}^2 dx + p_2(1-\frac{1}{2}\Theta_2)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2t}^2\, dx \\
& +(\dot{x}_s)^2(\kappa-1) \leqslant V_2+V_5 \leqslant p_1(1+\frac{1}{2}\Theta_1)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1t}^2 dx \\
& + p_2(1+\frac{1}{2}\Theta_2)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2t}^2\, dx + (\dot{x}_s)^2(\kappa+1),
\end{aligned} \tag{6.4.23}
$$

and also

$$
\begin{aligned}
& p_1(1-\frac{1}{2}\Theta_1)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1tt}^2 dx + p_2(1-\frac{1}{2}\Theta_2)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2tt}^2\, dx \\
& +(\ddot{x}_s)^2(\kappa-1) \leqslant V_3+V_6 \leqslant p_1(1+\frac{1}{2}\Theta_1)\int_0^{x_0} e^{\frac{-\mu x}{\eta_1}} z_{1tt}^2 dx \\
& + p_2(1+\frac{1}{2}\Theta_2)\int_0^{x_0} e^{\frac{-\mu x}{\eta_2}} z_{2tt}^2\, dx + (\ddot{x}_s)^2(\kappa+1).
\end{aligned} \tag{6.4.24}
$$

Hence, from (6.4.11), $\kappa > 1$ and (6.4.16) is satisfied, there exists $\sigma > 0$ such that

$$
\sigma\left(|\mathbf{z}|^2_{H_t^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2\right) \leqslant V \leqslant \frac{1}{\sigma}\left(|\mathbf{z}|^2_{H_t^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2\right), \tag{6.4.25}
$$

where, for a function $\mathbf{z} \in H^2((0,x_0);\mathbb{R}^2)$, $|\mathbf{z}|_{H_t^2((0,x_0);\mathbb{R}^2)}$ is defined by

$$
|\mathbf{z}|_{H_t^2((0,x_0);\mathbb{R}^2)} = \left(|\mathbf{z}|^2_{L^2((0,x_0);\mathbb{R}^2)} + |\mathbf{z}_t|^2_{L^2((0,x_0);\mathbb{R}^2)} + |\mathbf{z}_{tt}|^2_{L^2((0,x_0);\mathbb{R}^2)}\right)^{1/2}, \tag{6.4.26}
$$

with $\mathbf{z}_t$ and $\mathbf{z}_{tt}$ defined as (6.4.12)–(6.4.15). Let us point out that from (6.4.12)–(6.4.15), there exists $C > 0$ such that

$$
\frac{1}{C}|\mathbf{z}|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant |\mathbf{z}|_{H_t^2((0,x_0);\mathbb{R}^2)} \leqslant C|\mathbf{z}|_{H^2((0,x_0);\mathbb{R}^2)}, \tag{6.4.27}
$$

if $(|\mathbf{z}|^2_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|^2) < 1/C$. It follows from (6.4.25) and (6.4.27) that $\beta > 0$ can be taken sufficiently small such that inequality (6.4.18) holds provided (6.4.19) is satisfied. This concludes the proof of Lemma 6.4.1. $\Box$

Before proving Theorem 6.4.1, we introduce the following density argument, which shows that it is enough to prove the exponential decay of V along any C^3 solutions of the system.

Lemme 6.4.2. *Let V be a C^1 and nonnegative functional on $C^0([0,T];H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$. If there exist $\delta > 0$ and $\gamma > 0$ such that for any $(\mathbf{z},x_s) \in C^3([0,T]\times[0,x_0];\mathbb{R}^2)) \times C^3([0,T];\mathbb{R})$ solution of* (6.3.3)*–*(6.3.4)*, with associated initial condition $(\mathbf{z}^0,x_{s0})$ satisfying $\left|\mathbf{z}^0\right|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \delta$ and $|x_{s0} - x_0| \leqslant \delta$, one has*

$$
\frac{dV(\mathbf{z}(t,\cdot),x_s(t))}{dt} \leqslant -\frac{\gamma}{2}V(\mathbf{z}(t,\cdot),x_s(t)), \tag{6.4.28}
$$

then (6.4.28) *also holds in a distribution sense for any $(\mathbf{z},x_s) \in C^0([0,T];H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$ solution of* (6.3.3)*–*(6.3.4) *such that the associated initial condition $(\mathbf{z}^0,x_{s0})$ satisfies $\left|\mathbf{z}^0\right|_{H^2((0,x_0);\mathbb{R}^2)} < \delta$ and $|x_{s0} - x_0| < \delta$.*

Proof of Lemma 6.4.2. Let V be a C^1 and nonnegative functional on $C^0([0,T];H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$ and let $(\mathbf{z},x_s) \in C^0([0,T];H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$ be solution of (6.3.3)–(6.3.4) with associated initial condition $\left|\mathbf{z}^0\right|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \delta$ and $|x_{s0} - x_0| \leqslant \delta$. Let $(\mathbf{z}^{0n},x_{s0}^n) \in H^4((0,x_0);\mathbb{R}^2) \times (0,L), n \in \mathbb{N}$ be a sequence of functions that satisfy the fourth order compatibility conditions and

$$
|\mathbf{z}^{0n}|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \delta, \quad |x_{s0}^n - x_0| \leqslant \delta, \tag{6.4.29}
$$

such that $\mathbf{z}^{0n}$ converges to $\mathbf{z}^0$ in $H^2((0,x_0);\mathbb{R}^2)$ and x_{s0}^n converges to x_{s0}. From Remark 6.3.1, there exists a unique solution $(\mathbf{z}^n, x_s^n) \in C^0([0,T]; H^4((0,x_0);\mathbb{R}^2)) \times C^4([0,T];\mathbb{R})$ to (6.3.3)–(6.3.4) corresponding to the initial condition $(\mathbf{z}^{0n}, x_{s0}^n)$ and for any $t \in [0,T]$, we have

$$|\mathbf{z}^n(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s^n(t) - x_0| \leqslant C(T)\left(|\mathbf{z}^{0n}|_{H^2((0,x_0);\mathbb{R}^2)} + |x_{s0}^n - x_0|\right). \tag{6.4.30}$$

Hence, from (6.4.29) and the third equation of (6.3.3), the sequence $(\mathbf{z}^n, x_s^n)$ is bounded in $C^0([0,T]; H^2((0,x_0);\mathbb{R}^2)) \times C^1([0,T];\mathbb{R})$. By [178, Corollary 4], we can extract a subsequence, which we still denote by $(\mathbf{z}^n, x_s^n)$ that converges to $(\mathbf{u}, y_s)$ in $(C^0([0,T]; C^1([0,x_0]; R^2)) \cap C^1([0,T]; C^0([0,x_0]; R^2))) \times C^1([0,T];\mathbb{R})$, which is a classical solution of (6.3.3)–(6.3.5). If we define

$$J(\mathbf{u}) = \begin{cases} +\infty, & \text{if } \mathbf{u} \notin L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)), \\ |\mathbf{u}|_{L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2))}, & \text{if } \mathbf{u} \in L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)), \end{cases} \tag{6.4.31}$$

then J is lower semi-continuous and we have

$$J(\mathbf{u}) \leqslant \varliminf_{n\to+\infty} |\mathbf{z}^n|_{C^0([0,T];H^2((0,x_0);\mathbb{R}^2))}, \tag{6.4.32}$$

thus from (6.4.30) and the convergence of $(\mathbf{z}^{0n}, x_{s0}^n)$ in $H^2((0,x_0);\mathbb{R}^2) \times \mathbb{R}$, we have $J(\mathbf{u}) \in \mathbb{R}$ and $\mathbf{u} \in L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2))$. Moreover, as $(\mathbf{u}, y_s)$ is a solution to (6.3.3)–(6.3.5), we get the extra regularity $\mathbf{u} \in C^0([0,T]; H^2((0,x_0);\mathbb{R}^2))$. Hence, from the uniqueness of the solution given by Lemma 6.3.1, $\mathbf{u} = \mathbf{z}$ and consequently $y_s = x_s$, which implies that $(\mathbf{z}^n, x_s^n)$ converges to $(\mathbf{z}, x_s)$ in $(C^0([0,T]; C^1([0,x_0];\mathbb{R}^2)) \cap C^1([0,T]; C^0([0,x_0];\mathbb{R}^2))) \times C^1([0,T];\mathbb{R})$. Now, we define $V^n(t) := V(\mathbf{z}^n(t,\cdot), x_s^n(t))$. Note that $V(t) = V(\mathbf{z}(t,\cdot), x_s(t))$ is continuous with time t and well-defined as, from Lemma 6.3.1, $\mathbf{z} \in C^0([0,T]; H^2((0,x_0);\mathbb{R}^2))$. As $(\mathbf{z}^n, x_s^n)$ belongs to $C^0([0,T]; H^4((0,x_0);\mathbb{R}^2)) \times C^4([0,T];\mathbb{R})$ and is thus C^3, and as it is a solution of (6.3.3)–(6.3.4) with initial condition satisfying (6.4.29), we have from (6.4.28)

$$\frac{dV^n}{dt} \leqslant -\frac{\gamma}{2} V^n, \tag{6.4.33}$$

thus V^n is decreasing on $[0,T]$. Therefore

$$V^n(t) - V^n(0) \leqslant -\frac{\gamma t}{2} V^n(t), \ \forall t \in [0,T], \tag{6.4.34}$$

which implies that

$$\left(1 + \frac{\gamma t}{2}\right) V^n(t) \leqslant V^n(0), \ \forall t \in [0,T]. \tag{6.4.35}$$

Using the lower semi-continuity of J, by the continuity of V and the convergence of $(\mathbf{z}^{0n}, x_{s0}^n)$ in $H^2((0,x_0);\mathbb{R}^2) \times \mathbb{R}$, we have

$$\left(1 + \frac{\gamma t}{2}\right) V(t) \leqslant V(0), \ \forall t \in [0,T]. \tag{6.4.36}$$

Note that instead of approximating $(\mathbf{z}^0, x_{s0})$, we could have approximated $(\mathbf{z}(s,\cdot), x_s(s))$ where $s \in [0,T)$ and follow the same procedure as above. Therefore we have in fact for any $s \in [0,T)$

$$\left(1 + \frac{\gamma(t-s)}{2}\right) V(t) \leqslant V(s), \ \forall t \in [s,T], \tag{6.4.37}$$

thus for any $0 \leqslant s < t \leqslant T$

$$\frac{V(t) - V(s)}{t - s} \leqslant -\frac{\gamma}{2} V(t), \tag{6.4.38}$$

which implies that (6.4.28) holds in the distribution sense. This ends the proof of Lemma 6.4.2. □

We now state our final lemma, which gives a sufficient condition so that V defined by (6.4.3)–(6.4.9) satisfies a useful estimate along any C^3 solutions.

Lemme 6.4.3. *Let V be defined by (6.4.3)–(6.4.9). If the matrix $\mathbf{A}$ defined by (6.4.59)–(6.4.64) is positive definite, then for any $T>0$, there exists $\delta_1(T)>0$ such that for any $(\mathbf{z},x_s)\in C^3([0,T]\times[0,x_0];\mathbb{R}^2))\times C^3([0,T];\mathbb{R})$ solution of (6.3.3)–(6.3.5) satisfying $\left|\mathbf{z}^0\right|_{H^2((0,x_0);\mathbb{R}^2)}\leqslant\delta_1(T)$ and $|x_{s0}-x_0|\leqslant\delta_1(T)$,*

$$\frac{dV(\mathbf{z}(t,\cdot),x_s(t))}{dt}\leqslant-\frac{\mu}{2}V(\mathbf{z}(t,\cdot),x_s(t))+O\left((|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}+|x_s-x_0|)^3\right),\ \forall\ t\in[0,T]. \tag{6.4.39}$$

Here and hereafter, $O(s)$ means that there exist $\varepsilon>0$ and $C_1>0$, both independent of $\mathbf{z}$, x_s, T and $t\in[0,T]$, such that

$$(s\leqslant\varepsilon)\Longrightarrow(|O(s)|\leqslant C_1 s).$$

To prove this lemma, we differentiate V with respect to time along any C^3 solutions and perform several estimates on the different components of V. For the sake of simplicity, for any $\mathbf{z}\in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$, we denote from now on $|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ by $|\mathbf{z}|_{H^2}$.

Proof of Lemma 6.4.3. Let V be given by (6.4.3)–(6.4.9) and $T>0$. Let us assume that $(\mathbf{z},x_s)$ is a C^3 solution to the system (6.3.3)–(6.3.5), with initial condition $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)}\leqslant\delta_1(T)$ and $|x_{s0}-x_0|\leqslant\delta_1(T)$ respectively with $\delta_1(T)>0$ to be chosen later on. Let us examine the different components of the Lyapunov function. We start by studying V_1,V_2 and V_3 which can be treated similarly as in [13, Section 4.4]. Differentiating V_1 along the solution $(\mathbf{z},x_s)$ and integrating by parts, noticing (6.4.10), we have

$$\begin{aligned}\frac{dV_1}{dt}&=-2\int_0^{x_0}\left(p_1e^{\frac{-\mu x}{\eta_1}}z_1(1+z_1-x\frac{\dot{x}_s}{x_0})\frac{x_0}{x_s}z_{1x}+p_2e^{\frac{-\mu x}{\eta_2}}z_2(1-z_2+x\frac{\dot{x}_s}{x_0})\frac{x_0}{L-x_s}z_{2x}\right)dx\\&=-\mu V_1-\left[p_1e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s}z_1^2+p_2e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L-x_s}z_2^2\right]_0^{x_0}+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right).\end{aligned} \tag{6.4.40}$$

From (6.3.3), we have

$$\begin{aligned}&z_{1tt}+(1+z_1-x\frac{\dot{x}_s}{x_0})z_{1tx}\frac{x_0}{x_s}+(z_{1t}-x\frac{\ddot{x}_s}{x_0})z_{1x}\frac{x_0}{x_s}+z_{1t}\frac{\dot{x}_s}{x_s}=0,\\&z_{2tt}+(1-z_2+x\frac{\dot{x}_s}{x_0})z_{2tx}\frac{x_0}{L-x_s}-(z_{2t}-x\frac{\ddot{x}_s}{x_0})z_{2x}\frac{x_0}{L-x_s}-z_{2t}\frac{\dot{x}_s}{L-x_s}=0.\end{aligned} \tag{6.4.41}$$

Therefore, similarly to (6.4.40), we can obtain

$$\frac{dV_2}{dt}=-\mu V_2-\left[p_1e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s}z_{1t}^2+p_2e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L-x_s}z_{2t}^2\right]_0^{x_0}+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right). \tag{6.4.42}$$

From (6.4.41) and using (6.3.3), we get

$$\begin{aligned}&z_{1ttt}+(1+z_1-x\frac{\dot{x}_s}{x_0})z_{1ttx}\frac{x_0}{x_s}+2(z_{1t}-x\frac{\ddot{x}_s}{x_0})z_{1tx}\frac{x_0}{x_s}+\frac{\dot{x}_s}{x_s}(z_{1tt}+z_{1t}\frac{\dot{x}_s}{x_s})\\&+(z_{1tt}-x\frac{\dddot{x}_s}{x_0})z_{1x}\frac{x_0}{x_s}+z_{1tt}\frac{\dot{x}_s}{x_s}+z_{1t}\frac{\ddot{x}_sx_s-(\dot{x}_s)^2}{x_c^2}=0,\\&z_{2ttt}+(1-z_2+x\frac{\dot{x}_s}{x_0})z_{2ttx}\frac{x_0}{L-x_s}-2(z_{2t}-x\frac{\ddot{x}_s}{x_0})z_{2tx}\frac{x_0}{L-x_s}+\frac{\dot{x}_s}{L-x_s}(-z_{2tt}+z_{2t}\frac{\dot{x}_s}{L-x_s})\\&-(z_{2tt}-x\frac{\dddot{x}_s}{x_0})z_{2x}\frac{x_0}{L-x_s}-z_{2tt}\frac{\dot{x}_s}{L-x_s}-z_{2t}\frac{\ddot{x}_s(L-x_s)+(\dot{x}_s)^2}{(L-x_s)^2}=0.\end{aligned} \tag{6.4.43}$$

Then differentiating V_3 along the system solutions and using (6.4.43), we have

$$\begin{aligned}\frac{dV_3}{dt}\leqslant&-\left[p_1e^{-\frac{\mu x}{\eta_1}}\frac{x_0}{x_s}(z_{1tt}^2)(1+z_1-x\frac{\dot{x}_s}{x_0})\right]_0^{x_0}-\left[p_2e^{-\frac{\mu x}{\eta_2}}\frac{x_0}{L-x_s}z_{2tt}^2(1-z_2+x\frac{\dot{x}_s}{x_0})\right]_0^{x_0}\\&-\mu\min\left(\frac{x_0}{x_s},\frac{L-x_0}{L-x_s}\right)V_3-\mu\int_0^{x_0}\left(\frac{x_0}{x_s}p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}^2z_1-\frac{L-x_0}{L-x_s}p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}^2z_2\right)dx\end{aligned}$$

$$
\begin{aligned}
&+\mu\int_0^{x_0}\left(\frac{x_0}{x_s}p_1e^{-\frac{\mu x}{\eta_2}}xz_{1tt}^2\frac{\dot{x}_s}{x_0}-\frac{L-x_0}{L-x_s}p_2e^{-\frac{\mu x}{\eta_1}}xz_{2tt}^2\frac{\dot{x}_s}{x_0}\right)dx\\
&-3\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}^2\frac{\dot{x}_s}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}^2\frac{\dot{x}_s}{L-x_s}\right)dx\\
&-\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}^2z_{1x}\frac{x_0}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}^2z_{2x}\frac{x_0}{L-x_s}\right)dx \qquad (6.4.44)\\
&-4\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}(z_{1t}-x\frac{\ddot{x}_s}{x_0})z_{1tx}\frac{x_0}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}(z_{2t}-x\frac{\ddot{x}_s}{x_0})z_{2tx}\frac{x_0}{L-x_s}\right)dx\\
&-2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}(z_{1tt}+z_{1t}\frac{\dot{x}_s}{x_s})\frac{\dot{x}_s}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}(z_{2tt}-z_{2t}\frac{\dot{x}_s}{L-x_s})\frac{\dot{x}_s}{L-x_s}\right)dx\\
&-2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}z_{1t}\frac{\ddot{x}_sx_s-(\dot{x}_s)^2}{x_c^2}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}z_{2t}\frac{\ddot{x}_s(L-x_s)+(\dot{x}_s)^2}{(L-x_s)^2}\right)dx\\
&-2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}(z_{1tt}-x\frac{\dddot{x}_s}{x_0})z_{1x}\frac{x_0}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}(z_{2tt}-x\frac{\dddot{x}_s}{x_0})z_{2x}\frac{x_0}{L-x_s}\right)dx.
\end{aligned}
$$

Observe that, while previously all the cubic terms in $\mathbf{z}$ could be bounded by $|\mathbf{z}|_{H^2}^3$, here in the last line in (6.4.44) we have $\dddot{x}_s$ which is proportional to $\mathbf{z}_{tt}(t,x_0)$ and cannot be roughly bounded by the $|\mathbf{z}|_{H^2}$ norm. To overcome this difficulty, we transform these terms using Young's inequality and we get

$$
\begin{aligned}
&2\int_0^{x_0}\left(p_1e^{-\frac{\mu x}{\eta_1}}z_{1tt}(x\frac{\dddot{x}_s}{x_0})z_{1x}\frac{x_0}{x_s}-p_2e^{-\frac{\mu x}{\eta_2}}z_{2tt}(x\frac{\dddot{x}_s}{x_0})z_{2x}\frac{x_0}{L-x_s}\right)dx\\
&\quad\leqslant C\,|\mathbf{z}(t,\cdot)|_{C^1([0,x_0];\mathbb{R}^2)}\,(z_{1tt}(t,x_0)+z_{2tt}(t,x_0))^2+O\left(|\mathbf{z}(t,\cdot)|_{C^1([0,x_0];\mathbb{R}^2)}\,|\mathbf{z}|_{H^2}^2\right),
\end{aligned}
\qquad (6.4.45)
$$

where C denotes a constant, independent of $\mathbf{z}$, x_s, T and $t\in[0,T]$. Note that the first term on the right is now proportional to $\mathbf{z}_{tt}^2(t,x_0)$ with a proportionality coefficient $C\,|\mathbf{z}(t,\cdot)|_{C^1([0,x_0];\mathbb{R}^2)}$ that, by Sobolev inequality, can be made sufficiently small provided that $|\mathbf{z}|_{H^2}$ is sufficiently small and thus can be dominated by the boundary terms. More precisely, from (6.4.44) and (6.4.45) we have

$$
\begin{aligned}
\frac{dV_3}{dt}\leqslant&-\mu V_3-\left[p_1e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s}(z_{1tt}^2)\right]_0^{x_0}-\left[p_2e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L-x_s}z_{2tt}^2\right]_0^{x_0}\\
&+O\left(|\mathbf{z}|_{H^2}\right)\left(z_{1tt}^2(t,x_0)+z_{2tt}^2(t,x_0)\right)+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right).
\end{aligned}
\qquad (6.4.46)
$$

Let us now deal with the term V_4 that takes into account the position of the jump. In the following, we use notations $\mathbf{z}(0)$ and $\mathbf{z}(x_0)$ instead of $\mathbf{z}(t,0)$ and $\mathbf{z}(t,x_0)$ for simplicity. We have

$$
\begin{aligned}
\frac{dV_4}{dt}=&-\int_0^{x_0}\bar{p}_1e^{\frac{-\mu x}{\eta_1}}(1+z_1-x\frac{\dot{x}_s}{x_0})z_{1x}(x_s-x_0)\frac{x_0}{x_s}\,dx+\int_0^{x_0}\bar{p}_1e^{\frac{-\mu x}{\eta_1}}z_1\dot{x}_s\,dx\\
&-\int_0^{x_0}\bar{p}_2e^{\frac{-\mu x}{\eta_2}}(1-z_2+x\frac{\dot{x}_s}{x_0})z_{2x}(x_s-x_0)\frac{x_0}{L-x_s}\,dx\\
&+\int_0^{x_0}\bar{p}_2e^{\frac{-\mu x}{\eta_2}}z_2\dot{x}_s\,dx+2\kappa\dot{x}_s(x_s-x_0) \qquad (6.4.47)\\
=&-(x_s-x_0)\left[\bar{p}_1e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s}z_1+\bar{p}_2e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L-x_s}z_2\right]_0^{x_0}-\mu(V_4-\kappa(x_s-x_0)^2)\\
&+\frac{z_1(x_0)+z_2(x_0)}{2}\left(\int_0^{x_0}\bar{p}_1e^{-\frac{\mu x}{\eta_1}}z_1\,dx\right)+\frac{z_1(x_0)+z_2(x_0)}{2}\left(\int_0^{x_0}\bar{p}_2e^{-\frac{\mu x}{\eta_2}}z_2\,dx\right)\\
&+\kappa(z_1(x_0)+z_2(x_0))(x_s-x_0)+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right).
\end{aligned}
$$

According to Young's inequality, for any positive ε_1 and ε_2, we have

$$\begin{aligned}\frac{z_1(x_0)+z_2(x_0)}{2}\left(\int_0^{x_0}\bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1\, dx\right) &\leqslant \frac{\varepsilon_1}{4}\left(\frac{z_1(x_0)+z_2(x_0)}{2}\right)^2+\frac{1}{\varepsilon_1}\left(\int_0^{x_0}\bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1\, dx\right)^2,\\ \frac{z_1(x_0)+z_2(x_0)}{2}\left(\int_0^{x_0}\bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2\, dx\right) &\leqslant \frac{\varepsilon_2}{4}\left(\frac{z_1(x_0)+z_2(x_0)}{2}\right)^2+\frac{1}{\varepsilon_2}\left(\int_0^{x_0}\bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2\, dx\right)^2.\end{aligned} \tag{6.4.48}$$

Then using the boundary condition (6.3.4) and Cauchy-Schwarz inequality, (6.4.47) becomes

$$\begin{aligned}\frac{dV_4}{dt}\leqslant & -\mu V_4-\bar{p}_1(x_s-x_0)\frac{x_0}{x_s}\left((e^{-\frac{\mu x_0}{\eta_1}}-k_1)z_1(x_0)+b_1(x_s-x_0)\right)\\ & -\bar{p}_2(x_s-x_0)\frac{x_0}{L-x_s}\left((e^{-\frac{\mu x_0}{\eta_2}}-k_2)z_2(x_0)+b_2(x_s-x_0)\right)\\ & +(\varepsilon_1+\varepsilon_2)\frac{z_1^2(x_0)+z_2^2(x_0)}{8}+\max\left\{\frac{\Theta_1}{\varepsilon_1},\,\frac{\Theta_2}{\varepsilon_2}\right\}V_1\\ & +\kappa(x_s-x_0)(z_1(x_0)+z_2(x_0))+\mu\kappa(x_s-x_0)^2+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right).\end{aligned} \tag{6.4.49}$$

Let us now consider V_5. From (6.4.8) and (6.4.41), one has similarly

$$\begin{aligned}\frac{dV_5}{dt}= & -\int_0^{x_0}\bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1tx}\dot{x}_s\frac{x_0}{x_s}\, dx+\int_0^{x_0}\bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1t}\ddot{x}_s\, dx\\ & -\int_0^{x_0}\bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2tx}\dot{x}_s\frac{x_0}{L-x_s}\, dx+\int_0^{x_0}\bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2t}\ddot{x}_s\, dx+2\kappa\ddot{x}_s\dot{x}_s+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right)\\ = & -\dot{x}_s\left[\bar{p}_1 e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s}z_{1t}+\bar{p}_2 e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L-x_s}z_{2t}\right]_0^{x_0}-\mu(V_5-\kappa(\dot{x}_s)^2)\\ & +\frac{z_{1t}(x_0)+z_{2t}(x_0)}{2}\left(\int_0^{x_0}\bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1t}\, dx\right)+\frac{z_{1t}(x_0)+z_{2t}(x_0)}{2}\left(\int_0^{x_0}\bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2t}\, dx\right)\\ & +\kappa(z_{1t}(x_0)+z_{2t}(x_0))\dot{x}_s+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right).\end{aligned}$$

By differentiating (6.3.4) with respect to time, we have

$$\begin{aligned}z_{1t}(0)&=k_1 z_{1t}(x_0)-b_1\dot{x}_s,\\ z_{2t}(0)&=k_2 z_{2t}(x_0)-b_2\dot{x}_s,\end{aligned} \tag{6.4.50}$$

and therefore using Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}\frac{dV_5}{dt}\leqslant & -\mu V_5-\bar{p}_1\dot{x}_s\frac{x_0}{x_s}\left((e^{-\frac{\mu x_0}{\eta_1}}-k_1)z_{1t}(x_0)+b_1\dot{x}_s\right)\\ & -\bar{p}_2\dot{x}_s\frac{x_0}{L-x_s}\left((e^{-\frac{\mu x_0}{\eta_2}}-k_2)z_{2t}(x_0)+b_2\dot{x}_s\right)\\ & +(\varepsilon_1+\varepsilon_2)\frac{z_{1t}^2(x_0)+z_{2t}^2(x_0)}{8}+\max\left\{\frac{\Theta_1}{\varepsilon_1},\,\frac{\Theta_2}{\varepsilon_2}\right\}V_2\\ & +\kappa\dot{x}_s(z_{1t}(x_0)+z_{2t}(x_0))+\mu\kappa(\dot{x}_s)^2+O\left((|\mathbf{z}|_{H^2}+|x_s-x_0|)^3\right).\end{aligned} \tag{6.4.51}$$

Furthermore, by differentiating (6.4.50) with respect to time, we have

$$\begin{aligned}z_{1tt}(0)&=k_1 z_{1tt}(x_0)-b_1\ddot{x}_s,\\ z_{2tt}(0)&=k_2 z_{2tt}(x_0)-b_2\ddot{x}_s,\end{aligned} \tag{6.4.52}$$

and therefore using also (6.4.43), one has

$$\begin{aligned}\frac{dV_6}{dt} =& -\int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1ttx}\ddot{x}_s \frac{x_0}{x_s}\,dx + \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} z_{1tt}\dddot{x}_s\,dx \\ &- \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2ttx}\ddot{x}_s \frac{x_0}{L-x_s}\,dx + \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} z_{2tt}\dddot{x}_s\,dx + 2\kappa\dddot{x}_s\ddot{x}_s + \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \ddot{x}_s\Big(x\frac{\dddot{x}_s}{x_0}\Big) z_{1x}\frac{x_0}{x_s}dx \\ &- \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \ddot{x}_s\Big(x\frac{\dddot{x}_s}{x_0}\Big) z_{2x}\frac{x_0}{L-x_s}\,dx + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right) \\ =& -\ddot{x}_s\left[\bar{p}_1 e^{\frac{-\mu x}{\eta_1}}\frac{x_0}{x_s} z_{1tt} + \bar{p}_2 e^{\frac{-\mu x}{\eta_2}}\frac{x_0}{L-x_s} z_{2tt}\right]_0^{x_0} - \mu(V_6 - \kappa(\ddot{x}_s)^2) \\ &+ \frac{z_{1tt}(x_0) + z_{2tt}(x_0)}{2}\left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt}\,dx\right) + \frac{z_{1tt}(x_0) + z_{2tt}(x_0)}{2}\left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt}\,dx\right) \\ &+ \kappa(z_{1tt}(x_0) + z_{2tt}(x_0))\ddot{x}_s + \int_0^{x_0} \bar{p}_1 e^{\frac{-\mu x}{\eta_1}} \ddot{x}_s\Big(x\frac{\dddot{x}_s}{x_0}\Big) z_{1x}\frac{x_0}{x_s}dx \\ &- \int_0^{x_0} \bar{p}_2 e^{\frac{-\mu x}{\eta_2}} \ddot{x}_s\Big(x\frac{\dddot{x}_s}{x_0}\Big) z_{2x}\frac{x_0}{L-x_s}\,dx + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right).\end{aligned}$$

Note that, as above for V_3, here appears again $\dddot{x}_s$ which is proportional to $\mathbf{z}_{tt}(t, x_0)$ and cannot be bounded by $|\mathbf{z}|_{H^2}$. We therefore use Cauchy-Schwarz and Young's inequalities as previously and the boundary condition (6.4.52), to get

$$\begin{aligned}\frac{dV_6}{dt} \leqslant& -\mu V_6 - \bar{p}_1\ddot{x}_s\frac{x_0}{x_s}\left((e^{-\frac{\mu x_0}{\eta_1}} - k_1)z_{1tt}(x_0) + b_1\ddot{x}_s\right) \\ &- \bar{p}_2\ddot{x}_s\frac{x_0}{L-x_s}\left((e^{-\frac{\mu x_0}{\eta_2}} - k_2)z_{2tt}(x_0) + b_2\ddot{x}_s\right) \\ &+ (\varepsilon_1 + \varepsilon_2)\frac{z_{1tt}^2(x_0) + z_{2tt}^2(x_0)}{8} + \max\left\{\frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2}\right\} V_2 \\ &+ \kappa\ddot{x}_s(z_{1tt}(x_0) + z_{2tt}(x_0)) + \mu\kappa(\ddot{x}_s)^2 + O\left(|\mathbf{z}|_{H^2}\right)\left(z_{1tt}^2(x_0) + z_{2tt}^2(x_0)\right) \\ &+ O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right).\end{aligned} \tag{6.4.53}$$

Hence, from (6.4.40), (6.4.49) and the boundary conditions (6.3.4), we have

$$\begin{aligned}\frac{dV_1}{dt} + \frac{dV_4}{dt} \leqslant& -\mu(V_1 + V_4) \\ &+ \max\left\{\frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2}\right\} V_1 \\ &+ \left[\frac{x_0}{x_s}p_1(k_1^2 - e^{-\frac{\mu x_0}{\eta_1}}) + \frac{\varepsilon_1 + \varepsilon_2}{8}\right] z_1^2(x_0) \\ &+ \left[\frac{x_0}{L-x_s}p_2(k_2^2 - e^{-\frac{\mu x_0}{\eta_2}}) + \frac{\varepsilon_1 + \varepsilon_2}{8}\right] z_2^2(x_0) \\ &+ \left[-2\frac{x_0}{x_s}p_1 b_1 k_1 - \frac{x_0}{x_s}\bar{p}_1(e^{-\frac{\mu x_0}{\eta_1}} - k_1) + \kappa\right] z_1(x_0)(x_s - x_0) \\ &+ \left[-2\frac{x_0}{L-x_s}p_2 b_2 k_2 - \frac{x_0}{L-x_s}\bar{p}_2(e^{-\frac{\mu x_0}{\eta_2}} - k_2) + \kappa\right] z_2(x_0)(x_s - x_0) \\ &+ \left[\frac{x_0}{x_s}p_1 b_1^2 + \frac{x_0}{L-x_s}p_2 b_2^2 - \frac{x_0}{x_s}\bar{p}_1 b_1 - \frac{x_0}{L-x_s}\bar{p}_2 b_2 + \mu\kappa\right](x_s - x_0)^2 \\ &+ O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right).\end{aligned} \tag{6.4.54}$$

Let us now select ε_1 and ε_2 as follows:

$$\varepsilon_1 = 2\frac{\Theta_1}{\mu}, \quad \varepsilon_2 = 2\frac{\Theta_2}{\mu}, \tag{6.4.55}$$

where Θ_1 and Θ_2 are defined in (6.4.17). Then (6.4.54) can be rewritten in the following compact form:

$$\frac{dV_1}{dt} + \frac{dV_4}{dt} \leqslant -\frac{\mu}{2}V_1 - \mu V_4 - \mathbf{Z}^T \mathbf{A}_0 \, \mathbf{Z} + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right). \tag{6.4.56}$$

This expression involves the quadratic form $\mathbf{Z}^T \mathbf{A}_0 \, \mathbf{Z}$ with the vector $\mathbf{Z}$ defined as

$$\mathbf{Z} = (z_1(x_0) \;\; z_2(x_0) \;\; (x_s - x_0))^T. \tag{6.4.57}$$

and the matrix $\mathbf{A}_0$ satisfies

$$\mathbf{A}_0 = \mathbf{A} + O(|x_s - x_0|), \tag{6.4.58}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{6.4.59}$$

with

$$a_{11} = p_1(e^{-\frac{\mu x_0}{\eta_1}} - k_1^2) - \frac{\varepsilon_1 + \varepsilon_2}{8}, \tag{6.4.60}$$

$$a_{13} = a_{31} = p_1 b_1 k_1 + \frac{\bar{p}_1}{2}(e^{-\frac{\mu x_0}{\eta_1}} - k_1) - \frac{\kappa}{2}, \tag{6.4.61}$$

$$a_{22} = \frac{x_0}{L - x_0} p_2(e^{-\frac{\mu x_0}{\eta_2}} - k_2^2) - \frac{\varepsilon_1 + \varepsilon_2}{8}, \tag{6.4.62}$$

$$a_{23} = a_{32} = \frac{x_0}{L - x_0} p_2 b_2 k_2 + \frac{x_0}{L - x_0} \frac{\bar{p}_2}{2}(e^{-\frac{\mu x_0}{\eta_2}} - k_2) - \frac{\kappa}{2}, \tag{6.4.63}$$

$$a_{33} = -p_1 b_1^2 - \frac{x_0}{L - x_0} p_2 b_2^2 + \bar{p}_1 b_1 + \frac{x_0}{L - x_0} \bar{p}_2 b_2 - \mu\kappa. \tag{6.4.64}$$

Similarly, from (6.4.42) and (6.4.51), we get

$$\frac{dV_2}{dt} + \frac{dV_5}{dt} \leqslant -\frac{\mu}{2}V_2 - \mu V_5 - \mathbf{Z}_t^T \mathbf{A}_0 \, \mathbf{Z}_t + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right), \tag{6.4.65}$$

while from (6.4.46) and (6.4.53), we have

$$\frac{dV_3}{dt} + \frac{dV_6}{dt} \leqslant -\frac{\mu}{2}V_3 - \mu V_6 - \mathbf{Z}_{tt}^T \mathbf{A}_1 \, \mathbf{Z}_{tt} + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right) \tag{6.4.66}$$

with

$$\mathbf{A}_1 = \mathbf{A}_0 + \begin{pmatrix} O(|\mathbf{z}|_{H^2}) & 0 & 0 \\ 0 & O(|\mathbf{z}|_{H^2}) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.4.67}$$

If $\mathbf{A}$ is positive definite, from (6.4.58) and (6.4.67) and by continuity, $\mathbf{A}_0$ and $\mathbf{A}_1$ are also positive definite provided that $|\mathbf{z}|_{H^2}$ and $|x_s - x_0|$ are sufficiently small. Hence, from (6.4.56), (6.4.65), (6.4.66) and Lemma 6.3.1, there exists $\delta_1(T) > 0$ such that, if $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \delta_1(T)$ and $|x_{s0} - x_0| \leqslant \delta_1(T)$, one has

$$\frac{dV}{dt} \leqslant -\frac{\mu}{2}V + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right), \tag{6.4.68}$$

which ends the proof of Lemma 6.4.3. □

Let us now prove Theorem 6.4.1.

Proof of Theorem 6.4.1. From Lemma 6.4.1 and Lemma 6.4.2, all it remains to do is to show that for any $\gamma > 0$, under conditions (6.2.16) there exist μ, p_1, p_2, $\bar{p}_1$ and $\bar{p}_2$ satisfying (6.4.16) and such that V given by (6.4.3)-(6.4.9) decreases exponentially with rate $\gamma/2$ along any C^3 solution of the system (6.3.3)–(6.3.5). Using Lemma 6.4.3 we first show that for any $\gamma > 0$ there exists $\mu > \gamma$, and positive parameters p_1, p_2, $\bar{p}_1$ and $\bar{p}_2$ satisfying (6.4.16) and such that the matrix $\mathbf{A}$ defined by (6.4.59)-(6.4.64) is positive definite, which implies that (6.4.39) holds. Then, we show that this implies the exponential decay of V with decay rate $\gamma/2$ along any C^3 solution of (6.3.3)–(6.3.5).

Let us start by selecting p_1 and p_2 as

$$p_1 = \frac{\bar{p}_1}{2b_1}, \quad p_2 = \frac{\bar{p}_2}{2b_2}. \tag{6.4.69}$$

Then the cross terms (6.4.61), (6.4.63) of the matrix $\mathbf{A}$ become

$$a_{13} = a_{31} = \frac{\bar{p}_1}{2} e^{-\frac{\mu x_0}{\eta_1}} - \frac{\kappa}{2}, \quad a_{23} = a_{32} = \frac{x_0}{L - x_0} \frac{\bar{p}_2}{2} e^{-\frac{\mu x_0}{\eta_2}} - \frac{\kappa}{2}. \tag{6.4.70}$$

Let $\bar{p}_1$ and $\bar{p}_2$ be selected as

$$\bar{p}_1 = \kappa e^{\frac{\mu x_0}{\eta_1}}, \quad \bar{p}_2 = \kappa \frac{L - x_0}{x_0} e^{\frac{\mu x_0}{\eta_2}}. \tag{6.4.71}$$

Then we have

$$a_{13} = a_{31} = 0, \quad a_{23} = a_{32} = 0 \tag{6.4.72}$$

such that $\mathbf{A}$ can now be rewritten as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}. \tag{6.4.73}$$

Moreover from (6.4.69) and (6.4.71), we get

$$a_{33} = \frac{\bar{p}_1}{2} b_1 + \frac{x_0}{L - x_0} \frac{\bar{p}_2}{2} b_2 - \mu\kappa = \frac{\kappa}{2} b_1 e^{\frac{\mu x_0}{\eta_1}} + \frac{\kappa}{2} b_2 e^{\frac{\mu x_0}{\eta_2}} - \mu\kappa. \tag{6.4.74}$$

As conditions (6.2.16) are strict inequalities, by continuity it follows that we can select $\mu > \gamma$ such that these conditions (6.2.16) are still satisfied with μ instead of γ such that

$$\mu e^{-\frac{\mu x_0}{\eta_1}} < b_1 < \frac{\mu e^{-\frac{\mu x_0}{\eta_1}}}{1 - e^{-\frac{\mu x_0}{\eta_1}}}, \quad \mu e^{-\frac{\mu x_0}{\eta_2}} < b_2 < \frac{\mu e^{-\frac{\mu x_0}{\eta_2}}}{1 - e^{-\frac{\mu x_0}{\eta_2}}}, \tag{6.4.75}$$

this together with (6.4.74) gives

$$a_{33} > 0. \tag{6.4.76}$$

From (6.4.17), (6.4.55), (6.4.60), (6.4.62), (6.4.69) and (6.4.71), we have

$$a_{11} = \frac{\kappa}{2b_1}(1 - k_1^2 e^{\frac{\mu x_0}{\eta_1}}) - \frac{\kappa}{2\mu^2}\left[b_1(e^{\frac{\mu x_0}{\eta_1}} - 1) + b_2(e^{\frac{\mu x_0}{\eta_2}} - 1)\right], \tag{6.4.77}$$

$$a_{22} = \frac{\kappa}{2b_2}(1 - k_2^2 e^{\frac{\mu x_0}{\eta_2}}) - \frac{\kappa}{2\mu^2}\left[b_1(e^{\frac{\mu x_0}{\eta_1}} - 1) + b_2(e^{\frac{\mu x_0}{\eta_2}} - 1)\right]. \tag{6.4.78}$$

Then, under assumptions (6.2.16), it can be checked that

$$a_{11} > 0, \quad a_{22} > 0. \tag{6.4.79}$$

This implies that $\mathbf{A}$ is positive definite.
Thus from Lemma 6.4.3, for any $T > 0$, there exists $\delta_1(T) > 0$ such that for any $(\mathbf{z}, x_s) \in C^3([0,T] \times [0,x_0];\mathbb{R}^2)) \times C^3([0,T];\mathbb{R})$ solution of (6.3.3)–(6.3.5) satisfying $\left|\mathbf{z}^0\right|_{H^2((0,L);\mathbb{R}^2)} \leqslant \delta_1(T)$ and $|x_{s0} - x_0| \leqslant \delta_1(T)$, one has

$$\frac{dV}{dt} \leqslant -\frac{\mu}{2} V + O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right). \tag{6.4.80}$$

Now let us remark that from condition (6.4.75) we have

$$\max\left(2\frac{b_1\eta_1}{\mu}e^{\frac{\mu x_0}{\eta_1}}\left(1-e^{-\frac{\mu x_0}{\eta_1}}\right),\ 2\frac{L-x_0}{x_0}\frac{b_2\eta_2}{\mu}e^{\frac{\mu x_0}{\eta_2}}\left(1-e^{-\frac{\mu x_0}{\eta_2}}\right)\right)<2. \tag{6.4.81}$$

Therefore, there exists $\kappa>1$ such that

$$\max\left(2\kappa\frac{b_1\eta_1}{\mu}e^{\frac{\mu x_0}{\eta_1}}\left(1-e^{-\frac{\mu x_0}{\eta_1}}\right),\ 2\kappa\frac{L-x_0}{x_0}\frac{b_2\eta_2}{\mu}e^{\frac{\mu x_0}{\eta_2}}\left(1-e^{-\frac{\mu x_0}{\eta_2}}\right)\right)<2, \tag{6.4.82}$$

which means from (6.4.69) and (6.4.71) that (6.4.16) is satisfied. Hence from (6.4.80) and Lemma 6.4.1, since $\mu>\gamma$, there exists $\delta_0(T)\leqslant\delta_1(T)$ such that, if $\left|\mathbf{z}^0\right|_{H^2((0,x_0);\mathbb{R}^2)}\leqslant\delta_0(T)$ and $|x_{s0}-x_0|\leqslant\delta_0(T)$, then

$$\frac{dV}{dt}\leqslant-\frac{\gamma}{2}V \tag{6.4.83}$$

along the C^3 solutions of the system (6.3.3)–(6.3.5). Thus from Lemma 6.4.2, (6.4.83) holds along the $C^0([0,T];H^2((0,x_0);\mathbb{R}^2))\times C^1([0,T];\mathbb{R})$ solutions of (6.3.3)–(6.3.5) in a distribution sense.
So far $\delta_0(T)$ may depend on T, while δ^* in Theorem 6.4.1 does not depend on T. The only thing left to check is that we can find δ^* independent of T such that if $\left|\mathbf{z}^0\right|_{H^2((0,x_0);\mathbb{R}^2)}\leqslant\delta^*$ and $|x_{s0}-x_0|\leqslant\delta^*$, then (6.4.83) holds on $(0,T)$ for any $T>0$. As the constant β involved in Lemma 6.4.1 does not depend on T, there exists $T_1>0$ such that

$$\beta^{-2}e^{-\frac{\gamma}{2}T_1}<\frac{1}{2}. \tag{6.4.84}$$

As $T_1\in(0,+\infty)$, from Lemma 6.3.1, we can choose $\delta_0(T_1)>0$ satisfying $C(T_1)\delta_0(T_1)<\beta/2$, such that for every $x_{s0}\in(0,L)$ and $\mathbf{z}^0\in H^2((0,x_0);\mathbb{R}^2)$ satisfying the compatibility conditions (6.3.7)–(6.3.8) and

$$|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)}\leqslant\delta_0(T_1),\quad |x_{s0}-x_0|\leqslant\delta_0(T_1),$$

there exists a unique solution $(\mathbf{z},x_s)\in C^0([0,T_1];H^2((0,x_0);\mathbb{R}^2))\times C^1([0,T_1];\mathbb{R})$ to the system (6.3.3)–(6.3.5) satisfying

$$|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}+|x_s(t)-x_0|<\beta \tag{6.4.85}$$

and such that (6.4.83) holds on $(0,T_1)$ in a distribution sense. From (6.4.85), Lemma 6.4.1 and (6.4.84),

$$|\mathbf{z}(T_1,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}\leqslant\delta_0(T_1),\quad |x_s(T_1)-x_0|\leqslant\delta_0(T_1). \tag{6.4.86}$$

Moreover, the compatibility conditions hold now at time $t=T_1$ instead of $t=0$. Thus, from Lemma 6.3.1 there exists a unique $(\mathbf{z},x_s)\in C^0([T_1,2T_1];H^2((0,x_0);\mathbb{R}^2))\times C^1([T_1,2T_1];\mathbb{R})$ solution of (6.3.3)–(6.3.5) on $[T_1,2T_1]$ and (6.4.83) holds on $(T_1,2T_1)$ in a distribution sense. One can repeat this analysis on $[jT_1,(j+1)T_1]$ where $j\in\mathbb{N}^*\setminus\{1\}$. Setting $\delta^*=\delta_0(T_1)$, we get that (6.4.83) holds on $(0,T)$ for any $T>0$ in a distribution sense along the $C^0([0,T];H^2((0,x_0);\mathbb{R}^2))\times C^1([0,T];\mathbb{R})$ solutions of the system (6.3.3)–(6.3.5). In fact, it also implies the global existence and uniqueness of $(\mathbf{z},x_s)\in C^0([0,+\infty);H^2((0,x_0);\mathbb{R}^2))\times C^1([0,+\infty);\mathbb{R})$ solution of (6.3.3)–(6.3.5) and the fact that (6.4.83) holds on $(0,+\infty)$. This concludes the proof of Theorem 6.4.1. □

6.5 Extension to a general convex flux

We can in fact extend this method to a more general convex flux. Let $f\in C^3(\mathbb{R})$ be a convex function, and consider the equation

$$\partial_t y+\partial_x(f(y))=0. \tag{6.5.1}$$

For this conservation law, the Rankine-Hugoniot condition becomes

$$\dot{x}_s=\frac{f(y(t,x_s(t)^+))-f(y(t,x_s(t)^-))}{y(t,x_s(t)^+)-y(t,x_s(t)^-)}, \tag{6.5.2}$$

and, let (y^*, x_0) be an entropic shock steady state of (6.5.1)–(6.5.2), without loss of generality we can assume that $y^*(x_0^+) = -1$ and $y^*(x_0^-) = 1$, thus $f(1) = f(-1)$. Then, for any $x_0 \in (0, L)$, we have the following result:

Theorem 6.5.1. *Let $f \in C^3(\mathbb{R})$ be a convex function such that $f(1) = f(-1)$ and assume in addition that*

$$f'(1) \geqslant 1 \text{ and } |f'(-1)| \geqslant 1. \tag{6.5.3}$$

Let $\gamma > 0$. If the following conditions hold

$$b_1 \in \left(\frac{2\gamma e^{-\gamma x_0}}{f'(1) + |f'(-1)|}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}} \right),\ b_2 \in \left(\frac{2\gamma e^{-\gamma (L-x_0)}}{f'(1) + |f'(-1)|}, \frac{\gamma e^{-\gamma (L-x_0)}}{1 - e^{-\gamma (L-x_0)}} \right), \tag{6.5.4a}$$

$$k_1^2 < e^{-\gamma x_0} \left(1 - f'(1) \frac{b_1}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma (L-x_0)}}{\gamma e^{-\gamma (L-x_0)}} \right) \right), \tag{6.5.4b}$$

$$k_2^2 < e^{-\gamma (L-x_0)} \left(1 - |f'(-1)| \frac{b_2}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma (L-x_0)}}{\gamma e^{-\gamma (L-x_0)}} \right) \right), \tag{6.5.4c}$$

then the steady state (y^*, x_0) of the system (6.5.1), (6.5.2), (6.2.3), (6.2.8) *is exponentially stable for the H^2-norm with decay rate $\gamma/4$.*

One can use exactly the same method as previously. We give in Appendix 6.7.2 a way to adapt the proof of Theorem 6.4.1.

Remark 6.5.1. *One could wonder why we require condition* (6.5.3)*. This condition ensures that there always exist parameters b_i and k_i satisfying* (6.5.4)*.*

6.6 Conclusion and Open problems

The stabilization of shock-free regular solutions of quasilinear hyperbolic systems has been the subject of a large number of publications in the recent scientific literature. In contrast, there are no results concerning the Lyapunov stability of solutions with jump discontinuities, although they occur naturally in the form of shock waves or hydraulic jumps in many applications of fluid dynamics. For instance, the inviscid Burgers equation provides a simple scalar example of a hyperbolic system having natural solutions with jump discontinuities. The main contribution of this chapter is precisely to address the issue of the boundary exponential feedback stabilization of an unstable shock steady state for the Burgers equation over a bounded interval. Our strategy to solve the problem relies on introducing a change of variables which allows to transform the scalar Burgers equation with shock wave solutions into an equivalent 2×2 quasilinear hyperbolic system having shock-free solutions over a bounded interval. Then, by a Lyapunov approach, we show that, for appropriately chosen boundary conditions, the exponential stability in H^2-norm of the steady state can be achieved with an arbitrary decay rate and with an exact exponential stabilization of the desired shock location. Compared with previous results in the literature for classical solutions of quasilinear hyperbolic systems, the selection of an appropriate Lyapunov function is challenging because the equivalent system is parameterized by the time-varying position of the jump discontinuity. In particular, the standard quadratic Lyapunov function used in the book [13] has to be augmented with suitable extra terms for the analysis of the stabilization of the jump position. Based on the result, some open questions could be addressed. Could these results be generalized to any convex flux, especially when (6.5.3) is not satisfied? As we show the rapid stabilization result, is it possible to obtain finite time stabilization? Could we replace the left/right state at the shock by measurements nearby or by averages close to the shock? If not, could the error on both the state and shock location be bounded?

Acknowledgement

The authors would like to thank Tatsien Li and Sébastien Boyaval for their constant support. They would like also to thank National Natural Science Foundation of China (No. 11771336), ETH-ITS, ETH-FIM, ANR project Finite4SoS (No.ANR 15-CE23-0007) and the French Corps des IPEF for their financial support.

6.7 Appendix

6.7.1 Proof of Lemma 6.3.1

Proof. We adapt the fixed point method used in [13, Appendix B] (see also [118, 142]). We first deal with the case where

$$T \in (0, \min(x_0,\, L - x_0)). \tag{6.7.1}$$

For any $\nu > 0$, $x_{s0} \in \mathbb{R}$ and $\mathbf{z}^0 \in H^2((0,x_0);\mathbb{R}^2)$, let $C_\nu(\mathbf{z}^0, x_{s0})$ be the set of

$$\mathbf{z} \in L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2)) \cap W^{1,\infty}((0,T);H^1((0,x_0);\mathbb{R}^2)) \cap W^{2,\infty}((0,T);L^2((0,x_0);\mathbb{R}^2))$$

such that

$$|\mathbf{z}|_{L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2))} \leqslant \nu, \tag{6.7.2}$$

$$|\mathbf{z}|_{W^{1,\infty}((0,T);H^1((0,x_0);\mathbb{R}^2))} \leqslant \nu, \tag{6.7.3}$$

$$|\mathbf{z}|_{W^{2,\infty}((0,T);L^2((0,x_0);\mathbb{R}^2))} \leqslant \nu, \tag{6.7.4}$$

$$\mathbf{z}(\cdot, x_0) \in H^2((0,T);\mathbb{R}^2)), \quad |\mathbf{z}(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2))} \leqslant \nu^2, \tag{6.7.5}$$

$$\mathbf{z}(0,\cdot) = \mathbf{z}^0, \tag{6.7.6}$$

$$\mathbf{z}_t(0,\cdot) = -A(\mathbf{z}^0, \cdot, x_s(\mathbf{z}(\cdot,x_0))(0))\mathbf{z}^0_x, \tag{6.7.7}$$

where we write $x_s(\mathbf{z}(\cdot,x_0))(t)$ in order to emphasize its dependence on $\mathbf{z}(\cdot,x_0)$ in the following proof and

$$x_s(\mathbf{z}(\cdot,x_0))(t) =: x_{s0} + \int_0^t \frac{z_1(s,x_0) + z_2(s,x_0)}{2}\, ds. \tag{6.7.8}$$

In (6.7.7),

$$A(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \begin{pmatrix} a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) & 0 \\ 0 & a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) \end{pmatrix} \tag{6.7.9}$$

with

$$a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \left(1 + z_1(t, x) - x\frac{z_1(t, x_0) + z_2(t, x_0)}{2x_0}\right) \frac{x_0}{x_s(\mathbf{z}(\cdot, x_0))(t)}, \tag{6.7.10}$$

$$a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \left(1 - z_2(t, x) + x\frac{z_1(t, x_0) + z_2(t, x_0)}{2x_0}\right) \frac{x_0}{L - x_s(\mathbf{z}(\cdot, x_0))(t)}. \tag{6.7.11}$$

The set $C_\nu(\mathbf{z}^0, x_{s0})$ is not empty and is a closed subset of $L^\infty((0,T); L^2((0,L);\mathbb{R}^2))$ provided that $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \delta$ and $|x_{s0} - x_0| \leqslant \delta$, with δ sufficiently small (see for instance [13, Appendix B]). Let us define a mapping:

$$\begin{aligned} \mathcal{F} : C_\nu(\mathbf{z}^0, x_{s0}) &\longrightarrow L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)) \cap W^{1,\infty}((0,T); H^1((0,x_0);\mathbb{R}^2)) \\ &\qquad\qquad\qquad\qquad \cap W^{2,\infty}((0,T); L^2((0,x_0);\mathbb{R}^2)) \\ \mathbf{v} = (v_1, v_2)^T &\longmapsto \mathcal{F}(\mathbf{v}) = \mathbf{z} = (z_1, z_2)^T \end{aligned} \tag{6.7.12}$$

where $\mathbf{z}$ is the solution of the linear hyperbolic equation

$$\mathbf{z}_t + A(\mathbf{v}, x, x_s(\mathbf{v}(\cdot, x_0))(t))\mathbf{z}_x = 0, \tag{6.7.13}$$

$$\mathbf{z}(0, x) = \mathbf{z}^0(x), \tag{6.7.14}$$

with boundary conditions

$$z_1(t, 0) = k_1 z_1(t, x_0) + b_1\psi(t), \tag{6.7.15}$$

$$z_2(t, 0) = k_2 z_2(t, x_0) + b_2\psi(t), \tag{6.7.16}$$

where

$$\psi(t) = x_0 - x_s(\mathbf{v}(\cdot, x_0))(t). \tag{6.7.17}$$

In the following, we will treat z_1 in details. For the sake of simplicity, we denote

$$f_1(t, x) := a_1(\mathbf{v}(t, x), x, x_s(\mathbf{v}(\cdot, x_0))(t)). \tag{6.7.18}$$

It is easy to check from (6.7.10) that if ν is sufficiently small, then $f_1(t, x)$ is strictly positive for any $(t, x) \in [0, T] \times [0, x_0]$. Let us now define the characteristic curve $\xi_1(s; t, x)$ passing through (t, x) as

$$\begin{aligned} \frac{d\xi_1(s; t, x)}{ds} &= f_1(s, \xi_1(s; t, x)), \\ \xi_1(t; t, x) &= x. \end{aligned} \tag{6.7.19}$$

One can see that for every $(t, x) \in [0, T] \times [0, x_0]$, $\xi_1(\cdot; t, x)$ is uniquely defined on some closed interval in $[0, T]$. From (6.7.1), only two cases can occur (see Figure 6.2): If $\xi_1(t; 0, 0) < x \leqslant x_0$, there exists $\beta_1 \in [0, x_0]$ depending on (t, x) such that

$$\beta_1 = \xi_1(0; t, x). \tag{6.7.20}$$

If $0 < x < \xi_1(t; 0, 0)$, there exists $\alpha_1 \in [0, t]$ depending on (t, x) such that

$$\xi_1(\alpha_1; t, x) = 0, \tag{6.7.21}$$

and in this case, there exists $\gamma_1 \in [0, x_0]$ depending on α_1 such that

$$\gamma_1 = \xi_1(0; \alpha_1, x_0). \tag{6.7.22}$$

Moreover, we have the following lemma which will be used in the estimations hereafter (the proof can be found at the end of this appendix).

Lemme 6.7.1. *There exist $\nu_0 > 0$ and $C > 0$ such that, for any T satisfying (6.7.1), for any $\nu \in (0, \nu_0]$ and for a.e. $t \in (0, T)$, we have*

$$|f_1(t,\cdot)|_0 \leqslant C, \quad |f_{1x}(t,\cdot)|_0 \leqslant C\nu, \quad |f_{1t}(t,\cdot)|_0 \leqslant C\nu, \tag{6.7.23}$$

$$|\partial_x \xi_1(s;t,\cdot)|_0 \leqslant C, \quad |\partial_t \xi_1(s;t,\cdot)|_0 \leqslant C, \quad s \in [0,t], \tag{6.7.24}$$

$$|\partial_x \beta_1(t,\cdot)|_0 \leqslant C, \quad |\left(\partial_x \beta_1(t,\cdot)\right)^{-1}|_0 \leqslant C, \tag{6.7.25}$$

$$|\partial_t \beta_1(t,\cdot)|_0 \leqslant C, \quad |\left(\partial_t \beta_1(t,\cdot)\right)^{-1}|_0 \leqslant C, \tag{6.7.26}$$

$$|\partial_x \alpha_1(t,\cdot)|_0 \leqslant C, \quad \left|\left(\partial_x \alpha_1(t,\cdot)\right)^{-1}\right|_0 \leqslant C, \tag{6.7.27}$$

$$|\partial_x \gamma_1(t,\cdot)|_0 \leqslant C, \quad \left|\left(\partial_x \gamma_1(t,\cdot)\right)^{-1}\right|_0 \leqslant C, \tag{6.7.28}$$

$$\int_0^T |\partial_{tt}\beta_1(t,x_0)|^2\, dt \leqslant C\nu, \tag{6.7.29}$$

$$\int_0^{x_0} |\partial_{xx}\alpha_1(t,x)|^2\, dx \leqslant C\nu, \tag{6.7.30}$$

$$\int_0^{x_0} |\partial_{xx}\beta_1(t,x)|^2\, dx \leqslant C\nu, \tag{6.7.31}$$

$$\int_0^{x_0} |\partial_{xx}\gamma_1(t,x)|^2\, dx \leqslant C\nu. \tag{6.7.32}$$

In these inequalities, and hereafter in this section, $|f|_0$ denotes the C^0-norm of a function f with respect to its variable and C may depend on x_0, x_{s0}, ν_0, k_1, k_2, b_1 and b_2, but is independent of ν, T, $\mathbf{v}$ and $\mathbf{z}$.

Our goal is now to use a fixed point argument to show the existence and uniqueness of the solution to (6.3.3)–(6.3.5). Firstly, we show that for ν and δ sufficiently small, $\mathcal{F}$ maps $C_\nu(\mathbf{z}^0, x_{s0})$ into itself, i.e.,

$$\mathcal{F}(C_\nu(\mathbf{z}^0, x_{s0})) \subset C_\nu(\mathbf{z}^0, x_{s0}).$$

Then, in a second step, we prove that $\mathcal{F}$ is a contraction mapping.

1) $\mathcal{F}$ maps $C_\nu(\mathbf{z}^0, x_{s0})$ into itself.

For any $\mathbf{v} \in C_\nu(\mathbf{z}^0, x_{s0})$, let $\mathbf{z} = \mathcal{F}(\mathbf{v})$, we prove that $\mathbf{z} \in C_\nu(\mathbf{z}^0, x_{s0})$. By the definition of $\mathcal{F}$ in (6.7.12), using the method of characteristics, we can solve (6.7.13) to (6.7.16) for z_1 and obtain that

$$z_1(t,x) = \begin{cases} k_1 z_1^0(\gamma_1) + b_1 \psi(\alpha_1), & 0 < x < \xi_1(t;0,0), \\ z_1^0(\beta_1), & \xi_1(t;0,0) < x < x_0. \end{cases} \tag{6.7.33}$$

Obviously $\mathbf{z}$ verifies the properties (6.7.6)–(6.7.7). Next, we prove that $\mathbf{z}$ verifies the property (6.7.5). Using the change of variables and from (6.7.26), we have

$$\int_0^T z_1(t,x_0)^2 dt = \int_0^T z_1^0(\beta_1(t,x_0))^2 dt \leqslant C \int_0^{x_0} (z_1^0(x))^2 dx. \tag{6.7.34}$$

In (6.7.34) and hereafter, C denotes various constants that may depend on x_0, x_{s0}, ν_0, k_1, k_2, b_1 and b_2, but are independent of ν, T, $\mathbf{v}$ and $\mathbf{z}$. Similarly, by (6.7.26), we obtain

$$\int_0^T z_{1t}(t,x_0)^2 dt = \int_0^T (z_{1x}^0(\beta_1(t,x_0))\partial_t\beta_1(t,x_0))^2 dt \leqslant C \int_0^T (z_{1x}^0(x))^2 dx. \tag{6.7.35}$$

From (6.7.29) and using Sobolev inequality, one has

$$\begin{aligned} \int_0^T z_{1tt}(t,x_0)^2 dt &= \int_0^T (z_{1xx}^0(\beta_1(t,x_0))(\partial_t\beta_1(t,x_0))^2 + z_{1x}^0(\beta_1(t,x_0))\partial_{tt}\beta_1(t,x_0))^2 dt \\ &\leqslant C \int_0^{x_0} (z_{1xx}^0(x))^2 dx + 2|z_{1x}^0|_0^2 \int_0^T (\partial_{tt}\beta_1(t,x_0))^2\, dt \\ &\leqslant C |z_1^0|^2_{H^2((0,x_0);\mathbb{R})}. \end{aligned} \tag{6.7.36}$$

Combining (6.7.34)–(6.7.36), we get

$$|z_1(\cdot, x_0)|_{H^2((0,T);\mathbb{R})} \leqslant C|z_1^0|_{H^2((0,x_0);\mathbb{R})}. \tag{6.7.37}$$

Applying similar estimate to z_2 gives

$$|z_2(\cdot, x_0)|_{H^2((0,T);\mathbb{R})} \leqslant C|z_2^0|_{H^2((0,x_0);\mathbb{R})}. \tag{6.7.38}$$

From (6.7.37) and (6.7.38), we can select δ sufficiently small such that

$$|\mathbf{z}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)} \leqslant \nu^2, \tag{6.7.39}$$

which shows both the regularity and the boundedness property (6.7.5). We can again use the method of characteristics to prove properties (6.7.2)–(6.7.4). For a.e. $t \in (0,T)$,

$$z_{1x}(t,x) = \begin{cases} k_1 z_{1x}^0(\gamma_1)\partial_x\gamma_1 + b_1\dot{\psi}(\alpha_1)\partial_x\alpha_1, & 0 < x < \xi_1(t;0,0), \\ z_{1x}^0(\beta_1)\partial_x\beta_1, \quad \xi_1(t;0,0) < x < x_0. \end{cases} \tag{6.7.40}$$

$$z_{1xx}(t,x) = \begin{cases} \begin{aligned} &k_1 z_{1x}^0(\gamma_1)\partial_{xx}\gamma_1 + k_1 z_{1xx}^0(\gamma_1)(\partial_x\gamma_1)^2 \\ &+ b_1\ddot{\psi}(\alpha_1)(\partial_x\alpha_1)^2 + b_1\dot{\psi}(\alpha_1)\partial_{xx}\alpha_1 \end{aligned}, & 0 < x < \xi_1(t;0,0), \\ z_{1x}^0(\beta_1)\partial_{xx}\beta_1 + z_{1xx}^0(\beta_1)(\partial_x\beta_1)^2, \quad \xi_1(t;0,0) < x < x_0. \end{cases} \tag{6.7.41}$$

Note that the last equation is true in distribution sense but shows that $z_1 \in L^\infty((0,T);H^2((0,x_0);\mathbb{R}))$. We first estimate $|\mathbf{z}|_{L^\infty((0,T);H^2((0,x_0);\mathbb{R}^2))}$. From (6.7.8) and (6.7.17), using Sobolev inequality, we get

$$|\psi|_0 \leqslant |x_{s0} - x_0| + C|\mathbf{v}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)}, \tag{6.7.42}$$

$$|\dot{\psi}|_0 \leqslant C|\mathbf{v}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)}, \tag{6.7.43}$$

$$|\ddot{\psi}|_0 \leqslant C|\mathbf{v}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)}. \tag{6.7.44}$$

From (6.7.33), (6.7.40) and (6.7.41), we can compute directly using (6.7.25), (6.7.27)–(6.7.28) and (6.7.30)–(6.7.32) that

$$\begin{aligned} \int_0^{x_0} z_1^2 dx &\leqslant \left(\left|(\partial_x\beta_1(t,\cdot))^{-1}\right|_0 + 2k_1^2\left|(\partial_x\gamma_1(t,\cdot))^{-1}\right|_0\right)\int_0^{x_0}(z_1^0(x))^2dx + 2b_1^2x_0|\psi|_0^2, \\ &\leqslant C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |x_{s0} - x_0|^2 + |\mathbf{v}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}), \end{aligned} \tag{6.7.45}$$

$$\begin{aligned} \int_0^{x_0} z_{1x}^2 dx &\leqslant \left(|\partial_x\beta_1(t,\cdot)|_0 + 2k_1^2|\partial_x\gamma_1(t,\cdot)|_0\right)\int_0^{x_0}(z_{1x}^0(x))^2dx + 2x_0b_1^2|\dot{\psi}|_0^2|\partial_x\alpha_1(t,\cdot)|_0^2 \\ &\leqslant C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\mathbf{v}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}), \end{aligned} \tag{6.7.46}$$

$$\begin{aligned} \int_0^{x_0} z_{1xx}^2 dx &\leqslant \left(2|\partial_x\beta_1(t,\cdot)|_0^3 + 4k_1^2|\partial_x\gamma_1(t,\cdot)|_0^3\right)\int_0^{x_0}(z_{1xx}^0)^2dx \\ &\quad + 2\left|z_{1x}^0\right|_0^2\int_0^{x_0}|\partial_{xx}\beta_1(t,x)|^2dx + 4k_1^2\left|z_{1x}^0\right|_0^2\int_0^{x_0}|\partial_{xx}\gamma_1(t,x)|^2dx \\ &\quad + 4b_1^2|\partial_x\alpha_1(t,x)|_0^4\int_0^{x_0}|\ddot{\psi}(\alpha_1(t,x))|^2dx + 4b_1^2|\dot{\psi}|_0^2\int_0^{x_0}|\partial_{xx}\alpha_1(t,x)|^2dx \\ &\leqslant C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\mathbf{v}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}). \end{aligned} \tag{6.7.47}$$

Combining (6.7.45)–(6.7.47), we obtain

$$|z_1(t,\cdot)|_{H^2((0,x_0);\mathbb{R})} \leqslant C(|z_1^0|_{H^2((0,x_0);\mathbb{R})} + |x_{s0} - x_0| + |\mathbf{v}(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2)}), \tag{6.7.48}$$

Similarly, one can get

$$|z_2(t,\cdot)|_{H^2((0,x_0);\mathbb{R})} \leqslant C(|z_2^0|_{H^2((0,x_0);\mathbb{R})} + |x_{s0} - x_0| + |\mathbf{v}(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2)}). \tag{6.7.49}$$

Noticing from $\mathbf{v} \in C_\nu(\mathbf{z}^0, x_{s0})$ that
$$|\mathbf{v}(\cdot, x_0)|_{H^2((0,T);\mathbb{R}^2)} \leqslant \nu^2,$$
thus by selecting δ and $\nu \in (0, \nu_0]$ sufficiently small, in addition to the previous hypothesis on δ, we have indeed
$$|\mathbf{z}(t, \cdot)|_{H^2((0,x_0);\mathbb{R}^2)} \leqslant \nu, \quad \text{a.e.} \quad t \in (0, T), \tag{6.7.50}$$
which proves (6.7.2). The same method as to prove (6.7.50) enables us to show that z_1 verifies also (6.7.3) and (6.7.4). One only has to realize that
$$z_{1t}(t, x) = \begin{cases} k_1 z_{1x}^0(\gamma_1)\partial_t \gamma_1 + b_1 \dot{\psi}(\alpha_1)\partial_t \alpha_1, & 0 < x < \xi_1(t; 0, 0), \\ z_{1x}^0(\beta_1)\partial_t \beta_1, & \xi_1(t; 0, 0) < x < x_0. \end{cases}$$
$$z_{1tt}(t, x) = \begin{cases} k_1 z_{1x}^0(\gamma_1)\partial_{tt}\gamma + k_1 z_{1xx}^0(\gamma_1)(\partial_t\gamma_1)^2 + b_1\ddot{\psi}(\alpha_1)(\partial_t\alpha_1)^2 + b_1\dot{\psi}(\alpha_1)\partial_{tt}\alpha_1, & 0 < x < \xi_1(t; 0, 0), \\ z_{1x}^0(\beta_1)\partial_{tt}\beta_1 + z_{1xx}^0(\beta_1)(\partial_t\beta_1)^2, & \xi_1(t; 0, 0) < x < x_0. \end{cases}$$
$$z_{1tx}(t, x) = \begin{cases} k_1 z_{1x}^0(\gamma_1)\partial_x(\partial_t\gamma_1) + k_1 z_{1xx}^0(\gamma_1)(\partial_x\gamma_1\partial_t\gamma_1) & \\ \quad + b_1\ddot{\psi}(\alpha_1)(\partial_t\alpha_1\partial_x\alpha_1) + b_1\dot{\psi}(\alpha_1)\partial_x(\partial_t\alpha_1), & 0 < x < \xi_1(t; 0, 0), \\ z_{1x}^0(\beta_1)\partial_x(\partial_t\beta_1) + z_{1xx}^0(\beta_1)(\partial_x\beta_1\partial_t\beta_1), & \xi_1(t; 0, 0) < x < x_0, \end{cases}$$
and to estimate $\int_0^{\xi_1(t;0,0)} |\partial_{tt}\alpha_1|^2 dx$, $\int_{\xi_1(t;0,0)}^{x_0} |\partial_{tt}\beta_1|^2 dx$, $\int_0^{\xi_1(t;0,0)} |\partial_{tt}\gamma_1|^2 dx$, $\int_0^{\xi_1(t;0,0)} |\partial_x(\partial_t\alpha_1)|^2 dx$, $\int_{\xi_1(t,0,0)}^{x_0} |\partial_x(\partial_t\beta_1)|^2 dx$ and $\int_0^{\xi_1(t;0,0)} |\partial_x(\partial_t\gamma_1)|^2 dx$ similarly as in (6.7.30)–(6.7.32) using the fact that $\mathbf{v}$ belongs to $L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)) \cap W^{1,\infty}((0,T); H^1((0,x_0);\mathbb{R}^2)) \cap W^{2,\infty}((0,T); L^2((0,x_0);\mathbb{R}^2))$ with bound ν in these norms.
We can clearly perform similar estimates for z_2. Consequently there exist δ and $\nu_1 \in (0, \nu_0]$ sufficiently small depending only on C such that, for any $\nu \in (0, \nu_1]$, $\mathbf{z} = \mathcal{F}(\mathbf{v})$ verifies properties (6.7.2)–(6.7.7) and therefore $\mathcal{F}(C_\nu(\mathbf{z}^0, x_{s0})) \subset C_\nu(\mathbf{z}^0, x_{s0})$.

2) $\mathcal{F}$ is a contraction mapping.
Next, we prove that $\mathcal{F}$ is a contraction mapping satisfying the following inequality:
$$\begin{aligned} &|\mathcal{F}(\mathbf{v}) - \mathcal{F}(\bar{\mathbf{v}})|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + M|\mathcal{F}(\mathbf{v})(\cdot, x_0) - \mathcal{F}(\bar{\mathbf{v}})(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)} \\ \leqslant &\frac{1}{2}|\mathbf{v} - \bar{\mathbf{v}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + \frac{M}{2}|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)}, \end{aligned} \tag{6.7.51}$$
where $M > 0$ is a constant. We start with z_1, and with the estimate of $|z_1 - \bar{z}_1|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}$. For any chosen $\mathbf{v}$ and $\bar{\mathbf{v}}$ from $C_\nu(\mathbf{z}^0, x_{s0})$, without loss of generality, we may assume that $\xi_1(t; 0, 0) < \bar{\xi}_1(t; 0, 0)$, where $\bar{\xi}_1$ is the characteristic defined in (6.7.19) associated to $\bar{\mathbf{v}}$. From (6.7.33), we have
$$\begin{aligned} &\int_0^{x_0} |z_1(t, x) - \bar{z}_1(t, x)|^2 \, dx \\ = &\int_0^{\xi_1(t;0,0)} |k_1 z_1^0(\gamma_1) - k_1 z_1^0(\bar{\gamma}_1) + b_1\psi(\alpha_1) - b_1\bar{\psi}(\bar{\alpha}_1)|^2 \, dx \\ &+ \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2 \, dx + \int_{\bar{\xi}_1(t;0,0)}^{x_0} |z_1^0(\beta_1) - z_1^0(\bar{\beta}_1)|^2 \, dx. \end{aligned} \tag{6.7.52}$$
From the definition of ψ in (6.7.17) and (6.7.8), using Sobolev and Cauchy-Schwarz inequalities, we have
$$\begin{aligned} &\int_0^{\xi_1(t;0,0)} |b_1\psi(\alpha_1) - b_1\bar{\psi}(\bar{\alpha}_1)|^2 \, dx \\ = &\int_0^{\xi_1(t;0,0)} b_1^2 \left| \int_0^{\alpha_1} \frac{v_1(s, x_0) + v_2(s, x_0)}{2} \, ds - \int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s, x_0) + \bar{v}_2(s, x_0)}{2} \, ds \right|^2 dx \\ \leqslant &C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C|\bar{\mathbf{v}}(\cdot, x_0)|^2_{H^2((0,T);\mathbb{R}^2)} \int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2 \, dx. \end{aligned} \tag{6.7.53}$$

By the definition of γ_1 in (6.7.22) and the corresponding definition of $\bar{\gamma}_1$ and using (6.7.24), we obtain

$$\int_0^{\xi_1(t;0,0)} |k_1 z_1^0(\gamma_1) - k_1 z_1^0(\bar{\gamma}_1)|^2 \, dx \leqslant C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2 \, dx. \tag{6.7.54}$$

Combining (6.7.52)–(6.7.54), we get

$$\begin{aligned}
&\int_0^{x_0} |z_1(t,x) - \bar{z}_1(t,x)|^2 \, dx \\
\leqslant & C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})} + |\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)}) \int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2 \, dx \\
& + |z_1^0|^2_{H^2((0,x_0);\mathbb{R})} \int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2 \, dx + \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \bar{\psi}(\bar{\alpha}_1))|^2 \, dx \\
& + C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}.
\end{aligned} \tag{6.7.55}$$

We estimate each term in (6.7.55) separately. By the definition of β_1 in (6.7.20) and the corresponding definition of $\bar{\beta}_1$, we have

$$\int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2 \, dx = \int_{\bar{\xi}_1(t;0,0)}^{x_0} |\xi_1(0;t,x) - \bar{\xi}_1(0;t,x)|^2 \, dx. \tag{6.7.56}$$

Now, let us estimate $|\xi_1(0;t,x) - \bar{\xi}_1(0;t,x)|$. From the definition of x_s in (6.7.8) and the definitions of ξ_1 and $\bar{\xi}_1$, see (6.7.19), we get for any $s \in [0,t]$ that

$$\begin{aligned}
&|\xi_1(s;t,x) - \bar{\xi}_1(s;t,x)| \\
=& \Big| \int_s^t f_1(\theta, \xi_1(\theta;t,x)) \, d\theta - \int_s^t \bar{f}_1(\theta, \bar{\xi}_1(\theta;t,x)) \, d\theta \Big| \\
\leqslant & \int_s^t \Bigg(\left| \left(1 + v_1(\theta,\xi_1) - \xi_1 \frac{v_1(\theta,x_0) + v_2(\theta,x_0)}{2x_0} \right) \frac{x_0}{x_s(\mathbf{v}(\cdot,x_0))(\theta) x_s(\bar{\mathbf{v}}(\cdot,x_0))(\theta)} \right| \\
& \qquad \cdot \int_0^\theta \left| \frac{v_1(\alpha,x_0) - \bar{v}_1(\alpha,x_0) + v_2(\alpha,x_0) - \bar{v}_2(\alpha,x_0)}{2} \right| d\alpha \Bigg) \, d\theta \\
& + \int_s^t \left| \frac{x_0}{x_s(\bar{\mathbf{v}}(\cdot,x_0))(\theta)} \right| \cdot \left| v_1(\theta,\xi_1) - \bar{v}_1(\theta,\bar{\xi}_1) + \bar{\xi}_1 \frac{\bar{v}_1(\theta,x_0) + \bar{v}_2(\theta,x_0)}{2x_0} - \xi_1 \frac{v_1(\theta,x_0) + v_2(\theta,x_0)}{2x_0} \right| d\theta \\
\leqslant & C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)} + C\nu \int_s^t |\xi_1(\theta;t,x) - \bar{\xi}_1(\theta;t,x)| \, d\theta \\
& + C \int_s^t |v_1(\theta, \bar{\xi}_1(\theta;t,x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta;t,x))| d\theta.
\end{aligned} \tag{6.7.57}$$

From (6.7.57), we get for $\nu \in (0, \nu_0]$ sufficiently small and for $\bar{\xi}_1(t;0,0) < x \leqslant x_0$ that

$$\begin{aligned}
|\xi_1(\cdot;t,x) - \bar{\xi}_1(\cdot;t,x)|_{C^0([0,t];\mathbb{R})} \leqslant & C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)} \\
& + C \int_0^t |v_1(\theta, \bar{\xi}_1(\theta;t,x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta;t,x))| \, d\theta.
\end{aligned} \tag{6.7.58}$$

Thus, from (6.7.56) and (6.7.58) we have

$$\begin{aligned}
\int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2 \, dx \leqslant & C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\
& + C \int_{\bar{\xi}_1(t;0,0)}^{x_0} \left(\int_0^t |v_1(\theta, \bar{\xi}_1(\theta;t,x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta;t,x))| \, d\theta \right)^2 dx \\
\leqslant & C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_0^t \int_{\bar{\xi}_1(t;0,0)}^{x_0} |v_1(\theta,\bar{\xi}_1(\theta;t,x)) - \bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x))|^2\,dx\,d\theta \\
\leqslant & C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} \\
&+ C|v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}.
\end{aligned}
\tag{6.7.59}
$$

The last inequality is obtained using the change of variable $y = \bar{\xi}_1(\theta;t,x)$, well-defined for $0 \leqslant \theta \leqslant t \leqslant T$ and $\bar{\xi}_1(t;0,0) < x \leqslant x_0$. Let us now estimate $|\alpha_1 - \bar{\alpha}_1|_{L^2((0,\xi_1(t;0,0));\mathbb{R})}$. Without loss of generality, we may assume that $\alpha_1 \leqslant \bar{\alpha}_1$. By definition of α_1 in (6.7.21) and the corresponding definition of $\bar{\alpha}_1$, we have

$$
\int_{\alpha_1}^t f_1(s,\xi_1(s;t,x))ds = x = \int_{\bar{\alpha}_1}^t \bar{f}_1(s,\bar{\xi}_1(s;t,x))ds. \tag{6.7.60}
$$

Hence, similarly to (6.7.57), we get

$$
\begin{aligned}
|\alpha_1 - \bar{\alpha}_1| \leqslant & \frac{1}{\inf\limits_{(t,x)\in[0,T]\times[0,x_0]}|f_1(t,x)|}\int_{\bar{\alpha}_1}^t \left|f_1(s,\xi_1(s;t,x)) - \bar{f}_1(s,\bar{\xi}_1(s;t,x))\right|ds \\
\leqslant & C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)} + C\nu\int_{\bar{\alpha}_1}^t |\xi_1(\theta;t,x) - \bar{\xi}_1(\theta;t,x)|\,d\theta \\
&+ C\int_{\bar{\alpha}_1}^t |v_1(\theta,\bar{\xi}_1(\theta;t,x)) - \bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x))|d\theta.
\end{aligned}
\tag{6.7.61}
$$

Similarly to the proof of (6.7.58), for $\nu \in (0,\nu_0]$ sufficiently small, we can obtain that (note that $\xi_1(s;t,x)$ and $\bar{\xi}_1(s;t,x)$ for any $s \in [\bar{\alpha}_1,t]$ are well defined as we assume that $\alpha_1 \leqslant \bar{\alpha}_1$)

$$
\begin{aligned}
|\xi_1(\cdot;t,x) - \bar{\xi}_1(\cdot;t,x)|_{C^0([\bar{\alpha}_1,t];\mathbb{R})} \leqslant & C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)} \\
&+ C\int_{\bar{\alpha}}^t |v_1(\theta,\bar{\xi}_1(\theta;t,x)) - \bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x))|\,d\theta.
\end{aligned}
\tag{6.7.62}
$$

Using this inequality in (6.7.61) and performing similarly as in (6.7.59), we can obtain

$$
\int_0^{\xi_1(t;0,0)} |\alpha_1 - \bar{\alpha}_1|^2\,dx \leqslant C|\mathbf{v}(\cdot,x_0) - \bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C|v_1 - \bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \tag{6.7.63}
$$

Let us now focus on the estimation of the term $\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\,dx$ in (6.7.55). Using the compatibility condition (6.3.7), we have

$$
\begin{aligned}
&\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\,dx \\
=&\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - z_1^0(0) + z_1^0(0) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\,dx \\
=&\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1) - z_1^0(0) + k_1 z_1^0(x_0) + b_1(x_0 - x_{s0}) - (k_1 z_1^0(\bar{\gamma}_1) + b_1\bar{\psi}(\bar{\alpha}_1))|^2\,dx \\
\leqslant & C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\beta_1|^2\,dx + C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |x_0 - \bar{\gamma}_1|^2\,dx \\
&+ C\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} \left|\int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s,x_0) + \bar{v}_2(s,x_0)}{2}\,ds\right|^2\,dx.
\end{aligned}
\tag{6.7.64}
$$

We first estimate $\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\beta_1|^2\,dx$. As $\xi_1(s;t,x)$ is increasing with respect to $s \in [0,t]$, we have

$$
|\beta_1| < |\xi_1(\bar{\alpha}_1;t,x)| = |\xi_1(\bar{\alpha}_1;t,x) - \bar{\xi}_1(\bar{\alpha}_1;t,x)| \leqslant |\xi_1(\cdot;t,x) - \bar{\xi}_1(\cdot;t,x)|_{C^0([\bar{\alpha}_1,t];\mathbb{R})}, \tag{6.7.65}
$$

then by (6.7.62) and performing the same proof as in (6.7.59), we get

$$\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\beta_1|^2\,dx \leqslant C|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)} + C|v_1-\bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \tag{6.7.66}$$

Let us now look at the second term in (6.7.64), from (6.7.24) and the definition of $\bar{\gamma}_1$, we have

$$\begin{aligned}\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |x_0-\bar{\gamma}_1|^2\,dx &= \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\xi}_1(0;0,x_0)-\bar{\xi}_1(0;\bar{\alpha}_1,x_0)|^2\,dx\\ &\leqslant |\partial_t\bar{\xi}_1|_0^2 \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\,dx \leqslant C\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\,dx. \end{aligned}\tag{6.7.67}$$

It is easy to deal with the last term in (6.7.64), one has

$$\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} \left|\int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s,x_0)+\bar{v}_2(s,x_0)}{2}\,ds\right|^2 dx \leqslant C|\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)} \int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\,dx. \tag{6.7.68}$$

Thus, we only have to estimate $\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\,dx$. Noticing that for any fixed (t,x), the characteristic $\bar{\xi}_1(s;t,x)$ is increasing with respect to $s\in[\bar{\alpha}_1,t]$ and that $\xi_1^{-1}(\cdot;t,x)(\beta_1)=0$, we obtain

$$\bar{\alpha}_1 < \bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1) - \xi_1^{-1}(\cdot;t,x)(\beta_1).$$

Moreover,

$$\begin{aligned}\beta_1 &= x + \int_t^{\xi_1^{-1}(\cdot;t,x)(\beta_1)} f_1(s;\xi_1(s;t,x))\,d\theta,\\ \beta_1 &= x + \int_t^{\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1)} \bar{f}_1(s;\bar{\xi}_1(s;t,x))\,d\theta.\end{aligned}$$

Then similarly as for (6.7.61), we can prove that

$$\begin{aligned}\left|\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1)-\xi_1^{-1}(\cdot;t,x)(\beta_1)\right| \leqslant& C|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|_{L^2(0,T)}\\ &+ C\nu\int_{\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1)}^t |\xi_1(\theta;t,x)-\bar{\xi}_1(\theta;t,x)|\,d\theta\\ &+ C\int_{\bar{\xi}_1^{-1}(\cdot;t,x)(\beta_1)}^t |v_1(\theta,\bar{\xi}_1(\theta;t,x))-\bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x))|d\theta.\end{aligned}$$

Thus, similarly as in the proof for (6.7.63), we get

$$\begin{aligned}\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |\bar{\alpha}_1|^2\,dx \leqslant& C|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}\\ &+ C|v_1-\bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}.\end{aligned}\tag{6.7.69}$$

Finally, using estimations (6.7.66) and (6.7.67)–(6.7.69), (6.7.64) becomes

$$\begin{aligned}&\int_{\xi_1(t;0,0)}^{\bar{\xi}_1(t;0,0)} |z_1^0(\beta_1)-(k_1z_1^0(\bar{\gamma}_1)+b_1\psi(\bar{\alpha}_1))|^2\,dx\\ \leqslant& C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}+|\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)})\Big(|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}\\ &\qquad + |v_1-\bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}\Big).\end{aligned}\tag{6.7.70}$$

Combining (6.7.55), (6.7.59), (6.7.63) and (6.7.70), we get

$$\begin{aligned}
&|z_1-\bar{z}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}\\
&\leqslant C(|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}+|\bar{\mathbf{v}}(\cdot,x_0)|^2_{H^2((0,T);\mathbb{R}^2)})\left(|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}+|v_1-\bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}\right)\\
&\qquad\qquad+C|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}. \qquad (6.7.71)
\end{aligned}$$

We are left with estimating $|\mathbf{z}(\cdot,x_0)-\bar{\mathbf{z}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}$ in order to obtain (6.7.51). Here we give the estimation for z_1. Using (6.7.58), we get

$$\begin{aligned}
&\int_0^T|z_1(t,x_0)-\bar{z}_1(t,x_0)|^2\,dt\\
=&\int_0^T|z_1^0(\xi_1(0;t,x_0))-z_1^0(\bar{\xi}_1(0;t,x_0))|^2\\
\leqslant&|z_{1x}^0|_0^2\int_0^T|\xi_1(0;t,x_0)-\bar{\xi}_1(0;t,x_0)|^2\,dt\\
\leqslant&C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}\\
&+C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}\int_0^T\int_0^t|v_1(\theta,\bar{\xi}_1(\theta;t,x_0))-\bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x_0))|^2\,d\theta\,dt\\
\leqslant&C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}\\
&+C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}\int_0^T\int_\theta^T|v_1(\theta,\bar{\xi}_1(\theta;t,x_0))-\bar{v}_1(\theta,\bar{\xi}_1(\theta;t,x_0))|^2\,dt\,d\theta\\
\leqslant&C|z_1^0|^2_{H^2((0,x_0);\mathbb{R})}\left(|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|^2_{L^2((0,T);\mathbb{R}^2)}+|v_1-\bar{v}_1|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}\right). \qquad (6.7.72)
\end{aligned}$$

The last inequality is obtained by changing the variable $y=\bar{\xi}_1(\theta;t,x_0)$. Similar estimates can be done for z_2. Hence, from (6.7.71) and (6.7.72), there exists $M>0$ such that for δ sufficiently small and $\nu\in(0,\nu_2]$, where $\nu_2\in(0,\nu_1]$ is sufficiently small and depends only on C, we have

$$\begin{aligned}
&|\mathbf{z}-\bar{\mathbf{z}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))}+M|\mathbf{z}(\cdot,x_0)-\bar{\mathbf{z}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}\\
\leqslant&\frac{1}{2}|\mathbf{v}-\bar{\mathbf{v}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))}+\frac{M}{2}|\mathbf{v}(\cdot,x_0)-\bar{\mathbf{v}}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}.
\end{aligned} \qquad (6.7.73)$$

Hence $\mathcal{F}$ is a contraction mapping and has a fixed point $\mathbf{z}\in C_\nu(\mathbf{z}^0,x_{s0})$, i.e., there exists a unique solution $\mathbf{z}\in C_\nu(\mathbf{z}^0,x_{s0})$ to the system (6.3.3)–(6.3.5). Noticing (6.7.8), we get that $x_s\in C^1([0,T];\mathbb{R})$. To get the extra regularity $\mathbf{z}\in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$, we adapt the proof given by Majda [152, p.44-46]. There, the author used energy estimates method for an initial value problem. Using this method for our boundary value problem, we have to be careful with the boundary terms when integrating by parts. Substituting $\mathbf{v}$ by $\mathbf{z}$ in $\psi(t)$ and $f_1(t,x)$ in the expression of z_{1x}, z_{1xx} in (6.7.40) and (6.7.41), noticing (6.7.2)–(6.7.4) and computing similar estimates as in (6.7.46) and (6.7.47), we can obtain the "hidden" regularity $\mathbf{z}_x(\cdot,x_0)\in L^2((0,T);\mathbb{R}^2)$ and $\mathbf{z}_{xx}(\cdot,x_0)\in L^2((0,T);\mathbb{R}^2)$ together with estimates on $|\mathbf{z}_x(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}$ and $|\mathbf{z}_{xx}(\cdot,x_0)|_{L^2((0,T);\mathbb{R}^2)}$, which are sufficient to take care of the boundary terms when integrating by parts. This concludes the proof of the existence and uniqueness of a classical solution $x_s(t)\in C^1([0,T];\mathbb{R})$ and $\mathbf{z}\in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$ in $C_\nu(\mathbf{z}^0,x_{s0})$ to the system (6.3.3)–(6.3.5) for T satisfying (6.7.1).
The estimate (6.3.10) for $|\mathbf{z}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ part can be obtained from estimates (6.7.48)–(6.7.49) by firstly replacing $\mathbf{v}$ with $\mathbf{z}$ and then applying (6.7.37)–(6.7.38). Noticing the definition of x_s in (6.7.8) and applying (6.7.37)–(6.7.38) again, the estimate for the $|x_s(t)-x_0|$ part follows.
Next, we show the uniqueness of the solution in $C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$. Suppose that there is another solution $\tilde{\mathbf{z}}\in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$, we prove that $\tilde{\mathbf{z}}\in C_\nu(\mathbf{z}^0,x_{s0})$, for δ sufficiently small. To that end, assume that $\mathbf{z}(t,\cdot)=\tilde{\mathbf{z}}(t,\cdot)$ for any $t\in[0,\tau]$ with $\tau\in[0,T]$. If $\tau\neq T$, by (6.3.10), for δ sufficiently small and as $\tilde{\mathbf{z}}\in C^0([0,T];H^2((0,x_0);\mathbb{R}^2))$, one can choose $\tau'\in(\tau,T)$ small enough such that $\tilde{\mathbf{z}}\in C_\nu(\mathbf{z}(\tau),x_s(\tau))$ with T is replaced by $\tau'-\tau$ and by considering τ as the new initial time. Thus, $\mathbf{z}(t,\cdot)=\tilde{\mathbf{z}}(t,\cdot)$ for any $t\in[0,\tau']$.

As $|\tilde{\mathbf{z}}(t,\cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ is uniformly continuous on $[0,T]$, and as, moreover C and ν do not depend on T, we can repeat this process and finally get $\mathbf{z}(t,\cdot) = \tilde{\mathbf{z}}(t,\cdot)$ on $[0,T]$.
For general $T>0$, one just needs to take T_1 satisfying (6.7.1) and, noticing that C and ν do not depend on T_1, one can apply the above procedure at most $[T/T_1]+1$ times. This concludes the proof of Lemma 6.3.1. $\square$

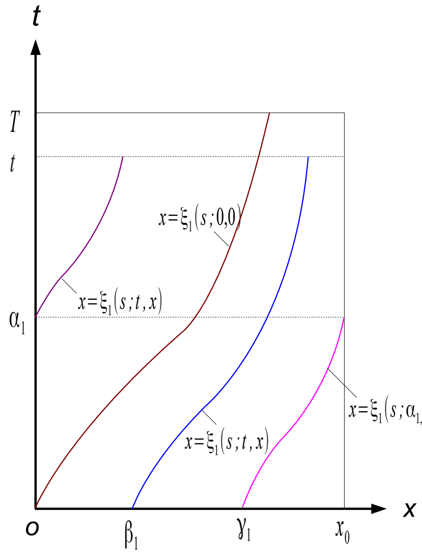

Figure 6.2 – Demonstration of the characteristics.

Proof of Lemma 6.7.1. From (6.7.19), we have

$$\begin{cases} \dfrac{\partial^2 \xi_1(s;t,x)}{\partial s \partial x} = f_{1x} \dfrac{\partial \xi_1(s;t,x)}{\partial x}, \\ \dfrac{\partial \xi_1(t;t,x)}{\partial x} = 1, \end{cases} \tag{6.7.74}$$

and

$$\begin{cases} \dfrac{\partial^2 \xi_1(s;t,x)}{\partial s \partial t} = f_{1x} \dfrac{\partial \xi_1(s;t,x)}{\partial t}, \\ \dfrac{\partial \xi_1(s;s,x)}{\partial s} + \dfrac{\partial \xi_1(s;s,x)}{\partial t} = 0. \end{cases} \tag{6.7.75}$$

Thus,

$$\partial_x \xi_1(s;t,x) = e^{-\int_s^t f_{1x}(\theta,\xi_1(\theta;t,x))d\theta}, \tag{6.7.76}$$

$$\partial_t \xi_1(s;t,x) = -f_1(t,x) e^{-\int_s^t f_{1x}(\theta,\xi_1(\theta;t,x))d\theta}. \tag{6.7.77}$$

From (6.7.76)–(6.7.77) and noticing $\beta_1 = \xi_1(0;t,x)$, we have

$$\frac{\partial \beta_1}{\partial t} = -\, f_1(t,x) e^{-\int_0^t f_{1x}(\theta,\xi_1(\theta;t,x))d\theta}, \quad \frac{\partial \beta_1}{\partial x} = e^{-\int_0^t f_{1x}(\theta,\xi_1(\theta;t,x))\, d\theta}. \tag{6.7.78}$$

From (6.7.76), noticing $\xi_1(\alpha_1; t, x) = 0$ and by chain rules, we have

$$\frac{\partial \alpha_1}{\partial x} = -\frac{1}{f_1(\alpha_1, 0)} e^{-\int_{\alpha_1}^t f_{1x}(s, \xi_1(s;t,x))\, ds}, \tag{6.7.79}$$

and as $\gamma_1 = \xi_1(0; \alpha_1, x_0)$, we obtain from (6.7.77) that

$$\frac{\partial \gamma_1}{\partial x} = \frac{d\gamma_1}{d\alpha_1}\frac{\partial \alpha_1}{\partial x} = \frac{f_1(\alpha_1, x_0)}{f_1(\alpha_1, 0)} e^{-\int_0^{\alpha_1} f_{1x}(s,\xi_1(s;\alpha_1,x_0))\, ds - \int_{\alpha_1}^t f_{1x}(s,\xi_1(s;t,x))\, ds}. \tag{6.7.80}$$

Observe that for a.e. $s \in (0, T)$ and $x \in [0, x_0]$,

$$|v_1(s, x)| \leqslant \left| \int_\theta^x v_{1x}(s, l) dl \right| + |v_1(s, \theta)|, \ \forall \ \theta \in [0, x_0] \tag{6.7.81}$$

and as v_1 is H^1 in x and its L^2-norm is bounded by ν, there exists θ such that $|v_1(s, \theta)| \leqslant \nu/\sqrt{x_0}$, therefore

$$|v_1(s, x)| \leqslant C\nu, \tag{6.7.82}$$

and similarly as v_1 is H^2 in x with the same bound and v_{1t} is in $L^\infty((0, T); H^1((0, x_0); \mathbb{R}))$ with bound ν from (6.7.3)

$$\begin{aligned} x \in [0, x_0], \ \ |v_{1x}(s, x)| \leqslant & C\nu, \text{ for a.e. } s \in (0, T), \\ x \in [0, x_0], \ \ |v_{1t}(s, x)| \leqslant & C\nu, \text{ for a.e. } s \in (0, T). \end{aligned} \tag{6.7.83}$$

From the expression of f_1 defined in (6.7.18) and using (6.7.76)–(6.7.80) and (6.7.82)–(6.7.83), after some direct computations, estimates (6.7.23)–(6.7.28) can be obtained. We now demonstrate the estimate (6.7.29) in details, while (6.7.30)–(6.7.32) can be treated in a similar way, thus we omit them. From (6.7.78), we have

$$\partial_{tt}\beta_1 \ = \ \Big(- f_{1t}(t, x) + f_1(t, x)\Big(f_{1x}(t, x) + \int_0^t f_{1xx}(\theta, \xi_1(\theta; t, x))\partial_t \xi_1(\theta; t, x)\, d\theta\Big)\Big) e^{-\int_0^t f_{1x}(\theta,\xi_1(\theta;t,x))\, d\theta}.$$

Looking at (6.7.18), as $\mathbf{v}$ is only in $L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)) \cap W^{1,\infty}((0, T); H^1((0, x_0); \mathbb{R}^2))$ $\cap\, W^{2,\infty}((0, T); L^2((0, x_0); \mathbb{R}^2))$, this equation is expressed a priori formally in the distribution sense. Thus, we have to be careful when we estimate (6.7.29). By (6.7.18) and using estimates (6.7.23), (6.7.24), we get by Cauchy-Schwarz inequality together with the change of variable $y = \xi_1(\theta; t, x_0)$ that

$$\begin{aligned} \int_0^T |\partial_{tt}\beta_1(t, x_0)|^2\, dt &\leqslant C\nu + C \int_0^T \left| \int_0^t f_{1xx}(\theta, \xi_1(\theta; t, x_0))\partial_t \xi_1(\theta; t, x_0)\, d\theta \right|^2 dt \\ &\leqslant C\nu + C \int_0^T \int_0^t v_{1xx}^2(\theta, \xi_1(\theta; t, x_0))\partial_t^2 \xi_1(\theta; t, x_0)\, d\theta dt \\ &= C\nu + C \int_0^T \int_\theta^T v_{1xx}^2(\theta, \xi_1(\theta; t, x_0))\partial_t^2 \xi_1(\theta; t, x_0)\, dt d\theta \\ &\leqslant C\nu + C \int_0^T \int_0^{x_0} v_{1xx}^2(\theta, y)\, dy d\theta \\ &\leqslant C\nu. \end{aligned} \tag{6.7.84}$$

□

6.7.2 Proof of Theorem 6.5.1

First observe that, after the change of variables (6.3.1), (6.3.2), the new equations are

$$
\begin{aligned}
&z_{1t} + \left(f'(1) + (f'(z_1+1) - f'(1)) - x\frac{\dot{x}_s}{x_0}\right) z_{1x}\frac{x_0}{x_s} = 0,\\
&z_{2t} + \left(-f'(-1) + (f'(-1) - f'(z_2-1)) + x\frac{\dot{x}_s}{x_0}\right) z_{2x}\frac{x_0}{L-x_s} = 0,\\
&\dot{x}_s(t) = \frac{f'(1)z_1(t,x_0) - f'(-1)z_2(t,x_0)}{2 + (z_1(t,x_0) - z_2(t,x_0))}\\
&\quad + \frac{(f(z_1(t,x_0)+1) - f'(1)z_1(t,x_0) - f(1)) - (f(z_2(t,x_0)-1) - f'(-1)z_2(t,x_0) - f(-1))}{2 + (z_1(t,x_0) - z_2(t,x_0))}
\end{aligned}
\tag{6.7.85}
$$

and the boundary conditions remain given by (6.3.4). Note that in (6.7.85) the expression of $\dot{x}_s$ can actually be written as

$$
\dot{x}_s(t) = \frac{f'(1)z_1(t,x_0) - f'(-1)z_2(t,x_0)}{2} + O\left(|\mathbf{z}(t,x_0)|^2\right). \tag{6.7.86}
$$

Thus, to prove Theorem 6.5.1, if suffices to show Theorem 6.4.1 with (6.7.85) instead of (6.3.3). We still define the Lyapunov function candidate as previously by (6.4.3)–(6.4.9). Then Lemma 6.4.1 and Lemma 6.4.2 remain unchanged. To adapt Lemma 6.4.3, one can check that, when differentiating V_1, V_2 and V_3 along the C^3 solutions of (6.7.85), (6.3.4) with associated initial conditions and noticing that under assumption $f(-1) = f(1)$, one has $f'(-1) \leqslant 0, f'(1) \geqslant 0$ from the property of convex function, we obtain as previously (6.4.40), (6.4.42) and (6.4.46) but with $f'(1)p_1$ instead of p_1 and $|f'(-1)|p_2$ instead of p_2 in the boundary terms and μV_i being replaced by $\mu\min(f'(1), |f'(-1)|)V_i$. Then, from (6.5.3) and dealing with V_4, we finally get:

$$
\begin{aligned}
\frac{dV_1}{dt} + \frac{dV_4}{dt} \leqslant & -\mu(V_1+V_4)\\
&+ \max\left\{\frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2}\right\} V_1\\
&+ \left[\frac{x_0}{x_s}p_1(k_1^2 - e^{-\frac{\mu x_0}{\eta_1}})f'(1) + \frac{\varepsilon_1+\varepsilon_2}{8}f'(1)^2\right] z_1^2(x_0)\\
&+ \left[\frac{x_0}{L-x_s}p_2(k_2^2 - e^{-\frac{\mu x_0}{\eta_2}})|f'(-1)| + \frac{\varepsilon_1+\varepsilon_2}{8}|f'(-1)|^2\right] z_2^2(x_0)\\
&+ f'(1)\left[-2\frac{x_0}{x_s}p_1b_1k_1 - \frac{x_0}{x_s}\bar{p}_1(e^{-\frac{\mu x_0}{\eta_1}} - k_1) + \kappa\right] z_1(x_0)(x_s - x_0)\\
&+ |f'(-1)|\left[-2\frac{x_0}{L-x_s}p_2b_2k_2 - \frac{x_0}{L-x_s}\bar{p}_2(e^{-\frac{\mu x_0}{\eta_2}} - k_2) + \kappa\right] z_2(x_0)(x_s - x_0)\\
&+ \left[\frac{x_0}{x_s}p_1b_1^2f'(1) + \frac{x_0}{L-x_s}p_2b_2^2|f'(-1)| - \frac{x_0}{x_s}\bar{p}_1b_1f'(1) - \frac{x_0}{L-x_s}\bar{p}_2b_2|f'(-1)| + \mu\kappa\right](x_s - x_0)^2\\
&+ O\left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3\right),
\end{aligned}
\tag{6.7.87}
$$

and a similar expression for $V_2 + V_5$ and $V_3 + V_6$ as previously. Thus Lemma 6.4.3 still holds but with $\mathbf{A}$ now defined by

$$
a_{11} = p_1(e^{-\frac{\mu x_0}{\eta_1}} - k_1^2)f'(1) - \frac{\varepsilon_1+\varepsilon_2}{8}f'(1)^2, \tag{6.7.88}
$$

$$
a_{13} = a_{31} = f'(1)p_1b_1k_1 + f'(1)\frac{\bar{p}_1}{2}(e^{-\frac{\mu x_0}{\eta_1}} - k_1) - f'(1)\frac{\kappa}{2}, \tag{6.7.89}
$$

$$
a_{22} = \frac{x_0}{L-x_0}p_2(e^{-\frac{\mu x_0}{\eta_2}} - k_2^2)|f'(-1)| - \frac{\varepsilon_1+\varepsilon_2}{8}|f'(-1)|^2, \tag{6.7.90}
$$

$$a_{23} = a_{32} = |f'(-1)|\frac{x_0}{L-x_0}p_2b_2k_2 + |f'(-1)|\frac{x_0}{L-x_0}\frac{\bar{p}_2}{2}(e^{-\frac{\mu x_0}{\eta_2}} - k_2) - |f'(-1)|\frac{\kappa}{2}, \tag{6.7.91}$$

$$a_{33} = -p_1b_1^2f'(1) - \frac{x_0}{L-x_0}p_2b_2^2|f'(-1)| + \bar{p}_1b_1f'(1) + \frac{x_0}{L-x_0}\bar{p}_2b_2|f'(-1)| - \mu\kappa. \tag{6.7.92}$$

instead of (6.4.60)–(6.4.64). We can then choose p_1, p_2, $\bar{p}_1$, $\bar{p}_2$ as previously by (6.4.69)–(6.4.71) and $\mathbf{A}$ becomes again diagonal with the expression of its elements given by

$$a_{33} = \frac{\kappa}{2}f'(1)b_1e^{\frac{\mu x_0}{\eta_1}} + \frac{\kappa}{2}|f'(-1)|b_2e^{\frac{\mu x_0}{\eta_2}} - \mu\kappa \tag{6.7.93}$$

$$a_{11} = \frac{\kappa f'(1)}{2b_1}(1 - k_1^2e^{\frac{\mu x_0}{\eta_1}}) - \frac{\kappa f'(1)^2}{2\mu^2}\left[b_1(e^{\frac{\mu x_0}{\eta_1}} - 1) + b_2(e^{\frac{\mu x_0}{\eta_2}} - 1)\right], \tag{6.7.94}$$

$$a_{22} = \frac{\kappa|f'(-1)|}{2b_2}(1 - k_2^2e^{\frac{\mu x_0}{\eta_2}}) - \frac{\kappa|f'(-1)|^2}{2\mu^2}\left[b_1(e^{\frac{\mu x_0}{\eta_1}} - 1) + b_2(e^{\frac{\mu x_0}{\eta_2}} - 1)\right], \tag{6.7.95}$$

instead of (6.4.74), (6.4.77) and (6.4.78) respectively. Then to prove Theorem 6.4.1 with (6.7.85) instead of (6.3.3), we only need to show now that under assumption (6.5.4) there exists $\mu > \gamma$ and $\kappa > 1$ such that $a_{ii} > 0, i = 1, 2, 3$ and such that (6.4.16) holds where $\Theta_i, i = 1, 2$ are still defined by (6.4.17). But this can be checked exactly as in the proof of Theorem 6.4.1. With condition (6.5.3), one can now check as in Remark 6.2.4 that there always exist parameters b_i and k_i such that conditions (6.5.4) are satisfied.

Chapter 7

Boundary feedback stabilization of hydraulic jumps

This chapter is taken from the following article (also referred to as [19]):

Georges Bastin, Jean-Michel Coron, Amaury Hayat, and Peipei Shang. Boundary feedback stabilization of hydraulic jumps. IFAC J. Syst. Control, 7:100026, 10, 2019.

Abstract. In an open channel, a hydraulic jump is an abrupt transition between a torrential (supercritical) flow and a fluvial (subcritical) flow. In this chapter, hydraulic jumps are represented by discontinuous shock solutions of hyperbolic Saint-Venant equations. Using a Lyapunov approach, we prove that we can stabilize the state of the system in H^2-norm as well as the hydraulic jump location, with simple feedback boundary controls and an arbitrary decay rate, by appropriately choosing the gains of the feedback boundary controls.

7.1 Introduction and main result

Nonlinear hyperbolic equations are well-known to give rise to discontinuities in finite time that are physically meaningful. Hydraulic jump is one of the most known example. A hydraulic jump is a phenomenon that frequently occurs in open channel flow, such as rivers and spillways. It describes a transition between a torrential (or supercritical) regime and a fluvial (or subcritical) regime, i.e., an abrupt transition between a fast flow and a slow flow with a higher height. As a consequence, a part of the initial kinetic energy of the flow is converted into an increase in potential energy, while some energy is irreversibly lost through turbulence and heat. This phenomenon can be seen not only in rivers and spillways but also in air flows of the atmosphere. This is for instance believed to explain the phenomenon of "Morning Glory cloud" [44] and may be at the origin of some gliders' crashes [129]. Hydraulic jumps are important not only because they occur naturally but also because they are sometimes engineered on purpose and are very useful in hydraulic applications to dissipate energy in water and prevent in this way the erosion of the streambed or damages on hydraulic installations [104]. However, when studying the flow equations, the stabilization of hydraulic jumps is seldom considered and almost all the studies focus on the stabilization of the dynamics of the fluvial regime [12, 13, 16, 18, 52, 71, 98, 137]. In this chapter, we explicitly address the issue of the stabilization of a hydraulic jump represented by a discontinuous shock solution of the flow equations, switching from the torrential regime to the fluvial regime. In other words, the two eigenvalues of the hyperbolic system modeling the shallow water are both positive in the torrential regime and one of them changes sign and switches to a negative value in the fluvial regime. Our goal is to achieve the stability of the channel with a general class of local feedback controls at the boundary. Fundamentally, the stabilization of shock steady states for hyperbolic systems, while being very interesting, has rarely been studied. One can refer to [20] and [165] for the scalar case and to our knowledge, no such result exists for systems. By a Lyapunov approach we prove the exponential H^2-stability of the steady state, with an arbitrary decay rate and with an exact exponential stabilization of the desired location of the hydraulic jump.

We consider a channel with a rectangular cross section with constant width, which is taken to be 1 without loss of generality. We denote by $Q(t,x)$ the flux and $H(t,x)$ the water depth, where t and x are, respectively, the time and space independent variables as usual. As the channel has a finite length $L>0$, the spatial domain is bounded and noted $[0,L]$. The Saint-Venant model which, neglecting friction, consists in a continuity equation and an equilibrium of forces, is written as

$$\begin{aligned}&\partial_t H + \partial_x Q = 0,\\ &\partial_t Q + \partial_x\left(\frac{gH^2}{2}+\frac{Q^2}{H}\right) = 0.\end{aligned} \tag{7.1.1}$$

We are interested with solution trajectories $(H(t,x),Q(t,x))^{\intercal}$ that may have a jump discontinuity at some point $x_s(t)\in(0,L)$ and are classical otherwise. Thus, in order to close the system, we need a relationship between Q and H before and after this jump. From the Rankine-Hugoniot condition applied to (7.1.1), two quantities are conserved through the jump in the jump's referential: the flux Q and the momentum $gH^2/2+Q^2/H$. This gives the following relationships at the jump $x_s(t)$:

$$\begin{aligned}&[Q]_-^+ = \dot{x}_s[H]_-^+,\\ &[Q]_-^+\dot{x}_s = \left[\frac{Q^2}{H}+\frac{1}{2}gH^2\right]_-^+,\end{aligned} \tag{7.1.2}$$

where, as usual, $\dot{x}_s$ denotes the time derivative of x_s, i.e., the speed of the jump. These relationships can be reformulated as:

$$\dot{x}_s = \frac{[Q]_-^+}{[H]_-^+}, \tag{7.1.3}$$

$$\text{and}\quad ([Q]_-^+)^2 = [H]_-^+\left[\frac{Q^2}{H}+\frac{1}{2}gH^2\right]_-^+, \tag{7.1.4}$$

where we define for any bounded function f in a neighbourhood of x_s: $[f]_-^+ = f(x_s^+(t)) - f(x_s^-(t))$. This relation (7.1.4) can be regarded as the generalisation for non-stationary states of the well-known Bélanger equation (7.1.8) below.

Our goal is to stabilize the steady states of the system (7.1.1), (7.1.3) and (7.1.4) where a (single) hydraulic jump occurs, meaning that the flow switches from the torrential regime to the fluvial regime with a discontinuity in height. Therefore, such steady states $((H^*,Q^*)^{\intercal},x_s^*)$ satisfy the following conditions:

1. Q^* is constant and positive, $x_s^*\in(0,L)$ and

$$H^* = \begin{cases}H_1^*, & x\in[0,x_s^*),\\ H_2^*, & x\in(x_s^*,L],\end{cases} \tag{7.1.5}$$

   where H_1^*, H_2^* are positive constants.

2. The steady state flow is in the torrential regime before the jump and in the fluvial regime after the jump. This means that in the torrential regime the two system eigenvalues are positive,

$$\lambda_1 = \frac{Q^*}{H_1^*} - \sqrt{gH_1^*} > 0,\quad \lambda_2 = \frac{Q^*}{H_1^*} + \sqrt{gH_1^*} > 0,\quad \text{for } x\in[0,x_s^*), \tag{7.1.6}$$

   while there is one positive and one negative eigenvalue in the fluvial regime [13],

$$-\lambda_3 = \frac{Q^*}{H_2^*} - \sqrt{gH_2^*} < 0,\quad \lambda_4 = \frac{Q^*}{H_2^*} + \sqrt{gH_2^*} > 0,\quad \text{for } x\in(x_s^*,L]. \tag{7.1.7}$$

   In particular this implies that $H_1^* < H_2^*$.

3. Furthermore, the Rankine-Hugoniot conditions applied to (7.1.1) in the stationary case are equivalent to the following well-known Bélanger equation [42]

$$\frac{H_2^*}{H_1^*} = \frac{-1 + \sqrt{1 + 8\frac{(Q^*)^2}{g(H_1^*)^3}}}{2}. \tag{7.1.8}$$

Physical remarks:

— The switch from the torrential regime to the fluvial regime corresponds to a transition (shock) between a state where the system (7.1.1) has two positive eigenvalues and a state where the system has one positive and one negative eigenvalue. As we will see later (from Theorem 7.1.1 together with (7.1.6) and (7.1.7)), this transition (shock) induces a discontinuity not only for the eigenvalue that changes sign but also for the eigenvalue that keeps the same sign. More precisely, if we denote by λ_s the eigenvalue that changes sign, then $\lambda_s(x_s^-(t)) > 0 > \lambda_s(x_s^+(t))$ for all $t > 0$. And if we denote by λ_c the eigenvalue that does not change sign, then $\lambda_c(x_s^-(t)) \neq \lambda_c(x_s^+(t))$ for all $t > 0$. We point out that smooth transitions could happen around critical equilibria or when source terms are considered (see [58], [102]). Such smooth transitions are also related to coupling conditions for networks for the transition from supersonic to subsonic fluid states, such as natural gas pipeline transportation systems that have been analyzed in [96].

— Note that when the solutions are classical, the formulation (7.1.1) of the Saint-Venant equations with the level H and the flux Q as state variables is equivalent to the alternative formulation with the level H and the velocity Q/H that is obtained by replacing the equilibrium of forces by an energy equation and is used for instance in [13, 16, 110]. When the solutions are not classical however, the two formulations are not equivalent anymore and this can be seen by looking at the stationary states: the formulation (7.1.1) in level and flux is compatible with shock and discontinuity of $H^*(x)$ while the version with the energy equation is not. This is logical as there is a pointwise loss of energy in the hydraulic jump, which implies that the energy conservation does not hold anymore.

— From (7.1.3), the location of the shock x_s may be moving around its initial location and potentially all along the channel. This can be seen in practical phenomena such as tidal bores. The main challenge of this work is to also stabilize this location when stabilizing the state of the system. This is not obvious as one can see that for given heights and flux (H_1^*, H_2^*, Q^*) satisfying (7.1.6)–(7.1.8), any shock location $x_s^* \in [0, L]$ induces an admissible steady state $((H^*, Q^*)^\mathsf{T}, x_s^*)$, where H^* is given by (7.1.5). Thus the steady states are not isolated and therefore not asymptotically stable in open loop. Indeed, any small perturbation on x_s^* corresponds to another steady state with the same heights and flux at the two ends.

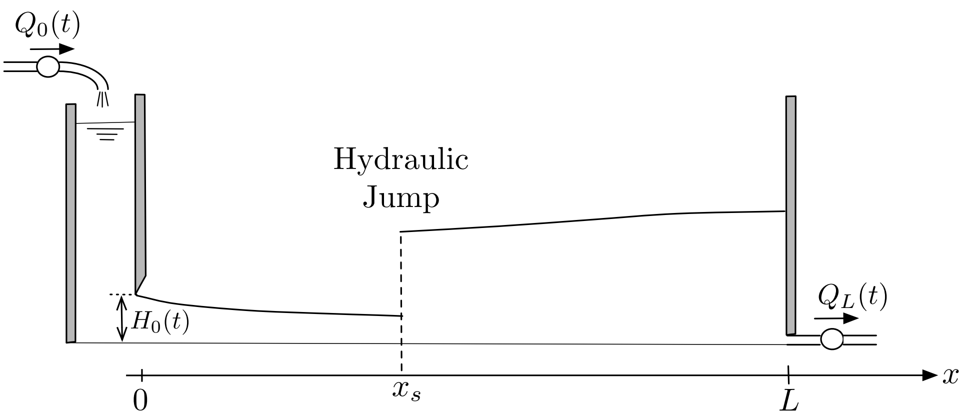

Figure 7.1 – Open channel with a hydraulic jump and three control devices : the gate opening $H_0(t)$ and the inflow $Q_0(t)$ and outflow $Q_L(t)$ which are driven by two pumps.

As illustrated in Figure 7.1, let us consider a channel which is equipped with devices allowing a feedback control on $H(t,0) = H_0(t)$, $Q(t,L) = Q_L(t)$ and $Q(t,0) \approx Q_0(t)$ (quasi-steady state approximation). Let the set point for the control be a steady state $((H^*,Q^*)^\intercal, x_s^*)$ defined as previously by (7.1.5)-(7.1.8). We assume that static boundary feedback control laws are selected so that the boundary conditions can be written in the following general form:

$$\begin{pmatrix} H(t,0) - H_1^* \\ Q(t,0) - Q^* \\ Q(t,L) - Q^* \end{pmatrix} = G \begin{pmatrix} Q(t,x_s^-) - Q^* \\ Q(t,x_s^+) - Q^* \\ H(t,x_s^-) - H_1^* \\ x_s - x_s^* \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ G_4(H(t,L) - H_2^*) \end{pmatrix} \tag{7.1.9}$$

where $G = (G_1, G_2, G_3)^\intercal : \mathbb{R}^4 \to \mathbb{R}^3$ and $G_4 : \mathbb{R} \to \mathbb{R}$ are of class C^2 and satisfy

$$G(\mathbf{0}) = \mathbf{0}, \quad G_4(0) = 0, \quad G_4'(0) = -\lambda_4. \tag{7.1.10}$$

Obviously, by (7.1.5), the steady state $((H^*,Q^*)^\intercal, x_s^*)$ satisfies the boundary conditions (7.1.9), as $H^*(0) = H_1^*$ and $H^*(L) = H_2^*$. Note that this boundary feedback is quite simple to implement as it only requires a pointwise measure of $H(t,L)$, $x_s(t)$, $H(t,x_s^-)$, $Q(t,x_s^+)$ and $Q(t,x_s^-)$.
In order to state the main stability result of this article, we first introduce the following notations:

$$D(x,\gamma) = \operatorname{diag}\left(\frac{s_i(1 - s_i\frac{\lambda_i}{\lambda_4})}{b_i} e^{\frac{\gamma}{x_i\lambda_i}(x_s^* - x)}, i \in \{1,2,3\} \right),$$

$$\widetilde{D}(\gamma) = \operatorname{diag}\left(\left(\sum_{j=1}^{3} e^{\frac{\gamma x_s^*}{x_i\lambda_i} - \frac{\gamma x_s^*}{x_j\lambda_j}} \right) \left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)^2, i \in \{1,2,3\} \right),$$

$$K = \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} & -\frac{\lambda_1}{\lambda_2-\lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1-\lambda_2} & -\frac{\lambda_2}{\lambda_1-\lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3+\lambda_4} \end{pmatrix} G'(\mathbf{0}) \begin{pmatrix} 1 & 1 & 0 \\ \frac{\lambda_1}{\lambda_4} & \frac{\lambda_2}{\lambda_4} & 1 + \frac{\lambda_3}{\lambda_4} \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.1.11}$$

$$d = \frac{1}{H_1^* - H_2^*},$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} & -\frac{\lambda_1}{\lambda_2-\lambda_1} & 0 \\ \frac{\lambda_2\lambda_1}{\lambda_1-\lambda_2} & -\frac{\lambda_2}{\lambda_1-\lambda_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_3+\lambda_4} \end{pmatrix} G'(\mathbf{0}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $s_1 = s_2 = 1$, $s_3 = -1$, $x_1 = x_2 = 1$, $x_3 = x_s^*/(L - x_s^*)$ and $x_4 = x_s^*/(x_s^* - L)$.
We consider the following initial condition

$$H(0,x) = H_0(x), \quad Q(0,x) = Q_0(x), \quad x_s(0) = x_{s,0} \tag{7.1.12}$$

where $x_{s,0} \in (0,L)$ and $(H_0(x), Q_0(x))^\intercal \in H^2((0,x_{s,0});\mathbb{R}^2) \cap H^2((x_{s,0},L);\mathbb{R}^2)$. We assume that the initial condition satisfies the first order compatibility conditions derived from (7.1.9), (see [13] for a proper definition of the first order compatibility condition which is omitted here for the sake of simplicity).
Now, we give the following definition:

Definition 7.1.1. The steady state $((H^*,Q^*)^\intercal, x_s^*)$ is locally exponentially stable for the H^2-norm with decay rate γ, if there exists $\delta^* > 0$ and $C^* > 0$ such that for any initial data $(H_0(x), Q_0(x))^\intercal \in H^2((0,x_{s,0});\mathbb{R}^2) \cap H^2((x_{s,0},L);\mathbb{R}^2)$ and $x_{s,0} \in (0,L)$ satisfying

$$|(H_0 - H_1^*, Q_0 - Q^*)^\intercal|_{H^2((0,x_{s,0});\mathbb{R}^2)} + |(H_0 - H_2^*, Q_0 - Q^*)^\intercal|_{H^2((x_{s,0},L);\mathbb{R}^2)} \leq \delta^*, \tag{7.1.13}$$

$$|x_{s,0} - x_s^*| \leq \delta^*, \tag{7.1.14}$$

and the corresponding first order compatibility conditions derived from (7.1.9), and for any $T > 0$, the system (7.1.1), (7.1.3), (7.1.4), (7.1.9) and (7.1.12) has a unique solution $(H, Q)^\mathsf{T} \in C^0([0,T]; H^2((0, x_s(t)); \mathbb{R}^2) \cap H^2((x_s(t), L); \mathbb{R}^2))$ and $x_s \in C^1([0,T])$ and

$$\begin{aligned}
&|(H(t,\cdot) - H_1^*, Q(t,\cdot) - Q^*)^\mathsf{T}|_{H^2((0,x_s(t));\mathbb{R}^2)} \\
&\qquad\qquad + |(H(t,\cdot) - H_2^*, Q(t,\cdot) - Q^*)^\mathsf{T}|_{H^2((x_s(t),L);\mathbb{R}^2)} + |x_s(t) - x_s^*| \\
&\qquad \leq C^* e^{-\gamma t}\Big(|(H_0 - H_1^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((0,x_{s,0});\mathbb{R}^2)} \\
&\qquad\qquad + |(H_0 - H_2^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((x_{s,0},L);\mathbb{R}^2)} + |x_{s,0} - x_s^*|\Big), \quad \forall t \in [0,T).
\end{aligned} \tag{7.1.15}$$

Remark 7.1.1. *A function f in $C^0([0,T]; H^2((0,x_s(t)); \mathbb{R}^2) \cap H^2((x_s(t), L); \mathbb{R}^2))$ is a function f in $C^0([0,T]; L^2((0,L); \mathbb{R}^2))$ such that, if one defines*

$$f_1(t,x) := f(t, x_s(t)x),\ t \in (0,T),\ x \in (0,1), \tag{7.1.16}$$

$$f_2(t,x) := f(t, L + (x_s(t) - L)x),\ t \in (0,T),\ x \in (0,1), \tag{7.1.17}$$

then f_1 and f_2 are both in $C^0([0,T]; H^2((0,1); \mathbb{R}^2))$. The transformation $f \to (f_1, f_2)$ enables us to reduce the problem to a time-invariant domain and to define the stability of a function $f \in C^0([0,T]; H^2((0,x_s(t)); \mathbb{R}^2) \cap H^2((x_s(t), L); \mathbb{R}^2))$, a function that is piecewise H^2 with a discontinuity that is potentially moving. This transformation will also be used later on in the analysis of the problem (see (7.2.4) below).

Based on Definition 7.1.1, we have the following theorem.

Theorem 7.1.1. *For any given steady state $((H^*, Q^*)^\mathsf{T}, x_s^*)$ of the system (7.1.1) satisfying (7.1.5)-(7.1.8) and the boundary conditions (7.1.9), for any $\gamma > 0$,*

if for $i = 1,2,3$

$$\begin{aligned}
b_i &\in \left(\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)(1 - e^{-\frac{\gamma}{x_i\lambda_i}x_s^*})}, \frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)} \right), \quad \text{if} \quad s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right) < 0, \\
b_i &\in \left(\frac{-\gamma e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)}, \frac{-\gamma x_i e^{-\frac{\gamma}{x_i\lambda_i}x_s^*}}{3ds_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)(1 - e^{-\frac{\gamma}{x_i\lambda_i}x_s^*})} \right), \quad \text{if} \quad s_i\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right) > 0,
\end{aligned} \tag{7.1.18}$$

and if the matrix

$$D(x_s^*, \gamma) - K^\mathsf{T} D(0,\gamma) K - \left(\sum_{k=1}^{3} \frac{2d^2}{\gamma^2} b_k s_k (1 - s_k \frac{\lambda_k}{\lambda_4})(e^{\frac{\gamma x_s^*}{x_k\lambda_k}} - 1) \right) \widetilde{D}(\gamma) \tag{7.1.19}$$

is positive definite, with $(b_1, b_2, b_3)^\mathsf{T}$, D, $\widetilde{D}$ and K defined in (7.1.11), then the steady state $((H^*, Q^*)^\mathsf{T}, x_s^*)$ is locally exponentially stable for the H^2-norm with decay rate $\gamma/4$.

Remark 7.1.2. *Note that it is not obvious that there always exists G such that K and $(b_1, b_2, b_3)^\mathsf{T}$ defined in (7.1.11) satisfy (7.1.18)-(7.1.19). We will prove in details that such G indeed exists in Appendix 7.5.*

7.2 Well-posedness of the system

In this section, we prove the well-posedness of the Saint-Venant equations (7.1.1) with the hydraulic jump conditions (7.1.3) and (7.1.4), the boundary feedback control conditions (7.1.9) and initial condition (7.1.12). We have the following well-posedness theorem.

Theorem 7.2.1. *For any $T > 0$, there exists $\delta(T) > 0$ such that, for any given initial condition* (7.1.12) *satisfying the first order compatibility conditions and*

$$|(H_0 - H_1^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((0,x_{s,0});\mathbb{R}^2)} + |(H_0 - H_2^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((x_{s,0},L);\mathbb{R}^2)} \le \delta(T), \tag{7.2.1}$$

$$|x_{s,0} - x_s^*| \le \delta(T), \tag{7.2.2}$$

the system (7.1.1), (7.1.3), (7.1.4), (7.1.9) *and* (7.1.12) *has a unique solution* $(H,Q)^\mathsf{T} \in C^0([0,T]; H^2((0,x_{s,0});\mathbb{R}^2) \cap H^2((x_{s,0},L);\mathbb{R}^2))$ *and* $x_s \in C^1([0,T])$. *Moreover, the following estimate holds for any* $t \in [0,T]$

$$\begin{aligned}
&|(H(t,\cdot) - H_1^*, Q(t,\cdot) - Q^*)^\mathsf{T}|_{H^2((0,x_s(t));\mathbb{R}^2)} \\
&\qquad\qquad + |(H(t,\cdot) - H_2^*, Q(t,\cdot) - Q^*)^\mathsf{T}|_{H^2((x_s(t),L);\mathbb{R}^2)} + |x_s(t) - x_s^*| \\
&\quad \le C(T)\Big(|(H_0 - H_1^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((0,x_{s,0});\mathbb{R}^2)} \\
&\qquad\qquad + |(H_0 - H_2^*, Q_0 - Q^*)^\mathsf{T}|_{H^2((x_{s,0},L);\mathbb{R}^2)} + |x_{s,0} - x_s^*|\Big).
\end{aligned} \tag{7.2.3}$$

Proof. One can see that the shock location x_s depends on t in general. In order to avoid the time-varying domains $[0, x_s(t)]$ and $[x_s(t), L]$, under the assumption that $x_s \in C^0([0,T])$, we perform, as in [79, 142], a transformation of the space coordinate x which allows to define new state variables on the fixed domain $[0, x_s^*]$ as follows:

$$\begin{aligned}
H_1(t,x) &= H(t, x\frac{x_s}{x_s^*}), \\
Q_1(t,x) &= Q(t, x\frac{x_s}{x_s^*}), \\
H_2(t,x) &= H(t, L + x\frac{x_s - L}{x_s^*}), \\
Q_2(t,x) &= Q(t, L + x\frac{x_s - L}{x_s^*}).
\end{aligned} \tag{7.2.4}$$

Let us denote by h_i and q_i the deviations

$$h_i = H_i - H_i^*, \quad q_i = Q_i - Q^*, \quad i = 1,2. \tag{7.2.5}$$

Then, the system (7.1.1), (7.1.3) and (7.1.4) is equivalent to the following 4×4 system, which is diagonalisable by blocks and defined on $\mathbb{R}^+ \times [0, x_s^*]$:

$$\begin{aligned}
&\partial_t h_1 - \left(x\frac{\dot{x}_s}{x_s^*}\right)\frac{x_s^*}{x_s}\partial_x h_1 + \frac{x_s^*}{x_s}\partial_x q_1 = 0, \\
&\partial_t q_1 + \left(\frac{2(q_1+Q^*)}{h_1+H_1^*} - x\frac{\dot{x}_s}{x_s^*}\right)\frac{x_s^*}{x_s}\partial_x q_1 + \left(g(h_1+H_1^*) - \frac{(q_1+Q^*)^2}{(h_1+H_1^*)^2}\right)\frac{x_s^*}{x_s}\partial_x h_1 = 0, \\
&\partial_t h_2 + \left(x\frac{\dot{x}_s}{x_s^*}\right)\frac{x_s^*}{L-x_s}\partial_x h_2 - \frac{x_s^*}{L-x_s}\partial_x q_2 = 0, \\
&\partial_t q_2 - \left(\frac{2(q_2+Q^*)}{h_2+H_2^*} - x\frac{\dot{x}_s}{x_s^*}\right)\frac{x_s^*}{L-x_s}\partial_x q_2 - \left(g(h_2+H_2^*) - \frac{(q_2+Q^*)^2}{(h_2+H_2^*)^2}\right)\frac{x_s^*}{L-x_s}\partial_x h_2 = 0,
\end{aligned} \tag{7.2.6}$$

where

$$\dot{x}_s = \frac{q_2(t,x_s^*) - q_1(t,x_s^*)}{h_2(t,x_s^*) - h_1(t,x_s^*) + H_2^* - H_1^*} \tag{7.2.7}$$

and with, from the jump condition (7.1.4), the following boundary condition at $x = x_s^*$:

$$(q_2 - q_1)^2 = (h_2 - h_1 + H_2^* - H_1^*)\left(\frac{(q_2+Q^*)^2}{h_2+H_2^*} + \frac{g}{2}(h_2+H_2^*)^2 - \frac{(q_1+Q^*)^2}{h_1+H_1^*} - \frac{g}{2}(h_1+H_1^*)^2\right). \tag{7.2.8}$$

Now, we introduce the following Riemann coordinates

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} h_1 \\ q_1 \\ h_2 \\ q_2 \end{pmatrix} \tag{7.2.9}$$

with

$$S_1 = \begin{pmatrix} \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \\ 1 & 1 \end{pmatrix}^{-1}, \quad S_2 = \begin{pmatrix} -\frac{1}{\lambda_3} & \frac{1}{\lambda_4} \\ 1 & 1 \end{pmatrix}^{-1} \tag{7.2.10}$$

and λ_i defined in (7.1.6), (7.1.7). Then the system (7.2.6) can be rewritten as

$$\mathbf{u}_t + (\Lambda(x_s) + A(\mathbf{u}, x_s) + x\dot{x}_s B(x_s))\mathbf{u}_x = 0, \tag{7.2.11}$$

where

$$\Lambda = \begin{pmatrix} \frac{x_s^*}{x_s}\lambda_1 & 0 & 0 & 0 \\ 0 & \frac{x_s^*}{x_s}\lambda_2 & 0 & 0 \\ 0 & 0 & \frac{x_s^*}{L - x_s}\lambda_3 & 0 \\ 0 & 0 & 0 & -\frac{x_s^*}{L - x_s}\lambda_4 \end{pmatrix} \tag{7.2.12}$$

and where A, B are two matrices of class C^2 that can be obtained by direct computations (omitted here for simplicity) and such that A satisfies $A(\mathbf{0}, x_s) = 0$. Using the change of coordinates (7.2.9), equation (7.2.7) becomes:

$$\dot{x}_s = \frac{u_1(t, x_s^*) + u_2(t, x_s^*) - u_3(t, x_s^*) - u_4(t, x_s^*)}{\sum\limits_{i=1}^{3} \frac{u_i(t, x_s^*)}{\lambda_i} - \frac{u_4(t, x_s^*)}{\lambda_4} + (H_1^* - H_2^*)} \tag{7.2.13}$$

and the boundary condition (7.2.8) becomes:

$$\frac{2Q^*}{H_2^*}(u_3 + u_4) - \frac{2Q^*}{H_1^*}(u_1 + u_2) + (gH_2^* - \frac{Q^{*2}}{H_2^{*2}})(\frac{u_4}{\lambda_4} - \frac{u_3}{\lambda_3}) - (gH_1^* - \frac{Q^{*2}}{H_1^{*2}})(\frac{u_1}{\lambda_1} + \frac{u_2}{\lambda_2}) = O\left(|\mathbf{u}(t, x_s^*)|^2\right). \tag{7.2.14}$$

Here and hereafter, $O(s)$ (with $s \geq 0$) means that for any $\varepsilon > 0$, there exists $C_1 > 0$ such that

$$(s \leq \varepsilon) \Longrightarrow (|O(s)| \leq C_1 s).$$

With the expression of the eigenvalues given by (7.1.6) and (7.1.7), (7.2.14) becomes

$$\lambda_4 u_4(t, x_s^*) = \lambda_1 u_1(t, x_s^*) + \lambda_2 u_2(t, x_s^*) + \lambda_3 u_3(t, x_s^*) + O\left(|\mathbf{u}(t, x_s^*)|^2\right). \tag{7.2.15}$$

Using (7.2.4), (7.2.5), (7.2.9) and (7.2.15), the boundary conditions (7.1.9) now become

$$\begin{pmatrix} u_1(t,0) \\ u_2(t,0) \\ u_3(t,0) \end{pmatrix} = \mathcal{B}\left(\begin{pmatrix} u_1(t, x_s^*) \\ u_2(t, x_s^*) \\ u_3(t, x_s^*) \end{pmatrix}, u_4(t,0), x_s - x_s^* \right), \tag{7.2.16}$$

where $\mathcal{B} = (B_1, B_2, B_3)^\mathsf{T} : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ is of class C^2 and where B_1 and B_2 are defined by

$$B_1 = (\lambda_2 G_1(\mathbf{u}(t, x_s^*), x_s) - G_2(\mathbf{u}(t, x_s^*), x_s)) \frac{\lambda_1}{\lambda_2 - \lambda_1}, \tag{7.2.17}$$

$$B_2 = (\lambda_1 G_1(\mathbf{u}(t, x_s^*), x_s) - G_2(\mathbf{u}(t, x_s^*), x_s)) \frac{\lambda_2}{\lambda_1 - \lambda_2}. \tag{7.2.18}$$

To define B_3, from the boundary conditions (7.1.9) and the change of variables (7.2.5), (7.2.9), we have

$$u_3(t,0) = \frac{-\lambda_4\lambda_3}{\lambda_3+\lambda_4}\left(\frac{u_4(t,0)}{\lambda_4} - \frac{u_3(t,0)}{\lambda_3}\right) + \frac{\lambda_3}{\lambda_3+\lambda_4}G_3(\mathbf{u}(t,x_s^*),x_s) \\ - \frac{\lambda_3}{\lambda_3+\lambda_4}G_4\left(\frac{u_4(t,0)}{\lambda_4} - \frac{u_3(t,0)}{\lambda_3}\right). \quad (7.2.19)$$

From condition (7.1.10), applying the implicit function theorem, one obtains

$$B_3 = \mathcal{F}(u_4(t,0), G_3(\mathbf{u}(t,x_s^*),x_s)) \quad (7.2.20)$$

in a neighborhood of $\mathbf{u}=0$ with

$$\mathcal{F}(0,0)=0, \quad \partial_1\mathcal{F}(0,0)=0, \quad \partial_2\mathcal{F}(0,0) = \frac{\lambda_3}{\lambda_3+\lambda_4}, \quad (7.2.21)$$

where $\partial_i\mathcal{F}$, $i=1,2$, denote the partial derivative of $\mathcal{F}$ with respect to its i-th variable.

Remark 7.2.1. *For simplicity, in* (7.2.17)-(7.2.20)*, we have used the following slight abuse of notation adapted from* (7.1.9)*:*

$$G_i(\mathbf{u}(t,x_s^*),x_s) = G_i\begin{pmatrix} u_1(t,x_s^*)+u_2(t,x_s^*) \\ u_3(t,x_s^*)+u_4(t,x_s^*) \\ \dfrac{u_1(t,x_s^*)}{\lambda_1}+\dfrac{u_2(t,x_s^*)}{\lambda_2} \\ x_s - x_s^* \end{pmatrix}, \quad i=1,2,3. \quad (7.2.22)$$

From (7.2.4)*,* (7.2.5)*,* (7.2.9) *and* (7.2.15)*, one can see that, as expressed in* (7.2.16)*, $\mathcal{B}$ only depends on $u_i(t,x_s^*)$, $i=1,2,3$, $u_4(t,0)$ and $x_s-x_s^*$ because from* (7.2.15) *$u_4(t,x_s^*)$ can be considered as a function of $u_i(t,x_s^*)$, $i=1,2,3$.*

The initial condition (7.1.12) becomes

$$\begin{aligned} &\mathbf{u}(0,x) = \mathbf{u}_0(x) = (u_{10}(x),u_{20}(x),u_{30}(x),u_{40}(x))^{\mathsf{T}}, \\ &x_s(0) = x_{s,0} \end{aligned} \quad (7.2.23)$$

that satisfies the first order compatibility conditions corresponding to (7.2.16). Thus, to study the well-posedness of (7.1.1), (7.1.3), (7.1.4), (7.1.9) and (7.1.12) is equivalent to study the well-posedness of (7.2.11), (7.2.13), (7.2.15)-(7.2.23). We have the following lemma from which one can easily obtain Theorem 7.2.1.

Lemme 7.2.1. *For any $T>0$, there exists $\delta(T)>0$ such that, for any $x_{s,0}\in(0,L)$ and $\mathbf{u}_0\in H^2((0,x_s^*);\mathbb{R}^4)$ satisfying the first order compatibility conditions and*

$$|\mathbf{u}_0|_{H^2((0,x_s^*);\mathbb{R}^4)} \le \delta(T) \text{ and } |x_{s,0}-x_s^*| \le \delta(T), \quad (7.2.24)$$

the system (7.2.11)*,* (7.2.13)*,* (7.2.15)*-*(7.2.23) *has a unique solution $\mathbf{u}\in C^0([0,T];H^2((0,x_s^*);\mathbb{R}^4))$ and $x_s\in C^1([0,T])$. Moreover, the following estimate holds for any $t\in[0,T]$*

$$|\mathbf{u}(t,\cdot)|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_s(t)-x_s^*| \le C(T)(|\mathbf{u}_0|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_{s,0}-x_s^*|). \quad (7.2.25)$$

Remark 7.2.2. *For the proof of Lemma 7.2.1, we refer to [20, Appendix], where the well-posedness of a 2×2 nonlinear hyperbolic system coupled with an ODE was studied. But the proof there can be easily adapted to the 4×4 nonlinear hyperbolic system coupled with an ODE. Noticing that $A(\mathbf{0},x_s)=0$ and that, from* (7.2.13)*, $\dot{x}_s=0$ when $\mathbf{u}=\mathbf{0}$, one has*

$$\Lambda(x_s) + A(\mathbf{u},x_s) + x\dot{x}_s B(x_s) = \Lambda(x_s)$$

when $\mathbf{u}=\mathbf{0}$. Thus, (7.2.11) *is indeed strictly hyperbolic provided that $|\mathbf{u}|_{C^0([0,T];H^2((0,x_s^*);\mathbb{R}^4))}$ is small enough and can be diagonalized in a neighbourhood of $\mathbf{u}=\mathbf{0}$. Then we can perform similar fixed point argument as in [20, Appendix] by carefully estimating the related norms of the solution along the characteristic curves. The C^1 regularity of x_s is then obtained directly from* (7.2.13)*. We omit the details.*

This completes the proof of Theorem 7.2.1. □

7.3 Exponential stability of the steady state for the H^2-norm

In this section we prove Theorem 7.1.1.

Proof of Theorem 7.1.1. It is worth noticing that due to the equivalence of the system (7.1.1), (7.1.3), (7.1.4), (7.1.9) and the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16), one only needs to prove the exponential stability of the null-steady state of the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16) for the H^2-norm. Motivated by [50], see also [13, Section 4.4], and by [20], we introduce the following Lyapunov function:

$$V(\mathbf{u}, x_s) = V_1(\mathbf{u}) + V_2(\mathbf{u}) + V_3(\mathbf{u}) + V_4(\mathbf{u}, x_s) + V_5(\mathbf{u}, x_s) + V_6(\mathbf{u}, x_s), \tag{7.3.1}$$

where:

$$V_1(\mathbf{u}) = \int_0^{x_s^*} \sum_{i=1}^4 p_i e^{-\frac{\mu}{x_i \lambda_i} x} u_i^2 dx, \tag{7.3.2}$$

$$V_2(\mathbf{u}) = \int_0^{x_s^*} \sum_{i=1}^4 p_i e^{-\frac{\mu}{x_i \lambda_i} x} u_{it}^2 dx, \tag{7.3.3}$$

$$V_3(\mathbf{u}) = \int_0^{x_s^*} \sum_{i=1}^4 p_i e^{-\frac{\mu}{x_i \lambda_i} x} u_{itt}^2 dx, \tag{7.3.4}$$

$$V_4(\mathbf{u}, x_s) = \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i \lambda_i} x} u_i(t,x)(x_s - x_s^*) dx + C_0 (x_s - x_s^*)^2, \tag{7.3.5}$$

$$V_5(\mathbf{u}, x_s) = \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i \lambda_i} x} u_{it}(t,x) \dot{x}_s dx + C_0 (\dot{x}_s)^2, \tag{7.3.6}$$

$$V_6(\mathbf{u}, x_s) = \int_0^{x_s^*} \sum_{i=1}^3 \frac{p_i'}{\lambda_i} e^{-\frac{\mu}{x_i \lambda_i} x} u_{itt}(t,x) \ddot{x}_s dx + C_0 (\ddot{x}_s)^2, \tag{7.3.7}$$

where p_i and C_0 are positive constants that shall be determined later on, while p_i' are constants, not necessarily positive, which will also be determined later on. Besides we impose $C_0 > 3/2$ and we recall that $x_1 = x_2 = 1$, $x_3 = x_s^*/(L - x_s^*)$ and $x_4 = x_s^*/(x_s^* - L)$. In the following we may denote for simplicity $V_i := V_i(\mathbf{u}, x_s)$ and $|\mathbf{u}|_{H^2} := |\mathbf{u}(t,\cdot)|_{H^2((0,x_s^*);\mathbb{R}^4)}$ in the computations. Similarly to what is done in [20], from the Cauchy-Schwarz inequality and as $C_0 > 3/2$, it can be shown that the Lyapunov function V considered here is equivalent to $(|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2$ provided that $|\mathbf{u}|_{H^2} + |x_s - x_s^*|$ is small enough and that

$$\max_i \left(\frac{p_i'^2 x_i}{\mu \lambda_i p_i} (1 - e^{-\frac{\mu}{x_i \lambda_i} x_s^*}) \right) < 2. \tag{7.3.8}$$

This means that, under condition (7.3.8), there exists $\bar{\rho} > 0$ and $\bar{C}$ such that, for every $T > 0$ and $\mathbf{u} \in C^0([0,T]; H^2((0,x_s^*);\mathbb{R}^4))$ and for every $x_s \in C^1([0,T])$ solution of the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16), if $|\mathbf{u}|_{H^2} + |x_s - x_s^*| \leq \bar{\rho}$

$$\frac{1}{\bar{C}} (|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2 \leq V(\mathbf{u}, x_s) \leq \bar{C} (|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2. \tag{7.3.9}$$

This can be proved by direct estimations (see [20] for more details).
From the boundary condition (7.2.16), as $\mathcal{B}$ is of class C^2, we have

$$\mathbf{v}(t,0) = \partial_1 \mathcal{B}(\mathbf{0},0,0)\mathbf{v}(t,x_s^*) + \partial_2 \mathcal{B}(\mathbf{0},0,0) u_4(t,0) + \partial_3 \mathcal{B}(\mathbf{0},0,0)(x_s - x_s^*) + O((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2), \tag{7.3.10}$$

where $\mathbf{v} = (u_1, u_2, u_3)^\intercal$ is the vector of the components of $\mathbf{u}$ on which the feedback (7.2.16) applies. This notation is practical as it isolates u_1, u_2 and u_3 from u_4 on which we have no control and whose boundary condition is imposed by the condition (7.2.15). In (7.3.10), the notation $\partial_1 \mathcal{B}$ is the 3×3 Jacobian matrix of the vector-valued function $\mathcal{B}$ with respect to its first variable which is a 3-D vector (see the expression of $\mathcal{B}$ in (7.2.16)). From (7.2.17)-(7.2.21), one can check that $\partial_2\mathcal{B}(\mathbf{0},0,0)\equiv\mathbf{0}$. Moreover, from (7.2.17)–(7.2.20), noticing (7.2.21), it can be verified that the matrix K and the vector $(b_1,b_2,b_3)^{\mathsf{T}}$ defined in (7.1.11) satisfy

$$K=(k_{ij})_{(i,j)\in\{1,2,3\}^2}=\partial_1\mathcal{B}(\mathbf{0},0,0),\quad \partial_3\mathcal{B}(\mathbf{0},0,0)=(b_1,b_2,b_3)^{\mathsf{T}}. \tag{7.3.11}$$

Let $\bar{T}>0$ be given and let $x_{s,0}\in(0,L)$ and $\mathbf{u}_0\in H^2((0,x_s^*);\mathbb{R}^4)$ satisfying the first order compatibility conditions and (7.2.24). Let $\mathbf{u}\in C^0([0,\bar{T}];H^2((0,x_s^*);\mathbb{R}^4))$ and $x_s\in C^1([0,\bar{T}])$ be the solution of the system (7.2.11), (7.2.13), (7.2.15)-(7.2.23). Let us start with the case where $\mathbf{u}$ is of class C^3. Taking the time derivative of V_1 along this solution, we obtain

$$\frac{dV_1}{dt}=-\mu V_1-\left[\sum_{i=1}^{4}p_ix_i\lambda_ie^{-\frac{\mu}{x_i\lambda_i}x}u_i^2\right]_0^{x_s^*}+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right). \tag{7.3.12}$$

By differentiating (7.2.11), similarly as (7.3.12), we can obtain

$$\frac{dV_2}{dt}=-\mu V_2-\left[\sum_{i=1}^{4}p_ix_i\lambda_ie^{-\frac{\mu}{x_i\lambda_i}x}u_{it}^2\right]_0^{x_s^*}+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right). \tag{7.3.13}$$

Now, let us deal with the V_3 term. To that end, we derive from (7.2.11) that

$$\begin{aligned}&\mathbf{u}_{ttt}+(\Lambda(x_s)\mathbf{u}_{ttx}+2\dot{x}_s\Lambda'(x_s)\mathbf{u}_{tx}+(\Lambda''(x_s)(\dot{x}_s)^2+\Lambda'(x_s)\ddot{x}_s)\mathbf{u}_x\\&+(A(\mathbf{u},x_s)\mathbf{u}_x)_{tt}+x\dddot{x}_sB(x_s)\mathbf{u}_x+2x\ddot{x}_s(B(x_s)\mathbf{u}_x)_t+x\dot{x}_s(B(x_s)\mathbf{u}_x)_{tt}=0.\end{aligned} \tag{7.3.14}$$

Thus,

$$\begin{aligned}\frac{dV_3}{dt}=&-\mu V_3-\left[\sum_{i=1}^{4}p_ix_i\lambda_ie^{-\frac{\mu}{x_i\lambda_i}x}u_{itt}^2\right]_0^{x_s^*}-\int_0^{x_s^*}\sum_{i=1}^{4}2p_ie^{-\frac{\mu}{x_i\lambda_i}x}x\dddot{x}_su_{itt}\left(\sum_{j=1}^{4}B_{ij}u_{jx}\right)dx\\&+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right).\end{aligned} \tag{7.3.15}$$

We observe that now $\dddot{x}_s$ appears. As $\dddot{x}_s$ is proportional to $\mathbf{u}_{tt}(x_s^*)$, it can not be bounded by $|\mathbf{u}|_{H^2}$. However, we can use Young's inequality to compensate it with the boundary terms. Using (7.3.10), one has

$$\begin{aligned}\frac{dV_3}{dt}\le&-\mu V_3-\sum_{i=1}^{4}\left(\left(p_ix_i\lambda_ie^{-\frac{\mu x_s^*}{x_i\lambda_i}}+O(|\mathbf{u}|_{H^2})\right)u_{itt}^2(x_s^*)-u_{itt}^2(0)\right)\\&+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right).\end{aligned} \tag{7.3.16}$$

Differentiating (7.3.5), from (7.2.11), one has

$$\begin{aligned}\frac{dV_4}{dt}=&(x_s-x_s^*)\int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_{it}(t,x)dx\\&+\dot{x}_s\int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_i(t,x)dx+2C_0\dot{x}_s(x_s-x_s^*)\\=&-(x_s-x_s^*)\int_0^{x_s^*}\sum_{i=1}^{3}x_ip_i'e^{-\frac{\mu}{x_i\lambda_i}x}u_{ix}(t,x)dx\\&+d\left(u_1(x_s^*)+u_2(x_s^*)-u_3(x_s^*)-u_4(x_s^*)\right)\int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_i(t,x)dx\\&+2dC_0(x_s-x_s^*)\left(u_1(x_s^*)+u_2(x_s^*)-u_3(x_s^*)-u_4(x_s^*)\right)\\&+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right),\end{aligned} \tag{7.3.17}$$

where we recall that $d = (H_1^* - H_2^*)^{-1} < 0$ is defined in (7.1.11). Thus, integrating by parts and using (7.2.15),

$$\begin{aligned}\frac{dV_4}{dt} &= -(x_s - x_s^*)\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_i(t,x)\right]_0^{x_s^*} - \mu(V_4 - C_0(x_s - x_s^*)^2) \\ &\quad + d\left(u_1(x_s^*)\left(1 - \frac{\lambda_1}{\lambda_4}\right) + u_2(x_s^*)\left(1 - \frac{\lambda_2}{\lambda_4}\right) - u_3(x_s^*)\left(1 + \frac{\lambda_3}{\lambda_4}\right)\right) \\ &\quad \left(2C_0(x_s - x_s^*) + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_i(t,x)dx\right) + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right). \end{aligned} \tag{7.3.18}$$

Similarly for V_5, from (7.2.11), one has

$$\begin{aligned}\frac{dV_5}{dt} &= -\dot{x}_s\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_{it}(t,x)\right]_0^{x_s^*} - \mu(V_5 - C_0(\dot{x}_s)^2) \\ &\quad + d\left(\sum_{i=1}^{3}\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)s_i u_{it}(x_s^*)\right)\left(2C_0\dot{x}_s + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_{it}(t,x)dx\right) \\ &\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right). \end{aligned} \tag{7.3.19}$$

By (7.3.14), for V_6, one has

$$\begin{aligned}\frac{dV_6}{dt} &= -\ddot{x}_s\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_{itt}(t,x)\right]_0^{x_s^*} - \mu(V_6 - C_0(\ddot{x}_s)^2) \\ &\quad + d\left(\sum_{i=1}^{3}\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)s_i u_{itt}(x_s^*)\right)\left(2C_0\ddot{x}_s + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_{itt}(t,x)dx\right) \\ &\quad - \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}x\dddot{x}_s\left(\sum_{j=1}^{4}B_{ij}u_{jx}\right)(x_s - x_s^*)\,dx + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right). \end{aligned} \tag{7.3.20}$$

Dealing with the $\dddot{x}_s$ term in (7.3.20) similarly as for V_3, we have

$$\begin{aligned}\frac{dV_6}{dt} &= -\ddot{x}_s\sum_{i=1}^{3}\left(\left(x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x_s^*} + O(|\mathbf{u}|_{H^2})\right)u_{itt}(x_s^*) - x_i p_i' u_{itt}(0)\right) - \mu(V_6 - C_0(\ddot{x}_s)^2) \\ &\quad + d\left(\sum_{i=1}^{3}\left(1 - s_i\frac{\lambda_i}{\lambda_4}\right)s_i u_{itt}(x_s^*)\right)\left(2C_0\ddot{x}_s + \int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_{itt}(t,x)dx\right) \\ &\quad + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right). \end{aligned} \tag{7.3.21}$$

Note that $V_2 + V_5$ has the same structure as $V_1 + V_4$ with u_i and $x_s - x_s^*$ being replaced by u_{it} and $\dot{x}_s$ respectively. The same applies for $V_3 + V_6$ by replacing u_i and $x_s - x_s^*$ in $V_1 + V_4$ with u_{itt} and $\ddot{x}_s$ respectively. Hence, we only need to analyze $V_1 + V_4$. From (7.3.12) and (7.3.18), recalling that $s_i = 1$ if $i \in \{1, 2\}$ and $s_3 = -1$, one has

$$\begin{aligned}\frac{d(V_1 + V_4)}{dt} &= -\left[\sum_{i=1}^{4} p_i x_i \lambda_i e^{-\frac{\mu}{x_i\lambda_i}x} u_i^2\right]_0^{x_s^*} - \mu(V_1 + V_4) \\ &\quad - (x_s - x_s^*)\left[\sum_{i=1}^{3} x_i p_i' e^{-\frac{\mu}{x_i\lambda_i}x} u_i\right]_0^{x_s^*} + \mu C_0(x_s - x_s^*)^2 \end{aligned}$$

$$+ d\left(\sum_{i=1}^{3} u_i(x_s^*)s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)\right)\left(2C_0(x_s-x_s^*)+\int_0^{x_s^*}\sum_{i=1}^{3}\frac{p_i'}{\lambda_i}e^{-\frac{\mu}{x_i\lambda_i}x}u_i(t,x)dx\right)$$
$$+ O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right). \tag{7.3.22}$$

Using now the boundary conditions (7.3.10) and (7.2.15), (7.3.22) becomes

$$\begin{aligned}
\frac{d(V_1+V_4)}{dt} =& -\mu(V_1+V_4)\\
&-\mathbf{v}(x_s^*)^\mathsf{T}\left(F(x_s^*,\mu)-K^\mathsf{T}F(0,\mu)K\right)\mathbf{v}(x_s^*)-\frac{x_4p_4}{\lambda_4}e^{-\frac{\mu}{x_4\lambda_4}x_s^*}(\lambda_1u_1(x_s^*)+\lambda_2u_2(x_s^*)+\lambda_3u_3(x_s^*))^2\\
&-\lambda_4|x_4|p_4u_4^2(0)+\sum_{i=1}^{3}x_ip_i\lambda_ib_i^2(x_s-x_s^*)^2+2\sum_{i=1}^{3}x_ip_i\lambda_ib_i\left(\sum_{j=1}^{3}k_{ij}u_j(x_s^*)(x_s-x_s^*)\right)\\
&-\left(\sum_{i=1}^{3}u_i(x_s^*)(x_s-x_s^*)\left(x_ip_i'e^{-\frac{\mu}{x_i\lambda_i}x_s^*}-2dC_0s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)\right)-\sum_{j=1}^{3}k_{ij}u_j(x_s^*)(x_s-x_s^*)x_ip_i'\right)\\
&+\sum_{i=1}^{3}x_ip_i'b_i(x_s-x_s^*)^2+\mu C_0(x_s-x_s^*)^2\\
&+d\left(\sum_{i=1}^{3}u_i(x_s^*)s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)\right)\left(\int_0^{x_s^*}\sum_{j=1}^{3}\frac{p_j'}{\lambda_j}e^{-\frac{\mu}{x_j\lambda_j}x}u_j(t,x)dx\right)\\
&+O\left((|\mathbf{u}|_{H^2}+|x_s-x_s^*|)^3\right),
\end{aligned}\tag{7.3.23}$$

where

$$F(x,\mu)=\mathrm{diag}\left(\lambda_ip_ix_ie^{-\frac{\mu}{x_i\lambda_i}x}, i\in\{1,2,3\}\right). \tag{7.3.24}$$

We observe that, except from the last product proportional to d, a quadratic form in $(\mathbf{v}(x_s^*)^\mathsf{T}, u_4(0), x_s-x_s^*)$ appears. Using successively the Young and Cauchy-Schwarz inequalities to deal with the last product, and noticing that

$$\int_0^{x_s^*}e^{-\frac{\mu}{x_i\lambda_i}x}dx=\frac{\lambda_ix_i}{\mu}(1-e^{-\frac{\mu}{x_i\lambda_i}x_s^*}), \tag{7.3.25}$$

we get that, for any $j\in\{1,2,3\}$,

$$\begin{aligned}
&d\left(\sum_{i=1}^{3}u_i(x_s^*)s_i\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)\right)\left(\int_0^{x_s^*}\frac{p_j'}{\lambda_j}e^{-\frac{\mu}{x_j\lambda_j}x}u_j(t,x)dx\right)\leq\\
&\frac{\varepsilon_j}{\mu}\left(\frac{p_j'^2x_j(1-e^{-\frac{\mu}{x_j\lambda_j}x_s^*})}{\lambda_jp_j}\right)\left(\int_0^{x_s^*}p_je^{-\frac{\mu}{x_j\lambda_j}x}u_j^2(t,x)dx\right)+\frac{d^2}{4\varepsilon_j}\left(\sum_{i=1}^{3}u_i(x_s^*)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)s_i\right)^2.
\end{aligned}$$

Using again the Cauchy-Schwarz inequality, we get that

$$\frac{d^2}{4\varepsilon_j}\left(\sum_{i=1}^{3}u_i(x_s^*)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)s_i\right)^2\leq\frac{d^2}{4\varepsilon_j}\left(\sum_{i=1}^{3}u_i^2(x_s^*)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)^2\left(\sum_{j=1}^{3}e^{\frac{\mu x_s^*}{x_i\lambda_i}-\frac{\mu x_s^*}{x_j\lambda_j}}\right)\right). \tag{7.3.26}$$

Therefore, combining (7.3.23)-(7.3.26), one has

$$\begin{aligned}
\frac{d(V_1+V_4)}{dt}\leq& -\mu(V_1+V_4)-\mathbf{v}(x_s^*)^\mathsf{T}\Big(F(x_s^*,\mu)-K^\mathsf{T}F(0,\mu)K\\
&-\frac{d^2}{4}\left(\sum_{k=1}^{3}\frac{1}{\varepsilon_k}\right)\mathrm{diag}\left(\left(\sum_{j=1}^{3}e^{\frac{\mu x_s^*}{x_i\lambda_i}-\frac{\mu x_s^*}{x_j\lambda_j}}\right)\left(1-s_i\frac{\lambda_i}{\lambda_4}\right)^2\right)_{i\in\{1,2,3\}}\Big)\mathbf{v}(x_s^*)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{x_4 p_4}{\lambda_4} e^{-\frac{\mu}{x_4\lambda_4}x_s^*}(\lambda_1 u_1(x_s^*) + \lambda_2 u_2(x_s^*) + \lambda_3 u_3(x_s^*))^2 - \lambda_4 |x_4| p_4 u_4^2(0) \\
&+ \left(\mu C_0 + \sum_{i=1}^{3} (x_i p_i \lambda_i b_i^2 + x_i p_i' b_i)\right)(x_s - x_s^*)^2 \\
&+ \sum_{i=1}^{3} \Big(\frac{\varepsilon_i}{\mu} \left(\frac{p_i'^2 x_i (1 - e^{-\frac{\mu}{\lambda_i x_i}x_s^*})}{\lambda_i p_i} \right) \left(\int_0^{x_s^*} p_i e^{-\frac{\mu}{x_i \lambda_i}x} u_i^2(t,x) dx \right) \Big) \\
&+ \sum_{j=1}^{3} \Big(2dC_0 s_j \left(1 - s_j \frac{\lambda_j}{\lambda_4}\right) - x_j p_j' e^{-\frac{\mu}{x_j\lambda_j}x_s^*} \\
&+ \sum_{i=1}^{3} (2x_i p_i \lambda_i b_i k_{ij} + x_i p_i' k_{ij}) \Big) u_j(x_s^*)(x_s - x_s^*) \\
&+ O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).
\end{aligned} \tag{7.3.27}
$$

In order to obtain an exponential decay, we first choose ε_i such that

$$
\frac{1}{\varepsilon_i} = \frac{2p_i'^2 x_i (1 - e^{-\frac{\mu}{x_i\lambda_i}x_s^*})}{\mu^2 \lambda_i p_i}, \quad i = 1, 2, 3. \tag{7.3.28}
$$

Therefore, (7.3.27) becomes

$$
\begin{aligned}
\frac{d(V_1 + V_4)}{dt} \leq &-\frac{\mu}{2} V_1 - \mu V_4 - \mathbf{v}(x_s^*)^\intercal \left(F(x_s^*, \mu) - K^\intercal F(0,\mu) K - \frac{d^2}{4} \left(\sum_{k=1}^{3} \frac{1}{\varepsilon_k} \right) \widetilde{D}(\mu) \right) \mathbf{v}(x_s^*) \\
&- \frac{x_4 p_4}{\lambda_4} e^{-\frac{\mu}{x_4\lambda_4}x_s^*}(\lambda_1 u_1(x_s^*) + \lambda_2 u_2(x_s^*) + \lambda_3 u_3(x_s^*))^2 - \lambda_4 |x_4| p_4 u_4^2(0) \\
&+ \Big(\mu C_0 + \sum_{i=1}^{3} (x_i p_i \lambda_i b_i^2 + x_i p_i' b_i)\Big)(x_s - x_s^*)^2 \\
&+ \sum_{j=1}^{3} \left(2dC_0 s_j \left(1 - s_j \frac{\lambda_j}{\lambda_4}\right) - x_j p_j' e^{-\frac{\mu}{x_j\lambda_j}x_s^*} + \sum_{i=1}^{3} (2x_i p_i \lambda_i b_i k_{ij} + x_i p_i' k_{ij}) \right) u_j(x_s^*)(x_s - x_s^*) \\
&+ O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right).
\end{aligned} \tag{7.3.29}
$$

We clearly see now two terms proportional to V_1 and V_4 respectively that will bring the exponential decay, and a quadratic form in $(\mathbf{v}(x_s^*)^\intercal, u_4(0), x_s - x_s^*)$ appears. In order to simplify the quadratic form by cancelling the cross terms, we choose

$$
p_i' = 2\frac{dC_0 s_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right) e^{\frac{\mu}{x_i\lambda_i}x_s^*}}{x_i}, \quad i = 1, 2, 3. \tag{7.3.30}
$$

Observe that from (7.1.18), one always has $b_i x_i p_i' < 0$ for $i = 1, 2, 3$, thus we can choose

$$
p_i = -\frac{p_i'}{2b_i\lambda_i} > 0, \quad i = 1, 2, 3. \tag{7.3.31}
$$

Therefore we have, using (7.3.30), (7.3.31) and Young's inequality

$$
\begin{aligned}
\frac{d(V_1 + V_4)}{dt} \leq &- \frac{\mu}{2} V_1 - \mu V_4 - \mathbf{v}(x_s^*)^\intercal \Big(F(x_s^*, \mu) - K^\intercal F(0,\mu) K - \frac{d^2}{4} \left(\sum_{k=1}^{3} \frac{1}{\varepsilon_k} \right) \widetilde{D}(\mu) \\
&- \operatorname{diag}\left(\frac{3|x_4| p_4 \lambda_i^2}{\lambda_4} e^{-\frac{\mu}{x_4\lambda_4}x_s^*} \right)_{i \in \{1,2,3\}} \Big) \mathbf{v}(x_s^*) \\
&- \lambda_4 |x_4| p_4 u_4^2(0) + \left(\mu C_0 - \frac{1}{2} \sum_{i=1}^{3} |x_i p_i' b_i| \right)(x_s - x_s^*)^2
\end{aligned}
$$

$$+ O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right). \tag{7.3.32}$$

Observe that the conditions (7.1.18) and (7.1.19) are satisfied for $\gamma > 0$, but as the inequalities are strict, there exists $\mu > \gamma$ such that (7.1.18) and (7.1.19) are also verified with μ instead of γ. We choose such μ and using (7.3.30), one can see that

$$\left(\mu C_0 - \frac{1}{2}\sum_{i=1}^{3}|x_i p_i' b_i|\right) < 0, \tag{7.3.33}$$

and one can also check from (7.2.17)–(7.2.20), (7.3.28), (7.3.30), (7.3.31), conditions (7.1.18) and (7.1.19) that the matrix defined by

$$F(x_s^*, \mu) - K^\intercal F(0,\mu)K - \frac{d^2}{4}\left(\sum_{k=1}^{3}\frac{1}{\varepsilon_k}\right)\widetilde{D}(\mu) \tag{7.3.34}$$

is positive definite. This implies that there exists $p_4 > 0$ such that the quadratic form in $\mathbf{v}(x_s^*)$ in (7.3.32) is non-positive. Therefore

$$\frac{d(V_1+V_4)}{dt} \leq -\frac{\mu}{2}V_1 - \mu V_4 + O\left((|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^3\right). \tag{7.3.35}$$

As $\mu > \gamma$, at least if $|\mathbf{u}|_{H^2} + |x_s - x_s^*|$ is small enough which can be guaranteed from Lemma 7.2.1 by requiring $\delta(\bar{T})$ small enough

$$\frac{d(V_1+V_4)}{dt} \leq -\frac{\gamma}{2}(V_1+V_4), \tag{7.3.36}$$

thus,

$$\frac{dV}{dt} \leq -\frac{\gamma}{2}V. \tag{7.3.37}$$

We have derived (7.3.37) under the assumption that the trajectories of (7.2.11), (7.2.13), (7.2.15) and (7.2.16) are of class C^3, but one can use a density argument to generalize the result for trajectories in $C^0([0,\bar{T}]; H^2((0,x_s^*);\mathbb{R}^4))$ by noticing that γ does not depend on any C^2 or C^3-norm of $\mathbf{u}$. The inequality (7.3.37) is then understood in the distribution sense. One can refer to [20] or [13, Comment 4.6] for more details.

By the equivalence between the Lyapunov function V and $(|\mathbf{u}|_{H^2} + |x_s - x_s^*|)^2$ if this last quantity is small, we get immediately the exponential stability of the null steady state of the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16) for the H^2-norm with decay rate $\gamma/4$. It remains to check that under assumption (7.1.19), (7.3.8) holds with p_i' and p_i defined as (7.3.30) and (7.3.31). Indeed,

$$\max_i\left(\frac{p_i'^2 x_i}{\mu\lambda_i p_i}(1 - e^{-\frac{\mu}{x_i\lambda_i}x_s^*})\right) < \frac{4C_0}{3}, \tag{7.3.38}$$

therefore there exists $C_0 > 3/2$ such that the condition (7.3.8) is satisfied.

So far $\delta(\bar{T})$ depends on $\bar{T}$, we next prove that for any given $T > 0$, we can choose δ^* independent of T such that (7.3.37) holds on $(0,T)$ as required in Definition 7.1.1.

Let us now assume that $x_{s,0} \in (0,L)$ and $\mathbf{u}_0 \in H^2((0,x_s^*);\mathbb{R}^4)$ satisfying the first order compatibility conditions and

$$|\mathbf{u}_0|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_{s,0} - x_s^*| < \bar{\rho} \text{ and } V(\mathbf{u}_0, x_{s,0}) \leq \nu, \tag{7.3.39}$$

where $\nu > 0$ is going to be chosen small enough. Then, for any $t \in [0,\bar{T}]$, at least if $\nu > 0$ is small enough, from (7.2.25), (7.3.9) and (7.3.37),

$$|\mathbf{u}(t)|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_s(t) - x_s^*| < \bar{\rho} \text{ and } V(\mathbf{u}(t), x_s(t)) \leq \nu. \tag{7.3.40}$$

Using (7.3.40) for $t = \bar{T}$ one can keep going on $[\bar{T}, 2\bar{T}]$ and then on $[2\bar{T}, 3\bar{T}]$, etc. So we get that, for every $j = 1, 2, 3, \ldots$,

$$V(\mathbf{u}(t), x_s(t)) \leq \nu,\ t \in [(j-1)\bar{T}, j\bar{T}], \tag{7.3.41}$$

$$(|\mathbf{u}(t)|_{H^2((0,x_s^*);\mathbb{R}^4)} + |x_s(t) - x_s^*|) < \bar{\rho},\ t \in [(j-1)\bar{T}, j\bar{T}], \tag{7.3.42}$$

$$\frac{dV}{dt} \leq -\frac{\gamma}{2} V \text{ in the distribution sense on } (0, j\bar{T}). \tag{7.3.43}$$

Noticing (7.2.9), there exists a δ^* such that if (7.1.13)-(7.1.14) hold, one has (7.3.39). Thus, noticing also that for any $T > 0$ there exists $j \in \mathbb{N}$ such that $(0, T) \subset (0, j\bar{T})$, one gets that the steady state $((H^*, Q^*)^\intercal, x_s^*)$ is locally exponentially stable for the H^2-norm with decay rate $\gamma/4$. The proof of Theorem 1.1 is thus complete. □

Remark 7.3.1. *Given the assumptions of Theorem 7.1.1, it is obvious that this stability result is robust with respect to small variations of G in the feedback control. However, it is actually also robust with respect to small variations of G_4. Indeed, if $|G_4'(0) + \lambda_4|$ is sufficiently small but with a bound independent of the state $(H, Q)^\intercal$ and x_s, we can still define $\mathcal{B}$ as in (7.2.17)–(7.2.20) using the implicit function theorem. Then looking at (7.3.10), $\partial_2\mathcal{B}(\mathbf{0}, 0, 0) \neq \mathbf{0}$, but for any $\delta > 0$, $|\partial_2\mathcal{B}(\mathbf{0}, 0, 0)| < \delta$ provided $|G_4'(0) + \lambda_4|$ is sufficiently small. Then all the additional terms about $u_4^2(0)$ and $u_i^2(x_s^*)$, $i = 1, 2, 3$ will be compensated by the fact that $p_4 > 0$ in (7.3.29) and that $|G_4'(0) + \lambda_4|$ is sufficiently small. The rest of the proof is the same as in the case where $G_4'(0) = -\lambda_4$.*

7.4 Conclusion

In this article, we have considered the problem of the boundary feedback stabilization of an open channel with a hydraulic jump. We focused on the case where the channel has a rectangular cross section without friction or slope. The channel dynamics are modelled by a version of the homogeneous Saint-Venant equations with the water level H and the flow rate Q as state variables. The hydraulic jump is represented by a discontinuous shock solution of the system. The main contribution of this chapter is to analyze the boundary feedback stabilization of the system with a general class of static feedback controls that require pointwise measurements of the level and the flux at the boundary and in the immediate vicinity of the hydraulic jump. In order to prove the well-posedness of the system, we first introduce a change of variables which allows to transform the Saint-Venant equations with shock wave solutions into an equivalent 4×4 quasilinear hyperbolic system which is parametrized by the jump position but has shock-free solutions. Then, by a Lyapunov approach, we show that, for the considered class of boundary feedback controls, the exponential stability in H^2-norm of the steady state can be achieved with an arbitrary decay rate and with an exponential stabilization of the desired location of the hydraulic jump. Compared with previous results in the literature for classical solutions of quasilinear hyperbolic systems, the H^2-Lyapunov function introduced in [50] (see also [13, Section 4.4]) has to be augmented with suitable extra terms for the analysis of the stabilization of the jump position. In the case where the cross section is irregular and with friction or slope, the jump stabilization issue is much more challenging and remains an open problem.

7.5 Appendix

In this appendix we prove that that there always exists G such that K and $(b_1, b_2, b_3)^\intercal$ defined in (7.1.11) satisfy (7.1.18)-(7.1.19). Let us first point out that, for every $K \in \mathbb{R}^{3\times3}$, there exists a linear map $G : \mathbb{R}^4 \to \mathbb{R}^3$ such that the third equation of (7.1.11) holds. Hence it remains only to show that there always exist K and $(b_1, b_2, b_3)^\intercal$ satisfying (7.1.18) and (7.1.19). In the special case where $K = \text{diag}(k_i, i \in \{1, 2, 3\})$, the condition that the matrix defined in (7.1.19) is positive definite becomes

$$k_i^2 < e^{-\frac{\gamma}{x_i\lambda_i} x_s^*} D_i, \quad \forall i \in \{1, 2, 3\}, \tag{7.5.1}$$

with

$$D_i := 1 - \frac{2d^2 b_i}{\gamma^2 s_i (1 - s_i \frac{\lambda_i}{\lambda_4})} \left(\sum_{k=1}^{3} b_k s_k (1 - s_k \frac{\lambda_k}{\lambda_4})(e^{\frac{\gamma x_s^*}{x_k \lambda_k}} - 1) \right) (\sum_{j=1}^{3} e^{\frac{\gamma x_s^*}{x_i \lambda_i} - \frac{\gamma x_s^*}{x_j \lambda_j}})(1 - s_i \frac{\lambda_i}{\lambda_4})^2. \tag{7.5.2}$$

Let us look at a limiting case in (7.1.18) and take $b_i = -\gamma e^{-\gamma x_s^*/(x_i \lambda_i)}/3ds_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right)$. Then we have

$$D_i = 1 - \frac{2}{9} \left(\sum_{k=1}^{3} (1 - e^{-\frac{\gamma x_s^*}{x_k \lambda_k}}) \right) (\sum_{j=1}^{3} e^{-\frac{\gamma x_s^*}{x_j \lambda_j}}). \tag{7.5.3}$$

We denote $y = \left(\sum_{k=1}^{3} e^{-\frac{\gamma x_s^*}{x_k \lambda_k}} \right)$. Thus we get

$$D_i : = 1 - \frac{2}{3} y + \frac{2}{9} y^2. \tag{7.5.4}$$

This is a second order polynomial with negative discriminant, thus D_i is always strictly positive. As D_i depends continuously on b_i, this implies that there exist $K = \text{diag}(k_i, i \in \{1, 2, 3\})$ and $(b_1, b_2, b_3)^\mathsf{T}$, satisfying (7.1.18) and (7.1.19).

Acknowledgement

The authors would like to thank Tatsien Li and Sébastien Boyaval for their constant support. They would like also to thank National Natural Science Foundation of China (No. 11771336), ETH-ITS, ETH-FIM, ANR project Finite4SoS (No.ANR 15-CE23-0007) and the French Corps des IPEF for their financial support.

Part III

PI Controllers

Chapter 8

PI controllers for 1-D nonlinear transport equation

This chapter is taken from the following article (also referred to as [59]):

Jean-Michel Coron and Amaury Hayat. PI controllers for 1-D nonlinear transport equation. Preprint, 2018.

Abstract. In this chapter, we introduce a method to obtain necessary and sufficient stability conditions for systems governed by 1-D nonlinear hyperbolic partial-differential equations with closed-loop integral controllers, when the linear frequency analysis cannot be used anymore. We study the stability of a general nonlinear transport equation where the control input and the measured output are both located on the boundaries. The principle of the method is to extract the limiting part of the stability from the solution using a projector on a finite-dimensional space and then use a Lyapunov approach. This chapter improves a result of Trinh, Andrieu and Xu, and gives an optimal condition for the design of the controller. The results are illustrated with numerical simulations where the predicted stable and unstable regions can be clearly identified.

8.1 Introduction

Stabilisation of systems with Proportional-Integral (PI) controllers has been well-studied in more recent decades as it is the most famous boundary control in engineering applications. The use of PI controllers in practical applications goes back to the end of the 18th century with the Perier brothers' pump regulator [72, Pages 50-51 and figure 231, Plate 26], [25, Chapter 2] and later on with Fleeming Jenkin's regulator studied by Maxwell in [156]. Of course these regulators were not yet referred as PI control but in practice they worked similarly. Mathematically the PI control was studied first by Minorsky at the beginning of the 20th century for finite-dimensional systems [159]. In the last decades, the stability of 1-D linear systems with PI control has been well-investigated both for finite-dimensional systems [4, 6] and infinite-dimensional systems (see for instance [21, 84, 130, 169, 194, 199, 200] for hyperbolic systems) and is now very well-known. For infinite-dimensional nonlinear systems, however, only few results are known comparatively, most of them conservative [13, Theorem 2.10], [194]. From a mathematical point of view, dealing nonlinear systems is a very challenging and interesting question. From a practical point of view, it can be seen as a necessity as numerous physical systems are based on infinite dimensional nonlinear models that are sometimes linearized afterward. The intuitive belief that the stability condition for a nonlinear system should be the same as the stability condition for its linearized counterpart when close to the equilibrium is wrong in general, as shown for example in [61].
The reason for this gap in knowledge between linear and nonlinear systems in infinite dimension is that the main method to obtain the stability of 1-D linear systems with PI control is the frequency (or spectrum) analysis (e.g. [200]), a powerful tool based on the Spectral Mapping Property which gives, among other

things, the limit of stability from the differential operator's eigenvalues (e.g. [143, 161, 176]). This powerful tool is not anymore available when dealing with nonlinear systems. Thus, most studies use a Lyapunov approach instead that has the advantage of enabling robust results [47, 119] but as a counterpart is often conservative, meaning that the stability conditions raised are only sufficient and not necessary. Among the necessary and sufficient conditions one can refer for instance to [13, Theorem 2.9]. Another point to mention is that, for nonlinear systems, the exponential stability in the different topologies are not equivalent [61].
In this article we introduce a method to obtain a necessary and sufficient condition on the stability. We study the general scalar transport equation with a PI boundary controller which was studied in [194], and in which the authors obtained a sufficient, although conservative, stability condition.
Not only is this equation interesting in itself [27], as it covers for instance the inviscid Burgers equation around a non-zero constant steady-state, which can be used as a basic model for fluid flows or road trafic, but it is also interesting as, even if it is the most simple nonlinear evolution equation, it already has some of the key features of nonlinear hyperbolic models whose stabilization has been quite studied in the recent years using various methods [13, 106, 114]. This problem has an associated linearized problem where the first eigenvalues making the system unstable are discrete and in finite number. We first extract from the solution of the nonlinear problem the part that would be associated to these eigenvalues in the linear case, using a projector on a finite-dimensional space. In the linearized problem this projected part of the solution is the limiting factor on the stability and it is therefore natural to think that it can also be the limiting factor in the non-linear case. Besides, we know precisely the dynamic of this projection and we can control precisely its decay. Then, a key point is to find a good Lyapunov function for the remaining part of the solution. As the remaining part of the solution is not the limiting factor, the Lyapunov function can be conservative with no harm provided that it gives a sufficient condition that goes beyond the limiting condition corresponding to the projected part.

8.2 Stability of non-linear transport equation with PI boundary condition

We are interested with the following problem

$$\partial_t z + \lambda(z)\partial_x z = 0, \tag{8.2.1}$$
$$z(0,t) = -k_I I(t), \tag{8.2.2}$$
$$\dot{I} = z(L,t), \tag{8.2.3}$$

where λ is a C^2 function with $\lambda(0) = \lambda_0 > 0$ and k_I is a constant. Let $T > 0$, one can show that the system is well-posed in $C^0([0,T], H^2(0,L)) \times C^2([0,T])$ for initial conditions small enough and sufficiently regular. More precisely one has [194]

Theorem 8.2.1. *Let $T > 0$. There exists $\delta(T) > 0$ such that for any $\phi_0 \in H^2(0,L)$ satisfying $|\phi_0|_{H^2} \leq \delta$, the system* (8.2.1)–(8.2.3) *with initial condition (ϕ_0, I_0) such that*

$$I_0 = -k_I^{-1}\phi_0(0),\ \ \phi_0(L) = k_I^{-1}\lambda(\phi_0(0))\phi_0'(0), \tag{8.2.4}$$

has a unique solution $(\phi, I) \in C^0([0,T], H^2(0,L)) \times C^2([0,T])$. Moreover there exists $C(T) > 0$ such that

$$|\phi(t,\cdot)|_{H^2} \leq C(T)\left(|\phi_0(\cdot)|_{H^2}\right). \tag{8.2.5}$$

The interest of this system comes from the fact that it is the most simple nonlinear system with a proportional integral control. However it already constitutes a challenge and, to our knowledge, the most advanced result so far is the following result developed in the recent years [194]:

Theorem 8.2.2. *If $0 < k_I < \lambda(0)\Pi(2-\sqrt{2})/2L$, then the nonlinear system* (8.2.1)–(8.2.3) *is exponentially stable for the H^2 norm, where*

$$\Pi(x) = \sqrt{x(2-x)}e^{-x/2}. \tag{8.2.6}$$

Note that $\Pi(2-\sqrt{2})/2 \approxeq 0.34$. In [194] it is also shown that this result is conservative. In order to study this system, it is interesting to compare it with the corresponding linear case namely the case where λ does not depend on z and (8.2.1) is replaced by

$$\partial_t z + \lambda_0 \partial_x z = 0. \tag{8.2.7}$$

In this case, a necessary and sufficient condition for the stability can be simply obtained from the frequency analysis, by looking at the eigenvalues of the system (8.2.7), (8.2.2), (8.2.3). It is easy to see that these eigenvalues satisfy the following equation [194].

$$k_I + \varrho e^{\frac{\varrho L}{\lambda_0}} = 0. \tag{8.2.8}$$

This implies from [24] that the linear system (8.2.7), (8.2.2), (8.2.3) is exponentially stable if and only if

$$k_I \in \left(0, \frac{\pi\lambda_0}{2L}\right). \tag{8.2.9}$$

In the nonlinear case, it is not possible anymore to use a frequency analysis method. One has to use other methods, as for instance the Lyapunov method, which is one of the most famous as it guarantees some robustness of the result. This method was for instance used in [194] to prove Theorem 8.2.2. However, this method is often conservative as, except in simple cases, it is often difficult to find the right Lyapunov function leading to an optimal condition. As stated in the introduction, we tackle this problem by extracting from the solution the part that limits the stability with a projector and apply our Lyapunov function to the remaining part. Our main result is the following

Theorem 8.2.3. *The nonlinear system* (8.2.1)–(8.2.3) *is exponentially stable for the H^2 norm if*

$$k_I \in \left(0, \frac{\pi\lambda(0)}{2L}\right). \tag{8.2.10}$$

The sharpness of this nonlinear result is suggested from the linear condition (8.2.9). This sharpness can also be illustrated by the following proposition

Proposition 8.2.4. *There exists $k_1 > \pi\lambda(0)/2L$, such that for any $k_I \in (\pi\lambda(0)/2L, k_1)$ the nonlinear system* (8.2.1)–(8.2.3) *is unstable for the H^2 norm.*

In Section 8.3 we introduce a new Lyapunov function that can be seen as a good Lyapunov function for this system but we show why it still leads to a conservative result. In Section 8.4 we introduce a projector to extract from the solution the limiting part for the stability. In Section 8.5 we prove Theorem 8.2.3 and Proposition 8.2.4 using the Lyapunov function and the projector respectively introduced in Section 8.3 and Section 8.4. In Section 8.6 we illustrate these results with a numerical simulation.

8.3 A quadratic Lyapunov function

In this section we first introduce a new Lyapunov function for the system (8.2.1)–(8.2.3). This Lyapunov function can be seen as a good candidate to study the stability for the H^2 norm, but, although it already gives a sufficient condition relatively close to the linear condition (8.2.9), we will show that it is not enough to achieve the optimal condition (8.2.10), which will be the motivation for the next section. As this part is only here to motivate the method of this chapter, we will give a sketch of proof for a Lyapunov function equivalent to the L^2 norm, but the same would apply for a similar Lyapunov function equivalent to the H^2 norm (see Section 8.5).

Let us define $V_0 : L^2(0,L) \times \mathbb{R} \to \mathbb{R}$ by

$$V_0(Z,I) := \int_0^L f(x) e^{-\frac{\mu}{\lambda_0}x} Z^2(x)dx + \left(\int_0^L \alpha Z dx + \beta I\right)^2, \tag{8.3.1}$$

where f is a positive C^1 function to be determined later on and α and β are non-zero constants to be determined later on as well. For any $(Z, I) \in L^2(0,L) \times \mathbb{R}$ one has from Cauchy-Schwarz inequality:

$$\begin{aligned}&\frac{\min\{f(x)e^{-\frac{\mu}{\lambda_0}x} : x \in [0,L]\}}{L}\left(\int_0^L Zdx\right)^2 + \alpha^2\left(\int_0^L Zdx\right)^2 + 2\beta\alpha I\left(\int_0^L Zdx\right) + (\beta I)^2\\ &\leq V_0(Z,I) \leq C_1\left(|Z|^2_{L^2(0,L)} + \beta I^2\right).\end{aligned} \tag{8.3.2}$$

Using that for any $p > 0$, there exists $n_1 \in \mathbb{N}^*$ such that

$$(p+1)a^2 + b^2 - 2ab \geq \frac{p}{n_1}\left(a^2 + b^2\right),\ \forall\ (a,b) \in \mathbb{R}^2, \tag{8.3.3}$$

there exists $C_2 > 0$ such that

$$\frac{1}{C_2}\left(|Z|^2_{L^2(0,L)} + \beta I^2\right) \leq V_0(Z,I) \leq C_2\left(|Z|^2_{L^2(0,L)} + \beta I^2\right). \tag{8.3.4}$$

Thus, our function V_0 is equivalent to the norm on $L^2(0,L) \times \mathbb{R}$ defined by $|(Z,I)| = \left(|Z|^2_{L^2(0,L)} + \beta I^2\right)$. It is therefore enough to find $f \in C^1([0,L],(0,+\infty))$, α and β such that V_0 is exponentially decreasing along all $C^0([0,T],H^2\times\mathbb{R})$ solutions of system (8.2.1)–(8.2.3) to prove that the null steady-state of the system (8.2.1)–(8.2.3) is exponentially stable for the L^2 norm. Let $T > 0$, and let (z, I) be a $C^3([0,T]\times[0,L]) \times C^3([0,T])$ solution of the system (8.2.1)–(8.2.3) (we could get the result for $C^0([0,T],H^2\times\mathbb{R})$ later on by density as in [20, Section 4], this will not be done in this section as it is only a sketch proof). Let us denote $V_0(z(x,\cdot), I(t))$ by $V_0(t)$. Differentiating V_0 with respect to t, using (8.2.1), (8.2.3) and integrating by parts one has

$$\begin{aligned}\frac{dV_0}{dt} =& -\left[\lambda(z(t,x))f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)\right]_0^L + \int_0^L \lambda(0)f'(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx - \mu\int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx\\ &+ \mu\int_0^L \frac{\lambda_0 - \lambda(z(t,x))}{\lambda_0}f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx\\ &+ \int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}\frac{\partial\lambda}{\partial z}z_xz^2 + (\lambda(z(t,x)) - \lambda(0))f'(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx\\ &+ 2\left(\int_0^L \alpha zdx + \beta I(t)\right)\left(-[\alpha\lambda z]_0^L + \int_0^L \alpha\frac{\partial\lambda}{\partial z}z_xzdx + \beta z(t,L)\right).\end{aligned} \tag{8.3.5}$$

Thus using (8.2.2), one has

$$\begin{aligned}\frac{dV_0}{dt} =& -\lambda(z(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}z^2(t,L) - \mu\int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx + \lambda(z(t,0))f(0)I^2(t)k_I^2\\ &- \int_0^L(-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx + \mu\int_0^L \frac{\lambda_0 - \lambda(z(t,x))}{\lambda_0}f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx\\ &+ 2\left(\int_0^L \alpha zdx + \beta I(t)\right)(-\alpha\lambda_0 z(t,L) + \beta z(t,L)\ -\alpha k_I I(t)\lambda(z(t,0)) - \alpha(\lambda(z(t,L)) - \lambda_0)z(t,L))\\ &+ \int_0^L f(x)\frac{\partial\lambda}{\partial z}z_xe^{-\frac{\mu}{\lambda_0}x}z^2 + (\lambda(z(t,x)) - \lambda(0))f'(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx\\ &+ 2\left(\int_0^L \alpha zdx + \beta I(t)\right)\int_0^L \alpha\frac{\partial\lambda}{\partial z}z_xzdx.\end{aligned} \tag{8.3.6}$$

We can now choose $\beta = \lambda_0\alpha$. Equation (8.3.6) becomes

$$
\begin{aligned}
\frac{dV_0}{dt} &= -\lambda(z(t,L))e^{-\frac{\mu}{\lambda_0}L}f(L)z^2(t,L) - \mu\left(\int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx + I(t)^2\right)\\
&+ (\lambda(0)f(0)k_I^2+\mu)I^2(t) - \int_0^L (-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx\\
&- 2\int_0^L \alpha^2 k_I\lambda(0)zI(t)dx - 2\alpha^2\lambda(0)^2k_I I^2(t)\\
&- 2\int_0^L \alpha^2 k_I(\lambda(z(t,0))-\lambda(0))zI(t)dx\\
&- 2\alpha^2(\lambda(z(t,0))-\lambda(0))(\lambda(z(t,0))+\lambda(0))k_I I^2(t)\\
&+ (\lambda(z(t,0))-\lambda(0))f(0)k_I^2 I^2(t)\\
&+ 2\left(\int_0^L \alpha z dx + \beta I(t)\right)(-\alpha(\lambda(z(t,L))-\lambda_0)z(t,L))\\
&+ \int_0^L f(x)\frac{\partial\lambda}{\partial z}z_x e^{-\frac{\mu}{\lambda_0}x}z^2 + (\lambda(z)-\lambda(0))f'(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx\\
&+ 2\left(\int_0^L \alpha z dx + \beta I(t)\right)\int_0^L \alpha\frac{\partial\lambda}{\partial z}z_x z dx.
\end{aligned}
\tag{8.3.7}
$$

Using the equivalence between V_0 and $(|z(t,\cdot)|_{L^2}+|I|)^2$, there exists a constant $C_3>0$, maybe depending continuously on μ but positive for $\mu\in[0,\infty)$ such that

$$
\mu\left(\int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx + I(t)^2\right) \geq \mu C_3 V_0,
\tag{8.3.8}
$$

and as λ is C^1, (8.3.7) can be simplified in

$$
\begin{aligned}
\frac{dV_0}{dt} \leq &-\mu C_3 V_0 - \lambda(z(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}z^2(t,L)\\
&- Q + O\left((|z(t,\cdot)|_{H^2}+|I(t)|)^3\right),
\end{aligned}
\tag{8.3.9}
$$

where $O(r)$ means that there exist $\eta>0$ and $C>0$, both independent of ϕ, I, T and $t\in[0,T]$, such that

$$
(|r|\leq\eta) \Longrightarrow (|O(r)|\leq C_1|r|),
$$

and where Q is the quadratic form defined by

$$
\begin{aligned}
Q :=\, & I^2(t)\left(2\alpha^2\lambda(0)^2k_I - \lambda(0)f(0)k_I^2 - \mu\right)\\
&+ \int_0^L (-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x}z^2(t,x) + 2\alpha^2 z k_I\lambda(0)I(t)dx.
\end{aligned}
\tag{8.3.10}
$$

To ensure the decay of V_0, we would like to make this quadratic form in z and I positive definite with $f>0$. This implies that f is decreasing and $k_I>0$. Then, bringing all the terms inside the integral we would need the discriminant to be positive, i.e.

$$
\frac{1}{L}\left(2\alpha^2\lambda(0)^2k_I - \lambda(0)f(0)k_I^2 - \mu\right)(-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x} - \alpha^4k_I^2\lambda(0)^2 > 0
\tag{8.3.11}
$$

If we place ourselves in the limiting favourable case where Q is only semi-definite positive, and $f(L)=\mu=0$, one has

$$
f'(x)\left(2\alpha^2\lambda(0)k_I - f(0)k_I^2\right) = -L\alpha^4k_I^2.
\tag{8.3.12}
$$

Thus f' is constant and, as $f(L)=0$,

$$-2\lambda(0)\alpha^2 f(0)k_I + f^2(0)k_I^2 + L^2\alpha^4 k_I^2 = 0. \tag{8.3.13}$$

With $\lambda(0)=\lambda_0$, this equation has a positive solution if and only if

$$4\alpha^4 k_I^2\left(\lambda_0^2 - k_I^2 L^2\right) \geq 0. \tag{8.3.14}$$

This is equivalent to $|k_I| \leq \lambda_0/L$. This is the limiting case, to get Q definite positive and V_0 exponentially decreasing we would need to add $V_{0,1}(t)=V_0(z_t,\dot{I})$ and $V_{0,2}(t)=V(z_{tt},\ddot{I})$ to make the Lyapunov function equivalent to the H^2 norm to deal with $O(|z(t,\cdot)|_{H^2}+|I(t)|)$ as in Section 8.5, and we would get the following sufficient condition: $k_I\in(0,\lambda_0/L)$ which is better than the condition given by Theorem 8.2.2, but conservative compared to the necessary condition (8.2.9). This motivates the next section.

8.4 Extracting the limiting part of the solution

In this section we introduce the projector that will enable us to extract from the solution the limiting part for the stability. We start by introducing the operator A,

$$A\begin{pmatrix}\phi\\ I\end{pmatrix} := \begin{pmatrix}-\lambda_0\phi_x\\ \phi(L)\end{pmatrix} \tag{8.4.1}$$

defined on the domain $\mathcal{D}(A)=\{(\phi,I)^{\mathsf{T}}|\phi\in H^2(0,L), I\in\mathbb{R}, \phi(0)=-k_I I\}$. And we note that looking for solutions to the linearized problem (8.2.7), (8.2.2), (8.2.3) can be seen as looking for solutions $(\phi,I)^{\mathsf{T}}\in C^0([0,T],\mathcal{D}(A))$ to the differential problem

$$\begin{pmatrix}\dot{\phi}\\ \dot{I}\end{pmatrix} = A\begin{pmatrix}\phi\\ I\end{pmatrix}. \tag{8.4.2}$$

As mentioned in Section 8.2, we know that any eigenvalue ϱ of this operator satisfies (8.2.8) which, denoting $\varrho\lambda_0^{-1}=\sigma_\varrho + i\omega_\varrho$ with $(\sigma_\varrho,\omega_\varrho)\in\mathbb{R}^2$, is equivalent to

$$\begin{aligned}\lambda_0 e^{\sigma_\varrho L}\left(\omega_\varrho\sin(\omega_\varrho L)-\sigma_\varrho\cos(\omega_\varrho L)\right) =& k_I,\\ \omega_\varrho\cos(\omega_\varrho L)+\sigma_\varrho\sin(\omega_\varrho L) =& 0.\end{aligned} \tag{8.4.3}$$

Assuming (8.2.9), there is a unique solution to (8.4.3) that also satisfies $\omega\in(-\pi/2L,\pi/2L)$ [192, Page 22]. We denote by ϱ_1 the corresponding eigenvalue. In [192] it was shown that this eigenvalue and its conjugate are the eigenvalues with the largest real part and are the limiting factor to the stability in the linear case. Although we do not need this claim in what follows, it explains why we consider this eigenvalue. We suppose that $\omega:=\omega_{\varrho_1}\neq 0$. The special case $\omega_{\varrho_1}=0$ is simpler can be treated similarly (see Remark 8.4.1).

We introduce the following operator:

$$p := \begin{pmatrix}p_1\\ p_2\end{pmatrix} \in \mathcal{L}(\mathcal{D}(A), \mathrm{Span}\{e^{-\frac{\varrho_1}{\lambda_0}x}, e^{-\frac{\bar{\varrho}_1}{\lambda_0}x}\}) \tag{8.4.4}$$

defined by

$$\begin{aligned}p_1\begin{pmatrix}\phi\\ I\end{pmatrix} :=& \alpha_1\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}I\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\ &+\bar{\alpha}_1\left(\int_0^L \phi(x)e^{\frac{\bar{\varrho}_1}{\lambda_0}x}dx + \lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0}L}I\right)e^{-\frac{\bar{\varrho}_1}{\lambda_0}x},\end{aligned} \tag{8.4.5}$$

$$p_2\begin{pmatrix}\phi\\ I\end{pmatrix} :=\frac{\alpha_1}{\varrho_1}\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}I\right)e^{-\frac{\varrho_1}{\lambda_0}L} + \frac{\bar{\alpha}_1}{\bar{\varrho}_1}\left(\int_0^L \phi(x)e^{\frac{\bar{\varrho}_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0}L}I\right)e^{-\frac{\bar{\varrho}_1}{\lambda_0}L}, \tag{8.4.6}$$

where $\bar{z}$ stands for the conjugate of z and $\alpha_1 := \varrho_1/(\varrho_1 L+\lambda_0)$. Here we used a slight abuse of notation and the notation $e^{-\frac{\varrho_1}{\lambda_0}x}$ outside the brackets refers actually to the function $x\to e^{-\frac{\varrho_1}{\lambda_0}x}$ defined on $[0,L]$. One can see that p is real even though ϱ_1 is complex, as p is the sum of a function and its conjugate. Denoting $\varrho_1\lambda_0^{-1}=\sigma+i\omega$, the formulation (8.4.5)–(8.4.6) is equivalent to

$$\begin{aligned}
p_1\begin{pmatrix}\phi\\ I\end{pmatrix} &= \left(\int_0^L \phi(x)e^{\sigma x}\cos(\omega x)dx+\lambda_0 I(t)\cos(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\alpha_1)\sin(\omega x)e^{-\sigma x}+\mathrm{Im}(\alpha_1)\cos(\omega x)e^{-\sigma x}\right)\\
&\quad+\left(\int_0^L \phi(x)e^{\sigma x}\sin(\omega x)dx+\lambda_0 I(t)\sin(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\alpha_1)\cos(\omega x)e^{-\sigma x}-\mathrm{Im}(\alpha_1)\sin(\omega x)e^{-\sigma x}\right).\\
p_2\begin{pmatrix}\phi\\ I\end{pmatrix} &= \left(\int_0^L \phi(x)e^{\sigma x}\cos(\omega x)dx+\lambda_0 I(t)\cos(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\frac{\alpha_1}{\varrho_1})\sin(\omega L)e^{-\sigma L}+\mathrm{Im}(\frac{\alpha_1}{\varrho_1})\cos(\omega x)e^{-\sigma x}\right)\\
&\quad+\left(\int_0^L \phi(x)e^{\sigma x}\sin(\omega x)dx+\lambda_0 I(t)\sin(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\frac{\alpha_1}{\varrho_1})\cos(\omega L)e^{-\sigma L}-\mathrm{Im}(\frac{\alpha_1}{\varrho_1})\sin(\omega L)e^{-\sigma L}\right).
\end{aligned}\tag{8.4.7}$$

However in the following, for simplicity, we will keep the complex formulation. We first show that p commutes with the operator A given by (8.4.1). Indeed one can check that, with

$$p_{1,\varrho_1} := \alpha_1\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}I\right)e^{-\frac{\varrho_1}{\lambda_0}x}, \tag{8.4.8}$$

one has

$$\begin{aligned}
p_{1,\varrho_1}\left(A\begin{pmatrix}\phi\\ I\end{pmatrix}\right) &= p_{1,\varrho_1}\left(\begin{pmatrix}-\lambda_0\phi_x\\ \phi(L)\end{pmatrix}\right)=\alpha_1\left(-\lambda_0\int_0^L \phi_x(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}\phi(L)\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\
&=\alpha_1\left(-\lambda_0\phi(L)e^{\frac{\varrho_1}{\lambda_0}L}+\lambda_0\phi(0)+\varrho_1\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}\phi(L)\right)e^{-\frac{\varrho_1}{\lambda_0}x}.
\end{aligned}\tag{8.4.9}$$

Using that $(\phi,I)^{\intercal}$ belongs to the space $\{(\phi,I)\in L^2(0,L)\times\mathbb{R}|\phi(0)=-k_I I\}$, together with (8.2.8), one gets that

$$\begin{aligned}
p_{1,\varrho_1}\left(A\begin{pmatrix}\phi\\ I\end{pmatrix}\right) &=\alpha_1\varrho_1\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}I\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\
&=-\lambda_0\left(p_{1,\varrho_1}\begin{pmatrix}\phi\\ I\end{pmatrix}\right)_x.
\end{aligned}\tag{8.4.10}$$

As $\bar{\varrho}_1$ also verifies (8.2.8) we get the same for $p_{1,\bar{\varrho}_1}$, which is defined as p_{1,ϱ_1} in (8.4.8) with ϱ_1 instead of ϱ_1. Thus from (8.4.5) and (8.4.8)

$$p_1\left(A\begin{pmatrix}\phi\\ I\end{pmatrix}\right)=\left(A\left(p\begin{pmatrix}\phi\\ I\end{pmatrix}\right)\right)_1. \tag{8.4.11}$$

Then from (8.2.8) and (8.4.6), one easily gets that, for any $(\phi,I)\in L^2(0,L)\times\mathbb{R}$, $p_2((\phi,I)^{\intercal})=-k_I^{-1}p_1((\phi,I)^{\intercal})(0)$, thus $p\begin{pmatrix}\phi\\ I\end{pmatrix}\in\mathcal{D}(A)$ and

$$p\left(A\begin{pmatrix}\phi\\ I\end{pmatrix}\right)=A\left(p\begin{pmatrix}\phi\\ I\end{pmatrix}\right). \tag{8.4.12}$$

Now, we show that p is a projector, meaning that $p \circ p = p$. To avoid overloading the computations, we denote

$$d_1 = \alpha_1 \left(\int_0^L \phi(x) e^{\frac{\varrho_1}{\lambda_0} x} dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0} L} I \right), \tag{8.4.13}$$

and $\bar{d}_1$ is defined similarly with $\bar{\varrho}_1$ instead of ϱ_1. Therefore one has

$$\begin{aligned} p_1 \left(p \begin{pmatrix} \phi \\ I \end{pmatrix} \right) = \alpha_1 & \left(\int_0^L d_1 + \bar{d}_1 e^{(\frac{\varrho_1}{\lambda_0} - \frac{\bar{\varrho}_1}{\lambda_0})x} dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0} L} \left(d_1 \frac{e^{-\frac{\varrho_1}{\lambda_0} L}}{\varrho_1} + \bar{d}_1 \frac{e^{-\frac{\bar{\varrho}_1}{\lambda_0} L}}{\bar{\varrho}_1} \right) \right) e^{-\frac{\varrho_1}{\lambda_0} x} \\ + \bar{\alpha}_1 & \left(\int_0^L \bar{d}_1 + d_1 e^{(\frac{\bar{\varrho}_1}{\lambda_0} - \frac{\varrho_1}{\lambda_0})x} dx + \lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0} L} \left(d_1 \frac{e^{-\frac{\varrho_1}{\lambda_0} L}}{\varrho_1} + \bar{d}_1 \frac{e^{-\frac{\bar{\varrho}_1}{\lambda_0} L}}{\bar{\varrho}_1} \right) \right) e^{-\frac{\bar{\varrho}_1}{\lambda_0} x}. \end{aligned} \tag{8.4.14}$$

Integrating and using (8.2.8), one has

$$\begin{aligned} p_1 \left(p \begin{pmatrix} \phi \\ I \end{pmatrix} \right) &= \alpha_1 \left(d_1 L + \lambda_0 \bar{d}_1 \frac{e^{(\frac{\varrho_1}{\lambda_0} - \frac{\bar{\varrho}_1}{\lambda_0})L} - 1}{\varrho_1 - \bar{\varrho}_1} - \lambda_0 \frac{k_I}{\varrho_1} \left(-\frac{d_1}{k_I} - \frac{\bar{d}_1}{k_I} \right) \right) e^{-\frac{\varrho_1}{\lambda_0} x} \\ &+ \bar{\alpha}_1 \left(\bar{d}_1 L + \lambda_0 d_1 \frac{e^{(\frac{\bar{\varrho}_1}{\lambda_0} - \frac{\varrho_1}{\lambda_0})L} - 1}{\bar{\varrho}_1 - \varrho_1} - \lambda_0 \frac{k_I}{\bar{\varrho}_1} \left(-\frac{d_1}{k_I} - \frac{\bar{d}_1}{k_I} \right) \right) e^{-\frac{\bar{\varrho}_1}{\lambda_0} x} \\ &= \alpha_1 \left(d_1 L + \lambda_0 \frac{d_1}{\varrho_1} + \lambda_0 \bar{d}_1 \frac{e^{(\frac{\varrho_1}{\lambda_0} - \frac{\bar{\varrho}_1}{\lambda_0})L} - 1}{\varrho_1 - \bar{\varrho}_1} + \lambda_0 \frac{\bar{d}_1}{\varrho_1} \right) e^{-\frac{\varrho_1}{\lambda_0} x} \\ &+ \bar{\alpha}_1 \left(\bar{d}_1 L + \lambda_0 \frac{\bar{d}_1}{\bar{\varrho}_1} + \lambda_0 d_1 \frac{e^{(\frac{\bar{\varrho}_1}{\lambda_0} - \frac{\varrho_1}{\lambda_0})L} - 1}{\bar{\varrho}_1 - \varrho_1} + \lambda_0 \frac{d_1}{\bar{\varrho}_1} \right) e^{-\frac{\bar{\varrho}_1}{\lambda_0} x}. \end{aligned} \tag{8.4.15}$$

But, still from (8.2.8), observe that

$$\frac{e^{(\frac{\varrho_1}{\lambda_0} - \frac{\bar{\varrho}_1}{\lambda_0})L} - 1}{\varrho_1 - \bar{\varrho}_1} = \frac{\left(-\frac{k_I}{\varrho_1} \right) \left(-\frac{\bar{\varrho}_1}{k_I} \right) - 1}{\varrho_1 - \bar{\varrho}_1} = -\frac{1}{\varrho_1}, \tag{8.4.16}$$

and recall that $\alpha_1 = \varrho_1 / (\varrho_1 L + \lambda_0)$, thus

$$p_1 \left(p \begin{pmatrix} \phi \\ I \end{pmatrix} \right) = d_1 e^{-\frac{\varrho_1}{\lambda_0} x} + \bar{d}_1 e^{-\frac{\bar{\varrho}_1}{\lambda_0} x} = p_1 \begin{pmatrix} \phi \\ I \end{pmatrix}. \tag{8.4.17}$$

Besides we have from (8.2.8) and (8.4.6)

$$\begin{aligned} p_2 \left(p \begin{pmatrix} \phi \\ I \end{pmatrix} \right) &= -k_I p_1 \left(p \begin{pmatrix} \phi \\ I \end{pmatrix} \right) (0) \\ &= -k_I p_1 \begin{pmatrix} \phi \\ I \end{pmatrix} (0) = p_2 \begin{pmatrix} \phi \\ I \end{pmatrix}. \end{aligned} \tag{8.4.18}$$

Therefore $p \circ p = p$. As p is a linear application, this implies in particular that

$$p \left(\begin{pmatrix} \phi \\ I \end{pmatrix} - p \begin{pmatrix} \phi \\ I \end{pmatrix} \right) = 0. \tag{8.4.19}$$

Thus, let $(\phi, I)^\intercal \in \mathcal{D}(A)$, if we define $\phi_1 = p_1(\phi, I)^\intercal$, $I_1 := p_2(\phi, I)^\intercal$ and $\phi_2 := \phi - \phi_1$ and $I_2 := I - I_1$, one has from (8.4.19) and (8.4.5), as $\alpha_1 \neq 0$

$$\int_0^L \phi_2(x) e^{\frac{\varrho_1}{\lambda_0} x} dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0} L} I_2 = \int_0^L \phi_2(x) e^{\frac{\bar{\varrho}_1}{\lambda_0} x} dx + \lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0} L} I_2 = 0. \tag{8.4.20}$$

Thus

$$\int_0^L \phi_2(x)\left(e^{\frac{\varrho_1}{\lambda_0}(x-L)} - e^{\frac{\bar{\varrho}_1}{\lambda_0}(x-L)}\right)dx = 0. \tag{8.4.21}$$

Or equivalently, denoting as previously $\varrho_1\lambda_0^{-1} = \sigma + i\omega$,

$$\int_0^L \phi_2(x)e^{\sigma(x-L)}\sin(\omega(x-L))dx = 0. \tag{8.4.22}$$

Remark 8.4.1. — *In the special case $\omega = 0$, we can define p similarly as previously but with $\alpha_1 = \bar{\alpha}_1 = 1/2$ instead. Then (8.4.12) still holds, but, as $\varrho_1 = \bar{\varrho}_1$, p is now a projector on the one-dimensional space* Span$\{e^{-\frac{\varrho_1}{\lambda_0}x}\}$ *and is defined by*

$$p_1((\phi, I)^\top) = \left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}I\right)e^{-\frac{\varrho_1}{\lambda_0}x}, \tag{8.4.23}$$

and $p_2((\phi, I)^\top) = \varrho_1^{-1}p_1((\phi, I)^\top)(L)$. Nevertheless (8.4.22) still holds and is straightforward. Indeed, we can still define $(\phi_1, I_1)^\top = p((\phi, I)^\top)$ and $(\phi_2, I_2) = (\phi - \phi_1, I - I_1)$, and, as $\omega = 0$, (8.4.22) holds directly.

— *Note that, when $\omega \neq 0$, (8.4.20) contains two equations, as p is a projector on a space of dimension 2. Therefore another relation can be inferred from (8.4.20) in addition to (8.4.22), namely*

$$\int_0^L \phi_2(x)e^{\sigma(x-L)}\cos(\omega(x-L))dx = -\lambda_0 I_2. \tag{8.4.24}$$

However this relation will not be used in the following.

8.5 Exponential stability analysis

In this section we use the results of the above sections to prove Theorem 8.2.3. We first separate the solution of the system in a projected part and a remaining part using the projector defined in Section 8.4. Then we use the Lyapunov function defined in Section 8.3 to deal with the remaining part.

Linear case We first prove Theorem 8.2.3 using our method in the linear case. This could seem senseless as the result in the linear case is already known. However, the purpose is to make clear how this method works while keeping the computations simple. Then we will show that this method still works when adding the nonlinearities. Let $T > 0$ and let ϕ be a solution of the linear system (8.2.7), (8.2.2), (8.2.3) in $C^1([0, L] \times [0, T])$. Using the last section, we define the following functions

$$\begin{pmatrix}\phi_1(t,x)\\ I_1(t,x)\end{pmatrix} = p\begin{pmatrix}\phi(t,x)\\ I(t)\end{pmatrix}, \tag{8.5.1}$$

$$\begin{pmatrix}\phi_2(t,x)\\ I_2(t,x)\end{pmatrix} = \begin{pmatrix}\phi(t,x)\\ I(t)\end{pmatrix} - \begin{pmatrix}\phi_1(t,x)\\ I_1(t)\end{pmatrix}. \tag{8.5.2}$$

We expect to have extracted from (ϕ, I) the limiting factor for the stability that is now contained in (ϕ_1, I_1). The function (ϕ_1, I_1) is a simple projection on a space of finite dimension, it has therefore a simple dynamic and is easy to control, while we will use our Lyapunov function introduced earlier in Section 8.3 to deal with (ϕ_2, I_2). In other words we will consider the following total Lyapunov function

$$V(t) = V_1(t) + V_2(t), \tag{8.5.3}$$

where V_1 is a Lyapunov function for (ϕ_1, I_1) to be defined and $V_2(t) = V_0(\phi_2(t, \cdot), I(t))$. Recall that the definition of V_0 is given in (8.3.1).

Let us look at ϕ_1. From the definition of p_1 given by (8.4.5), $p_1 = p_{1,\varrho_1} + p_{1,\bar{\varrho}_1}$ where p_{1,ϱ_1} is given by (8.4.8) and $p_{1,\bar{\varrho}_1}$ is given by the same definition with $\bar{\varrho}_1$ instead of ϱ_1. Similarly $p_2 = p_{2,\varrho_1} + p_{2,\bar{\varrho}_1}$ with

$$p_{2,\varrho_1}((\phi, I)^\intercal) = \frac{p_{1,\varrho_1}((\phi, I)^\intercal)(L)}{\varrho_1}, \tag{8.5.4}$$

and $p_{2,\bar{\varrho}_1}$ defined similarly but with $\bar{\varrho}_1$ instead of ϱ_1. Therefore we can define

$$\begin{pmatrix} \phi_{\varrho_1}(t,x) \\ I_{\varrho_1}(t) \end{pmatrix} := p_{\varrho_1} \begin{pmatrix} \phi(t,x) \\ I(t) \end{pmatrix} := \begin{pmatrix} p_{1,\varrho_1}(\phi, I)^\intercal(t,x) \\ p_{2,\varrho_1}(\phi, I)^\intercal(t,L) \end{pmatrix}, \tag{8.5.5}$$

and we can define its conjugate $(\phi_{\bar{\lambda}1}, I_{\bar{\lambda}1})^\intercal$ similarly. Thus we can decompose $(\phi_1, I_1)^\intercal$ in

$$\begin{pmatrix} \phi_1(t,x) \\ I_1(t) \end{pmatrix} = \begin{pmatrix} \phi_{\varrho_1}(t,x) \\ I_{\varrho_1}(t) \end{pmatrix} + \begin{pmatrix} \phi_{\bar{\varrho}_1}(t,x) \\ I_{\bar{\varrho}_1}(t) \end{pmatrix}. \tag{8.5.6}$$

Let us now define $V_1(t)$ by

$$V_1(t) := \int_0^L |\phi_{\varrho_1}(t,x)|^2 dx + |I_{\varrho_1}(t)|^2, \tag{8.5.7}$$

then

$$|\phi_1|_{L^2}^2 + |I_1|^2 \leq 4|\phi_{\varrho_1}|_{L^2}^2 + 4|I_{\varrho_1}|^2 \leq 4V_1. \tag{8.5.8}$$

Differentiating V_1 one has

$$\frac{dV_1}{dt} = \int_0^L 2\mathrm{Re}\left(\partial_t \phi_{\varrho_1} \phi_{\bar{\varrho}_1}\right) dx + 2\mathrm{Re}\left(\dot{I}_{\varrho_1} I_{\bar{\varrho}_1}\right). \tag{8.5.9}$$

From (8.4.5), (8.4.6), and (8.5.1)

$$\begin{pmatrix} \partial_t \phi_{\varrho_1}(t,x) \\ \dot{I}_{\varrho_1}(t) \end{pmatrix} = p_{\varrho_1} \begin{pmatrix} \partial_t \phi(t,x) \\ \dot{I}(t) \end{pmatrix} = p_{\varrho_1} \left(A \begin{pmatrix} \phi(t,x) \\ I(t) \end{pmatrix} \right), \tag{8.5.10}$$

Observe that the commutation property (8.4.11) still holds with p_{ϱ_1} instead of p, and that p_{ϱ_1} is still a linear operator. Therefore,

$$\frac{dV_1}{dt} = 2\mathrm{Re}(\varrho_1)\lambda_0 \int_0^L |\phi_{\varrho_1}|^2 dx + 2\mathrm{Re}(\varrho_1)|I_{\varrho_1}(t)|^2. \tag{8.5.11}$$

As $\mathrm{Re}(\varrho_1) < 0$ from (8.2.10) and (8.2.8),

$$\frac{dV_1}{dt} \leq -2|\mathrm{Re}(\varrho_1)| \min(\lambda_0, 1) V_1 \tag{8.5.12}$$

which will imply the exponential decay.

Let us now look at V_2. From (8.2.7), (8.2.2), (8.2.3), (8.4.6), (8.5.2), (ϕ_2, I_2) is also a solution to the linear system (8.2.7), (8.2.2), (8.2.3). Thus acting similarly as in Section 8.3, (8.3.5)–(8.3.9), we have

$$\begin{aligned} \frac{dV_{2,1}}{dt} \leq & -\mu C_3 V_{2,1} - \lambda(\phi(t,L)) f(L) e^{-\frac{\mu}{\lambda_0} L} \phi_2^2(t,L) - I_2^2(t) \left(2\alpha^2 \lambda(0)^2 k_I - \lambda(0) f(0) k_I^2 - \mu\right) \\ & - \int_0^L (-\lambda(0) f'(x)) e^{-\frac{\mu}{\lambda_0} x} \phi_2^2(t,x) dx + 2\alpha^2 k_I \lambda(0) \phi_2 I(t) dx. \end{aligned} \tag{8.5.13}$$

If we look now at the quadratic form in I_2 and ϕ_2 that appears, we can see that it is exactly the same as previously in (8.3.10). However, since ϕ_2 is the complementary of ϕ_1 in ϕ, we now have an additional information on ϕ_2 given by (8.4.22). Thus, denoting again this quadratic form by Q, and using (8.4.22) we have

$$\begin{aligned}Q &= \int_0^L (-\lambda_0 f'(x))e^{-\frac{\mu}{\lambda_0}x}\phi_2^2(t,x)dx + 2\alpha^2 k_I \lambda_0 I(t)\int_0^L \phi_2(1-\kappa\theta(x))dx \\ &\quad + I^2(t)\left(2\alpha^2\lambda_0^2 k_I - \lambda_0 f(0)k_I^2 - \mu\right) \\ &\geq \inf_{x\in[0,L]}(-\lambda_0 f'(x))e^{-\frac{\mu}{\lambda_0}L}\left(\int_0^L \phi_2^2(t,x)dx\right) \\ &\quad - 2\alpha^2 k_I\lambda_0|I(t)|\left(\int_0^L \phi_2^2 dx\right)^{1/2}\left(\int_0^L (1-\kappa\theta(x))^2dx\right)^{1/2} \\ &\quad + I^2(t)\left(2\alpha^2\lambda_0^2 k_I - \lambda_0 f(0)k_I^2 - \mu\right)\end{aligned} \tag{8.5.14}$$

where

$$\theta(x) := e^{\sigma(x-L)}\sin(\omega(x-L)) \tag{8.5.15}$$

and κ is a constant that can be chosen arbitrarily. As the right-hand side is now a quadratic form in $|\phi_2|_{L^2}$ and I, a sufficient condition for Q to be positive is

$$\begin{aligned}&\inf_{x\in[0,L]}(-\lambda_0 f'(x))e^{-\frac{\mu}{\lambda_0}L}\left(2\alpha^2\lambda_0^2 k_I - \lambda_0 f(0)k_I^2 - \mu\right) \\ &> \left(\alpha^2 k_I\lambda_0\right)^2\left(\int_0^L (1-\kappa\theta(x))^2dx\right).\end{aligned} \tag{8.5.16}$$

Of course we have all interest in choosing κ such that it minimizes the integral of $(1-\kappa\theta(x))^2$. We have

$$\int_0^L (1-\kappa\theta(x))^2dx = \kappa^2\left(\int_0^L \theta^2(x)dx\right) - 2\kappa\left(\int_0^L \theta(x)dx\right) + L. \tag{8.5.17}$$

This is a second order polynomial in κ thus, assuming $\omega \neq 0$, its minimum is

$$\begin{aligned}&L + \frac{\left(\int_0^L \theta(x)dx\right)^2}{\left(\int_0^L \theta^2(x)dx\right)^2}\left(\int_0^L \theta^2(x)dx\right) - 2\frac{\left(\int_0^L \theta(x)dx\right)}{\left(\int_0^L \theta^2(x)dx\right)}\left(\int_0^L \theta(x)dx\right) \\ &= L - \frac{\left(\int_0^L \theta(x)dx\right)^2}{\left(\int_0^L \theta^2(x)dx\right)}.\end{aligned} \tag{8.5.18}$$

Choosing such κ, and f' constant, condition (8.5.16) becomes

$$\begin{aligned}&e^{-\frac{\mu}{\lambda_0}L}\lambda_0\frac{f(0)-f(L)}{L}\left(2\alpha^2\lambda_0^2 k_I - \mu - \lambda_0 f(0)k_I^2\right) \\ &- \left(\alpha^2 k_I\lambda_0\right)^2 L\left(1 - \frac{\left(\int_0^L \theta(x)dx\right)^2}{\left(L\int_0^L \theta^2(x)dx\right)}\right) > 0.\end{aligned} \tag{8.5.19}$$

which is equivalent to

$$\begin{aligned}&-\lambda_0^2 k_I^2 f^2(0) + \left(2\alpha^2\lambda_0^2 k_I - \mu + f(L)\lambda_0 k_I^2\right)\lambda_0 f(0) - (2\alpha^2\lambda_0^2 k_I - \mu)\lambda_0 f(L) \\ &- e^{\frac{\mu}{\lambda_0}L}\left(\alpha^2 k_I\lambda_0\right)^2 L^2\left(1 - \frac{\left(\int_0^L \theta(x)dx\right)^2}{L\left(\int_0^L \theta^2(x)dx\right)}\right) > 0.\end{aligned} \tag{8.5.20}$$

We place ourselves in the limiting case, when $\mu = 0$ and $f(L) = 0$. As the left-hand side is a second order polynomial in $f(0)$, there exists a positive solution $f(0)$ to the inequality if and only if

$$\left(1 - \frac{\left(\int_0^L \theta(x)dx\right)^2}{L\left(\int_0^L \theta^2(x)dx\right)}\right) k_I^2 L^2 < \lambda_0^2. \tag{8.5.21}$$

Under assumption (8.2.10) we can show that this is always verified, this is done in the Appendix. When $\omega = 0$, taking again f' constant and the limiting case where $f(L) = 0$ and $\mu = 0$, Q is definite positive provided that

$$\begin{aligned}
&- \lambda^2(0)k_I^2 f^2(0) + \left(2\alpha^2\lambda_0^2 k_I - \mu + f(L)\lambda_0 k_I^2\right)\lambda_0 f(0) \\
&- (2\alpha^2\lambda_0^2 k_I - \mu)\lambda_0 f(L) - e^{\frac{\mu}{\lambda_0}L}\left(\alpha^2 k_I \lambda_0\right)^2 L^2 > 0.
\end{aligned} \tag{8.5.22}$$

There exists a positive solution $f(0)$ to this inequality if and only if

$$k_I^2 < \left(\frac{\lambda_0}{L}\right)^2, \tag{8.5.23}$$

but, as ϱ_1 is real and k_I is positive, $k_I = -(\lambda_0/L)(\varrho_1 L/\lambda_0)e^{-\varrho_1 L/\lambda_0} < \lambda_0/L$, thus (8.5.23) is satisfied. Thus, by continuity, there always exists $\mu_1 > 0$ and f positive such that $Q > 0$ and therefore

$$\frac{dV_2}{dt} \leq -\mu_1 C_3 V_2 - (\lambda_0 f(L)e^{-\frac{\mu}{\lambda_0}L})\phi_2^2(t, L). \tag{8.5.24}$$

This implies from (8.5.3) and (8.5.12) that

$$\frac{dV}{dt} \leq -\min\left(2|\mathrm{Re}(\varrho_1)|, 2|\mathrm{Re}(\varrho_1)|\lambda_0, \mu_1 C_3\right) V \tag{8.5.25}$$

This shows the exponential decay for V. It remains now only to show that it also implies the exponential decay for (ϕ, I) in the L^2 norm. But from (8.3.4) and (8.5.7), V is equivalent to the norm $(|\phi_{\varrho_1}|_{L^2} + |I_{\varrho_1}| + |\phi_2|_{L^2} + |I_2(t)|)^2$. Besides, we have from (8.4.8), (8.5.7), and using Cauchy-Schwarz inequality

$$\begin{aligned}
V_1(t) &\leq \left(\int_0^L |\alpha_1 e^{-\frac{\varrho_1}{\lambda_0}x}|^2 dx + \left|\frac{\alpha_1 e^{\frac{\varrho_1}{\lambda_0}L}}{\varrho_1}\right|^2\right)\left|\left(\int_0^L e^{2\varrho_1 x}dx\right)^{1/2}\left(\int_0^L \phi^2 dx\right)^{1/2} + Ie^{\frac{\varrho_1}{\lambda_0}L}\right|^2 \\
&\leq C_5\left(|\phi(t,\cdot)|_{L^2}^2 + |I(t)|^2\right),
\end{aligned} \tag{8.5.26}$$

where C_5 is a constant that does not depend on I or ϕ. Also, from (8.3.4), (8.5.26) and noting that $\phi_2 = \phi - \phi_1$ and $I_2 = I - I_1$,

$$\begin{aligned}
V_2(t) &\leq C_2\left(|\phi_2(t,\cdot)|_{L^2}^2 + |I_2(t)|^2\right) \\
&\leq C_6\left(|\phi(t,\cdot)|_{L^2}^2 + |I(t)|^2\right).
\end{aligned} \tag{8.5.27}$$

Thus, from (8.5.26) and (8.5.27), there exists $C_7 > 0$ independent of ϕ and I such that

$$V(t) \leq C_7\left(|\phi(t,\cdot)|_{L^2} + |I(t)|\right), \forall\, t \in [0, T]. \tag{8.5.28}$$

And from (8.3.4), (8.5.8), and (8.5.25),

$$\begin{aligned}
&|\phi(t,\cdot)|_{L^2}^2 + |I(t)|^2 \\
&\leq 4V_1(t) + C_2 V_2(t) \\
&\leq \max(4, C_2)e^{-\gamma t}V(0).
\end{aligned} \tag{8.5.29}$$

Thus, there exists $C_8 > 0$ independent of ϕ and I such that

$$\left(|\phi(t,\cdot)|_{L^2} + |I(t)|\right) \leq C_8 e^{-\gamma t}\left(|\phi(0,\cdot)|_{L^2} + |I(t)|\right). \tag{8.5.30}$$

This concludes the proof of Theorem 8.2.3 in the linear case.

Nonlinear case In this paragraph we show how to adapt the previous method when the system is nonlinear instead. Let $T>0$ and let ϕ be a solution to the nonlinear system (8.2.1)–(8.2.3). We suppose in the following that

$$|\phi(t,\cdot)|_{H^2} \leq \varepsilon,\ \forall\, t\in[0,T], \tag{8.5.31}$$

with $\varepsilon\in(0,1)$ to be chosen later on. This assumption can be done as we are looking for a local result with respect to the perturbations (i.e. the initial conditions), and, from (8.2.5), for any $\varepsilon>0$ there exists $\delta>0$ such that if $|\phi_0|_{H^2}\leq\delta$ then (8.5.31) holds. Let us assume in addition that $\phi\in C^3([0,L]\times[0,T])$ (we will relax this assumption later on using a density argument). We define again $(\phi_1,I_1)^\intercal$ and $(\phi_2,I_2)^\intercal$ as in (8.5.1)–(8.5.2), and $(\phi_1,I_1)^\intercal$ can still be decomposed using (8.5.6). We consider the Lyapunov function

$$V(t)=V_1(t)+V_2(t), \tag{8.5.32}$$

where $V_2(t)=V_{2,1}(t)+V_{2,2}(t)+V_{2,3}(t)$, with $V_{2,k}(t)=V_0(\partial_t^{k-1}\phi_2(t,\cdot),\partial_t^{k-1}I(t))$ (the definition of V_0 is given in (8.3.1)) and V_1 is defined by

$$\begin{aligned} V_1(t) := &\int_0^L |\phi_{\varrho_1}(t,x)|^2+|\partial_t\phi_{\varrho_1}(t,x)|^2+|\partial_t^2\phi_{\varrho_1}(t,x)|^2dx\\ &+|I_{\varrho_1}(t)|^2+|\dot I_{\varrho_1}(t)|^2+|\ddot I_{\varrho_1}(t)|^2. \end{aligned} \tag{8.5.33}$$

Remark 8.5.1. *Note that, strictly speaking, both V_1 and V_2 can be expressed as a functional on time-independent functions belonging to $H^2(0,L)\times\mathbb{R}$, using for instance the following notations for $(\phi,I)\in H^2(0,L)\times\mathbb{R}$:*

$$\begin{aligned} &\dot I := \phi(t,L),\ \ddot I := -\lambda(\phi(L))\partial_x\phi(L),\\ &\partial_t\phi := -\lambda(\phi)\partial_x\phi,\ \partial_t^2\phi := -\lambda'(\phi)\left(\partial_x\phi\right)^2-\lambda(\phi)\partial_x^2\phi. \end{aligned} \tag{8.5.34}$$

Of course these notations correspond to the time-derivatives of the functions when (I,ϕ) is time-dependent and a solution of (8.2.1)–(8.2.3).

Using (8.5.33), there exists $\varepsilon_1\in(0,1)$ such that for $\varepsilon<\varepsilon_1$, one has

$$\begin{aligned} &\frac{1}{2}\min(1,\lambda_0^4)\left(|\phi_{\varrho_1}|^2_{H^2}+|I_{\varrho_1}|^2+|\dot I_{\varrho_1}|^2+|\ddot I_{\varrho_1}|^2\right)\\ &\leq V_1\leq 2\max(1,\lambda_0^4)\left(|\phi_{\varrho_1}|^2_{H^2}+|I_{\varrho_1}|^2+|\dot I_{\varrho_1}|^2+|\ddot I_{\varrho_1}|^2\right), \end{aligned} \tag{8.5.35}$$

and therefore

$$\begin{aligned} &|\phi_1|^2_{H^2}+|I_1|^2+|\dot I_1|^2+|\ddot I_1|^2\\ &\leq 4|\phi_{\varrho_1}|^2_{H^2}+4|I_{\varrho_1}|^2+4|\dot I_{\varrho_1}|^2+4|\ddot I_{\varrho_1}|^2\\ &\leq 8\max(1,\lambda_0^4)V_1. \end{aligned} \tag{8.5.36}$$

which is similar to (8.5.8).
Differentiating V_1 one has, similarly as in (8.5.9),

$$\begin{aligned} \frac{dV_1}{dt} = &\int_0^L 2\mathrm{Re}\left(\partial_t\phi_{\varrho_1}\phi_{\bar\varrho_1}\right)+2\mathrm{Re}\left(\partial_t^2\phi_{\varrho_1}\partial_t\phi_{\bar\varrho_1}\right)+2\mathrm{Re}\left(\partial_t^3\phi_{\varrho_1}\partial_t^2\phi_{\bar\varrho_1}\right)dx\\ &+2\mathrm{Re}\left(\dot I_{\varrho_1}I_{\bar\varrho_1}\right)+2\mathrm{Re}\left(\ddot I_{\varrho_1}\dot I_{\bar\varrho_1}\right)+2\mathrm{Re}\left(\dddot I_{\varrho_1}\ddot I_{\bar\varrho_1}\right). \end{aligned}. \tag{8.5.37}$$

From (8.4.5), (8.4.6), and (8.5.1)

$$\begin{pmatrix}\partial_t\phi_{\varrho_1}(t,x)\\ \dot I_{\varrho_1}(t)\end{pmatrix} = p_{\varrho_1}\begin{pmatrix}\partial_t\phi(t,x)\\ \dot I(t)\end{pmatrix} = p_{\varrho_1}\left(A_1\begin{pmatrix}\phi(t,x)\\ I(t)\end{pmatrix}\right), \tag{8.5.38}$$

where A_1 is now defined for any $(\phi, I)^\intercal \in \mathcal{D}(A)$ by

$$A_1 \begin{pmatrix} \phi \\ I \end{pmatrix} := \begin{pmatrix} -\lambda(\phi)\phi_x \\ \phi(L) \end{pmatrix} = A \begin{pmatrix} \phi \\ I \end{pmatrix} + \begin{pmatrix} (\lambda_0 - \lambda(\phi))\,\phi_x \\ 0 \end{pmatrix}. \tag{8.5.39}$$

Thus using the commutation property with p_{ϱ_1},

$$\begin{aligned}
\begin{pmatrix} \partial_t \phi_{\varrho_1}(t,x) \\ \dot{I}_{\varrho_1}(t) \end{pmatrix} &= A \left(p_{\varrho_1} \begin{pmatrix} \phi(t,x) \\ I(t) \end{pmatrix} \right) + p_{\varrho_1} \begin{pmatrix} (\lambda_0 - \lambda(\phi))\,\partial_x \phi(t,x) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\lambda_0 (\phi_{\varrho_1})_x(t,x) \\ \phi_{\varrho_1}(t,L) \end{pmatrix} + \begin{pmatrix} \alpha_1 e^{-\frac{\varrho_1}{\lambda_0}x} \int_0^L (\lambda_0 - \lambda(\phi)) \partial_x(\phi(t,x)) e^{\frac{\varrho_1}{\lambda_0}x} dx \\ \frac{\alpha_1}{\varrho_1} e^{-\frac{\varrho_1}{\lambda_0}L} \int_0^L (\lambda_0 - \lambda(\phi)) \partial_x(\phi(t,x)) e^{\frac{\varrho_1}{\lambda_0}x} dx \end{pmatrix} \\
&= \varrho_1 \begin{pmatrix} \phi_{\varrho_1}(t,x) \\ I_{\varrho_1}(t) \end{pmatrix} + \begin{pmatrix} \alpha_1 e^{-\frac{\varrho_1}{\lambda_0}x} \int_0^L (\lambda_0 - \lambda(\phi)) \partial_x(\phi(t,x)) e^{\frac{\varrho_1}{\lambda_0}x} dx \\ \frac{\alpha_1}{\varrho_1} e^{-\frac{\varrho_1}{\lambda_0}L} \int_0^L (\lambda_0 - \lambda(\phi)) \partial_x(\phi(t,x)) e^{\frac{\varrho_1}{\lambda_0}x} dx \end{pmatrix}.
\end{aligned} \tag{8.5.40}$$

Besides, let $k \in \{0,1,2\}$, as λ is C^1, integrating by parts and using (8.2.2),

$$\begin{aligned}
&\left| \int_0^L (\lambda_0 - \lambda(\phi)) \partial_t^k \partial_x (\phi(t,x)) e^{\frac{\varrho_1}{\lambda_0}x} dx \right| \\
&= \Big| (\lambda_0 - \lambda(\phi(t,L))) \partial_t^k \phi(t,L) e^{\frac{\varrho_1}{\lambda_0}L} - (\lambda_0 - \lambda(\phi(t,0))) \partial_t^k \phi(t,0) \\
&- \int_0^L \frac{\varrho_1}{\lambda_0} (\lambda_0 - \lambda(\phi(t,x))) \partial_t^k \phi(t,x) e^{\frac{\varrho_1}{\lambda_0}x} \\
&- \lambda'(\phi(t,x)) \phi_t(t,x) \partial_t^k \phi(t,x) e^{\frac{\varrho_1}{\lambda_0}x} dx \Big| \\
&\leq \left| e^{\frac{\varrho_1}{\lambda_0}x} \right|_0 \frac{\varrho_1}{\lambda_0} \left(\int_0^L \left| \lambda_0 - \lambda(\phi(t,x)) - \lambda'(\phi(t,x)) \phi_t(t,x) \right|^2 dx \right)^{1/2} \\
&\times \left(\int_0^L |\partial_t^k \phi(t,x)|^2 dx \right)^{1/2} + O\left(\left|\partial_t^k \phi(t,L)\right| |\phi(t,L)| + |\partial_t^k \phi(t,0)| |\phi(t,0)| \right) \\
&\leq C_0 \left(|\partial_t^k \phi(t,L)| |\phi(t,L)| + |\partial_t^k I| |I| + (|\phi|_0 + |\phi_x|_0) \left| \partial_t^k \phi \right|_{L^2} \right),
\end{aligned} \tag{8.5.41}$$

where $|\cdot|_0$ denotes the C^0 norm or equivalently the L^∞ norm and C_0 is a constant independent of ϕ that depends only on λ, ϱ_1, L and k_I. Thus, using (8.5.41) with $k=0$, and noting that $|\phi|_0 + |\phi_x|_0$ can be bounded by $|\phi|_{H^2}$ from Sobolev inequality, the last term of (8.5.40) is a quadratic perturbation that can be bounded by $\left(|\phi|_{H^2}^2 + |I|^2 + \phi(t,L)^2\right)$. One can do similarly with the second and third time-derivative noticing that

$$\begin{aligned}
\begin{pmatrix} \partial_t^2 \phi \\ \ddot{I} \end{pmatrix} &= \begin{pmatrix} -\lambda_0 \partial_x (\partial_t \phi) \\ \partial_t \phi(t,L) \end{pmatrix} \\
&+ \begin{pmatrix} -\lambda'(\phi)\phi_t \phi_x + (\lambda_0 - \lambda(\phi)) \partial_x(\partial_t \phi) \\ 0 \end{pmatrix} \\
\begin{pmatrix} \partial_t^3 \phi \\ \dddot{I} \end{pmatrix} &= \begin{pmatrix} -\lambda_0 \partial_x (\partial_t^2 \phi) \\ \partial_t^2 \phi(t,L) \end{pmatrix} \\
&+ \begin{pmatrix} -\lambda''(\phi)(\phi_t)^2 \phi_x - 2\lambda'(\phi)\phi_t \phi_{tx} - \lambda'(\phi)\phi_{tt}\phi_x \\ +(\lambda_0 - \lambda(\phi)) \partial_x (\partial_t^2 \phi) \\ 0 \end{pmatrix},
\end{aligned} \tag{8.5.42}$$

and noticing that all the quadratic terms in ϕ involve at most a second derivative in ϕ. Thus as λ is C^2, all the quadratic terms belong to L^1 and their L^1 norm can be bounded by $|\phi|_{H^2}^2$. The L^1 norm of the third order derivative can be bounded by $(|\phi|_{H^2} + |\ddot{I}|^2 + (\partial_{tt}\phi(t,L))^2)$ using (8.5.41) and $k=2$. Therefore, noting from (8.4.5) that $|\partial_t^k \phi_1(t,L)| \leq 2|\bar{\varrho}_1||\partial_t^k I_{\varrho_1}|$, and as $\mathrm{Re}(\varrho_1) < 0$, as previously,

$$\begin{aligned}\frac{dV_1}{dt} \leq & -2|\mathrm{Re}(\varrho_1)|\min(\lambda_0,1)V_1 \\ & + O\left((|\phi_{\varrho_1}|_{H^2} + |\phi_2|_{H^2} + |I_{\varrho_1}| + |\dot{I}_{\varrho_1}| + |\ddot{I}_{\varrho_1}| + |I_2| + |\dot{I}_2| + |\ddot{I}_2|)^3\right) \\ & + C(|\phi_{\varrho_1}|_{H^2} + |I_{\varrho_1}|)(\phi_2^2(t,L) + \partial_t\phi_2^2(t,L) + \partial_t^2\phi_2^2(t,L)),\end{aligned} \tag{8.5.43}$$

where C is a positive constant that only depends on λ, ϱ_1, k_I, L. The first term will imply the exponential decay as previously, while there is now two others terms that will be compensated using V_2.

Let us now look at V_2. From (8.2.1)–(8.2.3), (8.2.8), (8.4.6), (8.5.2), and (8.5.40), (ϕ_2, I_2) is not anymore a solution to the original system but a solution to the following system

$$\begin{aligned}\partial_t\phi_2 + \lambda(\phi)\partial_x\phi_2 &= (\lambda_0 - \lambda(\phi))\partial_x\phi_1 + p_1\left(\begin{pmatrix}(\lambda_0 - \lambda(\phi))\partial_x\phi \\ 0\end{pmatrix}\right) \\ \phi_2(t,0) &= -k_I I_2(t) \\ \dot{I}_2 &= \phi_2(t,L).\end{aligned} \tag{8.5.44}$$

Thus acting again similarly as in Section 8.3, (8.3.5)–(8.3.9), and using (8.5.41), we have

$$\begin{aligned}\frac{dV_{2,1}}{dt} \leq & -\mu C_3 V_{2,1} - \lambda(\phi(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}\phi_2^2(t,L) - I_2^2(t)\left(2\alpha^2\lambda(0)^2 k_I - \lambda(0)f(0)k_I^2 - \mu\right) \\ & - \int_0^L (-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x}\phi_2^2(t,x) + 2\alpha^2 k_I\lambda(0)\phi_2 I(t)dx \\ & + O\left(\left(|\phi_{\varrho_1}|_{H^2} + |I_{\varrho_1}| + |\phi_2|_{H^2} + |I_2| + |\dot{I}_2| + |\ddot{I}_2|\right)^3\right) + C_{2,1}|\phi|_0\left|\phi_2^2(t,L)\right|,\end{aligned} \tag{8.5.45}$$

where $C_{2,1}$ is a positive constant independent of ϕ and I. The quadratic form in I_2 and ϕ_2 that appears is the same as in the linear case, thus, as in (8.5.13)–(8.5.23), and by continuity there exists $\mu_1 > 0$ and f positive such that the quadratic form is positive definite and therefore

$$\begin{aligned}\frac{dV_{2,1}}{dt} \leq & -\mu_1 C_3 V_{2,1} - (\lambda(\phi(t,L))f(L)e^{-\frac{\mu_1}{\lambda_0}L} - C_{2,1}|\phi|_0)\phi_2^2(t,L) \\ & + O\left(\left(|\phi_{\varrho_1}|_{H^2} + |I_{\varrho_1}| + |\phi_2|_{H^2} + |I_2(t)| + |\dot{I}_2(t)| + |\ddot{I}_2(t)|\right)^3\right).\end{aligned} \tag{8.5.46}$$

Let us now deal with $V_{2,2}$ and $V_{2,3}$. Observe that from (8.5.44), one has for $\phi_2 \in C^3$,

$$\begin{aligned}\partial_t^2\phi_2 + \lambda(\phi)\partial_x(\partial_t\phi_2) =& (\lambda_0 - \lambda(\phi))\partial_{tx}^2\phi_1 - p_1\left(\begin{pmatrix}(\lambda_0 - \lambda(\phi))\partial_{tx}^2\phi - \lambda'(\phi)\partial_t\phi\partial_x\phi \\ 0\end{pmatrix}\right) \\ & - \lambda'(\phi)\partial_t\phi\partial_x\phi_1 - \lambda'(\phi)\partial_t\phi\partial_x\phi_2, \\ \partial_t^3\phi_2 + \lambda(\phi)\partial_x(\partial_t^2\phi_2) =& (\lambda_0 - \lambda(\phi))\partial_{ttx}^3\phi_1 - \lambda'(\phi)\partial_t^2\phi\partial_x\phi_1 - 2\lambda'(\phi)\partial_t\phi\partial_{tx}^2\phi_1 \\ & - \lambda''(\phi)(\partial_t\phi)^2\partial_x\phi_1 \\ & - p_1\left(\begin{pmatrix}(\lambda_0 - \lambda(\phi))\partial_{ttx}^3\phi - \lambda''(\phi)(\partial_t\phi)^2\partial_x\phi - 2\lambda'(\phi)\partial_t\phi\partial_{tx}^2\phi - \lambda'(\phi)\partial_{tt}^2\phi\partial_x\phi \\ 0\end{pmatrix}\right) \\ & - 2\lambda'(\phi)\partial_t\phi\partial_{tx}\phi_2 - \lambda'(\phi)\partial_t^2\phi\partial_x\phi_2 - \lambda''(\phi)(\partial_t\phi)^2\partial_x\phi_2.\end{aligned} \tag{8.5.47}$$

and

$$\begin{aligned}\partial_t\phi_2(t,0) = -k_I\dot{I}_2(t), &\qquad \partial_t^2\phi_2(t,0) = -k_I\ddot{I}_2(t), \\ \ddot{I}_2 = \partial_t\phi_2(t,L), &\qquad \dddot{I}_2 = \partial_t^2\phi_2(t,L).\end{aligned} \tag{8.5.48}$$

From (8.5.41), $p_1((\lambda_0-\lambda(\phi))\partial^3_{ttx}\phi)$ can be bounded by $(|\phi|^2_{H^2}+|\ddot{I}|^2+(\partial^2_{tt}\phi(t,L))^2)$ and, from (8.4.5), $\partial_{ttx}\phi_1$ is proportional to $\partial_{tt}\phi_1$. Thus, as all the other terms in the right hand sides are quadratic perturbations and include at most a second order derivative, the L^1 norm of the right-hand sides can be bounded by $(|\phi_{\varrho_1}|^2_{H^2}+|\phi_2|^2_{H^2}+|I_{\varrho_1}|^2+|I_2|^2+|\dot{I}_2|^2+|\ddot{I}_2|^2+\phi^2(t,L)+(\partial_t\phi(t,L))^2+(\partial^2_{tt}\phi(t,L))^2)$, which is small compared to the first-order term in the left-hand sides. Therefore we have, as previously

$$\begin{aligned}\frac{dV_{2,k}}{dt} &\leq -\mu_1 C_3 V_{2,k} - \lambda(\phi(t,L))f(L)e^{-\frac{\mu_1}{\lambda_0}L}(\partial_t^k\phi_2)^2(t,L)\\ &-(\partial_t^k I_2)^2(t)\left(2\alpha^2\lambda(\phi(t,0))^2k_I-\lambda(\phi(t,0))f(0)k_I^2-\mu_1\right)\\ &-\int_0^L(-\lambda_0 f'(x))e^{-\frac{\mu_1}{\lambda_0}x}(\partial_t^k\phi_2)^2(t,x)+2\alpha^2k_I\lambda(\phi(t,0))\partial_t^k\phi_2 I(t)dx\\ &+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|I_{\varrho_1}|+|\phi_2|_{H^2}+|I_2|+|\dot{I}_2|+|\ddot{I}_2|\right)^3\right)\\ &+C_{2,k}|\phi|_0\left|(\partial_t^{k-1}\phi_2)(t,L)\right|^2,\ \text{for } k=2,\ 3,\end{aligned} \tag{8.5.49}$$

where $C_{2,k}$ are positive constants independent of ϕ and I. Besides, from (8.4.22), for $k=2$, 3,

$$\int_0^L \partial_t^{k-1}\phi_2(t,x)\left(e^{\sigma(x-L)}\sin(\omega(x-L))\right)dx=0, \tag{8.5.50}$$

Thus we can perform exactly as for $V_{2,1}$ and consequently

$$\begin{aligned}\frac{dV_{2,k}}{dt} &\leq -\mu_1C_3V_{2,k}-\left(\lambda(\phi(t,L))f(L)e^{-\frac{\mu_1}{\lambda_0}L}-C_{2,k}|\phi|_0\right)\left|\partial_t^{k-1}\phi_2(t,L)\right|^2\\ &+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|I_{\varrho_1}|+|\phi_2|_{H^2}+|I_2|+|\dot{I}_2|+|\ddot{I}_2|\right)^3\right),\ \text{for } k=2,3,\end{aligned} \tag{8.5.51}$$

thus, from (8.5.24) and (8.5.51),

$$\begin{aligned}\frac{dV_2}{dt} &\leq -\mu_1C_3V_2-\sum_{k=1}^3\left(\lambda(\phi(t,L))f(L)e^{-\frac{\mu_1}{\lambda_0}L}-C_{2,k}|\phi|_0|\partial_t^{k-1}\phi|_{L^2}\right)\left|\partial_t^{k-1}\phi_2(t,L)\right|^2\\ &+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|I_{\varrho_1}|+|\phi_2|_{H^2}+|I_2|+|\dot{I}_2|+|\ddot{I}_2|\right)^3\right).\end{aligned} \tag{8.5.52}$$

Thus, from (8.5.32) and (8.5.43),

$$\begin{aligned}\frac{dV}{dt} &\leq -\min\left(2|\mathrm{Re}(\varrho_1)|,2|\mathrm{Re}(\varrho_1)|\lambda_0,\mu_1C_3\right)V\\ &-\left(\lambda(\phi(t,L))f(L)e^{-\frac{\mu_1}{\lambda_0}L}-C_4|\phi|_{H^2}\right)\left(|\phi_2(t,L)|^2+|\partial_t\phi_2(t,L)|^2+\left|\partial_t^2\phi_2(t,L)\right|^2\right)\\ &+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|I_{\varrho_1}|+|\dot{I}_{\varrho_1}|+|\ddot{I}_{\varrho_1}|+|\phi_2|_{H^2}+|I_2(t)|+|\dot{I}_2(t)|+|\ddot{I}_2(t)|\right)^3\right).\end{aligned} \tag{8.5.53}$$

But from (8.3.4) and (8.5.35), V is equivalent to the norm $\Big(|\phi_{\varrho_1}|_{H^2}+|I_{\varrho_1}|+|\dot{I}_{\varrho_1}|+|\ddot{I}_{\varrho_1}|+|\phi_2|_{H^2}+|I_2(t)|$ $+|\dot{I}_2(t)|+|\ddot{I}_2(t)|\Big)^2$. Using Cauchy-Schwarz inequality and (8.4.8)–(8.5.33) as previously,

$$\begin{aligned}V_1(t) &\leq \left(\int_0^L|\alpha_1e^{-\frac{\varrho_1}{\lambda_0}x}|^2dx+\left|\frac{\alpha_1e^{\frac{\varrho_1}{\lambda_0}L}}{\varrho_1}\right|^2\right)\sum_{k=1}^3\left|\left(\int_0^L e^{2\varrho_1x}dx\right)^{1/2}\left(\int_0^L\partial_t^{k-1}\phi^2dx\right)^{1/2}+\partial_t^{k-1}Ie^{\frac{\varrho_1}{\lambda_0}L}\right|^2\\ &\leq C_5\left(|\phi(t,\cdot)|^2_{H^2}+|I(t)|^2+|\dot{I}(t)|^2+|\ddot{I}(t)|^2\right),\end{aligned} \tag{8.5.54}$$

where C_5 is a constant that does not depend on I or ϕ. Then using, (8.3.4), (8.5.26), noting that $\phi_2 = \phi - \phi_1$ and $I_2 = I - I_1$,

$$\begin{aligned} V_2(t) &\leq C_2 \left(|\phi_2(t,\cdot)|_{H^2}^2 + |I_2(t)|^2 + |\dot{I}_2(t)|^2 + |\ddot{I}_2(t)|^2 \right) \\ &\leq C_6 \left(|\phi(t,\cdot)|_{H^2}^2 + |I(t)|^2 + |\dot{I}(t)|^2 + |\ddot{I}(t)|^2 \right), \end{aligned} \tag{8.5.55}$$

which implies that

$$\begin{aligned} &\left(|\phi_{\varrho_1}|_{H^2} + |I_{\varrho_1}| + |\dot{I}_{\varrho_1}| + |\ddot{I}_{\varrho_1}| + |\phi_2|_{H^2} + |I_2(t)| + |\dot{I}_2(t)| + |\ddot{I}_2(t)| \right) \\ &= O\left(|\phi|_{H^2} + |I| + |\dot{I}| + |\ddot{I}| \right). \end{aligned} \tag{8.5.56}$$

But from (8.2.2)–(8.2.3) and Sobolev inequality,

$$\left(|\phi|_{H^2} + |I| + |\dot{I}| + |\ddot{I}| \right) = O\left(|\phi|_{H^2} \right). \tag{8.5.57}$$

Therefore, from (8.5.25), (8.5.56)–(8.5.57), and (8.5.31), there exists $\gamma > 0$ and $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $\varepsilon \in (0, \varepsilon_2)$, one has

$$\frac{dV}{dt} \leq -\gamma V. \tag{8.5.58}$$

This shows the exponential decay for V. As in the linear case, it remains now only to show that it also implies the exponential decay for ϕ in the H^2 norm. Note that, compared to the linear case where we showed the exponential decay of (ϕ, I) in the L^2 norm, here we can actually show the exponential decay of ϕ in the H^2 norm as it is implied by the exponential decay of (ϕ, I) in the H^2 norm from (8.5.57). Observe first that from (8.5.26)–(8.5.27) and (8.5.57) there exists $C_7 > 0$ independent of ϕ and I such that

$$V(t) \leq C_7 |\phi(t,\cdot)|_{H^2}, \forall\, t \in [0, T]. \tag{8.5.59}$$

And from (8.3.4), (8.5.8), and (8.5.58),

$$\begin{aligned} &|\phi(t,\cdot)|_{H^2} + |I(t)| + |\dot{I}(t)| + |\ddot{I}(t)| \\ &\leq 4\max(1, \lambda_0^2) V_1(t) + C_2 V_2(t) \\ &\leq \max(4, 4\lambda_0^2, C_2) e^{-\gamma t} V(0). \end{aligned} \tag{8.5.60}$$

Thus, there exists $C_8 > 0$ independent of ϕ and I such that

$$|\phi(t,\cdot)|_{H^2} \leq C_8 e^{-\gamma t} \left(|\phi(0,\cdot)|_{H^2} \right). \tag{8.5.61}$$

So far ϕ is assumed to be of class C^3, however since this inequality only involves the H^2 norm of ϕ, this can be extended to any solution $(\phi, I) \in C^0([0,T], H^2(0,L)) \times C^1([0,T])$ of the system (8.2.1)–(8.2.3) (see for instance [20] for more details). This concludes the proof of Theorem 8.2.3. We now prove Proposition 8.2.4, which follows rapidly from the proof of Theorem 8.2.3.

Proof of Proposition 8.2.4. From (8.8.15) in the Appendix, one can see that (8.5.21) still holds with $k_I = \pi\lambda_0/2L$. Thus by continuity there exists $k_1 > \pi\lambda_0/2L$ such that for any $k_I \in (\pi\lambda_0/2L, k_1)$ (8.5.21) still holds and consequently the quadratic form Q given by (8.5.14) is still definite positive. Suppose now by contradiction that the system is stable for the H^2 norm. Then for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for any initial condition $(\phi_0, I_0) \in H^2(0,L) \times \mathbb{R}$ such that $(|\phi_0|_{H^2} + |I_0|) \leq \delta_1$ and satisfying the compatibility condition $I_0 = -k_I^{-1}\phi_0(0)$ and $\phi(L) = k_I^{-1}\lambda(\phi_0(0))\phi_0'(0)$, the associated solution (ϕ, I) is defined on $[0, +\infty)$ and

$$(|\phi|_{H^2} + |I|) \leq \varepsilon,\ \forall\, t \in [0, +\infty). \tag{8.5.62}$$

Let $\Theta > 0$, from (8.5.11) and (8.5.52), using that $Q > 0$,

$$\begin{aligned}\frac{dV_1 - \Theta V_2}{dt} &\geq 2\text{Re}(\varrho_1)\min(\lambda_0, 1)V_1 + \mu\Theta C_3 V_2 \\ &+ \left(\Theta f(L)\lambda_0 e^{-\mu\frac{L}{\lambda_0}} - C_9(1+\Theta)\left(|\phi|_{H^2} + |I| + |\dot{I}| + |\ddot{I}|\right)\right)\left(\sum_{k=1}^{k=3}\left|(\partial_t^{k-1}\phi_2)(t,L)\right|^2\right) \\ &+ O\left(\left(|\phi_{\varrho_1}|_{H^2} + |I_{\varrho_1}| + |\phi_2|_{H^2} + |I_2(t)| + |\dot{I}_2(t)| + |\ddot{I}_2(t)|\right)^3\right),\end{aligned} \tag{8.5.63}$$

where C_9 is a constant independent of ϕ and I. We can choose (ϕ_0, I_0) satisfying the compatibility conditions and $\Theta > 0$ such that $c := (V_1 - \Theta V_2)(0) > 0$, and $(|\phi_0|_{H^2} + |I_0|) \leq \delta$ with δ to be chosen. Actually Θ only depends on the ratio between V_1 and V_2 thus it can be made independent of δ by simply rescaling $|\phi_0|_{H^2}$ and $|I_0|$. Using (8.5.63) and (8.5.57) there exists $\gamma_2 > 0$ and $\varepsilon > 0$ such that, if $(|\phi|_{H^2} + |I|) \leq \varepsilon$, then

$$\frac{dV_1 - \Theta V_2}{dt} \geq \gamma_2(V_1 - \Theta V_2). \tag{8.5.64}$$

Thus, from (8.5.62) and the stability hypothesis, we can choose $\delta > 0$ such that (8.5.64) holds. This implies that

$$(V_1 - \Theta V_2)(t) \geq ce^{\gamma_2 t},\ \forall\ t \in [0, +\infty), \tag{8.5.65}$$

which contradicts (8.5.62). This ends the proof of Proposition 8.2.4. □

Remark 8.5.2. *This last proof is limited by the limit value of k_I for which Q is not positive definite anymore. This is due to the fact that we have only extracted the first limiting eigenvalues from the solution. It is natural to think that we could apply the same method to extract a finite number of eigenvalues instead and separate (ϕ, I) in (ϕ_1, I_1), its projection on a n-dimensional space, and (ϕ_2, I_2). Then we would deduce more constraints like* (8.4.22) *on (ϕ_2, I_2), which would increase the upper bound of k_I for which Q defined in* (8.5.14) *is definite positive, and thus the bound k_1 for which Proposition* (8.2.4) *holds, and maybe, by increasing this number of eigenvalues, prove that this proposition holds for arbitrary large k_1.*

8.6 Numerical simulations

In this section we give a numerical simulation that illustrates Theorem 8.2.3 and Proposition 8.2.4. In this example we use $\lambda(z) = 1 + z$, this corresponds to the study of the Burgers equation around the constant steady-state with value 1, as in this case $y = 1 + z$ is solution to the Burgers equation $\partial_t y + y\partial_x y = 0$.

8.7 Conclusion

In this article we studied the exponential stability of a general nonlinear transport equation with integral boundary controllers and we introduced a method to obtain an optimal stability condition through a Lyapunov approach, by extracting first the limiting part of the stability from the solution using a projector on a finite-dimension space. We believe that this method could be used for many other systems and could be useful in the future as, for many nonlinear systems governed by partial differential equations, the stability conditions that are known today are only sufficient and may still be improved.

8.8 Appendix

In this section we prove (8.5.21) under assumption (8.2.10). Note that this is equivalent to

$$\frac{\left(\int_0^L \theta(x)dx\right)^2}{L\left(\int_0^L \theta^2(x)dx\right)} > 1 - \frac{\lambda_0^2}{k_I^2 L^2}. \tag{8.8.1}$$

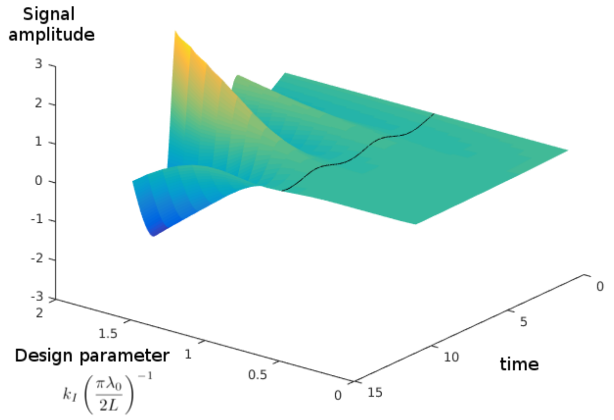

Figure 8.1 – Example of numerical simulations of $\phi(t,0)$ with respect to t varying between 0 and 10 for various values of k_I between $0.1k_{I,c}$ to $2k_{I,c}$, where $k_{I,c} = \pi\lambda_0/2L$ is the critical value of Theorem 8.2.3 and Proposition 8.2.4. The black line represents the trajectory for $k_I = k_{I,c}$. On the left k_I is larger and the system is unstable, and on the right k_I is smaller and the system is stable. As expected, the exponential decay observed in the stable region is the same as the exponential decay given by the real part of ρ_1. The system parameters are chosen such that $\lambda(z) = 1+z$, $\lambda_0 = L = 1$, and $\phi_0(x) = 0.1$ on $[0, L/2]$ and $\phi_0(L) = 0$ so that ϕ_0 satisfies the compatibility conditions (8.2.4) for any $k_I \in [0.1k_{I,c}, 2k_{I,c}]$. The simulations are obtained by a finite-difference method.

By definition of ϱ_1 (see Section 8.4) and (8.4.3), we have

$$\sigma = -\frac{\omega}{\tan(\omega L)}, \tag{8.8.2}$$

and using (8.4.3) and (8.8.2)

$$\frac{\lambda_0}{k_I L} = -\frac{\sin(\omega L)}{\omega L} e^{\frac{\omega L}{\tan(\omega L)}}. \tag{8.8.3}$$

Condition (8.8.1) thus becomes

$$\frac{\left(\int_0^L \theta(x)dx\right)^2}{L\left(\int_0^L \theta^2(x)dx\right)} + \frac{\sin^2(\omega L)}{(\omega L)^2} e^{2\frac{\omega L}{\tan(\omega L)}} - 1 > 0. \tag{8.8.4}$$

From (8.2.8) and the definition of θ given by (8.5.15), we have

$$\int_0^L \theta(x)dx = \frac{\omega}{\sigma^2 + \omega^2}. \tag{8.8.5}$$

Using (8.8.2),

$$\left(\int_0^L \theta(x)dx\right)^2 = \frac{\sin^4(\omega L)}{\omega^2}. \tag{8.8.6}$$

Similarly we have

$$\int_0^L \theta^2(x)dx = \frac{\sigma e^{-2\sigma L}(\sigma\cos(2\omega L) - \omega\sin(2\omega L)) + (\omega^2+\sigma^2) - \sigma^2 - e^{-2\sigma L}(\sigma^2+\omega^2)}{4\sigma(\sigma^2+\omega^2)}. \tag{8.8.7}$$

Therefore, using again (8.8.2) and the fact that $(1+\tan^{-2}(\omega L)) = \sin^{-2}(\omega L)$,

$$\begin{aligned}\int_0^L \theta^2(x)dx = &\left(\frac{\frac{\cos^2(\omega L)}{\sin^2(\omega L)} e^{2\frac{\omega L}{\tan(\omega L)}}(\cos^2(\omega L)+\sin^2(\omega L)) - e^{2\frac{\omega L}{\tan(\omega L)}}\frac{1}{\sin^2(\omega L)}+1}{-4\omega\sin^{-2}(\omega L)}\right)\tan(\omega L)\\ =&\frac{(e^{2\frac{\omega L}{\tan(\omega L)}}-1)}{4\omega}\sin^2(\omega L)\tan(\omega L).\end{aligned} \tag{8.8.8}$$

Therefore using (8.8.6) and (8.8.8), condition (8.8.4) becomes

$$\frac{4\sin^2(\omega L)}{(\omega L)\tan(\omega L)(e^{2\frac{\omega L}{\tan(\omega L)}}-1)} + \frac{\sin^2(\omega L)}{(\omega L)^2} e^{2\frac{\omega L}{\tan(\omega L)}} - 1 > 0, \tag{8.8.9}$$

which is equivalent to

$$\left(2\frac{\frac{2\omega L}{\tan(\omega L)}}{(e^{\frac{2\omega L}{\tan(\omega L)}}-1)} + e^{\frac{2\omega L}{\tan(\omega L)}}\right)\frac{\sin^2(\omega L)}{(\omega L)^2} - 1 > 0. \tag{8.8.10}$$

Note that, under assumption (8.2.10) and from the definition of ϱ_1, $\omega L \in (-\pi/2, \pi/2)$, which implies that $2(\omega L)/\tan(\omega L) \in (0,2)$. Hence, let us study the function $g : X \to (2X/(e^X-1)+e^X)$ on $(0,2)$. Taking its derivative one has

$$g'(X) = \frac{(e^X-1)(2+e^X(e^X-1)) - 2Xe^X}{(e^X-1)^2}. \tag{8.8.11}$$

Taking again the derivative of the numerator of the right-hand side of (8.8.11), one has

$$\begin{aligned}((e^X-1)(2+e^X(e^X-1)) - 2Xe^X))' =&(e^X-1)(e^X(e^X-1)+e^{2X})\\ &+ e^X(2+e^X(e^X-1)) - 2e^X - 2Xe^X.\end{aligned} \tag{8.8.12}$$

Thus using that $X < e^X - 1$ on $(0,+\infty)$ and in particular on $(0,2)$, we get

$$((e^X-1)(2+e^X(e^X-1)) - 2Xe^X))' > (e^X-1)(e^X(e^X-1) + 2e^{2X} - 2e^X) > 0. \tag{8.8.13}$$

Hence g' is non-decreasing on $(0,2)$. But, from (8.8.11), $g'(0) = 0$, therefore g is non-decreasing on $(0,2)$. As $\lim_{X\to 0} g(X) = 3$, we have

$$\left(2\frac{\frac{2\omega L}{\tan(\omega L)}}{(e^{2\frac{\omega L}{\tan(\omega L)}}-1)} + e^{2\frac{\omega L}{\tan(\omega L)}}\right)\frac{\sin^2(\omega L)}{(\omega L)^2} - 1 \geq 3\frac{\sin^2(\omega L)}{(\omega L)^2} - 1, \tag{8.8.14}$$

and, as $x \to \sin(x)/x$ is positive and decreasing on $[0,\pi/2]$, we have

$$\left(2\frac{\frac{2\omega L}{\tan(\omega L)}}{(e^{2\frac{\omega L}{\tan(\omega L)}}-1)} + e^{2\frac{\omega L}{\tan(\omega L)}}\right)\frac{\sin^2(\omega L)}{(\omega L)^2} - 1 \geq \frac{12}{\pi^2} - 1 > 0. \tag{8.8.15}$$

Hence (8.8.10) holds and therefore condition (8.8.1) holds as well. This ends the proof of (8.5.21) under assumption (8.2.10).

Acknowledgments

The authors would like to thank Peipei Shang and Sébastien Boyaval for their valuable remarks. The authors would also like to thank the ETH-FIM and the ETH-ITS for their support and their warm welcome. Finally, the authors would also like to thank the ANR project Finite 4SoS ANR 15-CE23-0007.

Chapter 9

PI controllers for the general Saint-Venant equations

This chapter is taken from the following article (also referred to as [108]):

Amaury Hayat. PI controllers for the general Saint-Venant equations. Preprint, 2019.

Abstract. We study the exponential stability in the H^2 norm of the nonlinear Saint-Venant (or shallow water) equations with arbitrary friction and slope using a single Proportional-Integral (PI) control at one end of the channel. Using a local dissipative entropy we find a simple and explicit condition on the gain the PI control to ensure the exponential stability of any steady-states. This condition is independent of the slope, the friction coefficient, the length of the river, the inflow disturbance and, more surprisingly, can be made independent of the steady-state considered. When the inflow disturbance is time-dependant and no steady-state exist, we still have the Input-to-State stability of the system, and we show that changing slightly the PI control enables to recover the exponential stability of slowly varying trajectories.

9.1 Introduction

Discovered in 1871, the Saint-Venant equations [10] (or 1-D shallow water equations) are among the most famous equations in fluid dynamics and have been investigated in hundreds of studies. Their richness, although being quite simple, has made them become a major tool in practice for many industrial goal, the most famous being probably the regulation of navigable rivers. They are the ground model for such purpose in France and Belgium. Regulation of rivers is a major issue, for navigation, freight transport, renewable energy production, but also for safety reasons, especially as several nuclear plants all around the world are implanted close to rivers. For these reasons, the stability of the steady-states of the Saint-Venant equations has been, and is still, a major issue.

Many results were obtained in the last decades. In 1999, the robust stability of the homogeneous linearized Saint-Venant equations was shown using a Lyapunov approach and proportional feedback controllers [52]. Later the stability of the homogeneous nonlinear Saint-Venant equations was achieved, still using proportional feedback controllers. In 2008, through a semi-group approach [83], the stability of the inhomogeneous nonlinear Saint-Venant equation was shown for sufficiently small friction and slope (or equivalently sufficiently small canal), and these results were successfully applied to real data sets from the Sambre river in Belgium. More recently, in [16] the authors have given sufficient conditions to stabilize the nonlinear Saint-Venant equations with arbitrary friction for the H^2 norm but no slope using again proportional feedback controllers, and in [110] with both arbitrary friction and slope. This last result being proved by exhibiting an explicit local entropy for the nonlinear inhomogeneous Saint-Venant equations.
It is worth mentioning that other stability results have also been obtained in less classical cases or with less

classical feedbacks. For instance, in [19] was shown the rapid stabilization of the homogeneous nonlinear Saint-Venant equations when a shock (e.g. a hydraulic jump) occurs in the target steady-state. Such shock induces new difficulties and the presence of shocks can limit in general the controllability and the stability in weaker norms of hyperbolic systems with boundary controls [2, 34]. Also, several results (e.g. [65]) were obtained using a backstepping approach, a very powerful method based on a Volterra transformation, developed mainly for PDE in [128], and generalized recently with a Fredholm transformation for hyperbolic systems [60, 202, 203]. One may look at [110] for a more detailed survey about this method and its use for the Saint-Venant equations. However, backstepping gives rise to non-local and non-static feedback laws that are likely to be harder to implement, and, to our knowledge, have not been implemented yet.

Most of the previous results were performed with static proportional feedback controllers. When it comes to industrial applications, however, the proportional integral (PI) control is by far the most popular regulator. It is used for instance for the regulation of the Sambre and Meuse river in Belgium [13, Chapter 8]. The reason behind such preference is the robustness of the PI control with off-set errors [6, Chap. 11.3]. An example can be found in [84] where the authors show the interest of adding an integral term to a proportionnal control on a linear and homogeneous system, and exhibit coherent experimental result.
For these reasons, the PI controller has fed a wide literature, at least when used on finite dimensional systems. However, despite their indisputable practical interest, PI controllers for nonlinear infinite dimensional systems have shown hard to handle mathematically and even studying simple systems give sometimes rise to lengthy proofs with relatively sophisticated tools [59]. While the behaviour and the stability of linearized equations with PI controller has been well understood in the past, partly thanks to spectral tools like the spectral mapping theorem (e.g. [143, 161] for hyperbolic systems), no such tools exist for nonlinear systems and the stability of the nonlinear Saint-Venant equations has remained a challenge until today. Among the existing linear result using a spectral approach one can refer to [200, 201] where the authors find a sufficient condition for the stabilization of the linearized inhomogeneous Saint-Venant equations. Necessary and sufficient conditions for the linearized homogeneous Saint-Venant equations are given in [13, Section 2.2.4.1, 3.4.4]. In [64] the authors find a necessary and sufficient condition for a linear scalar equation and show the difficulty of finding good conditions for the nonlinear equation, while in [22] the authors deal with 2×2 systems. Among the existing nonlinear results one can refer to [188] in the case where the operator without PI control generates an exponentially stable semi-group, [193] where the authors find a sufficient condition for the nonlinear homogeneous Saint-Venant equations, [13, 2.2.4.2] where the authors find a necessary and sufficient condition also for the nonlinear homogeneous Saint-Venant equations, while [13, 5.4.4,5.5] give a sufficient condition for the inhomogeneous Saint-Venant equations for a single channel or a network, but in the particular case of constant steady-states only, which simplifies their analysis [106]. Strictly speaking this last result was derived for the linearized system but with a Lyapunov approach which can easily be generalized to the nonlinear system. More recently, and to our knowledge this is the most advanced result, [17] gave a sufficient condition of stability for the inhomogeneous Saint-Venant equations with an arbitrary friction and river length but only in the absence of slope, using a Lyapunov approach.

In this chapter, we consider the stabilization of the general nonlinear Saint-Venant equations with a single boundary PI control. We give a simple and explicit condition on the parameters of the PI controller such that any steady-state is exponentially stable for the H^2 norm. While stability results in inhomogeneous and nonlinear systems often imply a limit length for the domain, depending on the source term, above with we are unable to guarantee any stability ([11, 83, 106, 107] or [13, Chap. 6]), this result holds whatever the friction, the slope, and the length of the channel. Besides, our condition is independent of the slope, the friction coefficient, the river length, and, more surprisingly, can be made independent of the steady-state considered. Finally, when there is no slope this condition is less restrictive that the condition obtained in [17] and when there is no friction or slope this condition coincides with the necessary and sufficient spectral condition of stability for the linearized system given in [22] and [13, Theorem 2.7].
The case where the inflow disturbances are time dependent and no steady-states exists was seldom considered in the literature. However, it is in fact unlikely that the industrial target state is a real steady-state as the inflow disturbance often depends on time in practice, even though only slowly. Therefore, in the more general framework of slowly time-varying target states, we show the Input-to-State Stability (ISS) of the system with

respect to the variation of the inflow disturbance. Finally, we show that if we allow the controller to depend on the target state, by changing slightly the PI controller, we can ensure the exponential stability of slowly-varying target trajectories that are the natural target trajectories to consider when there is no steady-state of the system.
This paper is organised as follows: in Section 9.2 we give a description of the nonlinear Saint-Venant equations, we introduce the time-varying target trajectories together with some definitions and existence results, then we state our main results. In Section 9.3 we prove our main result, Theorem 9.2.3, that deals with the exponential stability of time-varying state. In the Appendix, we show that Corollary 2 dealing with the exponential stability of steady-states, and Theorem 9.2.4 showing the ISS of the system with respect to the variation of the inflow disturbance, are both deduced from the proof of Theorem 9.2.3.

9.2 Model description

We consider the following nonlinear Saint-Venant equations for a rectangular channel with arbitrary slope and friction.

$$\begin{aligned} &\partial_t H + \partial_x (HV) = 0, \\ &\partial_t V + V\partial_x V + g\partial_x H + \left(\frac{kV^2}{H} - C(x)\right) = 0. \end{aligned} \tag{9.2.1}$$

Here, k is an arbitrary nonnegative friction coefficient and C denotes the slope, which is assumed to be a C^2 function, with $C(x) := -gdB/dx$ where B is the bathymetry and g the acceleration of gravity. We are interested in systems where the water flow uphill is a given function, unknown and imposed by external conditions, for instance a flow coming from another country, while the water flow downhill is controlled through a hydraulic installation. Therefore we have the following boundary conditions,

$$\begin{aligned} H(t,0)V(t,0) &= Q_0(t), \\ H(t,L)V(t,L) &= U(t), \end{aligned} \tag{9.2.2}$$

where $U(t)$ is a control feedback and $Q_0(t)$ is the incoming flow, which is a given (and unknown) function. Here L denotes the length of the water channel. In practical situations, the formal control $U(t)$ can be expressed by a simple linear model [17]

$$U(t) = v_G(H(t,L) - U_1(t)), \tag{9.2.3}$$

where $U_1(t)$ is the elevation of the gate of the dam, which is the real control input that can be chosen, while v_G is a constant depending on the parameters of the gate (potentially unknown as well).

Usually, the industrial goal of such system is to stabilize the level of the water at the end point $H(t,L)$, called control point, to a target value $H_c > 0$. On the other hand, the usual mathematical goal in such problem is to stabilize a target steady-state (H^*, V^*), potentially nonuniform [13][Preface]. However, in the present problem (9.2.1)–(9.2.2), it is clear that, when Q_0 is not constant, it is impossible to aim at stabilizing any steady-state and one needs to aim at stabilizing other target trajectories. Therefore, we define the following target trajectory (H_1, V_1) that we aim stabilizing as the solution of

$$\begin{aligned} &\partial_t H_1 + \partial_x (H_1 V_1) = 0, \\ &\partial_t V_1 + V_1\partial_x V_1 + g\partial_x H_1 + \left(\frac{kV_1^2}{H_1} - C(x)\right) = 0, \\ &H_1(t,0)V_1(t,0) = Q_0(t), \\ &H_1(t,L) = H_c, \end{aligned} \tag{9.2.4}$$

with the initial condition

$$H_1(0,\cdot) = H^*(\cdot) \text{ and } V_1(0,\cdot) = V^*(\cdot), \tag{9.2.5}$$

where (H^*, V^*) is the (unique) steady-state solution of the system when Q_0 is constant, equal to $Q_0(0)$. Namely (H^*, V^*) is the solution of

$$
\begin{aligned}
&\partial_x(HV) = 0,\\
&V\partial_x V + g\partial_x H + \left(\frac{kV^2}{H} - C(x)\right) = 0,\\
&H(L) = H_c,
\end{aligned}
\tag{9.2.6}
$$

with condition at $x = 0$

$$
H^*(0)V^*(0) = Q_0(0). \tag{9.2.7}
$$

We are now going to show that the trajectory (H_1, V_1) exists for any time and satisfies some bounds.

Existence and bounds of the target trajectory Instead of studying directly our target trajectory (H_1, V_1) we first construct an intermediary family of functions (H_0, V_0). We defined previously (H^*, V^*) as the steady-state associated to a constant flow rate $Q_0 \equiv Q_0(0)$, i.e. (H^*, V^*) is the solution of the ODE problem (9.2.6) with initial condition $H^*(0)V^*(0) = Q_0(0)$. But in fact at each time $t^* \in \mathbb{R}_+^*$, we can define a steady-state $(H^*_{t^*}, V^*_{t^*})$ associated to a constant flow rate $Q_0 \equiv Q_0(t^*)$, i.e. $(H^*_{t^*}, V^*_{t^*})$ is the solution of the ODE problem (9.2.6) with initial condition satisfying

$$
H^*_{t^*}(0)V^*_{t^*}(0) = Q_0(t^*). \tag{9.2.8}
$$

This problem could seem peculiar as all conditions should be imposed exclusively in 0 or in L to ensure the well-posedness. However looking at the first equation of (9.2.6), the problem (9.2.6), (9.2.8) is in fact equivalent to a single ODE on $H^*_{t^*}$ with boundary condition $H^*_{t^*}(L) = H_c$ and $V^*_{t^*}$ defined by $V^*_{t^*} = Q_0(t)/H^*_{t^*}$. Thus for each $t^* \in [0, +\infty)$ such function exists on $[0, L]$, is unique and C^3 provided that the state stays in the fluvial regime (or subcritical regime), i.e. $gH^*_{t^*} > V^{*2}_{t^*}$ on $[0, L]$, which, for a given H_c, is equivalent to a bound on $Q_0(t^*)$ (see [110] for more details). As we are interested in stabilizing physical trajectories in the fluvial regime, we assume that this assumption is satisfied in the following and that there exist $\alpha > 0$ and $H_\infty > 0$ independent of $t^* \in [0, \infty)$ such that

$$
\begin{aligned}
H^*_{t^*} &< \frac{1}{2}H_\infty \text{ on } [0, L],\\
gH^*_{t^*} - V^{*2}_{t} &> 2\alpha \text{ on } [0, L].
\end{aligned}
\tag{9.2.9}
$$

For a given H_c, this is again equivalent to imposing a bound Q_∞ on $\|Q_0\|_{L^\infty(0,\infty)}$, from (9.2.6) and (9.2.8), which would be more logical. However, for convenience, we will still use H_∞ and α in the following. This assumptions is quite physical, especially as in practical situation the river is in fluvial regime and $Q_0(t)$ is often periodic or quasi-periodic. This gives a family of one-variable functions indexed by a parameter t^*, which can also be seen as the two-variable functions $(H_0, V_0)\ : (t, x) \to (H^*_t(x), V^*_t(x))$. Besides, from (9.2.7), as (H^*_t, V^*_t) is the solution of a system of ODE with a parameter t, the two variable functions (H_0, V_0) therefore belongs to $C^3([0, +\infty) \times (0, L))$ (see [105][Chap. 5, Cor. 4.1]). And from its definition, one can note that $(H_0(0, \cdot), V_0(0, \cdot)) = (H^*, V^*)$. Now that we have introduced this intermediary family of functions, we can show the existence of the target trajectory (H_1, V_1) and we have the following Input-to-State Stability (ISS) result (see [183] for a definition of ISS for finite dimensional systems, [117, Chap 1, Chap 3] for a generalization to first-order hyperbolic PDE and [170] for the use of Lyapunov function to achieve ISS on time-varying hyperbolic systems),

Proposition 9.2.1. *There exist positive constants c_1, c_2 such that if $\partial_t Q_0 \in C^2([0, \infty))$, there exist $\mu > 0$, $\nu > 0$ and $\delta > 0$ such that if $\|\partial_t Q_0\|_{C^2([0,+\infty))} \leq \delta$, then for any $(H_1^0, V_1^0) \in H^2((0, L), \mathbb{R}^2)$ such that*

$$
\|H_1^0 - H^*\|_{H^2(0,L)} + \|V_1^0 - V^*\|_{H^2(0,L)} \leq \nu,
$$

the system (9.2.4) *with initial condition* (H_1^0, V_1^0) *has a unique solution* $(H_1, V_1) \in C^0([0,+\infty), H^2(0,L))$ *which satisfies the following ISS inequality*

$$
\begin{aligned}
&\|H_1(t,\cdot) - H_0(t,\cdot)\|_{H^2(0,L)} + \|V_1(t,\cdot) - V_0(t,\cdot)\|_{H^2(0,L)} \\
&\leq c_1(\|H_1^0 - H^*\|_{H^2(0,L)} + \|V_1^0 - V^*\|_{H^2(0,L)})e^{-\frac{\mu t}{2}} + c_2\left(\int_0^t |\partial_t Q_0(s) + \partial_{tt}^2 Q_0(s) + \partial_{ttt}^3 Q_0(s)|e^{\frac{\mu s}{2}} ds\right) e^{-\frac{\mu t}{2}}.
\end{aligned}
\tag{9.2.10}
$$

This result is shown in Appendix 9.5.1, and a definition of the C^2 norm is recalled in Remark 9.2.1. Note that Q_0 is supposed to be bounded, which is quite physical, but there is no additional requirement on this bound besides the physical assumption given by Q_∞ of remaining in the fluvial regime. This is important as in practical situations the value of the incoming flow can change a lot, even though slowly.

Here, we choose to stabilize the trajectory (H_1, V_1) associated to $H_1^0 = H^*$ and $H_1^0 = V^*$. As we will see, this target trajectory can be seen as the natural trajectory to stabilize as it satisfies the industrial goal $H(t,L) = H_c$ and it coincides with the steady-state solution when Q_0 is a constant. In this last case Q_0 and H_c are imposed and H^* and $V^* = Q_0/H^*$ are thus fully determined using (9.2.6). But one can note from (9.2.10) that, in fact, the behaviour of (H_1, V_1) at large time does not depend on the initial condition (H_1^0, V_1^0) in (9.2.5), provided that it is close in H^2 norm to (H^*, V^*).

Remark 9.2.1. *The same ISS result can be shown replacing the* H^2 *norm in Proposition 9.2.1 by the* H^p *norm where* $p \in \mathbb{N}^* \setminus \{1\}$, *with the condition* $\|\partial_t Q_0\|_{C^p([0,+\infty))} \leq \delta$ *instead of* $\|\partial_t Q_0\|_{C^2([0,+\infty))} \leq \delta$. *This is shown in Appendix 9.5.1. We define here the* C^p *norm for a function* $U \in C^p(I)$, *where* I *is an interval, as*

$$
\|U\|_{C^p(I)} := \max_{i\in[0,p]} (\|\partial_t^i U\|_{L^\infty(I)}) \tag{9.2.11}
$$

Thus, from Proposition 9.2.1 and (9.2.9), there exists a constant $\delta > 0$ such that, if $\|\partial_t Q_0\|_{C^2([0,\infty))} < \delta$, then $(H_1, V_1) \in C^0([0,+\infty), H^2(0,L))$ and

$$
H_1(t,x) < H_\infty, \ \forall\ (t,x) \in [0,+\infty) \times [0,L], \tag{9.2.12}
$$

$$
gH_1(t,x) - V_1^2(t,x) > \alpha, \ \forall\ (t,x) \in [0,+\infty) \times [0,L]. \tag{9.2.13}
$$

Besides, when Q_0 is a constant, it is easy to check that $(H_0, V_0) = (H^*, V^*)$ is also solution of (9.2.4)–(9.2.5). Thus, from the uniqueness of the solution of (9.2.4)–(9.2.5), $(H_1, V_1) = (H^*, V^*)$ and therefore we recover a steady-state. This illustrates that (H_1, V_1) can be seen as the natural target state when Q_0 is not a constant anymore. Moreover, from (9.2.4), stabilizing (H_1, V_1) also satisfies the industrial goal by stabilizing $H(t,L)$ on the value H_c.

Control design and main result As mentioned in the introduction, a usual type of controller used in pratice to reach this aim is the proportional-integral (PI) controller. It has the advantage of eliminating the offset coming from constant load disturbances, which can usually appear in these systems as the command on the gate's level are only known up to some constant incertainties. A generic PI controller is given by

$$
U_1(t) = k_p(H_c - H(t,L)) + k_I Z, \tag{9.2.14}
$$

where k_p and k_I are coefficients that can be designed and Z accounts for the integral term, i.e.

$$
\dot{Z} = H_c - H(t,L). \tag{9.2.15}
$$

With such controller, and using (9.2.3), the boundary conditions (9.2.2) become (9.2.15) and

$$
\begin{aligned}
&H(t,0)V(t,0) = Q_0(t), \\
&H(t,L)V(t,L) = v_G(1+k_p)H(t,L) - v_G k_p H_c - v_G k_I Z,
\end{aligned}
\tag{9.2.16}
$$

In Corollary 2 we show that this boundary control can be used to stabilize exponentially a steady-state when Q_0 is a constant. In Theorem 9.2.4 we show that this control can also provide an Input-to-State Stability property with respect to $\partial_t Q_0$. However, this control (9.2.14) cannot be used to stabilize a dynamic target trajectory (H_1, V_1), as there is no function $Z_1 \in C^1([0,+\infty))$ such that (H_1, V_1, Z_1) is a solution of (9.2.1), (9.2.15), (9.2.16) while (H_1, V_1) is a solution of (9.2.4). Therefore, when stabilizing a dynamic target trajectory, one has to add an additional term and use

$$U_1(t) = k_p(H_c - H(t,L)) + k_I Z - f(t), \tag{9.2.17}$$

where $f(t) := H_1(t,L)V_1(t,L)/v_G$. The boundary conditions (9.2.2) become then

$$\begin{aligned} &H(t,0)V(t,0) = Q_0(t),\\ &H(t,L)V(t,L) = H_1V_1(t,L) + v_G(1+k_p)(H(t,L)-H_c) - v_G k_I Z, \end{aligned} \tag{9.2.18}$$

where we have actually changed Z and re-define $Z := Z - k_p/k_I$, which still satisfies the equation (9.2.15). This new control (9.2.17) assumes that $V_1(t,L)$ is known at least up to a constant, as $H_1(t,L) = H_c$ and additional constants can be incorporated into Z. When no knowledge on the target state is available besides H_c, it is impossible to stabilize exponentially the system, and the best one can get is the Input-to-State Stability which is given by Theorem 9.2.4. However in the following we will keep working with (9.2.17) and (9.2.18) to show Theorem 9.2.3 and the exponential stability of the system, as the proof of Theorem 9.2.4 and Corollary 2 which uses only the control (9.2.14) and (9.2.16) are easily deduced from the proof of Theorem 9.2.3.

We introduce the first-order compatibility conditions associated to the boundary conditions (9.2.18) for an initial condition (H^0, V^0, Z^0).

$$\begin{aligned} &H^0(0)V^0(0) = Q_0(0),\\ &H^0(L)V^0(L) = H_1V_1(0,L) + v_G(1+k_p)(H^0(L)-H_c) - k_I Z^0,\\ &-\partial_x(H^0(0)V^0(0) + g\frac{gH^0(0)^2}{2}) - (k(V^0)^2(0) - CH^0(0)) = Q_0'(0),\\ &-\partial_x(H^0(L)V^0(L) + g\frac{gH^0(L)^2}{2}) - (k(V^0)^2(L) - CH^0(L)) = \partial_t(H_1V_1)(0,L)\\ &\qquad - v_G(1+k_p)\partial_x(H^0(L)V^0(L)) + k_I(H^0(L)-H_c). \end{aligned} \tag{9.2.19}$$

With such compatibility conditions the system (9.2.1), (9.2.15), (9.2.18) is well-posed and we have the following theorem due to Wang [198][Theorem 2.1] :

Theorem 9.2.2. *Let $T > 0$, and assume that $\|\partial_t Q_0\|_{C^3([0,+\infty))} \leq \delta(T)$, such that (H_1, V_1) is well-defined and belongs to $C^0([0,T], H^3(0,L))$. There exists $\nu(T) > 0$ such that for any $(H^0, V^0, Z^0) \in (H^2((0,L)))^2 \times \mathbb{R}$ satisfying*

$$\|H^0(\cdot) - H_1(0,\cdot)\|_{H^2(0,L)} + \|V^0(\cdot) - V_1(0,\cdot)\|_{H^2(0,L)} + |Z^0| \leq \nu(T), \tag{9.2.20}$$

and satisfying the compatibility conditions (9.2.19)*, the system* (9.2.1)*,* (9.2.15)*,* (9.2.18) *has a unique solution $(H, V, Z) \in (C^0([0,T], H^2((0,L))))^2 \times C^1([0,T])$. Moreover there exists a positive constant $C(T)$ such that*

$$\begin{aligned} &\|H(t,\cdot) - H_1(t,\cdot)\|_{H^2(0,L)} + \|V(t,\cdot) - V_1(t,\cdot)\|_{H^2(0,L)} + |Z|\\ &\qquad \leq C(T)\left(\|H^0(\cdot) - H_1(0,\cdot)\|_{H^2(0,L)} + \|V^0(\cdot) - V_1(0,\cdot)\|_{H^2(0,L)} + |Z^0|\right). \end{aligned} \tag{9.2.21}$$

To apply the result from [198], note that Z can be seen as a third component of the hyperbolic system with a null propagation speed, a constant initial condition Z^0 and $Z(t)$ being thus its value everywhere on $[0,L]$ including at the boundaries.

Remark 9.2.2. *If, in addition, $(H^0, V^0) \in H^3((0,L);\mathbb{R}^2)$, then the unique solution (H, V, Z) given by Theorem 9.2.2 belongs to $C^0([0,T], H^3((0,L);\mathbb{R}^2)) \times C^2([0,T])$ and there exists a constant $C(T)$ such that*

$$\begin{aligned} &\|H(t,\cdot) - H_1(t,\cdot)\|_{H^3(0,L)} + \|V(t,\cdot) - V_1(t,\cdot)\|_{H^3(0,L)} + |Z| \\ &\qquad \le C(T)\left(\|H^0(\cdot) - H_1(0,\cdot)\|_{H^3(0,L)} + \|V^0(\cdot) - V_1(0,\cdot)\|_{H^3(0,L)} + |Z^0|\right). \end{aligned} \tag{9.2.22}$$

We recall the definition of exponential stability

Definition 9.2.1. We say that a trajectory (H_1, V_1) is exponentially stable for the H^2 norm if there exists $\nu > 0$, $C > 0$ and $\gamma > 0$ such that for any $T > t_0 \ge 0$ and any (H^0, V^0, Z^0) satisfying

$$\|H^0(\cdot) - H_1(t_0,\cdot)\|_{H^2(0,L)} + \|V^0(\cdot) - V_1(t_0,\cdot)\|_{H^2(0,L)} + |Z^0| \le \nu, \tag{9.2.23}$$

and the compatibility conditions (9.2.19), the system (9.2.1), (9.2.15), (9.2.18) with initial condition (H^0, V^0, Z^0) at t_0 has a unique solution $(H, V, Z) \in (C^0([t_0,T], H^2((0,L))))^2 \times C^1([t_0,T])$ and,

$$\begin{aligned} &\|H(t,\cdot) - H_1(t,\cdot)\|_{H^2(0,L)} + \|V(t,\cdot) - V_1(t,\cdot)\|_{H^2(0,L)} + |Z| \\ &\le Ce^{-\gamma t}\left(\|H^0(\cdot) - H_1(t_0,\cdot)\|_{H^2(0,L)} + \|V^0(\cdot) - V_1(t_0,\cdot)\|_{H^2(0,L)} + |Z^0|\right), \ \forall\, t \in [t_0, +\infty). \end{aligned} \tag{9.2.24}$$

Remark 9.2.3. *From* (9.2.4) *and Sobolev inequality, this exponential stability implies in particular the exponential convergence of $H(t,L)$ to H_c.*

We can now state the main results of this article

Theorem 9.2.3. *There exists $\delta > 0$ such that if $\|\partial_t Q_0\|_{C^3([0,+\infty))} \le \delta$, then the trajectory (H_1, V_1) given by* (9.2.4) *of system* (9.2.1), (9.2.15), (9.2.18) *is exponentially stable for the H^2 norm if :*

$$\begin{gathered} k_p > -1 \text{ and } k_I > 0, \\ \text{or } \ k_p < -1 - \frac{gH_1(t,L) - V_1^2(t,L)}{v_G V_1(t,L)} \text{ and } k_I < 0. \end{gathered} \tag{9.2.25}$$

This result is proved in Section 9.3. The main idea of the proof consist in finding a local convex and dissipative entropy for the system (9.2.1), (9.2.15), (9.2.18).

In particular, in the case where Q_0 is constant, we can use the static boundary control (9.2.14), and we have the following corollary :

Corollary 2. *If Q_0 is constant, then the steady-state (H^*, V^*) of the system* (9.2.1), (9.2.15), (9.2.16) *given by* (9.2.6)–(9.2.7) *is exponentially stable for the H^2 norm if :*

$$\begin{gathered} k_p > -1 \text{ and } k_I > 0, \\ \text{or } \ k_p < -1 - \frac{gH^*(L) - V^{*2}(L)}{v_G V^*(L)} \text{ and } k_I < 0. \end{gathered} \tag{9.2.26}$$

Proof. This is a particular case of Theorem 9.2.3. To see this, note, as mentioned earlier, that when Q_0 is constant, then $(H_1, V_1) = (H^*, V^*)$. Then, observe that $f(t)$ given in (9.2.17) is a constant that can be added in Z (i.e. we can re-define $Z := Z - f(t)$, which still satisfies (9.2.15)). □

Remark 9.2.4. *In the literature, results about PI control of the Saint-Venant equations sometimes leave the step of modeling the spillway and use a generic formulation of the PI control on the outflow rate of the form*

$$H(t,L)V(t,L) = k_1(H(t,L) - H_c) - k_2 Z, \tag{9.2.27}$$

where Z is the integral term, still given by (9.2.15)*. Note that, with these notations, the sufficient condition of Corollary 2 becomes*

$$\begin{aligned} &k_p > 0 \text{ and } k_I > 0, \\ \text{or } \ &k_p < -\frac{gH^*(L) - V^{*2}(L)}{V^*(L)} \text{ and } k_I < 0. \end{aligned} \tag{9.2.28}$$

which is a known result in the linear case using a spectral approach. Theorem 9.2.3 and Corollary 2 show that this result remains true when the system is nonlinear, using a Lyapunov approach.

Remark 9.2.5. *When the system is homogeneous, conditions* (9.2.26) *are optimal (necessary and sufficient) [22], [13, Section 2.2.4.1].*

This approach uses very little knowledge of the state of the system, as we only measure the height at the boundary $x = L$. In practical situation, however, we may have also little knowledge of the target trajectory (H_1, V_1) or the input disturbance $Q_0(t)$ and we only know H_c. In this case we cannot use a controller of the form (9.2.18), but only a static controller of the form (9.2.16), namely

$$H(t, L)V(t, L) = v_G(1 + k_p)H(t, L) - v_G k_p H_c - v_G k_I Z. \tag{9.2.29}$$

In this case, it is impossible to aim at stabilizing the target trajectory (H_1, V_1), but we still have the Input-to-state Stability with respect to the input disturbance $\partial_t Q_0$,

Theorem 9.2.4. *There exists $\nu > 0$, $\delta > 0$, $\gamma > 0$ and C, such that if $\|\partial_t Q_0\|_{C^2([0,+\infty))} \leq \delta$, then for any $T > 0$ and $(H^0, V^0) \in (H^2(0, L))^2$ such that*

$$\|H^0 - H^*\|_{H^2(0,L)} + \|V^0 - V^*\|_{H^2(0,L)} \leq \nu,$$

the system (9.2.1)*,* (9.2.15)*,* (9.2.16) *with initial condition (H^0, V^0) has a unique solution $(H, V) \in C^0([0, T], H^2(0, L))$ which satisfies the following ISS inequality*

$$\begin{aligned} &\|H(t, \cdot) - H_0(t, \cdot)\|_{H^2(0,L)} + \|V(t, \cdot) - V_0(t, \cdot)\|_{H^2(0,L)} \\ &\leq Ce^{-\gamma t} \left(\|H^0 - H^*, V^0 - V^*\|_{H^2(0,L)} + \int_0^t |\partial_t Q_0(s) + \partial_{tt}^2 Q_0(s) + \partial_{ttt}^3 Q_0(s)| e^{\gamma s} ds \right). \end{aligned} \tag{9.2.30}$$

The proof is given in Appendix 9.5.2 and is a consequence from the proof of Theorem 9.2.3.

In Section 9.3 we prove Theorem 9.2.3.

9.3 Exponential stability for the H^2 norm

This section is divided in three parts. First we transform the system through a change of variables. Then we state three lemma, useful for the analysis. Finally we prove Theorem 9.2.3.

9.3.1 A change of variables

For any solution of (9.2.1), (9.2.15), (9.2.18) we define the perturbation as

$$\begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} H - H_1 \\ V - V_1 \end{pmatrix}. \tag{9.3.1}$$

Let us assume that there exists $\nu \in (0, \nu_0)$ to be selected later on, such that

$$\|H^0(\cdot) - H_1(0, \cdot)\|_{H^2(0,L)} + \|V^0(\cdot) - V_1(0, \cdot)\|_{H^2(0,L)} + |Z^0| \leq \nu. \tag{9.3.2}$$

The boundary conditions (9.2.18) can be written in the following form

$$\begin{aligned} v(t,0) &= \mathcal{B}_1(h(t,0),t), \\ v(t,L) &= \mathcal{B}_2(h(t,L),Z,t), \end{aligned} \tag{9.3.3}$$

with

$$\begin{aligned} \partial_1\mathcal{B}_1(0,t) &= -\frac{V_1(t,0)}{H_1(t,0)}, \\ \partial_1\mathcal{B}_2(0,0,t) &= \frac{v_G(1+k_p)-V_1(t,L)}{H_1(t,L)}, \\ \partial_2\mathcal{B}_2(0,0,t) &= -\frac{v_G k_I}{H_1(t,L)}. \end{aligned} \tag{9.3.4}$$

We introduce the following change of variables :

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v+\sqrt{\frac{g}{H_1}}h \\ v-\sqrt{\frac{g}{H_1}}h \end{pmatrix}. \tag{9.3.5}$$

Note that this change of variables is very similar to the change of variables used in [11, 110] with the only difference that (H_1, V_1) is not a steady-state anymore. It corresponds to the transformation in Riemann coordinates for the perturbations. Indeed, denoting S, F and G by

$$S(x,t) = \begin{pmatrix} \sqrt{\frac{g}{H_1(t,x)}} & 1 \\ -\sqrt{\frac{g}{H_1(t,x)}} & 1 \end{pmatrix}, \tag{9.3.6}$$

$$F\begin{pmatrix} H \\ V \end{pmatrix} = \begin{pmatrix} V & H \\ g & V \end{pmatrix}, \quad G\begin{pmatrix} H \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{kV^2}{H} - C(x) \end{pmatrix}, \tag{9.3.7}$$

and using (9.2.1), (9.2.15), (9.2.18), (9.2.4), (9.3.1)–(9.3.5), one has

$$\begin{aligned} \partial_t u_1 + \Lambda_1(\mathbf{u},x,t)\partial_x u_1 + l_1(\mathbf{u},x,t)\partial_x u_2 + B_1(\mathbf{u},x,t) = 0, \\ \partial_t u_1 - \Lambda_2(\mathbf{u},x,t)\partial_x u_2 + l_2(\mathbf{u},x,t)\partial_x u_1 + B_2(\mathbf{u},x,t) = 0, \end{aligned} \tag{9.3.8}$$

where,

$$A(\mathbf{u},x,t) = \begin{pmatrix} \Lambda_1(\mathbf{u},x,t) & l_1(\mathbf{u},x,t) \\ l_2(\mathbf{u},x,t) & \Lambda_2(\mathbf{u},x,t) \end{pmatrix} = S(x,t)F\left(S^{-1}(x,t)\mathbf{u} + \begin{pmatrix} H_1(t,x) \\ V_1(t,x) \end{pmatrix}\right)S^{-1}(x,t), \tag{9.3.9}$$

$$\begin{aligned} B(\mathbf{u},x,t) = \begin{pmatrix} B_1(\mathbf{u},x,t) \\ B_2(\mathbf{u},x,t) \end{pmatrix} &= S(x,t)F\left(S^{-1}(x,t)\mathbf{u} + \begin{pmatrix} H_1(t,x) \\ V_1(t,x) \end{pmatrix}\right)\left(\begin{pmatrix} \partial_x H_1(t,x) \\ \partial_x V_1(t,x) \end{pmatrix} + \partial_x(S^{-1})\mathbf{u}\right) \\ &+ S\partial_t\begin{pmatrix} H_1(t,x) \\ V_1(t,x) \end{pmatrix} + S(x,t)G\left(S^{-1}(x,t)\mathbf{u} + \begin{pmatrix} H_1(t,x) \\ V_1(t,x) \end{pmatrix}\right) - \partial_t S(x,t)S^{-1}(x,t)\mathbf{u}, \end{aligned} \tag{9.3.10}$$

and thus

$$\Lambda_1(0,x,t) = V_1 + \sqrt{gH_1}, \quad \Lambda_2(0,x,t) = V_1 - \sqrt{gH_1}, \tag{9.3.11}$$

$$l_1(0,x,t) = B_1(\mathbf{0},x,t) = 0, \quad l_2(0,x,t) = B_2(\mathbf{0},x,t) = 0, \tag{9.3.12}$$

$$\begin{aligned} \frac{\partial B_1}{\partial_u}(0,x,t) &= \gamma_1(t,x)u_1(t,x) + \gamma_2(t,x)u_2(t,x), \\ \frac{\partial B_2}{\partial_u}(0,x,t) &= \delta_1(t,x)u_1(t,x) + \delta_2(t,x)u_2(t,x). \end{aligned} \tag{9.3.13}$$

where

$$
\begin{aligned}
\gamma_1 &= \frac{3}{4}\sqrt{\frac{g}{H_1}}H_{1x} + \frac{3}{4}V_{1x} + \frac{kV_1}{H_1} - \frac{kV_1^2}{2H_1^2}\sqrt{\frac{H_1}{g}} \\
\gamma_2 &= \frac{1}{4}\sqrt{\frac{g}{H_1}}H_{1x} + \frac{1}{4}V_{1x} + \frac{kV_1}{H_1} + \frac{kV_1^2}{2H_1^2}\sqrt{\frac{H_1}{g}} \\
\delta_1 &= -\frac{1}{4}\sqrt{\frac{g}{H_1}}H_{1x} + \frac{1}{4}V_{1x} + \frac{kV_1}{H_1} - \frac{kV_1^2}{2H_1^2}\sqrt{\frac{H_1}{g}} \\
\delta_2 &= -\frac{3}{4}\sqrt{\frac{g}{H_1}}H_{1x} + \frac{3}{4}V_{1x} + \frac{kV_1}{H_1} + \frac{kV_1^2}{2H_1^2}\sqrt{\frac{H_1}{g}}.
\end{aligned} \tag{9.3.14}
$$

And for the boundary conditions, there exists $\nu_1 \in (0, \nu_0)$ such that for any $\nu \in (0, \nu_1)$, one has :

$$
\begin{aligned}
&u_1(t,0) = \mathcal{D}_1(u_2(t,0),t), \\
&u_2(t,L) = \mathcal{D}_2(u_1(t,L),Z,t), \\
&\dot{Z} = \frac{(u_1(t,L) - u_2(t,L))}{2}\sqrt{\frac{H_1(t,L)}{g}},
\end{aligned} \tag{9.3.15}
$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are C^2 functions and

$$
\begin{aligned}
\partial_1\mathcal{D}_1(0,t) &= -\frac{\lambda_2(0)}{\lambda_1(0)}, \\
\partial_1\mathcal{D}_2(0,0,t) &= -\frac{\lambda_1(L) - v_G(1+k_p)}{\lambda_2(L) + v_G(1+k_p)}, \\
\partial_2\mathcal{D}_2(0,0,t) &= -2\frac{v_G k_I\sqrt{\frac{g}{H_1(t,L)}}}{v_G(1+k_p) + \lambda_2(t,L)}.
\end{aligned} \tag{9.3.16}
$$

Expression (9.3.14) is simply a computation, very similar to what it done in [110] for instance, while the derivation of (9.3.15) and (9.3.16) are detailed in the appendix.

Remark 9.3.1. *Obviously, from the change of variables* (9.3.1)–(9.3.5)*, the exponential stability of the system* (9.2.1)*,* (9.2.15)*,* (9.2.18) *is equivalent to the exponential stability of the steady-state* $\mathbf{u}^* = 0$ *for the system* (9.3.8)*,* (9.3.15)*.*

As the operator A, given by (9.3.9), is a C^2 function in $\mathbf{u}$, t and x (and in particular C^1) and as, from (9.3.13) and (9.2.13), $\Lambda_1(\mathbf{0},x,t) > 0 > \Lambda_2(\mathbf{0},x,t)$, there exists $\nu_2 \in (0,\nu_1)$ and $E \in C^1(\mathcal{B}_{\nu_2}\times(0,L)\times[0,+\infty);\mathcal{M}_2(\mathbb{R}))$, where $\mathcal{B}_{\nu_2} \subset \mathbb{R}^2$ is the disc of radius ν_2, such that for any $\|\mathbf{u}(t,\cdot)\|_{H^2(0,L)} \leq \nu_2$,

$$
\begin{aligned}
E(\mathbf{u}(t,x),x,t)A(\mathbf{u}(t,x),x,t) &= D(\mathbf{u}(t,x),x,t)E(\mathbf{u}(t,x),x,t), \\
E(\mathbf{0},x,t) &= Id,
\end{aligned} \tag{9.3.17}
$$

where $D(\mathbf{u}(t,x),x,t) = (D_i(\mathbf{u}(t,x),x,t))_{i\in 1,2}$ is a diagonal matrix and Id is the identity matrix. Before going any further, let us note a few useful properties of these functions. For simplicity in the following we will denote for any $n \in \mathbb{N}^*$ and any function $U \in L^\infty((0,T)\times(0,L);\mathbb{R}^n)$ (resp. $L^\infty((0,L);\mathbb{R}^n)$)

$$
\begin{aligned}
&\|U\|_\infty := \|U\|_{L^\infty((0,T)\times(0,L);\mathbb{R}^n)}, \\
&(\text{resp.}\|U\|_\infty := \|U\|_{L^\infty((0,L);\mathbb{R}^n)}).
\end{aligned} \tag{9.3.18}
$$

We may also denote $\|\mathbf{u}\|_{H^2(0,L)}$ instead of $\|\mathbf{u}(t,\cdot)\|_{H^2(0,L)}$ to lighten the expressions. From the definition of A given in (9.3.9), and from (9.2.13), for $\|\mathbf{u}\|_{H^2(0,L)} \leq \nu_2$, there exists a constant C_1 depending only on H_∞,

α and ν_2 such that we have the following estimates

$$\begin{aligned}
&\max\left(\|\partial_t(A(\mathbf{u}(t,x),x,t)-A(\mathbf{0},x,t))\|_\infty, \|\partial_t(D(\mathbf{u}(t,x),x,t)-D(\mathbf{0},x,t))\|_\infty, \|\partial_t(E(\mathbf{u}(t,x),x,t))\|_\infty\right)\\
&\leq C_1\left(\|\mathbf{u}\|_\infty(\|\partial_t H_1\|_\infty+\|\partial_t V_1\|_\infty)+\|\partial_t\mathbf{u}\|_\infty\right),\\
&\max\left(\|\partial_t(A(\mathbf{u}(t,x),x,t)-A(\mathbf{0},x,t))\|_\infty, \|\partial_t(D(\mathbf{u}(t,x),x,t)-D(\mathbf{0},x,t)), \|\partial_t(E(\mathbf{u}(t,x),x,t))\|_\infty\right)\\
&\leq C_1\left(\|\mathbf{u}\|_\infty(\|\partial_x H_1\|_\infty+\|\partial_x V_1\|_\infty)+\|\partial_x\mathbf{u}\|_\infty\right).
\end{aligned} \tag{9.3.19}$$

For E and D, this comes from the fact that E and D are C^∞ functions with respect to the coefficients of A (note that D is the matrix of eigenvalues of A), and that $A \in C^2(\mathcal{B}_{\eta_0}; C^1([0,+\infty)\times[0,L]))$.

9.3.2 Three useful lemma

We introduce now three lemma, which will be useful in the following analysis. The first one is a classical result about Lyapunov functions,

Lemme 9.3.1. *Let $V\ :(H^2(0,L))^2\times\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+^*$ such that there exists a constant $c>0$ such that*

$$c\left(\|\mathbf{U}\|_{H^2(0,L)}+|z|\right)\leq V(\mathbf{U},z,t)\leq\frac{1}{c}\left(\|\mathbf{U}\|_{H^2(0,L)}+|z|\right),\ \forall\ (U,z,t)\in(H^2(0,L))^2\times\mathbb{R}\times\mathbb{R}_+. \tag{9.3.20}$$

If there exists $\gamma>0$ and $\delta>0$ such that, for any solution $(\mathbf{u},Z)$ of the system (9.3.8), (9.3.15) *with initial conditions satisfying $\|\mathbf{u}(0,\cdot)\|_{H^2(0,L)}+|Z(0)|\leq\delta$,*

$$\frac{d}{dt}\left[V(\mathbf{u}(t,\cdot),t)\right]<-\gamma V(\mathbf{u}(t,\cdot),t) \tag{9.3.21}$$

in a distribution sense, then the system (9.3.8), (9.3.15) *is exponentially stable for the H^2 norm and V is called a Lyapunov function for the system* (9.3.8), (9.3.15).

This first lemma reduces the problem of proving the exponential stability to finding a Lyapunov function V for the system (9.3.8), (9.3.15). A proper definition of a differential inequality in a distribution sense as in (9.3.21) can be found in [106]. To lighten this article we do not give a proof of this classical lemma, although a proof for a very similar case (Lyapunov function that does not depend explicitly on time and for the C^1 norm instead) can be found for instance in [106][Proposition 2.1], and is easily extended to this case.

The second Lemma is a variation of a result shown in [110] that gives a local entropy of the Saint-Venant equations. Let us first introduce the following function φ defined by

$$\begin{aligned}
\varphi_1(t,x)&=\exp\left(\int_0^x\frac{\gamma_1}{\lambda_1}dx\right),\\
\varphi_2(t,x)&=\exp\left(-\int_0^x\frac{\delta_2}{\lambda_2}dx\right),\\
\varphi(t,x)&=\frac{\varphi_1(t,x)}{\varphi_2(t,x)},
\end{aligned} \tag{9.3.22}$$

where λ_1 and λ_2 are defined by

$$\lambda_1(t,x)\ :=\Lambda_1(\mathbf{0},x,t)>0,\ \ \lambda_2(t,x)\ :=-\Lambda_2(\mathbf{0},x,t)>0. \tag{9.3.23}$$

We can now state the following lemma

Lemme 9.3.2. *There exists $\delta_0>0$ such that if $\|\partial_t H_1\|_{L^\infty((0,+\infty)\times(0,L))}\leq\delta_0$, the function $\lambda_2\varphi/\lambda_1$ is solution on $[0,L]$ to the following equation*

$$\partial_x f=\left|\frac{\varphi\gamma_2}{\lambda_1}+\frac{\varphi^{-1}\delta_1}{\lambda_2}f^2+\sqrt{\frac{g}{H_1}}\partial_t H_1\right|,\forall\ x\in[0,L],\ \ t\in[0,+\infty), \tag{9.3.24}$$

and for any $x \in [0, L]$ and any $t \in [0, +\infty)$,

$$\left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\varphi^{-1}\delta_1}{\lambda_2} f^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1\right) > 0. \tag{9.3.25}$$

The proof is given in the Appendix.

Eventually, we introduce our last Lemma, which seems very natural and is stated here to lighten the proof of Theorem 9.2.3.

Lemme 9.3.3. *There exists $l > 0$ and $C > 0$ such that if $\|\partial_t Q_0\|_{C^3([0,+\infty))} \leq l$, then*

$$\max\left(\|\partial_t H_1\|_{C^1([0,+\infty),L^\infty(0,L))}, \|\partial_t V_1\|_{C^1([0,+\infty),L^\infty(0,L))}\right) < C\|\partial_t Q_0\|_{C^3([0,+\infty))}. \tag{9.3.26}$$

This is a consequence of the ISS property (Proposition 9.2.1) and Remark 9.2.1 with $p = 3$, the relations (9.2.4), and Sobolev inequality. Thanks to this Lemma, we now only need to find a bound on $\partial_t H_1$ and $\partial_t V_1$ instead of a bound on $\partial_t Q_0$ in the proof of Theorem 9.2.3.

9.3.3 Proof of Theorem 9.2.3

We can now prove Theorem 9.2.3.

Proof of Theorem 9.2.3. From Theorem (9.2.2), Remark 9.3.1, and Lemma 9.3.1, one only needs to find a Lyapunov function $V : (H^2(0, L))^2 \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+^*$ satisfying (9.3.20) and (9.3.21). We define the following candidate :

$$V_a(\mathbf{U}, z, t) := \int_0^L f_1(t, x)e^{-\mu x}(E(\mathbf{U}(x), x, t)\mathbf{U})_1^2(t, x) + f_2(t, x)e^{\mu x}(E(\mathbf{U}(x), x, t)\mathbf{U})_2^2(t, x)dx + qz^2, \tag{9.3.27}$$

where f_1, f_2 are positive and bounded functions which will be defined later on, and μ and q are positives constant which will also be defined later on. Recall that E is still given by (9.3.17). Let $T > t_0 \geq 0$ and $(\mathbf{u}^0, Z^0) \in H^2(0, L) \times \mathbb{R}$ satisfying the compatibility condition (9.2.19) and such that

$$\left(\|\mathbf{u}^0\|_{H^2(0,L)} + |Z^0|\right) < \nu, \tag{9.3.28}$$

where ν is a constant to be chosen later on but such that $\nu < \min(\nu_2, \nu(T))$. Recall that $\nu(T)$ is given by Theorem 9.2.2. From Theorem 9.2.2 there exists a unique solution $\mathbf{u} \in C^0([t_0, T], H^2(0, L))$. We suppose in addition that $(\mathbf{u}^0, Z^0) \in H^3(0, L)$, and that (9.3.28) also hold for the H^3 norm instead of the H^2 norm in u. From Remark 9.2.2, $(\mathbf{u}, Z) \in C^0([t_0, T] \times H^3(0, L)) \times C^3([t_0, T])$. This assumption is here to allow us to compute easily the derivative of $\mathbf{u}$ but will be relaxed later on by density.

Let $\delta \in (0, \delta_0)$ to be chosen later on, with δ_0 is given by Lemma 9.3.2, and assume that

$$\max(\|\partial_t H_1\|_{C^1([t_0,\infty);L^\infty(0,L))}, \|\partial_t V_1\|_{C^1([t_0,\infty);L^\infty(0,L))} < \delta. \tag{9.3.29}$$

As this is the only assumption on H_1 and V_1, we can assume from now on that $t_0 = 0$ without loss of generality.

Looking at (9.3.27), V_a is indeed a function defined on $H^2(0, L) \times \mathbb{R} \times \mathbb{R}_+$, but for notational ease we will denote $V_a(t) := V_a(\mathbf{u}(t, \cdot), Z(t), t)$, where $Z(t)$ is given by (9.2.15), and $E := E(\mathbf{u}(t, x), x, t)$. Similarly we introduce

$$\begin{aligned}
V_b(\mathbf{U},t) \;&:= \int_0^L f_1 e^{-\mu x}(E(\mathbf{U}(x),x,t)\mathbf{I}(\mathbf{U},x,t))_1^2 + f_2 e^{\mu x}(E(\mathbf{U}(x),x,t)\mathbf{I}(\mathbf{U},x,t))_2^2 dx + q\frac{H_1(t,L)}{4g}(U_1(L)-U_2(L))^2,\\
V_c(\mathbf{U},t) \;&:= \int_0^L f_1 e^{-\mu x}(E(\mathbf{U}(x),x,t)\mathbf{J}(\mathbf{U},x,t))_1^2 + f_2 e^{\mu x}(E(\mathbf{U}(x),x,t)\mathbf{J}(\mathbf{U},x,t))_2^2 dx\\
&\quad + q\left(\sqrt{\frac{H_1(t,L)}{4g}}(I_1(t,L)-I_2(t,L)) + \frac{\partial_t H_1(t,L)}{4}\sqrt{\frac{1}{gH_1(t,L)}}(U_1(L)-U_2(L))\right)^2,
\end{aligned} \tag{9.3.30}$$

where

$$\begin{aligned}
\mathbf{I}(\mathbf{U},x,t) \;&:= (A(\mathbf{U},x,t)\partial_x(\partial_t\mathbf{U}) + (\partial_t A(\mathbf{U},x,t) + \partial_{\mathbf{U}}A(\mathbf{U},x,t).\partial_t\mathbf{U})\partial_x\mathbf{U} + \partial_t B\,(\mathbf{U},x,t) + (\partial_{\mathbf{U}}B(\mathbf{U},x,t))(\partial_t\mathbf{U})),\\
\mathbf{J}(\mathbf{U},x,t) \;&:= A(\mathbf{U},x,t)\partial_x(\partial_{tt}^2\mathbf{U}) + (\partial_{\mathbf{U}}A(\mathbf{U},x).\partial_{tt}^2\mathbf{U})\partial_x\mathbf{U} + (\partial_{\mathbf{U}}B(\mathbf{U},x))(\partial_{tt}^2\mathbf{U})\\
&\quad + (\partial_{tt}^2 A(\mathbf{U},x,t) + 2\partial_{\mathbf{U}}(\partial_t A(\mathbf{U},x,t)).\partial_t\mathbf{U})\partial_x\mathbf{U}\\
&\quad + 2\partial_t A(\mathbf{U},x,t)\partial_x(\partial_t\mathbf{U}) + 2\partial_{\mathbf{U}}A(\mathbf{U},x).\partial_t\mathbf{U}\partial_x(\partial_t\mathbf{U}) + ((\partial_{\mathbf{U}}^2 A(\mathbf{U},x).\partial_t\mathbf{U}).\partial_t\mathbf{U})\partial_x\mathbf{U}\\
&\quad + \partial_{tt}^2 B(\mathbf{U},x) + 2\partial_{\mathbf{U}}(\partial_t B(\mathbf{U},x)).\partial_t\mathbf{U} + (\partial_{\mathbf{U}}^2 B(\mathbf{U},x).\partial_t\mathbf{U})(\partial_t\mathbf{U}).
\end{aligned} \tag{9.3.31}$$

Observe that for a solution $\mathbf{u}$ of (9.3.8), and using the expression of Z given by (9.2.15), the expressions of $V_b(\mathbf{u}(t,\cdot),t)$ and $V_c(\mathbf{u}(t,\cdot),t)$ become

$$\begin{aligned}
V_b(\mathbf{u}(t,\cdot),t) \;&:= \int_0^L f_1(t,x)e^{-\mu x}(E\partial_t\mathbf{u})_1^2(t,x) + f_2(t,x)e^{\mu x}(E\partial_t\mathbf{u})_2^2(t,x)dx + q(\dot{Z}(t))^2,\\
V_c(\mathbf{u}(t,\cdot),t) \;&:= \int_0^L f_1(t,x)e^{-\mu x}(E\partial_{tt}^2\mathbf{u})_1^2(t,x) + f_2(t,x)e^{\mu x}(E\partial_{tt}^2\mathbf{u})_2^2(t,x)dx + q(\ddot{Z}(t))^2,
\end{aligned} \tag{9.3.32}$$

which justifies the expression chosen for (9.3.30) and (9.3.31). We also note for notational ease $V_b(t) := V_b(t,\mathbf{u}(t,\cdot))$ and $V_c(t) := V_c(t,\mathbf{u}(t,\cdot))$. Finally we denote $V := V_a + V_b + V_c$. We start now by dealing with V_a, Differentiating $t \to V_a(t)$ with respect to time, using (9.3.8), (9.3.17) and integrating by parts, one has

$$\begin{aligned}
\dot{V}_a =& -2\int_0^L f_1(t,x)e^{-\mu x}(E\mathbf{u})_1\left[(EA(\mathbf{u},x,t)\partial_x\mathbf{u})_1 + (EB)_1(\mathbf{u},x,t)\right]\\
&+ f_2(t,x)e^{\mu x}(E\mathbf{u})_2\left[(EA(\mathbf{u},x,t)\partial_x\mathbf{u})_2 + (EB)_2(\mathbf{u},x,t)\right]dx\\
&+ \int_0^L \partial_t(f_1)e^{-\mu x}(E\mathbf{u})_1^2 + \partial_t(f_2)e^{\mu x}(E\mathbf{u})_2^2 dx\\
&+ 2\int_0^L f_1 e^{-\mu x}(E\mathbf{u})_1\left((\partial_t E + \partial_{\mathbf{u}}E.\partial_t\mathbf{u})\,\mathbf{u}\right)_1 + f_2 e^{\mu x}\left((\partial_t E + \partial_{\mathbf{u}}E.\partial_t\mathbf{u})\,\mathbf{u}\right)_2 dx + 2qZ(t)\dot{Z}(t)\\
=& -2\int_0^L f_1(t,x)e^{-\mu x}(E\mathbf{u})_1\left[D_1(\mathbf{u},x,t)\left(\partial_x(E\mathbf{u}) - (\partial_x E + \partial_{\mathbf{U}}E.\partial_x\mathbf{u})\mathbf{u}\right)_1\right]\\
&+ f_2(t,x)e^{\mu x}(E\mathbf{u})_2\left[D_2(\mathbf{u},x,t)\left(\partial_x(E\mathbf{u}) - (\partial_x E + \partial_{\mathbf{U}}E.\partial_x\mathbf{u})\mathbf{u}\right)_2\right]dx\\
&+ \int_0^L \partial_t(f_1)e^{-\mu x}(E\mathbf{u})_1^2 + \partial_t(f_2)e^{\mu x}(E\mathbf{u})_2^2 dx - 2\int_0^L f_1 e^{-\mu x}(E\mathbf{u})_1(EB)_1(\mathbf{u},x,t) + f_2 e^{\mu x}(E\mathbf{u})_2(EB)_2(\mathbf{u},x,t)dx\\
&+ 2\int_0^L f_1 e^{-\mu x}(E\mathbf{u})_1\left((\partial_t E + \partial_{\mathbf{u}}E.\partial_t\mathbf{u})\,\mathbf{u}\right)_1 + f_2 e^{\mu x}\left((\partial_t E + \partial_{\mathbf{u}}E.\partial_t\mathbf{u})\,\mathbf{u}\right)_2 dx + 2qZ(t)\dot{Z}(t),
\end{aligned} \tag{9.3.33}$$

$$
\begin{aligned}
\dot{V}_a =& -\left[f_1 e^{-\mu x} D_1 (E\mathbf{u})_1^2 + D_2 f_2 e^{\mu x} (E\mathbf{u})_2^2\right]_0^L \\
& - \int_0^L (E\mathbf{u})_1 e^{-\mu x} \left((-\partial_x (D_1 f_1) - f_1 \partial_u (D_1).\partial_x \mathbf{u})(E\mathbf{u})_1 - 2 f_1 D_1 ((\partial_x E + \partial_{\mathbf{U}} E.\partial_x \mathbf{u})\mathbf{u})_1\right) \\
& + (E\mathbf{u})_2 e^{\mu x} \left((-\partial_x (D_2 f_2) - f_2 \partial_u (D_2).\partial_x \mathbf{u})(E\mathbf{u})_2 - 2 f_2 D_2 ((\partial_x E + \partial_{\mathbf{U}} E.\partial_x \mathbf{u})\mathbf{u})_2\right) dx \\
& + \int_0^L \partial_t (f_1) e^{-\mu x} (E\mathbf{u})_1^2 + \partial_t (f_2) e^{\mu x} (E\mathbf{u})_2^2 dx - 2 \int_0^L f_1 e^{-\mu x} (E\mathbf{u})_1 (EB)_1 (\mathbf{u}, x, t) + f_2 e^{\mu x} (E\mathbf{u})_2 (EB)_2 (\mathbf{u}, x, t) \\
& + 2 \int_0^L f_1 e^{-\mu x} (E\mathbf{u})_1 \left((\partial_t E + \partial_{\mathbf{u}} E.\partial_t \mathbf{u})\, \mathbf{u}\right)_1 + f_2 e^{\mu x} \left((\partial_t E + \partial_{\mathbf{u}} E.\partial_t \mathbf{u})\, \mathbf{u}\right)_2 dx \\
& - \mu \int_0^L D_1 f_1 e^{-\mu x} (E\mathbf{u})_1^2 - D_2 f_2 e^{\mu x} (E\mathbf{u})_2^2 dx + 2qZ(t)\dot{Z}(t).
\end{aligned}
\tag{9.3.34}
$$

In order to simplify this expression, observe that from (9.3.9), (9.3.17) and (9.3.23), $D_1(\mathbf{0}, x, t) = \lambda_1(t, x)$ and $D_2(\mathbf{0}, x, t) = -\lambda_2(t, x)$, using the fact that D is C^1 in $\mathbf{u}$, and using (9.3.19) and (9.3.29) there exists $C > 0$ depending only on H_∞ and α, ν and δ such that

$$
\|D_i - \operatorname{sgn}(D_i(\mathbf{0}, x, t))\lambda_i\|_\infty \leq \|C\mathbf{u}\|_\infty, \tag{9.3.35}
$$

$$
\|\partial_x D_i + \partial_{\mathbf{u}} D_i.\partial_x \mathbf{u} - \operatorname{sgn}(D_i(\mathbf{0}, x, t))\partial_x \lambda_i\|_\infty \leq C\left(\|\partial_x \mathbf{u}\|_\infty + \|\mathbf{u}\|_\infty\right),\ i \in \{1, 2\}, \tag{9.3.36}
$$

and

$$
\|\partial_x E\|_\infty \leq C\left(\|\mathbf{u}\|_\infty\right), \tag{9.3.37}
$$

$$
\|\partial_t E + \partial_{\mathbf{u}} E \partial_t \mathbf{u}\|_\infty \leq C\left(\|\mathbf{u}\|_\infty + \|\partial_t \mathbf{u}\|_\infty\right). \tag{9.3.38}
$$

Thus, using this together with (9.3.34)

$$
\begin{aligned}
\dot{V}_a \leq& -\left[f_1 e^{-\mu x} D_1 (E\mathbf{u})_1^2 + D_2 f_2 e^{\mu x} (E\mathbf{u})_2^2\right]_0^L \\
& - \int_0^L (E\mathbf{u})_1^2 e^{-\mu x} (-\partial_x (\lambda_1 f_1) - \partial_t (f_1)) + (E\mathbf{u})_2^2 e^{\mu x} (\partial_x (\lambda_2 f_2) - \partial_t (f_2)) dx \\
& - 2 \int_0^L f_1 e^{-\mu x} (E\mathbf{u})_1 (EB)_1 (\mathbf{u}, x, t) + f_2 e^{\mu x} (E\mathbf{u})_2 (EB)_2 (\mathbf{u}, x, t) dx \\
& - \mu \int_0^L \lambda_1 f_1 e^{-\mu x} (E\mathbf{u})_1^2 + \lambda_2 f_2 e^{\mu x} (E\mathbf{u})_2^2 dx + 2qZ(t)\dot{Z}(t) \\
& + C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right) \int_0^L (E\mathbf{u})_1^2 + (E\mathbf{u})_2^2 dx \\
& + C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)^2 \int_0^L |(E\mathbf{u})_1| + |(E\mathbf{u})_2| dx,
\end{aligned}
\tag{9.3.39}
$$

where C is a constant that may change between lines but only depends on ν, an upper bound of δ (for instance δ_0), μ, H_∞ and α. Note that C is continuous in $\mu \in [0, \infty)$, thus it can be made independent of μ by imposing an upper bound on μ, for instance $\mu \in (0, 1]$. Finally, from the second equation of (9.3.17), and the fact that E is C^1 in $\mathbf{u}$, there exists a continuous function r_1 such that, for any vector $\mathbf{v} \in \mathbb{R}^2$

$$
E(\mathbf{u}(t, x), x, t)\mathbf{v} - \mathbf{v} = (\mathbf{u}(t, x).r_1(\mathbf{u}(t, x), x, t))\mathbf{v},\ \forall\ (t, x) \in [0, T] \times [0, L]. \tag{9.3.40}
$$

As $E(\mathbf{u}(t, x), x, t)$ is locally a C^∞ function of the coefficients of A, r_1 is bounded on $\mathcal{B}_{\nu_2} \times [0, L] \times [0, T]$ by a bound that only depends on ν_2, H_∞ and α. Thus there exists a constant $\bar{C}$ depending only on ν_2, H_∞ and α such that

$$
\frac{1}{\bar{C}} \|\mathbf{v}\|_{L^2((0,L);\mathbb{R}^2)} \leq \|E\mathbf{v}\|_{L^2((0,L);\mathbb{R}^2)} \leq \bar{C} \|\mathbf{v}\|_{L^2((0,L);\mathbb{R}^2)}. \tag{9.3.41}
$$

Thus, using this together with the fact that D_1 and D_2 are C^1 with $\mathbf{u}$, (9.3.27), and Young's inequality and then Cauchy-Schwarz inequality on the last integral term,

$$\begin{aligned}
\dot{V}_a \leq & -\left[f_1 e^{-\mu x}\lambda_1 (E\mathbf{u})_1^2 - \lambda_2 f_2 e^{\mu x}(E\mathbf{u})_2^2\right]_0^L \\
& - \int_0^L (E\mathbf{u})_1^2 e^{-\mu x}(-\partial_x(\lambda_1 f_1) - \partial_t(f_1)) + (E\mathbf{u})_2^2 e^{\mu x}(\partial_x(\lambda_2 f_2) - \partial_t(f_2))dx \\
& - 2\int_0^L f_1 e^{-\mu x}(E\mathbf{u})_1(EB)_1(\mathbf{u},x,t) + f_2 e^{\mu x}(E\mathbf{u})_2(EB)_2(\mathbf{u},x,t)dx \\
& - \mu \min_{x\in[0,L]}(\lambda_1,\lambda_2)V_a + \mu \min_{x\in[0,L]}(\lambda_1,\lambda_2)qZ^2(t) + 2qZ(t)\dot{Z}(t) \\
& + C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)\|\mathbf{u}\|^2_{L^2(0,L)} \\
& + C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)^3 + C\|\mathbf{u}\|_\infty(|\mathbf{u}(t,0)^2| + |\mathbf{u}(t,L)^2|).
\end{aligned} \tag{9.3.42}$$

Now, as E and B are C^2 with $\mathbf{u}$ and continuous with x and t, and as $B(\mathbf{0},x,t)=0$, there exists a continuous function $r_2 \in C^0(\mathcal{B}_{\nu_2}\times[0,T]\times[0,L];\mathbb{R}^{n\times n\times n})$ such that,

$$(EB)(\mathbf{u}(t,x),x,t) = \partial_{\mathbf{u}}(EB)(\mathbf{0},x,t).\mathbf{u}(t,x) + (r_2(\mathbf{u},x,t).\mathbf{u}(t,x))\mathbf{u}(t,x),\ \forall\ t\in[0,T]\times[0,L]. \tag{9.3.43}$$

Note that from (9.3.10), r_2 is bounded on $\mathcal{B}_{\nu_2}\times[0,L]\times[0,T]$ by a constant that only depends on ν_2, δ, H_∞ and α. From (9.3.10) and (9.3.17) $\partial_{\mathbf{u}}(EB)(\mathbf{0},x,t) = \partial_{\mathbf{u}}B(\mathbf{0},x,t)$. Besides, from (9.3.17), E is invertible and C^1, thus an inequality similar to (9.3.17) holds for E^1, and $\mathbf{u} = E^{-1}(E\mathbf{u})$. Therefore, using (9.3.43) together with (9.3.40), the fact that r_1 and r_2 are bounded, and the expression of $\partial_{\mathbf{u}}B(\mathbf{0},x,t)$ given in (9.3.13)–(9.3.14), one has

$$\begin{aligned}
\dot{V}_a \leq & -\left[f_1 e^{-\mu x}\lambda_1 u_1^2 - \lambda_2 f_2 e^{\mu x}u_2^2\right]_0^L \\
& - \int_0^L (E\mathbf{u})_1^2 e^{-\mu x}(-\partial_x(\lambda_1 f_1) - \partial_t(f_1)) + (E\mathbf{u})^2 e^{\mu x}(\partial_x(\lambda_2 f_2) - \partial_t(f_2))dx \\
& - 2\int_0^L f_1 e^{-\mu x}\gamma_1 (E\mathbf{u})_1^2 + f_2 e^{\mu x}\delta_2 (E\mathbf{u})_2^2 + \left(\gamma_2 f_1 e^{-\mu x} + \delta_1 f_2 e^{\mu x}\right)(E\mathbf{u})_1(E\mathbf{u})_2 dx \\
& - \mu \min_{x\in[0,L]}(\lambda_1,\lambda_2)V_a + \mu \min_{x\in[0,L]}(\lambda_1,\lambda_2)qZ^2(t) + 2qZ(t)\dot{Z}(t) \\
& + C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)\|\mathbf{u}\|^2_{L^2(0,L)} \\
& + C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)^3 + C\|\mathbf{u}\|_\infty(|\mathbf{u}(t,0)^2| + |\mathbf{u}(t,L)^2|).
\end{aligned} \tag{9.3.44}$$

As $\mathcal{D}_1$ and $\mathcal{D}_2$ are of class C^2, denoting for simplicity $k_2 := \partial_1\mathcal{D}_1(0,t)$, $k_1 := \partial_1\mathcal{D}_2(0,0,t)$ and $k_3 := -\partial_2\mathcal{D}_2(0,0,t)$, and using (9.3.15)

$$\begin{aligned}
\dot{V}_a \leq & -\mu \min_{x\in[0,L]}(\lambda_1,\lambda_2)V_a + \left[f_1\lambda_1 k_2^2 - \lambda_2 f_2\right]u_2^2(t,0) \\
& - I_1(u_1(t,L),Z(t)) - \int_0^L I_2((E\mathbf{u})_1,(E\mathbf{u})_2)dx \\
& + C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)\Big(\|\mathbf{u}\|^2_{L^2(0,L)} + \left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)^2 + (|\mathbf{u}(t,0)^2| + |\mathbf{u}(t,L)^2|)\Big),
\end{aligned} \tag{9.3.45}$$

where I_1 and I_2 denote the following quadratic forms

$$\begin{aligned}
I_1(x,y) =& \left(\lambda_1 f_1(L) e^{-\mu L} - \lambda_2 f_2(L) e^{\mu L} k_1^2\right) x^2 + \left(q\sqrt{\frac{H_1}{g}} k_3 - \lambda_2 f_2(L) e^{\mu L} k_3^2 - \mu \min_{x\in[0,L]}(\lambda_1,\lambda_2) q\right) y^2 \\
&+ (2\lambda_2 f_2(L) e^{\mu L} k_3 k_1 - q\sqrt{\frac{H_1}{g}}\,(k_1-1))xy, \\
I_2(x,y) =& \left((-\lambda_1 f_1)_x + 2f_1\gamma_1(t,x) - \partial_t f_1\right) e^{-\mu x} x^2 + \left((\lambda_2 f_2)_x + 2 f_2 \delta_2(t,x) - \partial_t f_2\right) e^{\mu x} y^2 \\
&+ 2\left(\gamma_2 f_1 e^{-\mu x} + \delta_1 f_2 e^{\mu x}\right) xy.
\end{aligned} \tag{9.3.46}$$

We can perform similarly with V_b and V_c, to do this observe that $\partial_t \mathbf{u}$ and $\partial_{tt}^2 \mathbf{u}$ are respectively solutions of

$$\partial_t(\partial_t \mathbf{u}) + A(\mathbf{u},x,t)\partial_x(\partial_t \mathbf{u}) + (\partial_{\mathbf{u}} B(\mathbf{u},x,t))(\partial_t \mathbf{u}) + (\partial_t A(\mathbf{u},x,t) + \partial_{\mathbf{u}} A(\mathbf{u},x,t).\partial_t \mathbf{u})\partial_x \mathbf{u} + \partial_t B\,(\mathbf{u},x,t) = 0 \tag{9.3.47}$$

$$\begin{aligned}
&\partial_t(\partial_{tt}^2 \mathbf{u}) + A(\mathbf{u},x,t)\partial_x(\partial_{tt}^2 \mathbf{u}) + (\partial_{\mathbf{u}} A(\mathbf{u},x).\partial_{tt}^2 \mathbf{u})\partial_x \mathbf{u} + (\partial_{\mathbf{u}} B(\mathbf{u},x))(\partial_{tt}^2 \mathbf{u}), \\
&+ 2\partial_{\mathbf{u}}(\partial_t A(\mathbf{u},x,t)).\partial_t \mathbf{u})\partial_x \mathbf{u} + (\partial_{tt}^2 A(\mathbf{u},x,t) + 2\partial_{\mathbf{u}} A(\mathbf{u},x).\partial_t \mathbf{u} \partial_x(\partial_t \mathbf{u}) + \partial_t A(\mathbf{u},x,t)\partial_x(\partial_t \mathbf{u}) \\
&+ ((\partial_{\mathbf{u}}^2 A(\mathbf{u},x).\partial_t \mathbf{u}).\partial_t \mathbf{u})\partial_x \mathbf{u} + \partial_{tt}^2 B(\mathbf{u},x) + \partial_{\mathbf{u}}(\partial_t B(\mathbf{u},x)).\partial_t \mathbf{u} + (\partial_{\mathbf{u}}^2 B(\mathbf{u},x).\partial_t \mathbf{u})(\partial_t \mathbf{u}) = 0,
\end{aligned} \tag{9.3.48}$$

which are very similar to (9.3.8), as they only differ by quadratic perturbations or terms involving a time derivative of (H_1, V_1). We get then

$$\begin{aligned}
\dot V =& \dot V_a + \dot V_b + \dot V_c \leq -\mu \min_{x\in[0,L]}(\lambda_1,\lambda_2) V + \left[f_1\lambda_1 k_2^2 - \lambda_2 f_2\right]\left(u_2^2(t,0) + (\partial_t u_2(t,0))^2 + (\partial_{tt}^2 u_2(t,0))^2\right) \\
&- I_1(u_1(t,L), Z) - I_1(\partial_t u_1(t,L), \dot Z) - I_1(\partial_{tt}^2 u_1(t,L), \ddot Z) \\
&- \int_0^L I_2((E\mathbf{u})_1, (E\mathbf{u})_2) + I_2((E\partial_t \mathbf{u})_1, (E\partial_t \mathbf{u})_2) + I_2((E\partial_{tt}^2 \mathbf{u})_1, (E\partial_{tt}^2 \mathbf{u})_2) dx \\
&+ C\left(\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty\right)\Big(\|\mathbf{u}\|_{L^2(0,L)}^2 + \|\partial_t \mathbf{u}\|_{L^2(0,L)}^2 + \|\partial_{tt}^2 \mathbf{u}\|_{L^2(0,L)}^2 + (\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty)^2 \\
&+ |u_2(t,0)^2| + (|u_1(t,L)| + |Z|)^2 + |\partial_t u_2(t,0)^2| + (|\partial_t u_1(t,L)| + |\dot Z|)^2 + |\partial_{tt}^2 u_2(t,0)^2| + (|\partial_{tt}^2 u_1(t,L)| + |\ddot Z|)^2\Big) \\
&+ C\delta\left(|u_2(t,0)|^2 + (|u_1(t,L)| + |Z|)^2 + |\partial_t u_2(t,0)|^2 + (|\partial_t u_1(t,L)| + |\dot Z|)^2\right) + C\delta V.
\end{aligned} \tag{9.3.49}$$

The two last terms come from the successive differentiations of the boundary conditions (9.3.15), together with (9.3.29), or the terms in (9.3.47)–(9.3.48) involving a time derivative of A or B. One can see that three identical quadratic form appears in the integral in $((E\partial_t^i \mathbf{u})_1, (E\partial_t^i \mathbf{u})_2)$, $i = 0,1,2$, as well as three identical quadratic form at the boundaries in $(\partial_t^i u_1(t,L), \partial_t^i Z)$, $i = 0,1,2$, and three identical terms proportional respectively to $(\partial_t^i u_2(t,0))$, $i = 0,1,2$. Thus a sufficient condition to have V decreasing strictly would be that the square terms and the forms that appear at the boundaries are negative-definite and the quadratic form in the integral is negative, i.e. the three following conditions :

1. **Condition at** 0

$$\frac{\lambda_2 f_2(0)}{\lambda_1 f_1(0)} > k_2^2. \tag{9.3.50}$$

2. **Condition at** L

$$\frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} > k_1^2, \tag{9.3.51a}$$

$$\left(\lambda_1 f_1(L) - \lambda_2 f_2(L) k_1^2\right)\left(q\sqrt{\frac{H_1}{g}} - \lambda_2 f_2(L) k_3\right) k_3 - \left(\lambda_2 f_2(L) k_3 k_1 - \frac{1}{2} q\sqrt{\frac{H_1}{g}}\,(k_1-1)\right)^2 > 0. \tag{9.3.51b}$$

3. **Condition from the integral**

$$
\begin{aligned}
&((-\lambda_1 f_1)_x + 2f_1\gamma_1(t,x) - \partial_t f_1) > 0, && (9.3.52a)\\
&((-\lambda_1 f_1)_x + 2f_1\gamma_1(t,x) - \partial_t f_1)\,((\lambda_2 f_2)_x + 2f_2\delta_2(t,x) - \partial_t f_2) && \\
&-(\gamma_2 f_1 + \delta_1 f_2)^2 > 0,\ \forall\ (t,x) \in [0,T]\times(0,L). && (9.3.52b)
\end{aligned}
$$

Let assume for the moment that (9.3.50)–(9.3.52) are satisfied for any $\delta \in (0,\delta_3)$ where δ_3 is a positive constant. Then, as the inequalities (9.3.50)–(9.3.52) are strict, by continuity there exist $\mu > 0$ such that the square terms and the quadratic forms I_1 at the boundaries and the quadratic forms I_2 in the integral are positive definite. And there exists $\nu_3 \in (0,\nu_2)$ and $\delta_4 \in (0,\delta_3)$ such that, for any $\nu \in (0,\nu_3)$, and any $\delta \in (0,\delta_4)$,

$$\dot{V} \leq -\mu \min_{[0,L]}(\lambda_1,\lambda_2)V + C\delta V + C\left((\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty)^2\right), \tag{9.3.53}$$

where C is a positive constant depending only on the system. Note that here, the cubic boundary terms that appeared in (9.3.49) have been compensated by the strictly negative quadratic boundary terms, taking ν sufficiently small and using (9.2.21). Choosing $\delta_5 \in (0,\delta_4)$ such that $\delta_5 < \mu \min_{[0,L]}(\lambda_1,\lambda_2)/4C$, for any $\delta \in (0,\delta_5)$ one has

$$\dot{V} \leq -\frac{3}{4}\mu \min_{[0,L]}(\lambda_1,\lambda_2)V + C\left((\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty)^2\right). \tag{9.3.54}$$

Now, if we assume in addition that (9.3.20) hold, using (9.2.21), and Sobolev inequality, there exists $\nu_4 \in (0,\nu_3]$ such that, for any $\nu \in (0,\nu_4)$,

$$C\left((\|\mathbf{u}\|_\infty + \|\partial_x \mathbf{u}\|_\infty)^2\right) \leq \frac{\mu}{4}\min_{[0,L]}(\lambda_1,\lambda_2)V, \tag{9.3.55}$$

thus, setting $\gamma = \mu \min_{[0,L]}(\lambda_1,\lambda_2)$,

$$\dot{V} \leq -\frac{\gamma}{2}V. \tag{9.3.56}$$

which shows the exponential decay of V and ends the proof of Theorem 9.2.3.

In other words, all that remains to do is to find f_1, f_2 and q such that (9.3.50)–(9.3.52) are satisfied and such that V satisfies (9.3.20). In order to find such function we are now going to use Lemma 9.3.2. To understand the link between Lemma 9.3.2 and the three conditions (9.3.50)–(9.3.52), observe that the condition (9.3.52) give rise to a differential inequation, which, as it will appear later on, is linked to the differential equation solved by Lemma 9.3.2. Then (9.3.50) and (9.3.51) can be seen as boundary conditions/values of the solution of this differential inequation.

From Lemma 9.3.2, we know that there exists a solution on $[0,L]$ to equation (9.3.24), namely $\lambda_2\varphi/\lambda_1$. Therefore, as $[0,L]$ is a compact set, there exists ε_1 such that for any $\varepsilon \in [0,\varepsilon_1)$ there exists a solution $f_\varepsilon(t,x)$ to the following system

$$
\begin{aligned}
\partial_x f_\varepsilon(t,x) &= \left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2}(f_\varepsilon)^2 + \sqrt{\frac{g}{H_1}}\partial_t H_1\right) + \varepsilon,\\
f_\varepsilon(0) &= \frac{\lambda_2(t,0)}{\lambda_1(t,0)} + \varepsilon,
\end{aligned} \tag{9.3.57}
$$

and moreover $(t,x,\varepsilon) \to f_\varepsilon(t,x)$ is of class C^0 and $\partial_x f_\varepsilon(t,x)$ as well. This is a classical result on ODE due to Peano (see e.g. [105][Chap. 5, Th 3.1]). From (9.3.57), $\partial_t f_\varepsilon$ satisfies the following equation

$$\partial_x\partial_t f_\varepsilon = 2\frac{\delta_1}{\varphi\lambda_2}f_\varepsilon\partial_t f_\varepsilon + \left(\frac{\varphi\gamma_2}{\lambda_1}\right)_t + \left(\frac{\delta_1}{\varphi\lambda_2}\right)_t f_\varepsilon^2 + \sqrt{\frac{g}{H_1}}\partial_{tt}^2 H_1 - \frac{1}{2}\sqrt{\frac{g}{H_1^3}}(\partial_t H_1)^2. \tag{9.3.58}$$

We used here that, from Proposition 9.2.1 and Remark 9.2.1, $(H_1, V_1) \in C^0([0,+\infty); H^3(0,L))$, and from (9.2.4), $\partial_t\partial_x H_1 = -\partial_x^2(HV)$ and $\partial_t\partial_x V_1 = \partial_x\left(-V_1\partial_x V_1 - g\partial_x H_1 - (kV_1^2/H_1 - gC)\right)$. Thus $\partial_{tt}^2 H_1$ belongs to $C^0([0,T]\times H^1(0,L))$, and $(\gamma_1,\gamma_2,\delta_1,\delta_2)$ belong to $C^1([0,T]\times H^1(0,L))$. Using (9.3.58), we have

$$\begin{aligned}\partial_t f_\varepsilon(t,x) =& \partial_t f_\varepsilon(t,0)\exp\left(\int_0^x 2\frac{\delta_1}{\varphi\lambda_2} f_\varepsilon(t,y)dy\right)\\&+\int_0^x \exp\left(\int_y^x 2\frac{\delta_1}{\varphi\lambda_2}f_\varepsilon(t,\omega)d\omega\right)\left(\left(\frac{\varphi\gamma_2}{\lambda_1}\right)_t + \left(\frac{\delta_1}{\varphi\lambda_2}\right)_t f_\varepsilon^2 + \sqrt{\frac{g}{H_1}}\partial_{tt}^2 H_1 - \frac{1}{2}\sqrt{\frac{g}{H_1^3}}(\partial_t H_1)^2\right)dy.\end{aligned} \tag{9.3.59}$$

Instead of seeing the function f_ε as a solution of an ODE with a parameter t, one can see it as a solution of an ODE with parameters λ_1, λ_2, γ_2, δ_1, $\partial_t H_1$ and ε that we denote $g_\varepsilon(x,\lambda_1,\lambda_2,\gamma_1,\delta_1,\partial_t H_1)$. From [105][Theorem 2.1] g_ε is continuous with these parameters and with ε. But from (9.2.12), (9.2.13), and (9.3.29), all these parameters are bounded and therefore belong to a compact set when $t\in[0,+\infty)$. Thus,

$$\varepsilon \to g_\varepsilon(x,\lambda_1(t),\lambda_2(t),\gamma_1(t),\delta_1(t),\partial_t H_1(t)) = f_\varepsilon(t,x) \tag{9.3.60}$$

is uniformly continuous in ε for $(t,x)\in[0,\infty)\times[0,L]$. This, together with (9.3.59) implies that

$$\begin{aligned}&\left|\int_0^x \exp\left(\int_y^x 2\frac{\delta_1}{\varphi\lambda_2}f_\varepsilon(t,\omega)d\omega\right)\partial_t(\partial_y(H_1V_1))dy\right|\\&\leq C_0\max\left(\|\partial_t H_1\|_{C^1([0,+\infty);L^\infty(0,L))}, \|\partial_t V_1\|_{C^1([0,+\infty);L^\infty(0,L))}\right),\end{aligned} \tag{9.3.61}$$

where C_0 is a constant that only depends on L, H_∞, α, and is continuous with $\varepsilon\in[0,\varepsilon_1)$.
Similarly there exists a constant $C_1>0$ depending only on L, H_∞ and α such that

$$\|\partial_t\varphi_1\|_{L^\infty((0,+\infty)\times(0,L)}\leq C_1\max\left(\|\partial_t H_1\|_{C^1([0,+\infty);L^\infty(0,L))}, \|\partial_t V_1\|_{C^1([0,+\infty);L^\infty(0,L))}\right), \tag{9.3.62}$$

and similarly for φ_2. This, together with the definition of λ_1 and λ_2 given by (9.3.23), (9.3.59), and using the continuity of $\varepsilon\to f_\varepsilon$ on $[0,\varepsilon_1)$ (recall that this continuity is uniform with respect to $(t,x)\in[0,+\infty)\times[0,L]$), we get that there exists $C>0$ depending only on H_∞, α and ε and continuous with ε on $[0,\varepsilon_1)$ such that

$$|\partial_t f_\varepsilon(t,x)| \leq (|\partial_t f_\varepsilon(t,0)| + \max\left(\|\partial_t H_1\|_{C^1([0,+\infty);L^\infty(0,L))}, \|\partial_t V_1\|_{C^1([0,+\infty);L^\infty(0,L))}\right))C(\varepsilon). \tag{9.3.63}$$

But, from (9.3.57) $\partial_t f_\varepsilon(t,0) = (\lambda_2/\lambda_1)_t$, thus using (9.3.29) we obtain

$$|\partial_t f_\varepsilon(t,x)| \leq \delta C_2(\varepsilon), \tag{9.3.64}$$

where C_2 is again a constant that only depends on ε, α and H_∞ and is continuous with ε on $[0,\varepsilon_1)$. We can now restrict ourselves to $\varepsilon\in[0,\varepsilon_1/2]$ and then C_2 can be chosen independent of ε by simply taking its maximum on $[0,\varepsilon_1/2]$. Recall that from Lemma 9.3.2 we have, $f_0=\varphi\lambda_2/\lambda_1$, and

$$\left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2}f_0^2 + \sqrt{\frac{g}{H_1}}\partial_t H_1\right) > 0. \tag{9.3.65}$$

Recall that we still have not chosen the bound $\delta\in(0,\delta_0)$ on $\|\partial_t H_1\|_{C^1([0,\infty);L^\infty(0,L))}$ and $\|\partial_t V_1\|_{C^1([0,\infty);L^\infty(0,L))}$ given in (9.3.29). From the assumptions on k_p and k_I, i.e. (9.2.25), and (9.3.16), and recalling that $k_1 = \partial_1\mathcal{D}_2(0,0,t)$ and $k_3 = -\partial_2\mathcal{D}_2(0,0,t)$, one has

$$k_1^2 < \left(\frac{\lambda_1(L)}{\lambda_2(L)}\right)^2, \quad k_3>0. \tag{9.3.66}$$

Thus, using (9.3.23),

$$\eta_1 := \min\left(\left(\frac{1}{|k_1|} - \frac{\lambda_2(L)}{\lambda_1(L)}\right), 1 - \frac{\lambda_2(L)}{\lambda_1(L)}\right) > 0. \tag{9.3.67}$$

As $\varepsilon \to f_\varepsilon(t,x)$ is uniformly continuous with ε for $(t,x) \in [0,\infty) \times [0,L]$, there exists $\varepsilon_2 \in (0, \varepsilon_1/2)$ such that for any $(t,x) \in [0,\infty) \times [0,L]$

$$|f_{\varepsilon_2}(t,x) - f_0(t,x)| \leq \varphi(t,L)\eta_1, \tag{9.3.68}$$

and

$$\left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2} f_{\varepsilon_2}^2 + \sqrt{\frac{g}{H_1}}\partial_t H_1\right) > 0. \tag{9.3.69}$$

Note that ε_2 depends *a priori* on δ from (9.3.69). However, from Lemma 9.3.2 we can in fact choose ε_2 independent of δ and depending only on an upper bound of δ (for instance δ_0 given by Lemma 9.3.2). This is important as, in the following, we will choose a δ that may depends on ε.

We select f_1 and f_2 in the following way :

$$\begin{aligned} f_1(t,x) &= \frac{\varphi_1^2}{\lambda_1 f_{\varepsilon_2}(t,x)} > 0, \\ f_2(t,x) &= \varphi_2^2 \frac{f_{\varepsilon_2}(t,x)}{\lambda_2} > 0, \end{aligned} \tag{9.3.70}$$

and we can now check that the condition (9.3.52) is verified for δ small enough as

$$(-\lambda_1 f_1)_x = -2\frac{(\varphi_1)_x \lambda_1 f_1}{\varphi_1} + \varphi_1^2 \frac{\partial_x f_{\varepsilon_2}(t,x)}{f_{\varepsilon_2}^2(t,x)}. \tag{9.3.71}$$

Thus from (9.3.22)

$$-(\lambda_1 f_1)_x + 2\gamma_1 f_1 = \varphi_1^2 \frac{\partial_x f_{\varepsilon_2}}{f_{\varepsilon_2}^2} \tag{9.3.72}$$

and similarly

$$\begin{aligned} (\lambda_2 f_2)_x + 2\delta_2 f_2 &= (\varphi_2^2 f_{\varepsilon_2}(t,x))_x - (\varphi_2^2)_x f_{\varepsilon_2}(t,x) \\ &= \varphi_2^2 \partial_x f_{\varepsilon_2}. \end{aligned} \tag{9.3.73}$$

Therefore, from (9.3.57), (9.3.72), and (9.3.73), one has

$$\begin{aligned} (-(\lambda_1 f_1)_x + 2\gamma_1 f_1 - \partial_t f_1)((\lambda_2 f_2)_x + 2\delta_2 f_2 - \partial_t f_2) = \left(\frac{\varphi_1\varphi_2}{f_{\varepsilon_2}}\right)^2 \left(\left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2} f_{\varepsilon_2}^2 + \sqrt{\frac{g}{H_1}}\partial_t H_1\right) + \varepsilon_2\right)^2 \\ - \partial_x f_{\varepsilon_2}\left(\frac{\varphi_1^2}{f_{\varepsilon_2}^2}\partial_t f_2 + \varphi_2^2 \partial_t f_1\right) + (\partial_t f_1)(\partial_t f_2). \end{aligned} \tag{9.3.74}$$

But we have

$$\partial_t f_1 = 2\frac{(\partial_t \varphi_1)\varphi_1}{\lambda_1 f_{\varepsilon_2}} - \left(\frac{\partial_t \lambda_1}{\lambda_1^2 f_{\varepsilon_2}} + \frac{\partial_t f_{\varepsilon_2}}{\lambda_1 f_{\varepsilon_2}^2}\right)\varphi_1^2, \tag{9.3.75}$$

and besides, from (9.2.4) and (9.3.29), there exists $C_3 > 0$ depending only on α and H_∞, and an upper bound of δ (for instance δ_0), such that

$$\max(\|H_{1x}\|_{L^\infty((0,+\infty)\times(0,L)}, \|V_{1x}\|_{L^\infty((0,+\infty)\times(0,L)}) \leq C_3. \tag{9.3.76}$$

Thus, using (9.3.14) and (9.3.23), there exists $C_4 > 0$ depending only on α and H_∞, and δ_0 (but not on δ) such that

$$\max(\|\varphi_1\|_{L^\infty((0,+\infty)\times(0,L)}, \|\varphi^{-1}\|_{L^\infty((0,+\infty)\times(0,L)}) < C_4, \tag{9.3.77}$$

and similarly for φ_2. Observe now that, from $f_0 = \lambda_2\varphi/\lambda_1$ and (9.3.77), $|f_0|$ and $1/|f_0|$ can be bounded by a constant depending only on α, H_∞, and δ_0. Thus from (9.3.68)

$$1/C_5 \leq \|f_{\varepsilon_2}\|_{L^\infty((0,+\infty)\times(0,L)} \leq C_5, \tag{9.3.78}$$

where C_5 only depends on α, H_∞ and δ_0. And therefore, from (9.3.23), (9.3.64), (9.3.62), and (9.3.78) one has

$$|\partial_t f_1| \leq C_6 \delta, \tag{9.3.79}$$

and similarly

$$|\partial_t f_2| \leq C_7 \delta, \tag{9.3.80}$$

where C_6 and C_7 are constants that only depend on α, H_∞ (and δ_0). We now select the bound on $\max(|\partial_t H_1|, |\partial_t V_1|)$: we select $\delta_3 \in (0, \delta_0)$ such that, for any $\delta \in [0, \delta_3]$ and any $(t,x) \in [0,\infty) \times [0,L]$,

$$C_6 C_5^2 C_4^2 \delta < \varepsilon_2, \tag{9.3.81}$$

and

$$\begin{aligned}
&\varepsilon_2^2 + 2\varepsilon_2 \inf_{x\in[0,L],t\in[0,+\infty),\varepsilon\in(0,\varepsilon_2)} \left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2} f_\varepsilon^2 + \sqrt{\frac{g}{H_1}}\partial_t H_1 \right) \\
&> \left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2} X^2 + \sqrt{\frac{g}{H_1}}\delta + \varepsilon_2 \right) \left(C_6 \frac{\varphi_1^2}{X^2} + C_7 \varphi_2^2 \right) \left(\frac{X}{\varphi_1\varphi_2} \right)^2 \delta \\
&+ 2\sqrt{\frac{g}{H_1}} \left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\varphi^{-1}\delta_1}{\lambda_2} X^2 \right) \delta + \left(\frac{X}{\varphi_1\varphi_2} \right)^2 C_7 C_6 \delta^2,
\end{aligned} \tag{9.3.82}$$

for any $x \in [0,L]$ and any $X \in [1/C_5, C_5]$. Observe that this is obviously possible as $\varepsilon_2 > 0$ and, when $\delta_3 = 0$, (9.3.82) is verified and the inequality is strict. Then, from (9.3.22), (9.3.77), (9.3.74), (9.3.78)–(9.3.82),

$$\begin{aligned}
(-(\lambda_1 f_1)_x + 2\gamma_1 f_1 - \partial_t f_1)((\lambda_2 f_2)_x + 2\delta_2 f_2 - \partial_t f_2) &> \left(\frac{\varphi_1\varphi_2}{f_{\varepsilon_2}} \right)^2 \left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2} f_{\varepsilon_2}^2 \right)^2 \\
&= \left(\frac{\gamma_2}{\lambda_1} f_1 + \frac{\delta_1}{\lambda_2} f_2 \right)^2,
\end{aligned} \tag{9.3.83}$$

which is exactly the second inequality of (9.3.52). Besides, from (9.3.25) and (9.3.81),

$$\begin{aligned}
(-(\lambda_1 f_1)_x + 2\gamma_1 f_1 - \partial_t f_1) =& \varphi_1^2 \frac{\partial_x f_{\varepsilon_2}}{f_{\varepsilon_2}^2} - \partial_t f_1 \\
=& \frac{\varphi_1^2}{f_{\varepsilon_2}^2} \left(\left(\frac{\varphi\gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi\lambda_2} f_{\varepsilon_2}^2 + \sqrt{\frac{g}{H_1}}\partial_t H_1 \right) + \varepsilon_2 - \frac{\partial_t f_1 f_{\varepsilon_2}^2}{\varphi_1^2} \right) \\
&> 0.
\end{aligned} \tag{9.3.84}$$

We can now check that (9.3.50) and (9.3.51) are also verified thanks to the choice of ε_2 and η_1. Indeed, using (9.3.57) and (9.3.16), one has

$$\frac{\lambda_2(0) f_2(t,0)}{\lambda_1(0) f_1(t,0)} = f_{\varepsilon_2}^2(t,0) = \left(\frac{\lambda_2(0)}{\lambda_1(0)} + \varepsilon_2 \right)^2 > \left(\frac{\lambda_2(0)}{\lambda_1(0)} \right)^2 = k_2^2. \tag{9.3.85}$$

This explains our choice of initial condition for f_{ε_2}. Now, from (9.3.68), one has

$$\frac{\lambda_1(t,L) f_1(t,L)}{\lambda_2(t,L) f_2(t,L)} = \frac{\varphi^2(t,L)}{f_{\varepsilon_2}^2(L)} > \frac{1}{\left(\frac{\lambda_2(t,L)}{\lambda_1(t,L)} + \eta_1 \right)^2}, \tag{9.3.86}$$

and from the definition of η_1 given by (9.3.67),

$$\eta_1 + \frac{\lambda_2(L)}{\lambda_1(L)} = \min\left(\frac{1}{|k_1|}, 1 \right). \tag{9.3.87}$$

Therefore

$$\frac{\lambda_1(t,L) f_1(t,L)}{\lambda_2(t,L) f_2(t,L)} > \max(k_1^2, 1), \tag{9.3.88}$$

and in particular the condition (9.3.51a) is verified. Let us now look at condition (9.3.51b). So far we have not selected the positive constant q. We want to show that there exists $q > 0$ such that the condition (9.3.51b) is satisfied. Observe that the left-hand side of (9.3.51b) can be seen as a polynomial in q, and the condition (9.3.51b) can be rewritten as

$$\begin{aligned} P(q) &:= -\frac{q^2}{4}\frac{H_1}{g}(k_1-1)^2 + q\sqrt{\frac{H_1}{g}}k_3\left(\lambda_1 f_1(L) - \lambda_2 f_2(L)(k_1^2 - k_1(k_1-1))\right) - (\lambda_1 f_1(L))(\lambda_2 f_2(L))k_3^2 \\ &= -\frac{q^2}{4}\frac{H_1}{g}(k_1-1)^2 + q\sqrt{\frac{H_1}{g}}k_3(\lambda_1 f_1(L) - \lambda_2 f_2(L)k_1)) - (\lambda_1 f_1(L))(\lambda_2 f_2(L))k_3^2 > 0. \end{aligned} \tag{9.3.89}$$

From (9.3.88) $\lambda_1 f_1(t,L) > \lambda_2 f_2(t,L)k_1$ and from (9.3.66) $k_3 > 0$. Thus the real roots of P are positive if they exist. This implies that there exists a positive constant q such that (9.3.51b) is satisfied if the discriminant of P is positive. Denoting its discriminant by Δ,

$$\Delta = \frac{H_1}{g}k_3^2\lambda_2^2 f_2(t,L)^2\left[\left(\frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} - k_1\right)^2 - \left(\frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)}\right)(k_1-1)^2\right]. \tag{9.3.90}$$

Let us introduce $h\ : X \to (X-k_1)^2 - X(k_1-1)^2$. The function h is a second order polynomial with a positive dominant coefficient and observe that its roots are k_1^2 and 1. Thus h is increasing strictly on $[\max(k_1^2, 1), +\infty)$. Hence, using (9.3.88),

$$\begin{aligned} \Delta &= \frac{H_1}{g}k_3^2\lambda_2^2 f_2(t,L)^2 h\left(\frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)}\right) \\ &> \frac{H_1}{g}k_3^2\lambda_2^2 f_2(t,L)^2 h(\max(k_1^2,1)) = 0. \end{aligned} \tag{9.3.91}$$

This proves that there exists $q > 0$ such that (9.3.51b) is satisfied, and we select such q. All it remains to do now is to show that the function $(\mathbf{U}, z) \to V(t, \mathbf{U}, z)$, which is now entirely selected, satisfies (9.3.20).

From (9.2.13) and (9.2.12) we know that for any $(t,x) \in [0,\infty) \times [0,L]$,

$$\sqrt{gH_\infty} > \lambda_2 > \alpha,\ 2\sqrt{gH_\infty} > \lambda_1 > \alpha. \tag{9.3.92}$$

Besides, from the definition of φ_1 and φ_2 given by (9.3.22), (9.3.14) and the bound (9.2.13), (9.2.12), there exists a constant C_8 that only depends on δ, α and H_∞ such that

$$\frac{1}{C_8} \leq \|\varphi_1\|_\infty \leq C_8,\ \frac{1}{C_8} \leq \|\varphi_2\|_\infty \leq C_8. \tag{9.3.93}$$

Thus, using that $f_0 = \lambda_2\varphi/\lambda_1$, (9.3.70), (9.3.68), (9.3.93), and (9.3.92), there exists $c_1 > 0$ constant independent of $\mathbf{U}$ and z such that, for any $(\mathbf{U}, z) \in H^2(0,L) \times \mathbb{R}$,

$$c_1\left(\|\mathbf{U}\|_{H^2(0,L)} + |z|\right) \leq V(t,(\mathbf{U},z)) \leq \frac{1}{c_1}\left(\|\mathbf{U}\|_{H^2(0,L)} + |Z|\right) \forall\ t \in [0,+\infty), \tag{9.3.94}$$

which is exactly (9.3.20). This concludes the proof of Theorem 9.2.3. □

9.4 Conclusion

In this chapter we gave simple conditions on the design of a single PI controller to ensure the exponential stability of the nonlinear Saint-Venant equations with arbitrary friction and slope in the H^2 norm. These conditions apply when the inflow is an unknown constant, in that case the system has steady-states and any of them are stable. But they also apply when the inflow is time-dependant and slowly variable. In that case, no steady-states exists and one has to stabilize other target states. When the values of the target state are known at end of the river, we have exponential stability of the target state. Otherwise, we have the Input-to-State stability with respect to the variation of the inflow disturbance. In this study we only considered static feedback laws, it would be interesting to know if one could obtain better results with feeback laws that can change with time, even with only a finite number of modes. This would require to study a system with switching behaviors, and some results are known in the linear case [131, 132]. These sufficient conditions are found using a local quadratic entropy and, to the best of our knowledge, are less restrictive than any of the conditions that existed so far, even in the linear case. In [22] it was shown that, in absence of friction and slope, these conditions were optimal for the linear case. However, so far there is no answer when there is some slope or friction and whether these conditions are optimal or not would be a very interesting issue for a further study. Its possible application to a network of channels would also be a matter of interest. Finally, many stabilizing devices for finite dimensional systems also use a PID control with an additional derivative term. It has been shown in [64] that this control cannot ensure exponential stability for an homogeneous hyperbolic equation. It would be an interesting question to know whether a filtering on the derivative term could enable to recover the stability for infinite dimensional system and whether this would enable a faster stabilization than the PI control.

Acknowledgment

The author would like to thank Jean-Michel Coron for his constant support and his advices. The author would like to thank Sebastien Boyaval for many fruitful discussions. The author wishes also to thank Eric Demay, Peipei Shang, Shengquan Xiang and Christophe Zhang for fruitful discussions. Finally the author would like to thank the ANR project Finite4SoS (ANR 15-CE23-0007) and the french Corps des IPEF.

9.5 Appendix

9.5.1 Proof of Proposition 9.2.10

This appendix uses many computations that are very similar to the computations in Section 9.3, but in a simpler way. Thus, in order to avoid writing two times the same thing and to keep the proof relatively short, some steps might be quicker in this appendix. Let $T_1 > 0$ and to be chosen later on. As $(H_0(0), V_0(0))$ satisfies (9.2.9), there exists $\nu_a > 0$ such that for $\nu \in (0, \nu_a)$, $F((H_1^0, V_1^0)^T)$ has two distinct nonzero eigenvalues. Recall that F is given by (9.3.7) and that that ν is the bound on $\|H_1^0 - H_0(0), V_1^0 - V_0(0)\|_{H^2(0,L)}$. Besides, from (9.2.8) $(H_0(t,\cdot), V_0(t,\cdot))$ can be seen as the solution of a system of ODE with a parameter t in the initial condition. Thus, as $\partial_t Q_0 \in C^2([0,+\infty))$ and the slope C satisfies $C \in C^2([0,L])$, using (9.2.6) and [105][Chap. 5, Theorem 3.1], $(H_0, V_0) \in C^3([0,T_1] \times C^3([0,L]))$ and there exists a constant C depending only on H_∞, α and an upper bound of δ, such that,

$$\|\partial_t^i H_0, \partial_t^i V_0\|_{C^2([0,L])} \leq C \left| \sum_{n=1}^{i} \partial_t^n Q_0 \right|, \ \forall\ i \in [1,3], \tag{9.5.1}$$

and in particular

$$\|\partial_t H_0, \partial_t V_0\|_{C^2([0,T_1];C^2([0,L]))} \leq C \|\partial_t Q_0\|_{C^2(0,+\infty)}. \tag{9.5.2}$$

Thus [198][Theorem 2.1] can still be used on $(H_1 - H_0)$ and there exist $\delta_0(T_1) > 0$ and $\nu_0(T_1) \in (0, \nu_a)$ such that, if $\nu \in (0, \nu_0(T_1))$ and $\delta \in (0, \delta_0(T_1))$, there exists a unique solution $(H_1, V_1) \in C^0([0,T_1]; H^2(0,L))^2$ to the system (9.2.4)–(9.2.5). Besides (H_1, V_1) satifsfies an estimate as (9.2.21) but with (H_1, V_1) instead of (H, V) and (H_0, V_0) instead of (H_1, V_1). We denote by $C(T_1)$ the associated constant. Let us define $h_1 := H_1 - H_0$ and $v_1 := V_1 - V_0$. We transform $(h_1, v_1)^T$ into $\mathbf{w} = (w_1, w_2)^T$ using the change of variables defined by (9.3.1)–(9.3.5) with H_0 and V_0 instead of H_1 and V_1. Thus we obtain

$$
\begin{aligned}
\partial_t \mathbf{w} + A_0(\mathbf{w},x)\partial_x \mathbf{w} + B_0(\mathbf{w},x) + S_0\begin{pmatrix}\partial_t H_0\\ \partial_t V_0\end{pmatrix} = 0,\\
w_1(t,0) =&\mathcal{H}_1(w_2(t,0), Q_0(t)-Q_0(0)),\\
w_2(t,L) =&\mathcal{H}_2(w_2(t,L)),
\end{aligned}
\tag{9.5.3}
$$

where A_0, B_0 and S_0 have the same expression as A, B and S (given by(9.3.9), (9.3.10), (9.3.6)) but with (H_0, V_0) instead of (H_1, V_1). Similarly we define

$$
\lambda_1^0 = V_0 + \sqrt{gH_0},\ \lambda_2^0 = \sqrt{gH_0} - V_0,
\tag{9.5.4}
$$

and φ^0, defined as φ but with (H_0, V_0) instead of (H_1, V_1). Similarly as in Appendix 9.5.3,

$$
\mathcal{H}_2'(0) = -\lambda_1^0(L)/\lambda_2^0(L),\ \mathcal{H}_1'(0) = -\lambda_2^0(0)/\lambda_1^0(0),
\tag{9.5.5}
$$

which is of the form (9.3.15) with $v_G = 0$ and $Z = 0$. Before going any further, note that we can perform the same computations as in Section 2 with no problem, as the proof in Section 9.3 only used Proposition 9.2.1 to get that (H_1, V_1) exists for any time and that (9.2.13) and Lemma 9.3.3 hold, but we will see now that such claims are true for H_0 and V_0. The existence of (H_0, V_0) was already shown in section 9.2 and (9.2.9) is exactly (9.2.13) with (H_0, V_0) instead of (H_1, V_1). Finally, (9.5.2) is exactly the equivalent of Lemma 9.3.3 for (H_0, V_0). We define now the Lyapunov fonction candidate $V := V_a(\mathbf{w}(t,x),t) + V_b(\mathbf{w}(t,x),t) + V_c(\mathbf{w}(t,x),t) + V_d(\mathbf{w}(t,x),t)$ where V_a, V_b and V_c are defined in (9.3.27), (9.3.30), with f_1 and f_2 chosen as $f_1 := (\varphi_1^0)^2/(\lambda_1^0\eta)$ and $f_2 := (\varphi_2^0)^2\eta/(\lambda_2^0)$, where η is a function such that there exists a constant $\varepsilon > 0$ independent of $\mathbf{w}$ such that

$$
\begin{aligned}
\eta' &= \left|\frac{\gamma_2^0}{\lambda_1^0} + \frac{\delta_1^0}{\lambda_2^0}\eta^2\right| + \varepsilon, \forall\ x \in [0,L],\\
\eta(0) &= \frac{\lambda_2^0(0)}{\lambda_1^0(0)}\varphi^0(0) + \varepsilon.
\end{aligned}
\tag{9.5.6}
$$

Note that η exists as, for any $t \in [0,+\infty)$, $(\varphi(t,\cdot)^0\lambda_2^0(t\cdot)/\lambda_1^0(t\cdot))$ is a solution of

$$
\partial_x f = \left|\frac{\gamma_2^0}{\lambda_1^0} + \frac{\delta_1^0}{\lambda_2^0}f^2\right|, \forall\ x \in [0,L],
\tag{9.5.7}
$$

this can be proved as in Lemma 9.3.2, and this case was actually shown in [110]. Note that from (9.2.6), (9.2.8) and (9.2.9), $(H_0)_x$ and $(V_0)_x$ can be bounded by above and by below by constants that only depends on H_∞, α and an upper bound of Q_0 (which can also be expressed only with H_∞, α from (9.2.9)). Therefore, looking at their definition, the function f_1 and f_2 can also be bounded by above and below by constants that only depends on H_∞, α and ε. Thus there exist $c_1 > 0$ and $c_2 > 0$ depending only on H_∞ and α, ε and μ such that

$$
c_1\|h_1(t,\cdot), v_1(t,\cdot)\|_{H^2(0,L)}^2 \leq V(t) \leq c_2\|h_1(t,\cdot), v_1(t,\cdot)\|_{H^2(0,L)}^2, \forall\ t \in [0,T_1].
\tag{9.5.8}
$$

Consequently, by differentiating V exactly as in (9.3.33)–(9.3.49), and from (9.5.3), we obtain that there exists $\mu > 0$, $\nu_1 \in (0,\nu_0(T_1))$ and $\delta_3 > 0$ such that, for any $\|h_1(0,\cdot), v_1(0,\cdot)\|_{H^2(0,L)} \leq \nu_1$, and $\|\partial_t Q_0\|_{C^2([0,\infty))} \leq \delta$, where $\delta \in (0,\delta_3)$,

$$
\begin{aligned}
\dot{V} \leq& -\mu V + \int_0^L 2f_1 w_1(S_0\begin{pmatrix}\partial_t H_0\\ \partial_t V_0\end{pmatrix})_1 + 2f_2 w_2(S_0\begin{pmatrix}\partial_t H_0\\ \partial_t V_0\end{pmatrix})_2 dx,\\
&+ \int_0^L 2f_1 \partial_t w_1(S_0\begin{pmatrix}\partial_{tt}^2 H_0\\ \partial_{tt}^2 V_0\end{pmatrix})_1 + 2f_2 \partial_t w_2(S_0\begin{pmatrix}\partial_{tt}^2 H_0\\ \partial_{tt}^2 V_0\end{pmatrix})_2 dx,\\
&+ \int_0^L 2f_1 \partial_{tt}^2 w_1(S_0\begin{pmatrix}\partial_{ttt}^3 H_0\\ \partial_{ttt}^3 V_0\end{pmatrix})_1 + 2f_2 \partial_{tt}^2 w_2(S_0\begin{pmatrix}\partial_{ttt}^3 H_0\\ \partial_{ttt}^3 V_0\end{pmatrix})_2 dx.
\end{aligned}
\tag{9.5.9}
$$

Thus, using Cauchy-Schwarz inequality, (9.5.8), and (9.5.1) there exists $C_1 > 0$ depending only on H_∞, α and an upper bound of μ such that

$$\dot{V}(t) \leq -\mu V(t) + C_1|\partial_t Q_0(t) + \partial_{tt}^2 Q_0(t) + \partial_{ttt}^3 Q_0(t)|V^{1/2}(t), \ \forall\, t \in [0, T_1]. \tag{9.5.10}$$

and in particular

$$\dot{V}(t) \leq -\mu V(t) + C_1\|\partial_t Q_0\|_{C^2([0,t])} V^{1/2}(t), \ \forall\, t \in [0, T_1]. \tag{9.5.11}$$

Let us define $V_{eq} := (C_1\delta/\mu)^2$. From (9.5.11), if $V(t) > 2V_{eq}$, then there exists a constant $k > 0$ such that $\dot{V}(t) < -kV^{1/2}(t)$. We now choose δ such that $\sqrt{2}C_1\delta/(\mu\sqrt{c_1}) < \nu_1$. Thus, from (9.5.11) and as c_1, c_2, C_1 and μ do not depend on T_1, we can choose T_1 large enough such that

$$V(T_1) \leq 2V_{eq} \leq c_1\nu_1^2, \tag{9.5.12}$$

which implies that

$$\|h_1(T_1,\cdot), v_1(T_1,\cdot)\|_{C^2(0,L)} \leq \nu_1 \tag{9.5.13}$$

and therefore there exists a unique solution $(h_1, v_1) \in C^0([T_1, 2T_1], H^2(0, L))$, with initial condition $(h_1(T_1,\cdot), v_1(T_1,\cdot))$ (we use the same existence Theorem ([198][Theorem 2.1])) and, noting that $V(T_1) \leq 2V_{eq}$ implies $V(2T_1) \leq 2V_{eq}$, this analysis still hold. We can do similarly for any $[nT_1, (n+1)T_1]$ with $n \in \mathbb{N}$, thus, as $(H_0, V_0) \in C^0([0, +\infty), H^2(0, L))$, there exists a unique solution $(H_1, V_1) \in C^0([0, +\infty), H^2(0, L))$ and (9.5.10) holds for any $t \in [0, +\infty)$. Therefore denoting $g(t) = V(t)e^{\mu t}$, we deduce from (9.5.10) that

$$g'(t) \leq C_1|\partial_t Q_0(t) + \partial_{tt}^2 Q_0(t) + \partial_{ttt}^3 Q_0(t)|e^{\frac{\mu t}{2}}\sqrt{g(t)}. \tag{9.5.14}$$

Thus

$$V^{1/2}(t) \leq V^{1/2}(0)e^{-\frac{\mu t}{2}} + \frac{C_1}{2}\left(\int_0^t |\partial_t Q_0(s) + \partial_{tt}^2 Q_0(s) + \partial_{ttt}^3 Q_0(s)|e^{\frac{\mu s}{2}} ds\right)e^{-\frac{\mu t}{2}}. \tag{9.5.15}$$

This implies the ISS property

$$\begin{aligned}\|h_1(t,\cdot), v_1(t,\cdot)\|_{H^2((0,L);\mathbb{R}^2)} \leq & \sqrt{\frac{c_2}{c_1}}\|h_1(0,\cdot), v_1(0,\cdot)\|_{H^2((0,L);\mathbb{R}^2)}e^{-\frac{\mu t}{2}} \\ & + \frac{C_1}{2\sqrt{c_1}}\left(\int_0^t |\partial_t Q_0(s) + \partial_{tt}^2 Q_0(s) + \partial_{ttt}^3 Q_0(s)|e^{\frac{\mu s}{2}} ds\right)e^{-\frac{\mu t}{2}}.\end{aligned} \tag{9.5.16}$$

This ends the proof of Proposition 9.2.1. To extend this proof to the H^p norm for $p > 2$, note that using the same argument (9.5.2) holds with the $C^p([0, T_1]; C^3([0, L]))$ norm in the left-hand side and the C^p norm in the right-hand side. We can can define $V_3, ..., V_p$ on $H^p(0, L) \times \mathbb{R} \times \mathbb{R}_+$ as in (9.3.30) such that $V_k(\mathbf{w}(t, x), t) = V_a(\partial_t^k \mathbf{w}(t, x), t)$, for any $k \in [3, p]$. Then (9.5.8) holds with $V := V_a + V_b + V_c + V_3 + ...V_p$ and the H^p norm, and the rest can done done identically.

9.5.2 Proof of Theorem 9.2.4

Theorem 9.2.4 result from the proof of Theorem 9.2.3. Note that the boundary conditions (9.2.16) can be written under the form (9.2.18) with (H_0, V_0) instead of (H_1, V_1) where the only difference is that Z satisfies now

$$\dot{Z} = H_c - H(t, L) + \frac{f(t)}{v_G k_I}, \tag{9.5.17}$$

where $f(t) = H_c\partial_t V_0(t, L)$. The rest of the proof can be conducted as in Appendix 9.5.1 for (H_1, V_1), with *a priori* two differences : (H, V) satisfies the boundary conditions of the form (9.2.18) and not of the form given in (9.2.4), and $\dot{Z}$ satisfies (9.5.17) instead of (9.2.15). However, note that in Appendix 9.5.1 the only assumption used on the boundary conditions of the transformed system is that they are of the form (9.3.3), which is still the case here. Thus, the only difference with Appendix 9.5.1 are some additional terms when $\dot{Z}$ is used, which is in the boundary terms in the derivative of the Lyapunov function. There exists therefore $\delta_4 > 0$ and $\nu_2 > 0$ such that, for any $\|h_1(0,\cdot), v_1(0,\cdot)\|_{H^2(0,L)} \leq \nu_2$, and $\|\partial_t Q_0\|_{C^2([0,\infty))} \leq \delta$, where $\delta \in (0, \delta_4)$,

$$\dot{V}(t) \leq -\frac{\gamma}{2}V(t) + C_1|\partial_t Q_0(t) + \partial_{tt}^2 Q_0(t) + \partial_{ttt}^3 Q_0(t)|V^{1/2} + 2qZf(t) + 2q\dot{Z}f'(t) + 2q\ddot{Z}f''(t), \tag{9.5.18}$$

where C_1 is a constant only depending on H_∞, α, ν_2 and δ_4. Using Lemma 9.3.3, there exists a constant $C > 0$ depending only on H_∞, α, ν_2 and δ_4 such that

$$\dot{V} \leq -\frac{\gamma}{2}V + CV^{1/2}|\partial_t Q_0(t) + \partial_{tt}^2 Q_0(t) + \partial_{ttt}^3 Q_0(t)|. \tag{9.5.19}$$

The same argument as in Appendix 9.5.1, (9.5.14)–(9.5.16), implies directly the ISS property (9.2.30).

9.5.3 Boundary conditions (9.3.15) and (9.3.16)

In this appendix we justify the boundary conditions (9.3.15) with (9.3.16) after the change of variables. From the boundary conditions (9.3.3) in the physical coordinate (h, v), together with the definition of u_1 and u_2 given in (9.3.5), one has at $x = L$

$$\begin{aligned} u_1(t,L) &= \mathcal{B}_2(h(t,L), Z(t), t) + \sqrt{\frac{g}{H_1}}h(t,L) =: \mathcal{F}_1(h(t,L), Z(t), x, t), \\ u_2(t,L) &= \mathcal{B}_2(h(t,L), Z(t), t) - \sqrt{\frac{g}{H_1}}h(t,L) =: \mathcal{F}_2(h(t,L), Z(t), x, t). \end{aligned} \tag{9.5.20}$$

From its definition, $\mathcal{F}_1$ is C^1 and, from (9.3.4), and (9.2.21), there exists $\nu_1 \in (0, \nu_0)$ such that, for any $t \in [0, \infty)$, $\partial_1 \mathcal{F}_0(0, Z(t), t) \neq 0$. Thus $\mathcal{F}_1$ is locally invertible with respect to its first variable, thus there exists $\nu_2 \in (0, \nu_1)$ such that $h(t, L) = \mathcal{F}_1^{-1}(u_1(t,L), Z(t), t)$, where $\mathcal{F}_1^{-1}$ denotes the inverse with respect to the first variable. Besides, as $\mathcal{F}_1$ is of class C^2 with respect to the two first variables, $\mathcal{F}_1^{-1}$ is also of class C^2. Then, using (9.5.20)

$$u_2(t,L) = \mathcal{F}_2(\mathcal{F}_1^{-1}(u_1(t,L), Z(t), t), Z(t), t) =: \mathcal{D}_2(u_1(t,L), Z(t), t). \tag{9.5.21}$$

and, using (9.3.4),

$$\begin{aligned} \partial_1 \mathcal{D}_2(0,0,t) &= \partial_1 \mathcal{F}_2(0,0,t)\partial_1(\mathcal{F}_1^{-1})(0,0,t) \\ &= \frac{\partial_1 \mathcal{F}_2(0,0,t)}{\partial_1 \mathcal{F}_1(0,0,t)} = \frac{\partial_1 \mathcal{B}_2(0,0,t) - \sqrt{\frac{g}{H_1}}}{\partial_1 \mathcal{B}_2(0,0,t) + \sqrt{\frac{g}{H_1}}} \\ &= -\frac{\lambda_1(L) - v_G(1+k_p)}{\lambda_2(L) + v_G(1+k_p)}. \end{aligned} \tag{9.5.22}$$

Now, as $\partial_2 \mathcal{F}_1^{-1}(0,0,t) = -\partial_2 \mathcal{F}_1(0,0,t)/\partial_1 \mathcal{F}_1(0,0,t)$, using (9.3.4),

$$\begin{aligned} \partial_2 \mathcal{D}_2(0,0,t) &= \partial_1 \mathcal{F}_2(0,0,t)\partial_2(\mathcal{F}_1^{-1})(0,0,t) + \partial_2 \mathcal{F}_2(0,0,t) \\ &= -\partial_1 \mathcal{F}_2(0,0,t)\frac{\partial_2 \mathcal{F}_1(0,0,t)}{\partial_1 \mathcal{F}_1(0,0,t)} + \partial_2 \mathcal{F}_2(0,0,t) \\ &= \partial_2 \mathcal{B}_2(0,0,t)\left(1 - \frac{\partial_1 \mathcal{B}_2(0,0,t) - \sqrt{\frac{g}{H_1}}}{\partial_1 \mathcal{B}_2(0,0,t) + \sqrt{\frac{g}{H_1}}}\right) \\ &= -\frac{v_G k_I}{H_1(t,L)}\left(\frac{2\sqrt{gH_1(t,L)}}{v_G(1+k_p) + \lambda_2(t,L)}\right). \end{aligned} \tag{9.5.23}$$

The same can be done in $x = 0$ in a slightly easier way, as $\mathcal{B}_1$ does not depends on Z. This gives (9.3.15) and (9.3.16).

9.5.4 Proof of Lemma 9.3.2

In this appendix we prove Lemma 9.3.2. The proof is very similar to the proof given in [110] in the special case where (H_1, V_1) is a steady state. However, it happens that the proof actually does not need the relation

$(H_1V_1)_x = 0$ which is no longer true when (H_1, V_1) is not a steady-state. Let $f = (\lambda_2\varphi/\lambda_1)$, we have from (9.3.22) :

$$\begin{aligned}
\partial_x f =& \frac{\varphi}{\lambda_1^2}\left(\lambda_1\partial_x\lambda_2 - \lambda_2\partial_x\lambda_1 + \lambda_2\gamma_1 + \lambda_1\delta_2\right)\\
=& \frac{\varphi}{\lambda_1^2}\Bigg((V_1+\sqrt{gH_1})(-V_{1x}+\frac{\sqrt{gH_1}}{2H_1}H_{1x}) - (-V_1+\sqrt{gH_1})(V_{1x}+\frac{\sqrt{gH_1}}{2H_1}H_{1x})\\
&+(\sqrt{gH_1}-V_1)\left(\frac{3}{4}\sqrt{\frac{g}{H_1}}H_{1x}+\frac{3}{4}V_{1x}+\frac{kV}{H_1}-\frac{kV_1^2}{2H_1^2}\sqrt{\frac{H_1}{g}}\right)\\
&+(V_1+\sqrt{gH_1})\left(-\frac{3}{4}\sqrt{\frac{g}{H_1}}H_{1x}+\frac{3}{4}V_{1x}+\frac{2kV}{H_1}+\frac{kV_1^2}{2H_1^2}\sqrt{\frac{H_1}{g}}\right)\Bigg)\\
=& \frac{\varphi}{\lambda_1^2}\left(\sqrt{gH_1}\left(-2V_{1x}+\frac{3}{2}V_{1x}+\frac{2kV}{H_1}\right) - V_1\left(\frac{3}{2}\sqrt{\frac{g}{H_1}}H_{1x}-\frac{kV_1^2}{H_1^2}\sqrt{\frac{H_1}{g}}-\sqrt{\frac{g}{H_1}}H_{1x}\right)\right)\\
=& \frac{\varphi}{\lambda_1^2}\left(\frac{2kV}{H_1}\sqrt{gH_1}+\frac{kV_1^2}{H_1^2}\sqrt{\frac{H_1}{g}}V_1+\frac{1}{2}\sqrt{\frac{g}{H_1}}\partial_t H_1\right).
\end{aligned} \tag{9.5.24}$$

And on the other hand :

$$\begin{aligned}
\left(\frac{\varphi\gamma_2}{\lambda_1}+\frac{\delta_1}{\lambda_2\varphi}f^2\right) &= \frac{\varphi}{\lambda_1^2}\left(\lambda_1\gamma_2+\lambda_2\delta_1\right)\\
&= \frac{\varphi}{\lambda_1^2}\left(\frac{2kV}{H_1}\sqrt{gH_1}+\frac{kV_1^2}{H_1^2}\sqrt{\frac{H_1}{g}}V_1+V_1\sqrt{\frac{g}{H_1}}\frac{H_{1x}}{2}+V_{1x}\frac{\sqrt{gH_1}}{2}\right)\\
&= \frac{\varphi}{\lambda_1^2}\left(\frac{2kV}{H_1}\sqrt{gH_1}+\frac{kV_1^2}{H_1^2}\sqrt{\frac{H_1}{g}}V_1-\frac{1}{2}\sqrt{\frac{g}{H_1}}\partial_t H_1\right).
\end{aligned} \tag{9.5.25}$$

Thus from (9.5.24) and (9.5.25)

$$\partial_x f = \left(\frac{\varphi\gamma_2}{\lambda_1}+\frac{\delta_1}{\lambda_2\varphi}f^2+\sqrt{\frac{g}{H_1}}\partial_t H_1\right). \tag{9.5.26}$$

And there exists δ_0 such that, if $\|\partial_t H_1\|_{L^\infty((0,+\infty)\times(0,L)}\delta_0$,

$$\frac{\varphi}{\lambda_1^2}\left(\frac{2kV_1}{H_1}\sqrt{gH_1}+\frac{kV_1^2}{H_1^2}\sqrt{\frac{H_1}{g}}V_1+\sqrt{\frac{g}{H_1}}\partial_t H_1\right) > 0,\ \forall\ x\in[0,L],\ t\in[0,+\infty), \tag{9.5.27}$$

and, from (9.5.24) and (9.5.26),

$$\partial_x f = \left|\frac{\varphi\gamma_2}{\lambda_1}+\frac{\delta_1}{\lambda_2\varphi}f^2+\sqrt{\frac{g}{H_1}}\partial_t H_1\right|, \tag{9.5.28}$$

this ends the proof of Lemma 9.3.2.

Bibliography

[1] Ole Morten Aamo. Disturbance rejection in 2×2 linear hyperbolic systems. IEEE Trans. Automat. Control, 58(5):1095–1106, 2013.

[2] Fabio Ancona, Alberto Bressan, and Giuseppe Maria Coclite. Some results on the boundary control of systems of conservation laws. In Hyperbolic problems: Theory, numerics, applications. Proceedings of the ninth international conference on hyperbolic problems, Pasadena, CA, USA, March 25–29, 2002, pages 255–264. Berlin: Springer, 2003.

[3] Fabio Ancona and Andrea Marson. On the attainable set for scalar nonlinear conservation laws with boundary control. SIAM J. Control Optim., 36(1):290–312, 1998.

[4] Karl Johan Åström and Tore Hägglund. PID controllers: theory, design, and tuning, volume 2. Instrument society of America Research Triangle Park, NC, 1995.

[5] Karl Johan Åström, Tore Hägglund, and Karl J Astrom. Advanced PID control, volume 461. ISA-The Instrumentation, Systems, and Automation Society Research Triangle, 2006.

[6] Karl Johan Åström and Richard M. Murray. Feedback systems. Princeton University Press, Princeton, NJ, 2008. An introduction for scientists and engineers.

[7] Jean Auriol and Florent Di Meglio. Minimum time control of heterodirectional linear coupled hyperbolic PDEs. Automatica J. IFAC, 71:300–307, 2016.

[8] Abdellahi Bechir Aw and Michel Rascle. Resurrection of "second order" models of traffic flow. SIAM J. Appl. Math., 60(3):916–938, 2000.

[9] Andras Balogh and Miroslav Krstic. Infinite dimensional backstepping-style feedback transformations for a heat equation with an arbitrary level of instability. European journal of control, 8(2):165–175, 2002.

[10] Adhémar Barré de Saint-Venant. Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. Comptes Rendus de l'Académie des Sciences, 53:147–154, 1871.

[11] Georges Bastin and Jean-Michel Coron. On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval. Systems & Control Letters, 60(11):900–906, 2011.

[12] Georges Bastin and Jean-Michel Coron. On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval. Systems & Control Letters, 60(11):900–906, 2011.

[13] Georges Bastin and Jean-Michel Coron. Stability and Boundary Stabilisation of 1-D Hyperbolic Systems. Number 88 in Progress in Nonlinear Differential Equations and Their Applications. Springer International, 2016.

[14] Georges Bastin and Jean-Michel Coron. Stability and boundary stabilization of 1-D hyperbolic systems, volume 88 of Progress in Nonlinear Differential Equations and their Applications. Birkhäuser/Springer, [Cham], 2016. Subseries in Control.

[15] Georges Bastin and Jean-Michel Coron. Stability and boundary stabilization of 1-d hyperbolic systems. Springer, 2016.

[16] Georges Bastin and Jean-Michel Coron. A quadratic Lyapunov function for hyperbolic density-velocity systems with nonuniform steady states. Systems & Control Letters, 104:66–71, 2017.

[17] Georges Bastin and Jean-Michel Coron. Exponential stability of PI control for Saint-Venant equations with a friction term. December 2018. working paper or preprint.

[18] Georges Bastin, Jean-Michel Coron, and Brigitte d'Andréa Novel. On Lyapunov stability of linearised Saint-Venant equations for a sloping channel. Networks and Heterogeneous Media, 4(2):177–187, 2009.

[19] Georges Bastin, Jean-Michel Coron, Amaury Hayat, and Peipei Shang. Boundary feedback stabilization of hydraulic jumps. IFAC J. Syst. Control, 7:100026, 10, 2019.

[20] Georges Bastin, Jean-Michel Coron, Amaury Hayat, and Peipei Shang. Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation. Math. Models Methods Appl. Sci., 29(2):271–316, 2019.

[21] Georges Bastin, Jean-Michel Coron, and Simona Oana Tamasoiu. Stability of linear density-flow hyperbolic systems under PI boundary control. Automatica J. IFAC, 53:37–42, 2015.

[22] Georges Bastin, Jean-Michel Coron, and Simona Oana Tamasoiu. Stability of linear density-flow hyperbolic systems under PI boundary control. Automatica J. IFAC, 53:37–42, 2015.

[23] Harry Bateman. Some recent researches on the motion of fluids. Monthly Weather Review, 43(4):163–170, 1915.

[24] Richard Bellman and Kenneth L. Cooke. Differential-difference equations. Academic Press, New York-London, 1963.

[25] Stuart Bennett. A history of control engineering, 1930-1955. Number 47. IET, 1993.

[26] Saad Bennis. Hydraulique et hydrologie. PUQ, 2007.

[27] Sébastien Blandin, Xavier Litrico, Maria Laura Delle Monache, Benedetto Piccoli, and Alexandre Bayen. Regularity and Lyapunov stabilization of weak entropy solutions to scalar conservation laws. IEEE Trans. Automat. Control, 62(4):1620–1635, 2017.

[28] Sebastien Blandin, Dan Work, Paola Goatin, Benedetto Piccoli, and Alexandre Bayen. A general phase transition model for vehicular traffic. SIAM journal on Applied Mathematics, 71(1):107–127, 2011.

[29] François Bouchut and Sébastien Boyaval. A new model for shallow viscoelastic fluids. Math. Models Methods Appl. Sci., 23(8):1479–1526, 2013.

[30] François Bouchut and Sébastien Boyaval. Unified derivation of thin-layer reduced models for shallow free-surface gravity flows of viscous fluids. Eur. J. Mech. B Fluids, 55(part 1):116–131, 2016.

[31] François Bouchut, Anne Mangeney-Castelnau, Benoît Perthame, and Jean-Pierre Vilotte. A new model of Saint Venant and Savage-Hutter type for gravity driven shallow water flows. C. R. Math. Acad. Sci. Paris, 336(6):531–536, 2003.

[32] Dejan M. Bošković, Andras Balogh, and Miroslav Krstić. Backstepping in infinite dimension for a class of parabolic distributed parameter systems. Math. Control Signals Systems, 16(1):44–75, 2003.

[33] Alberto Bressan and Giuseppe Maria Coclite. On the boundary control of systems of conservation laws. SIAM J. Control Optim., 41(2):607–622, 2002.

[34] Alberto Bressan and Giuseppe Maria Coclite. On the boundary control of systems of conservation laws. SIAM J. Control Optim., 41(2):607–622, 2002.

[35] J. M. Burgers. Application of a model system to illustrate some points of the statistical theory of free turbulence. Nederl. Akad. Wetensch., Proc., 43:2–12, 1940.

[36] J. M. Burgers. A mathematical model illustrating the theory of turbulence. In Advances in Applied Mechanics, pages 171–199. Academic Press, Inc., New York, N. Y., 1948. edited by Richard von Mises and Theodore von Kármán,.

[37] Johannes Martinus Burgers. On the resistance experienced by a fluid in turbulent motion. In Proc. KNAW, volume 26, pages 582–604, 1923.

[38] Ch. I. Byrnes, D. S. Gilliam, and V. I. Shubov. On the global dynamics of a controlled viscous Burgers' equation. J. Dynam. Control Systems, 4(4):457–519, 1998.

[39] Christopher I. Byrnes and Alberto Isidori. New results and examples in nonlinear feedback stabilization. Systems Control Lett., 12(5):437–442, 1989.

[40] Vincenzo Capasso. Mathematical structures of epidemic systems, volume 97 of Lecture Notes in Biomathematics. Springer-Verlag, Berlin, 2008. With a foreword by Simon A. Levin, Corrected reprint of the 1993 original.

[41] Carlos Castro, Francisco Palacios, and Enrique Zuazua. An alternating descent method for the optimal control of the inviscid Burgers equation in the presence of shocks. Math. Models Methods Appl. Sci., 18(3):369–416, 2008.

[42] Hubert Chanson. Hydraulics of open channel flow. Butterworth-Heinemann, 2004.

[43] M. Hanif Chaudhry. Open-Channel Flow. Springer Publishing Company, Incorporated, 2nd edition, 2007.

[44] Reginald H. Clarke. The Morning Glory: An atmospheric hydraulic jump. Journal of Applied Meteorology, 11(2):304–311, 1972.

[45] Julian D. Cole. On a quasi-linear parabolic equation occurring in aerodynamics. Quart. Appl. Math., 9:225–236, 1951.

[46] Jean-Michel Coron. On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain. SIAM J. Control Optim., 37(6):1874–1896, 1999.

[47] Jean-Michel Coron. Control and nonlinearity, volume 136 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007.

[48] Jean-Michel Coron and Georges Bastin. Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the C^1-norm. SIAM Journal on Control and Optimization, 53(3):1464–1483, 2015.

[49] Jean-Michel Coron and Georges Bastin. Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the C^1-norm. SIAM J. Control Optim., 53(3):1464–1483, 2015.

[50] Jean-Michel Coron, Georges Bastin, and Brigitte d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. SIAM Journal on Control and Optimization, 47(3):1460–1498, 2008.

[51] Jean-Michel Coron and Brigitte d'Andréa Novel. Stabilization of a rotating body beam without damping. IEEE Trans. Automat. Control, 43(5):608–618, 1998.

[52] Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. A Lyapunov approach to control irrigation canals modeled by Saint-Venant equations. In CD-Rom Proceedings, Paper F1008-5, ECC99, Karlsruhe, Germany, pages 3178–3183, 1999.

[53] Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. IEEE Transactions on Automatic Control, 52(1):2–11, 2007.

[54] Jean-Michel Coron, Brigitte d'Andrea Novel, and Georges Bastin. Penser globalement, agir localement. RECHERCHE-PARIS-, 417:82, 2008.

[55] Jean-Michel Coron, Jonathan de Halleux, Georges Bastin, and Brigitte d'Andréa Novel. On boundary control design for quasilinear hyperbolic systems with entopies as lyapunov functions. In Proceedings of the 41st IEEE, Conference on Decision and Control, Las Vegas, Nevada USA, December 2002, pages 3010–3014, 2002.

[56] Jean-Michel Coron, Sylvain Ervedoza, Shyam Sundar Ghoshal, Olivier Glass, and Vincent Perrollaz. Dissipative boundary conditions for 2×2 hyperbolic systems of conservation laws for entropy solutions in BV. J. Differential Equations, 262(1):1–30, 2017.

[57] Jean-Michel Coron, Sylvain Ervedoza, Shyam Sundar Ghoshal, Olivier Glass, and Vincent Perrollaz. Dissipative boundary conditions for 2×2 hyperbolic systems of conservation laws for entropy solutions in BV. J. Differential Equations, 262(1):1–30, 2017.

[58] Jean-Michel Coron, Oliver Glass, and Zhiqiang Wang. Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed. SIAM J. Control Optim., 48(5):3105–3122, 2009/10.

[59] Jean-Michel Coron and Amaury Hayat. PI controllers for 1-D nonlinear transport equation. 2018. working paper or preprint.

[60] Jean-Michel Coron, Long Hu, and Guillaume Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. Automatica J. IFAC, 84:95–100, 2017.

[61] Jean-Michel Coron and Hoai-Minh Nguyen. Dissipative boundary conditions for nonlinear 1-D hyperbolic systems: sharp conditions through an approach via time-delay systems. SIAM J. Math. Anal., 47(3):2220–2240, 2015.

[62] Jean-Michel Coron and Hoai-Minh Nguyen. Dissipative boundary conditions for nonlinear 1-d hyperbolic systems: sharp conditions through an approach via time-delay systems. SIAM Journal on Mathematical Analysis, 47(3):2220–2240, 2015.

[63] Jean-Michel Coron and Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. Arch. Ration. Mech. Anal., 225(3):993–1023, 2017.

[64] Jean Michel Coron and Simona Oana Tamasoiu. Feedback stabilization for a scalar conservation law with PID boundary control. Chin. Ann. Math. Ser. B, 36(5):763–776, 2015.

[65] Jean-Michel Coron, Rafael Vazquez, Miroslav Krstic, and Georges Bastin. Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. SIAM J. Control Optim., 51(3):2005–2035, 2013.

[66] Jean-Michel Coron and Zhiqiang Wang. Output feedback stabilization for a scalar conservation law with a nonlocal velocity. SIAM J. Math. Anal., 45(5):2646–2665, 2013.

[67] Jean-Michel Coron and Shengquan Xiang. Small-time global stabilization of the viscous Burgers equation with three scalar controls. working paper or preprint, March 2018.

[68] Constantine M. Dafermos. Hyperbolic conservation laws in continuum physics, volume 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, third edition, 2010.

[69] Jean Le Rond d'Alembert. Essai d'une nouvelle théorie de la résistance des fluides. David l'aîné, 1752.

[70] J. de Halleux, C. Prieur, J.-M. Coron, B. d'Andréa Novel, and G. Bastin. Boundary feedback control in networks of open channels. Automatica J. IFAC, 39(8):1365–1376, 2003.

[71] Jonathan de Halleux, Christophe Prieur, Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. Boundary feedback control in networks of open channels. Automatica. A Journal of IFAC, the International Federation of Automatic Control, 39(8):1365–1376, 2003.

[72] Riche de Prony. Nouvelle architecture hydraulique, seconde partie. Firmin Didot, Paris, 1796.

[73] Serghei S Demidov. Création et développement de la théorie des équations différentielles aux dérivées partielles dans les travaux de j. d'alembert. Revue d'histoire des sciences, 35(1):3–42, 1982.

[74] Lane Desborough and Randy Miller. Increasing customer value of industrial control performance monitoring-honeywell's experience. In AIChE symposium series, number 326, pages 169–189. New York; American Institute of Chemical Engineers; 1998, 2002.

[75] Florent Di Meglio, Miroslav Krstic, and Rafael Vazquez. A backstepping boundary observer for a class of linear first-order hyperbolic systems. In 2013 European Control Conference (ECC), pages 1597–1602, July 2013.

[76] Florent Di Meglio, Rafael Vazquez, and Miroslav Krstic. Stabilization of a system of $n+1$ coupled first-order hyperbolic linear PDEs with a single boundary input. IEEE Trans. Automat. Control, 58(12):3097–3111, 2013.

[77] Ababacar Diagne, Mamadou Diagne, Shuxia Tang, and Miroslav Krstic. Backstepping stabilization of the linearized Saint-Venant-Exner Model: Part II- output feedback. In 2015 54th IEEE Conference on Decision and Control (CDC), pages 1248–1253, Dec 2015.

[78] Ababacar Diagne, Mamadou Diagne, Shuxia Tang, and Miroslav Krstic. Backstepping stabilization of the linearized *saint-venant-exner* model. Automatica J. IFAC, 76:345–354, 2017.

[79] Mamadou Diagne, Peipei Shang, and Zhiqiang Wang. Feedback stabilization for the mass balance equations of an extrusion process. IEEE Trans. Automat. Control, 61(3):760–765, 2016.

[80] Mamadou Diagne, Shuxia Tang, Ababacar Diagne, and Miroslav Krstic. Control of shallow waves of two unmixed fluids by backstepping. Annual Reviews in Control, 44:211–225, 2017.

[81] Markus Dick, Martin Gugat, and Günter Leugering. Classical solutions and feedback stabilization for the gas flow in a sequence of pipes. Netw. Heterog. Media, 5(4):691–709, 2010.

[82] Markus Dick, Martin Gugat, and Günter Leugering. A strict H^1-Lyapunov function and feedback stabilization for the isothermal Euler equations with friction. Numer. Algebra Control Optim., 1(2):225–244, 2011.

[83] Valérie Dos Santos and Christophe Prieur. Boundary control of open channels with numerical and experimental validations. IEEE transactions on Control systems technology, 16(6):1252–1264, 2008.

[84] Valérie Dos Santos, Georges Bastin, Jean-Michel Coron, and B. d'Andréa Novel. Boundary control with integral action for hyperbolic systems of conservation laws: stability and experiments. Automatica J. IFAC, 44(5):1310–1318, 2008.

[85] Valérie Dos Santos Martins, Yongxin Wu, and Mickael Rodrigues. Design of a proportional integral control using operator theory for infinite dimensional hyperbolic systems. IEEE Transactions on Control Systems Technology, 22(5):2024–2030, 2014.

[86] Peter M. Dower and Peter M. Farrell. On linear control of backward pumped raman amplifiers. IFAC Proceedings Volumes, 39(1):547–552, 2006.

[87] Leonhard Euler. Principe généraux du mouvement des fluides, volume 11. 1755.

[88] Andrew Russell Forsyth. Theory of differential equations. 1. Exact equations and Pfaff's problem; 2, 3. Ordinary equations, not linear; 4. Ordinary linear equations; 5, 6. Partial differential equations. Six volumes bound as three. Dover Publications, Inc., New York, 1959.

[89] A. V. Fursikov. Stabilizability of a quasilinear parabolic equation by means of boundary feedback control. Mat. Sb., 192(4):115–160, 2001.

[90] Jean-Charles Galland, Nicole Goutal, and Jean-Michel Hervouet. Telemac: A new numerical model for solving shallow water equations. Advances in Water Resources, 14(3):138–148, 1991.

[91] Jean-Frédéric Gerbeau and Benoit Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. Discrete Contin. Dyn. Syst. Ser. B, 1(1):89–102, 2001.

[92] Ivor Grattan-Guinness and Steven Engelsman. The manuscripts of Paul Charpit. Historia Mathematica, 9(1):65–75, 1982.

[93] James M. Greenberg and Tatsien Li. The effect of boundary damping for the quasilinear wave equation. J. Differential Equations, 52(1):66–75, 1984.

[94] Martin Gugat, Markus Dick, and Günter Leugering. Gas flow in fan-shaped networks: classical solutions and feedback stabilization. SIAM J. Control Optim., 49(5):2101–2117, 2011.

[95] Martin Gugat and Michaël Herty. Existence of classical solutions and feedback stabilization for the flow in gas networks. ESAIM Control Optim. Calc. Var., 17(1):28–51, 2011.

[96] Martin Gugat, Michael Herty, and Siegfried Müller. Coupling conditions for the transition from supersonic to subsonic fluid states. Netw. Heterog. Media, 12(3):371–380, 2017.

[97] Martin Gugat and Günter Leugering. Global boundary controllability of the de St. Venant equations between steady states. Annales de l'Institut Henri Poincaré. Analyse Non Linéaire, 20(1):1–11, 2003.

[98] Martin Gugat, Günter Leugering, Klaus Schittkowski, and E. J. P. Georg Schmidt. Modelling, stabilization, and control of flow in networks of open channels. In Online optimization of large scale systems, pages 251–270. Springer, Berlin, 2001.

[99] Martin Gugat, Günter Leugering, Simona Tamasoiu, and Ke Wang. H^2-stabilization of the isothermal Euler equations: a Lyapunov function approach. Chinese Annals of Mathematics. Series B, 33(4):479–500, 2012.

[100] Martin Gugat, Günter Leugering, and Ke Wang. Neumann boundary feedback stabilization for a nonlinear wave equation: A strict H^2-Lyapunov function. Math. Control Relat. Fields, 7(3):419–448, 2017.

[101] Martin Gugat and Rüdiger Schultz. Boundary feedback stabilization of the isothermal Euler equations with uncertain boundary data. SIAM J. Control Optim., 56(2):1491–1507, 2018.

[102] Martin Gugat and Stefan Ulbrich. The isothermal Euler equations for ideal gas with source term: product solutions, flow reversal and no blow up. J. Math. Anal. Appl., 454(1):439–452, 2017.

[103] Richard Haberman. Mathematical models. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1977. Mechanical vibrations, population dynamics, and traffic flow, An introduction to applied mathematics.

[104] Willi H Hager. Energy dissipators and hydraulic jump, volume 8 of Water Science and Technology Library. Springer, 1992.

[105] Philip Hartman. Ordinary differential equations. John Wiley & Sons, Inc., New York-London-Sydney, 1964.

[106] Amaury Hayat. Exponential stability of general 1-D quasilinear systems with source terms for the C^1 norm under boundary conditions. Preprint, 2017.

[107] Amaury Hayat. On boundary stability of inhomogeneous 2×2 1-D hyperbolic systems for the C^1 norm. Preprint, 2017.

[108] Amaury Hayat. PI controllers for the general Saint-Venant equations. January 2019. working paper or preprint.

[109] Amaury Hayat and Peipei Shang. Exponential stability of density-velocity systems with boundary conditions and source term for the h^2 norm. working paper or preprint, 2019.

[110] Amaury Hayat and Peipei Shang. A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope. Automatica J. IFAC, 100:52–60, 2019.

[111] Eberhard Hopf. The partial differential equation $u_t+uu_x = \mu u_{xx}$. Comm. Pure Appl. Math., 3:201–230, 1950.

[112] Long Hu, Florent Di Meglio, Rafael Vazquez, and Miroslav Krstic. Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. IEEE Trans. Automat. Control, 61(11):3301–3314, 2016.

[113] Long Hu and Guillaume Olive. Minimal time for the exact controllability of one-dimensional first-order linear hyperbolic systems by one-sided boundary controls. working paper or preprint, January 2019.

[114] Long Hu, Rafael Vazquez, Florent Di Meglio, and Miroslav Krstic. Boundary exponential stabilization of 1-d inhomogeneous quasilinear hyperbolic systems. arXiv preprint arXiv:1512.03539, 2015.

[115] R. E. Kalman. Contributions to the theory of optimal control. Bol. Soc. Mat. Mexicana (2), 5:102–119, 1960.

[116] R. E. Kalman. Mathematical description of linear dynamical systems. J. SIAM Control Ser. A, 1:152–192 (1963), 1963.

[117] Iasson Karafyllis and Miroslav Krstic. Input-to-State Stability for PDEs. Springer, 2018.

[118] Tosio Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Rational Mech. Anal., 58(3):181–205, 1975.

[119] Hassan K. Khalil. Nonlinear systems. Macmillan Publishing Company, New York, 1992.

[120] Shigeo Kida. Asymptotic properties of Burgers turbulence. J. Fluid Mech., 93(2):337–377, 1979.

[121] Daniel E Koditschek. Adaptive techniques for mechanical systems. Proc. 5th. Yale University Conference, page pp. 259–265, 1987.

[122] Vilmos Komornik. Stabilisation frontière rapide de systèmes distribués linéaires. C. R. Acad. Sci. Paris Sér. I Math., 321(4):433–437, 1995.

[123] Vilmos Komornik. Rapid boundary stabilization of linear distributed systems. SIAM J. Control Optim., 35(5):1591–1613, 1997.

[124] Miroslav Krstic. On global stabilization of Burgers' equation by boundary control. Systems Control Lett., 37(3):123–141, 1999.

[125] Miroslav Krstic, Lionel Magnis, and Rafael Vazquez. Nonlinear stabilization of shock-like unstable equilibria in the viscous Burgers PDE. IEEE Trans. Automat. Control, 53(7):1678–1683, 2008.

[126] Miroslav Krstic and Andrey Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. Systems Control Lett., 57(9):750–758, 2008.

[127] Miroslav Krstic and Andrey Smyshlyaev. Boundary control of PDEs, volume 16 of Advances in Design and Control. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. A course on backstepping designs.

[128] Miroslav Krstic and Andrey Smyshlyaev. Boundary Control of PDEs: A Course on Backstepping Designs, volume 16 of Advances in Design and Control. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[129] Joachim Kuettner and Rolf F. Hertenstein. Observations of mountain induced rotors and related hypotheses: A review. In 10th Conference on Mountain Meteorology and MAP Meeting, 2002.

[130] Pierre-Olivier Lamare and Nikolaos Bekiaris-Liberis. Control of 2×2 linear hyperbolic systems: backstepping-based trajectory generation and PI-based tracking. Systems Control Lett., 86:24–33, 2015.

[131] Pierre-Olivier Lamare, Antoine Girard, and Christophe Prieur. Lyapunov techniques for stabilization of switched linear systems of conservation laws. In 52nd IEEE Conference on Decision and Control, pages 448–453, Dec 2013.

[132] Pierre-Olivier Lamare, Antoine Girard, and Christophe Prieur. Switching rules for stabilization of linear systems of conservation laws. SIAM J. Control Optim., 53(3):1599–1624, 2015.

[133] Peter D. Lax. Hyperbolic systems of conservation laws. II. Comm. Pure Appl. Math., 10:537–566, 1957.

[134] Peter D. Lax. Hyperbolic systems of conservation laws and the mathematical theory of shock waves. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[135] Philippe LeFloch. Explicit formula for scalar nonlinear conservation laws with boundary condition. Math. Methods Appl. Sci., 10(3):265–287, 1988.

[136] Antonio Lepschy, Gian Antonio Mian, and Umberto Viaro. Feedback control in ancient water and mechanical clocks. IEEE Transactions on Education, 35(1):3–10, Feb 1992.

[137] Günter Leugering and E. J. P. Georg Schmidt. On the modelling and stabilization of flows in networks of open canals. SIAM Journal on Control and Optimization, 41(1):164–180, 2002.

[138] Tatsien Li. Global classical solutions for quasilinear hyperbolic systems, volume 32 of RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.

[139] Tatsien Li. Exact controllability for quasilinear hyperbolic systems and its application to unsteady flows in a network of open canals. Mathematical Methods in the Applied Sciences, 27(9):1089–1114, 2004.

[140] Tatsien Li. Controllability and observability for quasilinear hyperbolic systems, volume 3 of AIMS Series on Applied Mathematics. American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.

[141] Tatsien Li, Bopeng Rao, and Zhiqiang Wang. Exact boundary controllability and observability for first order quasilinear hyperbolic systems with a kind of nonlocal boundary conditions. Discrete and Continuous Dynamical Systems. Series A, 28(1):243–257, 2010.

[142] Tatsien Li and Wen Ci Yu. Boundary value problems for quasilinear hyperbolic systems. Duke University Mathematics Series, V. Duke University, Mathematics Department, Durham, NC, 1985.

[143] Mark Lichtner. Spectral mapping theorem for linear hyperbolic systems. Proc. Amer. Math. Soc., 136(6):2091–2101, 2008.

[144] Michael J. Lighthill. Viscosity effects in sound waves of finite amplitude. In Surveys in mechanics, pages 250–351 (2 plates). Cambridge, at the University Press, 1956.

[145] Xavier Litrico and Vincent Fromion. Modeling and control of hydrosystems. Springer Science & Business Media, 2009.

[146] Xavier Litrico, Vincent Fromion, and Jean-Pierre Baume. Tuning of robust distant downstream PI controllers for an irrigation canal pool. II: Implementation issues. Journal of irrigation and drainage engineering, 132(4):369–379, 2006.

[147] Xavier Litrico, Vincent Fromion, Jean-Pierre Baume, and Manuel Rijo. Modelling and PI control of an irrigation canal. In European Control Conference (ECC), 2003, pages 850–855. IEEE, 2003.

[148] Xavier Litrico, Pierre-Olivier Malaterre, Jean-Pierre Baume, Pierre-Yves Vion, and José Ribot-Bruno. Automatic tuning of pi controllers for an irrigation canal pool. Journal of irrigation and drainage engineering, 133(1):27–37, 2007.

[149] Weijiu Liu. Boundary feedback stabilization of an unstable heat equation. SIAM J. Control Optim., 42(3):1033–1043, 2003.

[150] Aleksandr Mikhaĭlovich Lyapunov. The general problem of the stability of motion. PhD thesis, Kharkov University, 1892.

[151] Aleksandr Mikhaĭlovich Lyapunov. The general problem of the stability of motion. Taylor & Francis, Ltd., London, 1992. Translated from Edouard Davaux's French translation (1907) of the 1892 Russian original and edited by A. T. Fuller, With an introduction and preface by Fuller, a biography of Lyapunov by V. I. Smirnov, and a bibliography of Lyapunov's works compiled by J. F. Barrett, Lyapunov centenary issue, Reprint of Internat. J. Control **5**5 (1992), no. 3 [MR1154209 (93e:01035)], With a foreword by Ian Stewart.

[152] Andrew Majda. Compressible fluid flow and systems of conservation laws in several space variables, volume 53 of Applied Mathematical Sciences. Springer-Verlag, New York, 1984.

[153] Frédéric Marbach. Control in fluid mechanics and boundary layers. PhD thesis, Université Pierre et Marie Curie - Paris VI, September 2016.

[154] Sophie Marbach, Hiroaki Yoshida, and Lydéric Bocquet. Osmotic and diffusio-osmotic flow generation at high solute concentration. I. mechanical approaches. The Journal of Chemical Physics, 146(19):194701, 2017.

[155] Fabien Marche. Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. Eur. J. Mech. B Fluids, 26(1):49–63, 2007.

[156] James Clerk Maxwell. On governors. Proceedings of the Royal Society of London, (16):270–283, 1868.

[157] Otto Mayr. The Origins of Feedback Control. The M.I.T. Press, Cambridge, Massachusetts, and London, England, 1970.

[158] Otto Mayr. Maxwell and the origins of cybernetics. Isis, 62(4):425–444, 1971.

[159] Nicolas Minorsky. Directional stability of automatically steered bodies. Naval Engineers Journal, 32(2), 1922.

[160] Javier Murillo and P García-Navarro. Energy balance numerical schemes for shallow water equations with discontinuous topography. Journal of Computational Physics, 236:119–142, 2013.

[161] Aloisio Freiria Neves, Hermano de Souza Ribeiro, and Orlando Lopes. On the spectrum of evolution operators generated by hyperbolic systems. J. Funct. Anal., 67(3):320–344, 1986.

[162] Ol'ga Arsen'eva Oleĭnik. Discontinuous solutions of non-linear differential equations. Uspehi Mat. Nauk (N.S.), 12(3(75)):3–73, 1957.

[163] Oliver Paish. Micro-hydropower: status and prospects. Proceedings of the Institution of Mechanical Engineers, Part A: Journal of Power and Energy, 216(1):31–40, 2002.

[164] Vincent Perrollaz. Asymptotic stabilization of entropy solutions to scalar conservation laws through a stationary feedback law. Ann. Inst. H. Poincaré Anal. Non Linéaire, 30(5):879–915, 2013.

[165] Vincent Perrollaz. Asymptotic stabilization of stationnary shock waves using a boundary feedback law. arXiv preprint arXiv:1801.06335, 2018.

[166] Benoit Perthame. Equations de transport non linéaires et systèmes hyperboliques. Cours DEA Analyse Numérique, Paris VI., 2003.

[167] Benoit Perthame. Contrôle de modèles dynamiques. Cours MAP434, Ecole Polytechnique., 2019.

[168] Sebastian Pfaff and Stefan Ulbrich. Optimal boundary control of nonlinear hyperbolic conservation laws with switched boundary data. SIAM J. Control Optim., 53(3):1250–1277, 2015.

[169] Seppo A. Pohjolainen. Robust multivariable PI-controller for infinite-dimensional systems. IEEE Trans. Automat. Control, 27(1):17–30, 1982.

[170] Christophe Prieur and Frédéric Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. Mathematics of Control, Signals, and Systems, 24(1-2):111–134, 2012.

[171] Christophe Prieur, Joseph Winkin, and Georges Bastin. Robust boundary control of systems of conservation laws. Mathematics of Control, Signals, and Systems, 20(2):173–197, 2008.

[172] Tie Hu Qin. Global smooth solutions of dissipative boundary value problems for first order quasilinear hyperbolic systems. Chinese Ann. Math. Ser. B, 6(3):289–298, 1985. A Chinese summary appears in Chinese Ann. Math. Ser. A **6** (1985), no. 4, 514.

[173] Jeffrey Rauch and Michael Taylor. Exponential decay of solutions to hyperbolic equations in bounded domains. Indiana Univ. Math. J., 24:79–86, 1974.

[174] Jean-Pierre Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. SIAM J. Control Optim., 45(3):790–828, 2006.

[175] Jean-Pierre Raymond. Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations. J. Math. Pures Appl. (9), 87(6):627–669, 2007.

[176] Michael Renardy. On the type of certain C_0-semigroups. Comm. Partial Differential Equations, 18(7-8):1299–1307, 1993.

[177] David L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. SIAM Rev., 20(4):639–739, 1978.

[178] Jacques Simon. Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. (4), 146:65–96, 1987.

[179] Marshall Slemrod. A note on complete controllability and stabilizability for linear control systems in Hilbert space. SIAM J. Control, 12:500–508, 1974.

[180] Marshall Slemrod. Boundary feedback stabilization for a quasilinear wave equation. In Control theory for distributed parameter systems and applications (Vorau, 1982), volume 54 of Lect. Notes Control Inf. Sci., pages 221–237. Springer, Berlin, 1983.

[181] Andrey Smyshlyaev and Miroslav Krstic. Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations. IEEE Trans. Automat. Control, 49(12):2185–2202, 2004.

[182] Andrey Smyshlyaev, Thomas Meurer, and Miroslav Krstic. Further results on stabilization of shock-like equilibria of the viscous Burgers PDE. IEEE Trans. Automat. Control, 55(8):1942–1946, 2010.

[183] Eduardo D. Sontag. Input to state stability: basic concepts and results. In Nonlinear and optimal control theory, volume 1932 of Lecture Notes in Math., pages 163–220. Springer, Berlin, 2008.

[184] Ying Tang, Christophe Prieur, and Antoine Girard. Singular perturbation approximation by means of a H^2 Lyapunov function for linear hyperbolic systems. Systems Control Lett., 88:24–31, 2016.

[185] René Taton. Euler et d'Alembert. In Zum Werk Leonhard Eulers, pages 95–117. Springer, 1984.

[186] René Taton. D'Alembert, Euler et l'Académie de Berlin. Dix-Huitième Siècle, 16(1):55–68, 1984.

[187] Tomomasa Tatsumi and Hiroshi Tokunaga. One-dimensional shock turbulence in a compressible fluid. J. Fluid Mech., 65:581–601, 1974.

[188] Alexandre Terrand-Jeanne, Vincent Andrieu, Valérie Dos Santos Martins, and Cheng-Zhong Xu. Adding integral action for open-loop exponentially stable semigroups and application to boundary control of PDE systems. 2018.

[189] Laetitia Thevenet, Jean-Marie Buchot, and Jean-Pierre Raymond. Nonlinear feedback stabilization of a two-dimensional Burgers equation. ESAIM Control Optim. Calc. Var., 16(4):929–955, 2010.

[190] Emmanuel Trélat. Contrôle optimal. Mathématiques Concrètes. [Concrete Mathematics]. Vuibert, Paris, 2005. Théorie & applications. [Theory and applications].

[191] Roberto Triggiani. Boundary feedback stabilizability of parabolic equations. Appl. Math. Optim., 6(3):201–220, 1980.

[192] Ngoc-Tu Trinh. Etude sur le contrôle/régulation automatique des systèmes non-linéaires hyperboliques. PhD thesis, Université de Lyon, 2017.

[193] Ngoc-Tu Trinh, Vincent Andrieu, and Cheng-Zhong Xu. PI regulation control of a fluid flow model governed by hyperbolic partial differential equations. In Proceeding of the International Conference on System Engineering, Coventry, England, 2015.

[194] Ngoc-Tu Trinh, Vincent Andrieu, and Cheng-Zhong Xu. Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations. IEEE Trans. Automat. Control, 62(9):4527–4536, 2017.

[195] John Tsinias. Sufficient Lyapunov-like conditions for stabilization. Math. Control Signals Systems, 2(4):343–357, 1989.

[196] Rafael Vazquez, Jean-Michel Coron, Miroslav Krstic, and Georges Bastin. Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, pages 1329–1334, 2011.

[197] Rafael Vazquez, Miroslav Krstic, and Jean-Michel Coron. Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system. 2011 50th IEEE Conference on Decision and Control and European Control Conference, pages 4937–4942, 2011.

[198] Zhiqiang Wang. Exact controllability for nonautonomous first order quasilinear hyperbolic systems. Chinese Ann. Math. Ser. B, 27(6):643–656, 2006.

[199] Cheng-Zhong Xu and Hamadi Jerbi. A robust PI-controller for infinite-dimensional systems. Internat. J. Control, 61(1):33–45, 1995.

[200] Cheng-Zhong Xu and Gauthier Sallet. Proportional and integral regulation of irrigation canal systems governed by the st venant equation. IFAC Proceedings Volumes, 32(2):2274–2279, 1999.

[201] Cheng-Zhong Xu and Gauthier Sallet. Multivariable boundary PI control and regulation of a fluid flow system. Math. Control Relat. Fields, 4(4):501–520, 2014.

[202] Christophe Zhang. Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback. October 2018. working paper or preprint.

[203] Christophe Zhang. Finite-time internal stabilization of a linear 1-D transport equation. January 2019. working paper or preprint.

[204] Y.-C. Zhao. Classical Solutions for Quasilinear Hyperbolic Systems. PhD thesis.

Stabilization of 1D nonlinear hyperbolic systems by boundary controls

Abstract:
This thesis is devoted to study the stabilization of nonlinear hyperbolic systems of partial differential equations. The main goal is to find boundary conditions ensuring the exponential stability of the system. In a first part, we study general systems that we aim at stabilizing in the C^1 norm by introducing a certain type of Lyapunov functions. Then we take a closer look at systems of two equations and we compare the results with the stabilization in the H^2 norm. In a second part we study a few physical equations: Burgers' equation and the density-velocity systems, which include the Saint-Venant equations and the Euler isentropic equations. Using a local dissipative entropy, we show that these systems can be stabilized with very simple boundary controls which, remarkably, do not depend directly on the parameters of the system, provided some physical admissibility condition. Besides, we develop a way to stabilize shock steady-states in the case of Burgers' and Saint-Venant equations. Finally, in a third part, we study proportional-integral (PI) controllers, which are very popular in practice but seldom understood mathematically for nonlinear infinite dimensional systems. For scalar systems we introduce an extraction method to find optimal conditions on the parameters of the controller ensuring the stability. Finally, we deal with the Saint-Venant equations with a single PI control.

Keywords: stabilization, partial differential equations, inhomogeneous, nonlinear, Boundary feedback control, Lyapunov functions, entropy, Saint-Venant equations, Burgers' equation.

Stabilisation de systèmes hyperboliques non-linéaires en dimension un d'espace

Résumé:
Cette thèse est consacrée à l'étude de la stabilisation des systèmes d'équations aux dérivées partielles hyperboliques non-linéaires. L'objectif principal est de trouver des conditions de bords garantissant la stabilité exponentielle du système. Dans une première partie on s'intéresse à des systèmes généraux qu'on cherche à stabiliser en norme C^1 en introduisant un certain type de fonctions de Lyapunov, puis on regarde plus précisément les systèmes de deux équations pour lesquels on peut comparer nos résultats avec la stabilisation en norme H^2. On s'intéresse ensuite à quelques équations physiques: l'équation de Burgers et les systèmes densité-vélocité, dont font partie les équations de Saint-Venant et les équations d'Euler isentropiques. A l'aide d'une entropie locale dissipative, on montre qu'on peut stabiliser les systèmes densité-vélocité par des contrôles aux bords simples et, étonnement, ces contrôles ne dépendent pas explicitement des paramètres du système, pourvu qu'ils soient physiquement admissibles. Par ailleurs, on développe une méthode pour stabiliser les états-stationnaires avec un choc dans le cas de l'équation de Burgers et des équations de Saint-Venant. Enfin, dans une troisième partie on s'intéresse aux contrôles proportionnels-intégraux (PI), très utilisés en pratique mais mal compris mathématiquement dans le cas des systèmes non-linéaires de dimension infinie. Pour les systèmes d'une seule équation on introduit une méthode d'extraction pour trouver des conditions optimales de stabilité sur les paramètres du contrôle. Finalement on traite le cas des équations de Saint-Venant avec un unique contrôle PI.

Mot clés: stabilisation, équations aux dérivées partielles, inhomogénéités, non-linéaire, contrôles aux bords, fonctions de Lyapunov, entropie, équations de Saint-Venant, équation de Burgers.