Asymptotic expansion for Stokes prturbed problems - Évaluation of singular integrals in Electromagnetism.

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Asymptotic expansion for Stokes perturbed problems - Evaluation of singular integrals In Electromagnetism

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Résumé

La premième partie a pour but l'établissement d'un développement asymptotique pour la solution du problème de Stokes avec une petite perturbation du domaine. Dans ce travail, nous avons appliqué la théorie du potentiel. On a écrit les solutions du problème non-perturbé et du problème perturbé sous forme d'opérateurs intégraux. En calculant la différence, et en utilisant des propriétés liées aux noyaux des opérateurs on a établi un développement asymptotique de la solution.
L'objectif principal de la deuxième partie de ce rapport est de déterminer les termes d'ordre élevé du développement asymptotique des valeurs propres et fonctions propres pour l'opérateur de Stokes dus aux changements d'interface de l'inclusion. Dans la troisième partie, nous proposons une méthode pour l'évaluation des integrales singulières provenant de la mise en oeuvre de la méthode des éléments finis de frontière en électromagnetisme. La méthode que nous adoptons consiste en une réduction récursive de la dimension du domaine d'intégration et aboutit à une représentation de l'intégrale sous la forme d'une combinaison linéaire d'intégrales à une dimension dont l'intégrand est régulier et qui peuvent s'évaluer numériquement mais aussi explicitement. Pour la discrétisation du domaine, des triangles plans sont utilisés ; par conséquent, nous évaluons des intégrales sur le produit de deux triangles. La technique que nous avons développée nécessite de distinguer entre diverses configurations géométriques

Mots clés: perturbation, développement asymptotique, problème de Stokes.

Abstract

This thesis contains three main parts. The first part concerns the derivation of an asymptotic expansion for the solution of Stokes resolvent problem with a small perturbation of the domain. Firstly, we verify the continuity of the solution with respect to the small perturbation δ via the stability of the density function. Secondly, we derive the asymptotic expansion of the solution, after deriving the expansion of the density function. The procedure is based on potential theory for the Stokes problem in connection with the boundary integral equation method, and geometric properties of the perturbed boundary. The main objective of the second part on this report, is to present a schematic way to derive high-order asymptotic expansions for both eigenvalues and eigenfunctions for the Stokes operator caused by small perturbations of the boundary. Also, we rigorously derive an asymptotic formula which is in some sense dual to the leading-order term in the asymptotic expansion of the perturbations in the Stokes eigenvalues due to interface changes of the inclusion. The implementation of the boundary element method requires the evaluation of integrals with a singular integrand. A reliable and accurate calculation of these integrals can in some cases be crucial and difficult. In the third part of this report we propose a method of evaluation of singular integrals based on recursive reductions of the dimension of the integration domain. It leads to a representation of the integral as a linear combination of one-dimensional integrals whose integrand is regular and that can be evaluated numerically and even explicitly. The Maxwell equations are used as a model equation, but these results can be used for the Laplace and the Helmholtz equations in 3-D. For the discretization of the domain we use planar triangles, so we evaluate integrals over the product of two triangles. The technique we have developped requires to distinguish between several geometric configurations.

Key words: perturbation, asymptotic expansion, Stokes problem.

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Contents

Introduction

This thesis has three main objectives:

- The first goal is to derive an asymptotic expansion for the solution of a transmission Stokes problem with a small perturbation on the interface.
- The second goal is to present a schematic way to derive high-order asymptotic expansions for both eigenvalues and eigenfunctions for the Stokes operator caused by small perturbations of the boundary
- The third one is to provide and develop a new method of precise evaluation of singular integrals derived from integral equation methods.

Perturbation problem

Perturbation theory is the study of the effects of small disturbances. It's basic idea is to obtain an approximate solution of a mathematical problem by exploiting the presence of a small parameter. An introduction to perturbation theory can be found in [26], [29] or [56].

A theoretical approach of this problem remains of interest for two main reasons: to study the behavior of the solution of perturbation problem, and to derive the perturbation approximation. The derived approximation formula can be applied in the theory of inverse problem as well. It allows to recover mechanical properties by inverting the displacement data, for example to recover the geometric features of the domain of perturbation, or the reconstruction of the location of the anomalies.

Stokes transmission problem

The Stokes resolvent system can be obtained by applying the Laplace transform to the system of the continuity and Navier-Stokes equations which describes the moderate Reynolds number flow of a viscous incompressible fluid (for details see [34], [33], [58]). So the study of this problem has a key role in the understanding of the biomechanics of blood flow, the Brownian motion observed by Brown, the motion of swimming microorganisms, and other biological or physico-chemical phenomena.

On the other hand, the potential theory for the Stokes resolvent system was developed by Varnhorn in [60] and [58]. In addition, the fundamental solution for the system of equations in $\mathbb{R}^3$ was obtained by McCraken in [43]. Also the Dirichlet and Neumann problems for the Stokes resolvent equations on bounded and exterior domains in $\mathbb{R}^n$ have been studied recently in [15] and [57]. A mixed boundary value problem for the same equations has been treated in [32]. These autors, construct a solution of these problems in the form of appropriate potentials (it can be represented as single layer potentials, double layer potentials, or their linear combination), with the unknown source densities defined *via* integral equations on the boundary of the domain. In order to show the existence of a solution (u, p) of the boundary value problem, the method of boundary inregral equations based on layer potentials is used. Potential theory, can also be used to study the transmission problem of the Stokes resolvent equation, which occurs in the case of contact between two materials with different physical characteristics. In a recent work [47], Mitrea and Wright have used layer potential methods to develop a powerful analysis of a transmission boundary value problems in arbitrary Lipschitz domains. In other works [35], Kohr et Wendlend used layer potential techniques to treat transmission boundary problems for the Stokes and Brinkman operators on C^1 and Lipschitz domains.

In this work, we consider a boundary value transmission problem for the Stokes system. Let $\Omega \subset \mathbf{R}^3$ be an open bounded domain with a connected Lipschitz boundary $\partial\Omega$ and suppose that Ω contains an inhomogeneity in the form of an open set D with a connected $\mathcal{C}^2$-boundary ∂D. We assume that there exists a constant $c_0 > 0$ such that $\inf_{x\in D} dist(x, \partial\Omega) > c_0$ which means that D is away from the boundary $\partial\Omega$.

Let us consider a boundary value transmission problem for the Stokes system:

$$\begin{cases} (-\Delta+\kappa^2)u+\nabla q=0 & \text{in} \quad \Omega\setminus\overline{D} \\ (-\Delta+\tilde{\kappa}^2)u+\nabla q=0 & \text{in} \quad D \\ \nabla\cdot u=0 & \text{in} \quad \Omega \\ u|_+-u|_-=0 & \text{on} \quad \partial D \\ \mu\frac{\partial}{\partial n}(u,q)|_+-\tilde{\mu}\frac{\partial}{\partial n}(u,q)|_-=0 & \text{on} \quad \partial D \\ u=g & \text{on} \quad \partial\Omega \\ \int\limits_\Omega q=0, \end{cases} \tag{1}$$

where μ and $\tilde{\mu}$ are positive constants related to the physical properties of Ω and D. Here we take κ^2 and $\tilde{\kappa}^2$ in $\mathbb{C}\setminus\{z\in\mathbb{R},\ z\leq 0\}$. Let $\kappa=\sqrt{\kappa^2}$ and $\tilde{\kappa}=\sqrt{\tilde{\kappa}^2}$ be particular square roots, which have positive real parts (*i.e.* Re $\kappa>0$, Re $\tilde{\kappa}>0$).

Let ν be the outward unit normal to ∂D and ν_Ω be the outward unit normal to $\partial\Omega$. Then, we denote by $\frac{\partial}{\partial n}$ the conormal derivative.

In this thesis, we propose to show the existence and the uniqueness of the solution of (1) and to express it in the form of linear combinations of potential operators.

Asymptotic expansion

We consider ∂D_δ a $\delta-$perturbation of ∂D defined by

$$\partial D_\delta=\{\tilde{x}=x+\delta\nu(x); x\in\partial D, 0<\delta<<1\} \tag{2}$$

and we denote by D_δ the domain bounded by ∂D_δ. Define (u_δ,q_δ) the solution of the transmission problem for the Stokes system with a small boundary perturbation D_δ:

$$\begin{cases} (-\Delta+\kappa^2)u_\delta+\nabla q_\delta=0 & \text{in} \quad \Omega\setminus\overline{D_\delta} \\ (-\Delta+\tilde{\kappa}^2)u_\delta+\nabla q_\delta=0 & \text{in} \quad D_\delta \\ \nabla\cdot u_\delta=0 & \text{in} \quad \Omega \\ u_\delta|_+-u_\delta|_-=0 & \text{on} \quad \partial D_\delta \\ \mu\frac{\partial}{\partial n}(u_\delta,q_\delta)|_+-\tilde{\mu}\frac{\partial}{\partial n}(u_\delta,q_\delta)|_-=0 & \text{on} \quad \partial D_\delta \\ u_\delta=g & \text{on} \quad \partial\Omega \\ \int\limits_\Omega q_\delta=0. \end{cases} \tag{3}$$

We aim to derive an asymptotic expansion of $(u_\delta - u)|_{\Omega_0}$ and $(u_\delta - u)|_{D_0}$ as δ tends to zero, where D_0 is any close subset of $D \cap D_\delta$ and Ω_0 is any subset of $\Omega \setminus (D \cup D_\delta)$. The procedure is based on potential theoretical considerations in connection with a boundary integral equations method, and geometric properties of the perturbed boundary. This work is submitted for publication [13].

Asymptotic behaviors for eigenvalues and eigenfunctions associated to the Stokes operator in the presence of a small boundary perturbations

The field of eigenvalue problems under shape perturbation has been an active research area for several decades. The main objective of the second part in this report is to present a schematic way to derive high-order asymptotic expansions for both eigenvalues and eigenfunctions for the Stokes operator caused by small perturbations of the boundary. Also, we rigorously derive an asymptotic formula which is in some sense dual to the leading-order term in the asymptotic expansion of the perturbations in the Stokes eigenvalues due to interface changes of the inclusion. This is inspired from the fact that the Stokes system can be viewed as the incompressible limit [7]. Thus we rigourously extended some results elaborated in [5] to the Stokes case. The properties of eigenvalue problems under shape deformations have been a subject of comprehensive studies such as [2, 3, 10, 12, 29, 49] and the area continues to carry great importance [5, 8, 28, 30, 46, 63]. A substantial portion of these investigations discusses the properties of smoothness and analyticity of eigenvalues and eigenfunctions with respect to perturbations.

Let us consider the following eigenvalue problem for the Stokes system with homogeneous boundary conditions:

$$\begin{cases} -\Delta v + \nabla p = \lambda v & \text{in } \Omega \\ \nabla . v = 0 & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega \end{cases} \tag{4}$$

with $\Omega \subset \mathbb{R}^3$ a bounded open domain with boundary of class C^2. Here $v = (v_1, v_2, v_3)$ denotes the velocity field while the scalar function p is the pressure.

We assume that the boundary $\partial\Omega$ is subjected to a small, smooth deformation and that the

boundary of the deformed domain Ω_δ is given by:

$$\partial\Omega_\delta := \{\tilde{x} = x + \delta h(x)\nu(x), \quad x \in \partial\Omega\} \tag{5}$$

where $\nu(x)$ is the outward normal vector on $\partial\Omega$ and $h(x)$ is a real function in $C^2(\partial\Omega)$ that satisfies

$$\|h(x)\|_{C^2(\partial\Omega)} < 1. \tag{6}$$

Obviously, the domain Ω_δ is of class C^2 and the Dirichlet eigenvalue problem for the Stokes system can be defined in Ω_δ as well.

In this work, we derive the asymptotic expansion of eigenvalues, eigenfunctions and the eigenpressures solutions to the Stokes system:

$$\begin{cases} -\Delta v_\delta + \nabla p_\delta = \lambda_\delta v_\delta & \text{in } \Omega_\delta \\ \nabla . v_\delta = 0 & \text{in } \Omega_\delta \\ v_\delta = 0 & \text{in } \partial\Omega_\delta. \end{cases} \tag{7}$$

Here we suppose that the eigenvalue λ_0 is simple. Then the eigenvalue λ_δ is simple and is near to λ_0 associated to the normalized eigenfunction v_δ.
To the best of our knowledge, this is the first work to rigorously investigate the Stokes eigenvalue problem in the presence of the perturbation and derive (formally) high-order terms in the asymptotic expansion of $\lambda_\delta - \lambda_0$ and $v_\delta - v_0$ when $\delta \to 0$. However, by the same method, one can derive asymptotic formula for the Neumann problem as well.
Zuazua and Ortega have proved in [49] the regularity of the eigenvalues and eigenfunctions of the Stokes system with respect to the perturbation parameter, by using the Lyapunov-Schmidt method. Their proofs are essentially inspired in the work of J. Albert [2, 3] for the Laplace operator. Our analysis and uniform asymptotic formulas of eigenvalues and eigenfunctions, which are represented by the single-layer potential involving the Green function, are considerably different from those in [28, 49].
These results were published in [16].

Singular integrals in Electromagnetism

Integral equation methods are an alternative for solving linear partial differential equations. It consists in transforming the partial differential equation posed in a domain, into an integral

equation posed on the edge. It is in particular a method adapted for diffraction problems that are posed in an unbounded domain (see [25]) and for which the consideration of the radiation condition is essential. We are interested in solving time-harmonic scattering problems by the integral aquation method. In order to ensure a unique solution to the boundary value problem we will take the constant impedance operator, known as standard or Leontovich impedence boundary condition. This approximation does not depend on the incident angle at all.
Once the boundary problem has been converted into a boundary integral equation, it is necessary to discretize the edge of the domain using plane polygons. Here we deal with plane triangles and Rao-Wilton-Glisson basis functions $\mathbf{f}_k$. In this thesis, we are particularly interested in the singular integrals resulting from the discretization of the variational problem of Maxwell 3-D equation, namely:

$$\iint_{\Gamma_h} \mathbf{f}_i(x) \cdot \mathbf{f}_j(y) \frac{1}{|x-y|} dxdy \tag{8}$$

which comes from the potential of the single layer. This integral has been treated by Marc Lenoire and Nicolas Salles in [40] with a purely analytic method.
The second one comes from a double layer potential can be written as follows:

$$\iint_{\Gamma_h} [\mathbf{f}_i(x) \times \mathbf{f}_j(y)] \cdot \frac{x-y}{|x-y|^{1+\xi}} dxdy; \xi = 0 \text{ or } 2. \tag{9}$$

This singular character of the integrand makes the precise evaluation of these integrals difficult. It has been the subject of a large number of publications. The method presented in this thesis allows us to evaluate analytically and with great precision, these singular integrals in the variational case. Note that if $\xi = 0$ the integral is not singular but the method we propose is still valid.

To better understand this singular integrand problem, in the variational case, it should be noted that the domain of integration for these integrals with "problem", is the product of two elements of the mesh, ie in 3-D, the product of two triangles S and T. When the two triangles are distant, it is possible to evaluate the integral with standard numerical integration methods. But the two problematic cases are the singular case, which occurs when the intersection of the two triangles is not empty(adjacent triangles and triangles with a common vertex). These two cases require separate treatment and this is a field of research still very active.

Reduction process

The presented method makes it possible to reduce a 4-D integral to a linear combination of mono-dimensional integrals whose integrand is regular. It is then possible to numerically evaluate these 1-D integrals, but also explicitly.

This method of reduction relies on several formulas that make it possible to reduce the dimension of the integration domain. So, repeating three times using these formulas we obtain regular 1-D integrals. These formulas are based on the homogeneity of the integrand; therefore, there are two conditions to check:
1) - The integrand must be homogeneous. We must therefore break the integrand into a part homogeneous, which is treated with our method, and a regular part, which can be treated with usual numerical integration methods.
2) - Flat polygons must be used to discretize the boundary of the domain. This in order to maintain the homogeneity of the integrand during the successive stages of reduction.

Organization of this thesis

This thesis is divided into three parts, and composed of eight chapters, corresponding to the goals introduced above.
The first part includes three chapters (Chapter 1, 2, 3) with the following contents:

Chapter 1 is devoted to introduce the notations and some preliminary results of layer-potential theory for Stokes problem and perturbed geometry.

Chapter 2 show the existence and uniqueness of the solutions of a Dirichlet transmission Stokes problem and presents it in form of potential operators.

In chapter 3 we first, verify the continuity of the solution with respect to the small perturbation δ *via* the stability of the density function. Secondly, we derive the asymptotic expansion of the solution, after deriving the expansion of the density function.

The second part states the results that have been published in the journal *Math Phys Anal Geom*, see [16].

In chapter 4, we derive asymptotic expansions for both eigenvalues and eigenfunctions for the Stokes operator caused by small perturbations of the boundary. Also, we rigorously derive an asymptotic formula.

In the third part we develop a method for the evaluation of integrals with a singular integrand derived from the application of a boundary element method on Maxwell's equation.

In chapter 5, the time-harmonic scattering problem of electromagnetic waves is described as a system of equations with boundary conditions. The uniqueness of the solution of this problem is shown thanks to a variational formulation. It is also concerned with the discretization of the problem with help of Rao-Wilton-Glisson basis functions for mesh triangulation.

The next chapter, is dedicated to the presentation of the method. The integration of homogeneous functions leads to formulas for reducing the size of the integration domain.

In chapter 7, we presents the evaluation of singular integrals when the triangles are Adjacent triangles, and the evaluation of singular integrals when the triangles have common vertex in chapter 8.

We present numerical results in the last chapter.

We have put the proofs for some inequalities and auxiliary results concerning the first part in Appendix A, the final expressions of the mono-dimensional integrals in Appendix B and some preliminary results about Fredholm alternative and integral formulation Maxwell equation in Appendix C.

Part I

Asymptotic expansion for the solution of a Stokes transmission problem with a small perturbation of the domain

CHAPTER 1

Preliminary results

Introduction

In this chapter, we present geometric definitions, notations and basic results which will be used in the next chapters as well as the main properties of the layer potential operators associated to the Stokes resolvent equation on a bounded Lipschitz domains in $\mathbb{R}^3$.
The outline of this chapter is as follows. The first section contains the setting for basic notations, definitions and related funtional spaces. Section 1.2 is devoted to present some geometric properties of a boundary perturbation. In section 1.3, we describe layer potential techniques for solving the Stokes resolvent problems on a bounded Lipschitz domains in $\mathbb{R}^3$. We then present in 1.3.1 the fundamental solution for Stokes resolvent systems and we define in 1.3.2 the properties of the associated layer potential operators.

1.1 Euclidean spaces

This section contains some notations and definitions concerning Euclidean spaces and functional spaces, wich plays a segnificant role all along this work.
For $x, y \in \mathbb{R}^3$, $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ let $x \cdot y := \sum_{i=1}^{3} x_i y_i$ be the scalar product of x, y and $|x| := \sqrt{x \cdot x}$ be the Euclidean norm of x.
Denote by $|x - y|$ the distance between two points x and y of $\mathbb{R}^3$ given by the formula

$$|x - y| = [\sum_{i=1}^{3} (x_i - y_i)^2]^{1/2}.$$

More generally, $|u|$ stands for the modulus of the vector u and we have $|u| = (\sum_i u_i^2)^{1/2}$. Similarly, the distance between two subsets A and B of $\mathbb{R}^3$ is denoted by $\mathrm{dist}(A, B)$ and we have

$$\mathrm{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|.$$

Definition 1.1.1. *(*Derivatives*)*

1. *The gradient is denoted by $\nabla = (\partial_1, \partial_2, \partial_3)$, where $\partial_j = \frac{\partial}{\partial_j}$ denotes the partial derivative with respect to the j-th coordinate $(x_j)_{j=1,2,3}$.*

2. *For all multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote the α-derivative of f by*

$$D^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} f}{\partial x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}},$$

   where the order of this derivative is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

3. *For vector-valued function $v = (v_1, v_2, v_3)$, $\nabla \cdot v = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$ defines the divergence of v.*

Definition 1.1.2. *(*The space C^k*)*
Given a non-negative integer k and an open domain $D \subset \mathbb{R}^3$, let $C^k(D)$ denote the space of all functions u defined in D which have all their derivatives $D^\alpha u$ of order $|\alpha| \leq k$ continuous in D. To simplify the notation, we set $C^0(D) \equiv C(D)$.
Define $C^k(\overline{D})$ as the space of all $u \in C^k(D)$ such that $D^\alpha u$ can be extended from D to a continuous function on $\overline{D}$, the closure of the set D, for all $|\alpha| \leq k$. $C^k(\overline{D})$ can be equipped with the norm

$$\|\phi\|_{C^k(\overline{D})} = \max_{i=0,\ldots,k} \sup_D |D^i \phi|.$$

Definition 1.1.3. *(*Lebesgue spaces*)*
Let $L^2(D)$ be the Lebesgue space of square integrable functions on D and let $L^2(\partial D)^3$ be the space of vector fields $\phi = (\phi_1, \phi_2, \phi_3) : \mathbf{R}^3 \to \mathbf{R}^3$ whose components ϕ_i belong to $L^2(\partial D)$. We have

$$\|\phi\|_{(L^2(\partial D))^d} = \sqrt{\int_{\partial D} (|\phi_1(x)|^2 + |\phi_2(x)|^2 + |\phi_3(x)|^2) d\sigma(x)}.$$

We recall now the definition of curvature of a curve and definitions of mean and Gauss curvature of a surface that will be appeared in the expansion terms of the area element in chapter 3. For a complete description, we refer to [9] and [55].

Definition 1.1.4. *(*Curvature*)*
Consider D a bounded open set in $\mathbb{R}^2$ with a connected boundary ∂D, $a, b \in \mathbb{R}$ with $a < b$, and $X(t) : [a, b] \to \mathbb{R}^2$ an arclength parametrization of ∂D. Namely, X is a C^2-function satisfying $|X'(t)| = 1$ for all $t \in [a, b]$ and

$$\partial D := \{x = X(t), t \in [a, b]\}.$$

Then the outward unit normal to ∂D, $\nu(x)$, is given by $\nu(x) = R_{\frac{-\pi}{2}} X'(t)$, where $R_{\frac{-\pi}{2}}$ is the rotation by $-\pi/2$; the tangent vector at x, $T(x) = X'(t)$, and $X'(t) \perp X''(t)$. Set the curvature $\tau(x)$ to be

$$X''(t) = \tau(x)\nu(x).$$

Definition 1.1.5. *(*Gaussian and mean curvature*)*
Let k_1 and k_2 be the principal curvatures of a regular surface S at a point P. Define

1. *The Gaussian curvature of S at P as the product $K = k_1 k_2$;*
2. *The mean curvature of S at P is the average $H = \frac{\kappa_1 + \kappa_2}{2}$ of the principal curvatures.*

We also remark that for a bounded domain D with boundary of class $C^{2,\alpha}$, the outward unit normal ν is a $C^{1,\alpha}$ function and the following properties hold:

Lemma 1.1.1. *([36], Lemma 6.16)*
There exists a positive constant L such that

$$|\nu(x) \cdot \{x - y\}| \leq L|x - y|^2,$$

and

$$|\nu(x) - \nu(y)| \leq L|x - y|,$$

for all $x, y \in \partial D$.

1.2 Small perturbation of an interface

Let $D \subset \mathbb{R}^3$ be a bounded domain with connected boundary ∂D of class C^2. For a fixed constant $\delta \in \mathbb{R}$, we consider ∂D_δ as a $\delta-$perturbation of ∂D defined by

$$\partial D_\delta = \{\tilde{x} = x + \delta\nu(x); x \in \partial D, 0 < \delta << 1\}. \tag{1.1}$$

We denote by D_δ the domain bounded by ∂D_δ and $d\sigma_\delta(\widetilde{x})$ the surface element of ∂D_δ at $\widetilde{x}$. The result presented below is given in [31] with more details.

Lemma 1.2.1. *The surface elements $d\sigma$ on ∂D and $\mathrm{d}\sigma_\delta$ on ∂D_δ are related by the formula*

$$\mathrm{d}\sigma_\delta(\widetilde{x}) = \sum_{n=0}^{2} \delta^n \sigma_n(x) \mathrm{d}\sigma(x), \tag{1.2}$$

where σ_n are bounded functions regardless of n. In particular,

$$\sigma_0(x) = 1, \sigma_1(x) = -2H(x), \sigma_2 = K(x),\ x \in \partial D, \tag{1.3}$$

with H and K denote the mean and the Gaussian curvature of ∂D respectively, as introduced in Definition 1.1.5.

Since ∂D is parallel to ∂D_δ, we can conclude that

$$\widetilde{\nu}(\widetilde{x}) = \nu(x),\ x \in \partial D,\ \widetilde{x} \in \partial D_\delta. \tag{1.4}$$

We then have for $\widetilde{x}, \widetilde{y} \in \partial D_\delta$:

$$\widetilde{x} - \widetilde{y} = x - y + \delta(\nu(x) - \nu(y)),$$

and

$$|\widetilde{x} - \widetilde{y}|^2 = |x - y|^2 (1 + 2\delta \frac{\langle x - y, \nu(x) - \nu(y)\rangle}{|x - y|^2} + \delta^2 \frac{|\nu(x) - \nu(y)|^2}{|x - y|^2}).$$

We also introduce two functions F and G for $x \neq y \in \partial D$ defined by:

$$E(x, y) := \frac{\langle x - y, \nu(x) - \nu(y)\rangle}{|x - y|^2}, G(x, y) := \frac{|\nu(x) - \nu(y)|^2}{|x - y|^2}.$$

Since ∂D is of class C^2, there exists a constant C depending only on ∂D such that:

$$|E(x, y)| + |G(x, y)|^{\frac{1}{2}} \leq C, \forall x, y \in \partial D.$$

Hence, we obtain:

$$|\widetilde{x} - \widetilde{y}| = |x - y| \sqrt{1 + 2\delta E(x, y) + \delta^2 G(x, y)} := |x - y| \sum_{n=0}^{\infty} \delta^n L_n(x, y),$$

where the serie converges absolutely and uniformly and the first two terms are:

$$L_0(x, y) = 1, L_1(x, y) = E(x, y).$$

1.3 Potential theory

This section presents layer potential theory for the Stokes resolvent system. The main sources used in the preparation of this section are [57] [58] [59] and [60]

1.3.1 Stokes resolvent system

1.3.1.1 Fundamental solution

Let consider a bounded Lipschitz domain $D \subset \mathbf{R}^3$ with connected boundary ∂D and a given complex constant κ^2 in $\mathbb{C}^*$ where $\mathbb{C}^* = \mathbb{C} \setminus \{z \in \mathbb{R},\ z \leq 0\}$. Let $\kappa = \sqrt{\kappa^2}$ be particular square root, which have positive real parts (*i.e.* Re $\kappa > 0$). We denote by $\Gamma^\kappa = (\Gamma^\kappa_{ij})_{i,j=1,2,3}$ and $F^\kappa = (F^\kappa)_{i=1,2,3}$ the fundamental tensor and vector of the following Stokes resolvent system:

$$\begin{cases} (-\Delta + \kappa^2)\Gamma^\kappa_{ij}(x,y) + \partial_j F^\kappa_i(x,y) = \delta_{ij}\delta_y(x) & \text{on} \quad D, \\ \partial_i \Gamma^\kappa_{ij}(x,y) = 0 & \text{on} \quad D, \end{cases} \tag{1.5}$$

where δ_y is the Dirac distribution with mass at y and δ_{ij} is the Kronecker symbol. The components of $(\Gamma^\kappa, F^\kappa)$ are obtained in these forms (see [57] [58]):

$$\begin{cases} \Gamma^\kappa_{ij}(x,y) &= \frac{1}{4\pi}\left\{\frac{\delta_{ij}}{|x-y|}e_1(\kappa|x-y|) + \frac{(x-y)_i(x-y)_j}{|x-y|^3}e_2(\kappa|x-y|)\right\}, \\ F^\kappa_i(x,y) &= \frac{1}{4\pi}\frac{(x-y)_i}{|x-y|^3}, \end{cases} \tag{1.6}$$

where

$$e_1(\epsilon) = \sum_{n=0}^{\infty} \frac{(n+1)^2}{(n+2)!}(-\epsilon)^n = exp(-\epsilon)(1+\epsilon^{-1}+\epsilon^{-2}) - \epsilon^{-2},$$

$$e_2(\epsilon) = \sum_{n=0}^{\infty} \frac{1-n^2}{(n+2)!}(-\epsilon)^n = exp(-\epsilon)(-1-3\epsilon^{-1}-3\epsilon^{-2}) + 3\epsilon^{-2}.$$

Here and throughout this report we use the Einstein convention for the summation notation omitting the summation sign for the indices appearing twice.

1.3.1.2 Stress and pressure tensors

Now, let introduce for $x, y \in \partial D$, the stress tensor $\mathcal{S}$ associated to the fundamental tensors $(\Gamma^\kappa, F^\kappa)$ defined by:

$$\mathcal{S}^\kappa_{ijk}(x,y) = -F^\kappa_j(\hat{x})\delta_{ik} + \frac{\partial \Gamma^\kappa_{ij}(\hat{x})}{\partial \hat{x}_k} + \frac{\partial \Gamma^\kappa_{ik}(\hat{x})}{\partial \hat{x}_j}, \quad i,j,k = 1,2,3. \tag{1.7}$$

where $\hat{x} = x - y = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $r = |\hat{x}|$. Combining (1.6) and (1.7), one can obtain:

$$\mathcal{S}^{\kappa}_{ijk}(x,y) = -\frac{1}{\omega_d}\left\{\delta_{ik}\frac{\hat{x}_j}{r^d}D_1(\kappa r) + (\delta_{kj}\frac{\hat{x}_i}{r^d} + \delta_{ij}\frac{\hat{x}_k}{r^d})D_2(\kappa r) + \frac{\hat{x}_i\hat{x}_j\hat{x}_k}{r^{d+2}}D_3(\kappa r)\right\} \tag{1.8}$$

where

$$D_1(t) = 8\frac{(\frac{t}{2})^{m+1}K_{m+1}(t)}{\Gamma(m)t^2} - \frac{6}{t^2} + 1,$$

$$D_2(t) = 8\frac{(\frac{t}{2})^{m+1}K_{m+2}(t)}{\Gamma(m)t^2} - \frac{6}{t^2} + 2\frac{(\frac{t}{2})^{m}K_{m}(t)}{\Gamma(m)},$$

$$D_3(t) = -16\frac{(\frac{t}{2})^{m+2}K_{m+2}(t)}{\Gamma(m)t^2} + \frac{30}{t^2},$$

with $m = \frac{3}{2}$, $t \in \mathbf{R}$, $\Gamma(m)$ is the Gamma function and K_n for $n \geq 0$ is the modified Bessel function of order n. For more details on these functions, see Appendix A.

The pressure tensor associated with the stress tensor $\mathcal{S}$ has the following components:

$$\Lambda^{\kappa}_{ik}(x,y) = -\frac{1}{4\pi}(2\frac{\delta_{ik}}{r^3} - \frac{\delta_{ik}\kappa^2}{r} - 6\frac{\hat{x}_i\hat{x}_k}{r^5}), \quad i,k = 1,2,3. \tag{1.9}$$

1.3.1.3 Decay behavior of the fundamental and the stress tensor

In order to study the decay behavior of $\Gamma^{\kappa}_{ij}(x,y)$ and $\mathcal{S}^{\kappa}_{ijk}(x,y)$ as $r = |x-y| \to 0$, we can apply the following decomposition:

$$\Gamma^{\kappa}_{ij}(x,y) = \Gamma^{0}_{ij}(x,y) + \Gamma_{ij}(x,y), \tag{1.10}$$

$$\mathcal{S}^{\kappa}_{ijk}(x,y) = \mathcal{S}^{0}_{ijk}(x,y) + \mathcal{S}_{ijk}(x,y), \tag{1.11}$$

where $\mathcal{S}^0$ and Γ^0 are weakly singular Stokes tensors (with respect to κ=0) and the remaining part $\mathcal{S}$ and Γ are continuous kernels, so that the continuity behavior of the corresponding surface potential is determined only by the Stokes tensors $\mathcal{S}^0$ and Γ^0. A detailed proof of the above relations can be found in [59] or [58]. Besides, we note that Γ^0 and $\mathcal{S}^0$ are fundamental tensors for the standard Stokes system [37].

This property will be very useful in the treatment of some boundary integral representation associated to a transmission system that will be analyzed in the next chapter.

1.3.2 Layer potential operators for the Stokes system

In this section, we present layer potential operators associated to the Stokes equation in the bounded Lipschitz domain $D \subset \mathbf{R}^3$. A special attention is devoted to the invertibility of the potential operators associated with the standard Stokes equation and the compactness of the complementary layer potential operators.

Definition 1.3.1. *For a density $\phi = (\phi_1, \phi_2, \phi_3) \in L^2(\partial D)^3$, we define the single-layer potential $S_D^\kappa \phi : \mathbf{R}^3 \backslash \partial D \to \mathbf{R}^3$ and the associated pressure potential, $V_D \phi : \mathbf{R}^3 \backslash \partial D \to \mathbf{R}$ as follows:*

$$S_{D,i}^\kappa \phi(x) := \int_{\partial D} \Gamma_{ij}^\kappa(x,y)\phi_j(y) d\sigma(y); i,j = 1,2,3, \tag{1.12}$$

$$V_D^\kappa \phi(x) := \int_{\partial D} F_j^\kappa(x,y)\phi_j(y) d\sigma(y); j = 1,2,3. \tag{1.13}$$

Likewise for a given density $\psi = (\psi_1, \psi_2, \psi_3) \in L^2(\partial D)^3$, we define the double-layer potential $\mathcal{D}_D^\kappa \psi : \mathbf{R}^3 \backslash \partial D \to \mathbf{R}^3$, by

$$\mathcal{D}_{D,i}^\kappa \psi(x) := \int_{\partial D} -\mathcal{S}_{ijk}^\kappa(x,y)\nu_k(y)\psi_j(y) d\sigma(y); i,j,k = 1,2,3, \tag{1.14}$$

where ν is the outward unit normal to ∂D. Moreover, the associated pressure potential $W_D^\kappa \psi : \mathbf{R}^3 \backslash \partial D \to \mathbf{R}$ is defined by

$$W_D^\kappa \psi(x) := \int_{\partial D} -\Lambda_{jk}(x,y)\nu_k(y)\psi_j(y) d\sigma(y); j,k = 1,2,3. \tag{1.15}$$

We can easily check that (S_D^κ, V_D^κ) and $(\mathcal{D}_D^\kappa, W_D^\kappa)$ are solutions of the Stokes system in $\mathbf{R}^3 \setminus \partial D$,

$$(-\Delta + \kappa^2) S_D^\kappa \phi(x) + \nabla V_D^\kappa \phi(x) = 0; \quad \nabla \cdot S_D^\kappa(x) = 0, \tag{1.16}$$

$$(-\Delta + \kappa^2) \mathcal{D}_D^\kappa \psi(x) + \nabla W_D^\kappa \psi(x) = 0; \quad \nabla \cdot \mathcal{D}_D^\kappa(x) = 0. \tag{1.17}$$

We need also to introduce the principal value of the double-layer potential (or the boundary version of the double layer potential).

Definition 1.3.2. *For $x \in \partial D$ and a density $\phi = (\phi_1, \phi_2, \phi_3) \in L^2(\partial D)^3$, the principal value of the double-layer potential is given by:*

$$\mathcal{K}_{D,i}^\kappa(\phi)(x) := p.v. \int_{\partial D} -\mathcal{S}_{ijk}^\kappa(x,y)\nu_k(y)\phi_j(y) d\sigma(y); i,j,k = 1,2,3, \tag{1.18}$$

and $(\mathcal{K}_D^\kappa)^*$ *the adjoint operator of* $\mathcal{K}_D^\kappa$*:*

$$(\mathcal{K}_{D,i}^\kappa)^*(\phi)(x) := p.v. \int_{\partial D} -\mathcal{S}_{ijk}^\kappa(x,y)\nu_k(x)\phi_j(y)d\sigma(y); i,j,k = 1,2,3. \tag{1.19}$$

where $p.v.$ *means the principal value of a singular integral.*

1.3.2.1 Trace and jump relations

Denote by $\frac{\partial}{\partial n}$ the conormal derivative defined by

$$\frac{\partial}{\partial n}(u,q) = (\nabla u + (\nabla u)^T)\nu - q\nu \text{ on } \partial D, \tag{1.20}$$

where $(\nabla u)^T$ denotes the matrix transposed of $\nabla u = (\partial_i u_k)_{i,k=1,2,3}$ (see [6], [14], [37] and [53]). In the lemma below we give trace relations for the single and double layer potential.

Theorem 1.3.1. *If* $D \subset \mathbf{R}^3$ *is a boundary Lipschitz domain with the boundary* ∂D*, then, for* $(\phi,\psi) \in L^2(\partial D)^3 \times L^2(\partial D)^3$ *one has.*

$$\mathcal{D}_D^\kappa(\psi)|_\pm = (\mp\frac{1}{2}I + \mathcal{K}_D^\kappa)(\psi), \tag{1.21}$$

$$\frac{\partial}{\partial n}(\mathcal{D}_D^\kappa(\psi),\, W_D^\kappa(\psi))|_+ = \frac{\partial}{\partial n}(\mathcal{D}_D^\kappa(\psi),\, W_D^\kappa(\psi))|_-, \tag{1.22}$$

$$\frac{\partial}{\partial n}(S_D^\kappa(\phi),\, V_D^\kappa(\phi))|_\pm = (\pm\frac{1}{2}I + (\mathcal{K}_D^\kappa)^*)(\phi), \tag{1.23}$$

$$S_D^\kappa(\psi)|_+ = S_D^\kappa(\psi)|_- = \mathcal{S}_D^\kappa(\psi). \tag{1.24}$$

Here and throughout this report, we use the subscripts $|_+$ (respectively $|_-$) for the limiting value of a field evaluated from the external side of D (or the internal side of D, respectively).

$$v|_\pm(x) := \lim_{D_\pm \ni y \to x} v(y).$$

The formulas (1.21), (1.22), (1.23) and (1.24) have been proved in [37] when $\kappa = 0$. Since $\mathcal{D}_D^\kappa - \mathcal{D}_D^0$ and $S_D^\kappa - S_D^0$ are smooth operators and according to (1.10) and (1.11), we can extend the result to the case $\kappa \neq 0$. Furthermore, using (1.21) and (1.23) we can easily show that the single and double layer potentials satisfy the following jump ralations on ∂D:

$$\mathcal{D}_D^\kappa(\psi)|_+ - \mathcal{D}_D^\kappa(\psi)|_- = -\phi \quad on \quad \partial D, \tag{1.25}$$

$$\frac{\partial}{\partial n}S_D^\kappa(\phi)|_+ - \frac{\partial}{\partial n}S_D^\kappa(\phi)|_- = \varphi \quad on \quad \partial D. \tag{1.26}$$

1.3.2.2 Decay behavior of layer potentials

Here we give another result about the decay behavior at infinity of the layer potentials proved by Varnhorn in [57] p.15 and [60].

Lemma 1.3.1. *For the single and double layer potentials $S_D^\kappa[\phi]$, $V_D^\kappa[\phi]$, $\mathcal{D}_D^\kappa[\psi]$ and $W_\Omega^\kappa[\psi]$, we have the following decay behavior as $|x| \to \infty$:*

i)
$$S_D^\kappa[\phi](x) = \mathrm{O}(|x|^{-1}), \quad as \ \ |x| \to \infty.$$

ii)
$$V_D^\kappa[\phi](x) = \mathrm{O}(|x|^{-2}), \quad as \ \ |x| \to \infty.$$

iii)
$$\mathcal{D}_D^\kappa[\psi](x) = \mathrm{O}(|x|^{-2}), \quad as \ \ |x| \to \infty.$$

iv)
$$W_D^\kappa[\psi](x) = \mathrm{O}(|x|^{-3}), \quad as \ \ |x| \to \infty.$$

1.3.2.3 Compactness of layer potential operators

According to theorem 2.27 in [36], the following lemma holds about the compactness of operators with kernels defined by (1.10) and (1.11).

Lemma 1.3.2. *For $x \in \partial D$, we define the operators :*

$$\mathcal{K}_{D,i}[\phi](x) := p.v. \int_{\partial D} -\mathcal{S}_{ijk}(x,y)\nu_k(y)\phi_j(y)d\sigma(y); \ \ i,j,k = 1,2,3, \tag{1.27}$$

$$\mathcal{K}^*_{D,i}[\phi](x) := p.v. \int_{\partial D} -\mathcal{S}_{ijk}(x,y)\nu_k(x_0)\phi_j(y)d\sigma(y); \ \ i,j,k = 1,2,3, \tag{1.28}$$

$$S_{D,i}[\phi](x) := \int_{\partial D} \Gamma_{ij}(x,y)\phi_j(y)d\sigma(y); \ \ i,j,k = 1,2,3, \tag{1.29}$$

$$S_D(\phi)|_+ = S_D(\phi)|_- = \mathsf{S}_D(\phi). \tag{1.30}$$

These operators have continuous kernel. Moreover, S_D, $\mathcal{K}_D$ and $\mathcal{K}^*_D$ are compact operators on $\mathcal{C}(\partial D)^3$ and on $L^2(\partial D)^3$.

1.3.2.4 Invertibility of layer potentials

We recall that the single layer potential associated with $\kappa = 0$ is defined as follows

$$S^0_{D,i}\phi(x) := \int_{\partial D} \Gamma^0_{ij}(x,y)\phi_j(y)d\sigma(y),\ i = 1, \cdots, 3.$$

where $D \subset \mathbf{R}^3$ is a bounded domain with boundary ∂D. It then follows from [37] the theorem 1.3.2.

Theorem 1.3.2.

$$\ker(\boldsymbol{S}^0_D) = \{c\nu : c \in \mathbf{R}\}. \tag{1.31}$$

Now, we give two lemmas about the invertibility of these operators on spaces that we are going to define.

Lemma 1.3.3. *[6] Let $L^2_0(\partial D) := \{f \in L^2(\partial D)^3; \int_{\partial D} f \cdot \nu = 0\}$ and define $H^1_0(\partial D)$ likewise. Then the following result holds.*

i) The operator $S^0_D : L^2_0(\partial D) \to H^1_0(\partial D)$ is invertible.

ii) The operator $\lambda I + \mathcal{K}^0_D$ and $\lambda I + (\mathcal{K}^0_D)^*$ are invertible on $L^2(\partial D)^3$ for $|\lambda| > 1/2$.

Lemma 1.3.4. *[37] Let $D \subset \mathbf{R}^3$ be a bounded domain with boundary ∂D. Then, we have: $(1/2)I + \mathcal{K}^\kappa_D : L^2_0(\partial D) \to L^2_0(\partial D)$ is invertible and so is $(1/2)I + (\mathcal{K}^0_D)^* : L^2_0(\partial D) \to L^2_0(\partial D)$.*

Conclusion

By referring to previous works, we recalled theorems and definitions on which our work in the next chapters, will be based.

CHAPTER 2

Resolution of the Stokes transmission problem

Introduction

This chapter is devoted to resolving the Dirichlet transmission Stokes problem and presenting its solution in form of a linear combination of potential operators. In the next section 2.1, we construct the solution of this problem, in the form of appropriate potentials and we reduce the transmission problem to a system of boundary integral equations. Then, in section 2.2, we prove unicity of the unknown source densities which implies unicity of the solution.

2.1 Boundary integral representation

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a connected Lipschitz boundary $\partial\Omega$ and suppose that Ω contains an inhomogeneity D with a $\mathcal{C}^2$-boundary ∂D.We assume that there exists a constant $c_0 > 0$ such that $\inf_{x\in D} dist(x, \partial\Omega) > c_0$ which means that D is away from the boundary $\partial\Omega$.
Using arguments given in the previous chapter, we are looking for a solution $(u,\, q)$ to the transmission Stokes system with Dirichlet boundary condition:

$$\begin{cases} (-\Delta + \kappa^2)u + \nabla q = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ (-\Delta + \tilde{\kappa}^2)u + \nabla q = 0 & \text{in} \quad D \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \\ u|_+ - u|_- = 0 & \text{on} \quad \partial D \\ \mu \frac{\partial}{\partial n}(u,q)|_+ - \tilde{\mu} \frac{\partial}{\partial n}(u,q)|_- = 0 & \text{on} \quad \partial D \\ u = g & \text{on} \quad \partial\Omega \\ \int_\Omega q = 0, \end{cases} \tag{2.1}$$

where μ and $\tilde{\mu}$ are positive constants related to the physical properties of Ω and D. Note that κ^2 and $\tilde{\kappa}^2$ are complex numbers in $\mathbb{C}^*$ such that $\mathbb{C}^* = \mathbb{C} \setminus \{z \in \mathbb{R},\ z \leq 0\}$.
Let $\kappa = \sqrt{\kappa^2}$ and $\tilde{\kappa} = \sqrt{\tilde{\kappa}^2}$ be particular square roots, which have positive real parts (*i.e.* Re $\kappa > 0$, Re $\tilde{\kappa} > 0$).
From Gauss's law theorem, we have the compatibility condition:

$$\int_{\partial\Omega} g \cdot \nu_\Omega = 0,$$

where ν_Ω is the outward unit normal to $\partial\Omega$. Let ν be the outward unit normal to ∂D.
In order to solve (2.1), We adopte a method of boundary integral equations, wich consist on transforming the transmission problem posed in $\Omega \setminus \overline{D}$ and in D, into integral equations defined on the boundarys ∂D and $\partial\Omega$. This transition from a volume problem to a surface problem is carried out using layer potentials. Indeed, according to (1.16) and (1.17) in chapter (1), for some triplet $(\phi, \psi, \theta) \in L^2(\partial D)^3 \times L^2(\partial D)^3 \times L^2(\partial\Omega)^3$, the boundary integral representation:

$$\begin{cases} u = S_D^{\tilde{\kappa}}[\phi] & \text{in } D, \\ u = S_D^{\kappa}[\psi] + \mathcal{D}_\Omega^{\kappa}[\theta] & \text{in } \Omega \setminus \overline{D}, \end{cases} \tag{2.2}$$

$$\begin{cases} q = V_D^{\tilde{\kappa}}[\phi] & \text{in } D, \\ q = V_D^{\kappa}[\psi] + W_\Omega^{\kappa}[\theta] & \text{in } \Omega \setminus \overline{D}, \end{cases} \tag{2.3}$$

solves

$$\begin{cases} (-\Delta + \kappa^2)u + \nabla q = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ (-\Delta + \tilde{\kappa}^2)u + \nabla q = 0 & \text{in} \quad D \\ \nabla \cdot u = 0 & \text{in} \quad \Omega. \end{cases}$$

So, we will look for a solution $(u,\ q)$ to the transmission problem (2.1) in the form of (2.2)-(2.3). Otherwise, we will look for $(\phi, \psi, \theta) \in L^2(\partial D)^3 \times L^2(\partial D)^3 \times L^2(\partial\Omega)^3$ satisfying the boundary conditions:

$$\begin{cases} u|_+ - u|_- = 0 & \text{on} \quad \partial D \\ \mu \frac{\partial}{\partial n}(u,q)|_+ - \tilde{\mu} \frac{\partial}{\partial n}(u,q)|_- = 0 & \text{on} \quad \partial D \\ u = g & \text{on} \quad \partial\Omega \end{cases}$$

Remark 2.1.1. *For the same problem, several boundary integral representations can be obtained.*

2.2 Existence and unicity of the solution

To be a solution to the problem (2.1), the representation (2.2)-(2.3) must satisfy the boundary conditions. According to the equations (1.21)-(1.24), (ϕ, ψ, θ) satisfy the integral equations

$$\begin{pmatrix} S_D^{\tilde{\kappa}}|_+ & -S_D^{\kappa}|_- & -\mathcal{D}_\Omega^{\kappa} \\ \tilde{\mu}(-\frac{1}{2}I + (\mathcal{K}_D^{\tilde{\kappa}})^*) & -\mu(\frac{1}{2}I + (\mathcal{K}_D^{\kappa})^*) & -\mu \frac{\partial}{\partial n}\mathcal{D}_\Omega^{\kappa} \\ 0 & S_D^{\kappa} & \frac{1}{2}I + \mathcal{K}_\Omega^{\kappa} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}. \tag{2.4}$$

Denote by A^κ the matrix corresponding to the left hand side of (2.4).
In order to study the existence and unicity of the solution of the system (2.4), we introduce the space $H(\partial D)$ defined by:

$$H(\partial D) := \{(\phi, \psi) \in L^2(\partial D)^3 \times L^2(\partial D)^3; \phi - \psi \in L_0^2(\partial D)\}$$

We annonce the theorem.

Theorem 2.2.1. *Let $(\phi, \psi, \theta) \in H(\partial D) \times L_0^2(\partial\Omega)$ be the unique solution of (2.4). Then (u, q) represented by (2.2)-(2.3) is the unique solution of (2.1).*

This is equivalent to the invertibility of A^κ. We represent A^κ by a sum of two matrices A^{inv} and A^{com} defined by

$$A^{com} = \begin{pmatrix} \mathsf{S}_D & -\mathsf{S}_D & -\mathcal{D}_\Omega^{\kappa} \\ \tilde{\mu}(\mathcal{K}_D)^* & -\mu(\mathcal{K}_D)^* & -\mu \frac{\partial}{\partial n}\mathcal{D}_\Omega^{\kappa} \\ 0 & S_D^{\kappa} & \mathcal{K}_\Omega \end{pmatrix}, \tag{2.5}$$

and

$$A^{inv} = \begin{pmatrix} S_D^0 & -S_D^0 & 0 \\ \tilde{\mu}(-\frac{1}{2}I + (\mathcal{K}_D^0)^*) & -\mu(\frac{1}{2}I + (\mathcal{K}_D^0)^*) & 0 \\ 0 & 0 & \frac{1}{2}I + \mathcal{K}_\Omega^0 \end{pmatrix}. \tag{2.6}$$

We know that S_D^κ is compact on $L^2(\partial\Omega)$ and that $\mathcal{D}_\Omega^\kappa$ and $\frac{\partial}{\partial n}\mathcal{D}_\Omega^\kappa$ which appear in the third column of A^{com} are also compact on $L^2(\partial D)$ since ∂D and $\partial\Omega$ do not intersect. According to lemma 1.3.2, A^{com} is compact. Hence, it sufficies to prove that A^{inv} is invertible and that A^κ is injective to show the invertibility of A^κ according to the Fredholm alternative.

2.2.1 Invertibility of A^{inv}

We prove this first result.

Lemma 2.2.1. *The operator $A^{inv} : H(\partial D) \times L_0^2(\partial\Omega) \to H_0^1(\partial D) \times L^2(\partial D)^3 \times L_0^2(\partial\Omega)$ is invertible.*

Proof. Let $(f, h, g) \in H_0^1(\partial D) \times L^2(\partial D)^3 \times L_0^2(\partial\Omega)$, the solution (ϕ, ψ, θ) of the system

$$A^{inv} \begin{pmatrix} \phi \\ \psi \\ \theta \end{pmatrix} = \begin{pmatrix} f \\ h \\ g \end{pmatrix},$$

is given by

$$\begin{aligned} \phi &= (S_D^0)^{-1}(f) + \psi \\ \psi &= \tfrac{1}{\tilde{\mu}-\mu}(-\tfrac{1}{2}\tfrac{1+\lambda}{1-\lambda}I + (\mathcal{K}_D^0)^*)^{-1}[h - \tilde{\mu}(-\tfrac{1}{2}I + (\mathcal{K}_D^0)^*)(S_D^0)^{-1}(f)] \\ \theta &= (\tfrac{1}{2}I + \mathcal{K}_\Omega^0)^{-1}(g) \end{aligned} \tag{2.7}$$

where $0 < \lambda = \frac{\mu}{\tilde{\mu}}$. As stated in Lemmas 1.3.3 and 1.3.4, the operators S_D^0, $-\frac{1}{2}\frac{1+\lambda}{1-\lambda}I + (\mathcal{K}_D^0)^*$ and $\frac{1}{2}I + \mathcal{K}_\Omega^0$ are invertible and the proof is then completed. □

2.2.2 Injectivity of A^{inj}

Now, we focus on to the injectivity of A^κ and we give theorem.

Theorem 2.2.2. *The operator $A^\kappa : H(\partial D) \times L_0^2(\partial\Omega) \to H_0^1(\partial D) \times L^2(\partial D)^3 \times L_0^2(\partial\Omega)$ is injective .*

Proof. Suppose that there exists $(\phi_0, \psi_0, \theta_0) \in H(\partial D) \times L^2_0(\partial\Omega)$ such that $A^\kappa(\phi_0, \psi_0, \theta_0)^T = (0,0,0)^T$ and consider (u, q) defined by

$$\begin{cases} u = S_D^{\tilde{\kappa}}[\phi_0] & , \quad q = V_D^{\tilde{\kappa}}[\phi_0] & \text{in } D, \\ u = S_D^{\kappa}[\psi_0] + \mathcal{D}_\Omega^{\kappa}[\theta_0] & , \quad q = V_D^{\kappa}[\psi_0] + W_\Omega^{\kappa}[\theta_0] & \text{in } \Omega \setminus \overline{D}. \end{cases}$$

We can see that the couple $(u,\ q)$ is a solution of (2.1) with $g = 0$. Then, using integration by parts [37] on $D_- = D$ and $D_+ = \Omega \setminus \overline{D}$, we obtain

$$2\int\limits_{D_-} Eu : Eu + \tilde{\kappa}^2 \int\limits_{D_-} u.u = \int\limits_{\partial D} \frac{\partial(u,q)}{\partial n}|_-.u|_-,$$

and

$$2\int\limits_{D_+} Eu : Eu + \kappa^2 \int\limits_{D_+} u.u = -\int\limits_{\partial D} \frac{\partial(u,q)}{\partial n}|_+.u|_+ + \int\limits_{\partial D} \frac{\partial(u,q)}{\partial n}|_-.u|_-,$$

where $E(u)$ is the deformation tensor define by

$$E(u) := \frac{1}{2}(\nabla u + (\nabla u)^T).$$

The system (2.1) garantees that $u|_- = 0$ on ∂D, $u|_- = u|_+$ and $\frac{\partial(u,q)}{\partial n}|_+ = \frac{\tilde{\mu}}{\mu}\frac{\partial(u,q)}{\partial n}|_-$ on ∂D. Thus, we have

$$\begin{aligned} 2\int\limits_{D_+} Eu : Eu + \tilde{\kappa}^2 \int\limits_{D_+} u.u &= -\int\limits_{\partial D} \frac{\tilde{\mu}}{\mu}\frac{\partial(u,q)}{\partial n}|_-.u|_- \\ &= -\frac{\tilde{\mu}}{\mu}[\int\limits_{D_-} 2Eu : Eu + \tilde{\kappa}^2 \int\limits_{D_-} u.u]. \end{aligned}$$

Therefore, we get:

$$\begin{aligned} &u = 0, \quad q = c_1, \quad E(u) = 0, \quad \text{on } \ D_+, \\ &u = 0, \quad q = c_2, \quad E(u) = 0, \quad \text{on } \ D_-, \end{aligned} \tag{2.8}$$

where c_1 and c_2 are real constants. Using the argument that $\int_\Omega q = 0$, we have

$$\int\limits_{\Omega} q = c_1 \int\limits_{D_-} + c_2 \int\limits_{D_+} = 0. \tag{2.9}$$

On the other hand, the fifth line of system (2.1) gives $c_1 = c_2\frac{\mu}{\tilde{\mu}}$. Taking account of (2.9), we obtain $c_1 = c_2 = 0$.

On the other hand, we define the solution to the system:

$$\begin{cases} (-\Delta + \tilde{\kappa}^2)v + \nabla p = 0 & \text{in} \quad \mathbf{R}^3 \setminus \overline{D} \\ \nabla \cdot v = 0 & \text{in} \quad \mathbf{R}^3 \setminus \overline{D} \\ v = 0 & \text{on} \quad \partial D \\ |v||\nabla v| = o(|x|^{-2}) & \text{as} \quad |x| \to +\infty \\ |v||p| = o(|x|^{-2}) & \text{as} \quad |x| \to +\infty. \end{cases} \tag{2.10}$$

We know that

$$S_D^{\tilde{\kappa}}(\phi_0)|_+ = S_D^{\tilde{\kappa}}(\phi_0)|_- = u|_- \text{on } \partial D.$$

By (2.8), we have $S_D^{\tilde{\kappa}}(\phi_0)|_+ = 0$. Then, $(v, p) = (S_D^{\tilde{\kappa}}(\phi_0), V_D^{\tilde{\kappa}}(\phi_0))$ is solution of exterior Dirichlet problem (2.10) which admits a unique solution [59]. Thus, we have $(v,\, p) = (0,\, 0)$ which implies

$$(S_D^{\tilde{\kappa}}(\phi_0), \mathbf{V}_D^{\tilde{\kappa}}(\phi_0)) = (0, 0) \text{ on } \mathbf{R}^3 \setminus \overline{D}.$$

The jump formula on ∂D yields:

$$\frac{\partial}{\partial n}(S_D^{\tilde{\kappa}}(\phi_0), V_D^{\tilde{\kappa}}(\phi_0))|_- - \frac{\partial}{\partial n}(S_D^{\tilde{\kappa}}(\phi_0), V_D^{\tilde{\kappa}}(\phi_0))|_+ = \phi_0 = 0; \;\; x \in \partial D.$$

Consider now $(v,\, p)$ the solution to the following interior Dirichlet problem:

$$\begin{cases} (-\Delta + \kappa^2)v + \nabla p = 0 & \text{in} \quad D \\ \nabla \cdot v = 0 & \text{in} \quad D \\ v = 0 & \text{on} \quad \partial D \end{cases} \tag{2.11}$$

We have

$$S_D^{\kappa}(\psi_0)|_- + \mathcal{D}_\Omega^{\kappa}(\theta_0)|_- = S_D^{\kappa}(\psi_0)|_+ + \mathcal{D}_\Omega^{\kappa}(\theta_0)|_+ = u|_+ \text{ on } \partial D.$$

Then, we get $S_D^{\kappa}(\psi_0)|_- + \mathcal{D}_\Omega^{\kappa}(\theta_0)|_- = 0$ on ∂D. Finally we obtain

$$(v, p) = (S_D^{\kappa}(\psi_0) + D_\Omega^{\kappa}(\theta_0),\, V_D^{\kappa}(\psi_0) + W_\Omega^{\kappa}(\theta_0))$$

a solution of (2.11).

However, interior Dirichlet problem (2.11) admits a unique solution (v, p) where the unknown p is determined up to an additive constant [59]. Thus, we get

$$(S_D^{\kappa}(\psi_0) + \mathcal{D}_\Omega^{\kappa}(\theta_0),\, V_D^{\kappa}(\psi_0) + W_\Omega^{\kappa}(\theta_0)) = (0, c).$$

On the other hand, the trace formula on ∂D give:

$$\begin{aligned}\frac{\partial}{\partial n}(v,p)|_+ - \frac{\partial}{\partial n}(v,p)|_- &= \frac{\partial}{\partial n}(S_D^\kappa(\psi_0),\, V^\kappa(\psi_0))|_+ + \frac{\partial}{\partial n}(\mathcal{D}_\Omega^\kappa(\theta_0), W_\Omega^\kappa(\theta_0))|_+ \\ &\quad -\frac{\partial}{\partial n}(S_D^\kappa(\psi_0),\, V^\kappa(\psi_0))|_- - \frac{\partial}{\partial n}(\mathcal{D}_\Omega^\kappa(\theta_0), W_\Omega^\kappa(\theta_0))|_- \\ &= \frac{\partial}{\partial n}(S_D^\kappa(\psi_0),\, V^\kappa(\psi_0))|_+ - \frac{\partial}{\partial n}(S_D^\kappa(\psi_0),\, V^\kappa(\psi_0))|_- \\ &= \psi_0 = c\nu.\end{aligned}$$

We deduce that $\psi_0 = 0$ on ∂D since $\psi_0 - \phi_0 \in L_0^2(\partial D)$. Therefore, we have $(S_D^\kappa(\psi_0), V_D^\kappa(\psi_0)) = (0,0)$ on $\mathbf{R}^3 \setminus \partial D$ which implies that $(S_D^\kappa(\psi_0), V_D^\kappa(\psi_0)) = (0,0)$ on D_+. Then, we get $(u,q) = (\mathcal{D}_\Omega^\kappa(\theta_0),\, W_\Omega^\kappa(\theta_0)) = (0,0)$ on D_+.

Now, we consider the solution to the following exterior problem :

$$\begin{cases} (-\Delta + \tilde{\kappa}^2)v + \nabla p = 0 & \text{in} \quad \mathbf{R}^3 \setminus \overline{\Omega} \\ \nabla \cdot v = 0 & \text{in} \quad \mathbf{R}^3 \setminus \overline{\Omega} \\ \frac{\partial}{\partial n}(v,p) = 0 & \text{on} \quad \partial\Omega \\ |v||\nabla v| = o(|x|^{-2}) & \text{as} \quad |x| \to +\infty \\ |v||p| = o(|x|^{-2}) & \text{as} \quad |x| \to +\infty. \end{cases} \tag{2.12}$$

Then, $(\mathcal{D}_\Omega^\kappa(\theta_0),\, W_\Omega^\kappa(\theta_0))$ is solution of (2.12) which is an exterior Neumann problem which has a unique solution. We deduce that $(v,q) = (0,0)$. We know that $(\mathcal{D}_\Omega^\kappa(\theta_0))|_- - (\mathcal{D}_\Omega^\kappa(\theta_0))|_+ = \theta_0 = 0$ on $\partial\Omega$ which implies that $(\mathcal{D}_\Omega^\kappa(\theta_0),\, W_\Omega^\kappa(\theta_0)) = (0,0)$ in $\mathbf{R}^3 \setminus \partial\Omega$ and in particularly in D_+. Then, we get $(u,q) = (S_D^\kappa[\psi_0], V_D^\kappa[\psi_0])$ on D_+.

We showed that $(\phi_0, \psi_0, \theta_0) = (0,0,0)$, which is equivalent to the injectivity of A^κ. □

Conclusion

Based on the properties shown in the first chapter, we have shown the existence and uniqueness of a solution for the Stokes transmission problem and we gave an explicit expression in the form of single and double layer operators.

CHAPTER 3

Asymptotic expansion

Introduction

The aim of this chapter is to give the expression of the unique solution of the perturbed problem and to derive high-order terms in asymptotic expansion of the displacement field resulting from a small perturbation of the interface.

3.1 Asymptotic expansion of the density functions

In the next section, we consider the perturbed transmission Stokes problem.
Let (u_δ, q_δ) be the solution to the perturbed problem:

$$\begin{cases} (-\Delta + \kappa^2)u_\delta + \nabla q_\delta = 0 & \text{in} \quad \Omega \setminus \overline{D_\delta} \\ (-\Delta + \tilde{\kappa}^2)u_\delta + \nabla q_\delta = 0 & \text{in} \quad D_\delta \\ \nabla \cdot u_\delta = 0 & \text{in} \quad \Omega \\ u_\delta|_+ - u_\delta|_- = 0 & \text{on} \quad \partial D_\delta \\ \mu \frac{\partial}{\partial n}(u_\delta, q_\delta)|_+ - \tilde{\mu} \frac{\partial}{\partial n}(u_\delta, q_\delta)|_- = 0 & \text{on} \quad \partial D_\delta \\ u_\delta = g & \text{on} \quad \partial\Omega \\ \int\limits_\Omega q = 0. \end{cases} \tag{3.1}$$

with ∂D_δ a $\delta-$perturbation of ∂D defined by

$$\partial D_\delta = \{\tilde{x} = x + \delta\nu(x); x \in \partial D, 0 < \delta << 1\} \tag{3.2}$$

Using theorem 2.2.1, (u_δ, q_δ) can be represented by:

$$\begin{cases} u_\delta = S^{\tilde{\kappa}}_{D_\delta}[\phi_\delta] & , \quad q_\delta = V^{\tilde{\kappa}}_{D_\delta}[\phi_\delta] & \text{in } D_\delta, \\ u_\delta = S^{\kappa}_{D_\delta}[\psi_\delta] + \mathcal{D}^{\kappa}_{\Omega}[\theta_\delta] & , \quad q_\delta = V^{\kappa}_{D_\delta}[\psi_\delta] + W^{\kappa}_{\Omega}[\theta_\delta] & \text{in } \Omega \setminus \overline{D_\delta}, \end{cases}$$

where $(\phi_\delta, \psi_\delta, \theta_\delta) \in H(\partial D_\delta) \times L^2_0(\partial\Omega)$ is the unique solution of the following system

$$\begin{cases} S^{\tilde{\kappa}}_{D_\delta}[\phi_\delta] - S^{\kappa}_{D_\delta}[\psi_\delta] - \mathcal{D}^{\kappa}_{\Omega}[\theta_\delta] = 0 & \text{on} \quad \partial D_\delta \\ \tilde{\mu}(-\frac{1}{2}I + (\mathcal{K}^{\tilde{\kappa}}_{D_\delta})^*)[\phi_\delta] - \mu(\frac{1}{2}I + (\mathcal{K}^{\kappa}_{D_\delta})^*)[\psi_\delta] - \mu\frac{\partial}{\partial n}\mathcal{D}^{\kappa}_{\Omega}[\theta_\delta] = 0 & \text{on} \quad \partial D_\delta \\ S^{\kappa}_{D_\delta}[\psi_\delta] + (\frac{1}{2}I + \mathcal{K}^{\kappa}_{\Omega})[\theta_\delta] = g & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

Let $\Psi_\delta(x) = x + \delta\nu(x)$ be the diffeomorphism from ∂D to ∂D_δ. The following estimates hold.

Lemma 3.1.1. *There exists a constant C depending only on D such that for any functions $(\phi_\delta, \psi_\delta, \theta_\delta) \in H(\partial D_\delta) \times L^2_0(\partial\Omega)$, we have:*

i)
$$\|S^{\tilde{\kappa}}_{D_\delta}[\phi_\delta] \circ \Psi_\delta - S^{\tilde{\kappa}}_{D}[\phi_\delta \circ \Psi_\delta]\|_{L^2(\partial D)^3} \leq C\delta\|\phi_\delta\|_{L^2(\partial D_\delta)^3}.$$

ii)
$$\|S^{\kappa}_{D_\delta}[\psi_\delta] \circ \Psi_\delta - S^{\kappa}_{D}[\psi_\delta \circ \Psi_\delta]\|_{L^2(\partial D)^3} \leq C\delta\|\psi_\delta\|_{L^2(\partial D_\delta)^3}.$$

iii)
$$\|(\mathcal{K}^{\tilde{\kappa}}_{D_\delta})^*[\phi_\delta] \circ \Psi_\delta - (\mathcal{K}^{\tilde{\kappa}}_{D})^*[\phi_\delta \circ \Psi_\delta]\|_{L^2(\partial D)^3} \leq C\delta\|\phi_\delta\|_{L^2(\partial D_\delta)^3}.$$

iv)
$$\|(\mathcal{K}^{\kappa}_{D_\delta})^*[\phi_\delta] \circ \Psi_\delta - (\mathcal{K}^{\kappa}_{D})^*[\phi_\delta \circ \Psi_\delta]\|_{L^2(\partial D)^3} \leq C\delta\|\phi_\delta\|_{L^2(\partial D_\delta)^3}.$$

Proof. The two last inequalities have been proven in Lemma 3.1 in [**?**]. Following the same steps, we can easily obtain the first and the second ones. □

Let us consider the asymptotic behavior of $S^{\tilde{\kappa}}_{D_\delta}[\phi_\delta]$, $S^{\kappa}_{D_\delta}[\psi_\delta]$, $\mathcal{D}^{\kappa}_{\Omega}[\theta_\delta]$, $(\mathcal{K}^{\tilde{\kappa}}_{D_\delta})^*[\phi_\delta]$ and $(\mathcal{K}^{\kappa}_{D_\delta})^*[\psi_\delta]$ on ∂D_δ, and $S^{\kappa}_{D_\delta}[\psi_\delta]$ on $\partial\Omega$ as $\delta \to 0$.
Denote by $\widetilde{\phi} = \phi_\delta \circ \Psi_\delta$ the vectorial function with components $\tilde{\phi}_{j(j=1,2,3)}$.
Besides, for all $\widetilde{x} \in \partial D_\delta$ and $r = x - y \neq 0$, we define the integral operators:

$$S^{\tilde{\kappa}}_{D_{\delta,i}}[\phi_\delta](\widetilde{x}) = \int_{\partial D_\delta} \Gamma^{\tilde{\kappa}}_{ij}(\widetilde{x}, \widetilde{y})\phi_{\delta,j}(\widetilde{y})d\sigma_\delta(\widetilde{y}),$$

$$S^{\kappa}_{D_{\delta,i}}[\psi_\delta](\widetilde{x}) = \int\limits_{\partial D_\delta} \Gamma^{\kappa}_{ij}(\widetilde{x},\widetilde{y})\psi_{\delta,j}(\widetilde{y})d\sigma_\delta(\widetilde{y}),$$

$$(\mathcal{K}^{\tilde{\kappa}}_{D_{\delta,i}})^*[\phi_\delta](\widetilde{x}) = p.v. \int\limits_{\partial D_\delta} -\mathcal{S}^{\tilde{\kappa}}_{ijk}(\widetilde{x},\widetilde{y})\widetilde{\nu}_k(\widetilde{x})\phi_{\delta,j}(\widetilde{y})d\sigma_\delta(\widetilde{y}),$$

$$(\mathcal{K}^{\kappa}_{D_{\delta,i}})^*[\psi_\delta](\widetilde{x}) = p.v. \int\limits_{\partial D_\delta} -\mathcal{S}^{\kappa}_{ijk}(\widetilde{x},\widetilde{y})\widetilde{\nu}_k(\widetilde{x})\psi_{\delta,j}(\widetilde{y})d\sigma_\delta(\widetilde{y}),$$

$$D^{\kappa}_{\Omega,i}[\theta_\delta](\widetilde{x}) = \int\limits_{\partial\Omega} -\mathcal{S}^{\kappa}_{ijk}(\widetilde{x},y)\nu_k(y)\theta_{\delta,j}(y)d\sigma(y),$$

and for $x \in \partial\Omega$

$$S^{\kappa}_{D_{\delta,i}}[\psi_\delta](x) = \int\limits_{\partial D_\delta} \Gamma^{\kappa}_{ij}(x,\widetilde{y})\psi_{\delta,j}(\widetilde{y})d\sigma_\delta(\widetilde{y}).$$

By Taylor expansions of $\Gamma^{\tilde{\kappa}}_{ij}(\widetilde{x}-\widetilde{y})$, $\Gamma^{\kappa}_{ij}(x-\widetilde{y})$, $\mathcal{S}_{ijk}(\widetilde{x},\widetilde{y})\widetilde{\nu}_k(\widetilde{x})$ and $\mathcal{S}_{ijk}(\widetilde{x},y)\nu_k(y)$, we obtain:

$$\Gamma^{\tilde{\kappa}}_{ij}(\widetilde{x},\widetilde{y}) = \Gamma^{\tilde{\kappa}}_{ij}(x,y) + \sum_{n=1}^{\infty}\delta^n \underbrace{\sum_{|\alpha|=n}\frac{(\nu(x)-\nu(y))^\alpha}{\alpha!}\nabla^\alpha\Gamma^{\tilde{\kappa}}_{ij}(x,y)}_{:=\Gamma^{\tilde{\kappa},n}_{ij}(x,y)},$$

$$\Gamma^{\kappa}_{ij}(x,\tilde{y}) = \Gamma^{\kappa}_{ij}(x,y) + \sum_{n=1}^{\infty}\delta^n \underbrace{\sum_{|\alpha|=n}\frac{(-\nu(y))^\alpha}{\alpha!}\nabla^\alpha_y\Gamma^{\kappa}_{ij}(x,y)}_{:=\Gamma'^{\kappa,n}_{ij}(x,y)},$$

$$-\mathcal{S}^{\kappa}_{ijk}(\widetilde{x},y)\nu_k(y) = \underbrace{-\mathcal{S}^{\kappa}_{ijk}(x,y)\nu_k(y)}_{:=\mathcal{H}_{ij}(x,y)} - \sum_{n=1}^{\infty}\delta^n \underbrace{\sum_{|\alpha|=n}\frac{(-\nu(x))^\alpha}{\alpha!}\nabla^\alpha_x\mathcal{S}^{\kappa}_{ijk}(x,y)\nu_k(y)}_{:=\mathcal{H}'^{n}_{ij}(x,y)},$$

$$-\mathcal{S}^{\kappa}_{ijk}(\widetilde{x},\widetilde{y})\widetilde{\nu}_k(\widetilde{x}) = \underbrace{-\mathcal{S}^{\kappa}_{ijk}(x,y)\nu_k(x)}_{:=\mathcal{H}_{ij}(x,y)} - \sum_{n=1}^{\infty}\delta^n \underbrace{\sum_{|\alpha|=n}\frac{(\nu(x)-\nu(y))^\alpha}{\alpha!}\nabla^\alpha_x\mathcal{S}^{\kappa}_{ijk}(x,y)\nu_k(y)}_{:=\mathcal{H}^{n}_{ij}(x,y)}.$$

We then introduce the operators with (1.2):

$$S^{\tilde{\kappa},n}_{D,i}[\phi_\delta](\widetilde{x}) = \sum_{m+q=n} \int_{\partial D} \Gamma^{\kappa,m}_{ij}(x,y)\sigma_q(x)\tilde{\psi}_j(y)d\sigma(y), \quad n \geqslant 0,$$

$$S'^{\kappa,n}_{D,i}[\psi_\delta](\widetilde{x}) = \sum_{m+q=n} \int_{\partial D} \Gamma'^{\kappa,m}_{ij}(x,y)\sigma_q(x)\tilde{\psi}_j(y)d\sigma(y), \quad n \geqslant 0,$$

$$\mathcal{D}^{\kappa,n}_{\Omega,i}[\theta_\delta](\widetilde{x}) = \int_{\partial\Omega} \mathcal{H}'^{\kappa,n}_{ij}(x,y)\theta_{\delta,j}(y)d\sigma(y), \quad n \geqslant 0,$$

$$\mathcal{K}^{\kappa,n}_{D,i}[\psi_\delta](\widetilde{x}) = p.v. \sum_{m+q=n} \int_{\partial D} \mathcal{H}^{\kappa,m}_{ij}(x,y)\sigma_q(x)\tilde{\psi}_j(x)d\sigma(x), \quad n \geqslant 0.$$

where $\tilde{\phi}(x) = \phi_\delta \circ \Psi(x)$ and $\tilde{\psi}(x) = \psi_\delta \circ \Psi(x)$.

Note that $\mathcal{S}^{\kappa,0}_D = \mathcal{S}'^{\kappa,0}_D = \mathcal{S}^\kappa_D$, $\mathcal{D}^{\kappa,0}_D = \mathcal{D}^\kappa_D$, $\mathcal{K}^{\kappa,0}_D = (\mathcal{K}^\kappa_D)^*$ and that $\sigma_q = 0$ for all $q \geq 3$. We can obtain the following theorem whose demonstration is similar to that of lemma 3.1.1.

Theorem 3.1.1. *Let $N \in \mathbf{N}$. There exists C depending on N and D such as for any $(\phi_\delta, \psi_\delta, \theta_\delta) \in H(\partial D_\delta) \times L^2_0(\partial\Omega)$:*

$$\|(\mathcal{K}^\kappa_{D_\delta})^*[\phi_\delta] \circ \Psi_\delta - (\mathcal{K}^\kappa_D)^*[\tilde{\phi}] - \sum_{n=1}^{N} \delta^n \mathcal{K}_D{}^{\kappa,n}[\widetilde{\phi}]\|_{(L^2(\partial D))^3} \leq C\delta^{N+1}\|\widetilde{\phi}\|_{(L^2(\partial D))^3}$$

$$\|\mathcal{S}^{\tilde{\kappa}}_{D_\delta}[\phi_\delta] \circ \Psi_\delta - \mathcal{S}^{\tilde{\kappa}}_D[\tilde{\phi}] - \sum_{n=1}^{N} \delta^n \mathcal{S}^{\tilde{\kappa},n}_D[\widetilde{\phi}]\|_{(L^2(\partial D))^3} \leq C\delta^{N+1}\|\widetilde{\phi}\|_{(L^2(\partial D))^3}$$

$$\|\mathcal{S}'^{\tilde{\kappa}}_{D_\delta}[\phi_\delta] - \mathcal{S}'^{\tilde{\kappa}}_D[\tilde{\phi}] - \sum_{n=1}^{N} \delta^n \mathcal{S}'^{\tilde{\kappa},n}_D[\widetilde{\phi}]\|_{(L^2(\partial D))^3} \leq C\delta^{N+1}\|\widetilde{\phi}\|_{(L^2(\partial D))^3}$$

$$\|\mathcal{D}^\kappa_{D_\delta}[\theta_\delta] - \mathcal{D}^\kappa_D[\theta_\delta] - \sum_{n=1}^{N} \delta^n \mathcal{D}^{\kappa,n}_D[\theta_\delta]\|_{(L^2(\partial\Omega))^3} \leq C\delta^{N+1}\|\theta_\delta\|_{(L^2(\partial\Omega))^3}$$

where $\widetilde{\phi} := \phi_\delta \circ \Psi_\delta$, $\widetilde{\psi} := \psi_\delta \circ \Psi_\delta$.

Let $(\phi^{(n)}, \psi^{(n)}, \theta^{(n)}), n \geq 1$ be the solution of the following system:

$$\begin{cases} \mathsf{S}_D^{\tilde{\kappa}}[\phi^{(n)}] - \mathsf{S}_D^{\kappa}[\psi^{(n)}] - \mathcal{D}_\Omega^{\kappa}[\theta^{(n)}] = \\ \sum_{m=0}^{n-1} -\mathsf{S}_D^{\tilde{\kappa},n-m}[\phi^{(m)}] + \mathsf{S}_D^{\kappa,n-m}[\psi^{(m)}] + \mathcal{D}_\Omega^{\kappa,n-m}[\theta^{(m)}] & \text{on} \quad \partial D \\ \\ \tilde{\mu}(-\frac{1}{2}I + (\mathcal{K}_D^{\tilde{\kappa}})^*)[\phi^{(n)}] - \mu(\frac{1}{2}I + (\mathcal{K}_D^{\kappa})^*)[\psi^{(n)}] - \mu\frac{\partial}{\partial n}D_\Omega^{\kappa}[\theta^{(n)}] = \\ \sum_{m=0}^{n-1} -\tilde{\mu}((\mathcal{K}_D^{\tilde{\kappa},n-m})^*)[\phi^{(m)}] + \mu((\mathcal{K}_D^{\kappa,n-m})^*)[\psi^{(m)}] + \mu\frac{\partial}{\partial n}\mathcal{D}_\Omega^{\kappa,n-m}[\theta^{(m)}] & \text{on} \quad \partial D \\ \\ S_D^{\kappa}[\psi^n] + (\frac{1}{2}I + \mathcal{K}_\Omega^{\kappa})[\theta^{(n)}] = \\ \delta_{0n}g + \sum_{m=0}^{n-1} -S_D^{\prime\kappa,n-m}[\psi^{(m)}] - \mathcal{K}_\Omega^{\kappa,n-m}[\theta^{(m)}] & \text{on} \quad \partial\Omega. \end{cases} \tag{3.4}$$

We recursively construct the sentence $(\phi^{(n)}, \psi^{(n)}, \theta^{(n)})\ \forall n \in \mathbb{N}^*$.

Let define

$$\phi^N = \sum_{n=0}^{N} \delta^n \phi^{(n)}, \psi^N = \sum_{n=0}^{N} \delta^n \psi^{(n)}, \theta^N = \sum_{n=0}^{N} \delta^n \theta^{(n)}, \tag{3.5}$$

where $(\phi^{(0)}, \psi^{(0)}, \theta^{(0)}) = (\phi, \psi, \theta)$ the unique solution of (2.4). Taking into account Theorem 3.1.1, one deduces from system (3.4) that:

$$\begin{cases} \sum_{n=0}^{N} \delta^n \mathsf{S}_D^{\tilde{\kappa},n}[\tilde{\phi} - \phi^N] - \sum_{n=0}^{N} \delta^n \mathsf{S}_D^{\kappa,n}[\tilde{\psi} - \psi^N] - \sum_{n=0}^{N} \delta^n \mathcal{D}_\Omega^{\kappa,n}[\theta_\delta - \theta^N] = O(\delta^{N+1}) & \text{on} \quad \partial D \\ \\ \sum_{n=0}^{N} \tilde{\mu}(-\frac{\delta_{0n}}{2}I + \delta^n(\mathcal{K}_D^{\tilde{\kappa},n})^*)[\tilde{\phi} - \phi^N] - \sum_{n=0}^{N} \mu(\frac{\delta_{0n}}{2}I + \delta^n(\mathcal{K}_D^{\kappa,n})^*)[\tilde{\psi} - \psi^N] \\ -\sum_{n=0}^{N} \mu\frac{\partial}{\partial n}\mathcal{D}_\Omega^{\kappa,n}[\theta_\delta - \theta^N] = O(\delta^{N+1}) & \text{on} \quad \partial D \\ \\ \sum_{n=0}^{N} \delta^n S_D^{\prime\kappa,n}[\tilde{\psi} - \psi^N] + \mathcal{D}_\Omega^{\kappa}[\theta_\delta - \theta^N] = O(\delta^{N+1}) & \text{on} \quad \partial\Omega. \end{cases}$$

Then we get the following lemma.

Lemma 3.1.2. *Let $N \in \mathbf{N}$, there exists C depending only on N and Ω such that*

$$\|\tilde{\phi} - \sum_{n=0}^{N} \delta^n \phi^{(n)}\|_{(L^2(\partial D))^3} \leq C\delta^{N+1},$$

$$\|\tilde{\psi} - \sum_{n=0}^{N} \delta^n \psi^{(n)}\|_{(L^2(\partial D))^3} \leq C\delta^{N+1},$$

$$\|\theta_\delta - \sum_{n=0}^{N} \delta^n \theta^{(n)}\|_{(L^2(\partial\Omega))^3} \leq C\delta^{N+1}$$

where$(\phi^{(n)}, \psi^{(n)}, \theta^{(n)})$ *are defined by the recursive relations (3.4).*

3.2 Asymptotic expansion for the solution of the perturbed problem

In this section, we develop the asymptotic behavior of $u_\delta - u$ as $\delta \to 0$. According to the boundary integral representation of the solution, given in section 3.1, we have:

$$u_\delta - u = \begin{cases} S^{\tilde{\kappa}}_{D_\delta}[\phi_\delta] - S^{\tilde{\kappa}}_{D}[\phi] & \text{in } D_0, \\ S^{\kappa}_{D_\delta}[\psi_\delta] - S^{\kappa}_{D}[\psi] + \mathcal{D}^{\kappa}_{\Omega}[\theta_\delta] - \mathcal{D}^{\kappa}_{\Omega}[\theta] & \text{in } \Omega_0, \end{cases} \tag{3.6}$$

where D_0 is any close subset of $D \cap D_\delta$ and Ω_0 is any subset of $\Omega \setminus (D \cup D_\delta)$.

We obtain with Taylor expansion of $\Gamma^{\tilde{\kappa}}_{ij}(x, \tilde{y})$ and (1.2):

$$S^{\tilde{\kappa}}_{D_\delta}[\phi_\delta](x) = \int_{\partial D_\delta} \Gamma^{\tilde{\kappa}}_{ij}(x, \tilde{y}) \tilde{\phi}_j(y) d\sigma_\delta(\tilde{y}) \tag{3.7}$$

$$= \int_{\partial D} \left(\Gamma^{\tilde{\kappa}}_{ij}(x, y) + \sum_{n=1}^{N} \delta^n \Gamma'^{\tilde{\kappa},n}_{ij}(x, y) \right) \left(\phi_j(y) + \sum_{n=1}^{N} \delta^n \phi^{(n)}_j(y) \right) \left(1 + \sum_{n=1}^{N} \delta^n \sigma^n(x) \right) d\sigma(y) + O(\delta^{N+1})$$

$$= \int_{\partial D} \Gamma^{\tilde{\kappa}}_{ij}(x, y)\phi(y) d\sigma(y) + \sum_{n=1}^{N} \delta^n \sum_{m+q+k=n} \int_{\partial D} \Gamma'^{\tilde{\kappa},m}_{ij}(x, y)\sigma_q(y)\phi^{(k)}(y) d\sigma(y) + O(\delta^{N+1})$$

$$= S^{\tilde{\kappa}}_{D}[\phi](x) + \sum_{n=1}^{N} \delta^n \sum_{m+q+k=n} \int_{\partial D} \Gamma'^{\tilde{\kappa},m}_{ij}(x, y)\sigma_q(y)\phi^{(k)}(y) d\sigma(y) + O(\delta^{N+1}) \ \forall x \in \partial D.$$

By the same way, $S^{\kappa}_{D_\delta}[\psi_\delta](x)$ for $x \in \partial D$ can be rewritten as follows:

$$S^{\kappa}_{D_\delta}[\psi_\delta](x) = \int_{\partial D_\delta} \Gamma^{\kappa}_{ij}(x, \tilde{y}) \tilde{\psi}_j(y) d\sigma_\delta(\tilde{y}) \tag{3.8}$$

$$= S^{\kappa}_{D}[\psi](x) + \sum_{n=1}^{N} \delta^n \sum_{m+q+k=n} \int_{\partial D} \Gamma'^{\kappa,m}_{ij}(x, y)\sigma_q(y)\psi^{(k)}(y) d\sigma(y).$$

Similarly, we have

$$\mathcal{D}^{\kappa}_{\Omega}[\theta_\delta](x) = \int_{\partial\Omega} -\mathcal{S}^{\kappa}_{ijk}(x,y)\nu_k(y)\theta_\delta(y)d\sigma(y) \tag{3.9}$$

$$= \int_{\partial\Omega} -\mathcal{S}^{\kappa}_{ijk}(x,y)\nu_k(y)\left(\theta_j(y) + \sum_{n=1}^{N}\delta^n\theta_j^{(n)}(y)\right)d\sigma(y) + O(\delta^{N+1}), x \in \Omega_0.$$

Then, we get the following result.

Theorem 3.2.1. *Define for $n \in \mathbf{N}$ the vector-valued functions $u^n = (u^n_i)_{i=1,2,3}$ as follows:*

$$u^n_i(x) = \begin{cases} \displaystyle\sum_{m+q+k=n}\int_{\partial D} \Gamma'^{\tilde{\kappa},m}_{ij}(x,y)\sigma_q(y)\phi^{(k)}_j(y)d\sigma(y) & \forall x \in D_0; \\ \displaystyle\sum_{m+q+k=n}\int_{\partial D} \Gamma'^{\kappa,m}_{ij}(x,y)\sigma_q(y)\psi^{(k)}_j(y)d\sigma(y) & \\ \displaystyle+\int_{\partial\Omega} -S^{\kappa}_{ijl}(x,y)\nu_l(y)\theta^{(n)}_j(y)d\sigma(y) & \forall x \in \Omega_0. \end{cases}$$

Then the following formula holds uniformly for $x \in \Omega_0 \cup D_0$:

$$u_\delta(x) - u(x) = \sum_{n=1}^{N}\delta^n u^n(x) + O(\delta^{N+1}). \tag{3.10}$$

The remainder $O(\delta^{N+1})$ depends on N, Ω and Ω_0.

3.3 Computation of the first order approximation

Let us compute the first order approximation of u_δ explicitly. If we put $(\phi^{(0)}, \psi^{(0)}, \theta^{(0)}) = (\phi, \psi, \theta)$ defined by (2.2)-(2.3), $(\phi^{(1)}, \psi^{(1)}, \theta^{(1)})$ is then defined by the following system:

$$\begin{cases} \mathsf{S}^{\tilde{\kappa}}_D[\phi^{(1)}] - \mathsf{S}^{\kappa}_D[\psi^{(1)}] - \mathcal{D}^{\kappa}_{\Omega}[\theta^{(1)}] = \\ -\mathsf{S}'^{\tilde{\kappa},1}_D[\phi^{(0)}] + \mathsf{S}^{\kappa,1}_D[\psi^{(0)}] + \mathcal{D}^{\kappa,1}_{\Omega}[\theta^{(0)}] & \text{on } \partial D \\ \\ \tilde{\mu}(-\frac{1}{2}I + (\mathcal{K}^{\tilde{\kappa}}_D)^*)[\phi^{(1)}] - \mu(\frac{1}{2}I + (\mathcal{K}^{\kappa}_D)^*)[\psi^{(1)}] - \mu\frac{\partial}{\partial n}\mathcal{D}^{\kappa}_{\Omega}[\theta^{(1)}] = \\ -\tilde{\mu}((\mathcal{K}^{\tilde{\kappa},1}_D)^*)[\phi^{(0)}] + \mu((\mathcal{K}^{\kappa,1}_D)^*)[\psi^{(0)}] + \mu\frac{\partial}{\partial n}\mathcal{D}^{\kappa,1}_{\Omega}[\theta^{(0)}] & \text{on } \partial D \\ \\ S^{\kappa}_D[\psi^{(1)}] + (\frac{1}{2}I + \mathcal{K}^{\kappa}_{\Omega})[\theta^{(1)}] = \\ -S'^{\kappa,1}_D[\psi^{(0)}] - \mathcal{K}^{\kappa,1}_{\Omega}[\theta^{(0)}] & \text{on } \partial\Omega. \end{cases}$$

Consider the vector-valued function $u^1 = (u_i^1)_{i=1,2}$ with:

$$u_i^1(x) = \begin{cases} \displaystyle\sum_{m+q+k=1} \int_{\partial D} \Gamma'^{\tilde{\kappa},m}_{ij}(x,y)\sigma^q(y)\phi_j^{(k)}(y)d\sigma(y) & \text{in } D_0; \\ \displaystyle\sum_{m+q+k=1} \int_{\partial D} \Gamma'^{\kappa,m}_{ij}(x,y)\sigma^q(y)\psi_j^{(k)}(y)d\sigma(y) & \\ \displaystyle + \int_{\partial\Omega} -S^{\kappa}_{ijl}(x,y)\nu_l(y)\theta_j^{(1)}(y)d\sigma(y) & \text{in } \Omega_0. \end{cases}$$

Therefore, u_i^1 takes the form:

$$u_i^1(x) = \int_{\partial D} \Gamma'^{\tilde{\kappa}}_{ij}(x,y)\tau(y)\phi_j^{(0)}(y)d\sigma(y) + \int_{\partial D} \Gamma'^{\tilde{\kappa}}_{ij}(x,y)\phi_j^{(1)}(y)d\sigma(y)$$

$$+ \int_{\partial D} \Gamma'^{\tilde{\kappa},1}_{ij}(x,y)\phi_j^{0}(y)d\sigma(y); \quad x \in D_0$$

and more explicitly:

$$u_i^1(x) = \int_{\partial D} \Gamma'^{\kappa}_{ij}(x,y)\tau(y)\phi_j^{(0)}(y)d\sigma(y) + \int_{\partial D} \Gamma'^{\kappa}_{ij}(x,y)\phi_j^{(1)}(y)d\sigma(y)$$

$$+ \int_{\partial D} \Gamma'^{\kappa,1}_{ij}(x,y)\phi_j^{(0)}(y)d\sigma(y) + \int_{\partial\Omega} -S^{\kappa}_{ijk}(x,y)\nu_k(y)\theta_j^{(1)}(y)d\sigma(y); \quad x \in \Omega_0.$$

Thus, we get the following formula for $x \in \Omega_0 \cup D_0$

$$u_\delta(x) - u(x) = \delta u^1(x) + O(\delta^2).$$

Conclusion

We present a result about the construction and justification of the asymptotic expansion of the solution of the transmission Stokes system with a boundary perturbation of an inclusion. We derive high-order terms in the asymptotic expansion of the solution with layer potential techniques. A reconstruction procedure to determine the localization of the inhomogeneity and the shape of the inclusion based on boundary measurements may be based on our results here.

The method, which is explicitly carried out here for the interior Dirichlet problem of the

Stokes operator in three dimensions, can also be used for many other boundary value problems, whenever a suitable potential theory is available, for example in the case of harmonic, elastic, or hydrodynamic boundary value problem.

Part II

Asymptotic expansion for Eigenvalues and Eigenfunctions in the Presence of Small Boundary Perturbations

CHAPTER 4

Asymptotic behaviors for eigenvalues and eigenfunctions associated to the Stokes operator with a small boundary perturbation

Introduction

The main objective of the second part on this report, is to present a schematic way to derive high-order asymptotic expansions for both eigenvalues and eigenfunctions for the Stokes operator caused by small perturbations of the boundary. Also, we rigorously derive an asymptotic formula which is in some sense dual to the leading-order term in the asymptotic expansion of the perturbations in the Stokes eigenvalues due to interface changes of the inclusion. The following is an outline of this chapter. In Section 4.1, we describe the main problem in this part. In Section 4.2, we develop a boundary integral formulation for solving the eigenvalue problem (4.4), and we present some preliminary results. Finally, Section 4.3 is dedicated to develop asymptotic expansion of the perturbations in the eigenvalues, in eigenfunctions and in eigenpressures. More precisely, in the Section 4.3.1, we present some basic results related to shape perturbation and we develop high-order terms in the expansion of the integral operator. In Section 4.3.2, we (formally) provide by layer potentials an asymptotic expansion for the perturbed eigenvalues if the unperturbed eigenvalue is simple. But in section 4.3.3, we rigourously derive the leading-order term for the perturbed Stokes eigenvalues in Hölder space and for both simple and multiple eigenvalues. Here the problems to be study are more general than the one introduced in Section 4.1. In Section 4.3.4, we end our

chapter by developing asymptotic expansion of the perturbations in the eigenfunctions and in eigenpressures using the same problem configurations.

4.1 Model problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded open domain with boundary of class C^2. We consider the following eigenvalue problem for the Stokes system with homogeneous boundary conditions:

$$\begin{cases} -\mu\Delta v + \nabla\tilde{p} = \tilde{\lambda} v & \text{in } \Omega \\ \nabla . v = 0 & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Here $v = (v_1, v_2, v_3)$ denotes the velocity field, μ the viscosity while the scalar function p is the pressure.

Now, if we suppose that μ is a positive constant, then the first equation in the above system is equivalent to:

$$-\Delta v + \nabla(\tilde{p}/\mu) = (\tilde{\lambda}/\mu)v \quad \text{in } \Omega.$$

Then, if μ is constant we have the following eigenvalue problem

$$\begin{cases} -\Delta v + \nabla p = \lambda v & \text{in}\Omega \\ \nabla . v = 0 & \text{in}\Omega \\ v = 0 & \text{in}\partial\Omega \end{cases} \tag{4.1}$$

with $p := \tilde{p}/\mu$ and $\lambda := \tilde{\lambda}/\mu$.

It is well known that this eigenvalue problem admits a sequence of a no decreasing positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ tending to infinity as $n \to +\infty$.

The eigenfunctions $\{v_n\}_{n\geq 1} \subset (H_0^1(\Omega))^3$ and the eigenpressures $\{p_n\}_{n\geq 1} \subset L^2(\Omega)$ may be taken so that $\{v_n\}_{n\geq 1}$ constitutes an orthonormal basis of

$$H(\Omega) := \{v \in (H_0^1(\Omega))^3 : \quad \nabla \cdot v = 0 \text{ in } \Omega\}.$$

The pressure p is determined up to an additive constant.

We assume that the boundary $\partial\Omega$ is subjected to a small, smooth deformation and that the boundary of the deformed domain Ω_δ is given by:

$$\partial\Omega_\delta := \{\tilde{x} = x + \delta h(x)\nu(x), \quad x \in \partial\Omega\} \tag{4.2}$$

where $\nu(x)$ is the outward normal vector on $\partial\Omega$ and $h(x)$ is a real function in $C^2(\partial\Omega)$ that satisfies

$$\|h(x)\|_{C^2(\partial\Omega)} < 1. \tag{4.3}$$

Obviously, the domain Ω_δ is of class C^2 and the Dirichlet eigenvalue problem for the Stokes system can be defined in Ω_δ as well.

In this chapter, we derive the asymptotic expansion of eigenvalues, eigenfunctions and the eigenpressures solutions to the Stokes system:

$$\begin{cases} -\Delta v_\delta + \nabla p_\delta = \lambda_\delta v_\delta & \text{in } \Omega_\delta \\ \nabla . v_\delta = 0 & \text{in } \Omega_\delta \\ v_\delta = 0 & \text{in } \partial\Omega_\delta. \end{cases} \tag{4.4}$$

Here we suppose that the eigenvalue λ_0 is simple. Then the eigenvalue λ_δ is simple and is near to λ_0 associated to the normalized eigenfunction v_δ.

4.2 Integral equations method

We now develop a boundary integral formulation for solving the perturbed eigenvalue problem (4.4). The components of the fundamental Stokes tensor $\Gamma = (\Gamma_{ij})_{i,j=1}^3$ and those of the associated pressure vector $P = (P_{ij})_{i,j=1}^3$, which determine the fundamental solution (Γ, P) of the Stokes system in $\mathbb{R}^3$, are given by (see for instance [6], [37])

$$\begin{cases} \Gamma_{ij}(\lambda, x) = -\frac{1}{4\pi}\frac{\delta_{ij} e^{\mathbf{i}\sqrt{\lambda}|x|}}{|x|} - \frac{1}{4\pi\lambda}\partial_{x_i}\partial_{x_j}(\frac{e^{\mathbf{i}\sqrt{\lambda}|x|}-1}{|x|}) \\ P_i(x) = -\frac{1}{4\pi}\frac{x_i}{|x|^3}, \end{cases} \tag{4.5}$$

where ∂x_i denotes $\partial/\partial x_i$ for $i = 1, 2, 3$ and $\mathbf{i}^2 = -1$. We recall that the i^{th} row Γ_i of Γ satisfies

$$\begin{cases} -\Delta \Gamma_i + \nabla P_i(x) - \lambda \Gamma_i = e^i \delta(x) & \text{in } \mathbb{R}^3 \\ \nabla . \Gamma_i = 0 & \text{in } \mathbb{R}^3; \end{cases} \tag{4.6}$$

in the sense of distributions and where $(e^i; i = 1, 2, 3)$ is the orthonormal basis of $\mathbb{R}^3$. Note that we used the Einstein convention for the summation notation omitting the summation sign for the indices appearing twice.

4.2.1 The potential theory for the Stokes system

Let us denote by $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ a complex vector-valued function with class $C^0(\partial\Omega)$.

The hydrodynamic single-layer potential with density $\varphi \in C^0(\partial\Omega)^3$ is the vector function $\mathcal{S}(\lambda)\varphi(x)$ defined by

$$\mathcal{S}(\lambda)\varphi(x) := \int_{\partial\Omega} \Gamma(\lambda, |x-y|)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^3 \backslash \partial\Omega. \tag{4.7}$$

The pressure term $\mathcal{Q}$ corresponding to the single layer potential is the function given by

$$\mathcal{Q}\varphi(x) := \int_{\partial\Omega} P(x,y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^3 \backslash \partial\Omega.$$

For a careful study of these potentials, one can refer to [34], [37], [60].

Taking into account the well known properties of Green function Γ, one obtains the result that the pair $(\mathcal{S}\varphi, \mathcal{Q}\varphi)$ are smooth functions in each of the domains Ω and $\mathbb{R}^3 \backslash \overline{\Omega}$ respectively. Also these functions are classical solutions to the Stokes system (4.1).

The continuity and jump relations of the Stokes surface potentials on the boundary $\partial\Omega$ are described in the following proposition (see [6], [17] pp. 41-42 or [60] p. 66):

Proposition 4.2.1. *Let $\varphi \in C^0(\partial\Omega)^3$ and let $\mathcal{S}$ denotes the surface potential defined in (4.7). Then on the boundary $\partial\Omega$ the following continuity and jump relations are satisfied:*

$$\begin{gathered} (\mathcal{S}(\lambda)\varphi)\big|_+ = (\mathcal{S}(\lambda)\varphi)|_- = \mathcal{S}(\lambda)\varphi \\ \tfrac{\partial \mathcal{S}(\lambda)(\varphi)}{\partial \nu}(x)\big|_\pm = \pm\tfrac{\varphi(x)}{2} + \textstyle\int_{\partial\Omega} \tfrac{\partial \Gamma(\lambda, |x-y|)}{\partial \nu(x)} \varphi(y)\, d\sigma(y). \end{gathered}$$

4.2.2 Boundary integral formulation

In this section, we give a boundary integral formulation in order to solve the eigenvalue problems (4.1) and (4.4).

Proposition 4.2.2. *Suppose h satisfies (4.3). Then, there exists $\delta_0 > 0$ such that the map $\Psi_\delta(x)$ defined by*

$$\Psi_\delta(x) = x + \delta h(x)\nu(x) \tag{4.8}$$

is a C^2- diffeomorphism from $\partial\Omega$ to $\partial\Omega_\delta$ for $\delta < \delta_0$. In addition, the following equality holds

$$\begin{aligned}\det\left(\nabla\Psi_\delta\right) &= 1 + \mathrm{tr}\nabla(\mathrm{h}\nu)\delta + \frac{1}{2}\big[\big(\mathrm{tr}\nabla(\mathrm{h}\nu)\big)^2 - \mathrm{tr}\big(\nabla(\mathrm{h}\nu)^2\big)\big]\delta^2 \\ &\quad + \det\left(\nabla(h\nu)\right)\delta^3,\end{aligned} \tag{4.9}$$

where tr *means the trace of a matrix. Moreover, we have* $\mathrm{tr}\nabla(\mathrm{h}\nu) = \mathrm{div}(\mathrm{h}\nu)$.

proof Recall that the function $h(x)$ is C^2 on $\partial\Omega$, then the map $\Psi_\delta(x)$ is also C^2. A simple calculation yields the equality (4.9). Consequently, for δ small enough the map $\Psi_\delta(x)$ is a C^2- diffeomorphism from $\partial\Omega$ to $\partial\Omega_\delta$.
We further denote Ψ_δ^{-1} the reciproque function of $\Psi_\delta(x)$. Thanks to Ψ_δ^{-1}, we can define the operator $\mathcal{A}_\delta(\lambda)$ as follows:

$$\mathcal{A}_\delta(\lambda)\varphi(x) = \big(\mathcal{S}_{\Omega_\delta}(\lambda)\varphi(\Psi_\delta^{-1})\big)(\Psi_\delta(x)), \quad \varphi \in (L^2(\partial\Omega))^3 \tag{4.10}$$

where $\mathcal{S}_{\Omega_\delta}(\lambda)$ is the hydrodynamic single-layer potential given by (4.7) when we have replaced the boundary $\partial\Omega$ by $\partial\Omega_\delta$.

For $i, j \in \{1, 2, 3\}$, we can define the j^{th}-component of the vector-valued function $\mathcal{A}_\delta(\lambda)$ as follows:

$$(\mathcal{A}^\delta(\lambda)\varphi)_j(x) = \Big(\big(\mathcal{S}_{\Omega_\delta}(\lambda)\varphi\big)_j(\Psi_\delta^{-1})\Big)(\Psi_\delta(x)), \quad \varphi \in (L^2(\partial\Omega))^3. \tag{4.11}$$

The j^{th}-component of the single-layer potential $\mathcal{S}_{\Omega_\delta}(\lambda)$ is given by

$$\big(\mathcal{S}_{\Omega_\delta}(\lambda)\varphi\big)_j(\tilde{x}) := \int_{\partial\Omega_\delta} \Gamma_{ji}(\lambda, |\tilde{x} - \tilde{y}|)\varphi_i(\tilde{y})\, d\sigma_\delta(\tilde{y}), \quad \tilde{x} \in \mathbb{R}^3\backslash\partial\Omega_\delta, j = 1, 2, 3, \tag{4.12}$$

where φ_i is the i^{th}-component of the vector-valued function φ. Using Proposition 4.2.2, relations (4.11)-(4.12) and the continuity relations given by Proposition 4.2.1, we obtain for $x \in \partial\Omega$ that

$$\big(\mathcal{A}^\delta(\lambda)\varphi\big)_j(x) = \int_{\partial\Omega} \Gamma_{ji}(\lambda, |\Psi_\delta(x) - \Psi_\delta(y)|)\det\left(\nabla\Psi_\delta(y)\right)\varphi_i(y)\, d\sigma(y), \quad j = 1, 2, 3. \tag{4.13}$$

Let $\mathcal{A}_0$ the operator defined as in (4.10) by

$$\mathcal{A}_0\phi = \mathcal{S}(\lambda)\phi,$$

where $\phi \in (L^2(\partial\Omega))^3$. Then, we have the following result, which is a slight variation of the Lemma 6.1 due to Ammari et al. [8, 10] for the scalar eigenvalue problem.

Proposition 4.2.3. *The operator-valued function $\mathcal{A}_0(\lambda) : H^{-1/2}(\partial\Omega)^3 \to H^{1/2}(\partial\Omega)^3$ is Fredholm of index zero in $\mathbb{C}\backslash i\mathbb{R}^-$. In addition the Dirichlet eigenvalues of the Stokes system (4.1) are exactly its real zeros.*

From Proposition 4.2.3 we know that if λ_0 is an eigenvalue of (4.1) then λ_0 is a real zero of $\mathcal{A}_0(\lambda)$. Moreover, for ϵ_0 small enough, the function $\mathcal{A}_0^{-1}(\lambda)$ is meromorphic in $\overline{D_{\epsilon_0}}(\lambda_0)$, where $D_{\epsilon_0}(\lambda_0)$ means the disc of center λ_0 and radius ϵ_0, and λ_0 is its unique pole in $\overline{D_{\epsilon_0}}$. Furthermore we have the following Laurent expansion:

$$\mathcal{A}_0^{-1}(\lambda) = (\lambda - \lambda_0)^{-1}\ell_0 + R_0(\lambda), \tag{4.14}$$

where $\ell_0 : Ker\mathcal{A}_0(\lambda_0) \to Ker\mathcal{A}_0(\lambda_0)$, and $R_0(\lambda)$ is a holomorphic function.

Our main results in this section are summarized in the following theorem.

Theorem 4.2.1. *Suppose that the eigenvalue λ_0 of (4.1) is with multiplicity 1. Then, there exist a positive constant $\delta_0(\epsilon_0)$ such that for $|\delta| < \delta_0$, the operator-valued function $\lambda \mapsto \mathcal{A}_\delta(\lambda)$ has a real zero $\lambda(\delta)$ in $\overline{D_{\epsilon_0}}(\lambda_0)$. This zero is exactly the eigenvalue of the perturbed eigenvalue problem (4.4), and is an analytic function with respect to δ in $]-\delta_0, \delta_0[$. It satisfies $\lambda(0) = \lambda_0$. Moreover, the following assertions hold:*

$$\begin{aligned} &\mathcal{A}_\delta^{-1}(\lambda) = (\lambda - \lambda(\delta))^{-1}\ell(\delta) + R_\delta(\lambda), \\ &\ell(\delta) : Ker(\mathcal{A}_\delta(\lambda(\delta)) \to Ker(\mathcal{A}_\delta(\lambda(\delta)), \end{aligned} \tag{4.15}$$

where $R_\delta(\lambda)$ is a holomorphic function with respect to $(\delta, \lambda) \in]-\delta_0, \delta_0[\times D_{\epsilon_0}(\lambda_0)$.

4.3 Asymptotic formula

4.3.1 High-order terms in the expansion of $\mathcal{A}_\delta$

We now present some basic results related to shape perturbation. The methods to be used here differ from those in [5, 8], but the expressions deduced are the same. To begin, let $(\tau_1(x), \tau_2(x))$ be the orthornormal basis of the tangent plan to the surface $\partial\Omega$ at a regular point x. Their cross product is then orthogonal to $\partial\Omega$ at the point x. By changing their order, we can assume that $\tau_1 \times \tau_2$ is a vector pointing towards the exterior of the surface $\partial\Omega$. Then dividing it by its length yields the unit normal vector $\nu(x)$, that is:

$$\nu_0(x) = \frac{\tau_1(x) \times \tau_2(x)}{|\tau_1(x) \times \tau_2(x)|}, \tag{4.16}$$

for $x \in \partial\Omega$. Evidently $\nu_0 = \nu$, where ν was introduced in section 4.1.

Set

$$\tau_1^\delta = \text{grad}\Psi_\delta \cdot \tau_1, \quad \text{and} \tau_2^\delta = \text{grad}\Psi_\delta \cdot \tau_2.$$

Using Proposition 4.2.2, we find that:

$$\tau_1^\delta = \tau_1 + \delta M\tau_1, \quad \text{and } \tau_2^\delta = \tau_2 + \delta M\tau_2, \tag{4.17}$$

where the (3×3)- matrix M is given by:

$$M = \begin{pmatrix} \partial_1(h\nu_1) & \partial_2(h\nu_1) & \partial_3(h\nu_1) \\ \partial_1(h\nu_2) & \partial_2(h\nu_2) & \partial_3(h\nu_2) \\ \partial_1(h\nu_3) & \partial_2(h\nu_3) & \partial_3(h\nu_3) \end{pmatrix}$$

with ν_i means the i-th $(i = 1, 2, 3)$ component of the vector ν.

For δ sufficiently small, one can see that the outward unit normal vector to $\partial\Omega_\delta$ is given by

$$\nu_\delta(x) = \frac{\tau_1^\delta(x) \times \tau_2^\delta(x)}{|\tau_1^\delta(x) \times \tau_2^\delta(x)|}, \tag{4.18}$$

for $x \in \partial\Omega$. Then, the following asymptotic expansion holds.

Proposition 4.3.1. *Let ν_0 be given by (4.16). Then, the outward unit normal $\nu_\delta(x)$ to $\partial\Omega_\delta$ at x, can be expanded uniformly as*

$$\nu_\delta(x) = \nu^{(0)}(x) + \sum_{n=1}^{\infty} \delta^n \nu^{(n)}(x), \quad x \in \partial\Omega,$$

where the vector-valued functions $\nu^{(n)}$ are uniformly bounded. In particular, for $x \in \partial\Omega$:

$$\nu^{(0)}(x) = \nu(x),$$

$$\nu^{(1)} = \frac{1}{|\tau_1 \times \tau_2|}\big[\tau_1 \times M\tau_2 + M\tau_1 \times \tau_2 - \big(\nu_0 \cdot (\tau_1 \times M\tau_2 + M\tau_1 \times \tau_2)\big)\nu_0\big].$$

Proof. Considering the expansions (4.17) for δ sufficiently small, the relation (4.18) becomes:

$$\nu_\delta = \frac{\mathbf{a} + \delta\mathbf{b} + \delta^2\mathbf{c}}{|\mathbf{a} + \delta\mathbf{b} + \delta^2\mathbf{c}|}, \tag{4.19}$$

where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vector-valued functions given by:

$$\mathbf{a} = \tau_1 \times \tau_2, \quad \mathbf{b} = M\tau_1 \times \tau_2 + \tau_1 \times M\tau_2, \text{ and } \mathbf{c} = M\tau_1 \times M\tau_2.$$

So that, by expanding the quotient (4.19) as δ tends to zero, we get the desired results.

Next, one can use Proposition 4.2.2 to get the uniformly convergent expansion for the surface element as follows:

Proposition 4.3.2. *Let $\tilde{y} = \Psi_\delta(y)$ where $\Psi_\delta(y)$ is given by (4.8) for $y \in \partial\Omega$. Then, the following expansion for the surface element $d\sigma_\delta(\tilde{y})$ holds uniformly for $y \in \partial\Omega$:*

$$d\sigma_\delta(\tilde{y}) = \det\left(\nabla\Psi_\delta\right) d\sigma(y) = \big(\sigma_0(y) + \sigma_1(y)\delta + \sigma_2(y)\delta^2 + \sigma_3(y)\delta^3\big) d\sigma(y), \tag{4.20}$$

where $\sigma_0 \equiv 1$, $\sigma_1(y) = \nabla \cdot (h\nu)$, $\sigma_2(y) = \frac{1}{2}\big[\big(\mathrm{tr}\nabla(\mathrm{h}\nu)\big)^2 - \mathrm{tr}\big(\nabla(\mathrm{h}\nu)^2\big)\big]$, and $\sigma_3(y) = \det\left(\nabla(h\nu)\right)$.

Set

$$\tilde{x} = \Psi_\delta(x), \quad x \in \partial\Omega \tag{4.21}$$

$$\tilde{y} = \Psi_\delta(y), \quad y \in \partial\Omega, \tag{4.22}$$

and define

$$\Theta(x, y) := \frac{1}{\delta}\big(\Psi_\delta(x) - \Psi_\delta(y) - (x - y)\big) \quad \text{for } \delta \neq 0. \tag{4.23}$$

Recall that Ψ_δ is a C^2 vector-valued function on $\partial\Omega$, then Θ is also a C^2 function. Moreover, the following holds.

Proposition 4.3.3. *The vector-valued function $\Theta(x, y)$ is C^2 on $\partial\Omega \times \partial\Omega$ and there exists a constant $C > 0$ that only depends on Ω and h such that:*

$$|\Theta(x, y)| \leq C|x - y|,$$
$$|\Theta(x, y) \cdot (x - y)| \leq C|x - y|^2,$$

for all $x, y \in \partial\Omega$.

Proof. Expression (4.23) shows that:

$$\Theta(x, y) \;=\; h(x)\nu(x) - h(y)\nu(y).$$

Since $\partial\Omega$ is a C^2 surface, there exists a constant $C' > 0$ such that:

$$|\nu(x) - \nu(y)| \leq C'|x - y|, \text{ and } |\nu(x) \cdot (x - y)| \leq C'|x - y|^2,$$

for all $x, y \in \partial\Omega$.
The last inequalities and the C^2 regularity of h yield the results of the proposition.

Now, by using (4.23) we obtain:

$$\tilde{x} - \tilde{y} = x - y + \delta\Theta(x, y), \quad (x, y) \in \partial\Omega \times \partial\Omega, \tag{4.24}$$

and the following results hold.

Lemma 4.3.1. *Let $r > 0$ be a fixed real and $m \geq 2$ be a fixed integer. The following asymptotic expansions*

$$\frac{e^{\boldsymbol{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|}}{|\tilde{x} - \tilde{y}|^m} = \frac{e^{\boldsymbol{i}\sqrt{\lambda}|x-y|}}{|x - y|^m}\Big(1 + \delta T_1^{(m)}(x, y) + \sum_{n\geq 2} \delta^n T_n^{(m)}(\lambda; x, y)\Big), \tag{4.25}$$

$$\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\frac{e^{\boldsymbol{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|} - 1}{|\tilde{x} - \tilde{y}|} = R_0^{(ij)}(\lambda; x, y) + \delta R_1^{(ij)}(\lambda; x, y) + \sum_{n\geq 2} \delta^n R_n^{(ij)}(\lambda; x, y) \tag{4.26}$$

hold uniformly for $(\lambda, x, y) \in B_r(0) \times \partial\Omega \times \partial\Omega$, where $B_r(0)$ is a ball in the complex plane of center zero and radius r. In addition the functions $T_n^{(m)}(\lambda; x, y)$ and $R_n^{(ij)}(\lambda; x, y)$ are smooth and bounded uniformly on $B_r(0) \times \partial\Omega \times \partial\Omega$. The first coefficients are given by:

$$R_0^{(ij)}(\lambda; x, y) := \partial_{x_i}\partial_{x_j}\frac{e^{\boldsymbol{i}\sqrt{\lambda}|x-y|} - 1}{|x - y|},$$

and

$$R_1^{(ij)}(\lambda; x, y) = (x-y)\cdot\Theta(x, y)\Big(-\lambda^2\delta_{ij} - 3\boldsymbol{i}\sqrt{\lambda}\frac{\delta_{ij}}{r} - \boldsymbol{i}\lambda^{3/2}\frac{r_i r_j}{r} + 7\lambda^2\frac{r_i r_j}{r^2} + 15\boldsymbol{i}\sqrt{\lambda}\frac{r_i r_j}{r^3}\Big)\frac{e^{\boldsymbol{i}\sqrt{\lambda}r}}{r^5}$$

$$+(x-y)\cdot\Theta(x, y)\Big(4\boldsymbol{i}\sqrt{\lambda}\frac{\delta_{ij}}{r} - 15\frac{\delta_{ij}}{r^2} - \lambda^2[-5\frac{r_i r_j}{r^2} + \frac{(r_i\theta_j + r_j\theta_i)}{(x - y)\cdot\Theta(x, y)} - 7\boldsymbol{i}\sqrt{\lambda}[-6\frac{r_i r_j}{r^3} + \frac{1}{r}\frac{(r_i\theta_j + r_j\theta_i)}{(x - y)\cdot\Theta(x, y)}]$$

$$+15[-7\frac{r_i r_j}{r^4} + \frac{1}{r^2}\frac{(r_i\theta_j + r_j\theta_i)}{(x - y)\cdot\Theta(x, y)}\Big)\frac{e^{\boldsymbol{i}\sqrt{\lambda}r}}{r^5}$$

$$+15\frac{(x - y)\cdot\Theta(x, y)}{r^7}\Big(\delta_{ij} - [-7\frac{r_i r_j}{r^2} + \frac{(r_i\theta_j + r_j\theta_i)}{(x - y)\cdot\Theta(x, y)}]\Big),$$

where $r = |x - y|$, $r_j = x_j - y_j$, θ_j means the j^{th} component of Θ, and δ_{ij} means the Kronecker index.

Proof. For $m = 2$ and $x \neq y$ we have:

$$|\tilde{x} - \tilde{y}|^2 = |x - y + \delta\Theta(x,y)|^2 \;=\; |x-y|^2 \left(1 + \delta c_1^{(2)}(x,y) + \delta^2 c_2^{(2)}(x,y)\right),$$

where

$$c_1^{(2)}(x,y) \;=\; \frac{2\Theta(x,y)\cdot(x-y)}{|x-y|^2}, \quad c_2^{(2)}(x,y) \;=\; \frac{\Theta(x,y)\cdot\Theta(x,y)}{|x-y|^2}.$$

Proposition 4.3.1 shows that $c_1^{(2)}(x,y)$ and $c_2^{(2)}(x,y)$ are bounded uniformly on $\partial\Omega \times \partial\Omega$.

For $m > 2$, we have:

$$|\tilde{x} - \tilde{y}|^m = |x - y + \delta\Theta(x,y)|^m = |x-y|^m \big|1 + \delta \frac{\Theta(x,y)}{|x-y|}\big|^m,$$

where $x \neq y$.
Using (4.23), Proposition 4.2.2 and Proposition 4.3.3, we see that the regular vector-valued function $(x,y) \mapsto \frac{\Theta(x,y)}{|x-y|}$ is well defined on $\partial\Omega \times \partial\Omega$, and it is independent of δ.
Therefore, we can expand:

$$|\tilde{x} - \tilde{y}|^m = c_0^{(m)}(x,y) + \sum_{n=1}^{\infty} \delta^n c_n^{(m)}(x,y) \quad \text{uniformly on } \partial\Omega \times \partial\Omega, \tag{4.27}$$

where the first coefficients $c_0^{(m)}(x,y) = |x-y|^m$, $c_1^{(m)}(x,y) = m < x-y, \Theta(x,y) > |x-y|^{m-2}$ and $c_2^{(m)}(x,y) = \frac{m}{2}|x-y|^m \big[|\frac{\Theta(x,y)}{|x-y|}|^2 + (m-2)\big(< \frac{x-y}{|x-y|^2}, \frac{\Theta(x,y)}{|x-y|} > \big)^2\big]$. Moreover, if m is even, then $c_n^{(m)}(x,y) = 0$ for $n \geq m+1$.
Now combining (4.27) for $m = 1$ with the well known asymptotic expansion of the exponential function, we immediately get

$$e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|} = \sum_{n=0}^{\infty} \delta^n K_n(\lambda; x, y) \quad \text{uniformly on } B_r(0) \times \partial\Omega \times \partial\Omega, \tag{4.28}$$

where the first coefficients $K_0(\lambda; x, y) = e^{\mathbf{i}\sqrt{\lambda}c_0^{(1)}} = e^{\mathbf{i}\sqrt{\lambda}|x-y|}$, and the coefficients $K_n(\lambda; x, y)$ can be deduced recursively from $c_n^{(1)}$.
Thanks to relations (4.24), (4.27) and (4.28), we can obtain the desired result given by (4.25) where the coefficients $T_n^{(m)}$ can be deduced easily from $c_n^{(m)}$ and K_n.

To prove relation (4.26), we firstly expand

$$\partial_{\tilde{x}_i}\partial_{\tilde{x}_j} \frac{e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|} - 1}{|\tilde{x}-\tilde{y}|} = \frac{1}{|\tilde{x}-\tilde{y}|}\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\big(e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|} - 1\big) + \partial_{\tilde{x}_i}\big(\frac{1}{|\tilde{x}-\tilde{y}|}\big)\partial_{\tilde{x}_j}\big(e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|} - 1\big) \tag{4.29}$$

$$+\partial_{\tilde{x}_j}\big(\frac{1}{|\tilde{x}-\tilde{y}|}\big)\partial_{\tilde{x}_i}\big(e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|}-1\big)+\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\big(\frac{1}{|\tilde{x}-\tilde{y}|}\big)\big(e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|}-1\big).$$

To simplify, let us denote $r=|x-y|$, $r_i=x_i-y_i$, $\tilde{r}=|\tilde{x}-\tilde{y}|$, and $\hat{r}_j$ defined by the relation

$$\partial_{\tilde{x}_j}\tilde{r}=\frac{\hat{r}_j}{\tilde{r}}. \tag{4.30}$$

Then,

$$\partial_{\tilde{x}_j}(\frac{1}{\tilde{r}^3})=-3\frac{\hat{r}_j}{\tilde{r}^5},\quad \text{and } \partial_{\tilde{x}_i}\partial_{\tilde{x}_j}(\frac{1}{\tilde{r}^3})=-3\frac{\delta_{ij}}{\tilde{r}^5}+15\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^7}. \tag{4.31}$$

Now, by using the following result

$$\partial_{\tilde{x}_j}\big(e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|}-1\big)=\partial_{\tilde{x}_j}(\mathbf{i}\sqrt{\lambda}\tilde{r})e^{\mathbf{i}\sqrt{\lambda}\tilde{r}}=\mathbf{i}\sqrt{\lambda}\frac{\hat{r}_j}{\tilde{r}}e^{\mathbf{i}\sqrt{\lambda}\tilde{r}}, \tag{4.32}$$

we get that

$$\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\big(e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|}-1\big)=\mathbf{i}\sqrt{\lambda}\big[\partial_{\tilde{x}_i}(\frac{\hat{r}_j}{\tilde{r}})e^{\mathbf{i}\sqrt{\lambda}\tilde{r}}+\frac{\hat{r}_j}{\tilde{r}}\partial_{\tilde{x}_i}e^{\mathbf{i}\sqrt{\lambda}\tilde{r}}\big]=\mathbf{i}\sqrt{\lambda}\big[\frac{\delta_{ij}}{\tilde{r}}-\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^3}+\mathbf{i}\sqrt{\lambda}\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^2}\big]e^{\mathbf{i}\sqrt{\lambda}\tilde{r}}. \tag{4.33}$$

To find the desired result in (4.26), we may use (4.31), (4.32) and (4.33) to see that the relation (4.29) verifies:

$$\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\frac{e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|}-1}{|\tilde{x}-\tilde{y}|^3}=\big[\mathbf{i}\sqrt{\lambda}\frac{\delta_{ij}}{\tilde{r}^4}-3\frac{\delta_{ij}}{\tilde{r}^5}-\lambda^2\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^5}-7\mathbf{i}\sqrt{\lambda}\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^6}+15\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^7}\big]e^{\mathbf{i}\sqrt{\lambda}\tilde{r}}+3\frac{\delta_{ij}}{\tilde{r}^5}-15\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^7}. \tag{4.34}$$

On the other hand, the components of the vectorial relation (4.24) can be given as follows

$$\tilde{x}_i-\tilde{y}_i=x_i-y_i+\delta\theta_i(x,y).\quad i=1,2,3 \tag{4.35}$$

where $\theta_i(x,y)$ means the i^{th} component of the vector-valued function $\Theta(x,y)$.
Then, by relations (4.30) and (4.35) we deduce that

$$\hat{r}_j=\tilde{x}_i-\tilde{y}_i=x_i-y_i+\delta\theta_i(x,y)=r_i+\delta\theta_i(x,y),\quad i=1,2,3. \tag{4.36}$$

Using both relations (4.24) and (4.36), we get the following expansion

$$\hat{r}_i\hat{r}_j=\alpha_0^{(ij)}+\alpha_1^{(ij)}\delta+\alpha_2^{(ij)}\delta^2, \tag{4.37}$$

where the first coefficients: $\alpha_0^{(ij)}=r_ir_j$, $\alpha_1^{(ij)}=r_i\theta_j+r_j\theta_i$ and $\alpha_2^{(ij)}=\theta_i\cdot\theta_j$.

Now regarding (4.27) and using the fact that $c_0^{(m)}\neq 0$ for each integer m. Then one can expand

$$\frac{1}{\tilde{r}^m}=\kappa_0^{(m)}(x,y)+\sum_{n=1}^{\infty}\delta^n\kappa_n^{(m)}(x,y)\quad \text{uniformly}, \tag{4.38}$$

where the first coefficients: $\kappa_0^{(m)}(x,y) = (c_0^{(m)})^{-1}(x,y)$, and $\kappa_1^{(m)}(x,y) = -\big(c_1^{(m)}(c_0^{(m)})^{-2}\big)(x,y)$.
By using (4.37)-(4.38), we get that

$$\frac{\hat{r}_i\hat{r}_j}{\tilde{r}^m} = \beta_{0,m}^{(ij)} + \delta\beta_{1,m}^{(ij)} + \sum_{n=2}^{\infty}\delta^n\beta_{n,m}^{(ij)}, \tag{4.39}$$

where the first coefficients: $\beta_{0,m}^{(ij)}(x,y) = \alpha_0^{(ij)}(c_0^{(m)})^{-1}(x,y)$, and
$\beta_{1,m}^{(ij)}(x,y) = -\alpha_0^{(ij)}\big(c_1^{(m)}(c_0^{(m)})^{-2}\big)(x,y) + \alpha_1^{(ij)}(c_0^{(m)})^{-1}(x,y)$.
To achieve the proof, we may insert all expansions (4.28), (4.38) (for $m \in \{4,5\}$), and (4.39)(for $m \in \{5,6,7\}$ into (4.34). We get that

$$\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\frac{e^{\mathbf{i}\sqrt{\lambda}|\tilde{x}-\tilde{y}|}-1}{|\tilde{x}-\tilde{y}|^3} = R_0^{(ij)}(x,y) + \delta R_1^{(ij)}(x,y) + \sum_{n=2}^{\infty}\delta^n R_n^{(ij)}(x,y)$$

where the first coefficient:

$$R_0^{(ij)}(\lambda;x,y) = \big(\mathbf{i}\sqrt{\lambda}\delta_{ij}\kappa_0^{(4)} - 3\delta_{ij}\kappa_0^{(5)} - \lambda^2\beta_{0,5}^{(ij)} - 7\mathbf{i}\sqrt{\lambda}\beta_{0,6}^{(ij)} + 15\beta_{0,7}^{(ij)}\big)K_0 + 3\delta_{ij}\kappa_0^{(5)} - 15\beta_{0,7}^{(ij)}$$

$$= \Big[\mathbf{i}\sqrt{\lambda}\delta_{ij}(c_0^{(4)})^{-1} - 3\delta_{ij}(c_0^{(5)})^{-1} - \lambda^2 r_i r_j(c_0^{(5)})^{-1} - 7\mathbf{i}\sqrt{\lambda}r_i r_j(c_0^{(6)})^{-1} + 15r_i r_j(c_0^{(7)})^{-1}\Big]e^{\mathbf{i}\sqrt{\lambda}c_0^{(1)}}$$

$$+3\delta_{ij}(c_0^{(5)})^{-1} - 15r_i r_j(c_0^{(7)})^{-1}.$$

Using the fact that $c_0^{(m)} = |x-y|^m$, we find that

$$R_0^{(ij)}(\lambda;x,y) := \partial_{x_i}\partial_{x_j}\frac{e^{\mathbf{i}\sqrt{\lambda}|x-y|}-1}{|x-y|^3}.$$

Based on (4.34), we find that

$$R_1^{(ij)}(\lambda;x,y) = \big(\mathbf{i}\sqrt{\lambda}\delta_{ij}\kappa_0^{(4)} - 3\delta_{ij}\kappa_0^{(5)} - \lambda^2\beta_{0,5}^{(ij)} - 7\mathbf{i}\sqrt{\lambda}\beta_{0,6}^{(ij)} + 15\beta_{0,7}^{(ij)}\big)K_1$$

$$+\big(\mathbf{i}\sqrt{\lambda}\delta_{ij}\kappa_1^{(4)} - 3\delta_{ij}\kappa_1^{(5)} - \lambda^2\beta_{1,5}^{(ij)} - 7\mathbf{i}\sqrt{\lambda}\beta_{1,6}^{(ij)} + 15\beta_{1,7}^{(ij)}\big)K_0 + 3\delta_{ij}\kappa_1^{(5)} - 15\beta_{1,7}^{(ij)}.$$

So that, by using the fact that $K_1 = \mathbf{i}\sqrt{\lambda}\Theta\cdot\frac{(x-y)}{r}e^{\mathbf{i}\sqrt{\lambda}c_0^{(1)}}$, we get

$$R_1^{(ij)}(\lambda;x,y) = \mathbf{i}\sqrt{\lambda}\Theta\cdot\frac{(x-y)}{r}\Big(\mathbf{i}\sqrt{\lambda}\delta_{ij}(c_0^{(4)})^{-1} - 3\delta_{ij}(c_0^{(5)})^{-1} - \lambda^2 r_i r_j(c_0^{(5)})^{-1} - 7\mathbf{i}\sqrt{\lambda}r_i r_j(c_0^{(6)})^{-1}$$

$$+15r_i r_j(c_0^{(7)})^{-1}\Big)e^{\mathbf{i}\sqrt{\lambda}c_0^{(1)}}$$

$$+\Big(\mathbf{i}\sqrt{\lambda}\delta_{ij}(c_0^{(4)})^{-2}c_1^{(4)} - 3\delta_{ij}(c_0^{(5)})^{-2}c_1^{(5)} - \lambda^2[-r_i r_j(c_0^{(5)})^{-2}c_1^{(5)} + (r_i\theta_j + r_j\theta_i)(c_0^{(5)})^{-1}]$$

$$-7\mathbf{i}\sqrt{\lambda}[-r_i r_j(c_0^{(6)})^{-2}c_1^{(6)} + (r_i\theta_j + r_j\theta_i)(c_0^{(6)})^{-1}] + 15[-r_i r_j(c_0^{(7)})^{-2}c_1^{(7)} + (r_i\theta_j + r_j\theta_i)(c_0^{(7)})^{-1}]\Big)e^{\mathbf{i}\sqrt{\lambda}c_0^{(1)}}$$

$$+3\delta_{ij}(c_0^{(5)})^{-2}c_1^{(5)} - 15[-r_ir_j(c_0^{(7)})^{-2}c_1^{(7)} + (r_i\theta_j + r_j\theta_i)(c_0^{(7)})^{-1}].$$

Using the explicit forms of the coefficients $c_0^{(m)}$ and $c_1^{(m)}$ given above, we immediately get the desired expression of $R_1^{(ij)}(\lambda; x, y)$. The other coefficients can be deduced easily by the same manner in terms of $c_n^{(m)}$.

Now using (4.35), we obtain for $i, j = 1, 2, 3$ that:

$$(\tilde{x}_i - \tilde{y}_i)(\tilde{x}_j - \tilde{y}_j) = \hat{g}_0(x, y) + \delta\hat{g}_1(x, y) + \delta^2\hat{g}_2(x, y), \tag{4.40}$$

where

$$\hat{g}_0(x, y) = (x_i - y_i)(x_j - y_j), \quad \hat{g}_1(x, y) = \theta_i(x, y)(x_j - y_j) + \theta_j(x, y)(x_i - y_i),$$

and

$$\hat{g}_2(x, y) = \theta_i(x, y) \cdot \theta_j(x, y).$$

Now, by using (4.20), (4.27) and (4.40), we immediately get

$$\frac{(\tilde{x}_i - \tilde{y}_i)(\tilde{x}_j - \tilde{y}_j)}{|\tilde{x} - \tilde{y}|^m} d\sigma_\delta(\tilde{y}) = \Big(\hat{T}_0(x, y) + \delta\hat{T}_1(x, y) + \sum_{n\geq 2} \delta^n \hat{T}_n(x, y)\Big) d\sigma(y), \tag{4.41}$$

where the Taylor coefficients $\hat{T}_n$ can be given explicitly with the aid of $\hat{g}_0, \hat{g}_1$, and $\hat{g}_2$.

Next, the following result holds.

Lemma 4.3.2. *The following uniform expansion holds on $\mathbb{R} \times \partial\Omega \times \partial\Omega$:*

$$(\tilde{x}_i - \tilde{y}_i)(\tilde{x}_j - \tilde{y}_j)|\tilde{x} - \tilde{y}|e^{i\sqrt{\lambda}|\tilde{x}-\tilde{y}|} d\sigma_\delta(\tilde{y}) = E_\delta(\lambda; x, y)\, d\sigma(y) = \sum_{n=0}^{\infty} \delta^n E_n(\lambda; x, y)\, d\sigma(y) \tag{4.42}$$

with

$$E_0(\lambda; x, y) = \hat{g}_0(x, y)|x - y|e^{i\sqrt{\lambda}|x-y|},$$

and the other coefficients E_n are deduced from those σ_n, $c_n^{(1)}$, K_n and $\hat{g}_n$.

Proof. From the proof of Lemma 4.3.1 (for $m = 1$) and from relations (4.20) and (4.40), one can get that

$$(\tilde{x}_i - \tilde{y}_i)(\tilde{x}_j - \tilde{y}_j)|\tilde{x} - \tilde{y}|e^{i\sqrt{\lambda}|\tilde{x}-\tilde{y}|} d\sigma_\delta(\tilde{y}) = \Big(\hat{g}_0(x, y) + \delta\hat{g}_1(x, y) +$$

$$\delta^2\hat{g}_2(x,y)\Big)\Big(\sum_{n=0}^{\infty}\delta^n c_n^{(1)}\Big)\Big(\sum_{n=0}^{\infty}K_n(\lambda;x,y)\Big)\Big(\sigma_0(y)+\delta\sigma_1(y)+$$

$$\delta^2\sigma_2(y)+\delta^3\sigma_3(y)\Big)\,d\sigma(y).$$

By collecting terms of equal powers in the above relation, one can deduce easily the uniform expansion (4.42) with

$$E_0(\lambda;x,y)=\hat{g}_0(x,y)|x-y|e^{\mathbf{i}\sqrt{\lambda}|x-y|}.$$

Now, from (4.5) we have

$$\Gamma_{ij}(\lambda,|x-y|)=\frac{e^{\mathbf{i}\sqrt{\lambda}r}}{4\pi}\big[\frac{\delta_{ij}}{r}-\frac{\hat{x}_i\hat{x}_j}{r^3}\big]+\frac{e^{\mathbf{i}\sqrt{\lambda}r}}{4\pi\lambda}\big[i\frac{\delta_{ij}}{r^4}-4i\frac{\hat{x}_i\hat{x}_j}{r^4}-3ir\hat{x}_i\hat{x}_j\big]$$

$$+\frac{1}{4\pi\lambda}\big[-3\frac{\delta_{ij}}{r^5}+15\frac{\hat{x}_i\hat{x}_j}{r^7}\big]+\frac{e^{i\lambda r}}{4\pi\lambda}\big[-3\frac{\delta_{ij}}{r^5}+15\frac{\hat{x}_i\hat{x}_j}{r^3}\big],$$

where $r=|x-y|$ and $\hat{x}_i=x_i-y_i$. Then, by inserting relations (4.25), (4.40), (4.41) and that given by Lemma 4.3.2 into above identity, we immediately get the following main results

Proposition 4.3.4. *Let the perturbed boundary $\partial\Omega_\delta$ defined by (4.2). Let $\tilde{x}$ and $\tilde{y}$ given by (4.21)-(4.22), and the surface element $d\sigma_\delta(\tilde{y})$ given by (4.20). Then, the components $\Gamma_{ij}, 1\le i,j\le 3$ of the fundamental Stokes tensor can be expanded uniformly as:*

$$\Gamma_{ij}(\lambda,|\tilde{x}-\tilde{y}|)\,d\sigma_\delta(\tilde{y})=\Big(\Gamma_{ij}^{(0)}(\lambda,|x-y|)+\delta\Gamma_{ij}^{(1)}(\lambda,|x-y|)\tag{4.43}$$

$$+\sum_{n\ge 2}\delta^n\Gamma_{ij}^{(n)}(\lambda,|x-y|)\Big)\,d\sigma(y),\quad y\in\partial\Omega$$

where the first coefficients:

$$\Gamma_{ij}^{(0)}(\lambda,|x-y|):=\frac{\delta_{ij}}{4\pi}\Big(T_0^{(1)}+\frac{i}{\lambda}T_0^{(4)}-3\frac{r_0}{\lambda^2}-3\frac{T_0^{(5)}}{\lambda^2}\Big)+\frac{1}{4\pi}(\frac{15}{\lambda^2}-1)T_0^{(3)}\hat{g}_0-\frac{4i}{4\pi\lambda}T_0^{(4)}\hat{g}_0$$

$$-\frac{3i}{4\pi\lambda}E_0+\frac{15}{4\pi\lambda^2}\hat{T}_0,$$

and

$$\Gamma_{ij}^{(1)}(\lambda,|x-y|):=\frac{\delta_{ij}}{4\pi}\Big(T_1^{(1)}+\frac{i}{\lambda}T_1^{(4)}-3\frac{r_1}{\lambda^2}-3\frac{T_1^{(5)}}{\lambda^2}\Big)+\frac{1}{4\pi}(\frac{15}{\lambda^2}-1)\big[T_0^{(3)}\hat{g}_1+T_1^{(3)}\hat{g}_0\big]$$

$$-\frac{4i}{4\pi\lambda}\big[T_0^{(4)}\hat{g}_1+T_1^{(4)}\hat{g}_0\big]-\frac{3i}{4\pi\lambda}E_1+\frac{15}{4\pi\lambda^2}\hat{T}_1.$$

In Proposition 4.3.4, the coefficients r_0 and r_1 are deduced from (4.27) for $m = 5$.

Now, introduce a sequence of components of integral operators $(\mathcal{A}_i^{(n)})_{n\geq 0}$, defined for any $\varphi \in L^2(\partial\Omega)^3$ by:

$$\big(\mathcal{A}^{(n)}\varphi\big)_i(x) = \int_{\partial\Omega} \Gamma_{ij}^{(n)}(x,y)\varphi_j(y)\, d\sigma(y), \quad \text{for } i,j \in \{1,2,3\} \text{ and } n \geq 0.$$

Using previous results, il is clear that we know explicitly the first terms $\mathcal{A}_i^{(0)}$, $\mathcal{A}_i^{(1)}$. For any positive integer N, we can by recursive method get the term $\mathcal{A}_i^{(N)}$. Then, the following theorem holds.

Theorem 4.3.1. *Let $\mathcal{A}^\delta$ be the operator defined by (4.10). Let $\Psi_\delta(x)$ be the diffeomorphism defined as in Lemma 4.2.2. Let N be a positive integer. There exists a positive constant C depending only on N, and $\|h\|_{C^2}$ such that for any $\tilde{\varphi} \in L^2(\partial\Omega_\delta)^3$ and $i,j \in \{1,2,3\}$, the i^{th}-component $\mathcal{A}_i^\delta$ defined by (4.13) satisfies:*

$$\Big\| \big(\mathcal{A}^\delta\tilde{\varphi}\big)_i o\Psi_\delta - \big(\mathcal{A}^{(0)}\varphi\big)_i - \sum_{n=1}^{N} \delta^n \big(\mathcal{A}^{(n)}\varphi\big)_i \Big\|_{L^2(\partial\Omega)} \leq C\delta^{N+1}\|\varphi\|_{L^2(\partial\Omega)^3},$$

where $\varphi = \tilde{\varphi} o \Psi_\delta$.

4.3.2 Asymptotic expansion of the eigenvalues

To develop asymptotic behaviors of eigenvalues and eigenfunctions with respect to the parameter of perturbation δ, we may use the results of Theorem 4.3.1. Then, the following asymptotic expansion related to the operator $\mathcal{A}_\delta$ appears clearly.

Proposition 4.3.5. *Suppose that we have all hypothesis of Theorem 4.3.1. Then, the operator $\mathcal{A}^\delta(\lambda)$ defined by (4.10) can be expanded uniformly for $x \in \partial\Omega$ as follows:*

$$\mathcal{A}^\delta(\lambda)\varphi = \mathcal{A}^{(0)}(\lambda)\varphi + \delta\mathcal{A}^{(1)}(\lambda)\varphi + \delta^2\mathcal{A}^{(2)}(\lambda)\varphi + \cdots; \quad \text{as } \delta \to 0, \tag{4.44}$$

where $\varphi \in L^2(\partial\Omega)^3$, the i^{th}-component of the first term is given by

$$\big(\mathcal{A}_i^{(0)}(\lambda)\varphi\big)_i(x) = \int_{\partial\Omega} \Gamma_{ij}^{(0)}(x,y)\varphi_j(y)\, d\sigma(y), \quad \text{for } i,j \in \{1,2,3\},$$

and more generally, the i^{th}-component of the term with order n is given by

$$\big(\mathcal{A}^{(n)}(\lambda)\varphi\big)_i(x) = \int_{\partial\Omega} \Gamma_{ij}^{(n)}(x,y)\varphi_j(y)\, d\sigma(y), \quad \text{for } i,j \in \{1,2,3\} \text{ and } n \geq 1.$$

The coefficients $\Gamma_{ij}^{(n)}$ are given by (4.43).

Now, based on the well known works [8, 10], we tray to give (formally) an asymptotic expansion related to the eigenvalues λ_δ if the eigenvalue λ_0 is simple. But, To get an explicit asymptotic formula for λ_δ we may develop our analysis in some Hölder space. This will be given in Theorem 4.3.2. Before that, let $a_j(\delta)$ denotes:

$$a_j(\delta) := \frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)^j (\mathcal{A}^\delta)^{-1}(\lambda)\partial_\lambda \mathcal{A}^\delta(\lambda) d\lambda. \tag{4.45}$$

The functions $a_j(\delta)$ is analytic in a complex neighborhood of 0 and satisfies: $\overline{a_j(\delta)} = a_j(\overline{\delta})$. The following main result holds.

Proposition 4.3.6. *Let λ_0 be a simple eigenvalue of the problem (4.1). Let the operator $\mathcal{A}^\delta$ be defined by (4.10). Then, there exists a small positive number δ_0 such that the eigenvalue $\lambda(\delta)$ is analytic in $]-\delta_0, \delta_0[$ and satisfies:*

$$\lambda(\delta) = \lambda_0 + \delta\lambda_1 + \sum_{n\geq 2} \lambda_n \delta^n, \tag{4.46}$$

where the first coefficients are (formally) given by:

$$\begin{cases} \lambda_1 = \frac{1}{2i\pi}\, tr \int_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)\big[(\mathcal{A}^{(0)})^{-1}(\lambda)\mathcal{A}^{(1)}(\lambda)(\mathcal{A}^{(0)})^{-1}(\lambda)\partial_\lambda \mathcal{A}^{(0)}(\lambda)\big]\, d\lambda, \\ \lambda_2 = \frac{1}{2i\pi}\, tr \int_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)\big[(\mathcal{A}^{(0)})^{-1}\mathcal{A}^{(1)}(\mathcal{A}^{(0)})^{-1}\partial_\lambda \mathcal{A}^{(1)} + (\mathcal{A}^{(0)})^{-1}\mathcal{A}^{(2)}(\mathcal{A}^{(0)})^{-1}\partial_\lambda \mathcal{A}^{(0)} \\ +(\mathcal{A}^{(0)})^{-1}\big(\mathcal{A}^{(1)}(\mathcal{A}^{(0)})^{-1}\big)^2 \partial_\lambda \mathcal{A}^{(0)}\big]\, d\lambda. \end{cases}$$

Proof. Let λ_δ be the eigenvalue of the problem (4.4). It is well known that λ_δ is also simple such that $\lambda_\delta \to \lambda_0$ as $\delta \to 0$.
On the other hand, if we take the curl of the first equation in (4.4), we see that there exists a function $w = w(v_\delta)$ called vorticity associated to v_δ such that:

$$\begin{cases} \Delta w + \lambda_\delta w = 0 & \text{in } \Omega \\ v_\delta = 0 & \text{in } \partial\Omega. \end{cases}$$

That is, w is an eigenfunction of the negative Laplacian, but with homogeneous boundary conditions on the velocity v_δ.
Moreover, let u_δ be the stream function for v_δ given as in Lemma 2.10 of [28]. Then, $w = \Delta u_\delta$ and $\nabla u_\delta = 0$ on $\partial\Omega$. Since u_δ is determined only up to a constant, we can then assume

that $u_\delta = 0$ on $\partial\Omega$.

Thus, u_δ satisfies the following eigenvalue problem for the Dirichlet biharmonic operator:

$$\begin{cases} \Delta\Delta u_\delta + \lambda_\delta \Delta u_\delta = 0 & \text{in } \Omega \\ u_\delta = 0 & \text{in } \partial\Omega. \end{cases} \tag{4.47}$$

Note that Temam [54] exploits the similar correspondence between the Stokes problem and the biharmonic problem in the proof of the regularity of solutions to the Stokes system and to justify several results. Moreover, as pointed out by Ashbaugh in [11], there is a similar correspondence between the eigenvalue problems for the Dirichlet Laplacian and system (4.47) with the boundary condition replaced by $\Delta u_\delta = 0$.

Then, one can exploit this correspondence to use the approach used in [10] in order to develop an asymptotic expansion for the eigenvalue.

On the other hand, it is well known [8, 10] that there exits a polynomial-valued function $\delta \mapsto \mathcal{Q}_\delta(\lambda)$ of degree 1, analytic in $]-\delta_0, \delta_0[$ and of the form:

$$\mathcal{Q}_\delta(\lambda) = \lambda - a_1(\delta)$$

such that the perturbation $\lambda_\delta - \lambda_0$ is precisely its zero. For the existence of $\mathcal{Q}_\delta$ one can follow the general approach used, for example, in [10] for the case of Laplace operator.

Writing:

$$\mathcal{Q}_\delta(\lambda_\delta - \lambda_0) = 0.$$

Then we have

$$\lambda_\delta - \lambda_0 = a_1(\delta).$$

Therefore, by (4.45) we have

$$\lambda_\delta - \lambda_0 = \frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)(\mathcal{A}^\delta)^{-1}(\lambda)\partial_\lambda \mathcal{A}^\delta(\lambda) d\lambda.$$

On the other hand, for δ in a small neighborhood of 0, the following Neumann series converges uniformly with respect to λ in ∂D_{ϵ_0}:

$$(\mathcal{A}^\delta)^{-1}(\lambda) = (\mathcal{A}^{(0)})^{-1}(\lambda) + \sum_{k=1}^{\infty} (\mathcal{A}^{(0)})^{-1}(\lambda)\big[\big(\mathcal{A}^{(0)}(\lambda) - \mathcal{A}^\delta(\lambda)\big)(\mathcal{A}^{(0)})^{-1}(\lambda)\big]^k.$$

So,

$$\lambda_\delta - \lambda_0 = \frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)(\mathcal{A}^{(0)})^{-1}(\lambda)\partial_\lambda \mathcal{A}^\delta(\lambda)\varphi \, d\lambda +$$

$$\frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0) \sum_{k=1}^{\infty} (\mathcal{A}^{(0)})^{-1}(\lambda)\big[\big(\mathcal{A}^{(0)}(\lambda) - \mathcal{A}^\delta(\lambda)\big)(\mathcal{A}^{(0)})^{-1}(\lambda)\big]^k \partial_\lambda \mathcal{A}^\delta(\lambda) \, d\lambda$$

By using (4.14), we find that $\dfrac{1}{2i\pi} tr \displaystyle\int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)(\mathcal{A}^{(0)})^{-1}(\lambda)\partial_\lambda \mathcal{A}^\delta(\lambda) \, d\lambda = 0.$

This result is a direct consequence of the fact that $\mathcal{R}_0(\lambda)$ and $\partial_\lambda \mathcal{A}^\delta(\lambda)$ are holomorphic in the variable λ.

Now we have:

$$\lambda_\delta - \lambda_0 = \frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)(\mathcal{A}^{(0)})^{-1}(\lambda)\big[\big(\mathcal{A}^{(0)}(\lambda) - \mathcal{A}^\delta(\lambda)\big)(\mathcal{A}^{(0)})^{-1}(\lambda)\big]\partial_\lambda \mathcal{A}^\delta(\lambda) \, d\lambda +$$

$$\frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)(\mathcal{A}^{(0)})^{-1}(\lambda)\big[\big(\mathcal{A}^{(0)}(\lambda) - \mathcal{A}^\delta(\lambda)\big)(\mathcal{A}^{(0)})^{-1}(\lambda)\big]^2 \partial_\lambda \mathcal{A}^\delta(\lambda) \, d\lambda +$$

$$\frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0) \sum_{k\geq 3} (\mathcal{A}^{(0)})^{-1}(\lambda)\big[\big(\mathcal{A}^{(0)}(\lambda) - \mathcal{A}^\delta(\lambda)\big)(\mathcal{A}^{(0)})^{-1}(\lambda)\big]^k \partial_\lambda \mathcal{A}^\delta(\lambda) \, d\lambda.$$

Inserting expression (4.44) into above relation, we may get:

$$\lambda_\delta - \lambda_0 = \frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)(\mathcal{A}^{(0)})^{-1}\big[\delta \mathcal{A}^{(1)}(\mathcal{A}^{(0)})^{-1} + \delta^2 \mathcal{A}^{(2)}(\mathcal{A}^{(0)})^{-1} + \cdots\big]\big(\partial_\lambda \mathcal{A}^{(0)} + \delta \partial_\lambda \mathcal{A}^{(1)}$$

$$+ \delta^2 \partial_\lambda \mathcal{A}^{(2)} + \cdots\big)\varphi \, d\lambda + \frac{1}{2i\pi} tr \int\limits_{\partial D_{\epsilon_0}} (\lambda - \lambda_0)(\mathcal{A}^{(0)})^{-1}\big[\delta \mathcal{A}^{(1)}(\mathcal{A}^{(0)})^{-1} +$$

$$\delta^2 \mathcal{A}^{(2)}(\mathcal{A}^{(0)})^{-1} + \cdots\big]^2\big(\partial_\lambda \mathcal{A}^{(0)} + \delta \partial_\lambda \mathcal{A}^{(1)} + \delta^2 \partial_\lambda \mathcal{A}^{(2)} + \cdots\big) \, d\lambda + \cdots$$

If we collect the same powers of δ, then we get the desired results.

4.3.3 Asymptotic formula in Hölder space

In section we suppose that any shape deformation like (4.2) occurs inside a bounded domain Ω in a Hölder space. Based on the works [5–8, 14], we may advance the asymptotic expansions for simple and/or multiple eigenvalues of the Stokes operator and we may give explicitly the fist term correction. Since the Stokes system can be viewed as the incompressible limit and as done in [7], our method in this section may be deeply based on the ones developed in [5] for the elastic case.

Suppose that in this section that $\Omega \subset \mathbb{R}^2$ is a bounded domain with $C^{1,1}$ boundary.Let D be an open subset of Ω such that $dist(\partial\Omega, D) \geq d_0 > 0$ representing an inclusion made of a different Newtonian fluid material. Assume that the boundary ∂D is of class $C^{2,1}$. We denote by σ_0 and σ_1 the stress tensor fields in $\Omega\backslash\overline{D}$ and D, respectively. We assume that both $\Omega\backslash\overline{D}$ and D are occupied by isotropic and homogeneous Newtonian fluids. Then, the tensors σ_0 and σ_1 may be given by

$$(\sigma_s)_{ijlk} = \mu_s(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) \quad \text{for } i,j,k,l = 1,2 \text{ and } s = 0,1, \tag{4.48}$$

where μ_0 and μ_1 are the viscosity constants of the flow in $\Omega\backslash\overline{D}$ and D, respectively. Given two (2×2) matrices A and B we denote by $A : B$ the contraction, i.e., $A : B = \sum_{ij} a_{ij}b_{ij}$. Now, it is useful to introduce the strain rate tensor $\mathcal{D}$ for the flow as follows:

$$\mathcal{D}(v) := \frac{1}{2}(\nabla v + (\nabla v)^T) = \Big(\frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\Big)_{1\leq i,j\leq 3}.$$

Let $\sigma_D := \sigma_0\chi_{\Omega\backslash\overline{D}} + \sigma_1\chi_D$. Then according to [14], one can rewrite the eigenvalue problem (4.1) as the generalized one:

$$\begin{cases} -\mathrm{div}(\sigma_D\mathcal{D}(v_0) - p_0 Id) = \lambda_0 v_0 & \text{in } \Omega \\ \nabla.v_0 = 0 & \text{in } \Omega \\ v_0 = 0 & \text{in } \partial\Omega \\ \|v_0\| = 1 \end{cases} \tag{4.49}$$

where $(v_0, p_0, \lambda_0) \in (H^2(\Omega))^2 \times H^1(\Omega) \times \mathbb{R}_+^*$ and Id means the identity. Here we refer the reader to ([14], Theorem IV.5.8) for more details about the regularity properties. On the other hand, if $\mathbb{S}$ mens the Stokes operator, it is well know that (see for example, [7, 14]) there

exist an orthogonal projection (the Leray projection) P such that:

$$\mathbb{S}(v) = P(-\Delta v), \quad \forall v \in (H^2(\Omega))^2 \cap H(\Omega).$$

In consequence, the Stokes operator enters the general framework about spectral properties of $-\Delta$.

The δ-perturbation, denoted by D_δ, of the domain D is given by

$$\partial D_\delta := \{\tilde{x} = x + \delta h(x)\nu(x), \quad x \in \partial D\} \tag{4.50}$$

where here $h(x)$ is assumed to be real function in $C^{1,1}(\partial D)$ that satisfies

$$\|h\|_{C^{1,1}(\partial D)} < c_0 \text{for some positive constant} c_0. \tag{4.51}$$

Let $\sigma_{D_\delta} := \sigma_0 \chi_{\Omega \setminus \overline{D}_\delta} + \sigma_1 \chi_{D_\delta}$. Therefore, (4.4) can be generalized to the following perturbed eigenvalue problem

$$\begin{cases} -\mathrm{div}(\sigma_{D_\delta}\mathcal{D}(v_\delta) - p_\delta Id) = \lambda_\delta v_\delta & \text{in } \Omega \\ \nabla . v_\delta = 0 & \text{in } \Omega \\ v_\delta = 0 & \text{in } \partial\Omega \\ \|v_\delta\| = 1 \end{cases} \tag{4.52}$$

where $(v_\delta, p_\delta, \lambda_\delta) \in (H^2(\Omega))^2 \times H^1(\Omega) \times \mathbb{R}_+^*$.
Our purpose in this section is to develop a rigourously asymptotic behavior of the eigenvalue of (4.52) assumed to be more precisely. The main result is the following.

Theorem 4.3.2. *Suppose that $\Omega \subset \mathbb{R}^2$ with class $C^{1,1}$. Let λ_0 be a simple eigenvalue of the problem (4.49). Assume that we have (4.51), and the pressure still the same if we have (4.50). Then, the first term correction in the asymptotic expansion of the eigenvalue λ_δ is given through the following formula:*

$$\lambda_\delta - \lambda_0 = \delta \int_{\partial D} h(x)\mathcal{V}[\mathcal{D}(v_0^e)](x) : \mathcal{D}(v_0^e(x))ds(x) + O(\delta^{1+\beta}),$$

where β some positif constant and $\mathcal{V}$ means the viscous moment tensor (VMT) given by $\mathcal{V}[\mathcal{D}(v_0^e)] = [\frac{1}{\sigma_1}(\mathcal{V}_1 \widehat{\nabla} v_0^e(x)\tau) \otimes \tau + (\frac{\sigma_0}{\sigma_1}\widehat{\nabla} v_0^e(x)\nu) \otimes \nu]$. Here, we denote by ν, τ respectively the outward normal vector and the tangent vector to ∂D and $\mathcal{V}_1$ a given operator expresses the transmission conditions for v_0.

Proof. To derive the corresponding formulae for the eigenvalues we will use an idea close to Osborn's theorem [50] which gives estimates for the convergence of the eigenvalues of a sequence of compact operators. For any $f \in L^2(\Omega)$, define the linear operator $T_\epsilon f = u_\epsilon$, where u_ϵ is the solution to the problem

$$\begin{cases} -\mathrm{div}(\sigma_{D_\delta}\mathcal{D}(u_\delta) - p_\delta Id) = f \quad \text{in } \Omega, \\ u_\delta = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.53}$$

For the same given function f, we define the linear operator $Tf = u_0$, where u_0 is the solution to the problem

$$\begin{cases} -\mathrm{div}(\sigma_D\mathcal{D}(u_0) - p_0 Id) = f \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.54}$$

Clearly T_ϵ and T are are compact self-adjoint operators. One may use standard energy estimates based on Korn and Poincaré inequalities to get these facts. Moreover, it is easy to see that $\{T_\delta; \delta \geq 0\}$ with $T_0 = T$ are collectively compact and that $T_\delta \to T$ pointwise as $\delta \to 0$. To get the last result, one may prove that $T_\delta f$ converges to Tf in $L^2(\Omega)$ for every $f \in L^2(\Omega)$. By assumption, in the presence of (4.50), there exist a function p such that $p_\delta = p_0 = p$. Then, let $u_\delta = T_\delta f$ and $u_0 = T_\delta f$. For any $w \in H^1(\Omega)$, we have:

$$\int_\Omega \nabla \cdot (\sigma_{D_\delta}\mathcal{D}(u_\delta) - pId).w = \int_\Omega f.w,$$

so,

$$\int_\Omega (\sigma_{D_\delta}\mathcal{D}(u_\delta) - pId) : \mathcal{D}(w) = -\int_\Omega f.w.$$

Similarly,

$$\int_\Omega (\sigma_D\mathcal{D}(u_0) - pId) : \mathcal{D}(w) = -\int_\Omega f.w.$$

Consequently, choosing $w = u_\delta - u_0$ and subtracting these two equations we get

$$\int_\Omega (\sigma_{D_\delta}\mathcal{D}(u_\delta) - \sigma_D\mathcal{D}(u_0)) : \mathcal{D}(u_\delta - u_0) = 0$$

which gives

$$\int_\Omega \sigma_{D_\delta}\mathcal{D}(u_\delta - u_0) : \mathcal{D}(u_\delta - u_0) = -\int_\Omega (\sigma_{D_\delta} - \sigma_D)\mathcal{D}(u_0) : \mathcal{D}(u_\delta - u_0).$$

Hence, by using successively Korn's inequality and Hölder's one, one can get

$$\|\nabla(u_\delta - u_0)\|_{L^2(\Omega)} \leq C\|\nabla u_0\|_{L^2(D_\delta \triangle D)},$$

where $\triangle$ denotes the symmetric difference. It then follows by the Poincaré inequality that

$$\|u_\delta - u_0\|_{H^1(\Omega)} \leq C\|\nabla u_0\|_{L^2(D_\delta \triangle D)}.$$

Consequently, using the last inequality and the fact that $|D_\delta \triangle D| \to 0$ as $\delta \to 0$ and that $\nabla u_0 \in L^2(\Omega)$ we obtain that $T_\delta \to T$ pointwise as $\delta \to 0$ in $L^2(\Omega)$.

Hence all hypotheses hold for the theorem of Osborn. Now if we set,

$$\widetilde{\mu}_0 = \frac{1}{\lambda_0} \quad \text{and } \widetilde{\mu}_\delta = \frac{1}{\lambda_\delta},$$

then according to the problem (4.53)(resp. (4.54)) we can see that $(\widetilde{\mu}_\delta, v_\delta)$ (resp.$(\widetilde{\mu}_0, v_0)$) is eigenpair of T_δ (resp. of T_0). So, a theorem of Osborn [50] yields

$$\left|\widetilde{\mu}_0 - \widetilde{\mu}_\delta - \prec (T_\delta - T)v_0, v_0 \succ \right| \leq \|(T - T_\delta)v_0\|^2_{L^2(\Omega)}, \tag{4.55}$$

where C is independent of δ and v_0 is the solution of (4.49). Furthermore, if v_δ is the solution to (4.52), then

$$\|v_0 - v_\delta\|_{L^2(\Omega)} \leq \|(T - T_\delta)v_0\|_{L^2(\Omega)}.$$

According to [5, 14] we can extend the regularity results obtained by De Giorgi and Nash in the scalar case [24] to get

$$\|v_\delta\|_{C^{1,\alpha}(\overline{D}_\delta)} + \|v_\delta\|_{C^{1,\alpha}(\Omega_{d_0/2} \setminus D_\delta)} \leq C, \quad \text{for some } \alpha > 0, \tag{4.56}$$

where C is a positif constant and for $d_0 > 0$ we have $\Omega_{d_0/2} := \{x \in \Omega : dist(x, \partial\Omega) > d_0/2\}$.

To compute the term $\prec (T_\delta - T)v_0, v_0 \succ$ appearing in (4.55), we may firstly recall that $Tv_0 = \widetilde{\mu}_0 v_0$ and $T_\delta v_0 = \tilde{u}_\delta$ where $\tilde{u}_\delta$ is the solution to

$$\begin{cases} -\nabla \cdot (\sigma_{D_\delta}\mathcal{D}(\tilde{u}_\delta) + pId) = v_0, & \text{in}\Omega \\ \tilde{u}_\delta = 0, & \text{on}\partial\Omega. \end{cases} \tag{4.57}$$

Putting $\tilde{v}_0 = \tilde{\mu}_0 v_0$, then

$$\begin{cases} -\nabla \cdot (\sigma_D\mathcal{D}(\tilde{v}_0) + pId) = v_0, & \text{in}\Omega \\ \tilde{v}_0 = 0, & \text{on}\partial\Omega. \end{cases} \tag{4.58}$$

From (4.57) and (4.58), we have

$$\begin{aligned}\prec (T_\delta - T)v_0, v_0 \succ &= \prec \tilde{u}_\delta - \tilde{v}_0, v_0 \succ = \widetilde{\mu}_0 \int\limits_\Omega v_0^2 + \int\limits_\Omega v_0 \tilde{u}_\delta dx \\ &= \widetilde{\mu}_0 \int\limits_\Omega (\sigma_D - \sigma_{D_\delta})\mathcal{D}(\tilde{u}_\delta) : \mathcal{D}(v_0)dx \\ &= \widetilde{\mu}_0 \int\limits_{D_\delta \setminus D} (\sigma_0 - \sigma_1)\mathcal{D}(\tilde{u}_\delta^i) : \mathcal{D}(v_0^e)dx - \widetilde{\mu}_0 \int\limits_{D \setminus D_\delta} (\sigma_0 - \sigma_1)\mathcal{D}(\tilde{u}_\delta^e) : \mathcal{D}(v_0^i)dx.\end{aligned}$$

Now, as done in [5] we put $x_t := x + th(x)\nu(x)$ for $x \in \partial D$ and $t \in [0, \delta]$. It is clearly that the Jacobian determinant of the change of variables $(x, t) \in \partial D \times [0, \delta] \mapsto x_t \in D_\delta \triangle D$ is $|h(x)| + O(\delta)$ for δ small enough, and hence we get

$$I_1 := \widetilde{\mu}_0 \int\limits_{D_\delta \setminus D} (\sigma_0 - \sigma_1)\mathcal{D}(\tilde{u}_\delta^i) : \mathcal{D}(v_0^e)dx = \widetilde{\mu}_0 \int\limits_0^\delta \int\limits_{\partial D \cap \{h>0\}} h(x)(\sigma_0$$

$$-\sigma_1)\mathcal{D}(\tilde{u}_\delta^i)(x_t) : \mathcal{D}(v_0^e)(x_t)ds(x)dt + O(\delta^2).$$

Moreover,

$$I_2 := -\widetilde{\mu}_0 \int\limits_{D \setminus D_\delta} (\sigma_0 - \sigma_1)\mathcal{D}(\tilde{u}_\delta^e) : \mathcal{D}(v_0^i)dx = -\widetilde{\mu}_0 \int\limits_0^\delta \int\limits_{\partial D \cap \{h<0\}} h(x)(\sigma_0$$

$$-\sigma_1)\mathcal{D}(\tilde{u}_\delta^e)(x_t) : \mathcal{D}(v_0^i)(x_t)ds(x)dt + O(\delta^2).$$

On the other hand, based on the result of Li and Nirenberg proved in [41], the following gradient estimate holds:

$$\|\tilde{u}_\delta\|_{C^{1,\alpha}(\overline{D}_\delta)} + \|\tilde{u}_\delta\|_{C^{1,\alpha}(\Omega_{d_0/2} \setminus D_\delta)} \leq C[\|\tilde{u}_\delta\|_{L^2(\Omega)} + \|v_0\|_{L^\infty(\Omega_{d_0/2})}], \tag{4.59}$$

where $\tilde{u}_\delta$ is the solution of (4.57), and the positive constant C is independent of δ. Remark that $\|\tilde{u}_\delta\|_{H^1(\Omega)} \leq C\|v_0\|_{L^2(\Omega)} \leq C$, it follows from (4.56) that

$$\|\tilde{u}_\delta\|_{C^{1,\alpha}(\overline{D}_\delta)} + \|\tilde{u}_\delta\|_{C^{1,\alpha}(\Omega_{d_0/2} \setminus D_\delta)} \leq C. \tag{4.60}$$

Now using Lemma 2.2 in [5], and the relations (4.58)-(4.61), one may obtain:

$$\|(T_\delta - T)v_0\|_{L^2(\Omega)} = \|\tilde{u}_\delta - \tilde{v}_0\|_{L^2(\Omega)} \leq C\delta^{\eta+1/2}. \tag{4.61}$$

On the other hand, we use the gradient estimates (4.57) and (4.58) for $\tilde{u}_\delta$ and v_0, to approximate

$$\mathcal{D}(\tilde{u}_\delta^i)(x_t) : \mathcal{D}(v_0^e)(x_t) = \mathcal{D}(\tilde{u}_\delta^i)(x_\delta) : \mathcal{D}(v_0^e)(x_\delta) + O(\delta^\alpha)$$

for δ sufficiently small. To proceed with our proof we may investigate the transmission conditions of v_0 along the interface ∂D. One can easily see from the equation in (4.49) that v_0 satisfies: $v_0^e = v_0^i$ and $(\sigma_0 \mathcal{D}(v_0^e))\nu = (\sigma_1 \mathcal{D}(v_0^i))\nu$ where $v_0^e = v_0|_{\Omega\setminus D}$ and $v_0^i = v_0|_{\overline{D}}$. Let τ be the unit tangential vector field to ∂D, and following the approach used in [5], one can find that

$$(\sigma_1 \mathcal{D}(v_0^i))\tau = (\mathcal{V}_1 \mathcal{D}(v_0^e))\tau \text{ on } \partial D, \tag{4.62}$$

where $\mathcal{V}_1 := 2\mu_0 I_4 + 2(\mu_1 - \mu_0) I_2 \otimes (\tau \otimes \tau)$. Here I_2 is the (2×2) identity matrix, I_4 is the identity 4-tensor and the viscosity constants μ_0, μ_1 are given in (4.48).

Applying the transmission conditions (4.62) for the function $\tilde{u}_\delta$, one may get

$$\mathcal{D}(\tilde{u}_\delta^i)(x_\delta) = \sigma_1^{-1}[(\sigma_1 \mathcal{D}(\tilde{u}_\delta^i)(x_\delta)\tau) \otimes \tau + (\sigma_1 \mathcal{D}(\tilde{u}_\delta^i)(x_\delta)\nu) \otimes \nu] = \sigma_1^{-1}[(\mathcal{V}_1 \mathcal{D}(\tilde{u}_\delta^e)(x_\delta)\tau) \otimes \tau$$

$$+(\sigma_0 \mathcal{D}(\tilde{u}_\delta^e)(x_\delta)\nu) \otimes \nu].$$

Now, regarding Lemma 3.2 in [5], which still applied to Stokes case, on may approximate

$$\mathcal{D}(\tilde{u}_\delta^i)(x_\delta) = \frac{\tilde{\mu}_0}{\sigma_1}[(\mathcal{V}_1 \mathcal{D}(v_0^e)(x)\tau) \otimes \tau + (\sigma_0 \mathcal{D}(v_0^e)(x)\nu) \otimes \nu] + O(\delta^{\frac{\alpha}{2(\alpha+2)}}). \tag{4.63}$$

Hence,

$$I_1 = \delta \widetilde{\mu}_0^2 \int\limits_{\partial D \cap \{h>0\}} h(x) \mathcal{V}[\mathcal{D}(v_0^e)](x) : \mathcal{D}(v_0^e)(x) ds(x) + O(\delta^{1+\frac{\alpha}{2(\alpha+2)}})$$

and

$$I_2 = \delta \widetilde{\mu}_0^2 \int\limits_{\partial D \cap \{h<0\}} h(x) \mathcal{V}[\mathcal{D}(v_0^e)](x) : \mathcal{D}(v_0^e)(x) ds(x) + O(\delta^{1+\frac{\alpha}{2(\alpha+2)}}),$$

where $\alpha > 0$ and $\mathcal{V}[\mathcal{D}(v_0^e)]$ deduced from (4.63). Therefore,

$$\prec (T_\delta - T)v_0, v_0 \succ = I_1 + I_2 = \delta \widetilde{\mu}_0^2 \int\limits_{\partial D} h(x) \mathcal{V}[\mathcal{D}(v_0^e)](x) : \mathcal{D}(v_0^e)(x) ds(x) + O(\delta^{1+\frac{\alpha}{2(\alpha+2)}}),$$

and by considering the above relation, (4.55) and (4.61) we obtain

$$\tilde{\mu}_0 - \tilde{\mu}_\delta = \delta \tilde{\mu}_0^2 \int\limits_{\partial D} h(x) \mathcal{V}[\mathcal{D}(v_0^e)](x) : \mathcal{D}(v_0^e)(x) ds(x) + O(\delta^{1+\beta})$$

with $\beta = \inf(\eta, \frac{\alpha}{2(\alpha+2)})$. To achieve the proof, we recall that

$$\lambda_\delta - \lambda_0 = \lambda_0\lambda_\delta(1/\lambda_0 - 1/\lambda_\delta) = \lambda_0\lambda_\delta(\tilde{\mu}_0 - \tilde{\mu}_\delta)$$

and we take into consideration that $\lambda_0\lambda_\delta = \lambda_0^2 + \delta(\lambda_0\lambda_1) + \delta^2(\lambda_0\lambda_2) + \cdots$.

Now we consider the case of a multiple eigenvalue. Let the operators T and T_δ be defined as in the proof of Theorem 4.3.2. We suppose that $\tilde{\mu}_0$ be a nonzero eigenvalue of T with multiplicity $m \geq 1$. Then, for δ small, T_δ has a set of m eigenvalues $\tilde{\mu}_\delta^j$ (counted according to their multiplicity) such that $\tilde{\mu}_\delta^j \to \tilde{\mu}_0$ for each $j = 1, \cdots, m$ as $\delta \to 0$. Let $\bar{\mu}_\delta = \frac{1}{m}\sum_{j=1}^m \tilde{\mu}_\delta^j$, and let $\{u_{0,1}, u_{0,2}, \cdots, u_{0,m}\}$ is an orthonormal basis for $Ker(T - \tilde{\mu}_0 I)$, then as written in (4.55) there exists a constant C (independent of δ) such that

$$\left|\widetilde{\mu}_0 - \bar{\mu}_\delta - \frac{1}{m}\sum_{j=1}^m \prec (T_\delta - T)u_{0,j}, u_{0,j} \succ \right| \leq C\|(T - T_\delta)u_{0,j}\|^2_{L^2(\Omega)}, \tag{4.64}$$

Moreover, for each $j = 1, \cdots, m$, there is an eigenfunction $u_{\delta,j}$ corresponding to $\tilde{\mu}_\delta^j$ such that $\|u_{\delta,j}\|_{L^2(\Omega)} = 1$, and

$$\|u_{\delta,j} - u_{0,j}\|_{L^2(\Omega)} \leq C\|(T - T_\delta)|_{Ker(T-\tilde{\mu}_0 I)}\|. \tag{4.65}$$

For more details, we can refer to [8, 30].

Suppose now that the eigenvalue λ_0 of the problem (4.49) is with geometric multiplicity m and let $v_{0,j}; \quad j = 1, \cdots, m$ be L^2-orthonormal eigenfunctions corresponding to λ_0. Let λ_δ^j be the eigenvalues of problem (4.52) generated by splitting from λ_0 and let v_δ^j be the associated eigenfunction (normalized with respect to L^2) such that $v_\delta^j \to v_{0,j}$ as $\delta \to 0$. Based on (4.64) and (4.65), one can use similar approaches as in the proof of Theorem 4.3.2 to get the following result.

Corollary 4.3.1. *Suppose that $\Omega \subset \mathbb{R}^2$ with class $C^{1,1}$. Let λ_0 be an eigenvalue, of the problem (4.49), with (geometric) multiplicity $m \geq 1$. Assume that we have (4.51), and the pressure still the same if we have (4.50). Then, there exist m eigenvalues λ_δ^j, $j = 1, \cdots, m$ (repeated according to their multiplicities) such that $\lambda_\delta^j \to \lambda_0$ as $\delta \to 0$ and the following asymptotic expansion holds:*

$$\frac{1}{\lambda_0} - \frac{1}{m}\sum_{j=1}^m \frac{1}{\lambda_\delta^j} = \frac{\delta}{m}\sum_{j=1}^m \int_{\partial D} h(x)\mathcal{V}[\mathcal{D}(v_{0,j}^e)](x) : \mathcal{D}(v_{0,j}^e)(x)ds(x) + O(\delta^{1+\beta}),$$

where β some positif constant and $\mathcal{V}$ given by Theorem 4.3.2.

4.3.4 Asymptotic formula for eigenfunctions and eigenpressures

This section is dedicated to develop asymptotic expansions for both eigenfunctions and eigenpressures associated to (4.4). Here we suppose that we have all hypothesis and conditions of the Section 1 and Section 2. Define the operator:

$$\mathcal{B}_\delta(\lambda)\varphi(x) = (\mathcal{W}(\lambda)\varphi)(\Psi_\delta^{-1})(\Psi_\delta(x)),$$

where $\varphi \in L^2(\partial\Omega)$, Ψ_δ given by Section 2.2, and $\mathcal{W}(\lambda)$ is the operator associated to hydrodynamic double layer potential [**?**, 34, 60]. Then, the following main result holds.

Theorem 4.3.3. *Suppose that we have (4.2) and let λ_0 be a simple eigenvalue of (4.1). Let $\mathcal{A}_\delta$ be the operator defined by (4.10), and $\mathcal{M}_\delta = \mathcal{A}_\delta + \mathcal{B}_\delta$. Let $\mathcal{O}_0$ be a bounded neighborhood of $\overline{\Omega}$ in $\mathbb{R}^3$. Then there exists a constant $\delta_1 > 0$ smaller than δ_0 such that the eigenfunction v_δ corresponding to the eigenvalue, λ_δ, in $(H^1(\Omega_\delta))^3 \cap H(\Omega_\delta)$ can be chosen to depend holomorphically in $(x, \delta) \in \mathcal{O}_0 \times]-\delta_1, \delta_1[$. Moreover this eigenfunction satisfies the following asymptotic formulae*

$$v_\delta(x) = v_0(x) + \sum_{n\geq 1} v_n(x)\delta^n, \tag{4.66}$$

where the function v_0 is the eigenfunction solution of (4.1) associated to λ_0. The terms v_n are computed from the Taylor coefficients of the operator valued function $\mathcal{M}_\delta$ and of those of the function $a(\delta) = (a_{ij}(\delta))_{1\leq i,j\leq 3}$.

Proof. From [34, 46] we deduce that there exist a continuous function $\varphi(t, \delta)$, which is analytic in $\mathbb{R}^2 \times]-\delta_0, \delta_0[$ and such that

$$v_\delta(x) = \mathcal{S}(\lambda_\delta)\varphi + \mathcal{W}(\lambda_\delta)\varphi, \quad x \in \Omega \tag{4.67}$$

solves the eigenvalue problem (4.4). Moreover, the function given by

$$U(\delta)(x) = \mathcal{M}(\lambda_\delta)\varphi(\Psi^{-1}, \delta)$$

satisfies the eigenvalue problem (4.4) in Ω_δ with the boundary conditions: $U(\delta)|_{\partial\Omega_\delta} = 0$. Here, $\mathcal{M}(\lambda_\delta)\varphi(\Psi^{-1}, \delta) = \mathcal{M}_\delta(\lambda)\varphi$.
Now, by (4.67), we deduce that $v_\delta(x) = U(\delta)(x) = \mathcal{M}(\lambda_\delta)\varphi(\Psi^{-1}, \delta)$ is jointly analytic with respect to (x, δ) in $\{\|x - \Psi_{\delta=0}(y)\| \leq z_0\} \times]-\delta_0, \delta_0[$, where z_0 is a positive constant. The function $v_\delta(x)$ is jointly analytic in the variables $(x, \delta) \in \mathcal{O}_0 \times]-\delta_0, \delta_0[$.

We shall now give the asymptotic expansion of the function $v_\delta(x)$ when δ tends to 0. Integral equation (4.7) gives us

$$v_\delta(x) = \int_{\partial\Omega} M(\lambda_\delta, |x - \Psi_\delta(y)|)\varphi(y,\delta)|\nabla\Psi_\delta(y)|d\sigma(y),$$

where M is the kernel of the operator $\mathcal{M}_\delta$. The perturbed eigenvalue λ_δ is in a small neighborhood of λ_0 for small values of δ. Then we have the following Taylor expansion

$$M(\lambda_\delta), |x - \Psi_\delta(y)|)|\nabla\Psi_\delta(y)| = M(\lambda_0, |x - \Psi(y)|)|\nabla\Psi(y)| + \sum_{k\geq 1}\delta^k M_k(x,y),$$

which holds uniformly in $x \in \overline{\mathcal{O}}_0$ and $y \in \partial\Omega$. The analyticity of the function $\varphi(y,\delta)$ with respect to δ immediately gives

$$\varphi(t,\delta) = \varphi_0(y) + \sum_{k\geq 1}\delta^k\varphi_k(y),$$

uniformly in $y \in \partial\Omega$. Substituting the last two asymptotic into (4.68) we find

$$v_\delta(x) = v_{\delta=0}(x) + \sum_{k\geq 1}\delta^k[\sum_{n=1}^{k}\int_{\partial\Omega}\varphi_{k-n}(y)M_n(x,y)d\sigma(y)].$$

The next result provide us with the asymptotic expansion of the eigenpressures.

Corollary 4.3.2. *Suppose that we have all hypothesis of Theorem (4.3.3). Then the eigenpressures p_δ solution of (4.4) have the following uniform asymptotic expansion:*

$$p_\delta(x) = p_0(x) + \sum_{n\geq 1} p_n(x)\delta^n, \tag{4.68}$$

where the function p_0 is the eigenpressure solution of (4.1) associated to λ_0. The terms p_n are computed from the Taylor coefficients λ_n and $v_n = (v_n^1, v_n^2, v_n^3)$ as follows:

$$p_n(.; x_i; .) = \int \Delta v_n^i dx_i + \sum_{k=0}^{n}\lambda_k\int v_{n-k}^i dx_i, \quad \textit{where } i = 1,2,3.$$

Proof. From system (4.4) we have

$$\nabla p_\delta = \Delta v_\delta + \lambda_\delta v_\delta.$$

Hence, we can expand the function p_δ in powers of δ as we have done for λ_δ and v_δ. Moreover, we have:

$$\partial_i p_\delta = \Delta v_\delta^i + \lambda_\delta v_\delta^i, \quad \text{for } i = 1, 2, 3. \tag{4.69}$$

To get the Taylor coefficients p_n, introduced in the formula (4.68), one can insert both asymptotic expansions (4.46) and (4.66) into relation (4.69), and integrate with the convenable variable.

Part III

Calculation of singularities in boundary integral method

CHAPTER 5

Boundary element method - Application to electromagnetism

Introduction

Here, we are interested in solving scattering problem of electromagnetic waves, by the boundary element method. We present the boundary integral method, for the Maxwell's equations. It allows reducing the external problem to a system of integral equations defined on the surface of an object and it takes into account the boundary conditions.
In the next section 5.1, we present mathematical model of scattering problem which includes Boundary conditions on the surface and radiation condition far from an object. We use the Stratton-Chu formulation to give a boundary integral formulation in section 5.1.2, then we write it under the variational formulation with a Leontovich Boundary Condition. The Section 5.2 will be dedicated to recall existence and uniqueness of a solution. In the last section, we discretize the border of the domain using plane triangles and Rao-Wilton-Glisson basis functions.

5.1 Scattering problem

The scattering problem is represented as a system of equations including boundary conditions on the surface of an object and radiation conditions far from an object. Here, we recall the uniqueness theorem and we intoduce an integral formulation of the problem.

5.1.1 Mathematical model

We consider the scattering problem of electromagnetic waves (**E**, **H**) by a perfect conducting body with a complex coating. Scattering waves take place when incident waves bounce off an obstacle in a variety of directions as depicted in figure 5.1. The amount of scattering waves that take place depends on the wavelength k_0 of the incident waves and structure of the object. We refer by Ω_- to the region of space embodying the scatterer, with a Lipschitz-

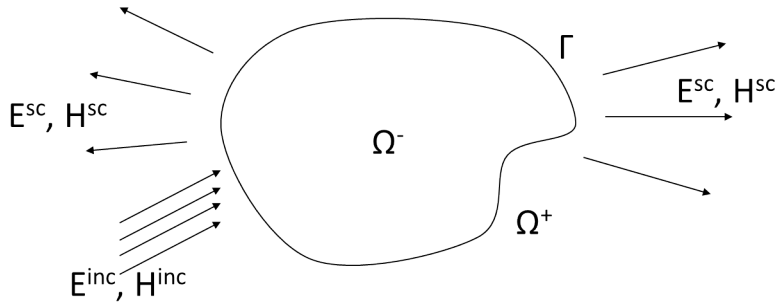

Figure 5.1: Scattering Problem

continuous boundary denoted by Γ, n is the exterior unit normal to Γ, pointing to the exterior. Electromagnetic waves propagate in $\Omega_+ = \mathbb{R}^n \backslash \overline{\Omega_-}$. We define the incident electromagnetic field by $(\mathbf{E}^{inc}, \mathbf{H}^{inc})$, the diffracted or scattered field $(\mathbf{E}^{sc}, \mathbf{H}^{sc})$ and the total electromagnetic fields (**E**, **H**) in Ω^+ as:

$$\begin{cases} \mathbf{E} = \mathbf{E}^{inc} + \mathbf{E}^{sc} \ \text{in}\Omega_+, \\ \mathbf{H} = \mathbf{H}^{inc} + \mathbf{H}^{sc} \ \text{in}\Omega_+. \end{cases} \tag{5.1}$$

Waves propagation medium is characterized by two physical quantities ϵ (electrical permittivity) and μ (magnetic permeability), where $\epsilon = \epsilon_0$ and $\mu = \mu_0$ for free space. The fields inside the coating are governed by a set of equations that take into account the detailed electromagnetic properties of the coating ϵ_r and μ_r.

The determination of the total electromagnetic field (**E**, **H**) induced in Ω_+ by an incident field on the impedance scatterer is governed by the following Maxwell equations:

$$\begin{cases} \mathbf{rot}\mathbf{E} + i\omega\mu\mathbf{H} = 0 \\ \mathbf{rot}\mathbf{H} - i\omega\epsilon\mathbf{E} = 0 \end{cases} \tag{5.2}$$

where ω is the pulsation associated to the time-harmonic electromagnetic fields defined as follows

$$\begin{cases} \mathbf{E}(x,t) = \Re(\mathbf{E}(x)e^{i\omega t}) \\ \mathbf{H}(x,t) = \Re(\mathbf{H}(x)e^{i\omega t}) \end{cases} \tag{5.3}$$

Let Z be the impedance operator. It depends on incident angle, medium thickness d and the physical characteristics ϵ_r and μ_r of the coating material. We consider boundary condition that binds the tangent electric and magnetic fields. The medium characteristics give an impedance at each point of the surface Γ.

$$\mathbf{E}_{tg} - Z(\mathbf{n} \times \mathbf{H}) = 0 \quad on\ \Gamma \tag{5.4}$$

where subscript tg denotes tangent component on the surface Γ defined as:

$$\mathbf{E}_{tg} = \mathbf{n} \times (\mathbf{E} \times \mathbf{n})$$

and $\mathbf{n}$ is the exterior normal vector to the surface.

The boundary condition (5.4) is called impedance boundary condition (IBC). The simplest form of which is known as Leontovich IBC or standard IBC (SIBC), where $Z = constant$ [38,51,52]. We should notice that asymptotic behavior of the fields $(\mathbf{E}, \mathbf{H})$ is depicted by the Silver-Müller radiation condition:

$$\lim_{r\to\infty} r(\mathbf{E} \times \mathbf{n}_r + \mathbf{H}) = 0. \tag{5.5}$$

where $r = |\mathbf{x}|$ and $\mathbf{n}_r = \dfrac{\mathbf{x}}{|\mathbf{x}|}$, $\mathbf{x} \in \mathbb{R}^3$.

So, we have the next problem:

Problem 5.1.1. *Find* $(\boldsymbol{E}, \boldsymbol{H})$ *such that*

$$\begin{cases} \mathbf{rot}\boldsymbol{E} + ik_0\mu\boldsymbol{H} = 0 \ \ in\ \Omega^+ \\ \mathbf{rot}\boldsymbol{H} - ik_0\epsilon\boldsymbol{E} = 0 \ \ in\ \Omega^+ \\ \boldsymbol{E}_{tg} - Z(\boldsymbol{n} \times \boldsymbol{H}) = 0 \ \ on\ \Gamma \\ \lim_{r\to\infty} r(\boldsymbol{E} \times \boldsymbol{n}_r + \boldsymbol{H}) = 0 \end{cases} \tag{5.6}$$

5.1.2 Integral formulation

The boundary integral method places their unknowns on Γ the boundary of Ω. It consists in expressing the electromagnetic field as a function of potentials defined on the boundary.
We will use the boundary integral representation to solve problem 5.1.1. We express the electromagnetic field as a function of potentials defined on Γ. The Stratton-Chu formulation (Appendix C eq. (C.30)) helps us to characterize the electromagnetic fields in terms of surface current densities. These current densities are uknowns in the integral formulation of the problem. We introduce current densities $\mathbf{J}$ and $\mathbf{M}$ on the boundary Γ as follows

$$\mathbf{M} = [\mathbf{E} \times \mathbf{n}]^+_- \ ; \ \mathbf{J} = [\mathbf{n} \times \mathbf{H}]^+_-,$$

where $[\,]^+_-$ denotes difference between upper (+) and lower (-) values of interface.
We introduce the variational form of the operators $(B - S)$, $(P + Q)$ and I as follows:

$$\langle (B-S)\mathbf{A}, \boldsymbol{\psi} \rangle = i \iint_\Gamma kG\mathbf{A} \cdot \boldsymbol{\psi} - \frac{1}{k} G \nabla_y \cdot \mathbf{A} \nabla_x \cdot \boldsymbol{\psi} dy dx \tag{5.7}$$

$$\langle (P+Q)\mathbf{A}, \boldsymbol{\psi} \rangle = \frac{1}{2} \int_\Gamma \boldsymbol{\psi} \cdot (\mathbf{n} \times \mathbf{A}) dx + \iint_\Gamma (\boldsymbol{\psi} \times \mathbf{A}) \cdot \nabla_x G dy dx \tag{5.8}$$

$$\langle I\mathbf{A}, \boldsymbol{\psi} \rangle = \int_\Gamma \mathbf{A} \cdot \boldsymbol{\psi} dx \tag{5.9}$$

where G is the Green kernel $G(x,y) = \frac{e^{-\imath k|x-y|}}{4\pi|x-y|}$.
Using the Stratton-Chu formulae, we can give a variationnal formulation of (5.6):
Find $(\mathbf{J}, \mathbf{M})$ such that:

$$< Z_0(B-S)\mathbf{J}, \mathbf{\Psi}_J > + < (P+Q)\mathbf{M}, \mathbf{\Psi}_J > = < E^{inc}, \mathbf{\Psi}_J >, \ \forall \mathbf{\Psi}_J, \tag{5.10}$$

$$- < (P+Q)\mathbf{J}, \mathbf{\Psi}_M > + < \frac{1}{Z_0}(B-S)\mathbf{M}, \mathbf{\Psi}_M > = < H^{inc}, \mathbf{\Psi}_M >, \ \forall \mathbf{\Psi}_M. \tag{5.11}$$

For more details see Appendix C. Note however that these two equations are completely equivalent.

5.1.3 Variational formulation

In order to insure a unique solution to the boundary value problem it is necessary to applay boundary condition. We recall impedance boundary condition (IBC) that relates current

densities $\mathbf{J}$ and $\mathbf{M}$

$$\mathbf{n} \times \mathbf{M} = Z\mathbf{J}. \tag{5.12}$$

The simplest IBC is **Leontovich IBC**, $Z = const$. Usually, it is taken for incident wave perpendicular to plane

$$a_0 = z_0\sqrt{\frac{\mu_r}{\epsilon_r}}\tan\left(\sqrt{\mu_r\epsilon_r}k_0 d\right) \quad (LIBC)$$

The impedance boundary condition (5.12) can be written

$$\mathbf{n} \times \mathbf{M} = a_0\mathbf{J}. \tag{5.13}$$

We multiply three dimensional LIBC (5.13) by test functions and integrate on the surface Γ, that gives us a weak form of the boundary condition. So, we take $\mathbf{\Psi}_J$ and $\mathbf{n} \times \mathbf{\Psi}_M$ as test functions to get the weak forms for (5.10) and (5.11), respectively

$$\int_\Gamma (\mathbf{n} \times \mathbf{M}) \cdot \mathbf{\Psi}_J ds = \int_\Gamma a_0\mathbf{J} \cdot \mathbf{\Psi}_J ds, \tag{5.14}$$

$$\int_\Gamma (\mathbf{n} \times \mathbf{M}) \cdot (\mathbf{n} \times \mathbf{\Psi}_M) ds = \int_\Gamma a_0\mathbf{J} \cdot (\mathbf{n} \times \mathbf{\Psi}_M) ds \tag{5.15}$$

We insert the LIBC into (5.10) through (5.14), by replacing the operator P that was defined in (5.8) then we obtain:

$$< P\mathbf{M}, \mathbf{\Psi}_J >= \frac{1}{2}\int_\Gamma \mathbf{\Psi}_J \cdot (\mathbf{n} \times \mathbf{M}) ds = \frac{a_0}{2}\int_\Gamma \mathbf{J} \cdot \mathbf{\Psi}_J ds$$

Using the formula of vector analysis

$$\mathbf{\Psi}_M \cdot (\mathbf{n} \times \mathbf{J}) = -\mathbf{J} \cdot (\mathbf{n} \times \mathbf{\Psi}_M),$$

we put weak form of LIBC (5.15) into 5.11 by replacing $< P\mathbf{J}, \mathbf{\Psi}_M >$. We get:

$$< P\mathbf{J}, \mathbf{\Psi}_M >= \frac{1}{2}\int_\Gamma \mathbf{\Psi}_M \cdot (\mathbf{n} \times \mathbf{J}) ds = -\frac{1}{2}\int_\Gamma (\mathbf{n} \times \mathbf{\Psi}_M) \cdot \mathbf{J} ds$$

$$= \frac{-1}{2a_0}\int_\Gamma (\mathbf{n} \times \mathbf{M}) \cdot (\mathbf{n} \times \mathbf{\Psi}_M) ds = \frac{-1}{2a_0}\int_\Gamma \mathbf{M} \cdot \mathbf{\Psi}_M ds \tag{5.16}$$

Now we define bilinear operator

$$A(U, \Psi) =< Z_0(B - S)\mathbf{J}, \mathbf{\Psi}_J > + \frac{1}{Z_0} < (B - S)\mathbf{M}, \mathbf{\Psi}_M >$$

$$+ < Q\mathbf{M}, \mathbf{\Psi}_J > - < Q\mathbf{J}, \mathbf{\Psi}_M > + \frac{a_0}{2} < \mathbf{J}, \mathbf{\Psi}_J > + \frac{1}{2a_0} < \mathbf{M}, \mathbf{\Psi}_M >$$

that combines integral equations (5.10)-(5.11), where operator P parts are replaced.

Finally, we introduce the following problem:

Problem 5.1.2. *Find* $U = (\mathbf{J}, \mathbf{M})$ *such that*

$$A(U, \Psi) = F(\Psi) \tag{5.17}$$

for all $\Psi = (\mathbf{\Psi}_J, \mathbf{\Psi}_M)$. *Where*

$$F(\Psi) = \int_\Gamma \mathbf{E}^{inc} \cdot \mathbf{\Psi}_J ds + \int_\Gamma \mathbf{H}^{inc} \cdot \mathbf{\Psi}_M ds.$$

5.2 Existence and uniqueness theorem

According to theorem C.1.4 from Appendix, there exists a unique solution of the problem (5.1.2), if the bilinear operator $A(U, \Psi)$ verifies the continuity and coercivity conditions.
For the sake of simplicity, we consider the operator $A(U, \Psi)$ as a sum of three bilinear operator

$$A(U, \Psi) = < Z_0(B - S)\mathbf{J}, \mathbf{\Psi}_J > + \frac{1}{Z_0} < (B - S)\mathbf{M}, \mathbf{\Psi}_M >$$

$$+ < Q\mathbf{M}, \mathbf{\Psi}_J > - < Q\mathbf{J}, \mathbf{\Psi}_M > + \frac{a_0}{2} < \mathbf{J}, \mathbf{\Psi}_J > + \frac{1}{2a_0} < \mathbf{M}, \mathbf{\Psi}_M >$$

Let us prove the continuity of $A(U, \Psi)$.

5.2.1 Continuity of the operator A

Lemma 5.2.1. *The bilinear operator* $A(U, \Psi)$ *is continuous on* V *for all* $\Psi \in V$.

Proof. : We have to show that there exists $C > 0$ such that for all $\Psi \in V$

$$|A(U, \Psi)| \leq C \|U\|_V \|\Psi\|_V$$

According to theorems (C.1.1), (C.1.2) and triangle inequality (property of a norm) we have

$$|A(U, \Psi)| \leq | < Z_0(B - S)\mathbf{J}, \mathbf{\Psi}_J > | + |Z_0^{-1}| \, | < (B - S)\mathbf{M}, \mathbf{\Psi}_M > | + | < Q\mathbf{M}, \mathbf{\Psi}_J > |$$

$$+|<Q\mathbf{J},\mathbf{\Psi}_M>|+\frac{|a_0|}{2}|<\mathbf{J},\mathbf{\Psi}_J>|+\frac{1}{2|a_0|}|<\mathbf{M},\mathbf{\Psi}_M>|\leq$$

$$\leq \|Z_0(B-S)\mathbf{J}\|_{-1/2,\mathrm{rot}_\Gamma}\|\mathbf{\Psi}_J\|_{-1/2,\mathrm{div}_\Gamma}$$

$$+|Z_0^{-1}|\|(B-S)\mathbf{M}\|_{-1/2,\mathrm{rot}_\Gamma}\|\mathbf{\Psi}_M\|_{-1/2,\mathrm{div}_\Gamma}+\|Q\mathbf{M}\|_{-1/2,\mathrm{rot}_\Gamma}\|\mathbf{\Psi}_J\|_{-1/2,\mathrm{div}_\Gamma}$$

$$+\|Q\mathbf{J}\|_{-1/2,\mathrm{rot}_\Gamma}\|\mathbf{\Psi}_M\|_{-1/2,\mathrm{div}_\Gamma}+\frac{a_0}{2}\|\mathbf{J}\|_{L^2(\Gamma)}\|\mathbf{\Psi}_J\|_{L^2(\Gamma)}+\frac{1}{2a_0}\|\mathbf{M}\|_{L^2(\Gamma)}\|\mathbf{\Psi}_M\|_{L^2(\Gamma)}$$

$$\leq C_1\|U\|_V\|\Psi\|_V$$

□

5.2.2 Coercivity of the operator A

Lemma 5.2.2. *Bilinear form $A(U,\Psi)$ verifies coercivity inequality for all $U\in V=[H^{-1/2}(div,\Gamma)\cap L^2(\Gamma)]^4$.*

Proof. : We have to show that there exist $\alpha>0$ such that

$$\Re[A(U,U^*)]\geq\alpha\|U\|_V^2-C\|U\|_H^2,\ \forall U\in V.$$

From [38], we know that there exists α_1 such that

$$\Re(A)=\Re(<Z_0(B-S)\mathbf{J},\mathbf{J}^*>)+\Re(<Z_0^{-1}(B-S)\mathbf{M},\mathbf{M}^*>)+\Re(<Q\mathbf{M},\mathbf{J}^*>)$$

$$-\Re(<Q\mathbf{J},\mathbf{M}^*>)+\Re(\frac{a_0}{2}\int_\Gamma\mathbf{J}\cdot\mathbf{J}^*ds)+\Re(\frac{1}{2a_0}\int_\Gamma\mathbf{M}\cdot\mathbf{M}^*ds)\geq$$

$$\geq\alpha\left(\|\mathbf{J}\|^2_{-1/2,\mathrm{div}_\Gamma}+\|\mathbf{M}\|^2_{-1/2,\mathrm{div}_\Gamma}\right)+\frac{\Re(a_0)}{2}\|\mathbf{J}\|^2_{L^2(\Gamma)}+\frac{\Re(a_0)}{2|a_0|^2}\|\mathbf{M}\|^2_{L^2(\Gamma)}$$

□

5.3 Discretization

We recall bilinear form that was formulated in the previous chapter. So we solve the problem

$$A(U,\Psi)=<\mathbf{E}^{inc},\mathbf{\Psi}_J>+<\mathbf{H}^{inc},\mathbf{\Psi}_M> \tag{5.18}$$

where

$$A(U,\Psi) =< Z_0(B-S)\mathbf{J}, \mathbf{\Psi}_J > + Z_0^{-1} < (B-S)\mathbf{M}, \mathbf{\Psi}_M >$$
$$+ < Q\mathbf{M}, \mathbf{\Psi}_J > - < Q\mathbf{J}, \mathbf{\Psi}_M > + \frac{a_0}{2} < \mathbf{J}, \mathbf{\Psi}_J > + \frac{1}{2a_0} < \mathbf{M}, \mathbf{\Psi}_M >$$

We assume that the initial boundary problem is defined in a 3-D domain. The boundary Γ of the domain is therefore a surface and can be discretized using triangles.

5.3.1 Triangular mesh element

The first step is to approach the surface of the obstacle by a surface Γ_h composed of finite number of two dimensional elements. These elements are triangular facets denoted by T_i for $i = 1$ to N_T:

$$\Gamma_h = \bigcup_{i=1}^{N_T} T_i.$$

We will call it an initial mesh (or original mesh).We denote by N_e the total number of edges of the mesh component Γ_h. The discretization of unknowns $\mathbf{J}$ and $\mathbf{M}$ should verify a condition-flow conservation of these currents. One way to ensure this is to use the basis functions of Rao-Wilton-Glisson introduced below.

5.3.2 Rao-Wilton-Glisson basis functions

We introduce local numbering of a triangle T. The vertices $(a_j^T)_{j=1,3}$ are arranged in clock-wise order. Triangle edges are numbered so that the edge T_j' connects vertices a_j^T and a_{j+1}^T.

Moreover, we give an orientation $\boldsymbol{\nu}_n$ to each edge n. Consider the two triangles sharing this edge. We note T_n^+ the triangle so that the direction of the edge n coincides with the forward direction (locally defined) of this triangle. For the other triangle, which we will denote T_n^-, the direction of the edge coincides with the indirect sense.

Each RWG basis function is associated with an edge and ensures the conservation of flux through this edge. If we denote $|T|$ the area of a triangle T, the n^{th} basis function is defined as follows:

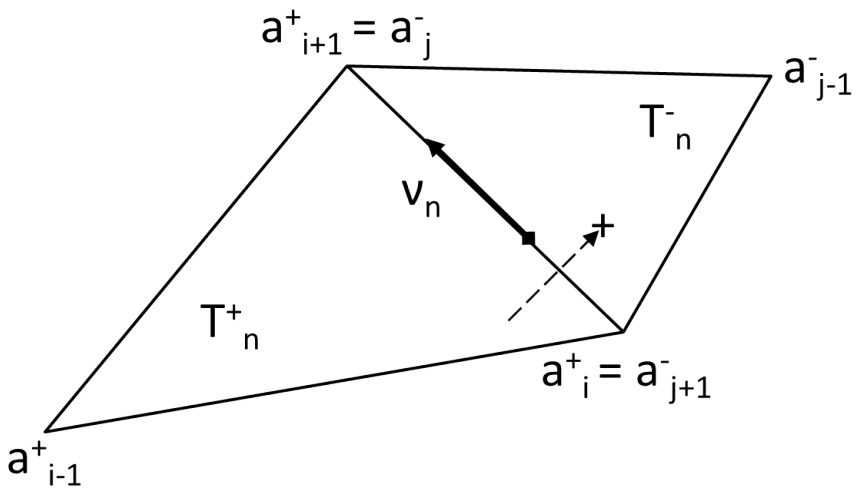

Figure 5.2: Triangles T_n^+ and T_n^- adjacent to edge n

Definition 5.3.1. *If n is the i^{th} local edge of triangle T_n^+ and the j^{th} of triangle T_n^- then:*

$$\mathbf{f}_n(x) = \begin{cases} \frac{l_n}{2|T_n^+|}(x - a_{i-1}^+) \;\; if \; x \in T_n^+ \\ \frac{l_n}{2|T_n^-|}(a_{j-1}^- - x) \;\; if \; x \in T_n^- \\ 0 \quad if \; x \notin T_n^+ \cup T_n^- \end{cases} \tag{5.19}$$

The density is proportional to $\mathrm{div}_\Gamma \mathbf{f}_n$*, where*

$$\mathrm{div}_\Gamma \mathbf{f}_n(x) = \begin{cases} +\frac{l_n}{|T_n^+|} \;\; on \; T_n^+ \\ -\frac{l_n}{|T_n^-|} \;\; on \; T_n^- \\ 0 \quad elsewhere. \end{cases} \tag{5.20}$$

5.3.3 Discrete problem

We decompose the electric and magnetic currents in the following form:

$$\mathbf{J}(y) = \sum_{i=1}^{N_e} J_i \mathbf{f}_i(y), \;\; \mathbf{M}(y) = \sum_{i=1}^{N_e} M_i \mathbf{f}_i(y).$$

Considering to these assumptions we are looking for approximated solution of the problem (5.18). We introduce the following discrete system

$$A^h(U_h, \Psi_h) = \sum_{i=1}^{N_e} < \mathbf{E}^{inc}, \mathbf{f}_i > + \sum_{i=1}^{N_e} < \mathbf{H}^{inc}, \mathbf{f}_i >, \tag{5.21}$$

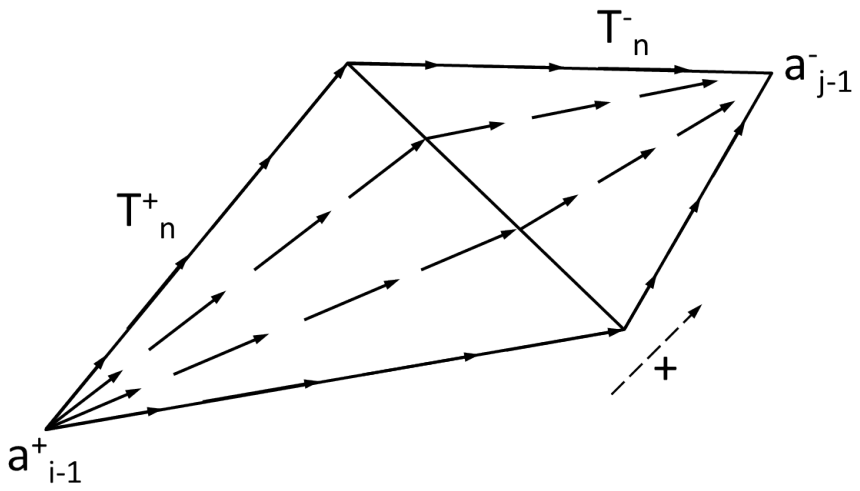

Figure 5.3: Rao-Wilton-Glisson basis function on triangle elements associated to the edge n

where

$$A^h(U_h,\Psi_h)=\sum_{i,j=1}^{N_e}<Z_0(B-S)\mathbf{f}_j,\mathbf{f}_i>J_j+Z_0^{-1}\sum_{i,j=1}^{N_e}<(B-S)\mathbf{f}_j,\mathbf{f}_i>M_j$$

$$+\sum_{i,j=1}^{N_e}<Q\mathbf{f}_j,\mathbf{f}_i>M_j-\sum_{i,j=1}^{N_e}<Q\mathbf{f}_j,\mathbf{f}_i>J_j+\frac{a_0}{2}\sum_{i,j=1}^{N_e}<\mathbf{f}_j,\mathbf{f}_i>J_j+\frac{1}{2a_0}\sum_{i,j=1}^{N_e}<\mathbf{f}_j,\mathbf{f}_i>M_j.$$

Since the basis function $\mathbf{f}_i$ is defined on two triangles, namely adjacent to the i^{th} edge , T_i^+ and T_i^-. The same for the basis function $\mathbf{f}_j$, they are defined on two triangles, T_j^+ and T_j^-. We define $[(B-S)]$ matrix elements

$$(B-S)_{i,j}=i\int\limits_{T_i^+\cup T_i^-}\int\limits_{T_j^+\cup T_j^-}kG(x,y)\mathbf{f}_j(y)\cdot\mathbf{f}_i(x)-\frac{1}{k}G(x,y)(\mathrm{div}_\Gamma\mathbf{f}_i(x))(\mathrm{div}_\Gamma\mathbf{f}_j(y))dxdy.$$

Analogically

$$Q_{i,j}=-i\int\limits_{T_i^+\cup T_i^-}\int\limits_{T_j^+\cup T_j^-}[\mathbf{f}_i(x)\times\mathbf{f}_j(y)]\cdot\nabla_\Gamma G(x,y)dydx.$$

For lonely integrals we have

$$I_{i,j}=\frac{a_0}{2}\sum_{t=1}^{N_{tr}}J_j\int\limits_{T_t}\mathbf{f}_j\cdot\mathbf{f}_i dx=\int\limits_{T_i^+\cup T_i^-}\mathbf{f}_j\cdot\mathbf{f}_i dx.$$

In the following, we will detail the evaluation of the double layer operator $Q_{i,j}$. The odher terms was evaluated in previous works.

5.4 Double layer operator

We denote by S the triangle T_i^+ and by T the triangle T_j^+. We detail the evaluation of $Q_{i,j}$ on $S \times T$ and it will be the same for other combinations of triangles. On each triangle, the current is written as a linear combination of three basis functions associated to three edges of a triangle.

$$\mathcal{K}_{i,j} = \iint_{S\times T} \mathbf{f}_i(x) \cdot \nabla_y G(x,y) \times \mathbf{f}_j(y)\, dx\, dy;\ 1 \le i,j \le 3$$

$$= \frac{1}{4|S||T|} \int_{S\times T} (x-a_i) \cdot \nabla_y G(x,y) \times (y-b_j)\, dx\, dy \tag{5.22}$$

where

$$G(x,y) = -\frac{e^{\iota k||x-y||}}{4\pi||x-y||},$$

and

$$\nabla_y G(x,y) = -\frac{(x-y)}{4\pi r^3} - \frac{k^2(x-y)}{8\pi r} - \frac{\iota k^3(x-y)}{8\pi} - \frac{(x-y)(1-\iota kr)}{4\pi} \Sigma_{n=0}^{\infty} \frac{(\iota k)^{n+3} r^n}{(n+3)!} \tag{5.23}$$

Let define the integral $\mathcal{R}_{i,j}$

$$\mathcal{R}_{i,j} = \int_{S\times T} (x-a_i) \cdot R(x,y) \times (y-b_j)\, dx\, dy;\ \ i,j = 1,2,3 \tag{5.24}$$

where

$$R(x,y) = -\frac{\iota k^3(x-y)}{8\pi} - \frac{(x-y)(1-\iota kr)}{4\pi} \Sigma_{n=0}^{\infty} \frac{(\iota k)^{n+3} r^n}{(n+3)!} \tag{5.25}$$

and

$$\mathcal{S}_{i,j} = \int_{S\times T} (x-a_i) \cdot \frac{(x-y)}{||x-y||^{\xi+1}} \times (y-b_j) dx dy;\ i,j = 1,2,3;\ \xi = 0,2 \tag{5.26}$$

The regular integral $\mathcal{R}_{i,j}$ can be calculated numerically with the usual known methods such as the Gauss method. In the following chapters, we propose a method for the analytical evaluation of the singular term.

Conclusion

We have established a system of equations (5.6) describing the scattering problem of a coated object with the boundary condition on Γ. We have introduced the Leontovich impedance boundary conditions (LIBC). Radiation conditions helped us to establish the uniqueness of solution. The Stratton-Chu formulation indicates how the volume problem reduced to the problem on a surface of an object. This constitutes the main concept of integral method. We set 3D LIBC (5.13) into EFIE and MFIE (5.10 - 5.11) and we introduce the problem (5.1.2). We discretized the surface of a three dimensional object by triangulation. We approximated the unknowns in terms of RWG basis functions. The singular integrals which intervene there, will be evaluated in the following parts.

CHAPTER 6

Presentation of the reduction method

Introduction

As we saw in the previous chapter, we must evaluate singular integrals derived from a double layer operator, arising from the boundary element method applied to resolve a scattering problem. There was some attempt to evaluate it with purely numerical methods but the result was not very accurate. The aim is to present a new approach for the explicit and precise calculation. We propose a method that does not require any numerical integration for the evaluation of singular integrals. It is based on a recursive reduction of an m-dimensional integral into a linear combination of (m-1)-dimensional integrals. It leads to a linear combination of 1-dimensional regular integrals with factors depending only on geometric quantities. These integrals can be evaluated either numerically or explicitly. This method makes it possible to obtain explicit formulae to evaluate singular integrals with a very high precision. It was introduced by M. Lenoir in 2006, see [39] to calculate a singular integral in the form of a single layer operator.

6.1 Presentation of the method

6.1.1 Integration of homogeneous functions

The reduction process is based on several simplification formulae presented in the next section. These formulae reduce the dimension of the integration domain. It can be used when

the integrand is homogeneous, or positively homogeneous.

Definition 6.1.1. *A function is said to be homogeneous of degree q or q-homogeneous, with respect to the variable $z \in \mathbb{R}^n$, if*

$$f(\lambda z) = \lambda^q f(z);\ \forall \lambda \in \mathbb{R};\ \lambda \neq 0. \tag{6.1}$$

Definition 6.1.2. *A function is said to be positively homogeneous of degree q, with respect to the variable $z \in \mathbb{R}^n$, if*

$$f(\lambda z) = \lambda^q f(z);\ \forall \lambda \in \mathbb{R};\ \lambda > 0. \tag{6.2}$$

Definition 6.1.3. *A function is said to be q-homogeneous with parameter if it is an homogeneous function with respect to a pair of variables (z, h), where z is the integration variable and h is a real parameter*

$$f(\lambda z, \lambda h) = \lambda^q f(z, h) \tag{6.3}$$

Theorem 6.1.1. *[39] Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be an homogeneous function of degree q with $q + n \neq 0$. Then,*

$$(q+n) \int_{\Omega} f(z)dz = \int_{\partial\Omega} (z|\vec{\nu}) f(z)\partial z \tag{6.4}$$

where $\vec{\nu}$ is the exterior normal to Ω, ∂z is the surface element on $\partial\Omega$, and $(z|\vec{\nu})$ is the inner scalar product.

Theorem 6.1.2. *[39] Let $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ be an homogeneous function of degree q referred to as an homogeneous function with parameter h.*

$$\int_{\Omega} f(z,h)dz = h^{q+n} \int_{\partial\Omega} (z|\vec{\nu}) \int_{h}^{+\infty} \frac{g(z,u)}{u^{q+n+1}} du\partial z \tag{6.5}$$

Theorem 6.1.3. *[39] Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ a positively homogeneous function of degree q with $q = -n$. We introduce an arbitrary homogeneous positive function $\theta : \Omega \to \mathbf{R}$ of degree $p > 0$. Then we have*

$$\int_{\Omega} f(z)dz = \frac{1}{p} \int_{\partial\Omega} (\overrightarrow{z}|\overrightarrow{\nu}) f(z) \ln(\theta(z)) ds_z \tag{6.6}$$

These formulae will allow us to reduce the dimension of the integration domain.

6.1.2 Reduction process

Consider the integral

$$\mathcal{I} = \int_{S\times T} f(z)\, dz,$$

where f is an homogeneous function of degree q, $z = (x, y)$ with $x \in S$ and $y \in T$ (see

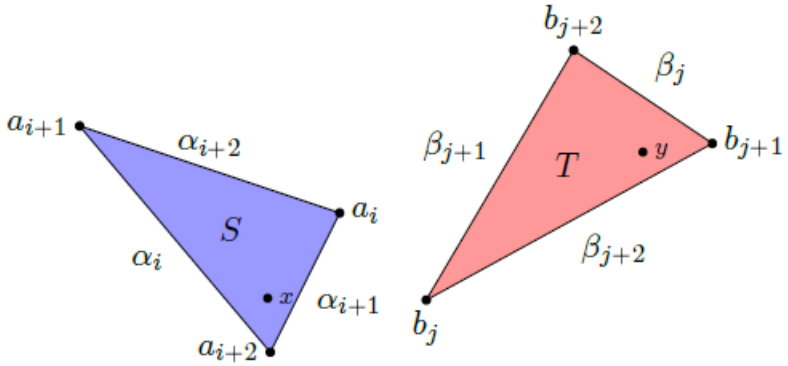

Figure 6.1: Integration domain

Figure (6.1)).

The initial 4-D integral is reduced to a linear combination of 3-D integrals of two types: integrals on domains of type $S \times \beta$ and others on domains $\alpha \times T$. The evolution of the integration domain during the reduction of $\alpha \times T$ is presented in Figure (6.3). The integral

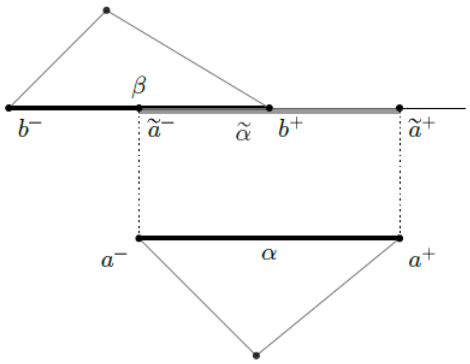

Figure 6.2: Projection of α on the support of β.

on $S \times \beta$ is reduced in a similar way. A second application of the formulae makes it possible to reduce this integral to two types of 2-D integrals: integrals on domains $S \times \beta$, with β a

vertex of the triangle T, and integrals on $\alpha \times T$. The last reduction allows to reduce to one dimensional integrals. The integral on $S \times \beta$ is reduced to a linear combination of integrals on $\alpha \times \beta$. The reduction of the integral on $\alpha \times \beta$ depends on the geometric configuration of the supports of the two segments. When their supports are intersecting, we obtain integrals on $a \times \beta$ and on $\alpha \times b$. When the supports of the segments are parallel (see Figure 6.2), we note $\widetilde{\alpha}$ and $\widetilde{a}$, the respective projections of α and a on the support of β. The integral on $\alpha \times \beta$ then reduces to integrals on $\widetilde{\alpha} \times b$ and integrals on $\widetilde{a} \times \beta$. All 1-D integrals are regular as we will see in the next chapters.

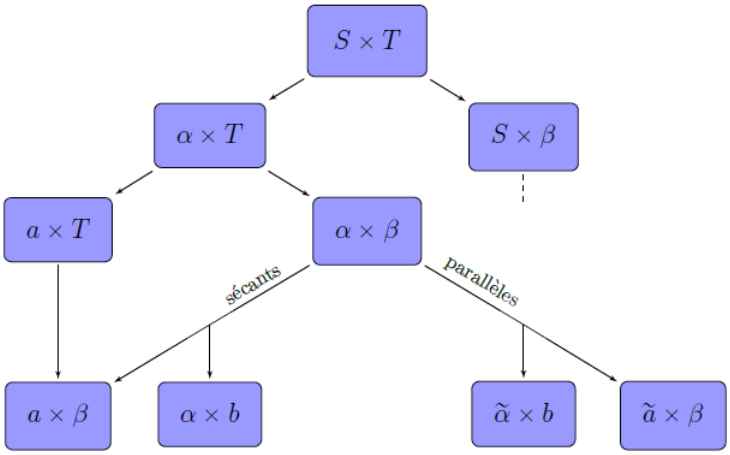

Figure 6.3: Evolution of the integration domain during the reduction process on the product of two coplanar triangles S and T.

6.1.3 Parametrization and flat polygons

It is important to note that the homogeneity of the integrand depends on the parameterization and the origin chosen in the computation of the integral. We will start by presenting a small calculation to detail the importance of the choice of the origin.
Let α and β be two segments of the plane (see Figure 6.4), we calculate the integral

$$\int_{\alpha\times\beta} \frac{x-y}{||x-y||} ds_x ds_y \tag{6.7}$$

with $x \in \alpha$ and $y \in \beta$.
An infinity of parametrizations is possible. But, in practice, only the use of the intersection

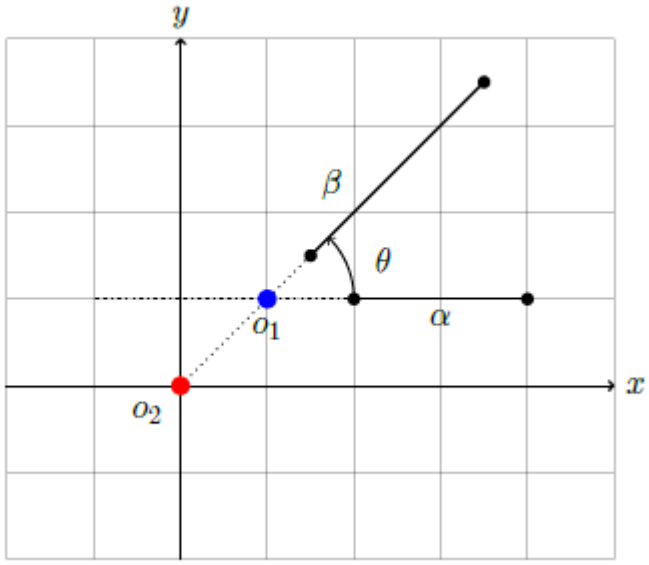

Figure 6.4: Influence of the origin on the homogeneity

of the supports of the two segments as origin ensures the homogeneity of the integer in (6.7).

• If we chose o_1 as origin in Figure (6.4), we obtain the following parametrization in terms of (s,t):

$$\begin{cases} x_1(s) = & s\,, \\ x_2(s) = & 0\,, \\ y_1(t) = & t\cos(\theta)\,, \\ y_2(t) = & t\sin(\theta). \end{cases}$$

which gives

$$\int_s \int_t \frac{s - t\cos(\theta) + t\sin(\theta)}{\sqrt{(s - t\cos(\theta))^2 + (t\sin(\theta))^2}} ds_x ds_y \tag{6.8}$$

where θ is a fixed angle. The integrand is homogeneous with respect to the pair of variables (s,t).

• If we chose o_2 as an origin, we notice that there is a constant that appears in the parameterization and we obtain

$$\begin{cases} x_1(s) = & s\,, \\ x_2(s) = & 1\,, \\ y_1(t) = & t\cos(\theta), \\ y_2(t) = & t\sin(\theta). \end{cases}$$

Then

$$\int_s \int_t \frac{s - t\cos(\theta) - 1 + t\sin(\theta)}{\sqrt{(s - t\cos(\theta))^2 + (1 - t\sin(\theta))^2}} ds_x ds_y \tag{6.9}$$

The integrand is no longer homogeneous.

We understand that it will therefore be very important during the process of reducing to carefully choose the origin in order to maintain the homogeneous character of the integrand. We will also see that a judicious choice of the origin makes it possible to simplify the calculations.

The reducing process of the integration's domain dimension, requires the use of flat polygons for discretization. In the contrary case, after a first step of reducing the dimension, the integrand is no more homogeneous. therefore, it is no longer possible to use formulae (6.4), (6.5) or (6.6) to continue reducing the dimension of the integration domain until one-dimensional integrals are obtained.
Indeed, the use of flat polygons ensures that the product of the two flat polygons is a polyhedron. So the boundary $\partial S \times T$ will be composed of hyperplanes on which the outgoing normal is constant.
Therefore, the scalar product present in formulae (6.4), (6.5) and (6.6) is piecewise constant and it will be possible to get the scalar product out of the integral, to reduce to an homogeneous integrand and thus repeat the process.

6.2 Geometric tools and Notation

We present some geometric notations needed for the calculations.

6.2.1 The mesh elements

Let the triangles S and T be defined (respectively) by their vertices a_i and b_j, $i, j = 1, 2, 3$ and its sides α_i respectively β_j, $i, j = 1, 2, 3$ such as α_i and β_j are opposed to a_i respectively b_j (see Figure 6.5). We introduce $|\alpha_i|$ and $|\beta_j|$ the length of the side α_i respectively β_j. Let $\vec{\alpha_i}$ and $\vec{\beta_j}$ be the vectors $(a_{i+2} - a_{i+1}) = (a_i^+ - a_i^-)$ respectively $(b_{i+2} - b_{i+1}) = (b_i^+ - b_i^-)$ (see Figure 6.6). We denote by λ_i and ϑ_j the outside normal vector to α_i respectively β_j.

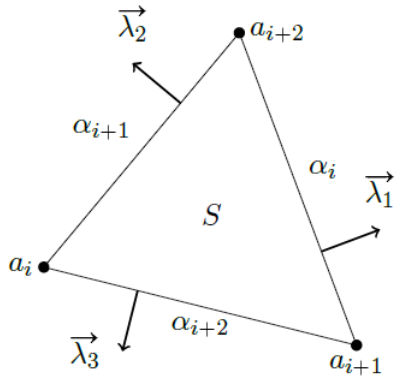

Figure 6.5: Definition of Triangle S

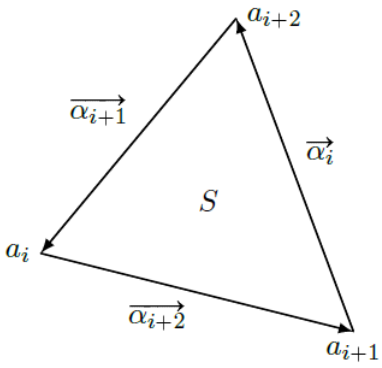

Figure 6.6: Orientation of the triangle S

6.2.2 Calculation of the projection

Note $p_i(x)$ the orthogonal projection (see Figure 6.7) of a point x on the side α_i of the triangle S and $g_i(x) = ||x - p_i(x)||$ the distance from x to the support of α_i. When $x = a_i$, we note that $p_i = p_i(a_i)$ and that $g_i = g_i(a_i)$ is the length of the height (see Figure 6.9). We will also use the signed distance from a point to the support of α_i which involves the outgoing normal to the triangle on α_i. This distance is defined by $\gamma_i = (p_i(x) - x|\vec{\lambda_i})$ (see Figure 6.9). We

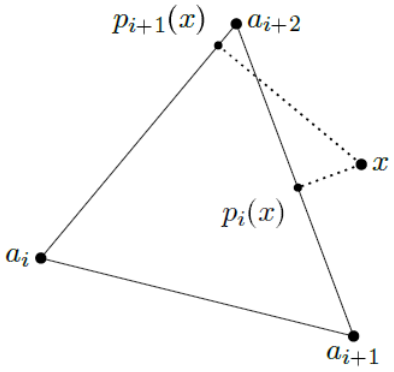

Figure 6.7: Projection

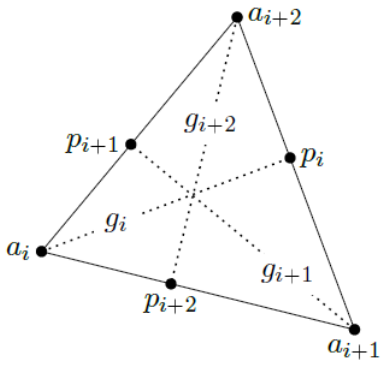

Figure 6.8: Projections and distances

use notations for the triangle T similar to those introduced for the triangle S. The orthogonal projection of y on the support of β_j, similar to $p_i(x)$, is denoted by $q_j(y)$ and the distance from y to the support of β_j is denoted by $d_j(y) = ||y - q_j(y)||$. We define the signed distance $\delta_j(y)$ from y to the support of β_j by $\delta_j(y) = (q_j(y) - y|\vec{\vartheta_j})$. Finally, $d_j = d_j(b_j)$ is the equivalent of g_i, namely the length of the height from b_j to β_j. (see Figure 6.9).

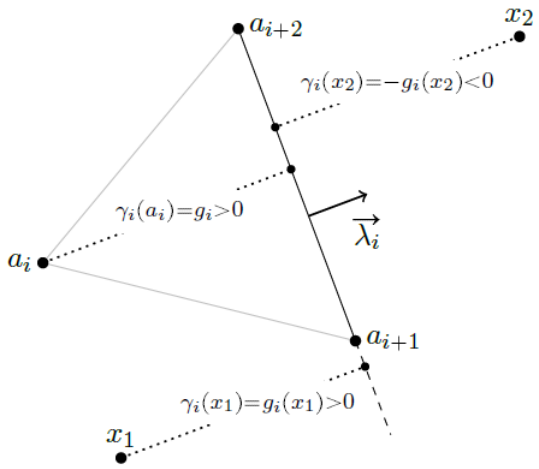

Figure 6.9: Distances signees

6.2.3 Calculation of abscissas

On side α (resp β) the abscissa s (resp t) is defined with respect to an origin o_α (resp o_β) and a unit vector $\frac{\vec{\alpha}}{|\alpha|}$ (resp $\frac{\vec{\beta}}{|\beta|}$). The abscissas of the ends $a^\pm$ and $b^\pm$ are respectively denoted by $s^\pm$ and $t^\pm$.

$$s^k = (a^k - o_\alpha|\frac{\vec{\alpha}}{|\alpha|})$$

$$t^l = (b^l - o_\beta|\frac{\vec{\beta}}{|\beta|})$$

6.3 Application of the reduction method

We propose to evaluate the integral

$$I(x,\, y) = \int\limits_{S\times\beta_j} \frac{x-y}{||x-y||^3}\, dx\, dy, \tag{6.10}$$

where S is a triangle, x a point in the triangle S, β_j is a segment in the same plane as S and y is a point in β_j. Let $b_j^+ = b_{j+1}$ and $b_j^- = b_{j+2}$ be the two extremities of β. The reduction process can be applied, provided that one takes some common point to β_j and S as origin, so that the function $f(z) = \frac{x-y}{||x-y||^3}$ is homogeneous on $S \times \beta_j$ with $q = 2,\ n = 3$. The origin can be taken anywhere on the support of β_j, but a wise choice is to choose one of the ends

of β_j as origin. Indeed, some of the distances being canceled, one will have a simplification. We choose for origin the point b^-. Formula (6.4) provides

$$I(x,\, y) = \frac{1}{5} \int\limits_{\partial(S\times\beta_j)} ((x,\, y)|\overrightarrow{\nu}) \frac{x-y}{||x-y||^3} \partial(x,\, y), \tag{6.11}$$

where ν is the outward normal at the border of the edge of $S \times \beta_j$ and $\partial(x,\, y)$ is the surface element along $\partial(S \times \beta_j)$. We have $\partial(S \times \beta_j) = (\partial S \times \beta_j) \cup (S \times \partial\beta_j)$.

$$((x,y)|\overrightarrow{\nu})_{b_j^+\times S} = |\beta_j|; \quad ((x,y)|\overrightarrow{\nu})_{|\alpha_i\times\beta_j} = \gamma_i(b_j^-)$$

where $|\beta_j|$ denotes the length of the segment β_j. Then, we get

$$I(x,\, y) = \frac{|\beta_j|}{5} \int\limits_S \frac{x-b_j^+}{||x-b_j^+||^3} ds_x + \sum_{i=1}^{3} \frac{\gamma_i(b_j^-)}{5} \int\limits_{\alpha_i\times\beta_j} \frac{x-y}{||x-y||^3}\, ds_x\, ds_y. \tag{6.12}$$

So we get a linear combinaison of 2-D integrals.

Conclusion

This chapter allowed to present the reduction formulae of the dimension of the integration domain, to explain the reduction process and also to emphasize the importance of the parametrization and the use of flat polygons on which our method is based. An important point in the following is the choice of the origin during the stages of reduction dimension of the integration domain, in order to maintain the homogeneity of the integrand

CHAPTER 7

Evaluation of a singular integral-Adjacent triangles

Introduction

In this chapter, we will apply the reduction process to evaluate singular integrals appearing in chapter 5, on the product of two triangular mesh element S and T, specially the singular integral

$$\mathcal{S}_{i,j} = \int\limits_{S\times T} (x-a_i)\cdot \frac{(x-y)}{||x-y||^{\xi+1}} \times (y-b_j)dxdy;\ i,j=1,2,3;\ \xi=0,2$$

Note that if $\xi = 0$ the integral $\mathcal{S}_{i,j}$ is not singular but the method we propose is still valid. Here we will deal with the case where triangles have an edge in common.

7.1 Barycentric basis functions

In order to simplify the expression of the integrand and to facilitate the application of the reduction formulae we will express the Rao-Wilton-Glisson basis functions as a linear combination of barycentric basis functions.
We begin by presenting the basis functions we will use and some of their properties.

Definition 7.1.1. *Let ϕ_i be a basis functions associated with the vertices of the triangle S:*

$$\phi_i(x) = 1 + (\overrightarrow{x-a_i}|\overrightarrow{e_i});i=1,2,3 \tag{7.1}$$

where

$$\vec{e_i} = \mu_1\overrightarrow{\alpha_{i+1}} + \mu_2\overrightarrow{\alpha_{i+2}},$$

with

$$\mu_1 = \frac{||\overrightarrow{\alpha_{i+2}}||^2 + (\overrightarrow{\alpha_{i+1}}|\overrightarrow{\alpha_{i+2}})}{||\overrightarrow{\alpha_{i+2}}||^2||\overrightarrow{\alpha_{i+1}}||^2 - (\overrightarrow{\alpha_{i+1}}|\overrightarrow{\alpha_{i+2}})^2},$$

$$\mu_2 = -\frac{||\overrightarrow{\alpha_{i+1}}||^2 + (\overrightarrow{\alpha_{i+1}}|\overrightarrow{\alpha_{i+2}})}{||\overrightarrow{\alpha_{i+2}}||^2||\overrightarrow{\alpha_{i+1}}||^2 - (\overrightarrow{\alpha_{i+1}}|\overrightarrow{\alpha_{i+2}})^2}.$$

The basis functions ϕ_i verify $\phi_i(a_i) = 1$, $\phi_i(a_{i+1}) = \phi_i(a_{i+2}) = 0$, besides

$$\phi_{i|\alpha_i} = 0, \phi_i(x)_{|\alpha_{i+1}} = 1 - \frac{x - a_i}{|\alpha_{i+1}|}, \phi_i(x)_{|\alpha_{i+2}} = 1 - \frac{x - a_i}{|\alpha_{i+2}|}. \tag{7.2}$$

The same, if the basis functions ψ are associated with the top of the triangle T:

$$\psi_j(y) = 1 + (\overrightarrow{y - b_j}|\overrightarrow{l_j}); j = 1, 2, 3 \tag{7.3}$$

We rewrite the Rao-Wiston-Glisson basis functions as a combination of barycentric basis functions and we use properties above to determine θ, κ, ζ, ρ, η and μ verifying:

$$\begin{aligned} (x - a_i) = &\quad \theta\phi_i(x) + \kappa\phi_{i+1}(x) + \zeta\phi_{i+2}(x), \\ (y - b_j) = &\quad \rho\psi_j(y) + \eta\psi_{j+1}(y) + \mu\psi_{j+2}(y). \end{aligned} \tag{7.4}$$

We obtain

$$\begin{aligned} (x - a_i) = &\quad 0 + (a_{i+1} - a_i)\phi_{i+1}(x) + (a_{i+2} - a_i)\phi_{i+2}(x), \\ (y - b_j) = &\quad 0 + (a_{j+1} - a_j)\psi_{j+1}(y) + (a_{j+2} - a_j)\psi_{j+2}(y). \end{aligned} \tag{7.5}$$

7.2 Decomposition into homogeneous integrands

We are interested in calculating the singular integral

$$S_{i,j} = \int\limits_{S\times T} (x - a_i) \cdot \frac{(x - y)}{||x - y||^{\xi+1}} \times (y - b_j)dxdy;\ i, j = 1, 2, 3;\ \xi = 0, 2 \tag{7.6}$$

Using (7.5), the integral (7.6) becomes

$$\begin{aligned} S_{i,j} = &\quad \int_{S\times T}(\kappa \times \eta) \cdot (\phi_{i+1}(x)\psi_{j+1}(y))\tfrac{(x,y)}{||x-y||^{\xi+1}}dxdy \\ + &\quad \int_{S\times T}(\kappa \times \mu) \cdot (\phi_{i+1}(x)\psi_{j+2}(y))\tfrac{(x-y)}{||x-y||^{\xi+1}}dxdy \\ + &\quad \int_{S\times T}(\zeta \times \eta) \cdot (\phi_{i+2}(x)\psi_{j+1}(y))\tfrac{(x-y)}{||x-y||^{\xi+1}}dxdy \\ + &\quad \int_{S\times T}(\zeta \times \mu) \cdot (\phi_{i+2}(x)\psi_{j+2}(y))\tfrac{(x-y)}{||x-y||^{\xi+1}}dxdy. \end{aligned} \tag{7.7}$$

So we have to evaluate four integrals of the same type:

$$I^{i,j} = \int\limits_{S\times T} (\phi_i(x)\psi_j(y))\frac{(x-y)}{||x-y||^{\xi+1}}dxdy;\ \xi = 0,2;\ i,j = 1,2,3 \tag{7.8}$$

Since the basis function is affine, the integrand is no longer homogeneous, so we are led to decompose it into a linear part and a constant part by making a change of origin:

$$\begin{aligned}\phi_i(x) &= 1 + (\overrightarrow{o-a_i}|\overrightarrow{e_i}) + (\overrightarrow{x-o}|\overrightarrow{e_i}) = \phi_i(o) + (\underline{x}|\overrightarrow{e_i});\ i = 1,2,3,\\ \psi_j(y) &= 1 + (\overrightarrow{o-b_j}|\overrightarrow{l_j}) + (\overrightarrow{y-o}|\overrightarrow{l_j}) = \psi_j(o) + (\underline{y}|\overrightarrow{l_j});\ j = 1,2,3.\end{aligned} \tag{7.9}$$

where o is the new origin, $\underline{x} = x - o$ and $\underline{y} = y - o$. More generally, we note $\underline{\alpha}$ and $\underline{S}$ the segment α and the triangle S after this change of origin. According to (7.8) and (7.9), we constate that the integral $I^{i,j}$ must be decomposed into four integrals whose integrands are homogeneous.

$$\begin{aligned}I^{i,j} = &\int\limits_{\underline{S}\times\underline{T}} \frac{(\underline{x}|\overrightarrow{e_i})(\underline{y}|\overrightarrow{l_j})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}}dxdy + \phi_i(o)\int\limits_{S\times T} \frac{(\underline{x}|\overrightarrow{e_i})(\underline{y}|\overrightarrow{l_j})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}}dxdy\\ &+\psi_j(o)\int\limits_{S\times T} \frac{(\underline{x}|\overrightarrow{e_i})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}}dxdy + \psi_j(o)\phi_i(o)\int\limits_{S\times T} \frac{(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}}dxdy.\end{aligned} \tag{7.10}$$

Once this decomposition has taken place, we obtain homogeneous integrands we can then apply the reduction method intermediate of the formulae (6.4) or (6.5). During the calculation, after each step of reduction, we will focus on expressing the results in terms of ϕ_i and ψ_j in order to benefit the best of the properties (7.2) of the basic functions. We distinguish two specific case where a singularity occurs. The two triangles S and T have a common stop or a common vertex.

7.3 Three-dimensional reduction

We are dealing with the case of triangles with an edge in common(see Figure 7.1), so there are two common vertices. For the first reduction, we take for origin one of the common vertices $o = a_1 = b_1$. Let $a_2 = b_2$ be the second common vertex and a_3 ,b_3 be the remaining vertices. We propose to calculate the integral

$$I^{1,1} = \int\limits_{S\times T} (\phi_1(x)\psi_1(y))\frac{(x-y)}{||x-y||^{\xi+1}}dxdy;\ \xi = 0,2. \tag{7.11}$$

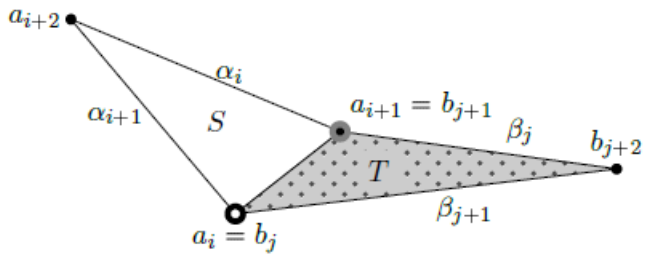

Figure 7.1: Adjacent triangles

It will be wise to choose the vertex $a_1 = b_1$, as origin o. In this case the basis functions are written

$$\phi_1(x) = 1 + (\underline{x}|\overrightarrow{e_1}); \quad \psi_1(y) = 1 + (\underline{y}|\overrightarrow{l_1}). \tag{7.12}$$

The equation (7.11) becomes

$$I^{1,1} = \int\limits_{\underline{S}\times\underline{T}} \frac{(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dxdy + \int\limits_{\underline{S}\times\underline{T}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dxdy$$
$$+ \int\limits_{\underline{S}\times\underline{T}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dxdy + \int\limits_{\underline{S}\times\underline{T}} \frac{\underline{x}-\underline{y}}{||\underline{x}-\underline{y}||^{1+\xi}} dxdy. \tag{7.13}$$

The four integrands are homogeneous of degrees $q = 1-\xi, q = 2-\xi, q = 1-\xi$ and $q = -\xi$ with $n = 4$. We apply the formula (6.4), it gives the following result

$$I^{1,1} = \sum_{i=1}^{3} \frac{\gamma_i(a_1)}{5-\xi} \int\limits_{\underline{\alpha_i}\times\underline{T}} \frac{(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \sum_{j=1}^{3} \frac{\delta_j(b_1)}{5-\xi} \int\limits_{\underline{S}\times\underline{\beta_j}} \frac{(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dxds_y$$
$$+ \sum_{i=1}^{3} \frac{\gamma_i(a_1)}{6-\xi} \int\limits_{\underline{\alpha_i}\times\underline{T}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \sum_{j=1}^{3} \frac{\delta_j(b_1)}{6-\xi} \int\limits_{\underline{S}\times\underline{\beta_j}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dxds_y$$
$$+ \sum_{i=1}^{3} \frac{\gamma_i(a_1)}{5-\xi} \int\limits_{\underline{\alpha_i}\times\underline{T}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \sum_{j=1}^{3} \frac{\delta_j(b_1)}{5-\xi} \int\limits_{\underline{S}\times\underline{\beta_j}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dxds_y$$
$$+ \sum_{i=1}^{3} \frac{\gamma_i(a_1)}{4-\xi} \int\limits_{\underline{\alpha_i}\times\underline{T}} \frac{\underline{x}-\underline{y}}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \sum_{j=1}^{3} \frac{\delta_j(b_1)}{4-\xi} \int\limits_{\underline{S}\times\underline{\beta_j}} \frac{\underline{x}-\underline{y}}{||\underline{x}-\underline{y}||^{1+\xi}} dxds_y, \tag{7.14}$$

where $\gamma_i(x)$ and $\delta_j(y)$ are the signed distances respectively of x to the support of α_i and of y to the support of β_j (see Figure 6.9).

In particular $\gamma_i(x) = 0$ for all x belonging to the support of α_i. $\gamma_i(a_i) = g_i$ and $\delta_j(b_j) = d_j$ are the lengths of the height from the vertex a_i or b_j to the support of α_i (respectively β_j). Thus, the formula will be simplified and we get

$$\begin{aligned}
I^{1,1} &= \tfrac{g_1}{5-\xi}\int_{\underline{\alpha_1}\times\underline{T}} \tfrac{(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \tfrac{d_1}{5-\xi}\int_{\underline{S}\times\underline{\beta_1}} \tfrac{(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dx ds_y \\
&+\tfrac{g_1}{6-\xi}\int_{\underline{\alpha_1}\times\underline{T}} \tfrac{(\underline{x}|\overrightarrow{e_1})(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \tfrac{d_1}{6-\xi}\int_{\underline{S}\times\underline{\beta_1}} \tfrac{(\underline{x}|\overrightarrow{e_1})(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dx ds_y \\
&+\tfrac{g_1}{5-\xi}\int_{\underline{\alpha_1}\times\underline{T}} \tfrac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \tfrac{d_1}{5-\xi}\int_{\underline{S}\times\underline{\beta_1}} \tfrac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dx ds_y \\
&+\tfrac{g_1}{4-\xi}\int_{\underline{\alpha_1}\times\underline{T}} \tfrac{\underline{x}-\underline{y}}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + \tfrac{d_1}{4-\xi}\int_{\underline{S}\times\underline{\beta_1}} \tfrac{\underline{x}-\underline{y}}{||\underline{x}-\underline{y}||^{1+\xi}} dx ds_y.
\end{aligned} \tag{7.15}$$

We express the results in terms of ϕ_1 and ψ_1 in order to benefit the best of the properties (7.2) of the basic functions.

$$(\underline{x}|\overrightarrow{e_1}) = \phi_1(x) - 1; \quad (\underline{y}|\overrightarrow{l_1}) = \psi_1(y) - 1$$

we obtain

$$\begin{aligned}
I^{1,1} &= \tfrac{g_1}{5-\xi}\int_{\alpha_1\times T} \tfrac{\psi_1(y)(x-y)}{||x-y||^{1+\xi}} ds_x dy - \tfrac{g_1}{5-\xi}\int_{\alpha_1\times T} \tfrac{(x-y)}{||x-y||^{1+\xi}} ds_x dy \\
&+\tfrac{d_1}{5-\xi}\int_{S\times\beta_1} \tfrac{\psi_1(y)(x-y)}{||x-y||^{1+\xi}} dx ds_y - \tfrac{d_1}{5-\xi}\int_{S\times\beta_1} \tfrac{(x-y)}{||x-y||^{1+\xi}} dx ds_y + \tfrac{g_1}{6-\xi}\int_{\alpha_1\times T} \tfrac{\phi_1(x)\psi(y)(x-y)}{||x-y||^{1+\xi}} ds_x dy \\
&-\tfrac{g_1}{6-\xi}\int_{\alpha_1\times T} \tfrac{\phi_1(x)(x-y)}{||x-y||^{1+\xi}} ds_x dy - \tfrac{g_1}{6-\xi}\int_{\alpha_1\times T} \tfrac{\psi_1(y)(x-y)}{||x-y||^{1+\xi}} ds_x dy + \tfrac{g_1}{6-\xi}\int_{\alpha_1\times T} \tfrac{(x-y)}{||x-y||^{1+\xi}} ds_x dy \\
&+\tfrac{d_1}{6-\xi}\int_{S\times\beta_1} \tfrac{\phi_1(x)\psi(y)(x-y)}{||x-y||^{1+\xi}} dx ds_y - \tfrac{d_1}{6-\xi}\int_{S\times\beta_1} \tfrac{\phi_1(x)(x-y)}{||x-y||^{1+\xi}} dx ds_y - \tfrac{d_1}{6-\xi}\int_{S\times\beta_1} \tfrac{\psi_1(y)(x-y)}{||x-y||^{1+\xi}} dx ds_y \\
&+\tfrac{d_1}{6-\xi}\int_{S\times\beta_1} \tfrac{(x-y)}{||x-y||^{1+\xi}} dx ds_y + \tfrac{g_1}{5-\xi}\int_{\alpha_1\times T} \tfrac{\phi_1(x)(x-y)}{||x-y||^{1+\xi}} ds_x dy - \tfrac{g_1}{5-\xi}\int_{\alpha_1\times T} \tfrac{(x-y)}{||x-y||^{1+\xi}} ds_x dy \\
&+\tfrac{d_1}{5-\xi}\int_{S\times\beta_1} \tfrac{\phi_1(x)(x-y)}{||x-y||^{1+\xi}} dx ds_y - \tfrac{d_1}{5-\xi}\int_{S\times\beta_1} \tfrac{(x-y)}{||x-y||^{1+\xi}} dx ds_y + \tfrac{g_1}{4-\xi}\int_{\alpha_1\times T} \tfrac{x-y}{||x-y||^{1+\xi}} ds_x dy \\
&+\tfrac{d_1}{4-\xi}\int_{S\times\beta_1} \tfrac{x-y}{||x-y||^{1+\xi}} dx ds_y.
\end{aligned}$$

Note that there are terms that vanish because of the fact that $\phi_{1_{|\alpha_i}} = 0, \psi_{i_{|\beta_i}} = 0$.
We have integrals in three dimensions we reapply the procedure to reduce dimention to two.

7.4 Two-dimensional reduction

The integrands are no more homogeneous. It is possible to continue the reduction by taking the second common vertex as the origin $a_2 = b_2 = o$. We reexpress the basis function as in (7.12) and we obtain

$$\phi_1(x) = \phi_1(a_2) + (\underline{x}|\overrightarrow{e_1}) = (\underline{x}|\overrightarrow{e_1}); \;\; \psi_1(x) = \psi_1(b_2) + (\underline{y}|\overrightarrow{l_1}) = (\underline{y}|\overrightarrow{l_1})$$

we get

$$I^{1,1} = (\frac{g_1}{5-\xi} - \frac{g_1}{6-\xi}) \int_{\underline{\alpha_1}\times\underline{T}} \frac{(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy + (\frac{g_1}{6-\xi} - 2\frac{g_1}{5-\xi} + \frac{g_1}{4-\xi}) \int_{\underline{\alpha_1}\times\underline{T}} \frac{(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x dy$$
$$+(\frac{d_1}{5-\xi} - \frac{d_1}{6-\xi}) \int_{\underline{S}\times\underline{\beta_1}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dx ds_y + (\frac{d_1}{4-\xi} + \frac{d_1}{6-\xi} - \frac{2d_1}{5-\xi}) \int_{\underline{S}\times\underline{\beta_1}} \frac{(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dx ds_y.$$

We apply the formula (6.4) a second time then, we get

$$I^{1,1} = (\frac{g_1}{5-\xi} - \frac{g_1}{6-\xi}) \frac{d_2}{4-\xi} \int_{\underline{T}} \frac{(\underline{y}|\overrightarrow{l_1})(\underline{a_3}-\underline{y})}{||\underline{a_3}-\underline{y}||^{1+\xi}} dy + (\frac{g_1}{5-\xi} - \frac{g_1}{6-\xi}) \frac{d_2}{4-\xi} \int_{\underline{\alpha_1}\times\underline{\beta_2}} \frac{(\underline{y}|\overrightarrow{l_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x ds_y$$
$$+(\frac{g_1}{6-\xi} - 2\frac{g_1}{5-\xi} + \frac{g_1}{4-\xi}) \frac{|\alpha_1|}{3-\xi} \int_{\underline{T}} \frac{(\underline{a_3}-\underline{y})}{||\underline{a_3}-\underline{y}||^{1+\xi}} ds_x dy + (\frac{g_1}{6-\xi} - 2\frac{g_1}{5-\xi} + \frac{g_1}{4-\xi}) \frac{d_2}{3-\xi} \int_{\underline{\alpha_1}\times\underline{\beta_2}} \frac{(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x ds_y$$
$$+(\frac{d_1}{5-\xi} - \frac{d_1}{6-\xi}) \frac{|\beta_1|}{4-\xi} \int_{\underline{S}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{b_3})}{||\underline{x}-\underline{b_3}||^{1+\xi}} dx ds_y + (\frac{d_1}{5-\xi} - \frac{d_1}{6-\xi}) \frac{g_2}{4-\xi} \int_{\underline{\alpha_2}\times\underline{\beta_1}} \frac{(\underline{x}|\overrightarrow{e_1})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} dx ds_y$$
$$+(\frac{d_1}{6-\xi} - \frac{d_1}{5-\xi}) \frac{|\beta_1|}{4-\xi} \int_{\underline{S}} \frac{(\underline{x}-\underline{b_3})}{||\underline{x}-\underline{b_3}||^{1+\xi}} ds_x + (\frac{d_1}{4-\xi} + \frac{d_1}{6-\xi} - \frac{d_1}{5-\xi}) \frac{g_2}{4-\xi} \int_{\underline{\alpha_2}\times\underline{\beta_1}} \frac{(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{1+\xi}} ds_x ds_y.$$

Recall that $\phi_1(x) = (\underline{x}|\overrightarrow{e_1})$; $\psi_1(x) = (\underline{y}|\overrightarrow{l_1})$, thanks to what we obtain

$$I^{1,1} = (\frac{|S|}{5-\xi} - \frac{|S|}{6-\xi}) \frac{2}{4-\xi} P^{\psi_1}(a_3, T, \xi) + (\frac{g_1}{5-\xi} - \frac{g_1}{6-\xi}) \frac{d_2}{4-\xi} Q(\alpha_1, \beta_2, \xi)$$
$$+(\frac{|S|}{6-\xi} - 2\frac{|S|}{5-\xi} + \frac{|S|}{4-\xi}) \frac{2}{3-\xi} P(a_3, T, \xi) + (\frac{g_1}{6-\xi} - 2\frac{g_1}{5-\xi} + \frac{g_1}{4-\xi}) \frac{d_2}{3-\xi} Q(\alpha_1, \beta_2, \xi)$$
$$-(\frac{|T|}{5-\xi} - \frac{|T|}{6-\xi}) \frac{2}{4-\xi} P^{\phi_1}(b_3, S, \xi) - (\frac{d_1}{5-\xi} - \frac{d_1}{6-\xi}) \frac{g_2}{4-\xi} Q^{1,\phi_1}(\beta_1, \alpha_2, \xi)$$
$$-(\frac{|S|}{6-\xi} - \frac{|S|}{5-\xi}) \frac{2}{4-\xi} P(b_3, S, \xi) + (\frac{d_1}{4-\xi} + \frac{d_1}{6-\xi} - \frac{d_1}{5-\xi}) \frac{g_2}{4-\xi} Q(\alpha_2, \beta_1, \xi),$$

where

$$P(a, T, \xi) = \int_T \frac{(a-y)}{||a-y||^{1+\xi}} ds_x ds_y, \quad Q(\alpha_i, \beta_j, \xi) = \int_{\alpha_i\times\beta_j} \frac{(x-y)}{||x-y||^{1+\xi}} ds_x ds_y,$$

$$P^{\psi_q}(a, T, \xi) = \int_T \frac{\psi_q(y)(a-y)}{||a-y||^{1+\xi}} dy, \quad Q^{1,\psi_q}(\alpha_i, \beta_j, \xi) = \int_{\alpha_i\times\beta_j} \frac{\psi_q(y)(x-y)}{||x-y||^{1+\xi}} ds_x ds_y.$$

We give also the other terms calculated with the same procedure:

$$I^{1,2} = P(a_3, T, \xi)(\frac{2|S|}{(5-\xi)(3-\xi)} - \frac{2|S|}{(5-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(3-\xi)} + \frac{2|S|}{(6-\xi)(4-\xi)})$$
$$+P(b_3, S, \xi)(-\frac{2|T|}{(5-\xi)(3-\xi)} + \frac{2|T|}{(5-\xi)(4-\xi)} + \frac{2|T|}{(6-\xi)(3-\xi)} - \frac{2|T|}{(6-\xi)(4-\xi)})$$
$$+Q(\alpha_1, \beta_2, \xi)(\frac{g_1 d_2}{(5-\xi)(3-\xi)} - \frac{g_1 d_2}{(5-\xi)(4-\xi)} - \frac{g_1 d_2}{(6-\xi)(3-\xi)} + \frac{g_1 d_2}{(6-\xi)(4-\xi)})$$
$$+Q(\alpha_2, \beta_1, \xi)(\frac{d_1 g_2}{(5-\xi)(3-\xi)} - \frac{d_1 g_2}{(6-\xi)(3-\xi)} + \frac{d_1 g_2}{(6-\xi)(4-\xi)} - \frac{d_1 g_2}{(5-\xi)(4-\xi)})$$
$$+Q^{1,\psi_2}(\alpha_2, \beta_1, \xi)(\frac{d_1 g_2}{(5-\xi)(4-\xi)} - \frac{d_1 g_2}{(6-\xi)(4-\xi)}) + P^{\psi_2}(a_3, T, \xi)(\frac{2|S|}{(5-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(4-\xi)})$$
$$-Q^{1,\phi_1}(\beta_1, \alpha_2, \xi)(\frac{d_1 g_2}{(6-\xi)(4-\xi)} - \frac{d_1 g_2}{(6-\xi)(5-\xi)}) + P^{\phi_1}(b_3, S, \xi)(-\frac{2|T|}{(6-\xi)(4-\xi)} + \frac{2|T|}{(6-\xi)(5-\xi)})$$
$$+Q^{\phi_1,\psi_2}(\alpha_2, \beta_1, \xi)(\frac{d_1 g_2}{(6-\xi)(5-\xi)}), \tag{7.16}$$

$$\begin{gathered}I^{1,3}=P(b_3,S,\xi)(-\tfrac{2|T|}{(5-\xi)(4-\xi)}+\tfrac{2|T|}{(6-\xi)(4-\xi)})-P^{\phi_1}(b_3,S,\xi)(\tfrac{2|T|}{(6-\xi)(5-\xi)})\\+P^{\psi_3}(a_3,T,\xi)(\tfrac{2|S|}{(5-\xi)(4-\xi)}-\tfrac{2|S|}{(6-\xi)(4-\xi)})+Q^{1,\psi_3}(\alpha_1,\beta_2,\xi)(\tfrac{d_2g_1}{(5-\xi)(4-\xi)}-\tfrac{d_2g_1}{(6-\xi)(4-\xi)})\\+Q^{1,\psi_3}(\alpha_2,\beta_1,\xi)(\tfrac{d_1g_2}{(5-\xi)(4-\xi)}-\tfrac{d_1g_2}{(6-\xi)(4-\xi)})+Q^{\phi_1,\psi_3}(\alpha_2,\beta_1,\xi)(\tfrac{d_1g_2}{(6-\xi)(5-\xi)})\end{gathered}\tag{7.17}$$

$$\begin{gathered}I^{2,2}=P(a_3,T,\xi)(\tfrac{2|S|}{(6-\xi)(3-\xi)}-\tfrac{4|S|}{(6-\xi)(4-\xi)}+\tfrac{2|S|}{(5-\xi)(6-\xi)})+P^{\psi_2}(a_3,T,\xi)(\tfrac{2|S|}{(6-\xi)(4-\xi)}-\tfrac{2|S|}{(6-\xi)(5-\xi)})\\+Q(\alpha_1,\beta_2,\xi)(\tfrac{g_1d_2}{(6-\xi)(3-\xi)}-\tfrac{2g_1d_2}{(6-\xi)(4-\xi)}+\tfrac{g_1d_2}{(6-\xi)(5-\xi)})+Q^{1,\phi_2}(\beta_2,\alpha_1,\xi)(-\tfrac{d_2g_1}{(6-\xi)(4-\xi)}+\tfrac{d_2g_1}{(6-\xi)(5-\xi)})\\+P(b_3,S,\xi)(-\tfrac{2|T|}{(6-\xi)(3-\xi)}+\tfrac{4|T|}{(6-\xi)(4-\xi)}-\tfrac{2|T|}{(6-\xi)(5-\xi)})+P^{\phi_2}(b_3,S,\xi)(\tfrac{2|T|}{(6-\xi)(5-\xi)}-\tfrac{2|T|}{(6-\xi)(4-\xi)})\\+Q(\alpha_2,\beta_1,\xi)(\tfrac{d_1g_2}{(6-\xi)(3-\xi)}-\tfrac{2d_1g_2}{(6-\xi)(4-\xi)}+\tfrac{d_1g_2}{(6-\xi)(5-\xi)})+Q^{1,\psi_2}(\alpha_2,\beta_1,\xi)(-\tfrac{d_1g_2}{(6-\xi)(5-\xi)}+\tfrac{d_1g_2}{(6-\xi)(4-\xi)})\end{gathered}\tag{7.18}$$

$$\begin{gathered}I^{2,3}=P(b_3,S,\xi)(\tfrac{2|T|}{(6-\xi)(5-\xi)}-\tfrac{2|T|}{(6-\xi)(4-\xi)})+P^{\psi_3}(a_3,T,\xi)(\tfrac{2|S|}{(6-\xi)(4-\xi)}-\tfrac{2|S|}{(6-\xi)(5-\xi)})\\+P^{\phi_2}(b_3,S,\xi)(-\tfrac{2|T|}{(6-\xi)(5-\xi)})+Q^{1,\psi_3}(\alpha_1,\beta_2,\xi)(\tfrac{d_2g_1}{(6-\xi)(4-\xi)}-\tfrac{d_2g_1}{(6-\xi)(5-\xi)})\\+Q^{1,\psi_3}(\alpha_2,\beta_1,\xi)(\tfrac{d_1g_2}{(6-\xi)(4-\xi)}-\tfrac{d_1g_2}{(6-\xi)(5-\xi)})+Q^{\phi_2,\psi_3}(\alpha_1,\beta_2,\xi)(\tfrac{d_2g_1}{(6-\xi)(5-\xi)})\end{gathered}\tag{7.19}$$

$$\begin{gathered}I^{3,3}=P^{\psi_3}(a_3,T,\xi)(\tfrac{2|S|}{(6-\xi)(5-\xi)})+P^{\phi_3}(b_3,S,\xi)(-\tfrac{2|T|}{(6-\xi)(5-\xi)})\\+Q^{\phi_3,\psi_3}(\alpha_1,\beta_2,\xi)(\tfrac{d_2g_1}{(6-\xi)(5-\xi)})+Q^{\phi_3,\psi_3}(\alpha_2,\beta_1,\xi)(\tfrac{d_1g_2}{(6-\xi)(5-\xi)})\end{gathered}\tag{7.20}$$

the symmetrical terms

$$\begin{gathered}I^{2,1}=P(a_3,T,\xi)(\tfrac{2|S|}{(5-\xi)(3-\xi)}-\tfrac{2|S|}{(5-\xi)(4-\xi)}-\tfrac{2|S|}{(6-\xi)(3-\xi)}+\tfrac{2|S|}{(6-\xi)(4-\xi)})\\+P(b_3,S,\xi)(-\tfrac{2|T|}{(5-\xi)(3-\xi)}+\tfrac{2|T|}{(5-\xi)(4-\xi)}+\tfrac{2|T|}{(6-\xi)(3-\xi)}-\tfrac{2|T|}{(6-\xi)(4-\xi)})\\+Q(\alpha_1,\beta_2,\xi)(\tfrac{g_1d_2}{(5-\xi)(3-\xi)}-\tfrac{g_1d_2}{(5-\xi)(4-\xi)}-\tfrac{g_1d_2}{(6-\xi)(3-\xi)}+\tfrac{g_1d_2}{(6-\xi)(4-\xi)})\\+Q(\alpha_2,\beta_1,\xi)(\tfrac{d_1g_2}{(5-\xi)(3-\xi)}-\tfrac{d_1g_2}{(6-\xi)(3-\xi)}+\tfrac{d_1g_2}{(6-\xi)(4-\xi)}-\tfrac{d_1g_2}{(5-\xi)(4-\xi)})\\+Q^{1,\psi_1}(\alpha_1,\beta_2,\xi)(\tfrac{d_2g_1}{(6-\xi)(4-\xi)}-\tfrac{d_2g_1}{(6-\xi)(5-\xi)})-Q^{1,\phi_2}(\beta_2,\alpha_1,\xi)(\tfrac{d_1g_2}{(5-\xi)(4-\xi)}-\tfrac{d_1g_2}{(6-\xi)(4-\xi)})\\+P^{\psi_1}(a_3,T,\xi)(\tfrac{2|S|}{(6-\xi)(4-\xi)}-\tfrac{2|S|}{(6-\xi)(5-\xi)})+P^{\phi_2}(b_3,S,\xi)(-\tfrac{2|T|}{(5-\xi)(4-\xi)}+\tfrac{2|T|}{(6-\xi)(4-\xi)})\\+Q^{\phi_2,\psi_1}(\alpha_1,\beta_2,\xi)(\tfrac{d_2g_1}{(6-\xi)(5-\xi)})\end{gathered}\tag{7.21}$$

$$\begin{gathered}I^{3,1}=P(a_3,T,\xi)(\tfrac{2|S|}{(5-\xi)(4-\xi)}-\tfrac{2|S|}{(6-\xi)(4-\xi)})+P^{\phi_3}(b_3,S,\xi)(\tfrac{2|T|}{(6-\xi)(4-\xi)}-\tfrac{2|T|}{(4-\xi)(4-\xi)})\\+P^{\psi_1}(a_3,T,\xi)(\tfrac{2|S|}{(5-\xi)(6-\xi)})+Q^{1,\phi_3}(\beta_1,\alpha_2,\xi)(\tfrac{d_1g_2}{(6-\xi)(4-\xi)}-\tfrac{d_1g_2}{(5-\xi)(4-\xi)})\\+Q^{1,\phi_3}(\beta_2,\alpha_1,\xi)(\tfrac{d_2g_1}{(6-\xi)(4-\xi)}-\tfrac{d_2g_1}{(5-\xi)(4-\xi)})+Q^{\phi_3,\psi_1}(\alpha_1,\beta_2,\xi)(\tfrac{d_2g_1}{(6-\xi)(5-\xi)})\end{gathered}\tag{7.22}$$

$$
\begin{aligned}
I^{3,2} = P(a_3,T,\xi)\left(\frac{2|S|}{(6-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(5-\xi)}\right) + P^{\phi_3}(b_3,S,\xi)\left(\frac{2|T|}{(6-\xi)(5-\xi)} - \frac{2|T|}{(6-\xi)(4-\xi)}\right) \\
+P^{\psi_2}(a_3,T,\xi)\left(\frac{2|S|}{(6-\xi)(5-\xi)}\right) + Q^{1,\phi_3}(\beta_2,\alpha_1,\xi)\left(\frac{d_2 g_1}{(6-\xi)(5-\xi)} - \frac{d_2 g_1}{(6-\xi)(4-\xi)}\right) \\
+Q^{1,\phi_3}(\beta_1,\alpha_2,\xi)\left(-\frac{d_1 g_2}{(6-\xi)(4-\xi)} + \frac{d_1 g_2}{(6-\xi)(5-\xi)}\right) + Q^{\phi_3,\psi_2}(\alpha_2,\beta_1,\xi)\left(\frac{d_1 g_2}{(6-\xi)(5-\xi)}\right)
\end{aligned} \tag{7.23}
$$

where

$$
Q^{\phi_p,\psi_q}(\alpha_i,\beta_j,\xi) = \int\limits_{\alpha_i\times\beta_j} \frac{\phi_p(x)\psi_q(y)(y-x)}{||x-y||^{1+\xi}} ds_x ds_y
$$

7.5 Reduction to one dimensional integrals

7.5.1 Calculation of $P(a,T,\xi)$

we will evaluate

$$
P(a,T,\xi) = \int\limits_T \frac{(a-y)}{||a-y||^{1+\xi}} ds_x ds_y, \tag{7.24}
$$

with $a = a_3$

7.5.1.1 Case where a does not belong to the support of T

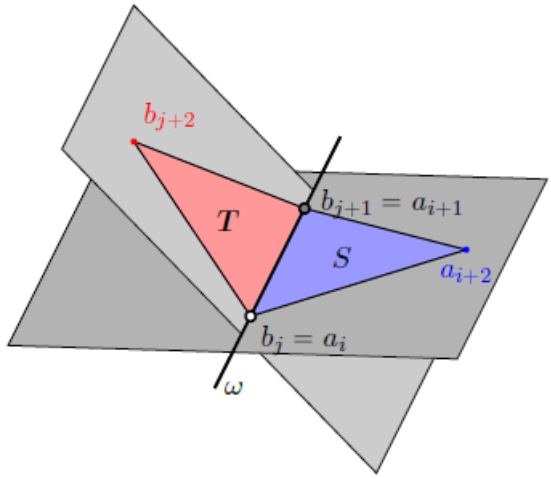

Figure 7.2: Triangles in intersecting planes

In this case the reduction formulae are not usable. In order to operate the reduction, it is necessary to break P into two parts.

$$P(a,T,\xi)=\int\limits_T \frac{(a-\hat{a})}{(h^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}ds_x ds_y+\int\limits_T \frac{(\hat{a}-y)}{(h^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}ds_x ds_y \tag{7.25}$$

where $\hat{a}$ be the projection of a into the support of T and h the distance $h=|a-\hat{a}|$ as in Figure (7.3). Taking $\hat{a}$ as a new origin we obtain homogeneous integrands of degrees $-1-\xi$ and $-\xi$ with parameter h. We can then apply the formula (6.5) which gives

$$P(a,T,\xi)=(a-\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})S_1(\hat{a},\beta_j,h,\xi)+\sum_{j=1}^{3}\delta_j(\hat{a})R_2(\hat{a},\beta_j,h,\xi) \tag{7.26}$$

where S_1 and R_2 are defined by

$$\begin{aligned} S_1(a,\beta,h,\xi) &= \textstyle\int_\beta h^{1-\xi}\int_h^{+\infty}\frac{du}{u^{2-\xi}(u^2+||a-y||^2)^{\frac{1+\xi}{2}}}ds_y \\ &= \textstyle\int_\beta f_1(h,||a-y||,\xi). \end{aligned} \tag{7.27}$$

and

$$\begin{aligned} R_2(a,\beta,h,\xi) &= \textstyle\int_\beta (a-y)h^{2-\xi}\int_h^{+\infty}\frac{du}{u^{3-\xi}(u^2+||a-y||^2)^{\frac{1+\xi}{2}}}ds_y \\ &= \textstyle\int_\beta (a-y) f_2(h,||a-y||,\xi) \end{aligned} \tag{7.28}$$

where

$$f_n(h,s,\xi)=h^{n-\xi}\int\limits_h^{+\infty}\frac{du}{u^{n+1-\xi}\left(u^2+s^2\right)^{\frac{1+\xi}{2}}} \tag{7.29}$$

$S_1(a,\beta,h,\xi)$ and $R_2(a,\beta,h,\xi)$ are calculated in (B.2) and (B.24).

7.5.1.2 Case where a belongs to the support of T

Since a belongs to the support of T we can consider a as an origin and get homogeneous integrand with $q=-\xi$ and $n=2$ but we must distinguish two different cases in order to apply the suitable reduction formula.

• If $n+q=0$, we apply (6.6) and we get

$$P(a,T,\xi)=\sum_{j=1}^{3}\frac{\delta_j}{2-\xi}\mathcal{L}(a,\beta_j) \tag{7.30}$$

where

$$\mathcal{L}(a,\beta_j)=\int\limits_{\beta_j}\frac{a-y}{||a-y||^3}\ln(||a-y||)ds_y \tag{7.31}$$

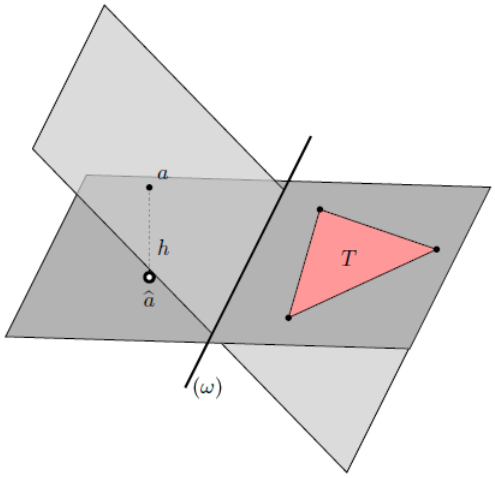

Figure 7.3: Projection of a on the support of T

• If $n + q \neq 0$, we apply (6.4) and we obtain

$$P(a, T, \xi) = \sum_{j=1}^{3} \frac{\delta_j}{2 - \xi} S(a, \beta, h, \xi)) \tag{7.32}$$

where

$$S(a, \beta, \xi) = \int_{\beta} \frac{a - y}{||a - y||^{1+\xi}} ds_y \tag{7.33}$$

see appendix (B.2)

7.5.2 Calculation of $Q(\alpha, \beta, \xi)$

Here we deal with

$$Q(\alpha, \beta, \xi) = \int_{\alpha \times \beta} \frac{(x - y)}{||x - y||^{1+\xi}} ds_x ds_y;\ \xi = 0, 2 \tag{7.34}$$

7.5.2.1 Case where α and β are neither parallel nor secants

First of all, we must find a new origin to ensure homogeneity of the integrand. Let $\breve{\alpha}$ be the projection of α on the plane parallel to α going through β and let o be the intersection of $\breve{\alpha}$ with β as in Figure (7.4). We will consider o as a new origin. Define $z = \breve{a} - a$ and $h = ||z||$. We express Q as a sum of two integrals whose integrands are homogeneous of

degrees $q = -1 - \xi$ and $-\xi$.

$$Q(\alpha,\beta,\xi) = \int\limits_{\check{\alpha}\times\beta} \frac{(x-\check{x})}{(h^2+||\check{x}-y||^2)^{\frac{1+\xi}{2}}} ds_x ds_y + \int\limits_{\check{\alpha}\times\beta} \frac{(\check{x}-y)}{(h^2+||\check{x}-y||^2)^{\frac{1+\xi}{2}}} ds_x ds_y \tag{7.35}$$

where $\check{x}$ is a point $\hat{\alpha}$. This decomposition allows us to apply the formula (6.5) and gives

$$Q(\alpha,\beta,\xi) = (x-\check{x})[\sum_{k=\pm} ks^k S_1(\check{a}^k,\beta,h,\xi) + \sum_{l=\pm} lt^l S_1(b^l,\check{\alpha},h,\xi)]$$

$$+[\sum_{k=\pm} ks^k R_2(\check{a}^k,\beta,h,\xi) - \sum_{l=\pm} lt^l R_2(a,\beta,h,\xi)] \tag{7.36}$$

where the abscissae are evaluated by $s^k = (\check{a}^k - o|\frac{\vec{\alpha}}{|\alpha|})$ and $t^l = (b^l - o|\frac{\vec{\beta}}{|\beta|})$.

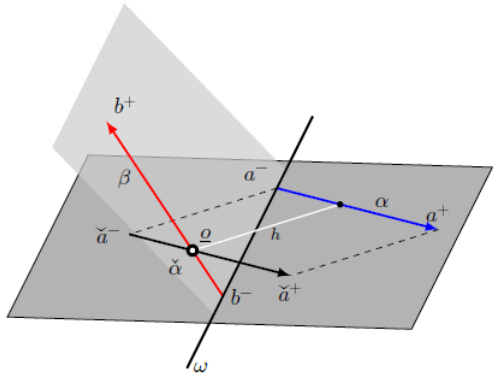

Figure 7.4: Segments neither parallel nor secants

7.5.2.2 Case where α and β are secants

Let $o = \alpha \cap \beta$ be the new origin. Note that can be $o = a_2 = b_2$ or $o = a_3 = b_3$. In this case the integrand is homogeneous of degree $q = -\xi$ and $n = 2$. We must distinguish two cases:
• If $q + n \neq 0$ the formula (6.4) gives

$$Q(\alpha,\beta,0) = \int\limits_{\alpha\times\beta} \frac{(x-y)}{||x-y||} ds_x ds_y$$

$$= \sum_{k=\pm} \frac{ks^k}{2} S(a^k,\beta,0) - \sum_{l=\pm} \frac{lt^l}{2} S(b^l,\alpha,0). \tag{7.37}$$

• If $q+n=0$ we apply (6.6) and we get

$$Q(\alpha,\beta,2)=\int\limits_{\alpha\times\beta}\frac{(x-y)}{||x-y||^3}ds_xds_y$$
$$=\sum_{k=\pm}ks^k\mathcal{L}(a^k,\beta)-\sum_{l=\pm}lt^l\mathcal{L}(b^l,\alpha) \tag{7.38}$$

7.5.2.3 Case where α and β are parallel

When the supports of α and β are parallel, we must use the second formula (6.5), called with parameter. This is possible by orthogonally projecting one of the segments (we choose β) on the support of the other (α) (see Figure 7.5). Thus, the distance d between the supports of the two segments, appears and plays the role of the parameter. Let $\tilde{\beta}$ and $\tilde{y}$, be the respective orthogonal projections of β and y on the support of α, $\vec{z}=\tilde{y}-y$ and $d=||\tilde{y}-y||$ the distance from α to the support of β. Then, the integral $Q(\alpha,\beta,\xi)$ becomes

$$Q(\alpha,\beta,\xi)=\vec{z}\int\limits_{\alpha\times\tilde{\beta}}\frac{1}{(d^2+||x-\tilde{y}||^2)^{\frac{1+\xi}{2}}}ds_xds_{\tilde{y}}+\int\limits_{\alpha\times\tilde{\beta}}\frac{(x-\tilde{y})}{(d^2+||x-\tilde{y}||^2)^{\frac{1+\xi}{2}}}ds_xds_{\tilde{y}} \tag{7.39}$$

Taking any point of α as a new origine, for exemple one of the two vertices of the triangle

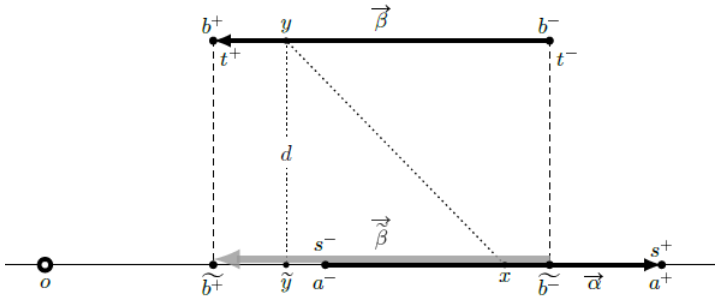

Figure 7.5: Segments are parallel

a^- and a^+, we get two homogeneous integrands with parameter d and degrees $q=-1-\xi$ respectively $q=-\xi$.

$$\begin{aligned}&Q(\alpha,\beta,\xi)=\vec{z}[s^+\textstyle\int_{\tilde{\beta}}d^{1-\xi}\int_d^{+\infty}\frac{du}{u^{2-\xi}(u^2+||a^+-\tilde{y}||^2)^{\frac{1}{2}}}ds_{\tilde{y}}\\&+\textstyle\sum_{l=\pm}lt^l\int_\alpha d^{1-\xi}\int_d^{+\infty}\frac{du}{u^{2-\xi}(u^2+||x-\tilde{b}^l||^2)^{\frac{1}{2}}}ds_x]\\&+s^+\textstyle\int_{\tilde{\beta}}d^{2-\xi}(a^+-y)\int_d^{+\infty}\frac{du}{u^{3-\xi}(u^2+||a^+-y||^2)^{\frac{1}{2}}}ds_{\tilde{y}}\\&+[\textstyle\sum_{l=\pm}lt^l\int_\alpha d^{2-\xi}(x-\tilde{b}^l)\int_d^{+\infty}\frac{du}{u^{3-\xi}(u^2+||x-\tilde{b}^l||^2)^{\frac{1}{2}}}ds_x]\end{aligned} \tag{7.40}$$

which is equivalent to

$$Q(\alpha,\beta,\xi)=\sum_{l=\pm} lt^l[\vec{z}S_1(\tilde{b}^l,\beta,d,\xi)-R_2(\tilde{b}^l,\beta,h,\xi)]$$

$$+s^+[S_1(a^+,\tilde{\beta},d,\xi)+R_2(a^+,\tilde{\beta},d,\xi)] \tag{7.41}$$

7.5.3 Calculation of $P^{\psi_q}(a,T,\xi)$

To calculate the singular integral

$$P^{\psi_q}(a,T,\xi)=\int_T \frac{\psi_q(y)(a-y)}{||a-y||^{1+\xi}}dy, \tag{7.42}$$

we are inspired by the section (7.5.1). There are two different geometrical cases.

7.5.3.1 Case where a does not belong to the support of T

Note that $a=a_3$ is a vertex of S. To maintain the homogeneity of the integrands the origin must belong to the support of the triangle T. So we consider $\hat{a}$ the projection of a on the support of T (see Figure 7.3) and we break $P^{\psi_q}(a,T,\xi)$ into two parts

$$P^{\psi_q}(a,T,\xi)=(a-\hat{a})\int_T \frac{\psi_q(y)}{(h^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy+\int_T \frac{\psi_q(y)(\hat{a}-y)}{(h^2+||\hat{a}-y||^2)^{1+\xi}}, \tag{7.43}$$

where $h=|a-\hat{a}|$ and $\xi=0,2$. Next, we express $\psi_q(y)$ with the new origin $o=\hat{a}$,

$$\psi_q(y)=\psi_q(\hat{a})+(y|l_q),$$

which gives

$$P^{\psi_q}(a,T,\xi)=(a-\hat{a})[\psi_q(\hat{a})\int_T \frac{1}{(h^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy+\int_T \frac{(y|l_q)}{(h^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy]$$

$$+\psi_q(\hat{a})\int_T \frac{(\hat{a}-y)}{(h^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy+\int_T \frac{(y|l_q)(\hat{a}-y)}{(h^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy. \tag{7.44}$$

We notice that the four intergrands are homogeneous with parameter h and degrees (respectively) $q=-1-\xi, q=-\xi, q=-\xi$ and $q=1-\xi$. We can then apply (6.5) and we

obtain

$$\begin{array}{c} P^{\psi_q}(a,T,\xi) = (a-\hat{a})\psi_q(\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})\int_{\beta_j} h^{1-\xi}\int_h^{+\infty}\frac{du}{u^{2-\xi}(u^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy \\ +(a-\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})\int_{\beta_j} h^{2-\xi}(y|l_q)\frac{du}{u^{3-\xi}(u^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy \\ +\psi_q(\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})\int_{\beta_j} h^{2-\xi}(\hat{a}-y)\int_h^{+\infty}\frac{du}{u^{3-\xi}(u^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy \\ +\sum_{j=1}^{3}\delta_j(\hat{a})\int_{\beta_j} h^{3-\xi}(y|l_q)(\hat{a}-y)\int_h^{+\infty}\frac{du}{u^{4-\xi}(u^2+||\hat{a}-y||^2)^{\frac{1+\xi}{2}}}dy. \end{array} \tag{7.45}$$

To simplify the formula, we express the basic functions in this form $(y|l_q) = \psi_q(y) - \psi_q(\hat{a})$ and we delete the terms that vanish, thanks to (7.2)

$$\begin{array}{c} P^{\psi_q}(a,T,\xi) = (a-\hat{a})\psi_q(\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})\int_{\beta_j} f_1(h,||\hat{a}-y||,\xi) - f_2(h,\hat{a}-y,\xi)ds_y \\ +(a-\hat{a})[\delta_{q+1}(\hat{a})\int_{\beta_{q+1}}\psi_q(y)f_2(h,\hat{a}-y,\xi)ds_y + \delta_{q+2}(\hat{a})\int_{\beta_{q+2}}\psi_q(y)f_2(h,\hat{a}-y,\xi)ds_y] \\ +\psi_q(\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})\int_{\beta_j}(\hat{a}-y)(f_2(h,||\hat{a}-y||,\xi) - f_3(h,\hat{a}-y,\xi))ds_y \\ +\delta_{q+1}(\hat{a})\int_{\beta_{q+1}}\psi_q(y)(\hat{a}-y)f_3(h,||\hat{a}-y||,\xi)ds_y + \delta_{q+2}(\hat{a})\int_{\beta_{q+2}}\psi_q(y)(\hat{a}-y)f_3(h,\hat{a}-y,\xi)ds_y \end{array} \tag{7.46}$$

otherwise we can write

$$\begin{array}{c} P^{\psi_q}(a,T,\xi) = (a-\hat{a})\psi_q(\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})(S1(\hat{a},\beta,h,\xi) - S2(\hat{a},\beta,h,\xi)) \\ +(a-\hat{a})[\delta_{q+1}(\hat{a})S_2^{\psi_q}(\hat{a},\beta_{q+1},h,\xi) + \delta_{q+2}(\hat{a})S_2^{\psi_q}(\hat{a},\beta_{q+2},h,\xi)] \\ +\psi_q(\hat{a})\sum_{j=1}^{3}\delta_j(\hat{a})R_2(\hat{a},\beta_j,h,\xi) - R_3(\hat{a},\beta_j,h,\xi) \\ +\delta_{q+1}(\hat{a})D_3^{\psi_q}(\hat{a},\beta_{q+1},h,\xi) + \delta_{q+2}(\hat{a})D_3^{\psi_q}(\hat{a},\beta_{q+2},h,\xi) \end{array} \tag{7.47}$$

where

$$D_n^{\psi_q}(a,\beta_p,h,\xi) = \delta_{q+1}(a)\int_{\beta_p}\psi_\kappa(y)(a-y)f_n(h,a-y,\xi)ds_y \tag{7.48}$$

7.5.3.2 Case where a belongs to the support of T

It suffices to consider a as origin to have homogeneity.

$$P^{\psi_q}(a,T,\xi) = \int_T \frac{\psi_q(y)(a-y)}{||a-y||^{1+\xi}}dy, \tag{7.49}$$

Since $o = a$ is the new origin, $\psi_q(y)$ can be written

$$\psi_q(y) = \psi_q(a) + (y|l_q)$$

We obtain

$$P^{\psi_q}(a,T,\xi) = \psi_q(a)\int_T \frac{(a-y)}{||a-y||^{1+\xi}}dy + \int_T \frac{(y|l_q)(a-y)}{||a-y||^{1+\xi}}dy \tag{7.50}$$

Integrands are homogeneous of degrees $q = -\xi$ and $q = 1 - \xi$. We apply (6.4) and we get

$$P^{\psi_q}(a,T,\xi) = \psi_q(a)\sum_{j=1}^{3}\frac{\delta_j(a)}{2-\xi}\int\limits_{\beta_j}\frac{(a-y)}{||a-y||^{1+\xi}}ds_y + \sum_{j=1}^{3}\frac{\delta_j(a)}{3-\xi}\int\limits_{\beta_j}\frac{(y|l_q)(a-y)}{||a-y||^{1+\xi}}dy \quad (7.51)$$

We replace $(y|l_q)$ by $(y|l_q) = \psi_q(y) - \psi_q(a)$, then we get

$$P^{\psi_q}(a,T,\xi) = \psi_q(a)\sum_{j=1}^{3}\frac{\delta_j(a)}{2-\xi}\int\limits_{\beta_j}\frac{(a-y)}{||a-y||^{1+\xi}}ds_y$$

$$+\sum_{j=1}^{3}\frac{\delta_j(a)}{3-\xi}\int\limits_{\beta_j}\frac{\psi_q(y)(a-y)}{||a-y||^{1+\xi}}ds_y - \psi_q(a)\sum_{j=1}^{3}\frac{\delta_j(a)}{3-\xi}\int\limits_{\beta_j}\frac{(a-y)}{||a-y||^{1+\xi}}ds_y \quad (7.52)$$

which is equivalent to

$$P^{\psi_q}(a,T,\xi) = \sum_{j=1}^{3}\frac{\delta_j(a)}{3-\xi}S^{\psi_q}(a,\beta_j,\xi)$$

$$+\psi_q(a)\sum_{j=1}^{3}[\frac{\delta_j(a)}{2-\xi}S(a,\beta_j,\xi) - \frac{\delta_j(a)}{3-\xi}S(a,\beta_j,\xi)] \quad (7.53)$$

7.5.4 Calculation of $Q^{1,\psi_q}(\alpha,\beta,\xi)$

We want to evaluate the singular integral on two edges

$$Q^{1,\psi_q}(\alpha,\beta,\xi) = \int\limits_{\alpha\times\beta}\frac{\psi_q(y)(x-y)}{||x-y||^{1+\xi}}ds_x ds_y \quad (7.54)$$

where $\beta \neq \alpha \neq \alpha_3$, $\beta \neq \beta_3 \neq \beta_q$. We are going to study two different cases.

7.5.4.1 Case where α and β are neither parallel nor secants

Let $\breve{\alpha}$ be the projection of α on the plane containing β and parallel to α, $\breve{x}$ a point in $\breve{\alpha}$, $h = |a - \breve{a}|$ and $\vec{z} = x - \breve{x}$ (see Figure 7.4). We decompose Q^{1,ψ_q} in two parts

$$Q^{1,\psi_q}(\alpha,\beta,\xi) = \vec{z}\int\limits_{\breve{\alpha}\times\beta}\frac{\psi_q(y)}{(h^2+||\breve{x}-y||^2)^{\frac{1+\xi}{2}}}ds_{\breve{x}}ds_y + \int\limits_{\breve{\alpha}\times\beta}\frac{\psi_q(y)(\breve{x}-y)}{(h^2+||\breve{x}-y||^2)^{\frac{1+\xi}{2}}}ds_{\breve{x}}ds_y \quad (7.55)$$

We will follow the same approach as previously. We consider $o = \breve{\alpha}\cap\beta$ the new origin and we replace in the equation (7.55) the basic function $\psi_q(y)$ by the equivalent expression

$$\psi_q(y) = \psi_q(o) + (y|l_q).$$

Once we have homogeneous integrands, the reduction formula (6.5) gives

$$
\begin{aligned}
&Q^{1,\psi_q}(\alpha,\beta,\xi) = \psi_q(o)[\textstyle\sum_{k=\pm} ks^k R_2(\hat{a}^k,\beta,h,||\hat{a}^k-y||,\xi) - R_2(b^l,\alpha,h,||\hat{x}-b^l||,\xi)]\\
&+\textstyle\sum_{k=\pm} ks^k[D_3^{\psi_q}(\hat{a}^k,\beta_j,||\hat{a}^k-y||,\xi) - \psi_q(o)R_3(\hat{a}^k,\beta,||\hat{a}^k-y||,\xi)]\\
&+\textstyle\sum_{l=\pm} lt^l[(-\psi_q(b^l)+\psi_q(o))R_3(b^l,\alpha,||\hat{x}-b^l||,\xi)]\\
&+\vec{z}\psi_q(o)[\textstyle\sum_{k=\pm} ks^k S_1(\hat{a}^k,\beta,h,\xi) + \sum_{l=\pm} lt^l S_1(\hat{b}^l,\alpha,h,\xi)]\\
&+\vec{z}\textstyle\sum_{k=\pm} ks^k[S_2^{\psi_q}(\hat{a}^k,\beta,h,||\hat{a}^k-y||,\xi) - \psi_q(o)S_2(\hat{a}^k,\beta,h,||\hat{a}^k-y||,\xi)]\\
&+\vec{z}\textstyle\sum_{l=\pm} lt^l[(\psi_q(b^l)-\psi_q(o))S_2(b^l,\hat{\alpha},h,||\hat{x}-b^l||,\xi)]
\end{aligned}
$$

7.5.4.2 Case where α and β are secants

Let $o = \alpha \cap \beta$ be the new origin. The basis function $\psi_q(y)$ can be expressed as following

$$\psi_q(y) = \psi_q(o) + (y|l_q)$$

which gives

$$Q^{1,\psi_q}(\alpha,\beta,\xi) = \psi_q(o)\int\limits_{\alpha\times\beta}\frac{(x-y)}{||x-y||^{1+\xi}}ds_x ds_y + \int\limits_{\alpha\times\beta}\frac{(y|l_q)(x-y)}{||x-y||^{1+\xi}}ds_x ds_y \tag{7.56}$$

The integrands are homogeneous of degrees $q=-\xi$ and $q=1-\xi$ with $n=2$.

• If $q+n\neq 0$, formula (6.4) gives

$$
\begin{aligned}
Q^{1,\psi_q}(\alpha,\beta,\xi) = \psi_q(o)[\sum_{k=\pm}\frac{ks^k}{2}S(a^k,\beta,0) - \sum_{l=\pm}\frac{lt^l}{2}S(b^l,\alpha,0)] + \sum_{k=\pm}\frac{ks^k}{3}S^{\psi_q(y)}(a^k,\beta,0)\\
-\psi_q(o)\sum_{k=\pm}\frac{ks^k}{3}S(a^k,\beta,0) - \sum_{l=\pm}\frac{lt^l}{3}(\psi_q(b^l)-\psi_q(o))S(b^l,\alpha,0),
\end{aligned}\tag{7.57}
$$

where

$$S^{\psi_q(y)}(a^k,\beta,0) = \int\limits_\beta\frac{\psi_q(y)(a^k-y)}{||a^k-y||}ds_y \tag{7.58}$$

• If $q+n=0$ it will be appropriate to apply formula (6.6), which leads us to this expression

$$
\begin{aligned}
Q^{1,\psi_q}(\alpha,\beta,\xi) = \psi_q(o)[\sum_{k=\pm}ks^k L(a^k,\beta) - \sum_{l=\pm}lt^l L(b^l,\alpha)] + \sum_{k=\pm}ks^k L^{\psi_q}(a^k,\beta)\\
-\sum_{k=\pm}ks^k\psi_q(o)L(a^k,\beta) - \sum_{l=\pm}lt^l(\psi_q(b^l)-\psi_q(o))L(b^l,\alpha)
\end{aligned}\tag{7.59}
$$

where

$$L^{\psi_q}(a^k,\beta)=\int\limits_{\beta}\frac{\psi_q(y)(a^k-y)}{||a^k-y||^3}ds_y \tag{7.60}$$

7.5.5 Calculation of $Q^{\varphi_p,\psi_q}(\alpha,\beta,\xi)$

Here we deal with the singular integral

$$Q^{\varphi_p,\psi_q}(\alpha,\beta,\xi)=\int\limits_{\alpha\times\beta}\frac{\phi_p(x)\psi_q(y)(y-x)}{||x-y||^{1+\xi}}ds_xds_y \tag{7.61}$$

7.5.5.1 Intersecting triangles

We project α on the plane parallel to α and going through β. Let $\hat{\alpha}$ be the projection of α and $\hat{x}$ the projection of x. Then, we provide an equivalent expression of the basis functions:

$$\phi_p(x)=1+\mu\frac{(x-a_p|\vec{\alpha})}{|\alpha|^2}=1+\mu\frac{(\hat{x}-\hat{a_p}|\vec{\hat{\alpha}})}{|\alpha|^2} \tag{7.62}$$

$$\psi_q(y)=1+\nu\frac{(y-b_q)|\vec{\beta}}{|\beta|^2} \tag{7.63}$$

where $\mu=\pm1(\mu=1$ if $a_p=a^+$ and $\mu=-1$ if $a_p=a^-)$. Likewise, $\nu=\pm1(\nu=1$ if $b_p=b^+$ and $\nu=-1$ if $b_p=b^-)$. We choose the intersection of the supports of $\hat{\alpha}$ and β, as origin and we note it o. we obtain :

$$\begin{aligned}\phi_p(x)&=1+\mu\frac{(o-\hat{a_p}|\vec{\alpha})}{|\alpha|^2}+\mu\frac{(\hat{x}-o|\vec{\alpha})}{|\alpha|^2}\\&=C_p+\mu\frac{(\hat{x}-o)}{\vec{\hat{\alpha}}}\end{aligned} \tag{7.64}$$

$$\begin{aligned}\psi_q(y)&=1+\nu\frac{(o-b_q|\vec{\beta})}{|\beta|^2}+\nu\frac{(y-o|\vec{\beta})}{|\beta|^2}\\&=C_q+\mu\frac{(y-o|\vec{\beta})}{|\beta|^2}\end{aligned} \tag{7.65}$$

where $C_p=1+\mu\frac{(o-\hat{a_p})|\vec{\alpha}}{|\alpha|^2}$ and $C_q=1+\nu\frac{(o-b_q)|\vec{\beta}}{|\beta|^2}$. Then, the expression of Q^{φ_p,ψ_q} becomes

$$Q^{\phi_p,\psi_q}(\alpha,\beta,\xi)=C_qC_q\int\limits_{\hat{\alpha}\times\beta}\frac{(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{1+\xi}{2}}}ds_{\hat{x}}ds_y+\frac{C_p\nu}{|\beta|^2}\int\limits_{\hat{\alpha}\times\beta}\frac{(y-o|\vec{\beta})(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{1+\xi}{2}}}ds_{\hat{x}}ds_y$$

$$+\frac{C_q\mu}{|\alpha|^2}\int\limits_{\hat{\alpha}\times\beta}\frac{(\hat{x}-o|\vec{\alpha})(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{1+\xi}{2}}}ds_{\hat{x}}ds_y+\frac{\mu\mu}{|\hat{\alpha}|^2|\beta|^2}\int\limits_{\hat{\alpha}\times\beta}\frac{(y-o|\vec{\beta})(\hat{x}-o|\vec{\alpha})(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{1+\xi}{2}}}ds_{\hat{x}}ds_y \tag{7.66}$$

In the formula below the integrands are homogeneous with variable h and of degrees $q=-\xi$, $q=1-\xi$, $1-\xi$ and $2-\xi$. Formula (6.5) gives

$$\begin{aligned}
&Q^{\phi_p,\psi_q}(\alpha,\beta,\xi)=C_pC_q[\textstyle\sum_{k=\pm}k\hat{s}^k\int_\beta(\hat{a}^k-y)h^{2-\xi}\int_h^{+\infty}\frac{du}{u^{3-\xi}(u^2+||\hat{a}^k-y||^2)^{\frac{1+\xi}{2}}}ds_y\\
&+\textstyle\sum_{l=\pm}l\hat{t}^l\int_{\hat{\alpha}}(\hat{x}-b^l)h^{2-\xi}\int_h^{+\infty}\frac{du}{u^{3-\xi}(u^2+||\hat{x}-b^l||^2)^{\frac{1+\xi}{2}}}ds_x]\\
&+\tfrac{C_p\nu}{|\beta|^2}[\textstyle\sum_{k=\pm}k\hat{s}^k\int_\beta(y-o|\vec{\beta})(\hat{a}^k-y)h^{3-\xi}\int_h^{+\infty}\frac{du}{u^{4-\xi}(u^2+||\hat{a}^k-y||^2)^{\frac{1+\xi}{2}}}ds_y\\
&+\textstyle\sum_{l=\pm}l\hat{t}^l\int_{\hat{\alpha}}(b^l-o|\vec{\alpha})(\hat{x}-b^l)h^{3-\xi}\int_h^{+\infty}\frac{du}{u^{4-\xi}(u^2+||\hat{x}-b^l||^2)^{\frac{1+\xi}{2}}}ds_x]\\
&+\tfrac{C_q\mu}{|\alpha|^2}[\textstyle\sum_{k=\pm}k\hat{s}^k\int_\beta(\hat{a}^k-o|\vec{\alpha})(\hat{a}^k-y)h^{3-\xi}\int_h^{+\infty}\frac{du}{u^{4-\xi}(u^2+||\hat{a}^k-y||^2)^{\frac{1+\xi}{2}}}ds_y\\
&+\textstyle\sum_{l=\pm}l\hat{t}^l\int_{\hat{\alpha}}(\hat{x}-o|\vec{\alpha})(\hat{x}-b^l)h^{3-\xi}\int_h^{+\infty}\frac{du}{u^{4-\xi}(u^2+||\hat{x}-b^l||^2)^{\frac{1+\xi}{2}}}ds_x]\\
&+\tfrac{\nu\mu}{|\alpha|^2|\beta|^2}[\textstyle\sum_{k=\pm}k\hat{s}^k\int_\beta(\hat{a}^k-o|\vec{\alpha})(y-o|\vec{\beta})(\hat{a}^k-y)h^{4-\xi}\int_h^{+\infty}\frac{du}{u^{5-\xi}(u^2+||\hat{a}^k-y||^2)^{\frac{1+\xi}{2}}}ds_y\\
&+\textstyle\sum_{l=\pm}l\hat{t}^l\int_{\hat{\alpha}}(\hat{x}-o|\vec{\alpha})(\hat{b}^l-o|\vec{\alpha})(\hat{x}-b^l)h^{4-\xi}\int_h^{+\infty}\frac{du}{u^{5-\xi}(u^2+||\hat{x}-b^l||^2)^{\frac{1+\xi}{2}}}ds_x]
\end{aligned} \tag{7.67}$$

which is equivalent to

$$\begin{aligned}
&Q^{\phi_p,\psi_q}(\alpha,\beta,\xi)=C_pC_q[\textstyle\sum_{k=\pm}k\hat{s}^kR_2(\hat{a}^k,\beta,h,\xi)\xi-\sum_{l=\pm}l\hat{t}^lR_2(b^l,\hat{\alpha},h,\xi)]\\
&+\textstyle\sum_{k=\pm}k\hat{s}^k[\frac{C_q\mu}{|\alpha|^2}(\hat{a}^k-o|\vec{\alpha})R_3(\hat{a}^k,\beta,h,\xi)+\frac{C_p\nu}{|\beta|^2}M_3(\hat{a}^k,o,\beta,h,\xi)]\\
&+\textstyle\sum_{l=\pm}l\hat{t}^l[-\frac{C_p\nu}{|\beta|^2}(b^l-o|\vec{\beta})R_3(b^l,\hat{\alpha},h,\xi)-\frac{C_q\mu}{|\alpha|^2}M_3(b^l,o,\alpha,h,\xi)]\\
&+\tfrac{\nu\mu}{|\alpha|^2|\beta|^2}[\textstyle\sum_{k=\pm}k\hat{s}^k(\hat{a}^k-o|\vec{\alpha})M_4(\hat{a}^k,o,\beta,h,\xi)-\sum_{l=\pm}l\hat{t}^l(b^l-o|\vec{\beta})M_4(b^l,o,\alpha,h,\xi)]
\end{aligned} \tag{7.68}$$

where

$$M_3(\hat{a}^k,o,\beta,h,\xi)=\int\limits_\beta(\hat{a}^k-o|\vec{\alpha})(\hat{a}^k-y)f_3(h,||\hat{a}^k-y||,\xi)ds_y \tag{7.69}$$

$$M_4(\hat{a}^k,o,\beta,h,\xi)=\int\limits_\beta(a-o|\vec{\alpha})(a-y)f_4(h,||\hat{a}^k-y||,\xi)ds_y \tag{7.70}$$

are given in Appendix (B).

7.5.5.2 Coplanar triangles

As we saw in section (7.5.2), we need to distinguish two geometrical configurations for the supports of the sides α and β.

Two intersecting segments

When the supports of α and β are intersecting, we choose the intersection of the supports, denoted o, as the origin. Then we use the same decompositions as (7.64) and (7.65).

$$\phi_p(y) = C_p + \mu\frac{(x-o|\vec{\alpha})}{|\alpha|^2} \tag{7.71}$$

$$\psi_q(y) = C_q + \mu\frac{(y-o|\vec{\beta})}{|\beta|^2} \tag{7.72}$$

In case where $\xi - 2 \neq 0$:

If we combine (7.71, 7.72) with (6.4) we obtain

$$\begin{aligned}
&Q^{\varphi_p,\psi_q}(\alpha,\beta,0) = \tfrac{C_pC_q}{2}[\textstyle\sum_{k=\pm} ks^k S(a^k,\beta,0) - \sum_{l=\pm} lt^l S(b^l,\alpha,0]\\
&+\tfrac{C_p\mu}{3|\beta|^2}[\textstyle\sum_{k=\pm} ks^k N(a^k,o,\beta,0) - \sum_{l=\pm} lt^l(b^l-o|\vec{\beta})S(b^l,\alpha,0)]\\
&+\tfrac{C_q\nu}{3|\alpha|^2}[\textstyle\sum_{k=\pm} ks^k(a^k-o|\vec{\alpha})S(a^k,\beta,0) - \sum_{l=\pm} lt^l N(b^k,o,\alpha,0)]\\
&+\tfrac{\nu\mu}{4|\alpha|^2|\beta|^2}\textstyle\sum_{k=\pm} ks^k[(a^k-o|\vec{\alpha})N(a^k,o,\beta,0) - \sum_{l=\pm} lt^l(b^l-o|\vec{\beta})N(b^k,o,\alpha,0)].
\end{aligned} \tag{7.73}$$

where

$$N(a^k,o,\beta,\xi) = \int_\beta \frac{(y-o|\vec{\beta})(a^k-y)}{||a^k-y||^{1+\xi}}ds_y \tag{7.74}$$

is given in Appendix (B)

Conclusion

This chapter presents the evaluation of the integral $\mathcal{S}_{i,j}$ when using affine basis functions in each element. We began by presenting the basis functions used and some of their properties that will simplify the calculations. We only dealt with the case of adjacent triangles. The case of triangles with common vertices presents no particular difficulty. This will be the subject of the next chapter.

CHAPTER 8

Triangular mesh elements with common vertex

In this chapter, we continue to apply the reduction process to evaluate singular integrals appearing in chapter 5, on the product of two triangular mesh element S and T. We deal with the case where triangles have a common vertex.

8.1 Reduction to dimension three

Here we deal with the case where the triangles have a vertex in common. Let $a_1 = b_1$ be the common vertex. We keep the same local orientation in S, then, $a_2 = a_1^+$ and $a_3 = a_1^-$. But, we take the opposite direction for T, so, $b_2 = b_1^-$ and $b_3 = b_1^-$.
we want to evaluate 9 different singular integrals in the form 7.8

$$I^{i,j} = \int\limits_{S\times T} (\phi_i(x)\psi_j(y))\frac{(x-y)}{||x-y||^{\xi+1}}dxdy;\ \xi = 0, 2;\ i, j = 1, 2, 3. \tag{8.1}$$

As already mentioned before the integral $I^{i,j}$ is not singular in the case $\xi = 0$ but the method still valid and it will not cost us extra calculations. It will be wise to choose the common vertex $a_1 = b_1$, as origin. In this case the basis functions are written

$$\phi_i(x) = \phi_i(a_1) + (\underline{x}|\overrightarrow{e_1});\quad \psi_j(y) = \psi_j(b_1) + (\underline{y}|\overrightarrow{l_1}). \tag{8.2}$$

Applying Theorem 6.4 we get

$$I^{1,3} = (\frac{g_1}{5-\xi} - \frac{g_1}{6-\xi})U^{1,\psi_3}(\alpha_1, T, \xi) + (\frac{d_1}{6-\xi} - \frac{d_1}{5-\xi})U^{\psi_3,1}(\beta_1, S, \xi)$$

$$-\frac{g_1}{6-\xi}U^{\psi_3,\phi_1}(\beta,S,\xi) \tag{8.3}$$

$$I^{2,1}=(\frac{g_1}{5-\xi}-\frac{g_1}{6-\xi})U^{\phi_2,1}(\alpha_1,T,\xi)+(\frac{d_1}{6-\xi}-\frac{d_1}{5-\xi})U^{1,\varphi_2}(\beta_1,S,\xi) +\frac{g_1}{6-\xi}U^{\phi_2,\psi_1}(\alpha_1,T,\xi) \tag{8.4}$$

$$I^{2,2}=\frac{g_1}{6-\xi}U^{\phi_2,\psi_2}(\alpha_1,T,\xi)-\frac{d_1}{6-\xi}U^{\psi_1,\phi_2}(\beta,S,\xi) \tag{8.5}$$

$$I^{2,3}=\frac{g_1}{6-\xi}U^{\phi_2,\psi_3}(\alpha_1,T,\xi)-\frac{d_1}{6-\xi}U^{\psi_3,\varphi_2}(\beta_1,S,\xi) \tag{8.6}$$

$$I^{3,1}=(\frac{g_1}{5-\xi}-\frac{g_1}{6-\xi})U^{\varphi_3,1}(\alpha_1,T,\xi)-(\frac{d_1}{5-\xi}-\frac{d_1}{6-\xi})U^{1,\varphi_3}(\beta_1,S,\xi) +\frac{g_1}{6-\xi}U^{\psi_3,\phi_1}(\alpha_1,T,\xi) \tag{8.7}$$

$$I^{3,2}=\frac{g_1}{6-\xi}U^{\phi_3,\psi_2}(\alpha_1,T,\xi)-\frac{d_1}{6-\xi}U^{\psi_2,\varphi_3}(\beta_1,S,\xi) \tag{8.8}$$

$$I^{3,3}=\frac{g_1}{6-\xi}U^{\phi_3,\psi_3}(\alpha_1,T,\xi)-\frac{d_1}{6-\xi}U^{\psi_3,\varphi_3}(\beta_1,S,\xi) \tag{8.9}$$

where $U^{1,\psi_q}(\alpha_i,T,\xi)$, $U^{\psi_q,1}(\beta_j,S,\xi)$ and $U^{\phi_p,\psi_q}(\alpha_i,T,\xi)$ are defined as following

$$U^{1,\psi_q}(\alpha_i,T,\xi)=\int\limits_{\alpha_i\times T}\frac{\psi_q(y)(x-y)}{||x-y||^{\xi+1}}ds_xdy \tag{8.10}$$

$$U^{\varphi_p,1}(\alpha,T,\xi)=\int\limits_{\alpha_i\times T}\frac{\varphi_p(x)(x-y)}{||x-y||^{\xi+1}}ds_xdy \tag{8.11}$$

$$U^{\varphi_p,\psi_q}(\alpha_i,T,\xi)=\int\limits_{\alpha_i\times T}\frac{\varphi_p(x)\psi_q(y)(x-y)}{||x-y||^{\xi+1}}ds_xdy \tag{8.12}$$

8.2 Reduction to dimension two

Let ω be the intersection of the supports of T and S. Here also we need to distinguish two different cases.

8.2.1 Intersection of two triangles w is secant with α

8.2.1.1 Computation of $U(\alpha, T, \xi)$

We want to evaluate the integral

$$U(\alpha, T, \xi) = \int\limits_{\alpha \times T} \frac{(x-y)}{||x-y||^{\xi+1}} ds_x dy; \;\; \xi = 0, 2. \tag{8.13}$$

We consider $\alpha \cap w = I$ as a new origin

$$= \sum_{j=1}^{3} \frac{d_j(I)}{3-\xi} \int\limits_{\alpha \times \beta_j} \frac{x-y}{||x-y||^{1+\xi}} ds_x ds_y + \sum_{k=\pm} \frac{k s^k}{3-\xi} \int\limits_{T} \frac{a^k - y}{||a^k - y||^{1+\xi}} dy$$

$$= \sum_{j=1}^{3} Q(\alpha, \beta_j, \xi) + \sum_{k=\pm} \frac{k s^k}{3-\xi} P(a^k, T, \xi)$$

$P(a^k, T, \xi)$ and $Q(\alpha, \beta_j, \xi)$ were already computated in B

8.2.1.2 Computation of U^{1,ψ_q}

Recall that $U^{1,\psi_q}(\alpha, T, \xi)$ is defined by

$$U^{1,\psi_q}(\alpha, T, \xi) = \int\limits_{\alpha \times T} \frac{\psi_q(y)(x-y)}{||x-y||^{\xi+1}} ds_x dy$$

and

$$\psi_q(y) = \psi_q(I) + (\underline{y} | \overrightarrow{l_q})$$

We get

$$U^{1,\psi_q}(\alpha, T, \xi) = \int\limits_{\alpha \times T} \frac{\psi_q(I)(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{\xi+1}} ds_x dy + \int\limits_{\alpha \times T} \frac{(\underline{y} | \overrightarrow{l_q})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{\xi+1}} ds_x dy$$

and then the reduction formula gives

$$U^{1,\psi_q}(\alpha, T, \xi) = \psi_q(I)(1 - \frac{3-\xi}{4-\xi}) U(\alpha, T, \xi)$$

$$+ \frac{1}{4-\xi} \Sigma_{k=\pm} k s^k P^{\psi_q}(a^k, T, \xi) + \sum_{j=1}^{3} \frac{\delta_j(I)}{4-\xi} Q^{1,\psi_q}(\alpha, \beta_j, \xi) \tag{8.14}$$

8.2.1.3 Computation of $U^{\varphi_p,1}$

Recall that $U^{\varphi_p,1}(\alpha,T,\xi)$ is defined by

$$U^{\varphi_p,1}(\alpha,T,\xi)=\int\limits_{\alpha\times T}\varphi_p(x)\frac{(x-y)}{||x-y||^{\xi+1}}ds_xdy$$

and

$$\varphi_p(x)=\varphi_p(I)+(\underline{x}|\overrightarrow{e_p})$$

We get

$$U^{\varphi_p,1}(\alpha,T,\xi)=\int\limits_{\alpha\times T}\frac{\varphi_p(I)(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{\xi+1}}ds_xdy+\int\limits_{\alpha\times T}\frac{(\underline{x}|\overrightarrow{e_p})(\underline{x}-\underline{y})}{||\underline{x}-\underline{y}||^{\xi+1}}ds_xdy$$

applying the reduction formula (6.4)

$$U^{\varphi_p,1}(\alpha,T,\xi)=\varphi_p(I)(1-\frac{3-\xi}{4-\xi})U(\alpha,T,\xi)$$

$$+\frac{1}{4-\xi}\sum_{k=\pm}\varphi_p(a^k)ks^kP(a^k,T,\xi)+\sum_{j=1}^{3}\frac{\delta_j(I)}{4-\xi}Q^{\varphi_p,1}(\alpha,\beta_j,\xi)\tag{8.15}$$

8.2.2 Intersection of two triangles w is parallel to α

When the support of α is parallel to w, the homogeneity of the integrand is lost. Then, the projection of α is carried out on the plane of the triangle T in order to decompose the norm $||x-y||$. Let $\hat{\alpha}$ and $\hat{a}^{\pm}$ be the respective projections of α and $a^{\pm}$ on the T plane . We denote by $\vec{z}=\hat{x}-x$ and $h=||x-\hat{x}||$ the distance between the segment α and $\hat{\alpha}$ which plays the role of the parameter.

8.2.2.1 Computation of $U(\alpha,T,\xi)$

$$U(\alpha,T,\xi)=\int\limits_{\hat{\alpha}\times T}\frac{(x-\hat{x})+(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}}ds_xdy\tag{8.16}$$

Thus, the integrand is homogeneous with respect to the pair formed by the integration variables $(\hat{x},y)$ and the parameter h. So we have

$$U(\alpha,T,\xi)=U_1(\hat{\alpha},T,h,\xi)-\vec{z}U_2(\hat{\alpha},T,h,\xi)$$

where

$$U_1(\hat{\alpha},T,h,\xi)=\int\limits_{\hat{\alpha}\times T}\frac{(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}}ds_{\hat{x}}dy$$

$$U_2(\hat{\alpha},T,h,\xi)=\int\limits_{\hat{\alpha}\times T}\frac{1}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}}ds_{\hat{x}}dy$$

The next step, we chose the extremity $\hat{a}^-$ of $\hat{\alpha}$ as a new origine, we apply the reduction formula (6.5) and we get

$$U(\alpha,T,\xi)=|\alpha|\int\limits_T(\hat{a}^+-y)f_3(h,||\hat{a}^+-y||)dy+\sum_{j=1}^{3}\delta_j(\hat{a}^-)\int\limits_{\hat{\alpha}\times\beta}f_2(h,||\hat{x}-y||)ds_{\hat{x}}ds_y$$

$$-\vec{z}[|\alpha|\int\limits_T f_2(h,||\hat{a}^+-y||,\xi)dy+\sum_{j=1}^{3}\delta_j(\hat{a}^-)\int\limits_{\hat{\alpha}\times\beta}f_2(h,||\hat{x}-y||)ds_{\hat{x}}ds_y]$$

otherwise

$$U(\alpha,T,\xi)=|\alpha|P_3(\hat{a}^+,T,h,\xi)+\sum_{j=1}^{3}\delta_j(\hat{a}^-)Q_2(\hat{\alpha},\beta,h,\xi)$$

$$-\vec{z}[|\alpha|P_2(\hat{a}^+,T,h,\xi)+\sum_{j=1}^{3}\delta_j(\hat{a}^-)Q_2(\hat{\alpha},\beta,h,\xi)]$$

8.2.2.2 Computation of $U^{1,\psi_q}(\alpha,T,\xi)$

We are inspired by what we did with $U(\alpha,T,\xi)$

$$U^{1,\psi_q}(\alpha,T,\xi)=\int\limits_{\alpha\times T}\frac{\psi_q(y)(x-y)}{||x-y||^{\xi+1}}ds_xdy$$

$$=U_2^{1,\psi_q}(\alpha,T,\xi)-\vec{z}U_1^{1,\psi_q}(\alpha,T,\xi)$$

where

$$U_1^{1,\psi_q}(\alpha,T,\xi)=\int\limits_{\hat{\alpha}\times T}\frac{\psi_q(y)}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}}ds_xdy$$

$$U_2^{1,\psi_q}(\alpha,T,\xi)=\int\limits_{\hat{\alpha}\times T}\frac{\psi_q(y)(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}}ds_xdy$$

We calculate each term separately. Let's start with evaluating $U_2^{1,\psi_q}(\hat{\alpha},T,\xi)$

$$U_2^{1,\psi_q}(\hat{\alpha},T,\xi)=\psi_q(\hat{a}^-)U_2(\hat{\alpha},T,h,\xi)+|\alpha|P_{4vect}^{\psi_q}(\hat{a}^+,T,h,\xi)$$

$$+\delta_{q+1}(\hat{a}^-)Q^{1,\psi_q}_{4vect}(\hat{\alpha},\beta_{q+1},h,\xi)+\delta_{q+2}(\hat{a}^-)Q^{1,\psi_q}_{4vect}(\hat{\alpha},\beta_{q+2},h,\xi)$$

$$-\psi_q(\hat{a}^-)[|\alpha|P_{4vect}(\hat{a}^+,T,h,\xi)+\sum_{j=1}^{3}\delta_j(\hat{a}^-)Q_{4vect}(\hat{\alpha},\beta_j,h,\xi)] \tag{8.17}$$

where

$$P^{\psi_q}_{4vect}(\hat{a}^+,T,h,\xi)=\int_T(\hat{a}^+-y)\psi_q(y)f_4(h,||\hat{a}^+-y||,\xi)dy$$

$$Q^{1,\psi_q}_{4vect}(\hat{\alpha},\beta_{q+1},h,\xi)=\int\limits_{\hat{\alpha}\times\beta_{q+1}}\psi_q(y)f_4(h,||\hat{x}-y||,\xi)ds_{\hat{x}}ds_y$$

$$P_{4vect}(\hat{a}^+,T,h,\xi)=\int_T(\hat{a}^+-y)f_4(h,||\hat{a}^+-y||,\xi)dy$$

$$Q_{4vect}(\hat{\alpha},\beta_{q+1},h,\xi)=\int\limits_{\hat{\alpha}\times\beta_{q+1}}f_4(h,||\hat{x}-y||,\xi)ds_{\hat{x}}ds_y$$

are performed below. On the other hand we have

$$U^{1,\psi_q}_1(\hat{\alpha},T,h,\xi)=\psi_q(\hat{a}^-)U_1(\hat{\alpha},T,h,\xi)+|\alpha|P^{\psi_q}_{3scal}(\hat{a}^+,T,h,\xi)$$

$$+\delta_{q+1}(\hat{a}^-)Q^{1,\psi_q}_{3scal}(\hat{\alpha},\beta_{q+1},h,\xi)+\delta_{q+2}(\hat{a}^-)Q^{1,\psi_q}_{3scal}(\hat{\alpha},\beta_{q+2},h,\xi)$$

$$-\psi_q(\hat{a}^-)[|\alpha|P_{3scal}(\hat{a}^+,T,h,\xi)+\sum_{j=1}^{3}\delta_j(\hat{a}^-)Q_{3scal}(\hat{\alpha},\beta,h,\xi)] \tag{8.18}$$

where

$$P^{\psi_q}_{3scal}(\hat{a}^+,T,h,\xi)=\int_T(\hat{a}^+-y)\psi_q(y)f_3(h,||\hat{a}^+-y||,\xi)dy$$

$$Q^{1,\psi_q}_{3scal}(\hat{\alpha},\beta_{q+1},h,\xi)=\int\limits_{\hat{\alpha}\times\beta_{q+1}}\psi_q(y)f_3(h,||\hat{x}-y||,\xi)ds_{\hat{x}}ds_y$$

$$P_{3scal}(\hat{a}^+,T,h,\xi)=\int_T(\hat{a}^+-y)f_3(h,||\hat{a}^+-y||,\xi)dy$$

8.2.2.3 Computation of $U^{\phi(p),\psi_q}(\alpha,T,\xi)$

We want to evaluate the singular integral

$$U^{\varphi_p,\psi_q}(\alpha_i,T,\xi)=\int\limits_{\alpha_i\times T}\frac{\varphi_p(x)\psi_q(y)(x-y)}{||x-y||^{\xi+1}}ds_xdy$$

$$= \int\limits_{\hat{\alpha}\times T} \frac{\varphi_p(x)\psi_q(y)(x-\hat{x})}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}} ds_{\hat{x}}dy \int\limits_{\hat{\alpha}\times T} \frac{\varphi_p(x)\psi_q(y)(\hat{x}-y)}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}} ds_{\hat{x}}dy$$

$$= -\vec{z}U_1^{\phi(p),\psi_q}(\hat{\alpha},T,\xi) + U_2^{\phi(p),\psi_q}(\hat{\alpha},T,\xi)$$

We calculate each term separately. Let's start with calculating $U_1^{\phi(p),\psi_q}(\hat{\alpha},T,\xi)$. We replace the basis functions in their equivalent expression with $\hat{a}^-$ as the origin, the integrant is homogeneous with parameter, we apply the formula (6.5) Which gives

$$U_1^{\phi(p),\psi_q}(\hat{\alpha},T,\xi) = \phi_p(\hat{a}^-)\psi_q(\hat{a}^-) \int\limits_{\hat{\alpha}\times T} \frac{1}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}} ds_{\hat{x}}dy + \int\limits_{\hat{\alpha}\times T} \frac{\varphi_p(\hat{a}^-)(\underline{y}|\overrightarrow{f_q})}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}} ds_{\hat{x}}dy$$

$$+ \int\limits_{\hat{\alpha}\times T} \frac{(\underline{x}|\overrightarrow{e_p})\psi_q(\underline{y})(\underline{y})}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}} ds_{\hat{x}}dy + \int\limits_{\hat{\alpha}\times T} \frac{(\underline{x}|\overrightarrow{e_p})(\underline{y}|\overrightarrow{f_q})}{(h^2+||\hat{x}-y||^2)^{\frac{\xi+1}{2}}} ds_{\hat{x}}dy$$

Conclusion

In this chapter, we gave formulae of evaluation of the integral $I^{i,j}$, on two triangles having a vertex in common.

CHAPTER 9

Numerical results

In the previous chapter we discretized the variational formulation of a three-dimentional scattering problem. Here we present our numerical results.

9.1 Radar Cross Section

In radiation theory it's well known that the energy intercepted by an object can be reflected, transmitted or absorbed through the target. We can assume that most of the energy is reflected. The spatial distribution of this energy depends on the size, shape and composition of the target, and on the frequency and nature of the incident wave. This distribution of energy is called scattering, and the target itself is often referred to as a scatterer. The radar cross section (RCS) of the body is a measure of the energy scattered in a particular direction for a given illumination [51].

Bistatic scattering is the name given to the situation when the scattering direction is not back toward the source of the radiation. If $\mathbf{E}$ and $\mathbf{H}$ represent fields scattered by an object illuminated by incident plane wave $\mathbf{E}^{inc}$ traveling in the direction of the unit vector $\mathbf{k}$, the *bistatic radar cross section* in the observation direction $\mathbf{r}$ is

$$\sigma(\mathbf{r}, \mathbf{k}) = \lim_{r \to \infty} 4\pi r^2 \frac{|\mathbf{E}|^2}{|\mathbf{E}^{inc}|^2}.$$

This cross section is defined as the area through which an incident plane wave carries sufficient power to produce, by omnidirectional radiation, the same scattered power density as that observed in a given far field direction. The *monostatic radar cross section* is defined

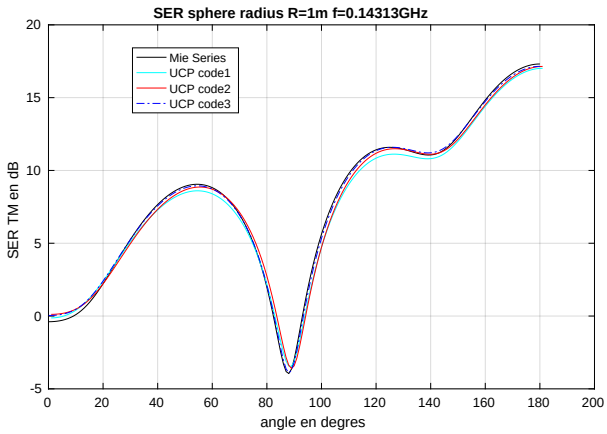

Figure 9.1: Bistattic RCS for a coated unit sphere, when $d = 0.05$, $r = 0.95$, $\mu_r = 1.0$, $\epsilon_r = 2$ and $\omega = 0.9$GHz

as the radar cross section observed in the back scattering direction, $\sigma(-\mathbf{k}, \mathbf{k})$.

Radar cross section is the measure of a target's ability to reflect radar signals in the direction of the radar receiver, i.e. it is a measure of the ratio of backscatter power per steradian (unit solid angle) in the direction of the radar (from the target) to the power density that is intercepted by the target.

The radar cross section of a target can be viewed as a comparison of the strength of the

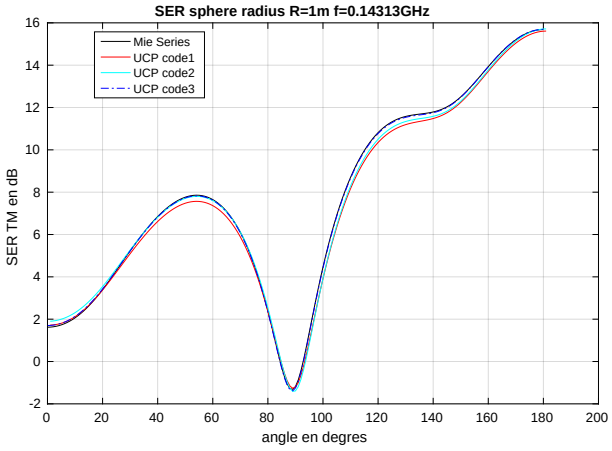

Figure 9.2: Bistattic RCS for a coated unit sphere, when $d = 0.0125$, $\mu_r = 1.0$, $\epsilon_r = 2$ and $\omega = 0.9$GHz

reflected signal from a target to the reflected signal from a perfectly smooth sphere.
The units for RCS are square meters. As RCS can span a wide range of values, a logarithmic decibel scale is also used with a typical reference value σ_{ref} equal to $1m^2$:

$$\sigma_{dBm^2} = 10 \ \log_{10}(\frac{\sigma}{\sigma_{ref}}) \tag{9.1}$$

9.2 Numerical results

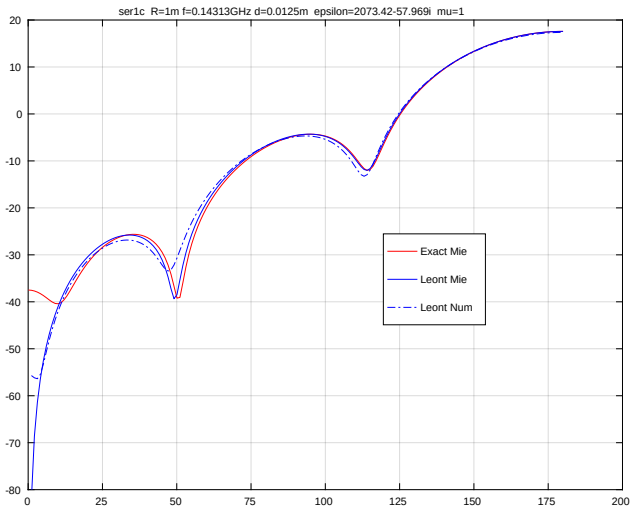

Figure 9.3: Bistattis RCS for a coated unit sphere

To illustrate our approach, we compute the RCS of a unit sphere coated with a thin dielectric layer. The radius of the inner conductor is denoted by r and the thickness of the coating by d. The material properties are described by a Leontovich Impedance $Z = const$.

$$Z = z_0 \sqrt{\frac{\mu_r}{\epsilon_r}} \tan\left(\sqrt{\mu_r \epsilon_r} k_0 d\right)$$

where μ_r and ϵ_r denote the magnetic permeability and electical permittivity of the coating. To validate the code, we test it on a sphere with different sphere-meshes and we compare it with Mie series. We take the sphere's radius is 1m and incident waves pulsation $\omega = 0.9GHz$. In figure (9.1), we plot the exact RCS calculated by Mie Series(continuous black line) and we compare it with our code results in different sphere mesh with inner radius $r = 0.95$ and coating thikness $d = 0.05$. In 9.1, the azure continuous line is the ucp code calculated on the mesh $\lambda/10$, red continuous line in our code calculated on the mesh $\lambda/15$ and the blue discontinue line is our code on the mesh $\lambda/20$.

In the second test, we decrease the coating thikness to $d = 0.125$ and we increase the inner sphere radius, for the same frequence as the previous test. In figure (9.2), we plot the exact RCS calculated by Mie Series(continuous black line) the red continuous line is our code calculated on the mesh $\lambda/10$, azure continuous line represent our code calculated on the mesh $\lambda/15$ and the blue discontinuous line represent our code on the mesh $\lambda/20$

In figure 9.3, we consider a coating thikness $d = 0.125$, a frequence $f = 014313GHz$, the same constant for the magnetic permiability $\mu_r = 1.0$ and electical permittivity $\epsilon_r = 2073.42$. In figure (9.3), we plot the exact RCS calculated by Mie Series(continuous red line) and the RCS calculated with our code

Physical constant

- $\pi \quad = 3.1415926535897932384...$;
- $c \quad = 2.99792458.10^8 ms^{-1}$ speed of light;
- $\mu_0 \quad = 4\pi.10^{-7}$ free space permeability in (henry/m);
- $\epsilon_0 \quad = 1/(\mu_0 c^2)$ free space permittivity in (farad/m);
- $Z_0 \quad = \sqrt{\mu_0/\epsilon_0} = \mu_0 c$ impedance in (ohm);
- $\omega \quad = 2\pi f$;
- $\lambda \quad = \frac{c}{f}$ wave length;
- $k_0 \quad = \omega\sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{\omega}{c}$ wave number;

CHAPTER A

Appendix A

A.1 Modified Bessel functions I and K

The differential equation

$$z^2\frac{\partial^2 u}{\partial z^2} + z\frac{\partial u}{\partial z} - (z^2+\nu^2)u = 0,$$

where ν is a real constant, is called the modified Bessel's equation and its solutions $I_\nu(z)$ and $K_\nu(z)$ are known as modified Bessel functions.
To have the formulas for $I_\nu(z)$ and $K_\nu(z)$ we need the following preliminary definitions ([1]).

Definition A.1.1. *The Euler constant (sometimes called Gamma) is defined as the limit of the expression*

$$\gamma = \lim_{n\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}-\ln n\right] = 0.5772156643.$$

The abbreviation $\ln$ *stands for the natural logarithm, also called the base* e *logarithm.*

Definition A.1.2. *(*Euler's formula*). For all complex numbers* z*, except the non-positive integers (*$z \neq 0, -1, -2, \ldots$*),*

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\ldots(z+n)}$$

If n *is a positive integer,the function is defined by* $\Gamma(n) = (n-1)!$.

Definition A.1.3. *(Modified Bessel function of the first kind)*

$$I_\nu(z) = \big(\frac{1}{2}z\big)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}.$$

We also have the following decay behaviour for $K_\nu(z)$ as ν is fixed $(\mathrm{Re}\nu > 0)$ and $z \to 0$

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(\frac{1}{2}z)^{-\nu}.$$

A.2 Approximation of unit normal vector

The following result can be found in [9].

Lemma A.2.1. *Let $\widetilde{\nu}(\widetilde{x})$ be the outward unit normal vector to $\partial\Omega_\delta$ at $\widetilde{x}$. Then the following formula holds*

$$\widetilde{\nu}(\widetilde{x}) = \big(1 + \mathcal{O}(\delta)\big)\nu(x) + \mathcal{O}(\delta)T(x). \tag{A.1}$$

Proof. Under the notations as in Chapter 5, it follows that $\widetilde{x} = \widetilde{X}(t) = X(t) + \delta\rho(x)\nu(x) = X(t)+\delta\rho(x)R_{\frac{-\pi}{2}}X'(t)$ is a parametrization of $\partial\Omega_\delta$ (where R stands for the rotation operator), $\nu(x) = R_{\frac{-\pi}{2}}X'(t)$, the tangent vector $T(x) = X'(t)$, $X'(t)\perp X''(t)$ and $X''(t) = \tau(x)\nu(x)$. Using the simplified notations $\rho(t)$ for $\rho(X(t))$ and $\rho'(t)$ for the tangential derivative of $\rho(t)$, then we have (A.1). □

A.3 Fundamental result of Riesz theory

Theorem A.3.1. *(Fredholm alternative). Let $T : X \to X$ be a linear compact operator and let*

$$\ker(I - T) = \{0\}.$$

Then $I - T$ has the bounded inverse $(I - T)^{-1} : X \to X$, where I is the identity operator.

Theorem A.3.2. *Let $A : X \to X$ be a compact linear operator on a normed space X. Then*

1. *$I - A$ injective $\Longleftrightarrow$ $(I - A)^{-1}$ surjective*
2. *$I - A$ bijective $\Longrightarrow$ $(I - A)^{-1}$ is bounded.*

Theorem A.3.3. *Let X be a normed space, $A : X \to X$ compact. If $\phi - A\phi = 0$ has only trivial solution $\phi = 0$ then $\phi - A\phi = f$ has a unique solution $\phi \in X$ for all $f \in X$, and ϕ depends continuously on f.*

Theorem A.3.4. *(see [36], Theorem 3.4). Let $A : X \to X$ be a compact linear operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective (and therefore also bijective), then the inverse operator $(I - A)^{-1} : X \to X$ is bounded, i.e., $I - A$ is an isomorphism.*

A.4 Singular kernel integrals

Denote by $D \subset \mathbb{R}^d$ a bounded domain with the boundary ∂D of class C^1. We consider the integral operator $\mathcal{P}$ defined as follows

$$\mathcal{P}[\phi] = \int_{\partial D} K(x, y)\phi(y)d\sigma(y). \tag{A.2}$$

Definition A.4.1. (Weakly singular kernel). *A kernel K is said to be weakly singular if it is defined and continuous for all $x, y \in \partial D, x \neq y$, and there exist positive constants M and $\alpha \in (0, d-1];\ d \geqslant 2$ such that*

$$|K(x, y)| \leq M|x - y|^{\alpha - d + 1}, \quad x, y \in \partial D, x \neq y.$$

The following theorem is a well-known theorem of Coifman-McIntosh-Meyer, ([44], Theorem 11). This theorem plays an important role in studying the boundedness of a class of singular kernel integrals.

Theorem A.4.1. *Let $n, m \geq 1$ be integers and suppose that $A : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function. Let $F : \mathbb{R}^m \to \mathbb{R}$ be an infinitely differentiable odd function. Then the antisymmetric kernel*

$$K(x, y) = F\left(\frac{A(x) - A(y)}{|x - y|}\right)|x - y|^{-n} \tag{A.3}$$

defines a bounded operator on $L^2(\mathbb{R}^n)$.

CHAPTER B

One dimensional integrales

B.1 Calculation of $S(a, \beta, \xi)$

$$S(a, \beta, \xi) = \int_{\beta} \frac{a - y}{||a - y||^{1+\xi}} ds_y \tag{B.1}$$

Let $q(a)$ be the projection of a on the support of β and $d = ||q(a) - a||$. Then

$$S(a, \beta, \xi) = (a - q(a)) \int_{t^-}^{t^+} \frac{1}{(d^2 + s^2)^{\frac{1+\xi}{2}}} ds + \int_{t^-}^{t^+} \frac{s}{(d^2 + s^2)^{\frac{1+\xi}{2}}} ds \tag{B.2}$$

where s is the abscissa on β which will be calculated as follows $s = y - q(a)$ and $t^{\pm}$ where $t^{\pm}$ are abscisses of the extremities of beta with $q(a)$ origin and $d = q(a) - a$.

B.2 Calculation of S_1, S_2 and S_3

• $\underline{S_1(a, \beta, h, \xi)}$

$$S_1(a, \beta, h, \xi) = \int_{\beta} f_1(h, ||a - y||, \xi) ds_y$$

$$= \int_{\beta} h^{1-\xi} \int_{h}^{+\infty} \frac{du}{u^{2-\xi}(u^2 + ||a - y||^2)^{\frac{1+\xi}{2}}} ds_y \tag{B.3}$$

where $f_n(h, s, \xi)$ were be defined in 7.29. Let $q(a)$ be the projection of a into the support of β, then

$$S_1(a, \beta, h, \xi) = \int_{t^-}^{t^+} h^{1-\xi} \int_h^{+\infty} \frac{du}{u^{2-\xi}(u^2 + d^2 + s^2)^{\frac{1+\xi}{2}}} ds_y \tag{B.4}$$

where $t^{\pm}$ are abscissae of the extremities of β with $q(a)$ origin and $d = q(a) - a$.

if $\xi = 0$

$$\begin{aligned} S_1(a, \beta, h, 0) &= \int_{t^-}^{t^+} h^1 \int_h^{+\infty} \frac{du}{u^2(u^2 + d^2 + s^2)^{\frac{1}{2}}} ds_y \\ &= \int_{t^-}^{t^+} \frac{\sqrt{h^2 + d^2 + s^2} - h}{d^2 + s^2} ds \\ &= \int_{t^-}^{t^+} \frac{1}{\sqrt{h^2 + d^2 + s^2} - h} ds \end{aligned} \tag{B.5}$$

$$S_1(a, \beta, h, 0) = [\frac{h}{s} - \frac{\sqrt{h^2 + s^2}}{s} + \lg(s + \sqrt{h^2 + s^2})]_{t^-}^{t^+} \tag{B.6}$$

for $\xi = 2$

$$\begin{aligned} S_1(a, \beta, h, 2) &= \int_{t^-}^{t^+} h^{-1} \int_h^{+\infty} \frac{du}{(u^2 + d^2 + s^2)^{\frac{3}{2}}} ds_y \\ &= \int_{t^-}^{t^+} \frac{1}{h(d^2 + s^2)} (1 - \frac{h}{\sqrt{h^2 + d^2 + s^2}}) ds \\ &= [\frac{1}{dh} \arctan(\frac{s}{d}) - \frac{1}{dh} \arctan(\frac{hs}{d\sqrt{h^2 + d^2 + s^2}})]_{t^-}^{t^+} \end{aligned} \tag{B.7}$$

- $S_2(a, \beta, h, \xi)$

$$S_2(a, \beta, h, \xi) = \int_{\beta} f_2(h, ||a - y||, \xi) ds_y$$

$$= \int_{\beta} h^{2-\xi} \int_{h}^{+\infty} \frac{du}{u^{3-\xi}(u^2 + ||a-y||^2)^{\frac{1+\xi}{2}}} ds_y \tag{B.8}$$

$$S_2(a,\beta,h,\xi) = \int_{t^-}^{t^+} h^{2-\xi} \int_{h}^{+\infty} \frac{du}{u^{3-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds \tag{B.9}$$

<u>if $\xi = 0$</u>

$$S_2(a,\beta,h,0) = \int_{t^-}^{t^+} h^2 \int_{h}^{+\infty} \frac{du}{u^3(u^2+d^2+s^2)^{\frac{1}{2}}} ds$$

$$[\frac{h^2+d^2}{2d^2}\text{arcsinh}(\frac{s}{\sqrt{h^2+d^2}}) - \frac{sh^2}{2d^2\sqrt{s^2+d^2}}\text{arcsinh}(\frac{\sqrt{d^2+s^2}}{h})]_{t^-}^{t^+} \tag{B.10}$$

<u>for $\xi = 2$</u>

$$S_2(a,\beta,h,2) = \int_{t^-}^{t^+} \int_{h}^{+\infty} \frac{du}{u(u^2+d^2+s^2)^{\frac{3}{2}}} ds$$

$$= \int_{t^-}^{t^+} -\frac{1}{(d^2+s^2)\sqrt{h^2+d^2+s^2}} + \frac{1}{(d^2+s^2)^{\frac{3}{2}}}\text{arcsinh}\frac{(\sqrt{d^2+s^2})}{h} ds$$

$$= [\frac{s}{d^2(\sqrt{d^2+s^2})}\text{arcsinh}(\frac{\sqrt{d^2+s^2}}{h}) - \frac{1}{d^2}\log(s+\sqrt{d^2+h^2+s^2})]_{t^-}^{t^+} \tag{B.11}$$

- <u>$S_3(a,\beta,h,\xi)$</u>

$$S_3(a,\beta,h,\xi) = \int_{\beta} f_3(h,||a-y||,\xi) ds_y$$

$$S_3(a,\beta,h,\xi) = \int_{t^-}^{t^+} h^{3-\xi} \int_{h}^{+\infty} \frac{du}{u^{4-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds \tag{B.12}$$

<u>if $\xi = 0$</u>

$$S_3(a,\beta,h,0) = \int_{t^-}^{t^+} \frac{2h^3 + (\sqrt{d^2+h^2+s^2})(d^2+s^2-2h^2)}{3(d^2+s^2)^2} ds$$

$$= \frac{1}{3}[\frac{h^3 s - h^2 s\sqrt{d^2+h^2 s^2}}{d^2(d^2+s^2)} + \frac{h^3 \arctan(\frac{s}{d})}{d^3}$$

$$-\frac{h^3}{d^3}\arctan(\frac{hs}{d\sqrt{d^2+h^2+s^2}}) + \lg(s+\sqrt{d^2+h^2+s^2})] \tag{B.13}$$

if $\xi = 2$

$$S_3(a,\beta,h,2) = \int_{t^-}^{t^+} \frac{d^2+s^2+2h^2(h-\sqrt{d^2+h^2+s^2})}{(d^2+s^2)^2\sqrt{d^2+h^2+s^2}} ds$$

$$= [\frac{s(-h+s\sqrt{d^2+h^2+s^2})}{d^2(d^2+s^2)} - \frac{h}{d^3}(\arctan(\frac{s}{d}) - \arctan(\frac{hs}{d\sqrt{d^2+h^2+s^2}}))] \tag{B.14}$$

B.3 Calculation of T_1, T_2 and T_3

• $T_1(a,\beta,h,\xi)$

$$T_1(a,\beta,h,\xi) = \int_{t^-}^{t^+} s f_1(h,\sqrt{d^2+s^2},\xi) ds \tag{B.15}$$

if $\xi = 0$

$$T_1(a,\beta,h,0) = \int_{t^-}^{t^+} sh \int_h^{+\infty} \frac{du}{u^2(u^2+||a-y||^2)^{\frac{1}{2}}} ds$$

if $\xi = 2$

$$T_1(a,\beta,h,2) = \int_{t^-}^{t^+} sh^{-1} \int_h^{+\infty} \frac{du}{(u^2+||a-y||^2)^{\frac{3}{2}}} ds$$

• $T_2(a,\beta,h,\xi)$

$$T_2(a,\beta,h,\xi) = \int_{t^-}^{t^+} s f_2(h,\sqrt{d^2+s^2},\xi) ds \tag{B.16}$$

if $\xi = 0$

$$T_1(a,\beta,h,0) = \int_{t^-}^{t^+} sh^2 \int_h^{+\infty} \frac{du}{u^3(u^2+||a-y||^2)^{\frac{1}{2}}} ds_y$$

$$= \int_{t^-}^{t^+} \frac{s\sqrt{d^2+h^2+s^2}}{2(d^2+s^2)} - \frac{sh^2}{2(d^2+s^2)^{\frac{3}{2}}} \text{arcsinh}(\frac{\sqrt{d^2+s^2}}{h}) ds$$

$$= [\frac{\sqrt{d^2+h^2+s^2}}{2} + \frac{h^2}{2\sqrt{d^2+s^2}} \text{arcsinh}(\frac{\sqrt{d^2+s^2}}{h})]_{t^-}^{t^+} \tag{B.17}$$

if $\xi = 2$

$$T_1(a,\beta,h,2) = \int_{t^-}^{t^+} s \int_h^{+\infty} \frac{du}{u(u^2+||a-y||^2)^{\frac{3}{2}}} ds$$

$$= \int_{t^-}^{t^+} -\frac{s}{(d^2+s^2)\sqrt{d^2+h^2+s^2}} + \frac{s}{(d^2+s^2)^{\frac{3}{2}}} \text{arcsinh}(\frac{\sqrt{d^2+s^2}}{h}) ds$$

$$[-\frac{1}{\sqrt{d^2+s^2}} \text{arcsinh} \frac{(\sqrt{d^2+s^2}}{h})]_{t^-}^{t^+} \tag{B.18}$$

• $T_3(a,\beta,h,\xi)$

$$T_3(a,\beta,h,\xi) = \int_{t^-}^{t^+} s f_3(h,\sqrt{d^2+s^2},\xi) ds \tag{B.19}$$

if $\xi = 0$

$$T_3(a,\beta,h,0) = \int_{t^-}^{t^+} \frac{s}{3(d^2+s^2)^2} (\sqrt{d^2+h^2+s^2}(d^2+s^2-2h^2)+2h^3) ds$$

$$[\frac{-h^3+(d^2+h^2+s^2)^{\frac{3}{2}}}{3(d^2+s^2)}]_{t^-}^{t^+} \tag{B.20}$$

if $\xi = 2$

$$T_3(a,\beta,h,2) = \int_{t^-}^{t^+} \frac{sd^2 + s^3 + 2sh(h - \sqrt{d^2+h^2+s^2})}{\sqrt{d^2+h^2+s^2}(d^2+s^2)^2}$$
$$[\frac{h - \sqrt{d^2+h^2+s^2}}{d^2+s^2}]_{t^-}^{t^+} \tag{B.21}$$

B.4 Calculation of R_1, R_2 and R_3

• $R_1(a,\beta,h,\xi)$

$$\begin{aligned} R_1(a,\beta,h,\xi) &= \textstyle\int_\beta (a-y) f_1(h,||a-y||,\xi) \\ &= \textstyle\int_\beta (a-y) h^{1-\xi} \int_h^{+\infty} \frac{du}{u^{2-\xi}(u^2+||a-y||^2)^{\frac{1+\xi}{2}}} ds_y \end{aligned} \tag{B.22}$$

• $R_2(a,\beta,h,\xi)$

$$R_2(a,\beta,h,\xi) = \int_\beta (a-y) f_2(h,||a-y||,\xi) ds_y$$

$$= (a - q(a)) \int_\beta f_2(h,||a-y||,\xi) ds_y + \int_\beta (q(a)-y) f_2(h,||a-y||,\xi) ds_y$$

$$= (a-q(a)) \int_{t^-}^{t^+} h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds_y - \frac{\vec{\beta}}{|\beta|} \int_{t^-}^{t^+} s h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds_y \tag{B.23}$$

wich is equivalent to

$$R_2(a,\beta,h,\xi) = (a - q(a)) S_2(a,\beta,h,\xi) - \frac{\vec{\beta}}{|\beta|} T_2(a,\beta,h,\xi) \tag{B.24}$$

• $R_2(a,\beta,h,\xi)$

$$R_3(a,\beta,h,\xi) = \int_\beta (a-y) f_3(h,||a-y||,\xi) ds_y$$

$$= (a - q(a)) \int_{\beta} f_3(h, ||a - y||, \xi) ds_y + \int_{\beta} (q(a) - y) f_3(h, ||a - y||, \xi) ds_y$$

$$= (a-q(a)) \int_{t^-}^{t^+} h^{3-\xi} \int_{h}^{+\infty} \frac{du}{u^{4-\xi}(u^2 + d^2 + s^2)^{\frac{1+\xi}{2}}} ds - \frac{\vec{\beta}}{|\beta|} \int_{t^-}^{t^+} s h^{3-\xi} \int_{h}^{+\infty} \frac{du}{u^{4-\xi}(u^2 + d^2 + s^2)^{\frac{1+\xi}{2}}} ds \tag{B.25}$$

wich is equivalent to

$$R_3(a, \beta, h, \xi) = (a - q(a)) S_3(a, \beta, h, \xi) - \frac{\vec{\beta}}{|\beta|} T_3(a, \beta, h, \xi) \tag{B.26}$$

B.5 Calculation of $S_2^{\psi_q}$

• $\underline{S_2^{\psi_q}(a, \beta, h, \xi)}$

$$S_2^{\psi_q}(a, \beta, h, \xi) = \int_{\beta} \psi_q(y) f_2(h, ||a - y||, \xi) ds_y \tag{B.27}$$

We remind that, on a stop β_q, the basis function ψ_q vanishes. So, the calculations will be performed only on $\beta = \beta_{q+1}$ and $\beta = \beta_{q+2}$.The basis function ψ_q on β can be written as follows

$$\psi_q|_\beta(y) = 1 - \frac{|y - b_q|}{|\beta|} \tag{B.28}$$

Which give

$$S_2^{\psi_q}(a, \beta, h, \xi) = \int_{\beta} f_2(h, ||y - a||, \xi) ds_y - \frac{1}{|\beta|} \int_{\beta} (y - b_q) f_2(h, ||y - a||, \xi) ds_y$$

$$= \int_{t^-}^{t^+} f_2(h, \sqrt{d^2 + s^2}, \xi) ds - \frac{\varepsilon}{|\beta|} \int_{t^-}^{t^+} (z - s) f_2(h, \sqrt{d^2 + s^2}, \xi) ds_y \tag{B.29}$$

where $\varepsilon = -1$ if $b_q = b^-$, $\varepsilon = 1$ if $b_q = b^+$ and z is the abscissa of b_q Finally, we obtain

$$S_2^{\psi_q}(a, \beta, h, \xi) = (1 - \frac{\varepsilon z}{|\beta|}) S_2(a, \beta, h, \xi) + \frac{\varepsilon}{|\beta|} T_2(a, \beta, h, \xi) \tag{B.30}$$

CHAPTER C

Integral formulation of Maxwell equation

C.1 Properties of integral operators

Some solutions of the problems of diffraction of electromagnetic waves can be expressed with the help of potential of single and dual layer defined on the surface of the obstacle. We present here the properties of the harmonic potential.

Definition C.1.1. *We introduce the potentials $(B-S)$ and Q, that are defined by*

$$(B-S)\mathbf{J} := \int_{\Gamma} \left(G(x,y)\mathbf{J}(y) + \frac{1}{k^2}\nabla_x G(x,y)\mathrm{div}_{\Gamma}\mathbf{J} \right) d\Gamma(y) \tag{C.1}$$

$$Q\mathbf{M} := \int_{\Gamma} \nabla_y G(x,y) \times \mathbf{M} d\Gamma(y) \tag{C.2}$$

and $G(x,y)$ is the Green kernel giving the outgoing solutions to the scalar Helmholtz equation:

$$G(x,y) := \frac{e^{-ik|x-y|}}{4\pi|x-y|}, \;\; \nabla_x G(x,y) := -\frac{(1+ik|x-y|)}{4\pi|x-y|^3} e^{-ik|x-y|}(\mathbf{x}-\mathbf{y}) \tag{C.3}$$

According to theorem 4.6 in [38](Chapter I, p.43) we have:

Theorem C.1.1. *The operator Q is continuous from $H^{-1/2}(\mathrm{div},\Gamma)$ to $H^{-1/2}(\mathrm{rot},\Gamma)$ and we have that:*

$$|(\mathbf{n}\times Q + \frac{I}{2})\mathbf{M}|_{-1/2,\mathrm{div}_{\Gamma}} \leq C|\mathbf{M}|_{-1/2,\mathrm{div}_{\Gamma}} \;\; \forall \mathbf{M} \in H^{-1/2}(\mathrm{div},\Gamma) \tag{C.4}$$

According to theorem 2.2 in [38](Chapter II, p.61) we have:

Theorem C.1.2. *The operator $(B-S)$ is an isomorphisme from $H^{-1/2}(\mathrm{div},\Gamma)$ to $H^{-1/2}(\mathrm{rot},\Gamma)$ and it verifies the inequality:*

$$\|(B-S)\boldsymbol{\phi}\|_{-1/2,\mathrm{rot}_\Gamma} \leq C\|\boldsymbol{\phi}\|_{-1/2,\mathrm{div}_\Gamma} \tag{C.5}$$

and the coercivity relation $\forall \boldsymbol{\phi} \in H^{-1/2}(\mathrm{div},\Gamma)$:

$$\Re(<\boldsymbol{\phi},(B-S)\boldsymbol{\phi}>) \geq C\|\boldsymbol{\phi}\|^2_{-1/2,\mathrm{div}_\Gamma} \tag{C.6}$$

C.1.1 Fredholm alternative

Let us mention the abstract theorems known as Fredholm alternative [48].

Theorem C.1.3. *Let V be a Hilbert space. Let H be a Hilbert space which contains V. Let $a(u,v)$ be continuous bilinear form on $V\times V$ which satisfies:*

$$\Re[a(u,v)] \geq \alpha\|u\|^2_V - c\|u\|^2_H, \alpha>0, \forall u\in V. \tag{C.7}$$

We consider the variational problem

$$a(u,v) = (g,v);\ \ \forall v\in V; g\in V^*.$$

Suppose that the injection of V into H is compact. Then the variational problem (VP) satisfies the Fredholm alternative i.e.
- either it admits a unique solution in V,
- or it has a finite dimensional kernel and a unique solution up to any element in this kernel, when the duality product of the right-hand side g vanishes on every element in this kernel.

We take the existence and uniqueness theorem from [48] (p.245, Theorem 5.6.1).

Theorem C.1.4. *(Existence and uniqueness): Let V and W be Hilbert spaces. Let $A(.,.)$ be a bilinear form continuous on $V\times V$ which satisfies*

$$\Re[A(u,\bar{u})] \geq \alpha\|u\|^2_V - C\|u\|^2_H,\ \ \alpha>0,\ \forall u\in V \tag{C.8}$$

where H is Hilbert space containing V. Let $B(q,v)$ be a bilinear form continuous on $W\times V$ which satisfies:

$$\sup_{\|u\|_V=1} |B(q,u)| \geq \beta\|q\|_W - C\|q\|_L,\ \ \beta>0,\ \forall q\in W \tag{C.9}$$

where L is a Hilbert space containing W.

Consider the following variational problem, with $g_1 \in V^*$ and $g_2 \in W^*$:

$$\begin{cases} A(u,v) & +B(p,v) = (g_1,v) \ \ \forall v \in V \\ B(q,u) & = (g_2,q) \ \ \forall q \in W \end{cases} \tag{C.10}$$

Denote by V_0 the kernel of the bilinear form B in V, i.e.

$$V_0 = \{u \in V, \ B(q,u) = 0, \ \forall q \in W\}$$

Suppose that the injection form V_0 into H is compact and the injection from W into L is compact. Suppose that there exists an element $u_{g_2} \in V$ such that:

$$B(q, u_{g_2}) = (g_2, q), \ \ \forall q \in V.$$

Then the variational problem (C.10) satisfies the Fredholm alternative, i.e.
-either it admits a unique solution in $V \times W$,
-or it admits a finite dimensional kernel, and a solution defined up to any element in this kernel, when the right-hand side (g_1, g_2) vanishes on any element in this kernel.

C.2 Integral formulation of the problem

We are looking for a presentation of the fields $(\mathbf{E}, \mathbf{H})$ outside the object in terms of electromagnetic currents $(\mathbf{J}, \mathbf{M})$ on the surface Γ [48]. We have Maxwell's equations on the surface Γ :

$$\begin{cases} \mathbf{rot}\mathbf{E} - ikZ_0\mathbf{H} = \mathbf{M}\delta_\Gamma \\ \mathbf{rot}\mathbf{H} + ikZ_0^{-1}\mathbf{E} = \mathbf{J}\delta_\Gamma \end{cases} \tag{C.11}$$

The superposition theorem lets us consider two cases. In the first case we suppose that $\mathbf{M} = 0$, in the second $\mathbf{J} = 0$. In both cases we want to get $\mathbf{E}$ and $\mathbf{M}$ expressions in terms of $\mathbf{J}$ and $\mathbf{M}$. After we combine these cases to get the general form of fields $\mathbf{E}$ and $\mathbf{M}$. In case $\mathbf{M} = 0$, we are looking for $\mathbf{E}$ in the form

$$\mathbf{E} = \mathbf{A} + \nabla V \tag{C.12}$$

with Lorentz gauge condition $\mathrm{div}\mathbf{A} = k^2 V$. The divergence of the second equation of (C.11) with support of gauge condition, shows that V satisfies scalar Helmholtz equation

$$ikZ_0^{-1}\mathrm{div}\mathbf{E} = \mathrm{div}\mathbf{J} \tag{C.13}$$

$$\mathrm{div}\mathbf{A} + \mathrm{div}\nabla V = \frac{1}{ikZ_0^{-1}}\mathrm{div}\mathbf{J} \tag{C.14}$$

$$\Delta V + k^2 V = -ik^{-1}Z_0\mathrm{div}\mathbf{J} \tag{C.15}$$

It can be expressed as a potential simple layer density of $-ik^{-1}Z_0\mathrm{div}\mathbf{J}$

$$V(x) = ik^{-1}Z_0 \int\limits_\Gamma G(x,y)\mathrm{div}_\Gamma \mathbf{J}(y)dy \tag{C.16}$$

In this expression G is the Green kernel, fundamental solution of Helmholtz equation, i.e. verifies Maxwell's equations and radiation condition.
We have that $\mathbf{A}$ verifies the next vector Helmholtz equation

$$\Delta\mathbf{A} + k^2\mathbf{A} = \nabla\mathrm{div}\mathbf{A} - \mathbf{rot}(\mathbf{rot}\mathbf{A}) + k^2\mathbf{A} \tag{C.17}$$

According to earlier condition the right side of the first term is equal to $k^2\nabla V$. When (C.12) allows us to express (C.17) as

$$k^2\nabla V - \mathbf{rot}(\mathbf{rot}\mathbf{A}) + k^2\mathbf{E} - k^2\nabla V = k^2\mathbf{E} - \mathbf{rot}(\mathbf{rot}\mathbf{E}) = -ikZ_0\mathbf{J}$$

The last equivalence derived from Maxwell's equations (C.11).
Then $\mathbf{A}$ is a potential of simple layer density $-ikZ_0\mathbf{J}$. So we can write

$$\mathbf{A}(x) = ikZ_0 \int\limits_\Gamma G(x,y)\mathbf{J}(y)dy \tag{C.18}$$

$$\mathbf{E}(x) = ikZ_0\left(\int\limits_\Gamma G(x,y)\mathbf{J}(y)dy + \frac{1}{k^2}\nabla_x \int\limits_\Gamma G(x,y)\mathrm{div}_\Gamma \cdot \mathbf{J}(y)dy\right) \tag{C.19}$$

$$\mathbf{H}(x) = \frac{1}{ikZ_0}\mathbf{rot}\mathbf{E} = \mathbf{rot}\int\limits_\Gamma G(x,y)\mathbf{J}(y)dy \tag{C.20}$$

We write them via the operators

$$\begin{cases} \mathbf{E}(x) = ikZ_0(B-S)\mathbf{J}(x) \\ \mathbf{H}(x) = -Q\mathbf{J}(x) \end{cases} \tag{C.21}$$

where $(B-S)$ and Q are the potentials defined in (C.1),(C.2):

$$(B-S)\mathbf{J}(x) = \int_\Gamma G(x,y)\mathbf{J}(y) + \frac{1}{k^2}\nabla_x G(x,y)\mathrm{div}_\Gamma \mathbf{J}(y)dy \tag{C.22}$$

$$Q\mathbf{J}(x) = -\mathbf{rot}_x \int_\Gamma G(x,y)\mathbf{J}(y)dy = \int_\Gamma \nabla_x G(x,y)\times \mathbf{J}(y)dy \tag{C.23}$$

By an argument of symmetry, the second case $\mathbf{J}=0$ derives the same way

$$\mathbf{E}(x) = \mathbf{rot}_x \int_\Gamma G(x,y)\mathbf{M}(y)dy \tag{C.24}$$

$$\mathbf{H}(x) = -ikZ_0^{-1}\left(\int_\Gamma G(x,y)\mathbf{M}(y)dy + \frac{1}{k^2}\nabla_y \int_\Gamma G(x,y)\mathrm{div}_\Gamma \mathbf{M}(y)dy\right) \tag{C.25}$$

Potentials express

$$\begin{cases} \mathbf{E}(x) = Q\mathbf{M}(x) \\ \mathbf{H}(x) = -ikZ_0^{-1}(B-S)\mathbf{M}(x) \end{cases} \tag{C.26}$$

Finally we combine $(C.21)$ and $(C.26)$ to get Stratton-Chu formula:

$$\begin{cases} \mathbf{E}(x) = ikZ_0(B-S)\mathbf{J}(x) - Q\mathbf{M}(x) \\ \mathbf{H}(x) = -Q\mathbf{J}(x) + ikZ_0^{-1}(B-S)\mathbf{M}(x) \end{cases} \tag{C.27}$$

These expressions are valid in external domain Ω^+. The impedance condition associates fields values on a boundary Γ. We have to determine external limit values on Γ. Using the classical jump relations, we can express the respective limiting boundary tangential values of $\mathbf{E}$ and $\mathbf{H}$ by

$$\begin{cases} \mathbf{E}_{tg}^{inc} = ikZ_0(B-S)\mathbf{J}(x) - Q\mathbf{M} + \frac{1}{2}\mathbf{E}_{tg} \\ \mathbf{H}_{tg}^{inc} = -Q\mathbf{J} + ikZ_0^{-1}(B-S)\mathbf{M}(x) + \frac{1}{2}\mathbf{H}_{tg} \end{cases} \tag{C.28}$$

We recall that on a boundary, we have

$$\mathbf{E}_{tg} = \mathbf{n}\times\mathbf{M}$$

$$\mathbf{H}_{tg} = -\mathbf{n}\times\mathbf{J}$$

where the subscript t designates the tangential component $\mathbf{E}_{tg} := \mathbf{n}\times(\mathbf{E}\times\mathbf{n})$ of the respective vector field on Γ.

Thus the electromagnetic field can be expressed in Ω^+ in terms of the equivalent currents and charges by the familiar Stratton-Chu formula

$$\begin{cases} \mathbf{E}_{tg}^{inc} = ikZ_0(B-S)\mathbf{J}(x) - Q\mathbf{M} + \frac{1}{2}\mathbf{n}\times\mathbf{M} \\ \mathbf{H}_{tg}^{inc} = -Q\mathbf{J} + ikZ_0^{-1}(B-S)\mathbf{M}(x) - \frac{1}{2}\mathbf{n}\times\mathbf{J} \end{cases} \tag{C.29}$$

Or in integral form

$$\begin{cases} \mathbf{E}_{tg}^{inc} = & ikZ_0 \int_\Gamma \left(G(x,y)\mathbf{J}(y) + \frac{1}{k^2}\nabla_x G(x,y)\mathrm{div}_\Gamma \mathbf{J} \right) dy \\ & + \frac{1}{2}\mathbf{n}\times\mathbf{M} - \int_\Gamma \nabla_x G(x,y)\times\mathbf{M} dy \\ \mathbf{H}_{tg}^{inc} = & -\frac{1}{2}\mathbf{n}\times\mathbf{J} - \int_\Gamma \nabla_x G(x,y)\times\mathbf{J} dy \\ & + ikZ_0^{-1}\int_\Gamma \left(G(x,y)\mathbf{M}(y) + \frac{1}{k^2}\nabla_x G(x,y)\mathrm{div}_\Gamma \mathbf{M} \right) dy \end{cases} \tag{C.30}$$

Later, to write variational formulation, we will multiply it by a test function and integrate over the surface Γ.

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