Essays on Allocation Procedures of Indivisibles

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Sur les procédures d'allocation et de décision collective en présences d'indivisibilités

Essays on Allocation Procedures of Indivisibles

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Résumé

Cette thèse modélise des mécanismes d'allocation de biens indivisibles dans des situations où prévalent des externalités dans les préférences. Ces externalités conduisent à définir les résultats des mécanismes comme des combinaisons complexes d'alternatives. Cette complexité rend difficile la collecte de préférences sur les résultats, ce qui pose problème pour l'étude normative des mécanismes tels que le mécxanisme concurrentiel ou le coeur. Une solution possible est de se limiter à une information partielle sur les préférences, et à conditionner l'analyse normative à des hypothèses précisant comment cette information partielle peut être étendue à une information complète sur les préférences. Ces hypothèses revenant à considérer un domaine restreint de préférence, une telle approche s'apperente à un mode d'analyse courant dans la théorie du Choix Social.

La thèse est organisée en quatre chapitres indépendants. Les trois premiers chapitres portent sur l'extension du modèle de Shapley-Scarf aux situations d'échanges entre coalitions, chaque membre d'une coalition étant sensible au bien-être des autres membres de la coalition. Les chapitres 1 et 2 sont consacrés à l'étude des domaines de préférences garantissant l'existence de différents types d'équilibre concurrentiel. Dans le chapitre 3, l'ensemble des biens est muni d'une structure géographique. La distribution des biens en différentes localisations permet de prendre en compte la distance entre partenaires comme source d'externalité dans les préférences. Nous identifions les domaines de préférences qui garantissent la non-vacuité du coeur, celui-ci pouvant être défini de différentes manières. Le chapitre 4 se démarque du modèle de Shapley-Scarf en considérant le cas de biens indivisibles publics. Nous montrons que cette situation s'apparente naturellement au problème du choix d'un comité formé de plusieurs membres, chacun étant choisi parmi des candidats à un poste spécifique. Nous caractérisons le domaine de préférences sur les comités pour lesquels le choix des membres poste

par poste est cohérent avec le choix direct d'un comité. Les méthodes de choix considérées sont basées sur la méthode du vote majoritaire.

Mots clés : biens indivisibles - Shapley-Scarf - externalités - coalitions - couples - distance - comités - vote majoritaire - restrictions de préférences

Abstract

This thesis focuses on the allocation of indivisible goods involving outcomes that are complex combinations of basic alternatives, complexity mainly arising from externality in preference. This complexity creates a difficulty with fully collecting the relevant information over preferences, and therefore creates also a difficulty when conducting a normative analysis of the different allocation mechanisms that may be implemented, such as the competitive mechanism or the core mechanism. A way to overcome the difficulty is collecting preferences over basic alternatives, and deriving normative conclusions are conditional to assumptions on how reported preferences are extended to preferences over outcomes. This route should be regarded as a specific approach to preference domain restriction which is well-known in Social Choice theory.

The thesis is organized in four independent chapters. The first three chapters focus on extensions of Shapley-Scarf or housing markets to the case where trades take place between coalitions, and where partners in coalition are sensitive to the well-being of all coalition members. Chapters 1 and 2 mainly investigate the domain restriction ensuring the existence of various types of competitive equilibrium. Chapter 3 endows the set of goods with a geographical structure. Distributing goods among several locations allow for considering distance to partners as a source of externality in preference. We identify domains of preference extensions which ensure the existence of various types of core allocations. Chapter 4 departs from the standard housing market by considering goods are pure public goods. We argue that this case naturally relates to the choice of a committee formed by several members, each selected from a specific set. We characterize preference domains over committees for which a well-defined seat-wise choice procedure based on majority voting is consistent with choosing a committee at once from majority voting, where consistency is given several possible meanings.

Keywords : housing market - Shapley-Scarf - coalitions - couples - externality - distance - committees - majority voting - preference restriction

Résumé en français

Problématique générale

De nombreux problèmes de décision collective sont résolus au moyen d'une procédure décentralisée dans laquelle les individus annoncent leurs préférences quant aux résultats possibles, ces préférences étant agrégées en un résultat collectif. Une littérature importante est consacrée à l'analyse positive et à l'évaluation normative des procédures centralisées de décision collective. Les préférences individuelles étant fondamentales dans la mise en oeuvre de ces procédures, une question essentielle est l'organisation pratique de la collecte de ces préférences. Une difficulté majeure de cette collecte peut résulter de la cardinalité de l'ensemble des choix possibles. Ceci est en particulier vrai dans le cas où cet ensemble est muni d'une structure spécifique par laquelle les choix peuvent être définis comme combinaisons d'alternatives, ou de décisions élémentaires.

Par exemple, considérons le cas où les choix sont des classements d'objets, de décisions ou de candidats. le premier cas est parfaitement illustré par le concours de l'eurovision. Dans ce concours, la position relative d'un pays, ainsi que la qualité perçue des prestations des pays ayant un rang similaire, importent pour les spectateurs intéressés par l'enjeu. En d'autres termes, les individus s'interessent autant au rang de leur champion qu'au classement de tous les candidats. Rien n'exclut a priori de collecter les préférencves portant sur tous les classements. Cependant, bien que possible en théorie, cette option soulève une évidente difficulté logistique. En 2018, le concours rassemblait 26 pays, permettant ainsi 26 ! classements possibles (sans compter les ex-aequos possibles). La solution retenue en pratique est de collecter une information partielle sur les préférences : les individus doivent attribuer 12, puis 10, 8, ..., 1 points à leur dix chansons préférées dans l'ordre décroissant de préférence. Les chansos sont

alors classées selon le nombre total de points qu'elles recueillent. Cependant, la vérification de toute propriété normative d'une procédure de décision collective pose problème si l'information sur les préférences individuelles n'est pas complète. Un exemple parmi d'autres est la propriété de non-manipulation. Les théorèmes bien connus de Gibbard (1973) et Satterthwaite (1975), qui concernent les fonctions de choix social univariées, impliquent de connaître les préférences individuelles sur les choix possibles, et ne peuvent être naturellement généralisées aux situations où ces préférences ne sont que partiellement identifiées. Considérons à nouveau le cas où les résultats sont des classements. Si les individus annoncent uniquement leur classement préféré, une procédure de décision collective est formellement définie comme une règle d'agrégation arrowienne, qui associe à tout ensemble de classement individuels, ou profil, un classement collectif. Etablir la non-manipulation d'une règle arrowienne requiert de formuler des hypothèses sur la manière où un classement peut être étendu à un classement de tous les classements. En d'autres termes, on doit au préalable choisir une règle d'extension des préférences, qui décrit comment le résultat préféré définit une préférence sur tous les résultats. Ainsi, la propriété de non-manipulation est définie conditionnellement au choix d'une règle d'extension spécifique.[1]

Un classement peut donc être conçu comme une combinaisons complexe de décision, ou alternatives, simples. De façon similaire, choisir un ensemble d'objets consiste à choisir une combinaison complexe d'alternatives élémentaires, la notion de complexité étant associée à la difficulté de collecter une information complète sur les préférences. La question logistique posée dans le cas où les résultats sont des classements prévaut tout autant lorsque les résultats sont des ensembles. Ainsi, la non-manipulation de fonctions de choix multivariées requiert des hypothèses sur la manières de générer une préférence sur les ensembles à partir d'une préférences sur les éléments de ces ensembles.[2]

Il existe de nombreuses situations dans lesquelles la collecte d'une information complète sur les préférences est trop coûteuse pour être concevable en pratique. Dans le cas d'un

1. Voir Bossert and Storcken (1992), Bossert et Sprumont (2014), Athanassoglu (2016) pour une analyse de la manipulation des règles d'agrégation arrowienne. D'autres propriétés normatives conditionnelles à une règle d'extension des préférences sont étudiées par Laffond and Lainé (2000), Lainé (2015), and Lainé, Özkes et Sanver (2016).

2. Voir Sprumont (1995) et Barberà (2011) pour une revue des résultats sur la manipulation des correspondances de choix social. Un cas particulier de choix multivarié est celui de l'élection d'une assemblée lorsque les électeurs déclarent leurs préférences sur les candidats (Benoît et Kornhauser (1994)).

référendum multiple, un résultat est défini comme un vecteur de coordonnées égales à zéro (non) ou un (oui), chaque électeur déclarant sa position sur chaque question plutôt que sur toutes les combinaisons de réponses aux différentes questions. Là encore, de nombreux travaux ont montré les problèmes résultant de cette absence d'information complète sur les préférences. Un sous-ensemble de ces problèmes est connu sous le vocable de paradoxes de la majorité composée.[3] Un autre problème est la violation possible de la propriété d'efficacité au sens de Pareto.[4] Tout comme pour la non-manipulation, la vérification de cette propriété requiert au préalable le choix d'une règle d'extension des préférences.

Cette thèse a pour objet l'étude de problèmes de choix collectifs ayant des résultats complexes et concernant l'allocation de biens purement indivisibles, la complexité des résultats résultant de l'existence d'une inter-dépendance entre les préférences. Une illustration éclairante est donnée par le cas d'une campagne de mobilité professionnelle, dont les élements constitutifs sont les suivants. Il existe un ensemble fini d'individus, chacun pourvu d'une position professionnelle et classant l'ensemble des positions ou emplois selon un ordre linéaire. Tout individu est susceptible de changer d'emploi, et donc tout emploi est susceptible de devenir vacant. De plus, chaque individu ne peut occuper qu'un emploi, et tout emploi doit être pourvu à l'isue de la campagne. Toute situation de ce type est un cas d'application du modèle de Shapley-Scarf (1974). Une réallocation des emplois peut être considérée comme un résultat simple dans le cas où les individus ne porte d'intérêt qu'à leur propre situation. En effet, le classement des emplois définit naturellement celui des allocations. Mais cette simplicié disparaît si les individus sont affectés par les emplois donnés à d'autres. ceci est le cas lorsque certains individus vivent en couple. De façon plus générale, on peut considérer l'existence d'une structure exogène de coalitions d'individus, la satisfaction d'un membre d'une coalition dépendant de celle de tous les autres membres de la coalition.

Dans ce cas comme dans ceux décrits plus haut, il paraît impossible en pratique de demander à chaque individu de classer l'ensemble des allocations d'emplois. Et si les individus annoncent seulement un classement des emplois, l'analyse de propriétés telles que l'efficacité au sens de Pareto, la stabilité au sens du Coeur, ou la non-manipulation requiert la définition

3. Voir Nurmi (1999), Lacy et Niou (2000), Laffond et Lainé (2006), (2009), (2012), (2013).
4. Voir Kadane (1972), Özkal-Sanver et Sanver (2006), Cuhadaroğlu et Lainé (2012), Benoît et Kornhauser (2010).

d'une règle étendant ce classement en une préférences sur l'ensemble des allocations. Dans la situation de mobilité professionnelle impliquant des couples, on pourrait concevoir l'idée d'une annonce de préférences jointes. Cependant cette possibilité est souvent exclue en pratique (notamment dans le cas des campagnes de mobilité des enseignants français).[5]

Le rôle joué par l'existence de préférences interdépendantes dans le modèle de Shapley-Scarf n'a pas à ce jour reçu une attention importante. Par contre, les problèmes posés par cette interdépendance ont été largement étudiés dans le cas des marchés d'appariement. En particulier, l'instabilité potentiellement créée par la présence de couples est bien documentée. Le lecteur pourra se référer à Sasaki et Toda (1996), Dutta et Masso (1997), Roth et Peranson (1999), Cantala (2004), Klaus et Klijn (2005), Echenique et Yenmez (2007), Klaus, Klijn et Masso (2007), Hafalir (2008), Mumcu et Saglam (2010), Bando (2012), Bando (2014), Fisher et Hafalir (2016).[6]

La thèse considère également un autre type de résultat complexe qui concerne de façon naturelle la question de la mobilité professionnelle des couples. Supposons que les emplois sont distribués entre différentes localités. Au-delà de la qualité donnée aux emplois potentiels, les individus comparent deux situations également selon la distance entre leur emploi et celui du partenaire. Nous proposons une modélisation de cette situation, dans laquelle les classement des emplois et le critère de la distance concourent à la construction d'une préférence sur les résultats (allocations). E nous montrons la sensibilité de cretaines propriétés normatives au choix des propriétés portant sur ces préférences.

Une troisième situation de choix collectif d'un résultat complexe est étudiée. Supposons qu'il existe deux types de biens indivisibles, chacun ayant la qualité d'un bien public pur. La question est de choisir collectivement un bien de chaque type. Les individus vivent en couple, chacun ayant des préférences sur un type spécifique de bien. A titre d'illustration, on peut considérer qu'un type de bien correspond à une discipline d'enseignement, et que chaque couple est formé par deux enseignants dans une même école (chacun en charge d'une des deux disciplines). Par ailleurs, deux couples différents enseignent dans deux écoles différentes. Dans un tel contexte, un bien représente par exemple un manuel d'enseignement qui, une fois choisi,

5. On peut observer que l'annonce de préférences jointes ne suffit pas à garantir l'existence d'une solution dans le Coeur, comme l'ont montré Doğan, Laffond, and Lainé (2011).

6. Voir Bando et al. (2016) pour une revue récente de la littérature.

sera imposé à toutes les écoles. L'autorité centrale, en charge de la décision, tient à prendre en considération le niveau de satisfaction des écoles, et demande à chaque enseignant (par discipline) d'annoncer sa préférence sur les manuels proposés. Ainsi, un résultat est une paire de manuels d'enseignement, et la procédure ne collecte pas directement l'information portant sur la préférence des écoles. D'un point de vue formel, ce problème est équivalent à celui consistant un choisir un comité formé par deux personnes, chacune en charge d'une fonction particulière. Par exemple, considérons une université offrant deux positions, chacune dans un département spécifique. Les électeurs sont les enseignants-chercheurs des deux département, chacune étant invité à se prononcer sur les candidats à la position dans ce département. Afin d'évaluer la qualité du recrutement du point de vue de l'université, on doit alors formuler des hypothèses sur la manières d'étendre les préférences exprimées dans chaque département en une préférence collective (des deux départements agrégés) sur les paires de candidats éligibles.

En définitive, l'analyse normative de ces trois classes de problème de choix collectif (mobilité professionnelle des couples, avec ou sans structure géographique, choix de biens publics de types multiples) requiert la définition d'une règle d'extension des préférences portant sur des parties de résulatts en préférences sur les résultats eux-mêmes. L'attention étant essentiellement portée sur le choix de cette règle d'extension, la problématique retenue dans cette thèse est

- pour le modèle de Shapley-Scarf, d'identifier les domaines de règles d'extension qui garantissent le respect de propriétés souhaitables (existence de solutions concurrentielles, existence de solutions stables au sens du Coeur),

- pour le modèle de choix d'un comité (biens publics multi-types), d'identifier les domaines de règles d'extension pour lesquelles le résultat peut être obtenu comme combinaison des choix par type de candidat.

Organisation de la thèse et résultats principaux

La thèse est organisée en quatre chapitres, chacun étant rédigé de sorte qu'il puisse être lu séparément. Les trois premires chapitres portent sur le modèle de Shapley-Scarf avec structure de coalitions, et la quatrième chapitre est consacré au problème du choix d'un comité.

Le chapitre 1 formalise un marché de Shapley-Scarf avec échanges entre coalitions. Le modèle originel de Shapley-Scarf (1974) est caractérisé par les éléments suivants. Il existe un ensemble d'individus et un ensemble de biens purement indivisibles, les deux ensembles ayant la même cardinalité finie. Chaque individu possède initialement un bien unique, et tous les biens sont initialement détenus, et les échanges sont organisés de sorte que chaque individu déteint un et un seul bien spécifique. Les préférences individuelles sur les biens sont représentées par un ordre linéaire. Un résultat bien connu est l'existence d'allocations dans le Coeur. De plus une allocation d'équilibre concurrentiel existe, et s'obtient comme le résultat d'un algorithme appelé top-trading-cycle (TTC), cet algorithme étant attribué à David Gale. Roth et Postlewaite (1977) montrent que cette allocation concurrentielle est unique, et qu'elle est également l'unique élement du Coeur strict (défini par domination faible). Nous supposons dans le chapitre 1 que l'ensemble des individus est partitionné en coalitions, cette partition étant exogène. Par ailleurs, chaque membre d'une coalition est indifférent entre deux allocations assignat les mêmes biens aux membres de cette coalition. Par contre, les biens assignés à ses partenaires importe. Dans une telle situation, il n'existe pas un concept naturel d'équilibre concurrentiel. En premier lieu, on peut interdire ou autoriser la transférabilité du revenu entre membres d'une même coalition. Et par ailleurs, on peut envisager différents modes de coopération au sein des coalitions. Nous distinguons trois types d'équilibre. A un équilibre égoiste, chaque individu maximise son utilité dans l'ensemble budgétaire de sa coalition. A un équilibre coopératif, à chaque coalition est assigné un vecteur de biens budgétairement réalisable et efficace au sens de Pareto. Enfin, à un équilibre coordonné, chaque membre d'une coalition maximise dans l'ensemble budgétairement possible sa satisfaction étant donnée l'allocation choisie pour ses partenaires. En considérant ces trois types d'équilibre et pour chacun en considérant la possibilité ou non d'un transfert de revenus entre membres d'une même coalition, on obtient en définitive six notions d'équilibre. Les équilibres forts (égoistes, coopératifs, coordonnés) sont associés à la non-transférabilité des revenus individuels, et les équilibres faibles (égoistes, coopératifs, coordonnés) sont associés à la transférabilité des revenus individuels. Nous offrons une étude complète des relations logiques entre ces différents concepts d'équilibre. De plus, nous montrons la possibilité d'absence de chaque type d'équilibre en l'abence de restriction sur les préférences. Enfin, nous introduisons deux restrictions de pré-

férence et nous montrons que chacune définit un domaine maximal pour un type spécifique d'équilibre. Le domaine des préférence coalition-monotones est maximal pour l'existence d'un équilibre égoiste fort, et le domaine de préférences faiblement lexicographiques est maximal pour l'existence d'un équilibre coopératif faible.

La propriété de coalition-monotonie est satisfaite si chaque membre d'une coalition a un ordre de préférence sur les biens et si ses préférences sur les allocations sont séparables par rapport aux ordres sur les biens de ses partenaires. Ainsi, en présence de coalition-monotonie, la collecte des ordres sur les biens fournit toute l'information nécessaire sur les préférences portant sur les allocations. De plus, cette propriété assure que le résultat de l'algorithme TTC est une allocation d'équilibre égoiste fort. cependant, et contrairement au modèle originel, l'algorithm TTC devient manipulable. La propriété lexicographique faible prévaut si, étant donné un sous-emsemble quelconque de biens, chaque individu identifie un bien prioritaire qu'il souhaite voir assigné à sa coalition, le membre de cette coalition recevant ce bien pouvant changer selon l'allocation résiduelle des biens attribués à la coalition. Nous montrons que sous l'hypothèse de préférences faiblement lexicographiques, il existe toujours un équilibre coopératif faible, obtenu comme résultat d'un algorithme TTC modifié (et manipulable).

Le chapitre 2 complète le chapitre 1 en établissant certains résultats supplémentaires concernant léquilibre égoiste fort. Nous montrons tout d'abord qu'il existe des allocations d'équilibre qui ne peuvent être obtenues comme résultat de l'algorithme TTC. ceci est une différence majeure entre notre modèle et le modèle originel de Shapley-Scarf. Cependant, il est facile de montrer que la structure de l'ensemble des allocations permet de définir toute allocation comme le résultat de la formation de cycles d'échanges successifs. Nous caractérisons les suites de cycles d'échanges conduisant à une allocation d'équilibre égoiste fort, et nous montrons que chaque suite est en fait définie au moyen de déviations légères des top-trading-cycles.

Le chapitre 3 reconsidère le modèle de Shapley-Scarf avec coalitions en munissant l'ensemble des biens d'une structure géographique. Le chapitre se limite au cas où les coalitions sont des couples. Les biens sont distribués de façon exogène entre différentes localités, et on retient la distance discrète (deux localisations quelqconques étant données à distance égale à l'unité si et seulement si elles sont différentes). Ayant à l'esprit les situations de mobilité pro-

fessionnelle, nous construisons un modèle permettant de capturer l'idée selon laquelle chaque membre d'un couple souhaitent travailler le plus près possible de son partenaire. La valorisation individuelle d'une situation incorpore la distance au partenaire ainsi que la qualité du bien attribué (notons que celle du bien attribué au partenaire n'est pas prise en compte). La prise en compte de la distance introduit une forme d'externalité dans les préférences. L'agrégation des deux critères est modélisée au moyen d'une règle d'extension séparable : entre deux allocations plaçant les conjoints à une même distance, un individu préfère celle lui attribuant le meilleur bien, et entre deux allocations lui attribuant un même bien, il préfère celle le plaçant à la plus petite distance du conjoint.

Nous étudions l'existence d'allocation stables au sens du Coeur, ainsi que la possibilité d'atteindre ces allocations au moyen d'un algorithme de type TTC. De plus, nous nous limitons à des concepts de Coeur dans lequels aucune coalition bloquante ne peut impliquer un individu sans son conjoint. Dans la mesure où deux conjoints peuvent avoir des intérêts divergents, la définition d'une coalition bloquante requiert de préciser comment ils peuvent s'entendre sur la comparaison entre allocations. Pour ce faire, nous introduisons la notion d'agrément de couple, qui consiste à étendre toute paire de préférences sur les allocations à un préordre commun sur les allocations. A chaque type d'agrément de couple est associé un concept spécifique de Coeur.

Nous montrons tout d'abord que lorsqu'aucune restriction ne prévaut sur les préférences, le Coeur peut être vide pour tout agrément de couple. Nous considérons alors deux restrictions, chacune correspondant à un cas polaire : les préférences avec priorité aux biens ("mon bien importe le plus, et pour tout bien qui m'est attribué, je préfère être proche de mon conjoint"), et les préférences avec priorité à la distance ("être proche de mon conjoint est ce qui m'importe le plus, et à distance donnée, je préfère recevoir un meilleur bien"). Nous étudions l'existence de chaque concept de Coeur pour chacune de ces deux classes de préférence et, pour chaque concept, nous construisons un algorithme spécifique qui conduit à une allocation du Coeur.

Le chapitre 4 est consacré au problème du choix d'un comité, déjà mentionné plus haut. Une société doit choisir un comité formé de deux membres, chacun occupant un poste particulier. I existe un ensemble distinct de candidats pour chaque poste. Nous supposons que chaque électeur a des préférences pour les candidats de chaque type, représentées par un ordre

linéaire. Ainsi, chaque électeur est caractérisé par deux ordres linéaires. Clairement, annoncer un ordre pour chaque type est insuffisant pour pleinement décrire les préférences sur les comités (même sous l'hypothèse de séparabilité). Et demander à chaque électeur ses préférences en matière de comité est problématqiue, compte-tenu de la taille des messages à transmettre. Par exemple, s'il existe dix candidats par type, il existe cent comités possibles, qu'il s'agirait de classer. Nous examinons la propension d'une méthode de sélection par type à choisir un résultat (comité) cohérent avec celui qui prévaudrait en cas d'information complète sur les préférences sur les comités. Dans la mesure où une méthode de vote apparaît comme naturelle dans ce contexte, nous nous limitons aux procédures fondées sur la préférence majoritaire.

Notre analyse suit ainsi une route très similaire à celle choisie pour la généralisation du modèle de Shapley-Scarf. Un candidat pour un poste peut être conçu comme une alternative simple, et un comité est une combinaison "complexe" (ici un vecteur) d'alternatives simples. De plus, les préférences sur les alternatives simples sont étendues à des préférences sur les résultats. ici, une règle d'extension associe à toute paire d'ordres linéaires un ordre sur les comités. Le sujet de ce chapitre est de caractériser les classes de règles d'extension garantissant la cohérence entre deux méthodes de vote majoritaire. La première, notée Maj, consiste à choisir le comité formé par les deux candidats choisis séparément selon la règle majoritaire, et la seconde consiste à choisir directement le comité.

Il est bien connu que Maj ne permet pas toujours d'identifier un vainqueur naturel (dit de Condorcet) s'il existe plus de deux candidats par type. Nous contournons cette difficulté en nous limitant aux seules préférences par type qui garantissent l'existence d'un vainqueur de Condorcet. Nous considérons alors deux types d'incohérence potentielle entre le choix direct du comité et le choix par type. Le premier, que nous appelons paradoxe du comité majoritaire, prévaut lorsque qu'un comité vainqueur de Condorcet existe (parmi tous les comités) mais n'est pas sélectionné par Maj. Le second, appelé paradoxe faible du comité majoritaire, prévaut lorsque l'une des deux situations suivantes est observée : le paradoxe du comité majoritaire existe, ou bien il n'existe pas de comité vainqueur de Condorcet. Nous montrons que sous l'hypothèse de neutralité (selon laquelle le nom des candidats ne joue aucun rôle dans la règle d'extension), la séparabilité de cette règle est nécessaire et suffisante à l'absence du paradoxe du comité majoritaire. Une règle d'extension est dite séparable si

pour tout type de siège sur lequel candidatent a et b, un électeur classe a devant b, alors il classe le comité (a, x) devant le comité (b, x) pour tout candidat x à l'autre siège. Par ailleurs, nous montrons qu'une condition nécessaire et suffisante (à nouveau sous l'hypothèse de neutralité) pour éviter le paradoxe faible est que tous les électeurs ont recours à la même règle d'extension, qui consiste à désigner un siège jugé prioritaire, et à classer les comités selon la préférence lexicographique définie en rapport avec cette priorité.

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Chapitre 1

Introduction

1.1 Overview

Many collective decision problems are resolved by using a centralized procedure in which individuals report their preferences over the possible outcomes, and these preferences are aggregated into a collective outcome. A huge literature has been devoted to the analysis of positive and normative criteria for evaluating centralized procedures. As full preferences are the key inputs for collective choice, a legitimate question is whether collecting preferences over outcomes is achievable in practice. One obvious difficulty stems from the cardinality of the choice set. This is especially true if outcomes are endowed with a specific structure, which leads to conceiving outcomes as combinations of basic alternatives.

For instance, suppose outcomes are rankings of objects, decisions, or contestants. A well-known example is the Eurovision song contest. Clearly, the relative ranking of each of the contestants, as well as the perceived quality of contestants similarly ranked matters for many citizens. In other words, voters are interested in not only the final rank of their champion but also the full ranking of contestants. Nothing prevents, in theory, asking all voters to report their preferences overall rankings. However, while possible in principle, this creates an obvious logistical problem. In 2018, 26 countries were competing, and this would imply that voters have to rank the $26!$ possible outcomes. The way out retained in the rules of the song contest is collecting only partial information about individual preferences : jury members are asked to give 12, 10, 8, ..., 1 point to their 10 favorite songs ranked in decreasing preference order. However, checking whether some normative property is satisfied may be problematic if only

partial information is collected about preferences. A relevant example is strategy-proofness. The seminal papers of Gibbard (1973) and Satterthwaite (1975) focus on single-valued social choice functions, and there is no natural way to extend their results to situations where voters do not report complete preferences over alternatives. Consider the case of rankings again as outcomes. Suppose voters report their preferred ranking only. Thus, the collective procedure is defined as an Arrovian aggregation rule. Establishing strategy-proofness for an aggregation rule requires making assumptions about how this ranking can be used to generate a ranking of all rankings. Hence, an extension rule has to be chosen to generate from a single best outcome a preference over all outcomes. Therefore, strategy-proofness becomes conditional to the choice of a specific preference extension.[1]

Rankings can be defined as complex combinations of basic alternatives. Similarly, outcomes defined as sets of alternatives are complex, where complexity should be defined as the difficulty of reporting complete preferences. Again, strategy-proofness for multi-valued collective choice functions requires assumptions on how preferences over single alternatives are extended to preferences over sets of alternatives.[2] Many other examples can be chosen where outcomes are too complex to make possible a full reporting of preferences. In multiple referenda, outcomes are defined as vectors with coordinates yes or no, and voters report issue-wise their preferred position instead of their preferences over all vectors. A large literature has been devoted to problems that may arise from the lack of information about preferences. One relates to compound-majority paradoxes.[3] Another may be the failure of Pareto optimality.[4] As for strategy-proofness, checking Pareto optimality or the existence of a paradox requires specifying how issue-wise preferences are extended to preferences over outcomes.

This thesis analyses collective choice problems with complex outcomes in the context of the allocation of purely indivisible goods, where complexity is created by externality in preferences. An illustrating example which will be extensively used in the sequel is given by

1. See Bossert and Storcken (1992), Bossert and Sprumont (2014), Athanassoglu (2016) for the analysis of strategy-proof Arrovian aggregation rules. Other properties for aggregation rules which explicitly use preference extensions for are considered in Laffond and Lainé (2000), Lainé (2015), and Lainé, Özkes and Sanver (2016).

2. See Sprumont (1995) and Barberà (2011) for surveys on strategy-proof social choice correspondences. A specific case with sets as outcomes is the choice of an assembly, where voters express preferences over one single candidate (Benoît and Kornhauser (1994)).

3. See Nurmi (1999), Lacy and Niou (2000), Laffond and Lainé (2006), (2009), (2011).

4. See Kadane (1972), Özkal-Sanver and Sanver (2006), Çuhadaroğlu and Lainé (2012), Benoît and Kornhauser (2010).

the problem of designing a centralized procedure organizing job mobility. Consider a finite set of individuals, each one being assigned a current job. Individuals rank all jobs according to some preference ordering, and we assume that individuals are entitled to change their job, that is getting a job initially assigned to somebody else. Relevant modeling of such a situation is the housing market introduced by Shapley and Scarf (1974). A reallocation of jobs among individuals can be considered as a simple outcome provided no individual pays attention to jobs assigned to other individuals. Indeed, individual rankings of goods coincide in that case with individual rankings of allocations. This is no longer true when externality prevails, that is when individuals derive utility from what is assigned to others. A typical situation with externality in preference is when individuals live in couples. More generally, externalities prevail when there is a coalition structure, all members of a coalition being sensitive to the well-being of all coalition members.

As for all other problems described above, it seems hardly possible in practice to ask individuals to report preferences overall job assignments. If individuals only report a ranking of jobs, establishing standard properties such as Pareto optimality, core stability or strategy-proofness requires assumptions about how these rankings over simple alternatives are extended to the complex outcomes defined as overall allocations. In the case of couples, one may promote the idea of submitting joint rankings of pairs of jobs, which can be seen as tractable. However, many real-life job mobility campaigns, such as those involving school teachers in France, do not allow for submitting joint preferences.[5]

The role played by preference externality in housing markets has not received much attention in the literature. This contrasts with the literature on two-sided markets. In particular, the potential instability created by couples is well-documented. The reader may refer to Sasaki and Toda (1996), Dutta and Masso (1997), Roth and Peranson (1999), Cantala (2004), Klaus and Klijn (2005), Echenique and Yenmez (2007), Klaus, Klijn and Masso (2007), Hafalir (2008), Mumcu and Saglam (2010), Bando (2012), Bando (2014), Fisher and Hafalir (2016).[6]

The thesis also considers a different type of complex outcome which fits with job mobility

5. Note that allowing for joint preferences do not ensure the existence of core stable allocations, as shown in Doğan, Laffond, and Lainé (2011).

6. See Bando et al. (2016) for a recent survey.

involving couples. Suppose current jobs are distributed across different locations. Beyond the intrinsic quality, individuals assign to jobs, the distance between jobs assigned to a couple is often a critical criterion in the comparison of allocations. How to incorporate this criterion in the analysis of housing markets is a question addressed in this thesis. When individuals only submit a ranking of jobs, assessing an allocation with respect to any normative property requires assumption on how individuals combine two criteria, job quality, and distance, in their preference over outcomes.

The third type of complex outcome arising from externality in preference is also considered. It departs from the classical housing market. Suppose there exist two types of indivisible goods, each good of each type being a pure public good. Moreover, suppose that all individuals are in a couple, and must collectively decide which good will be produced type-wise. For an illustration, consider a situation where each type relates to a topic taught in all secondary schools, and that each couple involves two teachers in the same school, each being in charge of one topic. In that context, a good of some type is a textbook that can be chosen for the relevant topic. Once chosen, a textbook will be used in all schools. Assume that what matters for the central authority is the well-being of schools. Quite naturally, teachers are asked to report their preference over textbooks for their own topic. In this setting, an outcome is thus a pair of textbooks, whereas the procedure does not collect all relevant information about school preferences over outcomes. Interestingly enough, this problem is equivalent to choosing a committee formed by two members, each belonging to one specific category. For instance, consider a university offering one position in two departments. Voters are staff members, each being asked to report preferences over candidates applying to her department. Assessing the quality of the recruitment for the university requires assumptions on how department-wise preferences can be extended to university preferences.

Hence, a normative analysis of each of these three collective decision-making problems (job mobility with couples, with or without a distance structure, multiple-type indivisible public goods) requires modeling a preference extension, which describes how reported preferences over elements of outcomes are extended to preferences over outcomes.

Focusing on preference extension as the primary object of investigation, the problematic chosen in the thesis is

- for the housing market with coalitions, identifying domains of preference extensions which ensure the fulfillment of appealing properties, such as the existence of competitive equilibrium solutions or core stable solutions,

- for the committee choice problem, identifying domains of preference extensions for which a well-defined choice procedure can be obtained as a combination of well-defined type-wise choice procedures.

1.2 Organization of the thesis and main results

The thesis is organized into four chapters. Each chapter is written so as to be self-contained. The first three chapters deal with housing markets with couples, while the fourth is devoted to the committee choice problem.

Chapter 1 introduce coalitional trade in housing or Shapley-Scarf markets. In standard Shapley-Scarf markets, there are as finitely many individuals as purely indivisible goods, and each is endowed with one indivisible good. Individual preferences over goods are linear orders. Trades can be organized so as to assign exactly one good to everybody. The pioneering result of Shapley and Scarf (1974) is the existence of allocations in the core. Moreover, a competitive equilibrium always exists and can be obtained as an outcome of a specific algorithm, named the Top-Trading-Cycles algorithm of David Gale. Furthermore, Roth and Postlewaite (1977) show that if indifference is ruled out, this algorithm yields a unique competitive allocation, which is also the unique strict core allocation. We assume that the set of individuals is partitioned into couples, each of the two partners having preferences defined over allocations (two allocations assigning the same bundle to the couple being indifferent). Thus, each individual pays attention to her partner's assigned good. In such a setting, there is no natural way to define a competitive equilibrium. First, we may allow or not income transfer among partners. Second, several types of cooperation may prevail in couples. We distinguish three types, each defining an equilibrium concept : at a selfish equilibria, each individual maximizes her utility within her coalition budget set ; at a cooperative equilibria, a budget constrained Pareto optimal bundle of goods is assigned to each couple ; at a coordinated equilibria, each partner maximizes her well-being over budget feasible allocations, given her partner's assigned

good. Mixing the type of cooperation with the type of budget set, one gets six notions of equilibrium. Strong (selfish, cooperative, coordinated) equilibria refer to the non-transferable income case, while weak (selfish, cooperative, coordinated) equilibria refer to the transferable income case. We provide a set-comparison of all equilibrium types. Moreover, we show that all types of competitive equilibrium may fail to exist if preferences over allocations are not restricted. Two preference domains are introduced and show that each is maximal for a specific equilibrium type : the domain of coalition responsive preferences is maximal for the existence of strong selfish equilibrium, while the domain of weakly lexicographic preferences is maximal for the existence of weak cooperative equilibrium.

Coalitional responsiveness holds if each partner has a ranking of goods, and its ranking over allocations is separable with respect to the two partners' rankings of goods. It follows that under coalition responsiveness, all the relevant information need to achieve a (strong selfish) equilibrium is reporting individual rankings over goods. Moreover, coalition responsiveness allows obtaining an equilibrium allocation as an outcome of the Top-Trading-Cycles algorithm. However, in contrast with standard Shapley-Scarf markets, this algorithm does not define a strategy-proof mechanism.

Weak lexicographic preferences mean that within any subset of (available) goods, each partner can identify a priority good assigned to the couple rather than having that good assigned to none. Moreover, who in the couple should receive the priority good may depend on the other good assigned to the couple. Clearly, weak lexicographic preferences cannot be identified by reporting individual rankings of goods. However, we prove that it allows obtaining an equilibrium as an outcome of a modification of the Top-Trading-Cycle algorithm (which also fails strategy-proofness).

Chapter 2 completes Chapter 1 with further results about strong equilibria. While the Top-Trading-Cycles algorithm always finds an equilibrium allocation under coalition responsive preferences, this allocation is not necessarily unique. As a consequence, and in contrast with standard Shapley-Scarf markets, there exist equilibrium allocations that do not arise from successive Top-Trading-Cycles. Nonetheless, the mere structure of Shapley-Scarf markets, even with a coalition structure in couples, ensures that every allocation can be defined as the outcome of a sequence of trading cycles. We characterize the set of those sequences associated

with each type of strong equilibrium allocation. Each of these sets is defined by making slight alterations of the properties of Top-Trading-Cycles.

Chapter 3 considers another extension of Shapley-Scarf markets where the set of individuals is partitioned into couples and goods are distributed among several locations. As in the first chapter, job mobility is a natural interpretation of our model, which aims at capturing the preference for living close to partners. Individuals' valuation of allocations incorporates the quality of their assigned job together with its distance to the partner's one. Hence, externality in preferences stems from distance rather than from partners' well-being. How preferences over allocation aggregate the two criteria is formalized by a preference extension, which is assumed to be separable : between two allocations placing the two goods at the same distance, individuals prefer the one with the better good for herself, and among two allocations assigning her the same good, she will prefer the one with a lower distance to partner.

The chapter investigates the existence of core allocations, and the possibility to achieve core allocations as outcomes of an algorithm similar to the Top-Trading-Cycles algorithm. We focus on notions of Core which do not break coalitions : no blocking coalition can involve one individual and not her partner. Moreover, since partners may have conflicting interests, defining a blocking coalition requires assumptions on how partners agree on how to compare allocations. We define the notion of couple agreement, which extends pairs of preferences to a quasi-ordering over allocations. We introduce several notions of core, each being related to a specific type of couple agreement.

In the case where no restriction prevails upon individual preferences, we show that the core may be empty for all types of couple agreement. Two domain restrictions are considered, each describing a polar type of individual preferences over allocations : priority-to-good preferences (my good matters first, and being given a good, I always prefer to be close to my partner) and priority-to-distance preferences (being close to my partner matters first and being at a given distance, I prefer to get a better good). We analyze the existence of each concept of core under these two types of preferences over allocation and, for specific types of couple agreements, we provide different modifications of Top-Trading-Cycles which always find a core allocation.

Chapter 4 is devoted to the committee choice problem. A society has to choose a committee formed by two members each belonging to a specific finite set of candidates. As already mentioned, this problem can also be described as a situation where the production of a pair of indivisible public goods (with different types) has to be collectively decided. We assume that each member of the society has preferences over candidates of each type. However, reporting type-wise preferences does not fully describes preferences over committees (even under the assumption of separability). Moreover, asking individuals to report preferences over committees is problematic. Indeed, if 10 candidates apply on each seat, individuals should rank 100 possible outcomes, which does not seem to be easily applicable. The chapter questions the capacity of a seat-wise choice procedure to select a committee that would also be chosen with full knowledge of preferences over committees. As voting is a natural choice method in this context, we focus on choice procedures based on majority voting.

The analysis follows a route very similar to the one chosen for analyzing Shapley-Scarf markets. Each candidate for a seat is conceived as a simple alternative, while a committee is a "complex" combination (here a vector) of simple alternatives. Preferences over simple alternatives are extended to preferences over outcomes by means of a preference extension. Here, a preference extension associates with type-wise preferences a ranking of all committees. The purpose of the chapter is characterizing the class of preference extensions for which no inconsistency arises between seat-wise majority voting (Maj) and direct majority voting.

It is well-known that Maj may not lead to a well-defined outcome if there are more than two candidates per seat. This difficulty is overcome by assuming the existence of a Condorcet winning candidate for each seat. Then two types of inconsistency are considered. The first one is defined as the majority committee paradox : A Condorcet winning committee exists and is not selected by Maj. The second one which is called the majority committee weak paradox : Either the majority committee paradox holds, or a Condorcet winner committee fails to exist.

We show that separability is a necessary and sufficient condition for a neutral preference extension (i.e., candidates' names play no role) to avoid the majority committee paradox. Separable preference extension means that if a and b are the two candidates for a seat, and if a voter ranks a above b, she will rank (a, x) above (b, x) for any candidate x for the

other seat. Moreover, we show that the domain of neutral preference extensions avoiding the majority committee weak paradox contains only one separable preference extension, namely the lexicographic preference extension. According to this extension, all voters agree on a seat as priority seat and compare committees according to their ranking of candidates for that priority seat whenever they differ and if both committees have the same candidate on the priority seat, compares them according to the ranking of candidates for the other seat.

Chapitre 2

Competitive Equilibria in Shapley-Scarf Markets with Coalitional Trade

Abstract : We investigate the existence and properties of several types of competitive equilibrium in generalized Shapley-Scarf markets where trades are organized among exogeneous coalitions. Coalitional trade generates externalities in individual valuations of allocations. Equilibrium concepts differ according to the definition of the budget constraint of coalitions, and to the type of cooperation prevailing among partners. We show that both existence and properties of equilibria crucially depend on the properties of individual preferences. We also derive maximal preference domains for the existence of two specific types of equilibrium.[1]

Keywords: Shapley Scarf markets - Competitive equilibrium - Coalition - Externalities

2.1 Introduction

Shapley and Scarf (1974) consider a market where purely indivisible goods are traded without money. There are finitely many individuals, each being endowed with an indivisible good. All agents have preferences over goods and can trade them in such a way that everyone ends up with exactly one good. Shapley and Scarf (1974) show that the core of such a market is non-empty and contains the set of competitive allocations. Moreover, they show that the set of allocations that are reached by means of the so-called Top-Trading-Cycles (TTC) algorithm

1. This is a joint work with Jean Lainé of CNAM-LIRSA, Paris, France

(introduced by David Gale) coincides with the set of competitive allocations. Individuals being single in a Shapley-Scarf market, they compare allocations according to the utility derived from their assigned good. However, there are many situations where trades are organized between group of agents, and where members of any group pay interest to the goods that are allocated to their partners as well as themselves. In this paper, we analyze a generalization of Shapley-Scarf markets to situations where trades are organized among coalitions.

Shapley-Scarf markets with coalitional trade provide a relevant framework for the analysis of job mobility, through which employees move from their current job to another one made available by their employer or by the market. Certain job mobility campaigns are organized as a centralized procedure, in which individuals report their preferences over available jobs, and jobs are reallocated in such a way that each individual ends up with one job, and no job is assigned to different individuals. Real-life centralized job mobility campaigns follow a "point procedure" : each available job is given a specific number of points, and each applicant to a job must be initially endowed with at least the number of points given to the job for her application to be admissible.[2] In a point procedure, points may be conceived as prices, which gather all the information relevant to a satisfactory allocation of jobs. Individual incomes are defined by the number of points given to their current job. This creates standard individual budget constraints, under which agents seek to maximize their satisfaction by being given their most preferred affordable job. An equilibrium situation is thus defined as a competitive price equilibrium of a Shapley-Scarf market. However, an important real-life aspect of job mobility results from the fact that some individuals live in couple. This obviously makes claims about job mobility much more complex to handle. Claiming for new jobs involves a bargaining process within couples : individual valuation of a job offer becomes conditional to the partner's situation. With this externality, individual valuations of allocations cannot reduce to individual valuations of individually offered jobs. As a consequence, defining a competitive equilibrium requires assumptions on how preferences over single jobs are extended

2. Such is the case for French teachers in primary and secondary schools. The assignment of points to a job may take into account many different features such as the level of demand pressure for the job, or the wish by the central authority to avoid offer shortages in some region. The number of points given to applicants may result from their own professional history, their current job, as well as their private data (marital status, number of children, ...). Nonetheless, a major feature explaining how many points are given to applicants is the nature of their current position.

to preferences over allocations.[3]

The aim of the paper is to investigate the existence and the properties of several types of competitive equilibrium in Shapley-Scarf market involving trades among coalitions. Equilibrium concepts differ according to the underlying definition of the budget constraint of coalitions, and to the type of cooperation prevailing among partners. A strong budget constraint relates to the impossibility to transfer income, or points, among partners, while a weak budget constraint allows for transferable income. In this case, an allocation is budget-feasible if the sum of prices of goods assigned to the coalition does not exceed the sum of prices of initially owned goods. Regarding the type of cooperation between partners, we retain three approaches. In the first one, individuals pay attention only to her own well-being given what is affordable by the coalition, leading to the concept of selfish equilibrium. The second considers an unspecified bargaining procedure among partners ending up at a budget-constrained Pareto optimum. This leads to the concept of cooperative equilibrium. The third formalizes a tacit collusion of Nash type in coalitions, where each partner maximizes her well-being over budget feasible allocations, given her partners' assigned goods. This leads to the concept of coordinated equilibrium. Mixing the two types of budget constraint with the three types of cooperation allows to define six types of competitive equilibrium.

We provide a set-comparison between all types of equilibrium. Moreover, we investigate their properties (individual rationality, Pareto efficiency, Core stability), and derive sufficient conditions for existence.

Roth and Postlewaite (1977) show that when indifferences are ruled out, the TTC algorithm yields a unique competitive allocation, which is also the unique strict core allocation.[4] A major difficulty regarding existence is that this is no longer true with coalitional trade. The same difficulty prevails in markets which multiple or multiple types of indivisible goods. Konishi, Quint and Wako (2001) show that even the core in such markets may be empty

3. This type of externalities has been mainly studied in two-sided markets. The reader may refer to Sasaki and Toda (1996), Dutta and Masso (1997), Roth and Peranson (1999), Cantala (2004), Klaus and Klijn (2005), Echenique and Yenmez (2007), Klaus, Klijn and Masso (2007), Hafalir (2008), Mumcu and Saglam (2010), Bando (2012), Bando (2014), Fisher and Hafalir (2016). See also Bando et al. (2016) for a recent survey.

4. When indifferences are allowed on preferences of agents, there are results that are more general. Wako (1984) shows that strict core is included in the set of competitive allocations. Wako (1991) shows that every non-competitive allocation is weakly dominated by some competitive allocation and the non-empty strict core is unique von Neumann-Morgenstern solution. Further properties of Shapley-Scarf markets may be found in Roth (1982), Roth (1984), Ma (1994), Bird (1994), Sönmez (1996), Abdulkadiroğlu and Sönmez (1999).

and competitive equilibrium allocations may fail to exist.[5] Doğan et al. (2011) show that a Shapley-Scarf market with couples may have an empty Core and provide a sufficient condition about preferences for Core existence. One major difference between their model and the present one is that they consider couple preferences rather than individual preferences.

Our main results may be summarized as follows :

- The domain of coalition responsive preferences over allocations is maximal for the existence of strong selfish equilibria. Coalition responsiveness holds if all individuals have a linear order over goods, and if any unilateral improvement of one partner's well-being benefits to all members of the coalition. Moreover, we show that with coalition responsive preferences, the TTC algorithm always ends up at a strong selfish equilibrium, and that this equilibrium allocation is also a strong cooperative and coordinated equilibrium allocation.

- Under transferable income, the domain of weak lexicographic preferences over allocations is maximal for the existence of weak cooperative equilibria. A preference is weak lexicographic if given any subset of available goods, there exists a unique good which should be assigned to the coalition, whom receiving it depending on the other goods allocated to the coalition. Moreover, we provide a constructive proof of existence based on a modification of the TTC algorithm. Furthermore, since weak cooperative equilibrium allocations belong to the coalitional Core, this result generalizes the main result of Doğan et al. (2011).

The paper is organized as follows. Section 2.1 is devoted to the formal model of Shapley-Scarf markets with coalitional trade. Alternative concepts of competitive equilibrium are defined in Section 2.2. Alternative preference domains are introduced in Section 2.3. All results are stated in section 3. In Section 3.1, we conduct a set-comparison analysis of equilibrium allocations. In Section 3.2 we investigate properties of competitive equilibria. Existence results are given in Section 3.3. In Section 3.4, we derive maximal preference domain for the existence of strong selfish and weak cooperative equilibria. Further comments conclude the paper.

5. See Papai (2003), Wako (2005), Papai (2007) and Klaus (2008) for generalizations of Shapley- Scarf Market with multiple or multiple types of indivisible goods. Mumcu and Saglam (2007) shows that the core of Shapley-Scarf market may be empty when preferences of agents are interdependent.

2.2 Coalitional trade in Shapley-Scarf Markets

2.2.1 Preliminaries

We consider a market without money, where a finite set $I = \{1, 2, ..., N\}$ of individuals collectively decide on how to trade purely indivisible goods in a finite set $G = \{1, 2, ..., N\}$. Individuals (resp. goods) are denoted by i, j, k (resp. x, y, z). An allocation σ is a bijection from I to G where $\sigma(i)$ is the good allocated to i. The set of allocations is denoted by Σ. There exists an initial allocation $\sigma^0 \in \Sigma$. Without loss of generality, σ^0 is defined by $\sigma^0(i) = i$ for all i. Under this assumption, allocations are permutations of I.

There exists an exogenous coalition structure, defined as a partition $\mathcal{C} = \{C_1, C_2, ..., C_M\}$ of I into non-empty sets. We define $C(i)$ as the element C of $\mathcal{C}$ such that $i \in C$. For coalition $C = \{i_1, ..., i_K\}$ with $i_1 < i_2 < ... < i_K$, we denote by $\sigma(C) = (\sigma(i_1), ..., \sigma(i_K))$ the vector of goods σ assigns to the members of C. Moreover, given any non-empty subset J of G with $|C| \leq |J|$, we define $\Sigma(C\ |_J) = \{\sigma \in \Sigma : \cup_{i \in C}\{\sigma(i)\} \subseteq J\}$ as the set of allocations assigning to each member of C a good in J.

Individuals have preferences over allocations represented by weak orders. For any finite set X, $\mathcal{W}(X)$ (resp. $\mathcal{L}(X)$) denotes the set of weak (resp. linear) orders over X. We denote the preferences of individual i by $R_i \in \mathcal{W}(\Sigma)$, with asymmetric part P_i. Given $\sigma, \sigma' \in \Sigma$, $\sigma R_i \sigma'$ means that σ is at least as good as σ' for i. An N-tuple $\pi = (R_i)_{i \in I}$ is called a profile, and Π stands for the set of profiles. We impose one property to preferences, called coalition selfishness. It states that members of each coalition pay attention only to the goods allocated to the coalition. Formally, coalition selfishness holds if $\forall C \in \mathcal{C}$, $\forall i \in C$, $\forall\ \sigma_1, \sigma_2, \sigma_1', \sigma_2' \in \Sigma$, $\sigma_1(C) = \sigma_1'(C)$ and $\sigma_2(C) = \sigma_2'(C)$ implies that $\sigma_1 R_i \sigma_2$ if and only if $\Leftrightarrow\ \sigma_1' R_i \sigma_2'$. Using coalition selfishness, for coalition $C = \{i_1, ..., i_K\}$ with $i_1 < i_2 < ... < i_K$, we sometimes write $(\sigma(i_1), ..., \sigma(i_K)) R_i (\sigma'(i_1), ..., \sigma'(i_K))$ if i ranks any allocation σ with $\sigma(C(i)) = (\sigma(i_1), ..., \sigma(i_K))$ no lower than any other allocation σ' with $\sigma'(C(i)) = (\sigma'(i_1), ..., \sigma'(i_K))$.

Given a profile π, an allocation σ is individually rational if $\forall i \in C$, $\sigma(C(i)) R_i \sigma^0(C(i))$. The set of individually rational allocations is denoted by $\Sigma(\sigma^0, \pi)$.

Definition 2.2.1 *A Shapley-Scarf market with coalitional trade is a triple* $\mathcal{E} =< N, \mathcal{C}, \pi >$ *where* N *is the number of goods and individuals,* $\mathcal{C}$ *is a coalition structure, and* π *is a profile satisfying coalition selfishness.*

Note that the original Shapley-Scarf market is a Shapley-Scarf market with coalitional trade where $\mathcal{C} = \{\{1\}, \{2\}, ..., \{N\}\}$.

2.2.2 Alternative concepts of competitive equilibrium

There is no natural notion of competitive equilibrium when externalities prevail in individual valuations. Different concepts of budget constraint, as well as different types of cooperation within coalitions, lead to different definitions of equilibrium.

We first introduce two notions of budget set. Define a price vector as an element $p = (p_x)_{x \in G}$ of $\mathbb{R}^N_+$.

Definition 2.2.2 *Let* p *be a price vector. The strong budget set of* i *for* p *is the subset of allocations* $B_i^S(p) = \{\sigma \in \Sigma : \forall j \in C(i),\ p_{\sigma(j)} \leq p_j\}$. *The weak budget set of* i *for* p *is the subset of allocations* $B_i^W(p) = \{\sigma \in \Sigma : \sum\limits_{j \in C} p_{\sigma(j)} \leq \sum\limits_{j \in C} p_j\}$.

Strong and weak budget sets are natural extensions of the standard Shapley-Scarf budget constraint. Each individual is endowed with a single good, and her income is the price of her initial endowment. Hence a good is affordable if its price is not greater than income. In case of coalitional trade, we may allow or not for income transfer among partners. If income transfers are not permitted, each member must be given a good priced lower than her initial good, as stated by the strong budget set. When income transfers are permitted, an allocation is affordable if the sum of prices of all goods allocated to the coalition does not exceed the total coalition income, as prescribed by the weak budget set. Observe that $B_i^S(p) = B_j^S(p)$ and $B_i^W(p) = B_j^W(p)$ for all $i, j \in C \in \mathcal{C}$. Therefore, we can write $B_i^S(p) = B_C^S(p)$ and $B_i^W(p) = B_C^W(p)$ for all $i \in C$. Furthermore, it is obvious that $B_j^S(p) \subseteq B_j^W(p)$ for all $j \in I$ and all $p \in \mathbb{R}^N_+$.

We introduce three alternative equilibrium concepts. Loosely speaking, an equilibrium allocation σ is such that partners cannot be better off with some other affordable allocation σ'.

However, in the case of coalitional trade, σ' may be identified either individually or collectively. Each equilibrium concept relates to a different approach to this issue. Given a price vector p together with a coalition C, we define a budget set $B_C(p)$ as an element of $\{B_C^W(p), B_C^S(p)\}$.

Definition 2.2.3 *A selfish equilibrium for $\mathcal{E}$ is a 2-tuple $(\sigma, p) \in \Sigma \times \mathbb{R}_+^N$ such that $\forall i \in I$, $\sigma \in \underset{B_{C(i)}(p)}{\arg\max} R_i$ where $\underset{B_{C(i)}(p)}{\arg\max} R_i = \{\sigma \in B_{C(i)}(p) : \sigma R_i \sigma', \forall \sigma' \in B_{C(i)}(p)\}$. If $B_{C(i)}(p) = B_{C(i)}^W(p)$, (σ, p) is a weak selfish equilibrium, and if $B_{C(i)}(p) = B_{C(i)}^S(p)$, (σ, p) is a strong selfish equilibrium.*

We denote respectively by $\mathbb{E}_{self}^S(\mathcal{E})$ and $\mathbb{E}_{self}^W(\mathcal{E})$ the sets of strong and weak selfish equilibria for $\mathcal{E}$, and we denote respectively by $E_{self}^S(\mathcal{E})$ and $E_{self}^W(\mathcal{E})$ the sets of strong and weak equilibrium allocations (i.e. allocations σ such that (σ, p) is a selfish equilibrium for some price vector p). A selfish equilibria relates to the lowest degree of collusion within coalitions : each individual pays interest only to her own well-being given what is affordable by the coalition.

Given a price vector p and a coalition C, a p-optimum for C is an allocation $\sigma \in B_C(p)$ such that there is no $\sigma' \in B_C(p)$ which verifies $\sigma' R_i \sigma$ for all $i \in C$ and $\sigma' P_j \sigma$ for some $j \in C$. If $B_C(p) = B_C^W(p)$, σ is a weak p-optimum for C, and if $B_C(p) = B_C^S(p)$, σ is a strong p-optimum for C. We denote by $\mathcal{O}_C^S(p)$ (resp. $\mathcal{O}_C^W(p)$) the set of strong (resp. weak) p-optima for C.

Definition 2.2.4 *A strong (resp. weak) cooperative equilibrium for $\mathcal{E}$ is a 2-tuple $(\sigma, p) \in \Sigma \times \mathbb{R}_+^N$ such that $\sigma \in \mathcal{O}_C^S(p)$ (resp. $\mathcal{O}_C^W(p)$) for all $C \in \mathcal{C}$.*

A cooperative equilibrium allocation results from some unspecified bargaining procedure among partners, which ends up at a budget-constrained Pareto optimal situation. We denote respectively by $\mathbb{E}_{coop}^S(\mathcal{E})$ and $\mathbb{E}_{coop}^W(\mathcal{E})$ the sets of strong and weak cooperative equilibria, and we denote respectively by $E_{coop}^S(\mathcal{E})$ and $E_{coop}^W(\mathcal{E})$ the sets of strong and weak cooperative equilibrium allocations.

Pick an allocation σ, a price vector p, and an individual i. The (σ, p)-restricted budget set for i is the set of allocations $B_i(\sigma, p) = \{\sigma' \in B_{C(i)}(p) : \forall j \in C \backslash \{i\}, \sigma'(j) = \sigma(j)\}$. A (σ, p)-restricted budget set for i contains all allocations which endow her partners with the same good as in σ and which give i a good making the coalition bundle budget-feasible.

If $B_{C(i)}(p) = B^S_{C(i)}(p)$, we write $B_i(\sigma, p) = B^S_i(\sigma, p)$, and if $B_{C(i)}(p) = B^W_{C(i)}(p)$, we write $B_i(\sigma, p) = B^W_i(\sigma, p)$. The best response of i to (σ, p) is the set $\Phi_i(\sigma, p) = \arg\max_{B_i(\sigma,p)} R_i$. If $B_i(\sigma, p) = B^S_i(\sigma, p)$, we obtain the strong best-response $\Phi^S_i(\sigma, p)$, and if $B_i(\sigma, p) = B^W_i(\sigma, p)$, we obtain the weak best-response $\Phi^W_i(\sigma, p)$.

Definition 2.2.5 *A strong (resp. weak) coordinated equilibrium for $\mathcal{E}$ is a 2-tuple $(\sigma, p) \in \Sigma \times \mathbb{R}^N_+$ such that $\forall i \in I$, $\sigma \in \Phi^S_i(\sigma, p)$ (resp. $\sigma \in \Phi^W_i(\sigma, p)$).*

Coordinated equilibria relate to a tacit Nash-type collusion scheme in coalitions. Given her partners' current goods, each individual maximizes her satisfaction under budget constraint. We denote respectively by $\mathbb{E}^S_{coor}(\mathcal{E})$ and $\mathbb{E}^W_{coor}(\mathcal{E})$ the sets of strong and weak coordinated equilibria. We denote respectively by $E^S_{coor}(\mathcal{E})$ and $E^W_{coor}(\mathcal{E})$ the associated sets of equilibrium allocations.

Observe that, as in standard Shapley-Scarf markets, the market clearing condition is embedded in the definition of allocations. The following examples illustrate the three types of equilibrium.

Example 2.2.1 *Consider $\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $\mathcal{C} = \{C_1, C_2\} = \{\{1, 2\}, \{3, 4\}\}$, and $\pi = (P_1, ..., P_4\}$ is a profile having the form below :*[6]

1	2	3	4
$(3, 4)$	$(1, 3)$	$(1, 2)$	$(1, 2)$
$(3, 2)$	$(1, 4)$	$(1, 3)$	$(3, 2)$
$(3, 1)$	$(1, 2)$	$(1, 4)$	$(4, 2)$
...	$(3, 4)$	...	...
	$(3, 2)$		
	$(3, 1)$		
	...		

Consider allocation $\sigma = (3, 4, 1, 2)$ and price vector $p = (1, ..., 1)$. Obviously, $B^S_C(p) = B^W_C(p) = \Sigma$ for all C. Moreover, there is no $\sigma' \neq \sigma$ in $B^S_i(p)$ such that $\sigma' R_i \sigma$ for all $i = 1, 3, 4$. Observe that $\sigma \in \mathcal{O}^S_{C_1}(p) \cap \mathcal{O}^S_{C_2}(p)$. Hence $\sigma \in E^S_{coop}(\mathcal{E}) \cap E^W_{coop}(\mathcal{E})$. Furthermore,

6. The individual's name appears in the first row. For each column labeled $i \in I$ represents agent i's preferences R_i over allocations. Allocations that are more preferred are listed above less preferred allocations. Note that for all $i = 1, 2$, we write $(\sigma(1), \sigma(2)) R_i (\sigma'(1), \sigma'(2))$ if i ranks any allocation σ with $\sigma(C_1) = (\sigma(1), \sigma(2))$ no lower than any other allocation σ' with $\sigma'(C_1) = (\sigma'(1), \sigma'(2))$ and for all $i = 3, 4$, we write $(\sigma(3), \sigma(4)) R_i (\sigma'(3), \sigma'(4))$ if i ranks any allocation σ with $\sigma(C_1) = (\sigma(3), \sigma(4))$ no lower than any other allocation σ' with $\sigma'(C_2) = (\sigma'(3), \sigma'(4))$.

$(\sigma, p) \notin \mathbb{E}^S_{self}(\mathcal{E}) \cup \mathbb{E}^W_{self}(\mathcal{E})$. *Indeed,* $p_3 \leq p_2$ *while* $(1,3)P_2(3,4)$. *Finally, let* $\sigma'' = (3,2,1,4)$. *Then* $(\sigma'', p) \notin \mathbb{E}^S_{coor}(\mathcal{E}) \cup \mathbb{E}^W_{coor}(\mathcal{E})$. *Indeed,* $(\sigma'', p) \in \mathbb{E}^W_{coor}(\mathcal{E})$ *only if* $p_1 + p_2 < p_3 + p_4$ *(otherwise, every* $\widetilde{\sigma}$ *with* $\widetilde{\sigma}(C_2) = (3,4)$ *is weakly budget feasible, while* $\widetilde{\sigma} P_3 \sigma''$*). Similarly,* $(\sigma'', p) \in \mathbb{E}^S_{coor}(\mathcal{E})$ *only if* $p_2 < p_4$.

Example 2.2.2 *Consider* $\mathcal{E} =< 8, \mathcal{C}, \pi >$ *where* $C_1 = \{1,2\}, C_2 = \{3,4\}, C_3 = \{5,6\}, C_4 = \{7,8\}$*, and where* π *has the form below :*

$$\left(\begin{array}{cccc} 1,2 & 3,4 & 5,6 & 7,8 \\ \hline (3,2) & (5,4) & (7,6) & (1,8) \\ (4,3) & (6,5) & (8,7) & (2,1) \\ ... & ... & ... & ... \end{array}\right)$$

Pick allocation $\sigma = (4,3,6,5,8,7,2,1)$ *and price vector* $p = (1,...,1)$. *As above,* $B^S_C(p) = B^W_C(p) = \Sigma$ *for all* C. *Note that* $\sigma R_1 \sigma'$ *for all* $\sigma' \in B^S_{C_1}(\sigma, p)$ *where* $\sigma'(2) = 3$. *Therefore* $\sigma \in \Phi^S_1(\sigma, p)$. *Similarly,* $\sigma P_2 \sigma'$ *for all* σ' *with* $\sigma'(1) = 4$. *Thus,* $\sigma \in \Phi^S_2(\sigma, p)$. *One also checks that* $\sigma \in \Phi^S_i(\sigma, p)$ *for all* $i = 3, ..., 8$. *Therefore* $(\sigma, p) \in \mathbb{E}^S_{coor}(\mathcal{E}) \cap \mathbb{E}^W_{coor}(\mathcal{E})$. *Finally,* $(\sigma, p) \notin \mathbb{E}^S_{coop}(\mathcal{E}) \cup \mathbb{E}^W_{coop}(\mathcal{E}) \cup \mathbb{E}^S_{self}(\mathcal{E}) \cup \mathbb{E}^W_{self}$, *since* $\sigma' = (3,2,5,4,7,6,1,8)$ P_i σ *for all* i *while being budget feasible for all coalitions.*

2.2.3 Restrictions upon preferences

We show below that both existence properties of equilibria are sensitive to the structure of preferences. Attention will be paid to three preference domains.

Definition 2.2.6 *A profile* $\pi = (R_i)_{i \in I}$ *is responsive if and only if for all* C *and all* $i \in C$*, there exists* $(\succ^i_j)_{j \in C} \in \mathcal{L}(G)^{|C|}$ *such that* $\forall \sigma, \sigma' \in \Sigma$ *with* $\sigma(C) \neq \sigma'(C)$*, we have*

$$\sigma P_i \sigma' \text{ if for all } j \in C, \text{ either } \sigma(j) \succ^i_j \sigma'(j) \text{ or } \sigma(j) = \sigma'(j).$$

We denote by $\Pi_R \subseteq \Pi$ the set of responsive profiles. Responsiveness holds if each individual has a linear order over goods for each of all her partners (including herself), and gets better off with a Pareto improving change according to these linear orders. Responsiveness relates the welfare of the coalition to that of its members. Observe that in definition 6, $\succ^i_j$ may not coincide with $\succ^j_j$: i may endow j with a ranking of goods that is not the one that j assigns to

herself. On the contrary, if each individual rank goods allocated to any partner as this partner does, any unilateral improvement of one partner's well-being will benefit to all members of the coalition. This special case is called coalition responsiveness.

Definition 2.2.7 *A profile $\pi = (R_i)_{i\in I}$ is coalition responsive if and only if it is responsive and $\forall C$, $\forall i, j, k \in C$, $\succ_i^j = \succ_i^k$.*

Under coalition responsiveness, we write $\succ_i = \succ_i^j$ for all $j \in C$. We denote by $\Pi_{CR} \subseteq \Pi_R$ the set of all coalition-based-responsive preferences profiles.[7]

Preferences are called joint if for all C and for all $i, j \in C$, we have $R_i = R_j$. Clearly, if preferences are joint and responsive, they are coalition responsive.

We now introduce lexicographic and weak lexicographic preferences. Denote by $\mathcal{J}$ (resp. $\mathcal{J}_z$ with cardinality $z > 0$) the set of all non-empty subsets of G (resp. with cardinality z).

Definition 2.2.8 *An individual i has lexicographic preferences if there exists a good-priority mapping $\gamma_i : \mathcal{J} \to G$ and a partner- priority mapping : $\mu_i : \mathcal{J} \to C(i)$ such that*

(1) $\gamma_i(J) \in J$ for all non-empty $J \subseteq \mathcal{J}$

(2) $\forall J \in \mathcal{J}$ and $\forall \sigma, \sigma' \in \Sigma(C(i) \mid_J)$ such that $\sigma(\mu_i(J)) = \gamma_i(J)$ and $\sigma'(\mu_i(J)) \neq \gamma_i(J)$, we have $\sigma P_i \sigma'$.

An individual i has lexicographic preferences if when facing any subset of goods J available for trade, there exists a unique good x (i.e., $\gamma_i(J) = x$) and a unique member $j \in C(i)$ (i.e., $\mu_i(J) = j$) such that i ranks any matching σ where j is given x (i.e., $\sigma(j) = x$)) above any other allocation σ' that allocates j some other good (i.e., $\sigma(j) \neq x$)). A profile is called lexicographic if it involves lexicographic preferences. As responsiveness, the lexicography property qualifies the link between the one-dimensional thinking that individuals frequently employ with the multi-dimensional nature of the coalitional allocations. Individuals with lexicographic preferences prioritize goods, and tieing each group J of goods to a member of the coalition, and they are better off when a higher priority good has been assigned to the

7. Klaus and Klijn (2005) show that coalition responsiveness (which they call responsiveness) plays an important role in the existence of stable matchings in two-sided markets with couples. See also Klaus, Klijn and Massò (2007).

appropriate member of their coalition. Once again, we allow partners in a coalition to have different ways to prioritize goods and partners. Π_L will stand for the set of all lexicographic preferences.[8]

Definition 2.2.9 *An individual i has weak lexicographic preferences if there exists a good-priority mapping $\gamma_i : \mathcal{J} \to G$ and a contingent partner-priority mapping $\lambda_i : \mathcal{J} \times \mathcal{J}_{|C(i)|} \to C(i)$ such that*

(1) $\gamma_i(J) \in J$ for all non-empty $J \subseteq \mathcal{J}$

(2) $\forall J \in \mathcal{J}$, $\forall J' \in \mathcal{J}_{|C(i)|}$ with $J' \subseteq J$, $\forall \sigma \in \Sigma(C(i)\ |_{J'})$ with $\sigma(\lambda_i(J \times J')) = \gamma_i(J)$, $\forall \sigma' \in \Sigma(C(i)\ |_J)$, we have $\sigma' R_i \sigma$ only if $\sigma'(\lambda_i(J \times J'')) = \gamma_i(J)$ for some $J'' \in \mathcal{J}_{|C(i)|} \backslash \{J'\}$ and $J'' \subseteq J$

Pick any subset J of goods. An individual with weak lexicographic preference always strictly prefers having a priority good in J assigned to a specific partner rather than having that good assigned to none. Moreover, who should receive the priority good may depend on the other goods assigned to the coalition. In contrast with lexicographic preference, weak lexicographic preferences rest upon a contingent partner-priority mapping. For an illustration where $C(i)$ is a couple, and where there are 4 goods, a lexicographic preference R_i may look like $(1,4)\ R_i\ (2,4)\ R_i\ (3,4)\ P_i\ (2,1)\ R_i\ (2,3)\ P_i$ ..., while a weak lexicographic preference R_i' may look like $(1,4)\ R_i'\ (4,2)\ R_i'\ (3,4)\ P_i'$ .... Observe that we may have $(3,2)\ R_i'\ (4,1)$. Indeed, good 4 is the priority good for i in G, but the second partner is the priority partner contingent to the fact that goods 1 and 4 are assigned to the couple. Hence, assigning good 4 to the first partner may bring a situation less preferred than another where good 4 is not assigned to the couple. Π_{WL} will stand for the set of weak lexicographic preference profiles. Clearly, $\Pi_L \subset \Pi_{WL}$.

8. Doğan et al. (2011) show that lexicographic preferences ensure the non-emptiness of the Core in a setting similar to the present one, the main difference being that all partners are assumed to share the same preference. We comment this result below.

2.3 Results

Results are organized in four sections. The first deals with set-comparison among equilibrium allocations. Properties of equilibrium allocations are stated in the second. Existence results are gathered in the third. While stated and proved for the specific case where coalitions are couples, these results can be extended to any coalition structure against some notational cost. Finally, we investigate in the fourth section the maximality of preference domains that ensure existence.

2.3.1 Set-comparison of equilibria

We begin with results comparing sets of equilibrium allocations for both strong and weak budget constraints.

Proposition 2.3.1 *For all* $\mathcal{E} =< N, \mathcal{C}, \pi >$,

1. $E^S_{self}(\mathcal{E}) \subseteq E^S_{coop}(\mathcal{E})$, *and* $E^W_{self}(\mathcal{E}) \subseteq E^W_{coop}(\mathcal{E})$
2. $E^S_{self}(\mathcal{E}) \subseteq E^S_{coor}(\mathcal{E})$, *and* $E^W_{self}(\mathcal{E}) \subseteq E^W_{coor}(\mathcal{E})$
3. *If* $\pi \in \Pi_{CR}$, *then* $E^S_{coop}(\mathcal{E}) \subseteq E^S_{coor}(\mathcal{E})$, *and* $E^W_{coop}(\mathcal{E}) \subseteq E^W_{coor}(\mathcal{E})$

Proof 2.3.1 *Let* $(\sigma, p) \in \mathbb{E}^S_{self}(\mathcal{E})$. *By definition,* $\sigma \in \arg\max_{B^S_{C(i)}(p)} R_i$ *for all* i, *then* $\sigma \in \mathcal{O}^S_C(p)$ *for all* C. *Therefore,* $\sigma \in E^S_{coop}(\mathcal{E})$. *The same argument applies to show that* $E^W_{self}(\mathcal{E}) \subseteq E^W_{coop}(\mathcal{E})$. *This proves assertion 1.*

If $(\sigma, p) \in \mathbb{E}^S_{self}(\mathcal{E})$, *then* $\arg\max_{B^S_{C(i)}(p)} R_i$ *for all* i. *For all* p, *we have* $B^S_i(\sigma, p) \subseteq B^S_{C(i)}(p)$ *for all* i. *Thus* $\sigma \in \cap_{i \in I} \Phi^S_i(\sigma, p)$ *for all* i, *and thus* $(\sigma, p) \in \mathbb{E}^S_{coor}(\mathcal{E})$. *The same argument holds for weak budget sets. This proves assertion 2.*

Finally, suppose that $\pi \in \Pi_{CR}$, *and let* $(\sigma, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$. *If* $\sigma \notin E^S_{coor}(\mathcal{E})$, *then* $(\sigma, p) \notin \mathbb{E}^S_{coor}(\mathcal{E})$. *It follows that there exist* i *and* $\sigma' \in B^S_{C(i)}(\sigma, p)$ *such that* $\sigma' P_i \sigma$ *and* $\sigma'(j) = \sigma(j)$ *for all* $j \in C \backslash \{i\}$. *Moreover,* $\pi \in \Pi_{CR}$ *implies that* $\sigma'(i) \succ_i \sigma(i)$. *Moreover, coalitional responsiveness implies* $\sigma' P_j \sigma$ *for all* $j \in C(i)$, *in contradiction with* $(\sigma, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$. *Since the same argument holds for weak budget sets, this shows assertion 3.*

Inclusions in Propositions 1.1 and 1.2 may be strict, even under preference restrictions, while those in Proposition 1.3 may no longer prevail if preferences are not coalition responsive.

Proposition 2.3.2 *There exists* $\mathcal{E} =< N, \mathcal{C}, \pi >$ *such that,*

1. $\pi \in \Pi_R \cap \Pi_L$, $E^S_{self}(\mathcal{E}) \neq E^S_{coop}(\mathcal{E})$, *and* $E^S_{self}(\mathcal{E}) \neq E^S_{coor}(\mathcal{E})$
2. $\pi \in \Pi_{CR}$ *and* $E^S_{self}(\mathcal{E}) \neq E^S_{coop}(\mathcal{E})$
3. $\pi \in \Pi_{CR}$ *and* $E^S_{coop}(\mathcal{E}) \neq E^S_{coor}(\mathcal{E})$
4. $\pi \in \Pi_R \cap \Pi_L$ *and* $E^S_{coop}(\mathcal{E}) \nsubseteq E^S_{coor}(\mathcal{E})$

Proof 2.3.2 *Consider the profile* π *given in example 1. Clearly,* π *can be completed so as to be responsive and lexicographic. We know that* $\sigma = (3, 4, 1, 2) \in E^S_{coop}(\mathcal{E})$. *However,* $\sigma \notin E^S_{self}(\mathcal{E})$. *Indeed,* $\sigma' P_2 \sigma$ *if* σ' *is any allocation such that* $\sigma(C_1) \in \{(1,3),(1,4),(1,2)\}$. *Since* $\sigma^0 \in B^S_{C_1}(p')$ *for all* p', *then* $\sigma \notin E^S_{self}(\mathcal{E})$. *Take* $p = (1, ..., 1)$. *Observe that* $\sigma P_1 \sigma'$ *for all* $\sigma' \in B^S_{C_1}(\sigma, p)$ *with* $\sigma'(2) = 4$. *Similarly,* $\sigma P_2 \sigma'$ *for all* $\sigma' \in B^S_{C_2}(\sigma, p)$ *with* $\sigma'(1) = 3$. *Thus,* $\sigma \in \Phi^S_1(\sigma, p) \cap \Phi^S_2(\sigma, p)$. *The reader will easily check that* $\sigma \in \Phi^S_i(\sigma, p)$, $\forall i = 3, 4$. *Therefore* $(\sigma, p) \in \mathbb{E}^S_{coor}(\mathcal{E})$. *This shows assertion 1.*

Pick $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\},\{3,4\}\}$, *and where* π *has the form*

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (4,2) & (2,4) & (1,3) & (3,1) \\ (2,4) & ... & ... & (1,3) \\ ... & & & ... \end{array}\right)$$

which can be completed to ensure $\pi \in \Pi_{CR}$, *with* $4 \succ_i 2 \succ_i ...$ *for* $i = 1, 2$, *and* $1 \succ_i 3 \succ_3 ...$, *for* $i = 3, 4$. *Let* $\sigma = (2, 4, 1, 3)$ *and* $p = (1, 1, 1, 1)$. *Obviously,* $\sigma \in \mathcal{O}^S_{C_1}(p) \cap \mathcal{O}^S_{C_2}(p)$. *Since* $\sigma \in B^S_{C(i)}(p)$ *for all* i, *then* $\sigma \in E^S_{coop}(\mathcal{E})$. *Now suppose that* $(\sigma, p') \in \mathbb{E}^S_{self}(\mathcal{E})$. *Since* $\sigma \in \arg\max_{B^S_{C_1}(p)} R_1$, $p'_4 > p'_1$. *Similarly,* $\sigma \in \arg\max_{B^S_{C_2}(p)} R_4$ *requires* $p'_1 > p'_4$, *a contradiction. This proves assertion 2.*

Pick $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\},\{3,4\}\}$, *and* π *has the form*

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (3,1) & (3,1) & (1,4) & (1,4) \\ (3,2) & (3,2) & (2,1) & (2,1) \\ (3,4) & (4,3) & (1,3) & (2,4) \\ (4,3) & ... & ... & ... \\ ... & & & \end{array}\right)$$

which can, as above, be completed in order to ensure $\pi \in \Pi_{CR}$, *with* $3 \succ_1 4 \succ_1 1 \succ_1 2$, $3 \succ_2 1 \succ_2 2 \succ_2 4$, $1 \succ_3 2 \succ_3 3 \succ_3 4$, *and* $1 \succ_4 4 \succ_4 3 \succ_4 2$. *Consider* $\sigma = (4,3,2,1)$ *and* $p = (1,1,1,1)$. *Clearly,* $\sigma \in B^S_{C(i)}(\sigma,p)$ *for all* i. *Since* $\sigma P_1 \sigma'$ *for all* $\sigma' \in B^S_1(\sigma,p)$ *with* $\sigma'(2) = 3$, *then* $\sigma \in \Phi^S_1(\sigma,p)$. *Furthermore,* $\sigma \in \Phi^S_2(\sigma,p)$ *since* $\sigma P_2 \sigma'$ *for all* $\sigma' \in B^S_2(\sigma,p)$ *with* $\sigma'(1) = 4$. *Similarly, we check that* $\sigma \in \Phi^S_i(\sigma,p)$ *for* $i = 3,4$. *Therefore,* $(\sigma,p) \in \mathbb{E}^S_{coor}(\mathcal{E})$. *Now suppose that* $(\sigma,p') \in \mathbb{E}^S_{coop}(\mathcal{E})$ *for some* p'. *Then* $\sigma \in \mathcal{O}^S_{C_1}(p')$ *requires* $p'_3 > p'_1$ *(since* $(3,2)$ P_1 $(4,3)$ *and* $(3,2)$ P_2 $(4,3)$*). Similarly, since* $(1,4))$ P_3 $(2,1)$ *and* $(1,4)$ P_4 $(2,1)$, *then* $\sigma \in \mathcal{O}^S_{C_2}(p')$ *implies* $p'_1 > p'_3$, *which is impossible. Thus,* $\sigma \notin E^S_{coop}(\mathcal{E})$. *This proves assertion 3.*

To prove assertion 4, pick $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\},\{3,4\}\}$, *and where* $\pi \in \Pi_R \cap \Pi_L$ *has the form*

$$\left(\begin{array}{ccc} 1 & 2 & 3,4 \\ \hline (1,3) & (1,2) & (2,4) \\ \dots & (1,3) & \dots \\ & \dots & \end{array}\right)$$

It is obvious that $(\sigma,p) \in \mathbb{E}^S_{coop}(\mathcal{E})$, *where* $\sigma = (1,3,2,4)$ *and* $p = (1,1,1,1)$. *Since* $\sigma^0 \in B^S_{C_1}(p')$ *for all* p' *and* $\sigma^0 P_2 \sigma$, *then* $\sigma \notin \Phi^S_2(\sigma,p')$ *for all* p'. *Thus* $\sigma \notin E^S_{coor}(\mathcal{E})$, *which proves assertion 4.*

Proposition 3 is the analog of Proposition 2 for weak equilibria.

Proposition 2.3.3 *There exists* $\mathcal{E} =< N, \mathcal{C}, \pi >$ *such that,*

1. $\pi \in \Pi_R \cap \Pi_L$, $E^W_{self}(\mathcal{E}) \neq E^W_{coop}(\mathcal{E})$, *and* $E^W_{self}(\mathcal{E}) \neq E^W_{coor}(\mathcal{E})$,
2. $\pi \in \Pi_{CR}$ *and* $E^W_{self}(\mathcal{E}) \neq E^W_{coop}(\mathcal{E})$,
3. $\pi \in \Pi_{CR}$ *and* $E^W_{coop}(\mathcal{E}) \neq E^W_{coor}(\mathcal{E})$
4. $\pi \in \Pi_R \cap \Pi_L$ *and* $E^W_{coop}(\mathcal{E}) \not\subseteq E^W_{coor}(\mathcal{E})$

Proof 2.3.3 *Consider again the responsive and lexicographic profile in example 1. We already know that* $(\sigma,p) \in E^W_{coop}(\mathcal{E})$, *where* $\sigma = (3,4,1,2)$, *and* $p = (1,1,1,1)$. *The same argument as in the proof of proposition 2.1 ensures that* $(\sigma,p) \in E^W_{coor}(\mathcal{E})$. *Since any allocation such that* $\sigma(C_1) = (1,2)$ *is weak budget feasible and* $(1,2)P_2(3,4)$, *there exists no* p' *such that* $(\sigma,p') \in E^W_{self}(\mathcal{E})$. *Therefore,* $\sigma \notin E^W_{self}(\mathcal{E})$ *, hence assertion 1.*

In order to prove assertion 2, consider the preference profile used in the proof of Proposition 2.2. Clearly, $(\sigma, p) \in E^W_{coop}(\mathcal{E})$*, where* $\sigma = (2,4,1,3)$*, and* $p = (1,1,1,1)$*. Now suppose that* $(\sigma, p') \in E^W_{self}(\mathcal{E})$ *for some* p'*. Since* $\sigma \in \arg\max_{B^W_{C_1}(p')} R_1$*, then one must have* $p'_4 + p'_2 > p'_1 + p'_2$*. Moreover weak budget feasibility of* σ *implies that* $p'_1 + p'_2 \geq p'_2 + p'_4$*, clearly a contradiction. Therefore* $\sigma \notin E^W_{self}(\mathcal{E})$*, hence assertion 2.*

Take the preference profile in the proof of Proposition 2.3. Consider $\sigma = (4,3,2,1)$ *and* $p = (1,1,1,1)$*. We know that* $(\sigma, p) \in \mathbb{E}^S_{coor}(\mathcal{E})$*. Since* $B^W_i(\sigma, p) = B^S_i(\sigma, p)$ *for all* i*,* $(\sigma, p) \in \mathbb{E}^W_{coor}(\mathcal{E})$*. Suppose that* $(\sigma, p') \in \mathbb{E}^W_{coop}(\mathcal{E})$ *for some* p'*. Since* $\sigma \in \mathcal{O}^W_{C_1}(p')$*, then we must have* $p'_3 + p'_2 > p'_1 + p'_2$*. Similarly,* $\sigma \in \mathcal{O}^W_{C_2}(p')$ *implies* $p'_1 + p'_4 > p'_3 + p'_4$*, which is impossible. Thus,* $\sigma \notin E^W_{coop}(\mathcal{E})$*, which shows assertion 3.*

Finally, the proof of assertion 4 is identical to the proof of Proposition 2.4.

We turn now to the comparison between strong and weak equilibria.

Proposition 2.3.4 *There exists* $\mathcal{E} =< N, \mathcal{C}, \pi >$ *such that*

1. $\pi \in \Pi_{CR}$*,* $E^W_{self}(\mathcal{E}) \nsubseteq E^S_{self}(\mathcal{E})$*, and* $E^W_{coop}(\mathcal{E}) \nsubseteq E^S_{coop}(\mathcal{E})$
2. $\pi \in \Pi_{CR}$*,* $E^S_{self}(\mathcal{E}) \nsubseteq E^W_{self}(\mathcal{E})$*, and* $E^S_{coop}(\mathcal{E}) \nsubseteq E^W_{coop}(\mathcal{E})$
3. $\pi \in \Pi_R \cap \Pi_L$*,* $E^W_{coop}(\mathcal{E}) \nsubseteq E^S_{coop}(\mathcal{E})$*, and* $E^W_{self}(\mathcal{E}) \nsubseteq E^S_{self}(\mathcal{E})$
4. $\pi \in \Pi_R \cap \Pi_L$*,* $E^S_{coop}(\mathcal{E}) \nsubseteq E^W_{coop}(\mathcal{E})$*, and* $E^S_{self}(\mathcal{E}) \nsubseteq E^W_{self}(\mathcal{E})$

Proof 2.3.4 *Pick* $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\}, \{3,4\}\}$*, and* $\pi \in \Pi_{CR}$ *has the form below.*

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (3,4) & (3,4) & (3,2) & (3,2) \\ \dots & \dots & (1,2) & (3,1) \\ & & \dots & (1,2) \\ & & & \dots \end{array}\right)$$

Let $\sigma = (3,4,1,2)$ *and* $p = (2,2,3,1)$*. Since* $p_1 + p_2 = p_3 + p_4$*, then* $\sigma \in B^W_{C(i)}(p)$ *for all* i*. Since* $p_2 + p_3 > p_3 + p_4$ *and* $p_1 + p_3 > p_3 + p_4$*, individuals* 3 *and* 4 *get with* σ *their first best in their weak budget set. Thus* $(\sigma, p) \in \mathbb{E}^W_{self}(\mathcal{E})$*. Now, strong budget feasibility of* σ *w.r.t.* p' *requires* $p'_1 = p'_3$ *and* $p'_2 = p'_4$*. Moreover,* $(\sigma, p') \in \mathbb{E}^S_{self}(\mathcal{E})$ *implies that* $p'_2 > p'_4$

(otherwise $\sigma \notin \arg\max_{B^S_{C_2}(p')} R_3$), in contradiction with budget feasibility. This shows that $E^W_{self}(\mathcal{E}) \not\subseteq E^S_{self}(\mathcal{E})$. A similar argument shows that $\sigma \in E^W_{coop}(\mathcal{E}) \backslash E^S_{coop}(\mathcal{E})$. This proves assertion 1.

Consider $\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\},\{3,4\}\}$, and $\pi \in \Pi_{CR}$ has the form below.

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (2,1) & (2,1) & (4,3) & (4,3) \\ (1,2) & (1,2) & (3,4) & (3,4) \\ \dots & \dots & \dots & \dots \end{array}\right)$$

Let $p = (3,1,3,1)$. It is easily checked that $(\sigma^0, p) \in \mathbb{E}^S_{self}(\mathcal{E})$. Moreover, $(\sigma^0, p') \in \mathbb{E}^W_{self}(\mathcal{E})$ for no price vector p', since this would require $p'_2 + p'_1 > p'_1 + p'_2$. By Proposition 1.1, $\sigma^0 \in E^S_{coop}(\mathcal{E})$, while, similarly to above, $(\sigma^0, p') \in \mathbb{E}^W_{coop}(\mathcal{E})$ implies $p'_2 + p'_1 > p'_1 + p'_2$. This proves assertion 2.

Consider $\mathcal{E} =< 6, \mathcal{C}, \pi >$, where $\mathcal{C} = \{C_1, C_2, C_3\} = \{\{1,2\},\{3,4\},\{5,6\}\}$, and where $\pi \in \Pi_R \cap \Pi_L$ such as below

$$\left(\begin{array}{ccc} 1,2 & 3,4 & 5,6 \\ \hline (3,4) & (1,6) & (1,2) \\ (3,2) & (5,6) & (5,2) \\ (3,5) & (3,6) & (4,2) \\ (3,6) & (4,6) & (3,2) \\ (3,1) & (2,6) & (6,2) \\ (1,4) & (1,4) & (1,6) \\ (1,2) & (5,4) & (5,6) \\ \dots & (3,4) & \dots \\ & \dots & \end{array}\right)$$

Let $\sigma = (3,4,5,6,1,2)$ and $p = (2,2,\frac{3}{2},\frac{5}{2},1,3)$. Since $p_3 + p_4 = p_1 + p_2 = p_5 + p_6$, then $\sigma \in B^W_{C(i)}(p)$ for all i. Moreover, since $p_3 + p_4 < p_1 + p_6$ while each $i \in C_1 \cup C_3$ gets a first-best allocation, then $(\sigma, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$. If $(\sigma, p') \in \mathbb{E}^S_{coop}(\mathcal{E})$ for some p', one must have by strong budget feasibility $p'_1 = p'_3 = p'_5$ and $p'_2 = p'_4 = p'_6$. Moreover, $\sigma \in \mathcal{O}^S_{C_2}(p')$ implies $p'_1 > p'_3$, in contradiction with budget feasibility. Thus $\sigma \in E^W_{coop}(\mathcal{E}) \backslash E^S_{coop}(\mathcal{E})$. Finally, it is easily seen that, since $p_3 + p_4 < p_1 + p_6$, $(\sigma, p) \in \mathbb{E}^W_{self}(\mathcal{E})$. Since strong budget feasibility for any p' requires $p'_1 = p'_3 = p'_5$ and $p'_2 = p'_4 = p'_6$, then any σ' with $\sigma'(C_2) = (1,6)$ belongs to $B^S_{C_2}(p')$. It follows from $\sigma' P_3 \sigma$ that $\sigma \in E^W_{self}(\mathcal{E}) \backslash E^S_{self}(\mathcal{E})$. This proves assertion 3.

Finally, pick $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\},\{3,4\}\}$, *and* $\pi \in \Pi_R \cap \Pi_L$ *such as below*

$$\left(\begin{array}{cc} 1,2 & 3,4 \\ \hline (2,1) & (4,3) \\ (2,3) & (4,2) \\ (2,4) & (4,1) \\ (1,2) & (3,4) \\ \dots & \dots \end{array}\right)$$

It is easy to check that $(\sigma^0, p) \in \mathbb{E}^S_{self}(\mathcal{E})$ *where* $p = (1,2,1,2)$. *By Proposition 1.1,* $(\sigma^0, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$. *Since* $\sigma = (2,1,4,3) \in \cap_{i\in I} B^W_{C(i)}(p')$ *for all price vectors* p', *and since* $\sigma P_i \sigma^0$ *for all* i, *then* $\sigma^0 \notin E^W_{coop}(\mathcal{E})$. *Moreover, we clearly have* $E^W_{self}(\mathcal{E}) = E^W_{coop}(\mathcal{E}) = \{\sigma\}$. *Therefore, again by Proposition 1.1,* $\sigma^0 \notin E^W_{self}(\mathcal{E})$. *This shows assertion 4.*

We end up this section with the comparison between weak and strong coordinated equilibria.

Proposition 2.3.5 *1.* $E^S_{coor}(\mathcal{E}) \subseteq E^W_{coor}(\mathcal{E})$ *for any* $\mathcal{E}$

2. There exists $\mathcal{E} =< N, \mathcal{C}, \pi >$ *with* $\pi \in \Pi_{CRL}$ *such that* $E^S_{coor}(\mathcal{E}) \neq E^W_{coor}(\mathcal{E})$

Proof 2.3.5 *Given any allocation* σ, I *admits a partition* $\{N_1, ..., N_K\}$ *into* K *non-empty sets such that for all* $1 \leq k \leq K$, $N_k = \{i^k_1, ..., i^k_{Z(k)}\}$ *and* $\sigma(i^k_1) = i^k_2$, ... , $\sigma(i^k_z) = i^k_{z+1}$, ... , $\sigma(i^k_{Z(k)-1}) = i^k_{Z(k)}$, *and* $\sigma(i^k_{Z(k)}) = i^k_1$. *Clearly, if* $\sigma \in \cap_{i\in I} B^S_{C(i)}(p)$ *for some price vector* p, *then for all* $1 \leq k \leq K$ *and all* $1 \leq z \leq Z(k)$, $p_{i^k_z} = p_{\sigma(i^k_z)}$. *If* $(\sigma, p) \in \mathbb{E}^S_{coor}(\mathcal{E})$, *strong budget feasibility implies* $p_i = p_{\sigma(i)}$ *for all* $i \in I$. *Then for all* i, *for all* σ' *such that* $\sigma'(j) = \sigma(j)$ *for all* $j \in C(i)\backslash\{i\}$, $\sigma' P_i \sigma$ *implies* $p_{\sigma'(i)} > p_i$ *(*). Now suppose that* $(\sigma, p) \notin \mathbb{E}^W_{coor}(\mathcal{E})$. *Note first that* $\sigma \in \cap_{i\in I} B^W_{C(i)}(p)$. *By definition of a weak coordinated equilibrium, there exists* $i^* \in I$ *and* $\sigma' \in \Sigma$ *with* $\sigma'(i) = \sigma(i)$ *for all* $i \in C(i^*)\backslash\{i^*\}$, *such that* $\sigma' P_{i^*} \sigma$, *and* $\sum_{i\in C(i)} p_{\sigma'(i)} \leq \sum_{i\in C(i)} p_i$. *Since* $p_i = p_{\sigma(i)}$ *for all* $i \in C(i^*)\backslash\{i^*\}$, *we get that* $p_{\sigma'(i^*)} \leq p_{i^*}$, *in contradiction with (*). This proves assertion 1.*

Pick $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\},\{3,4\}\}$, *and where* $\pi \in \Pi_{CRL}$ *has the form below*

$$\left(\begin{array}{cc} 1,2 & 3,4 \\ \hline (4,3) & (4,1) \\ (2,3) & ... \\ ... & \end{array}\right)$$

Consider $\sigma = (2,3,4,1)$ *and* $p = (3,1,3,2)$. *Since* $p_1 + p_4 \leq p_3 + p_4$ *and* $p_2 + p_3 \leq p_1 + p_2$, *then* $\sigma \in \cap_{i\in I} B^W_{C(i)}(p)$. *Moreover,* σ *is a first-best allocation for both* 3 *and* 4, *and thus* $\sigma \in \Phi^W_3(\sigma,p) \cap \Phi^W_4(\sigma,p)$. *Furthermore,* $p_3+p_4 > p_1+p_2$ *ensures that* $\sigma \in \Phi^W_1(\sigma,p) \cap \Phi^W_2(\sigma,p)$. *Therefore,* $(\sigma,p) \in \mathbb{E}^W_{coor}(\mathcal{E})$. *However,* $\sigma \notin E^S_{coor}(\mathcal{E})$. *Indeed, if* $\sigma \in \cap_{i\in I} B^S_{C(i)}(p')$ *for some* p', *then* $p'_4 \leq p'_3 \leq p'_2 \leq p'_1 \leq p'_4$, *and thus* $p'_4 = p'_3 = p'_2 = p'_1$. *This implies that* $\sigma' = (4,3,.,.) \in B^S_{C_1}(\sigma,p')$. *Since* $\sigma' P_1 \sigma$, *then* $\sigma \notin \Phi^W_1(\sigma,p')$. *Thus,* $\sigma \in E^W_{coor}(\mathcal{E}) \backslash E^S_{coor}(\mathcal{E})$, *which proves assertion 2.*

2.3.2 Properties of equilibria

Next, we consider three properties for equilibrium allocations, namely individual rationality, Core stability, and Pareto optimality.

2.3.2.1 Individual rationality

Proposition 2.3.6 *1. There exists* $\mathcal{E}$ *such that* $\pi \in \Pi_{CR}$ *and* $\sigma \in E^S_{coop}(\mathcal{E}) \cap E^S_{coor}(\mathcal{E}) \cap E^W_{coop}(\mathcal{E}) \cap E^W_{coor}(\mathcal{E})$ *is not individually rational.*

2. In any $\mathcal{E}$, $E^S_{self}(\mathcal{E}) \subseteq \Sigma(\sigma^0,\pi)$ *and* $E^W_{self}(\mathcal{E}) \subseteq \Sigma(\sigma^0,\pi)$.

Proof 2.3.6 *Consider* $\mathcal{E} =< 4,\mathcal{C},\pi >$ *where* $\mathcal{C} = \{C_1,C_2\} = \{\{1,2\},\{3,4\}\}$, *and where* π *is any coalition responsive profile having the form*

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (2,1) & (1,2) & (4,3) & (3,4) \\ (1,2) & (2,1) & (3,4) & (4,3) \\ ... & ... & ... & ... \end{array}\right)$$

Then $(\sigma,p) \in \mathbb{E}^S_{coor}(\mathcal{E}) \cap \mathbb{E}^S_{coop}(\mathcal{E}) \cap \mathbb{E}^W_{coop}(\mathcal{E}) \cap \mathbb{E}^W_{coor}(\mathcal{E})$ *where* $\sigma = (2,1,4,3)$ *and* $p = (1,1,1,1)$. *Clearly,* $\sigma^0 P_i \sigma$ *for* $i = 2,4$, *hence assertion 1.*

Pick any $\mathcal{E}$, *and any* $(\sigma,p) \in \mathbb{E}^S_{self}(\mathcal{E}) \cap \mathbb{E}^W_{self}(\mathcal{E})$. *By definition, either* $\sigma \in \arg\max_{B^S_{C(i)}(p)} R_i$ *or* $\sigma \in \arg\max_{B^W_{C(i)}(p)} R_i$, *while* $\sigma^0 \in B^S_{C(i)}(p) \cap B^W_{C(i)}(p)$ *for all* i. *This directly implies that* $\sigma R_i \sigma^0$ *for all* i, *hence assertion 2.*

The possibility of partners having different preferences over allocations plays a crucial role in Proposition 6.1, as pointed by

Proposition 2.3.7 *1. In any $\mathcal{E} =< N, \mathcal{C}, \pi >$ where π is a profile of joint preferences, $E^S_{coop}(\mathcal{E}) \subseteq \Sigma(\sigma^0, \pi)$ and $E^W_{coop}(\mathcal{E}) \subseteq \Sigma(\sigma^0, \pi)$.*

2. There exists $\mathcal{E} =< N, \mathcal{C}, \pi >$ where π is a profile of joint preferences, such that $E^S_{coor}(\mathcal{E}) \nsubseteq \Sigma(\sigma^0, \pi)$ and $E^W_{coor}(\mathcal{E}) \nsubseteq \Sigma(\sigma^0, \pi)$.

Proof 2.3.7 *Let $(\sigma, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$. If there exists $i \in C$ such that $\sigma^0(C) P_i \sigma(C)$, then $\sigma^0(C) P_j \sigma(C)$ for all $j \in C$. Since $\sigma^0 \in B^S_C(p)$, we contradict $\sigma \in \mathcal{O}_C(p)$, hence assertion 1.*

In order to prove assertion 2, consider $\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\}, \{3,4\}\}$, and π is as below

$$\left(\begin{array}{cc} 1,2 & 3,4 \\ \hline (4,1) & (2,3) \\ (1,2) & (3,4) \\ (2,3) & (4,1) \\ \dots & \dots \end{array}\right)$$

Let $\sigma = (2,3,4,1)$ and $p = (1,1,1,1)$. Clearly, $\sigma \in B^S_{C(i)}(p) \cap B^W_{C(i)}(p)$, $\sigma \in \cap_{i \in I} \Phi^S_i(\sigma, p)$, and $\sigma \in \cap_{i \in I} \Phi^W_i(\sigma, p)$. Hence $\sigma \in E^S_{coor}(\mathcal{E}) \cap E^W_{coor}(\mathcal{E})$. Since $\sigma^0 P_i \sigma$ for all $i \in I$, the conclusion follows.

2.3.2.2 Core stability

Pick any $\mathcal{E} =< N, \mathcal{C}, \pi >$. Given $\sigma, \sigma' \in \Sigma$, we define a $(\sigma', \sigma)-$blocking group as a non-empty subset S of I such that $\forall i \in S$, $\sigma' P_i \sigma$ and $\sigma'(i) \in S$.

We define the ***Coalitional Core*** of $\mathcal{E}$ as the subset $\Omega^{Coal}(\mathcal{E})$ of Σ containing all allocations σ for which there is no $(\sigma', \sigma)-$blocking group S for σ with $\forall i \in I$, $i \in S \Rightarrow C(i) \subseteq S$. Hence, Ω^{Coal} does not allow to blocking group to break coalitions.

Whenever breaking coalition can prevail, several concepts of Core can be considered. We focus here on two specific ones.

The ***Core*** of $\mathcal{E}$ is the subset $\Omega(\mathcal{E})$ of Σ containing all allocations σ for which there is no $(\sigma', \sigma)-$blocking group S such that $\forall i \notin S$, $\sigma'(i) = \sigma^0(i)$.

The ***Conservative Core*** of $\mathcal{E}$ is the subset $\Omega^{cons}(\mathcal{E})$ of Σ containing all σ such that if S is a $(\sigma', \sigma)-$ blocking coalition, then S is not a $(\sigma'', \sigma)-$blocking coalition for some σ'' with $\forall i \in S$, $\sigma'(i) = \sigma''(i)$.

Hence, $\Omega(\mathcal{E})$ refers to the standard concept of Core, in which all individuals not involved in a blocking group receive their initial good. Since individual preferences over allocations depend on what partners are endowed with, blocking may no longer be appealing in case partners outside of the blocking group are endowed with some other good. What $\Omega^{cons}(\mathcal{E})$ suggests is that a blocking group will form only if all its members can be better off regardless what individuals outside the group are endowed with. Observe that for any $\mathcal{E}$, $\Omega(\mathcal{E}) \subseteq \Omega^{coal}(\mathcal{E})$ and $\Omega(\mathcal{E}) \subseteq \Omega^{cons}(\mathcal{E})$.

Proposition 2.3.8 *There exists $\mathcal{E}$ with $\pi \in \Pi_{CR}$ such that $E^S_{self}(\mathcal{E}) \nsubseteq \Omega^{coal}(\mathcal{E})$ and $E^S_{self}(\mathcal{E}) \nsubseteq \Omega^{cons}(\mathcal{E})$.*

Proof 2.3.8 *Consider $\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $\mathcal{C} = \{\{1,2\},\{3,4\}\}$, and π is any coalition responsive profile having the form*

$$\left(\begin{array}{cc} 1,2 & 3,4 \\ \hline (2,1) & (4,3) \\ (1,2) & (3,4) \\ \dots & \dots \end{array}\right)$$

Then $(\sigma^0, p) \in \mathbb{E}^S_{self}(\mathcal{E})$ where $p = (2,1,2,1)$. Since I is a $(\sigma', \sigma^0)-$blocking group, with $\sigma' = (2,1,4,3)$, then $\sigma^0 \notin \Omega^{coal}(\mathcal{E}) \cup \Omega^{cons}(\mathcal{E})$.

As an immediate corollary, we get that a strong selfish equilibrium allocation may not be in the Core. Since $E^S_{self}(\mathcal{E}) \subseteq E^S_{coop}(\mathcal{E})$ by Proposition 1, then $E^S_{coop}(\mathcal{E}) \nsubseteq \Omega^{coal}(\mathcal{E})$ and $E^S_{coop}(\mathcal{E}) \nsubseteq \Omega^{cons}(\mathcal{E})$. Another consequence of Proposition 8 combined with Proposition 1 is that strong coordinated equilibrium allocations may be neither in the conservative Core, nor in the coalitional Core. Since $\Omega(\mathcal{E}) \subseteq \Omega^{coal}(\mathcal{E})$, this conclusion also holds for Core stability.

Proposition 2.3.9 *1. $E^W_{coop}(\mathcal{E}) \subseteq \Omega^{coal}(\mathcal{E})$ for all $\mathcal{E}$.*

2. There exists $\mathcal{E}$ such that $E^W_{coop}(\mathcal{E}) \nsubseteq \Omega^{cons}(\mathcal{E})$.

3. $E^W_{self}(\mathcal{E}) \subseteq \Omega(\mathcal{E})$ for all $\mathcal{E}$.

4. There exists $\mathcal{E}$ such that $\pi \in \Pi_{CR}$ and neither $E^W_{coor}(\mathcal{E}) \subseteq \Omega^{coal}(\mathcal{E})$ nor $E^W_{coor}(\mathcal{E}) \subseteq \Omega^{cons}(\mathcal{E})$.

Proof 2.3.9 *Pick $(\sigma, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$, and suppose that $\sigma \notin \Omega^{coal}(\mathcal{E})$. Thus, there exists $S \subseteq N$ and $\sigma' \in \Sigma$ such that $\sigma'(i) \in S$ for all $i \in S$, and $\sigma' P_i \sigma$ for all i with $C(i) \subseteq S$. Then, for any $C \subseteq S$ we have $\sum_{i \in C} p_{\sigma'(i)} > \sum_{i \in C} p_i$. This implies that $\sum_{i \in S} p_{\sigma'(i)} > \sum_{i \in S} p_i$, in contradiction with $\sigma'(i) \in S$ for all $i \in S$. Hence assertion 1.*

Consider $\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $\mathcal{C} = \{\{1,2\},\{3,4\}\}$, and π be such that

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (3,2) & (3,1) & (1,4) & (2,4) \\ (3,4) & \ldots & (1,2) & \ldots \\ (3,1) & & (2,4) & \\ \ldots & & \ldots & \end{array}\right)$$

Then $(\sigma, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$ where $\sigma = (3,1,2,4)$ and $p = (1,1,1,1)$. Check that $S = \{1,3\}$ is a $(\sigma', \sigma)-$blocking coalition for any $\sigma' \in \{(3,2,1,4),(3,4,1,2)\}$. Thus $\sigma \notin \Omega^{cons}(\mathcal{E})$. This proves assertion 2.

Suppose $(\sigma, p) \in \mathbb{E}^W_{self}(\mathcal{E})$ and $\sigma \notin \Omega(\mathcal{E})$. Let $S \subseteq I$ and let $\sigma' \in \Sigma$ such that $\sigma'(i) \in S$ for all $i \in S$ and $\sigma'(i) = i$ for all $i \in I \backslash S$. By construction, $\cup_{i \in S} \sigma'(i) = \cup_{i \in S} \{i\}$ (*). Now, if $\sigma' P_i \sigma$ for all $i \in S$, it follows that $\sum_{j \in C(i)} p_{\sigma'(j)} > \sum_{j \in C(i)} p_i$ for all $i \in S$. Hence $\sum_{i \in S} \sum_{j \in C(i)} p_{\sigma'(j)} > \sum_{i \in S} \sum_{j \in C(i)} p_i$, in contradiction with (*). This proves assertion 3.

For assertion 4, consider the market introduced in the proof of Proposition 8, and observe that $(\sigma^0, p) \in \mathbb{E}^W_{coor}(\mathcal{E})$, $\sigma^0 \notin \Omega^{coal}(\mathcal{E})$ and $\sigma^0 \notin \Omega^{cons}(\mathcal{E})$.

As a corollary of Proposition 9, we get that $E^W_{self}(\mathcal{E}) \subseteq \Omega^{coal}(\mathcal{E}) \cup \Omega^{cons}(\mathcal{E})$ for all $\mathcal{E}$, while we may have $E^W_{coop}(\mathcal{E}) \nsubseteq \Omega(\mathcal{E})$ (resp. $E^W_{coor}(\mathcal{E}) \nsubseteq \Omega(\mathcal{E})$) in some $\mathcal{E}$.

2.3.2.3 Pareto optimality

An allocation σ is Pareto optimal in $\mathcal{E} =< N, \mathcal{C}, \pi >$ if there is no $\sigma' \in \Sigma$ such that $\sigma' R_i \sigma$ for all i with $\sigma' P_j \sigma$ for some j. Since preferences are (coalition selfish) weak orders, we allow individuals to be indifferent between two different bundles assigned to their coalition. Observe that this is compatible with responsiveness. Indeed, the weak order $(1,2)\ P\ (1,3)\ P$

$(2,3)$ I $(3,2)$ P $(3,1)$ can be associated with linear orders $1 \succ 2 \succ 3$ for the 1st coordinate, and $2 \succ 3 \succ 1$ for the second. We show that the existence of indifference plays an important role for the Pareto optimality of equilibrium allocations.[9]

Say that an individual preference R_i is *discriminatory* if $\forall \sigma, \sigma'$ with $\sigma(C(i)) \neq \sigma'(C(i))$, either $\sigma P_i \sigma'$ or $\sigma' P_i \sigma$.

Proposition 2.3.10 *In any $\mathcal{E} =< N, \mathcal{C}, \pi >$ where π is discriminatory, every $\sigma \in E^W_{coop}(\mathcal{E})$ is Pareto optimal.*

Proof 2.3.10 *Let $(\sigma, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$ and suppose σ is not Pareto optimal. Thus, there exist σ' and $S \subseteq I$ such that $\sigma' P_i \sigma$ for all $i \in S$, while all $i \in I \backslash S$ are indifferent between σ and σ'. Since π is discriminatory, $\sigma'(i) = \sigma(i)$ for all $i \in I \backslash S$. Moreover, by definition of a weak cooperative equilibrium, $\sum_{j \in C(i)} p_{\sigma'(j)} > \sum_{j \in C(i)} p_j$ for all $i \in S$. By weak budget feasibility, $\sum_{j \in C(i)} p_j \geq \sum_{j \in C(i)} p_{\sigma(j)}$. Hence, $\sum_{i \in S} \sum_{j \in C(i)} p_{\sigma'(j)} > \sum_{i \in S} \sum_{j \in C(i)} p_{\sigma(j)}$. Since $\sigma'(i) = \sigma(i)$ for all $i \in I \backslash S$, $\sum_{i \notin S} \sum_{j \in C(i)} p_{\sigma'(j)} = \sum_{i \notin S} \sum_{j \in C(i)} p_{\sigma(j)}$. Thus, $\sum_{i \in I} p_{\sigma'(i)} > \sum_{i \in I} p_{\sigma(i)}$, which contradicts the fact that σ' is an allocation.*

Since by proposition 1, $E^W_{self}(\mathcal{E}) \subseteq E^W_{coop}(\mathcal{E})$ for all $\mathcal{E}$, we get as immediate consequence of proposition 10 that in any $\mathcal{E} =< N, \mathcal{C}, \pi >$ where π is discriminatory, $\sigma \in E^W_{self}(\mathcal{E})$ only if σ is Pareto optimal. A weak cooperative equilibrium allocation may be Pareto dominated with non-discriminatory preferences, even under coalition responsiveness, as illustrated in example 3.

Example 2.3.1 *$\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $\mathcal{C} = \{\{1,2\}, \{3,4\}\}$, and such that π is as below :*

9. It is already known that in standard Shapley-Scarf markets with indifference, an equilibrium allocation may be Pareto dominated. See Emmerson (1972), and Roth and Postlewaite (1977).

$$\left(\begin{array}{cc} 1,2 & 3,4 \\ \hline (1,4),(4,3) & (1,2) \\ (2,4) & (3,2) \\ (3,4) & (3,4) \\ (1,3) & (3,1) \\ (2,3) & (3,4) \\ (4,2) & (2,4) \\ (1,2) & (2,1) \\ (3,2) & (2,3) \\ (4,1) & (4,2)) \\ (2,1) & (4,3) \\ (3,1) & (1,4) \\ & (1,3) \end{array}\right)$$

It is easily checked that π is coalition responsive w.r.t. the following preferences over goods :

- *Individual 1 :* $4 \succ_1 1 \succ_1 2 \succ_1 3$

- *Individual 2 :* $4 \succ_2 3 \succ_2 2 \succ_2 1$

- *Individual 3 :* $1 \succ_3 3 \succ_3 2 \succ_3 4$

- *Individual 4 :* $2 \succ_4 4 \succ_4 1 \succ_4 3$

Let $p = (3,4,2,4)$. Since $p_1 + p_2 = p_1 + p_4$ and $p_3 + p_4 = p_3 + p_2$, $\sigma = (1,4,3,2)$ is weak budget feasible for both couples. Moreover, no allocation σ' with $\sigma'(C_2) = (1,2)$ is weak budget feasible for C_2, since $p_1 + p_2 > p_3 + p_4$. Thus σ is a first-best affordable allocation for C_2. Since σ is a first-best allocation for C_1, then $(\sigma, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$. However, σ is Pareto dominated by $\sigma' = (4,3,1,2)$.

To conclude this section, observe that in contrast with Proposition 10, a strong cooperative equilibrium allocation may be Pareto dominated even with discriminatory preferences. To see why, consider the market $\mathcal{E}$ defined in the proof of Proposition 8. We have $(\sigma^0, p) \in \mathbb{E}^S_{self}(\mathcal{E})$ while I is a $(\sigma', \sigma^0)-$blocking coalition, with $\sigma' = (2,1,4,3)$. Therefore, σ^0 is Pareto-dominated. Finally, since $(\sigma^0, p) \in \mathbb{E}^S_{coor}(\mathcal{E}) \cap \mathbb{E}^W_{coor}(\mathcal{E})$, a weak or strong coordinated equilibrium allocation can also be Pareto dominated with discriminatory preferences.

We can summarize the results in the following tables.

Set-comparison of equilibria

For both Strong and Weak	Cooperative	Coordinated
Selfish	$\subseteq$ ($\subsetneq$ even under CR and RL)	$\subseteq$ ($\subsetneq$ even under RL)
Cooperative		$\subsetneq$ under CR, $\not\subseteq$ under RL

Strong Selfish	$\not\subseteq$, $\not\supseteq$ even under CR	Weak Selfish
Strong Cooperative	$\not\subseteq$, $\not\supseteq$ even under CR	Weak Cooperative
Strong Coordinated	$\subseteq$ ($\subsetneq$ even under CR and RL)	Weak Coordinated

Properties of equilibria : Core Stability

	Core	Coalitional Core	Conservative Core
Selfish	Strong : NO Weak : YES	Strong : NO Weak : YES	Strong : NO Weak : YES
Cooperative	Strong : NO Weak : NO	Strong : NO Weak : YES	Strong : NO Weak : NO
Coordinated	Strong : NO Weak : NO	Strong : NO Weak : NO	Strong : NO Weak : NO

Properties of equilibria : Individual rationality (IR), Pareto optimality (PO)

	IR	PO
Selfish	YES	Strong : NO & Weak : YES if preferences are strict
Cooperative	NO YES if preferences are joint	Strong : NO & Weak : YES if preferences are strict
Coordinated	NO	NO

2.3.3 Existence of equilibria

2.3.3.1 Strong equilibria

We start with observing that a Shapley-Scarf market may not have a strong cooperative (resp. coordinated) equilibrium.

Example 2.3.2 *Consider* $\mathcal{E} =< 8, \mathcal{C}, \pi >$ *where* $C_1 = \{1,2\}, C_2 = \{3,4\}, C_3 = \{5,6\}$, $C_4 = \{7,8\}$, *and where* π *is such as below :*

$$\left(\begin{array}{cccc} 1,2 & 3,4 & 5,6 & 7,8 \\ \hline (5,2) & (3,2) & (4,6) & (5,8) \\ (1,4) & (3,7) & (1,6) & (7,8) \\ (1,2) & (3,4) & (5,6) & \dots \\ \dots & \dots & \dots & \end{array}\right)$$

By Proposition 7, $E^S_{coop}(\mathcal{E}) \subseteq \Sigma(\sigma^0, \pi)$. *It is easily checked that* $\Sigma(\sigma^0, \pi)$ *contains the 4 allocations* σ^0, $\sigma^1 = (1,4,3,2,5,6,7,8)$, $\sigma^2 = (5,2,3,4,1,6,7,8)$, *and* $\sigma^3 = (1,2,3,7,4,6,5,8)$. *Suppose that* $(\sigma^0, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$. *Then* $\sigma^0 \in \mathcal{O}^S_{C_2}(p)$ *requires* $p_7 > p_4$, *while* $\sigma^0 \in \mathcal{O}^S_{C_3}(p) \cap \mathcal{O}^S_{C_4}(p)$ *implies* $p_4 > p_5$ *and* $p_5 > p_7$. *Hence* $p_7 > p_4 > p_5 > p_7$, *clearly a contradiction. Similarly,*
- if $(\sigma^1, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$, *then* $\sigma^1 \in \mathcal{O}^S_{C_1}(p) \cap \mathcal{O}^S_{C_3}(p)$ *implies* $p_5 > p_1 > p_5$, *which is impossible.*
- if $(\sigma^2, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$, *then* $\sigma^2 \in \mathcal{O}^S_{C_2}(p) \cap \mathcal{O}^S_{C_3}(p) \cap \mathcal{O}^S_{C_4}$ *implies* $p_7 > p_4 > p_5 > p_7$, *which is impossible*
- if $(\sigma^3, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$, *then* $\sigma^3 \in \mathcal{O}^S_{C_1}(p) \cap \mathcal{O}^S_{C_2}(p)$ *implies* $p_4 > p_2 > p_4$, *again an impossibility.*
Therefore, $E^S_{coop}(\mathcal{E}) = \emptyset$.

Example 2.3.3 *Define* $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *such that* $\mathcal{C} = \{C_1, C_2\}$, $C_1 = \{1,2\}$, $C_2 = \{3,4\}$, *and* π *satisfies*

- $(3,1)R_1(4,1)R_1(2,1)$, $(3,2)R_1(4,2)R_1(1,2)$, $(1,3)R_1(2,3)R_1(4,3)$ *and* $(1,4)R_1(2,4)R_1(3,4)$
- $(1,3)R_2(1,2)R_2(1,4)$, $(2,3)R_2(2,4)R_2(2,1)$, $(3,4)R_2(3,1)R_2(3,2)$ *and* $(4,3)R_2(4,2)R_2(4,1)$
- $(3,x)R_3(y,x)$, $\forall x \in \{2,4\}$, $\forall y \in G\backslash\{3\}$ *and* $(2,1)R_3(x,1)$, $\forall x \in G\backslash\{2\}$.
- $(x,1)R_4(x,y), \forall x \in \{2,3,4\}$, $\forall y \in G\backslash\{1\}$ *and* $(1,4)R_4(1,x), \forall x \in G\backslash\{1,4\}$.

Suppose that $(\sigma, p) \in E^S_{coor}(\mathcal{E})$, *and consider the following cases :*

Case 1 : $\sigma(1) = 1$

If $\sigma(2) = 3$, *then* $3 \notin \{\sigma(3), \sigma(4)\}$. *Pick* σ' *such that* $\sigma'(3) = 3$ *and* $\sigma'(4) = \sigma(4)$. *Since* $(\sigma', p) \in B^S_3(\sigma, p)$ *and* $\sigma' R_3 \sigma$, *we contradict* $\sigma \in \Phi^S_3(\sigma, p)$. *If* $\sigma(2) = 2$, *then the definition of a coordinated equilibrium implies* $p_3 > p_1$ *and* $p_4 > p_1$. *We know that* $1 \notin \{\sigma(3), \sigma(4)\}$. *Pick* σ' *with* $\sigma'(4) = 1$ *and* $\sigma'(3) = \sigma(3)$. *Since* $(\sigma', p) \in B^S_4(\sigma, p)$ *and* $\sigma' R_4 \sigma$, *then* $\sigma \notin \Phi^S_4(\sigma, p)$, *a contradiction. Finally, if* $\sigma(2) = 4$, *pick* σ' *with* $\sigma'(2) = 2$

and $\sigma'(1) = \sigma(1)$. *Since* $(\sigma', p) \in B_2^S(\sigma, p)$ *and* $\sigma' R_2 \sigma$, *we get* $\sigma \notin \Phi_2^S(\sigma, p)$, *again a contradiction. Therefore, Case 1 is impossible.*

Case 2 : $\sigma(1) = 2$

If $\sigma(2) = 3$, *then* $\sigma \in B_1^S(\sigma, p)$ *ensures that* $\sigma' \in B_1^S(\sigma, p)$ *where* σ' *is any allocation with* $\sigma'(1) = 1$ *and* $\sigma'(2) = \sigma(2) = 3$. *Since* $\sigma' P_1 \sigma$, *we contradict* $\sigma \in \Phi_1^S(\sigma, p)$. *If* $\sigma(2) = 4$, *pick* σ' *with* $\sigma'(1) = 1$ *and* $\sigma'(2) = \sigma(2) = 4$. *Again,* $\sigma' \in B_1^S(\sigma, p)$ *and* $\sigma' R_1 \sigma$, *in contradicts with* $\sigma \in \Phi_1^S(\sigma, p)$. *If* $\sigma(2) = 1$, *the equilibrium property imposes that* $p_4 > p_1$ *and* $p_3 > p_1$. *Moreover,* $1 \notin \{\sigma(3), \sigma(4)\}$ *while any allocation* σ' *with* $\sigma'(4) = 1$ *and* $\sigma'(3) = \sigma(3)$ *is strong budget feasible for agent* 4. *Since* $\sigma' R_4 \sigma$, *we contradict* $\sigma \in \Phi_4^S(\sigma, p)$. *Therefore, Case 2 is impossible.*

Case 3 : $\sigma(1) = 3$

Then $3 \notin \{\sigma(3), \sigma(4)\}$ *while* $p_1 \geq p_3$. *If* $\sigma(4) \neq 1$, *then pick* σ' *with* $\sigma'(3) = 3$ *and* $\sigma'(4) = \sigma(4)$. *Since* $\sigma' \in B_3^S(\sigma, p)$ *and* $\sigma' R_3 \sigma$, *we contradict* $\sigma \in \Phi_3^S(\sigma, p)$. *Suppose* $\sigma(4) = 1$. *If* $\sigma(3) = 4$, *then* $\sigma(2) = 2$. *Moreover,* $\sigma \in \Phi_3^S(\sigma, p)$ *requires* $p_2 > p_3$ *and* $\sigma \in \Phi_2^S(\sigma, p)$ *requires* $p_4 > p_2$. *Thus* $p_4 > p_3$, *in contradiction with* $\sigma \in B_3^S(\sigma, p)$. *If* $\sigma(3) = 2$, *then* $\sigma(2) = 4$. *Pick* σ' *with* $\sigma'(1) = 1$ *and* $\sigma'(2) = \sigma(2) = 4$. *Since* $\sigma' \in B_1^S(\sigma, p)$ *and* $\sigma' R_1 \sigma$, *we contradict* $\sigma \in \Phi_1^S(\sigma, p)$. *Therefore, Case 3 is impossible.*

Case 4 : $\sigma(1) = 4$

Budget feasibility implies $p_1 \geq p_4$. *Suppose* $p_1 = p_4$. *Then* $\sigma \in \Phi_4^S(\sigma, p)$ *implies* $\sigma(4) = 1$. *If* $\sigma(2) = 3$ *and* $\sigma(3) = 2$ *then any* σ' *with* $\sigma'(1) = 1$ *and* $\sigma'(2) = \sigma(2) = 3$ *belongs to* $B_1^S(\sigma, p)$, *and* $\sigma' R_1 \sigma$ *contradicts* $\sigma \in \Phi_1^S(\sigma, p)$. *If* $\sigma(2) = 2$ *and* $\sigma(3) = 2$, *then* $\sigma \in \Phi_2^S(\sigma, p)$ *implies* $p_3 > p_2$. *Thus good* 2 *is affordable for individual 3, and this is not compatible with* $\sigma \in \Phi_3^S(\sigma, p)$. *This shows that* $p_1 > p_4$, *which implies that* $\sigma(4) \neq 1$. *This in turn implies that* $\sigma(3) = 3$ *and* $\sigma(2) = 1$, *in contradiction with* $\sigma \in \Phi_2^S(\sigma, p)$. *Thus, Case 4 is impossible. This shows that* $E_{coor}^S(\mathcal{E}) = \varnothing$.

Example 4 involves non-responsive preferences. We show that under coalition responsiveness, the well-known top-trading cycle algorithm always ends up at a strong selfish equilibrium allocation. We briefly recall how the algorithm works. A top-trading cycle (hereafter ttc) sequence $\mathcal{T}^{ttc}$ is a collection of sets $\{T^k\}_{k=1,...,K}$ where

- for all $k \in \{1, ..., K\}$, $T^k = \{i_1^k, ..., i_{M(k)}^k\}$ satisfies $i_{m+1}^k \succ_{i_m^k} i$ and $i_1^k \succ_{i_{M(k)}^k} i$ for all $m = 1, ..., M(k)$ and for all $i \in G_k = G \backslash \bigcup_{1 \leq k' < k} T^{k'}$

- $G_{K+1} = \varnothing$

A ttc allocation σ^{ttc} assigns goods to individuals consistently with a top-trading sequence. Hence, $\sigma^{ttc}(i_m^k) = i_{m+1}^k$ and $\sigma^{ttc}(i_{M(k)}^k) = i_1^k$ for all $k \in \{1, ..., K\}$ and all $m \in \{1, ..., M(k)\}$. A ttc allocation results from a multi-stage trade procedure where each agent is involved in one and only one stage, and is assigned her most preferred good among the goods available at that stage. It is well-known that standard Shapley-Scarf markets without indifference admit a unique ttc allocation.

Proposition 2.3.11 $E_{self}^S(\mathcal{E})$, $E_{coop}^S(\mathcal{E})$ *and* $E_{coor}^S(\mathcal{E})$ *are not empty in any* $\mathcal{E} =< N, \mathcal{C}, \pi >$ *where* $\pi \in \Pi_{CR}$.

Proof 2.3.11 *By definition of* Π_{CR}*, each individual* i *has a linear order* $\succ_i$ *over goods. Pick one ttc sequence* $T = \{T^k\}_{k=1,...,K}$*, and consider the associated ttc allocation* σ^{ttc}*. By construction, either* $\sigma^{ttc}(i) = i$ *or* $\sigma^{ttc}(i) \succ_i i$ *for all* $i \in I$*. By coalition responsiveness,* $\sigma^{ttc} \in \Sigma(\sigma^0, \pi)$*. Define for* $i \in I$ *the number* $k(i) = \{k \in \{1, ..., K\} : i \in T^k\}$*. Moreover, define* $p \in \mathbb{R}^N_+$ *by* $\forall i \in I$*,* $p_i = \frac{1}{k(i)}$*. Suppose that* $(\sigma^{ttc}, p) \notin \mathbb{E}^S_{self}(\mathcal{E})$*. Let* $H = \{i \in I : \sigma^{ttc} \notin \arg\max_{B^S_{C(i)}(p)} R_i\}$*, and let* $i^* \in H$ *with* $k(i^*) \leq k(i)$ *for all* $i \in H$*. Since there exists* $\sigma \in B^S_{C(i^*)}(p)$ *such that* $\sigma P_{i^*} \sigma^{ttc}$ *while* $\pi \in \Pi_{CR}$*, either* $\sigma(i^*) \succ_{i^*} \sigma^{ttc}(i^*)$ *or* $\sigma(j^*) \succ_{j^*} \sigma^{ttc}(j^*)$ *where* $j^* \in C(i^*)$*. If* $\sigma(i^*) \succ_{i^*} \sigma^{ttc}(i^*)$ *then* $\sigma(i^*) \notin G_{k(i^*)}$*. This implies that* $p_{\sigma(i^*)} > \frac{1}{k(i^*)} = p_{i^*}$*, and thus* $\sigma \notin B^S_{iC(i^*)}(p)$*, a contradiction. Similarly, if* $\sigma(j^*) \succ_{j^*} \sigma(j^*)$*, then* $\sigma(j^*) \notin G_{k(j^*)}$*, so that* $p_{\sigma(j^*)} > \frac{1}{k(j^*)} = p_{j^*}$*, which contradicts* $\sigma \in B^S_{C(i^*)}(p)$*. Therefore,* $(\sigma^{ttc}, p) \in \mathbb{E}^S_{self}(\mathcal{E})$*. By Proposition 1, we deduce that* $E^S_{coop}(\mathcal{E}) \neq \varnothing$ *and* $E^S_{coor}(\mathcal{E}) \neq \varnothing$*.*

Observe that responsiveness does not ensure the existence of strong selfish equilibria. For instance, consider $\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $\mathcal{C} = \{\{1, 2\}, \{3, 4\}\}$ and π is a responsive profile such as below

$$\left(\begin{array}{ccc} 1 & 2 & 3,4 \\ \hline (3,2) & (1,3) & (1,4) \\ (1,3) & (1,2) & (3,4) \\ (1,2) & ... & ... \\ ... & & \end{array}\right)$$

Note that $\Sigma(\sigma^0, p) = \{\sigma^0\}$, while $(\sigma^0, p) \in \mathbb{E}^S_{self}(\mathcal{E})$ implies $p_1 > p_3 > p_1$, which is impossible. By Proposition 6.2, we get $E^S_{self}(\mathcal{E}) = \varnothing$.

It is well-known that standard Shapley-Scarf market without indifference have a unique competitive equilibrium allocation, which is the outcome of a top-trading sequence. This is no longer true with coalitional trade, as shown by the following example.

Example 2.3.4 *Define* $\mathcal{E} =< 6, \mathcal{C}, \pi >$ *by* $\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$*, and*

$$\pi = \left(\begin{array}{ccc} 1,2 & 3,4 & 5,6 \\ \hline (2,3) & (4,5) & (6,1) \\ (3,2) & (5,4) & (1,6) \\ (1,3) & (3,5) & (5,1) \\ (1,2) & (3,4) & (5,6) \\ ... & ... & ... \end{array}\right)$$

where π *can be made coalition responsive w.r.t. linear orders over goods*

- $i = 1, 2$ *:* $3 \succ_i 2 \succ_i 1 \succ_i ...$
- $i = 3, 4$ *:* $5 \succ_i 4 \succ_i 3 \succ_i ...$
- $i = 5, 6$ *:* $1 \succ_i 6 \succ_i 5 \succ_i ...$

The ttc allocation is $\sigma^{ttc} = (3, 2, 5, 4, 1, 6)$*, and* $(\sigma^{ttc}, p) \in \mathbb{E}^S_{self}(\mathcal{E})$ *with* $p = (2, 1, 2, 1)$*. Now* $(\sigma, p) \in \mathbb{E}^S_{self}(\mathcal{E})$ *with* $\sigma = (2, 3, 4, 5, 6, 1)$ *and* $p = (1, 1, 1, 1, 1, 1)$*. Observe that* σ^{ttc} *is not Pareto optimal, since* $\sigma P_i \sigma^{ttc}$ *for all* i*.*

2.3.3.2 Weak equilibria

Under weak budget feasibility, coalition responsiveness no longer ensures the existence of selfish equilibria, as shown in the following example.

Example 2.3.5 *Consider* $\mathcal{E} =< 8, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$*,* π *is the joint and responsive (hence coalition responsive) profile below :*

$$\left(\begin{array}{cccc} 1,2 & 3,4 & 5,6 & 7,8 \\ \hline (7,2) & (2,4) & (3,6) & (7,4) \\ (7,3) & (2,1) & (8,6) & (7,5) \\ (1,2) & (5,4) & (3,8) & (8,4) \\ \ldots & (3,4) & (5,6) & (7,8) \\ & \ldots & \ldots & \ldots \end{array}\right)$$

By Proposition 6, $E^W_{self}(\mathcal{E}) \subseteq \Sigma(\sigma^0, \pi)$*. Moreover,* $\Sigma(\sigma^0, \pi)$ *contains the 4 allocations* σ^0*,* $\sigma^1 = (7, 3, 2, 1, 5, 6, 8, 4)$*,* $\sigma^2 = (1, 2, 5, 4, 3, 6, 7, 8)$*, and* $\sigma^3 = (1, 2, 3, 4, 8, 6, 7, 5)$*. If* $(\sigma^1, p) \in \mathbb{E}^W_{self}(\mathcal{E})$*, then one must have* $p_8 + p_6 > p_5 + p_6$ *and* $p_7 + p_5 > p_7 + p_8$*, which is impossible. Similarly,*

- *if* $(\sigma^2, p) \in \mathbb{E}^W_{self}(\mathcal{E})$*, one must have* $p_8 + p_4 > p_7 + p_8$ *and* $p_3 + p_7 > p_1 + p_2 > p_3 + p_4$
- *if* $(\sigma^3, p) \in \mathbb{E}^W_{self}(\mathcal{E})$*, then* $p_5 > p_3 > p_5$
- *if* $(\sigma^0, p) \in \mathbb{E}^W_{self}(\mathcal{E})$*, then* $p_3 > p_5 > p_3$

All situations being impossible, we conclude that $E^W_{self}(\mathcal{E}) = \varnothing$*.*

We show the existence of weak cooperative equilibria under the assumption of weak lexicographic preferences. Since weak cooperative equilibrium allocations belong to the coalitional Core (by proposition 9.1), we strengthen the main result of Doğan et al. (2011), which prove the non-emptiness of the coalitional Core under lexicographic joint preferences. The proof is given for a coalition structure involving couples, and, at the cost of additional notations, it can be adapted to any coalition structure. It is based on a modified version of the ttc algorithm. Given any i, we denote $C(i) = \{i, c(i)\}$. Pick any weak lexicographic profile π together with any non-empty subset J of goods. We define a J-ttc w.r.t. allocation σ as an ordered subset of individuals $T(J, \sigma) = \{i_k\}_{1 \leq k \leq K}$ such that :

— For all $1 \leq k \leq K - 1$, $\sigma(i_{k+1}) = \gamma_{i_k}(J)$, and $\gamma_{i_K}(J) \in \{\sigma(i_1), \sigma(c(i_1))\}$
— For all $1 \leq k \leq K$, $C(i_k) \cap T(J, \sigma) = \{i_k\}$

A J-ttc $T(J,\sigma)$ involves at most one partner in each couple. Individual in $T(J,\sigma)$ point to the individual endowed by σ with the (necessarily unique) priority good in J she wishes either for herself or for her partner. Moreover, the last individual i_K points either to i_1 or to $c(i_1)$. Clearly, one can always find a path $i_1 \to i_2 \to ... \to i_K$ such that either i_1 or $c(i_1)$ has priority good for individual i_K, and such that the priority good for each individual i_k, where $1 \leq k < K$, is the one owned by neither $i_{k'}$ nor $c(i_{k'})$ with $k' < k$. Therefore, a $J-$ttc exists for all J. Note that a $J-$ttc is a singleton $\{i\}$ if either $\sigma(i) = \gamma_i(J)$ or $\sigma(c(i)) = \gamma_i(J)$.

Given any J-ttc $T(J,\sigma)$ w.r.t. σ, the allocation $\sigma^{T(J,\sigma)}$ is defined by :

(1) $\forall C \in \mathcal{C}$ such that $C \cap T(J,\sigma) = \varnothing$, $\sigma^{T(J,\sigma)}(C) = \sigma(C)$

(2) For all $k \in \{1,...,K\}$, $\sigma^{T(J,\sigma)}(i_k) = \gamma_{i_k}(J)$

(3) For all $k \in \{1,...,K-1\}$, $\sigma^{T(J,\sigma)}(c(i_k)) = \sigma(c(i_k))$

(4) $\sigma^{T(J,\sigma)}(c(i_1)) = \left\{ \begin{array}{l} \sigma(i_1) \text{ if } \sigma(c(i_1)) = \sigma(c(i_1)) \\ \sigma(c(i_1)) \text{ if } \sigma(c(i_1)) = \sigma(i_1) \end{array} \right\}$

The allocation $\sigma^{T(J,\sigma)}$ endows individuals in $T(J,\sigma)$ with the priority good they each point to, and their partners, maybe except the first one, keeps their current good. The partner $c(i_1)$ of the first individual i_1 keeps also her current good if i_K gets the current good $\sigma(i_1)$ of i_1, and gets her partner's good $\sigma(i_1)$ if i_K gets $\sigma(c(i_1)$. For an illustration, consider the following example.

Example 2.3.6 *Consider* $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{\{1,2\},\{3,4\}\}$*, and where* π *is the weak lexicographic profile below*

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (4,3) & (3,1) & (2,1) & (1,2) \\ (2,4) & (2,3) & (3,2) & (3,2) \\ (4,1) & (3,4) & (2,4) & (4,2) \\ (3,1) & (2,4) & (1,3) & (4,1) \\ (2,3) & (4,1) & (4,1) & (3,4) \\ (2,1) & (1,2) & (2,1) & ... \\ ... & ... & ... & \end{array} \right)$$

There are two G-ttc w.r.t. σ^0, $T(G,\sigma^0) = \{1,4\}$ and $T'(G,\sigma^0) = \{2,3\}$, respectively leading to $\sigma^{T(G,\sigma^0)} = (4,1,3,2)$ and $\sigma^{T'(G,\sigma^0)} = (1,3,2,4)$. Note that the set of goods traded through a J-ttc may not coincide with the set of involved individuals : goods 2 and 4 are traded through $T(G,\sigma^0)$, which involves individuals 1 and 4. This only happens when the last individual in the cycle points to the good owned by the partner of the first individual in the cycle.

The modified ttc algorithm consists in forming successive J-ttc, starting with $J = G$, and continuing with nested subsets of goods and individuals, some goods being definitely assigned on the way. Formally, the algorithm operates as follows :

Stage 1 : Form a G-ttc T^1w.r.t. σ^0, and define $\sigma^1 = \sigma^{T^1(G,\sigma^0)}$. Then remove from G all priority goods involved in T^1, that is the set $L^1 = \{x \in G : x = \gamma_i(G)$ for some $i \in T^1\}$, and define $G^2 = G \backslash L^1$. Remove from I all individuals in T^1, and define $I^2 = I \backslash T^1$. Clearly, $|G^2| = N - |T^1| = |I^2|$. For each $i \in I^2$ with $c(i) \in T^1$, we adopt the following convention : $\gamma_i(G^2)$ is a good $x \in G^2$ which maximizes $R_{c(i)}$ in the subset of allocations $\Sigma_2 = \{\sigma \in \Sigma : \sigma(\{i, c(i)\}) \in \{(y, \sigma^1(c(i)), (\sigma^1(c(i), y))\}$ with $y \in G^2\}$. In case of indifference in $R_{c(i)}$, i uses her own preferences as tie-breaking criterion, or any arbitrary one in case i is also indifferent between several goods.

...

Stage s : Define $G^s = G \backslash \cup_{1 \leq s' \leq s-1} L^{s'}$ and $I^s = I \backslash \cup_{1 \leq s' \leq s-1} T^{s'}$. Define allocation σ^{s-1} by : for all $C \in \mathcal{C}$,

- If $C \subseteq I^s$, then $\sigma^{s-1}(C) = \sigma^0(C) = (i, c(i))$
- If $C \cap I^s \neq \varnothing$ and $C \cap T^{s'} \neq \varnothing$ for some $1 \leq s' \leq s-1$, then $\sigma^{s-1}(C) = \sigma^{s'}(C)$
- If $C \cap T^{s'} \cap T^{s''} \neq \varnothing$ for some $1 \leq s'' < s' \leq s-1$, then $\sigma^{s-1}(C) = \sigma^{s'}(C)$

Form a G^s-ttc $T^s \subseteq I^s$ w.r.t. σ^{s-1} , and define $\sigma^s = \sigma^{T^s(G^s, \sigma^{s-1})}$. Remove from G^s all priority goods in T^s, that is the set $L^s = \{x \in G^s : x = \gamma_i(G^s)$ for some $i \in T^s\}$, and define $G^{s+1} = G^s \backslash L^s$, $I^{s+1} = I^s \backslash T^s$. Clearly, $|G^{s+1}| = |G^s| - |T^s| = |I^{s+1}|$. Finally, for each $i \in I^{s+1}$ with $c(i) \in T^s$, define $\gamma_i(G^s)$ as a good $x \in G^s$ such that either $(g, \sigma^s(c(i))$ or $(\sigma^s(c(i), g))$ which maximizes $R_{c(i)}$ in $\Sigma_s = \{\sigma \in \Sigma : \sigma(\{i, c(i)\}) \in \{(y, \sigma^s(c(i)), (\sigma^s(c(i), y))$ with $y \in G^{s+1}\}$. In case of indifference, i uses her own preferences as tie-breaking criterion, or any arbitrary one in case i is also indifferent between several goods.

...

We proceed to the next step $s+1$ as long as $|G^{s+1}| > 0$. Since G is finite, and since at each step, at least one individual receives her final good, the algorithm terminates in $S \leq N$ steps. The final outcome σ^S is defined as follows : $\forall C = \{i, c(i)\}$ with $c(i) \in T^s$, $i \in T^{s'}$, and $s < s'$,

- $\sigma^S(C) = (\sigma^{s'}(i), \sigma^s(c(i)))$ if $\sigma^{s'}(C)$ maximizes $R_{c(i)}$ in $\Sigma_{s'}$
- $\sigma^S(C) = (\sigma^s(c(i)), \sigma^{s'}(i))$ if $(\sigma^s(c(i)), \sigma^{s'}(i))$ maximizes $R_{c(i)}$ in $\Sigma_{s'}$

The modified ttc algorithm works as follows. At each stage s, exactly one G^s-ttc T^s is formed, where G^s is the set of goods available for trade. Each individual i in T^s gets her priority good in restriction to G^s. Her partner keeps her current good, except maybe the partner of the first individual, who may get her partner's good. All assigned priority goods are removed from the set G^s, and are no longer traded throughout the algorithm, and members of T^s are removed from the current set of individuals. Moreover, the partner first involved in some cycle dictates her preference to her partner in all subsequent stages. This preference resumes to comparing allocations that keeps the priority good assigned to the couple (only in case of indifference are partner's preferences used as tie-breaking rule). Thus, every couple is involved in two trading cycles, one for each partner. Observe that the allocation of a priority good at some stage may not be the final one. Indeed, both priority goods are finally assigned

to the couple so as to maximize the well-being of the one first involved in some ttc.

Consider again example 8, showing the existence of two G-ttc w.r.t. σ^0. Pick $T^1 = \{1,4\}$, leading to allocation $\sigma^1 = (4,1,3,2)$. Then, at the next stage, $I^2 = \{2,3\}$, $G^2 = \{1,3\}$ and individual 2 considers all allocations that assign good 4 either to individual 1 or to herself, and good 2 to the other couple. Then individual 2 points to good 3, which makes individual 1 better off than with good 1. Similarly, individual 3 points to good 1, leading to the G^2-ttc $T^2 = \{2,3\}$. Since keeping good 4 (resp. 2) is the best option for individual 1 (resp. 4), the final outcome is $\sigma^S = (4,3,1,2) = \sigma^2$. If instead $T'^1 = \{2,3\}$ is formed, then $I^2 = G^2 = \{1,4\}$, and $\sigma^1 = (1,3,2,4)$. Individuals 1 and 4 both point to good 1, leading to $T'^2 = \{1\}$ and $\sigma'^2 = (1,3,2,4)$. Since getting good 1 (resp. keeping good 2) is the best option for individual 2 (resp. 3), the final outcome is $\sigma'^S = (3,1,2,4)$.

Proposition 2.3.12 *In all $\mathcal{E} =< N, \mathcal{C}, \pi >$ with $\pi \in \Pi_{WL}$, a final outcome of the modified ttc algorithm is a weak cooperative equilibrium allocation.*

Proof 2.3.12 *Define the price vector p by : for all $x \in G$, $p_x = \frac{1}{2^s}$ if and only if $x \in L^s$, where $1 \leq s \leq S$. First, observe that $p_i + p_{c(i)} = p_{\sigma^S(i)} + p_{\sigma^S(c(i))}$ for all i : σ^S makes all weak budget constraints binding. To see why, pick any $C = \{i, c(i)\}$, and suppose w.l.o.g. that $c(i) \in T^s$ and $i \in T^{s'}$, with $s < s'$. Suppose first that $\sigma^s(i) = i$. By definition of p, $p_{c(i)} = p_{\sigma^s(c(i))} = \frac{1}{2^s}$, and $p_i = p_{\sigma^{s'}(i)} = \frac{1}{2^{s'}}$. Since $\sigma^S(C) \in \{(\sigma^{s'}(i), \sigma^s(c(i)), (\sigma^s(c(i), \sigma^{s'}(i))$, then $p_i + p_{c(i)} = p_{\sigma^S(i)} + p_{\sigma^S(c(i))} = \frac{1}{2^s} + \frac{1}{2^{s'}}$. Similarly, suppose that $\sigma^s(i) = c(i)$. Since $i \in L^s$, then $p_i = p_{\sigma^s(c(i))} = \frac{1}{2^s}$, and $p_{c(i)} = p_{\sigma^{s'}(i)} = \frac{1}{2^{s'}}$, and the same conclusion follows.*

Suppose that $(\sigma^S, p) \notin \mathbb{E}^W_{coop}(\mathcal{E})$. Thus, there exist i and an allocation $\sigma \in B^W_{C(i)}(p)$ such that either $[\sigma P_i \sigma^S$ and $\sigma R_{c(i)} \sigma^S]$ or $[\sigma R_i \sigma^S$ and $\sigma P_{c(i)} \sigma^S]$. Suppose that $C(i)$ is such that $i \in T^s$ and $c(i) \in T^{s'}$, with $s' > s$. By construction, $\gamma_i(G^s) \in \{\sigma^S(i), \sigma^S(c(i))\}$. Since preferences are weak lexicographic, $\sigma R_i \sigma^S$ implies that either $\sigma^S(i) \in \{\sigma(i), \sigma(c(i)\}$ or there exists $x \notin G^s$ with $x \in \{\sigma(i), \sigma(c(i)\}$. In the latter case, we get $p_x \geq \frac{1}{2^{s-1}} \geq \frac{1}{2^s} + \frac{1}{2^{s'}} = p_i + p_{c(i)}$, in contradiction with $\sigma \in B^W_{C(i)}(p)$. In the former case, since $c(i)$ gets at stage s' the priority good in $G^{s'}$ according to i's preference (given that i already owns $\gamma_i(G^s)$), and since indifferences are eventually broken in favor of the preference of $c(i)$, there must exist $x' \notin G^{s'}$ with $x' \in \{\sigma(i), \sigma(c(i)\}$. Thus, $p_{\sigma(i)} + p_{\sigma(c(i))} \geq \frac{1}{2^s} + \frac{1}{2^{s'-1}} > \frac{1}{2^s} + \frac{1}{2^{s'}} = p_i + p_{c(i)}$, again in contradiction with $\sigma \in B^W_{C(i)}(p)$. This completes the proof.

As for strong selfish equilibria, a weak cooperative equilibrium allocation is not necessarily a final outcome of the modified ttc algorithm (under weak lexicographic preferences). This is illustrated by the following example.

Example 2.3.7 *Let $\mathcal{E} =< 6, \mathcal{C}, \pi >$ where $\mathcal{C} = \{\{1,2\}, \{3,4\}, \{5,6\}\}$, and where π is a (weak) lexicographic profile of joint preferences such as below*

$$\left(\begin{array}{ccc} 1,2 & 3,4 & 5,6 \\ \hline (3,2) & (2,4) & (1,6) \\ (3,5) & (1,4) & (5,6) \\ (3,1) & (3,4) & (.,6) \\ (1,2) & ... & ... \\ (1,.) & & \\ ... & & \end{array}\right)$$

Since $\gamma_3(G) = \gamma_4(G) = 4$ and $\gamma_5(G) = \gamma_6(G) = 6$, the first two stages of the modified ttc algorithm leads to removing goods 4 and 6, and, w.l.o.g. individuals 4 and 6, while others are staying at the initial allocation σ^0. Let $G^2 = \{1,2,3,5,6\}$, and $G^3 = \{1,2,3,5\}$. There is a unique G^3-ttc $T = \{1,3\}$, leading to allocation $\sigma^3 = (3,1,2,4,5,6)$, $G^4 = \{1,5\}$, and $I^4 = \{2,5\}$. The final outcome of the algorithm is $\sigma^4 = \sigma^S = (3,5,2,4,1,6)$. Now consider $\sigma = (3,2,1,4,5,6)$ and $p = (2,3,2,2,1,1)$. It is straightforward to check that $(\sigma,p) \in \mathbb{E}^W_{coop}(\mathcal{E})$, while σ cannot be generated by means of a sequence of modified ttc.

We turn now to the existence of weak coordinated equilibria. First, Shapley-Scarf markets may not admit such equilibria, as shown in the following example.

Example 2.3.8 *Consider $\mathcal{E} =< 4, \mathcal{C}, \pi >$ where $C_1 = \{1,2\}$, $C_2 = \{3,4\}$, and π is described below (two bundles appearing in the same cell being arbitrarily ranked) :*

$$\left(\begin{array}{ccc} 1 & 2 & 3,4 \\ \hline (3,2);(2,3) & (3,2);(2,3) & (3,1);(1,3) \\ (3,4);(4,3) & (3,4);(4,3) & (2,4);(4,2) \\ (1,3);(4,1) & (1,2);(2,1) & (1,2);(2,1) \\ (1,2);(2,1) & ... & (3,4);(4,3) \\ ... & & ... \end{array}\right)$$

Let $(\sigma,p) \in \mathbb{E}^W_{coor}(\mathcal{E})$, and consider all possible allocations :

Suppose first that $\sigma(C_1) \in \{(3,4),(4,3)\}$. Then $\sigma \in \Phi^W_1(\sigma,p) \cap \Phi^W_2(\sigma,p)$ implies $p_3 > p_1$. Indeed, suppose $p_1 \leq p_3$. If $\sigma(1) = 3$, then any allocation σ' with $\sigma'(C_1) = (3,2)$ is weak budget feasible while $\sigma' P_2 \sigma$, and $\sigma(1) = 4$, any allocation σ' with $\sigma'(C_1) = (2,3)$ is weak budget feasible while $\sigma' P_1 \sigma$. Similarly, $\sigma \in \Phi^W_3(\sigma,p) \cap \Phi^W_4(\sigma,p)$ implies $p_2 > p_3$ and $p_1 > p_4$. It follows that $p_2 > p_3 > p_1 > p_4$, which implies $p_1 + p_2 > p_3 + p_4$. Therefore, $\sigma \notin B^W_{C_2}(p)$, a contradiction.

Suppose $\sigma(1,2) \in \{(3,1),(1,3)\}$. If $\sigma(C_1) = (1,3)$, then $\sigma^0 \in B^W_{C_1}(p)$ and $\sigma^0 P_2 \sigma$ contradicts $\sigma \in \Phi^W_2(\sigma,p)$. If $\sigma(C_1) = (3,1)$, then $\sigma' \in B^W_{C_1}(p)$ and $\sigma' P_2 \sigma$ contradicts $\sigma \in \Phi^W_1(\sigma,p)$, where σ' is any allocation with $\sigma'(C_1) = (2,1)$.

Suppose $\sigma(1,2) \in \{(2,3),(3,2)\}$. If $\sigma(C_2) = (1,4)$, then $\sigma^0 \in B^W_{C_2}(p)$ and $\sigma^0 P_3 \sigma$ contradicts $\sigma \in \Phi^W_3(\sigma,p)$. If $\sigma(C_2) = (4,3)$, then $\sigma' \in B^W_{C_2}(p)$ and $\sigma' P_4 \sigma$ contradicts $\sigma \in \Phi^W_4(\sigma,p)$, where σ' is any allocation with $\sigma'(C_1) = (4,1)$.

Suppose $\sigma(1,2) \in \{(2,4),(4,2)\}$. *If* $\sigma(C_1) = (4,2)$, *then* $\sigma^0 \in B^W_{C_1}(p)$ *and* $\sigma^0 P_1 \sigma$ *contradicts* $\sigma \in \Phi^W_1(\sigma,p)$. *If* $\sigma(C_2) = (2,4)$, *then* $\sigma' \in B^W_{C_1}(p)$ *and* $\sigma' P_2 \sigma$ *contradicts* $\sigma \in \Phi^W_2(\sigma,p)$, *where* σ' *is any allocation with* $\sigma'(C_1) = (2,1)$.

Suppose $\sigma(1,2) \in \{(1,4),(4,1)\}$. *If* $\sigma(C_2) = (3,2)$, *then* $\sigma^0 \in B^W_{C_2}(p)$ *and* $\sigma^0 P_4 \sigma$ *contradicts* $\sigma \in \Phi^W_4(\sigma,p)$. *If* $\sigma(C_2) = (2,3)$, *then* $\sigma' \in B^W_{C_2}(p)$ *and* $\sigma' P_3 \sigma$ *contradicts* $\sigma \in \Phi^W_3(\sigma,p)$, *where* σ' *is any allocation with* $\sigma'(C_1) = (4,3)$.

Finally, suppose $\sigma(1,2) \in \{(1,2),(2,1)\}$. *If* $\sigma(C_1) = (1,2)$, $\sigma \in \Phi^W_1(\sigma,p)$ *implies* $p_3 > p_1$. *If* $\sigma(3,4) = (3,4)$, $\sigma \in \Phi^W_3(\sigma,p)$ *implies* $p_2 > p_3$. *If* $\sigma(3,4) = (4,3)$, *we also get* $p_2 > p_3$ *from* $\sigma \in \Phi^W_4(\sigma,p)$. *Thus,* $p_2 > p_3 > p_2$, *which is impossible. If* $\sigma(C_1) = (2,1)$, $\sigma \in \Phi^W_1(\sigma,p) \cap \Phi^W_2(\sigma,p)$ *implies* $p_3 > p_1$ *and* $p_4 > p_2$. *Furthermore,* $\sigma \in \Phi^W_3(\sigma,p) \cap \Phi^W_4(\sigma,p)$ *implies* $p_2 > p_3$. *Thus,* $p_2 > p_3 > p_1 > p_4 > p_2$, *which is impossible.*

As an immediate consequence of Proposition 1.3 combined with Proposition 12, we get a sufficient condition for the existence of weak coordinated equilibria :

Proposition 2.3.13 *In all* $\mathcal{E} =< N, \mathcal{C}, \pi >$ *where* $\pi \in \Pi_{WL} \cap \Pi_{CR}$, *the modified ttc algorithm terminates at a weak coordinated equilibrium allocation.*

Finally, a weak selfish equilibrium may fail to exist under (weak) lexicographic preferences, as shown by the following example.

Example 2.3.9 *Let* $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $C_1 = \{1,2\}$, $C_2 = \{3,4\}$,*and* π *is the lexicographic profile below*

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline (2,1) & (1,2) & (4,3) & (3,4) \\ (2,3) & \dots & (4,2) & \dots \\ (2,4) & & (4,1) & \\ (1,2) & & (3,4) & \\ \dots & & \dots & \end{array}\right)$$

By Proposition 6.2, $E^W_{self}(\mathcal{E}) \subseteq \Sigma(\sigma^0,\pi)$, *while* $\Sigma(\sigma^0,\pi)$ *has* σ^0 *as unique element. However,* $\sigma^0 \notin E^W_{self}(\mathcal{E})$. *Indeed, if* $(1,2) \in B^W_{C_1}(p)$ *for some price vector* p, *then* $(2,1) \in B^W_{C_1}(p)$ *while* $(2,1)\ P_1\ (1,2)$. *Therefore* $(1,2) \notin \arg\max_{B^W_{C_1}(p)} R_1$, *which shows that* $E^W_{self}(\mathcal{E}) = \varnothing$.

2.3.4 Maximal domains

We have shown the existence of strong selfish (resp. weak cooperative) equilibria under coalition responsive (resp. weakly lexicographic) preferences. An interesting question is whether these domains are maximal. A preference domain $D \subset \Pi$ is maximal for some property α if α is satisfied for all profiles selected from D, and is violated

when D is enlarged with any preference not in D. A related, although not equivalent, approach is followed by Doğan et al. (2011) for Shapley-Scarf markets with joint and discriminatory preferences. Given two linear orders P, P' over Σ, define the lexicographic index of P by $L(P) = \min\{d_K(P, P') : P' \text{ is lexicographic}\}$, where d_K stands for the Kemeny distance. Hence, the lexicographic index of P is the minimal number of pairwise comparisons of allocations to be reverted for the lexicographic property to hold. The minimal departure from the lexicographic property is associated with a lexicographic index 1. Doğan et al. (2011, Proposition 3 page 65) prove that the coalitional Core (and hence, by Proposition 9.1, the set of weakly cooperative equilibria) may be empty in some market where all preferences but one are lexicographic, and where the non-lexicographic preference has an index 1. Now consider profile π below involving 2 couples :

$$\left(\begin{array}{cc} 1,2 & 3,4 \\ \hline (2,3) & (4,3) \\ (4,3) & (4,2) \\ (3,1) & (4,1) \\ (1,3) & (1,3) \\ (4,1) & (2,3) \\ (2,1) & (3,4) \\ (1,2) & ... \\ ... & \end{array}\right)$$

Clearly, preferences of 1 and 2 have lexicographic index 1, while preferences of 3 and 4 are lexicographic. Moreover, preferences of 1 and 2 are weak lexicographic. By Proposition 12, there exists a weak cooperative equilibrium. And by Proposition 9.1 the coalitional Core is non-empty. This implies that the domain of lexicographic preferences is not maximal for the existence of weak cooperative equilibria (and coalitional Core). However, Proposition 14 shows maximality holds for weak lexicographic preferences if there are enough goods (see Appendix 5.1 for the proof). .

Proposition 2.3.14 *If $N \geq 10$, Π_{WL} is maximal for the existence of weak cooperative equilibria.*

We end up this section with showing that the coalition responsive domain is maximal for the existence of strong selfish equilibria. The proof of Proposition 15 is postponed to Appendix 5.2.

Proposition 2.3.15 *If $N \geq 8$, then Π_{CR} is maximal for the existence of strong selfish equilibria.*

2.4 Further comments

We conclude this paper with three additional comments.

(1) Proposition 11 states that the standard ttc algorithm always terminates at a strong selfish equilibrium when preferences are coalition responsive. Roth (1982) was the first to show that, in the original Shapley-Scarf market, the ttc algorithm defines a strategy-proof allocation mechanism. This is no longer the case with coalitional trade, even in the case of joint preferences, as shown by the following example.

Example 2.4.1 *Let* $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $C_1 = \{1, 2\}$, $C_2 = \{3, 4\}$, *and*

$$\pi = \begin{pmatrix} 1,2 & 3,4 \\ \hline (4,3) & (3,1) \\ (2,3) & (3,4) \\ (2,4) & ... \\ (4,2) & \\ ... & \end{pmatrix}$$

Clearly, one may complete π *so as to ensure coalition responsiveness (with linear orders over goods such as* $4 \succ_1 2 \succ_1 1 \succ_1 3$, $3 \succ_2 4 \succ_2 2 \succ_2 1$, $3 \succ_3 ...$, *and* $1 \succ_4 4 \succ_4 ...$. *The outcome of the ttc algorithm is* $\sigma^* = (4, 2, 3, 1)$. *If individual 1 reports any (coalition responsive) preference with* $(2, 3)$ *at top instead of her true preference, the ttc outcome becomes* $\sigma = (2, 4, 3, 1)$, *and misrepresenting is worthwhile since* $\sigma P_1 \sigma^*$.

Proposition 12 states that the modified ttc algorithm ends up at a weak cooperative equilibrium allocation if preferences are weakly lexicographic. A simple example shows that, as for the standard ttc algorithm, the modified one fails at satisfying strategy-proofness.

Example 2.4.2 *Let* $\mathcal{E} =< 8, \mathcal{C}, \pi >$ *where* $C_1 = \{1, 2\}, C_2 = \{3, 4\}, C_3 = \{5, 6\}, C_4 = \{7, 8\}\}$, *and*

$$\pi = \begin{pmatrix} 1,2 & 3,4 & 5,6 & 7,8 \\ \hline (5,3) & (1,4) & (1,6) & (3,8) \\ (.,3) & (1,.) & (5,6) & (7,8) \\ (1,2) & (2,.) & ... & ... \\ ... & (3,4) & ... & ... \end{pmatrix}$$

Clearly, π *can be completed in order to satisfy the weak lexicographic property. In a first stage of the modified algorithm, one can get allocation* $\sigma^1 = (3, 2, 1, 4, 5, 6, 7, 8)$. *Since all individuals but* 2 *point to their own goods at stage 2, one gets as final outcome* $\sigma^S = (2, 3, 1, 4, 5, 6, 7, 8)$. *If individual* 1 *reports any (lexicographic) preference with* $(5, 3)$ P_1 $(5, .)$ P_1 $(1, 2)$ $P_1 ...$ *instead of her true preference, one gets at stage 1* $\sigma'^1 = (5, 2, 3, 4, 1, 6, 7, 8)$. *At stage 2, individuals* 2 *and* 3 *point to each other, leading to* $\sigma'^2 = (5, 3, 2, 4, 1, 6, 7, 8)$, *which is the final outcome. Since* $\sigma'^2 P_1 \sigma^S$, *strategy-proof is violated.*

(2) Under weak lexicographic preferences, the modified ttc algorithm may not terminate at a weak selfish equilibrium allocation (see example 11). An open problem is

finding a domain restriction ensuring the existence of weak selfish equilibria. Note that assuming uniform preferences does not give such a domain, as show in

Example 2.4.3 *Consider* $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $C_1 = \{1,2\}, C_2 = \{3,4\}, C_3 = \{5,6\}, C_4 = \{7,8\}\}$, *and for all* $i \in N$, R_i *is such that* $(5,2)\ P_i\ (1,4)\ P_i\ (1,2)\ P_i\ (3,2)\ P_i\ (3,7)\ P_i\ (3,4)\ P_i\ (4,6)\ P_i\ (1,6)\ P_i\ (5,6)\ P_i\ (5,8)\ P_i\ (7,8)$ *... . Since all individuals have the same preferences, then any* $\sigma \in E^W_{coop}(\mathcal{E})$ *must be individually rational. To see why, note joint preferences imply that* $E^W_{coop}(\mathcal{E}) = E^W_{self}(\mathcal{E})$ *for all* $\mathcal{E}$. *Moreover, proposition 7.1 shows that* $E^W_{coop}(\mathcal{E}) \neq \varnothing$ *under joint preferences. The four individually rational allocations are :* σ^0, $\sigma^1 = (1,4,3,2,5,6,7,8)$, $\sigma^2 = (5,2,3,4,1,6,7,8)$, *and* $\sigma^3 = (1,2,3,7,4,6,5,8)$.

Then $(\sigma^0, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$ and $(\sigma^1, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$ both imply $p_5 > p_1 > p_5$ (since $\sigma' P_1 \sigma^0$ and $\sigma' P_2 \sigma^0$ for all σ' with $\sigma'(C_1) = (5,2)$, and since $\sigma'' P_5 \sigma^0$ and $\sigma'' P_6 \sigma^0$ for all σ'' with $\sigma''(C_3) = (1,6)$). Similarly, $(\sigma^2, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$ implies $p_7 > p_4 > p_5 > p_7$, while $(\sigma^3, p) \in \mathbb{E}^W_{coop}(\mathcal{E})$ implies $p_4 > p_2 > p_4$. Thus $E^W_{coop}(\mathcal{E}) = \varnothing$, and therefore $E^W_{self}(\mathcal{E}) = \varnothing$ by Proposition 1.

(3) It is well-known that in standard Shapley-Scarf markets, an allocation is a competitive equilibrium allocation if and only if it is the outcome of the ttc algorithm. Moreover, when preferences over goods are linear orders, there is a unique competitive allocation. We have shown that Shapley-Scarf markets with coalitional trade may have multiple strong and weak equilibrium allocations. This leads to the following questions :

- In all markets with coalition responsive preferences, the outcome of the ttc algorithm is a strong selfish equilibrium allocation. Since this outcome is necessarily unique under coalition responsive preferences, we may find strong selfish equilibria that are not obtained by means of the ttc algorithm. By construction, all these allocations are associated with a set of trading cycles. Can we characterize all sets of trading cycles that are associated with strong selfish equilibria ?

- In all markets with weakly lexicographic preferences, the outcome of the modified ttc algorithm is a weak cooperative equilibrium allocation, but the reverse implication does not hold (see example 9). Can we characterize weak cooperative equilibrium allocations which are not outcomes of the generalized ttc algorithm ?

2.5 Appendix

Proof of Proposition 14

Given $w, x, y, z \in G$, denote by $\{x,y\}^\delta\ R\ \{w,z\}^\delta$ a situation where either $(x,y)\ R\ (w,z)$ or $(y,x)\ R\ (w,z)$ or $(x,y)\ R\ (z,w)$ or $(y,x)\ R\ (z,w)$. If R is a preference that is not weakly lexicographic, P must satisfy one of the two following properties :

(1) there exist $w, x, y, z \in G$ such that one of the two conditions below holds

(1.1) $\{x,y\}^\delta\ P\ \{w,z\}^\delta\ P\ \{w,y\}^\delta$

(1.2) $\{x,y\}^\delta \ P \ \{x,z\}^\delta \ P \ \{w,y\}^\delta$

(2) there exist $v,w,x,y,z \in G$ such that $\{x,y\}^\delta \ P \ \{v,w\}^\delta \ P \ \{z,y\}^\delta$

Pick any number of goods $N \geq 8$, and pick a preference P satisfying property (1.1). Since one can assign this preference to any individual, there is no loss of generality with assuming that P is such that $\{2,5\}^\delta \ P \ \{1,4\}^\delta \ P \ \{1,2\}^\delta$.

Consider $\mathcal{E} =< 8, \mathcal{C}, \pi >$ where $C_1 = \{1,2\}, C_2 = \{3,4\}, C_3 = \{5,6\}, C_4 = \{7,8\}\}$ and

$$\pi = \begin{pmatrix} 1,2 & 3,4 & 5,6 & 7,8 \\ \hline (5,2) & (3,2) & (4,6) & (5,8) \\ (1,4) & (3,7) & (1,6) & (7,8) \\ (1,2) & (3,4) & (5,6) & ... \\ ... & ... & ... & \end{pmatrix}$$

For any $N > 8$, we can extend π by adding up couples where both partners rank σ^0 without harming the argument. Doğan et al. (2011, proposition 2, page 65) use π (up to a relabeling of goods) to show that $\Omega^{coal}(\mathcal{E}) = \varnothing$. By Proposition 9.1, we conclude that $E^W_{coop}(\widetilde{\mathcal{E}}) = \varnothing$.

Now pick any number of individuals $N \geq 10$, and pick a preference P satisfying property (1.2). Using the same trick as above, we can assume that $(9,2) \ P \ (9,7) \ P \ (1,2)$, and consider a market involving exactly 10 individuals. Consider $\mathcal{E}' =< 10, \mathcal{C}, \pi' >$ where $C_1 = \{1,2\}$, $C_2 = \{3,4\}$, $C_3 = \{5,6\}$, $C_4 = \{7,8\}\}$, $C_5 = \{9,10\}$, and

$$\pi' = \begin{pmatrix} 1,2 & 3,4 & 5,6 & 7,8 & 9,10 \\ \hline (9,2) & (3,7) & (4,6) & (2,8) & (7,10) \\ (9,7) & (3,5) & (1,6) & (9,8) & (1,10) \\ (1,2) & (3,4) & (5,6) & (7,8) & (9,10) \\ ... & ... & ... & ... & ... \end{pmatrix}$$

By Proposition 7.1, $E^W_{coop}(\mathcal{E}') \subseteq \Sigma(\sigma^0, \pi)$. The reader will check that $\Sigma(\sigma^0, \pi) = \{\sigma^0, ..., \sigma^6\}$, where

- $\sigma^1 = (9,2,3,5,4,6,7,8,1,10)$
- $\sigma^2 = (9,7,3,4,5,6,2,8,1,10)$
- $\sigma^3 = (9,7,3,5,4,6,2,8,1,10)$
- $\sigma^4 = (1,2,3,5,4,6,7,8,9,10)$
- $\sigma^5 = (1,2,3,5,4,6,7,8,9,10)$
- $\sigma^6 = (1,2,3,4,5,6,9,8,7,10)$

Consider $\sigma \in \{\sigma^0, \sigma^2, \sigma^6\}$. If $(\sigma, p) \in \mathbb{E}^W_{coop}(\mathcal{E}')$, then $\sigma \in \mathcal{O}^W_{C_2}(p) \cap \mathcal{O}^W_{C_3}(p)$ implies $p_5 > p_4 > p_5$, which is impossible. Similarly, consider $\sigma \in \{\sigma^1, \sigma^5\}$. If $(\sigma, p) \in \mathbb{E}^W_{coop}(\mathcal{E}')$, then $\sigma \in \mathcal{O}^W_{C_4}(p) \cap \mathcal{O}^W_{C_5}(p)$ implies $p_9 > p_7 > p_9$. If $(\sigma^4, p) \in \mathbb{E}^W_{coop}(\mathcal{E}')$, then $\sigma^4 \in \mathcal{O}^W_{C_1}(p) \cap \mathcal{O}^W_{C_5}(p)$ implies $p_9 > p_1 > p_9$.

Finally, suppose $(\sigma^3, p) \in \mathbb{E}^W_{coop}(\mathcal{E}')$. Since $\sigma^3 \in B^W_{C_1}(p) \cap B^W_{C_4}$, then $p_1 + p_2 \geq p_7 + p_9$ and $p_7 \geq p_2$. Thus $p_1 + p_2 \geq p_2 + p_9$, in contradiction with $\sigma \in \mathcal{O}^W_{C_1}(p)$. This proves that $E^W_{coop}(\mathcal{E}') = \varnothing$.

Finally, pick any $N \geq 8$, and consider a preference P satisfying property (2). Again by the same trick, we can assume that $(5,2)\ P\ (7,4)\ P\ (1,2)$, and restrict attention to a market involving exactly 8 goods. Consider $\mathcal{E}'' =< 8, \mathcal{C}, \pi'' >$ where $C_1 = \{1,2\}$, $C_2 = \{3,4\}$, $C_3 = \{5,6\}$, $C_4 = \{7,8\}$, and

$$\pi'' = \left(\begin{array}{cccc} 1,2 & 3,4 & 5,6 & 7,8 \\ \hline (5,2) & (3,2) & (4,6) & (5,8) \\ (7,4) & (3,5) & (1,6) & (1,8) \\ (1,2) & (3,7) & (5,6) & (7,8) \\ ... & (3,4) & ... & ... \\ & ... & & \end{array} \right)$$

Again, by Proposition 7.1, $E^W_{coop}(\mathcal{E}) \subseteq \Sigma(\sigma^0, \pi)$. The reader will check that $\Sigma(\sigma^0, \pi) = \{\sigma^0, ..., \sigma^6\}$, where

- $\sigma^1 = (5,2,3,7,4,6,1,8)$
- $\sigma^2 = (5,2,3,4,1,6,7,8)$
- $\sigma^3 = (7,4,3,2,1,6,5,8)$
- $\sigma^4 = (7,4,3,2,5,6,1,8)$
- $\sigma^5 = (1,2,3,7,4,6,5,8)$
- $\sigma^6 = (1,2,3,5,4,6,7,8)$

Then $(\sigma^1, p) \in \mathbb{E}^W_{coop}(\mathcal{E}'')$ implies $p_5 \leq p_1 \leq p_7$, which makes bundle $\{5,8\}$ budget feasible for couple $\{7,8\}$, in contradiction with $\sigma^1 \in \mathcal{O}^W_{C_4}(p)$. Moreover, $(\sigma^2, p) \in \mathbb{E}^W_{coop}(\mathcal{E}'')$ implies $p_5 > p_4 > p_5$, an impossibility. If $(\sigma^3, p) \in \mathbb{E}^W_{coop}(\mathcal{E}'')$, budget feasibility for all couples requires $p_1 = p_5$, in contradiction with $\sigma_3 \in \mathcal{O}^W_{C_1}(p)$. Moreover, $(\sigma^4, p) \in \mathbb{E}^W_{coop}(\mathcal{E}'')$ implies $p_5 > p_1 > p_5$. If $(\sigma^5, p) \in \mathbb{E}^W_{coop}(\mathcal{E}'')$, then budget feasibility requires $p_4 = p_5 = p_7$, while $\sigma^5 \in \mathcal{O}^W_{C_2}(p)$ requires $p_5 > p_4$, a contradiction. If $(\sigma^6, p) \in \mathbb{E}^W_{coop}(\mathcal{E}'')$, one has $p_4 = p_5$ by budget feasibility. Then $\sigma^6 \in \mathcal{O}^W_{C_1}(p) \cap \mathcal{O}^W_{C_4}(p)$ implies $p_5 > p_1 > p_7$, while $\sigma^6 \in \mathcal{O}^W_{C_2}(p)$ implies $p_2 > p_4$. Thus $p_1 + p_2 > p_4 + p_7$. Since $(7,4)$ is budget feasible for C_1, one contradicts $\sigma^6 \in \mathcal{O}^W_{C_1}(p)$. Finally, $\sigma^0 \notin \Omega^{coal}(\mathcal{E})$ (since I is a $(\sigma^3, \sigma^0)-$blocking coalition), we conclude by Proposition 9.1 that $\sigma^0 \notin E^W_{coop}(\mathcal{E}'')$. This shows that $E^W_{coop}(\mathcal{E}'') = \varnothing$, which completes the proof.

Proof of Proposition 15

Two possible violations of coalition responsiveness must be considered : either responsiveness does not hold, or responsiveness holds but coalition responsiveness does not. Consider the first type of violation. Pick a non-responsive preference R. By definition, there exist $w, x, y, z \in G$ such that one of the two following property is satisfied :

(1) either $(w,x)\ P\ (z,y)\ P\ (w,y)\ P\ (z,x)$ (1.1) or $(x,w)\ P\ (y,z)\ P\ (y,w)\ P\ (x,z)$ (1.2)

(2) either $(w,x)\ P\ (w,y)\ P\ (z,y)\ P\ (z,x)$ (2.1) or $(x,w)\ P\ (y,w)\ P\ (y,z)\ P\ (x,z)$ (2.2)

Suppose property (1.1) is satisfied. We can assume w.l.o.g. that $(5,2)\ P\ (1,4)\ P\ (5,4)\ P\ (1,2)$.

Let $N \geq 8$. Suppose for a while that neither $(2,4)$ nor $(4,2)$ is ranked between $(5,2)$ and $(1,2)$. Consider $\mathcal{E} =< 8, \mathcal{C}, \pi >$, where $C_1 = \{1,2\}$, $C_2 = \{3,4\}$, $C_3 = \{5,6\}$, $C_4 = \{7,8\}$, and

$$\pi = \left(\begin{array}{cccc} 1,2 & 3,4 & 5,6 & 7,8 \\ \hline (5,2) & (3,2) & (4,6) & (5,8) \\ (1,4) & (3,7) & (1,6) & (7,8) \\ (5,4) & (3,4) & (5,6) & ... \\ (1,2) & ... & ... & ... \\ ... & ... & ... & ... \end{array} \right)$$

As in the proof above, for any $N > 8$, we can extend π by adding up couples where both partners rank σ^0 without harming the argument. Observe that $\mathcal{E}$ is very similar to the market considered in example 4. By Proposition 7.1, $E^S_{coop}(\mathcal{E}) \subseteq \Sigma(\sigma^0, \pi)$. It is easy to check that $\Sigma(\sigma^0, \pi) = \{\sigma^0, ..., \sigma^4\}$ where

- $\sigma^1 = (1,4,3,2,5,6,7,8)$
- $\sigma^2 = (5,2,3,4,1,6,7,8)$
- $\sigma^3 = (1,2,3,7,4,6,5,8)$
- $\sigma^4 = (5,4,3,2,1,6,7,8)$

Using the same argument as in example 4, we get that $\sigma \in E^S_{coop}(\mathcal{E})$ only if $\sigma = \sigma^4$. If $(\sigma^4, p) \in \mathbb{E}^S_{coop}(\mathcal{E})$, then strong budget feasibility implies that $p_5 \leq p_1$. This implies that bundle $(5,2) \in B^S_{C_1}(p)$, while $(5,2)\ P_1\ (5,4)$ and $(5,2)\ P_2\ (5,4)$. This contradicts $\sigma^4 \in \mathcal{O}^S_{C_1}(p)$. Thus $E^S_{coop}(\mathcal{E}) = \varnothing$, and therefore $E^S_{coop}(\mathcal{E}) = \varnothing$. By Proposition 1.1, $E^S_{self}(\mathcal{E}) = \varnothing$.

Now if $(2,4)$ and/or $(4,2)$ is ranked in R_1 and in R_2 between $(5,2)$ and $(1,2)$, this leads to at most two additional individually rational allocations $\sigma^5 = (2,4,3,7,1,6,5,8)$ and $\sigma^6 = (4,2,3,7,1,6,5,8)$. Strong budget feasibility for σ^5 implies $p_1 = p_2 = p_4 = p_7 = p_5$, in contradiction with $\sigma^5 \in \mathcal{O}^S_{C_3}(p)$. Similarly, budget feasibility for σ^6 implies $p_1 = p_4 = p_5 = p_7$, in contradiction with $\sigma^6 \in \mathcal{O}^S_{C_1}(p)$.

If property (1.2) holds, the same proof applies, provided we swap goods in all bundles listed in π.

The reader will check that the same argument applies to property 2. Indeed, swapping bundles $(1,4)$ and $(5,4)$ in the preferences of individuals 1 and 2 does not alter the argument.

Consider now the case of a responsive preference that is not coalition responsive. This prevails when a coalition contains two individuals with mutually incompatible responsive preferences. Using the same trick as above, we can restrict attention to a

market containing exactly 8 goods.

Assume w.l.o.g. that $(1,4)\ P_1\ (5,4)\ P_1\ (1,2)$, while 2 has responsive preferences such that $(5,2)\ P_2\ (5,4)$. Consider the market $\mathcal{E}' =< 8, \mathcal{C}, \pi' >$, where $C_1 = \{1,2\}$, $C_2 = \{3,4\}$, $C_3 = \{5,6\}$, $C_4 = \{7,8\}$, and

$$\pi' = \left(\begin{array}{ccccc} 1 & 2 & 3,4 & 5,6 & 7,8 \\ \hline (1,4) & ... & (3,2) & (4,6) & (5,8) \\ (5,4) & (5,2) & (3,7) & (1,6) & (7,8) \\ (1,2) & ... & (3,4) & (5,6) & ... \\ ... & (5,4) & ... & ... & ... \\ ... & ... & ... & ... & ... \end{array} \right)$$

By Proposition 6.2, $E^S_{self}(\mathcal{E}') \subseteq \Sigma(\sigma^0, \pi')$. Moreover, $\Sigma(\sigma^0, \pi' \backslash R_2) = \{\sigma^0, ..., \sigma^3\}$ where

- $\sigma^1 = (1,4,3,2,5,6,7,8)$
- $\sigma^2 = (5,4,3,2,1,6,7,8)$
- $\sigma^3 = (1,2,3,7,4,6,5,8)$

Since R_2 is responsive, then $(1,2)\ P_2\ (1,4)$. Thus $\sigma^1 \notin \Sigma(\sigma^0, \pi')$, which implies $\sigma_1 \notin E^S_{self}(\mathcal{E}')$. If $(\sigma^2, p) \in \mathbb{E}^S_{self}(\mathcal{E}')$, we get from strong budget feasibility that $p_1 = p_5$ and $p_2 \geq p_4$. Thus $(5,4) \in B^S_{C_1}(p)$, and $(5,4) \notin \arg\max_{B^S_{C_1}(p)} R_2$, a contradiction. If $(\sigma^3, p) \in \mathbb{E}^S_{self}(\mathcal{E}')$, $(1,2) \in \arg\max_{B^S_{C_1}(p)} R_1$ implies $p_4 > p_2$. Since $(3,2) \in B^S_{C_2}(p)$, then $(3,7) \notin \arg\max_{B^S_3(p)} R_3$, a contradiction. If $(\sigma^0, p) \in \mathbb{E}^S_{self}(\mathcal{E}')$, we get again $p_4 > p_2$, and thus $(3,4) \notin \arg\max_{B^S_{C_2}(p)} R_3$, a contradiction. This shows that $E^S_{self}(\mathcal{E}') = \varnothing$. Observe that we can insert in R_1 any bundle containing good $x \in \{3,6,8\}$ without modifying the argument.

If $(5,4)\ P_1\ (5,2)\ P_1\ (1,2)$ we get $\sigma^4 = (5,2,3,4,1,6,7,8)$ as additional candidate for equilibrium. If $(\sigma^4, p) \in \mathbb{E}^S_{self}(\mathcal{E}')$, we get from $(3,4) \in \arg\max_{B^S_{C_2}(p)} R_3$ that $p_2 > p_4$. It follows that $(1,4) \in B^S_{C_1}(p)$. Since $(1,4)\ P_1\ (5,2)$, we contradict $(5,2) \in \arg\max_{B^S_{C_1}(p)} R_1$.

Suppose that $(2,4)$ and/or $(4,2)$ is ranked in R_1 at least as high as $(1,2)$. Consider bundle $(2,4)$, which creates one additional candidate $\sigma^5 = (2,4,3,7,1,6,5,8)$ for equilibrium. If $(\sigma^5, p) \in \mathbb{E}^S_{self}(\mathcal{E}')$, we get from $(3,7) \in \arg\max_{B^S_{C_2}(p)} R_3$ that $p_2 > p_4$, and from $(1,6) \in \arg\max_{B^S_{C_3}(p)} R_5$ that $p_4 > p_5$. Since $(1,6) \in B^S_{C_3}(p)$, $p_5 \geq p_1$. Thus $p_2 > p_1$, in contradiction with $(2,4) \in B^S_{C_1}(p)$. Consider bundle $(4,2)$, which creates candidate $\sigma^6 = (4,2,3,7,1,6,5,8)$. The argument just above leads to $p_4 > p_1$, in contradiction with $(4,2) \in B^S_{C_1}(p)$. This completes the proof.

Chapitre 3

Characterization of Competitive Equilibria in Shapley-Scarf Markets with Coalitional Trade

Abstract : Aslan and Lainé (2018) introduce three types of competitive equilibrium in Shapley-Scarf markets where trades are organized among couples of individuals. They prove that for each equilibrium type the Top-Trading-Cycles (TTC) algorithm always finds an equilibrium allocation. TTC corresponds to the construction of a specific sequence of trading cycles. Moreover, they show the existence of equilibrium allocations that are not TTC outcomes. In this note, we characterize for each equilibrium type the set of trading cycle sequences which implement equilibrium allocations.[1]

Keywords: Shapley Scarf markets - Competitive equilibrium - Top trading cycles

3.1 Introduction

Shapley-Scarf markets relate to barter systems involving purely indivisible goods (Shapley and Scarf, 1974). In such markets, there are finitely many individuals, each owning one specific indivisible good and ranking all goods according to some (weak) preference order. Trades ensure that everyone ends up with exactly one good. Shapley and Scarf (1974) show that the core of such a market is non-empty and contains the set of competitive allocations. Moreover, they show that the set of allocations that are reached by means of the so-called Top-Trading-Cycles (TTC) algorithm (attributed to David Gale) coincides with the set of competitive allocations.

TTC is a desirable algorithm to guide the trade : at every step, TTC first lets every remaining agent point to his most preferred remaining good and every remaining object point to its owner, then trades every cycle by letting agents in the cycle obtain the objects they point to. As an algorithm, it can be run by a simple computer code. As a

1. This is a joint work with Jean Lainé of CNAM-LIRSA, Paris, France

function from the set of problems to the set of solutions, TTC finds the unique strict core allocation (i.e., defined by weak domination)[2], and is the unique mechanism that satisfies individual rationality, strategy-proofness, and Pareto efficiency (Ma, 1994). These merits help TTC at becoming prominent in the market design literature. Many papers have extended TTC to other models including, to name a few, on-campus housing allocation (Abdulkadiroğlu and Sönmez, 1999), school choice (Abdulkadiroğlu and Sönmez (2003), Morrill (2013, 2015), kidney exchange (Roth et al.,(2004)), and tuition and worker exchange (Dur and Ünver (2019)).

TTC also plays an important role in Shapley-Scarf markets with coalition structure introduced in Aslan and Lainė (2018). They assume that

— the set of individuals is partitioned into couples, each individual paying attention to the good assigned to her partner,
— individuals have preferences over allocations defined by weak orders that make indifferent any two allocations assigning the same goods to their coalition,
— given any price vector, coalition budget sets contain all vectors of goods such that the one assigned to an individual is priced less than her initially owned one.[3]

Three concepts of competitive equilibrium are considered, each being related to the type of cooperation prevailing among partners. An allocation σ is a selfish equilibrium allocation there is a price vector for which each individual maximizes her utility within her coalition budget set. Moreover, σ is a cooperative equilibrium allocation if it assigns each coalition a budget-constrained Pareto optimal vector of goods. Furthermore, σ is a coordinated equilibrium allocation if it satisfies all coalition budget constraints and maximizes each individual's satisfaction given what her partners are endowed with. Interesting enough, TTC finds an equilibrium allocation which can be implemented as an equilibrium of all three types provided preferences over allocations are coalition responsive. Coalition responsiveness means the existence for each partner of a linear order over goods that is consistent with preferences over allocations in the following sense : getting a good better for one partner makes every coalition member better off. While TTC guarantees the existence of equilibria, they are not unique. Therefore, TTC no longer characterizes competitive equilibria.

This contrasts with standard Shapley-Scarf markets. Indeed, Roth and Postlewaite (1977) show that when indifference is ruled out, TTC yields a competitive allocation, which is also the unique strict core allocation.[4]

Since every allocation can be defined as a set of trading cycles, a natural question

2. Shapley and Scarf (1974) give a simple example showing that the core (defined by strict domination) may contain several elements. Roth and Postewaite (1977) show that when indifference is ruled out, the strict core reduces to a singleton, which coincides with the competitive allocation.

3. Aslan and Lainé (2018) also consider equilibrium concepts with transferable income among partners.

4. When indifferences are allowed on preferences of agents ; there are results that are more general. Wako (1984) shows that strict core is included in the set of competitive allocations. Wako (1991) shows that every non-competitive allocation is weakly dominated by some competitive allocation and the non-empty strict core is unique von Neumann-Morgenstern solution. Further properties of Shapley-Scarf markets may be found in Roth (1982), Roth (1984), Ma (1994), Bird (1994), Sönmez (1996), Abdulkadiroğlu and Sönmez (1999).

is finding for each type of equilibrium the sets of trading cycles that implement an equilibrium allocation. This is what this paper aims at. We characterize equilibrium trading sequences by means of three properties, namely top-balanceness, δ-balanceness and strong δ-balanceness. Each of these properties is slight alterations of the top-trading cycle properties.

The paper is organized as follows. Section 2.1 is devoted to the formal model of Shapley-Scarf markets with coalitional trade. Competitive equilibrium concepts are defined in Section 2.2. Coalition responsiveness is formalized in Section 2.3. Characterization results are presented in Section 3. Further comments conclude the paper.

3.2 Shapley-Scarf Markets with Coalitional Trade

3.2.1 Preliminaries

We consider a finite set $I = \{1, 2, ..., N\}$ of individuals confronting a set $G = \{1, 2, ..., N\}$ of purely indivisible goods. Individuals (resp. goods) are denoted by i, j, k (resp. x, y, z). An allocation σ is a bijection from I to G where $\sigma(i)$ is the good allocated to i. There exists an initial allocation $\sigma^0 \in \Sigma$. Without loss of generality, σ^0 is defined by $\sigma^0(i) = i$ for all i. Under this assumption, allocations are permutations of I. The set of allocations is denoted by Σ.

There exists an exogenous coalition structure, defined as a partition $\mathcal{C} = \{C_1, C_2, ..., C_M\}$ of I into non-empty sets. We define $C(i)$ as the element C of $\mathcal{C}$ such that $i \in C$. For coalition $C = \{i_1, ..., i_K\}$ with $i_1 < i_2 < ... < i_K$, we denote by $\sigma(C) = (\sigma(i_1), ..., \sigma(i_K))$ the vector of goods σ assigns to the members of C.

For any finite set X, $\mathcal{W}(X)$ (resp. $\mathcal{L}(X)$) denotes the set of weak (resp. linear) orders over X. Each individual i has preferences over allocations represented by $R_i \in \mathcal{W}(\Sigma)$, with asymmetric part P_i. Given $\sigma, \sigma' \in \Sigma$, $\sigma R_i \sigma'$ means that σ is at least as good as σ' for i. A profile is an N-tuple $\pi = (R_i)_{i \in I}$, and Π stands for the set of profiles. We assume preferences satisfy coalition selfishness : members of each coalition pay interest only to the goods allocated to the coalition. Formally, coalition selfishness holds if $\forall C \in \mathcal{C}$, $\forall i \in C$, $\forall\ \sigma_1, \sigma_2, \sigma_1', \sigma_2' \in \Sigma$, $\sigma_1(C) = \sigma_1'(C)$ and $\sigma_2(C) = \sigma_2'(C)$ implies that $\sigma_1 R_i \sigma_2 \Leftrightarrow \sigma_1' R_i \sigma_2'$. Using coalition selfishness, we equivalently write $(x_j)_{j \in C} R_i (y_j)_{j \in C}$ and $\sigma R_i \sigma'$, where $\sigma(C(i)) = (x_j)_{j \in C}$ and $\sigma'(C(i)) = (y_j)_{j \in C}$.

Definition 3.2.1 *A Shapley-Scarf market with coalitional trade is a triple $\mathcal{E} =< N, \mathcal{C}, \pi >$ where N is the number of goods and individuals, $\mathcal{C}$ is a coalition structure, and π is a profile satisfying coalition selfishness.*

Observe that a standard Shapley-Scarf market is a Shapley-Scarf market with coalitional trade where $\mathcal{C} = \{\{1\}, \{2\}, ..., \{N\}\}$.

3.2.2 Competitive Equilibrium concepts

We first define budget sets.

Definition 3.2.2 *Let $p = (p_x)_{x \in G} \in \mathbb{R}^N_+$ be a price vector. The budget set of i for p is the subset of allocations $B_i(p) = \{\sigma \in \Sigma : \forall j \in C,\ p_{\sigma(j)} \leq p_j\}$.*

Budget sets are natural generalizations of standard Shapley-Scarf budget constraint to coalitions. Each being endowed with a single good, her income is the price of her initial endowment, and a good is affordable if its price does not exceed the price of that single good. Budget sets impose this constraint to all coalition members, without allowing for income transfers. Observe that $B^S_i(p) = B^S_j(p)$ for all $i, j \in C \in \mathcal{C}$. Therefore, we can write $B^S_i(p) = B^S_C(p)$ for all $i \in C$.

There is no obvious equivalent of the standard notion of competitive equilibrium when within-coalition externalities are introduced in individual valuations. Different types of cooperation within coalitions allow for several alternative definitions of equilibrium. We introduce below the three equilibrium concepts which differ according to the type of cooperation that prevails between partners.

Definition 3.2.3 *A selfish equilibrium for $\mathcal{E}$ is a 2-tuple $(\sigma, p) \in \Sigma \times \mathbb{R}^N_+$ such that $\forall i \in I,\ \sigma \in \arg\max_{B_{C(i)}(p)} R_i$.*

We denote the set of selfish equilibria for $\mathcal{E}$ by $\mathbb{E}_{self}(\mathcal{E})$ and denote the set of selfish equilibrium allocations (i.e. allocations σ such that (σ, p) is a selfish equilibrium for some price vector p) by $E_{self}(\mathcal{E})$. Selfish equilibria relate to the lowest degree of collusion among partners : each individual pays attention only to her own well-being given what is affordable by the coalition.

Pick a price vector p and a coalition C. An allocation σ is a p-optimum for C if $\sigma \in B_C(p)$ and there is no $\sigma' \in B_C(p)$ such that $\sigma' R_i \sigma$ for all $i \in C$ and $\sigma' P_j \sigma$ for some $j \in C$. The set of p-optima for C is denoted by $\mathcal{O}_C(p)$.

Definition 3.2.4 *A cooperative equilibrium for $\mathcal{E}$ is a 2-tuple $(\sigma, p) \in \Sigma \times \mathbb{R}^N_+$ such that $\sigma \in \mathcal{O}_C(p)$.*

A cooperative equilibrium allocation results from some unspecified bargaining procedure among partners, which ends up at a budget-constrained Pareto optimal situation. We denote the set of cooperative equilibria by $\mathbb{E}_{coop}(\mathcal{E})$ and denote the set of cooperative equilibrium allocations by $E_{coop}(\mathcal{E})$.

Pick an allocation σ, a price vector p, and an individual i. The (σ, p)-restricted budget set for i is the set of allocations $B_i(\sigma, p) = \{\sigma' \in B_{C(i)}(p) : \forall j \in C \backslash \{i\}, \sigma'(j) = \sigma(j)\}$. A (σ, p)-restricted budget set for i contains all allocations which endow her partners with the same good as in σ and which give i a good making the coalition bundle budget-feasible. The best response of i to (σ, p) is the set $\Phi_i(\sigma, p) = \arg\max_{B_i(\sigma,p)} R_i$.

Definition 3.2.5 *A coordinated equilibrium for $\mathcal{E}$ is a 2-tuple $(\sigma, p) \in \Sigma \times \mathbb{R}^N_+$ such that $\forall i \in I$, $\sigma \in \Phi_i(\sigma, p)$.*

Coordinated equilibria relate to a tacit Nash-type collusion scheme in coalitions. Given her partners' current goods, each individual maximizes her satisfaction under budget constraint. We denote the set of coordinated equilibria by $\mathbb{E}_{coor}(\mathcal{E})$ and denote the associated set of equilibrium allocations by $E_{coor}(\mathcal{E})$.

3.2.3 Preferences restrictions

Aslan and Lainé (2018) show that the existence of equilibria is sensitive to the structure of preferences over allocations. Indeed, a selfish equilibrium may not exist under the full domain assumption. We define below coalition responsive preferences.

Definition 3.2.6 *A profile $\pi = (R_i)_{i \in I}$ is responsive if $\forall C \in \mathcal{C}$, $\forall i \in C$, there exists $(\succ^i_j)_{j \in C} \in \mathcal{L}(G)^{|C|}$ such that :*

$\forall \sigma, \sigma' \in \Sigma$ with $\sigma(C) \neq \sigma'(C)$, we have $\sigma P_i \sigma'$ if $\forall j \in C$, either $\sigma(j) \succ^i_j \sigma'(j)$ or $\sigma(j) = \sigma'(j)$.

We denote by $\Pi_R \subseteq \Pi$ the set of responsive profiles. Responsiveness holds if each individual has a linear order over goods for each of all her partners (including herself), and gets better off with a Pareto improving change according to these linear orders. Responsiveness relates the welfare of the coalition to the welfare of its members. Observe that in definition 6, $\succ^i_j$ may not coincide with $\succ^j_j$: i may assign j a ranking of goods that is not the one that j assigns to herself. On the contrary, if each individual rank goods allocated to any partner as this partner does, any unilateral improvement of one partner's well-being will benefit to all members of the coalition. This special case is called coalition responsiveness.

Definition 3.2.7 *A profile $\pi = (R_i)_{i \in I}$ is coalition responsive if and only if it is responsive and $\forall C$, $\forall i, j, k \in C$, $\succ^j_i = \succ^k_i$.*

We denote the set of all coalition-based-responsive profiles by Π_{CR}.

Coalition responsiveness plays an important role both in the non-emptiness and set-comparison of equilibrium allocation sets. This role is summarized in the two propositions below.

Proposition 3.2.1 *For any $\mathcal{E}$, $E_{self}(\mathcal{E}) \subseteq E_{coop}(\mathcal{E})$, and for any $\mathcal{E}$ with $\pi \in \Pi_{CR}$, $E_{coop}(\mathcal{E}) \subseteq E_{coor}(\mathcal{E})$. Moreover, there exists $\mathcal{E}$ with $\pi \in \Pi_{CR}$ such that $E_{self}(\mathcal{E}) \neq E_{coop}(\mathcal{E})$, and there exists $\mathcal{E}$ with $\pi \in \Pi_{CR}$ such that $E_{coop}(\mathcal{E}) \neq E_{coor}(\mathcal{E})$.*

Proof 3.2.1 *See Aslan and Lainé (2018), propositions 1 and 2.*

Proposition 3.2.2 *$E_{self}(\mathcal{E}) \neq \varnothing$ in any $\mathcal{E} =< N, \mathcal{C}, \pi >$ with $\pi \in \Pi_{CR}$.*

Proof 3.2.2 *See Aslan and Lainé (2018), proposition 11.*

Under coalition responsive preferences, the TTC algorithm always finds a selfish equilibrium allocation. By Proposition 1, it finds an equilibrium allocation for the other two concepts. In the sequel, we assume that all preference profiles are coalition responsive.

3.3 Characterization of Equilibria

It is well-known that in standard Shapley-Scarf markets, any competitive allocation "can be thought of as resulting from the method of top trading cycles" (Roth and Postlewaite 1977), page 135). However, this is no longer true with coalitional trade, even under the assumption of coalition responsive preferences. This is illustrated by the example below.

Example 3.3.1 *Let* $\mathcal{E} =< 4, \mathcal{C}, \pi >$ *where* $\mathcal{C} = \{C_1, C_2\} = \{\{1,2\},\{3,4\}\}$, *and*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \hline (2,1) & (2,1) & (4,3) & (4,3) \\ (1,2) & (1,2) & (3,4) & (3,4) \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Clearly π *can be completed so as to ensure coalition responsiveness, with linear orders over goods :*

- $i = 1, 2$ *:* $2 \succ_i 1 \succ_i \dots$
- $i = 3, 4$ *:* $4 \succ_i 3 \succ_i \dots$

The TTC outcome is σ^0, *and* (σ^0, p) *is a selfish equilibrium where* $p = (2,1,2,1)$. *Consider* $\sigma' = (2,1,4,3)$ *and* $p' = (1,1,1,1)$. *Since* $\sigma \in \arg\max_{B_j(p')} R_i$ *for all* $i \in I$, (σ', p') *is a selfish equilibrium.*

Allocation σ' in example 1 is associated with the set of trading cycles $1 \rightleftarrows 2$ and $3 \rightleftarrows 4$. Clearly, every allocation is associated with a set of trading cycles. Formally, a trading cycle associated with an allocation σ is defined as a non-empty subset $T^k = \{i_1^k, ..., i_{M(k)}^k\}$ of I such that $\sigma(i_m^k) = i_{m+1}^k$ for all $1 \leq m < M$ and $\sigma(i_{M(k)}^k) = i_1^k$. A trading sequence for an allocation σ is a partition $\mathcal{T}_\sigma = \{T^k\}_{k=1,...,K}$ of I into trading cycles.

The TTC outcome is associated with a specific trading sequence. Given an individual i and a subset of goods $G' \subseteq G$, given $n \in \{1, ..., N\}$, the $n-$th best good among G' according to a linear order $\succ_i$ is denoted by $n_{G'}(i)$.

Definition 3.3.1 *The top-trading sequence* $\mathcal{T}_\sigma^{ttc} = \{T^k\}_{k=1,...,K}$ *for allocation* σ *is the trading sequence defined by :*

— $T^1 = \{i_1^1, ..., i_{M(1)}^1\}$ *satisfies* $i_{m+1}^1 = 1_G(i_m)$ *and* $i_1^1 = 1_G(i_{M(1)})$ *for all* $m = 1, ..., M(1)$ *and for all* $i \in I_1 = I$.

— *for all* $k \in \{2, ..., K\}$, $T^k = \{i_1^k, ..., i_{M(k)}^k\}$ *satisfies* $i_{m+1}^k = 1_{I_k}(i_m)$ *and* $i_1^k = 1_{I_k}(i_{M(k)})$ *for all* $m = 1, ..., M(k)$ *and for all* $i \in I_k = I \backslash \bigcup_{1 \leq k' < k} T^{k'}$.

— $I_{K+1} = \varnothing$

Now, an allocation is a competitive equilibrium allocation in a standard Shapley-Scarf market if and only if its associated trading sequence is the top-trading one. Since a competitive allocation in Shapley-Scarf market with coalitions may be associated with a trading sequence that is not the top-trading one, the question we address is finding the set of all trading sequences associated with each equilibrium concept. All results are stated and proven for a coalition structure involving only couples. However, They can be generalized to any coalition structure with some (heavy) notational cost.

We first characterize the set of trading sequences for strong coordinated equilibrium allocations. Given any i, we denote $C(i) = \{i, j\}$.

Definition 3.3.2 *A trading sequence* $\mathcal{T}_\sigma = \{T^k\}_{k=1,...,K}$ *for allocation* σ *is top-balanced if and only if for all* $k, k' \in \{1, ..., K\}$ *and for all* $C = \{i, j\} \in \mathcal{C}$ *with* $i \in T^k$ *and* $j \in T^{k'}$ *where* $k' \geq k$ *such that either the condition* (a) *or* (b) *below are satisfied :*

(a) $\sigma(i) = 1_{G_k}(i)$ *and* $\sigma(j) = 1_{G_{k'}}(j)$

(b) $\sigma(i) = 2_{G_k}(i)$ *and* $\sigma(j) = 1_{G_k}(i) = 1_{G_{k'}}(j)$

where $G_1 = G$ *and* $G_k = G \backslash \bigcup_{1 \leq l < k} T^z$ *for all* $k \geq 2$

Top-balanceness is a natural extension of the top-trading property : it states that the only case where an individual i involved in some trading cycle T^k is not given her most preferred good according to $\succ_i$ among available ones is when both partners share the same most preferred available good when trading, while j obtains this good in some subsequent trading cycle $T^{k'}$ and i is assigned her second most preferred good among those available at stage k.

Top-balanceness characterizes trading sequences for strong coordinated equilibrium allocations.

Proposition 3.3.1 *For all* $\mathcal{E} =< N, \mathcal{C}, \pi >$ *where* $\pi \in \Pi_{CR}$, $\sigma \in E_{coor}(\mathcal{E})$ *if and only if* T_σ *is top-balanced.*

Proof 3.3.1 ***Necessary part :*** *Let* $\sigma \in E_{coor}(\mathcal{E})$, *then there exists a price vector* $p = (p_x)_{x \in G}$ *such that* $(\sigma, p) \in \mathbb{E}_{coor}(\mathcal{E})$. *Consider the trading sequence* $\mathcal{T}_\sigma = \{T^k\}_{k=1,...,K}$ *for* σ. *Since* $p_x = p_y$ *for all* $x, y \in T^k$ *and for all* $T^k \in T_\sigma$, *we can rank cycles* T^k *according to the decreasing order of price levels (i.e.* $\forall k, k' \in \{1, ..., K\}$, $[k < k'] \Leftrightarrow [p_x > p_y, \forall (x, y) \in T^k \times T^{k'}]$*). We will say hereafter that* $\mathcal{T}_\sigma$ *is* $p-$*ordered. Below, we inductively show that* $\mathcal{T}_\sigma$ *is top-balanced.*

First consider $k = 1$. Take any $i \in T^1$ so that $G_1 = G$. Let $c(i) \in T^{k'}$ where $k' \geq 1$.

Suppose $\sigma(i) = 1_G(i)$ and $\sigma(j) \neq 1_{G_{k'}}(j)$. Then, take an allocation σ' such that $\sigma'(i) = \sigma(i)$ and $\sigma'(j) = 1_{G_{k'}}(j)$. Note that $\sigma' \in B_j(\sigma, p)$ and by definition of coalitional responsive preferences $\sigma' P_i \sigma$ and $\sigma' \ P_j \ \sigma$. Hence, it contradicts with $\sigma \in E_{coor}(\mathcal{E})$. Now suppose that $\sigma(i) \neq 1_G(i)$ and $\sigma(j) \neq 1_G(i)$. Consider any allocation σ' with $\sigma'(i) = 1_G(i)$ and $\sigma'(j) = \sigma(j)$. It is obvious that $\sigma' \in B_i(\sigma, p)$. Again by definition of Π_{CR}, we have $\sigma' P_i \sigma$. It contradicts with $\sigma \in E_{coor}(\mathcal{E})$. Thus $\sigma(i) \neq 1_G(i) \Rightarrow \sigma(j) = 1_G(i)$. Similarly, suppose $1_{G_{k'}}(j) \neq 1_G(i)$. Consider any allocation σ'' with $\sigma''(i) = \sigma(i)$ and $\sigma''(j) = 1_{G_{k'}}(j)$. It obvious that $\sigma'' \in B_j(\sigma, p)$. Since $\pi \in \Pi_{CR}$ we have $\sigma'' P_j \sigma$. Hence, it contradicts with $\sigma \in E_{coor}(\mathcal{E})$. Thus $\sigma(i) \neq 1_G(i) \Rightarrow \sigma(j) = 1_G(i) = 1_{G_{k'}}(j)$. Now suppose $\sigma(i) \neq 2_G(i)$. Then, consider any allocation σ''' with $\sigma'''(j) = \sigma(j)$ and $\sigma'''(i) = 2_G(i)$. It obvious that $\sigma''' \in B_i(\sigma, p)$. Since $\pi \in \Pi_{CR}$, $\sigma''' P_i \sigma$. It contradicts with $\sigma \in \Phi_i(\sigma, p)$. Hence it contradicts with $\sigma \in E_{coor}(\mathcal{E})$. Thus either condition (a) or condition (b) must hold for $i \in T^1$ such that $j \in T^{k'}$ and $k' \geq 1$.

Next consider $k = 2$. Take any $i \in T^2$. We have $G_2 = G \setminus T^1$. Let $j \in T^{k'}$. If $k' = 1 < k$, then we get back to the arguments above. If $k' \geq k$, then any allocation σ' with $\{\sigma'(i), \sigma'(j)\} \subseteq G \setminus G_2 = T^1$, we have $\sigma' \notin B_i(p) = B_j(p)$ for all $i \in T^2$. So, the proof used for $i \in T^1$ where $G_1 = G$ clearly works for any $i \in T^2$ where $G_2 = G \setminus T^1$.

Finally, by induction over $k = 1, ..., K$ we can prove that any $i \in T^k$ and $j \in T^{k'}$ where $k' \geq k$, either (a) or (b) is satisfied.

Sufficiency part : *Let $\sigma \in \Sigma_G$ be such that $T_\sigma = \{T^k\}_{k=1,...,K}$ is top-balanced. Given any $i \in I$, define as the number $k(i) \in \{1, ..., K\}$ such that $i \in T^{k(i)}$. Let p be the price vector defined by $p_i = \frac{1}{k(i)}$ for all $i \in T^{k(i)}$. Suppose $(\sigma, p) \notin \mathbb{E}_{coor}(\mathcal{E})$. Thus, there exists some individuals for whom the allocation σ is not the most preferred (σ, p)-restricted budget feasible allocation. Let us denote the set of these individuals by H. Hence $H = \{i \in I : \sigma \notin \Phi_i(\sigma, p)\}$. Take an individual $i^* \in H$ who is the member of the earliest trading cycle among the individuals in H, and denote the partner of i^* by j^*. Then there exists an allocation $\sigma' \in B_{i^*}(\sigma, p)$ such that $\sigma' P_{i^*} \sigma$ while $\sigma'(j^*) = \sigma(j^*)$. Suppose $k(j^*) \geq k(i^*)$. Since T_σ is top-balanced, it implies either $\sigma(i^*) = 1_{G_{k(i^*)}}(i^*)$ or $\sigma(i^*) = 2_{G_{k(i^*)}}(i^*)$ and $\sigma(j^*) = 1_{G_{k(j^*)}}(j^*) = 1_{G_{k(i^*)}}(i^*)$. Therefore, $\sigma' P_{i^*} \sigma$ implies that $\sigma'(i^*) \notin G_{k(i^*)}$. Hence, $\sigma'(i^*) \in G_k$ where $k < k(i^*)$. Thus $p_{\sigma'(i^*)} > \frac{1}{k(i^*)} = p_{i^*}$ which contradicts $\sigma' \in B_i(\sigma, p)$. Finally, if $k(j^*) < k(i^*)$, then, top-balanceness implies $\sigma(i^*) = 1_{G_{k(i^*)}}(i^*)$ and the same argument applies.*

We turn now to cooperative equilibrium. Thanks to proposition 1, characterizing trading sequences for cooperative equilibria amounts to strengthen the top-balanceness property to $\delta-$balanceness. With words, $\delta-$balanceness holds if top-balanceness is satisfied and the following additional condition is satisfied. Suppose the first endowed partner i gets her second-best available good. By top-balanceness, this means that her first-best good is assigned to her partner j, and this good is j's first-best available one. Now suppose that j gives her good back to i and gets instead her second-best available good (in case more than one good is available when j trades). Then one of the two

partners would be worse off with this change.

Definition 3.3.3 *A trading sequence $T_\sigma = \{T^k\}_{k=1,...,K}$ for allocation σ is δ-balanced if and only if for all $k, k' \in \{1, ..., K\}$ and for all $C = \{i, j\} \in \mathcal{C}$ with $i \in T^k$ and $j \in T^{k'}$ where $k' \geq k$, either the condition (a) or (b) holds :*

(a) $\sigma(i) = 1_{G_k}(i)$ and $\sigma(j) = 1_{G_{k'}}(j)$

$(b.1)$ $\sigma(i) = 2_{G_k}(i)$ and $\sigma(j) = 1_{G_k}(i) = 1_{G_{k'}}(j)$

$(b.2)$ If $2_{G_{k'}}(j)$ exists, then $(2_{G_k}(i), 1_{G_{k'}}(j))P_l(1_{G_k}(i), 2_{G_{k'}}(j))$ for some $l \in \{i, j\}$

where $G_1 = G$ and $G_k = G \setminus \bigcup_{1 \leq l < k} T^z$ for all $k \geq 2$.

Proposition 3.3.2 *For all $\mathcal{E} =< N, \mathcal{C}, \pi >$ where $\pi \in \Pi_{CR}$, $\sigma \in E_{coop}(\mathcal{E})$ if and only if T_σ is δ-balanced.*

Proof 3.3.2 ***Necessary part :*** *Let $\sigma \in E_{coop}(\mathcal{E})$, then there exists a price vector $p \in \mathbb{R}^G_+$ such that $(\sigma, p) \in \mathbb{E}_{coop}(\mathcal{E})$. Consider the trading sequence $T_\sigma = \{T^k\}_{k=1,...,K}$ for σ. Since for all $T^k \in T_\sigma$ and for all $x, y \in T^k$, $p_x = p_y$, we can rank trading cycles in T_σ in such a way that T_σ is $p-$ordered. Below, we inductively show that T_σ is δ-balanced.*

First consider $k = 1$ and take $C = \{i, c(i) = j\}$. Let $i \in T^1$ so that $G_1 = G$. Let $j \in T^{k'}$ where $k' \geq 1$. We know from Aslan and Lainé (2018) that if $\pi \in \Pi_{CR}$ then $E_{coop}(\mathcal{E}) \subseteq E_{coor}(\mathcal{E})$. Then Proposition 3 implies that if $\sigma \in E_{coop}(\mathcal{E})$ then either condition (a) or condition $(b1)$ is satisfied. Thus, to complete the proof, one must show that condition $(b2)$ must hold. Suppose that $\sigma(i) = 2_{G_k}(i)$ and $\sigma(j) = 1_{G_k}(i) = 1_{G_{k'}}(j)$. Assume that $2_{G_{k'}}(j)$ exists and $(1_G(i), 2_{G_{k'}}(j))P_l(2_G(i), 1_{G_{k'}}(j))$ for all $l \in \{i, j\}$. Then, consider any allocation σ' such that $\sigma'(i) = 1_G(i)$ and $\sigma'(j) = 2_{G_{k'}}(j)$. σ' clearly belongs to $B^S_C(p)$ and $\sigma' P_l \sigma$ for all $l \in \{i, j\}$. Hence $\sigma \notin E_{coop}(\mathcal{E})$, a contradiction. This proves that condition $(b2)$ must hold.

Next consider $k = 2$. Let $i \in T^2$ so that $G_2 = G \setminus T^1$. Let $j \in T^{k'}$. If $k' < k$, then we get from construction above that either condition (a) or condition (b) must hold. If $k' \geq k$, then any allocation σ with $\{\sigma(i), \sigma(j)\} \subseteq G \setminus G_2 = T_1$, we have $\sigma \notin B_C(p)$. So, the arguments for $i \in T^1$ where $G_1 = G$ clearly works for any $i \in T^2$ where $G_2 = G \setminus T^1$.

Finally by iterating the same argument over $k = 3, ..., K$, we can prove that for any $i \in T^k$ and $j \in T^{k'}$ where $k' \geq k$, either (a) or (b) is satisfied.

Sufficiency part : *Let $\sigma \in \Sigma$ be such that $T_\sigma = \{T^k\}_{k=1,...,K}$ is δ-balanced. Given any $i \in I$, define as the number $k(i) \in \{1, ..., K\}$ such that $i \in T^{k(i)}$. Let p be the price vector defined by $p_i = \frac{1}{k(i)}$ for all $i \in T^{k(i)}$. Suppose $(\sigma, p) \notin \mathbb{E}_{coop}(\mathcal{E})$. Thus, there exists $C = \{i, c(i) = j\}$ such that $\sigma \notin \mathcal{O}_C(p)$. Consider $H = \{i \in I : \sigma \notin \mathcal{O}_{C(i)}(p)\}$. Take an individual $i^* \in H$ who is the member of the earliest trading cycle among the individuals in H, and denote the partner of i^* by j^*. Then, there exists an allocation $\sigma' \in B_C(p)$ such that $\sigma' P_{i^*} \sigma$ and $\sigma' P_{j^*} \sigma$. If $k(j^*) \geq k(i^*)$, $\pi \in \Pi_{CR}$ implies either $\sigma'(i^*) \succ_{i^*} \sigma(i^*)$ or $\sigma'(j^*) \succ_{j^*} \sigma(j^*)$. Suppose $\sigma'(j^*) \succ_{j^*} \sigma(j^*)$. From*

δ-balanceness of T_σ, we have $\sigma(j^*) = 1_{G_{k(j^*)}}(j^*)$ thus $\sigma'(j^*) \notin G_{k(j^*)}$, which contradicts $\sigma' \in B_C(p)$. If $\sigma'(i^*) \succ_{i^*} \sigma(i^*)$ and $\sigma(j^*) \succ_{j^*} \sigma'(j^*)$, δ-balanceness of T_σ implies $\sigma(i^*) \in \{1_{G_{k(i^*)}}(i^*), 2_{G_{k(i^*)}}(i^*)\}$. If $\sigma(i^*) = 1_{G_{k(i^*)}}(i^*)$ then $\sigma'(i^*) \notin G_{k(i^*)}$ and this contradicts $\sigma' \in B_C(p)$. If $\sigma(i^*) = 2_{G_{k(i^*)}}(i^*)$, then $(\sigma(i^*), \sigma(j^*)) = (2_{G_{k(i^*)}}(i^*), 1_{G_{k(j^*)}}(j^*))$ where $1_{G_{k(j^*)}}(i^*) = 1_{G_{k(i^*)}}(i^*)$. The definition of $B_C(p)$ together with δ-balanceness implies $(\sigma'(i^*), \sigma'(j^*)) = (1_{G_{k(i^*)}}(i^*), k_{G_{k(j^*)}}(i^*))$ where $k \geq 2$. Again by definition of δ-balanceness, we have $(2_{G_{k(i^*)}}(i^*), 1_{G_{k(j^*)}}(j^*))$ P_i $(1_{G_{k(i^*)}}(i^*), 2_{G_{k(j^*)}}(j^*))$ R_i $(1_{G_{k(i^*)}}(i^*), k_{G_{k(j^*)}}(i^*))$ for some $i \in \{i^*, j^*\}$, which contradicts $\sigma' P_i \sigma$ for all $i \in \{i^*, j^*\}$.

We introduce now the property of strong δ-balanced trading sequence, which slightly strengthen δ-balanceness, by stating that instead of one partner, both would get worse off with the change considered for δ-balanceness.

Definition 3.3.4 *The strong δ-balanced trading sequence $T_\sigma = \{T^k\}_{k=1,...,K}$ is strong δ-balanced if and only if for all $k, k' \in \{1, ..., K\}$ and for all $C = \{i, j\} \in \mathcal{C}$ with $i \in T^k$ and $j \in T^{k'}$ where $k' \geq k$, the condition (a') or (b') holds :*

(a') $\sigma(i) = 1_{G_k}(i)$ and $\sigma(j) = 1_{G_{k'}}(j)$

$(b'.1)$ $\sigma(i) = 2_{G_k}(i)$ and $\sigma(j) = 1_{G_k}(i) = 1_{G_{k'}}(j)$

$(b'.2)$ If $2_{G_{k'}}(j)$ exists, then $(2_{G_k}(i), 1_{G_{k'}}(j))$ P_l $(1_{G_k}(i), 2_{G_{k'}}(j))$ for all $l \in \{i, j\}$

where $G_1 = G$ and $G_k = G \setminus \bigcup_{1 \leq l < k} T^z$ for all $k \geq 2$.

Note that condition b' implies condition b for δ-balanced trading sequence.

Proposition 3.3.3 *For all $\mathcal{E} =< N, \mathcal{C}, \pi >$ where $\pi \in \Pi_{CR}$, $\sigma \in E_{self}(\mathcal{E})$ if and only if T_σ is strongly δ-balanced.*

Proof 3.3.3 ***Necessary part :*** *Let $\sigma \in E_{self}(\mathcal{E})$, then there exists a price vector $p \in \mathbb{R}^G_+$ such that $(\sigma, p) \in \mathbb{E}_{self}(\mathcal{E})$. Consider the trading sequence $T_\sigma = \{T^k\}_{k=1,...,K}$ for σ. Since for all k and for all $x, y \in T^k$, $p_x = p_y$, we can rank cycles T^k in such a way that T_σ is $p-ordered$. Below, we inductively show that T_σ is strongly δ-balanced. By proposition 4, we know that $\pi \in \Pi_{CR}$ implies $E_{self}(\mathcal{E}) \subseteq E_{coop}(\mathcal{E})$. Therefore, if $\sigma \in E_{self}(\mathcal{E})$ then either condition (a) or condition $(b'1)$ is satisfied. Thus, to complete the proof, one must show that condition $(b'.2)$ must hold.*

First consider $k = 1$. Let $i \in T^1$ so that $G_1 = G$. Let $j \in T^{k'}$ where $k' \geq 1$. Suppose that $\sigma(i) = 2_{G_k}(i)$ and $\sigma(j) = 1_{G_k}(i) = 1_{G_{k'}}(j)$. Assume that $2_{G_{k'}}(j)$ exists and $(1_G(i), 2_{G_{k'}}(j)) \succ_l (2_G(i), 1_{G_{k'}}(j))$ for some $l \in \{i, j\}$. Then, consider any allocation σ' such that $\sigma'(i) = 1_G(i)$ and $\sigma'(j) = 2_{G_{k'}}(j)$. σ' clearly belongs to $B_j(p)$ and $\sigma' P_j \sigma$. Hence $\sigma \notin E_{self}(\mathcal{E})$, a contradiction. This proves that condition $(b'.2)$ must hold.

Next consider $k = 2$. Let $i \in T^2$ so that $G_2 = G \setminus T^1$. Let $j \in T^{k'}$. If $k' < k$, then we get from construction above that either condition $(b'2)$ must hold. If $k' \geq k$, then any allocation σ with $\{\sigma(i), \sigma(j)\} \subseteq G \setminus G_2 = T_1$, we have $\sigma \notin B_i(p) = B_j(p)$. So, the proof used for $i \in T^1$ where $G_1 = G$ clearly works for any $i \in T^2$ where $G_2 = G \setminus T^1$.

Finally by iterating the same argument over $k = 3, ..., K$, we can prove that for any $i \in T^k$ and $j \in T^{k'}$ where $k' \geq k$, $(b'.2)$ is satisfied.

Sufficiency part : *Let $\sigma \in \Sigma$ be such that $T_\sigma = \{T^k\}_{k=1,...,K}$ is strongly δ-balanced. Given any $i \in I$, define as the number $k(i) \in \{1, ..., K\}$ such that $i \in T^{k(i)}$. Let p be the price vector defined by $p_i = \frac{1}{k(i)}$ for all $i \in T^{k(i)}$. Suppose that $(\sigma, p) \notin \mathbb{E}_{self}(\mathcal{E})$. Thus, there exists some individuals for whom the allocation σ is not the most preferred budget feasible allocation. Let us denote the set of these individuals by H.*

Take an individual $i^* \in H$ who is the member of the earliest trading cycle among the individuals in H. Then, there exists an allocation $\sigma' \in B_{i^*}(p)$ such that $\sigma' P_{i^*} \sigma$. Suppose $k(j^*) \geq k(i^*)$ where $C = \{i^*, j^* = c(i^*)\}$. $\pi \in \Pi_{CR}$ implies either $\sigma'(i^*) \succ_{i^*} \sigma(i^*)$ or $\sigma'(j^*) \succ_{j^*} \sigma(j^*)$.

Suppose that $\sigma'(j^*) \succ_{j^*} \sigma(j^*)$. From strongly δ-balanceness of T_σ, we have $\sigma(j^*) = 1_{G_{k(j^*)}}(j^*)$ implies that $\sigma'(j^*) \notin G_{k(j^*)}$. Thus $p_{\sigma'(i^*)} > \frac{1}{k(i^*)} = p_{i^*}$ which contradicts $\sigma' \in B_C(p)$.

If $\sigma'(i^*) \succ_{i^*} \sigma(i^*)$ and $\sigma(j^*) \succ_{j^*} \sigma'(j^*)$, strongly δ-balanceness of T_σ implies that $\sigma(i^*) \in \{1_{G_{k(i^*)}}(i^*), 2_{G_{k(i^*)}}(i^*)\}$. If $\sigma(i^*) = 1_{G_{k(i^*)}}(i^*)$ then $\sigma'(i^*) \notin G_{k(i^*)}$ and this contradicts $\sigma' \in B_C(p)$. If $\sigma(i^*) = 2_{G_{k(i^*)}}(i^*)$ then $(\sigma(i^*), \sigma(j^*)) = (2_{G_{k(i^*)}}(i^*), 1_{G_{k(j^*)}}(j^*))$ where $1_{G_{k(j^*)}}(i^*) = 1_{G_{k(i^*)}}(i^*)$. The definition of $B_C(p)$ together with strongly δ-balanceness implies $(\sigma'(i^*), \sigma'^*)) = (1_{G_{k(i^*)}}(i^*), k_{G_{k(j^*)}}(i^*))$, where $k \geq 2$. Again by definition of strongly δ-balanceness, we have $(2_{G_{k(i^*)}}(i^*), 1_{G_{k(j^*)}}(j^*))$ P_i $(1_{G_{k(i^*)}}(i^*), 2_{G_{k(j^*)}}(j^*))$ R_i $(1_{G_{k(i^*)}}(i^*), k_{G_{k(j^*)}}(i^*))$ for all $i \in \{i^*, j^*\}$, which contradicts $\sigma' P_{i^*} \sigma$ and the proof is complete.

3.4 Further comments

In this paper, we generalize Shapley-Scarf markets to the case where trades are organized between coalitions. We define three types of competitive equilibrium. Moreover, we characterize each type of equilibrium allocation in terms of the properties of their associated trading cycles. These properties explicitly describe rather slight deviations from the well-known top-trading cycles. They also provide a simple test allowing to check whether an allocation is competitive.

A common feature of equilibrium types is that budget sets do not allow for income transfers among partners. When such transfers are permitted, Aslan and Lainé (2018) show the existence of cooperative equilibria under the strong assumption of lexicographic preferences. The proof is based on an alteration of TTC. However, as in the present paper, there exist cooperative equilibrium allocations which cannot be obtained as outcomes of the modified TTC. Characterizing cooperative equilibrium allocations by their associated trading sequence remains an open problem.

Chapitre 4

Stability of Shapley Scarf Markets with Couple Agreements : When distance Matters

Abstract : This paper considers Shapley-Scarf markets where individuals live in couple and where the set of goods is endowed with a geographical structure. This structure aims at capturing the fact that couples prefer living together. Individual preferences over allocations are based on the satisfaction derived from the assigned job and the distance to partner. Several notions of core stability are introduced, each relating to the type of agreements prevailing among partners. We investigate the sensitivity of Core to the preference domain, the location structure and the initial distribution of goods.[1]

Keywords: Shapley-Scarf markets- Core - Distance - Preference extension

4.1 Introduction

This paper considers a generalization of Shapley-Scarf markets where the set of individuals is partitioned into couples and the set of goods is partitioned into geographical areas. In Shapley-Scarf markets, purely indivisible goods are traded without money. There are finitely many individuals, each owning an indivisible good, and trades are organized in such a way that everyone ends up with exactly one good. It is well-known since Shapley and Scarf (1974) that when there is no indifference, the Gale's Top-Trading-Cycles (TTC for short) algorithm admits a unique outcome which is a core stable allocation. Characterizations of the core are provided in Ma (1994), Svensson (1999), Takamiya (2001), and Miyagawa (2002). Furthermore, TTC is not only strategy-proof (Roth, 1982) but also coalitional strategy-proof (Bird (1984)), dominant strategy implementable (Mizukami and Wakayama (2007)), and Nash implementable when there are at least three individuals (Sönmez, 1996). Moreover, Abdulkadiroğlu and Sönmez (1998) show that one can generate all Pareto efficient allocations from

1. This is a joint work with Jean Lainé of CNAM-LIRSA, Paris, France

TTC by varying individuals' endowments.

Our primary purpose is investigating the existence of core allocations when individuals live in couples, and where goods are distributed among several locations. Moreover, we are interested in checking whether core stable allocations can be obtained as outcomes of a relevant adaptation of TTC.

Living in a couple creates an externality in the individual valuation of allocations. We assume that individuals compare allocations according to two criteria : the quality of their assigned good, and the distance between the two goods assigned to the couple. Moreover, we assume that preferences over allocations satisfy separability : among two allocations placing the goods at the same distance, an individual prefers the one assigning her a better good, and among two allocations assigning the same good to an individual, she prefers the one making the two goods closer to each other.

A natural interpretation of our model is organizing job mobility among employees through a centralized procedure where each individual report her preferences over all positions. In many real-life situations (e.g., the mobility of school teachers), employees live in couples, and the location of the partner's position matters a lot when evaluating different outcomes.

The potential instability created by the presence of couples has been extensively investigated in two-sided matching markets.[2] In contrast, very few similar studies prevail for Shapley-Scarf markets. Mumcu and Saglam (2007) show that the core of Shapley-Scarf market may be empty in the presence of externalities, individuals having preferences over allocations and not only goods. Hong and Park (2017) prove that egocentric preferences over allocations essentially preserve the properties of TTC.[3] Doğan et al. (2011) consider Shapley-Scarf markets where couples have joint preferences over pairs of goods and prove the non-emptiness of the Core when these joint preferences have a lexicographic structure.[4]. While the criterion of distance is sometimes mentioned as a source of externality in preferences in the literature on two-sided markets, the role played by an explicit location structure remains to be modeled for both two-sided and Shapley-Scarf markets. A notable exception for two-sided markets is Cantala (2004), where the set of goods are partitioned into different regions and a togetherness condition is formalized. Cantala shows that core existence is not guaranteed even when couples agree on the same ranking of regions.

Hence, to the best of our knowledge, this paper is the first at investigating core existence with an explicit treatment of distance in a Shapley-Scarf market. Distance determines how preferences over goods are transformed into preferences over alloca-

2. The reader may refer to Sasaki and Toda (1996), Dutta and Masso (1997), Roth and Peranson (1999), Cantala (2004), Klaus and Klijn (2005), Echenique and Yenmez (2007), Klaus, Klijn and Masso (2007), Hafalir (2008), Mumcu and Saglam (2010), Bando (2012), Kojima, Pathak and Roth (2013), Bando (2014), Ashlagi, Braverman and Hassidim (2014), Fisher and Hafalir (2016). See also Bando et al. (2016) for a recent survey.

3. Preferences of an individual are egocentric whenever any two allocations assign distinct houses to him, her ranking between the two allocations is determined her ranking of the goods she receives at those allocations.

4. Doğan et al. (2011) also show that their results extend to the case where individuals trade several goods, studied in Konishi, Quint and Wako (2001). See Papai (2003), Wako (2005), Papai (2007) and Klaus (2008) for generalizations of Shapley- Scarf Market with multiple or multiple types of indivisible goods.

tions in the following way. Each individual identifies an allocation with the good she is assigned together with the distance to her partner's good. Thus, comparing two allocations amounts to compare two vectors (rank of assigned good, distance to partner), and preferences over allocations are defined as weak orders over all those vectors.

Having defined preferences over allocations, we focus on a concept of core where no blocking coalition can "break couples" : no blocking coalition can involve one individual and not her partner. Since each individual has her own ranking of allocations, asserting a couple gets better off requires assumptions on how partners with different rankings of goods may agree on how to compare allocations. We assume the existence of an unspecified bargaining procedure extending partners' preferences over allocations into a quasi-ordering of allocations. This quasi-ordering is called couple agreement, and lists all mutually agreed binary comparisons of allocations. Hence, each possible way to generate a quasi-ordering of allocations from a couple of orderings of allocations defines a specific concept of core. We consider two classes of quasi-orderings.

- the (resp. strict) core couple agreement : a couple gets better off when both partners get better off (resp. none of the partners gets less well off and one gets better off),

- the (resp. strict) Minmax couple agreement : a couple gets better off when both the best treated and the worst treated partners get better off (resp. at least one of them gets better off, the other staying at worst as well off)

Our main results can be summarized as follows :

- The core may be empty for all types of couple agreement.

- If preferences over allocations give priority to goods (i.e., are lexicographic w.r.t. good quality), TTC finds an allocation which is in the core for core and strict core couple agreements. However, core existence for strict Minmax couple agreements is not guaranteed. Moreover, we provide a constructive proof of the existence of the core for Minmax couple agreement under a further restriction such that all partners are initially close to each other based on a modification of TTC.

- If preferences give priority to distance (i.e., are lexicographic w.r.t. the distance to partner), the core for Minmax couple agreement, and the core for core couple agreement always exists in a market where all partners are initially close to each other, while it may fail to exist even if the initial distribution of goods in such that all partners can be made close to each other by some reallocation. Moreover, we specify another modification of TTC which ends up an allocation in the core for strict core couple agreements if there are only two locations.

The paper is organized as follows. In Section 2, we formalize Shapley-Scarf markets with couple agreements and a location structure of goods. Having provided notations and basic definitions (section 2.1), preferences over allocations are formalized in section 2.2. Alternative couple agreements are introduced in section 2.3, and the core is defined in section 2.4. Results are stated in section 3. Section 3.1 deals with existence results for the case of preferences giving priority to goods, while those for preferences giving priority to distance are given in section 5.2. Properties of core allocations are investi-

gated in Section 5.3. For exposition clarity, proofs are postponed to an Appendix. We conclude the paper with some open questions.

4.2 Shapley-Scarf markets with couples and a location structure

4.2.1 Notations

For any finite set X, $\mathcal{W}(X)$ (resp. $\mathcal{L}(X)$, $\mathcal{Q}(X)$) denotes the set of weak orders (resp. linear orders, quasi-orderings) over X. If $\gamma \in \mathcal{W}(X)$ ranks good x as high as good y, we write $x\gamma y$. Moreover, the upper-contour set of x for γ is defined by $U(x,\gamma) = \{y \in G : y\gamma^+ x\}$, where γ^+ denote the asymmetric part of γ. Moreover, the rank of x in p_i is defined by $r(x,\gamma) = 1 + |U(x,\gamma)|$.

We consider a finite set of individuals I confronting a set of purely indivisible goods $G = \{1, 2, ..., n\}$, where $n \geq 2$ is even[5]. Individuals (resp. goods) are denoted by i, j, k (resp. x, y, z).

Individual preferences over goods are elements $\mathcal{L}(G)$, and p_i stands for individual i's preference. A profile over goods is an $n-$tuple $p = (p_1, ..., p_n) \in (\mathcal{L}(G))^n$, and Π stands for the set of all profiles over goods.

An allocation σ is a bijection from I to G. Hence, $\sigma(i)$ is the good allocated to individual i. The set of allocations is denoted by Σ. There exists an initial allocation $\sigma^0 \in \Sigma$. Without loss of generality, σ^0 is defined by $\sigma^0(i) = i$ for all i. Under this assumption, allocations are permutations of I.

The set of individuals is partitioned into couples. This partition is exogenous and denoted by $\mathbf{C} = \{C_h, h = 1, ..., n/2\}$. Individual i 's partner is usually denoted by i'. Given an allocation σ and a couple $C = \{i, i'\}$, we denote the (resp. ordered) set of goods that σ assigns to the members of C by $\sigma(C) = \{\sigma(i), \sigma(i')\}$ (resp. $\overrightarrow{\sigma}(C) = (\sigma(i), \sigma(i'))$.

A critical feature of the model is that goods differ in their location. Formally, we consider a partition $\mathbf{L} = \{L_l, l = 1, ..., L\}$ of G into non-empty subsets, each being interpreted as a location. Moreover, we retain the discrete metric d between goods, defined on $G \times G$ by $\forall l \in \{1, ..., L\}$, $\forall x \in L_l$, $\forall y \in L_{l'}$, $d(x,y) = 0$ if $l = l'$ and $d(x,y) = 1$ otherwise.

4.2.2 Preferences over allocations

The existence of couples creates an externality in individual valuations of allocations : the partner's assigned good matters. In order to formalize how preferences over

5. The assumption $| I |=| G |$ is made for notational simplicity. All results below are easily generalized to the case where $| I |>| G |$.

goods generate to preferences over allocations, we first introduce the concept of preference extension :

Definition 4.2.1 *A preference extension is a weak order* $\succsim$ *(with asymmetric part* $\succ$*) on* $\{1,...,n\} \times\{0,1\}$ *which satisfies :*

(a) $\forall k \neq k' \in \{1,...,n\}$, $t \in \{0,1\}$, $[k < k'] \Rightarrow [(k,t) \succ (k',t)]$

(b) $\forall k \in \{1,...,n\}$, $(k,0) \succ (k,1)$

We define the switch for $k \in \{1,...,n\}$ in $\succsim$ as the integer $\varphi(k,\succsim) = Max\{k' \in \{1,...,n\} : (k',0) \succ (k,1)\}$. Observe that separability implies $\varphi(k,\succsim) \geq k$. We define next two polar types of preference extensions.

Definition 4.2.2 - *The priority-to-good preference extension is the element* $\succ^G$ *of* $\mathcal{L}(\{1,...,n\} \times\{0,1\})$ *such that* $\forall k \in \{1,...,n\}$, $\varphi(k,\succ^G) = k$.

- *The priority-to-distance preference extension is the element* $\succ^D$ *of* $\mathcal{L}(\{1,...,n\} \times\{0,1\})$ *such that* $\forall k \in \{1,...,n\}$, $\varphi(k,\succ^D) = n$.

Note that there is no indifference if the preference extension is priority-to-good or priority-to-distance. To see why, pick $k,k' \in \{1,...,n\}$, $t,t' \in \{0,1\}$ and suppose $(k,t) \sim (k',t')$. If $\succ=\succ^G$, one must have $k = k'$. It follows from separability that $t = t'$. If $\succ=\succ^D$, one must have $t = t'$, and separability implies $k = k'$. Thus, $(k,t) \sim (k',t')$ if and only if $(k,t) = (k',t')$. An extension profile is an element $\succsim^n= (\succsim_i,,\succsim_i)$ of $\mathcal{W}(\{1,...,n\} \times \{0,1\})$.

Given individual i and allocation σ, we define $\Gamma(i,\sigma) = (r(\sigma(i),p_i), d(\sigma(i),\sigma(i'))$. A preference extension $\succsim_i$ generates the preference over allocations $R_i \in \mathcal{W}(\Sigma)$ defined as follows.

Definition 4.2.3 *A preference over allocations for* $(p_i,\succsim_i)$ *is the element* R_i *of* $\mathcal{W}(\Sigma)$ *(with asymmetric part* P_i*) such that for all* $\sigma,\gamma \in \Sigma$*, for all* $i \in I$*,* $\sigma R_i \gamma$ *if and only if* $\Gamma(i,\sigma) \succsim_i \Gamma(i,\gamma)$.

This way of defining preferences over allocations deserves several comments :

- An individual's preference over allocations is sensitive only to the goods assigned to her and her partner. This justifies the following notational abuse : instead of writing $\sigma R_i \gamma$, we will often write $(\sigma(i),\sigma(i'))R_i(\gamma(i),\gamma(i'))$.

- Each individual i compares allocations w.r.t. two criteria. The first is the satisfaction level obtained from her assigned good, measured by the rank $r(\sigma(i),p_i)$. The second is the distance $t \in \{0,1\}$ between her assigned good and the good assigned to her partner i' . Therefore, comparing σ and σ' reduces to comparing vectors $\Gamma(i,\sigma)$ and $\Gamma(i,\sigma')$. Observe that i does not consider the level of well-being reached by i' : only distance matters.

- Properties (a) and (b) in definition 1 relate to standard separability. Property (a) states that among two allocations making partners at the same distance to each

other, the one assigning to a higher ranked good is preferable. Property (b) states that among two allocations assigning the same good to i, the one is making i closer to i' is preferable.

- Individuals face a trade-off between the quality of their assigned good and the distance to partner. The switch towards distance is the maximum sacrifice an individual is willing to afford in order to be close to her partner. The trade-off between distance and good quality disappears for both priority-to-good and priority-to-distance preference extensions. In the former case, an individual will never accept to trade good quality against a reduction of distance. Conversely, in the latter case, an individual will not accept a gain in good quality against an increase in distance. Among intermediate cases between the good priority and distance priority preference extensions, a possible one refers to the following position : "As long as I get a good which is ranked high enough, I would not accept to lose it in exchange for a good closer to my partner. But if my good is of poor quality to myself, I would accept any other good closer to my partner".

The next definition summarizes the Shapley-Scarf Market with couples and a location structure :

Definition 4.2.4 *A market is a 7-tuple $\mathcal{E} = \{I, G, \mathbf{C}, \mathbf{L}, p, \succsim^n, \sigma^0\}$, where I is the set of individuals, G is the set of goods, $\mathbf{C}$ is a partition of individuals into couples, $\mathbf{L}$ is a partition of goods into locations, p is a profile over goods, $\succsim^n$ is an extension profile and σ^0 is the initial allocation.*

4.2.3 Couple agreements

Since partners may disagree in comparing allocations, we assume some bargaining prevails within couples. A couple agreement is a list of mutually agreed binary comparisons. More precisely, a couple agreement is obtained by aggregating partners' preferences over allocations to a quasi-ordering over allocations.

Definition 4.2.5 *A couple agreement is an application $\varepsilon : \mathcal{W}(\Sigma) \times \mathcal{W}(\Sigma) \to \mathcal{Q}(\Sigma)$ such that $\forall \sigma, \sigma' \in \Sigma$, $\forall C = \{i, i'\}$, $\forall (R_i, R_{i'}) \in \mathcal{W}(\Sigma) \times \mathcal{W}(\Sigma)$, $\sigma \varepsilon(R_i, R_{i'}) \sigma'$ and $\sigma' \varepsilon(R_i, R_{i'}) \sigma$ if $\sigma R_j \gamma$ and $\sigma' R_j \sigma$ for $j = i, i'$.*

For notational simplicity, we equivalently write $\sigma \varepsilon_C \sigma'$ and $\sigma \varepsilon(R_i, R_{i'}) \sigma'$ for a couple $C = \{i, i'\}$. We introduce below alternative couple agreements. We first introduce useful notations. Given two different 2-tuples (x, y) and (x', y') in $\{1, ..., n\}^2$, we write $(x, y) < (x', y')$ if and only if $x \leq y$ and $x' \leq y'$, and $(x, y) \ll (x', y')$ if and only if $x < y$ and $x' < y'$. Moreover, given a profile over allocations R, an allocation σ, and a couple $C = \{i, i'\}$, we define

- $\Gamma(\sigma, R, C) = (r(\sigma, R_i), r(\sigma, R_{i'}))$,
- $\Gamma^+(\sigma, R, C)) = \max\{r(\sigma, R_i), r(\sigma, R_{i'})\}$ and $\Gamma^-(\sigma, R, C)) = \min\{r(\sigma, R_i), r(\sigma, R_{i'})\}$,
- $\Gamma^\sim(\sigma, R, C)) = (\Gamma^-(\sigma, R, C), \Gamma^+(\sigma, R, C))$

Pick $\sigma, \sigma' \in \Sigma$, $C = \{i, i'\}$, and a profile $R = (R_1, ..., R_n) \in \mathcal{W}(\Sigma)^n$ of preferences over allocations.

Definition 4.2.6 *The Core couple agreement ε^{co} is defined by : $\sigma \varepsilon_C^{co} \sigma'$ if and only if $\Gamma(\sigma, R, C) \ll \Gamma(\sigma', R, C)$.*

The strict Core couple agreement ε^{Sco} is defined by : $\sigma \varepsilon_C^{Sco} \sigma'$ if and only if $\Gamma(\sigma, R, C) < \Gamma(\sigma', R, C)$.

Definition 4.2.7 *The Minmax couple agreement ε^{Mm} is defined by : $\sigma \varepsilon_C^{Mm} \sigma'$ if and only if $\Gamma^{\sim}(\sigma, R, C) \ll \Gamma^{\sim}(\sigma', R, C)$.*

The strict Minmax couple agreement ε^{SMm} is defined by : $\sigma \varepsilon_C^{SMm} \sigma'$ if and only if $\Gamma^{\sim}(\sigma, R, C) < \Gamma^{\sim}(\sigma', R, C)$.

Given two couple agreements ε and η, we say that ε is a refinement of η if and only if $\forall C = \{i, i'\} \in \mathbf{C}$, $\forall R \in \mathcal{W}(\Sigma)^n$, $\forall \sigma, \sigma' \in \Sigma$, $\sigma \varepsilon(R_i, R_{i'}) \sigma'$ if $\sigma\ \eta(R_i, R_{i'})\ \sigma'$. Observe that ε^{Mm} is a refinement of ε^{co}. Indeed, pick any $\sigma, \sigma' \in \Sigma$, any $C = \{i, i'\}$, and any profile R over allocations with $\sigma \varepsilon_C^{co} \sigma'$. By definition of ε_C^{co}, $r(\sigma, R_i) < r(\sigma', R_i)$ and $r(\sigma, R_{i'}) < r(\sigma', R_{i'})$. It follows that $\Gamma^+(\sigma, R, C) < \Gamma^+(\sigma', R, C)$ and $\Gamma^-(\sigma, R, C) < \Gamma^-(\sigma', R, C)$ which clearly implies $\Gamma^{\sim}(\sigma, R, C) \ll \Gamma^{\sim}(\sigma, R, C)$. Thus $\sigma \varepsilon_C^{SMm} \sigma'$. The reverse implication does not hold. Similarly, one gets that

- ε^{SMm} is refinement of ε^{Mm},
- ε^{SMm} is a refinement of ε^{Sco},
- ε^{Sco} is a refinement of ε^{co}.

4.2.4 The Core

Definition 4.2.8 *An allocation σ is blocked in $\mathcal{E}$ for couple agreement ε if there exist a set S of couples and an allocation σ' satisfying the following two properties :*

(1) $\forall C \in S$, $\sigma' \varepsilon_C \sigma$ and $\neg(\sigma \varepsilon_C \sigma')$

(2) $\sigma(\mathbf{C} \backslash S) = \sigma^0(\mathbf{C} \backslash S)$.

The Core for ε is the subset $\mathcal{C}(\mathcal{E}, \varepsilon)$ of allocations which are not blocked for ε.

For sake of simplicity, we use the following terminology : the (resp. strict) Core of $\mathcal{E}$ is $\mathcal{C}(\mathcal{E}, \varepsilon^{co})$ (resp. $\mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$), and the (resp. strict) Minmax Core of is $\mathcal{C}(\mathcal{E}, \varepsilon^{Mm})$ (resp. $\mathcal{C}(\mathcal{E}, \varepsilon^{SMm})$).

If σ is blocked for ε, S is called blocking coalition, and we say that S blocks σ with σ'. Observe that if ε is a refinement of η, $\mathcal{C}(\mathcal{E}, \varepsilon) \subseteq \mathcal{C}(\mathcal{E}, \eta)$.

4.3 Results

We will show with a simple example shows that TTC does not always find a core allocation whatever the prevailing couple agreement. For sake of completeness, we briefly

recall how TTC operates. Each individual i points to the individual endowed with the good i prefers the most. Since the number n of individuals is finite, at least one cycle of individuals can be formed. Then assign each individual in a cycle her most preferred good, and remove all individuals (and goods) involved in the cycle. By proceeding in the same way in all subsequent stages, the set of individuals is exhausted in at most n stages.

Example 4.3.1 *Let $\mathcal{E}$ be the market where $I = \{1, 1', 2, 2'\}$, $\mathbf{L} = \{L_1, L_2\}$, where $L_1 = \{1, 1'\}$, $L_2 = \{2, 2'\}$ and $C_i = \{i, i'\}$ for all $i = 1, 2$. Consider the profile over goods*

$$p = \left(\begin{array}{cccc} 1 & 1' & 2 & 2' \\ \hline 2 & 1' & 1 & 2' \\ 1 & \dots & 2 & \dots \\ \dots & & \dots & \end{array}\right)$$

Moreover suppose that preference extensions are defined by $(1,0) \succ_i (2,0) \succ_i (1,1) \succ_i (2,1)$ for all $i \in I$. It implies the profile R over allocations as follows :[6]

$$R = \left(\begin{array}{cccc} 1 & 1' & 2 & 2' \\ \hline (2,2') & (1,1') & (1,1') & (2,2') \\ (1,1') & \dots & (2,2') & \dots \\ (2,1'),(2,1') & & (1,2),(1,2') & \\ \dots & & \dots & \end{array}\right)$$

TTC ends up at allocation $\sigma = (2, 1', 1, 2')$. Observe that $\forall i \in I$, $\sigma^0 P_i \sigma$. Thus $S = \{1, 1', 2, 2'\}$ blocks σ with σ^0 for couple agreement ε^{co}. Since $\sigma \notin \mathcal{C}(\varepsilon^{co}, p)$ while all other couple agreements are refinements of ε^{co}, σ belongs to the Core for no couple agreement.

Actually, not only TTC fails at finding a core allocation, but the core may be empty for all couple agreements, as stated in

Proposition 4.3.1 *Let $n \geq 6$. There exists a market $\mathcal{E}$ such that $\mathcal{C}(\mathcal{E}, \varepsilon) = \varnothing$ for all $\varepsilon = \{\varepsilon^{co}, \varepsilon^{Sco}, \varepsilon^{Mm}, \varepsilon^{SMm}\}$.*

This negative results suggests restricting the domain of preferences over allocations. Hereafter, we focus on the polar cases of preferences giving priority either to goods or to distances.

6. Putting more than one allocation in the preference R_i of an agent i means that the agent i is indifferent between those allocations.

4.3.1 Priority-to-Good Preferences

Our first result is that priority-to-good preferences ensure the non-emptiness of the strict Core. Actually, the reason is that TTC finds a strict core allocation.[7]

Proposition 4.3.2 *$\mathcal{C}(\mathcal{E}, \varepsilon^{Sco}) \neq \varnothing$ in any market $\mathcal{E}$ where $\succsim_i = \succ_i^G$ for all $i \in I$.*

Since ε^{Sco} is a refinement of ε^{co}, Proposition 2 implies

Corollary 4.3.1 *$\mathcal{C}(\mathcal{E}, \varepsilon^{co}) \neq \varnothing$ in any market $\mathcal{E}$ where $\succsim_i = \succ_i^G$ for all $i \in I$.*

Roth and Postlewaite (1977) show that in standard Shapley-Scarf markets where indifferences are ruled out, there exists a unique strict core allocation, which is the outcome of TTC. This is no longer true in our setting, as shown by the following example.

Example 4.3.2 *Let $\mathcal{E}$ be the market where $I = \{1, 1', 2, 2'\}$, $\mathbf{L} = \{L_1, L_2\}$, $\succsim^n = (\succsim_1^G, ..., \succsim_n^G)$, $L_1 = \{1, 1'\}$, $L_2 = \{2, 2'\}$ and $C_i = \{i, i'\}$ for all $i = 1, 2$. Consider the profile over goods*

$$p = \begin{pmatrix} 1 & 1' & 2 & 2' \\ \hline 1 & 1 & 2 & 2 \\ 1' & 1' & 2' & 2' \\ ... & ... & ... & ... \end{pmatrix}$$

Since $\forall i \in I$, $\succsim_i = \succ_i^G$, one gets the profile over allocations

$$R = \begin{pmatrix} 1 & 1' & 2 & 2' \\ \hline (1,1') & (1',1) & (2,2') & (2',2) \\ (1,2);(1,2') & (2,1);(2',1)) & (2,1);(2,1') & (1,2);(1',2) \\ (1',1) & (1,1') & (2',2) & (2,2') \\ ... & ... & ... & ... \end{pmatrix}$$

TTC ends up at allocation $\sigma = (1, 1', 2, 2') \in \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$. However, there is another allocation $\sigma' = (1', 1, 2', 2) \in \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$.

We move to existence of Minmax and strict Minmax Cores. We first show the possible emptiness of strict Minmax Core.

Proposition 4.3.3 *There exists a market $\mathcal{E}$ where $\forall i \in I$, $\succsim_i = \succ_i^G$ and $\mathcal{C}(\mathcal{E}, \varepsilon^{SMm}) = \varnothing$.*

7. This result is similar to the one by Hong and Park (2017), since priority-to-good preference is a concept very similar to the one of egocentric preference.

It turns out that a positive result is established for the Minmax Core provided an additional restriction on the initial allocation : all partners are initially close. As a preliminary step, we define a modified version of the TTC algorithm (TTC1) algorithm for markets verifying this additional condition.

4.3.1.1 The TTC1 algorithm

Take any market $\mathcal{E}$ where $\succsim_i = \succ_i^G$ for all $i \in I$ and $d(\overrightarrow{\sigma^0}(C)) = 0$ for all $C \in \mathbf{C}$.

We begin with some useful basic notations and definitions. Given a non-empty subset G' of G, we denote the first-best good of individual i in G' according to p_i by $1_{G'}(i)$. Moreover, for any couple $C = \{i, i'\}$ and any $G' \subseteq G$, we define the first-best good for C in G' as the good $x(C, G')$ such that :

- $x(C, G') = 1_G(i)$ if $r(1_G(i), p_i) \leq r(1_{G'}(i'), p_{i'})$
- $x(C, G') = 1_{G'}(i')$ if $r(1_G(i), p_i) > r(1_{G'}(i'), p_{i'})$

The TTC1 algorithm (TTC1) operates as follows.

Stage 1 : Define the directed graph D^1 with couples as vertices such that $(C, C') \in D^1$ if and only if $x(C, G) \in \sigma^0(C')$. Observe that each couple points in D^1 to at least one couple and at most two couples. Since each vertex has a strictly positive out-degree, D^1 contains a cycle $T_1 = \{C_1, ..., C_K\}$ where :[8]

- $C_k \neq C_{k'}$ for all $k \neq k' \in \{1, ..., K\}$,
- $x(C_k, G) \in \sigma^0(C_{k+1})$ for all $k \in \{1, ..., K\}$ (with the convention $K + 1 \equiv 1$)

Define allocation σ^1 by :

- $\overrightarrow{\sigma^1}(C) = \overrightarrow{\sigma^0}(C)$ for all $C \in \mathbf{C} \backslash T_1$,
- $\overrightarrow{\sigma^1}(C_k) = (i_{k+1}, i'_{k+1})$ if $x(C_k, G) = 1_G(i_k) = i_{k+1}$ and $\overrightarrow{\sigma^1}(C_k) = (i'_{k+1}, i_{k+1})$ if $x(C_k, G) = 1_G(i'_k) = i_{k+1}$ for all $C_k = \{i_k, i'_k\} \in T_1$.

Define $\mathbf{C}^2 = \mathbf{C} \backslash T_1$ and $G^2 = G \backslash \cup_{C \in T_1} \sigma^0(C)$.

...

Stage s : Define the directed graph D^s with vertices in $\mathbf{C}^s$ such that $(C, C') \in D^s$ if and only if $x(C, G^s) \in \sigma^{s-1}(C')$. Then built cycle $T_s = \{C_1, ..., C_S\}$ such that :

- $C_s \neq C_{s'}$ for all $s \neq s' \in \{1, ..., S\}$,
- $x(C_s, G^s) \in \sigma^{s-1}(C_{s+1})$ for all $s \in \{1, ..., S\}$ (with the convention $S + 1 \equiv 1$)

Moreover, define allocation σ^s by :

- $\overrightarrow{\sigma^s}(C) = \overrightarrow{\sigma^{s-1}}(C)$ for all $C \in T_k$ where $k < s$,
- $\overrightarrow{\sigma^s}(C) = \overrightarrow{\sigma^{s-1}}(C)$ for all $C \in \mathbf{C}^s \backslash T_s$
- $\overrightarrow{\sigma^s}(C_s) = (i_{s+1}, i'_{s+1})$ if $x(C_s, G^s) = 1_{G^s}(i_s) = i_{s+1}$ and $\overrightarrow{\sigma^s}(C_s) = (i'_{s+1}, i_{s+1})$ if

8. Note that whenever several cycles coexist in a stage, they are mutually disjoint, and we form exactly one of them before proceeding to the next stage.

$x(C_s, G^s) = 1_{G^s}(i'_s) = i_{s+1}$ for all $C_s = \{i_s, i'_s\} \in T_s$

Define $\mathbf{C}^{s+1} = \mathbf{C}^{\mathbf{s}} \backslash T_s$ and $G^{s+1} = G^s \backslash \cup_{C \in T_s} \sigma^{s-1}(C)$.

Since $\mathbf{C}^{s+1} \subsetneq \mathbf{C}^{\mathbf{s}}$ for all $s \geq 1$, the algorithm ends in finitely many stages, until some stage Z where $\mathbf{C}^{Z+1} = G^{Z+1} = \varnothing$.

It should be obvious that TTC1 coincides with TTC when all individuals are singles. Considering couples instead of individuals as basic elements of cycles, the main difference with TTC relates to the definition of the first-best good for a couple : among the (at most) two goods that are first-best for one of the partners, we select the one with lowest rank in the relevant ordering of goods. In case of a tie, any of the two goods can be chosen.

TTC1 is illustrated by the following example.

Example 4.3.3 *Let $\mathcal{E}$ be the market where $I = \{1, 1', ..., 4, 4'\}$, $\mathbf{L} = \{L_1, L_2\}$ with $L_1 = \{1, 1', 2, 2'\}$, $L_2 = \{3, 3', 4, 4'\}$, $\mathbf{C} = \{C_1, ..., C_4\}$ with $C_i = \{i, i'\}$ for all $i = 1, 2, 3, 4$, $\succsim^n = (\succsim_1^G, ..., \succsim_n^G)$, and*

$$p = \left(\begin{array}{cccccccc} 1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' \\ \hline 1 & 1' & 2 & 2' & 1 & 2 & 1 & 2 \\ ... & ... & ... & ... & 2 & 4' & 2 & 3' \\ & & & & 4 & 1 & 3 & 1 \\ & & & & 3 & 1' & 4 & 1' \\ & & & & ... & 3 & ... & 4' \\ & & & & & ... & & ... \end{array} \right)$$

Stage 1 *: The first-best good for 1 is 1 and the first-best good for $1'$ is $1'$. Since $r(1, p_1) = r(1', p_{1'})$, we choose (arbitrarily) $x(C_1, G) = 1$. This allows to build $T_1 = \{C_1\}$, leading to allocation σ^1 defined by*

- $\overrightarrow{\sigma^1}(C) = \overrightarrow{\sigma^0}(C)$ *for all* $C \in \mathbf{C} \backslash T_1 = \{C_2, C_3, C_4\}$,
- $\sigma^1(1) = 1, \sigma^1(1') = 1'$

Thus, $\mathbf{C}^2 = \{C_2, C_3, C_4\}$ and $G^2 = G \backslash \cup_{C \in T_1} \sigma^0(C) = \{2, 2', 3, 3', 4, 4'\}$.

Stage 2 *: The first-best good for 2 in G^2 is 2 and the first-best good for $2'$ in G^2 is $2'$. Since $r(2, p_2) = r(2', p_{2'})$, we can take $x(C_2, G^2) = 2$. Hence cycle $T_2 = \{C_2\}$ and allocation σ^2 defined by*

- $\overrightarrow{\sigma^2}(C) = \overrightarrow{\sigma^1}(C)$ *for all* $C \in \{C_1, C_3, C_4\}$,
- $\sigma^2(2) = 2, \sigma^2(2') = 2'$

Thus, $\mathbf{C}^3 = \{C_3, C_4\}$ and $G^3 = G^2 \backslash \cup_{C \in T_2} \sigma^1(C) = \{3, 3', 4, 4'\}$.

Stage 3 *: The first-best good for 3 in G^3 is 4 and the first-best good for $3'$ in G^3 is $4'$. Since $r(4', p_{3'}) < r(4, p_3)$, $x(C_3, G^3) = 4'$. Similarly, $x(C_4, G^3) = 3'$. Hence cycle $T_3 = \{C_3, C_4\}$ and the final allocation σ^3 defined by*

- $\overrightarrow{\sigma^3}(C) = \overrightarrow{\sigma^2}(C)$ *for all* $C \in \{C_1, C_2\}$,

- $\sigma^3(3') = 4', \sigma^3(4') = 3'$
- $\sigma^3(3) = 4, \sigma^3(4) = 3$

We show that every outcome of TTC1 is a Minmax core allocation. As a consequence, we get the following proposition :

Proposition 4.3.4 *$\mathcal{C}(\mathcal{E}, \varepsilon^{Mm}) \neq \varnothing$ in any market $\mathcal{E}$ where $\succsim_i = \succ_i^G$ for all $i \in I$ and $d(\overrightarrow{\sigma^0}(C)) = 0$ for all $C \in \mathbf{C}$.*

4.3.2 Priority-to-Distance Preferences

We assume in the sequel that $\succsim_i = \succ_i^D$ for each agent i in any market. In contrast with the case of priority-to-good preferences, TTC may not terminate at a strict Core allocation under priority-to-distance preferences. To see why, consider again example 1, where preferences give priority to distance, and where the TTC outcome is not in the Core. Therefore, it is not in the strict Core.

Observe that partners of both couples are at initial distance 0. Moreover, given any initial distribution of goods among locations, trades can be organized so as to ensure that all partners are close. We say in that case that the market is 0-solvable. A classification of markets is obtained by considering the minimal number of couples with distant partners that trades can secure. Given any allocation σ in a market $\mathcal{E}$, we define $\Psi(\sigma, \mathcal{E}) = |\{\{i, i'\} \subset I : d(\sigma(i), \sigma(i')) = 1\}|$ as the number of couples whose partners remain distant at σ.

Definition 4.3.1 *A market $\mathcal{E}$ is s-solvable if $min\{\Psi(\sigma, \mathcal{E}), \sigma \in \Sigma\} = s$ where $s \leq n$.*

An s-solvable market is such that at least s couples will remain distant at any allocation.

We consider Core existence having in mind this classification of markets.

Proposition 4.3.5 *For all $s \geq 1$, there exists a s-solvable market $\mathcal{E}$ where $\succsim_i = \succ_i^D$ such that $\forall \varepsilon \in \{\varepsilon^{co}, \varepsilon^{Mm}, \varepsilon^{SMm}, \varepsilon^{Sco}\}$, $\mathcal{C}(\mathcal{E}, \varepsilon^{Sco}) = \varnothing$.*

The class of 0-solvable markets remains to be considered.

Proposition 4.3.6 *There exists a 0-solvable market $\mathcal{E}$ where $\succsim_i = \succ_i^D$ such that $\forall \varepsilon \in \{\varepsilon^{SMm}, \varepsilon^{Sco}\}$, $\mathcal{C}(\mathcal{E}, \varepsilon) = \varnothing$.*

The non-emptiness of $\mathcal{C}(\mathcal{E}, \varepsilon^{Mm})$ (hence of $\mathcal{C}(\mathcal{E}, \varepsilon^{co})$) for 0-solvable markets remains to be investigated. We conjecture that $\mathcal{C}(\mathcal{E}, \varepsilon^{Mm})$ is non-empty at least for 2-location markets. A partial answer to this conjecture is given for 0-solvable markets where all partners are initially close.

Proposition 4.3.7 *$\mathcal{C}(\mathcal{E}, \varepsilon^{Mm}) \neq \varnothing$ in any market $\mathcal{E}$ where $\succsim_i = \succ_i^D$ for all $i \in I$ and $\Psi(\sigma^0, \mathcal{E}) = 0$.*

However, a similar result no longer holds for the refinement ε^{SMm} of ε^{Mm}.

Proposition 4.3.8 *There exists a market $\mathcal{E}$ where $\succsim_i = \succ_i^D$ for all $i \in I$ and $\Psi(\sigma^0, \mathcal{E}) = 0$ such that $\mathcal{C}(\mathcal{E}, \varepsilon^{SMm}) = \varnothing$.*

We can ensure the existence of the strict Core when all partners are initially close under the additional assumption of two locations. First, we introduce a second modified Top-Trading-Cycles (TTC2) algorithm

4.3.2.1 The TTC2 algorithm

Take any market $\mathcal{E}$ with $\Psi(\sigma^0, \mathcal{E}) = 0$ and $\mid \mathbf{L} \mid = 2$.

Stage 1 : Form a subset $T_1 = \{i_1, ..., i_K\}$ of I verifying :

- $\forall i \in T_1$, $i' \notin T_1$,
- $\forall k \in \{1, ..., K-1\}$, $i_{k+1} = \arg\max_G p_{i_k}$ and $\arg\max_G p_{i_K} \in \{i_1, i_1'\}$.

It should be obvious that a set such as T_1 exists. Moreover, T_1 is not a top trading cycle. Indeed, while all individuals but the last one point to their successor's good as first-best good, the individual involved last may point to the good owned by the first individual's partner. Moreover, if an individual is involved in T_1, then her partner is not.

Call T_1-compatible an allocation σ^1 such that $\sigma^1(i) = i$ for all i with $\{i, i'\} \cap T_1 = \varnothing$, and for all $k \in \{1, ..., K\}$

- $\forall k \in \{1, ..., K\}$, $\sigma^1(i_k) = \arg\max_G p_{i_k}$,
- $\forall k \in \{2, ..., K\}$, if $d(\sigma^1(i_k), i_k') = 0$, $\sigma^1(i_k') = i_k'$,
- if $d(\sigma^1(i_1), i_1') = 0$, $\sigma^1(i_1') = i_1$ if $\sigma^1(i_K) = i_1'$, and $\sigma^1(i_1') = i_1'$ otherwise,
- $\forall k \in \{1, ..., K\}$, if $d(\sigma^1(i_k), i_k') = 1$, $d(\sigma^1(i_k), \sigma^1(i_k')) = 0$.

With words, allocation σ^1 is T_1-compatible if each individual i_k in T_1 gets her successor's good while all partners stay at distance 0. Moreover, in the case where i_k does not change the location of her good, her partner must keep her initial good. A small difficulty arises when the last individual i_K in T_1 does not point to i_1 but to the partner i_1' of i_1, and when i_1 does not change the location of her good. We solve it by assigning good i_1 to i_1'.

We claim that a T_1-compatible allocation exists. This is obvious if for all k, i_{k+1} and i_k' belong to the same location. Moreover, observe that the two sets $B^{1\to 2}(T_1) = \{i_k \in T_1 : i_k \in L_1 \text{ and } \arg\max_G p_{i_k} \in L_2\}$ and $B^{2\to 1}(T_1) = \{i_k \in T_1 : i_k \in L_2 \text{ and } \arg\max_G p_{i_k} \in L_1\}$ have the same cardinality. Pick any bijection $\Omega : B^{1\to 2}(T_1) \to B^{2\to 1}(T_1)$ and define σ^1 by

- for all $i_k \in B^{1\to 2}(T_1)$, $\sigma^1(i_k') = [\Omega(i_k)]'$,
- for all $i_k \in B^{2\to 1}(T_1)$, $\sigma^1(i_k') = [\Omega^{-1}(i_k)]'$,
- for all $i_k \in T_1 \backslash [B^{1\to 2}(T_1) \cup B^{2\to 1}(T_1)]$, $\sigma^1(i_k') = i_k'$,

- for all $i \in I \backslash T_1$, $\sigma^1(i) = i$ and $\sigma^1(i') = i'$.

In the case where $\sigma^1(i_K) = i'_1$, we proceed as follows. Suppose $i_1 \in B^{1\to 2}(T_1)$ and $\Omega(i_1) = j$. Then $\sigma^1(j') = i_1$. Similarly, if $i_1 \in B^{2\to 1}(T_1)$ and $\Omega^{-1}(i_1) = j$, then $\sigma^1(j') = i_1$.

It is straightforward to check that σ_1 is T_1-compatible. By construction, $\Psi(\sigma^0, \mathcal{E}) = 0$.

Remove T_1 from I and remove $H_1 = \{\sigma^1(i) : i \in T_1\}$ from G. Define $I_2 = I \backslash T_1$ and $G_2 = G \backslash H_1$. Observe that I_2 and G_2 may not coincide. Indeed, if $\sigma_1(i_K) = i'_1$, $i'_1 \in I_2 \backslash G_2$. Moreover $|I_2| = |G_2|$.

The set I_2 is partitioned into two sets :

- $I_2^s = \{i \in I_2 : i' \in T_1\}$ is the set of individual in I_2 whose partner i' assigned her final good in T_1.

- $I_2^f = \{i \in I_2 : \{i, i'\} \cap T_1 = \varnothing\}$ is the set of individuals in I_2 such that neither i nor i' trades at stage 1 (therefore $\sigma^1(j) = j$ for all $j \in I_2^f$).

For each $i \in I_2^s$, define $G_{2,i} = \{j \in G_2 : d(j, \sigma^1(i')) = 0\}$ as the set of available goods close to the one assigned to i' by σ^1.

For each $i \in I_2$, we define the linear order q_i^2 over G_2 by :

- $\forall i \in I_2^s$, (1) $\forall j \in G_{2,i}$, $\forall h \in G_2 \backslash G_{2,i}$, jq_i^2h, (2) $q_i^2 \mid_{G_{2,i}} = p_i \mid_{G_{2,i}}$, and (3) $q_i^2 \mid_{G_2 \backslash G_{2,i}} = p_i \mid_{G_2 \backslash G_{2,i}}$ [9],

- $\forall i \in I_2^f$, $q_i^2 = p_i \mid_{G_2}$.

Stage 2 : Form a subset $T_2 = \{i_1, ..., i_{K_2}\}$ of I_2 verifying :

- $\forall i \in T_2$, $i' \notin T_2$,

- $\forall k \in \{1, ..., K_2 - 1\}$, $\sigma^1(i_{k+1}) = \arg\max_{G_2} q_{i_k}$, and $\arg\max_{G_2} q_{i_{K_2}} \in \{\sigma^1(i_1), \sigma^1(i'_1)\}$.

Call (T_1, T_2)-compatible an allocation σ^2 such that :

- $\sigma^2(i) = \sigma^1(i) = i$ for all i with $\{i, i'\} \cap (T_1 \cup T_2) = \varnothing$,

- $\sigma^2(i) = \sigma^1(i)$ for all $i \in T_1$,

and for all $k \in \{1, ..., K_2\}$,

- $\sigma^2(i_k) = \arg\max_{G_2} q_{i_k}$,

- if $i_k \in I_2^f \cap T_2$ and $d(\sigma^2(i_k), i'_k) = 0$, $\sigma^2(i'_k) = \sigma^1(i'_k) = i'_k$,

- if $i_k \in I_2^f \cap T_2$ and $d(\sigma^2(i_k), i'_k) = 1$, $d(\sigma^2(i_{k+1}), \sigma^2(i'_k)) = 0$.

By the same argument as the one above, a (T_1, T_2)-compatible allocation always exists.

Remove T_2 from I_2 and remove $H_2 = \{\sigma^2(i) \in G_2 : i \in T_2\}$ from G_2. Define $I_3 = I_2 \backslash T_2$ and $G_3 = G_2 \backslash H_2$.

9. For a linear order p over the set X, we define $p \mid_{X'}$ over the set $X' \subseteq X$ such that for all $i, j \in X'$, $ip \mid_{X'} j$ if and only if ipj

The set I_3 is partitioned into two sets :

- $I_3^s = \{i \in I_3 : i' \in T_1 \cup T_2\}$ contains individuals in I_3 whose partner i' is assigned a good in either T_1 or T_2.

- $I_3^f = \{i \in I_3 : \{i, i'\} \cap T_1 \cup T_2 = \varnothing\}$ contains individuals in I_3 such that neither i nor i' trades at stage 1 or stage 2 (therefore $\sigma^2(j) = \sigma^2(j)$ for $j \in I_3^f$).

For each $i \in I_3^s$, define $G_{3,i} = \{j \in G_3 : d(j, \sigma^2(i')) = 0\}$ as the set of available goods close to the one assigned to i' by σ^2.

For each $i \in I_3$ define q_i^3 as the linear order over G_3 such that :

- $\forall i \in I_3^s$, (1) $\forall j \in G_{3,i}, \forall h \in G_3 \backslash G_{3,i}$, jq_i^3h, (2) $q_i^3 \mid_{G_{3,i}} = p_i \mid_{G_{3,i}}$, and (3) $q_i^3 \mid_{G_3 \backslash G_{3,i}} = p_i \mid_{G_3 \backslash G_{3,i}}$,

- $\forall i \in I_3^f$, $q_i^3 = p_i \mid_{G_3}$.

...

Stage s : Form a subset $T_s = \{i_1, ..., i_{K_s}\} \subset I_s$ of I_s such that :

- $\forall i \in T_s$, $i' \notin T_s$,

- $\forall l \in \{1, ..., K_s - 1\}$, $\sigma_s(i_{k+1}) = \arg\max_{G_s} q_{i_k}^s$, and $\arg\max_{G_s} q_{i_{K_s}}^s \in \{\sigma_{s-1}(i_1), \sigma_{s-1}(i_1')\}$.

Call σ^s as a $(T_1, T_2, ..., T_s)$-compatible allocation along the same line as for σ^2.

Remove T_s from I_s and remove $H_s = \{\sigma^s(i) \in G_s : i \in T_s\}$ from G_s Define $I_{s+1} = I_s \backslash T_s$ and $G_{s+1} = G_s \backslash H_s$.

...

Since a non-empty subset can be formed at each stage, the algorithm stops at stage s^* such that $I_{s^*+1} = G_{s^*+1} = \varnothing$. Denote the resulting $(T_1, T_2, ..., T_{s^*})$ compatible allocation by σ^*.

With words, the algorithm works as follows. Starting from a situation where all partners are close to each other, form a cycle T_1. For individuals not changing location when getting their successor's good, keep their partner at their initial endowment (so partners stay at distance 0). For individuals changing location when getting their successor's good, assign their partner a good initially owned by an individual j who does not trade in T_1, while her partner j' does and changes location in the opposite way. By construction, the number of trades in T_1 implying a move from L_1 to L_2 is equal to the number of trades in T_1 implying a move from L_2 to L_1. Thus such an allocation is feasible, and keeps all partners at distance 0.

Remove T_1 from the set of individuals, and make all goods traded in T_1 unavailable for further trade. Thus, we have two types of individuals : those who did not trade yet, and those whose partner has traded in T_1. Remark that they may be currently endowed with a good different from their initial one. Now each remaining individual i points to the individual j endowed with i's first-best available good defined as follows. If $i' \in T_1$, i's first-best good is the one maximizing her satisfaction given the location of her partner's good. Since priority is given to distance, this first-best must be in that location. If $i' \notin T_1$, i's first-best good is the one maximizing her satisfaction over

all available goods. This allows to build a new cycle T_2, and to define a new current allocation along the same lines as the previous one :

- all individuals in T_1 keep their current good (they do so forever)

- all individuals in T_2 get their first-best good (and keep it forever)

- each partner i' of an individual i in T_2 stays at status quo if i does not change location while trading, and gets a good in her partner's new location otherwise.

This process continues until there is no remaining individual (and good available for trade). The resulting allocation is denoted by σ^*.

The algorithm makes each couple C active in exactly two stages :

- In the first stage $s(C)$, one partner i gets as final good her best good among available ones, while i' either stays with good i' (or i), or gets a new good close to the one obtained by i.

- In the second stage $s'(C)$, i' gets as final good her first-best good among those available and close to the good obtained by i.

Note that in any stage s whenever there are two cycles T and T' and a couple $C = \{i, i'\}$ such that $i \in T$ and $' \in T'$, we form only one of them and proceed to the next stage. This implies that which cycle is formed may influence the final allocation.

To conclude, we illustrate TTC2 with the following example.

Example 4.3.4 *Let $\mathcal{E}$ be the market where $I = \{1, 1', ..., 4, 4'\}$, $\mathbf{L} = \{L_1, L_2\}$, $L_1 = \{1, 1', 4, 4'\}$, $L_1 = \{2, 2', 3, 3'\}$ and $C_i = \{i, i'\}$ for all $i = 1, 2, 3, 4,$. Moreover $\succsim_i = \succ_i^D$ for all $i \in I$. Consider the following profile over goods*

$$\left(\begin{array}{cccccccc} 1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' \\ \hline 2 & 1' & 1' & 1' & 4 & 1 & 2' & 1 \\ 1 & 4 & 3 & 3 & 4' & 2 & 1 & 1' \\ 2' & 3 & 4 & 4 & 2 & 4 & 1' & 3 \\ 1' & 4' & 2 & 2' & 2' & 2' & 2 & 3' \\ 3 & 1 & 1 & 3' & 3 & 1' & 4' & 2 \\ 3' & 2 & 3' & 4' & 3' & 3' & 3 & 4 \\ 4 & 3' & 4' & 2 & 1' & 4' & 3' & 4' \\ 4' & 2' & 2' & 1 & 1 & 3 & 4 & 2' \end{array}\right)$$

Stage 1 *: The first-best good for 1 is 2 and the first-best good for 2 and 1′is 1′ . One can form either $T_1 = \{1, 2\}$ or $T_1 = \{1'\}$. Suppose $T_1 = \{1, 2\}$ is chosen. This gives $\sigma^1(1) = 2$, $\sigma^1(2) = 1'$. Moreover, $B^{1\to 2}(T_1) = \{1\}$ and $B^{2\to 1}(T_1) = \{2\}$. Pick the bijection $\Omega : B^{1\to 2}(T_1) \to B^{2\to 1}(T_1)$ such that $\Omega(1) = 2$. Therefore $\sigma^1(1') = \sigma^0([\Omega(1)]') = 2'$ and $\sigma^1(2') = \sigma^0(1) = 1$. Note that σ^1 keeps all agents in C_1 and C_2 close to their partner. Moreover $\sigma^1(i) = \sigma^0(i) = i$ for all $i = 3, 3', 4, 4'$. We remove the set T_1 from I and remove the set $H_1 = \{1', 2\}$ from G. Hence $I_2 = \{1', 2', 3, 3', 4, 4'\}$ and $G_2 = \{1, 2', 3, 3', 4, 4'\}$.*

Note that $I_2^s = \{1', 2'\}$ *and* $I_2^f = \{3, 3', 4, 4'\}$.

Stage 2 *: In restriction to* G_2*, the first-best good for* $1'$ *in* L_2 *is* 3*, the first-best good for* 3 *is* 4 *and the first-best good for* 4 *is* $2'$ *which is owned by agent* $1'$*. Note that, since agent* 1 *gets a good in* L_2 *as a final good, agent* $1'$ *maximizes her satisfaction by getting her first best in* L_2*. Thus, one can form* $T_2 = \{1', 3, 4\}$*. We get* $B^{1\to 2}(T_2) = 4$ *and* $B^{2\to 1}(T_2) = 3$*. Pick the bijection* Ω *such that* $\Omega(4) = 3$*. Therefore* $\sigma^2(1') = 3$*,* $\sigma^2(3) = 4$ *and* $\sigma^2(4) = 2'$*. Furthermore,* $\sigma^2(3') = \sigma^1([\Omega^{-1}(3)]') = 4'$ *and* $\sigma^2(4') = \sigma^1([\Omega(4)]') = 3'$*. Note again that* σ^2 *keeps all agents in* C_1*,* C_2 *and* C_3 *close to their partner. We also have* $\sigma^2(i) = \sigma^1(i)$ *for all* $i = 1, 2, 2', 4, 4'$*. We remove the set* T_2 *from* I_2 *and remove the set* $H_2 = \{2', 3, 4\}$ *from the set* G_2*. Hence* $I_3 = \{2', 3', 4'\}$ *and* $G_3 = \{1, 3', 4'\}$.

Stage 3 *: The first-best available good for* $4'$ *in* L_2 *is* $3'$*. Since* $4'$ *currently owns the good* $3'$ *(i.e.,* $\sigma^2(4') = 3'$*), one can form* $T_3 = \{4'\}$ *and get* $\sigma_3(4') = 3'$*. Moreover* $\sigma^3(i) = \sigma^2(i)$ *for all* $i = 1, 1', 2, 2', 3, 3', 4$*. The set* T_3 *is removed from* I_3 *and* $H_3 = \{3'\}$ *is removed from* G_3*. Hence* $I_4 = \{2', 3'\}$ *and* $G_4 = \{1, 4'\}$.

Stage 4 *: The first-best available good for* $2'$ *in* L_1 *is* $4'$ *(owned by* $3'$*) and the first-best available good for* $3'$ *in* L_1 *is* 1 *(owned by* $2'$*). This gives* $T_4 = \{2', 3'\}$ *and* $\sigma^4(2') = 4'$*,* $\sigma^4(4') = 1$*. For all* $i = 1, 1', 2, 3, 4, 4'$*,* $\sigma^4(i) = \sigma^3(i)$*. The set* T_4 *is removed from* I_3 *and* $H_3 = \{1, 4'\}$ *is removed from* G_3*. Then,* $I_5 = G_5 = \varnothing$ *so the algorithm stops. Hence, one outcome of the algorithm* $\sigma^* = \sigma^4 = (2, 3, 1', 4', 4, 1, 2', 3')$.

Observe that at each stage other subsets can be chosen. Hence, one may generate several trading trajectories. The reader can check that five allocations can be obtained as TTC2 outcomes : $(2, 3, 1', 4, 2', 3', 4', 1)$*,* $(2, 3, 1', 4, 3', 2', 4', 1)$*,* $(1, 1', 3, 2, 4, 4', 2', 3')$*,* $(1, 1', 2, 3, 4, 4', 2', 3')$*,* $(1, 1', 3, 3', 4, 4', 2', 2)$*. This illustrates the multiplicity of TTC2 outcomes.*

We claim that all outcomes of TTC2 belong to $\mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$. As a consequence, we get the following proposition :

Proposition 4.3.9 $\mathcal{C}(\mathcal{E}, \varepsilon^{Sco}) \neq \varnothing$ *in any market* $\mathcal{E}$ *where* $\succsim_i = \succ_i^D$ *for all* $i \in I$*,* $|\mathbf{L}| = 2$ *and* $\Psi(\sigma^0, \mathcal{E}) = 0$.

4.3.3 Properties of The Core

Next, we consider two properties for the Core for each couple agreement : individual rationality and Pareto optimality. Given a profile R over allocations, an allocation σ is individually rational if $\sigma R_i \sigma^0$ for all individuals i. The set of individually rational allocations is denoted by $\Sigma(\sigma^0, R)$. It is fairly obvious that core allocations in Shapley-Scarf with singles are individually rational, since individuals can block by themselves. However, this is no longer true in our setting.

Proposition 4.3.10 *There exists* $\mathcal{E}$ *such that* $\mathcal{C}(\mathcal{E}, \varepsilon) \nsubseteq \Sigma(\sigma^0, R)$ *for all* $\varepsilon \in \{\varepsilon^{Mm}, \varepsilon^{SMm}, \varepsilon^{Sco}, \varepsilon^{co}\}$.

Let us turn to Pareto optimality. An allocation σ is Pareto optimal in $\mathcal{E}$ if there is no $\sigma' \in \Sigma$ such that $\sigma' R_i \sigma$ for all $i \in I$ with $\sigma' P_j \sigma$ for some $j \in I$. It is well-known that

core allocations in standard Shapley-Scarf markets may not be Pareto efficient (Roth and Postlewaite (1977)). Moreover, when there is no indifference, there exists a unique strict core allocation, which is Pareto efficient (and coincides with the competitive allocation). A similar result prevails in our setting.

Proposition 4.3.11 *(1) In any market $\mathcal{E}$, every $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$ with $\varepsilon \in \{\varepsilon^{Sco}, \varepsilon^{SMm}\}$ is Pareto optimal.*

(2) There exists a market $\mathcal{E}$ where $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{co})$ is not Pareto optimal.

As already mentioned, TTC defines a strategy-proof mechanism for standard Shapley-Scarf markets. A natural question is investigating whether the (modified) TTC defined for generalized markets also satisfy strategy-proofness. Proposition 2 states that the outcome of TTC belongs to the Core with priority-to-goods preferences. It is straightforward to check that the argument used in Roth (1982) can be transposed to establish strategy-proofness of TTC in our setting. Also for the case of priority-to-goods preferences, Proposition 4 shows that TTC1 always terminates at a Minmax Core allocation if all partners are initially close to each other. Moreover, TTC1 may not have a unique outcome. This creates a difficulty in defining strategy-proofness. However, one easily shows that strategy-proofness may be violated in restriction to profiles ensuring a unique TTC1 outcome.

Example 4.3.5 *Let $\mathcal{E}$ be the market where $I = \{1, 1', 2, 2', 3, 3'\}$, $\mathbf{L} = \{L_1, L_2, L_3\}$, $L_1 = \{1, 1'\}$, $L_2 = \{2, 2'\}$, $L_3 = \{3, 3'\}$, and $C_i = \{i, i'\}$ for all $i = 1, 2$. Preferences over goods are*

$$p = \begin{pmatrix} 1 & 1' & 2 & 2' & 3 & 3' \\ \hline 1 & 1' & 1 & 1 & 1 & 1 \\ \dots & \dots & 1' & 3' & 1' & 2' \\ & & 2 & \dots & 2 & 3' \\ & & 3 & & \dots & \dots \\ & & \dots & & & \end{pmatrix}$$

If $\succsim_i = \succ_i^G$ for all $i \in I$, the profile over allocations is defined by[10]

$$R = \begin{pmatrix} 1 & 1' & 2 & 2' & 3 & 3' \\ \hline (1,1') & (1,1') & (1,1') & (1',1) & (1,1') & (1',1) \\ \dots & \dots & (1, G\backslash 1') & (G\backslash 1', 1) & (1, G\backslash 1') & (G\backslash 1', 1) \\ & & (1',1) & (3,3') & (1',1) & (2,2') \\ & & (1', G\backslash 1) & (G\backslash 3, 3') & (1', G\backslash 1) & \dots \\ & & (2,2') & \dots & (2,2') & \\ & & (2, G\backslash 2') & & \dots & \\ & & (3,3') & & & \\ & & (3, G\backslash 3') & & & \end{pmatrix}$$

10. For any $G' \subseteq G$, any $x \in G$ and any $i \in I$, putting (x, G') in R_i means that individual i is indifferent between all (x, y) such that $y \in G'$.

It is easily checked that $\sigma = (1, 1', 3, 3', 2, 2')$ *is the unique TTC1 outcome. If individual* 2 *reports any preference over goods with good* 2 *at top, one gets* $G^2 = I\backslash\{1\}$ *and* $x(C_2, G^2) = 2$*. Thus* $T_2 = \{C_2\}$ *is the unique TTC1 at stage 2 and* $\sigma^2 = (1, 1', 2, 2', 3, 3')$*. In the third stage, we get* $x(C_3, \{3, 3'\}) = 3'$*. Thus* $T_3 = \{C_3\}$ *is the unique TTC1 at stage 3 and* $\sigma' = (1, 1', 2, 2', 3, 3')$ *is the unique TTC1 outcome. Since* $\sigma^{Z'} R_2 \sigma^Z$*, strategy-proofness is violated.*

In markets with two locations where all partners are initially close and preferences give priority to distance, Proposition 9 shows that TTC2 always terminates at a strict Core allocation. We show that TTC2 is manipulable for a specific way to extend preferences over allocations to preferences over sets of allocations. Given a linear order $\succ$ over a finite set X, and given any two non-empty subsets A and B of X, we say that A is better than B according to the Fishburn criterion if $a \succ b \succ c$ for all $a \in A\backslash B$, all $b \in A \cap B$ and all $c \in B\backslash A$ (Fishburn, 1972). The following example shows that TTC2 fails at satisfying strategy-proofness under the Fishburn criterion : there exists a 2-location market where all partners are initially close and have priority-to-distance preferences such that one individual can by misreporting her preference ensure a better set of outcomes of TTC2 in the Fishburn sense.

Example 4.3.6 *Let* $\mathcal{E}$ *be the market where* $I = \{1, 1', 2, 2', 3, 3', 4, 4'\}$*,* $\mathbf{L} = \{L_1, L_2\}$*,* $L_1 = \{1, 1', 3, 3', 4, 4'\}$*,* $L_2 = \{2, 2'\}$ *and* $C_i = \{i, i'\}$ *for all* $i = 1, 2, 3, 4$*. Preferences over goods are defined by*

$$p = \left(\begin{array}{cccccccc} 1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' \\ \hline 2 & 1' & 3 & 2' & 1 & 4 & 4 & 1 \\ 1 & \ldots & \ldots & \ldots & 3 & 4' & \ldots & 4' \\ \ldots & & & & & 3' & & \ldots \\ & & & & & \ldots & & \end{array}\right)$$

If $\succsim_i = \succ_i^D$ *for all* $i \in I$*, the profile* R *over allocations is defined by*

$$\left(\begin{array}{cccc} 1 & 1' & 2 & 2' \\ \hline (2, L_1\backslash\{2\}) & (L_1\backslash\{1'\}, 1') & (3, L_1\backslash\{3\}) & (L_1\backslash\{2'\}, 2') \\ (1, L_1\backslash\{1\}) & \ldots & \ldots & \ldots \\ \ldots & & & \\ & \ldots & & \end{array}\right)$$

$$\left(\begin{array}{cccc} 3 & 3' & 4 & 4' \\ \hline (1, L_1\backslash\{1\}) & (4', 4) & (4, 4') & (L_1\backslash\{1\}, 1) \\ (3, L_1\backslash\{3\}) & (4, 4') & \ldots & (4, 4') \\ \ldots & (L_1\backslash\{3'\}, 3') & & \ldots \\ & \ldots & & \end{array}\right)$$

At the first stage, one can form one of the four subsets $\{1,2,3\},\{1'\},\{2'\}$, *and* $\{4\}$. *It is easily checked that whichever set is chosen, there is a unique TTC2 outcome* $\sigma=(2,1',3,2',1,3',4,4')$. *Now if individual* $3'$ *reports an order over goods with* $4'$ *at top, this creates a new possible subset* $\{3',4',1,2\}$ *at the first stage. Observe that* σ *remains a TTC2 outcome. However, forming* $\{3',4',1,2\}$ *leads to an additional outcome* σ' *with* $\overrightarrow{\sigma'}(3,3')\in(4,4')$. *Since* $\sigma' P_{3'}\sigma$, *TTC2 is not strategy-proof in the Fishburn sense.*

4.4 Concluding comments

In this paper, we consider a generalization of Shapley-Scarf markets which naturally fits with the design of a centralized job mobility procedure involving couples. To the best of our knowledge, this paper is the first at explicitly involving a distance criterion in the preference structure. We assume that individuals retain two criteria in order to compare allocations : the quality of their assigned good and its distance to the partner's assigned good. Moreover, we focus on two polar cases, each giving priority to one of these two criteria.

Attention is mostly paid to the existence of core stable allocations. When assuming that no coalition breaking a couple is admissible, several refinements of the core can be considered, each corresponding to a specific type of agreement among partners (strict core, Minmax, and strict Minmax). Since the core may be empty without restriction upon preferences. Moreover, we focus on two polar cases, each giving priority either to the criterion of good quality or to the criterion of distance.

Several open questions remain to be addressed. In the case of priority-to-good preferences, we show that TTC always finds a core allocation and that a modified TTC finds a Minmax core allocation under the assumption that all partners are initially close to each other. Even under this assumption, the strict Minmax core may be empty. An unsolved question is the existence of Minmax core allocations when at least one couple involves initially distant partners. In the case of priority-to-distance preferences, our results show that the initial distribution of goods among locations matters. If no allocation can make all partners close to each other, the core may be empty. Even if one allocation at least can ensure all partners to be close to each other, the strict Core and the strict Minmax core may be empty. Whether Minmax core (and thus core) allocations always exist remains to be answered. However, we establish the non-emptiness of the Minmax core in the specific situation where all partners being initially close. If we further assume that there are exactly two locations, the strict core is also non-empty. An interesting issue is considering the case of at least three locations.

Finally, all results are based on a discrete distance. Obviously, this calls for an analysis of Shapley-Scarf markets with more general geographical structure.

4.5 Appendix

Proof of Proposition 1

Let $\mathcal{E}$ be the market where $I = \{1, 1', 2, 2', 3, 3'\}$, $\mathbf{L} = \{L_1, L_2\}$, $L_1 = \{1, 2, 3\}$, $L_2 = \{1', 2', 3'\}$, and $\mathbf{C} = \{C_1, C_2, C_3\}$, with $C_1 = \{1, 1'\}$, $C_2 = \{2, 2'\}$, $C_3 = \{3, 3'\}$. Consider the profile over goods

$$p = \begin{pmatrix} 1 & 1' & 2 & 2' & 3 & 3' \\ \hline 1' & 1' & 2' & 2' & 3' & 3' \\ 3' & 2' & 3' & 1' & 1 & 2' \\ 1 & 2 & 2 & 3 & 2 & 1' \\ 2 & 3 & 1 & 2 & 3 & 3 \\ 3 & 1 & 3 & 1 & ... & 2 \\ ... & ... & ... & ... & ... & ... \\ & & & & & \end{pmatrix}$$

Preference extensions are such that $(1,0) \succ_i (2,0) \succ_i (3,0) \succ_i (4,0) \succ_i ...$ for all $i \in I$. Clearly, this partial order can be completed so as to be separable. Let $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{co})$. Observe that at least one couple has to remain distant. Actually, σ must make exactly one couple being distant. Indeed, if σ make two couples C and C' distant, we have $\Gamma(i, \sigma) = \{(r, 1)\}_{1 \leq r \leq n}$ for all $i \in C \cup C'$. It is easily checked that there exists an allocation σ' with $\Gamma(i, \sigma') \in \{(1,0), (2,0), (3,0), (4,0)\}$ for all $i \in C \cup C'$. Since $\Gamma(i, \sigma') \succ \Gamma(i, \sigma)$ for all $i \in C \cup C'$, then we get $\sigma' P_i \sigma$ for all $i \in C \cup C'$. Hence $r(\sigma', R_i) < r(\sigma, R_i)$ for all $i \in C \cup C'$. So that $\Gamma(\sigma', R, C) \ll \Gamma(\sigma, R, C)$ and $\Gamma(\sigma', R, C') \ll \Gamma(\sigma, R, C')$. It implies $\sigma' \varepsilon_C^{co} \sigma$, $\sigma' \varepsilon_{C'}{}^{co} \sigma$ and $\rceil(\sigma \varepsilon_C \sigma')$, $\rceil(\sigma \varepsilon_{C'} \sigma')$. Therefore, C and C' blocks σ with σ', which is a contradiction.

Suppose that $d(\sigma(1), \sigma(1')) = 0$ and $d(\sigma(2), \sigma(2')) = 0$. Moreover, suppose $\{\sigma(2), \sigma(2')\} \subset L_1$. It is easily checked that since $d(\sigma(3), \sigma(3')) = 1$, then $\{C_2, C_3\}$ blocks σ with $\sigma' = (1, 1', 3', 2', 2, 3)$. Indeed, $2, 2'$ ranked higher the allocation that assigned them $\overrightarrow{\sigma}(C_2) = (3', 2')$ where $\Gamma(2, \sigma') = (2, 0)$ and $\Gamma(2', \sigma') = (1, 0)$ to another allocation σ which assigned them goods in L_1 since $\Gamma(i, \sigma) = (k, 0)$ where $k \geq 3$ for $i \in \{2, 2'\}$. Moreover $3, 3'$ ranked higher the allocation σ' such that $\overrightarrow{\sigma'}(C_3) = (2, 3)$ where $\Gamma(3, \sigma') = (3, 0)$ and $\Gamma(3', \sigma') = (4, 0)$ to any other allocation that make them distant. If $\{\sigma(1), \sigma(1')\} \subset L_1$, a similar argument shows that $\{C_1, C_3\}$ blocks σ with $\sigma' = (3', 1', 2, 2', 1, 3)$. Thus, $d(\sigma(1), \sigma(2)) = 0$ and $d(\sigma(3), \sigma(4)) = 0$ contradicts $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{co})$.

The rest of the proof follows the same line. If $d(\sigma(1), \sigma(1')) = 0$ and $d(\sigma(3), \sigma(3')) = 0$, then

- If $\{\sigma(3), \sigma(3')\} \subset L_1$, then $\{C_2, C_3\}$ blocks σ with $\sigma' = (1, 1', 2, 3, 3', 2')$.
- If $\{\sigma(1), \sigma(1')\} \subset L_1$, then $\{C_1, C_2\}$ blocks σ with $\sigma' = (1', 2', 1, 2, 3, 3')$.

Finally, if $d(\sigma(1), \sigma(1')) = 1$, then $d(\sigma(2), \sigma(2')) = d(\sigma(3), \sigma(3')) = 0$.

- If $\{\sigma(2), \sigma(2')\} \subset L_1$, then $\{C_1, C_2\}$ blocks σ with $\sigma' = (1, 2, 2', 1', 3, 3')$.
- If $\{\sigma(3), \sigma(3')\} \subset L_1$, then $\{C_1, C_3\}$ blocks σ with $\sigma' = (1, 3, 2, 2', 3', 1')$.

This proves that $\mathcal{C}(\mathcal{E}, \varepsilon^{co}) = \varnothing$.

Proof of Proposition 2

Take any market $\mathcal{E}$ where $\succsim_i = \succ_i^G$ for all $i \in I$. Denote by σ the outcome of TTC at some profile over goods p. We claim that $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$. Define $\{T_1, ..., T_K\}$ as the set of successive TTC. Suppose that $\sigma \notin \mathcal{C}(\varepsilon^{Sco}, p)$. Let S be a coalition blocking σ for ε^{Sco} with allocation γ. Let i^* be the individual such that $C = \{i^*, i^{*\prime}\} \in S$ and $i^* \in T_{k^*}$. Moreover for all $k \in \{1, ..., k^* - 1\}$ and for all i such that $i \in T_k$, we have $\{i, i'\} \notin S$.

By definition of a blocking coalition, $(r(\gamma, R_{i^*}), r(\gamma, R_{i^{*\prime}})) < (r(\sigma, R_{i^*}), r(\sigma, R_{i^{*\prime}}))$. Suppose first that $r(\gamma, R_{i^*}) < r(\sigma, R_{i^*})$ and $r(\gamma, R_{i^{*\prime}}) = r(\sigma, R_{i^{*\prime}})$. The definition of $\succ^G$ ensures that $r(\gamma(i^*), p_{i^*}) < r(\sigma(i^*), p_{i^*})$ and $d(\sigma(i^*), \sigma(i^{*\prime})) = d(\gamma(i^*), \gamma(i^{*\prime}))$. If $r(\gamma(i^*), p_{i^*}) < r(\sigma(i^*), p_{i^*})$ then $\{\gamma(i^*), \gamma(i^{*\prime})\} \in S$, while $\gamma(i^*) \in T_k$ for some $k < k^*$. This contradicts the definition of k^*. Secondly, consider the case where $r(\gamma, R_{i^*}) < r(\sigma, R_{i^*})$ and $r(\gamma, R_{i^{*\prime}}) < r(\sigma, R_{i^{*\prime}})$. There are two possibilities in this case. First, suppose $\sigma(i^*) \neq \gamma(i^*)$. Again the definition of $\succ^G$ ensures that $r(\gamma, p_{i^*}) < r(\sigma, p_{i^*})$, so that the same contradiction as above holds. Now, suppose $\sigma(i^*) = \gamma(i^*)$. Since $\sigma(i^*) = \gamma(i^*)$, $\sigma(i^*) \in S$. Iterating the same argument shows that $\gamma(i) = \sigma(i)$ for all $i \in T_{k^*}$. Since we have $r(\gamma, R_{i^*}) < r(\sigma, R_{i^*})$ and $\sigma(i^*) = \gamma(i^*)$, we must have $d(\gamma(i^*), \gamma(i^{*\prime})) < d(\sigma(i^*), \sigma(i^{*\prime}))$. Hence, $r(\gamma(i^{*\prime}), p_{i^{*\prime}}) < r(\sigma(i^{*\prime}), p_{i^{*\prime}})$. Suppose $i^{*\prime} \in T_{k_1}$ with $k_1 > k^*$. Since $i^{*\prime}$ gets with σ her first-best good among those available at stage k_1, $i_2 = \gamma(i^{*\prime}) \in T_{k_2}$ with $k_2 < k_1$, and $\{i_2, i_2'\} \in S$. Thus $r(\gamma, R_{i_2}) \leqslant r(\sigma, R_{i_2})$. It follows that either $\gamma(i_2) = \sigma(i_2)$ and $d(\gamma(i_2), \gamma(i_2')) \leq d(\sigma(i_2), \sigma(i_2'))$ or $\gamma(i_2) \in T_{k_3}$ with $k_3 < k_2$.

Consider the former case. Since $\gamma(i^{*\prime}) \in T_{k_2}$ there must exist $j \in T_{k_2}$ such that $i_3 = \gamma(j) \in T_{k_3}$ with $k_3 < k_2$ and $\{i_3, i_3'\} \in S$. Thus we have built a set $\{i_1, i_2, i_3\}$ such that $i_1 = i^{*\prime} \in T_{k_1}$, $i_2 \in T_{k_2}$, $i_3 \in T_{k_3}$, $k_3 < k_2 < k_1$, $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3$, and $\{\{i^*, i^{*\prime}\}, \{i_2, i_2'\}, \{i_3, i_3'\}\} \subseteq S$. Iterating the construction leads to a set $\{i_1, ..., i_Z\}$ such that for all $z \in \{1, ..., Z - 1\}$, $i_z \in T_{k_z}$, $\gamma(i_z) = i_{z+1}$, $\{\{i_1, i_1'\}, ..., \{i_Z, i_Z'\}\} \subseteq S$, and $T_{k_Z} = T_{k^*} < T_{k_{Z-1}} < ... < T_{k_1}$. This contradicts $\gamma(i) = \sigma(i)$ for all $i \in T_{k^*}$.

Proof of Proposition 3

Let $\mathcal{E}$ be the market where $I = \{1, 1', ..., 5, 5'\}$, $\mathbf{L} = \{L_1\}$, where $L_1 = I$ and $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5\}$, with $C_i = \{i, i'\}$ for all $i = 1, 2, 3, 4, 5$. Consider the following profile over goods

$$p = \left(\begin{array}{cccccccccc} 1 & 1' & 2 & 2' & 3 & 3' & 4 & 4' & 5 & 5' \\ \hline 4 & 1' & 2 & 1' & 3 & 4 & 3 & 4' & 5 & 5' \\ 5 & 2' & 5 & 5' & 5 & 2' & 5' & 5 & 5' & 5 \\ 1 & 5' & 1 & 5 & 4' & 5' & 5 & 3' & ... & ... \\ .. & ... & 5' & 3' & 5' & 3' & 2 & 5' & ... & ... \\ & & 1' & 2' & ... & ... & 4 & 1' & & \\ & & ... & ... & & & ... & ... & & \end{array}\right)$$

Since $\Gamma^{\sim}(\sigma^0, R, C_5) = (1,1)$, $\sigma \in \mathcal{C}(\varepsilon^{Mm}, p)$ only if $\overrightarrow{\sigma}(C_5) = (5,5')$. Moreover, observe that $\Gamma^{\sim}(\sigma^0, R, C_1) = (1,3)$, $\Gamma^{\sim}(\sigma^0, R, C_2) = \Gamma^{\sim}(\sigma^0, R, C_4) = (1,5)$ and $\Gamma^{\sim}(\sigma^0, R, C_3) = (1,4)$. Pick $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{Mm})$. One must have $\rfloor(\sigma^0 \varepsilon_C^{Mm} \sigma)$ for all $C \in \mathbf{C}$. Equivalently, one of the two conditions below must be satisfied :

(1) $\Gamma^{-}(\sigma, R, C) < \Gamma^{-}(\sigma^0, R, C)$

(2) $\Gamma^{-}(\sigma, R, C) \geq \Gamma^{-}(\sigma^0, R, C)$ and $\Gamma^{+}(\sigma, R, C) \leq \Gamma^{+}(\sigma^0, R, C)$

Since $\Gamma^{-}(\sigma^0, R, C) = 1$ for all C, condition (1) cannot hold. It follows that

- $\Gamma^{\sim}(\sigma, R, C_1) \in \{(1,1), (1,2), (1,3), (2,2)\}$

- $\Gamma^{\sim}(\sigma, R, C_2) \in \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (3,3), (3,4)\}$

- $\Gamma^{\sim}(\sigma, R, C_3) \in \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (3,3)\}$

- $\Gamma^{\sim}(\sigma, R, C_4) \in \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (3,3), (3,4)\}$

Since $\overrightarrow{\sigma}(C_5) = (5,5')$, one easily checks that :

- $\overrightarrow{\sigma}(C_1) \in \{(4,1'), (4,2'), (1,1')\}$

- $\overrightarrow{\sigma}(C_2) \in \{(2,1'), (2,3'), (2,2'), (1,1'), (1',3')\}$, - $\overrightarrow{\sigma}(C_3) \in \{(3,4), (3,2'), (3,3'), (4',4), (4',2'\}$,

- $\overrightarrow{\sigma}(C_4) \in \{(3,4'), (3,3'), (3,1'), (2,4'), (2,3'), (4,4')\}$.

Next, we identify all possible allocations satisfying the above requirements. This leads to the following computation : starting with a couple of goods assigned to C_1, write all possible couples of goods that can be assigned to C_2, then C_3 and finally C_4. We link any two compatible couples with an arrow.

- $(4,1') \to (2,3') \to (3,2') \to \varnothing$,

- $(4,1') \to (2,2') \to (3,3') \to \varnothing$,

- $(4,2') \to (2,1') \to (3,3') \to \varnothing$,

- $(4,2') \to (2,3') \to \varnothing$,

- $(4,2') \to (1,1') \to (3,3') \to (2,4')$,

- $(1,1') \to (2,3') \to (3,2') \to (4,4')$,

- $(1,1') \to (2,3') \to (4,4') \to \varnothing$,

- $(1,1') \to (2,3') \to (3,4) \to \varnothing$,

- $(1,1') \to (2,2') \to (3,3') \to (4,4')$,

- $(1,1') \to (2,2') \to (4,4') \to (3,3')$.

This shows the existence of 4 possible allocations : σ^0, $\sigma^1 = (1, 1', 2, 3', 3, 2', 4, 4', 5, 5')$, $\sigma^2 = (4, 2', 1, 1', 3, 3', 2, 4', 5, 5')$, and $\sigma^3 = (1, 1', 2, 2', 4', 4, 3, 3', 5, 5')$. Finally, it is straightforward to check that :

- $S = \{C_2, C_3\}$ blocks σ^0 for ε^{Mm} with σ^1 : indeed, $\Gamma^{\sim}(\sigma^1, R, C_2) = (1, 4) < \Gamma^{\sim}(\sigma^0, R, C_2) = (1, 5) \Rightarrow \sigma^1 \varepsilon^{Mm}_{C_2} \sigma^0$, and $\Gamma^{\sim}(\sigma^1, R, C_3) = (1, 2) < \Gamma^{\sim}(\sigma^0, R, C_3) = (1, 4) \Rightarrow \sigma^1 \varepsilon^{Mm}_{C_3} \sigma^0$.

- $S = \{C_1, C_2, C_4\}$ blocks σ^1 for ε^{Mm} with $\sigma^2 : \Gamma^{\sim}(\sigma^2, R, C_1) = (1, 2) < \Gamma^{\sim}(\sigma^1, R, C_1) = (1, 3) \Rightarrow \sigma^2 \varepsilon^{Mm}_{C_1}(p) \sigma^1$, $\Gamma^{\sim}(\sigma^2, R, C_2) = (1, 3) < \Gamma^{\sim}(\sigma^1, R, C_3) = (1, 4) \Rightarrow \sigma^2 \varepsilon^{Mm}_{C_2} \sigma^1$, and $\Gamma^{\sim}(\sigma^2, R, C_4) = (1, 4) < \Gamma^{\sim}(\sigma^1, R, C_4) = (1, 5) \Rightarrow \sigma^2 \varepsilon^{Mm}_{C_4} \sigma^1$.

- $S = \{C_3, C_4\}$ blocks σ^2 for ε^{Mm} with σ^3 : indeed, $\Gamma^{\sim}(\sigma^3, R, C_3) = (1, 3) < \Gamma^{\sim}(\sigma^2, R, C_3) = (1, 4) \Rightarrow \sigma^3 \varepsilon^{Mm}_{C_3} \sigma^2$, and $\Gamma^{\sim}(\sigma^3, R, C_4) = (1, 3)$ and $\Gamma^{\sim}(\sigma^2, R, C_4) = (1, 4)$ ensure $\sigma^3 \varepsilon^{Mm}_{C_4} \sigma^2$.

- $S = \{C_2, C_3\}$ blocks σ^3 for ε^{Mm} with $\sigma^1 : \Gamma^{\sim}(\sigma^1, p, C_2) = (1, 4) < \Gamma^{\sim}(\sigma^3, R, C_2) = (1, 5) \Rightarrow \sigma^1 \varepsilon^{Mm}_{C_2} \sigma^3$, and $\Gamma^{\sim}(\sigma^1, R, C_3) = (1, 2) < \Gamma^{\sim}(\sigma^3, R, C_3) = (1, 3) \Rightarrow \sigma^2 \varepsilon^{Mm}_{C_3} \sigma^3$.

This proves that $\mathcal{C}(\varepsilon^{Mm}, p) = \varnothing$.

Proof of Proposition 4

Take any market $\mathcal{E}$ where $\succsim_i = \succ_i^G$ for all $i \in I$ with $d(\overrightarrow{\sigma^0}(C)) = 0$ for all C. Consider a resulting allocation σ^Z of TTC1. We claim that $\sigma^Z \in \mathcal{C}(\mathcal{E}, \varepsilon^{SMm})$. To see why, first observe that for any $C \in \mathbf{C}$ and any $z = \{1, ..., Z\}$, $C \in T^z$ then $\sigma^z(C) = \sigma^Z(C)$ and $d(\sigma^Z(i), \sigma^Z(i')) = 0$ for all $C = \{i, i'\}$.

Suppose that $\sigma^Z \notin \mathcal{C}(\mathcal{E}, \varepsilon^{SMm})$, and let coalition S block σ^Z with σ. Pick $C = \{i_k, i'_k\} \in S \cap T^z$ be such that $S \cap [\cup_{1 \leq z' < z} T^{z'}] = \varnothing$. Assume without loss of generality that $x(C, G^z) = 1_{G^z}(i_k)$. By definition of ε^{SMm}, $\Gamma^-(\sigma, R, C) < \Gamma^-(\sigma^Z, R, C)$. Moreover, by definition of T^z, $r(\sigma^Z, p_{i_k}) \leq r(\sigma^Z, p_{i'_k})$. This implies that preferences giving priority to goods requires $r(\sigma, p_{i_k}) < r(\sigma^Z, p_{i_k})$ or $r(\sigma, p_{i_{k'}}) < r(\sigma^Z, p_{i_k})$. Hence one gets $\sigma(C) \cap G^{z'} \neq \varnothing$ for some $z' < z$. Thus $S \nsubseteq \mathbf{C}^z$, in contradiction with the definition of C.

Proof of Proposition 5

Consider the market $\mathcal{E}$ defined in the proof of proposition 1, where preference extensions can be completed so as to be priority-to-distance. Since $\mathcal{C}(\mathcal{E}, \varepsilon^{co}) = \varnothing$, the proof is complete for $s = 1$.

Suppose $s > 1$. Define $\bar{\mathcal{E}} = \{\bar{I}, \bar{G}, \bar{\mathbf{C}}, \bar{\mathbf{L}}, \bar{p}, \bar{\succsim}^{\bar{n}}, \bar{\sigma^0}\}$ as follows :

— $\bar{I} = I \cup \hat{I}$, where $I \cap \hat{I} = \varnothing$ and $\hat{I} = \{i_1, ..., i_{2(z-1)}\}$
— $\bar{\mathcal{C}} = \mathcal{C} \cup \hat{\mathcal{C}}$, where $\mathcal{C} \cap \hat{\mathcal{C}} = \varnothing$ and $\hat{\mathcal{C}} = \{\hat{\mathcal{C}}_h : 1 \leq h \leq z-1\}$ where $\hat{\mathcal{C}}_h = \{i_{2h-1}, i_{2h}\}$
— $\bar{\mathcal{L}} = \mathcal{L} \cup \hat{\mathcal{L}}$, where $\mathcal{L} \cap \hat{\mathcal{L}} = \varnothing$ and $\hat{\mathcal{L}} = \{\{i_h\} : 1 \leq h \leq 2(z-1)\}$
— $\bar{p} = (\bar{p}_i)_{i \in \bar{I}}$ is such that :
  — $\forall i, j, k \in I$, $[j\ p_i\ k] \Leftrightarrow [j\ \bar{p}_i\ k]$
  — $\forall i, j \in I$, $\forall k \in \hat{I}$, $[j\ \bar{p}_i\ k]$
  — $\forall i \in \hat{I}$, $r(i, p_i) = 1$

— $\forall i \in \bar{I}$, $\bar{\sigma^0}(i) = i$

Let $\sigma \in \mathcal{C}(\bar{\mathcal{E}}, \varepsilon^{co})$. From the definition of $\mathcal{E}$ together with the construction of $\bar{\mathcal{E}}$, one has : **(a)** $\exists h, k \in \{1,2,3\}$ with $h \neq k$ such that $\forall i \in C_h \cup C_k$, $d(\sigma(i), \sigma(c(i))) = 0$, **(b)** there exists a coalition S such that $S \cap (C_h \cup C_k) \neq \varnothing$ which blocks σ. Therefore $\mathcal{C}(\bar{\mathcal{E}}, \varepsilon^{co}) = \varnothing$.

Proof of Proposition 6

Let $\mathcal{E}$ be such that $I = G = \{1, 1', 2, 2', 3, 3', 4, 4', 5, 5', 6, 6'\}$, $\mathbf{C} = \{C_1, ..., C_6\}$, $C_i = \{i, i'\}$ with $i \in \{1, ..., 6\}$, $L_1 = \{1, ..., 6\}$ and $L_2 = \{1', ..., 6'\}$. Consider the profile over goods p where a set in some cell means that any ranking of the elements of this set can be chosen :[11]

$$p = \left(\begin{array}{cccccc} 1,1' & 2,2' & 3,3' & 4,4' & 5,5' & 6,6' \\ \hline L_2 & L_2 & L_2 & L_2 & L_2 & L_2 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 1 & 5 & 4 & 4 \\ 3 & 3 & 2 & 6 & 6 & 5 \\ ... & ... & ... & ... & ... & ... \end{array}\right)$$

Obviously, $\mathcal{E}$ is 0-solvable, with exactly 3 couples being assigned goods in L_2. Pick an allocation σ. By priority-to-distance preferences together with the definition of p, σ is Core stable only if 3 couples are assigned goods in L_2. Let z be the number of couples C in $\{C_1, C_2, C_3\}$ such that $\sigma(C) \subset L_2$. Observe that $z \in \{0, ..., 3\}$ and consider the following cases :

Case 1 : $z \in \{0, 3\}$

Case 1.1 : $z = 0$

If $1 \in \sigma(C_1)$, $\{C_1, C_3\}$ blocks σ for ε^{Sco} with $\sigma' = (1', 3', 2, 2', 3, 1, 4, 4', 5, 5', 6, 6')$ if $\sigma(3) = 3$ and with $\sigma'' = (1', 3', 2, 2', 1, 3, 4, 4', 5, 5', 6, 6')$ if either $\sigma(3') = 1$ or $1 \notin \sigma(C_3)$. If $1 \notin \sigma(C_1)$, $\{C_1, C_2\}$ blocks σ for ε^{Sco} with $\gamma = (1, 2, 1', 2', 3, 3', 4, 4', 5, 5', 6, 6')$.

Case 1.2 : $z = 3$

It follows that all 3 couples C in $\{C_4, C_5, C_6\}$ are such that $\sigma(C) \subset L_1$, and a similar argument applies.

Case 2 : $z \in \{1, 2\}$

Case 2.1 : $z = 1$

— Suppose $\sigma(C_1), \sigma(C_2) \subset L_1$.
If $2 \in \sigma(C_2)$, $\{C_1, C_2\}$ blocks σ for ε^{Sco} with $\gamma = (1, 2, 1', 2', 3, 3', 4, 4', 5, 5', 6, 6')$.
If $2 \notin \sigma(C_2)$, $\{C_1, C_2\}$ blocks σ for ε^{Sco} with $\gamma' = (1', 2', 2, 1, 3, 3', 4, 4', 5, 5', 6, 6')$.
— Suppose $\sigma(C_1), \sigma(C_3) \subset L_1$.

11. We use the following conventions : $L_2 p_i j$ if and only if $i p_i j$ for all $i \in L_2$.

If $1 \in \sigma(C_1)$, $\{C_1, C_3\}$ blocks σ for ε^{Sco} with $\delta = (1', 3', 2, 2', 3, 1, 4, 4', 5, 5', 6, 6')$ if $\sigma(3) = 3$, and with $\delta' = (1', 3', 2, 2', 1, 3, 4, 4', 5, 5', 6, 6')$ if either $\sigma(3') = 3$ or $3 \notin \sigma(C_3)$.
If $1 \notin \sigma(C_1)$, $\{C_1, C_3\}$ blocks σ for ε^{Sco} with $\lambda = (1, 3, 2, 2', 1', 3', 4, 4', 5, 5', 6, 6')$ if $\sigma(1) = 2$, and with $\lambda' = (3, 1, 2, 2', 1', 3', 4, 4', 5, 5', 6, 6')$ if either $\sigma(1') = 2$ or $2 \notin \sigma(C_1)$.

— Suppose $\sigma(C_2), \sigma(C_3) \subset L_1$.
  — Suppose $2 \in \sigma(C_2)$.
    If $1 \in \sigma(C_2)$, $\{C_2, C_3\}$ blocks $\sigma$for ε^{Sco} with $\rho = (1, 1', 2', 3', 3, 2, 4, 4', 5, 5', 6, 6')$ if $\sigma(3) = 3$ and with $\rho' = (1, 1', 2', 3', 2, 3, 4, 4', 5, 5', 6, 6')$ if either $\sigma(3') = 3$ or $3 \notin \sigma(C_3)$.
    If $\sigma(C_3) = \{1, 3\}$, $\{C_2, C_3\}$ blocks $\sigma$for ε^{Sco} with $\omega = (1, 1', 2, 3, 2', 3', 4, 4', 5, 5', 6, 6')$ if $\sigma(2) = 2$, with $\omega' = (1, 1', 3, 2, 2', 3', 4, 4', 5, 5', 6, 6')$ if $\sigma(2') = 2$.
    If $1 \in \sigma(C_3)$ and $3 \notin \sigma(C_3)$, $\{C_2, C_3\}$ blocks σ for ε^{Sco} with ρ. If $1 \notin \sigma(C_2) \cup \sigma(C_3)$, $\{C_2, C_3\}$ blocks σ for ε^{Sco} with ρ if $3 \in \sigma(C_2)$, with ω if $3 \notin \sigma(C_2)$ and $\sigma(2) = 2$, and with ω' if $3 \notin \sigma(C_2)$ and $\sigma(2') = 2$.
    If $1 \notin \sigma(C_3)$, $\{C_2, C_3\}$ blocks σ for ε^{Sco} with ρ if $\sigma(3) = 3$, and with ρ' if either $\sigma(3') = 3$ or $3 \notin \sigma(C_3)$.
  — Suppose $2 \notin \sigma(C_2)$.
    If $\sigma(2) = 1$, $\{C_2, C_3\}$ blocks σ for ε^{Sco} with ω'. If $\sigma(2') = 1$, $\{C_2, C_3\}$ blocks σ for ε^{Sco} with ω. If $1 \notin \sigma(C_2)$, $\{C_2, C_3\}$ blocks σ for ε^{Sco} with ω' if $\sigma(3') = 3$, and with ω if either $\sigma(3) = 3$ or $3 \notin \sigma(C_2)$

Case 2.2 : $z = 2$

It follows that 2 couples C in $\{C_4, C_5, C_6\}$ are such that $\sigma(C) \subset L_1$, and a similar argument applies.

Proof of Proposition 7

Take any market $\mathcal{E}$ where $\succsim_i = \succ_i^D$ for all $i \in I$ with $\sigma^0(i) = i$ for all $i \in I$ and $d(\overrightarrow{\sigma^0}(C)) = 0$ for all C. Consider a resulting allocation σ^Z of TTC1. We claim that $\sigma^Z \in \mathcal{C}(\mathcal{E}, \varepsilon^{SMm})$.

Suppose that $\sigma^Z \notin \mathcal{C}(\mathcal{E}, \varepsilon^{SMm})$, and let coalition S block σ^Z with σ. Pick $C = \{i_k, i'_k\} \in S \cap T^z$ be such that $S \cap [\cup_{1 \leq z' < z} T^{z'}] = \varnothing$. Assume w.l.o.g. that $x(C, G^z) = 1_{G^z}(i_k)$. By definition of ε^{SMm}, $\Gamma^-(\sigma, R, C) < \Gamma^-(\sigma^Z, R, C)$. Moreover, by definition of T^z, $r(\sigma^Z, p_{i_k}) \leq r(\sigma^Z, p_{i'_k})$. This implies that preferences with priority to distance require either $r(\sigma, p_{i_k}) < r(\sigma^Z, p_{i_k})$ or $r(\sigma, p_{i_{k'}}) < r(\sigma^Z, p_{i_k})$ (since $d(\sigma^Z(i_k), \sigma^Z(i'_k)) = 0$). Hence, one gets $\sigma(C) \cap G^{z'} \neq \varnothing$ for some $z' < z$. Thus $S \nsubseteq \mathbf{C}^z$, in contradiction with the definition of C.

Proof of Proposition 8

Let $\mathcal{E}$ be the market where $I = \{1, 1', ..., 5, 5'\}$, $\mathbf{L} = \{L_1, L_2\}$, $L_1 = \{1, 1', 2, 2', 3, 3', 4, 4'\}$, $L_2 = \{5, 5'\}$ and $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5\}$, with $C_i = \{i, i'\}$ for all $i = 1, 2, 3, 4, 5$. Moreover $\succsim_i = \succ_i^D$ for all $i \in I$ and $d(\overrightarrow{\sigma^0}(C)) = 0$ for all $C \in \mathbf{C}$. Consider the same profile

p in the proof of Proposition 3. All arguments in the proof of Proposition 3 also apply here. Hence we can conclude that $\mathcal{C}(\varepsilon^{Mm}, p) = \varnothing$.

Proof of Proposition 9

Suppose $\sigma^* \notin \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$, and let $S \subseteq \mathbf{C}$ and $\gamma \in \Sigma$ satisfying

(1) $\forall C \in S$, $\gamma \varepsilon_C^{Sco}(\succsim_p)\sigma$, and $\rceil(\sigma \varepsilon_C^{Sco} \gamma)$

(2) $\sigma(\mathbf{C} \backslash S) = \sigma^0(\mathbf{C} \backslash S)$.

Claim 1 : Let $C \in S$. If $C \cap T_1 \neq \varnothing$, then $\forall i \in T_1$, $\gamma(i) = \sigma^*(i)$ and $\{i, i'\} \in S$.

Proof of Claim 1 : Pick $\{j, j'\} \in S$ with $j \in T_1$. If $\gamma(j) \neq \sigma^*(j)$, $\gamma(j)\ p_j\ \sigma^*(j)$, which contradicts the definition of T_1. Thus $\gamma(j) = \sigma^*(j) = j_1$. This implies in turn that $\{j_1, j_1'\} \in S$ with $j_1 \in T_1$. The conclusion follows from iterating the argument for j_1, j_2, ... until all individual in T_1 are taken into account.

Claim 2 : Let $C \in S$. If $C \cap T_2 \neq \varnothing$, then $\forall i \in T_2$, $\gamma(i) = \sigma^*(i)$ and $\{i, i'\} \in S$.

Proof of Claim 2 : Pick $\{j, j'\} \in S$ with $j \in T_2$. If $\gamma(j) \neq \sigma^*(j)$, $\gamma(j)\ q_j^2\ \sigma^*(j)$, which implies $\gamma(j) \notin G_2$. This implies that $\gamma(j) \in \{\sigma^1(i) : i \in T_1\}$. In other words, either $\gamma(j) \in T_1$ or $\gamma(j)' \in T_1$. Since $j \notin T_1$, one contradicts claim 1. The conclusion follows from iterating the argument for all individuals in T_2.

Claim 3 : Let $s \in \{1, ..., s^*\}$ and let $C \in S$. If $C \cap T_s \neq \varnothing$, then $\forall i \in T_s$, $\gamma(i) = \sigma^*(i)$ and $\{i, i'\} \in S$.

Proof of Claim 3 : Induction hypothesis : Let $s \in \{1, ..., s^*\}$ and let $C \in S$. If $C \cap T_{s'} \neq \varnothing$ for some $s' < s$, then $\forall i \in T_{s'}$, $\gamma(i) = \sigma^*(i)$ and $\{i, i'\} \in S$.

Pick $\{j, j'\} \in S$ with $j \in T_s$. If $\gamma(j) \neq \sigma^*(j)$, $\gamma(j)\ q_j^s\ \sigma^*(j)$, which implies $\gamma(j) \notin G_s$. Moreover, $\{\gamma(i), \gamma(i)'\} \in S$. By the induction hypothesis, one cannot have $\gamma(i) \in T_{s'}$ for some $s' < s$. Thus $\gamma(i)' \in T_{s'}$ and at stage s' good $\gamma(i)$ has been assigned to some j' with $j \in T_{s'}$. Again by the induction hypothesis, $\{j, j'\} \in S$. Moreover, since $\gamma(j) \notin G_s$, $j' \in T_{s''}$ with $s'' < s$ and one gets $\gamma(j) = \sigma^*(j)$ and $\gamma(j') = \sigma^*(j')$, in contradiction with $\{j, j'\} \in S$.

For all $C \in \mathbf{C}$, we denote the stage s' such that $C \cap T_{s'} \neq \varnothing$ by $s'(C)$. Pick $C = \{j_1, j_1'\} \in S$ such that $s'(C) \leq s'(C')$ for all $C' \in S$. Assume w.l.o.g. that $j_2 = \gamma(j_1) \neq \sigma^*(j_1)$, and $j_1 \in T_{s_1}$.

Case 1 : $j_2 \in T_{s_2}$, and $s_2 = s(\{j_2, j_2'\})$

Since $j_2 = \gamma(j_1)$ and $j_1 \in S$, $\{j_2, j_2'\} \in S$. By Claim 3, $\gamma(i) = \sigma^*(i)$ for all $i \in T_{s_2}$, which makes $j_2 = \gamma(j_1)$ impossible.

Case 2 : $j_2 \in T_{s_2}$, and $s_2 = s'(\{j_2, j_2'\})$

Since $s'(C) \leq s'(C')$ for all $C' \in S$, $s_2 \geq s_1$. The algorithm ensures that $\widetilde{s}_2 = s(\{j_2, j_2'\}) < s_1$ (otherwise, $j_2 \in G_{s_1}$, in contradiction with $\sigma^*(j_1) \in T_{s_1} : j_1$ does not point towards her first-best good). It follows that :

- $j_2' \in T_{\widetilde{s}_2}$, $\{j_2, j_2'\} \in S$,

- good j_2 has been assigned to some individual h_2 with $h_2' \in T_{\widetilde{s}_2}$,

- $j_2 \notin G_{s_1}$: good j_2 has been assigned as final good before stage s_1.

By Claim 3, $\{j_2, j_2'\} \in S$ implies $\{h_2, h_2'\} \in S$, which contradicts $s'(C) \leq s'(C')$ for all $C' \in S$.

Proof of Proposition 10

Let $\mathcal{E}$ be the market where $I = \{1, 1', 2, 2'\}$, $\mathbf{L} = \{L_1, L_2\}$, $L_1 = \{1, 1'\}$, $L_2 = \{2, 2'\}$ and $C_i = \{i, i'\}$ for all $i = 1, 2$. Consider the profile over goods

$$p = \begin{pmatrix} 1 & 1' & 2 & 2' \\ \hline 1' & 1' & 2' & 2' \\ 2 & 1 & 1 & 2 \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Suppose that all preference extensions give priority to distance : $(1,0) \succ_i (2,0) \succ_i (1,1) \succ_i (2,1)$ for all $i \in I$. This gives the profile over allocations

$$R = \begin{pmatrix} 1 & 1' & 2 & 2' \\ \hline (1',1) & (1,1') & (2',2) & (2,2') \\ (2,2') & (1',1) & (1,1') & (2',2) \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Consider allocation $\sigma = (1', 1, 2', 2)$. Observe that $\Gamma^+(\sigma, R, C_i) = 2$ and $\Gamma^-(\sigma, R, C_i)) = 1$ for all $i = 1, 2$. This shows that $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{Mm})$. Moreover, $\Gamma^{\sim}(\gamma, R, C_i) < \Gamma^{\sim}(\gamma, R, C_i)$ implies $\Gamma^+(\gamma, R, C_i)) = \Gamma^-(\gamma, R, C_i)) = 1$, which is clearly impossible. Hence $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{Mm}) \cap \mathcal{C}(\mathcal{E}, \varepsilon^{sMm})$. Since $\mathcal{C}(\mathcal{E}, \varepsilon^{Mm}) \subseteq \mathcal{C}(\mathcal{E}, \varepsilon^{co})$ and $\mathcal{C}(\mathcal{E}, \varepsilon^{SMm}) \subseteq \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$, $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon)$for all $\varepsilon \in \{\varepsilon^{Mm}, \varepsilon^{SMm}, \varepsilon^{Sco}, \varepsilon^{co}\}$. However, $\sigma^0 P_i \sigma$ for $i = 1', 2'$.

Proof of Proposition 11

Proof of assertion (1) : Suppose $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$ is not Pareto optimal. So there exists $\sigma' \in \Sigma$ such that $\sigma' R_i \sigma$ for all $i \in I$ with $\sigma' P_j \sigma$ for some $j \in I$. It follows that $\Gamma(\sigma', R, C) = (r(\sigma', R_i), r(\sigma', R_{i'})) < \Gamma(\sigma, R, C) = (r(\sigma, R_i), r(\sigma, R_{i'}))$, which contradicts that $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$. Since $\mathcal{C}(\mathcal{E}, \varepsilon^{SMm}) \subseteq \mathcal{C}(\mathcal{E}, \varepsilon^{Sco})$, the proof is complete.

Proof of assertion (2) : Let $\mathcal{E}$ be the market where $I = \{1, 1', 2, 2'\}$, $\mathbf{L} = \{L_1\}$, $L_1 = \{1, 1', 2, 2'\}$ and $C_i = \{i, i'\}$ for all $i = 1, 2$. Consider the profile over goods

$$p = \begin{pmatrix} 1 & 1' & 2 & 2' \\ \hline 1' & 2' & 2 & 1 \\ \dots & 1 & \dots & 2' \\ & \dots & & \dots \end{pmatrix}$$

Suppose that preference extensions give priority to goods : $(1,0) \succ_i (1,1) \succ_i (2,0) \succ_i (2,1)$ for all $i \in I$. The profile over allocations is

$$R = \left(\begin{array}{cccc} 1 & 1' & 2 & 2' \\ \hline (1',1);(1',2);(1',2') & (1,2');(2,2');(1',2') & (2,1);(2,2');(2,1') & (1',1);(2,1);(2',1) \\ ... & (1',1);(2,1);(2',1) & ... & (1,2');(1',2');(2,2') \\ & ... & & ... \end{array} \right)$$

Pick allocation $\sigma = (1', 1, 2, 2')$. Observe that $\Gamma(\sigma, R, C_i) = (1, 2)$ for all $i = 1, 2$. The reader can easily check that $\sigma \in \mathcal{C}(\mathcal{E}, \varepsilon^{co})$ while allocation $\sigma' = (1', 2', 2, 1)$ Pareto dominates σ.

Chapitre 5

Choosing a Committee under Majority Voting

Abstract : We consider the elections of a seat-posted committee, and investigate the propensity of seat-wise majority voting to choose a committee that fulfills the majority will with respect to preferences over committees. Voters have seat-wise preferences and preferences over committees are derived from seat-wise preferences by means of a neutral preference extension. Neutrality means that the names of candidates do not play any role. The majority committee paradox refers to a situation where a Condorcet winner exists for each seat, and a Condorcet winner committee also exists but does not coincide with the combination of seat-wise Condorcet winners. The majority committee weak paradox refers to a situation where the combination of seat-wise Condorcet winners is not a Condorcet winner among committees. We characterize the domains of preference extensions immune to each of the paradoxes.[1]

Keywords: Committee election - Voting Paradoxes - Majority voting - Separable preferences

5.1 Introduction

Arrovian social choice theory provides a theoretical framework for evaluating social choice functions, which aggregates individual ordinal preferences over social alternatives, or candidates, into a collective outcome. In the case where the outcome is a single candidate, asking voters to report their ranking of candidates is not problematic. However, if a committee of several candidates is to be chosen, this informational requirement is hardly implementable in practice. Consider an election of a faculty council involving a dean, a vice-dean for research and a vice-dean for teaching. If there are four candidates per seat, fully expressing preferences means ranking the 64 possible outcomes. Clearly, as the number of seats or the number of candidates for each seat increase,

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referring to Arrovian social choice functions becomes less and less useful in practice. Designing a seat-wise procedure is a frequent solution that overcomes this difficulty. In a seat-wise procedure, voters report their preferences over candidates seat-wise, and candidates are selected seat-wise. It is well-known that a seat-wise procedure may not lead to the outcome that would prevail for a direct choice procedure where voters report their preferences over the outcomes. This happens when individual preferences exhibit complementarities among candidates, but this may even prevail with separable preferences which prohibits any sort of complementarity.

The potential inconsistency between seat-wise and direct procedures results from the fact that seat-wise preferences describe only partially preferences over outcomes. A rather rich literature dealing with this inconsistency and other potential drawbacks of seat-wise procedures deals with multiple referenda, which is equivalent to a committee choice problem with two candidates per seat. In this setting, each voter is characterized by an ideal committee, and simple majority voting provides a natural seat-wise choice procedure that we denote by Maj. Compound majority voting paradoxes studied in the literature express the fact that Maj may lead to outcomes exhibiting some undesirable properties. The Anscombe's paradox (Anscombe (1976), Wagner (1984), Laffond and Lainé (2013)) shows that a majority of voters may disagree with the outcome of Maj on a majority of seats. The multiple elections paradox (Brams et al. (1998), Scarsini (1998)) prevails when the winner for Maj receives zero votes in the direct elections (or, equivalently, may be ranked first by no voter). The Ostrogorski paradox (Ostrogorski (1913), Rae and Daudt (1976), Bezembinder and Van Acker (1985), Deb and Kelsey (1987), Kelly (1989), Shelley (1994), Laffond and Lainé (2006)) prevails when another outcome beats the one of Maj according to majority voting under the assumption that committees are compared by means of the Hamming distance criterion.[2] [3]

The Hamming distance criterion provides a specific way to relate seat-wise preferences and preferences over committees. Other ways can be considered, each referring to a particular preference extension. Formally, when there are only two candidates per seat, a preference extension rule maps each ideal committee to a (weak) ordering of committees. A usual property retained for a preference extension is separability : if a and b are the two candidates for some seat s, and if a voter ranks a above b, she will rank two committees identical in all seats but s according to her preference over a and b.[4] Kadane (1972) shows that even under the assumption of a separable extension, Maj may select a Pareto dominated committee. Moreover, Özkal-Sanver and Sanver (2006) show that if there are at least three seats, no anonymous seat-wise procedure

2. Hamming distance criterion in this specific setting simply means that voters prefer the committee(s) agreeing with her ideal on a higher number of seats.

3. Laffond and Lainé (2009) show that Maj always selects a Pareto optimal element in the Top-Cycle of the majority tournament among outcomes (Schwartz 1972) while Maj may select an outcome which does not belong to the Uncovered set (Miller (1977), Moulin (1986)). An overview of compound majority paradoxes in multiple referenda is provided in Laffond and Lainé (2010).

4. Lacy and Niou (2000) show that under a non-separable preference extension rule, Maj may select a Condorcet-loser outcome (i.e., an outcome majority defeated by all other outcomes). However, if separability holds, Maj always chooses the Condorcet winner outcome (i.e., the outcome majority defeating all other outcomes) whenever it exists (Kadane (1972)).

guarantees that a Pareto optimal committee will be chosen for all separable preference extensions. However, for the Hamming preference extension rule, Maj always produce a Pareto-optimal committee (Brams et al. 2007). Çuhadaroğlu and Lainé (2009) prove that under a mild richness assumption, the Hamming preference extension defines the largest domain of separable preference extensions for which Maj always picks a Pareto optimal outcome.[5]

All the above mentioned studies deal with the case of two candidates per seat. Less attention has been paid to situations where there are more than two candidates per seat. Benoît and Kornhauser (2010) generalize the result of Özkal-Sanver and Sanver (2006) : if any separable preference extension is admissible, and if there are at least three seats or when there are precisely two seats with more than two candidates per seat, a seat-wise procedure selects a Pareto optimal outcome if and only if it is dictatorial. While this strong result disqualifies seat-wise procedures in the full domain of separable preferences extensions, it suggests investigating whether they can perform better under some domain restrictions. This is the route followed in this paper, which addresses the following question : can we characterize the class of preference extensions under which Maj selects a Condorcet winner committee, that is a committee preferred to all other committees by a majority of voters ?

One difficulty is that Maj is not well-defined with at least three candidates per seat. Indeed, it is well-known that a Condorcet winner for each seat (i.e., a candidate preferred to all other candidates by a majority of voters) may fail to exist. However, well-known restrictions upon voters' preferences ensure the existence of a seat-wise Condorcet winner are single-peakedness and Sen's value restriction (Black (1948), Sen (1966)). We assume that preferences over candidates for each seat are such that a Condorcet winner exists, and we address the following problem : characterizing the preference extension domain for which the committee formed by all seat-wise Condorcet winners form a Condorcet winner among committees. If Maj selects a Condorcet winner committee, there is no inconsistency between seat-wise majority voting and direct majority voting (where voters rank committees). Hence, characterizing the preference extension domain that precludes this inconsistency solves the problem created by the Arrovian informational requirement : in order to fulfill the majority will for committees, it is sufficient to fulfill the majority will for each seat.

The inconsistency between seat-wise majority voting and direct majority voting arises in two cases, each related to a new voting paradox. The *majority committee paradox* prevails when a Condorcet winner committee exists and is not selected by Maj. The *majority committee weak paradox* prevails when either the majority committee paradox holds or a Condorcet winner committee fails to exist (while Maj is well-defined).

Under a neutrality assumption for preference extensions (meaning that candidates' names play no role), we characterize the preference extension domain immune to the majority committee paradox and the one immune to the majority committee weak

5. Within a similar setting, Laffond and Lainé (2012) show that Maj may fail at implementing a compromise, even under strong restrictions upon the seat-wise majority margin.

paradox in the case of two-seat committees. More precisely, we prove that separability is a necessary and sufficient condition for a neutral preference extension to avoid the majority committee paradox. Moreover, the domain of neutral preference extensions avoiding the majority committee weak paradox is much smaller, it reducing to a unique lexicographic preference extension. According to lexicographic preference extensions all voters agree on a seat as priority seat and compares committees according to their ranking of candidates for that priority seat whenever they differ and if both committees have the same candidate on the priority seat, compares them according to ranking of candidates for other seat.

Our results complement the ones obtained by Hollard and Le Breton (1996) and Vidu (1999, 2002). In the case of two candidates per seat, Hollard and Le Breton (1996) show that any separable tournament over committees can be achieved through seat-wise majority voting. This result is generalized in Vidu (1999) to the case of more than two candidates per seat. Moreover, Vidu (2002) shows that a similar result prevails even when seat-wise preferences are single-peaked (implying the existence of seat-wise Condorcet winners).

The paper is organized as follows. Section 2 provides basic notations and definitions. Majority committee paradoxes are formalized in section 3. Results are stated in section 4. We conclude with several comments about further research, with some preliminary results. All proofs are postponed to an Appendix.

5.2 The Model

5.2.1 Preliminaries

We consider two finite sets $\mathbb{C}_1$ and $\mathbb{C}_2$, with respective cardinalities C_1 and C_2. Sets $\mathbb{C}_1$ and $\mathbb{C}_2$ are interpreted as sets of candidates competing for seat-1 and for seat-2. We use letters a, b, c to denote arbitrary candidates for seat-1 and x, y, z to denote arbitrary candidates for seat-2. A *committee* $\mathcal{C}$ is an element of $\mathbb{C} = \mathbb{C}_1 \times \mathbb{C}_2$.

We also consider a finite set of voters $\mathcal{N} = \{1, ..., n, ..., N\}$ where $N \in \mathbb{N}$ is odd. For each $t \in \{1, 2\}$, every voter n has preferences over candidates in $\mathbb{C}_t$, called t-preferences, represented by a complete linear order P_n^t. The upper-contour set of a candidate $a \in \mathbb{C}_t$ for P_n^t is defined by $U(a, P_n^t) = \{b \in \mathbb{C}_t : bP_n^t a\}$. Moreover, the rank of a in P_n^t is defined by $r^t(a, P_n^t) = 1 + |U(a, P_n^t)|$. Given a finite set X, let $\mathcal{L}(X)$ denote the set of linear orders over X. Finally, for $t = 1, 2$, a t-profile $\pi^t = (P_n^t)_{n\in\mathcal{N}}$ is an element of $(\mathcal{L}(\mathbb{C}_t))^N$.

We call preference over committees, or in short preference, of voter n an element $P_n = (P_n^1, P_n^2)$ of $\mathcal{L}(\ \mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2)$. The P_n-rank vector of committee $\mathcal{C} = (a, x) \in \mathbb{C}$ is defined by $r(\mathcal{C}, P) = (r^1(a, P_n^1), r^2(x, P_n^2))$. A *profile* is an element $\pi = (P_n)_{n\in\mathcal{N}}$ of $(\mathcal{L}(\mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2))^N$.

5.2.2 Preference Extension

Seat-wise preferences over candidates and preferences are logically related. We assume that preferences over candidates for each seat are extended to preferences by means of a preference extension. Formally, *a preference extension* is a mapping δ from $\mathcal{L}(\mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2)$ to $\mathcal{L}(\mathbb{C})$. A *preference extension profile* is a vector $\delta^{\mathcal{N}} = (\delta_1, ..., \delta_N)$ of preference extensions. Given a profile $\pi = (P_n)_{n\in\mathcal{N}} \in (\mathcal{L}(\mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2))^N$, a preference extension profile $\delta^{\mathcal{N}}$ generates the extended profile $\delta^{\mathcal{N}}(\pi) = ((\delta_n(P_n))_{n\in\mathcal{N}} \in (\mathcal{L}(\mathbb{C}))^N$.

We retain two properties for preference extensions, neutrality and separability. Neutrality prevails if the names of candidates do not matter when comparing committees. In other words, only ranks given to candidates are taken into account.

Definition 5.2.1 *A preference extension δ is **neutral** if for all $P = (P^1, P^2)$, $P' = (P'^1, P'^2) \in \mathcal{L}(\mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2)$, and for all $\mathcal{C}, \mathcal{C}' \in \mathbb{C}$, if $[r(\mathcal{C}, P) = r(\mathcal{C}, P')$ and $r(\mathcal{C}', P) = r(\mathcal{C}', P')$] then $[\mathcal{C}\delta(P)\mathcal{C}' \Leftrightarrow \mathcal{C}\delta(P')\mathcal{C}']$.*

It follows from definition 1 that given a preference $P \in \mathcal{L}(\mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2)$, any neutral preference extension δ can be equivalently defined as the linear order $\gg_\delta$ over $\{1, ..., C_1\}\times\{1, ..., C_2\}$ by : for all $i, j \in \{1, ..., C_1\}\times\{1, ..., C_2\}$, $i \gg_\delta j$ if and only if there exists $\mathcal{C}, \mathcal{C}' \in \mathbb{C}$ such that $\mathcal{C}\delta(P)\mathcal{C}'$ where $r(\mathcal{C}, P) = i$ and $r(\mathcal{C}', P) = j$. Hence, *a neutral preference extension profile* can be equivalently defined by vector $\gg_{\delta^{\mathcal{N}}} = (\gg_{\delta_1}, ..., \gg_{\delta_N})$.

We denote the set of neutral preference extensions by Δ.

The following example illustrates how a neutral extension operates. Let $\mathbb{C}_1 = \{a, b\}$, $\mathbb{C}_2 = \{x, y\}$, $\mathcal{N} = \{1, 2, 3\}$, and consider the seat-wise profiles $\pi = (\pi^1, \pi^2) = ((P^1_n, P^2_n)_{n=1,2,3})$ defined below

$$\pi^1 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ \hline a & b & a \\ b & a & b \end{array}\right), \pi^2 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ \hline x & y & x \\ y & x & y \end{array}\right)$$

Let $\gg_{\delta^{\mathcal{N}}}$ be the following neutral preference extension profile

$$\gg_{\delta^{\mathcal{N}}} = \left(\begin{array}{ccc} \gg_{\delta_1} & \gg_{\delta_2} & \gg_{\delta_3} \\ \hline (1,1) & (1,1) & (1,1) \\ (2,1) & (2,1) & (1,2) \\ (1,2) & (1,2) & (2,1) \\ (2,2) & (2,2) & (2,2) \end{array}\right)$$

$\gg_{\delta^{\mathcal{N}}}$ combined with $\pi = (\pi^1, \pi^2)$ lead to the following extended profile

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{ccc} 1 & 2 & 3 \\ \hline (a,x) & (b,y) & (a,x) \\ (b,x) & (a,y) & (a,y) \\ (a,y) & (b,x) & (b,x) \\ (b,y) & (a,x) & (b,y) \end{array} \right)$$

Hereafter we refer to neutral preference extensions simply as preference extensions. The second property for preference extension rules we consider, called separability, holds if it is always preferable to assign a seat to a better candidate whoever the other committee member is. Hence, separability precludes any complementarity between candidates for different seats.

Definition 5.2.2 *A preference extension δ is separable if for all $a, b \in \mathbb{C}_1$, for all $x, y \in \mathbb{C}_2$, and for all $P = (P^1, P^2) \in \mathcal{L}(\mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2)$,*

- *(a,x) $\delta(P)$ (b,x) if and only if a P^1 b*
- *(a,x) $\delta(P)$ (a,y) if and only if x P^2 y.*

Under neutrality, separability is equivalently defined as follows :

- for all $(i_1, i_2) \neq (j_1, i_2) \in \{1, ..., C_1\} \times \{1, ..., C_2\}$, $(i_1, i_2) \gg_\delta (j_1, i_2)$ if $i_1 < j_1$
- for all $(i_1, i_2) \neq (i_1, j_2) \in \{1, ..., C_1\} \times \{1, ..., C_2\}$, $(i_1, i_2) \gg_\delta (i_1, j_2)$ if $i_2 < j_2$.

The set of neutral and separable extensions is denoted by Δ^{sep}. We introduce below two specific elements of this set :

Definition 5.2.3
- *The 1-lexicographic preference extension δ^{1Lex} is defined by :* $(1,1) \gg (1,2) \gg ... \gg (1,C_2) \gg ... \gg (C_1,1) \gg (C_1,2)... \gg (C_1,C_2)$.
- *The 2-lexicographic preference extension, δ^{2Lex} is defined by :* $(1,1) \gg (2,1) \gg ... \gg (C_1,1) \gg ... \gg (1,C_2) \gg (2,C_2)... \gg (C_1,C_2)$.

The following definition of a choice problem summarizes all the relevant features of a committee selection procedure :

Definition 5.2.4 *A choice problem is a 5-tuple $\mathcal{P} = (\mathbb{C}_1, \mathbb{C}_2, \mathcal{N}, \pi, \delta^{\mathcal{N}})$ where $\mathbb{C}_1$ and $\mathbb{C}_2$ are the set of candidates for seat-1 and seat-2 respectively, $\mathcal{N}$ is the set of voters with profile $\pi \in (\mathcal{L}(\ \mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2))^N$, and $\delta^{\mathcal{N}} \in \Delta^N$ is a preference extension profile.*

5.3 Majority Voting Paradoxes

We now formalize seat-wise and direct selection procedures based on simple majority voting. This requires formalizing two types of majority tournaments. Given $t \in \{1, 2\}$ together with a t-profile $\pi^t = (P_n^t)_{n \in \mathcal{N}} \in (\mathcal{L}(\mathbb{C}_t))^N$, the π^t-majority tournament is the complete and asymmetric binary relation $T(\pi^t)$ defined over $\mathbb{C}_t \times \mathbb{C}_t$ by : $\forall a, b \in \mathbb{C}_t$, $aT(\pi^t)b$ if and only if $|\{n \in \mathcal{N} : xP_n^t y\}| > \frac{N}{2}$. If $aT(\pi^t)b$, we say that candidate a defeats candidate b in π^t. Similarly, given a profile $\pi = (P_n)_{n \in \mathcal{N}} \in (\mathcal{L}(\mathbb{C}_1) \times \mathcal{L}(\mathbb{C}_2))^N$

together with a preference extension profile $\delta^{\mathcal{N}}$, the $\delta^{\mathcal{N}}(\pi)-$majority tournament is the complete and asymmetric binary relation $T(\delta^{\mathcal{N}}(\pi))$ defined over $\mathbb{C}\times\mathbb{C}$ by : $\forall\, \mathcal{C},\mathcal{C}' \in \mathbb{C}$, $\mathcal{C}\mathcal{T}(\delta^{\mathcal{N}}(\pi))\mathcal{C}'$) if $|\{n \in \mathbb{N} : \mathcal{C}\ \delta_n(P_n)\ \mathcal{C}'\}| > \frac{N}{2}$. If $\mathcal{C}\mathcal{T}(\delta^{\mathcal{N}}(\pi))\mathcal{C}'$, we say that committee $\mathcal{C}$ defeats committee $\mathcal{C}'$ in $\delta^{\mathcal{N}}(\pi)$. Moreover, $T(\pi^t)$ (resp. $T(\delta^{\mathcal{N}}(\pi))$) admits a (necessarily unique) Condorcet winner if there exists a candidate $c(T(\pi^t) \in \mathbb{C}_t$ (resp. a committee $c(T(\delta^{\mathcal{N}}(\pi)))$ that defeats all other candidates in π^t (resp. in $\delta^{\mathcal{N}}(\pi)$). We adopt the convention $c(T(\pi^t)) = \varnothing$ (resp. $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$) when the underlying tournament has no Condorcet winner.

The seat-wise procedure consists of selecting a candidate for each seat from the seat-wise majority tournaments $T(\pi^1)$ and $T(\pi^2)$. We assume that preferences over candidates are restricted so as to ensure that both tournaments $T(\pi^1)$ and $T(\pi^2)$ admit a Condorcet winner. The direct procedure on the other hand consists of selecting a committee from the majority tournament over committees $T(\delta^{\mathcal{N}}(\pi))$. The seat-wise and direct procedures are inconsistent if either there is a Condorcet winner among committees that is not the combination of seat-wise Condorcet winners, or if there is no Condorcet winner among committees. This leads to the following two definitions of voting paradoxes :

Definition 5.3.1 *The majority committee paradox occurs at choice problem $\mathcal{P}$ if and only if $T(\pi_1)$, $T(\pi_2)$, and $T(\delta^{\mathcal{N}}(\pi))$ each admit a Condorcet winner while $c(T(\pi_1)) \times c(T(\pi_2)) \neq c(T(\delta^{\mathcal{N}}(\pi)))$.*

The majority committee paradox is illustrated in the following simple example :

Example 5.3.1 *Let $\mathbb{C}_1 = \{a, b\}$, $\mathbb{C}_2 = \{x, y\}$, $\mathcal{N} = \{1, 2, 3\}$, and consider the seat-wise profiles $\pi = (\pi^1, \pi^2)$ defined below :*

$$\pi^1 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ \hline a & a & a \\ b & b & b \end{array}\right),\ \pi^2 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ \hline x & x & y \\ y & y & x \end{array}\right)$$

For $i = 1, 2, 3$ let $\gg_{\delta_i}$ be $(2,1) \gg_{\delta_i} (1,2) \gg_{\delta_i} (1,1) \gg_{\delta_i} (2,2)$, combined with π, leading to the following extended profile :

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{ccc} 1 & 2 & 3 \\ \hline (b,x) & (b,x) & (b,y) \\ (a,y) & (a,y) & (a,x) \\ (a,x) & (a,x) & (a,y) \\ (b,y) & (b,y) & (b,x) \end{array}\right)$$

Clearly, $c(T(\pi^1)) \times c(T(\pi^2)) = (a, x)$ and $c(T(\delta^{\mathcal{N}}(\pi)) = (b, x)$, thus the majority committee paradox holds.[6]

6. Note that the preference extensions used by the voters are not separable which turns out to be necessary and sufficient to avoid the majority committee paradox as will be shown in Theorem 1.

Definition 5.3.2 *The majority committee weak paradox occurs at choice problem $\mathcal{P}$ if and only if $T(\pi_1)$ and $T(\pi_2)$ both admit a Condorcet winner while $c(T(\pi_1)) \times c(T(\pi_2)) \neq c(T(\delta^{\mathcal{N}}(\pi))$.*

The majority paradox implies the weak majority paradox, while the opposite is not true. Moreover, it is straightforward to show that the majority paradox never prevails when there are two candidates per seat.

The next example will illustrate the majority committee weak paradox.

Example 5.3.2 *Let $\mathbb{C}_1 = \{a, b\}$, $\mathbb{C}_2 = \{x, y\}$, $\mathcal{N} = \{1, 2, 3\}$, and consider the seat-wise profiles $\pi = (\pi^1, \pi^2)$ defined below :*

$$\pi^1 = \begin{pmatrix} 1 & 2 & 3 \\ \hline a & b & a \\ b & a & b \end{pmatrix}, \ \pi^2 = \begin{pmatrix} 1 & 2 & 3 \\ \hline x & y & y \\ y & x & x \end{pmatrix}$$

Let $\gg_{\delta^{\mathcal{N}}}$ be such that

— $(1,1) \gg_{\delta_i} (2,1) \gg_{\delta_i} (1,2) \gg_{\delta_i} (2,2)$ *for* $i = 1, 3$.
— $(1,1) \gg_{\delta_3} (1,2) \gg_{\delta_3} (2,1) \gg_{\delta_3} (2,2)$.

Combination of $\gg_{\delta^{\mathcal{N}}}$ with π leads to the following extended profile :

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} 1 & 2 & 3 \\ \hline (a,x) & (b,y) & (a,y) \\ (b,x) & (b,x) & (b,y) \\ (a,y) & (a,y) & (a,x) \\ (b,y) & (a,x) & (b,x) \end{pmatrix}$$

(a,x) defeats (b,x), (b,x) defeats (a,y), (a,y) defeats (b,y), and finally, (b,y) defeats (a,x) in $\delta^{\mathcal{N}}(\pi)$, So, there is no Condorcet winner among committees while π_1 and π_2 both admit condorcet winners. Thus, majority committee weak paradox holds.[7]

5.3.1 Results

We define an extension domain as a non-empty subset $\mathcal{D}$ of Δ. First, we characterize the domain of preference extensions that are immune to the majority committee paradox in the following sense : $\mathcal{D} \subseteq \Delta$ is immune to majority committee paradox if and only if for all choice problems $\mathcal{P} = (\mathbb{C}_1, \mathbb{C}_2, \mathcal{N}, \pi, \delta^{\mathcal{N}})$ such that $c(T(\pi_1)) \neq \varnothing$, $c(T(\pi_2)) \neq \varnothing$ and $c(T(\delta^{\mathcal{N}}(\pi))) \neq \varnothing$; $\delta^{\mathcal{N}} \subseteq \mathcal{D}^N$ implies $c(T(\pi_1)) \times c(T(\pi_2)) = c(T(\delta^{\mathcal{N}}(\pi)))$. That is, a domain of preference extensions is immune to the majority committee paradox if and

7. Note that for each voter the preference extension used is either δ^{1Lex} or δ^{2Lex}, but voters are not unanimously using one or the other which makes a significant difference as we will show in Theorem 2.

only if at any choice problem where all the voters use a preference extension from this domain and both seat-wise and committee-wise Condorcet winners exist ; the combination of seat-wise Condorcet winners is the Condorcet winning committee.

Theorem 5.3.1 *A preference extension domain $\mathcal{D}$ is immune to the majority paradox if and only if $\mathcal{D} \subseteq \Delta^{sep}$.*

Our second result is a similar characterization for the weak majority weak paradox. Following an almost identical construction, we characterize the domain of preference extensions that are immune to the majority committee weak paradox in the following sense : $\mathcal{D} \subseteq \Delta$ is immune to majority committee paradox if and only if for all choice problems $\mathcal{P} = (\mathbb{C}_1, \mathbb{C}_2, \mathcal{N}, \pi, \delta^{\mathcal{N}})$ such that $c(T(\pi_1)) \neq \varnothing$ and $c(T(\pi_2)) \neq \varnothing$; $\delta^{\mathcal{N}} \subseteq \mathcal{D}^N$ implies $c(T(\pi_1)) \times c(T(\pi_2)) = c(T(\delta^{\mathcal{N}}(\pi)))$. That is, a domain of preference extensions is immune to the majority committee weak paradox if and only if at any choice problem where all the voters use a preference extension from this domain and both seat-wise Condorcet winners exist ; Condorcet winning committee exists and is equal to the combination of seat-wise Condorcet winners.

Theorem 5.3.2 *A preference extension domain $\mathcal{D}$ is immune to the majority weak paradox if and only if either $\mathcal{D} = \{\delta^{1Lex}\}$ or $\mathcal{D} = \{\delta^{2Lex}\}$.*

5.4 Further comments

At least three routes are opened to further research. The first is considering committee choice problems involving more than two seats. We strongly conjecture that results similar to Theorem 1 and Theorem 2 hold in such a more general setting. The second route aims at characterizing the domain of neutral preference extensions which is Condorcet decisive when there are at least 3 candidates per seat. A third route deals with the existence of Condorcet winning committee when seat-wise profiles admit a Condorcet winner, but without any requirement upon their mutual consistency. We provide very preliminary results on this issue. Call Condorcet decisive a preference domain $\mathcal{D}$ such that for all choice problems $\mathcal{P} = (\mathbb{C}_1, \mathbb{C}_2, \mathcal{N}, \pi, \delta^{\mathcal{N}})$ with $c(T(\pi^1)) \neq \varnothing$, $c(T(\pi^2)) \neq \varnothing$, $\delta^{\mathcal{N}} \in \mathcal{D}^N$ ensures $c(T(\delta^{\mathcal{N}}(\pi))) \neq \varnothing$. With words, a domain of preference extensions is Condorcet decisive if and only if a Condorcet winning committee exists at any choice problem where all the voters pick a preference extension in this domain and seat-wise Condorcet winners exist. We characterize the domain of preference extensions that are Condorcet decisive for the 2-candidate case ($|\mathbb{C}_1| = |\mathbb{C}_2| = 2$).

First, we consider the case of singleton domains.

Proposition 5.4.1 *Every neutral preference extension is Condorcet decisive.*

We turn now to domains containing more than one extension rule. First observe that the set of all 24 extension rules can be partitioned into 6 classes, where in each

classes, each element of the class can be obtained from another element by means of coordinate-wise permutations of ranks. The six classes are described below

Class 1	Class 2
$\delta_1 : (1,1) \gg_{\delta_1} (1,2) \gg_{\delta_1} (2,1) \gg_{\delta_1} (2,2)$	$\delta_2 : (1,1) \gg_{\delta_2} (2,1) \gg_{\delta_2} (1,2) \gg_{\delta_2} (2,2)$
$\delta_1^2 : (1,2) \gg_{\delta_1^2} (1,1) \gg_{\delta_1^2} (2,2) \gg_{\delta_1^2} (2,1)$	$\delta_2^2 : (1,2) \gg_{\delta_2^2} (2,2) \gg_{\delta_2^2} (1,1) \gg_{\delta_2^2} (2,1)$
$\delta_1^3 : (2,1) \gg_{\delta_1^3} (2,2) \gg_{\delta_1^3} (1,1) \gg_{\delta_1^3} (1,2)$	$\delta_2^3 : (2,1) \gg_{\delta_2^3} (1,1) \gg_{\delta_2^3} (2,2) \gg_{\delta_3^3} (1,2)$
$\delta_1^4 : (2,2) \gg_{\delta_1^4} (2,1) \gg_{\delta_1^4} (1,2) \gg_{\delta_1^4} (1,1)$	$\delta_2^4 : (2,2) \gg_{\delta_2^4} (1,2) \gg_{\delta_2^4} (2,1) \gg_{\delta_2^4} (1,1)$
Class 3	**Class 4**
$\delta_3 : (1,1) \gg_{\delta_3} (1,2) \gg_{\delta_3} (2,2) \gg_{\delta_3} (2,1)$	$\delta_4 : (1,1) \gg_{\delta_4} (2,1) \gg_{\delta_4} (2,2) \gg_{\delta_4} (1,2)$
$\delta_3^2 : (1,2) \gg_{\delta_3^2} (1,1) \gg_{\delta_3^2} (2,1) \gg_{\delta_3^2} (2,2)$	$\delta_4^2 : (1,2) \gg_{\delta_4^2} (1,1) \gg_{\delta_4^2} (2,2) \gg_{\delta_2^2} (2,1)$
$\delta_3^3 : (2,1) \gg_{\delta_3^3} (2,2) \gg_{\delta_3^3} (1,2) \gg_{\delta_3^3} (1,1)$	$\delta_4^3 : (2,1) \gg_{\delta_4^3} (2,2) \gg_{\delta_4^3} (1,1) \gg_{\delta_4^3} (1,2)$
$\delta_3^4 : (2,2) \gg_{\delta_3^4} (2,1) \gg_{\delta_3^4} (1,1) \gg_{\delta_3^4} (1,2)$	$\delta_4^4 : (2,2) \gg_{\delta_4^4} (2,1) \gg_{\delta_4^4} (1,2) \gg_{\delta_4^4} (1,1)$
Class 5	**Class 6**
$\delta_5 : (1,1) \gg_{\delta_5} (2,2) \gg_{\delta_5} (1,2) \gg_{\delta_5} (2,1)$	$\delta_6 : (1,1) \gg_{\delta_6} (2,2) \gg_{\delta_6} (2,1) \gg_{\delta_6} (1,2)$
$\delta_5^2 : (1,2) \gg_{\delta_5^2} (1,1) \gg_{\delta_5^2} (2,2) \gg_{\delta_5^2} (2,1)$	$\delta_6^2 : (1,2) \gg_{\delta_6^2} (1,1) \gg_{\delta_6^2} (2,2) \gg_{\delta_6^2} (2,1)$
$\delta_5^3 : (2,1) \gg_{\delta_5^3} (2,2) \gg_{\delta_5^3} (1,1) \gg_{\delta_5^3} (1,2)$	$\delta_6^3 : (2,1) \gg_{\delta_6^3} (2,2) \gg_{\delta_6^3} (1,1) \gg_{\delta_6^3} (1,2)$
$\delta_5^4 : (2,2) \gg_{\delta_5^4} (2,1) \gg_{\delta_5^4} (1,2) \gg_{\delta_5^4} (1,1)$	$\delta_6^4 : (2,2) \gg_{\delta_6^4} (2,1) \gg_{\delta_6^4} (1,2) \gg_{\delta_6^4} (1,1)$

We claim that focusing on the six representatives $\delta_1, \delta_2, \delta_2, \delta_2, \delta_5, \delta_6$ is sufficient for characterizing Condorcet decisive domains. Indeed, all proofs for these representatives can be duplicated to all other extension rules provided a relevant permutation of chosen profiles. For instance, Suppose profile $\pi = (P_n^1, P_n^2)$ is relevant for δ_1 and one wishes to replace δ_1 by δ_1^2. Define permutations α^1 and α^2 of $\{1,2\}$ such that $\alpha^1(1) = 1$, $\alpha^1(2) = 2$ and $\alpha^2(2) = 1$, $\alpha^2(1) = 2$. Clearly δ_1^2 is obtained from δ_1 as follows : for all $z, z', w, w' \in \{1,2\}$, $(z,z') \gg_{\delta_1} (w,w') \Leftrightarrow (\alpha^1(z), \alpha^2(z')) \gg_{\delta_1^2} (\alpha^1(w), \alpha^2(w'))$. Now define profile π' by $\forall\ n \in \mathcal{N}$,

$$\text{- } {P'}_n^1 = \begin{pmatrix} P_n^1 \\ -P_n^1 \end{pmatrix} \text{if} \begin{pmatrix} \alpha^1(1) = 1, \alpha^1(2) = 2 \\ \alpha^1(1) = 2, \alpha^1(2) = 1 \end{pmatrix}$$

$$\text{- } {P'}_n^2 = \begin{pmatrix} P_n^2 \\ -P_n^2 \end{pmatrix} \text{if} \begin{pmatrix} \alpha^2(1) = 1, \alpha^2(2) = 2 \\ \alpha^2(1) = 2, \alpha^2(2) = 1 \end{pmatrix}$$

Then $(\delta_1)^n(P_n)$ is equal to $(\delta_1^2)^n(P'_n)$.

We denote the set of all preference extension rules in the Class $i \in \{1, ..., 6\}$ by Δ_i.

Proposition 5.4.2 *A preference extension domain $\mathcal{D}$ is Condorcet decisive if and only if either $\mathcal{D} = \{\delta\}$ where δ is neutral or $\mathcal{D} \in \{\{\Delta_1, \Delta_3\}, \{\Delta_2, \Delta_4\}, \{\Delta_5, \Delta_6\}\}$.*

Characterizing Condorcet decisive domains for more than 2 candidates per seat is left as an open problem.

5.5 Appendix

Proof of theorem 1

The proof is organized in five lemmata. Consider a choice problem $\mathcal{P} = (\mathbb{C}_1, \mathbb{C}_2, \mathcal{N}, \pi, \delta^{\mathcal{N}})$. Let $\delta^{\mathcal{N}} \subseteq \mathcal{D}^N \subseteq (\Delta^{sep})^N$ and $c(T(\pi^1)) \times c(T(\pi^2)) = (a, x)$ and $c(T(\delta^{\mathcal{N}}(\pi))) = (b, y)$ with $(b, y) \neq (a, x)$. Moreover, suppose without loss of generality that $b \neq a$. Then, by separability, (a, y) is preferred to (b, y) by a majority of voters, which is impossible.

Now, take a preference extension domain $\mathcal{D}$ which is immune to the majority paradox and a preference extension $\delta \subseteq \mathcal{D}$. We will show that $\mathcal{D} \subseteq \Delta^{sep}$. Let $i, j \in \{1, ..., C_1\}$ and $k, l \in \{1, ..., C_2\}$.

Lemma 5.5.1 *$(1,1) \gg_\delta (i,k)$ for all $(i,k) \neq (1,1)$.*

Proof 5.5.1 *Suppose for a contradiction that $\delta \subseteq \mathcal{D}$ such that $\gg_\delta$ does not rank $(1,1)$ as top-element. Take a choice problem $\mathcal{P}$ be such that $\mathbb{C}_1 = \{a, b\}$, $\mathbb{C}_2 = \{x, y\}$. Moreover, suppose that $\pi = (\pi_1, \pi_2) = ((P_n^1, P_n^2)_{n \in \mathcal{N}})$ is such that $P_n^1 = P^1$ and $P_n^2 = P^2$ for all $n \in \mathcal{N}$ with respective tops a and x. Thus $c(T(\pi^1)) \times c(T(\pi^2)) = (a, x)$. Since $\gg_\delta$ does not rank $(1,1)$ first, then $c(T(\delta^{\mathcal{N}}(\pi))) \neq (a, x)$ is not immune to the majority paradox*

Lemma 5.5.2 *$\gg_\delta$ ranks either $(1,2)$ or $(2,1)$ second.*

Proof 5.5.2 *Suppose that $\gg_\delta$ ranks (i,k) second where $(i,k) \notin \{(1,2),(2,1)\}$.*

Case 1 : *Suppose $i \neq 1$ and $k \neq 1$. Let $N = 3$ and pick a profile $\pi = (\pi^1, \pi^2)$ such that*

$$\pi^1 = \left(\begin{array}{ccc} P_1^1 & P_2^1 & P_3^1 \\ \hline a & a & c \\ \dots & \dots & \dots \\ b & b & b \\ \dots & \dots & \dots \\ & & \end{array} \right), \pi^2 = \left(\begin{array}{ccc} P_1^2 & P_2^2 & P_3^2 \\ \hline x & z & x \\ \dots & \dots & \dots \\ y & y & y \\ \dots & \dots & \dots \\ & & \end{array} \right)$$

where b (resp. y) has rank i (resp. k) in all columns of π^1 (resp. π^2) and all voters use extension rule δ. We get the extended profile

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{ccc} \delta(P_1) & \delta(P_2) & \delta(P_3) \\ \hline (a,x) & (a,z) & (c,x) \\ (b,y) & (b,y) & (b,y) \\ \dots & \dots & \dots \end{array} \right)$$

Thus $c(T(\pi^1)) \times c(T(\pi^2)) = (a, x) \neq c(T(\delta^{\mathcal{N}}(\pi)) = (b, y)$.

Case 2 : *Suppose* $i = 1$ *and* $k \geq 3$. *Let* $N = 3$ *and pick a profile* $\pi = (\pi^1, \pi^2)$ *such that*

$$\pi^1 = \begin{pmatrix} \frac{P_1^1 \quad P_2^1 \quad P_3^1}{a \quad a \quad a} \\ \dots \quad \dots \quad \dots \\ \end{pmatrix}, \pi^2 = \begin{pmatrix} \frac{P_1^2 \quad P_2^2 \quad P_3^2}{y \quad z \quad x} \\ x \quad x \quad \dots \\ \dots \quad \dots \quad \dots \\ \dots \quad y \quad y \\ \dots \quad \dots \quad \dots \end{pmatrix}$$

where y *has rank* k *in the 2nd and 3rd column of* π^2 *and all voters use extension rule* δ. *We get the extended profile*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \frac{\delta(P_1) \quad \delta(P_2) \quad \delta(P_3)}{(a, y) \quad (a, z) \quad (a, x)} \\ \dots \quad (a, y) \quad (a, y) \\ \dots \quad \dots \quad \dots \end{pmatrix}$$

Thus $c(T(\pi^1)) \times c(T(\pi^2)) = (a, x) \neq c(T(\delta^{\mathcal{N}}(\pi)) = (a, y)$.

Case 3 : *Suppose* $i \geq 3$ *and* $k = 1$. *Let* $N = 3$ *and pick a profile* π *with* π^1 *and* π^2 *as below*

$$\pi^1 = \begin{pmatrix} \frac{P_1^1 \quad P_2^1 \quad P_3^1}{b \quad c \quad a} \\ a \quad a \quad \dots \\ \dots \quad \dots \quad \dots \\ \dots \quad b \quad b \\ \dots \quad \dots \quad \dots \end{pmatrix}, \pi^2 = \begin{pmatrix} \frac{P_1^2 \quad P_2^2 \quad P_3^2}{x \quad x \quad x} \\ \dots \quad \dots \quad \dots \end{pmatrix}$$

where b *has rank* i *in the 2nd and 3rd column of* π^1 *and all voters use extension rule* δ. *We get the extended profile*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \frac{\delta(P_1) \quad \delta(P_2) \quad \delta(P_3)}{(b, x) \quad (c, x) \quad (a, x)} \\ \dots \quad (b, x) \quad (b, x) \\ \dots \quad \dots \quad \dots \end{pmatrix}$$

Thus $c(T(\pi^1)) \times c(T(\pi^2)) = (a, x) \neq c(T(\delta^{\mathcal{N}}(\pi)) = (b, x)$, *a contradiction.*

From Lemma 1 and Lemma 2 we have showed that $\gg_\delta$ ranks $(1,1)$ first and either $(1,2)$ or $(2,1)$ second. We consider below the case where $\gg_\delta$ ranks $(1,1)$ first and $(1,2)$ second since it suffices to invert profiles for the case where $\gg_\delta$ ranks $(2,1)$ second.

Lemma 5.5.3 *If $\gg_\delta$ ranks $(1,1)$ first and $(1,2)$ second, $\forall i,k,l$, $(i,k) \gg_\delta (i,l)$ implies that $k < l$.*

Proof 5.5.3 *Suppose for a contradiction that there exist $i \in \{1,...,C_1\}$ such that $(i,l) \gg_\delta (i,k)$ with $l > k$. Now let $N = 3$ and pick a profile π with π^1 and π^2 such that*

$$\pi^1 = \left(\begin{array}{ccc} P_1^1 & P_2^1 & P_3^1 \\ \hline a & a & ... \\ ... & ... & ... \\ ... & ... & a \\ ... & ... & ... \\ \end{array}\right), \pi^2 = \left(\begin{array}{ccc} P_1^2 & P_2^2 & P_3^2 \\ \hline x & y & ... \\ y & x & ... \\ ... & ... & ... \\ ... & ... & x \\ ... & ... & ... \\ ... & ... & y \\ ... & ... & ... \\ \end{array}\right)$$

where a has rank i in the 3rd column of π^1, x has rank k, y has rank l in the 3rd column of π^2 and all voters use extension rule δ. We get the extended profile

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{ccc} \delta(P_1) & \delta(P_2) & \delta(P_3) \\ \hline (a,x) & (a,y) & ... \\ (a,y) & (a,x) & ... \\ ... & ... & ... \\ ... & ... & (a,y) \\ ... & ... & ... \\ ... & ... & (a,x) \\ \end{array}\right)$$

Thus $c(T(\pi^1)) \times c(T(\pi^2)) = (a,x) \neq c(T(\delta^{\mathcal{N}}(\pi)) = (a,y)$, a contradiction. Note that $i = 1$ (which implies $k \geq 3$) and $k = 1$ (which implies $i \geq 2$) are possible cases covered despite what π^1 and π^1 apparently suggest.

Lemma 5.5.4 *$(i,k) \gg_\delta (2,1)$ implies that $i = 1$.*

Proof 5.5.4 *By Lemma 3, $(i,k) \gg_\delta (2,1)$ implies $(i,1) \gg_\delta (2,1)$. Assume for a contradiction that $i \neq 1$ (Note $i = 2$ is ruled out by Lemma 3). Consider $(i^*,1) \gg_\delta (2,1)$ that leads to highest rank for $(i,1)$ (except $(1,1)$ of course). Now let $N = 3$ and pick a profile π with π^1 and π^2 such that*

$$\pi^1 = \left(\begin{array}{ccc} P_1^1 & P_2^1 & P_3^1 \\ \hline c & a & b \\ a & c & a \\ \dots & \dots & \dots \\ b & b & c \\ \dots & \dots & \dots \\ \end{array} \right), \pi^2 = \left(\begin{array}{ccc} P_1^2 & P_2^2 & P_3^2 \\ \hline x & x & x \\ \dots & \dots & \dots \\ \end{array} \right)$$

where b has rank i^* in the 1st and 2nd column of π^1 , c has rank i^* in the 3rd column of π^1 and all voters use extension rule δ. We get the extended profile

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{ccc} \delta(P_1) & \delta(P_2) & \delta(P_3) \\ \hline (c,x) & (a,x) & (b,x) \\ \dots & \dots & \dots \\ (b,x) & (b,x) & (c,x) \\ \dots & \dots & \dots \\ (a,x) & (c,x) & (a,x) \\ \dots & \dots & \dots \\ \end{array} \right)$$

where (b,x) can only be beaten by $(c,.)$ in $\delta(P_1)$ and $(a,.)$ in $\delta(P_2)$ by $(i^*,1) \gg_\delta (i',1)$ for all $i' \in \{2,...,C_1\}$ combined with Lemma 3. Thus $c(T(\pi^1)) \times c(T(\pi^2)) = (a,x) \neq c(T(\delta^{\mathcal{N}}(\pi)) = (b,x)$.

Lemma 5.5.5 *For all i,k,l, $(i,k) \gg_\delta (j,k)$ implies $i < j$.*

Proof 5.5.5 *Suppose for a contradiction that $(j,k) \gg_\delta (i,k)$ with $j > i$. Pick a profile π with π^1 and π^2 such that*

$$\pi^1 = \left(\begin{array}{ccc} P_{type1}^1 & P_{type2}^1 & P_{type3}^1 \\ \hline b & a & \dots \\ a & b & \dots \\ \dots & \dots & \dots \\ \dots & \dots & a \\ \dots & \dots & \dots \\ \dots & \dots & b \\ \dots & \dots & \dots \\ \end{array} \right), \pi^2 = \left(\begin{array}{ccc} P_{type1}^2 & P_{type2}^2 & P_{type3}^2 \\ \hline x & x & \dots \\ \dots & \dots & \dots \\ \dots & \dots & * \\ \dots & \dots & \dots \\ \end{array} \right)$$

where a (resp. b) has rank i (resp. j) in the 3rd column of π^1, W has rank k in the 3rd column of π^2 and all voters use extension rule δ. The extended profile is

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{ccc} \delta(P_{type1}) & \delta(P_{type2}) & \delta(P_{type3}) \\ \hline (b,x) & (a,x) & \dots \\ \dots & \dots & \dots \\ & (b,x) & \dots \\ \dots & \dots & \dots \\ \dots & \dots & (b,*) \\ \dots & \dots & \dots \\ \dots & \dots & (a,*) \\ \dots & \dots & \dots \\ & & \end{array} \right)$$

By lemma 4, (b,x) can only be beaten by $(a,*)$ in $\delta(P_{type2})$ and $(b,*)$ beats $(a,*)$ in $\delta(P_{type3})$. Now consider a profile that consists of m_2 $Type1$ voters, 1 $Type2$ voters and m_2 $Type3$ voters.

By putting each alternative in $\mathbb{C}_2$ to the k^{th} rank in exactly one $Type3$ voters preference P^2_{type3} without specifying further the preferences of $Type1$ and $Type2$ voters (they may have any preference that satisfies the above structure), we obtain π' with π'^1 and π'^2 and $\delta^N(\pi')$ with $c(T(\pi'^1)) \times c(T(\pi'^2)) = (a,x)$ and $c(T(\delta^{\mathcal{N}}(\pi)) = (b,x)$, which is a contradiction.

Observe that (b,x) beats any $(c,*)$ with $c \neq a$ for at least m_2 (from $Type1$ voters) $+1$ (from $Type2$ voters) voters and (b,x) beats any $(a,*)$ for at least m_2 (from $Type1$ voters) $+$ 1 (from $Type3$ voters) both of which constitutes a majority out of $2m_2 + 1$ voters, $c(T(\delta^{\mathcal{N}}(\pi)) = (b,x)$.

Combining Lemma 1 to 5 shows that $\mathcal{D} \subseteq \Delta^{sep}$.

Proof of theorem 2

The proof is organized in two lemmata. Consider a choice problem $\mathcal{P} = (\mathbb{C}_1, \mathbb{C}_2, \mathcal{N}, \pi, \delta^{\mathcal{N}})$.

Lemma 5.5.6 *Let $\delta^{\mathcal{N}} \subseteq \mathcal{D}^N \subseteq (\Delta^{sep})^N$. If a profile π is such that $c(T(\pi^1)) \neq \varnothing$ and $c(T(\pi^2)) \neq \varnothing$ then either $c(T(\pi_1)) \times c(T(\pi_2)) = c(T(\delta^{\mathcal{N}}(\pi)))$ or $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.*

Proof 5.5.6 *Let $c(T(\pi^1)) \times c(T(\pi^2)) = (a,x)$ and suppose $c(T(\delta^{\mathcal{N}}(\pi)) = (b,y)$ with $(b,y) \neq (a,x)$. Moreover, suppose without loss of generality that $b \neq a$. Then by separability, (b,x) is preferred to (a,x) by a majority of voters which is impossible.*

Since δ^{1Lex} and δ^{2Lex} are separable preference extensions, the sufficiency part in theorem 2 directly follows from Lemma 6.

We turn to the necessary part.

Now, take a preference extension domain $\mathcal{D}$ which is immune to the majority weak paradox and a preference extension $\delta \subseteq \mathcal{D}$. We will show that either $\mathcal{D} = \{\delta^{1Lex}\}$ or $\mathcal{D} = \{\delta^{2Lex}\}$. We already showed in Theorem 1 that if a preference extension domain $\mathcal{D}$ is immune to the majority paradox then $\mathcal{D} \subseteq \Delta^{sep}$. Since if a preference extension

domain $\mathcal{D}$ is immune to the majority weak paradox then $\mathcal{D}$ is immune to the majority paradox. Hence $\mathcal{D} \subseteq \Delta^{sep}$.

Let $i, j \in \{1, ..., C_1\}$ and $k, l \in \{1, ..., C_2\}$

Lemma 5.5.7 *Suppose there exists $\delta \in \mathcal{D}$ such that $(2,1) \gg_\delta (i,k)$ for all $(i,k) \in \{1,...,m\}^2 \backslash \{(1,1),(2,1)\}$. Then $\delta = \delta^{lex2}$.*

Proof 5.5.7 *Suppose that $(i,k) \gg_\delta (j,l)$ with $k > l$. It follows from separability that $i < j$. Let $N = 3$ and pick a profile π with π^1 and π^2 having the form below*

$$\pi^1 = \begin{pmatrix} P_1^1 & P_2^1 & P_3^1 \\ \hline a & a & ... \\ ... & b & ... \\ & ... & ... \\ & & b \\ & & ... \\ & & a \\ ... & ... & ... \end{pmatrix}, \pi^2 = \begin{pmatrix} P_1^2 & P_2^2 & P_3^2 \\ \hline x & y & ... \\ ... & x & ... \\ & ... & ... \\ & & x \\ & & ... \\ & & y \\ ... & ... & ... \end{pmatrix}$$

where a (resp. b) has rank j (resp. i) in the 3rd column of π^1, x (resp. y) has rank l (resp. k) in the 3rd column of π^2 and all voters use preference extension δ.

Clearly, $c(T(\pi^1)) \times c(T(\pi^2)) = (a,x)$. By separability, $\gg_\delta$ ranks $(1,1)$ first. By assumption, $(2,1) \gg_\delta (1,2)$ and $(i,k) \gg_\delta (j,l)$. This implies that (b,y) defeats (a,x) in $T(\delta^{\mathcal{N}}(\pi))$. Therefore, $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$, hence the majority weak paradox holds, a contradiction.

Lemma 5.5.8 *Suppose there exists $\delta \in \mathcal{D}$ such that $(1,2) \gg_\delta (i,k)$ for all $(i,k) \in \{1,...,m\}^2 \backslash \{(1,1),(1,2)\}$. Then $\delta = \delta^{lex1}$.*

Proof 5.5.8 *Suppose that $(i,k) \gg_\delta (j,l)$ with $i > j$. It follows from separability that $k < l$. Let $N = 3$ and pick a profile π^1 and π^2 having the form below*

$$\pi^1 = \begin{pmatrix} P_1^1 & P_2^1 & P_3^1 \\ \hline a & b & ... \\ ... & a & ... \\ & ... & ... \\ & & a \\ & & ... \\ & & b \\ ... & ... & ... \end{pmatrix}, \pi^2 = \begin{pmatrix} P_1^2 & P_2^2 & P_3^2 \\ \hline x & x & ... \\ ... & y & ... \\ & ... & ... \\ & & y \\ & & ... \\ & & x \\ ... & ... & ... \end{pmatrix}$$

where a (resp. b) has rank j (resp. i) in the 3rd column of π^1, x (resp. y) has rank l (resp. k) in the 3rd column of π^2 and all voters use extension rule δ.

Clearly, $c(T(\pi^1)) \times c(T(\pi^2)) = (a, x)$. By separability, $\gg_\delta$ ranks $(1, 1)$ first. By assumption, $(1, 2) \gg_\delta (2, 1)$ and $(i, k) \gg_\delta (j, l)$. This implies that (b, y) defeats (a, x) in $T(^{\mathcal{N}}(\pi))$. We get from Lemma 6 that $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$ and majority weak paradox holds. Observe that the argument used in the proof of Lemma 7 applies once we 'invert' profiles π^1 and π^2.

Pick any $\delta \subseteq \mathcal{D}$. By separability, either $(1, 1) \gg_\delta (1, 2) \gg_\delta ...$, or $(1, 1) \gg_\delta (2, 1) \gg_\delta ....$ We get from Lemma 7 and 8 that $\delta \in \{\delta^{1Lex}, \delta^{2Lex}\}$. Thus, either $\mathcal{D} = \{\delta^{1Lex}\}$, or $\mathcal{D} = \{\delta^{2Lex}\}$, or $\mathcal{D} = \{\delta^{1Lex}, \delta^{2Lex}\}$. Consider the last domain, and let $N = 3$ and pick a profile π with π^1 and π^2 having the form below

$$\pi_1 = \left(\begin{array}{ccc} P_1^1 & P_2^1 & P_3^1 \\ \hline a & b & a \\ b & a & b \\ c & c & c \\ ... & ... & ... \end{array}\right), \pi_2 = \left(\begin{array}{ccc} P_1^2 & P_2^2 & P_3^2 \\ \hline x & y & y \\ y & x & x \\ z & z & z \\ ... & ... & ... \end{array}\right)$$

Suppose preference extension profile $\delta^{\mathcal{N}} = (\delta^{lex2}, \delta^{lex1}, \delta^{lex2})$. The extended profile is

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{ccc} \delta^{lex2}(P_1) & \delta^{lex1}(P_2) & \delta^{lex2}(P_3) \\ \hline (a, x) & (b, y) & (a, y) \\ (b, x) & (b, x) & (b, y) \\ (c, x) & (b, z) & (c, y\ ) \\ (., x) & (b, .) & (., y) \\ (a, y) & (a, y) & (a, x) \\ (b, y) & (a, x) & (b, x) \\ (c, y) & (a, z) & (c, x) \\ ... & ... & ... \end{array}\right)$$

Thus (b, x) defeats (a, y) in $T(\delta^{\mathcal{N}}(\pi))$. Thus the weak paradox holds, a contradiction. This proves the necessary part.

Proof of proposition 1

Denote by Δ_{11} the set of all neutral preference extensions with rank vector $(1, 1)$ at top. Take any a choice problem $\mathcal{P} = (\mathbb{C}_1, \mathbb{C}_2, \mathcal{N}, \pi, \delta^{\mathcal{N}})$ where $\mathbb{C}_1 = \{a, b\}$, $\mathbb{C}_1 = \{x, y\}$. Pick a profile π with $\pi^1 = (P_n^1)_{n \in \mathcal{N}}$ and $\pi^2 = (P_n^2)_{n \in \mathcal{N}}$ where $N = n_1 + n_2 + n_3 + n_4$ having the form below :[8]

$$\pi = \left(\begin{array}{cccccccc} P_{n_1}^1 & P_{n_1}^2 & P_{n_2}^1 & P_{n_2}^2 & P_{n_3}^1 & P_{n_3}^2 & P_{n_4}^1 & P_{n_4}^2 \\ \hline a & y & b & x & a & x & b & y \\ b & x & a & y & b & y & a & x \end{array}\right)$$

8. $P_{n_i}^t$ stands for the t-preferences of n_i agents where $i = 1, 2, 3, 4$.

Lemma 5.5.9 *δ^{1Lex} and δ^{2Lex} are Condorcet decisive.*

Proof 5.5.9 *It is an immediate consequence of Theorem 2.*

Lemma 5.5.10 *Let $\delta \in \Delta_{11}$ be such that $(1,1) \gg_\delta (2,1) \gg_\delta (2,2) \gg_\delta (1,2)$. Then δ is Condorcet decisive.*

Proof 5.5.10 - *Consider profile π above, leading to the extended profile*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta(P_{n_1}) & \delta(P_{n_2}) & \delta(P_{n_3}) & \delta(P_{n_4}) \\ \hline (a,y) & (b,x) & (a,x) & (b,y) \\ (b,y) & (a,x) & (b,x) & (a,y) \\ (b,x) & (a,y) & (b,y) & (a,x) \\ (a,x) & (b,y) & (a,y) & (b,x) \end{pmatrix}$$

Suppose w.l.o.g. that $c(T(\pi_1)) \times c(T(\pi_2)) = (a,y)$. Thus, $n_1 + n_3 > n_2 + n_4$ (1) and $n_1 + n_4 > n_2 + n_3$ (2). For simplicity, hereafter, $(\mu, \mu') \in T(\delta^{\mathcal{N}}(\pi))$ is referred as $\mu \to \mu'$.

Suppose $(b,y) \to (a,y)$. Then $n_3 + n_4 > n_1 + n_2$, and thus $n_3 > n_2$. If $(a,x) \to (b,y)$ or $(b,x) \to (b,y)$, then $n_2 + n_3 > n_1 + n_4$ in contradiction with (2). Thus $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$ only if $(a,y) \to (b,y)$. If $(a,x) \to (a,y)$ or $(b,x) \to (a,y)$, then $n_2 + n_3 > n_1 + n_4$ in contradiction with (2). Thus either $c(T(\delta^{\mathcal{N}}(\pi))) = (a,y)$ (if $n_2 > n_3$) or $c(T(\delta^{\mathcal{N}}(\pi))) = (b,y)$ (if $n_3 > n_2$), and the proof is complete.

Lemma 5.5.11 *Let $\delta \in \Delta_{11}$ be such that $(1,1) \gg_\delta (1,2) \gg_\delta (2,2) \gg_\delta (2,1)$. Then δ is Condorcet decisive.*

Proof 5.5.11 *Consider profile π above, then we have following extended profile*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta(P_{n_1}) & \delta(P_{n_2}) & \delta(P_{n_3}) & \delta(P_{n_4}) \\ \hline (a,y) & (b,x) & (a,x) & (b,y) \\ (a,x) & (b,y) & (a,y) & (b,x) \\ (b,x) & (a,y) & (b,y) & (a,x) \\ (b,y) & (a,x) & (b,x) & (a,y) \end{pmatrix}$$

Suppose w.l.o.g. that $c(T(\pi_1)) \times c(T(\pi_2)) = (a,y)$. Thus, $n_1 + n_3 > n_2 + n_4$ (1), $n_1 + n_4 > n_2 + n_3$ (2). Suppose $(a,x) \to (a,y)$. Then $n_3 + n_4 > n_1 + n_2$, and thus $n_3 > n_2$. If $(b,b') \to (a,x)$ or $(b,x) \to (a,x)$, then $n_2 + n_4 > n_1 + n_4$ in contradiction with (1). Thus $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$ only if $(a,y) \to (a,x)$. If $(b,y) \to (a,y)$ or $(b,x) \to (a,b')$, then $n_2 + n_4 > n_1 + n_3$ in contradiction with (1). Thus either $c(T(\delta^{\mathcal{N}}(\pi))) = (a,y)$ (if $n_2 > n_3$) or $c(T(\delta^{\mathcal{N}}(\pi))) = (a,x)$ (if $n_3 > n_2$), and the proof is complete.

Combining lemma 9,10 and 11 show that every preference extension in Δ_{11} is Condorcet decisive. Take any preference extension $\delta \notin \Delta_{11}$ and suppose that δ is not Condorcet decisive. Suppose that $\gg_\delta$ ranks (i,k) at top, with $(i,k) \neq (1,1)$. Define permutations α^1 and α^2 of $\{1,2\}$ by :[9]

- if $i=2$, $\alpha^1(2)=1$ and $\alpha^1(1)=2$, and if $i=1$, $\alpha^1(1)=1$ and $\alpha^1(2)=2$

- if $k=2$, $\alpha^2(2)=1$ and $\alpha^2(1)=2$, and if $k=1$, $\alpha^2(1)=1$ and $\alpha^2(2)=2$

Define $\gamma \in \Delta_{11}$ by $\forall z,z',w,w' \in \{1,2\}$, $(z,z') \gg_\delta (w,w') \Leftrightarrow (\alpha^1(z),\alpha^2(z')) \gg_\gamma (\alpha^1(w),\alpha^2(w'))$.

Pick a profile $\pi=(P_1,...,P_N)$ such that $c(T(\delta^{\mathcal{N}}(\pi)))=\varnothing$. Now define profile π' by : for all $n \in \mathcal{N}$,

- $P'^1_n = \begin{pmatrix} P^1_n \\ -P^1_n \end{pmatrix}$ if $r = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

- $P'^2_n = \begin{pmatrix} P^2_n \\ -P^2_n \end{pmatrix}$ if $r' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Then $\delta^{\mathcal{N}}(\pi)$ with $\delta^{\mathcal{N}}=(\delta,...,\delta)$ and $\gamma^{\mathcal{N}}(\pi')$ with $\gamma^{\mathcal{N}}=(\gamma,...,\gamma)$ generate the same majority tournament.[10] This shows that there exists a rule in Δ_{11} that is not Condorcet decisive, a contradiction.

Proof of proposition 2

Lemma 5.5.12 *Let $i \neq j \in \{1,...,6\}$. If $\{i,j\} \notin \{\{1,3\},\{2,4\},\{5,6\}\}$, no domain containing $\{\delta_i,\delta_j\}$ is Condorcet decisive.*

Proof 5.5.12 *Let $\mathbb{C}_1=\{a,b\}$ and $\mathbb{C}_2=\{x,y\}$*

Pick a profile π with π_1 and π_2 having the form below

$$\pi = \left(\begin{array}{cccccccccccccccc} P^1_{n_1} & P^2_{n_1} & P^1_{n_2} & P^2_{n_2} & P^1_{n_3} & P^2_{n_3} & P^1_{n_4} & P^2_{n_4} & P^1_{n_5} & P^2_{n_5} & P^1_{n_6} & P^2_{n_6} & P^1_{n_7} & P^2_{n_7} & P^1_{n_8} & P^2_{n_8} \\ \hline a & y & a & y & b & x & b & y & a & x & a & y & b & x & b & y \\ b & x & b & x & a & y & a & x & b & y & b & x & a & y & a & x \end{array}\right)$$

(1) *Consider any domain containing $\{\delta_1,\delta_2\}$. Pick the profile in which $n_4=1$, $n_5=1$ $n_6=1$, $n_i=0$ for all $n=1,2,3,7,8$, and where individual with preference P_{n_4} uses extension δ_1 while individuals with preference P_{n_5} or P_{n_6} use extension δ_2.*[11] *We get,*

9. Through the permutations α^1 and α^2, a preference extension which ranks $(1,1)$ at top can be obtained from any preference extension which does not rank ranks $(1,1)$ at top.

10. We define $-P^1_n$ as follows : if $aP^1_n b$ then $b(-P^1_n)a$ for all $a,b \in \mathbb{C}_1$ and $-P^2_n$ is defined as follows : if $xP^1_n y$ then $y(-P^2_n)x$ for all $x,y \in \mathbb{C}_2$.

11. For sake of simplicity, if $n_k=1$, we will later denote by n_k the individual with preference P_{n_k}.

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_1(P_{n_1}) & \delta_2(P_{n_2}) & \delta_2(P_{n_3}) \\ \hline (b,y) & (a,x) & (a,y) \\ (b,x) & (b,x) & (b,y) \\ (a,y) & (a,y) & (a,x) \\ (a,x) & (b,y) & (b,x) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (a,y) \to (b,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(2) *Consider any domain containing* $\{\delta_1, \delta_4\}$. *Pick the profile where* $n_1 = 1$, $n_3 = 1$ $n_6 = 1$. *If individuals* n_1 *and* n_3 *use extension* δ_1 *while* n_6 *uses extension* δ_4, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_1(P_{n_1}) & \delta_1(P_{n_2}) & \delta_4(P_{n_3}) \\ \hline (a,x) & (b,x) & (a,y) \\ (a,y) & (b,y) & (b,y) \\ (b,x) & (a,x) & (b,x) \\ (b,y) & (a,y) & (a,x) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (a,y) \to (b,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(3) *Consider any domain containing* $\{\delta_1, \delta_5\}$. *Pick the profile in which* $n_1 = 1$, $n_7 = 1$, $n_8 = 1$. *If individual* n_1 *use the extension* δ_1 *while* n_7 *and* n_8 *use extension* δ_5, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_1(P_{n_1}) & \delta_1(P_{n_2}) & \delta_4(P_{n_3}) \\ \hline (a,x) & (b,x) & (b,y) \\ (a,y) & (a,y) & (a,x) \\ (b,x) & (b,y) & (b,x) \\ (b,y) & (a,x) & (a,y) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (a,y) \to (b,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(4) *Consider any domain containing* $\{\delta_1, \delta_6\}$. *Pick the profile in which* $n_1 = 1$, $n_6 = 1$, $n_8 = 1$. *If* n_1 *uses extension* δ_1 *while* n_6 *and* n_8 *use extension* δ_6, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_1(P_{n_1}) & \delta_6(P_{n_6}) & \delta_6(P_{n_8}) \\ \hline (a,x) & (a,y) & (b,y) \\ (a,y) & (b,x) & (a,x) \\ (b,x) & (b,y) & (a,y) \\ (b,y) & (a,x) & (b,x) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (a,y) \to (b,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(5) *Consider any domain containing* $\{\delta_2, \delta_3\}$. *Pick the profile in which* $n_1 = 1$, $n_2 = 1$, $n_8 = 1$. *If* n_1 *and* n_2 *use extension* δ_2 *while* n_8 *uses extension* δ_3, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_2(P_{n_1}) & \delta_2(P_{n_2}) & \delta_3(P_{n_8}) \\ \hline (a,x) & (a,y) & (b,y) \\ (b,x) & (b,y) & (b,x) \\ (a,y) & (a,x) & (a,x) \\ (b,y) & (b,x) & (a,y) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (a,y) \to (b,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(6) *Consider any domain containing* $\{\delta_2, \delta_5\}$. *Pick the profile in which* $n_1 = 1$, $n_2 = 1$, $n_7 = 1$. *If* n_1, n_2 *use extension* δ_2 *while agent* n_7 *uses extension* δ_5, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_2(P_{n_1}) & \delta_2(P_{n_2}) & \delta_5(P_{n_7}) \\ \hline (a,x) & (a,y) & (b,x) \\ (b,x) & (b,y) & (a,y) \\ (a,y) & (a,x) & (b,y) \\ (b,y) & (b,x) & (a,x) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (a,y) \to (b,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(7) *Consider any domain containing* $\{\delta_2, \delta_6\}$. *Pick the profile in which* $n_1 = 1$, $n_6 = 1$, $n_8 = 1$. *If* n_1 *uses extension* δ_2 *while agents* n_6 *and* n_8 *use the extension* δ_6. *We get,*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_2(P_{n_1}) & \delta_6(P_{n_6}) & \delta_6(P_{n_8}) \\ \hline (a,x) & (a,y) & (b,y) \\ (b,x) & (b,x) & (a,x) \\ (a,y) & (b,y) & (a,y) \\ (b,y) & (a,x) & (b,x) \end{pmatrix}$$

Therefore, $(a,x) \to (a,y) \to (a,x) \to (b,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(8) *Consider any domain containing* $\{\delta_3, \delta_4\}$. *Pick the profile in which* $n_1 = 1$, $n_6 = 1$, $n_7 = 1$. *If* n_1 *uses extension* δ_3 *while* n_6 *and* n_7 *use extension* δ_4, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_3(P_{n_1}) & \delta_4(P_{n_6}) & \delta_4(P_{n_7}) \\ \hline (a,x) & (b,x) & (a,y) \\ (a,y) & (a,x) & (b,y) \\ (b,y) & (a,y) & (b,x) \\ (b,x) & (b,y) & (a,x) \end{pmatrix}$$

Therefore, $(a,x) \to (a,y) \to (b,y) \to (b,x) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(9) *Consider any domain containing* $\{\delta_3, \delta_5\}$. *Pick the profile in which* $n_2 = 1$, $n_3 = 1$, $n_5 = 1$. *If* n_2 *and* n_3 *use extension* δ_3 *while* n_5 *uses extension* δ_5, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_3(P_{n_2}) & \delta_3(P_{n_3}) & \delta_5(P_{n_5}) \\ \hline (a,y) & (b,x) & (a,x) \\ (a,x) & (b,y) & (b,y) \\ (b,x) & (a,y) & (a,y) \\ (b,y) & (a,x) & (b,x) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (b,y) \to (a,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(10) *Consider any domain containing* $\{\delta_3, \delta_6\}$. *Pick the profile in which* $n_2 = 1$, $n_3 = 1$, $n_5 = 1$. *If* n_2 *and* n_3 *use extension* δ_3 *while* n_5 *uses extension* δ_6, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_3(P_{n_2}) & \delta_3(P_{n_3}) & \delta_6(P_{n_5}) \\ \hline (a,y) & (b,x) & (a,x) \\ (a,x) & (b,y) & (b,y) \\ (b,x) & (a,y) & (b,x) \\ (b,y) & (a,x) & (a,y) \end{pmatrix}$$

Therefore, $(a,x) \to (b,x) \to (b,y) \to (a,y) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(11) *Consider any domain containing* $\{\delta_4, \delta_5\}$. *Pick the profile in which* $n_2 = 1$, $n_3 = 1$, $n_8 = 1$. *If* n_2 *and* n_3 *use extension* δ_4 *while* n_8 *uses extension* δ_5, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_4(P_{n_2}) & \delta_4(P_{n_3}) & \delta_5(P_{n_8}) \\ \hline (a,y) & (b,x) & (b,y) \\ (b,y) & (a,x) & (a,x) \\ (b,x) & (a,y) & (b,x) \\ (a,x) & (b,y) & (a,y) \end{pmatrix}$$

Therefore, $(a,x) \to (a,y) \to (b,y) \to (b,x) \to (a,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

(12) *Consider any domain containing* $\{\delta_4, \delta_6\}$. *Pick the profile in which* $n_2 = 1$, $n_3 = 1$, $n_5 = 1$. *If* n_2 *and* n_3 *use the* δ_4 *while* n_5 *uses extension* δ_6, *we get*

$$\delta^{\mathcal{N}}(\pi) = \begin{pmatrix} \delta_4(P_{n_2}) & \delta_4(P_{n_3}) & \delta_6(P_{n_5}) \\ \hline (a,y) & (b,x) & (a,x) \\ (b,y) & (a,x) & (b,y) \\ (b,x) & (a,y) & (b,x) \\ (a,x) & (b,y) & (a,y) \end{pmatrix}$$

Therefore, $(b,x) \to (a,x) \to (a,y) \to (b,y) \to (b,x)$. *Hence* $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$.

This shows that a domain containing $\{\delta_i, \delta_j\}$ *is Condorcet decisive only if* $\{i,j\} \in \{\{1,3\},\{2,4\},\{5,6\}\}$**.**

Lemma 5.5.13 *No domain containing more than 2 extension rules in* $\{\delta_1, ..., \delta_6\}$ *is Condorcet decisive.*

Proof 5.5.13 *This is a straightforward consequence of lemma 12.*

Lemma 5.5.14 *Let* $i \neq j \in \{1, ..., 6\}$. *If* $\{i, j\} \in \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$, *the* $\mathcal{D} = \{\delta_i, \delta_j\}$ *is Condorcet decisive.*

Proof 5.5.14 *Let* $\mathbb{C}_1 = \{a, b\}$ *and* $\mathbb{C}_2 = \{x, y\}$.

Given $\{i, j\} \in \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$ *together with a profile* π, *denote by* N_i *(resp.* N_j*) the set of individuals using extension* δ_i *(resp.* δ_j*). Now, any profile* π *can be written as below*

$$\pi = \left(\begin{array}{cccccccccccccccc} P^1_{n_1} & P^2_{n_1} & P^1_{n_2} & P^2_{n_2} & P^1_{n_3} & P^2_{n_3} & P^1_{n_4} & P^2_{n_4} & P^1_{n_5} & P^2_{n_5} & P^1_{n_6} & P^2_{n_6} & P^1_{n_7} & P^2_{n_7} & P^1_{n_8} & P^2_{n_8} \\ \hline a & x & a & y & b & x & b & y & a & x & a & y & b & x & b & y \\ b & y & b & x & a & y & a & x & b & y & b & x & a & y & a & x \end{array}\right)$$

where N_i *contains all individuals with preferences* $P^1_{n_k}$, $P^2_{n_k}$ *with* $k = 1, 2, 3, 4$, *and* N_j *contains all individuals with preferences* $P^1_{n_k}$, $P^2_{n_k}$ *with* $k = 5, 6, 7, 8$.

(1) *Consider* $\{i, j\} = \{1, 3\}$. *We get the extended profile*

$$\delta^{\mathcal{N}}(\pi) = \left(\begin{array}{cccccccc} \delta_1(P_{n_1}) & \delta_1(P_{n_2}) & \delta_1(P_{n_3}) & \delta_1(P_{n_4}) & \delta_3(P_{n_5}) & \delta_3(P_{n_6}) & \delta_3(P_{n_7}) & \delta_3(P_{n_8}) \\ \hline (a,x) & (a,y) & (b,x) & (b,y) & (a,x) & (a,y) & (b,x) & (b,y) \\ (a,y) & (a,x) & (b,y) & (b,x) & (a,y) & (a,x) & (b,y) & (b,x) \\ (b,x) & (b,y) & (a,x) & (a,y) & (b,y) & (b,x) & (a,y) & (a,x) \\ (b,y) & (b,x) & (a,y) & (a,x) & (b,x) & (b,y) & (a,x) & (a,y) \end{array}\right)$$

Observe that

- $(a, x) \to (b, x)$ *if and only if* $(a, x) \to (b, y)$.
- $(a, y) \to (b, x)$ *if and only if* $(a, y) \to (b, y)$.

Suppose that $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$. *Then one of the following cases must hold :*

Case 1 : $(a, x) \to (a, y) \to \{(b, x), (b, y)\} \to (a, x)$. *Hence,* $(a, y) \to (b, x)$ *implies that* $n_1+n_2+n_5+n_6 > n_3+n_4+n_7+n_8$ *and* $(b, x) \to (a, x)$ *implies that* $n_3+n_4+n_7+n_8 > n_1 + n_2 + n_5 + n_6$ *which is a contradiction.*

Case 2 : $ay \to ax \to \{bx, by\} \to ay$. *As in the Case 1,* $(b, x) \to (a, y)$ *and* $(a, x) \to (b, x)$ *are mutually incompatible.*

(2) $\{i, j\} = \{2, 4\}$. *We get the extended profile*

$$\delta^{\mathcal{N}}(\pi) =$$

$$\left(\begin{array}{cccccccc} \delta_2(P_{n_1}) & \delta_2(P_{n_2}) & \delta_2(P_{n_3}) & \delta_2(P_{n_4}) & \delta_4(P_{n_5}) & \delta_4(P_{n_6}) & \delta_4(P_{n_7}) & \delta_4(P_{n_8}) \\ \hline (a,x) & (a,y) & (b,x) & (b,y) & (a,x) & (a,y) & (b,x) & (b,y) \\ (b,x) & (b,y) & (a,x) & (a,y) & (b,x) & (b,y) & (a,x) & (a,y) \\ (a,y) & (a,x) & (b,y) & (b,x) & (b,y) & (b,x) & (a,y) & (a,x) \\ (b,y) & (b,x) & (a,y) & (a,x) & (a,y) & (a,x) & (b,y) & (b,x) \end{array}\right)$$

Observe that

- $(ax) \to (ay)$ *if and only if* $(ax) \to (by)$.

- $(b,x) \to (a,y)$ *if and only if* $(b,a') \to (b,y)$.

Suppose that $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$. *Then one of the following cases must hold :*

Case 1 : $(a,x) \to (b,x) \to \{(a,y),(b,y)\} \to (a,x)$. *Hence,* $(b,x) \to (a,y)$ *implies that* $n_1+n_3+n_5+n_7 > n_2+n_4+n_6+n_8$ *and* $(a,y) \to (a,x)$ *implies that* $n_2+n_4+n_6+n_8 > n_1+n_3+n_5+n_7$ *which is a contradiction.*

Case 2 : $(b,x) \to (a,x) \to \{(a,y),(b,y)\} \to (b,x)$. *As in the Case 1,* $(a,x) \to (a,y)$ *and* $(a,y) \to (b,x)$ *are mutually incompatible.*

(3) $\{i,j\} = \{5,6\}$. *We get the extended profile*

$$\delta^{\mathcal{N}}(\pi) =$$

$$\left(\begin{array}{cccccccc} \delta_5(P_{n_1}) & \delta_5(P_{n_2}) & \delta_5(P_{n_3}) & \delta_5(P_{n_4}) & \delta_6(P_{n_5}) & \delta_6(P_{n_6}) & \delta_6(P_{n_7}) & \delta_6(P_{n_8}) \\ \hline (a,x) & (a,y) & (b,x) & (b,y) & (a,x) & (a,y) & (b,x) & (b,y) \\ (b,y) & (b,x) & (a,y) & (a,x) & (b,y) & (b,x) & (a,y) & (a,x) \\ (a,y) & (a,x) & (b,y) & (b,x) & (b,x) & (b,y) & (a,x) & (a,y) \\ (b,x) & (b,y) & (a,x) & (a,y) & (a,y) & (a,x) & (b,y) & (b,x) \end{array}\right)$$

Observe that

- $(a,x) \to (a,y)$ *if and only if* $(a,x) \to (b,x)$.

- $(b,y) \to (a,y)$ *if and only if* $(b,y) \to (b,x)$.

Suppose that $c(T(\delta^{\mathcal{N}}(\pi))) = \varnothing$. *Then one of the following cases must hold :*

Case 1 : $(b,y) \to (a,x) \to \{(a,y),(b,x)\} \to (b,y)$. *Hence,* $(a,y) \to (b,y)$ *implies that* $n_2+n_3+n_6+n_7 > n_1+n_4+n_5+n_8$ *and* $(a,x) \to (a,y)$ *implies that* $n1+n4+n5+n8 > n2+n3+n6+n7$ *which is a contradiction.*

Case 2 : $(a,x) \to (b,y) \to \{(a,y),(b,x)\} \to (a,x)$. *As in the Case 1,* $(b,y) \to (a,y)$ *and* $(a,y) \to (a,x)$ *are mutually incompatible.*

Combining proposition 1, Lemma 12, 13 and 14 completes the proof of proposition 2.

Chapitre 6

Conclusion

To the best of our knowledge, this thesis is the first study extending the Shapley-Scarf model of housing market to situations where trade takes places among groups of individuals, or coalitions. We think that our results highlight interesting aspects of real-life issues, such as designing job mobility campaigns involving couples. In all chapters we follow a unified approach, consisting at identifying preference domains which ensure the existence of well-known solutions to allocation problems, such as the competitive or the core solution. Preference domains are obtained by using preference extensions, which bridge the gap between partial information collected on preferences and completed information needed for these solutions to be well-defined and normatively evaluated. And the reason why only partial information is assumed available is that the existence of a coalition structure makes hard collecting full preferences.

This thesis paves the way to further research in many directions. Each chapter gathers open problems, that we briefly list below.

We have shown in the first chapter that the top-trading-cycles (TTC) algorithm always finds a strong equilibrium in Shapley-Scarf markets where the coalition structure is in couples. Moreover, strong equilibria are not necessarily outcomes of this algorithm, and we characterize strong equilibria in terms of trading cycles. A similar situation prevails for weak equilibria. Indeed, an algorithm similar to TTC one always finds a weak cooperative equilibrium allocation, but the reverse implication does not hold. A question opened for further research is characterizing weak equilibria. Another interesting issue is relating our model with Shapley-Scarf markets with multiple types of indivisible goods, such as the one developed in Konishi, Quint and Wako (2001).

In the third chapter, we introduce a distance structure beyond coalitional trade. When individuals are giving priority to goods, it is shown that if individuals giving priority to goods, TTC always terminates a core allocation. Moreover, we define a modified version TTC1 of TTC algorithm which finds a Minmax core allocation under the assumption that all partners are initially close to each other. The existence of Minmax core allocations when there exists at least one individual who is initially distant to her partner remains to be investigated. If individuals are giving priority to distance,

we show the existence of Minmax core only for the case of all partners is initially close to each other. Existence for any initial distribution of goods general remains to be considered. We also ensure the existence of the strict Core when all partners are initially close under the additional assumption of two locations by showing that another version TTC2 of TTC always terminates at strict Core allocations. Nevertheless, we show that TTC1 and TTC2 mechanisms are not strategy-proof. The existence of strategy-proof Core stable mechanisms also remains unsolved. Furthermore, the model could be enriched by considering metrics other than the discrete one, as well as important aspects which are to be considered in job mobility problems, such as preferences over locations or heterogenous skills which could restrict the set of individual applications.

The last chapter departs from the standard housing market by considering indivisible goods as public goods. We argue that this case naturally relates to the choice of a committee formed by several members, each selected from a specific set applying to a specific seat. Voters have seat-wise preferences and preferences over committees are derived from seat-wise preferences by means of a neutral preference extension. Under a neutrality assumption for preference extensions, we characterize the preference extension domain immune to the majority committee paradox and the one immune to the majority committee weak paradox in the case of two-seat committees. In this chapter, we focus on committee choice problems involving only two seats. Although we strongly conjecture that our results for two seats hold in a case for more than two seats, it still remains as an open problem. Both paradoxes relate to seat-wise preferences for which seat-wise Condorcet winners exist. Under this assumption, the natural question is whether combining seat-wise Condorcet winners provides a Condorcet winner among committees. However, we may question the existence of the latter without assuming the existence of seat-wise winners. Put differently, the question is identifying (maximal) Condorcet domains for committee choice. Chapter four gives a first answer is given in the case where there are two seats and only two candidates per seat. Considering more than two seats and at least three candidates per seat is the scope of current research. Finally, the consistency between seat-wise choice procedure and direct choice should be considered for other choice methods, such as scoring rules or compromise rules.

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Fatma ASLAN

Sur les procédures d'allocation et de décision collective en présences d'indivisibilités /Essays on Allocation Procedures of Indivisibles

Résumé : Cette thèse porte sur les mécanismes d'allocation de biens indivisibles en présence d'externalités dans les préférences individuelles. Ces extrernalités rendent difficile en pratique la collecte d'une information complète sur les préférences. Aussi, l'analyse normative des mécanismes d'allocation requiert de formuler des hypothèses sur la manière d'étendre l'information collectée aux préférences sur les allocations. Cette approche revient à définir des restrictions sur le domaine de préférences admissibles, une démarche bien connue de la théorie du choix social. Les trois premiers chapitres portent sur l'analyse du marché de Shapley-Scarf dans lequel les échanges sont organisés entre coalitions. Les chapitres 1 et 2 établissent des restrictions de domaine garantissant l'existence de différents types d'équilibre concurrentiel. Dans le chapitre 3, l'ensemble des biens est muni d'une géographie, ce qui permet de définir la distance entre partenaires comme source d'externalité. Nous identifions certains domaines de préférences qui assurent la non-vacuité de différents types de Coeur. Le chapitre 4 porte sur le cas de biens indivisibles publics purs. Nous montrons que le problème est formellement équivalent à celui du choix d'un comité dont les membres sont choisis dans des ensembles distincts. Nous caractérisons certains domaines de préférences sur les comités pour lesquels le choix majoritaire membre par membre est cohérent avec le choix majoritaire du comité dans son ensemble.
Mots clés : Shapley-Scarf, couples, externalité, distance, comité, choix majoritaire

Abstract : This thesis focuses on the allocation of indivisible goods in presence of externality in individual preferences. This externality creates a difficulty with collecting full information about preferences. Therefore, conducting a normative analysis of allocation mechanisms requires assumptions on how reported preferences can be extended to preferences over outcomes. This approach is in line with the literature on preference domain restriction well-known in Social Choice theory. The first three chapters focus on Shapley-Scarf markets where trades are organized among coalitions. Coalitional trade generates externalities in individual valuations of allocations. Chapters 1 and 2 investigate domain restrictions ensuring the existence of various types of competitive equilibrium. Chapter 3 endows the set of goods with a geographical structure and considers distance to partners as a source of externality in preference. We identify domains of preference extensions which guarantee the existence of various types of core allocations. Chapter 4 focuses the case of pure public indivisible goods, which is formally identical to choosing a committee formed by several members, each selected from a specific set. We characterize preference domains over committees for which a well-defined seat-wise choice procedure based on majority voting is consistent with choosing a committee at once from majority voting.
Keywords : Shapley-Scarf, couples, externality, distance, committees, majority voting