A combinatorial approach to Rauzy-type dynamics

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Abstract

Rauzy-type dynamics are group (or monoid) actions on a collection of combinatorial objects. The first and best known example concerns an action on permutations, associated to interval exchange transformations (IET) for the Poincaré map on compact orientable translation surfaces. The equivalence classes on the objects induced by the group action are related to components of the moduli spaces of Abelian differentials with prescribed singularities, and, in two variants of the problem, have been classified by Kontsevich and Zorich, and by Boissy, through methods involving both combinatorics, algebraic geometry, topology and dynamical systems.

In the first half of this thesis, we provide a purely combinatorial proof of both classification theorems. Our proof can be interpreted geometrically and the overarching idea is close to that of Kontsevich and Zorich, although the techniques are rather different. Not all Rauzy-type dynamics have a geometrical correspondence however, and some parts of this first proof do not seem to generalize well.

In the second half of the thesis we develop a new method, that we call the *labelling method*. This second method is not completely disjoint from the first one, but it the new crucial ingredient of considering a sort of 'monodromy' for the dynamics, in a way that we now sketch.

Many statements in this thesis are proven by induction. It is conceivable to prove, by induction, a classification theorem for unlabelled objects. However, as the labelling method will show, it is easier to prove two statements in parallel within the same induction, the one on the unlabelled objects, and an apparently much harder one, on the monodromy of the labelled objects. Although the final result is stronger than the initial aim, by virtue of the stronger inductive hypothesis, the method may work more easily.

This second approach extends to several other Rauzy-type dynamics. Our first step is to apply the labelling method to derive a second proof of the classification theorem for the Rauzy dynamics. Then we apply it to the study of two other Rauzy-type dynamics (one of which is strictly related to the Rauzy dynamics on non-orientable surfaces), and finally we inventory a surprisingly high number of Rauzy-type dynamics for which the labelling method seems to apply, and for which detailed proofs are left for future work.

Contents

Two prologs

Chapter 1

A combinatorial origin of the Rauzy dynamics: A counting problem for palindrome words

We open this manuscript by introducing an apparently simple counting problem, associated to palindrome words, where we will see the emergence of the notion of *Rauzy dynamics*, which is the main topic of the thesis.

We will study several instances of Rauzy dynamics, most of which are motivated by different problems in algebraic geometry and dynamical systems. Remarkably, several of these instances emerge also in this peculiar class of problems for palindrome words. More specifically, four distincts dynamics arise from the problem of palindrome words: the Rauzy dynamics for permutations and matchings, that will be studied in depth in sections 3.1.1 and 8 respectively, and a Rauzy-type dynamics for signed versions of permutations and matchings, that we will only briefly describe at the end of this section.

This thesis focuses on the study of Rauzy-type dynamics for what concerns *classification theorems*, which is the precise description of certain equivalence classes. The counting problem presented here appears just as a motivation which, differently than the standard theory of Rauzy induction, can be described briefly and in simple terms. The full problem on palindrome words will not be solved here, nor will it be studied comprehensively. In fact, we will do little more than showing that certain generating functions are determined by a linear system, and that the underlying matrix has a block decomposition. The blocks of this decomposition corresponds to the Rauzy classes we will try to classify here. The determination of the forementioned generating functions thus appears as a rather difficult task, that we perform here only at very small sizes.

1.1 The problem of palindrome words

Let $\mathcal{A}$ be a finite alphabet of cardinality q. For x a word[1] in $\mathcal{A}^*$, let $\overline{x}$ be the reverse word. Let $n \geq 1$, and $\mathcal{E}$ an ordering of the string $(x_1, \overline{x_1}, \cdots, x_n, \overline{x_n})$. For example, at $n = 3$ and renaming for simplicity $(x_1, \dots, x_3)$ as (x, y, z), a possible choice is $\mathcal{E} = x\,y\,z\,\overline{x}\,\overline{z}\,\overline{y}$.

[1]We adopt here the convention that the empty word ϵ is in $\mathcal{A}^*$ i.e $\mathcal{A}^* = \bigcup_{i=0}^{\infty} \mathcal{A}^i$.

Define $F_q[\mathcal{E}](\xi_1, \cdots, \xi_n)$ as the generating function of palindrome realisations of $\mathcal{E}$, i.e.

$$F_q[\mathcal{E}](\xi_1, \dots, \xi_n) = \sum_{\substack{x_1,\dots,x_n \in \mathcal{A}^* \\ \mathcal{E}=\overline{\mathcal{E}}}} \prod_{i=1}^{n} \xi_i^{|x_i|}$$

We will omit the subscript q when clear. For example, for $\mathcal{E} = x\,y\,z\,\overline{x}\,\overline{z}\,\overline{y}$ as above, the choice $(x, y, z) = (abbab, abbababbabab, bab)$ contributes with the monomial $\xi^5\eta^{12}\zeta^3$ to $F[\mathcal{E}](\xi)$, because the word

$$abbab\;abbababbabab\;bab\;babba\;bab\;bababbababba$$

is indeed a palindrome.

Alternatively, we can define $a^{\mathcal{E}}_{i_1,\dots,i_n}$ as the number of palindrome realisations of $\mathcal{E}$, in which $|x_k| = i_k$. Then F reads

$$F[\mathcal{E}](\xi_1, \dots, \xi_n) = \sum_{i_1,\dots,i_n=0}^{\infty} a^{\mathcal{E}}_{i_1,\dots,i_n} \xi_1^{i_1} \dots \xi_n^{i_n}\,.$$

Of course $a^{\mathcal{E}}_{0,\dots,0} = 1$ for all $\mathcal{E}$. In the non-trivial cases we have

Proposition 1. *When $N := i_1 + \cdots + i_n \geq 1$, we have $a^{\mathcal{E}}_{i_1,\dots,i_n} = q^{k(\mathcal{E};i_1,\dots,i_n)}$ with $1 \leq k(\mathcal{E}; i_1, \dots, i_n) \leq N$.*

Indeed, the function $k(\mathcal{E}; i_1, \dots, i_n)$ has an explicit characterisation, which requires a few definition, and the graphical notation of the following section.

We will now pass instead to the following:

Theorem 2. *$F[\mathcal{E}](\xi_1, \dots, \xi_n)$ is a family of formal power series fixed by the obvious B_n symmetry, (i.e., symmetry under $x \leftrightarrow \overline{x}$, and under $(x, \overline{x}, \xi) \leftrightarrow (y, \overline{y}, \eta)$), by $F[\varnothing] = 1$, and by the following linear recursions. First, the trivial*

$$F[x\mathcal{E}\overline{x}](\xi, ...) = \frac{1}{1 - q\xi}\, F[\mathcal{E}](...)\,. \tag{1.1}$$

Then, say that $\mathcal{E}$ starts with $x = x_1$ and ends with $\overline{y} = \overline{x_2}$. Let us define $\mathcal{E}(\overline{x}, y)$ as the ordering of the string $(\overline{x}, y, x_3, \overline{x_3}, \cdots, x_n, \overline{x_n})$ such that $\mathcal{E} = x\mathcal{E}(\overline{x}, y)\overline{y}$. Then

$$\begin{aligned} F[x\mathcal{E}(\overline{x}, y)\overline{y}](\xi, \eta, ...) = \;& F[\mathcal{E}(\overline{x}, xy)\overline{y}](\xi\eta, \eta, ...) && \textit{(for } x \preceq y\textit{)} \\ & + F[x\mathcal{E}(\overline{x}\overline{y}, y)](\xi, \xi\eta, ...) && \textit{(for } x \succeq y\textit{)} \\ & - F[\mathcal{E}(\overline{x}, x)](\xi\eta, ...) && \textit{(for } x \equiv y\textit{)} \end{aligned} \tag{1.2}$$

Finally, say that $\mathcal{E}$ starts with $x = x_1$, and let us define $\mathcal{E}(\overline{x})$ as the ordering of the string $(\overline{x}, x_2, \overline{x_2}, \cdots, x_n, \overline{x_n})$ such that $\mathcal{E} = x\mathcal{E}(\overline{x})$. Then

$$F[x\mathcal{E}(\overline{x})](\xi, ...)|_{\xi=0} = F[\mathcal{E}(\epsilon)](...)\,. \tag{1.3}$$

Proof. The symmetry statement, i.e. that the expression $F[\mathcal{E}](\xi_1, \dots, \xi_n)$ is invariant under replacing some x_i by its reverse, and is covariant by interchanging the name of two variables, and the initialisations, such as $F[\varnothing] = 1$ and equation (1.3), are obvious. So we focus on the main linear recursion, equation (1.2).

We have

$$\begin{aligned}
F[x\mathcal{E}\overline{x}](\xi,\xi_1,...) &= \sum_{\substack{x,x_1,\dots,x_n\in\mathcal{A}^*\\ \mathcal{E}=\overline{\mathcal{E}}}} \xi^{|x|}\prod_i \xi_i^{|x_i|} \qquad \text{since } x\mathcal{E}\overline{x}=\overline{x\mathcal{E}\overline{x}} \iff \mathcal{E}=\overline{\mathcal{E}}\\
&= \sum_x \xi^{|x|} \sum_{\substack{x_1,\dots,x_n\in\mathcal{A}^*\\ \mathcal{E}=\overline{\mathcal{E}}}} \prod_i \xi_i^{|x_i|}\\
&= \frac{1}{1-q\xi} \sum_{\substack{x_1,\dots,x_n\in\mathcal{A}^*\\ \mathcal{E}=\overline{\mathcal{E}}}} \prod_i \xi_i^{|x_i|} \qquad \text{since } |\mathcal{A}|=q\\
&= \frac{1}{1-q\xi}\, F[\mathcal{E}](\xi_1,...)
\end{aligned}$$

and

$$\begin{aligned}
F[x\mathcal{E}(\overline{x},y)\overline{y}](\xi,\eta,...) &= \sum_{\substack{x,y,x_1,\dots,x_n\in\mathcal{A}^*\\ \mathcal{E}=\overline{\mathcal{E}}}} \xi^{|x|}\eta^{|y|}\prod_i \xi_i^{|x_i|}\\
&= \sum_{\substack{x,y,x_1,\dots,x_n\in\mathcal{A}^*\\ x\preceq y\\ \mathcal{E}=\overline{\mathcal{E}}}} \xi^{|x|}\eta^{|y|}\prod_i \xi_i^{|x_i|} + \sum_{\substack{x,y,x_1,\dots,x_n\in\mathcal{A}^*\\ x\succeq y\\ \mathcal{E}=\overline{\mathcal{E}}}} \xi^{|x|}\eta^{|y|}\prod_i \xi_i^{|x_i|}\\
&\quad - \sum_{\substack{x,y,x_1,\dots,x_n\in\mathcal{A}^*\\ x\equiv y\\ \mathcal{E}=\overline{\mathcal{E}}}} \xi^{|x|}\eta^{|y|}\prod_i \xi_i^{|x_i|}
\end{aligned}$$

Let us analyse the case $x \preceq y$ (the case $x \succeq y$ is symmetric).

Since $x \preceq y$ and $\mathcal{E} = \overline{\mathcal{E}}$ i.e $x\mathcal{E}(\overline{x}, y)\overline{y}$ is a palindrome, we must have $\overline{y} = \overline{y'}\overline{x}$ for some $y' \preceq y$ and thus $y = xy'$. Replacing y with xy' we obtain

$$\begin{aligned}
\sum_{\substack{x,y,x_1,\dots,x_n\in\mathcal{A}^*\\ x\preceq y\\ \mathcal{E}=\overline{\mathcal{E}}}} \xi^{|x|}\eta^{|y|}\prod_i \xi_i^{|x_i|} &= \sum_{\substack{x,y',x_1,\dots,x_n\in\mathcal{A}^*\\ x\mathcal{E}(\overline{x},xy')\overline{y'}\overline{x}=\overline{x\mathcal{E}(\overline{x},xy')\overline{y'}\overline{x}}}} \xi^{|x|}\eta^{|x|+|y'|}\prod_i \xi_i^{|x_i|}\\
&= \sum_{\substack{x,y',x_1,\dots,x_n\in\mathcal{A}^*\\ \mathcal{E}(\overline{x},xy')\overline{y'}=\overline{\mathcal{E}(\overline{x},xy')\overline{y'}}}} \xi^{|x|}\eta^{|x|+|y'|}\prod_i \xi_i^{|x_i|}\\
&\text{since } x\mathcal{E}(\overline{x},xy')\overline{y'}\overline{x} = \overline{x\mathcal{E}(\overline{x},xy')\overline{y'}\overline{x}} \iff \mathcal{E}(\overline{x},xy')\overline{y'} = \overline{\mathcal{E}(\overline{x},xy')\overline{y'}}\\
&= \sum_{\substack{x,y',x_1,\dots,x_n\in\mathcal{A}^*\\ \mathcal{E}(\overline{x},xy')\overline{y'}=\overline{\mathcal{E}(\overline{x},xy')\overline{y'}}}} (\xi\eta)^{|x|}\eta^{|y'|}\prod_i \xi_i^{|x_i|}\\
&= F[\mathcal{E}(\overline{x},xy)\overline{y}](\xi\eta,\eta,...)
\end{aligned}$$

The case $x \equiv y$ is clearly required to cancel the double-counting in the case analysis $x \preceq y$ or $x \succeq y$. Its analysis is similar, and in fact simpler, and we omit it. $\square$

Remark 3. *Note that both the right-hand-side (RHS) terms of equations (1.1) and (1.3), and the third summand on the RHS of equation (1.2) have one word variable* *less (the triple $(x, \bar{x}, \xi)$ or the triple $(y, \bar{y}, \eta)$) w.r.t. the LHS, while the two other summands in (1.2) have the same number of variables. As a result, we can consider equations (1.1), (1.2) and (1.3) as defining an inhomogeneous linear system, for functions $F[\mathcal{E}]$ with $\mathcal{E}$ in n variables, assuming that the $F[\mathcal{E}]$'s with $\mathcal{E}$ in $n-1$ variables are known.*

Remark 4. *Although calculating $F[\mathcal{E}](\xi_1, \ldots, \xi_n)$ seems to be rather difficult in general, there are two rather simple facts that we can establish with no effort, which is*

$$\lim_{q \to 0} F_q[\mathcal{E}](\xi_1, \ldots, \xi_n) = 1\,; \tag{1.4}$$

$$\lim_{q \to 1} F_q[\mathcal{E}](\xi_1, \ldots, \xi_n) = \prod_{j=1}^{n} \frac{1}{1-\xi_j}\,. \tag{1.5}$$

The reason is that in an empty alphabet the only solution to the problem is $x_i = \epsilon$, while in an alphabet of a single character, say $\mathcal{A} = \{a\}$, all tentative solutions $x_i = a^{k_i}$ in fact produce palindrome words $\mathcal{E} = a^{2\sum_i k_i}$.

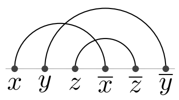
Conversely, even the first derivative w.r.t. q of the function F_q, evaluated at $q = 0$ or at $q = 1$, are non-trivial functions (yet again, related to the 'classical' Rauzy dynamics), as we shall discuss in a future.

Let us anticipate here that the derivative at $q = 0$ is related to the absolute contribution of square-tiled surfaces having a single horizontal cylinder to the Masur–Veech volume of ambient strata of Abelian differentials (some explanation of these notions are postponed to Chapter 3), as explained in [DGZZ16] (see also [DGZZ17] for related results). In this field, several interesting conjectures are still open, and only one sporadic result (namely, Conditional Theorem 2.10 and Theorem 2.11 in [DGZZ16]) has been proven so far, and with the help of very deep techniques and results, established in [CMZ16].

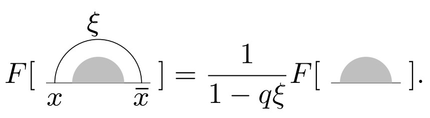
The only other value of q for which we have sensible results is a suitable limit $q \to \infty$, that we discuss briefly later on along this chapter.

1.2 Graphical representation

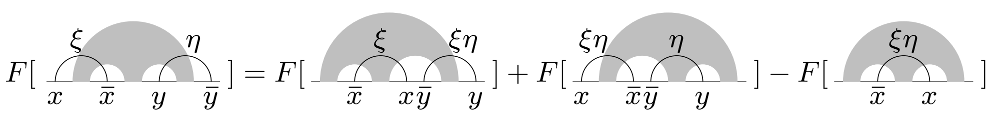
An expression $\mathcal{E}$ (up to B_n symmetry) can be represented by a matching over $[2n]$. In our example:

$\mathcal{E} = x\,y\,z\,\bar{x}\,\bar{z}\,\bar{y} =$ 

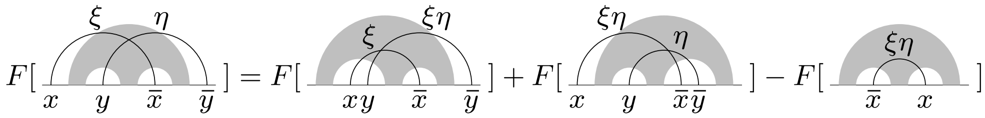
The recursions of theorem 2 can then be rewritten graphically. Equation (1.1) becomes:



If $\bar{x}$ is before y in $\mathcal{E}(\bar{x}, y)$, equation (1.2) becomes:

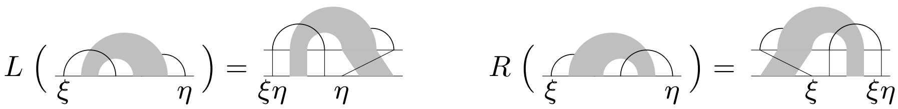


While, if y is before $\bar{x}$ in $\mathcal{E}(\bar{x}, y)$, equation (1.2) is written:



From the Remark 3, if we want to compute the $F[\mathcal{E}]$'s for $\mathcal{E}$ in n variables, we have to solve a linear inhomogeneous system of equations. As customary, we shall start by considering the associated homogeneous system. In this case, only equation (1.2) remains non-trivial, and simplifies to have only two terms on the RHS, instead of three.

Still, as we have seen, equation (1.2) is graphically represented in two different ways, depending on the mutual order of $\bar{x}$ and y in $\mathcal{E}(\bar{x}, y)$. However, the transformation rule is unique if we choose a more adapted graphical representation. Let $m = \mathcal{E}$ be a matching over $[2n]$ with parameters $(\xi_1, \ldots, \xi_n)$ on the arcs, and such that $m(1) \neq 2n$ (i.e. we are in the case of equation (1.2), not in the case of equation (1.1)). Then define the maps L and R on the parametrised matchings as:



and we just get the (unique and simple) equation $F[m] = F[R(m)] + F[L(m)] + \cdots$, where the dots stand for generating functions in $\mathcal{E}$'s of smaller length.

Let us now describe the function $k(\mathcal{E}; i_1, \ldots, i_n)$ appearing in Proposition 1. Call $N = i_1 + \cdots + i_n$, and define $\mathcal{E}^{[i]}$ as the matching with N arcs, obtained from the matching associated to $\mathcal{E}$ by replacing the a-th arc with a 'rainbow' of i_a parallel arcs (note that i_a may be zero). Call ω_N the matching consisting of N parallel arcs. For m_1 and m_2 matchings on the same number of arcs, call $\langle m_1 | m_2 \rangle_q := q^{C(m_1, m_2)}$, where $C(m_1, m_2)$ is the number of cycles in the diagram obtained by glueing m_1 (on the top half-plane) with the mirror image of m_2 (on the bottom half-plane), i.e. the customary scalar product in the Temperley–Lieb Algebra. Then $k(\mathcal{E}; i_1, \ldots, i_n) = C(\mathcal{E}^{[i]}, \omega_N)$, and $a^{\mathcal{E}}_{i_1,\ldots,i_n} = \langle \mathcal{E}^{[i]} | \omega_N \rangle_q$. This is easily seen from the definition of $a^{\mathcal{E}}_{i_1,\ldots,i_n}$: each of the $2N$ points on the horizontal line in the representation of $\mathcal{E}^{[i]}$ corresponds to a symbol in a word x_a or $\overline{x_a}$. Each arc of the matching $\mathcal{E}^{[i]}$ corresponds to the constraint that the h-th symbol of x_a, from the left, coincides with the h-th symbol from the right of $\overline{x_a}$, for some a and h. Each arc of the matching ω_N corresponds to the constraint that the h-th symbol of $\mathcal{E}$, counting from the left, coincides with the h-th symbol $\mathcal{E}$, counting from the right, for $1 \leq h \leq N = |\mathcal{E}|/2$. Overall, these constraints form cycles, each cycle corresponds to a list of constraints for symbols $a_i \in \mathcal{A}$, of the form $a_1 = a_2 = a_3 = \ldots = a_{2\ell} = a_1$, so that we have one independent free choice in $\mathcal{A}$ (i.e., a factor q) for each cycle. $\square$

This shows that the function F is also a formal power series in q, and that, at fixed q, the radius of convergence in the ξ_i's is no smaller than $1/q$. Indeed, as we will see later on, it is exactly $1/q$, for a rather trivial reason (it is no larger than $1/q$ in variable ξ_i, once that all the other ξ_j's are set to zero, and we have monotonicity for the ξ_i's in $\mathbb{R}^+$ because the series coefficients are non-negative).

The operators L and R act on the matching m, and on the weights $(\xi_1, \dots, \xi_n)$. The second action is a transvection on the logs, i.e., calling $\lambda_i = \ln \xi_i$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, the action is by a multiplication by a matrix $T^{(ij)}$ of the form

$$T^{(ij)} = \begin{array}{cc} \begin{matrix} & & & j & \end{matrix} & \\ \begin{pmatrix} 1 & 0 & 0 & \vdots & 0 \\ & 1 & 0 & 1 & \cdots \\ & & \ddots & 0 & 0 \\ & 0 & & \ddots & 0 \\ & & & & 1 \end{pmatrix} & \begin{matrix} \\ i \\ \\ \\ \\ \end{matrix} \end{array}$$

Thus, for what concerns these parameters, we have an action inside $\mathrm{SL}(n, \mathbb{Z})$. We will neglect this sector, and rather concentrate on the action of L and R on the matchings m with parameters $(\xi, \eta, \dots)$ omitted. This is the action that will be related to the 'Rauzy induction'.

Thus, we have $(2n-1)!!$ matchings, and an associated homogeneous linear system, schematically of the form $F[m] = F[L(m)] + F[R(m)]$. Iterating the substitution, we have $F[m] = F[LL(m)] + F[LR(m)] + F[RL(m)] + F[RR(m)]$, and so on, so that $F[m]$ is related to $F[m']$'s, for m' which are reachable by a sequence of L and R. Let us say that $m' \sim m$ in this case. As we will (easily) see in the following, if $m' = Lm$, there exists an integer $k < n$ such that $m = L^k m'$, and similarly for R, so that $\sim$ is in fact an equivalence relation, and the homogeneous linear system for the F's is decomposed in strongly-connected diagonal blocks (i.e., there is no triangular part). Thus, a very basic requirement before trying to determine our generating functions is to understand this block-decomposition of the linear system. Such a problem will correspond to the 'classification problem' for the associated 'Rauzy-type dynamics', the core problem of this thesis that we will attack in a few pages.

Nonetheless, before doing this, in the following section we want to give some small results on the original problem, of determining $F[\mathcal{E}]$, for $\mathcal{E}$ small enough.

1.3 Some enumerative results on $F_q[\mathcal{E}]$

Here we want to present a few results on non-trivial patterns at small size. Before doing this, we shall establish some triviality criteria, which allows us to focus on the most relevant patterns.

Proposition 5 (Reduction of descents)**.** *Let $\mathcal{E}$ have the form $\mathcal{E} = \cdots xy \cdots \bar{y}\bar{x} \cdots$, and let $\mathcal{E}'$ be the corresponding reduction $\mathcal{E}' = \cdots z \cdots \bar{z} \cdots$. Then*

$$F[\mathcal{E}](\dots, \xi, \eta, \dots) = \frac{1}{\xi - \eta} \left(\xi F[\mathcal{E}'](\dots, \xi, \dots) - \eta F[\mathcal{E}'](\dots, \eta, \dots) \right). \tag{1.6}$$

Proof. Indeed any solution to the problem for $\mathcal{E}'$, in which $|z| = k$, is associated to $k+1$ solutions to the problem for $\mathcal{E}$, one per $0 \le |x| \le k$ (and $|y| = k - |x|$). This is realised via the substitution $\zeta^k \to \xi^k + \xi^{k-1}\eta + \dots + \eta^k = (\xi^{k+1} - \eta^{k+1})/(\xi - \eta)$, so that the formula follows. □

Proposition 6 (Block decomposition)**.** *Let $\mathcal{E} = \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3$, with $\mathcal{E}_1$ involving x_1, …, x_k, $\mathcal{E}_3$ involving $\overline{x_1}$, …, $\overline{x_k}$, and $\mathcal{E}_3$ involving y_1, …, y_h, $\overline{y_1}$, …, $\overline{y_h}$. Then*

$$F[\mathcal{E}](\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_h) = F[\mathcal{E}_1\mathcal{E}_3](\xi_1, \dots, \xi_k) F[\mathcal{E}_2](\eta_1, \dots, \eta_h). \tag{1.7}$$

Proof. This is an immediate consequence of the fact that, for every choice of words, the lengths of $\mathcal{E}_1$ and $\mathcal{E}_3$ are equal, so that $\mathcal{E} = \overline{\mathcal{E}}$ iff $\mathcal{E}_1 = \overline{\mathcal{E}_3}$ and $\mathcal{E}_2 = \overline{\mathcal{E}_2}$. □

Let us now provide the explicit result for all patterns $\mathcal{E}$ of size at most 2, and for patterns of size 3 which are associated to permutations (i.e., of the form $\mathcal{E} = x_1 \dots x_n \overline{x_{\sigma(1)}} \dots \overline{x_{\sigma(n)}}$). We have:

Proposition 7.

$$F[x\bar{x}](\xi) = \frac{1}{1 - q\xi} \tag{1.8}$$

$$F[xy\bar{y}\bar{x}](\xi, \eta) = \frac{1}{(1 - q\xi)(1 - q\eta)} \tag{1.9}$$

$$F[xy\bar{x}\bar{y}](\xi, \eta) = F[x\bar{x}y\bar{y}](\xi, \eta) = \sum_{n,m=0}^{\infty} q^{\gcd(n,m)} \xi^n \eta^m \tag{1.10}$$

$$F[xyz\bar{x}\bar{y}\bar{z}](\xi, \eta, \zeta) = \sum_{n,m,p=0}^{\infty} q^{\gcd(n+m,m+p)} \xi^n \eta^m \zeta^p \tag{1.11}$$

$$F[xyz\bar{x}\bar{z}\bar{y}](\xi, \eta, \zeta) = \sum_{n,m,p=0}^{\infty} q^{\gcd(n,m+p)} \xi^n \eta^m \zeta^p \tag{1.12}$$

Proof. The first two equations are a direct application of equation (1.1). For the third equation, note that both functions satisfy, by equations (1.2) and (1.3),

$$F(\xi, \eta) = F(\xi\eta, \eta) + F(\xi, \xi\eta) - F[x\bar{x}](\xi\eta) \tag{1.13}$$

$$F(0, \eta) = F[y\bar{y}](\eta) \tag{1.14}$$

and that this fixes them univocally. This already proves that the two functions are equal, and we need now to determine their coefficients $a_{n,m}$.

Taking the claim in the proposition as an ansatz, we shall prove that only the function $a_{n,m} = q^{\gcd(n,m)}$ verifies the linear recursions above.

First of all, we rewrite equation (1.13) in terms of $a_{n,m}$ as

$$\sum_{n,m=0}^{\infty} a_{n,m} \xi^n \eta^m = \sum_{n,m=0}^{\infty} a_{n,m} \xi^{n+m} \eta^m + \sum_{n,m=0}^{\infty} a_{n,m} \xi^n \eta^{n+m} - \sum_{n=0}^{\infty} (q\,\xi\eta)^n.$$

Now, it is easily seen, by equation (1.14), that $a_{0,n} = a_{n,0} = q^n$, which coincides with $q^{\gcd(n,0)}$ under the customary convention $\gcd(n,0) = n$. Then, we see that $a_{n,n} = a_{0,n} + a_{n,0} - q^n = q^n$, thus we verify $a_{n,n} = q^n = q^{\gcd(n,n)}$. Then, we see that, for all $n, m \geq 1$, $a_{n+m,m} = a_{n,m}$ and $a_{n,n+m} = a_{n,m}$. As this is the same linear recursion involved in the Euclid algorithm for the g.c.d. of two positive integers, we can conclude that $a_{n,m} = q^{\gcd(n,m)}$ also in the case in which n and m are distinct positive integers.

For the patterns of size 3, it is easy to see that, up to symmetries and reductions provided by Proposition 6, there are only two cases left out of the $3! = 6$ permutations. For the case of equation (1.12), this is deduced from equation (1.10) and Propositions 5. For the case of equation (1.11), this is deduced from equations (1.12) and (1.9), which enter the recursion (1.2). The verification of the latter is similar to the analysis performed for equation (1.10), and we omit it here. □

Remark 8. *Note that the function in equation (1.10) has an alternate description. Call*

$$\Phi(\xi,\eta) = \sum_{\substack{i,j\geq 0 \\ \gcd(i,j)=1}} \xi^i \eta^j\,, \tag{1.15}$$

then

$$F(\xi,\eta) = 1 + \sum_{k\geq 1} q^k \Phi(\xi^k,\eta^k)\,. \tag{1.16}$$

The function $\Phi(\xi,\eta)$ in itself is apparently rather complicated. It is conceivable that it has already appeared in number theory, however we are not aware of any such occurrence. It can be related, in a complicated way, to an elementary function, as follows: call

$$\Phi_0(\xi,\eta) = \sum_{\substack{i,j\geq 0 \\ (i,j)\neq(0,0)}} \xi^i \eta^j = \frac{\xi+\eta-\xi\eta}{(1-\xi)(1-\eta)}\,, \tag{1.17}$$

then

$$\Phi_0(\xi,\eta) = \sum_{k} \Phi(\xi^k,\eta^k)\,. \tag{1.18}$$

This allows to check that

$$F(\xi,\eta)|_{q=1} = 1 + \frac{\xi+\eta-\xi\eta}{(1-\xi)(1-\eta)} = \frac{1}{(1-\xi)(1-\eta)}\,, \tag{1.19}$$

which is in agreement with Remark 4.

Now we shall discuss the limit $q \to \infty$ of the functions $F_q[\mathcal{E}]$. At this aim we need to first introduce a regularised version:

$$\tilde{F}_q[\mathcal{E}](\xi_1,\ldots,\xi_n) := F_q[\mathcal{E}](\xi_1/q,\ldots,\xi_n/q)\,. \tag{1.20}$$

For each pattern $\mathcal{E}$, $\tilde{F}_q[\mathcal{E}]$ is a power series in q^{-1}, with coefficients in $\mathbb{Q}(\xi_i)$ (rational functions in the ξ_i's). This is derived easily from the description of the function $a^{\mathcal{E}}_{i_1,\ldots,i_n}$ provided above. Let us discuss this fact more in detail in the notationally simpler situation, in which in $\mathcal{E}$ all the x_i variables appear first, and all the $\overline{x_j}$ arrive last. In this case $\mathcal{E}$ is encoded by a permutation $\sigma \in \mathfrak{S}_n$ ($\sigma(i) = j$ if $\overline{x_j}$ appears in position $2n+1-i$ in $\mathcal{E}$)[2]. For a vector $k = (k_1,\ldots,k_n) \in \mathbb{N}^n$, denote by $\sigma^{[k]}$ the permutation in $\mathfrak{S}_{\sum_i k_i}$ obtained by replacing the i-th edge of σ by k_i parallel edges. Call $C(\sigma)$ the number of cycles in σ. Then it is easy to see that

$$\tilde{F}_q[\sigma](\xi_1,\ldots,\xi_n) = \sum_{k\in\mathbb{N}^n} q^{C(\sigma^{[k]})-\sum_i k_i} \prod_i \xi_i^{k_i}\,; \tag{1.21}$$

and we have a non-positive power of q, because in a permutation there are at most as many cycles as edges. As the equality is obtained only in the case of the identity permutation, it is also not hard to establish that

Theorem 9.

$$\lim_{q\to\infty} \tilde{F}_q[\sigma](\xi_1,\ldots,\xi_n) = \sum_{\substack{\tau\subseteq\sigma \\ \tau\simeq \mathrm{id}_k}} \prod_{i\in\tau} \frac{\xi_i}{1-\xi_i}\,; \tag{1.22}$$

[2] Note however that we will use a different notation in the following chapters.

where the sum is over τ, subsets of the edges of σ of some size $0 \leq k \leq n$, such that the resulting permutation, after relabeling, coincides with the identity permutation of size k; the product over i edges of τ, instead, is performed with ξ_i corresponding to the original labels within σ.

As every singleton subset of σ is isomorphic to the identity of size 1, we have

$$\tilde{F}_q[\sigma](\xi_1, \dots, \xi_n) = 1 + \sum_{i=1}^{n} \frac{\xi_i}{1-\xi_i} + \cdots \tag{1.23}$$

where the dots are non-negative terms, formal power series in the ξ_j's and in $1/q$. This shows that the radius of convergence of $\tilde{F}_q$ in the ξ_j's is 1.

Example 10. *For $\sigma = (1, 3, 2)$ the sets τ are the list $\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{1, 2\}\}$, thus we have*

$$\begin{aligned}
\lim_{q\to\infty} \tilde{F}_q[\sigma](\xi, \eta, \zeta) &= 1 + \frac{\xi}{1-\xi} + \frac{\eta}{1-\eta} + \frac{\zeta}{1-\zeta} + \frac{\xi}{1-\xi}\frac{\eta}{1-\eta} + \frac{\xi}{1-\xi}\frac{\zeta}{1-\zeta} \\
&= \frac{\xi}{1-\xi} + \left(1 + \frac{\xi}{1-\xi}\right)\left(1 + \frac{\eta}{1-\eta} + \frac{\zeta}{1-\zeta}\right) \\
&= \frac{1}{1-\xi}\left(\xi + \frac{1-\eta\zeta}{(1-\eta)(1-\zeta)}\right) .
\end{aligned} \tag{1.24}$$

Now consider the linear recursion (1.2), rewritten in terms of the formal power series for $\tilde{F}$, and restricted to the order q^{-k}. On the LHS we have the coefficient $[q^{-k}]\tilde{F}_q[\sigma]$, while on the R.H.S. we have a linear combination of coefficients $[q^{-k'}]\tilde{F}_q[\sigma']$, for some k' and s', such that $k' \leq k$, $|\sigma'| \leq n$, and $k' + |\sigma'| \leq n + k - 1$.

Let us see why this is the case. To start with, it is trivial to see that it must be $k' \leq k$ and $|\sigma'| \leq n$, and thus, of course, $k' + |\sigma'| \leq n + k$. The crucial extra "$-1$" in the formula comes from the fact that on the RHS terms of equation (1.2) the function enters with an argument $\zeta := \xi\eta$. When rewritten in terms of the function $\tilde{F}_q$, this reads $\xi/q \times \eta/q = 1/q \times \xi\eta/q$. Thus, we can consider the Taylor expansion of $\tilde{F}_q$ also w.r.t. ζ, as any power of the parameter ζ also comes with an extra factor $1/q$. The terms of order 1 or more account for having terms with $k' < k$. The term of order zero corresponds to specialise the function on the RHS to $\zeta = 0$. However, as a simple generalisation of equation (1.3), specialising *any* edge-parameter to zero corresponds to study the expression in which the edge has been removed, thus it accounts for $n' = n - 1$.

This allows to establish the Taylor coefficients of $\tilde{F}_q[\sigma]$ recursively, up to some order k and for all permutations up to a given size n, starting from the zero-th order of equation (1.22), with no need of solving a linear system (i.e., no need of inverting the associated matrix).

Let us now come back to the original function $F_q[\sigma]$ (instead of $\tilde{F}_q[\sigma]$). The equation (1.21) now reads

$$F_q[\sigma](\xi_1, \dots, \xi_n) = \sum_{k\in\mathbb{N}^n} q^{C(\sigma^{[k]})} \prod_i \xi_i^{k_i} ; \tag{1.25}$$

and we have non-negative powers of q. In fact, the power is zero only for the vector $k = 0$, which is in agreement with Remark 4.

Now, suppose that $\gcd(k_1, \dots, k_n) = h > 1$. We can colour the edges of $\sigma^{[k]}$ in h colours, depending from their congruence classes modulo h, and as a result the cycles of $\sigma^{[k]}$ will be monochromatic (i.e., every cycle is composed of edges all of the same colour). This implies that $C(\sigma^{[k]}) = h\, C(\sigma^{[k/h]})$, so that we can write

$$\Phi_q[\sigma](\xi_1, \dots, \xi_n) := \sum_{\substack{k \in \mathbb{N}^n \\ \gcd(k_1,\dots,k_n)=1}} q^{C(\sigma^{[k]})} \prod_i \xi_i^{k_i} \,; \tag{1.26}$$

$$F_q[\sigma](\xi_1, \dots, \xi_n) = \sum_{h \geq 1} \Phi_{q^h}[\sigma](\xi_1^h, \dots, \xi_n^h)\,. \tag{1.27}$$

Note that this decomposition is in agreement with the exact enumeration discussed in Remark 8. Note also that the same decomposition holds for the rescaled versions of the functions, i.e.

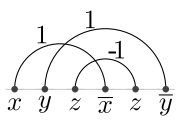
$$\tilde{\Phi}_q[\sigma](\xi_1, \dots, \xi_n) := \sum_{\substack{k \in \mathbb{N}^n \\ \gcd(k_1,\dots,k_n)=1}} q^{C(\sigma^{[k]}) - \sum_i k_i} \prod_i \xi_i^{k_i} \,; \tag{1.28}$$

$$\tilde{F}_q[\sigma](\xi_1, \dots, \xi_n) = \sum_{h \geq 1} \tilde{\Phi}_{q^h}[\sigma](\xi_1^h, \dots, \xi_n^h)\,. \tag{1.29}$$

1.4 A signed version of the problem

In our problem of palindrome words we require $\mathcal{E}$ to be an ordering of $(x_1, \overline{x_1}, \cdots, x_n, \overline{x_n})$. An obvious generalisation would be to continue requiring that each word variable x_i appears twice in $\mathcal{E}$, but however no longer ask that, for all i, both x_i and its reverse $\overline{x_i}$ be present in the string, while instead we could allow for two occurrences of x_i, and no occurrences of $\overline{x_i}$. An example of such an expression is $\mathcal{E} = x\,y\,z\,\overline{x}\,z\,\overline{y}$.

We have already described how to encode an expression $\mathcal{E}$ through a matching. In this more general framework, we still have matchings, and we also have a 'sign' $s_i \in \{+1, -1\}$ associated to each arc i. We use $s_i = +1$ if the words x_i and $\overline{x_i}$ are present in $\mathcal{E}$, and $s_i = -1$ if, on the contrary, the word x_i occurs twice in $\mathcal{E}$. In our example we would have the representation

$\mathcal{E} = x\,y\,z\,\overline{x}\,z\,\overline{y} =$ 

Theorem 2 generalise to:

Theorem 11. *$F[\mathcal{E}](\xi_1, \dots, \xi_n)$ is a formal power series fixed by the obvious symmetries, (i.e., symmetry under $x \leftrightarrow \overline{x}$ if $s_x = +1$, under $(x, \overline{x}, \xi) \leftrightarrow (y, \overline{y}, \eta)$ if $s_x = s_y = +1$, and under $(x, \xi) \leftrightarrow (y, \eta)$ if $s_x = s_y = -1$), by $F[\varnothing] = 1$, by the linear recursions in Theorem 2, plus their analogues:*

$$F[x\mathcal{E}x](\xi, \dots) = \frac{(1 + q\xi)}{1 - q\xi^2}\, F[\mathcal{E}](\dots)\,; \tag{1.30}$$

$$\begin{aligned} F[x\mathcal{E}(x,y)\overline{y}](\xi, \eta, \dots) = \quad & F[\mathcal{E}(x, xy)\overline{y}](\xi\eta, \eta, \dots) && \textit{(for } x \preceq y\textit{)} \\ & + F[x\mathcal{E}(yx, y)](\xi, \xi\eta, \dots) && \textit{(for } x \succeq y\textit{)} \\ & - F[\mathcal{E}(x, \overline{x})](\xi\eta, \dots) && \textit{(for } x \equiv y\textit{)} \end{aligned} \tag{1.31}$$

$$F[x\mathcal{E}(\bar{x},y)y](\xi,\eta,...) = \begin{aligned}[t] & F[\mathcal{E}(\mathcal{E}(\bar{x},y\bar{x})y]](\xi\eta,\eta,...) && \textit{(for } x \preceq y\textit{)} \\ & + F[x\mathcal{E}(\bar{x}y,y)](\xi,\xi\eta,...) && \textit{(for } x \succeq y\textit{)} \\ & - F[\mathcal{E}(\bar{x},x)](\xi\eta,...) && \textit{(for } x \equiv y\textit{)} \end{aligned} \tag{1.32}$$

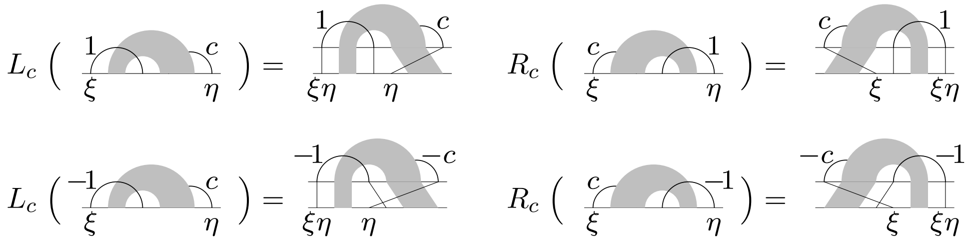
$$F[x\mathcal{E}(x,y)y](\xi,\eta,...) = \begin{aligned}[t] & F[\mathcal{E}(x,y\bar{x})y](\xi\eta,\eta,...) && \textit{(for } x \preceq y\textit{)} \\ & + F[x\mathcal{E}(\bar{y}x,y)](\xi,\xi\eta,...) && \textit{(for } x \succeq y\textit{)} \\ & - F[\mathcal{E}(x,x)](\xi\eta,...) && \textit{(for } x \equiv y\textit{)} \end{aligned} \tag{1.33}$$

$$F[x\mathcal{E}(x)](\xi,...)|_{\xi=0} = F[\mathcal{E}(\epsilon)](...)\,. \tag{1.34}$$

Similarly as was the case for Theorem 2, we have a subset of these equations that remain non-trivial once we restrict to the homogeneous part of the linear system. In this case we have the four equations (1.2), (1.31), (1.32) and (1.33).

Having to deal with four similar equations is annoying. Yet again, the graphical representation offers a way to compact it into two equations.

Let $m = \mathcal{E}$ be a matching over $[2n]$ with weight $(\xi_1, \ldots, \xi_n)$ and colors $c \in \{1, -1\}$ on the arcs and such that $m(1) \neq 2n$ (i.e. we are not in the case of equation (1.1) or (1.30)). Then define the maps L_c and R_c on the parametrised and colored matchings as:



Then, at the price of a case-differentiation of the operators L_c andd R_c above, we can formally write the unique and simple equation $F[m] = F[R_c(m)] + F[L_c(m)] + \cdots$, where again the dots stand for functions with shorter $\mathcal{E}$'s.

The two operators L_c and R_c are the operators of a Rauzy-type dynamics on permutations and on matchings as can be seen above however we will not describe it any further.

Chapter 2

Permutational diagram monoids and groups

Let us try to formalise the mathematical nature of the operators L and R (or L_c and R_c) which has appeared in the previous 'prolog' section. Our goal is to have a degree of flexibility which allows to put all of our Rauzy-type dynamics within the same framework. Also, by a formalisation, we will be able to stress the fundamental features of these operations, and leave aside the aspects which may be specific of a given realisation of the dynamics.

2.1 The main definition

Let X_I be a set of labeled combinatorial objects, with elements labeled from the set I. In the most general context I may be a *multi-set*, i.e. allow for repetitions. However, at the purposes of the present thesis, we will mainly concentrate on the case in which each label is repeated exactly once. For this reason, in the following we will assume that I can be totally ordered, and, in most circumstances, we shall identify it with the set $[n] = \{1, 2, \dots, n\}$ (and use the shortcut $X_n \equiv X_{[n]}$).[1] The symmetric group $\mathfrak{S}_n$ acts naturally on X_n, by producing the object with permuted labels.[2]

Vertex-labeled graphs (or digraphs, or hypergraphs) are a typical example. Extra structure may be added, e.g. in hypergraphs we can complement hyperedges with a cyclic ordering of the incident vertices, and in both graphs and hypergraphs we can specify a cyclic ordering of the incident edges at each vertex. This provides an embedding of the abstract graph on a surface, and may be turned into a prescription for drawing such a structure on a disk, although possibly with crossing (hyper-)edges.

Set partitions are a special case of hypergraphs (all vertices have degree 1). Matchings are a special case of partitions, in which all blocks have size 2. Permutations σ are a special case of matchings, in which each block $\{i, j\}$ has $i \in \{1, 2, \dots, n\}$ and $j = \sigma(i) + n \in \{n + 1, n + 2, \dots, 2n\}$.

[1] The use of I instead of just $[n]$ is a useful notation when considering substructures: if $x \in X_n$, and $x' \subseteq x$ w.r.t. some notion of inclusion, it may be conventient to say that $x' \in X_I$ for I a suitable subset of $[n]$, instead that the canonical one.

[2] For I a multiset, the natural action would be the one of $\mathfrak{S}_{|I|}/\mathrm{Aut}(I)$.

We will consider dynamics over spaces of this type, generated by operators of a special form that we now introduce:

Definition 12 (Monoid and group operators)**.** *We say that A is a* monoid operator *on set X_n, if, for the datum of a finite set Y_n, a map $a : X_n \to Y_n$, and a map $\alpha : Y_n \to \mathfrak{S}_n$, it consists of the map on X_n defined by*

$$A(x) = \alpha_{a(x)} x \,, \tag{2.1}$$

where the action αx is in the sense of the symmetric-group action over X_n. We say that A is a group operator *if, furthermore, $a(A(x)) = a(x)$.*

Said informally, the function a "poses a question" to the structure x. The possible answers are listed in the set Y. For each answer, there is a different permutation, by which we act on x. Actually, as anticipated, we only use a and α in the combination $A = \alpha_{a(\cdot)}$, so that the use of two symbols for the single function A is redundant. This choice is done for clarity in our applications, where the notation allows to stress that Y_n has a much smaller cardinality than X_n and $\mathfrak{S}_n$, i.e. very few 'answers' are possible. In our main application, $|X_n| = |\mathfrak{S}_n| = n!$ while $|Y_n| = n$. The asymptotic behaviour is similar ($|X_n|$ is at least exponential in n, while $|Y_n|$ is linear) in all of our applications.

Clearly we have:

Proposition 13. *Group operators are invertible.*

Proof. For a given value of n, let A be a group operator on the set X_n. The property $a(A(x)) = a(x)$ implies that, for all $k \in \mathbb{N}$, $A^k(x) = (\alpha_{a(x)})^k x$. Thus, for all x there exists an integer $d_A(x) \in \mathbb{N}^+$ such that $A^{d_A(x)}(x) = x$. More precisely, $d_A(x)$ is (a divisor of) the l.c.m. of the cycle-lengths of $\alpha_{a(x)}$. Call $d_A = \mathrm{lcm}_{x \in X_n} d_A(x)$ (i.e., more shortly, the l.c.m. over $y \in Y_n$ of the cycle-lengths of α_y). Then d_A is a finite integer, and we can pose $A^{-1} = A^{d_A - 1}$. The reasonings above show that A is a bijection on X_n, and A^{-1} is its inverse. □

Definition 14 (monoid and group dynamics)**.** *We call a* monoid dynamics *the datum of a family of spaces $\{X_n\}_{n\in\mathbb{N}}$ as above, and a finite collection $\mathcal{A} = \{A_i\}$ of monoid operators. We call a* group dynamics *the analogous structure, in which all A_i's are group operators.*

For a monoid dynamics on the datum $S_n = (X_n, \mathcal{A})$, we say that $x, x' \in X_n$ are strongly connected*, $x \sim x'$, if there exist words $w, w' \in \mathcal{A}^*$ such that $wx = x'$ and $w'x' = x$.*

For a group dynamics on the datum $S_n = (X_n, \mathcal{A})$, we say that $x, x' \in X_n$ are connected*, $x \sim x'$, if there exists a word $w \in \mathcal{A}^*$ such that $wx = x'$.*

Here the action wx is in the sense of monoid action. Being connected is clearly an equivalence relation, and coincides with the relation of being graph-connected on the Cayley Graph associated to the dynamics, i.e. the digraph with vertices in X_n, and edges $x \longleftrightarrow_i x'$ if $A_i^{\pm 1} x = x'$. An analogous statement holds for strong-connectivity, and the associated Cayley Digraph.

Definition 15 (classes of configurations)**.** *Given a dynamics as above, and $x \in X_n$, we define $C(x) \subseteq X_n$, the* class of x*, as the set of configurations connected to x, $C(x) = \{x' \,:\, x \sim x'\}$.*

Two natural distances on the classes of X_n can be associated to a group dynamics with generators $\mathcal{A} = \{A_i\}$.

Definition 16 (distance and alternation distance)**.** *Let σ, τ configurations of X_n in the same class. The* graph distance $d_G(\sigma, \tau)$ *is the ordinary graph distance in the associated Cayley Graph, i.e. $d_G(\sigma, \tau)$ is the minimum $\ell \in \mathbb{N}$ such that there exists a word $w \in \{A_i, A_i^{-1}\}^*$, of length ℓ, such that $\tau = w\sigma$. A* graph geodesic *is a w realising the minimum. We also define the* alternation distance $d(\sigma, \tau)$*, and* alternation geodesics*, as the analogous quantities, for words in the infinite alphabet $\{(A_i)^j\}_{j \geq 1}$.*

Example 17. *If $w = a^3\, b\, a^{-2}\, b^{-3}\, a$, and $wx = x'$, we know that the distance $d_G(x, x')$ is at most $3 + 1 + 2 + 3 + 1 = 10$, and the alternation distance $d(x, x')$ is at most 5.*

Contrarily to graph distance, the alternation distance is stable w.r.t. a number of combinatorial operations that we will perform on our configurations, and which are defined later on in the text or in future work (restriction to 'primitive classes', study of 'reduced dynamics', permutations in $\mathfrak{S}_\infty$, ...). These facts suggests that alternation distance is a more natural notion in this family of problems.

2.2 Group-decoration

Now, let G be a finite group, and X_I^G be a set of labeled combinatorial objects, such that we have a G^I action, i.e., for each element $i \in I$ it is defined a group-action of G, over the element i. As a result, we have a natural group action $\mathfrak{S}_n \ltimes G^n$ on X_n, which, for $(\sigma; g_1, \dots, g_n) \in \mathfrak{S}_n \ltimes G^n$, first, for $i = 1, \dots, n$, applies g_i to the i-th element of the object, and then permutes the objects $i \in I$ with the given permutation σ (the fact that we have a semi-direct product is easily established).

The notion of monoid and group operators generalises immediately, as well as the first few definitions, of classes, distance, etc., that we have provided above. Note in particular that, for A to be a group operator, we need (as before) that $a(A(x)) = a(x)$. However, now the α_a's are valued in $\mathfrak{S}_n \ltimes G^n$, instead that in $\mathfrak{S}_n$. So for A to be a group operator we need that not only the permutation σ is the same for x and for $A(x)$, but also the list $(g_1, \dots, g_n)$ of group-elements in G^I. Then, the conclusion of Proposition 13 remains valid, and the proof requires only minor modifications, in which the orders of the group elements g_i enter the l.c.m. We leave the details to the reader.

We mention in passing that the dynamics of L and R introduced in the previous section can be described in the 'simple' setting described in the paragraph above (i.e., with no group G). On the contrary, the dynamics of L_c and R_c introduced in the previous section can be described only in this more general setting, namely with the choice $G = \mathbb{Z}_2$ (an element $g \in \{+1, -1\}$ acts on a vertex i by multiplication on the parameter of its adjacent edge). However this thesis will not investigate this more general case futher.

2.3 The main goal

The goal of this thesis is to provide *classification theorems* for dynamics of this kind. In particular, we will be concerned with the classification of classes, and further combinatorial study of their structure, for four special group dynamics: The *Rauzy dynamics*, The *extended Rauzy dynamics*, the *Rauzy dynamics for matching* and the *Involution dynamics for matchings.*

Of these four dynamics, the first two were first introduced by Rauzy [Rau79], and their study was pioneered by Veech [Vee82] and Masur [Mas82] (the connected classes are called *Rauzy classes*). In these cases the group action is related to the interval exchange map on translation surfaces. Thus, on one side, its study is motivated by questions in dynamical systems. On the other side, the previously obtained classification of classes relies on notions and known results in algebraic geometry. Section 3.1.1 describes these two dynamics, and Section 3.2 gives a short account of these connections.

The classification theorems associated to these families have been provided (with some *caveat* discussed later on) in a series of papers, starting with the seminal work of Kontsevich and Zorich [KZ03], and followed by [Boi12], so our approach provides just an alternative derivation of these results. Nonetheless, it has two points of interest.

A first point is that, as our approach is quite different from the previous ones, along the way we happen to extract some extra information on the combinatorial structure of the Rauzy classes. An example, which is mentioned in Section 3.2.3, is the construction of 'many' representatives for each Rauzy class, extending previous results of Zorich [Zor08].

The second, more methodological point of interest is that our approach is completely combinatorial, self-contained, and in particular it makes no use of any facts from algebraic geometry. In this sense, it answers a question posed by Kontsevich and Zorich in [KZ03]:

> *The extended Rauzy classes can be defined in purely combinatorial terms [...] thus the problem of the description of the extended Rauzy classes, and hence, of the description of connected components of the strata of Abelian differentials, is purely combinatorial. However, it seems very hard to solve it directly. [...] we give a classification of extended Rauzy classes using not only combinatorics but also tools of algebraic geometry, topology and of dynamical systems.*

Note that the task of providing a purely combinatorial proof of these classification theorems has been also carried over, with different methods than ours, by [Fic16].

In the following sections of this thesis, we will address other variants of these group actions on combinatorial structures, that, by analogy, we name *Rauzy-type dynamics.* However, a number of these other versions do not have at the moment any algebraic or geometric interpretation. As such the proof employed for the classification theorem of the Rauzy dynamics which relies (though it does not transpire in the proof) on the underlying geometry of the problem (in a way made clear in section 3.2) does not generalise.

To surmount this difficulty, we develop in section 5, a very general proof method for the classification theorem of *Rauzy-type dynamics* that we call the *labelling* *method*. Applying the labelling method will provide, in sections 6, 7 respectively, a second proof of the classification theorem of the Rauzy dynamics as well as proofs of the classification theorems of the Involution dynamics for matchings. In section 8 we define the Rauzy dynamics for matchings for which the proof can be carried out and will be in a subsequent paper.

.Finally, we conclude the thesis by presenting a few Rauzy-type dynamics for which the labelling method can apply (though the method may need some tinkering for a few of those) but that we did not solve.

Part I

A first approach to Rauzy dynamics

Chapter 3

The Rauzy dynamics: introduction, context, results

3.1 Definition of the Rauzy dynamics

3.1.1 Three basic examples of dynamics

Let $\mathfrak{S}_n$ denote the set of permutations of size n, and $\mathfrak{M}_n$ the set of matchings over $[2n]$, thus with n arcs. Let us call ω the permutation $\omega(i) = n + 1 - i$.

A permutation $\sigma \in \mathfrak{S}_n$ can be seen as a special case of a matching over $[2n]$, in which the first n elements are paired to the last n ones, i.e. the matching $m_\sigma \in \mathfrak{M}_n$ associated to σ is $m_\sigma = \{(i, \sigma(i) + n) \,|\, i \in [n]\}$.

We say that $\sigma \in \mathfrak{S}_n$ is *irreducible* if $\omega\sigma$ doesn't leave stable any interval $\{1, \ldots, k\}$,

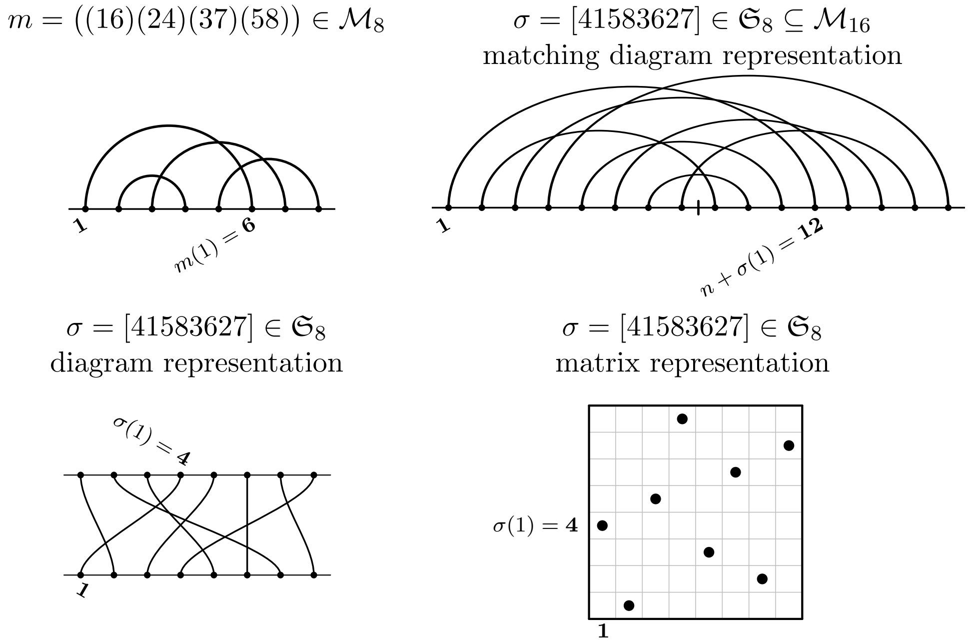


Figure 3.1: Diagram representations of matchings and permutations, and matrix representation of permutations.

for $1 \leq k < n$, i.e. if $\{\sigma(1), \dots, \sigma(k)\} \neq \{n-k+1, \dots, n\}$ for any $k = 1, \dots, n-1$. We also say that $m \in \mathfrak{M}_n$ is *irreducible* if it does not match an interval $\{1, \dots, k\}$ to an interval $\{2n-k+1, \dots, 2n\}$. Let us call $\mathfrak{S}_n^{\mathrm{irr}}$ and $\mathfrak{M}_n^{\mathrm{irr}}$ the corresponding sets of irreducible configurations.

We represent matchings over $[2n]$ as arcs in the upper half plane, connecting pairwise $2n$ points on the real line (see figure 3.1, top left). Permutations, being a special case of matching, can also be represented in this way (see figure 3.1, top right), however, in order to save space and improve readibility, we rather represent them as arcs in a horizontal strip, connecting n points at the bottom boundary to n points on the top boundary (as in Figure 3.1, bottom left). Both sets of points are indicised from left to right. We use the name of *diagram representation* for such representations.

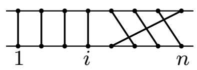
We will also often represent configurations as grids filled with one bullet per row and per column (and call this *matrix representation* of a permutation). We choose here to conform to the customary notation in the field of Permutation Patterns, by adopting the algebraically weird notation, of putting a bullet at the *Cartesian* coordinate (i, j) if $\sigma(i) = j$, so that the identity is a grid filled with bullets on the *anti-diagonal*, instead that on the diagonal. An example is given in figure 3.1, bottom right.

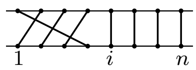
Let us define a special set of permutations (in cycle notation)

$$\gamma_{L,n}(i) = (i-1\ i-2\ \cdots\ 1)(i)(i+1)\cdots(n)\,; \tag{3.1a}$$

$$\gamma_{R,n}(i) = (1)(2)\cdots(i)(i+1\ i+2\ \cdots\ n)\,; \tag{3.1b}$$

i.e., in a picture in which the action is diagrammatic, and acting on structures $x \in X_n$ from below,

$\gamma_{L,n}(i)$: 

$\gamma_{R,n}(i)$: 

Of course, $\omega \gamma_{L,n}(i) \omega = \gamma_{R,n}(n+1-i)$.

The three group dynamics we analyse in this chapter and the next (chapter 4) and in a later one (chapter 8) are

- $\mathcal{M}_n$ **:** The space of configuration is $\mathfrak{M}_n^{\mathrm{irr}}$, irreducible n-arc matchings. There are two generators, L and R, with $a_L(m)$ being the index paired to 1, $a_R(m)$ the index paired to $2n$, $\alpha_{L,i} = \gamma_{L,2n}(i)$, and $\alpha_{R,i} = \gamma_{R,2n}(i)$ (a and α are as in Definition 12). See Figure 3.2, top.

- $\mathcal{S}_n$ **:** The space of configuration is $\mathfrak{S}_n^{\mathrm{irr}}$, irreducible permutations of size n. Again, there are two generators, L and R. If permutations are seen as matchings such that indices in $\{1, \dots, n\}$ are paired to indices in $\{n+1, \dots, 2n\}$, the dynamics coincide with the one given above. See Figure 3.2, middle.

- $\mathcal{S}_n^{\mathrm{ex}}$ **:** The space of configuration is $\mathfrak{S}_n^{\mathrm{irr}}$. Now we have four generators, L and R are as above, and L' and R' act as $L'\sigma = S(L(S\sigma))$, and $R'\sigma = S(R(S\sigma)$ where S is the anti-diagonal axial symmetry in the matrix representation. See Figure 3.2, middle and bottom.

More precisely, we study $\mathcal{S}_n$ and $\mathcal{S}_n^{\mathrm{ex}}$ here, and delay the study of $\mathcal{M}_n$ to section 8.

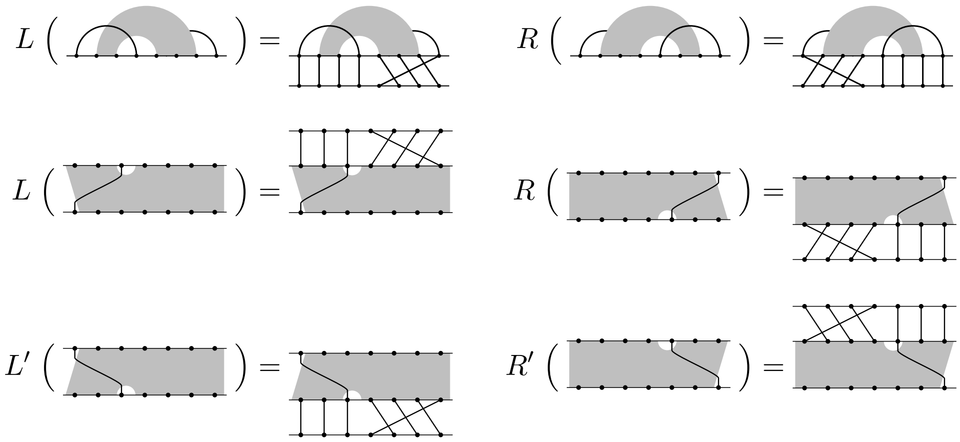


Figure 3.2: Our three main examples of dynamics, in diagram representation. Top: the $\mathcal{M}_n$ case. Middle: the $\mathcal{S}_n$ case. Bottom: in the $\mathcal{S}_n^{\text{ex}}$ case we have the same operators L and R as in the $\mathcal{S}_n$ case, plus the two operators L' and R'.

The motivation for restricting to irreducible permutations and matchings shall be clear at this point: a non-irreducible permutation is a grid with a non-trivial block-decomposition. The operators L and R only act on the first block (say, of size k), while, in $\mathcal{S}_n^{\text{ex}}$, the operators L' and R' only act on the last block (say, of size k'), so that the study of the dynamics trivially reduces to the study of the $\mathcal{S}_k$ dynamics, or of the direct product $\mathcal{S}_k \times \mathcal{S}_{k'}$, on these blocks (see figure 3.4).

This simple observation, however, comes with a discaimer: in our combinatorial operations on configurations, which produce the required induction steps in the classification theorem, we shall always guarantee that the outcome of our manipulations on irreducible configurations is still irreducible. Proving such a condition will occasionally be a subtle task.

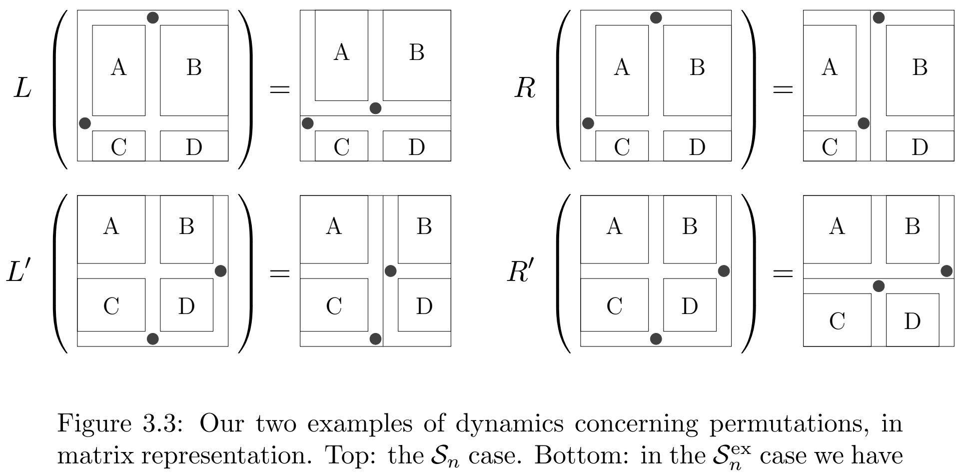


Figure 3.3: Our two examples of dynamics concerning permutations, in matrix representation. Top: the $\mathcal{S}_n$ case. Bottom: in the $\mathcal{S}_n^{\text{ex}}$ case we have the same operators L and R as in the $\mathcal{S}_n$ case, plus the two operators L' and R'. See also Figure 3.2.

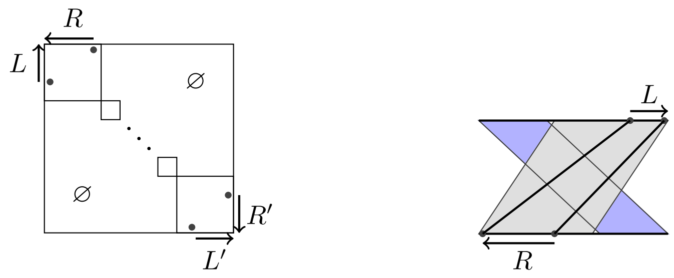


Figure 3.4: Left: A reducible permutation in matrix representation; the $\mathcal{S}^{\text{ex}}$ dynamic acts on the first block with L and R and on the last block with L' and R'. An analogous statement holds for the dynamics $\mathcal{S}$. Right: A reducible permutation in diagram representation; the $\mathcal{S}$ dynamics acts on the gray part with L and R, while leaving the blue part unchanged. An analogous statement holds for the dynamics $\mathcal{S}^{\text{ex}}$.

3.1.2 What kind of results?

In this thesis, we study collections X_n of discrete combinatorial objects, where n is a size parameter, and the cardinality of X_n is exponential or super-exponential (i.e., $\ln X_n = \Theta(n(\ln n)^\gamma))$ for some γ). We introduce two or more bijections, called 'operators', like the operators L, R, $\dots$ of the previous section, and define classes as the connected components of the associated Cayley Graph. It will come out that there is a super-polynomial number of classes (in the examples investigated here, roughly $\exp(\alpha\sqrt{n})$ for a certain α), the majority of them having super-exponential cardinalities, which are not 'round' numbers[1].

Thus, a natural question is: in which form can we expect to 'solve' such a problem, if the structure of the problem is apparently so wild, and any possible complete answer is most likely not encompassed by a compact formula?

We may expect, and in fact give here, results of the following forms:

- A *classification* of the classes, i.e., the identification of a natural labeling of the classes, and a criterium that, for a configuration of size n, gives the label of its class through an algorithm which is polynomial in n (and possibly linear). As a corollary, this would give an algorithm that, for any two configurations, determines in polynomial time if they are in the same class or not.[2]

- The exact characterisation of the Cayley Graph of the few *exceptional classes* that may exist, which in fact *do* have round formulas, and an intelligible structure. Interestingly, in the case of the Rauzy dynamics, we found two exceptional classes: the hyperelliptic one that has been understood since Rauzy [Rau79, sec. 4], and a second one, which is primitive only in the $\mathcal{S}$ dynamics (not in the $\mathcal{S}^{\text{ex}}$ one), and, to the best of our knowledge, was never described

[1] I.e., numbers which most likely do not have simple factorised formulas, contrarily e.g. to the existence of hook formulas for the number of standard and semi-standard Young tableaux, or MacMahon formulas for the number of plane partitions, and various other examples in Algebraic Combinatorics.

[2] We insist on the algorithm complexity aspect as, if we do not pose a complexity bound, the mere greedy exploration of the Cayley Graph trivialises the question at hand.

before. These two classes play an important role in the forementioned classification theorem, and their detailed study is performed in Appendix C.

- Upper/lower bounds on various interesting quantities, e.g. the cardinalities of the classes, or their diameter.

- Finally, what we consider the most important and original contribution of this part of the thesis, the elucidation of a combinatorial structure which is recursive in n: even if the structure of the classes at size n is intrinsically too complex for being described by a simple formula (see [Del13] for the most compact formulas known so far), nothing prevents in principle from the existence of a reasonably explicit description of this structure at size n, in terms of the structures at sizes $n' < n$. This interplay of structures at different sizes, obtained through peculiar "surgery" operations at the level of the corresponding Cayley Graphs (partially analogous to the well-established surgeries at the level of Riemann surfaces of [EMZ03, KZ03], see the following Section 3.2.3), appears at several stages in this proof of the classification theorem for the Rauzy dynamics.

3.1.3 Definition of the invariants

The main purpose of this thesis, is to characterise the classes appearing in the dynamics introduced above: of $\mathcal{S}_n$ and $\mathcal{S}_n^{\mathrm{ex}}$, in this part, and of $\mathcal{M}_n$, and the involution dynamics in the chapter 8 and 6 respectively. The characterisation is based roughly on three steps: (1) we identify some "exceptional classes", for which the structure of the configurations is completely elucidated; (2) we identify a collection of data structures which are invariant under the dynamics; (3) we prove, through a complex induction based on surgery, that these invariants are complete, i.e. we characterise the admissible invariant structures, and prove that configurations with the same invariant, and not in an exceptional class, are also in the same class.

In this section we sketch the step (2), i.e. we present the invariants. Some of the proofs are postponed.

3.1.3.1 Cycle invariant

Let σ be a permutation, identified with its diagram. An edge of σ is a pair (i^-, j^+), for $j = \sigma(i)$, where $-$ and $+$ denote positioning at the bottom and top boundary of the diagram. Perform the following manipulations on the diagram: (1) replace each edge with a pair of crossing edges; more precisely, replace each edge endpoint, say i^-, by a black and a white endpoint, i_b^- and i_w^- (the black on the left), then introduce the edges (i_b^-, j_w^+) and (i_w^-, j_b^+). (2) connect by an arc the points $i_w^\pm$ and $(i+1)_b^\pm$, for $i = 1, \dots, n-1$, both on the bottom and the top of the diagram; (3) connect by an arc the top-right and bottom-left endpoints, n_w^+ and 1_b^-. Call this arc the "-1 mark".

The resulting structure is composed of a number of closed cycles, and one open path connecting the top-left and bottom-right endpoints, that we call the *rank path*. If it is a cycle that goes through the -1 mark (and not the rank path), we call it the *principal cycle*. Define the length of an (open or closed) path as the number of top (or bottom) arcs (connecting a white endpoint to a black endpoint) in the path. These numbers are always positive integers (for $n > 1$ and irreducible permutations). The

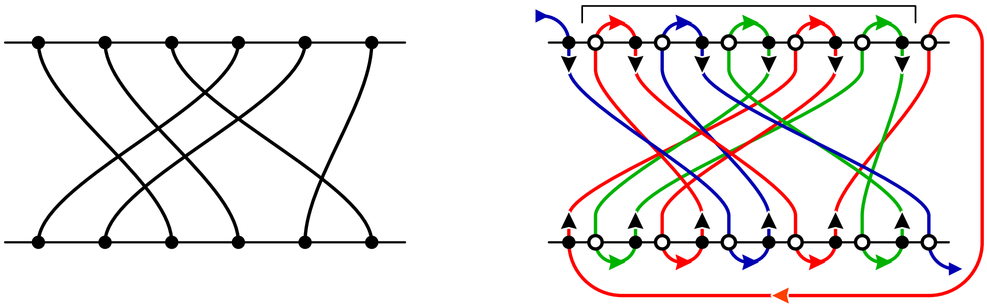


Figure 3.5: Left: an irreducible permutation, $\sigma = [451263]$. Right: the construction of the cycle structure. Different cycles are in different colour. The length of a cycle or path, defined as the number of top (or bottom) arcs, is thus 2 for red and violet, and 1 for blue. The green arrows are the endpoints of the rank path, which is in blue. As a result, in this example $\lambda(\sigma) = (2, 2)$ (for the cycles of color red and violet), $r(\sigma) = 1$ (corresponding to the rank path of color blue), and $\ell(\sigma) = 2$.

length r of the rank path will be called the *rank* of σ, and $\lambda = \{\lambda_i\}$, the collection of lengths of the cycles, will be called the *cycle structure* of σ. Define $\ell(\sigma)$ as the number of cycles in σ (this does not include the rank). See Figure 8.3, for an example.

Note that this quantity does *not* coincide with the ordinary path-length of the corresponding paths. The path-length of a cycle of length k is $2k$, unless it goes through the -1 mark, in which case it is $2k + 1$. Analogously, if the rank is r, the path-length of the rank path is $2r + 1$, unless it goes through the -1 mark, in which case it is $2r + 2$. (This somewhat justifies the name of "-1 mark" for the corresponding arc in the construction of the cycle invariant.)

In the interpretation within the geometry of translation surfaces, the cycle invariant is exactly the collection of conical singularities in the surface (we have a singularity of $2k\pi$ on the surface, for every cycle of length $\lambda_i = k$ in the cycle invariant, and the rank corresponds to the 'marked singularity' of [Boi12], see Section 3.2).

It is easily seen that

$$r + \sum_i \lambda_i = n - 1\,, \tag{3.2}$$

this formula is called the *dimension formula*. Moreover, in the list $\{r, \lambda_1, \ldots, \lambda_\ell\}$, there is an even number of even entries (This is part of Lemma 81 in Section 4.4.1, but it could also be proven easily already at this stage[3]).

We have

Proposition 18. *The pair (λ, r) is invariant in the $\mathcal{S}$ dynamics.*

Now connect also the endpoints n_w^- and 1_b^+ of the rank path. Call $\lambda' = \lambda \cup \{r\}$ the resulting collection of cycle lengths. It is easily seen that

$$\sum_i \lambda'_i = n - 1\,. \tag{3.3}$$

[3]This is a simple induction. Adding an edge at any given position changes the cycle invariant either by $\lambda_i \to a + 1 \oplus \lambda_i - a$ for some a, or by $\lambda_i \oplus \lambda_j \to \lambda_i + \lambda_j + 1$. A case analysis on the parity of λ_i, a and λ_j leads to our claim.

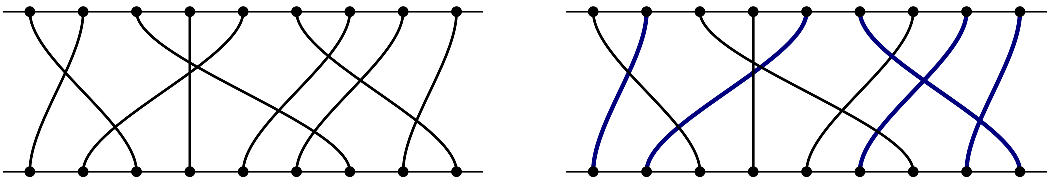


Figure 3.6: Left: an example of permutation, $\sigma = [\,251478396\,]$. Right: an example of subset $I = \{1, 2, 6, 8, 9\}$ (labels are for the bottom endpoints, edges in I are in blue). There are two crossings, out of the maximal number $\binom{|I|}{2} = 10$, thus $\chi_I = 8$ in this case, and this set contributes $(-1)^{|I|+\chi_I} = (-1)^{5+8} = -1$ to $A(\sigma)$.

We have

Proposition 19. *The quantity λ' is invariant in the $\mathcal{S}^{\mathrm{ex}}$ dynamics.*

These two propositions are proven in Section 4.1.1.

3.1.3.2 Sign invariant

For σ a permutation, let $[n]$ be identified to the set of edges (e.g., by labeling the edges w.r.t. the bottom endpoints, left to right). For $I \subseteq [n]$ a set of edges, define $\chi(I)$ as the number of pairs $\{i', i''\} \subseteq I$ of non-crossing edges. Call

$$\overline{A}(\sigma) := \sum_{I \subseteq [n]} (-1)^{|I|+\chi(I)} \tag{3.4}$$

the *Arf invariant* of σ (see Figure 3.6 for an example). Call $s(\sigma) = \mathrm{Sign}(\overline{A}(\sigma)) \in \{-1, 0, +1\}$ the *sign* of σ. We have

Proposition 20. *The sign of σ can be written as $s(\sigma) = 2^{-\frac{n+\ell}{2}} \overline{A}(\sigma)$, where ℓ is the number of cycles of σ. The quantity $s(\sigma)$ is invariant both in the $\mathcal{S}$ and $\mathcal{S}^{\mathrm{ex}}$ dynamics.*

The proof of invariance claimed in this proposition is given in Section 4.2.2, while the proof that $s(\sigma) = 2^{-\frac{n+\ell}{2}} \overline{A}(\sigma)$ is given in Section 4.4.1, namely in Lemma 81. In section 3.2 we motivate the name of Arf invariant by making explicit the connection to the associated quantity in the theory of translation surfaces (and, more generally, of Riemann surfaces).

Here we give the main idea in the proof of the invariance. The four operators of $\mathcal{S}^{\mathrm{ex}}$ are related by a dihedral symmetry of the diagram, and the expression for $\overline{A}(\sigma)$ is manifestly invariant under these symmetries, so it suffices to consider just one operator, say L. This operator uses an edge e of the diagram of σ to determine the permutation, then it moves a number of edge endpoints by a single position on the left, and the endpoint of a second edge f on the right.

Any function of the form $F(\sigma) = \sum_{I \subseteq [n]} f_\sigma(I)$ can be decomposed in the form

$$F(\sigma) = \sum_{I \subseteq [n] \smallsetminus \{e,f\}} \left(f_\sigma(I) + f_\sigma(I \cup \{e\}) + f_\sigma(I \cup \{f\}) + f_\sigma(I \cup \{e, f\}) \right).$$

Call $\tau = L(\sigma)$. It can be verified that, for the function $\overline{A}(\sigma)$,

$$f_\sigma(I) = f_\tau(I)\,, \qquad f_\sigma(I \cup \{e\}) = f_\tau(I \cup \{e\})\,,$$
$$f_\sigma(I \cup \{f\}) = f_\tau(I \cup \{e,f\})\,, \qquad f_\sigma(I \cup \{e,f\}) = f_\tau(I \cup \{f\})\,,$$

so that the sum of the four terms is invariant for any given $I \subseteq [n] \smallsetminus \{e,f\}$.

3.1.3.3 Cycles of length 1 and primitivity

We have stated in Section 3.1.3.1 that a certain graphical construction leads to the definition of a "cycle invariant". In this section we discuss how cycles of length 1 have an especially simple behaviour.

Establishing this simplifying feature is helpful in our proof of the main classification theorem. It goes in the direction of understanding as many as possible relevant combinatorial features of these classes, and allows us to rule out a large part of the massive numerics associated to this problem.

Definition 21 (descent and special descent)**.** *For a permutation σ in the dynamics $\mathcal{S}_n$, we say that the edge $(i, \sigma(i))$ is a* descent *if $\sigma(i+1) = \sigma(i) - 1$, and it is a* special descent *if $\sigma(1) = \sigma(i) - 1$ and $\sigma(i+1) = n$. For σ in the dynamic $\mathcal{S}_n^{\mathrm{ex}}$, we say that $(i, \sigma(i))$ is a* special descent *also if $\sigma^{-1}(i-1) = 1$ and $\sigma^{-1}(n) = \sigma(i) + 1$.*

Note that the descents of a permutation (special or not) are associated to cycles of length 1 (see figure 3.7 left). In particular, the number of descents is preserved by the dynamics. Say that σ is *primitive* if it has no descents. Hence in a given class C either all permutations are primitive (in this case we say that C is a *primitive class*), or none is. We define $\mathrm{prim}(\sigma)$, the *primitive of* σ, as the configuration obtained by removing descent- and special-descent-edges from σ (see figure 3.7 for an example in the $\mathcal{S}$ dynamics). We have

Proposition 22. *$\sigma \sim \tau$ iff $\mathrm{prim}(\sigma) \sim \mathrm{prim}(\tau)$ and $|\sigma| = |\tau|$.*

In other words, within a class we can 'move the descents freely'. In particular, this gives

Corollary 23. *The map $\mathrm{prim}(\,\cdot\,)$ is a homomorphism for the dynamics.*

I.e., if $\sigma \sim \tau$, then $\mathrm{prim}(\sigma) \sim \mathrm{prim}(\tau)$. In fact, much more is true (various other properties are discussed in Appendix B).

3.1.4 Exceptional classes

As we have outlined, a simple preliminary analysis allows to restrict to 'irreducible' and 'primitive' classes. Then, as anticipated, the invariants described above allow to characterise all primitive classes for the dynamics on irreducible configurations, *with two exceptions.* These two exceptional classes, for the $\mathcal{S}_n$ dynamics, are called Id_n and Id'_n. In the $\mathcal{S}_n^{\mathrm{ex}}$ dynamics, only Id_n remains exceptional and primitive (in the literature on Rauzy dynamics, Id_n is called the *'hyperelliptic class'*, because the Riemann surface associated to Id_n is hyperelliptic. Similarly Id'_n is often referred to as the *'hyperelliptic class' with a marked point*).

The most compact definition of Id_n and Id'_n is as being the classes containing id_n and id'_n, respectively, where id_n is the identity of size n, and, for $n \geq 3$, id'_n is the

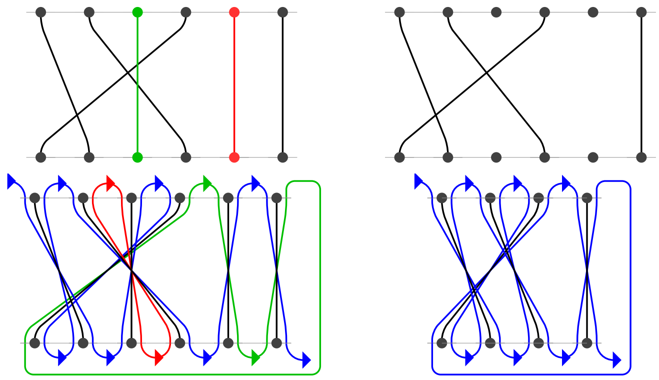
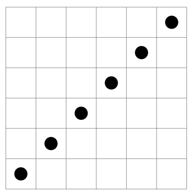
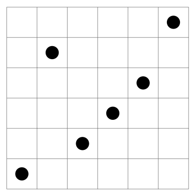


Figure 3.7: Top left: A non-primitive permutation $\sigma \in \mathcal{S}_6$ with a descent in red and the special descent in green. Its cycle invariant is $(\{1,1\}, r=3)$ as can be seen explicitly in the cycle construction shown on the bottom left of the figure. Top right: the permutation $\mathrm{prim}(\sigma)$, the primitive of σ. Its cycle invariant is $(\varnothing, r=3)$, as can be seen on the bottom right of the figure.

permutation σ of size n such that $\sigma(1) = 1$, $\sigma(2) = n-1$, $\sigma(n) = n$ and $\sigma(i) = i-1$ elsewhere, for example

$\mathrm{id}_6 =$  $\mathrm{id}'_6 =$ 

The cycle and sign invariants of these classes depend from their size mod 4, and are described in Table 3.1. The classes Id_n and Id'_n are atypical w.r.t. other classes, in various respects:

- They have 'small' cardinality. More precisely, at size n, $|\mathrm{Id}_n| = 2^{n-1} - 1$ and $|\mathrm{Id}'_n| = (2^{n-2} + n - 2)$ (compare this to the fact that all other classes have size $\geq \exp(c\, n \ln n)$, with $c \geq \frac{2}{3}$, i.e. at least size $\sim n!^{2/3}$).
- Contrarily to all other classes, the Cayley Graph associated to the dynamics can be described in a compact way, in terms of one or more complete binary

	n even	n odd
(λ, r) of Id_n	$(\varnothing, n-1)$	$(\{\frac{n-1}{2}\}, \frac{n-1}{2})$
(λ, r) of Id'_n	$(\{\frac{n-2}{2}, \frac{n-2}{2}\}, 1)$	$(\{n-2\}, 1)$

$n \mod 8$	0	1	2	3	4	5	6	7
s of Id_n	+	0	−	−	−	0	+	+
s of Id'_n	+	+	0	−	−	−	0	+

Table 3.1: Cycle, rank and sign invariants of the exceptional classes. The sign $s \in \{-1, 0, +1\}$ is shortened into $\{-, 0, +\}$.

n	Id	Id$'$	non-exceptional classes					
4	$\varnothing\|3-$							
5	$2\|2$	$3\|1-$						
6	$\varnothing\|5+$	$22\|1$	$\varnothing\|5-$					
7	$3\|3+$	$5\|1+$	$2\|4$	$4\|2$	$3\|3-$	$5\|1-$		
8	$\varnothing\|7+$	$33\|1+$	$22\|3$	$32\|2$	$42\|1$	$33\|1-$	$\varnothing\|7+$	$\varnothing\|7-$

Table 3.2: List of invariants (λ, r, s) for $n \leq 8$, for which the corresponding class in the $\mathcal{S}_n$ dynamics exists. We shorten s to $\{-,+\}$ if valued $\{-1,+1\}$, and omit it if valued 0.

trees of a certain height, in the case of Id'_n complemented by 'few' other nodes and transitions (a linear number).

- The configurations of these classes have a simple structure, labeled by a certain array of integers, and related to the position of the configuration in the Cayley graph. This structure makes easy to verify, in linear time, if a configuration is in Id_n, in Id'_n, or in none of the above, and thus allows to restrict the quest for a classification to non-special classes.

These results are illustrated at length in Appendix C.

3.1.5 The classification theorems

For the case of the $\mathcal{S}$ dynamics, we have a classification involving the cycle structure $\lambda(\sigma)$, the rank $r(\sigma)$ and sign $s(\sigma)$ described in Section 3.1.3

Theorem 24. *Two permutations σ and σ' are in the same class iff they have the same number of descents, and* $\mathrm{prim}(\sigma)$, $\mathrm{prim}(\sigma')$, *are in the same class.*

A permutation σ is primitive iff $\lambda(\sigma)$ has no parts of size 1 (note that the rank may be 1). Besides the exceptional classes Id *and* Id$'$, *which have cycle and sign invariants described in Table 3.1, the number of primitive classes with cycle invariant (λ, r) (no $\lambda_i = 1$) depends on the number of even elements in the list $\{\lambda_i\} \cup \{r\}$, and is, for $n \geq 9$,*

zero, *if there is an odd number of even elements;*

one, *if there is a positive even number of even elements; the class then has sign 0.*

two, *if there are no even elements at all. The two classes then have non-zero opposite sign invariant.*

For $n \leq 8$ the number of primitive classes with given cycle invariant may be smaller than the one given above, and the list in Table 3.2 gives a complete account.

As a consequence, two primitive permutations σ and σ', not of Id *or* Id$'$ *type, are in the same class iff they have the same cycle and sign invariant.*

For the case of the $\mathcal{S}^{\mathrm{ex}}$ dynamics, we have only two invariants left, as defined in Section 3.1.3: the cycle structure $\lambda'(\sigma)$ and the sign $s(\sigma)$. Recall that $\lambda'(\sigma) = \lambda(\sigma) \cup \{r(\sigma)\}$, when the same configuration is considered under the $\mathcal{S}$ dynamics. Note however that a permutation that is primitive and of rank 1 in $\mathcal{S}$, is not primitive in $\mathcal{S}^{\text{ex}}$. In particular, the class Id'_n, which has rank 1 for all n, is non-primitive in $\mathcal{S}^{\text{ex}}$. We have

Theorem 25. *Two permutations σ and σ' are in the same class iff they have the same number of descents, and* $\text{prim}(\sigma)$, $\text{prim}(\sigma')$*, are in the same class.*

A permutation σ is primitive iff $\lambda'(\sigma)$ has no parts of size 1. Besides class Id*, which has invariant as in the previous theorem (the rank is just added to λ'), the number of primitive classes with cycle invariant λ' (no $\lambda'_i = 1$) depends on the number of even elements in the list $\{\lambda'_i\}$, and is, for $n \geq 8$,*

zero, *if there is an odd number of even elements;*

one, *if there is a positive even number of even elements;*

two, *if there are no even elements at all. The two classes have non-zero opposite sign invariant.*

For $n \leq 7$ the number of primitive classes with given cycle invariant may be smaller than the one given above, and the list in Table 3.3 gives a complete account.

As a consequence, two primitive permutations σ and σ', not of Id *type, are in the same class iff they have the same cycle and sign invariant.*

We will first obtain Theorem 194 and then Theorem 25 as an almost-straighforward corollary. Curiously enough, historically, the first and fundamental article [KZ03] proved Theorem 25, and it was only a few years later that the proof technique was adapted from the $\mathcal{S}^{\text{ex}}$ case to $\mathcal{S}$, and the article [Boi12] proved Theorem 194.

On the contrary, within our techniques, it is both (a bit) easier to prove Theorem 194 than Theorem 25, if we had to do both of them from scratch, and is considerably easier to prove Theorem 25 as corollary of 194, while we are not aware of a simple derivation of Theorem 194 from Theorem 25.

3.1.6 Surgery operators

The hardest part in our proof of Theorem 194, concerning the dynamics $\mathcal{S}_n$, is the proof by induction that all the classes with a given admissible set of invariants are non-empty and connected. Classes of different rank (in the three cases $r = 1$, $r = 2$ and $r > 2$) are treated differently, so we do this with 3 operators, that we call q_1, q_2 and T, for these three cases, respectively. These operators produce (representants within) classes of given rank at size n from (representants within) classes at smaller sizes, by performing local manipulations of the configurations,

n	Id	non-exceptional classes	
4	3−		
5	22		
6	5+	5−	
7	33+	24	33−

Table 3.3: List of invariants (λ', s) for $n \leq 7$, for which the corresponding class in the $\mathcal{S}_n^{\text{ex}}$ dynamics exists. We shorten s to $\{-, +\}$ if valued $\{-1, +1\}$, and omit it if valued 0.

which, in the language of translation surfaces (see section 3.2.3), correspond in a hidden way to the insertion of a 'handle' or of a 'cylinder' in a Riemann surface. Thus we call them *surgery operators.* In the next paragraphs we make a preliminary discussion on which combinatorial properties shall such operators have, in order to provide the appropriate tool in our proof's scheme.

First of all, for each operator X, we will require that it is a homomorphism, i.e., that for all pairs of configurations in the same class, $\sigma \sim \tau$, we have $X(\sigma) \sim X(\tau)$. As a consequence, $\bar{X}(C)$, the class obtained by applying X to any configuration in C, is well-defined.

Then, we have a simple yet crucial constraint on the action the operators may possibly have, given by the dimension formula, equation (3.2): for a permutation σ of size n with cycle invariant (λ, r), we have $r + \sum_i \lambda_i = n - 1$. We know in retrospective, from the statement of Theorem 194 (see Section 4.4.1, Lemma 81 for a proof), that the number of even elements in $\lambda \cup \{r\}$ must be even, so that the cycle invariant shall change consistently under the action of the surgery operator.

For example, suppose that we aim to construct an operator X such that, for every σ with invariant (λ, r, s), $X(\sigma)$ has invariant $(\lambda, r+1, s)$. This operator would be a viable candidate for constructing an induction. However, the number of even elements in the two lists $\{\lambda_i\} \cup \{r\}$ and $\{\lambda_i\} \cup \{r+1\}$ have different parity, so that we know in advance that such an operator X cannot exist. The best hope is to identify instead an operator – it will be our operator T – such that, for every σ with invariant (λ, r, s), $T(\sigma)$ has invariant $(\lambda, r+2, s)$. Such an operator increases the size of a configuration by 2 (by the dimension formula), while preserving the forementioned parity constraint.

The classes in the image of T have rank $r > 2$, thus for the cases of rank 1 and rank 2 we need to define two other operators, q_1 and q_2. For the first one, tentatively, we may hope to have an operator such that, if σ has invariant (λ, r, s) and size n, then $q_1(\sigma)$ has invariant $(\lambda', 1, s)$ and size $n + 1$[4] Again, by equation (3.2) we have that $1 + \sum \lambda'_i = r + \sum \lambda_i + 1$. What should we hope as a possible natural choice of λ'? The answer is deceivingly simple, namely $\lambda' := \lambda \cup \{r\}$, i.e. we add a new cycle, of length r, in order to satisfy equation (3.2).

Finally, it remains the case of q_2. This operator works on a similar basis as q_1, with the following difference: if σ has invariant (λ, r, s) and size n, then $q_2(\sigma)$ has invariant $(\lambda'', 2, 0)$ and size $n+1$ with $\lambda'' := \lambda \cup \{r-1\}$. Note that the sign invariant of the image class is set to zero, regardless of the sign in the pre-image. This is consistent with the statement of the theorem: the new rank is 2, so the number of even entries in $\lambda \cup \{r\}$ becomes positive (while keeping its parity due to our definition of q_2).

Which properties of these operators shall we establish?

As we will prove the theorem by induction, we can assume that at size $n - 1$ and $n - 2$ the classes are fully charaterized by the triple (λ, r, s) (where $s = s(\sigma) \in \{-1, 0, 1\}$), as described in the statement of the theorem, and then concentrate on size n.

For the operator T, we need the following to hold: let C be some class with invariant (λ, r, s), with $r > 2$, and let B be the (unique by induction) class with invariant $(\lambda, r - 2, s)$, then $\bar{T}(B) = C$. The mere existence of this operator, once

[4] We may consider also a larger increase of size, as was for T, but let us consider the simplest scenario first.

established that it is a homomorphism, would provide the inductive step, from size $n-2$ to size n, for every class with rank more than 2. Indeed the operator $\bar{T}$ is a bijection between classes of invariant (λ, r, s) and classes of invariant $(\lambda, r+2, s)$.

For q_1, we have a slightly more complex requirement. Let C be a class with invariant $(\lambda, 1, s)$. For every i an integer appearing in λ with positive multiplicity, let $\lambda(i) = \lambda \setminus \{i\}$. Define $\Delta(C, i)$ as the (unique by induction) class with invariant $(\lambda(i), i, s)$. Clearly, for all such i, $\bar{q}_1(\Delta(C, i))$ has invariant $(\lambda, 1, s)$. Moreover, it turns out that no other classes B are such that $\bar{q}_1 B$ has this invariant. Now, establishing that q_1 is a homomorphism is not enough. We need to further establish that $\bar{q}_1(\Delta(C, i)) = C$ for every i, ruling out the possibility that, e.g., if $\sigma \in \Delta(C, i)$ and $\tau \in \Delta(C, j)$, then $q_1(\sigma) \not\sim q_1(\tau)$ despite the fact that they have the same invariants. A crucial simplifying property of the operator q_1 we will introduce is that q_1 is a injective map that is almost surjective, because, when $q_1^{-1}(\tau)$ is not defined, then $q_1^{-1}(R^{-1}\tau)$ is.

Finally, it remains the case of q_2. This operator works on a similar basis as q_1, but the loss of memory of the sign invariant introduces a further subtlety. Now we need the following. Let C have invariant $(\lambda, 2, 0)$. The set of (λ', r', s') such that it may be $\bar{q}_2(C') = C$ for some C' with invariant (λ', r', s') is a bit complicated. First of all, its elements have the form $(\lambda(i-1), i, s)$ for $i \in \lambda$ and $s \in \{0, \pm 1\}$. Then, if there are no even cycles in $\lambda(i-1)$, we have $s \in \{\pm 1\}$, otherwise $s = 0$. For i and s in the set as above (depending on λ), define $\mathcal{E}(C, i, s)$ as the class with invariant $(\lambda(i-1), i, s)$ (by induction, there is at most one class with such invariant). Thus $\bar{q}_2(\mathcal{E}(C, i, s))$ has invariant $(\lambda, 2, 0)$. We ask that $\bar{q}_2(\mathcal{E}(C, i, s)) = C$ for every such i and s. Again, besides the list we have just described, there are no other classes B such that $\bar{q}_2 B$ has invariant $(\lambda, 2, 0)$. Similarly as was the case for q_1, q_2 is injective and almost surjective.

The construction of these operators would produce the core of the induction steps. Some further subtleties are in order, though. In particular, the existence of exceptional classes requires to verify the bevaviour of classes $T(\mathrm{Id}_n), q_1(\mathrm{Id}_n), q_2(\mathrm{id}_n)$ and $T(\mathrm{Id}'_n), q_1(\mathrm{Id}'_n), q_2(\mathrm{Id}'_n)$, at all sizes n, as these could create a proliferation of new classes, even if the properties outlined above are determined for non-exceptional classes. For example, in such a case $\{T^{2k}(\mathrm{Id}_4), T^{2(k-1)}(\mathrm{Id}_8), \dots T^2(\mathrm{Id}_{4k}), \mathrm{Id}_{4k+4}\}$ could be $k+1$ different classes, according to Table 3.1, all with invariant $(\varnothing, k+1, 1)$. Luckily enough, what really happens (and we manage to prove) is that for each of the three operators $X \in \{\bar{T}, \bar{q}_1, \bar{q}_2\}$, and each of the two families of exceptional classes $C \in \{\mathrm{Id}_n, \mathrm{Id}'_n\}$, for n above a finite (small) value, there exists a non-exceptional class D such that $X(C) = X(D)$.

In our proof, it is convenient to establish separately this last fact (this is done in Section 4.4.2), and the behaviour of the operators on non-exceptional classes (this is done in Sections 4.3.1 and 4.3.2).

Table 4.4, which provides the worked-out procedure of the induction at size 11, may be illuminating.

3.2 Connection with the geometry of translation surfaces

In this section we illustrate how the combinatorial operators and invariants introduced in Section 2 are related to certain operations, called *Rauzy–Veech induction*, acting on interval exchange transformations associated to the Poincaré map of an interval on a translation surface. This correspondance implies that the classification of equivalence classes w.r.t. the combinatorial operators is equivalent to the classification of strata of translation surfaces with a given set of conical singularities.

A first part reviews the geometrical background (and explains all the terms used in the paragraph above), while a second part defines the geometrical version of the invariants (illustrated in Section 3.1.3) that turned out to provide a complete set in the classification theorem. Finally a last part describes the geometrical interpretation of our combinatorial surgery operators (illustrated in Section 3.1.6).

Note that this section introduce the minimal amount of terminology required to make sense of the invariants and the surgery operators in the geometrical context. The interested reader may find a more extensive account in the three surveys [Zor06], [Yoc07] and [FM14], which develop respectively the theory of (quasi-)flat surfaces, of interval exchange transformations and of Teichmüller spaces (the three texts have a large amount of overlap between them).

3.2.1 Strata of translation surfaces

A *translation surface* is a connected compact oriented surface M of genus g, equipped with a Riemannian metric which is flat except for a discrete set S of conical singularities, and has trivial SO(2)-holonomy. In this context, trivial SO(2)-holonomy means that the parallel transport of a tangent vector along any closed curve (avoiding the singularities) brings the vector back to itself (instead of having it rotated). A condition for this to happen is that the angle of each sigularity is a multiple of 2π. Indeed, for 'small' closed curves, which encircle one conical singularity of angle α while staying far from all the others, the parallel transport of a vector would rotate the vector by the angle α, which is defined modulo 2π.

Additionally, the trivial holonomy allows us to choose a vertical direction. Choose one (non-singular) point on the surface, and one vector x in the tangent space at that point. Then, one can define a vector field via the parallel transport of x. The resulting parallel vector field is defined only outside of the singularities, but it can be extended to the singularities in a multivalued way: in a singularity of angle $2\pi(d_i+1)$ it would take $d_i + 1$ distinct values.

Equivalently, a translation surface with the choice of a vertical direction can be described as a triplet (M, U, S), where M is a topological connected surface, $S = \{s_1, \dots, s_k\}$ is the discrete subset of singularities of M, and $U = \{U_i, z_i\}$ is an atlas on $M \smallsetminus S$ such that every transition function is a translation:

$$\forall i,j \qquad \begin{array}{rccc} z_i \circ z_j^{-1} : & z_j(U_i \cap U_j) & \to & z_i(U_i \cap U_j) \\ & z & \mapsto & z + c \end{array}$$

and for each $s_i \in S$ there exists a neighborhood that is isometric to a cone of angle $2\pi(d_i + 1)$. Since $z_i = z_j + c$ for all pairs of intersecting U_i and U_j, we can define on each chart a holomorphic complex 1-form $\omega_{z_i} = dz_i$, and thus the abelian differential

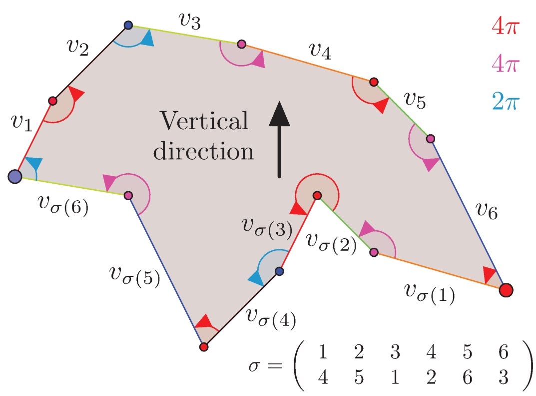


Figure 3.8: From the datum of n vectors $v_1, \dots, v_n$ in $\mathbb{C}$ and a permutation $\sigma \in \mathfrak{S}_n$, we construct a $2n$-gon. Then every pair of parallel sides of the polygon are identified by translation. The conical singularities of the associated translation surface are represented directly on the figure, by small angles, and the stratum $H(d_1^{m_1}, \dots, d_k^{m_k})$ to which the surface belongs can be evinced from this representation: the vertices of the polygon are partitioned into equivalence classes (due to the identification of the sides of the polygon by translation) represented by the coloured arcs in the figure. Each class is associated to a singularity, and the angle of each singularity is the sum of the angles around the vertices in its equivalence class. We can also directly compute the angle of a singularity, divided by 2π, by counting the number of arcs of the given colour on any of the two broken lines $P_\pm$ of the polygon (the two arcs around the vertices 0 and $\sum_{i=1}^n v_i$, there where P_+ and P_- meet, are not counted). In this example there are three equivalence classes, denoted in blue, red and magenta. The angle of the blue, red and magenta classes are 2π, 4π and 4π, respectively. Indeed, there are 1 blue, 2 red and 2 magenta arcs both on the top and on the bottom broken lines.

ω on $M \smallsetminus S$. This abelian differential ω extends to the points $s_i \in S$, where it has zeroes of degree d_i.

Conversely, given a non-zero abelian differential ω on a compact connected Riemann surface M, with a finite set of zeroes $S = \{s_1, \dots s_k\}$, of degree $d_1, \dots, d_k$, we define a *translation atlas* as follows. Let (U, ζ) be a chart containing $p \in M \smallsetminus S$ such that $\omega_\zeta = \Phi(\zeta) d\zeta$. From this, one can obtain a chart (U, ξ) of adapted local coordinates at p with $\omega_\xi = d\xi$ on U by defining $\xi = \int_p^z \Phi(w)dw$. Likewise, we can show that a neighbourhood of the zero s_i is isometric to an Euclidean cone of angle $2\pi(d_i + 1)$, by finding local coordinates η for which $\omega_\eta = \eta^{d_i} d\eta$.

A geometric construction of a translation surface can be done as follows. Choose n distinct vectors $v_1, \dots, v_n$ of $\mathbb{C}$ and a permutation $\sigma \in \mathfrak{S}_n$ and construct the two broken lines (i.e., polygonal chains) P_+ and P_- in $\mathbb{C}$, with vertices

$$P_+ = (0, v_1, v_1 + v_2, \dots, v_1 + \dots + v_n)$$

and

$$P_- = (0, v_{\sigma(n)}, v_{\sigma(n)} + v_{\sigma(n-1)}, \dots, v_{\sigma(n)} + \dots + v_{\sigma(1)}) \,.$$

Occasionally, we will use u_1, u_2, …, u_n as synonym of $v_{\sigma(n)}$, $v_{\sigma(n-1)}$, …, $v_{\sigma(1)}$.

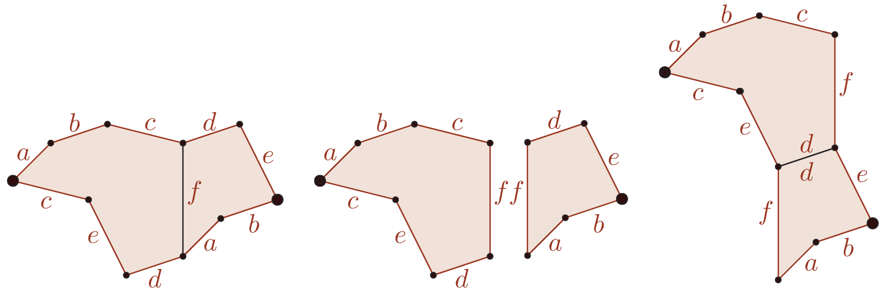


Figure 3.9: A cut and paste move.

If the vectors v_i satisfy certain inequalities (see Definition 26 below), P_+ and P_- intersect only at their endpoints, so the concatenation of P_+ and the reverse of P_- defines a polygon embedded in the plane, with $2n$ sides, which are naturally paired: if $\sigma(i) = j$, the j-th side of P_+ has the same direction and orientation as the $(n+1-i)$-th side of P_-. These pairs of sides can thus be identified, so that the resulting surface has no border. By this procedure, illustrated in figure 3.8, we obtain a translation surface.

It is known that any translation surface can be represented in such a way, and under the canonical choice of the vertical direction for the vector field, the corresponding abelian differential is the canonical dz of the complex plane. As explained on figure 3.8, the set of conical singularities can be directly read on the polygon.

For $g \geq 2$, the moduli space of abelian differentials H_g is the space of pairs (M, ω), where M a compact connected Riemann surface of genus g and ω is a non-zero abelian differential, with the following identification: the points (M, ω) and (M', ω') are equivalent if there is an analytic isomorphism $f : M \to M'$ such that $f^*\omega' = \omega$. H_g is a complex orbifold of dimension $4g-3$.

Let $d_1 \leq \cdots \leq d_k$ and $m_1, \ldots, m_k$ be sequences of nonnegative integers. We denote by $H(d_1^{m_1}, \ldots, d_k^{m_k})$ the subspace of H_g such that for all points (M, ω) the abelian differential ω has m_i zeroes of degrees d_i, for $i = 1, \ldots, k$. It was shown by Masur and Veech that the stratum $H(d_1^{m_1}, \ldots, d_k^{m_k})$ is a complex orbifold of dimension $n = 2g - 1 + \sum_i m_i$. By the Gauss–Bonnet formula, if M has genus g then the degrees of the conical singularities must verify

$$\sum_{i=1}^{k} m_i d_i = 2g - 2\,. \tag{3.5}$$

In fact, H_g can be partitioned in terms of the strata $H(d_1^{m_1}, \ldots, d_k^{m_k})$ for d_i's and m_i's such that (3.5) is satisfied.

In term of polygons, two translation surfaces S_1 and S_2 with the same area are in the same stratum $H(d_1^{m_1}, \ldots, d_k^{m_k})$ if and only if there exists a sequence of cut and paste moves from S_1 to S_2 (see figure 3.9). The 'if' part is obvious, while the 'only if' part is a result of [Mas82].

Let M be a translation surface with a vertical direction, and let I be a horizontal open segment on M. The *first return map* T (or *Poincaré map*) of the translation flow[5] from I is an *interval exchange transformation* (IET). An IET is a one-to-

[5] I.e., the flow associated to the vertical foliation with singularities at the conical points defined

one map ϕ from $I \setminus \{x_1, \dots, x_k\}$ to $I \setminus \{x'_1, \dots, x'_k\}$ that is piecewise of the form $\phi(z) = z + c_i$, i.e., there exists some minimal set of points $\{x_1, \dots, x_k\} \subset I$, and $\{x'_1, \dots, x'_k\} \subset I' \equiv \phi(I)$, such that, if we subdivide $I \setminus \{x_1, \dots, x_k\}$ into $k+1$ maximal open subintervals $\{I_1, \dots, I_{k+1}\}$, of lengths $\lambda_1, \dots, \lambda_{k+1}$, and $I' \setminus \{x'_1, \dots, x'_k\}$ into subintervals $\{I'_1, \dots, I'_{k+1}\}$ of the same lengths, the map ϕ is identified by the datum of $\tau \in \mathfrak{S}_{k+1}$, and of $(\lambda_1, \dots, \lambda_{k+1}) \in (\mathbb{R}_+)^{k+1}$, with a convention on the labelings as shown in the example of Figure 3.10 (i.e., the intervals of $I' \setminus \{x'_1, \dots, x'_k\}$, in their natural order from left to right, have lengths $\lambda_{\tau(k+1)}, \lambda_{\tau(k)}, \dots, \lambda_{\tau(1)}$).

Consequently, also a first return map T is parametrised by the same combinatorial datum, of the permutation $\tau \in \mathfrak{S}_{k+1}$, and a "vector of lengths" $\lambda = (\lambda_1, \dots, \lambda_{k+1})$ (see figure 3.11). Moreover, the permutation τ is irreducible (i.e., it has no non-trivial blocks) since the vertical foliation is uniquely ergodic (see [Vee82] and [Mas82] for a proof of the unique ergodicity).

The construction of a polygon with identified sides for a given datum of (τ, λ), with some useful geometric properties, involves the notion of *suspension*:

Definition 26 (Suspension)**.** *Let σ be an irreducible permutation of size n and λ a vector of lengths. We say that the vectors $(v_i)_{i=1,\dots,n} \in \mathbb{C}^n$ are a* suspension data *of (σ, λ) if*

- *$\forall\, i$, $\mathfrak{Re}(v_i) = \lambda_i$.*
- *$\forall\, i < n$, $\sum_{j=1}^{i} \mathfrak{Im}(v_j) > 0$ and $\sum_{j=1}^{i} \mathfrak{Im}(u_j) < 0$.*

(Recall, $u_j = v_{\sigma(n+1-j)}$). Given a suspension datum $(v_i)_i$ of (σ, λ) we can build the translation surface $S = (\sigma, (v_i)_i)$ and choose the interval $]0, \sum_{i=1}^{n} \lambda_i[$ on the real line so that the interval exchange map T on I is encoded exactly by (σ, λ). (See figures 3.12 and 3.13). In this case we say that S is a suspension of (σ, λ). Veech proved that almost any translation surface could be obtained as a suspension.

by taking the integral curve of the parallel vector field i.e. defined by $\phi_t(x) = x + it$

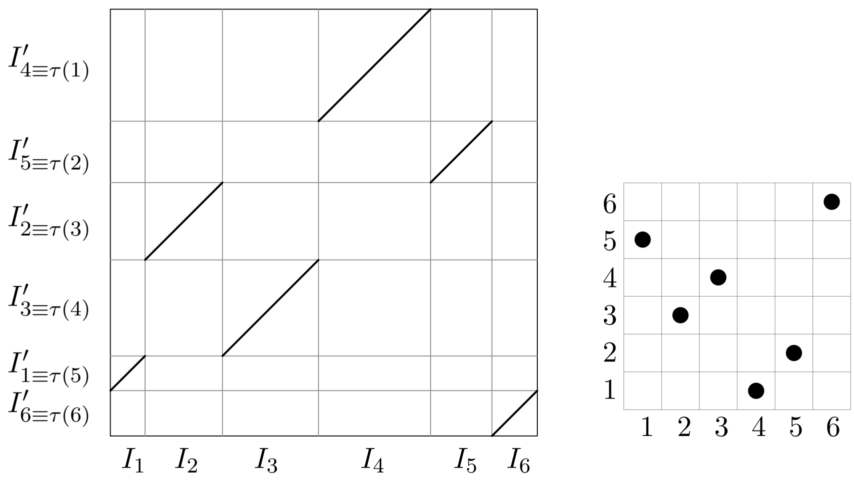


Figure 3.10: Left: an IET ϕ. Right: the associated permutation $\tau = \tau(\phi)$. We have a bullet at coordinate (i, j) if $\tau(i) = j$. Note that there is a reflexion along the horizontal axis in the correspondence between τ and the obvious diagram interpretation of the graphics of ϕ, i.e. τ describes the graphics of $-\phi(x)$.

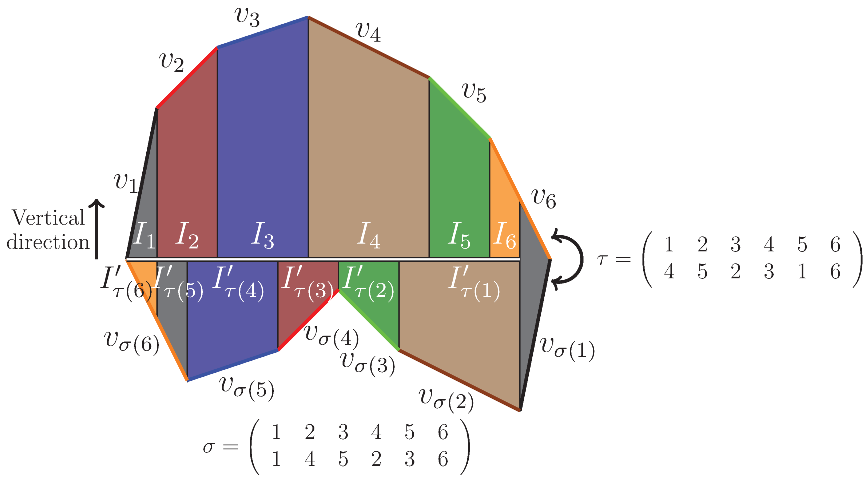


Figure 3.11: Let σ and $(v_i)_{1\leq i\leq n}$ define a translation surface and choose a horizontal open interval I. The interval exchange map T is obtained by following the vertical flow from almost each point of I, until it returns for the first time to I. The image by T of a point of I is not defined if its vertical flow meets a singularity before reaching I, and these are the points x_i mentioned in the text. In the figure, the top sub-intervals (I_i) of I are numbered from left to right, and the bottom ones $(I'_{\tau(j)})$ are numbered with j increasing from right to left.

Let $(M,\omega) \in H(d_1^{m_1},\dots,d_k^{m_k})$, and introduce the shortcut $n = 2g-1+\sum_i m_i$ for the dimension of the stratum. In analogy with the previous section, we state that the Rauzy class of (M,ω) is the subset of $\mathfrak{S}_n$ containing all the permutations which are the combinatorial datum of some interval exchange map T over an interval I on (M,ω).

If $(\sigma,(v_i)_i)$ is a polygon representation of $S=(M,\omega)$, then the Rauzy class of S contains σ, since S can be obtained as a suspension of (σ,λ) by Veech theorem, and is thus the class $C(\sigma)$, in the notations of the previous section.

More generally, on $H(d_1^{m_1},\dots,d_k^{m_k})$ we have a notion of connectivity due to the

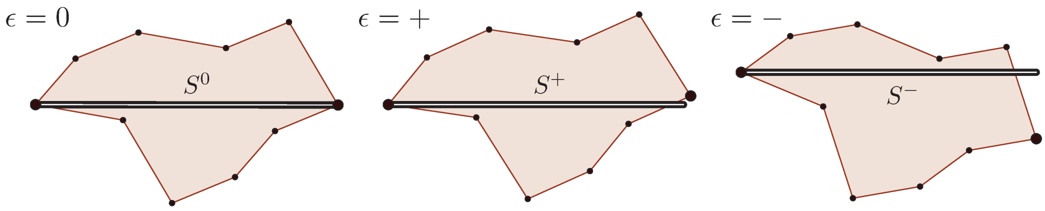


Figure 3.12: Let (σ,λ) describe an IET. The three figures describe three suspesions of (σ,λ), $(v_i^\epsilon)_{1\leq i\leq n}$ for $\epsilon = 0,+,-$. The second condition in the definition of a suspension means that for the translation surface $S^\epsilon = (\sigma,(v_i^\epsilon)_i)$ the top and bottom broken lines remain always above and below the real line, except possibly for their last segment. In the figure, the endpoint of the broken lines P_+ and $P-$ are on, above or below the real axis, for $\epsilon = 0$, $+$ and $-$, respectively.

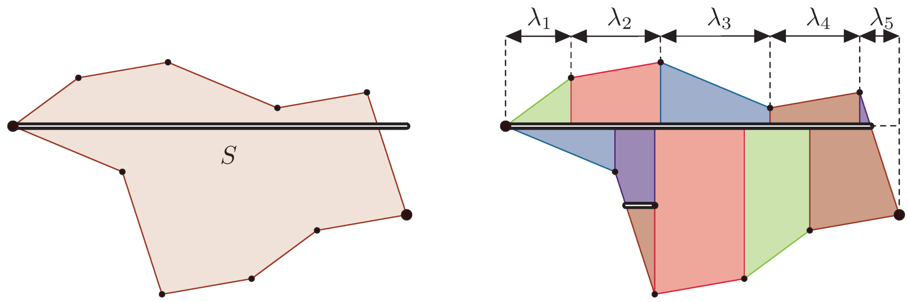


Figure 3.13: As illustrated by Figure 3.12, and here on the left picture, in the construction of the translation surface $S = (\sigma, (v_i)_i)$ from the suspension $(v_i)_i$ of (σ, λ), the interval I can be in part outside the polygon. In this case, the construction of the interval exchange map involves a small subtlety, detailed on the right picture. We cut the portion of I outside the polygon, and translate it according to the identification of opposite sides. When this is done, the interval exchange map has a clear behaviour also on the right-most interval I_n.

orbifold structure of this space. Veech proved the two following things :

- Two points (M, ω) and (M', ω') of the stratum $H(d_1^{m_1}, \dots, d_k^{m_k})$ are in the same connected component if and only if they are in the same extended Rauzy class.
- Two permutations $\tau, \tau' \in \mathfrak{S}_n$ are in the same extended Rauzy class if and only if they are connected with respect to the extended Rauzy dynamics $\mathcal{S}_n^{\text{ex}}$.

Thus, if (M, ω) and $(M', \omega') \in H(d_1^{m_1}, \dots, d_k^{m_k})$ are suspensions of (τ, λ) and (τ', λ'), respectively, we have $(M, \omega) \sim (M', \omega')$ iff $\tau \sim \tau'$ (the two $\sim$ symbols are different, the first one is the connectivity within the orbifold, the second one is the connectivity on the Cayley graph of the Rauzy extended dynamics), and consequently the extended Rauzy classes completely determine the connected components of the strata of H_g.

Let us now interpret the combinatorial definition of the extended Rauzy classes, given in Section 3.1.1, in terms of the *Rauzy–Veech induction*. Let M be a translation surface with a vertical direction, I an open interval and $T = (\tau, \lambda)$ the associated interval exchange map. We label the top and bottom sub-intervals induced by T as in Figure 3.11 above. The Rauzy–Veech induction concerns the study of the possible configurational changes in the structure of the IET, when the surface (M, ω) is kept fixed, while the interval I is modified. A *right step* of the Rauzy–Veech induction consists in shortening the interval I, from its right-end, by removing the shortest sub-interval among I_n and $I'_{\tau(1)}$. If we call J the new (total) interval, we have

$$\begin{cases} J = I \setminus I'_{\tau(1)} & \text{if } |I_n| > |I'_{\tau(1)}| \\ J = I \setminus I_n & \text{if } |I_n| < |I'_{\tau(1)}| \end{cases}$$

The case $|I_n| = |I'_{\tau(1)}|$ never occurs if the λ_i's are generic in $(\mathbb{R}_+)^n$ (and thus they do not satisfy any linear relation with coefficients in $\mathbb{Q}$), so we omit any discussion of this possibility. Conforming to the customary language, in the first case of the analysis above we say that T is *of right type 0*, and in the second case we say that T is *of right type 1*. We also shorten right type with R-type.

Let us call $R_{\text{RV-ind}}$ the operation that sends the 'old' IET T to the 'new' one, $T_1 = (\tau_1, \lambda_1) = R_{\text{RV-ind}}(T)$. We use this verbose notation in order to make a distinction with operators L and R defined in Section 3.1.1, whose role here is as follows: if T is of R-type 0, then $\tau_1 = R(\tau)$, while if T is of R-type 1, then $\tau_1 = L(\tau)$ (see figure 3.14).

Likewise, a *left step* of the Rauzy–Veech induction consists in shortening the interval I from the left-end. We have

$$\begin{cases} J = I \smallsetminus I'_{\tau(n)} & \text{if } |I_1| > |I'_{\tau(n)}| \\ J = I \smallsetminus I_1 & \text{if } |I_1| < |I'_{\tau(n)}| \end{cases}$$

and we say that T is *of left type 0* in the first case and *of left type 1* in the second case. Let us call $T_2 = (\tau_2, \lambda_2) = L_{\text{RV-ind}}(T)$ the resulting new IET. Then, the map $L_{\text{RV-ind}}$ has the property that $\tau_2 = R'(\tau)$ or $L'(\tau)$ depending on the left-type of T, still with L' and R' defined as in Section 3.1.1.

The extended Rauzy dynamic $\mathcal{S}_n^{\text{ex}}$ acts on a permutation σ, while the Rauzy–Veech induction acts on IET (σ, λ). As we have seen, the two operations are strictly related. Finally, we can define a third version of the operation, an induction based on cut and paste operations (which we also call the Rauzy–Veech induction), that acts on a suspension $(\sigma, (v_i)_i)$.

Given a suspension $(\sigma, (v_i)_i)$ of (σ, λ), a right step of the Rauzy–Veech induction for suspensions shortens the interval I into an interval J as it would for (the right step of) the Rauzy–Veech induction for IETs, then identifies a triangle within the

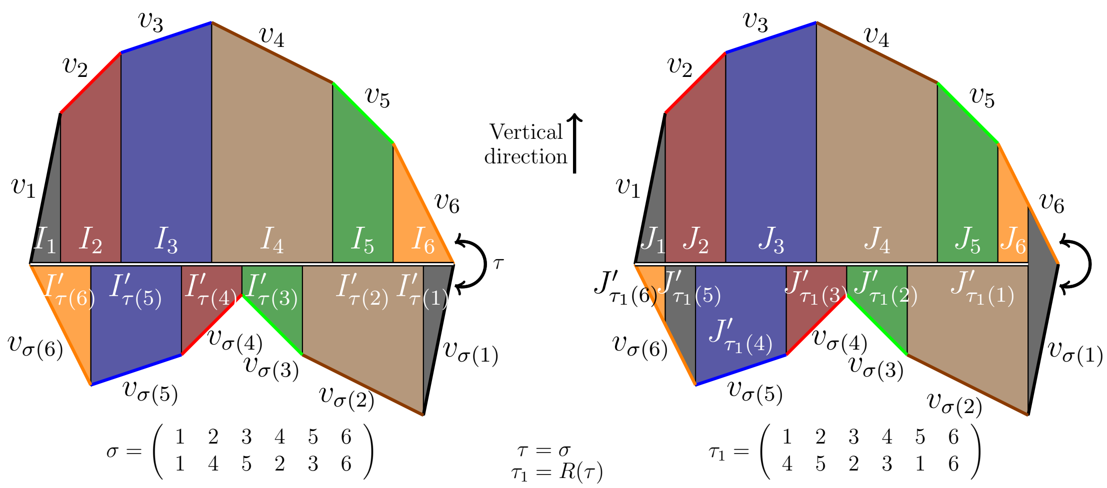
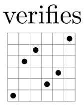
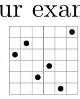


Figure 3.14: We apply the right step of the Rauzy–Veech induction on the surface shown on the left. In this case $T = (\tau, \lambda)$ is of right type 0, since $|I_6| > |I'_{\tau(1)}|$, thus $J = I \smallsetminus I'_{\tau(1)}$. The new interval exchange map $T_1 = (\tau_1, \lambda_1)$ on J verifies $\tau_1 = R(\tau)$, namely, in our example,

$\tau =$  and $\tau_1 = R(\tau) =$  .

polygon, namely, the one with vertices $p = \sum_{i=1}^{n} v_i$, $p - v_n$ and $p - v_{\sigma(1)}$, and pastes it back on the polygon, accordingly to the identification of the sides of the polygon and of the triangle, either on top of the broken line P_+, or below the broken line P_- in case of a Rauzy induction of right type 1 or 0, respectively. (see Figure 3.15).

Likewise, a left step cuts the left-end of the interval, and deplaces a triangle on the far left, with vertices 0, v_1 and $v_{\sigma(n)}$, in a way that depends on the type of the left step of the Rauzy induction.

The relation of this version of the induction with the previous ones is based on the following fact. If $(\sigma, (v_i)_i)$ is a suspension of (σ, λ), $(\sigma_1, (v_i^1)_i)$ is the surface obtained by the Rauzy–Veech induction for suspensions, and (σ_1, λ_1) is the IET obtained by Rauzy–Veech induction for IETs. Then, $(\sigma_1, (v_i^1)_i)$ is a suspension of (σ_1, λ_1).

As we have mentioned above, in [Vee82] Veech showed that the extended Rauzy classes were generated by the four operators L, R, L', R' that we have introduced, and, more generally, that for every finite word w in the alphabet $\{L_{\text{RV-ind}}, R_{\text{RV-ind}}\}$, and every word ω of the same length in the alphabet $\{0, 1\}$, there exists an IET T such that the induction performed according to the word w visits IET's T_i of type ω_i (namely, of left-type or right-type ω_i, according to w_i), and this crucial result is at the basis of our combinatorial study of the classes determined by the action of L, R, L' and R' on permutations. In other words, if we call Π_1 the projection from a suspension S to the associated IET $T = (\tau, \lambda)$, and Π_2 the projection from T to the associated permutation τ, we have the commuting diagram

$$
\begin{array}{lccc}
\text{Rauzy–Veech induction for suspensions:} & S = (\tau, (v_i)_i) & \xrightarrow{\sim \text{ w.r.t. } L_{\text{RV-ind}};\ R_{\text{RV-ind}}} & S' = (\tau', (v_i')_i) \\
& \Big\downarrow \Pi_1 & & \Big\downarrow \Pi_1 \\
\text{Rauzy–Veech induction for IETs:} & T = (\tau, \lambda) & \xrightarrow{\sim \text{ w.r.t. } L_{\text{RV-ind}};\ R_{\text{RV-ind}}} & T' = (\tau', \lambda') \\
& \Big\downarrow \Pi_2 & & \Big\downarrow \Pi_2 \\
\text{Combinatorial operators:} & \tau & \xrightarrow{\sim \text{ w.r.t. } L,R;\ L',R'} & \tau'
\end{array}
\tag{3.6}
$$

The (standard) Rauzy classes defined in Section 3.1.1 also have a geometric interpretation. Given a stratum $H(d_1^{m_1}, \dots, d_k^{m_k})$, we denote by $H(d_1^{m_1}, \dots, \bar{d_i}^{m_i}, \dots, d_k^{m_k})$ the space with a marked singularity of degree d_i. A point (M, ω) is in $H(d_1^{m_1}, \dots, \bar{d_i}^{m_i}, \dots, d_k^{m_k})$ if, for (M, ω) represented by a polygon as in Figure 3.8, the angle of the singularity of the equivalence class containing the vertex 0 is $2\pi(d_i + 1)$. Thus $H(d_1^{m_1}, \dots, d_k^{m_k})$ is partitioned into k parts, one for every possible value for the angle of the marked singularity. For example, in the figure 3.8 the angle around the singularity passing through $(0, 0)$ is 2π, so the corresponding translation surface (M, ω) in the figure is in $H(\bar{0}^1, 1^2)$.

Finally the *Rauzy–Veech induction* is also defined and it corresponds to the induction where only right steps are allowed. It should be clear from figure 3.15 that the degree of the singularity around the vertex 0 does not change after a right move of the Rauzy–Veech induction for suspensions, since the cut and paste at the

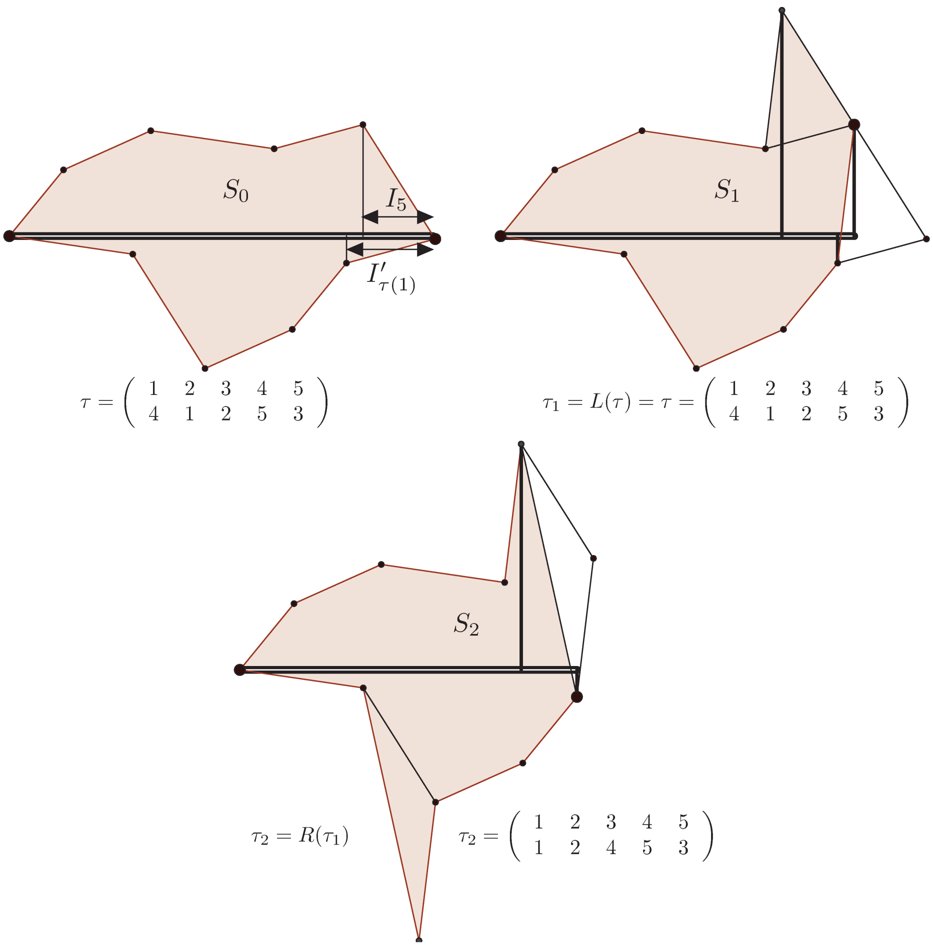


Figure 3.15: We apply two times the right version of the Rauzy–Veech induction on the left figure. In the first case $T = (\tau, \lambda)$ is of right type 1, since $|I_5| < |I'_{\tau(1)}|$, thus $J = I \setminus I_5$. The new interval exchange map $T_1 = (\tau_1, \lambda_1)$ on J verifies $\tau_1 = L(\tau)$. We cut the triangle on the right part of S_0 and paste it on the top broken line, since we are in type 1. In this way we obtain the suspension S_1 of (τ_1, λ_1). Then, with a second right-step of the Rauzy–Veech induction for suspensions, we obtain the suspension S_3 from S_2. In this case the step is of type 0, thus $\tau_2 = R(\tau_1)$.

level of the triangle can never change the colour class of the left-most vertex, thus the colour class of the left-most vertex is an invariant of the (non-extended) Rauzy dynamics, and this corresponds to the marking.

In analogy to the forementioned result of Veech, Boissy proved in [Boi12] that the standard Rauzy classes are in one-to-one correspondence with the connected components of the strata of the moduli space of abelian differentials with a marked singularity, so that we have a commuting diagram analogous to the one in (3.6), justifying the study of the combinatorial dynamics $\mathcal{S}$.

To conclude this section, let us describe some more recent works closely connected to the classification of the connected components of the strata of the moduli space of abelian differentials of Kontsevich and Zorich and of Boissy [KZ03, Boi12].

In [Lan08], Lanneau classified the connected components of the strata of the moduli space of *quadratic differentials*, Then, Lanneau together with Boissy [BL09] have formulated a combinatorial definition of extended Rauzy classes for quadratic differentials. These are defined as IETs on half-translation surfaces and are once again in one-to-one correspondence with the connected components of the strata of quadratic differentials. In this context the combinatorial datum is no longer a permutation but a linear involution (i.e. a matching with a one marked point between two given arcs), and are related, to a certain extent, to the dynamics $\mathcal{M}$ of Section 3.1.1.

More recently, Boissy studied the *labelled Rauzy classes* in [Boi13], an object that we will also consider as it is a key component of the *labelling method* (refer to section 5 for more on this). Boissy also classified the connected components of the strata of the moduli space of *meromorphic differentials* in [Boi15]. Delecroix studied the cardinality of Rauzy classes in [Del13], and Zorich provided representatives of every Rauzy class in the form of Jenkins–Strebel differentials [Zor08]. Finally, in [Fic16] Fickenscher has presented a combinatorial proof, independent of the one presented here, of the Kontsevich-Zorich-Boissy classification of Rauzy classes.

3.2.2 Geometric interpretation of the invariants

In this section we define the geometric version of two invariants of the strata, and show that they correspond to our combinatorial definitions of the *cycle-* and the *sign-*, or *arf-invariant* presented in Section 3.1.3. We start by working at the level of the connected components of a stratum $H(d_1^{m_1}, \dots, \bar{d_i}^{m_i}, \dots, d_k^{m_k})$ of abelian differentials with a marked singularity.

The first invariant that we introduce consists of an integer partition with one marked part, and can be represented by the list

$$(\underbrace{d_1+1, \dots, d_1+1}_{m_1 \text{ times}}, \dots, \underbrace{d_i+1, \dots, d_i+1}_{m_i-1 \text{ times}}, \dots, \underbrace{d_k+1, \dots, d_k+1}_{m_k \text{ times}}; d_i+1)$$

Each entry $d_i + 1$ is associated to a conical singularity, and corresponds to its angle, divided by 2π. As said above, in the standard (non-extended) dynamics there is a singularity passing through the vertex $(0, 0)$ of the polygon representation, and its degree does not change in the dynamics (because the Rauzy–Veech induction is acting on the other endpoint of the interval i.e. only the right step of the induction is allowed for the standard dynamics), and this accounts for the fact that one entry, the one after the semicolon, is singled out. This invariant corresponds to the cycle invariant (λ, r) of any permutation of the associated Rauzy class, as is apparent from what we said above. More precisely, let $(M, \omega) \in H(d_1^{m_1}, \dots, \bar{d_i}^{m_i}, \dots, d_k^{m_k})$, let $(\sigma, (v_i)_{1\le i\le n})$ be a suspension data of (M, ω) and let C be the (standard) Rauzy class associated to the connected component containing (M, ω). As we said before, $\sigma \in C$, since the translation surface is a suspension of (σ, μ) for some $\mu \in (\mathbb{R}_+)^n$.

The results now follows from Figure 3.16, which indicates the correspondance between the angles around the vertices in the polygon and the counting of top (or bottom) arcs of the cycle invariant in the permutation.

The second invariant is more subtle, and corresponds to the parity of the spin structure of the surface, that we now describe. Let $\gamma : S^1 \to M$ be a smooth closed curve on M, avoiding the conical singularities. Using as a reference our fixed parallel vector field (i.e., the vertical direction), we can define the *Gauss map* G

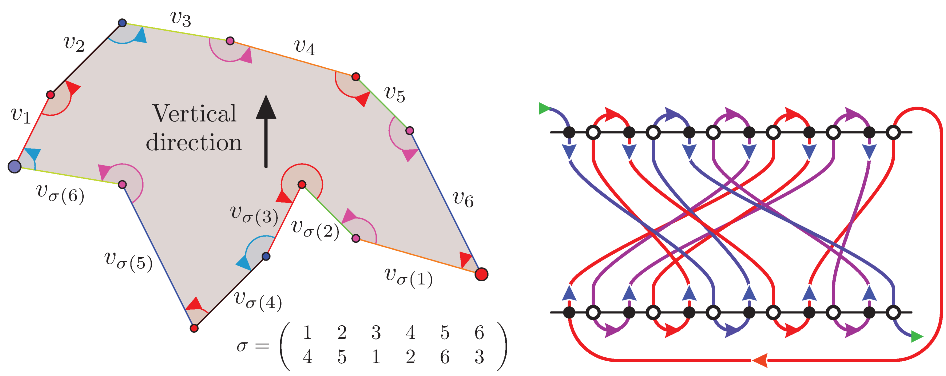


Figure 3.16: On the left, the polygon construction associated to the permutation $\sigma = (4, 5, 1, 2, 6, 3)$, and a certain set of v_i's. The arcs around the singularities are represented, with colours denoting their classes. On the right, the cycle invariant construction associated to the same permutation. One can see that the arcs (from left to right) of the top broken line of the polygon correspond to the top arcs (from left to right) of the cycle invariant construction, while the arcs (now from right to left) of the bottom broken line of the polygon correspond to the bottom arcs (still from left to right) of the cycle invariant construction. As a result, the angle of a singularity divided by 2π corresponds to the length of a cycle.

from γ to $[0, 2\pi[\simeq S^1$ as follows: to every point of the curve, $\gamma(x)$, we associate the angle $\theta(x) \in S^1$ corresponding to the angle between the tangent vector $\gamma'(x)$ and the vector field in x. The index $\mathrm{ind}(\gamma)$, called the *degree* of the Gauss map, is the integer counting (with sign) the number of times the curve $G \cdot \gamma$ is wrapped around S^1.

This allows us to define:

Definition 27 (Parity of the spin structure)**.** *Let $(\alpha_i, \beta_i)_{1\leq i\leq g}$ be a collection of smooth closed curves avoiding the singularities, and representing a symplectic basis for the homology $H_1(M, \mathbb{Z})$. The* parity of the spin structure *of M is the quantity*

$$\Phi(M) = (-1)^{\sum_{i=1}^{g}(\mathrm{ind}(\alpha_i)+1)(\mathrm{ind}(\beta_i)+1)} \tag{3.7}$$

It follows from [Ati71] that this is an invariant of the connected components of the stratum, and from [Joh80] that it is well-defined and independent of the choice of symplectic basis. More precisely, Johnson in [Joh80] shows that the quantity is the *Arf invariant* of a certain quadratic form $\Phi : H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$. We will describe this second point of view in a few paragraphs, and we now proceed to construct Φ concretely in the special case of translation surfaces.

Let c be a cycle of $H_1(M, \mathbb{Z}_2)$, and let γ be a smooth simple closed curve avoiding the singularities, and representing c. We define the function $\Phi : H_1(S, \mathbb{Z}_2) \to \mathbb{Z}_2$ as

$$\Phi(c) = \mathrm{ind}(\gamma) + 1 \pmod 2.$$

This function is only well-defined when all the zeroes of ω have even degree (i.e. $(M, \omega) \in H(2d_1^{m_1}, \ldots, 2\bar{d}_i^{m_i}, \ldots, 2d_k^{m_k})$), otherwise two curves representing the same cycle c might have opposite index.

Suppose we are in the even-degree case. For c and c' distinct cycles, and γ and γ' representing c and c' respectively, and in generic mutual position (i.e., they may cross, but only in generic way), we define the bilinear *intersection form* $\Omega(c, c')$ as the number of intersections between γ and γ', mod 2.

The following theorem (obtained as a corollary of a theorem of [Joh80]) certifies that our function is a quadratic form:

Theorem 28. *The function Φ is well-defined on $H_1(M, \mathbb{Z}_2)$. It is a quadratic form associated to the bilinear intersection form Ω in the following sense:*

$$\forall\ c, c' \qquad \Phi(c + c') = \Phi(c) + \Phi(c') + \Omega(c, c')\,.$$

For our quadratic form $\Phi : H_1(S, \mathbb{Z}_2) \to \mathbb{Z}_2$ the Arf invariant is defined as

$$\operatorname{arf}(\Phi) = \frac{1}{\sqrt{|H_1(S, \mathbb{Z}_2)|}} \sum_{x \in H_1(S,\mathbb{Z}_2)} (-1)^{\Phi(x)} = 2^{-g} \sum_{x \in H_1(M,\mathbb{Z}_2)} (-1)^{\Phi(x)}\,. \tag{3.8}$$

Let $(a_i, b_i)_{1 \le i \le g}$ be a sympletic basis of $H_1(S, \mathbb{Z}_2)$, then it can be shown that

$$\operatorname{arf}(\Phi) = (-1)^{\sum_{i=1}^{g} \Phi(a_i)\Phi(b_i)}\,. \tag{3.9}$$

Let $(\alpha_i, \beta_i)_{1 \le i \le g}$ be a family of smooth closed curves (avoiding the singularities) representing the family of cycles $(a_i, b_i)_{1 \le i \le g}$. Since we defined our quadratic form by $\Phi(c) = \operatorname{ind}(\gamma) + 1 \mod 2$ for any cycle c and curve γ representing c, by comparing (3.7) and (3.9) we have

$$\operatorname{arf}(\Phi) = \Phi(S)\,,$$

i.e., the parity of the spin structure is the Arf invariant of Φ.

Let us now describe how to obtain our definition of the Arf invariant, given in section 4.2.1, starting from the formula in (3.8).

Let (M, ω) be a translation surface defined as the suspension $(\sigma, (v_i)_i)$ of an IET T. Say that $\sigma(i) = j$. For $x \in [0, 1]$ sufficiently small, there exists a curve γ_{n+1-j} from the point $p_i^- = v_{\sigma(n)} + v_{\sigma(n-1)} + \cdots + x\, v_{\sigma(i)}$ on P_- to the point $p_j^+ = v_1 + v_2 + \cdots + x\, v_j$ on P_+, such that the Gauss map G discussed above remains within the interval $[-\pi/2, \pi/2]$. Indeed, this fact is true for all x, whenever $j < n$, but only for x small enough for $j = n$ and when the suspension has $\epsilon \neq 0$ w.r.t. the notations of Figure 3.12. This can be seen from the existence of a positive δ such that a strip of width δ around the interval I leaves the points p_i^- and the points p_i^+ on opposite half-planes. Figure 3.17 provides an example of this construction.

Because of the identification of the sides of the polygon, the γ_i's are closed curves. As $G \cdot \gamma_i(x) \in [-\pi/2, \pi/2]$, it is easily seen that they all have index zero. Let $(c_i)_i$ be the family of cycles represented by the curves $(\gamma_i)_i$, then it can be shown that

Lemma 29. *The family of cycles $\{c_1, \ldots, c_n\}$ is a generating family of $H_1(M, \mathbb{Z}_2)$.*

For a permutation π, define the symmetric matrix $K(\pi)$ to be zero on the diagonal, and elsewhere

$$\left(K(\pi)_{ij}\right)_{1 \le i,j \le n} = \begin{cases} 1 & \text{if } (\pi(i) - \pi(j))(i - j) < 0 \\ 0 & \text{if } (\pi(i) - \pi(j))(i - j) > 0 \end{cases}$$

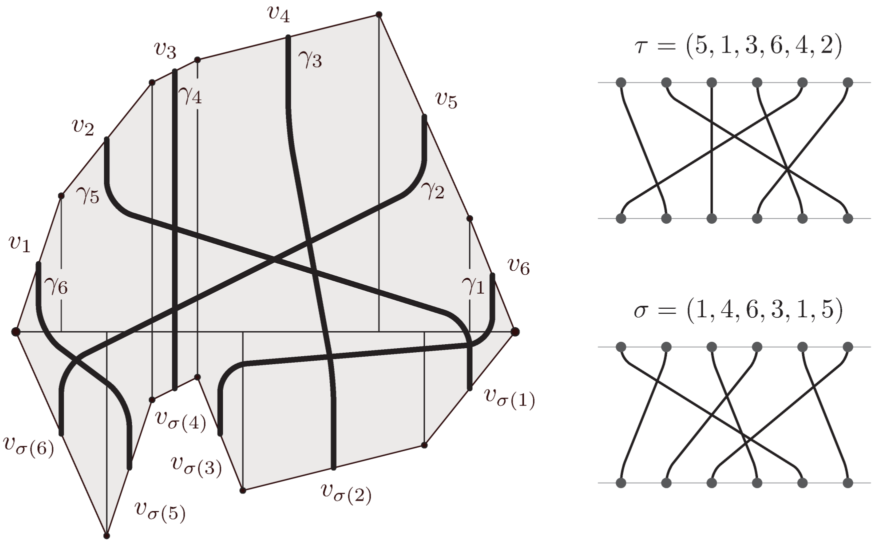


Figure 3.17: On the left, the polygon construction associated to the permutation $\sigma = (1, 4, 6, 3, 1, 5)$ with a collection of simple smooth closed curves $(\gamma_i)_{1 \leq i \leq n}$. On the right, the permutation $\tau = (5, 1, 3, 6, 4, 2)$ represents the collection of curves $(\gamma_i)_{1 \leq i \leq n}$: two closed curves γ_i and γ_j intersect if and only if the edges $(i, \tau(i))$ and $(j, \tau(j))$ of τ intersect, since the set of curves $(\gamma_i)_{1 \leq i \leq n}$ and the permutation τ have the same topology. The permutation σ is the reverse of τ, i.e. it is obtained from τ by multiplying with the longest permutation $\omega = (n, n-1, \ldots, 1)$, thus two closed curves γ_i and γ_j intersect if and only if the edges $(n-i+1, \sigma(n-i+1))$ and $(n-j+1, \sigma(n-j+1))$ of σ do *not* intersect.

Then, for all c_i and c_j $(i \neq j)$, $\Omega(c_i, c_j) = K(\tau)_{i,j}$ (see figure 3.17 for this correspondence). Defining σ as the (right-)reverse of τ, i.e. $\sigma = \tau\omega$ where $\omega = (n, n-1, \ldots, 1)$ is the longest permutation, we also have $\Omega(c_i, c_j) = 1 - K(\sigma)_{i,j}$. From the fact, proven above, that $\mathrm{ind}(\gamma_j) = 0$, we get $\Phi(c_j) = \mathrm{ind}(\gamma_j) + 1 \mod 2 = 1$.

For a set $I \subseteq \{1, \ldots, n\}$ we define $\Phi(I) = \Phi(\sum_{i \in I} c_i)$ and $\chi'_I(\pi) = \sum_{\{i,j\} \subset I} K(\pi)_{ij}$ as the number of pairs $i < j$ of edge labels, both in I, which are crossing in π. In Section 3.1.3.2 we already have defined the analogous quantity $\chi_I(\pi) = \sum_{(i,j) \subset I} (1 - K(\pi)_{ij})$, as the number of pairs $i < j$ in I which do not cross in π. Then we have

Proposition 30. *If the surface S is in the Rauzy class $C(\sigma)$, for all $I \subset \{1, \ldots, n\}$ we have*

$$\Phi(I) = |I| + \chi'(I)(\sigma\omega) = |I| + \chi(I)(\sigma)\,. \tag{3.10}$$

Proof. Call again $\tau = \sigma\omega$. We have, by iterated application of Theorem 28,

$$\Phi(I) := \Phi\Big(\sum_{i \in I} c_i\Big) = \sum_{i \in I} \Phi(c_i) + \sum_{(i,j) \subseteq I} K(\tau)_{ij}\,. \tag{3.11}$$

Recall that, on one side, $\Phi(c_i) = 1$ for all i, and, on the other side, $\sum_{(i,j) \subseteq I} K(\tau)_{ij}$ is the definition of $\chi'(I)(\tau)$. Thus

$$\Phi(I) = |I| + \chi'(I)(\tau)\,. \tag{3.12}$$

Similarly for σ, as $K(\tau)_{ij} = 1 - K(\sigma)_{ij}$ and $\sum_{(i,j)\subseteq I}(1 - K(\sigma)_{ij})$ corresponds to the definition of $\chi(I)(\sigma)$,

$$\Phi(I) = |I| + \chi(I)(\sigma)\,. \tag{3.13}$$

$\square$

As a result, we can rewrite the definition (3.4) of $\overline{A}(\sigma)$ as

$$\bar{A}(\sigma) := \sum_{I\subseteq\{1,\dots,n\}} (-1)^{|I|+\chi(I)} = \sum_{I\subseteq\{1,\dots,n\}} (-1)^{\Phi(I)} \tag{3.14}$$

which, identifying sets I with cycles $\sum_{i\in I} c_i$, is our definition (3.9) of the Arf invariant, up to an overall factor that we shall now discuss. Let us start by observing:

Lemma 31. *Let $(M,\omega) \in H(d_1^{m_1},\dots,\bar{d}_i^{\,m_i},\dots,d_k^{m_k})$ be a translation surface of genus g represented by a permutation σ of size n. Finally, let $k' = \sum_{j=1}^{l} m_j$ be the total number of singularities, then*

$$n = 2g + k' - 1\,. \tag{3.15}$$

Proof. This follows from the Gauss–Bonnet and the dimension formulas

$$\sum_{j=1}^{k} m_j d_j = 2g - 2\,; \qquad\qquad r + \sum_{j=1}^{k} m_j \lambda_j = n - 1\,; \tag{3.16}$$

with $r = \bar{d}_i + 1$ and $\lambda_j = d_j + 1$ for the non-marked singularities z_j, since

$$n - 1 = r + \sum_{j=1}^{k} m_j d_j + 1 = 2g - 2 + \sum_{j=1}^{l} m_j = 2g - 2 + k'$$

$\square$

Note that the integer $n \equiv 2g + k' - 1$ on the two sides of (3.15) is exactly the dimension of the complex orbifold $H(d_1^{m_1},\dots,\bar{d}_i^{\,m_i},\dots,d_k^{m_k})$. We have

Proposition 32.

$$\bar{A}(\sigma) = 2^{k'-1+g}\,\mathrm{arf}(\Phi)\,.$$

Proof. We already know that $\{c_i,\dots,c_n\}$ is a generating set. Since the basis of $H_1(S,\mathbb{Z}_2)$ has size $2g$, and $n = 2g + k' - 1$, whenever $k' > 1$ this family is linearly dependent on $\mathbb{Z}_2$, thus there exists an index j and a set I not containing j such that $c_j = \sum_{i\in I} c_i$. Call $X = \{1,\dots,\hat{j},\dots,n\}$. Denoting by Δ the symmetric difference of two sets, we can rearrange the summands in $\bar{A}(\sigma)$ as

$$\begin{aligned}
\bar{A}(\sigma) &= \textstyle\sum_{J\subseteq X}(-1)^{\Phi(J)} + \sum_{J\subseteq X}(-1)^{\Phi(J\cup\{j\})}\\
&= \textstyle\sum_{J\subseteq X}(-1)^{\Phi(J)} + \sum_{J\subseteq X}(-1)^{\Phi(J\Delta I)}\\
&= \textstyle 2\sum_{J\subseteq X}(-1)^{\Phi(J)}\,,
\end{aligned}$$

where in the last passage we use the fact that, as J ranges over subsets of X and $I \subseteq X$, then also $J' = J\Delta I$ ranges over subsets of X.

Hence by induction, performing $k' - 1$ steps, and up to relabeling the indices of the cycles so that the remaining family $(c_1,\dots,c_{2g})$ forms a basis,

$$\bar{A}(\sigma) = 2^{k'-1} \sum_{J\subseteq\{1,\dots,2g\}} (-1)^{\Phi(J)} = 2^{k'-1+g} 2^{-g} \sum_{x\in H_1(M,\mathbb{Z}_2)} (-1)^{\Phi(x)} = 2^{k'-1+g}\mathrm{arf}(\Phi)\,.$$

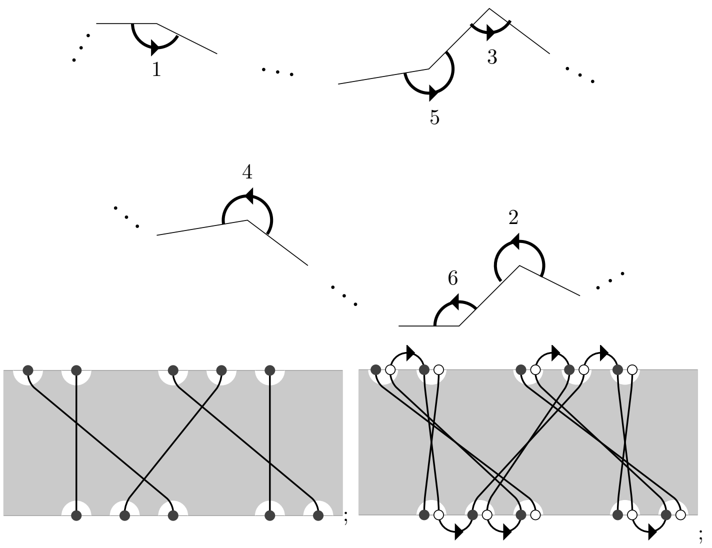


Figure 3.18: Top: a translation surface with a conical singularity of angle 6π. Bottom: the associated permutation.

$\square$

Recall that in Proposition 20 we claimed that $s(\sigma) = 2^{-\frac{n+k'-1}{2}}\bar{A}(\sigma)$ (this fact is proven later on), thus, when Φ is defined, that is, for $(M,\omega) \in H(2d_1^{m_1}, \dots, 2\bar{d}_i^{\,m_i}, 2d_k^{m_k})$, this quantity is equal to $\mathrm{arf}(\Phi)$, since $\frac{n+k'-1}{2} = \frac{2g-2+2k'}{2} = k'-1+g$ by Lemma 31.

When $(M,\omega) \notin H(2d_1^{m_1}, \dots, 2\bar{d}_i^{\,m_i}, 2d_k^{m_k})$, i.e. when ω has some zeroes of odd degree, the function Φ is not defined. Nonetheless the function $s(\sigma)$ remains well-defined, and its value is 0 in this case (see the following Lemma 81 for a proof).

In summary, we have

$$
\begin{array}{lcl}
s(\sigma) \in \{\pm 1\} & \Longleftrightarrow & s(\sigma) = \mathrm{arf}(\Phi) \text{ and } (M,\omega) \in H(2d_1^{m_1}, \dots, 2\bar{d}_i^{\,m_i}, 2d_k^{m_k})\,; \\
s(\sigma) \in \{0\} & \Longleftrightarrow & \text{An even number of conical singularities with angle multiple of } 4\pi.
\end{array}
$$

3.2.3 Surgery operators

In this section we describe two families of 'geometric surgery operations' which can be performed on translation surfaces, and more notably on suspensions. Some special cases of these will correspond to the combinatorial surgery operators, introduced in Section 3.1.6, and crucially used in the main body of the this part of the thesis (section 4.4).

For this purpose it will be useful to introduce a convention on the visualisation of conical singularities. We have seen that the degree d_i of a zero z_i can be computed directly on the translation surface, either by adding the angles of the polygon which are associated to the singularity (then dividing by 2π, and subtracting 1) or also, more conveniently, just by counting the number of top (or bottom) arcs around the vertices of the singularity, minus 1 (cf. figure 3.8 and figure 3.18). In order to visualise a singularity, say at z_i, it is useful to consider a small neighbourhood, say of radius $\epsilon \ll |v_i|$. Such a neighbourhood is isometric to a cone of angle $2\pi(d_i+1)$, i.e. the glueing of $2(d_i+1)$ half-disks of radius ϵ. If, in a picture, we take care of drawing some trajectories of the vertical foliation, we can trade half-disks for angular sectors of circle with smaller angles α_j, so that, in total, we can squeeze the $2\pi(d_i+1)$ cone into an ordinary disk (by choosing α_j's such that $\sum_{j=1}^{2(d_i+1)} \alpha_j = 2\pi$), and still provide a picture in which the foliation is reconstructed unambiguously. Such a representation is illustrated in figure 3.19.

We now introduce a family of suspensions, with certain special properties guaranteeing that the behaviour of the geometric surgery operations is more simple than what is the case on a generic suspension: (recall, u_j are synonims for $v_{\tau(n+1-j)}$)

Definition 33 (Regular suspension)**.** *We say that a suspension $S = (\tau; v_1, \ldots, v_n)$ is* regular*, and* of shift y *if*

- $-\frac{\pi}{2} < \arg(u_1) < 0 < \arg(u_2) < \arg(u_3) < \cdots < \arg(u_n) < \frac{\pi}{2}$,
- $\mathfrak{Im}(v_1 + v_2 + \cdots + v_n) = \mathfrak{Im}(u_1 + u_2 + \cdots + u_n) = -y$, *with* $y \geq 0$,
- $\tau(n) = n$.

An example is given in Figure 3.20. In particular, a regular suspension, with reference to the notation of Figure 3.12, has $\epsilon \in \{0, -\}$.

3.2.3.1 Adding a cylinder

The first family of geometric surgery operators consists in *breaking up a zero* (of the abelian differential ω), as discussed, e.g., in [KZ03, sect. 4.2] and [EMZ03]. Geometrically, this consists in adding a cylinder on one given singularity of the translation surface.

Let S be a translation surface, represented as a polygon with identified sides, and let z_i be a zero of degree d_i. A *saddle connection* from z_i to z_i is a geodesic (i.e., in our case, a straight line segment which may wrap along the identified sides of the

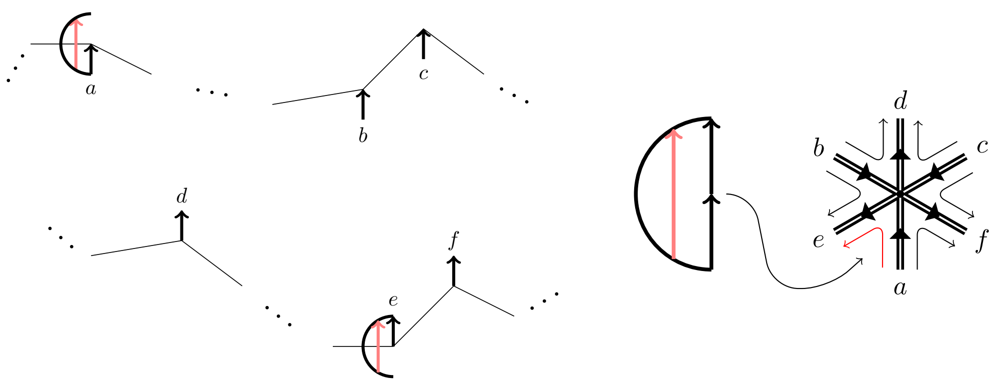


Figure 3.19: On the left picture, the black arrows represent the leaves of the foliation on the critical point z_i. Since z_i has conical angle 6π, there are 6 half-discs (of radius ϵ) forming together the neighbourhood of z_i. One such half-disc, adjacent to the leaves a and e, is represented. On the right picture, we have a representation of the leaves of the vertical foliation in the neighbourhood of the singularity z_i. Each angular sector in the picture is of angle $\pi/3$, although it corresponds to a half-disc of radius ϵ. The red trajectory, associated to a leaf passing in a neighbourhood of the singularity, helps in visualising the glueing procedure.

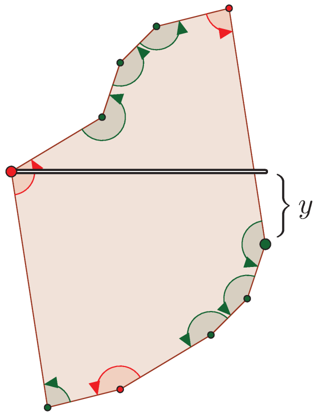


Figure 3.20: A regular suspension for $\tau = (23145)$. This permutation has rank 1, and one cycle of length 3, i.e. the marked singularity has degree zero (red angles) and there is a single further singularity, of degree two (green angles).

polygon) connecting a vertex of the equivalence class of z_i on the top broken line to a vertex of the equivalence class of z_i on the bottom broken line. So, for a zero of degree $\ell = d + 1$, there are ℓ^2 families of saddle connections, one per top/bottom choice of angle, with segments in the same family differing for the choice of the winding.

In order to break up a zero z_i, we choose one such saddle connection, in one such family, and "add a cylinder". This operator may be denoted as $\mathcal{Q}_{i,h,k,s}(v)$, with i an index associated to the singularity, $h, k \in \{1, \dots, d_i + 1\}$ associated to the choice of top/bottom angle, s associated to the choice of saddle connection within the family, and v a two-dimensional vector with positive scalar product with the versor of the saddle connection.

This construction is especially simple if, for the given polygon S and triple (i, h, k) as above, there exists one saddle connection s_0 in the family which consists just in a straight segment, and doesn't go through the sides of the polygon (if it exists, it is unique). I.e., if the segment in the plane between the two angles h and k of the polygon is contained within the polygon. In this case adding a cylinder corresponds, in the polygon representation, to adding a parallelogram, with one pair of opposite sides associated to the saddle connection, and the other pair associated to a vector v, which is thus inserted (at the appropriate position) in the list of v_j's describing the polygon $S = (\tau; v_1, \dots, v_n)$. This also adds one edge to the permutation τ, or, in other words, it adds the same vector v both at a position in the list $(v_j)_j$ and in the list $(u_j)_j$. The procedure in this simplified situation is described in Figure 3.21. In this situation we denote the corresponding operator as $\mathcal{Q}_{i,h,k}(v)$, i.e. we omit the index $s = s_0$ for the choice of saddle connection.

When, instead, we choose a saddle connection with a non-trivial winding, in the polygon representation we shall add the vector v at several places along the list of v_j's. The other copies are associated to the introduction of false singularities (i.e. zeroes of ω of degree zero, i.e., in the permutation, edges associated to descents). This does not change the topology of the surface, but gives non-primitive representants, or forces to study further manipulations which allow to remove false singularities. We will not discuss this analysis here.

Let us describe in more detail why the construction depicted above is legitimate, now at the level of the translation surface as a whole (and not of the polygon representation). The saddle connection on the surface is a closed curve, since its endpoints are identified in the polygonal construction. Neglecting the identification, it consists of some vector w, if the trajectory is followed from vertex h to k, in the local flat metric. We cut the surface along this curve, obtaining a new surface S' with two boundary components homeomorphic to circles. Finally, we construct a cylinder by taking a parallelogram with sides $(+v, +w, -v, -w)$, gluing its two $\pm w$ sides to the two boundaries of S', and its two $\pm v$ sides among themselves. This gives the desired surface $S_{\text{new}} = \mathcal{Q}_{i,h,k,s}(v)S$.

This operation breaks up the singularity z_i of degree d_i into two singularities z and z' of degrees d and d' such that $d + d' = d_i$. The value of d and d' depends on which pair of vertices (in the equivalence class of z_i) are connected by the geodesic. In particular, for each h there exists exactly one value k such that the pair (d, d') is attained, and similarly for k (see figure 3.21).

This operation is especially clear in the neighbourhood of z_i: the introduction of the cylinder separates the singularity into two new singularities, each one with its own neighbourhood represented by a concatenation of half-planes (cf. Figure 3.22).

In order to avoid the complicancy on the polygon representation deriving by the presence of winding geodesics (and the resulting introduction of false singularities), we will remark the crucial property:

Proposition 34. *If S is a regular suspension of shift y for a permutation τ of rank r, $\bar{r}$ is the singularity associated to the rank, r_0 is the top angle at the beginning of the rank cycle, and $k < r$ is the k-th bottom angle of the rank cycle, there exists one saddle connection in the family (r_0, k) which is contained within the polygon, and, for all v with argument in a non-empty range and $\mathfrak{Im}(v) < y$, $\mathcal{Q}_{\bar{r},r_0,k}(v)S$ is a regular suspension of shift $y - \mathfrak{Im}(v)$.*

The range for $\arg(v)$ is as follows. If the new edge of the permutation is added to the list $(u_j)_j$ at j, we shall have

$$\theta_- < \arg(v) < \theta_+ \quad \theta_- = \begin{cases} \arg(u_j) & j > 1 \\ 0 & j = 1 \end{cases} \quad \theta_+ = \begin{cases} \arg(u_{j+1}) & j < n \\ \frac{\pi}{2} & j = n \end{cases} \tag{3.17}$$

This construction is illustrated in Figure 3.23.

The two surgery operators q_1 and q_2 correspond to $\mathcal{Q}_{\bar{r},r_0,k}(v)$ as in Proposition 34, for k being such that the degree d_i of the marked singularity is divided into d and $d_i - d$, with the degree of the new marked singularity being $d = 0$ or 1, for q_1 and q_2, respectively.

We note in passing that the construction described so far, of breaking up a zero, can be made local, i.e. can be realised in a way such that the flat metric does not change outside of some neighbourhood of z_i (see [KZ03, sect. 4.2] for details). This, however, and at difference with Proposition 34, yet again comes at the price of introducing false singularities in the polygon (and permutation) representation.

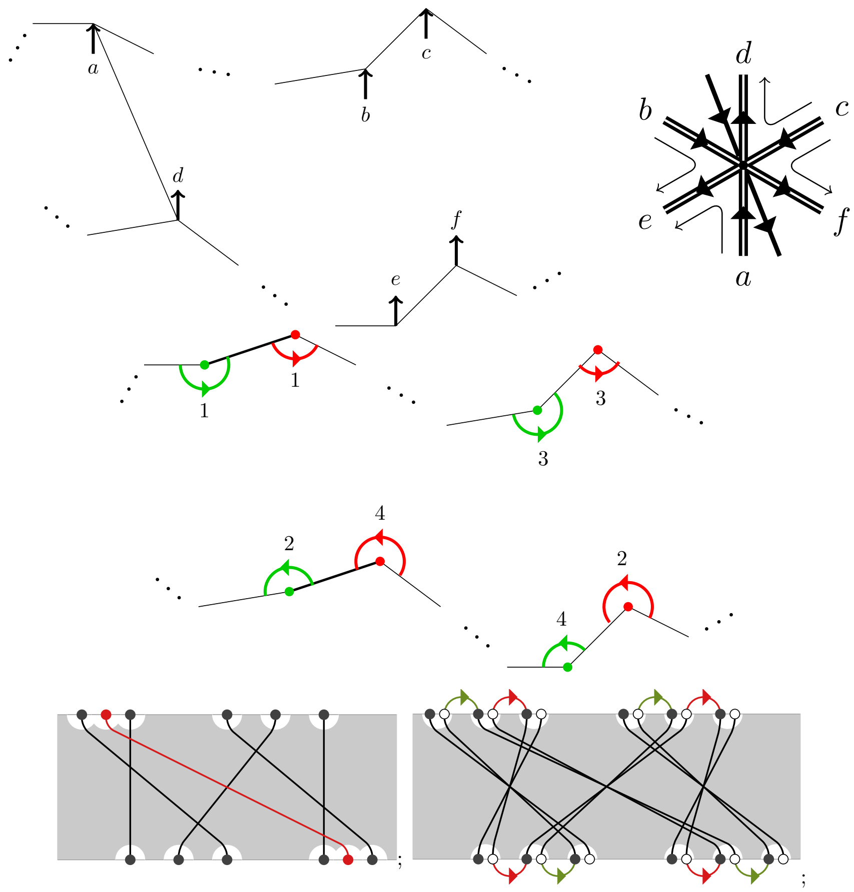


Figure 3.21: Top left: a saddle connection between two vertices of the conical singularity of angle 6π on the translation surface. Top right: the same geodesic in a neighbourhood of z_i. Middle: we cut the surface along the geodesic and glue the two boundary of a cylinder to the two newly formed boundaries of the surface. Adding the cylinder breaks up the singularity of angle 6π into two singularities of angle 4π (in green and red respectively). Bottom: the projection on the permutation of this procedure consists in adding an edge (in red in the figure) in between two arcs belonging to the same cycle (cf figure 3.18 for the drawing of the cycle invariant).

3.2.3.2 Adding a handle

Our second family of geometric surgery operations $\mathcal{T}$ corresponds to adding a handle, and will be related to the operator T. Let us start by describing this operation on the translation surface. Let z_i be a singularity. We choose a direction $\theta \in [0, 2(d_i+1)\pi]$ and a length ρ, and consider the arc of geodesic γ starting from z_i with direction θ and length ρ. Assume that this arc does not contain any singularity except for its starting point. We call w the corresponding vector. We cut the surface S along the vector w, and call S' the resulting surface. Choose a vector v with positive scalar product with the versor of w (, i.e., with an angle within $[0, \pi]$ w.r.t. the local metric at the starting point), and construct a cylinder, by folding a parallelogram $(+v, +w, -v, -w)$ analogously to how was the case for $\mathcal{Q}$, i.e. gluing its two $\pm w$ sides

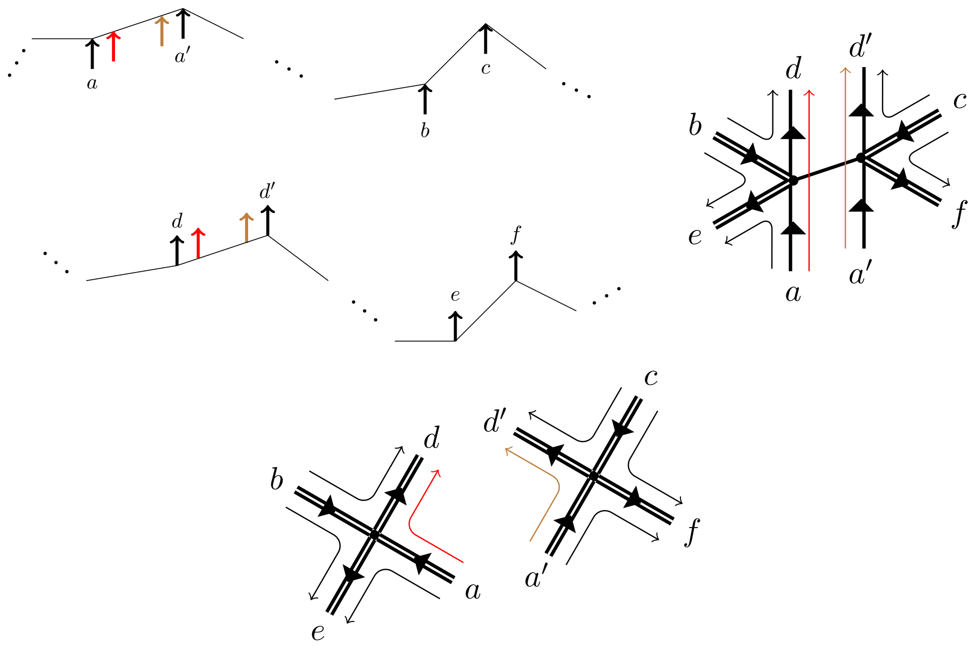


Figure 3.22: Top right: adding the cylinder breaks up the singularity of angle 6π in two, as is shown globally on the polygon, on the left, and in a neighbourhood of the saddle connection, on the right. Bottom: after this operation, we see that there are two singularities in this case each of total angle 4π (neighbourhoods centered around each of them are shown). In red and brown, two leaves of a foliation parallel to the saddle connection, passing next to the singularities, help in visualising the local modification. In this case, in order to simplify the drawing, we have a description of the local neighbourhoods which corresponds to the case in which the saddle connection is itself vertical.

to the two boundaries of S', and its two $\pm v$ sides among themselves. This gives the desired surface $S_{\text{new}} = \mathcal{T}_{i,w}(v)S$. This procedure is illustrated, in a somewhat simplified picture, in Figure 3.24.

Let us describe this now at the level of the polygon representation. Say that the direction θ is within the angle x on the top boundary of the polygon, and call P the endpoint of w. Similarly to the case of $\mathcal{Q}$, if P is on the boundary of the polygon, and the vector w is completely contained within the polygon (i.e., it goes from x to P without passing through the identified sides), then the construction is especially simple also at the level of the polygon: it does not require the introduction of false singularities, and consists in the introduction of a parallelogram. More precisely, if P is on the bottom side u_j of the polygon, the operation consists of two parts: first, it cuts the side u_j into two consecutive and collinear sides u_j' and u_j'', thus adding (temporarily) a false singularity of angle 2π on P, and then it inserts a cylinder following w, which is now a saddle connection between two angles of the polygon (just as in $\mathcal{Q}$, but with the two angles associated to distinct singularities).

By this operation we have added a handle to the surface S on the singularity z_i. If the degree of z_i on S is d_i, then its degree on S_{new} is $d_i + 2$, as explained in figure 3.25. Seen in a neighbourhood of the singularity, this operation corresponds

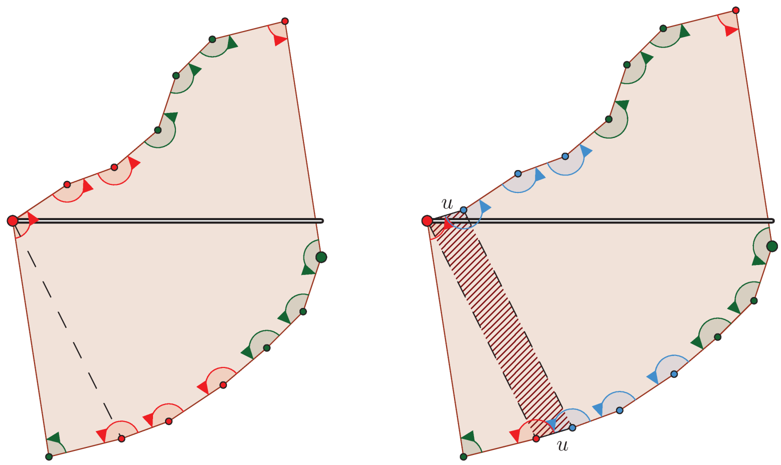


Figure 3.23: Left: a regular suspension for $\tau = (413256)$. This permutation has rank 3 (red angles), thus the marked singularity is of degree 2, and one cycle of length 3 (green), i.e. a singularity of degree 2. Right: the result of applying the operator $\mathcal{Q}$, which breaks the marked singularity into a marked singularity of degree zero and a new singularity of degree 2.

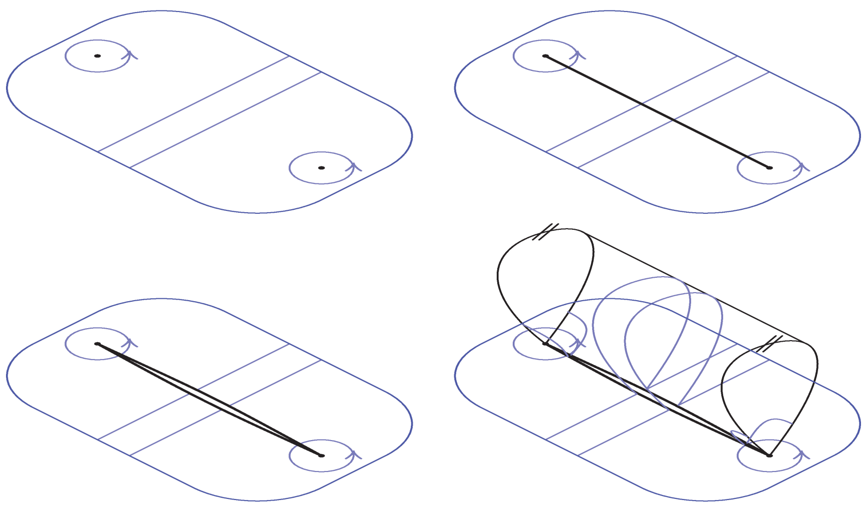


Figure 3.24: Top left: two singularities on a portion of a translation surface. In this case, in order to simplify the drawing, they are both false-singularities. A small circle around each of them, and a couple of typical leaves of the vertical foliation, are shown. Top right: a saddle connection between the two singularities is added. Bottom left: the surface is cut along this line. Bottom right: a handle is added along these boundaries. Note how both the circles around the singularities and the leaves are modified. In particular, now we have a unique singularity, in this case of angle 6π, as the arcs on the cylinder are connected in such a way to concatenate the two circles.

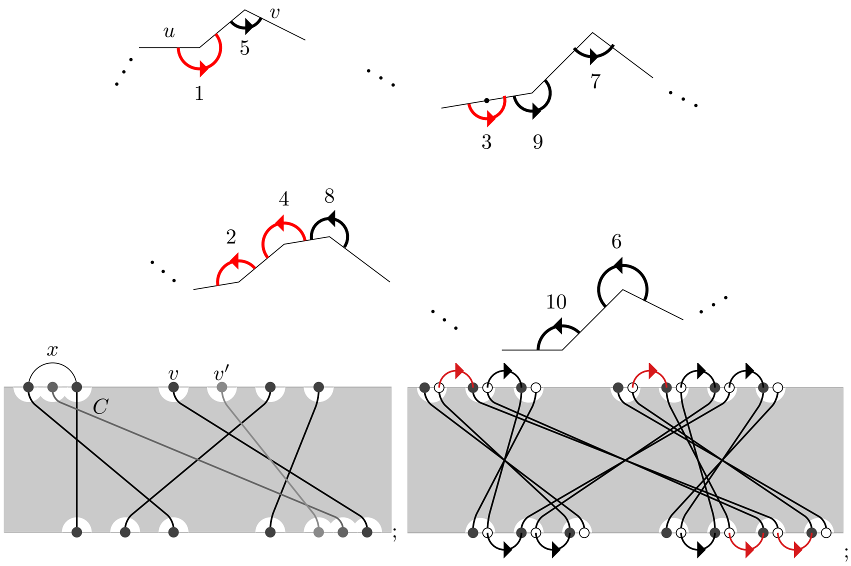


Figure 3.25: Top : In A portion of a translation surface S' corresponding to an operator $\mathcal{T}$ acting on the example surface S of Figure 3.18. For what concerns the construction of small circles around the singularities, while in S we have an arc from u to v, here instead the arcs labeled with indices from 1 to 5 ultimately connect u to v, while also encircling the newly introduced false sigularity, and the rest remains unchanged. The four arcs in red (two on the top broken line, and two on the bottom) are added to the same cycle, and thus increase by 2 the degree of the corresponding singularity, in this example passing from angle 6π to angle 10π. Bottom: the procedure at the level of permutations. Let us describe the surgery procedure in details, comparing the language of permutations with the geometric counterpart:

to adding four angular sectors between two consecutive angular sectors (see figure 3.26), and, in particular, the false singularity that was temporarily added is now merged to the original singularity z_i.

If the point P is not on a side of the polygon, we can proceed by a slight generalisation of the first part of the operation: choose a bottom side u_j, call P' and P'' its endpoints, and suppose that the triangle (P, P', P'') can be added/removed on the top/bottom copies of the edge (i.e., u_j on the bottom and $v_{\tau^{-1}(n+1-j)}$ on top), so to obtain an equivalent representation of the polygon, with the introduction of a unique false singularity (this happens when the segments (P, P') and (P, P'') on the plane do not cross the boundary of the polygon). If we do so, we are in the same situation as before, i.e. we have decomposed $u_j = u'_j + u''_j$, with $u'_j = (P', P)$ and $u''_j = (P, P'')$, the only difference being the fact that u'_j and u''_j are not collinear.

In order to describe the procedure at the level of the polygon, and with a generic point P as above (not necessarily on the boundary), we need to supplement the choice of x and j to the notation, so we will write $S_{\text{new}} = \mathcal{T}_{x,j,w}(v)S$.

Similarly as was the case with adding a cylinder, if we follow this procedure on a regular suspension, and the point P is chosen appropriately, we can have a regular suspension as outcome, as illustrated in the following:

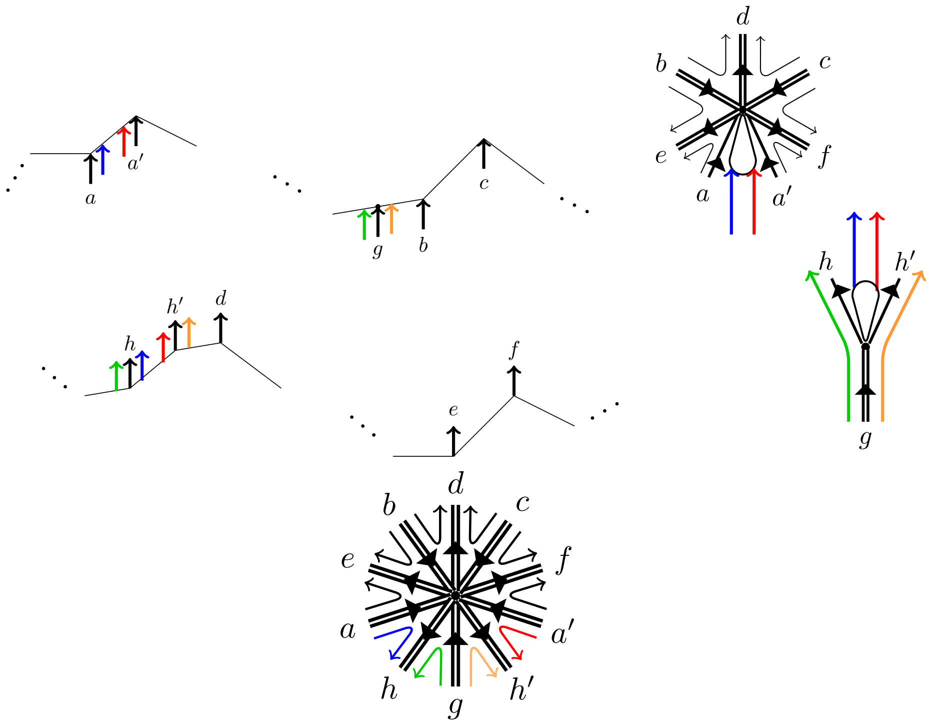


Figure 3.26: Adding the point P on the boundary v of the polygon divides this side in two, thus P becomes a false singularity of angle 2π (i.e. a zero of degree 0). The two half-disks are denoted here by the presence of a green and an orange leaf in the vertical foliation. Adding the cylinder glues together the two singularities, resulting in this case in a 10π singularity, whose neighbourhood, obtained by the insertion of the parallelogram with blue and red leaves on the two singularities (see top-right), is isomorphic to the neighbourhood illustrated at the bottom of the image.

Proposition 35. *Given a side $u_j = (P', P'')$ on the bottom side of a regular suspension, excluded the leftmost one, define $\theta_\pm$ as in (3.17). Consider the triangle Δ with one side u_j, and the other two sides with slopes $\theta_\pm$. For each P in the interior of Δ, if we add the triangle (P, P', P'') to the polygon, below u_j, and remove the corresponding copy of the triangle below $v_{\tau^{-1}(n+1-j)}$, we obtain a new suspension which is regular, has the same shift parameter, and has a false singularity in P.*

Proposition 36. *Let S be a regular suspension of shift y, for a permutation τ of rank r, $\bar{r}$ is the singularity associated to the rank, r_0 is the top angle at the beginning of the rank cycle, $1 \leq j \leq n-1$ is the index of a vector on the bottom side, excluded the left-most one.*

Let notations be as in Proposition 35, with S' being the outcome polygon, and w be the saddle connection connecting r_0 to P, by passing through no sides of the polygon S'. Let v be a vector with $\arg(u'_j) < \arg(v) < \arg(u''_j)$, and $\mathfrak{Im}(v) < y$. Then the suspension $\mathcal{T}_{r_0,j,w}(v)S$ is regular with shift $y - \mathfrak{Im}(v)$.

This construction is illustrated in Figure 3.27.

Our T operator is a special case of this operation, corresponding to j being the

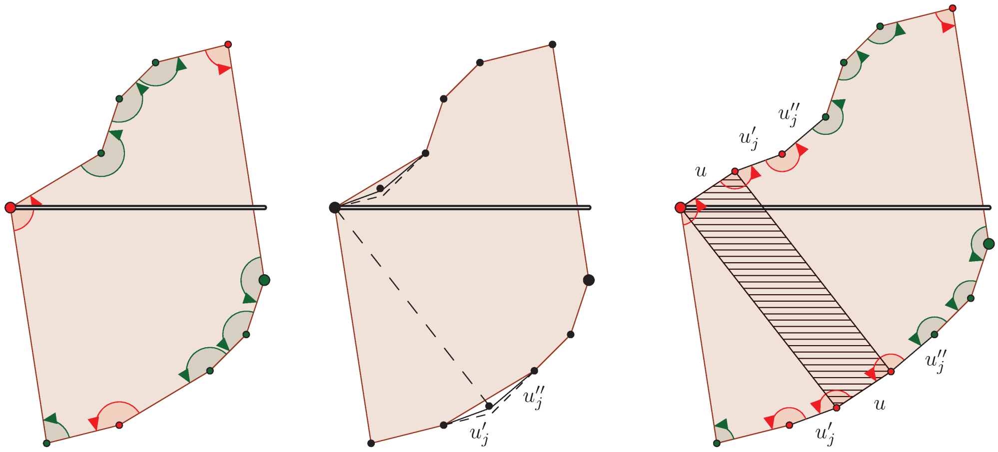


Figure 3.27: Left: a regular suspension S for $\tau = (23145)$ (the same as in figure 3.20). Middle: the construction of S' described by Proposition 35. Right: the operation $\mathcal{T}_{r_0,j,w}(v)S$ described by Proposition 36 (this suspension is as in the left part of figure 3.23). The resulting permutation, $\tau = (413256)$, has the same cycle structure as the starting one, except for the rank, which has increased by 2.

right-most vector of the polygon, on the bottom side. We can define, more generally, operators $\{T_j\}_{1 \leq j \leq n-1}$, of which $T \equiv T_1$ is the special case above. For what concerns the invariants, all of these T_j behave in the same way: the degree of the marked singularity increases by 2, and all the rest of the invariant stays put.

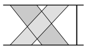
3.2.3.3 Regular suspensions in a given Rauzy class

Our classification theorem, based on the analysis of surgery operators $q_{1,2}$ and T, and the results of the present section, in particular Propositions 34 and 36, allow to describe large families of regular suspensions (with no false singularities) which are certified to be within one given (primitive) Rauzy class.

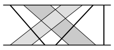
For a class C, let us call $\mathcal{R}(C)$ the set of such regular suspensions. Thus we have $\mathcal{R}(C) = \{S\}$, with $S = (\tau; v_1, \dots, v_n)$ satisfying certain properties, which we now explicitate. We describe explicitly $\mathcal{R}(\mathrm{Id}_n)$ and $\mathcal{R}(\mathrm{Id}'_n)$, and, for C a non-exceptional primitive class with invariant (r, λ, s), we describe $\mathcal{R}(C) \equiv \mathcal{R}(r, \lambda, s)$ recursively in terms of the invariant.

We say that $(v_1, \dots, v_n)$ is a *regular vector* if it satisfies the first two conditions of Definition 33. We can then anticipate a corollary of the results presented in this section and in the body of the paper:

Corollary 37. *The surfaces in the sets $\mathcal{R}(C)$ below are all regular suspensions:*



$$\mathcal{R}(\mathrm{Id}_n) = \Big\{ \Big(\square ; v_1, \dots, v_n \Big) \textit{ with } (v_1, \dots, v_n) \textit{ regular} \Big\} \tag{3.18}$$



$$\mathcal{R}(\mathrm{Id}'_n) = \Big\{ \Big(\square ; v_1, \dots, v_n \Big) \textit{ with } (v_1, \dots, v_n) \textit{ regular} \Big\} \tag{3.19}$$

$$\mathcal{R}(1, \lambda, s) = \bigcup_{\ell \mid \exists \lambda_i = \ell} \{\mathcal{Q}_{\bar{r}, r_0, 1}(v) S\}_{S \in \mathcal{R}(\ell, \lambda \setminus \ell, s);\ v \textit{ as in Prop. 34}} \tag{3.20}$$

$$\mathcal{R}(2, \lambda, 0) = \bigcup_{\substack{\ell \mid \exists \lambda_i = \ell \\ s = 0, \pm 1}} \{\mathcal{Q}_{\bar{r}, r_0, 2}(v) S\}_{S \in \mathcal{R}(\ell+1, \lambda \setminus \ell, s);\ v \textit{ as in Prop. 34}} \tag{3.21}$$

$$\mathcal{R}(r, \lambda, s) = \bigcup_{j=1}^{n-1} \{\mathcal{T}_{r_0, j, w}(v) S\}_{S \in \mathcal{R}(r-2, \lambda, s);\ w \textit{ and } v \textit{ as in Props. 35 and 36}} \quad r \geq 3 \tag{3.22}$$

where, in (3.21), the set of invariants which contribute to the set-union is as follows: if $\lambda \setminus \ell$ has a positive number of even cycles, then $s = 0$, otherwise, $s = \pm 1$.

(The domains in the sums are explained in detail in the following Corollaries 75 and 77).

This result is complementary to the work present in an article of Zorich [Zor08], in which he constructs representatives of every connected component of every strata, consisting of Jenkins–Strebels differentials. One advantage of our approach is the fact that the representatives given in Corollary 37 are a large number (the smallest asymptotics for a non-exceptional class, corresponding to the iterated application of $\mathcal{T} \circ \mathcal{Q}_2$, is of the order of $n!!$). A second advantage is that each representative has a simple interpretation in terms of nested insertions of handles and cylinders.

3.2.4 A summary of terminology

We end this section by collecting a list of notions which appear both in our approach and in the geometric construction (but, sometimes, under different names). First, we recall in words some notational shortcuts for the strata which were used in [KZ03] and [Boi12]:

- They call $H^{\text{hyp}}(2g-2)$ and $H^{\text{hyp}}(g-1, g-1)$ the *hyperelliptic classes*, which, by Lemma 31, correspond to Id_n with $n = 2g$ and Id_n with $n = 2g+1$, respectively.
- They call $H^{\text{even}}(2d_1^{m_1}, \dots, 2d_k^{m_k})$ and $H^{\text{odd}}(2d_1^{m_1}, \dots, 2d_k^{m_k})$ the two connected components of the stratum $H(2d_1^{m_1}, \dots, 2d_k^{m_k})$ with Arf invariant $+1$ and -1 respectively.
- We shall call $H^{\text{even}}(2d_1^{m_1}, \dots, 2\bar{d}_i^{m_i}, \dots, 2d_k^{m_k})$ and $H^{\text{odd}}(2d_1^{m_1}, \dots, 2\bar{d}_i^{m_i}, \dots, 2d_k^{m_k})$ the two connected components of the marked stratum $H(2d_1^{m_1}, \dots, 2\bar{d}_i^{m_i}, \dots, 2d_k^{m_k})$ with Arf invariant 1 and -1 respectively (in analogy with their notation).
- Finally, we use symbols $H(d_1^{m_1}, \dots, d_k^{m_k})$ (respectively $H(d_1^{m_1}, \dots, \bar{d}_i^{m_i}, \dots, d_k^{m_k})$) only when a positive number of the d_i's are odd. In this case, these symbols denote a stratum (respectively, a marked stratum) with some zeroes of odd degree, so that the Arf invariant is not defined, and the stratum is connected.

Then, Table 3.4 gives a list of correspondences between the terminology adopted in [KZ03] and [Boi12], related to the geometric notions associated to the strata, and the one, issued from the combinatorial approach, used in this part of the thesis.

Geometric and topological objects	**Combinatorial objects**
Translation surface with vertical direction Riemann surface with abelian differential (M,ω)	Suspension data $(\sigma,(v_i)_i)$
Conical singularity of angle $2\pi(d_i+1)$ Zero of ω of degree d_i False singularity (zero of ω of degree 0)	Cycle of length $\lambda_i = d_i+1$ descent (cycle of length 1)
Marked conical singularity of angle $2\pi(d+1)$ Marked zero of ω of degree d	Rank path of length $r = d+1$
Arf invariant $\mathrm{arf}(\Phi)$	Arf (or sign) invariant $s(\sigma)$
$H^{\mathrm{hyp}}(2g-2)$ $H^{\mathrm{hyp}}(g-1,g-1)$	Id_n with $n=2g$ even Id_n with $n=2g+1$ odd
$H^{\mathrm{hyp}}(2g-2,\bar{1})$ $H^{\mathrm{hyp}}(g-1,g-1,\bar{1})$	Id'_n with $n=2g+1$ odd Id'_n with $n=2g+2$ even
Connected component in $\mathcal{S}$ $H^{\mathrm{even}}(2d_1^{m_1},\ldots,2\bar{d_i}^{m_i},\ldots,2d_k^{m_k})$ Connected component in $\mathcal{S}$ $H^{\mathrm{odd}}(2d_1^{m_1},\ldots,2\bar{d_i}^{m_i},\ldots,2d_k^{m_k})$ Connected component in $\mathcal{S}$ $H(d_1^{m_1},\ldots,\bar{d_i}^{m_i},\ldots,d_k^{m_k})$	Rauzy class with invariant $(\lambda, r=2d_i+1, s=+1)$ $\lambda = ((2d_1+1)^{m_1}\cdots(2d_i+1)^{m_i-1}$ $\cdots(2d_k+1)^{m_k})$ as above, with $s=-1$ Rauzy class with invariant $(\lambda, r=d_i+1, s=0)$ $\lambda = ((d_1+1)^{m_1}\cdots(d_i+1)^{m_i-1}$ $\cdots(d_k+1)^{m_k})$
Connected component in $\mathcal{S}^{\mathrm{ex}}$ $H^{\mathrm{even}}(2d_1^{m_1},\ldots,2d_k^{m_k})$ Connected component in $\mathcal{S}^{\mathrm{ex}}$ $H^{\mathrm{odd}}(2d_1^{m_1},\ldots,2d_k^{m_k})$ Connected component in $\mathcal{S}^{\mathrm{ex}}$ $H(d_1^{m_1},\ldots,d_k^{m_k})$	Extended Rauzy class with invariant $(\lambda, s=+1)$ $\lambda = ((2d_1+1)^{m_1}\cdots(2d_k+1)^{m_k})$ as above, with $s=-1$ Extended Rauzy class with invariant $(\lambda, s=0)$ $\lambda = ((d_1+1)^{m_1}\cdots(d_k+1)^{m_k})$
Bubbling a handle Adding a cylinder	T operator q_1 and q_2 operators

Table 3.4: Correspondence between the terminology adopted in [KZ03] and [Boi12] and the one used in this part of the thesis. Id'_n (i.e the hyperelliptic class with a marked point) was studied in [BL12].

Chapter 4

A first proof of the Rauzy dynamics for permutation.

4.1 Some basic facts

4.1.1 Properties of the cycle invariant

In this section we come back to the cycle invariant introduced in Section 3.1.3.1. In particular, we shall prove Propositions 202 and 199, concerning the fact that (λ, r) is invariant in the $\mathcal{S}$ dynamics, and λ is invariant in the $\mathcal{S}^{\text{ex}}$ dynamics. These facts are especially evident in the diagrammatic representation, which we illustrate in Figure 9.11.

The idea is that the operators of the dynamics, i.e. the permutations $\gamma_{L,n}(i)$ and $\gamma_{R,n}(i)$ defined in (3.1), perform local modifications on portions of paths, without changing their lengths. For all cycles (or the rank path) except for at most two of them, this is completely evident: each arc of the cycles is deformed by (say) the permutation $\gamma_{L,n}(i)$ in a way which can be retracted without changing the topology of the connections, this in turns certifying that the length of the involved cycle does not change. (We recall that the length is defined as the number of top or bottom arcs). Furthermore, it also preserves the path-lengths of these cycles.

For the paths passing from fours special endpoints (marked as A, B, C and D in the figure), the invariance of the cycle structure holds through a more subtle mechanism, involving the exchange of two arcs, and, for what concerns path-lengths, by a crucial use of the -1 mark. The way in which the topology of connections among these four points is modified by the permutation $\gamma_{L,n}(i)$ is hardly explained in words, but is evident from Figure 9.11. This proves that the list $\lambda' = \lambda \cup \{r\}$ is invariant. We shall prove also that, in the $\mathcal{S}_n$ dynamics, the rank length is also preserved (and thus, by set difference, λ is). This comes from the fact that the two endpoints of the rank path cannot move under the action of L and R (while they *would* move under L' and R'). Figure 9.11 shows that path lengths are preserved also if the paths are coloured (e.g., in the figure, if the lengths of the green and purple paths in σ are ℓ_1 and ℓ_2, respectively, they are still ℓ_1 and ℓ_2 in $L\sigma$). As we perform local modifications on portions of paths adjacent to arcs, these do not affect the rank-path endpoints, and thus do not affect the rank-path 'colour'. From this we deduce that also the length r of the rank path is invariant. This completes the proof.

Note that the length of the principal cycle (i.e. the cycle going through the -1

mark) is *not* preserved. Also, is not preserved the boolean value [does the rank-path pass through the -1 mark]. Indeed, in the figure, the purple path goes through the mark in σ, while the green path does, in $L\sigma$. Nonetheless, these quantities evolve in a rather predictable way. If we annotate which cycles/paths go through the arcs in the top-right portion of the diagram (namely, at the right of $\sigma(1)$), we can observe that iterated powers of L make a circular shift on these labels (yet again, represented by the green and purple colours on the arcs), and this immediately reflects on which cycle (or the rank path) goes through the -1 mark. This fact is analysed in more detail in Section 4.1.3 below.

4.1.2 Standard permutations

As we have seen in the introduction, irreducibility is a property of classes: either all permutations of a class are irreducible or none is. It is a trivial and graphically evident property, as the block structure is clear both from the matrix representation of σ, and from the diagram representation of σ. There is a further, less obvious, characterisation of irreducibility, which leads us to the definition of 'standard permutations'. Both notions are already present in the literature. For example, standard permutations are introduced by Rauzy [Rau79], they play a crucial role in [KZ03] (and various other papers), and their enumerations are investigated in [Del13].

Definition 38 (standard permutation)**.** *The permutation σ is* standard *if $\sigma(1) = 1$.*

(This definition is slightly different from the common one, where σ is standard if $\sigma(1) = 1$ and $\sigma(n) = n$. As we will see below, in Proposition 46, this is a minor problem, because the two notions are easily related.) As mentioned above, we have:

Lemma 39. *A class $C \subseteq \mathcal{S}_n$ is irreducible if and only if it contains a standard permutation.*

This lemma is proven towards the end of this subsection. Before doing this, we need to introduce *zig-zag paths* (see also Figure 4.2).

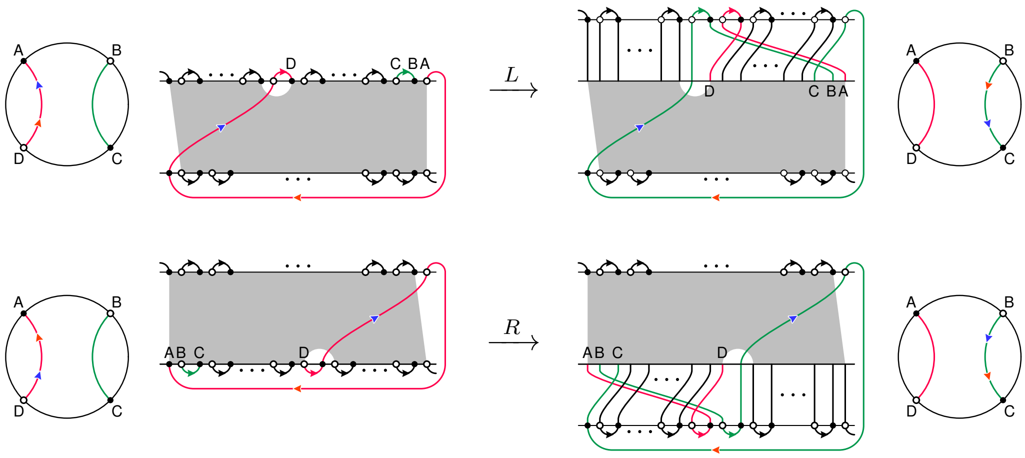


Figure 4.1: Invariance of $\lambda(\sigma) \cup \{r\}$ in $\mathcal{S}_n$, and of $\lambda'(\sigma)$ in $\mathcal{S}_n^{\mathrm{ex}}$, illustrated for the operators L (top) and R (bottom). For $\mathcal{S}_n^{\mathrm{ex}}$, the cases of operators L' and R' are deduced analogously.

Definition 40. *A set of edges* $((i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell))$ *is a* L zig-zag path *if* $i_1 = 1$*, the indices satisfy the pattern of inequalities*

$$j_{2b} > j_{2b-1}\,, \qquad i_{2b} > n - j_{2b-1} + 1\,, \qquad i_{2b} > i_{2b+1}\,, \qquad n - j_{2b+1} + 1 > i_{2b}\,, \tag{4.1}$$

and either $i_\ell = n$ *or* $j_\ell = 1$*. The analogous structure starting with* $j_1 = n$ *is called* R zig-zag path.

It is easy to see that, if σ has a L zig-zag path, no set $[s] \subsetneq [n]$ can have $\sigma[s] = [n, \ldots, n-s+1]$, as, from the existence of the path, there must exist either a pair $(u, v) = (i_{2b+1}, j_{2b+1})$ such that $u \leq s$, $n - s + 1 > v$ and $v = \sigma(u)$, or a pair $(u, v) = (j_{2b}, i_{2b})$ such that $u \geq n-s+1$, $s < v$, and $v = \sigma^{-1}(u)$ (this is seen in more detail below). A similar argument holds for R zig-zag paths. Thus zig-zag paths provide a concise certificate for irreducibility.

We now illustrate an algorithm that, for a given $\sigma \in \mathfrak{S}_n$, produces in linear time either a certificate of reducibility, by exhibiting the top-left block of the permutation, or a certificate of irreducibility in the form of a canonically-chosen L zig-zag path, that we shall call the *greedy* L *zig-zag path* of the permutation. A completely analogous algorithm gives the greedy R zig-zag path. This algorithm will imply the following:

Lemma 41. *A permutation* σ *is irreducible if and only if it has a zig-zag path.*

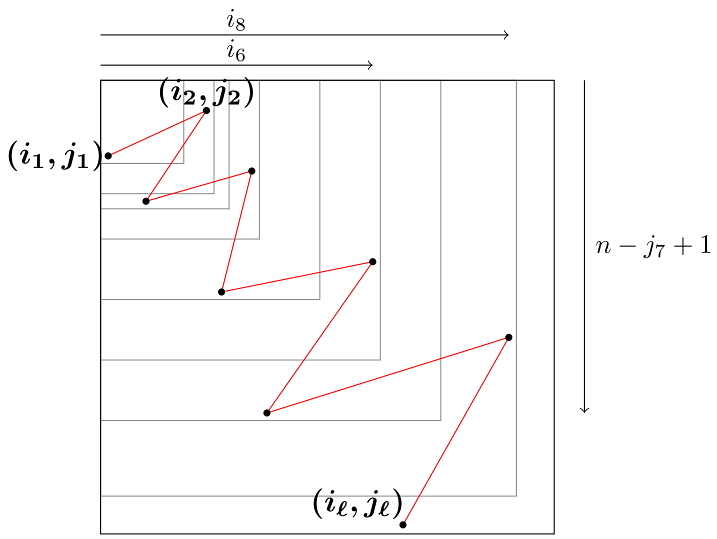


Figure 4.2: Structure of a zig-zag path, in matrix representation. It is constructed as follows: for each pair (i_a, j_a), put a bullet at the corresponding position, and draw the top-left square of side $\max(i_a, j_a)$. Draw the segments between bullets (i_a, j_a) and (i_{a+1}, j_{a+1}) (here in red). The resulting path must connect the top-left boundary of the matrix to the bottom-right boundary. If the bullets are entries of σ, then a top-left $k \times k$ square cannot be a block of the matrix-representation of σ, because, in light of the inequalities (4.1), one of the two neighbouring rectangular blocks (the $k \times (n-k)$ block to the right, or the $(n-k) \times k$ block below) must contain a bullet of the path, and thus be non-empty.

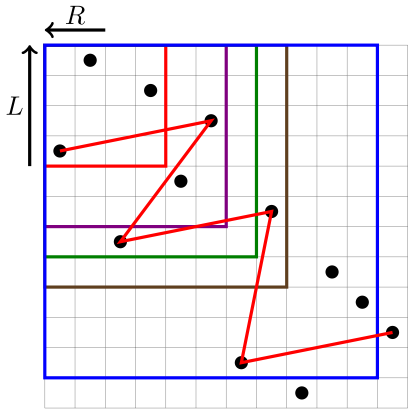


Figure 4.3: We construct the shortest L zig-zag path of σ in a greedy way: we take $(i_1, j_1) = (1, \sigma(1))$, then draw the square (here in red) and take (i_2, j_2) the rightmost point above row j_1. If it were inside the square, the permutation would have been reducible. It is not the case here, so we draw a larger square (here in violet), and take (i_3, j_3) as the lowest point to the left of column i_2. We continue this way, up to when the point (i_6, j_6) is reached. As $i_6 = n$, we terminate by certifying the irreducibility of σ.

Proof. For $X = \{x_a\} \subseteq [n]$, we use $\sigma(X)$ as a shortcut for $\{\sigma(x_a)\}$. Set $J_1 := \sigma(\{1\})$ and $i_1 = 1$. Now, either σ is reducible, and has top-left block of size 1, or $j_1 := \max(J_1) < n$. If $j_1 = 1$, then $\big((i_1, j_1)\big)$ is a L zig-zag path. If $j_1 > 1$, consider the set $I_1 = \sigma^{-1}(\{n, n-1, \ldots, n-j_1+1\})$. Either the largest element of this set is $n-j_1+1$, in which case we certify reducibility of σ with a first block of size $n - j_1 + 1$, or it is some index $i_2 > n - j_1 + 1$. Say $\sigma(i_2) = j_2 < j_1$. If $i_2 = n$, then $\big((i_1, j_1), (i_2, j_2)\big)$ is a L zig-zag path. If $i_2 < n$, consider the set $J_2 = \sigma\{i_1 + 1 = 2, \ldots, i_2\}$. Either the smallest element of this set is $n - i_2 + 1$, in which case we certify reducibility of σ, and with top-left block of size i_2, or it is some index j_3 such that $n - j_3 + 1 > i_2$. Say $i_3 := \sigma^{-1}(j_3) < i_2$. Continue in this way. At each round we query the image or pre-image of values in some interval $(i_a + 1, i_a + 2, \ldots, i_{a+1})$, or same with j, so the total amount of queries is bounded by $2n$. At the various rounds at which we do not halt we have $i_{b+2} > i_b$ and $j_{b+2} < j_b$, As the indices i_a, j_a are in the range $[n]$, the algorithm must terminate in at most $\sim n$ rounds, and can only terminate either because we have found a reducible block, or because $i_{2b} = n$ for some b, or because $j_{2b-1} = 1$ for some b. This proves both the lemma and the statement on the linearity of the algorithm. $\square$

Figure 4.3 illustrates an example of the construction of the greedy L zig-zag path, in matrix representation, and the caption illustrates the run of the algorithm.

Let us analyse a bit more carefully the complexity. One can see that the worst case is for irreducibility, and is attained on the permutation σ such that the vectors $(i_{a+1}, j_{a+1}) - (i_a, j_a)$ of the greedy L or R zig-zag path are $\ldots (3,1)$, $(-1,-3)$, $(3,1)$, $(-1,-3)\ldots$, with an exception at the first and last elements (either a $(3,1)$ replaced by $(2,1)$, or a $(-1,-3)$ replaced by $(-1,-2)$, depending on parities). In this case the algorithm takes $n-1$ rounds, and overall performs all of the $2n$ queries.

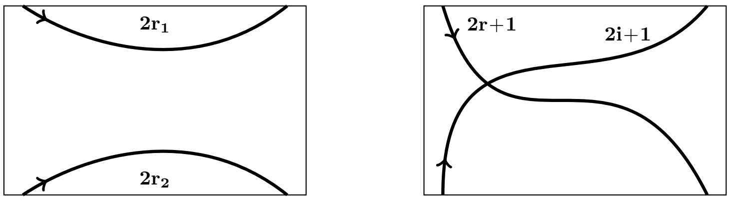


Figure 4.4: Left: a schematic representation of a permutation of type $H(r_1, r_2)$. Right: a representation of a permutation of type $X(r, i)$. These configurations have rank $r_1 + r_2 - 1$ and r, respectively.

An analogous construction can be done by starting with $j_1 = n$. By this, in the reducible case we find the same certificating block, and in the irreducible case we construct the greedy R zig-zag path.

One can see that at least one among the two greedy paths (L and R) has minimal length among all zig-zag paths, and that the length of the two greedy paths differ by at most 1. We call this value the *level* of σ, and denote it by the symbol $\ell v(\sigma)$.

Of course, an irreducible permutation σ of level 1 is a standard permutation. More is true:

Lemma 42. *An irreducible permutation σ is at alternation distance at most $\ell v(\sigma)-1$ from a standard permutation.*

Proof. Say that the level of σ is $\ell > 1$, and is realised by its greedy L zig-zag path (the argument for R is symmetric). So we have indices $j_2 < j_1$ and $i_1 = 1 < i_2$.

The action of L^{n-j_2} moves the bullet (i_2, j_2) in position (i_2, n) without affecting any other edge of the path. Indeed, because of the inequalities satisfied by the indices of the greedy L zig-zag path, $j_a < j_1$ for all $a \geq 3$. Then, the path obtained by dropping the first edge is a R zig-zag path of level $\ell - 1$ for the new configuration $\sigma' = L^{n-j_2}\sigma$, and the two configurations σ, σ' are nearest neighbour for alternation distance. The reasoning in the other case is analogous, with $R \leftrightarrow L$ and $i_a \leftrightarrow n - j_a + 1$. $\square$

Now Lemma 39 follows as an immediate corollary.

4.1.3 Standard families

Here we introduce a property of permutations which is not invariant under the dynamics, nonetheless it is useful in a combinatorial decomposition of the classes, both for classification and enumeration purposes.

Definition 43. *A permutation σ is* of type H *if the rank path goes through the -1 mark, and* of type X *otherwise. In the case of a type-X permutation, we call* principal cycle *the cycle going through the -1 mark.*

The choice of the name is done for mnemonic purposes. Imagine to cut the cycle invariant at the -1 mark. Then we have two open paths. In a type-H permutation, these paths have a $\asymp$ connectivity pattern (which reminds of a H), and in a type-X permutation they have a $\times$ pattern (which reminds of a X).

A more refined definition is as follows

Definition 44. *A permutation σ is* of type $H(r_1, r_2)$ *if it is of type H, and the portions of the rank path before and after the -1 mark have path-length $2r_1$ and $2r_2$, respectively (so the rank length is $r = r_1 + r_2 - 1$). It is* of type $X(r, i)$ *if it is of type X, has rank r, and the principal cycle has path-length $2i + 1$ (i.e., it contributes a i entry to the cycle structure λ).*

See Figure 4.4 for a schematic illustration.

A standard permutation is a notion with a simple definition. More subtle is the associated notion:

Definition 45 (standard family)**.** *Let σ be a standard permutation. The collection of $n - 1$ permutations $\{\sigma^{(i)} := L^i\sigma\}_{0 \leq i \leq n-2}$ is called the* standard family *of σ.*

The properties established in Section 4.1.1 ultimately imply the following statement:

Proposition 46 (Properties of the standard family)**.** *Let σ be a standard permutation, and $S = \{\sigma^{(i)}\}_i = \{L^i(\sigma)\}_i$ its standard family. The latter has the following properties:*

1. *Every $\tau \in S$ has $\tau(1) = 1$;*
2. *The $n - 1$ elements of S are all distinct;*
3. *There is a unique $\tau \in S$ such that $\tau(n) = n$;*

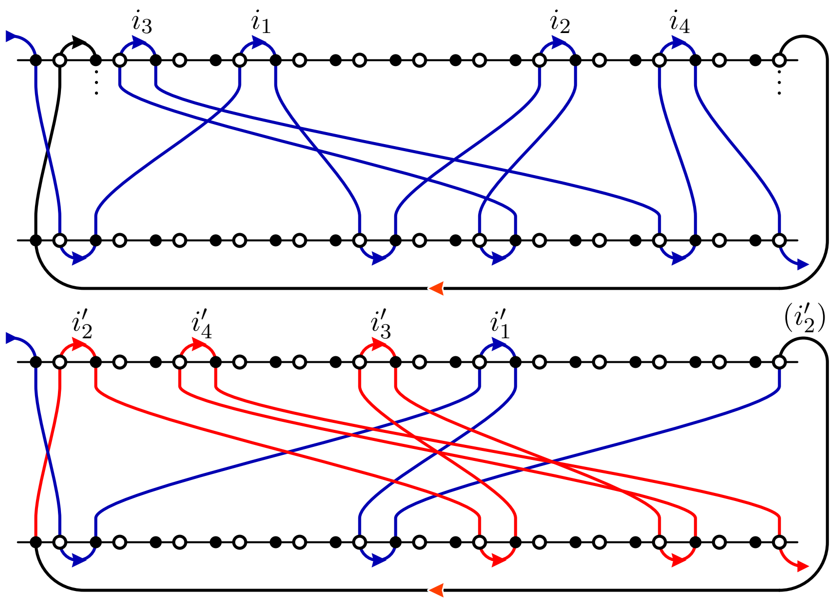


Figure 4.5: Illustration of the proof of an aspect of the fourth property in Proposition 46. Top: portions of a configuration σ, with rank 4, of type $X(4, *)$. The top arcs in the rank are at positions $i_1, \ldots, i_4$, and numbered according to their order along the path. Bottom: the result of applying L^{n-i_2} to σ. The new configuration is of type $H(2, 3)$, as the blue and red paths have path-length 4 and 6, respectively.

4. *Let m_i be the multiplicity of the integer i in λ (i.e. the number of cycles of length i), and r the rank. There are $i\,m_i$ permutations of S which are of type $X(r,i)$, and 1 permutation of type $H(r-j+1,j)$, for each $1 \leq j \leq r$.*[1]

5. *Among the permutations of type $X(r,i)$ there is at least one τ with $\tau^{-1}(2) < \tau^{-1}(n)$.*

Proof. The first three statements are obvious.

The fourth statement is based on the definitions of Section 4.1.1. Let us analyse the top arcs: the cycle or path going through the -1 mark is also the one going through the rightmost arc, thus, as the family $L^i(\sigma)$, for $1 \leq i \leq n-1$, corresponds to a cyclic shift of the 'colours' on these arcs, we deduce that each cycle/path goes through the -1 mark a number of times identical to its length value, contributing to the cycle invariant. I.e., there are $i\,m_i$ permutations in S of type $X(r,i)$, and r permutations of type H, as claimed. In order to prove that, among the latter, we have exactly one permutation per type $H(r-j+1,j)$, for $1 \leq j \leq r$, we need to add some more structure. Let us number the top arcs of the rank path in σ from 1 to r, following the order by which they are visited by the path, starting from the top-left corner. Then it becomes clear that $L^i(\sigma)$ has type $H(r-j+1,j)$ exactly when the arc $n-i$ is the $(r-j+1)$-th arc of the rank path (this is also illustrated in Figure 4.5)

Let us now pass to the fifth statement. As we are in type X, we have at least one cycle. Let $a_1, \dots, a_k$ be the arcs associated to a cycle of length k. For $j = 1, \dots, k$ we consider the edge $(\ell_{j,1}, \sigma(\ell_{j,1}))$ incident to the left endpoint of the arc a_j, and the edge $(\ell_{j,2}, \sigma(\ell_{j,2}) = \sigma(\ell_{j,1})+1)$ incident to the right endpoint. Since we have a cycle (and not the rank path), at least one of these arcs (say a_j) must have $\ell_{j,1} > \ell_{j,2}$, because the ℓ_j's form a cyclic sequence of distinct integers, thus must have at least one descent in this list.

This reasoning may not work if we deal with the principal cycle, as we may have only one descent, in correspondence of the -1 mark. There are two possible workarounds to this apparent problem. The first one is that we can start by looking at a configuration τ_0 of type H (there must be at least one such configuration in the family, because the rank is at least 1). In this configuration no cycle is the principal cycle, and we can choose one arc per cycle with the required property. The second workaround is the fact that the reasoning doesn't apply only when the principal cycle has only one descent, and in correspondence of the -1 mark, but in fact in this very case we just have $\tau^{-1}(2) < \tau^{-1}(n)$ with no need of further analysis.

Let us continue the analysis of the generic case, of an arc a_j with $\ell_{j,1} > \ell_{j,2}$. Then $\sigma' = L^{n-a_j}(\sigma)$ has a principal cycle of length k, and these two edges, in σ', become $(\ell_{j,1}, n)$ and $(\ell_{j,2}, 2)$ (see figure 4.6). This allows to establish the fifth property. $\square$

Corollary 47. *As a consequence of the property (5), let τ' be the permutation resulting from removing the edge $(1,1)$ from a permutation τ as in (5). Then τ' is irreducible, and there exists ℓ (namely, $\ell = \tau^{-1}(2)-2$) such that $R^\ell(\tau')$ is standard.*

This corollary will be of crucial importance in establishing the appropriate conditions of the surgery operators q_1 and q_2, outlined in Section 3.1.6. (More precisely it will

[1] Note that, as $\sum_i i\,m_i + r = n-1$ by the dimension formula (3.2), this list exhausts all the permutations of the family.

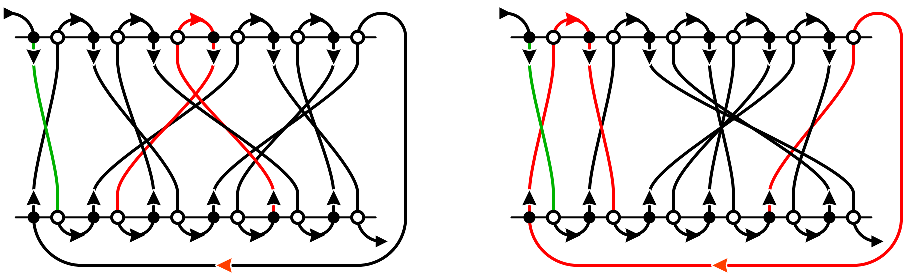


Figure 4.6: Illustration of the proof of the fifth property in Proposition 46. Left: a configuration σ, with cycle invariant $(3, 1; 1)$, of type $X(1, 1)$ (in order to produce a small but generic example, we have taken a non-primitive configuration). The cycle of length 3 is not the principal cycle, thus it must have an arc such that $\ell_{j,1} > \ell_{j,2}$. One such arc, and the two incident edges, are denoted in red. Right: the configuration σ' obtained by applying L the appropriate number of times, so to make this arc go through the -1 mark. At this point, if we remove the left-most edge, the resulting configuration is irreducible.

be a key element of the proof of Theorems 79 and 80.)

Proof. For τ as in the proof of the property (5), note that removing the edge $(1, 1)$ provides an irreducible permutation τ'. Moreover it is clear that τ' has level $\ell v(\tau') = 2$, as in fact $R^{\ell_{j,2}-2}(\tau')$ is standard.[2] □

4.1.4 Reduced dynamics and boosted dynamics

Some parts of our proof are just obtained from the inspection of certain given (finite) patterns. How can such a simple ingredient be compatible with a classification of classes of arbitrary size? A crucial notion is the definition of 'reduced' and 'boosted' dynamics, i.e., the analysis of the behaviour of *patterns* in configurations under the dynamics.[3]

Given a permutation σ, we can partition the set of edges into two colors: black and gray. Say that $c : [n] \to \{\text{black}, \text{gray}\}$ describes this colouration. The permutation τ corresponding to the restriction of σ to the black edges is called the *reduced permutation* for (σ, c). Conversely, call $\hat{c}$ a data structure required to reconstruct σ from the pair $(\tau, \hat{c})$. A choice for $\hat{c}$ is as an unordered list of quadruples, $(i, k|j, h)$. Such a quadruple means that there is a gray edge with bottom endpoint being the k-th gray point at the right of the i-th black point, and top endpoint being the h-th gray point at the left of the j-th black point. These quadruples evolve in a simple way under the action of L and R (i.e., they evolve consistently with the base values i and j, while k and h stay put). As a corollary, if we consider the dynamics on τ as an edge-labeled permutation, we can reconstruct the evolution of $\hat{c}$ from the evolution of τ.

Call *pivots* of σ the two edges $(1, \sigma(1))$ and $(\sigma^{-1}(n), n)$, if we work in the $\mathcal{S}_n$ dynamics. [If we are in $\mathcal{S}_n^{\text{ex}}$, we call pivot also $(n, \sigma(n))$ and $(\sigma^{-1}(1), 1)$]

[2]We have an exponent $\ell_{j,2} - 2$, instead of $\ell_{j,2} - 1$ because we have removed the edge $(1, 1)$.

[3]Here 'pattern' is said in the sense of Permutation Pattern Theory, and is clarified in the following.

For a pair (σ, c), we say that σ is *proper* if no gray edge of σ is a pivot. In this case the dynamics on τ extends to what we call the *boosted dynamics* on σ, defined as follows: for every operator H (i.e., $H \in \{L, R\}$ for $\mathcal{S}_n$ and $H \in \{L, L', R, R'\}$ for $\mathcal{S}_n^{\mathrm{ex}}$), we define $\alpha_H(\sigma, c)$ as the smallest positive integer such that $H^{\alpha_H(\sigma,c)}(\sigma)$ is proper, and, for a sequence $S = H_k \cdots H_2 H_1$ acting on τ, the sequence $B(s)$, the *boosted sequence* of S, acting on σ is $B(S) = H_k^{\alpha_k} \cdots H_2^{\alpha_2} H_1^{\alpha_1}$ for the appropriate set of α_j's.

In other words, the boosted dynamics is better visualised as the appropriate notion such that the following diagram makes sense: for $(\tau, \hat{c})$ an edge-labeled permutation with colouring data, as described above, and calling Φ the operator that reconstructs (σ, c) from it, we have

$$\begin{tikzcd}
(\tau, \hat{c}) \arrow[r, "S"] \arrow[d, "\Phi"] & (\tau', \hat{c}') \arrow[d, "\Phi"] \\
(\sigma, c) \arrow[r, "B(S)"] & (\sigma', c')
\end{tikzcd}$$

Working in the reduced dynamics gives concise certificates of connectedness: we can prove that $\sigma \sim \sigma'$ by showing the existence of a triple $(\tau, \hat{c}, S)$ that allows to reconstruct the full diagram above. The idea is that one can often show the existence of a given pair (τ, S) (of finite size), and a family $\{\hat{c}_n\}_{n\in\mathbb{N}}$, which allows to prove the connectedness of families $\{\sigma_n \sim \sigma'_n\}_{n\in\mathbb{N}}$. This is the method used in section 4.3.1 for the proof of the main theorem 67 on the T operator.

In a slightly more general form, instead of having a finite sequence S, we could have three finite sequences S_-, S_0 and S_+ such that the connectedness of σ_n and σ'_n is proven through the sequence $S_- S_0^n S_+$. We use this generalised pattern only once, in order to prove explicitly the connectedness of two permutations inside a given class, in a case which is not covered by any of the 'big theorems' on the surgery operators (see Section 4.4.3, Lemma 89)

4.1.5 Square constructors for permutations

In this section we define a rather general surgery operation on permutations, which behaves in a simple way under the Rauzy dynamics:

Definition 48 (Square constructor)**.** *Let τ be a permutation matrix of size k, and let $0 \leq i, j \leq k$, with $i + j > 0$. The pair i, j describes a way to decompose τ into four (possibly empty) rectangular blocks, $\tau = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, so that A and B have i rows, and A and C have j columns. For $h, \ell \in \{1, \ldots, n\}$, and σ a permutation of size n with $\ell = \sigma(h)$), the* square constructors *$C_{\tau,i,j}^{\mathrm{col}-h}$ and $C_{\tau,i,j}^{\mathrm{row}-\ell}$, acting on σ, produce the same configuration σ', of size $n + k + 1$, as described in Figure 4.7.*

Lemma 49 (reduced dynamics for constructors)**.** *Let (τ, i, j) as above. Let σ and σ' be two permutations of the same size, with one coloured edge (say, red and the rest is black). Let $(h, \sigma(h))$ be the only entry of σ marked in red and $(h', \sigma'(h'))$ be the only entry of σ' marked in red. If $\sigma \sim \sigma'$ for the dynamic $\mathcal{S}$, with colours as above, then $C_{\tau,i,j}^{\mathrm{col}-h}(\sigma) \sim C_{\tau,i,j}^{\mathrm{col}-h'}(\sigma')$.*

Proof. This is a case of reduced dynamics, where, in $C_{\tau,i,j}^{\mathrm{col}-h}(\sigma)$, the entries of τ plus the extra entry at the bottom-right corner of block A are gray. The red point is the point at the top-left of block A, this reduced permutation is (σ, h) where h is the index of the red entry. Thus we can define a bijection ϕ between $C^{\text{col}-h}_{\tau,i,j}(\sigma)$ and (σ, h). Now suppose σ and σ' are connected by the sequence of operators S, i.e. $S(\sigma) = \sigma'$. Then we need to show that $B(S)(\phi^{-1}(\sigma, h)) = \phi^{-1}(\sigma', h')$.

By transitivity of connectedness, it is not necessary to consider arbitrary sequences, it is enough to consider $\sigma' = L\sigma$ and $\sigma' = R\sigma$. The symmetry of the definition of the square constructor allows to consider just one case, and we choose L.

If the red point is not a pivot, then neither are the gray edges, and $B(L) = L$, so $L(\phi^{-1}(\sigma, h)) = \phi^{-1}(\sigma', h')$ as wanted. Thus we only need to consider the case when the red point is a pivot. In such a case, (i.e. the red point is (h, n)), then $B(L) = L^{i+1}$ and $B(L)(\phi^{-1}(\sigma, h)) = \phi^{-1}(L(\sigma), h)$ as wanted (see figure 4.8). $\square$

Lemma 50 (square transportation)**.** *Let* (τ, i, j) *as above. Then, the action of the constructor at different locations gives equivalent configurations, i.e. for all* σ *of size* n *and all* $h, h' \in [n]$ *we have* $C^{\text{col}-h}_{\tau,i,j}(\sigma) \sim C^{\text{col}-h'}_{\tau,i,j}(\sigma)$.

Proof. We start by proving that the lemma holds for $(h, h') = (1, \sigma^{-1}(n))$. This is not hard to see. In fact, $R^{j+1}C^{\text{col}-1}_{\tau,i,j}(\sigma)$ and $L^{i+1}C^{\text{row}-n}_{\tau,i,j}(\sigma)$ are the same configuration, namely the one illustrated in Figure 4.9.

Now consider the reduced dynamics as in Lemma 49. Let (σ, c_L) the colouring of σ with the red entry on the L-pivot, and (σ, c_R) the one with the red entry on the R-pivot. Let ϕ be the map of the reduced dynamics. We have just proven that $\phi^{-1}(\sigma, c_L) \sim \phi^{-1}(\sigma, c_R)$. Let E be the operator that exchanges (σ, c_L) and (σ, c_R),

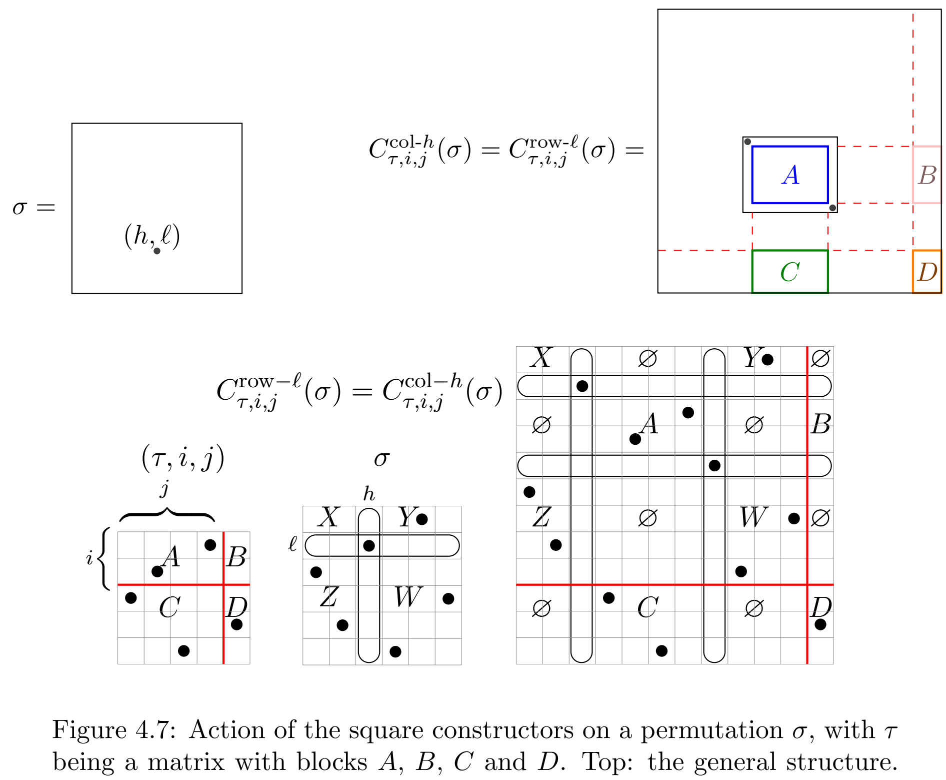


Figure 4.7: Action of the square constructors on a permutation σ, with τ being a matrix with blocks A, B, C and D. Top: the general structure. Bottom: an example.

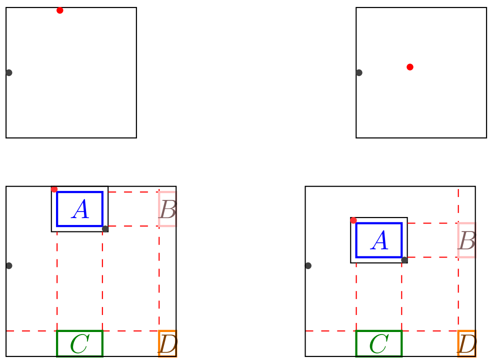
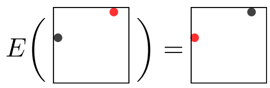


Figure 4.8: Top left: (σ, h). Top right: $(L(\sigma), h)$. Bottom left: $C^{\mathrm{col}-h}_{\tau,i,j}(\sigma)$. Bottom right: $B(L)(C^{\mathrm{col}-h}_{\tau,i,j}(\sigma))$.

i.e.



By Lemma 39 we know that, in the reduced dynamics, we have a sequence S in the dynamics such that $S\sigma = \sigma'$, and σ' is standard. In this configuration, either the L-pivot is red, or exactly one configuration in the associated standard family has a red R-pivot. In the first case, E can be applied to any configuration in the standard family. This shows that, for σ standard, all configurations (σ, c_j) for the red point on the j-th column are connected to (σ, c_L), by a sequence of the form $L^{-k}EL^k$, and thus are connected among themselves by a sequence of the form $L^{-h}EL^{h-k}EL^k$. Lemma 49 allows us to conclude, for σ' a standard permutation, and, by conjugating with the sequence S, for σ a generic permutation in the class (see Figure 4.10).

□

Corollary 51. *Let (τ, i, j) as above. Let σ and σ' such that $\sigma \sim \sigma'$ for the dynamic $\mathcal{S}$, then, for every $1 \leq h, h' \leq n$, $C^{\mathrm{col}-h}_{\tau,i,j}(\sigma) \sim C^{\mathrm{col}-h'}_{\tau,i,j}(\sigma')$.*

A further corollary of this lemma is the statement of Proposition 22, for which it suffices to take A a diagonal matrix and B, C, D empty blocks.

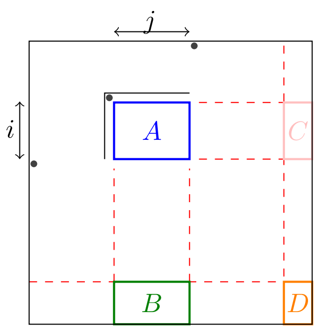


Figure 4.9: Structure of a configuration σ', such that $R^{-j-1}\sigma' = C^{\mathrm{col}-1}_{\tau,i,j}(\sigma)$ and $L^{-i-1}\sigma' = C^{\mathrm{row}-n}_{\tau,i,j}(\sigma)$.

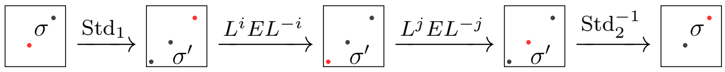


Figure 4.10: The sequence of operations which allow to transport the red mark from one position to another. Here Std is the standardasing sequence of the (uncoloured) configuration σ.

4.2 The sign invariant

4.2.1 Arf functions for permutations

For σ a permutation in $\mathfrak{S}_n$, let

$$\chi(\sigma) = \#\{1 \leq i < j \leq n \mid \sigma(i) < \sigma(j)\} \tag{4.2}$$

i.e. $\chi(\sigma)$ is the number of pairs of non-crossing edges in the diagram representation of σ.

Let $E = E(\sigma)$ be the subset of n edges in $\mathcal{K}_{n,n}$ described by σ. For any $I \subseteq E$ of cardinality k, the permutation $\sigma|_I \in \mathfrak{S}_k$ is defined in the obvious way, as the one associated to the subgraph of $\mathcal{K}_{n,n}$ with edge-set I, with singletons dropped out, and the inherited total ordering of the two vertex-sets.

Define the two functions

$$A(\sigma) := \sum_{I \subseteq E(\sigma)} (-1)^{\chi(\sigma|_I)}\,; \qquad \overline{A}(\sigma) := \sum_{I \subseteq E(\sigma)} (-1)^{|I| + \chi(\sigma|_I)}\,. \tag{4.3}$$

When σ is understood, we will just write χ_I for $\chi(\sigma|_I)$. The quantity A is accessory in the forthcoming analysis, while the crucial fact for our purpose is that the quantity $\overline{A}$ is invariant both in the $\mathcal{S}_n$ and in the $\mathcal{S}_n^{\mathrm{ex}}$ dynamics. We will prove this in the following subsection.

As a result of a well-known property of Arf functions, rederived in Section 4.4.1 (Lemma 81), the quantity $\overline{A}(\sigma)$ is either zero or $\pm$ a power of 2. For now, we content ourselves with the definition

Definition 52. *For σ a permutation, define $s(\sigma)$, the* sign *of σ, as the quantity*

$$s(\sigma) := \mathrm{Sign}(\overline{A}(\sigma)) \in \{0, \pm 1\}\,. \tag{4.4}$$

However, we will see (in lemma 81) that $s(\sigma) = 2^{-\frac{n+\ell}{2}} \overline{A}(\sigma)$, where ℓ is the number of cycles (the rank path is not counted) .

4.2.2 Calculating with Arf functions

Evaluating the functions A and $\overline{A}$ on a generic 'large' permutation, starting from the definition, seems a difficult task, as we have to sum an exponentially large number of terms. However, as the name of Arf function suggests, these quantities are combinatorial counterparts of the classical Arf invariant for manifolds (in this case, the translation surfaces where the Rauzy dynamics is defined, see Section 3.2.2 for more details), and inherits from them the covariance associated to the change of basis in the corresponding quadratic form, and its useful consequences.

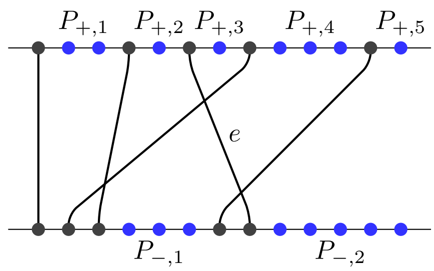


Figure 4.11: A subset of edges in a permutation, as in Definition 53. The represented edge set is E', the blue vertices at the bottom constitute the set V''_-, while those at the top are V''_+. The two partitions denoted by the symbols $P_{\pm,i}$ are the coarsest partitions which are compatible with E'. We cannot show the full matrix Q for such a big example, but we can give one row, for the edge which has the label e in the drawing. The row Q_e reads $(Q_e)_{11,12,\dots,15,21,\dots,25} = (1,1,0,0,0,\ 0,0,1,1,1)$.

We will exploit this to some extent, in a simple and combinatorial way, that we now explicitate.

The point is that we will *not* try to evaluate Arf functions of large configurations starting from scratch. We will rather compare the Arf functions of two (or more) configurations, which differ by a finite number of edges, and establish linear relations among their Arf functions. Yet another tool is proving that the Arf function of a given configuration is zero, by showing that it contains some finite *pattern* that implies this property.

In order to have the appropriate terminology for expressing this strategy, let us define the following:

Definition 53. *Call $V_\pm$ the two vertex-sets of $\mathcal{K}_{n,n}$. For σ a permutation with edge set E, and $E' \subseteq E$ of cardinality m, call $V''_\pm$ the subset of $V_\pm$ not containing the endpoints of E'. A pair of partitions $P_+ = (P_{+,1}, \dots, P_{+,h}) \in \mathcal{P}(V''_+)$ and $P_- = (P_{-,1}, \dots, P_{-,k}) \in \mathcal{P}(V''_-)$ are said to be* compatible with E' *if each block contains indices which are consecutive in $[n]$.*

Define the $m \times (hk)$ matrix valued in GF_2

$$Q_{e,ij} := \begin{cases} 1 & \text{edge } e \in E' \text{ does not cross the segment connecting } P_{-,i} \text{ to } P_{+,j}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

For $v \in \mathrm{GF}_2^d$, let $|v|$ be the number of entries equal to 1. Similarly, identify v with the corresponding subset of $[d]$. Given such a construction, introduce the following functions on $(\mathrm{GF}_2)^{hk}$

$$A_{\sigma,E',P}(v) := \sum_{u \in (\mathrm{GF}_2)^{E'}} (-1)^{\chi_u + (u,Qv)}\,; \quad \overline{A}_{\sigma,E',P}(v) := \sum_{u \in (\mathrm{GF}_2)^{E'}} (-1)^{|u| + \chi_u + (u,Qv)}\,. \tag{4.6}$$

The construction is illustrated in Figure 4.11.

Let us comment on the reasons for introducing such a definition. The quantities $A_{\sigma,E',P}(v)$ allows to sum together many contributions to the function A, which behave all in the same way once the subset restricted to E' is specified. In a way similar to the philosophy behind the reduced dynamics of Section 4.1.4, our goal is to have E' of fixed size, while $E \smallsetminus E'$ is arbitrary and of unbounded size, so that the verification of our properties, as it is confined to the matrix Q, involves a finite data structure.

Indeed, let us split in the natural way the sum over subsets I that defines A and $\overline{A}$, namely

$$\sum_{I \subseteq E} f(I) = \sum_{I' \subseteq E'} \sum_{I'' \subseteq E \smallsetminus E'} f(I' \cup I'')$$

For I and J two disjoint sets of edges, call $\chi_{I,J}$ the number of pairs $(i,j) \in I \times J$ which do not cross. Then clearly

$$\chi_{I \cup J} = \chi_I + \chi_J + \chi_{I,J}$$

Now let $u(I') \in \{0,1\}^{E'}$ be the vector with entries $u_e = 1$ if $e \in I'$ and 0 otherwise. Let $m(I'') = \{m_{ij}(I'')\}$ be the $k \times h$ matrix describing the number of edges connecting the intervals $P_{-,i}$ to $P_{+,j}$ in σ, and $v(I'') = \{v_{ij}(I'')\}$, $v_{ij} \in \{0,1\}$, as the parities of the m_{ij}'s. Call I''_{ij} the restriction of I'' to edges connecting $P_{-,i}$ and $P_{+,j}$. Clearly, $\chi_{I',I''} = \sum_{ij} \chi_{I',I''_{ij}} = \sum_{e,ij} u_e Q_{e,ij} m_{ij}$, which has the same parity as the analogous expression with v's instead of m's. Now, while the m's are in $\mathbb{N}$, the vector v is in a linear space of finite cardinality, which is crucial for allowing a finite analysis of our expressions.

As a consequence,

$$A(\sigma) = \sum_{I \subseteq E(\sigma) \smallsetminus E'(\sigma)} (-1)^{\chi_I} A_{\sigma,E',P}(v(I))\,; \tag{4.7}$$

$$\overline{A}(\sigma) = \sum_{I \subseteq E(\sigma) \smallsetminus E'(\sigma)} (-1)^{|I|+\chi_I} \overline{A}_{\sigma,E',P}(v(I))\,. \tag{4.8}$$

Then we have two criteria for establishing relations among Arf functions

Proposition 54. *Let σ, E' and P as above. If for all $v \in (\mathrm{GF}_2)^{hk}$ we have $A_{\sigma,E',P}(v) = 0$, then $A(\sigma) = 0$. The same holds for $\overline{A}$.*

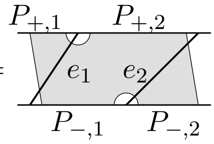
Proposition 55. *Let σ and τ be permutations (possibly of different size), with edge sets $E \cup E'_\sigma$ and $E \cup E'_\tau$ respectively. Let $P = (P_-, P_+)$ be a pair of partitions of size $|E|$ compatible with both E'_σ and E'_τ. If there exists $K \in \mathbb{Q}$ such that for all $v \in (\mathrm{GF}_2)^{hk}$ we have $A_{\sigma,E'_\sigma,P}(v) = K\, A_{\tau,E'_\tau,P}(v)$, then $A(\sigma) = K\, A(\tau)$. The same holds with one or both of the A's replaced by $\overline{A}$. Analogous statements hold for linear combinations of Arf functions associated to more than two configurations.*[4]

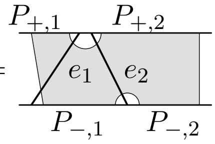
The proof of these propositions is an immediate consequence of equations (4.7) and (4.8).

For this purpose of our main classification theorems, we need four facts which are specialisations of the propositions above. One of them establish the invariance of function $\overline{A}$ under the dynamics, the other three relate the Arf functions on configurations obtained from the "surgery operators" sketched in Section 3.1.6 and discussed in Section 4.3, and a few further, simpler manipulations.

[4]We refer here to the obvious generalisation, of the form, for σ_j having edge-set $E \cup E'_{\sigma_j}$, $P = (P_-, P_+)$ a pair of partitions of size $|E|$ compatible with all E'_{σ_j}'s, if there exist $K_j \in \mathbb{Q}$ s.t. for all $v \in (\mathrm{GF}_2)^{hk}$ we have $\sum_j K_j A_{\sigma_j,E'_{\sigma_j},P}(v) = 0$, then $\sum_j K_j A(\sigma_j) = 0$, and similarly with some of the A's replaced by $\overline{A}$'s.

Proposition 56 (Invariance of the sign)**.**

$$\overline{A}\Bigg(\tau = $$ 

$$\Bigg) = \overline{A}\Bigg(\sigma = $$ 

$$\Bigg) \tag{4.9}$$

Proof. We have in this case

$$Q_\tau = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}; \qquad\qquad Q_\sigma = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}; \tag{4.10}$$

Checking that the conditions of Proposition 55 are met, with $K = 1$, is a straightforward calculation. The resulting function of v for τ is

$$\begin{aligned}
\overline{A}_{\tau,\{e_1,e_2\},P}(v) &= \sum_{u \in (\mathrm{GF}_2)^2} (-1)^{|u| + \chi_u + (u, Qv)} \\
&= (-1)^{0+0+0} + (-1)^{1+0+(0\ 1\ 0\ 1)\cdot v} + (-1)^{1+0+(1\ 1\ 0\ 0)\cdot v} \\
&\qquad\qquad + (-1)^{0+1+((0\ 1\ 0\ 1)+(1\ 1\ 0\ 0))\cdot v} \\
&= 1 - (-1)^{(0\ 1\ 0\ 1)\cdot v} - (-1)^{(1\ 1\ 0\ 0)\cdot v} - (-1)^{(1\ 0\ 0\ 1)\cdot v}
\end{aligned}$$

and likewise the resulting function of v for σ is

$$\begin{aligned}
\overline{A}_{\sigma,\{e_1,e_2\},P}(v) &= \sum_{u \in (\mathrm{GF}_2)^2} (-1)^{|u| + \chi_u + (u, Qv)} \\
&= (-1)^{0+0+0} + (-1)^{1+0+(0\ 1\ 0\ 1)\cdot v} + (-1)^{1+0+(1\ 0\ 0\ 1)\cdot v} \\
&\qquad\qquad + (-1)^{0+1+((0\ 1\ 0\ 1)+(1\ 0\ 0\ 1))\cdot v} \\
&= 1 - (-1)^{(0\ 1\ 0\ 1)\cdot v} - (-1)^{(1\ 0\ 0\ 1)\cdot v} - (-1)^{(1\ 1\ 0\ 0)\cdot v}
\end{aligned}$$

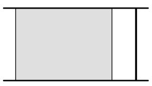
thus we have

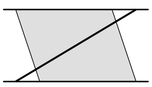
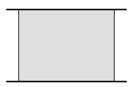
$$\overline{A}_{\tau,\{e_1,e_2\},P}(v) = \overline{A}_{\sigma,\{e_1,e_2\},P}(v) \qquad \text{for all } v \in \{0,1\}^4. \tag{4.11}$$

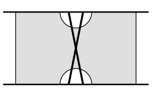
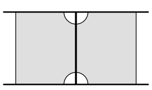
This proposition implies the invariance of $\overline{A}$ under the operation L, as $\tau = L\sigma$. The invariance under R, L' and R' is deduced from the symmetry of the definition of A and $\overline{A}$ under the dihedral group on permutation diagrams.

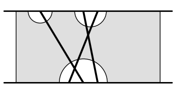
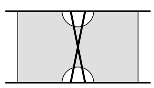
It is convenient to introduce the notation $\vec{A}(\sigma) = \left(\begin{smallmatrix} \overline{A}(\sigma) \\ A(\sigma) \end{smallmatrix}\right)$. We have

Proposition 57.

$$\vec{A}\Bigg($$  $$\Bigg) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{A}\Bigg($$  $$\Bigg); \tag{4.12}$$

$$\vec{A}\Bigg($$  $$\Bigg) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \vec{A}\Bigg($$  $$\Bigg); \tag{4.13}$$

$$\vec{A}\Bigg($$  $$\Bigg) = 2\vec{A}\Bigg($$  $$\Bigg); \tag{4.14}$$

$$\vec{A}\Bigg($$  $$\Bigg) = \vec{A}\Bigg($$  $$\Bigg). \tag{4.15}$$

These relations are all straightforward applications of Proposition 55, with matrices Q of rather small dimension. Equation (7.32), involving a larger matrix Q, may be derived with a shortcut: on the LHS, the sum over $u \in (\mathrm{GF}_2)^3$ contains the four terms contributing to the RHS, where the left-most edge is absent, and four other terms, which cancel pairwise for obvious reasons, even with no need for analysing the matrices Q in detail. $\square$

Equation (4.12) implies

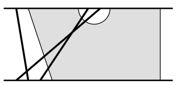
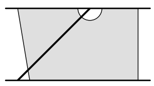
Corollary 58.

$$\vec{A}(\mathrm{id}_n) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2^{\frac{n+1}{2}} \begin{pmatrix} \cos\frac{(n+1)\pi}{4} \\ \sin\frac{(n+1)\pi}{4} \end{pmatrix}$$
$$= (-1)^{\lfloor \frac{n+1}{4} \rfloor} \begin{cases} \begin{pmatrix} 2^{\frac{n+1}{2}} \\ 0 \end{pmatrix} & n \equiv -1 \pmod 4; \\ \begin{pmatrix} 2^{\frac{n}{2}} \\ 2^{\frac{n}{2}} \end{pmatrix} & n \equiv 0 \pmod 4; \\ \begin{pmatrix} 0 \\ 2^{\frac{n+1}{2}} \end{pmatrix} & n \equiv 1 \pmod 4; \\ \begin{pmatrix} -2^{\frac{n}{2}} \\ 2^{\frac{n}{2}} \end{pmatrix} & n \equiv 2 \pmod 4; \end{cases} \tag{4.16}$$

In particular, $\overline{A}(\mathrm{id}_4) = -4 < 0$ and $\overline{A}(\mathrm{id}_6) = 8 > 0$, this fact is used in section 4.4.2.

Equations (7.23) and (7.32) imply

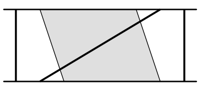
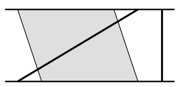
Corollary 59.

$\overline{A}\Big(\sigma =$  $\Big) = 2\overline{A}\Big(\tau =$  $\Big)$ (4.17)

This corollary states that, for the operator T defined in Section 4.3.1, $\overline{A}(T\tau) = 2\,\overline{A}(\tau)$.

Equations (4.12) and (7.22) imply

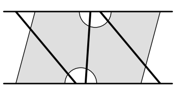
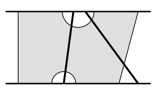
Corollary 60.

$\overline{A}\Big(\sigma =$  $\Big) = 2\,\overline{A}\Big(\tau =$  $\Big)$. (4.18)

This corollary states that, for the operator q_1 defined in Section 4.3.2, $\overline{A}(q_1\tau) = 2\,\overline{A}(\tau)$, because in fact $q_1\sigma = R\tau$.

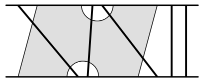
More generally, we have

Proposition 61.

$\vec{A}\Big(\sigma =$  $\Big) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{A}\Big(\tau =$  $\Big)$. (4.19)

that gives, using twice equation (4.12) in Proposition 57,

Corollary 62.

$$\overline{A}\Big(\ $$  $$\ \Big) = 0\,. \tag{4.20}$$

This corollary states that, for the operator q_2 defined in Section 4.3.2, $\overline{A}(q_2\tau) = 0$.

The proof of Proposition 61 is done again by applying Proposition 55. We have in this case

$$Q_\sigma = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\,, \qquad\qquad Q_\tau = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\,, \tag{4.21}$$

and in particular $Q_\sigma = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) Q_\tau$. Checking that the conditions of Proposition 55 are met, with $K = 2$ and 0 in the two cases, is a straightforward calculation. The resulting function $\overline{A}$ of v, as calculated for the RHS, is

$$1 + (-1)^{1+(\,1\;0\;0\;1\,)\cdot v} + (-1)^{1+(\,1\;0\;1\;0\,)\cdot v} + (-1)^{3+(\,0\;0\;1\;1\,)\cdot v} \tag{4.22}$$

while for the LHS we have

$$\begin{aligned} 1 &+ (-1)^{1+(\,0\;0\;1\;1\,)\cdot v} + (-1)^{1+(\,1\;0\;0\;1\,)\cdot v} + (-1)^{1+(\,1\;0\;1\;0\,)\cdot v} \\ &+ (-1)^{3+(\,1\;0\;1\;0\,)\cdot v} + (-1)^{3+(\,1\;0\;0\;1\,)\cdot v} + (-1)^{3+(\,0\;0\;1\;1\,)\cdot v} + (-1)^{6} \end{aligned} \tag{4.23}$$

which, after some simplifications, reduces to twice the expression in (4.22).

The calculation for function A is analogous. For example, for the LHS we have

$$\begin{aligned} 1 &+ (-1)^{(\,0\;0\;1\;1\,)\cdot v} + (-1)^{(\,1\;0\;0\;1\,)\cdot v} + (-1)^{(\,1\;0\;1\;0\,)\cdot v} \\ &+ (-1)^{1+(\,1\;0\;1\;0\,)\cdot v} + (-1)^{1+(\,1\;0\;0\;1\,)\cdot v} + (-1)^{1+(\,0\;0\;1\;1\,)\cdot v} + (-1)^{3}\,, \end{aligned} \tag{4.24}$$

an expression which is identically zero. □

4.3 Surgery operators

In this section we introduce and study the 'surgery operators' which have been outlined in Section 3.1.6, and whose geometric interpretation has been described in Section 3.2.3. These operators have a crucial role in the proof of the classification theorem. In particular, we introduce the notion:

Definition 63 (Pullback function). *Let $f : X_n \to X_{n+1}$ be a function, $\sim$ be an equivalence relation on X_n, with classes Y_n, and compatible with f, and let $\tilde{f}$ be the function which leads to the commuting diagram*

$$\begin{array}{ccc} X_n & \xrightarrow{\;f\;} & X_{n+1} \\ \big\downarrow{\scriptstyle \cdot/\sim} & & \big\downarrow{\scriptstyle \cdot/\sim} \\ Y_n & \xrightarrow{\;\tilde{f}\;} & Y_{n+1} \end{array} \tag{4.25}$$

Call $Y(x)$ the function which associates to each $x \in X_n$ its class in Y_n. We say that f is a pullback function *if, for all $x' \in X_{n+1}$, and for all $y \in Y_n$ such that $\tilde{f}(y) = Y(x)$, there exists a $x \in X_n$ such that $Y(x) = y$ and $f(x) = x'$.*

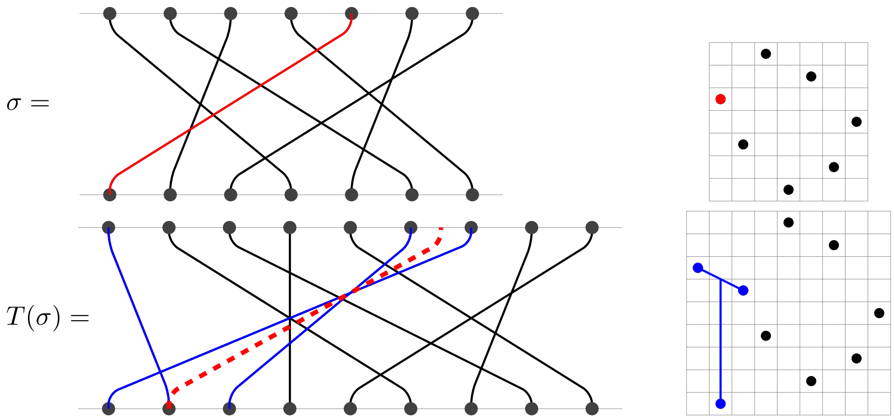


Figure 4.12: An example of the T operator

In other words, each diagram on the left can be completed to the diagram on the right:

$$
\forall \quad \begin{array}{ccc} & & x' \\ & & \big\downarrow{\scriptstyle \cdot/\sim} \\ y & \xrightarrow{\tilde{f}} & y' \end{array} \qquad\qquad \exists\, x \text{ s.t.} \quad \begin{array}{ccc} x & \xrightarrow{f} & x' \\ \big\downarrow{\scriptstyle \cdot/\sim} & & \big\downarrow{\scriptstyle \cdot/\sim} \\ y & \xrightarrow{\tilde{f}} & y' \end{array} \tag{4.26}
$$

Our operators $\bar{T}$, $\bar{q}_1$ and $\bar{q}_2$ are functions on the sets of (non-empty) classes of a give size, and with rank in a certain range. In Theorems 67, 79 and 80 we will establish that they are pullback functions w.r.t. the equivalence relation given by the invariant.

4.3.1 Operator T

We define a first operator, in terms of the 'square constructors' defined in Section 4.1.5.

Definition 64. *We define the T operator $T : \mathfrak{S}_n \to \mathfrak{S}_{n+2}$ as $T = C^{\mathrm{col}-1}_{\tau,0,1}$ with $\tau = (\,\overline{\bullet\,|}\!\!\!\top\,)$.*

Figure 4.12 illustrates this with an example.

Remark 65. *From Corollary 51, we know that if $\sigma \sim \sigma'$, then $T(\sigma) \sim T(\sigma')$.*

As a consequence, we can define $\bar{T}$ as the map from classes at size n to classes at size $n+2$, such that $\bar{T}(C) = C'$ if there exists $\sigma \in C$ with $T(\sigma) \in C'$.

Lemma 66. *Let C be a class with invariant (λ, r, s). Then $\bar{T}(C)$ has invariant $(\lambda, r+2, s)$.*

Proof. In light of Corollary 59, we know that the sign invariant does not change. For what concerns the cycle invariant, Figure 4.13 illustrates our claim, which holds more generally also for the constructor $C^{\mathrm{col}-\ell}_{\tau,0,1}$, with arbitrary ℓ. $\square$

Finally, as outlined in Section 3.1.6, we need to establish the following crucial property:

Theorem 67 (Pullback of T)**.** *For every non-exceptional class C at size $n+2$ with invariant (λ, r, s), and $r > 2$, there exists an irreducible σ at size n such that $T(\sigma) \in C$. Or equivalently, $\bar{T}$ is surjective onto the set of (primitive irreducible) non-exceptional classes with rank larger than 2, and more specifically it is a pullback function.*

The proof of this theorem takes the largest part of this subsection. Before starting this proof, we need to set some notation for dealing with reduced permutations, as explained in Section 4.1.4, which are of arbitrary size, but with a finite number of 'black' edges (and an arbitrarily large number of 'gray' ones).

In the in-line notation for permutations – i.e., the string $(\sigma(1), \dots, \sigma(n))$ – we will use the notation $*$ for a sublist of arbitrary length, possibly zero, and the symbol n to denote the maximal element, while indices such as h, k, ... will be used for generic elements. For example, the permutation $\sigma = (1, 4, 2, 5, 6, 8, 9, 3, 7)$ is one of those of the form $(1, *, k, k+1, *, n, *)$, because it starts with 1, the consecutive 5 and 6 are candidate for k and $k+1$, and $n = 9$ comes after them. Note that, when wild characters like k are used, a permutation can be of a given form in more than one way (e.g., this would have been the case for $\sigma = (1, 4, 2, 5, 6, 7, 9, 3, 8)$, as both 5 and 6 are possible choices for k). In some specially taylored expressions this never happens. This would have been the case, for example, for the pattern $(1, *, 2, k, k+1, *, n, *)$ as the explicit 2 forces k to admit at most one realisation.

Notation 1. *We want to represent graphically permutations in reduced dynamics, explicitating only the pattern τ of black edges, and, of the auxiliary structure $\hat{c}$, which pairs of consecutive black-edge endpoints have no gray endpoints in between them. In a matrix diagram, we use red lines between consecutive rows and columns*[5] *to denote the certified absence of gray points (in absence of a red line, there may or may not be gray points in between).*

Such a data structure, when "not too big", is also conveniently encoded by an in-line expression of the form mentioned above. For example, the in-line patterns

[5] And/or before the first/after the last row and column.

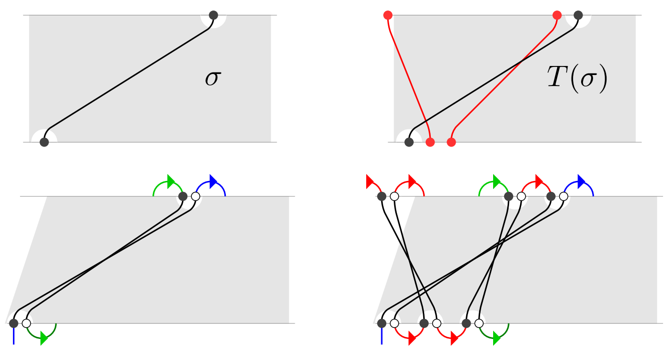
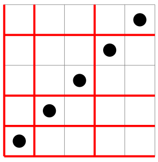
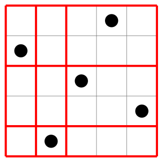


Figure 4.13: Left: the permutation σ. Right: the permutation $T\sigma$. It is clear that the only change in the cycle invariant is the addition of two to the rank length. On the bottom, the edges are drawn 'doubled', as in the diagram construction of the cycle invariant.

$(1, 2, *, 3, k, *, (k+1), *)$ and $(k, 1, (k-1), *, n, *, 2)$ are represented graphically as

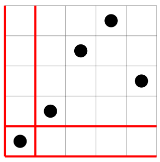
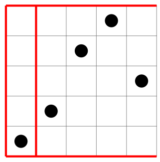
$$(1, 2, *, 3, k, *, (k+1), *) \ : \qquad\qquad (k, 1, (k-1), *, n, *, 2) \ : \tag{4.27}$$

(4.28)

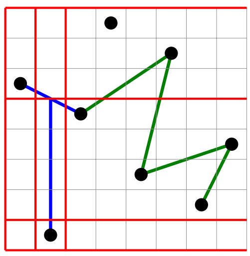
We will use extensively this graphical convention in the following proof of Theorem 67.

In particular, in this section, all our patterns are of the form $(1, 2, \dots)$, so we have two red lines at the bottom, and two at the left. This implies that we know the position of the L-pivot, which is the $\sigma(1) = 1$ entry. If (and only if) we do not have a red line on top, we do not know the position of the R-pivot. Nonetheless, up to an operator L^a for some a, we can move all the gray entries above the top-most black entry, and in turns trade the horizontal red line at position 2 for a red line on top. For example,


$\sim$


From this moment on, in the reduced dynamics we will always have a red line on the left and top side of the diagram, and a full control on the position of both pivots. I.e., in the reduced dynamics the pivots are always both 'black'.

Our aim is to prove that, for a collection of patterns covering all possible cases of the theorem (of standard non-exceptional primitive irreducible configurations of rank at least 3), in the reduced dynamics we can reach a pattern that certifies the presence of a T-structure, which, when removed, leaves with an irreducible configuration. The presence of a T-structure is easily verified (in our drawings, we use blue construction lines to evidentiate it). Irreducibility after T^{-1} is certified by a zig-zag path (that we represent in green), which reaches red lines at its endpoints (for certifying the absence of an irreducible block constituted exclusively of gray entries). For primitivity, in principle there is nothing to check, as the number of descents is preserved both by the dynamics, and by removing the T. Nonetheless, for reasons explained below, we will keep track explicitly of possible descents involving black edges. A typical "winning outcome" of the dynamics, with the associated construction lines, is as follows:



We call such a pattern a T-*structure certificate*.

The irreducibility of the outcome is not for granted. It is not uncommon that shorter and simpler candidate sequences have to be rejected for lack of this property. An example which is instructive in retrospective is the pattern associated to case 4.1 in Table 4.1. Naïvely, one could have guessed that a good choice of sequence is $\bar{R}\bar{L}^3$, just as for case 2. This indeed produces a T-structure, however the pattern resulting after removing the structure is not irreducible (we have a block of size 4, followed by one of size 2). This forced us to search for the longer sequence appearing in the table.

Patterns are stable by inclusion. If we add points to a pattern τ in correspondence of crossing of non-red lines, producing a larger pattern τ', and τ could reach a certificate as above, also τ' can, this because the relevant features of both the T-structure and the zig-zag path are related to red lines, which do not interfere with the insertion.

The relation in the other direction is slightly more involved: if τ' can reach a certificate, and a point p in τ' is neither used in the T-structure certificate, nor used as a pivot in the sequence, then $\tau = \tau' \smallsetminus p$ can reach a certificate.

We will use this relation mostly in the second form. In some cases, when we get a pattern τ' from our case analysis, instead of producing a related certificate, we will make the ansatz that a certificate exists also under a certain reduction, and produce the associated (generally smaller) certificate. These reductions will also allow to merge together a number of branches in the case analysis.

Now we can state the following

Proposition 68. *Table 4.1 lists triples (τ, τ', S) such that $S\tau = \tau'$. (The "names" for the triples are for future reference).*

Gray bullets denote arbitrary blocks, possibly empty, while gray bullets with a black bullet inside denote arbitrary non-empty blocks.

All τ' are T-structure certificates, with the exception of pattern 7, which has a zig-zag path if and only if at least one of the two gray bullets is a non-empty block.

All sequences S have alternation length at most 6, all zig-zag paths in the certificates have length at most 5.

Curiously, as a corollary (not useful at our purposes), we get that patterns 3.1 and 4.1 are connected, as well as 5.1, 5.2 and 6.1, fact that was not obvious *a priori*.

We are now ready for proving our theorem.

Proof of Theorem 67. Our aim is to prove that, for each irreducible primitive non-exceptional class C of rank at least 3, there exists a configuration σ' in the image of the operator T, such that its preimage is also primitive and irreducible.

The proof will go as follows: we describe how to break the problem into a finite number of cases, ameaneable to reduced dynamics, and in fact each of a form considered in Proposition 68. To this end it is convenient to assume that, to start with, we have some reference configuration σ which is standard (in the sense of Section 4.1.2). In particular, its in-line expression starts with $(1, 2, \dots)$. This is legitimate because, from the results of Section 4.1.2 (see Lemma 39), we know that each irreducible class has at least one standard family. As a matter of fact (and for what we know from Appendix C) the only exceptional configuration of this form of rank at least 3 is id_n. Our case analysis is tree-like, and we will see that this case emerge from *a single branch* of our tree, though this was not granted in advance. As it was

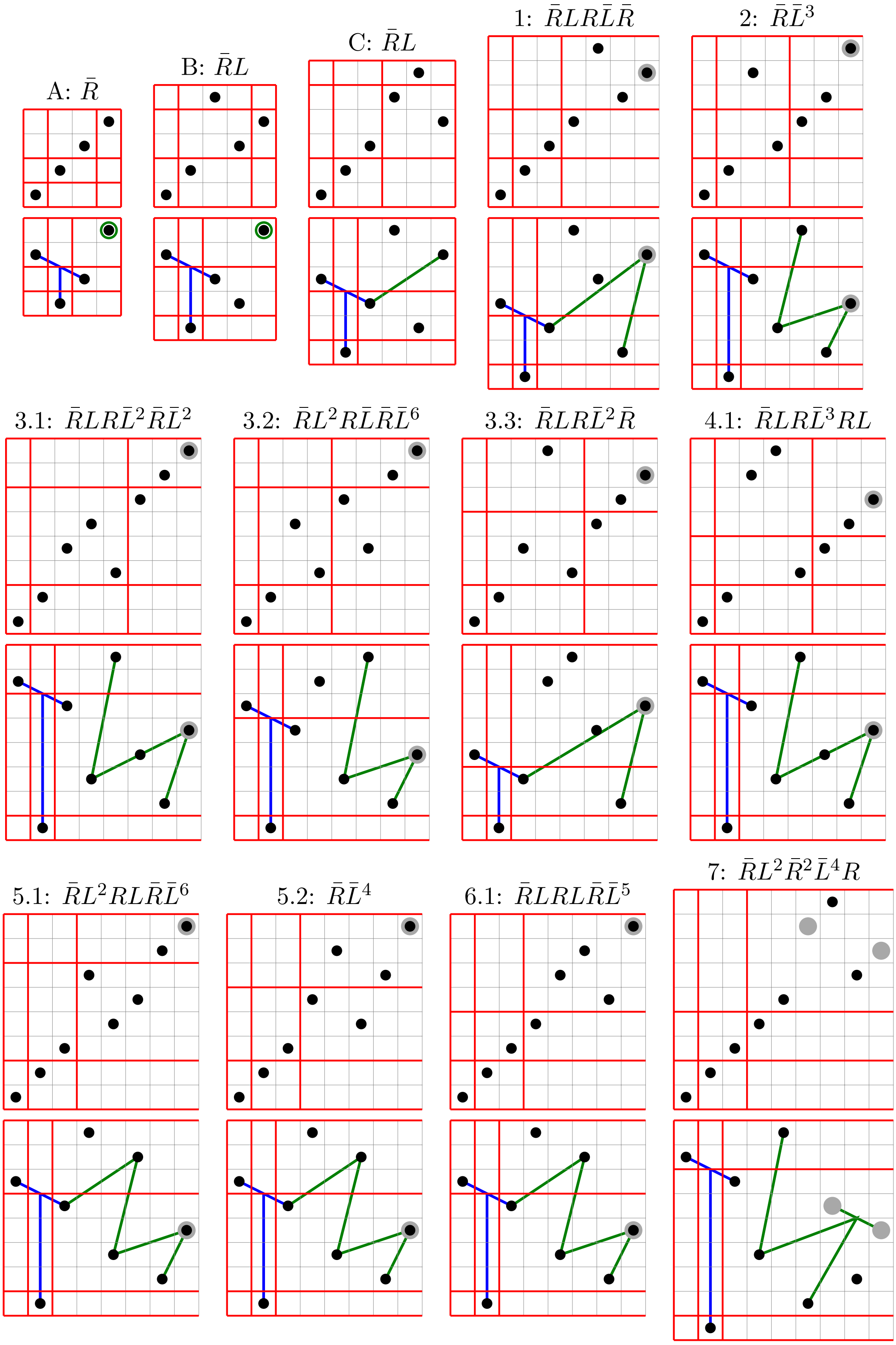


Table 4.1: Triples as is Proposition 68.

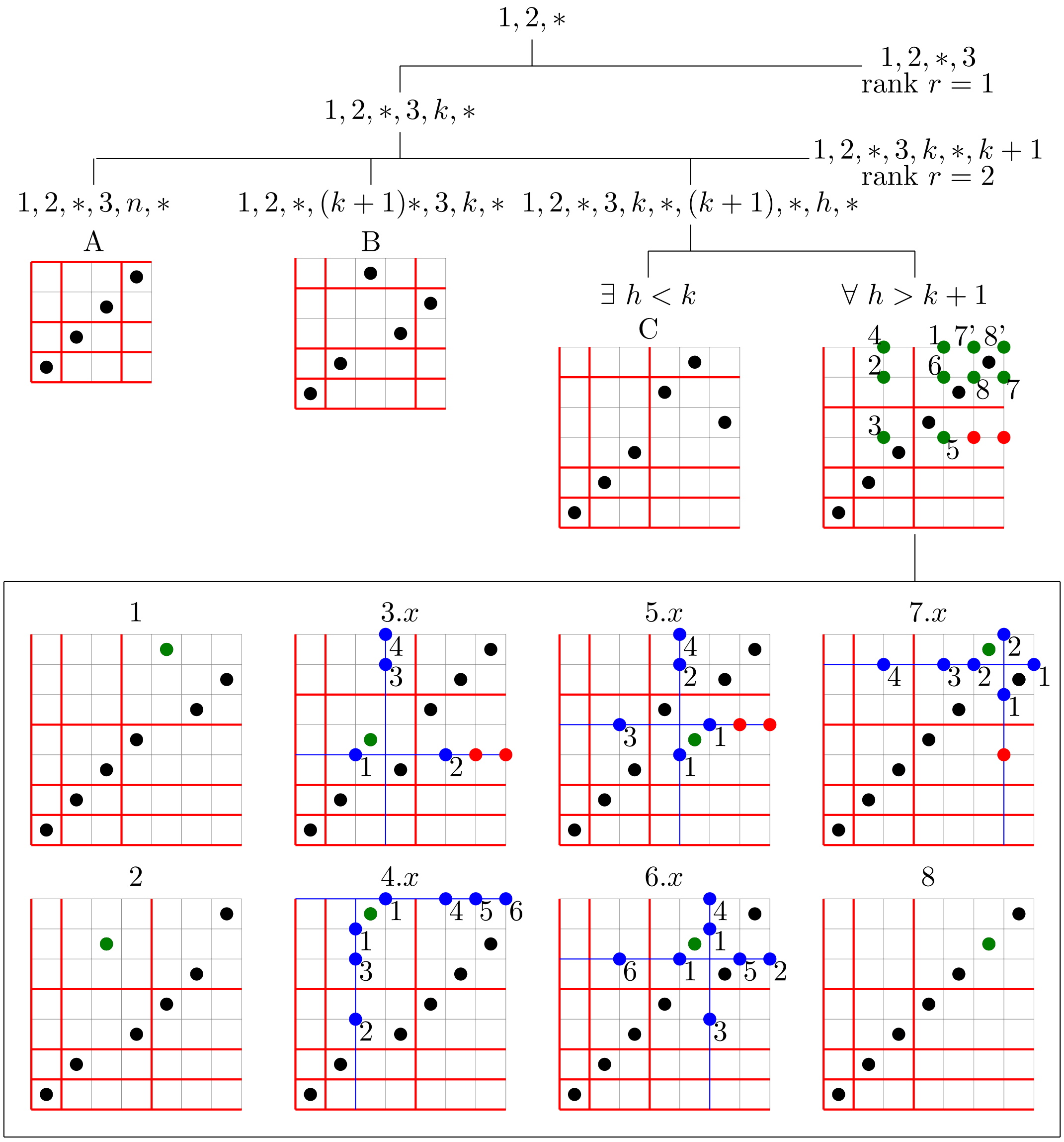


Figure 4.14: Decomposition of cases in the proof of Theorem 67. Numbers on blue bullets correspond to different values of x in case label.

obvious that it had to come from at least one branch, the relevant feature is that the tree is finite, and in particular that there is a finite number of branches leading to id_n configurations (instead, e.g., of one branch per value of n), which would have made the reasoning more cumbersome, if not impractical.

Figure 4.14 summarises the main steps of the case decomposition, which we also describe in words in the following paragraphs.

As we said, by Lemma 39, we can suppose that σ is a standard permutation with $\sigma(2) = 2$, i.e., in in-line notation, with the pattern $\sigma = (1, 2, *)$. We now investigate the possible preimages of 3.

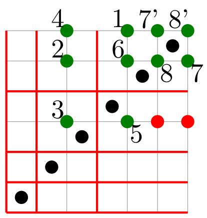
1. If 3 is at the end i.e. $\sigma = (1, 2, *, 3)$, then σ has rank 1 and the theorem does not apply.

2. Otherwise $\sigma = 1, 2, *, 3, k, *$ for some $k > 3$.
   - 2.1. If $k = n$ i.e. $\sigma = (1, 2, *, 3, n, *)$, we are in case 'A' of Figure 4.14, which is solved by the homonymous case in Proposition 68.
   - 2.2. Otherwise $k < n$ and we can investigate the possible positions of the image of $k + 1$.
     - 2.2.1. If $\sigma = (1, 2, *, (k+1), *, 3, k, *)$, we are in case 'B', which yet again is solved by the homonymous case in Proposition 68.
     - 2.2.2. If $\sigma = (1, 2, *, 3, k, *, (k+1))$ then σ has rank 2 and the theorem does not apply.
     - 2.2.3. Otherwise $\sigma = (1, 2, *, 3, k, *, (k+1), *, h, *)$ for one or more h:
       - 2.2.3.1. If there exists such a h with $3 < h < k$, we are in case 'C'.
       - 2.2.3.2. Otherwise, we shall analyse with care the case $\sigma = (1, 2, *, 3, k, *, (k+1), *, h, *)$ with all entries after $k + 1$ being larger than $k + 1$.

We have reached a case that requires a better control on the extra data structure $\hat{c}$ of the reduced dynamics. At all intersections of non-red lines, we may or may not have a non-empty block of points. We will take some order on these points, and continue a case analysis of the form "given that the first $a - 1$ blocks are empty, and the a-th one has at least one point,. . . ".

In some of these cases, adding one such block produces a pattern which may have a descent involving the black points, thus we need to further refine the analysis with a further entry certifying the absence of such descents. Although not necessary by itself (as we control the number of descents in certificates), this is important for the following reason: differently from the primitive case, at every given size there are several non-primitive exceptional standard configurations with rank at least 3, which thus would proliferate in all sorts of branches in the decomposition tree, making the analysis inconclusive. It is restriction to primitive configurations that allows to concentrate the identity configurations of all sizes into a single branch (indeed, a result of Appendix C is that id_n is the only standard permutation in Id_n starting with $(1, 2, *)$).

The following picture, which is the pertinent crucial node in the tree of Figure 4.14, represents the position of the blocks, by green bullets. The red bullets correspond to blocks which are certified to be empty, by the fact that, as we said, all entries at the right of $k + 1$ are larger than $k + 1$.



We have ten candidate blocks, labeled $\{1, 2, \ldots, 7, 8; 7', 8'\}$, that we now analyse. First of all, it is easily seen that cases $7'$ and $8'$ coincide, up to relabeling, to cases 7 and 8, respectively. Then, cases 1 and 2 can be analysed in one stroke, by adding one entry in the corresponding block, while cases from 3 to 7 require a finer analysis in order to ensure primitivity, and thus involve the insertion of two entries, one in the corresponding block, the other one in an available block among those splitting the descent. We have thus several cases, labeled from 3.1 to 7.4 (4 cases for blocks 3, 5, 7 and 6 cases for blocks 4 and 6). The case of block 8 is the one that reduces to the sole configuration id_n.

All cases from 1 to 7.4 correspond to some case appearing in Proposition 68 (which explains the naming of cases in the proposition). Sometimes this occurs after dropping one entry. The following scheme summarizes this correspondence:

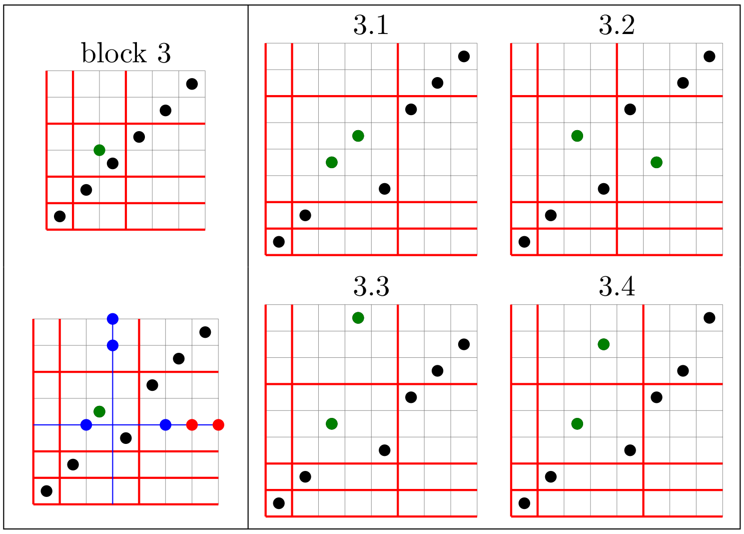
$$
\begin{array}{ll}
1 & \langle \mathbf{1} \rangle \\
2 & \langle \mathbf{2} \rangle \\
3 & \left\{ \begin{array}{lll} 3.1 & & \langle \mathbf{3.1} \rangle \\ 3.2 & & \langle \mathbf{3.2} \rangle \\ 3.3 & & \langle \mathbf{3.3} \rangle \\ 3.4 & \text{[drop col 3]} & \langle 2 \rangle \end{array} \right. \\
4 & \left\{ \begin{array}{lll} 4.1 & & \langle \mathbf{4.1} \rangle \\ 4.2 & & \langle 3.3 \rangle \\ 4.3 & \text{[drop col 4]} & \langle 2 \rangle \\ 4.4 & \text{[drop col 3]} & \langle 1 \rangle \\ 4.5 & \text{[drop col 8]} & \langle 2 \rangle \\ 4.6 & \text{[drop col 7]} & \langle 2 \rangle \end{array} \right.
\end{array}
\qquad
\begin{array}{ll}
5 & \left\{ \begin{array}{lll} 5.1 & & \langle \mathbf{5.1} \rangle \\ 5.2 & & \langle \mathbf{5.2} \rangle \\ 5.3 & & \langle 3.2 \rangle \\ 5.4 & \text{[drop col 6]} & \langle 1 \rangle \end{array} \right. \\
6 & \left\{ \begin{array}{lll} 6.1 & & \langle \mathbf{6.1} \rangle \\ 6.2 & \text{[drop col 7]} & \langle 1 \rangle \\ 6.3 & & \langle 5.2 \rangle \\ 6.4 & \text{[drop col 5]} & \langle 1 \rangle \\ 6.5 & \text{[drop col 8]} & \langle 1 \rangle \\ 6.6 & \text{[drop col 6]} & \langle 2 \rangle \end{array} \right. \\
7 & \left\{ \begin{array}{lll} 7.1 & & \langle \mathbf{7} \rangle \\ 7.2 & & \langle \mathbf{7} \rangle \\ 7.3 & & \langle 6.4 \rangle \\ 7.4 & \text{[drop col 8]} & \langle 2 \rangle \end{array} \right.
\end{array}
$$

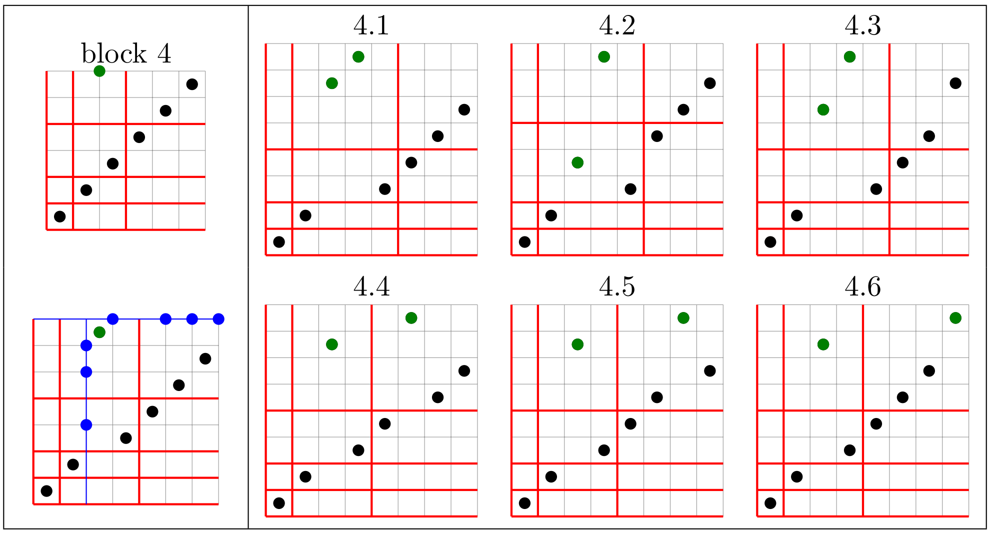
On the left column, we have the case in the decomposition. Possibly, the annotation "[drop col $\cdot$]" specifies that we have to drop one entry, in the given column. In "$\langle \cdot \rangle$" we put the label of Table 4.1 (in boldface, if it is the first appearence in the list).

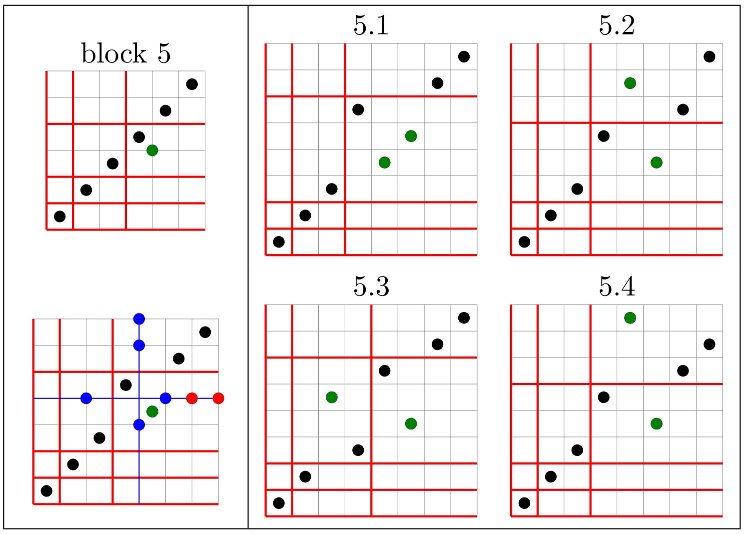
The two cases 7.1 and 7.2 correspond to pattern 7 in which one or the other gray block is certified to be non-empty, which is a sufficient condition for having a zig-zag path certificate.

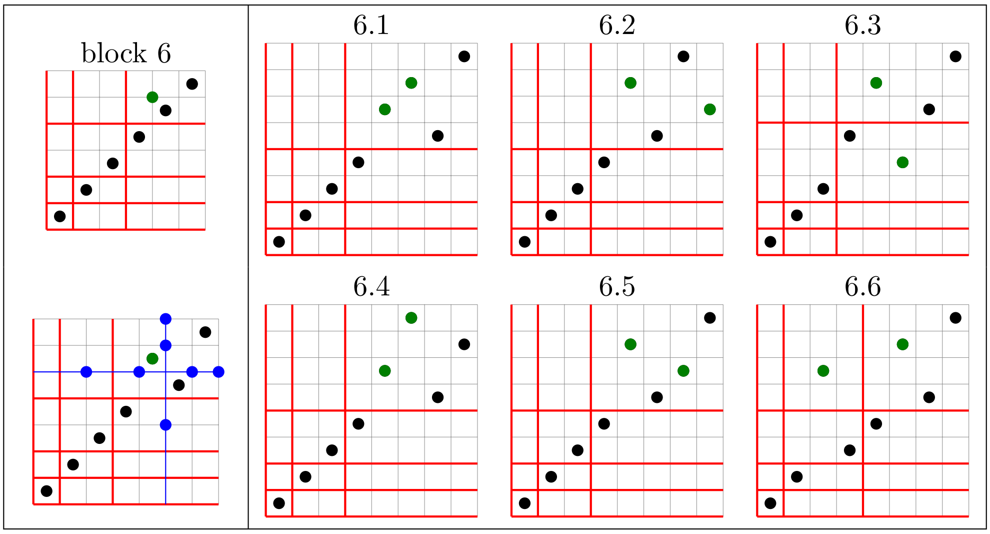
The lower part of Figure 4.14 specifies which is which among all subcases 3.1, …, 7.4 associated to blocks 3 to 7 (the same information is also carried in a more detailed way in the tables at the end of this subsection: we remind the configuration associated to the block, analyse the possible positions for a further entry splitting the descent, using blue construction-lines, and then list the corresponding cases).

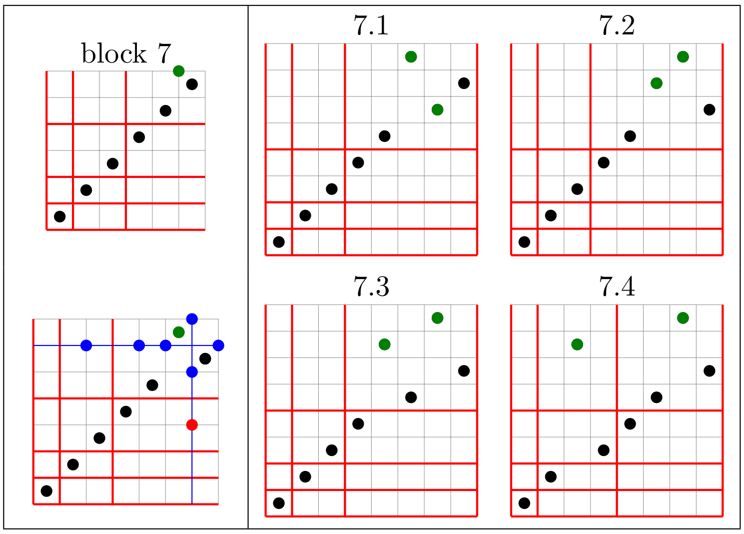
As we anticipated, if all blocks $\{1, 2, \dots, 7; 7'\}$ are empty, and only blocks 8 (and $8'$) are possibly non-empty, then the permutation is id_n for $n \geq 6$, and the theorem does not apply, since the class Id_n is exceptional. $\square$


block 3
3.1
3.2
3.3
3.4


block 4
4.1
4.2
4.3
4.4
4.5
4.6


block 5
5.1
5.2
5.3
5.4





4.3.2 Operators q_1 and q_2

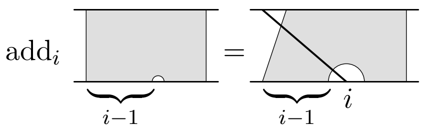
We have seen that $\bar{T}\left(\mathfrak{S}_{n-2}^{(\mathrm{prim})}\right) \to \mathfrak{S}_{n}^{(\mathrm{prim})}|_{\mathrm{rank}>2}$, it is surjective on this set, and is a pullback function. In order to produce a complete decomposition, we need to introduce two more operators, q_1 and q_2, such that

$$\bar{q_1}\left(\mathfrak{S}_{n-1}^{(\mathrm{prim})}\right) \to \mathfrak{S}_{n}^{(\mathrm{prim})}|_{\mathrm{rank}=1}\,; \qquad \bar{q_2}\left(\mathfrak{S}_{n-1}^{(\mathrm{prim})}\right) \to \mathfrak{S}_{n}^{(\mathrm{prim})}|_{\mathrm{rank}=2}\,. \tag{4.29}$$

We do this in this section.

Definition 69. *Let σ be a permutation of size n. We define $\mathrm{add}_i(\sigma)$ for $i = 1, \dots, n$ to be the permutation of size $n+1$ $\mathrm{add}_i(\sigma) := \sigma(1)+1, \dots, \sigma(i-1)+1, 1, \sigma(i)+1, \dots, \sigma(n)+1$.*

I.e., $\mathrm{add}_i(\sigma)$ is described by the diagrammatic manipulation below:



The two operators q_1 and q_2 are defined in terms of add_i's for suitable i's. In order to endow them with the appropriate properties, we shall investigate the difference between the cycle invariants of σ and $\mathrm{add}_i(\sigma)$. This requires a case analysis based on the 'type' of permutations (among H- and X-type, see Definition 43). This is presented in Table 4.2, and illustrated in the following paragraph.

Proposition 70. *Let $i \in \{1,2\}$. Let σ be a primitive permutation with invariant (λ, r) with $r > i$.*

- *If σ has type $X(r,j)$ then there exists exactly one index ℓ such that* $\mathrm{rank}(\mathrm{add}_\ell(\sigma)) = i$.
- *If σ has type $H(r_1,r_2)$ with $r_2 \neq i$ then there exists exactly one index ℓ such that* $\mathrm{rank}(\mathrm{add}_\ell(\sigma)) = i$.
- *If σ has type $H(r_1,r_2)$ with $r_2 = i$ then there exist exactly two indices $\{\ell, m\} = \{1, \sigma^{-1}(n)\}$ such that* $\mathrm{rank}(\mathrm{add}_\ell(\sigma)) = \mathrm{rank}(\mathrm{add}_m(\sigma)) = i$*; moreover* $\mathrm{add}_\ell(\sigma)$ *and* $\mathrm{add}_m(\sigma)$ *are in the same class, as* $\mathrm{add}_\ell(\sigma) = R^{-1}(\mathrm{add}_m(\sigma))$.

Proof. This emerges from the analysis of Table 4.2. We discuss the three propositions one by one, with reference to the 6 rows of the table.

- If $X(r,j)$, it can be checked that the only possibility for having rank i is case ④ with $s = r - i$. The reason why we need $r > i$ is because, e.g. if $r = 1$, either $s = 0$ or 1, but both eventualities are impossible, since cycles of length 0 do not exist and cycles of length 1 are not allowed in primitive permutations.
- If $H(r_1,r_2)$ with $r_2 \neq i$, it can be checked that the only possibility is case ③ with $s = r_2 - i$, if $r_2 > i$, and case ① with $s = r_1$, if $r_2 = 1$. (In fact, the only case with $r_2 < i$ is when $i = 2$ and $r_2 = 1$).
- If $H(r_1,r_2)$ with $r_2 = i$, then there are exactly two possibilities: case ③ of the table with $s = 0$, and case ① with $s = r_1$. Clearly we go from one target configuration to another with the operator $R^{\pm 1}$, since these only differ by the edge $(\alpha, 1)$ which is $(1,1)$ in case ③ and $(\sigma^{-1}(n), 1)$ in case ①.

□

The previous proposition shows that for configurations σ with rank i, there exists one and only one configuration τ and index ℓ such that $\sigma = \mathrm{add}_\ell(\tau)$, with an exceptional case in which there are two such configurations, which are however easily shown to be in the same class. We can thus just "break the tie" for the third situation in Proposition 70 (see Figure 4.15 for an illustration), and give the following definition:

Definition 71. *Let $i \in \{1,2\}$. Let τ be a primitive permutation with rank $r > i$. Define $q_i(\tau)$ to be the unique* $\mathrm{add}_\ell(\tau)$ *such that* $\mathrm{rank}(\mathrm{add}_\ell(\tau)) = i$*, if we are in one of the first two situations of Proposition 70, and $q_i(\tau) = \mathrm{add}_{\tau^{-1}(n)}(\tau)$ if we are in the third situation.*

Note that the edge $(\ell, 1)$ in $q_i(\sigma)$ is never $(1,1)$. This is true by a further investigation of the table, for the first two situations, and holds at sight for the third situation, exactly because of our choice of convention (if it were, this would mean

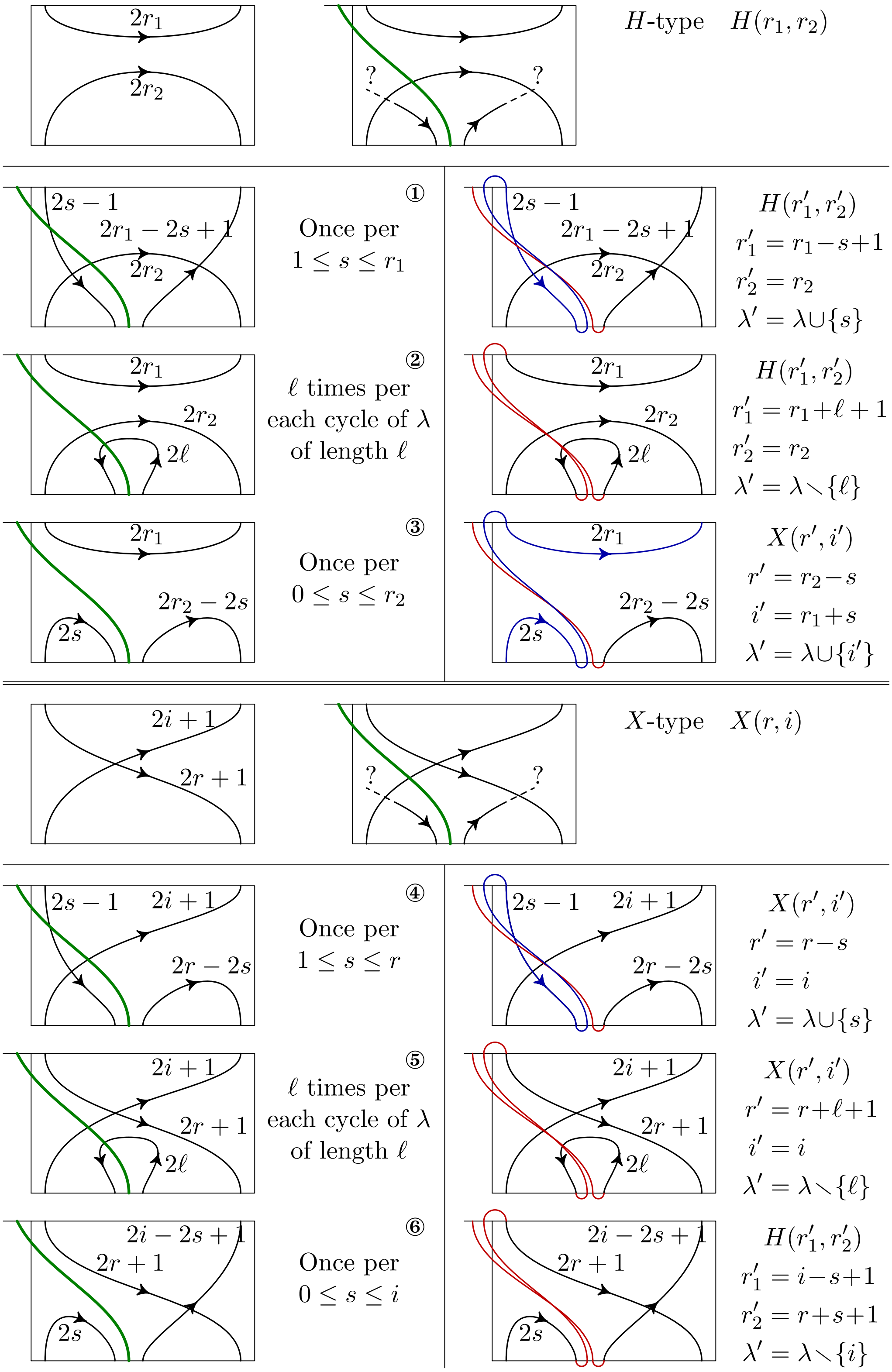


Table 4.2: Modification to the cycle invariant of a configuration produced by add_i. In green, the newly-added edge. In red, parts which get added to the rank path. In blue, parts which are singled out to form a new cycle.

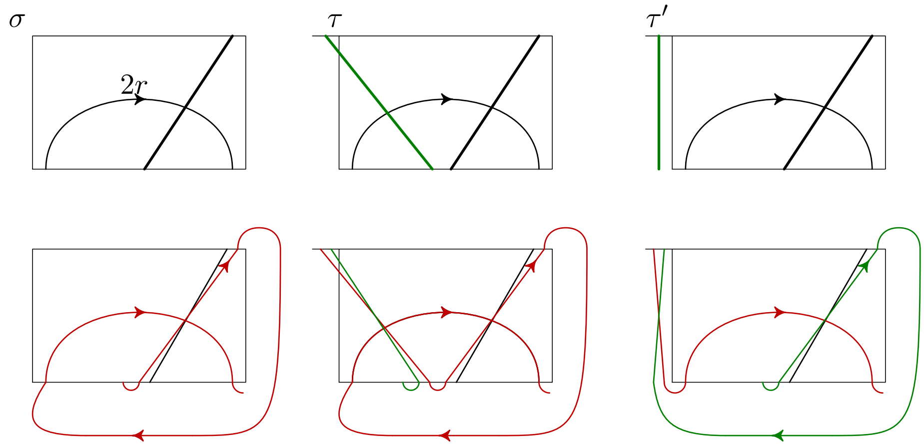


Figure 4.15: The two possible consistent definitions of q_r on a configuration σ of type $H(r', r)$, mentioned in the third case of Proposition 70. The two configurations τ and τ' are such that $\tau = R\tau'$. Our choice is to define $q_r\sigma = \tau$.

that $\sigma^{-1}(n) = 1$, which implies that σ is reducible). Thus the edge $(i, 1)$ is never a pivot, a fact that will be useful when we pass to a reduced dynamics after the action of a surgery operator q_i.

Now that we have defined our q_i's, we need to show that they satisfy the properties outlined in Section 3.1.6, i.e. the statement of the following lemma:

Lemma 72. *Let σ and σ' be configurations in the same class. Then $q_i(\sigma) \sim q_i(\sigma')$ for both $i = 1, 2$.*

Proof. Clearly both q_i's are injective. They are actually 'almost bijective', as the number of preimages is at most 2 (and, for what it matters, in large size there is a unique preimage in the majority of configurations).

Let τ and τ' be primitive permutations in the same class C, and let S be a sequence such that $S(\tau) = \tau'$. Call $\sigma = q_i(\tau)$. The reduced permutation of σ where the edge $(\ell, 1)$ is the only gray edge is τ. Since this edge is not a pivot, we can use the boosted dynamics and define $\sigma' = B(S)(\sigma)$, and we have that $\sigma' = q_i(\tau')$.

Indeed, by definition of $B(S)$, removing the edge $(\ell', 1)$ in σ' gives τ', and $(\ell', 1)$ is not a pivot, which coincides exactly with our characterisation of $q_i(\tau')$ (since σ has rank i, so has σ'). Thus $q_i(\sigma) \sim q_i(\sigma')$. $\square$

As a consequence of this lemma, analogously to what we have done for the operator T, we can define the two maps $\bar{q}_1$ and $\bar{q}_2$ on classes.

First of all, we observe the triviality

Lemma 73. $\bar{q}_1(\mathrm{Id}_n) = \mathrm{Id}'_{n+1}$.

(this is seen by direct inspection of the canonical representatives, as $q_1(\mathrm{id}_n) = R^{-2}\,\mathrm{id}'_{n+1}$).

Then, yet again, these maps have a definite behaviour for what concerns the invariants. Let us introduce the useful notation

Notation 2. *Let λ be a cycle invariant, and let $j \in \lambda$. We use $\lambda(j)$ as a shortcut for $\lambda \setminus \{j\}$.*

Then:

Lemma 74. *Let C be a class with invariant (λ, r, s) with $r > 1$. Then $C' = \bar{q}_1(C)$ has invariant $(\lambda \cup \{r\}, 1, s)$.*

Proof. The claim on the sign comes as an application of Corollary 60. Indeed, as seen in Proposition 56, the sign is a class invariant, so we can take $\sigma \in C'$ to be standard with $\sigma(n) = n$. Since σ has rank 1, we know that $\sigma(2) = n - 1$. This is the permutation σ on the LHS of Corollary 60, and removing the edge $(1, 1)$, i.e. taking the preimage of $R^{-1}\sigma$ under q_1, we obtain the permutation τ on the RHS of Corollary 60 (indeed, we have $q_1(\tau) = R\sigma$ because we are in the third case of the definition of q_1).

For what concerns cycle and rank invariants, this comes from a straighforward inspection (cf. again Table 4.2). $\square$

Corollary 75. *Let σ be a permutation with invariant $(\lambda, 1, s)$ then either $\tau := q_1^{-1}(\sigma)$ or $\tau' := q_1^{-1}(R\sigma)$ or both are defined. Each of these configurations have invariant $(\lambda(i), i, s)$ for some $i \in \lambda$.*

Lemma 76. *Let C be a class with invariant (λ, r, s) with $r > 2$. Then $\bar{q}_2(C)$ has invariant $(\lambda \cup \{r - 1\}, 2, 0)$.*

Proof. The claim on the sign comes as an application of Corollary 62. Again we can choose a standard representative with $\sigma(n) = n$. Then there must exist j such that $\sigma' = R^j(\sigma)$ has $\sigma'(n) = n$ and $\sigma'(n - 1) = n - 1$. Since σ' has rank 2, this configuration is of the form of the LHS permutation of Corollary 62, and removing the edge $(\sigma'^{-1}(1), 1)$, which corresponds to take the preimage of the surgery operator q_2, i.e. produces a permutation of smaller size (call it τ) such that $q_2(\tau) = \sigma'$ (or $q_2(\tau) = R(\sigma')$ if $\sigma'^{-1}(1) = 1$). This proves that, regardless from the sign of τ, the sign of σ' is zero.

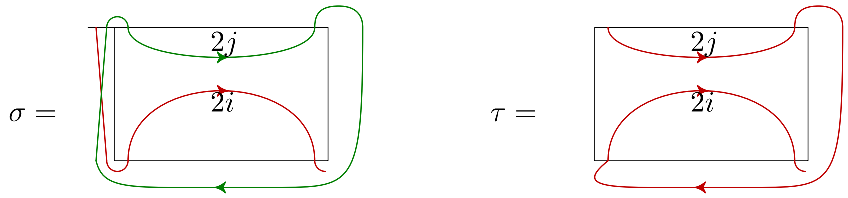
Again, for what concerns cycle and rank invariants, this comes from a straighforward inspection. $\square$

Corollary 77. *Let σ be a permutation with invariant $(\lambda, 2, 0)$ then either $q_2^{-1}(\sigma)$ or $q_2^{-1}(R\sigma)$ are defined. This configuration has invariant $(\lambda(i - 1), i, s)$ for some $i \in \lambda$ and $s \in \{-1, 0, +1\}$, more precisely,*

- *if $\lambda(i - 1)$ has a positive number of even cycles, then $s = 0$;*
- *if $\lambda(i - 1)$ has no even cycles, then $s = \pm 1$.*

At this point, we are left with the investigation of the pullback properties of our operators.

Lemma 78. *Let $\sigma = (1, \sigma(2), \ldots, \sigma(n))$ be a permutation of type $X(i, j)$ with cycle and rank invariant (λ, i). Then $\tau = (\sigma(2) - 1, \ldots, \sigma(n) - 1)$ has type $H(j, i)$ and invariant $(\lambda(j), r')$, with $r' = i + j - 1$.*



Proof. This is the reverse implication of case ③ in Table 4.2, specialised to $s = 0$. $\square$

Theorem 79 (Pullback of $\bar{q}_1$)**.** *Let C be a class with invariant $(\lambda, 1, s)$ then for every $j \in \lambda$ there exists a class B_j with invariant $(\lambda(j), j, s)$ such that $\bar{q}_1(B_j) = C$. In other words, $\bar{q}_1$ is a pullback function w.r.t. the equivalence relation given by the invariant, i.e. $C \xrightarrow{\cdot/\sim} (\lambda, r, s)$.*

Proof. First of all, let us remark that, by Corollary 75, the set of invariants $(\lambda(j), j, s)$ coincides with the set of preimages of $(\lambda, 1, s)$ under $\widetilde{\bar{q}_1}$.[6]

Now, we take a standard family of C. Then, by Lemma 46, for every $j \in \lambda$ there exists $\sigma_j = (1, \sigma_j(2) \dots)$ of type $X(1, j)$ such that $\tau_j = (\sigma_j(2) - 1, \dots, \sigma_j(n) - 1)$ is irreducible.

Moreover, τ_j has invariant $(\lambda(j), j, s)$. We know this, for cycle and rank, from Lemma 78 (in the case $i = 1$), and, for the sign, from Lemma 74, the fact that $q_1(\tau_j) = R(\sigma_j)$, and that σ_j has sign s. Thus for every j the class $B_j \ni \tau_j$ satisfies $\bar{q}_1(B_j) = C$. $\square$

For operator q_2, the facts we can establish at this point are a bit weaker, the trouble coming from the "loss of memory" on the sign, which becomes 0 regardless of what was its value on the preimage. As a result, we will need some extra work in the induction step of the next section, while at this point we establish the following:

Theorem 80 (Pullback of $\bar{q}_2$)**.** *Let C be a class with invariant $(\lambda, 2, 0)$. Then for every $j \in \lambda$ there exists a class B_j with invariant $(\lambda(j), j+1, s)$, for some s, such that $\bar{q}_2(B_j) = C$. In other words, $\bar{q}_2$ is a pullback function w.r.t. the equivalence relation given by the cycle invariant, i.e. $C \xrightarrow{\cdot/\sim} (\lambda, r)$.*

Proof. The proof is analogous to the one above (with the discussion of the sign left out).

By Corollary 77, the set of cycle invariants $(\lambda(j), j+1)$ coincides with the set of preimages of $(\lambda, 2)$ under $\widetilde{\bar{q}_2}$.

We take a standard family of C, then by Lemma 46 for every $j \in \lambda$ there exists $\sigma_j = (1, \sigma_j(2) \dots)$ with type $X(2, j)$ such that $\tau_j = (\sigma_j(2) - 1, \dots, \sigma_j(n) - 1)$ is irreducible.

Moreover, τ_j has cycle and rank invariant $(\lambda(j-1), j)$ by Lemma 78 (in the case $i = 2$). Since $q_2(\tau_j) = R(\sigma_j)$, for every j the class $B_j \ni \sigma'_j$ satisfies $\bar{q}_2(B_j) = C$. $\square$

[6] Where $\widetilde{\cdot}$ is the notion introduced in the definition of pullback function.

4.4 The induction

This section is devoted to the main induction of this part of the thesis, that implies the classification theorem. The induction mainly works at the level of non.exceptional classes, so, before doing this, we need to exclude the proliferation of distint classes with the same invariant due to the action of our surgery operators on exceptional classes. This requires to establish 'fusion lemmas', i.e. lemmas of the form, for a given exceptional class $I = \{I_n\}$, for $n \geq n_0$, there exists non-exceptional classes $C = \{C_n\}$, such that a certain surgery operator X, acting on I, gives the same class as the action on C (in formulas, $\exists\, C \;:\; X(C) \sim X(I)$). In the paragraph above, in principle, I may be Id or Id$'$, while X may be T, q_1 or q_2. However, we will see that not all of these cases need to be considered.

In proving these lemmas, we will make the ansatz that the classes C_n at different $n \geq n_0$ have one representative with a 'nice structure' in n, e.g. a certain permutation of size n_0 and an identity block of size $n - n_0$ in a specific position, and that the sequence connecting this representative to a canonical representative of the exceptional class, once written in terms of dynamics operators and their inverses (so that the variable-size identity block is never broken up along the dynamics), is the same for all sizes.

In particular, we need to establish that the forementioned representative is in fact in a class C which is non-exceptional. This can be done in two ways, either by using our knowledge of the structure of *all* configurations in the special classes Id and Id$'$, presented in Appendix C, or by using a result, also proven in Appendix C, stating that both Id and Id$'$ have a unique standard family (in particular, in each of these classes there is only one standard permutation π with $\pi(1) = 1$ and $\pi(n) = n$, namely $\pi = \mathrm{id}_n$ for Id_n and $\pi = \mathrm{id}'_n$ for Id'_n), so that, if the representative is standard, the check is straightforward. The second method is more compact, so we will adopt it in this section.

The very last lemma of this form will be based on a more general ansatz (see the pattern in the following Definition 90). The sequence[7] $w_n \in \{L, R, \bar{L}, \bar{R}\}^*$ is not the same for all $n \geq n_0$, but rather has the form $w_n = w_{\mathrm{end}} w^{n-n_0} w_{\mathrm{start}}$. Indeed, instead of having a 'large' identity block of size $n - n_0$, we have two 'large' simple blocks, of size $n - n_0 - k$ and k, and the iterated sequence S in the middle of S_n is used to change the value of k by one.

Such intelligible patterns allow to give unified proofs of these fusion lemmas which hold for all n large enough, as desired. We admit that these patterns could have hardly been guessed without the aid of computer search. We have been lucky in the respect that not only our ansatz holds, but also the sequences implementing our ansatz are the shortest ones connecting the two guessed representatives, so that, once the 'good' representatives have been found by the computer at finite size, we could invent a proof just by analysing the structure of the numerically-found geodesic path in the Cayley graph.

4.4.1 Simple properties of the invariant

Here we prove a few facts on the invariants, that we have previously stated without proof. A stronger characterisation will emerge from the main induction, however it

[7] Recall that we use the shortcuts $\bar{L}$, $\bar{R}$ for L^{-1} and R^{-1}.

is instructive to deduce the following lemma, at the light of the previous results, and with a small extra effort.

Lemma 81. *Let C be a class with invariant (λ, r, s) and let ℓ be the number of cycles (not including the rank). The following statements hold:*

1. *The list $\{r, \lambda_1, \ldots, \lambda_\ell\}$ contains an even number of even entries.*

2. *The sign of the class, $s(C) = \mathrm{Sign}(\overline{A}(C))$ can be written as $s(C) = 2^{-\frac{n+\ell}{2}}\overline{A}(C)$.*

3. *The sign $s(C)$ is zero if and only if some of the entries r and λ_j of the cycle invariant (λ, r) are even.*

Proof. First of all, the statements can be verified for the (exceptional) class of the identity, since computing (by induction) the cycle invariant (λ, r) of id_n is an easy task, while the calculation of the Arf invariant was done in Corollary 58. The results are summarised in Table 4.3, top.

Then, the statements follow for the other exceptional class, Id'_n, by using the fact that $\bar{q}_1(\mathrm{Id}_n) = \mathrm{Id}'_{n+1}$ (Lemma 73) and the action of $\bar{q}_1$ on the Arf invariant of a class (Corollary 60). The results are summarised in Table 4.3, bottom.

Finally, the statements follow inductively for all non-exceptional classes, for the three cases of rank equal to 1, 2, or at least 3, using the pullback results for the operators $\bar{q}_1$, $\bar{q}_2$ and $\bar{T}$ (Theorems 79, 80 and 67, respectively) and the action of the operators on the cycle invariant of a class (Lemma 74, 76 and 66 respectively), established in Section 4.3, and on the Arf invariant of a class (Corollary 60, 62 and 59 respectively), established in Section 4.2.

In order to illustrate how the argument works, let us analyse it in full detail in the case of operator T. Let C be a class of size n with invariant (λ, r, s), with $r \geq 3$. Then, by the pullback theorem for T (Theorem 67), there exists a class B such that $\bar{T}(B) = C$, and B shall have invariant $(\lambda, r-2, s)$ in agreement with Lemma 66. Clearly T preserves the parity of the number of even entries in the cycle invariant (only r has changed, and by 2), so, as by induction B has an even number of even cycles, this also holds for C.

By induction we know that $s(B) = 2^{-\frac{n-2+\ell}{2}}\overline{A}(B)$ where $\ell = \ell(B)$ is the number of cycles in B. However, we have just noticed that $\ell(C) = \ell(B) = \ell$. By Corollary 59, $\overline{A}(C) = 2\overline{A}(B)$, thus $2^{-\frac{n+\ell}{2}}\overline{A}(C) = 2 \times 2^{-\frac{n+\ell}{2}}\overline{A}(B) = 2^{-\frac{n-2+\ell}{2}}\overline{A}(B) = s(B)$. Hence, since we have $s(C) = s(B)$ by Lemma 66, we have consistently $s(C) = \mathrm{Sign}(\overline{A}(C)) = 2^{-\frac{n+\ell}{2}}\overline{A}(C)$.

Finally, by induction $s(B)$ is zero if and only if some of the entries among $r-2$ and λ_j are even. As we have just established that $s(C) = s(B)$ and that C has as many even cycles as B, the statement holds also for C. □

The results in Table 4.3 imply the following simple fact

Lemma 82. *For all $n \geq 4$, the cycle invariants (λ, r) of $\bar{T}(\mathrm{Id}_n)$ and of $\bar{T}(\mathrm{Id}'_n)$ are distinct.*

Proof. Just combine Table 4.3 and Lemma 66. □

n	2	3	4	5	6	7	8	9
(λ, r) of Id_n	$(\varnothing, 1)$	$(\{1\}, 1)$	$(\varnothing, 3)$	$(\{2\}, 2)$	$(\varnothing, 5)$	$(\{3\}, 3)$	$(\varnothing, 7)$	$(\{4\}, 4)$
$\overline{A}(\mathrm{Id}_n)$	-2	-4	-4	0	8	16	16	0
(λ, r) of Id'_n			$(\{1,1\}, 1)$	$(\{3\}, 1)$	$(\{2,2\}, 1)$	$(\{5\}, 1)$	$(\{3,3\}, 1)$	$(\{7\}, 1)$
$\overline{A}(\mathrm{Id}'_n)$			-8	-8	0	16	32	32

Table 4.3: The cycle and sign invariants of Id_n and Id'_n. The pattern of (λ, r) is repeated with period 2, while the pattern of $\overline{A}$ is repeated with period 8. We include also the non-primitive classes Id_2 and Id'_4, in order to illustrate the mechanism on smaller values of n.

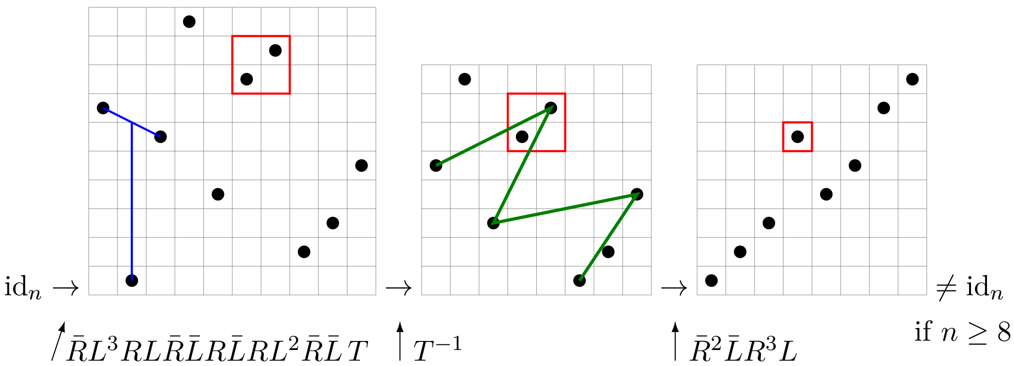


Figure 4.16: The three configurations involved in the proof of Lemma 83.

4.4.2 Fusion lemmas for T

Lemma 83 (Fusion lemma for (T, Id))**.** *Let $n \geq 8$. There exists some non-exceptional class A_n such that $\bar{T}(A_n) = \bar{T}(\mathrm{Id}_n)$.*

Proof. We just searched (by computer) a sequence such that $T(\mathrm{id}_n)$ is transformed into a $\sigma = T(\tau)$ where $\tau \notin \mathrm{Id}_n$. For $n \geq 7$ the configuration $\bar{R}L^3RL\bar{R}\bar{L}R\bar{L}RL^2\bar{R}\bar{L}\,T(\mathrm{id}_n)$ is shown in Figure 4.16, left, where the red square is an identity of size $n-6$. This configuration has a T-structure and its pre-image w.r.t. T is the configuration shown in Figure 4.16, middle. Finally, by applying the algorithm of standardization to τ, we see that $\tau_s = \bar{R}^2\bar{L}R^3L(\tau)$ is the standard permutation shown in Figure 4.16, right, where the red square is an identity of size $n-7$. When this block is non-empty, this configuration has $\tau_s(1) = 1$ and $\tau_s(n) = n$, but it is not id_n, from which we conclude. □

Lemma 84 (Fusion lemma for (T, Id'))**.** *Let $n \geq 9$. There exists some non-exceptional class C_n such that $\bar{T}(C_n) = \bar{T}(\mathrm{Id}'_n)$.*

Proof. We searched a sequence such that $T(\mathrm{id}'_n)$ is transformed into a $\sigma = T(\tau)$ where $\tau \notin \mathrm{Id}'_n$. The configuration $\bar{R}LR\bar{L}^2\bar{R}\bar{L}^3RL^2RL\bar{R}\bar{L}\,T(\mathrm{id}'_n)$ is shown in Figure 4.17, left, where the red square is an identity of size $n-7$. This configuration has a T-structure and its pre-image w.r.t. T is the configuration shown in Figure 4.17, middle. Finally, by applying the algorithm of standardization to τ we see that $\tau_s = \bar{R}^2\bar{L}^2\bar{R}L^2R(\tau)$ is the standard permutation shown in Figure 4.17, right,

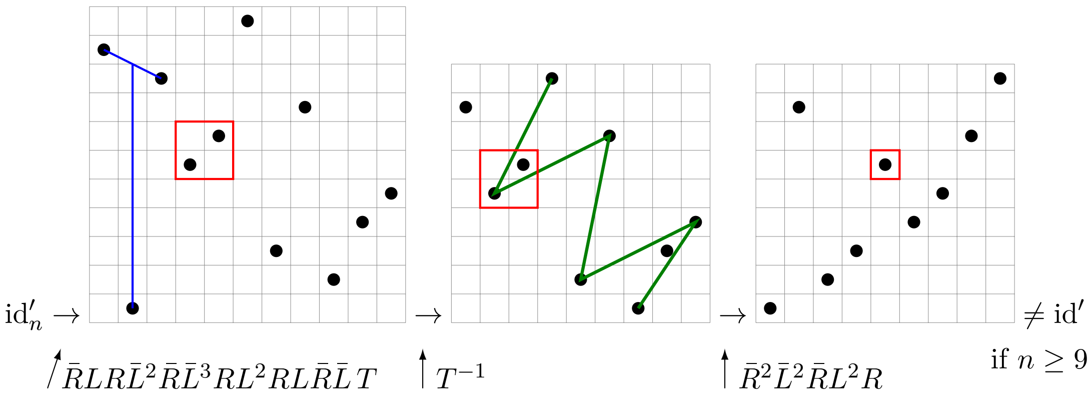


Figure 4.17: The three configurations involved in the proof of Lemma 84.

where the red square is an identity of size $n-8$. When this block is non-empty, this configuration has $\tau_s(1) = 1$ and $\tau_s(n) = n$, but it is not id'_n, from which we conclude. □

4.4.3 Fusion lemmas for q_1 and q_2

In this section we shall see that there is no proliferation of classes due to the action of q_1 and q_2 on exceptional classes of large size Id_n and Id'_n. This makes, in principle, $2 \times 2 = 4$ cases. However, we know from Lemma 73 that $\bar{q}_1(\mathrm{Id}_n) = \mathrm{Id}'_{n+1}$, while neither q_1 nor q_2 apply to Id'_n, because these classes have rank 1, so this rules out three cases out of four, and we just need the following:

Lemma 85 (Fusion lemma for (q_2, Id))**.** *Let $n \geq 6$. There exists some non-exceptional class B_n such that $\bar{q}_2(B_n) = \bar{q}_2(\mathrm{Id}_n)$.*

Proof. We searched a sequence such that $q_2(\mathrm{id}_n)$ is transformed into a $\sigma = q_2(\tau)$ where $\tau \notin \mathrm{Id}_n$. The configuration $RL\bar{R}L^3\, q_2(\mathrm{id}_n)$ is shown in Figure 4.18 (second image), where the red square is an identity of size $n-6$ (so it is possibly empty). Its pre-image w.r.t. q_2 is the configuration shown in Figure 4.18 (third). Then, it can be easily verified that for $n \geq 6$ the standard permutation $\tau_s = \bar{L}^3\bar{R}^3(\tau)$ shown in Figure 4.18 (fourth), is neither id_n nor id'_n. □

However, we are not done still. We need to deal with the problem, anticipated in Section 4.3.2, that the theorem on the pullback of $\bar{q}_2$, Thm. 80, is less precise than what we need, as it does not rule out the possibility that classes of different sign remain non-connected after the application of q_2 (which sets the sign to zero), in which case we would have two classes with the same invariant $(\lambda, 2, 0)$ one coming from a sign $s = +1$, the other from $s = -1$. For this we have a trivial preparatory observation:

Lemma 86. *Let $\lambda \cup \{2\}$ be an integer partition, with an even number of even parts. The two following facts are equivalent:*

1. *There exists a value of j such that $(\lambda \setminus j) \cup \{j+1\}$ has a positive even number of even parts.*

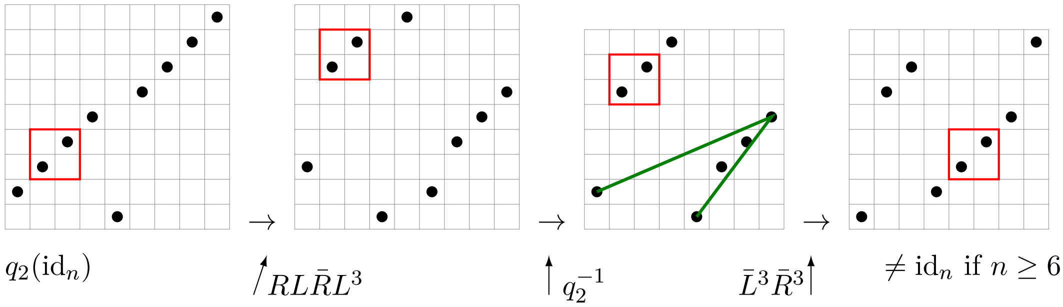


Figure 4.18: The four configurations involved in the proof of Lemma 85. Recall that, whenever $\sigma(1) \neq 1$, $q_2^{-1}\sigma$ is obtained by removing the entry of σ in the bottom-most row.

2. *λ does not consist of a single (even) part.*

Proof. Clearly λ has at least one even part. If it has also some odd part, its value is a valid candidate for j. If it has three or more even parts, all parts of λ are valid candidates. On the other side, if $\lambda = \{2k\}$, then $j = 2k$ is the only candidate, and results in $(\lambda \setminus j) \cup \{j+1\} = \{2k+1\}$. $\square$

This is used in the following (again, we use $\lambda(j) \equiv \lambda \setminus j$)

Lemma 87 (Fusion of sign for q_2, ordinary case)**.** *Let C be a class with invariant $(\lambda, 2, 0)$. Then, if λ does not consists of a single cycle of even length, there exists a value of j, and a class B_j with invariant $(\lambda(j), j+1, 0)$, such that $\bar{q}_2(B_j) = C$.*

Proof. By Theorem 80 we know that, given a class with invariant $(\lambda, 2, 0)$, then, for all $j \in \lambda$, there exists a class B_j with cycle invariant $(\lambda(j), j+1)$, such that $\bar{q}_2(B_j) = C$. By the previous lemma, if $\lambda \neq \{2k\}$, there exists one such j with $(\lambda \setminus j) \cup \{j+1\}$ containing even parts, thus, by Lemma 81, the corresponding class must also have $s(B_j) = 0$, as claimed. $\square$

In light of this lemma, the only possible instance of proliferation of disjoint classes which is still open, for which we need a separate argument, corresponds to the very special partitions λ escaping the characterisation above, i.e., referring to notations as in Lemma 87, the λ's consisting of a single cycle of even length. These special cases are analysed in the following Lemma 89. We observe the trivial fact

Lemma 88. *Define $C'_k = \bar{T}^{k-1}(\mathrm{Id}_4)$ and $C''_k = \bar{T}^{k-2}(\mathrm{Id}_6)$. Then C'_k has invariant $(\varnothing, r = 2k+1, -1)$ and C''_k has invariant $(\varnothing, r = 2k+1, +1)$.*

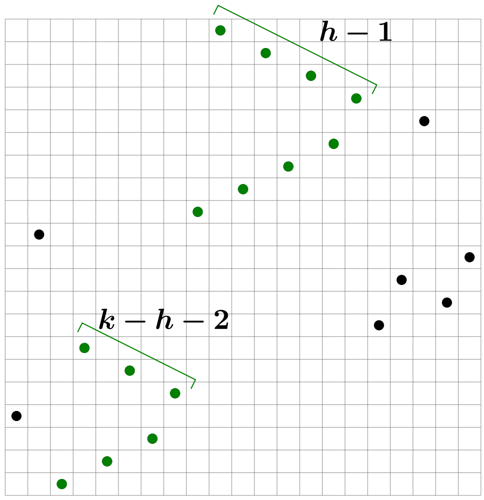
Proof. By corollary 58 Id_4 has invariant $(\varnothing, 3, +1)$ while Id_6 has invariant $(\varnothing, 5, -1)$. Then, the statement follows from the properties of $\bar{T}$ implied by Corollary 59. $\square$

Then we have

Lemma 89 (Fusion of sign for q_2, special case)**.** *For $k \geq 3$, define $C'_k = \bar{T}^{k-1}(\mathrm{Id}_4)$ and $C''_k = \bar{T}^{k-2}(\mathrm{Id}_6)$, and similarly $c'_k = T^{k-1}(\mathrm{id}_4)$ and $c''_k = T^{k-2}(\mathrm{id}_6)$. We have $q_2(c'_k) \sim q_2(c''_k)$, i.e. $\bar{q}_2(C'_k) = \bar{q}_2(C''_k)$.*

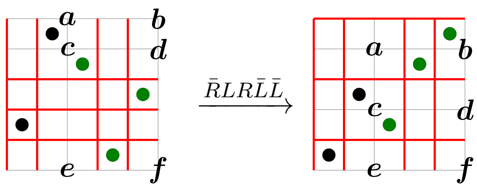
We split the proof in a few lemmas. We start with a definition.

Definition 90. *For $k \geq 3$ and $0 \leq h \leq k-3$, call $C_{k,h}$ the following configuration*



We have

Proposition 91.



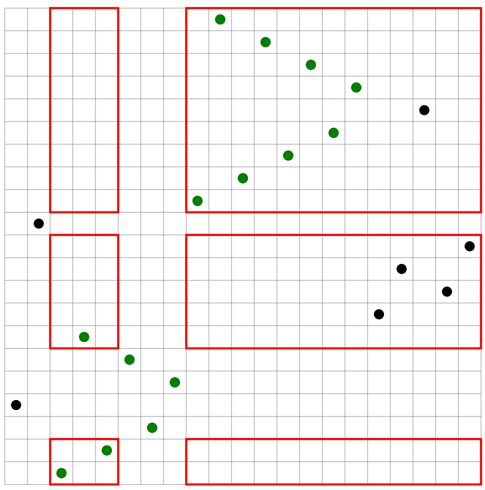
(4.30)

Note, the sequence above does *not* require to be boosted. In other words, the exponents in the boosted sequence are $\alpha_j \in \{\pm 1\}$.[8]

Corollary 92. *For $k \geq 4$ and $0 \leq h < k-3$,*

$$C_{k,h} \xrightarrow{\bar{R}L R\bar{L}\bar{L}} C_{k,h+1}$$

It suffices to identify the blocks $a, \ldots, f$ in the boosted dynamics of Proposition 91 as follows:



[8]Also note that the green bullets on the right-hand-side are not the image of those on the left, they are just coloured for use in the following statement.

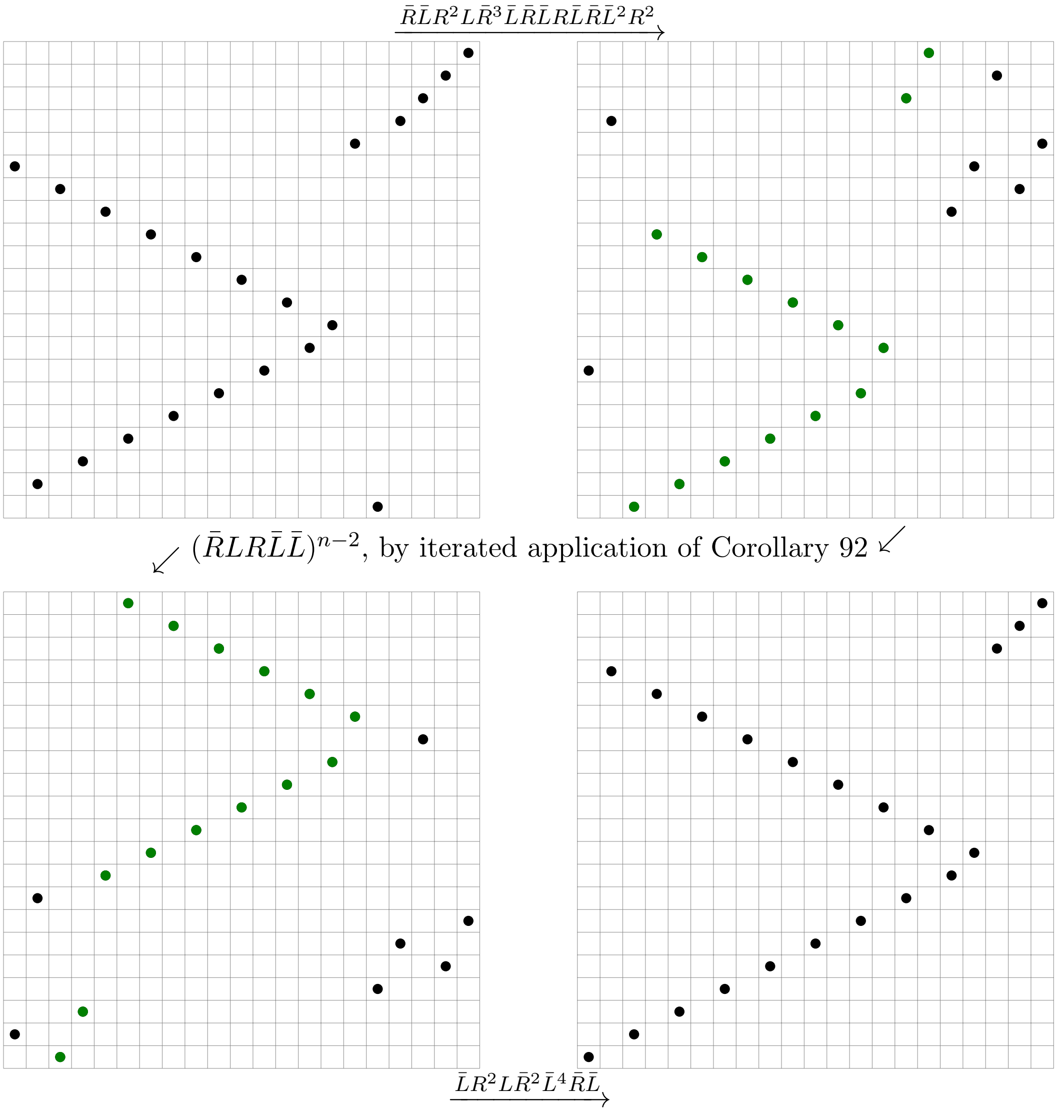


Figure 4.19: Illustration of Lemma 89. The configuration on the top left is $q_2 T^n(id_6)$, and the one on the bottom right is $\bar L q_2 T^{n+1}(id_4)$, for $n = 7$.

$\square$

Then we have the technical verifications

Proposition 93.

$$\bar R \bar L R^2 L \bar R^3 \bar L \bar R \bar L R \bar L \bar R \bar L^2 R^2 q_2(c'_k) = C_{k,0}\,.$$

Proposition 94.

$$\bar L R^2 L \bar R^2 \bar L^4 \bar R \bar L\, C_{k,k-3} = q_2(c''_k)\,.$$

(the configurations involved in these two propositions are shown in Figure 4.19), which, together with Corollary 92, imply Lemma 89.

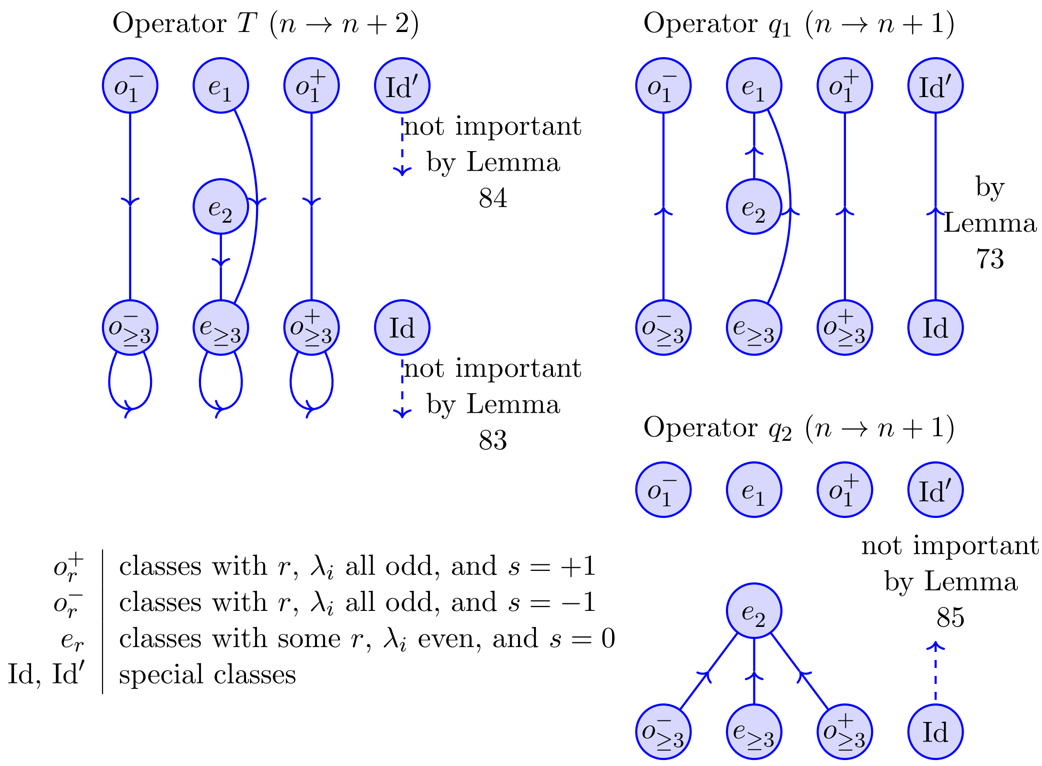


Figure 4.20: Inductive scheme for the application of operators T, q_1 and q_2.

4.4.4 Classification of Rauzy classes

We are now ready to collect the large number of lemmas established so far into a proof of Theorem 194.

Proof of Theorem 194. By the results of Appendix C, we have a full understanding of the two exceptional classes Id_n and Id'_n (and, of course, we know their invariant), so that we can concentrate solely on (primitive) non-exceptional classes.

We proceed by induction. The theorem is established by explicit investigation for classes up to $n = 8$. Then, for the inductive step, we suppose that non-exceptional classes of size $n-1$ and n are completely characterized by (λ, r, s), in agreement with the statement of Theorem 194, and investigate classes at size $n+1$.

Figure 4.20, in three copies for the three surgery operators $\bar{T}$, $\bar{q}_1$ and $\bar{q}_2$, illustrates a partition of the set of classes according to certain properties of the cycle invariant. This partition, despite containing 'only' 5 blocks for non-exceptional cases, is fine enough so that classes within the same block have a consistent behaviour w.r.t. all of our three surgery operators. These behaviours are represented through arrows. Recall that the operator $\bar{T}$ increases the size by 2, while operators $\bar{q}_{1,2}$ increase it by 1. The pullback theorems (Theorem 79, 80 and 67) of Section 4.3.1 and 4.3.2 justify the fact that, in order to prove that there exists *at least* one non-exceptional class per invariant, for the invariants listed in the theorem statement, it suffices to observe that, in the layer of largest size, all blocks of the partition have positive in-degree (when the three copies are considered altogether). This is steadily verified on the image.

For blocks which have in-degree higher than 1, we need our fusion lemmas to conclude that there exists *exactly* one non-exceptional class per invariant, for the invariants listed in the theorem statement.

In the following paragraph we recall which lemma justifies which arrow of the diagram, and how the fusion lemmas are used. We analyse the arrows in some order, according to the rank of the image of the operator.

We start with rank at least 3. By Theorem 67, $\bar{T}$ is a surjection from classes of size $n-1$ to classes of size $n+1$ with rank at least 3 other than Id_{n+1}. Notice however that the classes of size $n-1$ include Id_{n-1} and Id'_{n-1}. Nonetheless, by the 'fusion' Lemmas 83 and 84, there exist two non-exceptional classes, C_1 and C_2, of size $n-1$, such that $\bar{T}(\mathrm{Id}_{n-1}) = \bar{T}(C_1)$ and $\bar{T}(\mathrm{Id}'_{n-1}) = \bar{T}(C_2)$, thus $\bar{T}$ provides a natural bijection from the set of non-exceptional classes of size $n-1$ to the set of non-exceptional classes of size $n+1$ with rank at least 3.

Now we pass to rank 1. By (the easy) Lemma 73 $\bar{q}_1(\mathrm{Id}_n) = \mathrm{Id}'_{n+1}$, furthermore $\bar{q}_1$ cannot be applied to Id'_n since Id'_n has rank 1. Thus the case of exceptional classes has been dealt with. By Theorem 79, if there exist two classes C_1 and C_2 with invariant $(\lambda, 1, s)$ of size $n+1$, then this would imply that, for some i a part of λ, there exist also two classes of size n with invariant $(\lambda(i), i, s)$. So the induction step is proved for classes of rank 1.

Finally, we analyse classes of rank 2. Again the exceptional classes are ruled out, since Id'_n has rank 1, and, by the fusion Lemma 85, there exists C_1 such that $\bar{q}_2(\mathrm{Id}_n) = \bar{q}_2(C_1)$. So we are left only with the issue of the pullback of $\bar{q}_2$, among non-exceptional classes.

Theorem 80 tells us that for every C with invariant $(\lambda, 2, 0)$, for every $j \in \lambda$ there exists some class B_j with invariant $(\lambda(j), j+1)$ such that $\bar{q}_1(B_j) = C$, so there are two cases:

- One of these B_j has invariant $(\lambda(j), j+1, 0)$. In this case, as was above with q_1, C is the only class with invariant $(\lambda, 2, 0)$.
- None of these B_j has invariant $(\lambda(j), j+1, 0)$. In this case it could be that all the classes B_j^+ with invariant $(\lambda(j), j+1, 1)$ give a class C^+ with invariant $(\lambda, 2, 0)$ and all the classes B_j^- with invariant $(\lambda(j), j+1, -1)$ give a class C^-, distinct from C^+, with invariant $(\lambda, 2, 0)$. We need to rule out this possibility now, because this was not still excluded by the 'weak' Lemma 80.

  Indeed, at this point we have all the elements to conclude that this does not happen: by Lemma 87, if no B_j has invariant $(\lambda(j), j+1, 0)$ then λ consists of a single cycle of even length (say, of length $2k$), so the two candidate classes B^+ and B^- must have invariant $(\varnothing, 2k+1, +1)$ and $(\varnothing, 2k+1, -1)$ respectively. By the induction and Lemma 88 we know that $B^+ = \bar{T}^{k-2}(\mathrm{Id}_6)$ and $B^- = \bar{T}^{k-1}(\mathrm{Id}_4)$, and by Lemma 89 we know that for n large enough $\bar{q}_2(B^+) = \bar{q}_2(B^-)$.

This proves the induction step for classes of rank 2, and allows to conclude. □

Table 4.4 illustrates a typical step of the induction, at a size sufficiently large that all fusion lemmas are already in place, and all typical situations do occur.

Figure 4.21 shows the top-most part ($4 \leq n \leq 8$) of the full decomposition tree associated to the action of T, q_1 and q_2 operators on the classes. At difference with

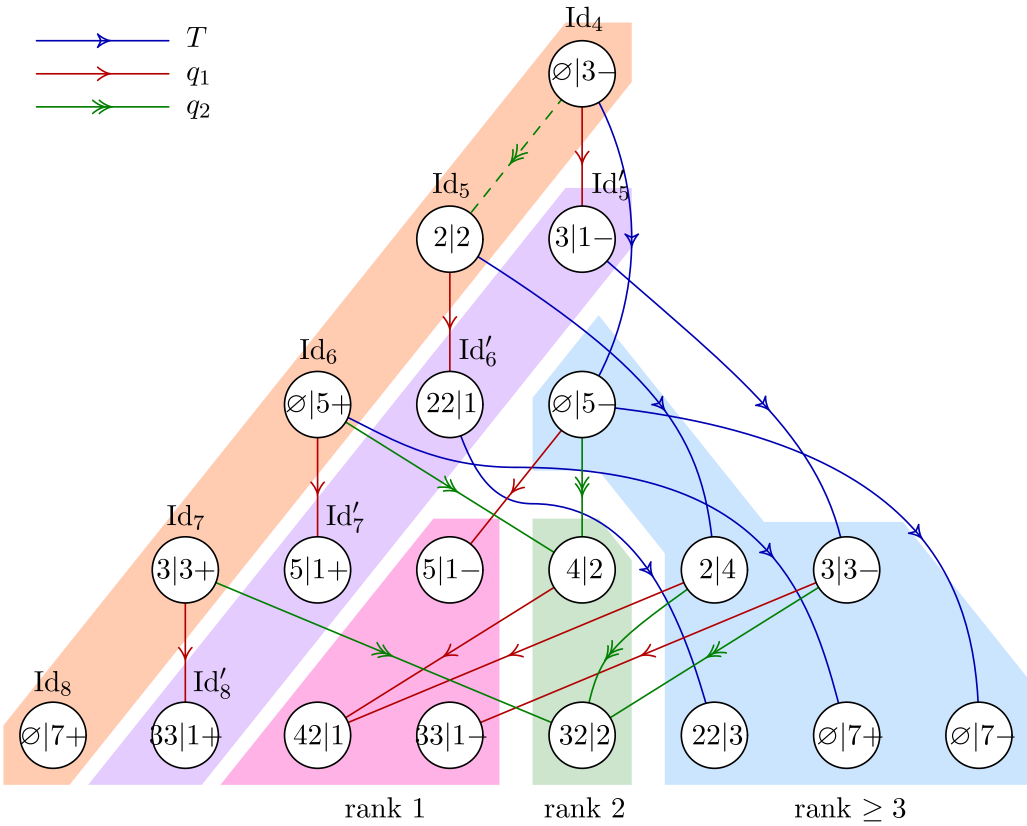


Figure 4.21: Full decomposition tree up to $n = 8$. Each balloon denotes a class, and contains its cycle and sign invariant, in the form $\lambda_1\lambda_2\cdots\lambda_\ell|rs$, with $s \in \{+,-\}$ and omitted if zero.

Table 4.4, this part is quite lacunary w.r.t. the general pattern. The relevant fact is that, for $n > 8$, there are no more exceptions (cf. the statements in Section 4.4.2 and 4.4.3, which hold under the hypotheses $n \geq n_0$ for certain values of n_0 all at most 8), so that this picture encompasses all of the small-size exceptional behaviour.

4.4.5 Classification of extended Rauzy classes

In this section we show how, by a crucial use of Lemma 46, the classification theorem for $\mathcal{S}_n^{\mathrm{ex}}$, Theorem 25 descends from the one for $\mathcal{S}_n$, Theorem 194, as a rather straightforward corollary.

The idea is that, by including more operators for the dynamics on the same set of configurations, we may only join classes. As, in fact, most of the invariants for $\mathcal{S}_n$ survive in $\mathcal{S}_n^{\mathrm{ex}}$ (essentially, only the rank is lost), as a matter of fact, and as we shall prove, there is no residual lack of connectivity: all classes of $\mathcal{S}_n$ with the same structure of invariants except for the rank ultimately join together in $\mathcal{S}_n^{\mathrm{ex}}$.

Let us define two involutions on $\mathfrak{S}_n$, S and $\bar{S}$. The first one is defined as

	$n = 9, 10$		$n = 11$
		new	$((5), 5, -)$; Id
	$((2), 6, 0)$;	$\bar{T} \to$	$((2), 8, 0)$;
	$((3), 5, +)$;	$\bar{T} \to$	$((3), 7, +)$;
	$((3), 5, -)$;	$\bar{T} \to$	$((3), 7, -)$;
by Lemma 83	$((4), 4, 0)$; Id	$\bar{T} \to$	$((4), 6, 0)$;
	$((4), 4, 0)$;		
	$((5), 3, +)$;	$\bar{T} \to$	$((5), 5, +)$;
	$((5), 3, -)$;	$\bar{T} \to$	$((5), 5, -)$;
	$((6), 2, 0)$;	$\bar{T} \to$	$((6), 4, 0)$;
	$((2, 2, 2), 2, 0)$;	$\bar{T} \to$	$((2, 2, 2), 4, 0)$;
by Lemma 84	$((7), 1, +)$;	$\bar{T} \to$	$((7), 3, +)$;
	$((7), 1, +)$; Id$'$		
	$((7), 1, -)$;	$\bar{T} \to$	$((7), 3, -)$;
	$((3, 2, 2), 1, 0)$;	$\bar{T} \to$	$((3, 2, 2), 3, 0)$;
	$((3, 2), 4, 0)$;	$\bar{q}_2 \to$	$((3, 3, 2), 2, 0)$;
	$((3, 3), 3, +)$;		
	$((3, 3), 3, -)$;		
	$((2, 2), 5, 0)$;	$\bar{q}_2 \to$	$((4, 2, 2), 2, 0)$;
	$((4, 2), 3, 0)$;		
by Lemma 89	$((), 9, +)$;	$\bar{q}_2 \to$	$((8), 2, 0)$;
	$((), 9, -)$;		
by Lemma 85	$((), 9, -)$; Id		
	$((), 9, +)$;	$\bar{q}_1 \to$	$((9), 1, +)$;
	$((), 9, -)$;	$\bar{q}_1 \to$	$((9), 1, -)$;
by Lemma 73	$((), 9, -)$; Id	$\bar{q}_1 \to$	$((9), 1, -)$; Id$'$
	$((5, 2), 2, 0)$;	$\bar{q}_1 \to$	$((5, 2, 2), 1, 0)$;
	$((2, 2), 5, 0)$;		
	$((4, 3), 2, 0)$;	$\bar{q}_1 \to$	$((4, 3, 2), 1, 0)$;
	$((4, 2), 3, 0)$;		
	$((3, 2), 4, 0)$;		
	$((3, 3), 3, +)$;	$\bar{q}_1 \to$	$((3, 3, 3), 1, +)$;
	$((3, 3), 3, -)$;	$\bar{q}_1 \to$	$((3, 3, 3), 1, -)$;

Table 4.4: A typical step of the induction, at a size sufficiently large that all fusion lemmas are already in place, and all typical situations do occur. All classes at size $n = 11$ (right column), except for Id_{11}, are obtained from classes at $n = 9$, under the action of $\bar{T}$, and classes at $n = 10$ and rank large enough, under the action of $\bar{q}_1$ (on classes with $r \geq 2$) and $\bar{q}_2$ (on classes with $r \geq 3$). For comparison, the list of classes for $n \leq 8$, and their construction through surgery operators, is shown in figure 4.21.

$S\sigma = \sigma^{-1}$, i.e., in matrix representation, it acts as a reflection along the main diagonal, and, in diagram representation, as a reflection along the horizontal axis. The operator S intertwines between the four operators in the $\mathcal{S}_n^{\rm ex}$ dynamics, i.e. $SLS = L'$ and $SRS = R'$. Analogously, we have $\bar{S}\sigma = \tau$ for $s(i) = j \Leftrightarrow \tau(n+1-i) = n+1-j$. Then, $\bar{S}$ acts on permutations in matrix representation as a reflection along the main anti-diagonal, and on permutations in diagram representation as a reflection along the vertical axis, and intertwines between the four operators in the $\mathcal{S}_n^{\rm ex}$ dynamics as $\bar{S}L\bar{S} = R'$ and $\bar{S}R\bar{S} = L'$.

Recall that, in Section 4.1.3, we defined permutations of type X and H, and the *principal cycle* in a type-X permutation as the cycle passing through the '-1 mark'. It is easy to establish

Proposition 95. *Let σ be an irreducible permutation of type H, with invariant (λ, r, s). Then $\sigma^{-1} = S\sigma$ is an irreducible permutation of type H, with invariant (λ, r, s).*

Let σ be an irreducible permutation of type X, with invariant (λ, r, s), and principal cycle of length $\bar{r}$. Let $\bar{\lambda} = \lambda \setminus \{\bar{r}\} \cup \{r\}$. Then $\sigma^{-1} = S\sigma$ is an irreducible permutation of type X, with invariant $(\bar{\lambda}, \bar{r}, s)$ and principal cycle of length r.

This follows from the diagrammatic construction of the invariants, which is symmetric w.r.t. the involution S, up to exchanging the lengths of the rank and of the principal cycle in type-X permutations. □

Recall that we define $\lambda' = \lambda \cup \{r\}$, and that we have determined that λ' is invariant in the $\mathcal{S}_n^{\rm ex}$ dynamics.

For the proof Theorem 25 we need two ingredients. On one side, we need to deal with exceptional classes. For Id'_n, it is especially easy: this class has rank 1 at all n, thus it is not primitive in the $\mathcal{S}^{\rm ex}$ dynamics, and shall be discarded. So we are left to prove that the class Id_n is not connected to other classes by the two further operators L' and R' of the dynamics. This is easily evinced from the results of Appendix C.1, in particular the organisation of the configurations in Id_n in a complete binary tree. The involution $\bar{S}$ acts just as the vertical symmetry on this tree. As a result, the class Id_n is closed under the action of L, R and $\bar{S}$, which, as $L' = \bar{S}R\bar{S}$ and $R' = \bar{S}L\bar{S}$, implies closure under the action of L, R, L' and R'.

On the other side, we shall prove that all the non-exceptional classes of $\mathcal{S}_n$ with same sign invariant, and cycle invariant $(\lambda' \setminus \{\lambda'_i\}, \lambda'_i)$, for i running over the different cycles of λ', are connected by the extended dynamics. A first example occurs at size 7, where we shall show that the classes of $\mathcal{S}_7$ with invariant $(\{2\}, 4, 0)$ and $(\{4\}, 2, 0)$ merge into a unique class with invariant $(\{2, 4\}, 0)$.

In fact, we shall prove an even stronger statement:

Theorem 96. *Any two configurations σ, σ' in the same class for the dynamics $\mathcal{S}_n^{\rm ex}$ are connected by a word of the form $w = w_1 (L')^k w_2$, with k an integer, and w_1, w_2 words in the $\mathcal{S}_n$ dynamics.*

Proof. In other words, let $\lambda' = \lambda \cup \{r_1, r_2\}$, with $r_1 \neq r_2$. We want to prove that the classes C_1 and C_2 of $\mathcal{S}_n$, with invariants $(\lambda \cup \{r_2\}, r_1, s)$ and $(\lambda \cup \{r_1\}, r_2, s)$, respectively, are connected in $\mathcal{S}_n^{\rm ex}$, and through a word w of the type above.

Let $\sigma \in C_1$ be a configuration in a standard family, and of type $H(r'_1, r''_1)$ (thus with $r'_1 + r''_1 = r_1$). From our classification theorem for $\mathcal{S}_n$ we can restrict to such configurations with no loss of generality. It is easily seen that $S\sigma$ is also standard, and has the same invariant in $\mathcal{S}_n$ (by Proposition 95), thus $\sigma \sim S\sigma$ in the $\mathcal{S}_n$ dynamics.

By Lemma 46, for all $r_2 \in \lambda$ there exists i such that $L^i\sigma$ has form $X(r_1, r_2)$, and thus (by Proposition 95) $SL^i\sigma = (L')^i S\sigma$ has form $X(r_2, r_1)$, and is thus in class C_2. This completes our proof. $\square$

As remarked above, the stronger Theorem 96 implies Theorem 25 as a corollary.

Part II

Classifying Rauzy-type dynamics via the labelling method

In this second part, we introduce a general method to prove classification theorems for Rauzy-type dynamics. We call this method the *labelling method*, and describe its mechanisms in the following chapter.

Then, in chapters 7 and 6 we will apply the labelling method to give another proof of the classification of the Rauzy dynamics $\mathcal{S}_n$ and $\mathcal{S}_n^{ex}$, as well as proof of the classification for a new Rauzy-type dynamics for matchings, with no geometric counterpart so far, that we name the *involution dynamics*, $\mathfrak{I}_n$. For the 'customary' Rauzy dynamics for matchings $\mathcal{M}_n$ (cf section 8), the one which is natural from the point of view of IET's on non-orientable surfaces, the labelling method definitely apply but the proof will be carried out in a subsequent paper.

Finally, in a third part of this manuscript, containing the remaining chapter 9 we present a collection of Rauzy-type dynamics for which the labelling method shall apply, i.e., for which the basic hypotheses are satisfied. We do not carry out the classification proofs, i.e. we do not control still all of the more subtle hypotheses for the method to work through, but we provide invariants of those dynamics, which may be complete (for some of these dynamics, the completeness of the invariants is compatible with numerical data collected so far, for some others we already know that the invariants are not complete, however we have small degeneracies, typically some phenomena compatible with analogs of missing the Arf invariant in the Rauzy dynamics for permutations).

Chapter 5

The labelling method

We place ourself in the framework of Chapter 2. Let X_n be the combinatorial set and (A_k) be the set of operators of the dynamics. Before introducing the labelling method, let us lay down a number of necessary definitions.

Definition 97 ((k,r)-coloring and reduction)**.** *Let $x \in X_n$ be a combinatorial structure, and k, r integers with $k+r=n$. A* (k,r)-coloring *c of x is a coloring of the n vertices of x into a black set of k vertices and a gray set of r vertices, such that, calling y the restriction of x to the black vertices, y is also a combinatorial structure (thus $y \in X_k$). We call y the* reduction *of x.*

For example, for a matching $m \in \mathcal{M}_n = X_{2n}$, in principle we shall color in black and gray the $2n$ points. However, in order for y to be a matching, in a valid (k,r)-coloring we need that the endpoints of an edge of the matching are either both black or both gray (in particular r is even).

Definition 98 (Boosted sequence and boosted dynamics)**.** *Let $x \in X_n$ and let y be the reduction of a (k,r)-coloring c of x. Let S be a sequence of operators in X_k such that $S(y) = y'$.*

If a sequence S' in X_n is such that y' is the reduction of $(x',c') = S'(x,c)$, we say that S' is a boosted sequence of S for (x,c). *See figure 5.1.*

A dynamics $((X_n)_n, (A_k)_k)$ has a boosted dynamics *if for every n, $x \in X_n$, (k,r)-coloring c of x and sequence S as above, there exists a boosted sequence S'.*

Note that, for a given (n,x,c,S) as above, the boosted sequence S' is not necessarily unique. This is rather obvious when the dynamics is a group dynamics, as in this case, even without boosting, the set of sequences $S : x_1 \to x_2$ is a coset (w.r.t. sequence concatenation) within the group of all possible sequences, i.e., if $Sx_1 = x_2$, every sequence $S' : x_2 \to x_2$ is such that $(S')^k S x_1 = x_2$ for all $k \in bZ$. Nonetheless, with abuse of notation, in our definition of the boosted dynamics we will often refer to 'the' boosted sequence S', by this referring to the most natural boosted sequence, w.r.t. some notion changing from case to case. This choice will often correspond to the shortest such sequence, and will be in a sense canonical, although our proofs do not rely on the existence of a canonical sequence, but merely require the existence statement.

As we said in the introduction, we represent combinatorial structures in X_n as n vertices on the real line (indicized from left to right) with the combinatorial structure (be it matching, set partition, graph or hypergraph) placed above in the upper half

$$
\begin{tikzcd}
 & & (x'', c'') \arrow[dl, bend left, "\text{red}"] \\
(x, c) \arrow[r, "S'"] \arrow[d, "\text{red}"] \arrow[urr, bend left, "S''"] & (x', c') \arrow[d, "\text{red}"] & \\
y \arrow[r, "S"] & y' &
\end{tikzcd}
$$

Figure 5.1: S' is a boosted sequence of S for (x, c), because red $\circ$ S' gives the same result as $S \circ$ red. Note that S' may be not unique: in the diagram we show a second boosted sequence of S for (x, c), namely S''. It is not necessarily the case that $(x', c') := S'(x, c)$ coincides with $(x'', c'') := S''(x, c)$, but merely that their reductions coincide (they must be both y').

plane. This is done in order to allow the dynamics operator to act graphically from below (they will be represented as permutations, contained within a horizontal strip), as is customarily the case for diagram algebras (like the partition algebra, the braid group, and so on)

This graphical representation is also useful in order to produce a certain construction on configurations, which we now describe. Place two new vertices, with label 0 and $n+1$, to the left and right of the existing vertices $1, \ldots, n$, respectively. Then, connect by an arc in the lower half plane the pair of adjacent vertices $(i, i+1)$, for $i = 0, \ldots, n$. This modification of the structure $x \in X_n$ will be called a *combinatorial structure with arcs*.

If we have a (k, r) coloring of x, in the construction of the combinatorial structure with arcs we will set the two extra points, 0 and $n+1$, as black.

Let $\Sigma = \{b_0 \ldots, b_n\}$ be an alphabet of distinct symbols. We define a *labelling of* $x \in X_n$ to be a bijection

$$\Pi_b \quad : \quad \{0, \ldots, n\} \quad \to \quad \Sigma$$

i.e., a labeling of the bottom arcs of x in the costruction above, with the symbols from Σ. See figure 5.2.

In the following treatment, once Π_b is given, it will be convenient to designate arcs either by their position $\beta \in \{0, \ldots, n\}$ or by their label $b \in \Sigma$, depending from the situation. As a convention, and in order to avoid confusion, we will use greek and latin letters as above in the two cases.

We need a definition for a consistent action of the dynamics on the labelling. Recall that in Chapter 2 we have defined an operator A of the dynamics as the datum of a pair α and a, with $a : X_n \to Y_n$ and $\alpha : Y_n \to \mathfrak{S}_n$, so that $Ax = \alpha_{a(x)}x$ is the symmetric-group action of α_a on x.

Definition 99 (Labelling and dynamics)**.** *For $x \in X_n$, let (x, Π_b) be a combinatorial structure with labelling. The operator $\hat{A}$ is the* labelled extension *of A if it acts on (x, Π_b) as follows:*

$$\hat{A}(x, \Pi_b) = (\alpha_{a(x)}(x), \Pi_b \circ \alpha'_{a'(x)})$$

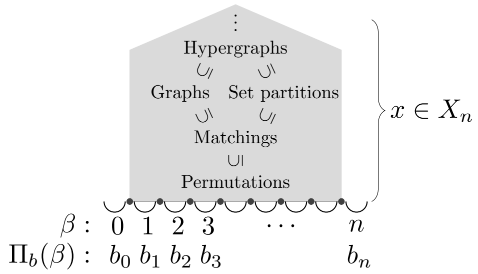


Figure 5.2: A representation of a combinatorial structure $x \in X_n$ with a labelling Π_b.

where,in analogy with the Definition 12, $a : X_n \to Y_n$*,* $\alpha : Y_n \to \mathfrak{S}_n$*,* $a' : X_n \to Y'_n$*,* $\alpha' : Y'_n \to \mathfrak{S}_{n+1}$*.*

Thus, in this case, not only A chooses a permutation α_a, depending on x, by which acting on the n vertices of x, but it also chooses a permutation $\alpha'_{a'}$, again depending only on x, by which acting on the set of $n+1$ labels, in both cases in the sense of the symmetric-group action.

Such a definition is not unique, but in most cases there will be a unique and simple choice which is natural. In particular, for group dynamics, we will search for a choice that makes the action of A on x and of $\hat{A}$ on (x, Π_b) (for any Π_b) cyclic of the same order.

When there will be no confusion, we will often write $\hat{A}$ for A.

Definition 100 (Labelling, vertices, and boosted dynamics)**.** *Let* $x \in X_n$*,* c *a* (k,r)*-coloring of* x*,* $y \in X_k$ *the corresponding reduction, and* Π_b *a labelling of* y*.*

We say that a gray vertex ℓ *of* x *is* within an arc β *of* y*, and write* $\ell \in b$*, if the black vertices of* x*, corresponding to the vertices* β *and* $\beta+1$ *of* y*, are the first black vertices to the left and to the right of* ℓ *(note that* β *and* $\beta+1$ *may be* 0 *and* $k+1$*, i.e. the external vertices added in the arc construction). We say that the labelling is* compatible with the boosted dynamics *if the following statement holds:*

Let x*,* c*,* y *and* Π_b *as above,* S *a sequence in* X_k*, and* S' *a boosted sequence of* S *for* (x,c)*. If* v *is a gray vertex of* x *within the arc* $b \in \Sigma$ *of* (y, Π_b)*, then the image of* v *in* $(x', c') = S'(x,c)$ *must be within the image of the arc* b *in* $(y', \Pi'_b) = S(y, \Pi_b)$*.*

In other words, if the labelling is compatible with the boosted dynamics, we can keep track of the positions of the gray vertices by understanding how the labelling of a reduced combinatorial structure $y \in X_k$ evolves in the extended labelled dynamics.

Definition 101. *Given a combinatorial structure* $x \in X_n$ *with a labelling* Π_b*, let* $L(x, \Pi_b)$ *be defined as*

$$L(x, \Pi_b) = \{\Pi'_b \mid \exists S \text{ with } S(x, \Pi_b) = (x, \Pi'_b)\}\,, \tag{5.1}$$

that is, the set of all the labellings that are reachable from (x, Π_b) *by a loop* S *of the dynamics.*

We say that the r-point monodromy of x is known *if the following problem is solved: for every ordered* r*-uple of arcs* $(b_1, \dots, b_r)$ *in the structure with arcs* (x, Π_b)*, and every* r*-uple* $(\beta_1, \dots, \beta_r)$ *of disctinct integers in* $\{0, \dots, n\}$*, we know whether* *there exists a $\Pi'_b \in L(x, \Pi_b)$, and thus an S such that $S(x, \Pi_b) = (x, \Pi'_b)$, with the property that $S(b_a) = \beta_a$ for all $a = 1, \dots, r$.*

The definition above is somewhat redundant. In fact it is clear that, for $\pi \in kS_{n+1}$,

$$L(x, \Pi_b \circ \pi) = L(x, \Pi_b)\, \pi \tag{5.2}$$

(and in particular, by choosing $\pi = \Pi_b^{-1}$, it suffices to describe the set L for the canonical left-to-right labelling). Also, if we have a group dynamics, the dependence from x is only through the class of x, more precisely, if S_0 is a sequence in the labelled dynamics from (x, Π_b) to (x', Π'_b), then

$$L(x, \Pi_b) = L(x', \Pi'_b) \tag{5.3}$$

because, for each $\Pi''_b \in L(x', \Pi'_b)$ realised with the sequence S, then $\Pi''_b \in L(x, \Pi_b)$ is realised with the sequence $S_0^{-1} S S_0$, and similarly with $x \leftrightarrow x'$ and $S_0 \leftrightarrow S_0^{-1}$.

The interest in these properties is highlighted in the crucial definition below, of the 'labelling method' as a whole:

Definition 102. *A dynamics $((X_n)_n, (A_k)_k)$ is* amenable to the labelling method *if:*

1. *It has a boosted dynamics.*
2. *There is a labelling compatible with the boosted dynamics.*
3. *For a suitable value of $r > 0$, the r-point monodromy of any class is known.*

When proving a classification theorem for a group dynamics,[1] we shall usually proceed in two steps: first, guessing the 'right' set Inv_n of invariants for the dynamics on X_n, and then, proving that the set is right. In order to do the latter, i.e. to prove that there exists exactly one class per invariant, we need to show two things:

existence For every $\mathrm{inv} \in \mathrm{Inv}_n$ there exists a combinatorial structure $x \in X_n$ with invariant inv.

completeness The invariant discriminates the classes, i.e. for every pair $x_1, x_2 \in X_n$ with the same invariant $\mathrm{inv} \in \mathrm{Inv}_n$ there exists a sequence S such that $S(x_1) = x_2$.

The first item of the list is often achieved relatively easily, by constructing a candidate $x \in X_n$ either directly or by induction. The difficulty of the construction depends mainly on how tractable and explicit the invariant set is.

The second part, which is sensibly more complicated, is where the labelling method comes forth.

The method is aimed at constructing an inductive step, so we assume that the invariant is complete up to sizes $n' < n$, and that the r-point monodromy problem

[1] In case of a monoid dynamics the goal needs to be modified accordingly, and there is no general recipe. If the Cayley digraphs of the classes has a strongly connected component, we can restrict the study to this portion of the classes, with small difference w.r.t. the group case (this occurs in section 9.3). Another possible sensible question consists in modifying the monoid dynamics into a group dynamics, by adding new or modifying existant operators, so that the weak-connectedness in the original Cayley digraph of the classes turns into ordinary connectedness in the new Cayley graph.

(with an appropriate value of r specified below) is solved for all configurations up to sizes $n' < n$, and we analyse how we can prove that the invariant is complete at size n. At a later moment (and we do not discuss this further in this chapter, as the proof strategy changes from case to case), we shall solve the r-point monodromy problem (with the same value of r) at size n, in light of the r-point monodromy problem at smaller size, and the completeness at size up to n.

So, we have two arbitrary x_1, $x_2 \in X_n$, combinatorial structure with the same invariant $\mathrm{inv} \in \mathrm{Inv}_n$, and we want to show that they are connected by the dynamics.

The first ingredient that we shall use is the notion of (k, r)-colorings and reductions. In certain circumstances (and surely when the r-point monodromy problem is solved) we can interpret (k, r)-colorings and reductions the other way around, i.e. as ways of embedding a class C_k at size k (or a portion of it) into a class C_n at size n. Different colourings may correspond to different embeddings, so the same (portion of) class C_k will appear inside C_n with several copies (and it may appear, again with several copies, within other classes C'_n, while also other classes C'_k may appear in C_n, and some portions of C_n may have no intelligible reductions to classes at smaller size).

Our fist goal is **find x'_1 and x'_2 in X_n,** connected to x_1 and x_2 respectively, with the following property: there exists two (k, r)-coloring[2] c_1 and c_2 of x'_1 and x'_2 respectively, such that $y_1 := \mathrm{red}(x'_1, c_1)$ and $y_2 := \mathrm{red}(x'_2, c_2)$ have the same invariant $\mathrm{inv}' \in \mathrm{Inv}_k$. Which, by induction hypothesis, implies that they are in the same class. That is, x'_1 and x'_2 are in (possibly different) embeddings of the same class. Call S_1 and S_2 the sequences such that $x'_1 = S_1 x_1$ and $x'_2 = S_2 x_2$.

Then, use induction to **find a sequence S such that** $S(y_1) = y_2$.

Since the dynamics is amenable to the labelling method, there exists a boosted dynamics and a labelling compatible with it. **Let us choose such a labelling.**

Let Π_b be a labelling of y_1. Say that the r gray vertices of (x'_1, c_1) are within the arcs with labels $b^1, \dots, b^r$ of (y_1, Π_b), in this order. In addition, the r gray vertices of (x'_2, c_2) are within the arcs with position, say, $\beta^1, \dots, \beta^r$ of y_2, in this order. **We will be concerned with these two r-uples.**

By the boosted dynamics, there exists a sequence S' which is **a boosted sequence of S** for (x'_1, c_1). Let $(x_3, c_3) = S'(x'_1, c_1)$. If it were $(x_3, c_3) = (x'_2, c_2)$, we would be done, but, as it has already been noted above, this needs not to be the case, and (x_3, c_3) may be contained in yet another embedded copy of the class inv' at size k. So, we know that (x_3, c_3) and (x'_2, c_2) are not equal in general, nonetheless the reduction of both of them is y_2.

By compatibility of the labelling with the boosted dynamics, **we know the location of the r gray vertices of** (x_3, c_3). Namely, by virtue of our notation (in which the alphabet Π_b changes alongside the dynamics), we have that the r gray vertices are within the arcs with labels $b^1, \dots, b^r$ of $(y_2, \Pi'_b) = S(y_1, \Pi_b)$. Thus x_3 and x'_2 possibly differ by the position of the r vertices which are gray in the colouring, that are within the arcs with position $(\Pi'_b)^{-1}(b^a)$ and β^a of y_2, for x_3 and x'_2, respectively, for $1 \le a \le r$.

Since the r-point monodromy problem is solved for y_2, **we know whether there**

[2] We shall adopt r to be the smallest value such that X_k is a valid combinatorial set, and in most of our applications it is just $r = 1$ or $r = 2$.

$$
\begin{tikzcd}
\text{size } n & x_1 \arrow[r, "S_1"] & x'_1 \arrow[rr, "S'_*S'"] \arrow[d, dashed] & & x'_2 \arrow[d, dashed] & x_2 \arrow[l, "S_2"'] \\
\begin{matrix}\text{size } n,\\ (k,r)\text{-colored}\end{matrix} & & (x'_1, c_1) \arrow[r, "S'"] \arrow[d, "\text{red}"] & (x_3, c_3) \arrow[r, "S'_*"] \arrow[d, "\text{red}"] & (x'_2, c_2) \arrow[d, "\text{red}"] & \\
\text{size } n-r & & y_1 \arrow[r, "S"] \arrow[d, dashed] & y_2 \arrow[r, "S_*"] \arrow[d, dashed] & y_2 \arrow[d, dashed] & \\
\begin{matrix}\text{size } n-r,\\ \text{labelled}\end{matrix} & & (y_1, \Pi_b) \arrow[r, "\hat{S}"] & (y_2, \Pi'_b) \arrow[r, "\hat{S}_*"] & (y_2, \Pi''_b) &
\end{tikzcd}
$$

Figure 5.3: Outline of the proof of connectivity between x_1 and x_2, using the labelling method. The sequence S sends y_1 to y_2, however the arcs containing the gray vertices of x'_1 may not be at their correct place, in order to match with those of x'_2. The sequence S_* corrects for this. Thus the boosted sequence S'_*S' sends x'_1 to x'_2, and x_1 and x_2 are connected.

exists a sequence S_* that sends the arcs with label b^a to the positions β^a. More precisely, a sequence $S_* \in L(y_2, \Pi'_b)$ such that the labeling $\Pi''_b = S_*(\Pi'_b)$ has $(\Pi''_b)^{-1}(b^a) = \beta_a$ for all $1 \leq a \leq r$.

If S_* exists, let S'_* be the boosted sequence of S_* for (x_3, c_3). Then, again by compatibility of the labelling with the boosted dynamics, we have $S'_*(x_3, c_3) = (x'_2, c_2)$. Thus we have shown that x_1 and x_2 are connected, namely

$$
x_2 = (S_2)^{-1} S'_* S' S_1 x_1 \tag{5.4}
$$

See figure 5.3

Conversely, **if S^* does not exist, the labelling method has failed.**

Let us discuss the possible difficulties one can encounter when trying to employ the labelling method. We can identify two main tasks: proving that the dynamics is amenable to the method, and then applying it to the 'proof that $x_1 \sim x_2$' outlined above. Each of the two parts has its own difficulties.

In the first part:

- Proving the existence of a boosted dynamics is often easy. The dynamics we study (at least in this thesis) are 'regular', in the sense that group operators in X_{n+1} and in X_n do not differ much from each other. Thus, given a sequence of operators $S = A_{i_k} \dots A_{i_1}$ on X_{n-r}, a boosted sequence on X_n often has the form $S' = A_{i_k}^{j_k} \dots A_{i_1}^{j_1}$ for some integer powers j_ℓ which are often easily evinced.

  This heuristic may not necessarily work in the case of a monoid dynamics, even worse, for a given sequence $S = A_{i_k} \dots A_{i_1}$ on X_{n-r}, it may be not clear which powers j_ℓ make the sequence $S' = A_{i_k}^{j_k} \dots A_{i_1}^{j_1}$ well-defined in X_n.

- Proving that there is a labelling compatible with the boosted dynamics is normally straigthforward, and the permutations $\alpha'_{a'(\cdot)}(\cdot)$ are rather similar to $\alpha_{a(\cdot)}(\cdot)$.

- Solving the r-point monodromy problem is the most difficult sub-task. In all our applications, the structure of the set $L(x, \Pi_b)$ depends heavily on the invariant set, and we mostly have statements of the like "the label b can be sent to b' if and only if b and b' are isomorphic, w.r.t. some notion of homorphism on the invariant". However, the proof is performed by induction at size n, **after** the induction step for the classification theorem has been established at size n. I.e. we suppose both statements true at size $n' < n$ then we apply the labelling method to prove the classification theorem at size n, and finally we work out the monodromy problem at size n, in this very order for all of our applications.

For the second part there are two steps which can necessitate an important amount of non-automatised work.

At the very beginning, we shall certify that we can reach configurations x'_1 and x'_2 which admit a coloring and a reduction of the appropriate form. Doing this can be very trick, and the amount of difficulty mostly depends on the structure of the invariant set. For example, in the Rauzy dynamics for permutations (Chapters 7), a whole section (Sections 7.6) has to be dedicated to it. Conversely, in the involution dynamics (Section 6), this step only requires a small verification.

Exceptional classes also cause problems, as they often break the "one class per invariant" landscape (which is modified into "one non-exceptional class per invariant"). As a result, even with the induction hypothesis at hand, we may have that y_1 and y_2 share the same invariant but are not in the same class, because one of the two is in an exceptional class. When we had to face this issue, we could solve it by choosing x'_1 and x'_2 appropriately so that y_1 and y_2 are easily checked to not be in an exceptional class. However, in the cases we have studied so far, certifying this has required the complete characterisation of the structure of the exceptional classes, a knowledge which may remain elusive in more complicated dynamics.

The step concerning the sequence S_* also needs a certain amount of care. Indeed, as we have anticipated, the solution to the r-point monodromy problem normally sounds like "a sequence of the desired form exists iff there is a homomorphism between two certain r-rooted structures which sends the roots of one structure to the roots of the other, in a given order". Now, as we said, on one side it could be difficult to prove the statement. But, also importantly, on the other side it could be difficult to exploit this rather implicit information, because the identification of a homomorphism may be a difficult problem for arbitrary families of structures. We will occasionally need skilful constructions in order to work with structures in which the isomorphism problem has been simplified to a tractable form.

For the rest, and assuming that the sequence S_* with the appropriate properties always exists (otherwise, as we said, we would be in trouble), all the other steps are elementary, and follow a non-ambiguous roadmap, with no need of case-to-case inventions.

Chapter 6

An involutive Rauzy dynamics: $\mathfrak{I}_n$

We introduce in this section a new Rauzy-type dynamics for matchings called the *involution dynamics* and noted $\mathfrak{I}_n$. The classification of the Rauzy classes for the involution dynamics is our first example of application of the labelling method described in section 5.

Our choice to start with the involution dynamics $\mathfrak{I}_n$ rather than the already studied Rauzy dynamics $\mathcal{S}_n$ originates from the ease with which the labelling method applies to $\mathfrak{I}_n$. As we signaled at the end of chapter 5, several difficulties often appears when carrying out a classification proof using the labelling method. In the case of the $\mathfrak{I}_n$ dynamics, however, all the steps are simple and straightforward.

As such, the $\mathfrak{I}_n$ dynamics is a proof of concept for the labelling method. Moreover the structure of a 'labelling method'-type classification proof is highlighted in this instance instead of lost in the specific intricacies of the dynamics.
There are two reasons that explains this simplicity:

- There are no exceptional classes.
- There is only one invariant: a multigraph whose vertices are the arcs of the matching and in which two vertices are connected by an multiedge if and only if their associated arcs are adjacent in the matching. Moreover, the labelling of the bottom arcs corresponds to the labelling of the edges of the graph, thus the set $L(x, \Pi_b)$ (cf definition 102) is in bijection with the automorphism group of the edge-set of the graph.

Those properties simplify the two most difficult parts of the labelling method: the first step of the labelling method and the characterisation of the set $L(x, \Pi_b)$ (i.e. the r-point monodromy). Indeed, the absence of exceptional classes and the existence of a single invariant make the first step almost trivial.

Concerning the characterisation of the set $L(x, \Pi_b)$ for the whole dynamics $\mathfrak{I}_n$ the issue is actually very intricate. Indeed working by induction on graph automorphisms can go wrong very quickly.

However we circumvent the problem by studying a subset of the Rauzy classes of $\mathfrak{I}_n$ for which the characterisation is easy due to the special features of their multigraph invariant. We also establish that the classification of those classes is sufficient to classify the whole Rauzy classes of $\mathfrak{I}_n$. Therefore we avoid the need to prove the characterisation of the set $L(x, \Pi_b)$ in general.

$m = ((17)(24)(38)(56)) \in \mathcal{M}_8$ $\qquad$ $m = ((17)(24)(38)(56)) \in \mathcal{M}_8$

diagram representation $\qquad$ chord diagram representation

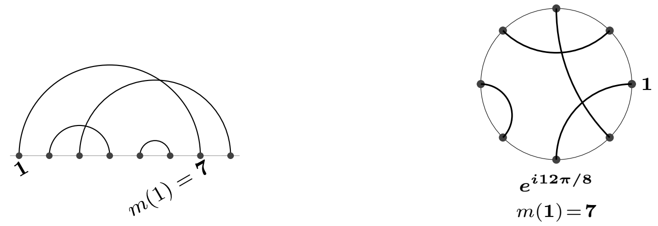


Figure 6.1: Diagram and chord diagram representations of a matching.

For the sake of completeness, in the appendix D we still work out the set $L(x, \Pi_b)$ in general, proving a result strong enough to carry out the induction for the whole set of Rauzy classes. However this will not be needed, in light of the work-around to the problem developed in this section.

In comparison to this relatively easy application of the labelling method, in the $\mathcal{S}_n$ dynamics we have two exceptional classes (Id and Id$'$), three invariants (irreducibility, cycle invariant and arf invariant) and while the set $L(x, \Pi_b)$ does have a description in term of the cyclic shifts of labels of the arcs of the cycles of the cycle invariant it remains complicated.

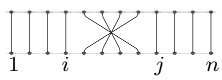
Before defining the involution dynamics, we introduce another representation of matchings over $[2n]$ that we call the *chord diagram* representation. Given a matching over $[2n]$, we represent the chord diagrams over $[2n]$ as arcs in the conformal disk model, connecting pairwise the $2n$-th roots of unity (see figure 6.1).

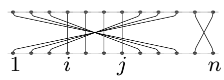
Let us define a special set of permutations:

$$\forall i < j, \quad \gamma_{A^I,n}(i,j) = \quad 1, \dots, i, j-1, j-2, \dots, i+1, j, \dots n \tag{6.1}$$

$$\begin{aligned} \forall i < j, \quad \gamma_{A^E,n}(i,j) = \quad & j \bmod (n) + 1, (j+1) \bmod (n) + 1, \dots, (j+i-2) \bmod (n) + 1, \\ & i, i+1, \dots, j, \\ & (i-1) \bmod (n), (i-2) \bmod (n), \dots, (i-n+j) \bmod (n) \end{aligned} \tag{6.2}$$

I.e in a picture:

$\gamma_{A^I,n}(i,j)$: 

$\gamma_{A^E,n}(i,j)$: 

Let us present the definition of the dynamics. We analyse two main dynamics $\mathfrak{I}_n$ and $\mathfrak{I}_n^{ex}$. We will also define a generalisation to set partitions in section 9.1.

$\mathfrak{I}_n$: The space of configuration is $\mathfrak{M}_n$, the matchings over $[2n]$. There are n generators, $(A_i^I)_i$ one per arc, with $\alpha_{a_{A_i^I(m)}} = \gamma_{A^I,2n}(a_i)$ where a_i is the ith arc of m. See Figure 6.2, Left.

$\mathfrak{I}_n^{ex}$: The space of configuration is $\mathfrak{M}_n$. Now we have $2n$ generators, the $(A_i^I)_i$ as above, and the operators $(A_i^E)_i$, also one per arc, with $\alpha_{a_{A_i^E(m)}} = \gamma_{A^E,2n}(a_i)$ where a_i is the ith arc of m. See Figure 6.2, right.

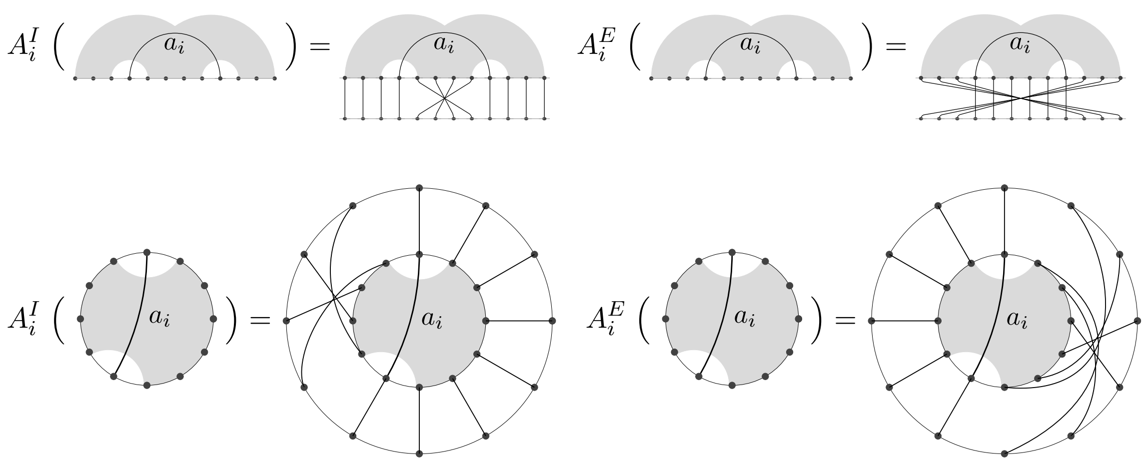


Figure 6.2: Top left: the $\mathfrak{I}_n$ case in diagram representation. Bottom left: the $\mathfrak{I}_n$ case in chord diagram representation. Top and bottom right: in the $\mathfrak{I}_n^{ex}$ case we have the same operators as in the $\mathfrak{I}_n$ case plus the operators $(A_i^E)_i$.

It is clear from the chord diagram representation that the operators A_i^I and A_i^E are 'dual' in the following sense: A_i^I performs a reversing involution inside the arc a_i in the anticlockwise direction while A_i^E does so in the clockwise direction.

As mentioned above, in contrast to the $\mathcal{S}_n$ and $\mathcal{M}_n$ dynamics, there is no need to restrict to some definitions of an 'irreducible' matching. This will simplify the proof in the step 1 of the labelling method since we will not have to justify that the two matchings y_1 and y_2 of size $2n-2$ (reduction of x_1' and x_2' by graying the first arc) are irreducible.

This does not mean that we cannot find some definition of irreducibility: we will identify in the following section a subset of matchings called cup-block matchings for which the involution dynamics factors into a direct product $\times_i \mathfrak{I}_{k_i}$ on the blocks of the matching.

However this property will be reflected on the graph invariant: a matching will be part of the subset if and only if its graph invariant has a 1-cut. Thus the situation is very different from the Rauzy dynamics cases where the irreducibility is not encoded in the cycle invariant and has to be justified separately.

6.1 Definition of the invariant

In this section, we present the only invariant of the involution dynamics for matchings: the multigraph invariant $\mathfrak{G}(m)$.

Let m be a matching, identified with its diagram and let $(a_i)_{1\leq i\leq n}$ be the arcs of m. We say that two arcs $a_i = (x_1, y_1)$, $a_j = (x_2, y_2)$ are adjacent if $|x_1 - x_2| = 1$ or $|y_1 - y_2| = 1$ or $|y_1 - x_2| = 1$ or $|x_1 - y_2| = 1$ and we define the adjacence number of (a_i, a_j) to be the number of time the two arcs are adjacent.

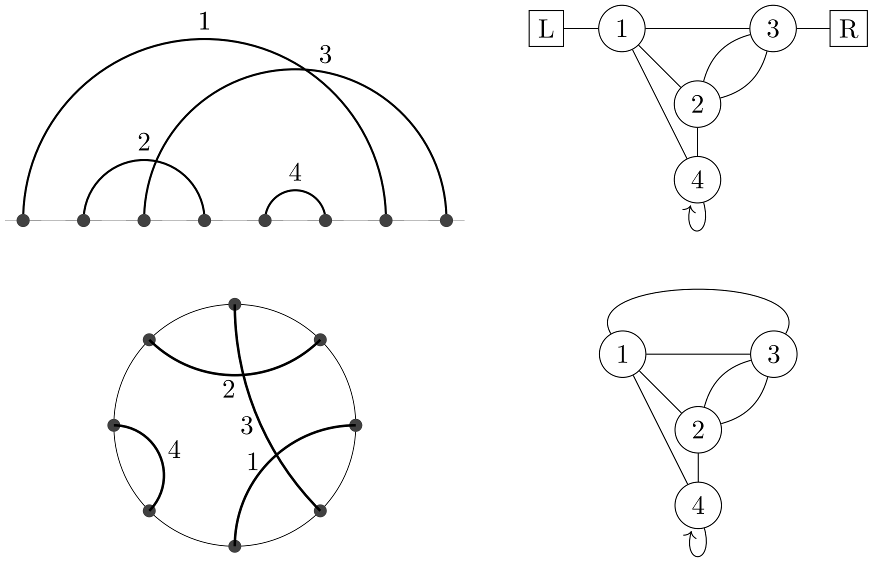


Figure 6.3: Top left: the matching $m = ((1,7),(2,4),(3,8),(5,6))$ in its diagram representation with the arcs labelled from 1 to n. Top right: the construction of the graph invariant $\mathfrak{G}(m)$. The left vertex 1 and the right vertex 3 have only degree 3 while the others have degree 4. There is a multi-edge between vertices 2 and 3 since the arc 3 is adjacent twice to the arc 2.
Bottom left: the matching in its chord diagram representation. Bottom right: The construction of the graph invariant $\mathfrak{G}'(m)$. It is clear that in the chord diagram representation the arcs 1 and 3 are adjacent thus we join the vertices 1 and 3 in $\mathfrak{G}(m)$ the resulting graph is $\mathfrak{G}'(m)$.

Construct the multigraph $\mathfrak{G}(m)$ in the following way: The vertex set $V(\mathfrak{G}(m))$ is $(a_i)_i$ with the addition of two labelled vertices: the vertex L and the vertex R. The vertex L is connected by an edge with the leftmost vertex $a_1 = (1, m(1))$ and the vertex R is connected by an edge with the rightmost vertex $a_\ell = (m(2n), 2n)$. Two vertices a_i, a_j are connected by a multi-edge if and only if they are adjacent and the degree of the multi-edge is the adjacence number of (a_i, a_j). A loop can arise if the vertex $a_i = (j, j+1)$ for some j.

Thus every vertex a_i has degree 4 (loops count for 2) since an arc is adjacent to four (possibly equal) arcs (two per endpoint). The leftmost and rightmost vertices a_1 and a_ℓ are an exception to this rule as one of their endpoints is on the left and right extremities of the matching respectively. However they are connected once to the vertices L and R respectively so their degree is indeed 4. The vertices L and R have obviously degree 1. Confer to figure 6.3 top for an example of the construction.

We have

Proposition 103. *The multigraph $\mathfrak{G}(\cdot)$ with its two fixed vertices L and R is invariant in the $\mathfrak{I}$ dynamics.*

Proof. The idea is that the operators of the dynamics, i.e. the permutations $\gamma_{A^I,2n}(a_i)$ where $a_i = (x, y)$ is an arc, perform local modifications on the endpoints of the arcs, without changing their adjacency relation. Thus the multigraph $\mathfrak{G}$ is preserved.

For all pair of endpoints excepts two, this is completely evident: a pair of adjacent endpoint $(j, j+1)$ are either sent to $(j, j+1)$ if they are outside of the arc a_i or to $(j'+1, j')$ for some j' if they are inside.

For the two pairs of endpoint $(x, x+1)$ and $(y-1, y)$ the situation is a more complicated: $(x, x+1)$ is sent to $(x, y-1)$ and $(y-1, y)$ is sent to $(x+1, y)$. Let a_{i_1} and a_{i_2} be the two arcs with one endpoint being $x+1$ and $y-1$ respectively. In m, a_{i_1} and a_i are adjacent in $(x, x+1)$ and a_{i_2} and a_i are adjacent in $(y-1, y)$. In $A_i^I(m)$, the situation is reversed a_{i_1} and a_i are adjacent in $(y-1, y)$ and a_{i_2} and a_i are adjacent in $(x+1, x)$ since the endpoint $x+1$ (respectively $y-1$) of a_{i_1} (of a_{i_2}) in m was sent to $y-1$ $(x+1)$ in $A_i^I(m)$.

Finally the leftmost vertex and the rightmost vertex are always connected to L and R respectively since the left endpoint 1 and the right endpoint $2n$ of the matching are always sent to 1 and $2n$ respectively.

The figure 6.4 illustrate all the cases. □

In section 6.3, we introduce the labelling of the bottom arcs (cf definition of a combinatorial structure with arc). This will enable us to prove instead a stronger statement: proposition 109. More specifically, for any two connected matchings m, m' we will construct an isomorphism g between the invariant graphs $\mathfrak{G}(m)$ and $\mathfrak{G}(m')$ that will depend on the sequence S of operators connecting m to m'.

By the previous proposition if m and m' are in the same class then their graph invariant $\mathfrak{G}(m)$ and $\mathfrak{G}(m')$ are isomorphic. Thus we must leave no ambiguity on the construction of $\mathfrak{G}(m)$:

Let us indicize (from 0 to $n+1$) the vertices in the order they appear in the matching starting from the left: the vertex i corresponds to the i-th arc of the matching starting from the left. The two fixed vertices L and R are respectively indicized 0 and $n+1$.

Likewise edges are indicized from 0 to $2n$ in the order they appear in the matching starting from the left: the edge i corresponding to the adjacence of the endpoints i and $i+1$ of a pair of arcs i. The edges incident to L and R are respectively indicized 0 and $2n$.

This gives us two bijections

$$\phi_m : [0, n+1] \to V$$

and

$$\psi_m : [0, 2n] \to E$$

that construct the graph $\mathfrak{G}(m) = (V, E)$.

The construction of the invariant for the $\mathfrak{I}^{ex}$ dynamics results from a small modification of the multigraph invariant $\mathfrak{G}$ for $\mathfrak{I}$, which we now describe.

Let us consider the graph $\mathfrak{G}(m)$ and call a and b be the vertices connected to the vertices L and R respectively. Then delete the vertices L and R and connect by an edge a and b. Call $\mathfrak{G}^{ex}(m)$ the resulting multigraph. Clearly $\mathfrak{G}^{ex}(m)$ corresponds to the construction detailed in the beginning of this section but with the adjacency relation of the chord diagram representation instead of the diagram representation. See figure 6.3 bottom.

Then we have

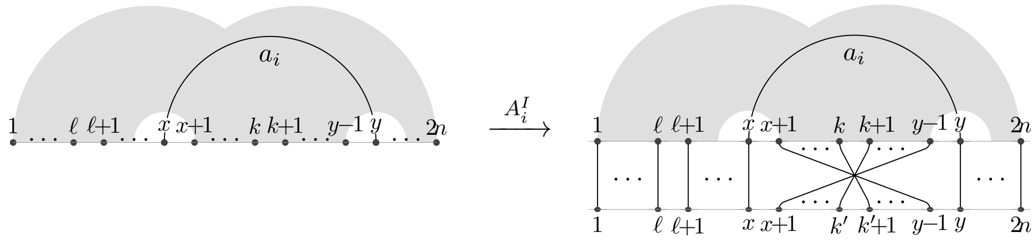


Figure 6.4: Invariance of $\mathfrak{G}(m)$ in $\mathfrak{I}_n$ illustrated for an operator A_i^I (top). It is clear from the figure that the adjacency relation between arcs are preserved.

Proposition 104. *The multigraph $\mathfrak{G}^{ex}(\cdot)$ is invariant in the $\mathfrak{I}^{ex}$ dynamics.*

Proof. It is clear from the chord diagram representation (but not so much in the diagram representation) that for all i the operators A_i^I and A_i^E are symmetric in their action (as shown in figure6.2 bottom), thus just like A_i^I (and as will be proved in proposition 109 in the case of A_i^I) A_i^E performs local modifications the endpoints of the arcs, without changing their adjacency relation.

However, in contrast with A_i^I, the operator A_i^E does no leave invariant the vertices a and b connected L and R in $\mathfrak{G}(m)$: 1 is not necessarily sent to 1 nor is $2n$ sent to $2n$. Nevertheless, the adjacency between a and b (in the chord diagram) is preserved by A_i^E just like any other adjacency. That explains the addition of an edge between those two vertices in $\mathfrak{G}(m)^{ex}$ and the removing of L and R since they were introduced as a marking. □

6.2 The classification theorems

For the $\mathfrak{I}$ dynamics the classification theorem involves the multigraph invariant $\mathfrak{G}$ described in the previous section and is:

Theorem 105. *The number of classes with multigraph invariant $\mathfrak{G}$ depends on the structure of the multigraph, and is*

one, *if $\mathfrak{G}$ is connected, 4-regular except for the two vertices L and R, which are of degree 1, with self-loops that count for 2 in the degree of the vertex.*

zero, *otherwise.*

For the $\mathfrak{I}^{ex}$ dynamics the classification theorem involves the multigraph invariant $\mathfrak{G}^{ex}$ described in the previous section and is:

Theorem 106. *The number of classes with multigraph invariant $\mathfrak{G}^{ex}$ depends on the structure of the multigraph, and is*

one, *if $\mathfrak{G}^{ex}$ is connected, 4-regular, with self-loops that count for 2 in the degree of the vertex.*

zero, *otherwise.*

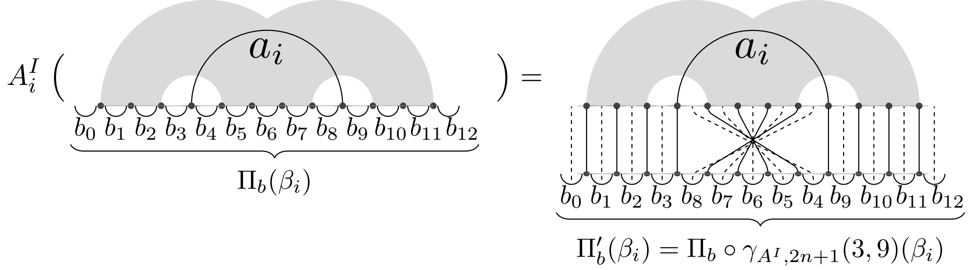


Figure 6.5: The action of the dynamics on the labels of the bottom arcs. In this case $a_i = (4, 9)$, thus $\gamma_{A^I,2n+1}(3, 9)(\beta_i) = \beta_i$ if $\beta_i \leq 3$ or $\beta_i \geq 9$ and $\gamma_{A^I,2n+1}(3, 9)(\beta_i) = 9 - (\beta_i - 4) - 1$ if $4 \leq \beta_i \leq 8$.

6.3 Amenability of the labelling method for $\mathfrak{I}_n$

In this section we introduce the notions necessary to show that $\mathfrak{I}_n$ is amenable to the labelling method. Therefore, our task is to define a boosted dynamics and a labelling Π_b compatible with it as well as to provide a quantified description of the set $L(m, \Pi_b)$ for any matching m.

Following the construction of the *combinatorial structure with arcs* of section 5 we enrich the matchings over $[2n]$ of $2n + 1$ bottoms arcs connecting adjacent vertices. Let $\Sigma = \{b_0 \ldots, b_{2n}\}$ be an alphabet, we define a *labelling of* $m \in \mathfrak{M}_n$ to be a bijection

$$\Pi_b \quad : \quad \{0, \ldots, 2n\} \quad \to \quad \Sigma$$

that labels the bottom arcs of m.

The dynamics can naturally be extended to act on the labelling:

Definition 107 (Dynamics $\mathfrak{I}_n$ on the labelling)**.** *Let (m, Π_b) be a matching with a labelling. Then $A_i^I(m, \Pi_b) = (A_i^I(m), \Pi_b \circ \gamma_{A^I,2n+1}(x - 1, y))$, where $a_i = (x, y)$ is the i-th arc of m (given our definition of a labelling $\gamma_{A^I,2n+1}(x - 1, y)$ acts on $\{0, \ldots, 2n\}$ rather than $\{1, \ldots, 2n + 1\}$). See figure 6.5*

The labelling Π_b of the bottom arcs of a matching m naturally provides a labelling $\Pi_b \circ \psi^{-1}$ of the edges of $\mathfrak{G}(m)$. See figure 6.6 for an example. When there will be no confusion we will often write Π_b for $\Pi_b \circ \psi^{-1}$ and thus say that Π_b is a labelling of $E(\mathfrak{G}(m))$. The labelling of the edges of the graph also yields a labelling of the vertices of the graph since every vertex is uniquely identified by the labels of its incident edges. Thus Π_b can designate a labelling of both the edge set and the vertex set of $\mathfrak{G}(m)$.

Let us recall the definition of a graph isomorphism adapted to our context.

Definition 108. *We say that $g : E(\mathfrak{G}(m)) \to E(\mathfrak{G}(m'))$ bijection is an* isomorphism *of the edge-set of* $\mathfrak{G}(m)$ *if $\exists f : V(\mathfrak{G}(m)) \to V(\mathfrak{G}(m))$ bijection (fixing the vertices L and R) such that $e \in V(\mathfrak{G}(m))$ connects $u, v \in V(\mathfrak{G}(m))$ if and only if $g(e)$ connects $f(u)$ and $f(v)$.*

We say that $f : V(\mathfrak{G}(m)) \to V(\mathfrak{G}(m'))$ bijection (fixing the vertices L and R) is an isomorphism of the vertex-set of $\mathfrak{G}(m)$ *if $\exists g : E(\mathfrak{G}(m)) \to E(\mathfrak{G}(m))$ bijection*

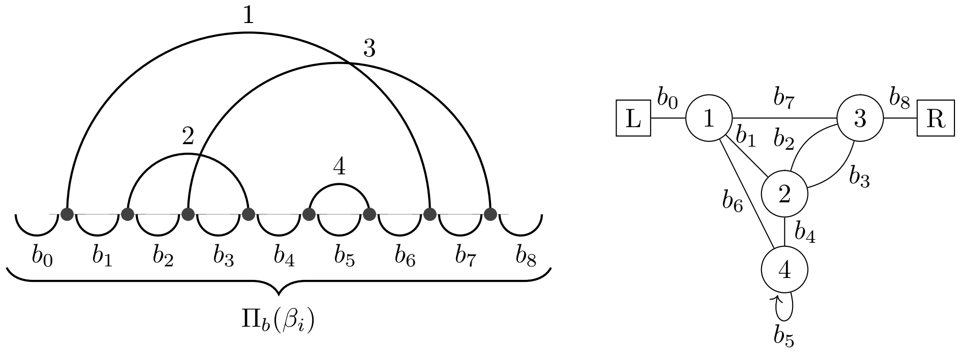


Figure 6.6: Left: the matching $m = ((1,7),(2,4),(3,8),(5,6))$ with a labelling Π_b. Right: the construction of the graph invariant $\mathfrak{G}(m)$ with the edge-labelling inherited from the labelling Π_b of m.

such that $e \in V(\mathfrak{G}(m))$ connects $u, v \in V(\mathfrak{G}(m))$ if and only if $g(e)$ connects $f(u)$ and $f(v)$.

Clearly the existentially quantified f is an isomorphism of the vertex-set and is unique. Thus to every isomorphism of the edge set, we associate a unique isomorphism of the vertex set.

Likewise the existentially quantified g is an isomorphism of the edge-set. However it is not unique: if a multi-edges $e = \{e_1, \dots, e_k\}$ connects two vertices u and v then any g' such that $g = g'$ on $E(\mathfrak{G}(m)) \smallsetminus e$ and $g'(\{e_1, \dots, e_k\}) = g(\{e_1, \dots, e_k\})$ is valid. Thus if f is an isomorphism of the vertex set, any two isomorphisms g, g' of the edge set associated to it are equal up to permutations of the labels of the multi-edges.

We have finally introduced all the notions needed to carry out the proof of a stronger version of proposition 103. Indeed we have

Proposition 109. *Let (m, Π_b) be a matching with a labelling, let S be a sequence of operators and let $(m', \Pi'_b) = S(m, \Pi_b)$. Finally, let us define $g : [0, 2n] \to [0, 2n]$ such that $\Pi'_b = \Pi_b \circ g$. Then, $\mathfrak{G}(m)$ and $\mathfrak{G}(m')$ are isomorphic since $\psi_{m'} \circ g \circ \psi_m^{-1} : E(\mathfrak{G}(m)) \to E(\mathfrak{G}(m'))$ is an isomorphism of the edge set.*

Proof. We prove the statement for a single operator A_j^I and the rest follows by induction.

The proof for a single operator can be easily deduced from the proof of proposition 103 combined with the definition of the dynamics on the labelling.

Indeed, proposition 103 shows that the adjacency between arcs is preserved by the operators, thus we have $f : V(\mathfrak{G}(m)) \to V(\mathfrak{G}(m'))$ bijection such that $u, v \in V(\mathfrak{G}(m))$ are adjacent if and only if $f(u)$ and $f(v)$ are. Now by choice of our definition of the dynamics on the labelling (definition 107) $\Pi'_b = \Pi_b \circ g$ with g verifying: $e \in V(\mathfrak{G}(m))$ connects $u, v \in V(\mathfrak{G}(m))$ if and only if $g(e)$ connects $f(u)$ and $f(v)$. $\square$

We have announced at the beginning of this part that for any matching m and labelling Π_b the set $L(m, \Pi_b)$ was in bijection with the automorphism group of the edge-set of $\mathfrak{G}(m)$. Finally we can state this quantifying theorem for $L(m, \Pi_b)$.

Theorem 110. *Let (m, Π_b) be a matching with a labelling. The set $L(m, \Pi_b)$ is in bijection with the automorphism group of the edge-set of $\mathfrak{G}(m)$ in the following sense: $\Pi'_b \in L(m, \Pi_b)$ if and only if there exists (a unique) $g : E(\mathfrak{G}(m)) \to E(\mathfrak{G}(m))$ automorphism of $\mathfrak{G}(m)$ such that $\Pi'_b = \Pi_b \circ \psi_{m'}^{-1} \circ g \circ \psi_m$.*

By proposition 109 the 'only if' part is proven, the 'if' part will be handled during the induction after having proven the classification theorem 105.

Clearly the theorem solves the question of the r-point monodromy for every r, as the labels of r arcs $b_1, \ldots, b_r$ be can sent to the positions $\beta_1, \ldots, \beta_r$ respectively if and only if there is a graph automorphism that sends the edge $\Pi^{-1}(b_i)$ to the edge β_i for all i.

To conclude this section on the amenability of the dynamics $\mathfrak{I}_n$, we must define the boosted dynamics and prove its compatibility with the labelling.

Definition 111 (boosted dynamics for $\mathfrak{I}_n$)**.** *Let m be a matching and let m' be the reduction of a $(2k, 2r)$ coloring c of m. Let $(A^I_{i_j})_{j \le k}$ be the operators associated to the k black arcs of m, then the boosted sequence of an operator A^I_j for (m, c) is $B(A^I_j) = A^I_{i_j}$. Indeed, the reduction m'_1 of $(m_1, c_1) = A^I_{i_j}(m, c)$ verifies $m'_1 = A^I_j(m')$.*

Thus, by induction, we can define for every sequence S the boosted sequence $B(S)$ of S for (m, c).

Clearly our choice of labelling is compatible with the boosted dynamics:

Proposition 112 (Labelling compatible with boosted dynamics)**.** *Let $m \in \mathfrak{M}_n$ be a matching and let $m' \in \mathfrak{M}_k$ be the reduction of a $(2k, 2r)$-coloring c of m. Let Π_b be a labelling of m' and let S be a sequence of operator for m'. Let a_i be a gray arc of m with endpoints between the arcs with labels $b, b' \in \Sigma$ of (m', Π_b).*

Then in $(m_1, c_1) = B(S)(m, c)$ the gray arc a_i has its endpoints between the arcs with labels b, b of $(m'_1, \Pi'_b) = S(m', \Pi_b)$.

Proof. We prove the statement for a single operator A^I_j and the rest follows by induction. Let $(A^I_{i_j})$ be the operators associated to the k black arcs of m then $B(A^I_j) = A^I_{i_j}$ by definition of the boosted sequence. Consider an endpoint of a gray arc a of m that is between the bottom arc with label b in m'. There are two cases:

- The position of the bottom arc $\beta = \Pi_b^{-1}(b)$ is outside of the arc $a_p = (x, y)$ of m', i.e. $\beta < x$ or $\beta \geq y$.
- The position of the bottom arc $\beta = \Pi_b^{-1}(b)$ is inside the arc $a_p = (x, y)$ of m', i.e. $\beta \geq x$ and $\beta < y$.

In both cases it is clear from figure 6.7 that the gray endpoint in (m_1, c_1) will still be between the arc with label b of (m'_1, Π'_b).

□

6.4 Preparing the induction

This section assembles all the necessary propositions for the induction, so that the discussion of the inductive step can be made as compact as possible. More specifically, we start by proving the existence part of the labelling method in the subsection 6.4.1. Then the following subsection proves Proposition 119 that solve the (small) difficulty of the first step of the induction.

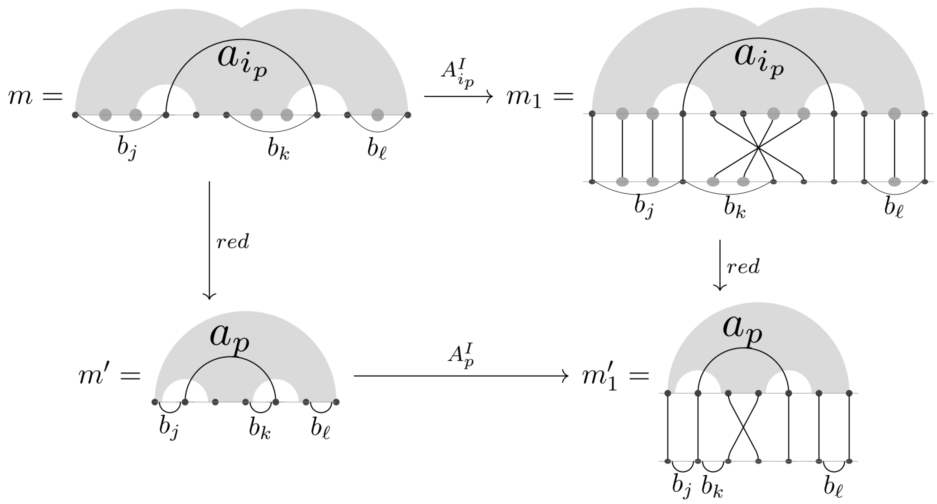


Figure 6.7: The gray vertices (represented bigger in the figure) of m are transported along the labels of the bottom arcs when applying the boosted dynamics.

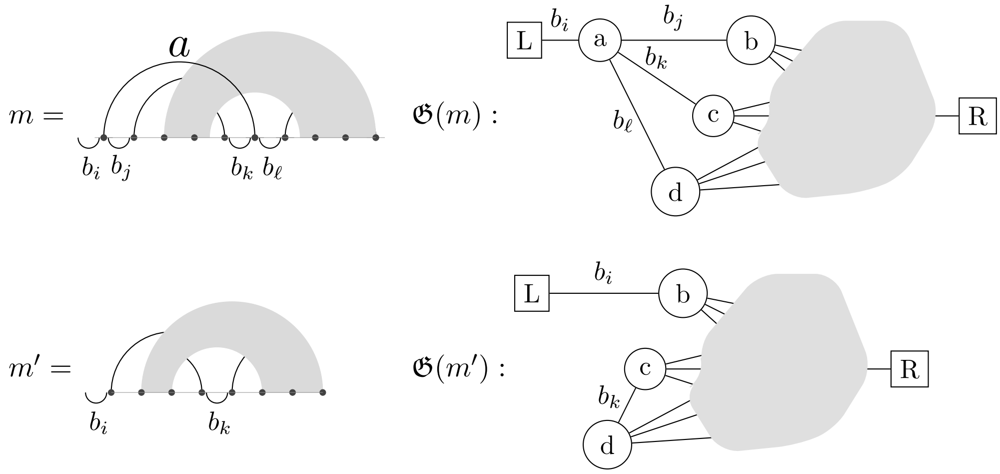


Figure 6.8: Change of the graph invariant when removing the first arc the matching m.

6.4.1 Existence of a matching for every graph invariant $\mathfrak{G}$

Let us consider what happens to the graph invariant when adding a new left-most arc to a matching m', or, symmetrically, when removing the left-most arc from a matching m.

Proposition 113. *Let m be a matching, and let m' be the matching obtained by removing the first arc $(1, m(1))$ of m. The graph invariant of $\mathfrak{G}(m')$ is obtained from $\mathfrak{G}(m)$ as follows:*

Let a be the vertex adjacent to L. If $m(1) = 2$, $\mathfrak{G}(m)$ has a loop at a, and is connected with a simple edge to a vertex b associated to the edge $(3, m(3))$. Then $\mathfrak{G}(m')$ is obtained by removing a together with its incident edges, and connecting L to b. Otherwise, call b the vertex associated to the arc $(2, m(2))$, which is thus

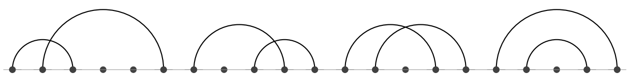


Figure 6.9: The four types of double arcs. Form left to right: type A, B, C, D

adjacent to a, and call c and d its other two neighbours (here it is understood that it could be $d = c$, or $c = b$, or both). Then $\mathfrak{G}(m')$ is obtained by removing a together with its incident edges, adding one edge between L and b, and a second edge between c and d. The construction is represented in figure 6.8.

We now prove, by induction, the existence part of the labelling method. Let us suppose that, up to size $n-1$, for every graph invariant there exists a matching with such invariant. Let $\mathfrak{G}$ be a graph invariant at size n, let a be the neighbour of L and b, c and d be the neighbours of a (again, possibly some items of the list $\{a, b, c, d\}$ may be repeated). Consider $\mathfrak{G}'$, the graph obtained from the procedure of Proposition 113. In particular, this graph must have one (or possibly more) edges connecting c and d. By induction there exists a matching m' with invariant $\mathfrak{G}'$. Let $(i, i+1)$ be the arc associated to the edge (or one of the edges) connecting c and d. Then, adding an arc with one endpoint to the far left, and the other one between i and $i+1$ provides a matching with invariant $\mathfrak{G}$.

6.4.2 Key proposition for the first step

We have considered so far the dynamics $\mathfrak{I}_n$ acting on the whole set $\mathfrak{M}_m$. At this point we restrict the study to a subset of matchings, which is closed for the dynamics, and is thus constituted by subset of the set of Rauzy classes for $\mathfrak{I}_n$. It will turn out that: (1) performing the classification on this subset will be considerably easier than performing directly the full classification; (2) the full classification will follow, at the end, as a remarkably simple corollary, through the observation that the dynamics $\mathfrak{I}_{2n}$ on the subset has implications on the dynamics $\mathfrak{I}_n$ on the full set.

Let us first define this subset of matchings.

Definition 114. *Let $m \in \mathfrak{M}_{2n}$ be a matching with an even number of arcs. We say that m is a* double-matching configuration*, and write $m \in \mathfrak{M}_n^{\mathrm{d}}$, if there exists a pairing of the arcs of m such that, for every pair $(a, b) = ((i, j), (j', j'))$ (also called a* double-arc*), one of the four conditions hold:*

type A: *$a = (i, j)$ and $b = (i-1, i+1)$, with $j - i \geq 2$.*

type B: *$a = (i, j)$ and $b = (j-1, j+1)$, with $j - i \geq 2$.*

type C: *$a = (i, j)$ and $b = (i+1, j+1)$, with $j - i \geq 2$.*

type D: *$a = (i, j)$ and $b = (i+1, j-1)$, with $j - i \geq 3$.*

(in all cases, it is intended that the role of a and b can be interchanged).

Note that, in the graph invariant of such a configuration, the two vertices of a pair of arcs are connected by a double edge. Also note that, if $(a, b) = ((i, i+2), (i+1, i+3))$, the double-arc is simultaneously of type A, B and C, while in all other

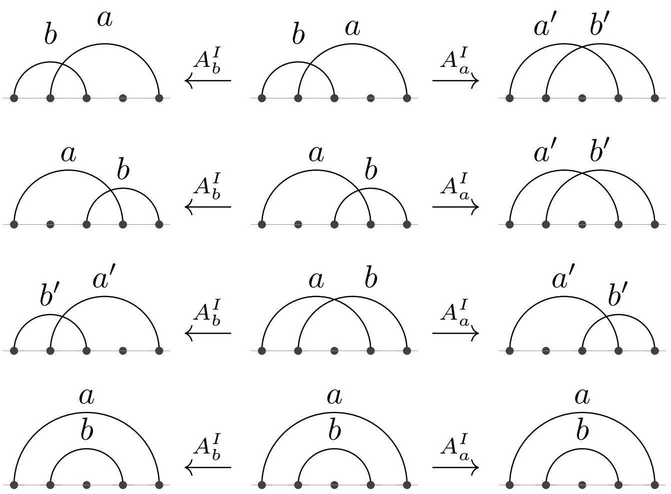


Figure 6.10: The change of type of a double-arc (a, b) under the operations A^I_a and A^I_b.

circumstances the type is unique. The forementioned special case, however, is of little interest, as it corresponds to a trivial part of the dynamics.

We can characterise the set of double-matchings via the associated graph invariants:

Proposition 115. *A matching $m \in \mathfrak{M}_{2m}$ allows for at most a single pairing of the arcs which makes it a double-matching $m \in \mathfrak{M}^{\mathrm{d}}_n$. It allows for such a pairing if and only if the vertices of $\mathfrak{G}(m)$, excluding $\{L, R\}$, can be covered by disjoint cycles of length 2 (i.e., if it has a perfect matching of double-edges, that is a perfect matching in which the edges choosen for the matching are all double edges).*

Proof. We have already observed that pairs of arcs in a double-matching provide double-edges in the graph invariant, so, in order to have n disjoint pairs, it is necessary and sufficient to have n vertex-disjoint pairs of double-edges. This characterisation allows to easily establish the uniqueness. If we have a triple-edge, we are in the forementioned trivial situation of $(a, b) = ((i, i+2), (i+1, i+3))$: unicity still holds, and the three possible choices of cycle-coverings are associated to the three 'types' that this pair is simultaneously. In a connected graph of maximal degree 4 and average degree smaller than 4, double-edges occur either isolated, or in linear chains. If the linear chain has odd length, it admits a unique covering by 2-cycles, otherwise the graph admits no coverings at all.

Definition 116. *In the following we will often consider the reduction m' of a double-matching (m, c) where c is a $(n-4, 4)$-coloring in which the left-most double-arc of m is grayed. We call $r : \mathfrak{M}^{\mathrm{d}}_n \to \mathfrak{M}^{\mathrm{d}}_{n-1}$ this operation, i.e. define $r(m) = m'$.*

Lemma 117. *Let m be a double-matching and (a, b) a double-arc. The image of (a, b) by either A^I_a or A^I_b is also a double-arc. The change of type are illustrated in figure 6.10 and listed in table 6.1.*

Proposition 118. *The set $\mathfrak{M}^{\mathrm{d}}_n$ is invariant under the dynamics $\mathfrak{I}_n$.*

Operator \ Type	A	B	C	D
A_a^I	C	C	B	D
A_b^I	A	B	A	D

Operator \ Type	A	B	C	D
A_c^I	A	B	D	C
$A_{c'}^I$	B	A	C	D

Table 6.1: The first table lists the type of the image of a double-arc of a given type, after the action of its two operators A_a^I and A_b^I. (See figure 6.10 for a proof)
The second table lists the type of the image of a double-arc of a given type, after the action an operator A_c^I associated to an arc with a single endpoint between i and j (with notations as in Definition 114) or an operator $A_{c'}^I$ associated to an arc that contain the double-arc.

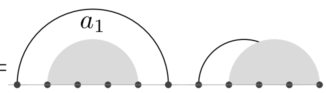
Indeed, we have already described, in Table 6.1, how the image of a double-arc by the involution operators of itself or of another double-arc is a double-arc.

From now on, we will concentrate our effort on proving a proposition for the first step of the induction (of the labelling method):

Proposition 119 (Key proposition for the first step)**.** *Let m_1 and m_2 be two double-matchings with the same invariant $\mathfrak{G}$. Then there exist sequences S_1 and S_2 such that $r(S_1(m_1))$ and $r(S_2(m_2))$ have the same graph invariant $\mathfrak{G}'$.*

Let us introduce a few definitions and lemmas before starting the proof.

Definition 120. *A matching m is* cup-block *if $m([1, \dots, m(1)]) = [1, m(1)]$. See below:*

$m =$ 

Proposition 121. *If a matching m is* cup-block *then the dynamics $\mathfrak{I}_n$ factorizes into the product $\mathfrak{I}_{m(1)} \times \mathfrak{I}_{n-m(1)}$ (i.e. the class is composed only of cup-block matchings, with the same value of $m(1)$, and the two factors in the product above commute).*

Proof. The statement is obvious in light of the rules of the dynamics.

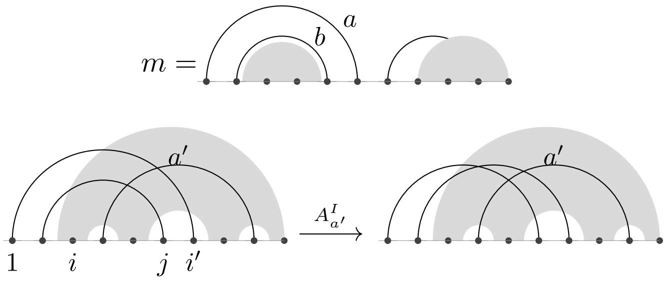
Proposition 122. *Let m be a double-matching. Exactly one of the two following statements holds:*

- *There exists S such that the left-most double-arc of $m' = S(m)$ is of type B.*
- *m is cup-block, and the left-most double-arc is of type D.*

Proof. Let (a, b) be the left-most double-arc of m.

- If (a, b) is of type B, then we are done.
- If (a, b) is of type C, then its image under A_a^I is of type B, as show in figure 6.10.

- If (a,b) is of type A, then its image under $A^I_b A^I_a$ is of type B (see again figure 6.10).
- Consider now the case in which (a,b) is of type D. If m is cup-block, then these two arcs are fixed by the dynamics. Otherwise, there exists an arc c crossing a and b, and the image of (a,b) under $A^I_{a'}$ is of type C, so we conclude as above. These two cases are illustrated by the following images:



□

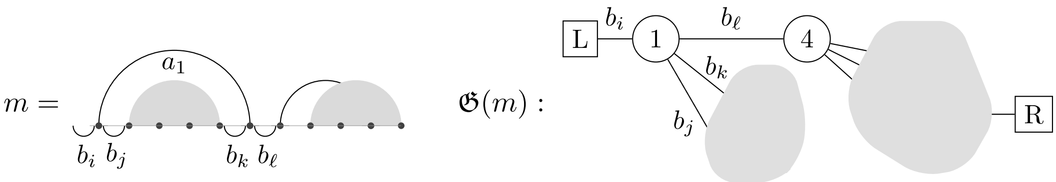
For simplicity we call m a *double-matching of type B* if its left-most double-arc is of type B.

We have thus extablished that, in $\mathfrak{M}^{\mathrm{d}}_n$, there are two types of Rauzy classes: those where all the double-matchings are cup-block (and have a first double-arc of type D), and those where a double-matching of type B can be found, and is in fact at distance at most 3. These two types of classes will be mostly treated in a slightly different way in the proofs to follows.

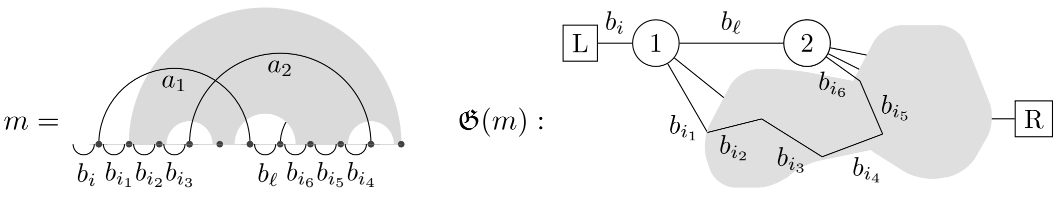
For the graph invariant $\mathfrak{G}$ to be complete we need to prove that at least those two types of Rauzy classes are discriminated. This result is established in corollary 125.

Lemma 123. *A matching m is cup-block if and only if the edge $e = (m(1), m(1)+1)$ of the graph $\mathfrak{G}(m)$ is a 1-cut separating the vertices L and R.*

Proof. $\boxed{\Longrightarrow}$ This follows from the following figure:



$\boxed{\Longleftarrow}$ By contraposition, if m is not cup-block there is an arc $a_2 = (x, m(x))$ such that $x < m(1)$ and $m(x) > m(1)$ then the sequence of edges $(1,2), \dots, (x-1,x), (m(x)-1, m(x)), (m(x)-2), m(x)-1), \dots, (m(1)+1, m(1)+2)$ forms a path from vertex a_1 to the vertex connected to a_1 by the edge e. Thus e is part of a cycle, and cannot be a 1-cut edge. See figure below:



□

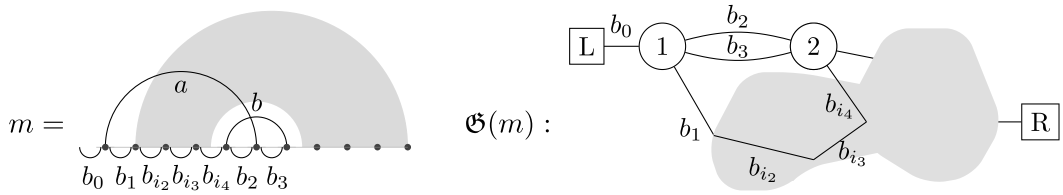
Notation 3. *Let $(\mathfrak{G}, \Pi_b)$ be a labelled graph invariant associated to a double-matching m, and let (a, b) be the two vertices of the left-most double-arc. We say that Π_b is a 0-5-labelling if:*

The edges incident to a are labelled $b_0, \ldots, b_3$ and b_0 is the label of the edge connecting L and a.

The two remaining edges incident to b (a and b are connected by a double edge by definition of a double-arc) are labelled b_4, b_5.

Lemma 124. *Let m be a double-matching such that the (a, b) is the left-most double-arc, it is of type B, and $a = (1, j)$. Let $(\mathfrak{G}(m), \Pi_b)$ be the graph invariant of (m, Π_b), equipped with a 0-5-labelling. Then none of the edges labelled b_1, b_2 and b_3 incident to the vertex associated to the arc a are 1-cut edges.*

Proof. Since (a, b) is of type B, the edges b_2 and b_3 are multiedges connecting $a = (1, j)$ with $b = (j-1, j+1)$, thus none of them can be a 1-cut edge. Let a_3 be the other vertex incident to b_1, then the sequence of edges $b_1, (2, 3) \ldots (j-2, j-1) b_2$ form a cycle on vertex a_1, thus the edge b_1 cannot be a 1-cut edge. See figure below:



□

Corollary 125. *Let m and m' be two double-matchings such that the first double-arc of m is of type B and m' is cup-block. Then $\mathfrak{G}(m)$ and $\mathfrak{G}(m')$ are not isomorphic.*

Proof. The proof is a consequence of Lemmas 123 and 124 □.

The last step before proving proposition 119 is to understand how the graph invariant changes when removing the left-most double-arc.

Proposition 126 (Change of graph invariant: type B)**.** *Let m be a double-matching of type B, let (a, b) be its left-most double-arc, and let $a = (1, m(1))$. Let $m' = r(m)$. Then the graph invariant $\mathfrak{G}(m')$ is obtained from $\mathfrak{G}(m)$ as follows:*

Let c be the vertex adjacent to a, different from L and b, and let d, e be the two vertices adjacent to b, different from a, (but possibly coincident), then:

- *Delete the vertices a and b.*
- *Connect by a simple edge L to c, and d to e.*

The construction is represented in figure 6.11.

Definition 127. *Let $\mathfrak{G}$ be a graph invariant and $f = (d, e)$ an edge of $\mathfrak{G}$, we define $C_e(\mathfrak{G})$ to be the graph obtain by the reverse operation of proposition 126. More precisely:*

- *Delete the edge f and $g = (L, c)$.*

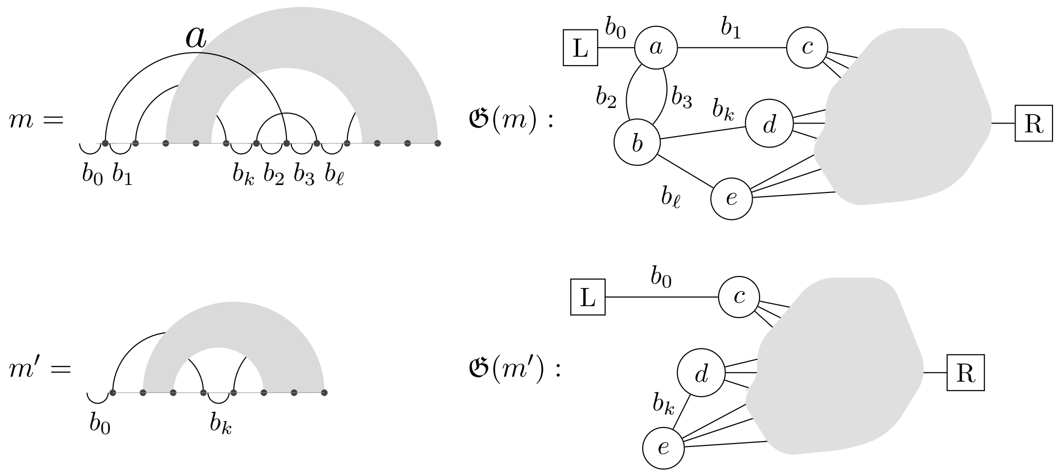


Figure 6.11: Change of the graph invariant when removing the first double-arc of type B of the double-matching m.

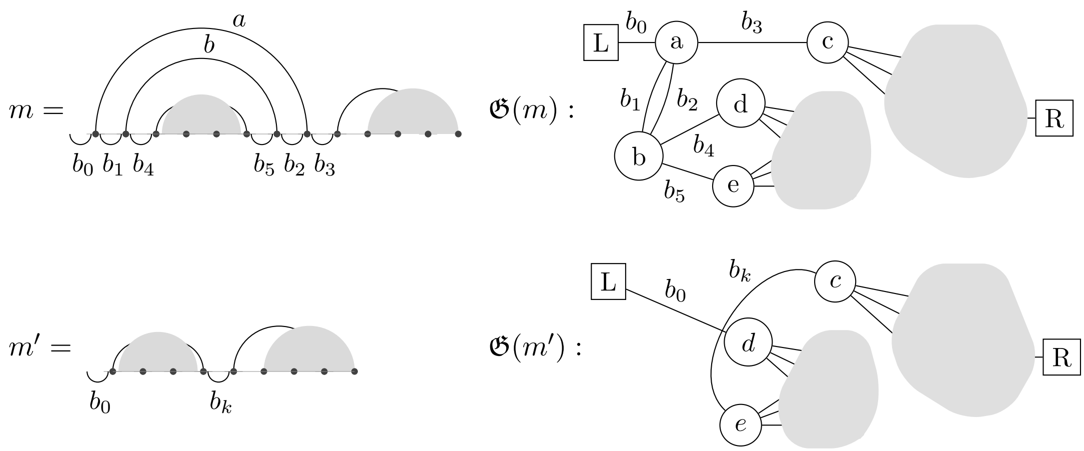


Figure 6.12: Change of the graph invariant when removing the first double-arc of type D of the cup-block double-matching m.

- *Add two vertices a and b.*
- *Connect a to L and c, b to d and e and add a double edge between a and b*

e.g. in figure 6.11 $C_{b_k}(\mathfrak{G}(m') = \mathfrak{G}(m)$

Given a graph invariant $\mathfrak{G}$ (of a double-matching of type B), call $r(\mathfrak{G})$ the graph invariant obtained by this procedure. Then we have:

Corollary 128. *Let m_1 and m_2 be two double-matchings of type B with the same graph invariant $\mathfrak{G}$. Then $r(m_1)$ and $r(m_2)$ have the same graph invariant $r(\mathfrak{G})$.*

This proves Proposition 119 for the classes of type B. Let us now consider cup-block classes:

Proposition 129 (Change of graph invariant: Type cup-block)**.** *Let m be a cup-block double matching, and let (a, b) be its left-most double-arc, with $a = (1, m(1))$. Let $m' = r(m)$. Then the graph invariant of $\mathfrak{G}(m')$ is obtained from $\mathfrak{G}(m)$ as follows:*

Let c be the vertex adjacent to a, different from L and b, and let d, e be the two vertices adjacent to b, different from a, (but possibly coincident). If they are distinct, fix the labels as by setting $d = (3, m(3))$. Then:

- *Delete the vertices a and b.*
- *Connect by a simple edge L to d and c to e.*

The construction is represented in figure 6.12.

We could have exchanged the role of d and e, obtaining a generally different graph, but with the same characteristics. Thus it is clear that, for given a graph $\mathfrak{G}$, there are two (possibly isomorphic) graphs that can be obtained when applying the procedure, by connecting L to either of the two vertices adjacent to b. We summarize this result below.

Corollary 130. *Let $(\mathfrak{G}, \Pi_b)$ be a 0-5-labelled graph invariant. Let (m, Π_b) be a matching with labelled graph invariant $(\mathfrak{G}, \Pi_b)$ and let $m' = r(m)$.*

Then the graph $\mathfrak{G}(m')$ depends on which b_i verifies $\Pi_b^{-1}(b_i) = 2$ for $i \in \{4, 5\}$. The full description is specified in figure 6.13.

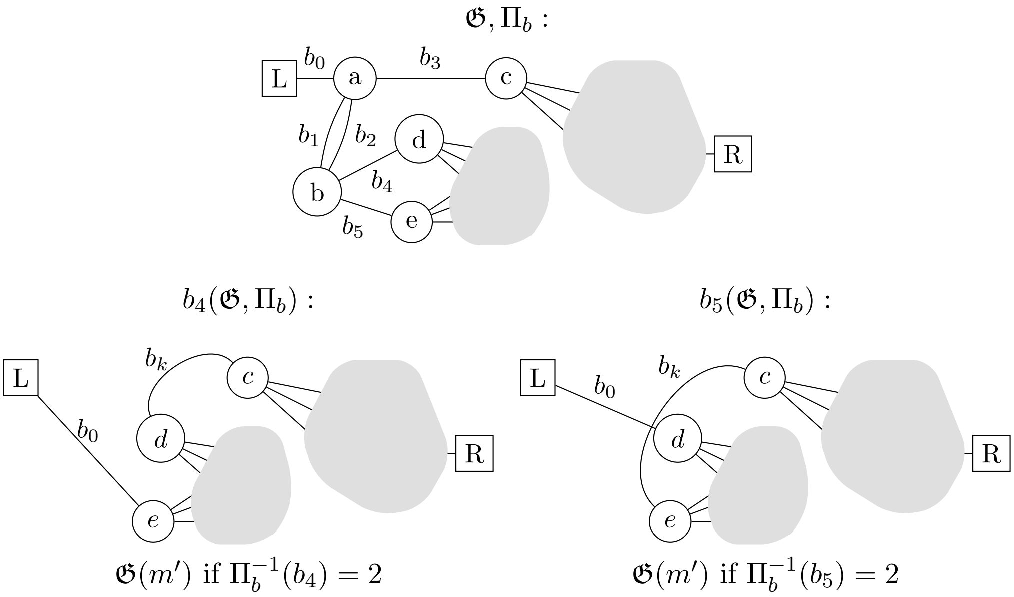


Figure 6.13: The two possible graph invariants associated to m'.

Definition 131. *Keeping the notation of the previous corollary, we define $b_i(\mathfrak{G}, \Pi_b)$ for $4 \leq i \leq 5$ to be the graph associated to a double-matching $m' = r(m)$ such that (m, Π_b) is a labelled cup-block double-matching and $\Pi_b^{-1}(b_i) = 2$. See figure 6.13.*

Next we establish that, in a given class, we can always find two matchings m_4 and m_5 with invariant $(\mathfrak{G}, \Pi_b)$ such that the reductions $r(m_i)$ have graph invariant $b_i(\mathfrak{G}, \Pi_b)$ for $4 \leq i \leq 5$. This is clearly the case since, given a matching m, the action of A_a^I, the operator associated to the left-most arc, will swap b_4 and b_5:

Corollary 132. *Let $(\mathfrak{G}, \Pi_b)$ be a 0-5-labelled graph invariant. Let (m, Π_b) be a double matching with labelled graph invariant $(\mathfrak{G}, \Pi_b)$ and let $m' = r(m)$. Finally let (a, b) be the left-most double-arc of m.*

If $\Pi_b^{-1}(b_4) = 2$, then $(m', \Pi_b') = A_a^I(m, \Pi_b)$ has $\Pi_b^{-1}(b_5) = 2$, and vice-versa.

We can now finally prove the content of proposition 119.

Proof of proposition 119. Let m_1 and m_2 be two double-matchings with the same graph invariant $\mathfrak{G}$. By Proposition 121 and corollary 125 they are either both connected to a double-matching of type B or both cup-block.

In the first case, let S and S' be the sequence such that $S(m_1)$ and $S(m_2)$ are of type B, then by Corollary 128 $r(S(m_1))$ and $r(S(m_2))$ have the same graph invariant.

In the second case, let Π_b be a 0-5 labelling of $\mathfrak{G}$ and let Π_b^1 and Π_b^2 be the corresponding labelling for m_1 and m_2 respectively. $r(m_1)$ has graph invariant $b_i(\mathfrak{G})$ for $i = 4$ or $i = 5$ then by Corollary 132 either $r(m_2)$ or $r(A_a^i(m_2))$ (or both, if d and e coincide) has the correct graph invariant. □

6.5 Carrying out the induction

In the present section, we proceed to the proof of Theorems 105 and 110, which will be performed by induction. More precisely we prove by induction the two following statements:

- Every pair m_1, m_2 of matchings with the same graph invariant $\mathfrak{G}$ are connected.
- Theorem 110.

By induction the statements are true for size smaller than n.
Let us prove the inductive step $\boxed{n - 1 \implies n.}$

• The first statement is proved using labelling method. Let m_1 and m_2 be two double-matchings with the same graph invariant $\mathfrak{G}$.

- By proposition 119 there exists m_1' and m_2' with graph invariant $\mathfrak{G}'$ such that $m_1' = r(x_1)$ and $m_2' = r(x_2)$, with x_1 and x_2 connected to m_1 and m_2 respectively. For $i = 1, 2$, let c_i be the $(2n - 4, 4)$-coloring of x_i such that the reduction of (x_i, c_i) is m_i'.
- By induction (using the first statement), there is a sequence S such that $S(m_1') = m_2'$, since these two configurations have the same graph invariant.
- The boosted dynamics and a labelling compatible with it have been defined in Definition 111 and 107, and the proof of compatibility has been carried out in Proposition 112.
- Let Π_b be a 0-5-labelling for m_1'. The two gray endpoints of the left-most double-arc of (x_1, c_1) are within the arcs with labels b_0 and b_i, for some i.
- By the boosted dynamics, there exists S', a boosted sequence of S, for (x_1, c_1). Let $(x_3, c_3) = S'(x_1, c_1)$. By definition of a boosted sequence the reduction of (x_3, c_3) is m_2'.

- By compatibility of the labelling with the boosted dynamics, the two gray endpoints of (x_3, c_3) are within the bottom arcs with label b_0 and b_i of $(m'_2, \Pi'_b) = S(m'_1, \Pi_b)$.

- If m_1 and m_2 are cup-block, then $x_3 = x_2$ since the invariant is cup-block the first double-arc has type D and fixed position $(1, m_1(1))$. Thus the gray double-arc of (x_3, c_3) is at the same position of the gray double-arc of (x_2, c_2) that is position $(1, m_1(1) = m_2(1) = x_1(1) = x_2(1) = x_3(1))$.

- Otherwise, by proposition 109, $\Pi'_b = \Pi_b \circ g$ with g an automorphism of the edge-set of $\mathfrak{G}'$. Since L is a fixed vertex, the edge labelled b_0 connecting L to a must also be fixed by g, thus $(\Pi'_b)^{-1}(b_0) = 0$. On the other hand then the gray endpoints of (x_2, c_2) are within the bottom arcs with position 0 and β of m'_2.

- Let us show that there is an edge-automorphism g' of $\mathfrak{G}'$ such that if $\Pi''_b = \Pi'_b \circ g'$ then $(\Pi''_b)^{-1}(b_i) = \beta$. In other words, let $e = (u, v)$ be the edge of $\mathfrak{G}'$ associated to the label b_i of (m_2, Π'_b) and let $f = (u', v')$ be the edge of $\mathfrak{G}'$ associated to the position β of m'_2, we want to prove that there is an automorphism sending e to f.

  Consider $\mathfrak{G}(x_3) = C_e(\mathfrak{G}')$ and $\mathfrak{G}(x_2) = C_f(\mathfrak{G}')$ where C_ℓ is the constructor defined in Definition 127. Since $kG(x_3)$ and $kG(x_2)$ are isomorphic, there is ϕ such that $\phi(\mathfrak{G}(x_3)) = \mathfrak{G}(x_2)$. Clearly ϕ send the pair of (u, v) to the pair (u', v') (possibly reversing them) by construction. Thus restriction of ϕ_r to the vertex set of $\mathfrak{G}'$ yields an automorphism of edges that send e to f.

- Let $\Pi''_b = \Pi'_b \circ \phi_r$, by theorem 110 we have a sequence S_1 such that $S_1(m'_2, \Pi'_b) = (m'_2, \Pi''_b)$ and as a result the boosted sequence S'_1 sends the gray double arc of (x_3, c_3) is send to the positions $0, \beta$. It only remains to prove that it is still of type B and we will have proven that $S'_1(x_3, c_3) = (x_2, c_2)$.

  However this is obvious by inspection of table 6.1. Indeed given a matching m of type B, the only way for the leftmost double arc of type B to change its type is to apply the operator associated to $(1, m(1))$ or that of another arc that contains it. In our case such operator never appears in the boosted sequence S' and S_1' since the leftmost double-arc is grayed and since it is the leftmost arc it is not contain in any other arc.

The final step to conclude the classification is to establish, by induction, Theorem 110 for classes inside $\mathfrak{M}_n^{\mathrm{d}}$. The results at the end of Appendix D are the key components of this proof.

If the left-most arc can be made of type B, the result of the reduction is a connected graph (see Figure 6.11), which is an invariant for a class inside $\mathfrak{M}_{n-1}^{\mathrm{d}}$. We can now use Proposition 216 in order to construct the largest part of the desired automorphism group for G, and note that the non-trivial coset associated to the $\mathfrak{S}_{e,e'}$ factor on the RHS of the proposition is achieved by the action of the operator corresponding to the arc $(m(1) - 1, m(1) + 1)$ of the left-most double arc.

If we are in the cup-block case, then since the dynamics is factorized the result of the reduction can be divided into two graphs, each of which is an invariant of a class at smaller size (see the graphs G_1 and G_2 in the Appendix D above Proposition

217). We can now use Proposition 217 in order to construct the largest part of the desired automorphism group for G, namely $\mathrm{Aut}_+(G_1) \times \mathrm{Aut}_+(G_2)$, and note that there are possibly two non-trivial generators for cosets, one associated to the $\mathfrak{S}_{e,e'}$ factor on the RHS of the proposition, the other one associated to a map in $\mathrm{Aut}(G)$ exchanging L and R, if it exists. These two generators are achieved by the combined action of the operators corresponding to the arc $(1, m(1))$ and $(2, m(1) - 1)$.

6.6 From $\mathfrak{I}$ on $\mathfrak{M}_n^{\mathrm{d}}$ to $\mathfrak{I}$ on $\mathfrak{M}_n$

In this section we prove the classification for all classes. The key idea is that, given two matchings m_1, $m_2 \in \mathfrak{M}_n$ with the same graph invariant, we can find two double-matchings m_1', $m_2' \in \mathfrak{M}_n^{\mathrm{d}}$ with the same graph invariant such that a sequence S connecting m_1' to m_2' can be projected to a sequence connecting m_1 and m_2.

Let m_1 and m_2 be two matchings with the same graph invariant $\mathfrak{G}$. Introduce a labeling on m_1, which thus induces a labeling on the graph invariant. Pull back a labeling on m_2 which is compatible with the labeling of the graph. Now, transform $m_1 \in \mathfrak{M}_n$ into a $m_1' \in \mathfrak{M}_n^{\mathrm{d}}$, say by transforming all arcs, into type-B double-arcs, and make gray the short arcs of the double-arcs (i.e., if the double-arc is $(a, b) = ((i, j), (j - 1, j + 1))$, make gray the edge $(j - 1, j + 1)$). This splits the vertices of degree 4 of the graph, into pairs of vertices connected by a double-edge, and the labeling allows to keep track of which of the three possible splittings shall be perfomed at each vertex. Similarly, pull back this information on the splitting into an information on how to transform the arcs of m_2 into the double-arcs of m_2', which thus will be of type A, B, C or D depending on the splitting, and again has half of the edges in gray. Moreover on type A and B, the gray arc is the short one.

In the above we implicitly used the definition of a good 2-coloring.

Definition 133. *Let $m \in \mathfrak{M}_{2n}$ be a matching with n double arc let c be a (2n,2n)-coloring of m. If c two-color every double arc $da = (a, b)$ i.e. one arc is colored in gray and one in black, we say that c is a 2-coloring of m.*

Moreover we define a good 2-coloring c of m as a 2-coloring such that every double arc (i, j) of type A or B has the arc (i, j) black and the other (either $(i - 1, i + 1)$ or $(j - 1, j + 1)$) is gray.

The the reduction of good 2-coloring takes a double-matching with n double arcs $(i_1, j_1), \ldots, (i_n, j_n)$ into a matching with n normal arcs $(i_1, j_1), \ldots, (i_k, j_k), (i_1', j_1'), \ldots, (i_{n-k}', j_{n-k}')$.

Let us now construct a sequence connecting m_1 to m_2 from a sequence connecting m_1' to m_2'. Let us consider the following lemma:

Lemma 134. *let m be a double matching and c be a good 2-coloring of m. Let A_e^I be an operator of a black arc of (m, c) and define $(m', c') = A_e^I(m, c)$ then c' is a good 2-coloring of m'*

Proof. We only show that the double arc containing e is still correctly colored. The rest (i.e showing that the other double arcs remain correctly colored) is done similarly with the help of table 6.1. Let $da = (e, b)$ be the double arc (i, j).

If da has type A or B then c must be (i, j) by definition of a good coloring. Then by table 6.1 the image of da by A_e^I is a double arc of type C thus it is correctly colored (since both possibilities are fine)

If da has type C, then by table 6.1 the image of da by A^I_e is a double arc da' of type A or B. Nevertheless the coloring of the c is black thus the coloring is correct.

If da has type D, then by table 6.1 the image of da by A^I_e is a double arc of type D thus it is correctly colored (since both possibilities are fine). ☐

Then we have

Proposition 135. *Let m_1 and m_2 be two double matchings in the same class and let S be a shortest sequence such that $m_2' = S(m_1)$. Let c_1 be a good coloring of m_1 then there exists a sequence S' utilizing only operators from the black arcs of (m_1, c_1) such that $(m_3, c_3) = S(m_1, c_1)$ verifies the following properties:*

c_3 is a good coloring and m_2 can be obtained from m_3 by substituting some double arcs (i, j) of type B by double arcs (i, j) of type A or vice-versa.

Proof. First let us note that since S is a shortest sequence there are no operator A^I_b where b is the small arc of a double arc $da = (a, b)$ of type A or B since in that case $A^I_b = id$.

We prove the proposition by induction on the size of the sequence S. If the sequence is empty this is trivially true.

Inductive step: Let S be a sequence of size n, thus $S = A^I_c S_1$. By induction hypothesis, there exists S_1' such that $(m_3, c_3) = S_1'(m_1', c_1)$ verifies the hypothesis of the proposition: c_3 is good and $m_2 = S_1(m_1)$ can be recovered from m_3 by substituting some double arcs (i, j) of type B by double arcs (i, j) of type A or vice-versa.

Let us consider A^I_c where c is an arc of m_2. The propriety concerning the coloring was proven in the lemma above.

There are 3 cases:

- c is the big arc (i, j) of the double arc (i, j) of type A or B, then, by induction, c is also the big arc (i, j) of the double arc (i, j) of type A or B (possibly exchanged). Moreover it is colored in black in (m_3, c_3) by definition. Thus taking $S' = A^I_c S_1'$ is approriate for the proposition.

- c is an arc $da = (c, d)$ of type C. By induction, c is the same arc of da the double arc of type C of (m_3, c_3). If c is black in (m_3, c_3) then we take $S' = A^I_c S_1'$, however if d is black then we take $S' = A^I_d S_1'$. Clearly $A^I_d(m_3, c_3)$ and $A^I_c(m_3, c_3)$ are identical except for da that is exchanged from a double arc of type A into one of type B or conversely. Thus S' still verifies the properties of the proposition.

- c is an arc $da = (c, d)$ of type D. By induction, c is the same arc of da the double arc of type D of (m_3, c_3). If c is black in (m_3, c_3) then we take $S' = A^I_c S_1'$, however if d is black then we take $S' = A^I_d S_1'$. Clearly $A^I_d(m_3, c_3)$ and $A^I_c(m_3, c_3)$ are completely identical. Thus S' still verifies the properties of the proposition.

☐

We can finally prove the classification theorem for any matching.

Proof of the classification theorem. The existence part was proved in section 6.4.1 thus we only have to establish that two matchings m_1 and m_2 with the same graph invariant are connected.

Let (m'_1, c_1) and (m'_2, c_2) be two double matchings with good 2-coloring and same graph invariant such that the corresponding reduction are m_1 and m_2 respectively. By the classification theorem for double matchings there exists a sequence S connecting m'_1 and m'_2.

By proposition 135 there exists a sequence S' such that $(m'_3, c_3) = (m'_1, c_1)$ verifies the following properties: c_3 is a good 2-coloring, S' contains only operators on the black arcs and m'_2 and m'_3 are obtained from one another by possibly exchanging type A and B of some double arcs.

First we note that the reduction of (m'_3, c_3) is m_2 since the reduction of (m'_2, c_2) was also m_2 (and the small arcs of the double arc of type A or B are always gray).

Then S' is also a boosted sequence of (m'_1, c_1) for some sequence S_0 since we never used gray arcs.

Everything put together, we have $m_2 = S_0(m_1)$. □

Chapter 7

A second proof of the Rauzy dynamics for permutations

In this section we present a second proof of the classifcation of the Rauzy classes of the dynamics $\mathcal{S}_n$ by applying the labelling method. The $\mathcal{S}_n$ dynamics has already been introduced in section 3.1.1. In particular refer to section 3.1.3 for the definition of the invariants and section 3.1.5 for the statement of the classification theorem, Thm. 194.

As previously mentioned in our discussion on applicability of the labelling method to different Rauzy-type dynamics, the application of the method to the dynamics $\mathcal{S}_n$ is an arduous process, due to the existence of two distinct invariants, and of exceptional classes.

The proof can be organised into three majors parts: Section 7.2 discusses the amenability of $\mathcal{S}_n$ to the labelling method, Sections 7.3–7.6 give preliminar results useful for the main induction, and finally Section 7.7 presents the main induction.

7.1 Proof overview

A proof of classification via the labelling method, in the way we have presented it, always proceeds in two parts. First we prove the amenability of the dynamics $\mathcal{S}_n$ in section 7.2 (although, as usually occurs in this family of proofs, the proof of the r-point monodromy must be deferred to the main induction). Then we carry out the main induction in section 7.7, in which we apply the labelling method to prove the classification theorem and the r-point monodromy theorem.

In this section we describe the ideas behind the main induction.

In an instance where the labelling method can be applied 'smoothly' (as is the case with the dynamics $\mathfrak{I}_n$, cf. Section 6), the main induction is constituted of three statements: the *existence* and *completeness* part of the classifcation theorem, and then the r-point monodromy of the set $L(x, \Pi_b)$. Moreover, nothing prevents from presenting the main induction immediately after that the amenability has been established.

The dynamics $\mathcal{S}_n$ is not such a smooth instance, therefore the main induction, that we now detail, will include a certain number of unavoidable technical statements, and several sections (i.e. Sections 7.3 to 7.6) are needed in preparation of it.

Main induction: We demonstrate by induction on n the seven following statements:

1. **Proposition 136.** *Every class contains a shift-irreducible standard family (cf definition 161).*
2. **Theorem 137.** *Let σ be a permutation with cycle invariant (λ, r) and let ℓ be the number of cycles (not including the rank) of σ i.e. $\ell = |\lambda|$.*
   - *The list $\lambda \bigcup \{r\}$ has an even number of even parts.*
   - $\overline{A}(\sigma) = \begin{cases} \pm 2^{\frac{n+\ell}{2}} & \text{if there are no even parts in the list } \lambda \bigcup \{r\} \\ 0 & \text{otherwise.} \end{cases}$
3. **Proposition 138** (Existence part of the classification theorem)**.** *For every valid invariant (λ, r, s) (i.e., every invariant in the list of Thm. 194) there exists a permutation with invariant (λ, r, s).*
4. **Proposition 139** (First step of the labelling method)**.** *Let $\sigma, \sigma' \in \mathfrak{S}_n$ be two irreducible permutation with invariant (λ, r, s). There exist σ_1 and σ'_1, connected to σ and σ' respectively, with the following property: let c be the $(2n-2, 2)$-coloring of σ_1 where the edge $e = (\sigma_1^{-1}(1), 1)$ is grayed, and call τ the reduction of (σ_1, c); define the analogous quantities for σ'_1 (i.e., the coloring c', the edge $e' = (\sigma_1'^{-1}(1), 1)$, and the configuration τ'); then τ and τ' are irreducible, have the same invariant (λ', r', s'), and none of them is in an exceptional class.*
5. **Proposition 140** (Completeness part of the classification theorem)**.** *Every pair of permutations σ and σ' with the same invariant (λ, r, s) is connected.*
6. **Proposition 141.** *Every non exceptional class contains a I_2X-permutation (cf. Definition 164).*
7. **Theorem 154** on the 2-point monodromy problem for this dynamics.

The three major steps of the present proof are establishing Propositions 138 and 140 and Theorem 154.

As we commented at the end of Section 5, the most difficult step in the "seven-step" organisation of the abstract labelling method, when specialised to the $\mathcal{S}_n$ dynamics, is the first step. In this case, this passage takes the form of Proposition 139 within the main induction.

This proposition is difficult to establish because it requires that τ and τ' satisfy simultaneously four unrelated properties: being irreducible, having the same cycle invariant (λ', r'), having the same Arf invariant s', and not being in an exceptional class. Let us outline the steps of the proof of this proposition, within their logic sequence:

Cycle invariant: In Section 7.4 (Proposition 159), we observe that if σ_1 and σ'_1 are standard and are both of type $X(r, i)$ or of type $H(r_1, r_2)$, then their reductions have the same cycle invariant (λ', r').

Irreducibility: We introduce a special subset of standard families that we call the *shift-irreducible standard families* (see Definition 161). By definition, if σ is in a shift-irreducible standard family, then the reduction τ of (σ, c) is (almost always) irreducible, where c is the $(2n-2, 2)$-coloring of σ where edge $e = (\sigma^{-1}(1), 1)$ is grayed. (In fact, this occurs only 'almost always', but exceptions can be handled efficiently.)

Thus, by choosing σ_1 and σ'_1 in shift-irreducible standard families, we can take care simultaneously of irreducibility and of having the same cycle invariant. However, contrarily to the existence of standard families, proving that every irreducible permutation is connected to a shift-irreducible family is much harder. We establish that a special family of permutations, named *I_2X permutations*, guarantees that the associated standard family is shift-irreducible, and we dedicate the whole Section 7.6 to the construction of one such permutation for every (λ, r, s), which, on top of this, are not in an exceptional class (see corollary 165).

Then, in the main induction we use the stronger induction hypothesis of Proposition 141 (statement 6) in order to prove Proposition 136 (statement 1). Finally Proposition 141 (statement 6) is demonstrated by applying the completeness part of the classification theorem (Proposition 140, statement 5) to the existence of an I_2X permutation for every (λ, r, s) (proven in Section 7.6, Theorems 176 and 181).

Sign invariant: in Section 7.5 we prove Proposition 170. Applied to our case, it shows that if σ_1 and σ'_1 have invariant (λ, r, s) with $s \neq 0$ and are shift-irreducible standard permutation of type $X(r, i)$ then τ_1 and τ'_1 also have sign invariant s. There are two other cases for the types of permutation, for which the underlying mechanism is too technical for being described concisely, let us content here by saying that they make use of Theorem 137 and Proposition 172 (also established in Section 7.5), respectively.

Exceptional classes: finally we must certify that neither τ_1 nor τ'_1 are in an exceptional class. This is achieved by showing that, in a standard family, and for the reduction of the $(2n-2, 2)$-coloring in which the edge $(\sigma^{-1}, 1)$ is gray, at most one permutation of the family is in a exceptional class (a lemma established in the appendix discussing the structure of exceptional classes). As a result, we can always avoid this case.

7.2 Amenability of S_n to the labelling method

In this section we introduce the notions necessary to show that $\mathcal{S}_n$ is amenable to the labelling method. We have already defined a boosted dynamics in Section 4.1.4 thus our task is to determine a labelling Π_b compatible with it as well as to state a r-point monodromy theorem for set $L(\sigma, \Pi_b)$ for any permutation σ.

Let us overview the content of the section:

The first subsection introduces the definition of a special set of labellings, that we call *consistent labellings*, and proves a number of properties. In short, consistent labellings are labellings that respect the structure of the cycle invariant.

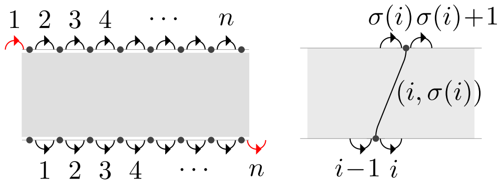
The second subsection extends the dynamics to the labelled case, proves its compatibility with the boosted dynamics (cf. Theorem 153), and introduces the theorem pertinent to the 2-point monodromy problem (cf. Corollary 155).

7.2.1 Preliminaries: Definition of a *consistent labelling*

Our notion of arcs departs slightly from the definition inherited from the combinatorial structure with arcs as we define both bottom and top arcs in our diagram representation of permutations.

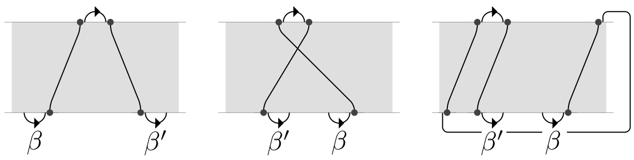
Let σ be a permutation of size n. The procedure to construct the cycle invariant (λ, r) (as described in Section 3.1.3.1) involves the introduction of $n-1$ top and bottom arcs connecting adjacent top and bottom vertices. As our proof requires it, we also add one top arc, to the left of the other top arcs (i.e to the left of the edge $(\sigma^{-1}(1), 1)$) and one bottom arc, to the right of the bottom arcs (i.e to the right of the edge $(n, \sigma(n))$). Those two arcs are clearly added at the two endpoints of the rank path.

We number the top and bottom arcs from left to right, and refer to them by their position: the bottom arc $\beta \in \{1, \ldots, n\}$ is the β-th bottom arc, counting from the left.



By convention, the variables used to name the positions of the top (bottom) arcs will be α (respectively β), in order not to make confusion with other parts of the diagram (for which we will use $i, j, \ldots$ or $x, y, ldots$).

Definition 142. *We say that two (bottom) arcs β, β' are* consecutive *(in a cycle) if they are inside the same cycle (or the rank path), and they are consecutive in the cyclic order induced by the cycle (or the total order induced by the path). This occurs in one of the three graphical patterns:*



In formulas:

$$\beta' = \sigma^{-1}(\sigma(\beta+1)+1) \text{ if } \sigma(\beta+1) < n \text{ and } \beta' = \sigma^{-1}(\sigma(1)+1) \text{ if } \sigma(\beta+1) = n$$

We define consecutive arcs for top arcs similarly.

Remark 143. *As we have seen above, when representing graphically the consecutive arcs, we need three figures depending on the different cases (edges crossing or not, and edges ending at the north-est corner of the diagram). However, these cases are treated in a very similar way, and, in the graphical explanation of our following properties, we shall mostly draw consecutive arcs by representing the case of non-crossing and non-corner edges, i.e. the left-most of the drawings above. It is intended that the underlying reasonings remain valid for the other cases.*

Next we define suitable alphabets used to label the top and bottom arcs of a permutation.

Notation 4. *For all j, let $\Sigma_{i,j} = \{b_{0,i,j}, \ldots, b_{i-1,i,j}\}$ and $\Sigma'_{i,j} = \{t_{0,i,j}, \ldots, t_{i-1,i,j}\}$ be a pair of alphabets which label the bottom arcs and the top arcs respectively of a cycle of length i, and let $\Sigma_r = \{b_0^{\mathrm{rk}}, \ldots, b_r^{\mathrm{rk}}\}$ and $\Sigma'_r = \{t_0^{\mathrm{rk}}, \ldots, t_r^{\mathrm{rk}}\}$ be the pair of alphabets used to label the bottom arcs and the top arcs respectively of the rank path.*

Note that the labels of the rank range from 0 to r, instead of form 0 to $r-1$, since we added a left-most top arc and a right-most bottom arc, which are now part of the rank, according to the construction of the arcs of a permutation outlined above.

Finally, we can introduce our notion of consistent labelling.

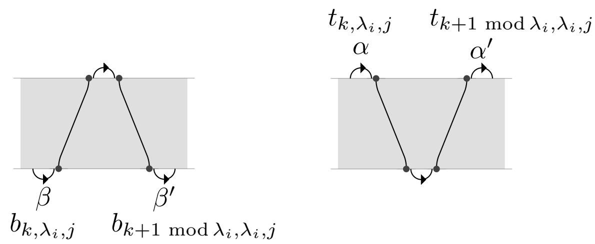
Definition 144 (Consistent labelling)**.** *Let σ be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)$ and define a* consistent labelling *to be a pair (Π_b, Π_t) of bijections:*

$$\begin{array}{rcccl} \Pi_b & : & \{1, \ldots, n\} & \to & \Sigma_b = \Sigma_r \cup \left[\bigcup_{i=1}^k \left(\bigcup_{j=1}^{m_i} \Sigma_{\lambda_i,j} \right) \right] \\ \Pi_t & : & \{1, \ldots, n\} & \to & \Sigma_t = \Sigma'_r \cup \left[\bigcup_{i=1}^k \left(\bigcup_{j=1}^{m_i} \Sigma'_{\lambda_i,j} \right) \right] \end{array}$$

such that

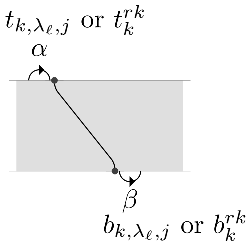
1. *Two arcs within the same cycle have labels within the same alphabet. Thus if $S_b = \{(\beta_k)_{1 \le k \le \lambda_i}\}$ and $S_t = \{(\alpha_k)_{1 \le k \le \lambda_i}\}$ are the sets of bottom (respectively top) arcs of a cycle of length λ_i, then $\Pi_b(S_b) = \Sigma_{\lambda_i,j}$ and $\Pi_t(S_t) = \Sigma'_{\lambda_i,j}$ for some $1 \le j \le m_i$.*

2. *Two consecutive arcs of a cycle of length λ_i have labels with consecutive indices: if β and β' are consecutive, then $\Pi_b(\beta) = b_{k,\lambda_i,j}$ for some $k < \lambda_i$ and $j \le m_i$ and $\Pi_b(\beta') = b_{k+1,\lambda_i,j}$, where $k+1$ is intended modulo λ_i. Likewise for top arcs.*



3. *The right bottom arc β and the top left arc α of an edge i are labeled by the same indices:*

$$\textit{if } \beta = i, \alpha = \sigma(i), \textit{ then } \left\{ \begin{array}{rcl} \Pi_t(\alpha) = t_{k,\lambda_\ell,j} & \Leftrightarrow & \Pi_b(\beta) = b_{k,\lambda_\ell,j} \\ \Pi_t(\alpha) = t_k^{\mathrm{rk}} & \Leftrightarrow & \Pi_b(\beta) = b_k^{\mathrm{rk}} \end{array} \right.$$



4. *Let $(\alpha_i)_{0\le i\le r}$ and $(\beta_i)_{0\le i\le r}$ be the top (respectively bottom) arcs of the rank ordered along the path (thus $\alpha_0 = 1$, $\beta_0 = \sigma^{-1}(1)$, $\alpha_1 = \sigma(\beta_0+1)+1 \dots$ $\alpha_r = \sigma(n)$, $\beta_r = n$). Then*

$$\forall\ 0 \le i \le r,\ \Pi_t(\alpha_i) = t_i^{\mathrm{rk}} \text{ and } \Pi_b(\beta_i) = b_i^{\mathrm{rk}}$$

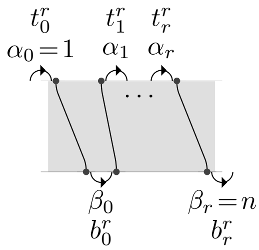


Figure 7.1 provides two examples of consistent labellings.

Lemma 145. *Let σ be a permutation and $\Pi_b : \{1,\dots,n\} \to \Sigma_b$ a labelling of bottom arcs verifying property 1, 2 and 4 of Definition 144. Then there exists a unique $\Pi_t : \{1,\dots,n\} \to \Sigma_t$ such that (Π_b,Π_t) is a consistent labelling.*

Proof. Let (Π_b,Π_t) a be a consistent labelling. Then by property 3 we must have $\Pi_t(\alpha) = \begin{cases} t_{i,\lambda_\ell,j} & \text{if } \Pi_b(\sigma^{-1}(\alpha)) = b_{i,\lambda_\ell,j} \\ t_i^{\mathrm{rk}} & \text{if } \Pi_b(\sigma^{-1}(\alpha)) = b_i^{\mathrm{rk}} \end{cases}$. This uniquely defines Π_t. □

This lemma implies that the data (σ,Π_b) or (σ,Π_t) are sufficient to reconstruct $(\sigma,(\Pi_b,\Pi_t))$. Thus, occasionally, we will consider just (σ,Π_b) rather than $(\sigma,(\Pi_b,\Pi_t))$.

Lemma 146. *Let $(\sigma,(\Pi_b,\Pi_t))$ be a permutation with a labelling. We have that (Π_b,Π_t) is a consistent labelling (i.e. it verifies properties 1 to 4) if property 2 is true for bottom arcs, property 3 is true and $\Pi_t(1) = t_0^{\mathrm{rk}}$.*

Proof. Clearly property 1 is implied by property 2 and 3, and property 4 is implied by property 2 and 3 applied to the rank, plus the fact that $\Pi_t(1) = t_0^{\mathrm{rk}}$, which, by property 2, fixes the order of the other labels of the rank. □

It is clear from the definition that two consistent labellings of a permutation can only differ by the two following operations: a cyclic shift of the labels within a cycle (due to property 2), and the permutation of the 'j' labels of two cycles of same size

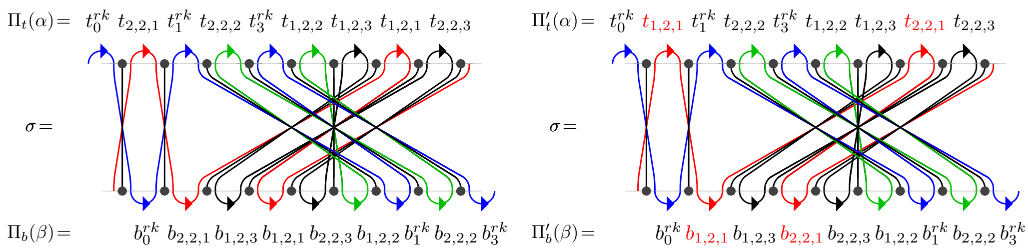


Figure 7.1: Two consistent labellings (Π_b,Π_t) and (Π'_b,Π'_t) of a permutation σ with cycle invariant $(\{2,2,2\},2)$. Following definition 147 we have $(\Pi'_b,\Pi'_t) = \mathrm{Sh}^1_{2,1}((\Pi_b,\Pi_t))$

(due to property 1). In particular, the labels of the rank, and their order, coincide in all consistent labelings of a given permutation (by property 4).

We state this property more formally in the following definition and proposition.

Definition 147. *Let σ be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)$, and define operators on consistent labelling $\Pi_b : \{1, \ldots, n\} \to \Sigma$*

- *The* shift operator $\mathrm{Sh}^m_{\lambda_i,j}$, *that shifts by m the labels of the jth cycle of length λ_i:*
$$\forall m \geq 1,\ \forall 1 \leq i \leq k,\ \forall 1 \leq j \leq m_i$$
$$\mathrm{Sh}^m_{\lambda_i,j}(\Pi_b)(\beta) = \begin{cases} b_{\ell+m,\lambda_i,j} & \text{if } \exists \ell \,/ \Pi_b(\beta) = b_{\ell,\lambda_i,j} \\ \Pi_b(\beta) & \text{otherwise.} \end{cases}$$

- *The* exchange operator $\mathrm{Ex}_{\lambda_i,j_1,j_2}$, *that exchanges the labels of the j_1th and j_2th cycles of length λ_i:*
$$\forall 1 \leq i \leq k,\ \forall 1 \leq j_1, j_2 \leq m_i$$
$$\mathrm{Ex}_{\lambda_i,j_1,j_2}(\Pi_b)(\beta) = \begin{cases} b_{\ell,\lambda_i,j_2} & \text{if } \exists \ell \,/ \Pi_b(\beta) = b_{\ell,\lambda_i,j_1} \\ b_{\ell,\lambda_i,j_1} & \text{if } \exists \ell \,/ \Pi_b(\beta) = b_{\ell,\lambda_i,j_2} \\ \Pi_b(\beta) & \text{otherwise} \end{cases}$$

Proposition 148 (Set of consistent labellings)**.** *Let σ be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)$. Two consistent labellings are obtained from one another by a sequence of shift and exchange operators.*

Lemma 149. *Let σ be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)$ and let Π be the set of consistent labellings. Then*
$$|\Pi| = \prod_{i=1}^{k} (m_i! \lambda_i^{m_i}) \,.$$

7.2.2 The amenability

Let us define two families of permutations (in cycle notation):
$$\gamma_{t,n}(i) = (1)(2)\cdots(i)(i+1\ n\ n-1\ \cdots\ i+2)$$
$$\gamma_{b,n}(i) = (2\ 3\ \cdots\ i-1\ 1)(i)(i+1)\cdots(n)$$

Note that, with this notation, $L(\sigma) = \gamma_{t,n}^{-1}(\sigma(1)) \circ \sigma$ and $R(\sigma) = \sigma \circ \gamma_{b,n}(\sigma^{-1}(n))$.

The dynamics $\mathcal{S}_n$ can be naturally extended in order to act also on the labelling.

Definition 150 (Action of the dynamics $\mathcal{S}_n$ on the labelling)**.** *Let $(\sigma, (\Pi_b, \Pi_t))$ be a permutation with a consistent labelling. Then $L(\sigma, (\Pi_b, \Pi_t)) = (L(\sigma), (\Pi_b, \Pi_t \circ \gamma_{t,n}(\sigma(1)))$ and $R(\sigma, (\Pi_b, \Pi_t)) = (R(\sigma), (\Pi_b \circ \gamma_{b,n}(\sigma^{-1}(n)), \Pi_t))$.*

Refer to Figure 7.2 for an illustration.

There are two good reasons for this to be 'the correct way' of extending the dynamics to the labelled case. First, it is compatible with the structure of the cycle invariant, as the image by the dynamics of a consistent labelling is another consistent labelling (this is the content of Theorem 151). Moreover, as is crucially needed by our methods, it is compatible with the boosted dynamics (see Theorem 153).

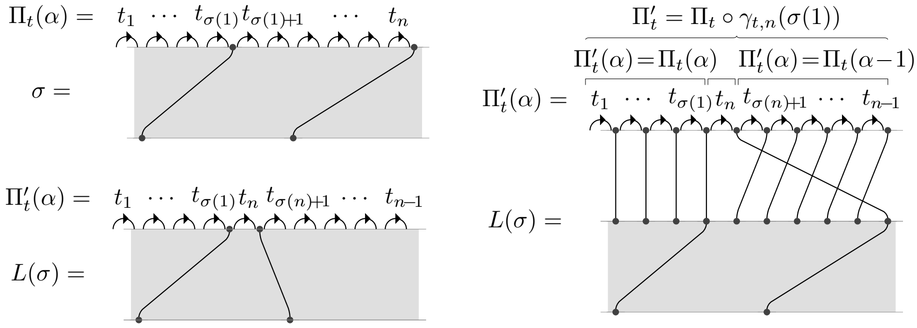


Figure 7.2: The action on the dynamics (with an L operator) on a consistent labelling.

Theorem 151. *Let $(\sigma, (\Pi_b, \Pi_t))$ be a permutation with a consistent labelling. Then $L(\sigma, (\Pi_b, \Pi_t))$ and $R(\sigma, (\Pi_b, \Pi_t))$ are permutations with a consistent labelling.*

Proof. By symmetry, we can consider just the action of the operator L. Let us show that $(L(\sigma), (\Pi'_b = \Pi_b, \Pi'_t = \Pi_t \circ \gamma_{t,n}(\sigma(1)))$ is a permutation with a consistent labelling. First note that

$$L(\sigma)(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \leq \sigma(1), \\ \sigma(i)+1 & \text{if } \sigma(1)+1 \leq \sigma(i) \leq n-1 \\ \sigma(1)+1 & \text{if } \sigma(i) = n, \end{cases}$$

By Lemma 146, we need to check that Π'_b verifies property 2, (Π'_b, Π'_t) verifies property 3, and $\Pi'_t(1) = t_0^{\mathrm{rk}}$.

Let $\alpha > 1$ be a top arc. There are three possibilities: $\alpha \leq \sigma(1)$, $\alpha = \sigma(1)+1$ and $\alpha \geq \sigma(1)+2$.

- If $\alpha \leq \sigma(1)$ then $\Pi'_t(\alpha) = \Pi_t(\alpha)$ by definition of Π'_t (cf. Figure 7.2 right). Let $\beta = L(\sigma)^{-1}(\alpha-1)$ and $\beta' = L(\sigma)^{-1}(\alpha)$ be the two consecutive bottom arcs associated to α in $L(\sigma)$. We also have $\beta = \sigma^{-1}(\alpha-1)$ and $\beta' = \sigma^{-1}(\alpha)$ by definition of $L(\sigma)$, since $\alpha \leq \sigma(1)$. Thus $\Pi'_b(\beta) = b_{i,\lambda_\ell,j}$ (or b_i^{rk}) and $\Pi'_b(\beta') = b_{i+1 \mod \lambda_\ell,\lambda_\ell,j}$ (or b_{i+1}^{rk}), by property 2 for (σ, Π_b), since $\Pi'_b = \Pi_b$ and β and β' are consecutive in σ.

  Likewise, $\Pi'_t(\alpha) = t_{i+1 \mod \lambda_\ell,\lambda_\ell,j}$ (or t_{i+1}^{rk}) and $\Pi'_b(\beta') = b_{i+1\lambda_\ell,\lambda_\ell,j}$ (or b_{i+1}^{rk}) by property 3 for $(\sigma, (\Pi_b, \Pi_t))$, since $\Pi'_b = \Pi_b$ and $\Pi'_t(\alpha) = \Pi_t(\alpha)$.

  Thus in this case $\Pi'_b = \Pi_b$ verifies property 2 and (Π'_b, Π'_t) verifies property 3 (see also figure 7.3).

  Note that the same reasoning applies for $\alpha = 1$, from which we deduce that $\Pi'_t(1) = t_0^{\mathrm{rk}}$ and $\Pi'_b(L(\sigma)^{-1}(1)) = b_0^{\mathrm{rk}}$.

- If $\alpha = \sigma(1)+1$, let $\alpha' = n$. Then $\Pi'_t(\alpha) = \Pi_t(\alpha')$ by definition of Π'_t (cf. Figure 7.2). Let $\beta' = L(\sigma)^{-1}(n)$ and $\beta' = L(\sigma)^{-1}(\alpha)$ be the two consecutive bottom arcs associated to α in $L(\sigma)$ (this is the special case of consecutive arcs, cf. the third figure in Definition 142). We have $\beta = \sigma^{-1}(n-1)$ and $\beta' = \sigma^{-1}(n)$ by definition of $L(\sigma)$, thus β and β' are also the two consecutive bottom arcs associated to α in σ.

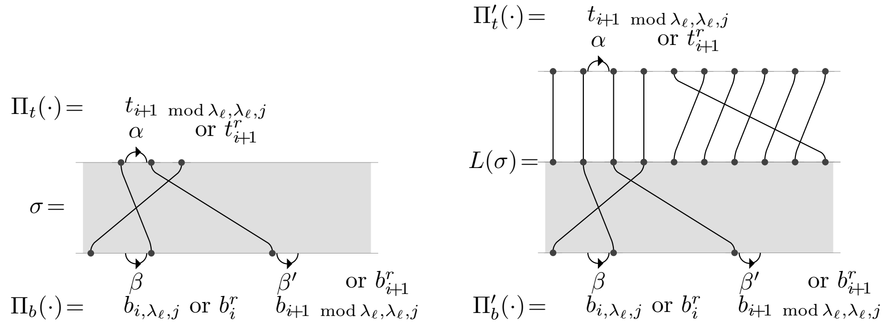


Figure 7.3: The proof that Π'_b verifies property 2 and (Π'_b, Π'_t) verifies property 3 for the case $\alpha \leq \sigma(1)$.

Thus $\Pi'_b(\beta) = b_{i,\lambda_\ell,j}$ (or b^{rk}_i) and $\Pi'_b(\beta') = b_{i+1 \bmod \lambda_\ell,\lambda_\ell,j}$ (or b^{rk}_{i+1}), by property 2 for (σ, Π_b), since $\Pi'_b = \Pi_b$ and β and β' are consecutive in σ.

Likewise, $\Pi'_t(\alpha) = t_{i+1 \bmod \lambda_\ell,\lambda_\ell,j}$ (or t^{rk}_{i+1}) and $\Pi'_b(\beta') = b_{i+1\lambda_\ell,\lambda_\ell,j}$ (or b^{rk}_{i+1}), by property 3 for $(\sigma, (\Pi_b, \Pi_t))$, since $\Pi'_b = \Pi_b$ and $\Pi'_t(\alpha = \sigma(1) + 1) = \Pi_t(\alpha' = n)$.

Thus, also in this case $\Pi'_b = \Pi_b$ verifies property 2 and (Π'_b, Π'_t) verifies property 3 (see also Figure 7.4).

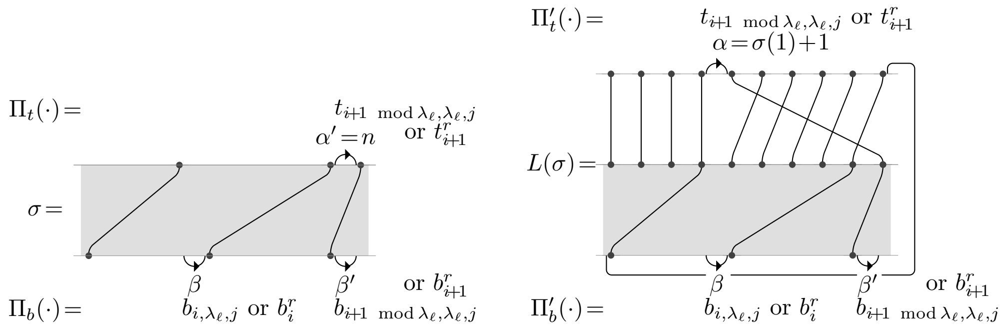


Figure 7.4: The proof that Π'_b verifies property 2 and (Π'_b, Π'_t) verifies property 3 for the case $\alpha = \sigma(1) + 1$.

- If $\alpha \geq \sigma(1) + 2$, let $\alpha' = \alpha - 1$. Then $\Pi'_t(\alpha) = \Pi_t(\alpha')$ by definition of Π'_t (cf. Figure 7.2, right). Let $\beta = L(\sigma)^{-1}(\alpha - 1)$ and $\beta' = L(\sigma)^{-1}(\alpha)$ be the two consecutive bottom arcs associated to α. We have $\beta = \sigma^{-1}(\alpha')$ and $\beta' = \sigma^{-1}(\alpha' - 1)$ by definition of $L(\sigma)$, thus β and β' are also the two consecutive bottom arcs associated to α in σ.

  Thus $\Pi'_b(\beta) = b_{i,\lambda_\ell,j}$ (or b^{rk}_i) and $\Pi'_b(\beta') = b_{i+1 \bmod \lambda_\ell,\lambda_\ell,j}$ (or b^{rk}_{i+1}), by property 2 for (σ, Π_b), since $\Pi'_b = \Pi_b$ and β and β' are consecutive in σ.

  Likewise $\Pi'_t(\alpha) = t_{i+1 \bmod \lambda_\ell,\lambda_\ell,j}$ (or t^{rk}_{i+1}) and $\Pi'_b(\beta') = b_{i+1\lambda_\ell,\lambda_\ell,j}$ (or b^{rk}_{i+1}) by property 3 for $(\sigma, (\Pi_b, \Pi_t))$, since $\Pi'_b = \Pi_b$ and $\Pi'_t(\alpha) = \Pi_t(\alpha')$.

□

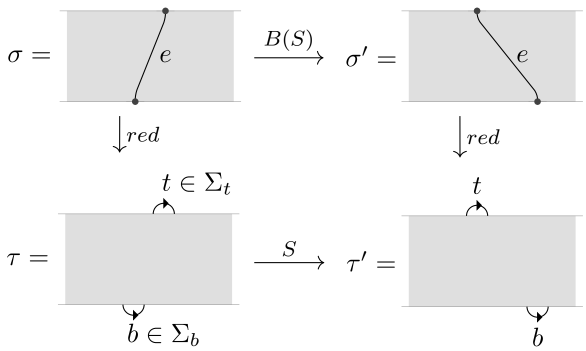


Figure 7.5: The gray edges of σ are transported along the labels of the arcs when applying the boosted dynamics. For instance, $e \in (b, t)$ in τ, Π.

Corollary 152. *Let (σ, Π_b) be a permutation with a consistent labelling, and let S be a 'loop' (i.e. a sequence in $\{L, R\}^*$ such that $S(\sigma) = \sigma$). Then $S(\Pi_b)$ is a consistent labelling on σ, and is obtained from Π_b by a sequence of shift and exchange operators.*

Proof. This is immediate from Theorem 151 and Proposition 148. $\square$

Let us now prove the compatibility of the labelling with the boosted dynamics.

Theorem 153 (The labelling is compatible with the boosted dynamics)**.** *Let $\sigma \in \mathfrak{S}_n$ be a permutation, and let $\tau \in \mathfrak{S}_k$ be the reduction of a $(2k, 2r)$-coloring c of σ. Let (Π_b, Π_t) be a consistent labelling of τ, and let S be a sequence of operators for the dynamics $\mathfrak{S}_k$, acting on $(\tau, (\Pi_b, \Pi_t))$ as described in Definition 150. Let e be a gray edge of σ with endpoints within the arcs with labels $t = \Pi_t(\alpha)$ and $b = \Pi_b(\beta)$ of τ, for some α and β.*

Then, in $(\sigma', c') = B(S)(\sigma, c)$ the gray edge e has its endpoints within the arcs with labels t and b of $(\tau', (\Pi'_b, \Pi'_t)) = S(\tau, (\Pi_b, \Pi_t))$. See Figure 7.5.

Proof. By induction on the length of the sequence, and by symmetry, we can consider just the case $S = L$.

Let us consider $\sigma \in \mathfrak{S}_n$, c a $(2k, 2r)$-coloring of σ, $\tau \in \mathfrak{S}_k$ the corresponding reduction and (Π_b, Π_t) a consistent labelling of τ, as in the statement of the theorem.

Let $e \in (t, b)$ be a gray edge of (σ, c), which is inserted within the top arc with label t and bottom arc with label b of $(\tau, (\pi_b, \Pi_t))$.

We must show that the gray edge e of $(\sigma', c') = B(L)(\sigma, c)$ is still within the arcs with labels t and b of $(\tau', \Pi'_b, \Pi'_t) = L(\tau, \Pi_b, \Pi_t)$.

First, let k_2 be the size of the block of gray edges immediately to the left of the right pivot (that is, the edge $(\sigma^{-1}(n), n)$). By the mechanisms of the boosted dynamics, in (σ, c) we have $B(L) = L^{k_2}$ (see Figure 7.6). We shall consider three cases, depending on the position of α, the arc labelled by t:

- If $\tau(1) < \alpha = \Pi_t^{-1}(t) < k$, then for $(\tau', (\Pi'_b, \Pi'_t)) = L(\tau, (\Pi_b, \Pi_t))$ we have $\Pi'^{-1}_t(t) = \alpha + 1$. Since $e = (i, \sigma(i))$ is within the arcs t and b of $(\tau, (\Pi_b, \Pi_t))$, we have $\sigma(1) < \sigma(i) < n - k_2$, thus $L^{k_2}(e) = (i, \sigma(i) + k_2)$ in $(\sigma', c') = L^{k_2}(\sigma, c)$. Therefore in (σ', c') the edge e is inserted within the arcs with labels t and b of $(\tau', (\Pi'_b, \Pi'_t))$. See Figure 7.6 with the edge e_1 inserted between t_1 and b_1.

- If $\alpha = \Pi_t^{-1}(t) < \tau(1)$, we are in a situation which is almost identical to the previous one, and we omit to discuss it.

- If $\alpha = \Pi_t^{-1}(t) = n$, then for $(\tau', (\Pi'_b, \Pi'_t)) = L(\tau, (\Pi_b, \Pi_t))$ we have $\Pi_t'^{-1}(t) = \tau(1) + 1$. Since $e = (i, \sigma(i))$ is inserted within t and b in $(\tau, (\Pi_b, \Pi_t))$, we have $n - k_2 \leq \sigma(i) < n$, thus $L^{k_2}(e) = (i, j)$ with j such that $s(1) < j \leq \sigma(1) + k_2$ in $(\sigma', c') = L^{k_2}(\sigma, c)$. Thus in (σ', c') the gray edge e is inserted within the arcs with labels t and b of τ'. See Figure 7.6 with the edge e_2 inserted within t_2 and b_2.

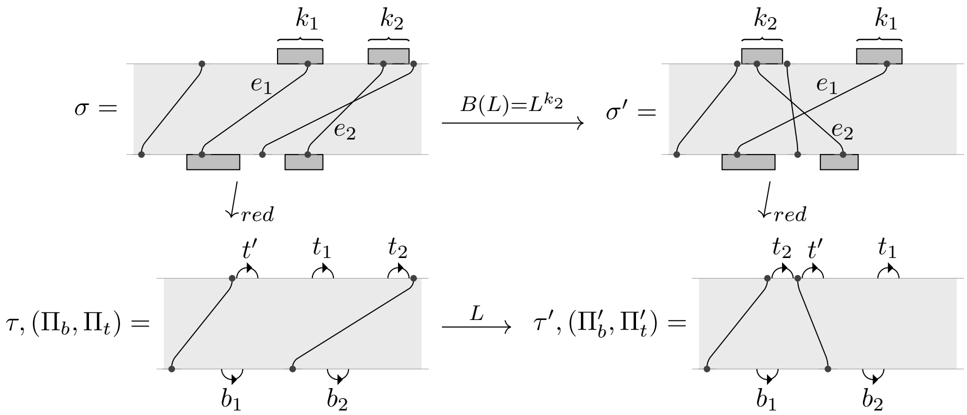


Figure 7.6: Illustration that the edges, inserted within the labels of a reduced permutation, follow those labels when applying the boosted dynamic.

□

The last main task of this section is to introduce the 2-point monodromy theorem. As announced in the overview section, Sec. 7.1, this theorem is the seventh (and last) statement in the organisation of the main induction, in Section 7.7.

Theorem 154. *Let (σ, Π_b) be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \dots, \lambda_k^{m_k}\}, r)$, equipped of a consistent labelling.*

Cycle 1-shift: Suppose λ does not contain any even cycle, or it has 2 or more even cycles. Let i be a cycle of length λ_i in λ. Then there exists a loop S in the dynamics such that $\Pi'_b = S(\Pi_b)$ is a consistent labeling, and verifies $\Pi'_b(\beta) = b_{\ell+1 \bmod \lambda_i, \lambda_i, i} \Leftrightarrow \Pi_b(\beta) = b_{\ell, \lambda_i, i}$. In other words, there exists a loop in the dynamics that shifts the labels of the cycle i by one (and thus, by taking powers, by any integer m). The positions of the other labels are, in principle, unknown, nonetheless they are constrained by the fact that Π'_b is also a consistent labelling.

Cycle 2-shift: Suppose now that λ has exactly one even cycle, of length λ_i. Then there exists a loop S such that $\Pi'_b = S(\Pi_b)$ verifies $\Pi'_b(\beta) = b_{\ell+2 \bmod \lambda_i, \lambda_i, 1} \Leftrightarrow \Pi_b(\beta) = b_{\ell, \lambda_i, 1}$. In other words, there exists a loop that shifts the labels of the unique even cycle by two (and thus by any even integer $2m$). This is consistent *with the fact that, if and only if λ_i is odd, then iterated shifts by* 2 *ultimately produce a shift by* 1.

Cycle jump: For any two cycles j_1, j_2 of the same length λ_i, there exists a loop S such that $\Pi'_b = S(\Pi_b)$ verifies $\Pi'_b(\beta) = b_{\ell,\lambda_i,j_2} \Leftrightarrow \Pi_b(\beta) = b_{\ell,\lambda_i,j_1}$. In other words, there exists a loop that sends the labels of the cycle j_2 on the positions of the labels of the cycle j_1, preserving their ordering.

It might not be clear a priori why such theorem implies the 2-point monodromy. Indeed, in our proof we will choose the label of the top arc to be t_0^{rk}, thus, since it is fixed by Theorem 151, we only need to consider the label of the bottom arc. In this case we have the immediate corollary:

Corollary 155 (2-point monodromy theorem)**.** *Let (σ, Π_b) be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \dots, \lambda_k^{m_k}\}, r)$ and with a consistent labelling. Let $(t, b) \in \Sigma_t \times \Sigma_b$ such that $t = t_0^{\mathrm{rk}}$.*

- *If $b = b_i^{\mathrm{rk}}$ for some i, there exists a loop S such that $\Pi'_b = S(\Pi_b)$ verifies $\Pi'_b(\beta) = b$ if and only if $\Pi_b(\beta) = b$.*
- *Let us now distinguish two cases:*
  (1) λ has no even cycles or has at least two of them. Then:
    - *If $b = b_{\ell,\lambda_i,j_1}$ for some triple (ℓ, λ_i, j_1), there exists a loop S such that $\Pi'_b = S(\Pi_b)$ verifies $\Pi'_b(\beta) = b$ if and only if $\Pi_b(\beta) = b_{\ell',\lambda_i,j_2}$ for some (ℓ', j_2).*

  (2) λ has exactly one even cycle, the cycle i of length λ_i. Then
    - *If $b = b_{\ell,\lambda_j,j_1}$ for some (ℓ, λ_j, j_1) with $j \neq i$, there exists a loop S such that $\Pi'_b = S(\Pi_b)$ verifies $\Pi'_b(\beta) = b$ if and only if $\Pi_b(\beta) = b_{\ell',\lambda_j,j_2}$ for some (ℓ', j_2).*
    - *If $b = b_{\ell,\lambda_i,1}$ for some $(\ell, \lambda_i, 1)$, there exists a loop S such that $\Pi'_b = S(\Pi_b)$ verifies $\Pi'_b(\beta) = b$ if $\Pi_b(\beta) = b_{\ell',\lambda_i,1}$ with $\ell' - \ell \equiv 0 \pmod 2$.*

In both cases, $\Pi_t(0) = \Pi'_t(0) = t_0^{\mathrm{rk}}$ since the labels of the rank are fixed in a consistent labelling.

Let us rephrase the content of this corollary. If we choose the label of the top arc to be t_0^{rk}, then if the bottom label is in the rank it is fixed, otherwise, if it is in a cycle, we can find a loop S such that the label can move to any bottom label of a cycle of the same length, with one exception: if the bottom label is in the unique cycle of even length. In this case, we can only find a loop S such that the label can move to the bottom labels of the cycle which has the same parity of the index.

Proof. By Theorem 151, if $b = b_i^{\mathrm{rk}}$ then, since Π'_b is a consistent labelling and the labels of the rank are fixed, $\Pi'_b(\beta) = b$ if and only if $\Pi_b(\beta) = b$.

If $b = b_{\ell,\lambda_i,j_1}$, by Theorem 151, if $\Pi'_b = S(\Pi_b)$ for some loop S, then it is a consistent labelling, thus we must have $\Pi'_b(\beta) = b$ if and only if $\Pi'_b(\beta) = b_{\ell',\lambda_i,j_2}$ for some ℓ' and j_2, this by Corollary 152.

The rest of the statement is a straightforward application of Theorem 154. $\square$

7.3 Cycle invariant and edge addition

This premilinary section study the change of the cycle invariant when inserting a few consecutives edges in a permutation. The results of this section will be used in sections 7.5,7.6,7.7.

Notation 5. *let σ be a permutation and let α be a top arc and β be a bottom arc, we define $\sigma|_{i,\alpha,\beta}$ to be the permutation obtained from σ by inserting $i \in \mathbb{N}$ consecutive and parallel edges within α and β. (see figure 7.7 for an example with $i = 1$).*

For the special case of $i = 2$ we call the two added edges a double-edge.

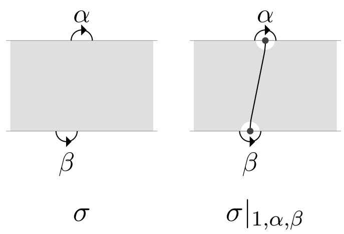


Figure 7.7: The insertion of one edge within the arcs α and β.

Proposition 156 (One edge insertion)**.** *Let σ be a permutation with cycle invariant (λ, r).*

Let $\sigma|_{1,\alpha,\beta}$ be the permutation resulting from the insertion of an edge within two arcs of two differents cycles of respective length ℓ and ℓ'. Then the cycle invariant of $\sigma|_{i,\alpha,\beta}$ is $(\lambda \setminus \{\ell, \ell'\} \bigcup \{\ell + \ell' + 1\}, r)$.

Let $\sigma|_{1,\alpha,\beta}$ be the permutation resulting from the insertion of an edge within an arc of the rank path and an arc of a cycle (principal cycle or not) of length ℓ. Then the cycle invariant of $\sigma|_{i,\alpha,\beta}$ is $(\lambda \setminus \{\ell\}, r + \ell + 1)$.

Proof. See figure 7.8.

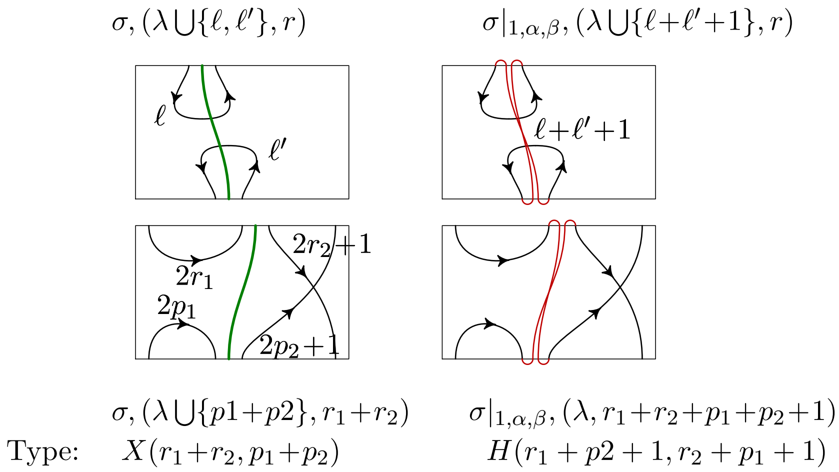


Figure 7.8: The first line represents the case: Top arc : any cycle. Bottom arc: any cycle. The second line represents the case: Top arc : rank path. Bottom arc: principal cycle.

Some cases are not represented in the figure. The missing cases are:

- Top arc: principal cycle. Bottom arc: any cycle.
- Top arc: any cycle. Bottom arc: principal cycle.
- Top arc: rank path. Bottom arc: any cycle.
- Top arc: principal cycle. Bottom arc: rank path.
- Top arc: any cycle. Bottom arc: rank path.

Their proof is nearly identical to the ones represented in the figure and are thus omitted. □

Proposition 157 (double-edge insertion)**.** *Let σ be a permutation with cycle invariant (λ, r).*

Inserting a double-edge within two arcs of the same cycle (or rank path) increase length of the cycle (or rank) by two.

Likewise inserting a double-edge within two arcs of two differents cycles (or rank path) increase the length of each by 1.

Proof. See figure 7.9. □

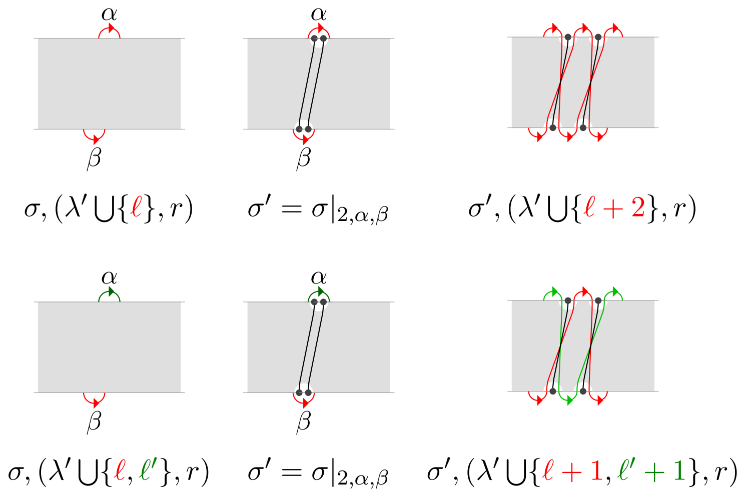


Figure 7.9: The first column represents the permutation σ and its the cycle invariant. The second column represents the permutation σ' obtained from the insertion of a double-edge within the arcs α and β of σ and the third column displays and demonstrates the new invariant of σ'.

7.4 Shift-irreducible standard family

In section 4.1.2 we introduced (or reintroduced since it was already well-know in the litterature) the notions of standard permutation and standard family. We ask the reader to refer to the basic facts concerning those notions as we will be developping in this section the more specific notion of *shift-irreducible standard family*.

Briefly a *shift-irreducible standard* family is a standard family (σ^i) for which all but two permutations are irreducible after removing the edge $((\sigma^i)^{-1}(1), 1)$. They are extremly useful for the main induction as they resolve the irreduciblity issue of proposition 139 in the proof overwiew section 7.1.

For ease of reference we reproduce below one proposition concerning the properties of the standard families. The original proposition 46 can be found page 67.

Proposition 158 (Properties of the standard family)**.** *Let σ be a standard permutation with cycle invariant (λ, r), and $S = \{\sigma^{(i)}\}_i = \{L^i(\sigma)\}_i$ its standard family. The latter has the following properties:*

1. *Every $\tau \in S$ has $\tau(1) = 1$;*
2. *The $n-1$ elements of S are all distinct;*
3. *There is a unique $\tau \in S$ such that $\tau(n) = n$;*
4. *Let m_i be the multiplicity of the integer i in λ (i.e. the number of cycles of length i), and r the rank. There are $i\, m_i$ permutations of S which are of type $X(r, i)$, and 1 permutation of type $H(r - j + 1, j)$, for each $1 \leq j \leq r$.*[1]

Let us introduce a convenient notation.

Notation 6. *Let σ be a permutation, we define $d(\sigma)$ to be the permutation obtained from σ by discarding the edge $(\sigma^{-1}(1), 1)$, thus if $\sigma = \sigma(1), \ldots, \sigma(\sigma^{-1}(1)), \ldots \sigma(n)$ then $d(\sigma) = \sigma(1) - 1, \ldots, \sigma(\widehat{\sigma^{-1}(1)}), \ldots, \sigma(n) - 1$.*

Finally we describe the cycle invariant (λ', r') of $d(\sigma^i)$ in function of the cycle invariant (λ, r) of σ^i.

Proposition 159. *Let $(\sigma^i)_i$ be a standard family with cycle invariant (λ, r) then $\forall i$*

- *If σ^i has type $X(r, j)$, $d(\sigma^i)$ has type $H(j, r)$ and cycle invariant $(\lambda \setminus \{j\}, r + j - 1)$.*
- *If σ^i has type $H(r_1, r_2)$, $d(\sigma^i)$ has type $X(r_2 - 1, r_1 - 1)$ and cycle invariant $(\lambda \bigcup \{r_1 - 1\}, r_2 - 1)$.*

Proof. The first case of this proposition was proven in lemma 78. The second case is proven likewise: This is the reverse implication of case ⑥ in table 4.2, specialised to $s = 0$. □

This proposition is very useful for the induction since it implies the following lemma

Corollary 160. *Let σ and σ' be two standard permutations σ and σ' have invariant (λ, r, s) and same type $X(r, i)$ or $H(r_1, r_2)$ then the reducted permutations $\tau = d(\sigma)$ and $\tau' = d(\sigma')$ have same cycle invariant.*

[1] Note that, as $\sum_i i\, m_i + r = n - 1$ by the dimension formula (3.2), this list exhausts all the permutations of the family.

This property (i.e same cycle invariant for τ and τ') was the first of the three properties we where looking for in our proposition 139 of the proof overview.

For the purpose of guarantying the irreducibility of τ and τ', we now define a more precise notion than just the standard family, we call it a *shift-irreducible standard family.*

Definition 161. *Let σ be a standard permutation and $S = (\sigma^i = L^i(\sigma))_{0\leq i\leq n-2}$ be its standard family. We say that S is shift-irreducible if*

$$\forall i \in \{0, \dots, n-2\} \setminus \{n-\sigma(2), n-\sigma(n)+1\} \;\; d(\sigma^i) \text{ is irreducible.}$$

In other words, a standard family S is shift-irreducible if every $d(\sigma^i)$ that can be irreducible is indeed irreducible. $d(\sigma^{n-\sigma(2)})$ and $d(\sigma^{n-\sigma(n)+1})$ are both always reducible since $\sigma^{n-\sigma(2)}(2) = n$ thus $d(\sigma^{n-\sigma(2)})(1) = n-1$ and $\sigma^{n-\sigma(n)+1}(n) = 2$ thus $d(\sigma^{n-\sigma(n)+1})(n-1) = 1$. Figure 7.10 provides an example of a shift-irreducible family.

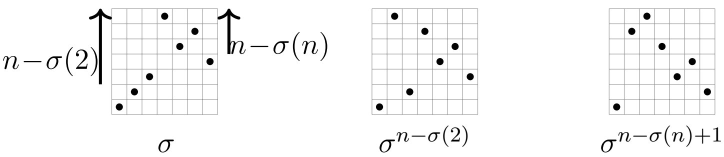


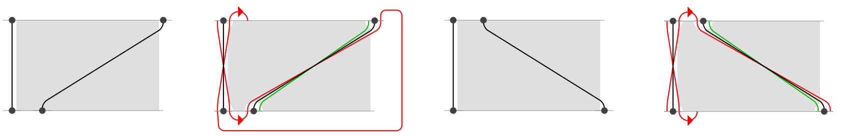
Figure 7.10: An example of a shift-irreducible family $S = (\sigma^i)_{0\leq i\leq n-2}$ with its two unavoidably reducible permutations $d(\sigma^{n-\sigma(2)})$ and $d(\sigma^{n-\sigma(n)+1})$. For a proof of the shift-irreducibility, the characterisation proposition 163 is helpful.

Proposition 162. *Let σ be a standard permutation with cycle invariant (λ, r), and $S = (\sigma^i = L^i(\sigma))_{0\leq i\leq n-2}$ its standard family. Then S is shift irreducible if and only if*

- *For every $i \in \lambda$ the image by d of the $i \cdot m_i$ permutations of S of type $X(r, i)$ are irreducible (where m_i is the multiplicity of i in λ).*
- *For all $1 < j < r$ the image by d of the permutation of S of type $H(r-j+1, j)$ is irreducible.*

Proof. By proposition 158, the permutations of S are exactly the im_i permutations of type $X(r, i)$ for every $i \in \lambda$ and the permutations of type $H(r-j+1, j)$ for all $1 \leq j \leq r$. Clearly the permutations of type $H(1, r)$ and $H(r, 1)$ are respectively $\sigma^{n-\sigma(2)}$ and $\sigma^{n-\sigma(n)+1}$ (see figure 7.11). Thus if the image by d of every permutation besides those two are irreducible the family is shift-irreducible and reciproquely if the family is shift-irreducible the image by d of every permutation besides those two are irreducible. □

As mentioned above the two permutations $\sigma^{n-\sigma(2)}$ and $\sigma^{n-\sigma(n)+1}$ -whose images by d are reducible- have type $H(1, r)$ and $H(r, 1)$. Thus $d(\sigma^{n-\sigma(2)})$ has cycle invariant $(\lambda \bigcup \{0\}, r-1)$ and $d(\sigma^{n-\sigma(n)+1})$ has cycle invariant $(\lambda \bigcup \{r-1\}, 0)$ by proposition 159. Neither are cycle invariants that we allow for in the classification theorem (we never consider cycles or rank of length 0 since it always implies reducibility).



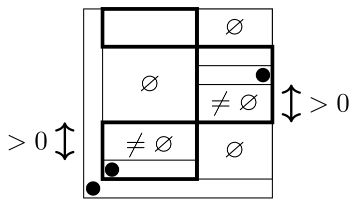
$\sigma^{n-\sigma(2)}$ has type $H(1,r)$ $\sigma^{n-\sigma(n)+1}$ has type $H(r,1)$

Figure 7.11: The two permutations $\sigma^{n-\sigma(2)}$ and $\sigma^{n-\sigma(n)+1}$ are of type $H(1,r)$ and $H(r,1)$ respectively.

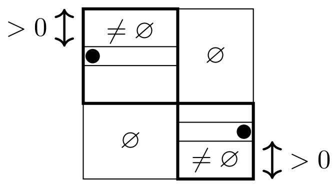
In other term, every single one of the permutations $d(\sigma^i)$ that can be used for the induction are irreducible if $(\sigma^i)_i$ is an shift-irreducible standard family.

It is interesting that the fruitful notion of shift-irreducible family has a very simple characterisation in term of just the permutation σ of the family with $\sigma(1) = 1$ and $\sigma(2) = 2$.

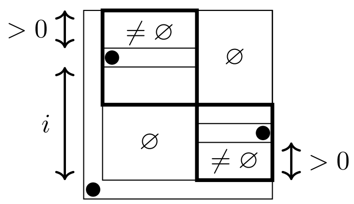
Proposition 163 (Characterisation of shift-irreducible families)**.** *Let σ be a standard permutation with $\sigma(2) = 2$ and $S = (\sigma^i = L^i(\sigma))_{0 \leq i \leq n-2}$ its standard family. Then S is shift-irreducible if and only if σ does not have the following form :*



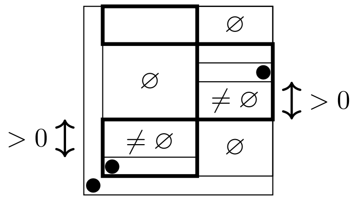
Proof. We show that S is not shift-irreducible if and only if σ has the form described in the proposition. If S is not shift-irreducible then it means that there exists a $i \in \{0, \ldots, n-2\} \setminus \{n - \sigma(2), n - \sigma(n) + 1\}$ such that $d(\sigma^i)$ is reducible. Thus $d(\sigma^i)$ has the following form



The blocks with $\neq \varnothing$ must not be empty otherwise we would have either $\sigma^i = \sigma^{n-\sigma(2)}$ or $\sigma^{n-\sigma(n)+1}$. σ^i has then the form



and since by definition $\sigma^i = L^i(\sigma)$, σ must have the form

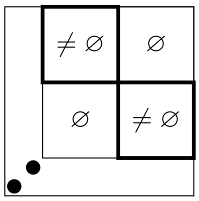


as predicted. The reverse implication is obtained by reversing the steps of the proof. □

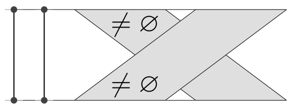
In light of this proposition we will often say *shift-irreducible permutation* to denote the permutation σ of the shift-irreducible standard family with $\sigma(1) = 1$ and $\sigma(2) = 2$. Moreover to prove the existence of a shift-irreducible standard family we will just construct a shift-irreducible permutation.

A particular type of shift-irreducible permutation is the *I_2X-permutation*:

Definition 164. *Let σ be a standard permutation with $\sigma(2) = 2$. We call σ a* I_2X-permutation *if it has the following form:*



equivalently



Corollary 165. *A I_2X-permutation is a shift-irreducible permutation.*

Proof. It is clear that a I_2X-permutation verifies the condition of the characterisation propositon 163. □

7.5 Arf invariant

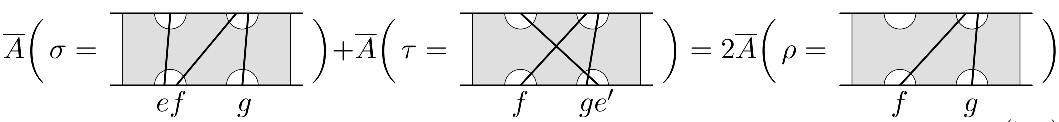
This section supplements section 4.2 by detemining new identities of the arf invariant that are required for the inductive proof using the labelling method. The proofs make use of the framework developed in section 4.2.2 and the reader should refer to it.

This section contains four statements of note: propositions 166, 170 and 172 will be used in a technical way during the induction and proposition 168 will be applied in section 7.6 to construct pairs of permutations with same cycle invariant but opposite sign invariant.

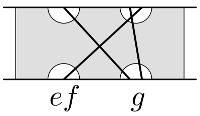
7.5.1 Arf relation for the induction

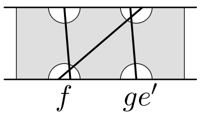
We have a first lemma, also involving the evaluation of $\overline{A}$ on three distinct configurations

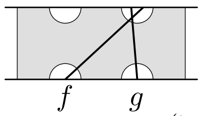
Proposition 166.



(7.1)

$$\overline{A}\Big(\ \sigma =$$ 

$$\Big)+\overline{A}\Big(\ \tau =$$ 

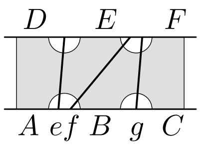
$$\Big) = 2\overline{A}\Big(\ \rho =$$ 

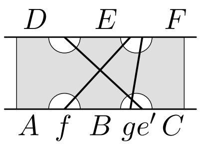
$$\Big) \tag{7.2}$$

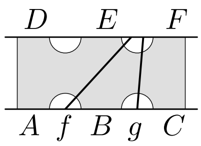
Proof. We will prove in detail the first identity. The proof of the second identity follows the same method and only the recapitulative table is presented.

We work in the framework of Proposition 55. Thus, we have to examine the matrices Q_σ, Q_τ and Q_ρ. In fact the configuration ρ, on top, has a coarser partition than configurations σ and τ (in other words, it was not necessary to draw an empty half-disk on its top-left corner), and we choose to keep the same partition in order to clarify the comparison.

As this application of our general strategy involves rather big matrices, we need to give names to the edges and the blocks of the partition. We choose as follows:

$$\sigma =$$ 

$$\tau =$$ 

$$\rho =$$ 

$$\tag{7.3}$$

Thus, the matrices Q have, as lines, subsets of $\{e, f, g\}$, $\{e', f, g\}$ and $\{f, g\}$, in the three cases, and, as columns, pairs in $\{A, B, C\} \times \{D, E, F\}$. For clarity of visualisation of the mechanism of the proof, we augment these matrices by two other columns, the cardinality of the subset and the associated factor χ.

A first fact is rather obvious. The subsets of $\{e, f, g\}$ not containing e contribute to the Arf invariant $\overline{A}_\sigma$ the same function as $\overline{A}_\rho$, as well as the subsets of $\{e', f, g\}$ not containing e' do for $\overline{A}_\tau$. Thus, if the lemma holds true, there must be a cancellation among the contributions of the four subsets of $\{e, f, g\}$ which do contain e, and the four subsets of $\{e', f, g\}$ which do contain e'.

The 4 pertinent rows of the matrices Q_σ and Q_τ are shown below:

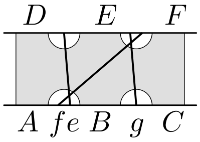
$$Q_\sigma : \quad \begin{array}{r|ccccccccc|cc} & AD & AE & AF & BD & BE & BF & CD & CE & CF & |\cdot| & \chi_\cdot \\ \hline \{e\} & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ \{e,f\} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\ \{e,g\} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 1 \\ \{e,f,g\} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 3 & 3 \\ \hline \end{array} \tag{7.4}$$

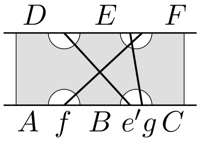
$$Q_\tau : \quad \begin{array}{r|ccccccccc|cc} & AD & AE & AF & BD & BE & BF & CD & CE & CF & |\cdot| & \chi_\cdot \\ \hline \{e'\} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \{e',f\} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ \{e',g\} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ \{e',f,g\} & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 & 1 \\ \hline \end{array} \tag{7.5}$$

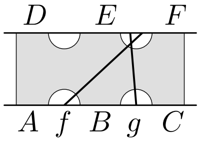
The contributions of two rows do 'match' if the rows of Q are identical, and the quantities $|J|+\chi_J$ have opposite parity (as they enter in the invariant as $(-1)^{|J|+\chi_J}$).

The contributions of the four rows of Q_σ and Q_τ do match by complementation (i.e., $\{e\}$ matches $\{e', f, g\}$, $\{e, f\}$ matches $\{e', g\}$, etc.).

For the second identity, we choose as follows:

$\tau =$ 

$\sigma =$ 

$\rho =$ 

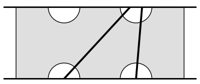
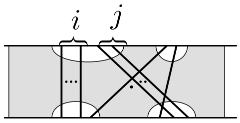
(7.6)

and the relevant part of the matrices Q_σ and Q_τ read as:

$$Q_\tau : \quad \begin{array}{r|ccccccccc|cc} & AD & AE & AF & BD & BE & BF & CD & CE & CF & |\cdot| & \chi. \\ \hline \{e\} & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ \{e,f\} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ \{e,g\} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 1 \\ \{e,f,g\} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 3 & 1 \\ \hline \end{array} \tag{7.7}$$

$$Q_\sigma : \quad \begin{array}{r|ccccccccc|cc} & AD & AE & AF & BD & BE & BF & CD & CE & CF & |\cdot| & \chi. \\ \hline \{e'\} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \{e',f\} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ \{e',g\} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\ \{e',f,g\} & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 3 & 1 \\ \hline \end{array} \tag{7.8}$$

Likewise, the contributions of the four rows of Q_σ and Q_τ do match by complementation (i.e., $\{e\}$ matches $\{e', f, g\}$, $\{e, f\}$ matches $\{e', g\}$, etc.). $\square$

Let $\sigma =$  , we define $\sigma_{i,j} =$ 

With this notation, the first equation of proposition 166 can be rewritten

$$\overline{A}(\sigma_{1,0}) + \overline{A}(\sigma_{0,1}) = 2\overline{A}(\sigma_{0,0}).$$

Lemma 167.

$$\overline{A}(\sigma_{2,0}) - \overline{A}(\sigma_{0,2}) = 2\left(\overline{A}(\sigma_{1,0}) - \overline{A}(\sigma_{0,1})\right) \tag{7.9}$$

$$\overline{A}(\sigma_{3,0}) + \overline{A}(\sigma_{0,3}) = 3\overline{A}(\sigma_{2,0}) + 3\overline{A}(\sigma_{0,2}) - 4\overline{A}(\sigma_{0,0}) \tag{7.10}$$

Proof. Let us start with equation 7.9, by applying proposition 166 to $\sigma_{1,0}$ and $\sigma_{0,1}$ we have :

$$\overline{A}(\sigma_{2,0}) + \overline{A}(\sigma_{1,1}) = 2\overline{A}(\sigma_{1,0}) \tag{7.11}$$

$$\overline{A}(\sigma_{0,2}) + \overline{A}(\sigma_{1,1}) = 2\overline{A}(\sigma_{0,1}) \tag{7.12}$$

Thus equation 7.11 minus equation 7.12 gives equation 7.9.

For equation 7.10, we apply proposition 166 to $\sigma_{2,0}$, $\sigma_{1,1}$ and $\sigma_{0,2}$:

$$\overline{A}(\sigma_{3,0}) + \overline{A}(\sigma_{2,1}) = 2\overline{A}(\sigma_{2,0}) \tag{7.13}$$

$$\overline{A}(\sigma_{2,1}) + \overline{A}(\sigma_{1,2}) = 2\overline{A}(\sigma_{1,1}) \tag{7.14}$$

$$\overline{A}(\sigma_{1,2}) + \overline{A}(\sigma_{0,3}) = 2\overline{A}(\sigma_{0,2}) \tag{7.15}$$

Now

$$
\begin{aligned}
\overline{A}(\sigma_{3,0}) + \overline{A}(\sigma_{0,3}) &= \overline{A}(\sigma_{3,0}) + \overline{A}(\sigma_{2,1}) - \left(\overline{A}(\sigma_{2,1}) + \overline{A}(\sigma_{1,2})\right) + \overline{A}(\sigma_{1,2}) + \overline{A}(\sigma_{0,3}) \\
&= 2\overline{A}(\sigma_{2,0}) + 2\overline{A}(\sigma_{0,2}) - 2\overline{A}(\sigma_{1,1}) && \text{by eqs 7.13,7.14,7} \\
&= 2\overline{A}(\sigma_{2,0}) + 2\overline{A}(\sigma_{0,2}) - \left(2\overline{A}(\sigma_{1,0}) - \overline{A}(\sigma_{2,0})\right) - \left(2\overline{A}(\sigma_{0,1}) - \overline{A}(\sigma_{0,2})\right) && \text{by eqs 7.11,7} \\
&= 3\overline{A}(\sigma_{2,0}) + 3\overline{A}(\sigma_{0,2}) - 2\left(\overline{A}(\sigma_{1,0}) + \overline{A}(\sigma_{0,1})\right) \\
&= 3\overline{A}(\sigma_{2,0}) + 3\overline{A}(\sigma_{0,2}) - 4\overline{A}(\sigma_{0,0}) && \text{by proposition}
\end{aligned}
$$

□

The main use of those two lemmas is to compare the Arf invariant of two configurations having the same cycle invariant and differing by just a few consecutives and parallel edges. The exact statement is the content of the next proposition.

Proposition 168 (Opposite sign)**.** *Let σ be a permutation with exactly two even cycles (one being possibly the rank). Let α be a top arc of the first cycle and β and β' be two consecutive bottom arcs of the second cycle.*

Define $\sigma|_{i,\alpha,\beta}$ and $\sigma|_{i,\alpha,\beta'}$ following notation 5 (as a reminder they are the two permutations obtained by adding i parallel and consecutive edges within α, β and within α, β' respectively). See figure 7.12 for the case $i = 1$ and σ has two even cycles (none being the rank path).

Then for $i \leq 3$, $\sigma|_{i,\alpha,\beta}$ and $\sigma|_{i,\alpha,\beta'}$ have invariant (λ', r', s) and $(\lambda', r', -s)$ respectively.

To be more precise :

1. *If σ has cycle invariant $(\lambda \bigcup \{2\ell, 2\ell'\}, r)$ we have*

	$i=1$	$i=2$	$i=3$
(λ, r, s) *of* $\sigma\|_{i,\alpha,\beta}$	$(\lambda \bigcup \{2\ell+2\ell'+1\}, r, s)$	$(\lambda \bigcup \{2\ell+1, 2\ell'+1\}, r, s)$	$(\lambda \bigcup \{2\ell+2\ell'+3\}, r, s)$
(λ, r, s) *of* $\sigma\|_{i,\alpha,\beta'}$	$(\lambda \bigcup \{2\ell+2\ell'+1\}, r, -s)$	$(\lambda \bigcup \{2\ell+1, 2\ell'+1\}, r, -s)$	$(\lambda \bigcup \{2\ell+2\ell'+3\}, r, -s)$

2. *If σ has cycle invariant $(\lambda \bigcup \{2\ell\}, 2r)$ we have*

	$i=1$	$i=2$	$i=3$
(λ, r, s) *of* $\sigma\|_{i,\alpha,\beta}$	$(\lambda, 2r+2\ell+1, s)$	$(\lambda \bigcup \{2\ell+1\}, 2r+1, s)$	$(\lambda, 2r+2\ell+3, s)$
(λ, r, s) *of* $\sigma\|_{i,\alpha,\beta'}$	$(\lambda, 2r+2\ell+1, -s)$	$(\lambda \bigcup \{2\ell+1\}, 2r+1, -s)$	$(\lambda, 2r+2\ell+3, -s)$

Proof. Let us start with the cycle invariant. For the case $i = 1$ and $i = 2$ this is a strict application of propositions 156 and 157 respectively.

For $i = 3$ note that $\sigma|_{3,\alpha,\beta}$ is obtained from $\sigma|_{1,\alpha,\beta}$ by adding a double-edge. In the first case $\sigma|_{1,\alpha,\beta}$ has cycle invariant $(\lambda \bigcup \{2\ell + 2\ell' + 1\}, r)$ and the double-edge is inserted on two arcs of the cycle of length $2\ell + 2\ell' + 1$ thus by proposition 157 the cycle invariant of $\sigma|_{3,\alpha,\beta}$ is $(\lambda \bigcup \{2\ell + 2\ell' + 3\}, r)$.

In the second case $\sigma|_{1,\alpha,\beta}$ has cycle invariant $(\lambda, 2r + 2\ell + 1)$ and the double-edge is inserted on two arcs of the rank path of length $2r + 2\ell + 1$ thus by proposition 157 the cycle invariant of $\sigma|_{3,\alpha,\beta}$ is $(\lambda, 2r + 2\ell + 3)$.

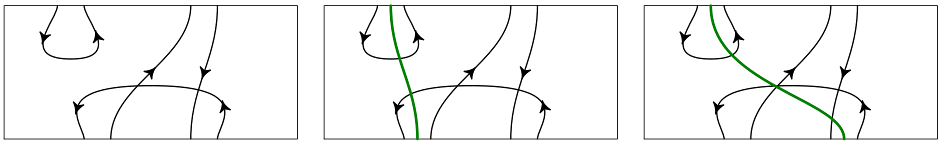


Figure 7.12: From left to right: the permutations σ, $\sigma|_{1,\alpha,\beta}$ and $\sigma|_{1,\alpha,\beta'}$. The proposition 168 says that $\sigma|_{1,\alpha,\beta}$ and $\sigma|_{1,\alpha,\beta'}$ have same cycle invariant and opposite sign invariant.

The same reasoning applies to $\sigma|_{3,\alpha,\beta'}$.

In the following the reasoning applies to both cases (σ has cycle invariant $(\lambda \bigcup\{2\ell, 2\ell'\}, r)$ and σ has cycle invariant $(\lambda \bigcup\{2\ell\}, 2r)$) so we will no longer differentiate.

By hypothesis on σ, λ has no even part thus it can be verified on the two tables that the cycle invariant of $\sigma|_{i,\alpha,\beta}$ and $\sigma|_{i,\alpha,\beta'}$ has no even cycle for $i \leq 3$.

Consequently, $\overline{A}(\sigma|_{i,\alpha,\beta}) = \pm 2^{\frac{n_i+\ell_i}{2}}$ and $\overline{A}(\sigma|_{i,\alpha,\beta'}) = \pm 2^{\frac{n_i+\ell_i}{2}}$ where n_i is the size of $\sigma|_{i,\alpha,\beta}$ and ℓ_i is the number of cycle (not including the rank) of $\sigma|_{i,\alpha,\beta}$ by theorem 137.

Let n be the size of σ and ℓ_0 be the number of cycles (not including the rank) of σ. Then by inspecting the tables we know that

$$\overline{A}(\sigma|_{1,\alpha,\beta}) = \pm 2^{\frac{n+\ell_0}{2}} = \pm \overline{A}(\sigma|_{1,\alpha,\beta'}) \tag{7.16}$$

$$\overline{A}(\sigma|_{2,\alpha,\beta}) = \pm 2^{\frac{n+\ell_0}{2}+1} = \pm \overline{A}(\sigma|_{2,\alpha,\beta'}) \tag{7.17}$$

and that

$$\overline{A}(\sigma) = 0 \tag{7.18}$$

by theorem 137 since σ has even cycles.

Moreover by applying proposition 166 to $\sigma|_{1,\alpha,\beta}, \sigma|_{1,\alpha,\beta'}$ and σ we have that

$$\begin{aligned} \overline{A}(\sigma|_{1,\alpha,\beta}) + \overline{A}(\sigma|_{1,\alpha,\beta'}) &= 2\overline{A}(\sigma) \\ &= 0 && \text{by eq 7.18} \end{aligned}$$

thus

$$\overline{A}(\sigma|_{1,\alpha,\beta}) = -\overline{A}(\sigma|_{1,\alpha,\beta'}) \tag{7.19}$$

Now by applying lemma 167 (equation 7.9) to $\sigma|_{2,\alpha,\beta}, \sigma|_{2,\alpha,\beta'}$ and $\sigma|_{1,\alpha,\beta}, \sigma|_{1,\alpha,\beta'}$ we have

$$\begin{aligned} \overline{A}(\sigma|_{2,\alpha,\beta}) - \overline{A}(\sigma|_{2,\alpha,\beta'}) &= 2(\overline{A}(\sigma|_{1,\alpha,\beta}) - \overline{A}(\sigma|_{1,\alpha,\beta'})) \\ &= 4\overline{A}(\sigma|_{1,\alpha,\beta}) && \text{by eq 7.19} \end{aligned}$$

Thus by equations 7.16 and 7.17 we must have

$$\overline{A}(\sigma|_{2,\alpha,\beta}) = -\overline{A}(\sigma|_{2,\alpha,\beta'}) \tag{7.20}$$

Finally by applying lemma 167 (equation 7.10) to $\sigma|_{3,\alpha,\beta}, \sigma_{3,\alpha,\beta'}$ and $\sigma|_{2,\alpha,\beta}, \sigma|_{2,\alpha,\beta'}$ and σ we have

$$\begin{aligned} \overline{A}(\sigma|_{3,\alpha,\beta}) + \overline{A}(\sigma|_{3,\alpha,\beta'}) &= 3(\overline{A}(\sigma|_{2,\alpha,\beta}) + \overline{A}(\sigma|_{2,\alpha,\beta'})) - 4\overline{A}(\sigma) \\ &= 0 && \text{By eqs 7.20 and 7.18} \end{aligned}$$

thus

$$\overline{A}(\sigma|_{3,\alpha,\beta}) = -\overline{A}(\sigma_{3,\alpha,\beta'}). \tag{7.21}$$

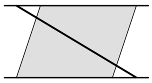
□

The key point of the proposition is really that $i \leq 3$, $\sigma|_{i,\alpha,\beta}$ and $\sigma|_{i,\alpha,\beta'}$ have same cycle invariant and opposite sign invariant. We will use this proposition to construct I_2X-permutation with same cycle invariant and opposite sign in the next section.

Remark 169. *The proof of proposition 168 makes use of theorem 137. This theorem will only be proved during the induction, thus every time we use proposition 168, we must check that we have already proved theorem 137 or indicate that the newly proven proposition is also dependent on theorem 137.*

Proposition 170. *Let $\tau, (\Pi_b, \Pi_t)$ be a permutation with a consistent labelling and invariant (λ, r, s), then*

$$\overline{A}\Big(\tau|_{1,t_0^{rk},b_r^{rk}} = \quad\Big) = 0 \tag{7.22}$$



More generaly for $0 \leq i \leq r$,

$$\overline{A}(\tau|_{1,t_0^{rk},b_{r-i}^{rk}}) = \begin{cases} 0, & \text{if } i \equiv 0 \mod 2 \\ 2\overline{A}(\tau) & \text{Otherwise.} \end{cases} \tag{7.23}$$

Note that the first equation corresponds to the case $i = 0$ of the second equation.

Proof. The first equation is a straighforward application of proposition 54. The second equation is derived from the first by induction on i. As noted above the base case ($i = 0$) of the induction is equation 7.22. For the inductive step, we suppose equation 7.23 true for i.

First case: if $i + 1$ is even. Then $\overline{A}(\tau|_{1,t_0^{rk},b_{r-i}^{rk}}) = 2\overline{A}(\tau)$ by induction, applying proposition 166, we have that

$$\overline{A}(\tau|_{1,t_0^{rk},b_{r-(i+1)}^{rk}}) + \overline{A}(\tau|_{1,t_0^{rk},b_{r-i}^{rk}}) = 2\overline{A}(\tau)$$

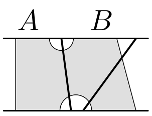
since the arcs $\beta = \Pi_b^{-1}(b_{r-(i+1)}^{rk})$ and $\beta' = \Pi_b^{-1}(b_{r-i}^{rk})$ are consecutive. Thus $\overline{A}(\tau|_{1,b_0^{rk},t_{r-(i+1)}^{rk}}) = 0$.

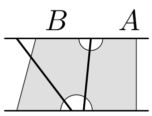
Second case: if $i + 1$ is odd is proved similarly. □

Now we have a relation that goes slightly beyond the framework of Proposition 55.

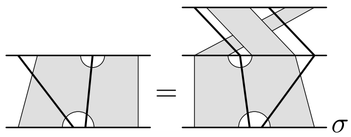
Lemma 171.

$$\vec{A}\Big(\sigma = \quad\Big) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \vec{A}\Big(\tau = \quad\Big) \tag{7.24}$$

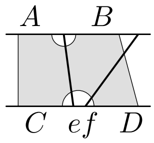


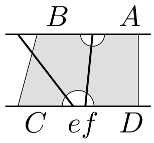


The letters A and B above the configurations denote that fact that, contrarily to what was the case up to this point, the gray part of the diagrams does not *denote the same* *collection of arcs, but rather the two configurations are related by the permutation depicted below*

$\tau =$ 

Proof. It is convenient to introduce labels in the configurations σ and τ:

$\sigma =$ 

$\tau =$ 

(7.25)

Similarly to the setting of Proposition 55, the Arf invariant involves a sum over subsets of the edges, and we can decompose this sum according to the restriction of the subset to edges e and f. We will denote by n_{AC}, n_{AD}, n_{BC} and n_{BD} the number of edges in the subset which connect the associated intervals. Consider the $\binom{n_{AC}+n_{AD}}{2}$ pairs of edges (x, x') in the subset, which reach the block A. Each such pair is either non-crossing both in σ and τ, or crossing both in σ and τ. A similar statement holds for the $\binom{n_{BC}+n_{BD}}{2}$ pairs of edges (y, y') which reach the block B. Now consider the $(n_{AC} + n_{AD})(n_{BC} + n_{BD})$ pairs of edges (x, y) in the subset, with x reaching A and y reaching B. Each such pair is non-crossing in exactly one configuration among σ and τ.

Remark that the statement of the lemma just reads

$$A(\sigma) = -\overline{A}(\tau) \qquad\qquad \overline{A}(\sigma) = -A(\tau) \tag{7.26}$$

Yet again, we have to examine the matrices Q_σ and Q_τ. These matrices have, as lines, subsets of $\{e, f\}$, and, as columns, pairs in $\{A, B\} \times \{C, D\}$. For clarity of visualisation of the mechanism of the proof, we augment these matrices by two other columns, the cardinality of the subset and the associated factor χ. However, because of the exchange of blocks A and B, it is now confusing to just state that there is some overall factor that goes for the ride. Let us instead pose that, for a given subset I not containing e or f, there is a factor $(-1)^{\chi_0}$ in the calculation of $A(\sigma)$. Thus, if we let $N = |I| = n_{AC} + n_{AD} + n_{BC} + n_{BD}$, the factor in $\overline{A}(\sigma)$ is $(-1)^{\chi_0+N}$. Similarly, if we call $P = (n_{AC} + n_{AD})(n_{BC} + n_{BD})$, the factor in $A(\tau)$ is $(-1)^{\chi_0+P}$. Thus, with abuse of notation, in the column for χ we will write quantities which are not necessarily χ, but rather an integer with the same parity as χ. We are now ready to present our matrices:

$$Q_\sigma : \qquad \begin{array}{r|cccc|cc} & AC & BC & AD & BD & |\cdot| & \chi. \\ \varnothing & 0 & 0 & 0 & 0 & N & \chi_0 \\ \{e\} & 1 & 0 & 0 & 1 & N+1 & \chi_0 \\ \{f\} & 1 & 1 & 0 & 0 & N+1 & \chi_0 \\ \{e,f\} & 0 & 1 & 0 & 1 & N+2 & \chi_0+1 \end{array} \tag{7.27}$$

$$Q_\tau : \qquad \begin{array}{r|cccc|cc} & AC & BC & AD & BD & |\cdot| & \chi. \\ \varnothing & 0 & 0 & 0 & 0 & N & \chi_0+P \\ \{e\} & 0 & 0 & 1 & 1 & N+1 & \chi_0+P \\ \{f\} & 0 & 1 & 1 & 0 & N+1 & \chi_0+P \\ \{e,f\} & 0 & 1 & 0 & 1 & N+2 & \chi_0+P+1 \end{array} \tag{7.28}$$

We shall inspect the contribution to $A(\sigma)$, $\overline{A}(\sigma)$, $A(\tau)$ and $\overline{A}(\tau)$ of the four terms (for the subsets of $\{e, f\}$) associated to a given subset I. We have

$A(\sigma)$	$(-1)^{\chi_0}$	$(1 + (-1)^{n_{BC}+n_{BD}+1} + (-1)^{n_{AC}}((-1)^{n_{BC}} + (-1)^{n_{BD}}))$
$\overline{A}(\tau)$	$(-1)^{\chi_0+P+N}$	$(1 + (-1)^{n_{BC}+n_{BD}+1} + (-1)^{n_{AD}+1}((-1)^{n_{BC}} + (-1)^{n_{BD}}))$
$\overline{A}(\sigma)$	$(-1)^{\chi_0+P}$	$(1 + (-1)^{n_{BC}+n_{BD}+1} + (-1)^{n_{AC}+1}((-1)^{n_{BC}} + (-1)^{n_{BD}}))$
$A(\tau)$	$(-1)^{\chi_0+N}$	$(1 + (-1)^{n_{BC}+n_{BD}+1} + (-1)^{n_{AD}}((-1)^{n_{BC}} + (-1)^{n_{BD}}))$

(7.29)

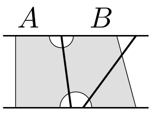
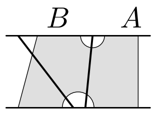
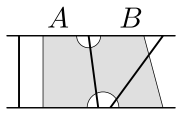
We can now prove equations (7.26). Note that

$$P + N = (n_{AC} + n_{AD} + 1)(n_{BC} + n_{BD} + 1) - 1\,. \tag{7.30}$$

If $n_{AC} + n_{AD}$ is odd, $P + N$ is odd, and the rows of the table match pairwise, up to having opposite sign, as desired. Similarly, if $n_{BC} + n_{BD}$ is odd, $P + N$ is odd, and the rows of the table match pairwise, up to having opposite sign (only the first two summands of each row are left in this case). In the remaining case, $N + P$ is even, and n_{AC} and n_{AD} have the same parity, as well as n_{BC} and n_{BD}. Only the last two summands of each row are left in this case, and again the rows of the table match pairwise, up to having opposite sign. This completes the proof. $\square$

We can now deduce a corollary, which is more relevant in the following, and that involves three distinct configurations

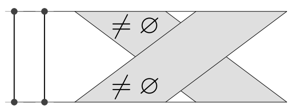
Proposition 172.

$\overline{A}\Big(\sigma =$  $\Big) + \overline{A}\Big(\tau =$  $\Big) = \overline{A}\Big(\rho =$  $\Big)$ (7.31)

Proof. Combine the first of equations (7.26), and equation (4.12) applied to ρ. $\square$

7.6 A I_2X-permutation for every (λ, r, s)

In this section, we will construct a non exceptional I_2X-permutation for every invariant (λ, r, s).



The tools we will use from the preceding sections are :

- The double-edge insertion proposition 157 to obtain the correct cycle invariant (λ, r).
- The opposite sign proposition 168 to obtain the correct sign invariant s.

Let us give an overview of the section before detailling the actual construction of a I_2X-permutation for every a given (λ, r, s).

Proposition 173 and 179 allows to add a (or a pair of) cycle of any given length into a permutation without breaking the property that is it I_2X.

Propositions 175 and 180 construct I_2X-permutations for some specific value of (λ', r'), more precisely for every r' and for some λ' of small cardinalities. We call those permutations *base permutations.*

Finally Propositions 168 and 174 allows us to produce a I_2X-permutation with invariant $(\lambda, r, -s)$ from a I_2X-permutation with invariant (λ, r, s).

The construction of a I_2X-permutation σ for a given (λ, r, s), proceeds in three steps. Of course at each step of the construction we garantee that the resulting permutations are still I_2X

1. We choose a base permutation σ^0 with invariant (λ', r) such that $\lambda' \subseteq \lambda$.

2. We add cycles on σ^0 until the new permutation σ^1 has invariant (λ, r).

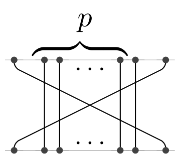
3. If σ^1 has no even cycle then the sign of σ^1 is ± 1 (refer to theorem 137 for a reminder of the relationship between the sign invariant s and the cycle invariant (λ, r))). Since we have no control on whether it is $+1$ or -1, we need to construct another permutation σ^2 with opposite sign so as to insure that either σ^1 or σ^2 has invariant (λ, r, s).

   This is done by using proposition 174 in most cases and by proposition 168 for a few remaining cases. The constuction is the subject of theorem 176.

   If σ^1 has even cycles then the sign of σ^1 is necessarily 0 and we are done. The construction is the subject of theorem 181

4. Finally we will verify (in a remark after the theorem) that the constructed permutations are not exceptional.

Let us define C_p for any $p \in \mathbb{N}$ the cross permutation

$C_p =$ 

Proposition 173 (Adding cycles 1)**.** *let σ be a permutation with cycle invariant (λ, r), we call $\sigma_i(C_p)$ the permutation obtained by replacing the ith edge of σ by a the cross permutation C_p (see figure 7.13). The cycle invariant (λ', r) of $\sigma_i(C_p)$ depends on p in the following way :*

- *if $p = 4k$ then $\lambda' = \lambda \bigcup \{p + 1\}$.*
- *if $p = 4k + 1$ then $\lambda' = \lambda \bigcup \{2k + 1, 2k + 1\}$.*
- *if $p = 4k + 2$ then $\lambda' = \lambda \bigcup \{p + 1\}$.*
- *if $p = 4k + 3$ then $\lambda' = \lambda \bigcup \{2k + 2, 2k + 2\}$.*

Proof. By induction on p. The base cases are for $p = 0$ and $p = 1$.

For those cases, the permutation σ with invariant (λ, r) becomes respectively $\sigma_i(C_0)$ with invariant $(\lambda \bigcup \{1\}, r)$ and $\sigma_i(C_1)$ with invariant $(\lambda \bigcup \{1, 1\}, r)$, as displayed in figure 7.14 :

Then the statement follows by induction from the insertion of a double-edge (the resulting change of the cycle invariant are described in proposition 157). $\square$

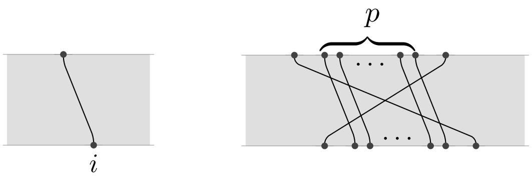
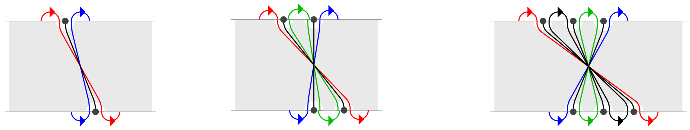


Figure 7.13: Left: a permutation σ. Right : the permutation $\sigma_i(C_p)$.

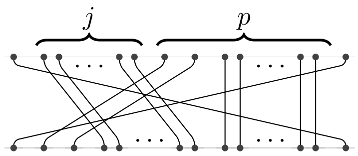


σ, (λ, r) $\quad$ $\sigma_i(C_0)$, $(\lambda \bigcup \{1\}, r)$ $\quad$ $\sigma_i(C_1)$, $(\lambda \bigcup \{1, 1\}, r)$

Figure 7.14: The base cases of the induction of the proof of proposition 173

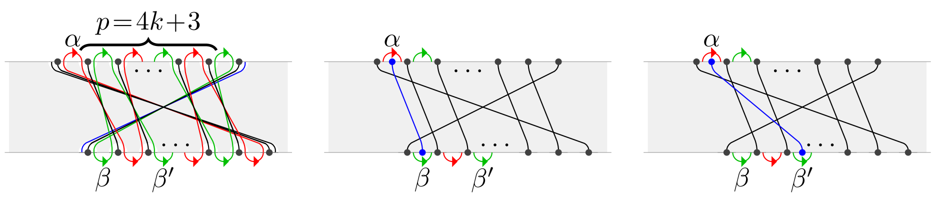
Note that if $p = 4k+3$ the two cycles of the C_p structure attached on $\sigma' = \sigma_i(C_p)$ are even. Let α be the first top arc of the C_p structure of σ' and β and β' its first and third bottom arcs (refer to figure 7.15 left). Then by proposition 168 for $j = 1, 2, 3$ the sign invariant of $\sigma'|_{j,\alpha,\beta}$ and $\sigma'|_{j,\alpha,\beta'}$ are opposite while their cycle invariants remain egal (figure 7.15 middle and right).

For clarity, let us call $C_{p,j}$ for any p, j the permutation:

$C_{p,j} =$ 

Then we have $\sigma'|_{j,\alpha,\beta} = \sigma_i(C_{p+j})$ and $\sigma'|_{j,\alpha,\beta'} = \sigma_i(C_{p,j})$ and our discussion implies the following statement:

Proposition 174 (Two opposite signs)**.** *Let σ be a permutation and let $p = 4k + j$ with $0 \leq j < 3$ and $k > 0$. Then $\sigma_i(C_p)$ and $\sigma_i(C_{p-(j+1),j+1})$ have invariant (λ, r, s) and $(\lambda, r, -s)$ respectively for some (λ, r, s).*



$\sigma' = \sigma_i(C_p)$ $\quad$ $\sigma'|_{1,\alpha,\beta} = \sigma_i(C_{p+1})$ $\quad$ $\sigma'|_{1,\alpha,\beta'} = \sigma_i(C_{p,1})$

Figure 7.15: Left: a permutation with a C_p structure containing two even cycles of even length. Middle and Right: By proposition 168 the two permutations have same cycle invariant and opposite sign.

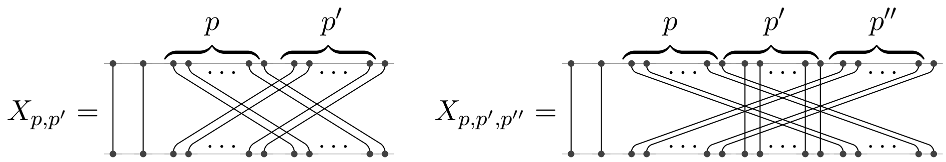
This statement will be crucial for our construction in theorem 176. Indeed, as outlined in the beginning of the section, we will construct for every (λ, r, s) a permutation with invariant (λ, r, s) by adding cycles on a base permutation with invariant $(\lambda' \subseteq \lambda, r)$. This construction will be performed through the means of proposition 173) .

Thus in the final step of this procedure, we construct $\sigma_1 = \sigma_i(C_p)$ with invariant (λ, r). In order to insure that our constructed permutation has sign invariant s, we also consider $\sigma_1' = \sigma_i(C_{p-(j+1),j+1})$ for a correct j. Then by proposition 174 either σ_1 or σ_1' will have invariant (λ, r, s).

The last ingredient of our proof of theorem 176 are the base permutations.

The following proposition provides the base permutations for the case where (λ, r, s) has no even cycle (first line of figure 7.16). It will also allow us to apply proposition 168 to obtain two permutations with opposite sign (second line of figure 7.16) for the few cases not covered by proposition 174.

Let us define the permutations $X_{p,p'}$, $X_{p,p',p''}$ for any p, p', p'' to be



Then we have:

Proposition 175 (Base permutations, no even cycle)**.** *For every $k \geq 1$.*

The permutations $X_{1,2k}$, $X_{2,1,2k}$ and $X_{2,2k}$ (described in the first line of figure 7.16 have cycle invariant $(\lambda = \{2k+1\}, r = 1)$, $(\lambda = \{2k+1\}, r = 3)$ and $(\lambda = \varnothing, r = 2k+3)$ respectively.

The permutations $X_{1,4k+3}$, $X_{2,1,4k+3}$ and $X_{2,4k+3}$ (described in the second line of figure 7.16) have cycle invariant $(\lambda = \{2k+2, 2k+2\}, r = 1)$, $(\lambda = \{2k+2, 2k+2\}, r = 3)$ and $(\lambda = \{2k+2\}, r = 2k+4)$ respectively.

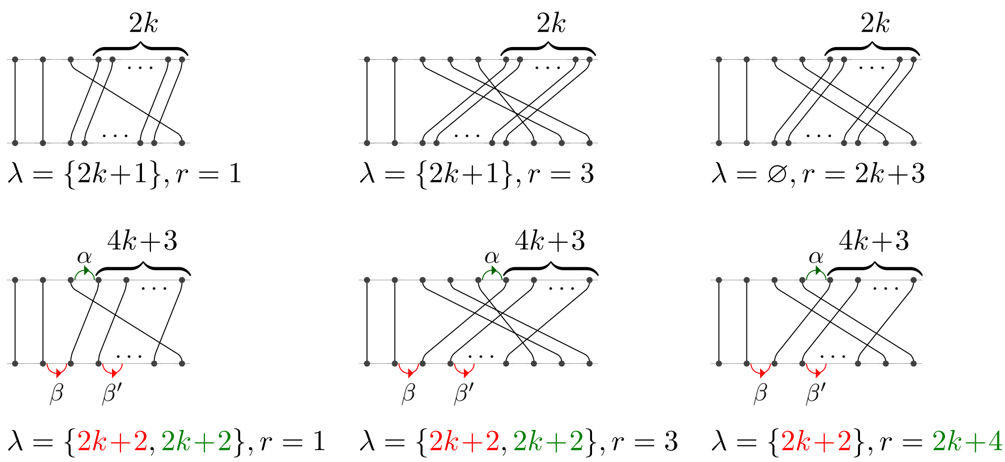
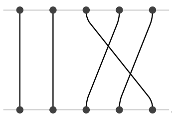
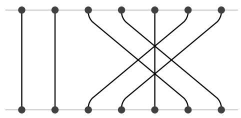
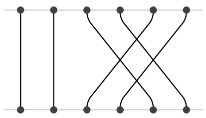
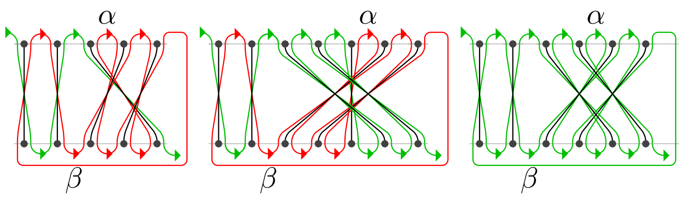


Figure 7.16: Two families of base permutations with their respective cycle invariant.

Proof. By induction on k. The base cases for the first family are respectively :

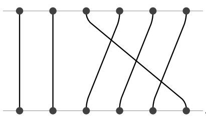
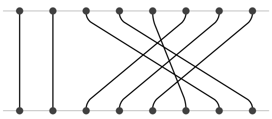
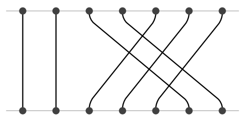
,  and  .

And they have cycle invariant (λ, r) : $(\{3\}, 1)$, $(\{3\}, 3)$ $(\varnothing, 5)$ as shown just below :

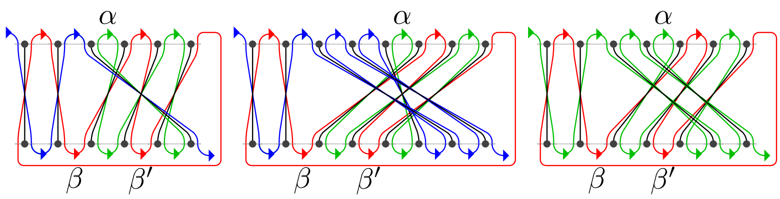


Then the statement follows by induction from the insertion a double-edge within α and β (the resulting change of the cycle invariant are described in proposition 157).

The base cases for the second family are respectively:

,  and  .

And they have cycle invariant (λ, r) : $\lambda = \{2, 2\}, r = 1$, $\lambda = \{2, 2\}, r = 3$, $\lambda = \{2\}, r = 4$ as shown just below :



Then the statement follows by induction from the insertion of two double-edges within α and β (the resulting change of the cycle invariant are described in proposition 157). $\square$

We can finally state and prove the first theorem of this section.

Theorem 176. *Let (λ, r, s) be an invariant with no even cycle, then there exists a I_2X-permutation with invariant (λ, r, s).*

Proof. We can always consider that the size of the permutation is at least 10, for smaller size the result can be obtained by automatic search.

Let (λ, r, s) be an invariant with no even cycles. Following the proof sketch of the introduction, we fist construct a base permutation σ^0 with invariant $(\lambda' \subseteq \lambda, r)$ then add cycles to obtain a I_2X-permutation σ_1 with invariant (λ, r) and finally we use either proposition 174 or proposition 168 to obtain two I_2X-permutations with opposite sign.

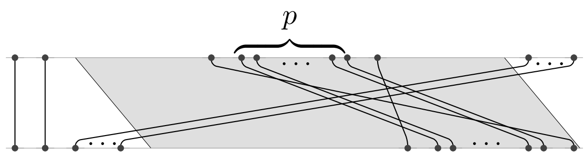


Figure 7.17: The permutation resulting from adding a C_p on the last edge of σ^0 or one of its successor. It is clearly also I_2X.

- If the rank $r = 1$, then the base permutation σ^0 with invariant $(\lambda' = \{2\ell+1\}, 1)$ for any $\ell \geq 1$ is exactly $X_{2\ell}$ according to proposition 175 (first line, first case of figure 7.16).
- If the rank $r = 3$, then the base permutation σ^0 with invariant $(\lambda' = \{2\ell+1\}, 3)$ for any $\ell \geq 1$ is exactly $X_{2,1,2\ell}$ according to proposition 175 (first line, second case of figure 7.16)).
- If the rank $r > 5$, then the base permutation σ^0 with invariant $(\lambda' = \varnothing, r)$ is exactly $X_{2,r-3}$ according to proposition 175 (first line, third case of figure 7.16 indeed r odd implies $r - 3$ is even)).

Next we add (by the means of proposition 173) cycles one by one and then by pair on the last edge of the current permutation σ^i so as to make sure that the last C_p attached is not a C_2. This is always possible unless $\lambda = \lambda'$ (in which case $\sigma_1 = \sigma^0$) or $\lambda' = \lambda \bigcup \{3\}$. In this case we only have to add a cycle a length 3 to the permutation and the procedure begins and ends with the attachement of a C_2: $\sigma_1 = \sigma^0_n(C_2))$.

The procedure must finish with the addition of a C_p with $p > 2$ in order to make the application of proposition 174 possible.

Note that since we always attach C_p on the last edge $(|\sigma^i|, \sigma^i(|\sigma^i|))$ of the current permutation σ^i, the successive permutations from σ^0 to σ_1 are all I_2X. See figure 7.17.

Let $2\ell_1 + 1 < \ldots < 2\ell_k + 1$ be the length and $(m_i)_{1\leq i\leq k}$ the multiplicity of the cycles to be added on σ^0 (i.e. the cycles of $\lambda \setminus \lambda'$). The procedure is divided into two steps:

• First we look at the parity of m_i from $i = 1$ to $i = k$ and if m_i is odd we attach a $C_{2\ell_i}$ on the last edge of the current permutation (which adds a cycle of length $2\ell_i + 1$ to the cycle invariant).

More precisely, let $i_1, \ldots, i_m$ be the indices such that the $(m_{i_j})_j$ are odd, we define:

$$\sigma^j = \sigma^{j-1}_{|\sigma^{j-1}|}(C_{2\ell_j}) \text{ for } 1 \leq j \leq m.$$

Let (λ^j, r) be the cycle invariant of σ^j, by proposition 173, we have $\lambda^j = \lambda^{j-1} \bigcup \{2\ell_j + 1\}$. Thus the multiplicities $(2m'_i)_{1\leq i\leq k}$ of the cycles of length $(2\ell_i + 1)_{1\leq i\leq k}$ to be added on σ^m are all even.

• In the second step, we attach a $C_{4\ell_i+1}$ on the last edge of the current permutation (which add two cycles of length $2\ell_i + 1$ to the cycle invariant) consecutively $m_i/2$ time for $i = 1$ to $i = k$.

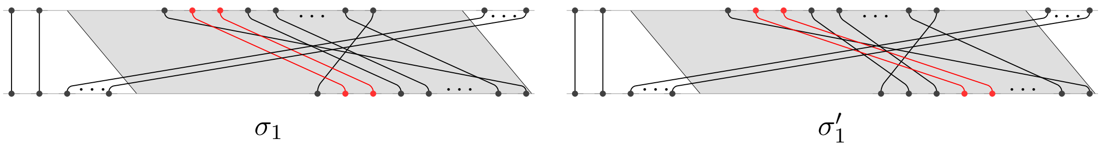


Figure 7.18: The two constructed I_2X-permutations $\sigma_1 = \sigma_{|\sigma|}(C_p)$ and $\sigma'_1 = \sigma_{|\sigma|}(C_{p-(2),2})$ with invariant $(\lambda, r, \pm s)$. In this case $p = 4k + 1$.

More specifically, we define $\sigma^{j,i}$ and its cycle invariant $(\lambda^{j,i}, r)$ by

$$\begin{cases} \sigma^{j,i} = \sigma^{j-1,i}_{|\sigma^{j-1,i}|}(C_{4\ell_i+1}) \text{ and } \lambda^{j,i} = \lambda^{j-1,i} \bigcup \{2\ell_i+1, 2\ell_i+1\} & \text{for } 1 \le j \le m'_i/2 \text{ and } 1 \le i \le k \\ \sigma^{0,i} = \sigma^{m'_{i-1}/2,i-1} \text{ and } \lambda^{0,i} = \lambda^{m'_{i-1}/2,i-1} & \text{for } 1 < i \le k, \\ \sigma^{0,1} = \sigma^m \text{ and } \lambda^{0,1} = \lambda^m. \end{cases}$$

Once again, the values of the cycle invariants are justified by proposition 173

Let us call the permutation obtained by the procedure σ_1.

By construction σ_1 has cycle invariant (λ, r) and the last added C_p by the procedure is C_2 if and only if there was just one cycle of length 3 to be added.

Let us now deal with the sign invariant.

We can divide the problem in two cases : The last C_p attached has $p > 2$ or not (the latter case corresponds to attaching exactly one C_2 or not attaching anything, thus remaining with the base permutation). Moreover, in accordance with the procedure, $p = 4k + j$, $k > 0$ and $0 \geq j < 3$. Thus p and k verify the condition

$*$ If the last C_p attached has $p > 2$. We have $\sigma_1 = \sigma_{|\sigma|}(C_p)$ for some σ (more specifically $\sigma = \sigma^{m'_k/2-1,k}$ the second to last permutation of the procedure) and σ_1 has the form shown in figure 7.17.

Moreover, in accordance with the procedure, $p = 4k + j$, $k > 0$ and $0 \geq j < 3$. Thus p verify the condition of proposition 174 and $\sigma'_1 = \sigma_{|\sigma|}(C_{p-(j+1),j+1})$ has invariant cycle (λ, r) and sign invariant opposite to σ_1, thus either one of them has invariant (λ, r, s). By construction σ'_1 is also I_2X. (See figure 7.18 for a representation of σ_1 and σ'_1 for $j = 2$).

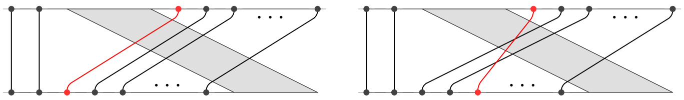
$*$ If no C_p or just one C_2 are attached. We will only consider the case no C_p are added, if a C_2 is added on the last edge, the reasoning is strictly identical since adding a C_p does not change the already existing cycles and arcs of the permutation (it only adds cycles and consecutive arcs). Futhermore among the three cases $X_{1,2\ell}$, $X_{2,1,2\ell}$ and $X_{2,r-3}$ we will only handle the first one, the other two being similar.

Let $\sigma^0 = \sigma_1 = X_{1,2\ell}$ for some $\ell \geq 2$ (this is always the case since otherwise $X_{1,2\ell}$ has size 5)

- If $2\ell = 4k$ then removing the first parallel edge of the 2ℓ consecutives and parallel edges of $\sigma_1 = X_{1,2\ell}$ we get $\tau = X_{1,4(k-1)+1}$ by proposition 168 and proposition 175 (second line first case of figure 7.16) the permutations $\tau|_{1,\alpha,\beta} = \sigma^1$ and $\tau|_{1,\alpha,\beta'}$ have same cycle invariant and opposite sign.

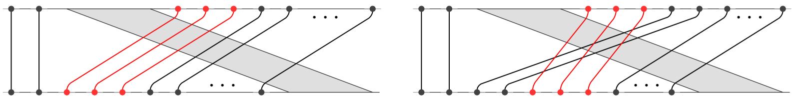
- If $2\ell = 4k + 2$ then removing the first three parallel edge of the 2ℓ consecutives and parallel edges of $\sigma_1 = X_{1,2\ell}$ we get $\tau = X_{1,4(k-1)+1}$ by proposition 168 and proposition 175(second line first case of fig 7.16) the permutations $\tau|_{3,\alpha,\beta} = \sigma^1$ and $\tau|_{3,\alpha,\beta'}$ have same cycle invariant and opposite sign.

Schematically the two cases (for $X_{1,2\ell}$, $X_{2,1,2\ell}$ and $X_{2,r-3}$ with or without a C_2 attached on the last edge) are :



$\tau|_{1,\alpha,\beta} = \sigma^1$ $\qquad$ $\tau|_{1,\alpha,\beta'} = \sigma^2$

and



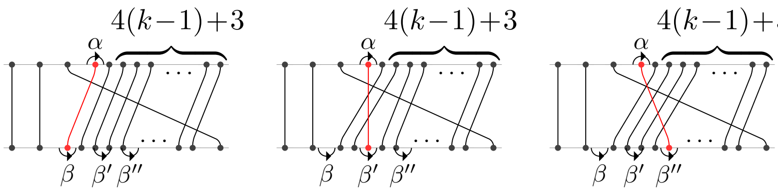
$\tau|_{3,\alpha,\beta} = \sigma^1$ $\qquad$ $\tau|_{3,\alpha,\beta'} = \sigma^2$

Finally the permutations obtained are clearly I_2X. ☐

Remark 177. *Theorem 176 makes use of proposition 168 directly and indirectly in its use of proposition 174. Therefore theorem 176 is also dependent on theorem 137 and we must check that we have proved theorem 137 before using theorem 176.*

Remark 178. *It is unfortunate that in the case* $r = 1$ *and* $\lambda = \{2\ell+1\}$ *one of the two permutations with invariant* $(\lambda, r, \pm s)$ *produced by theorem 176 is exactly* $X_{1,2\ell} = \mathrm{id}'_{2\ell+3}$ *i.e. a permutation from an exceptional class. We rectify this by constructing a third permutation with the same invariant as* $X_{1,2\ell}$*.*

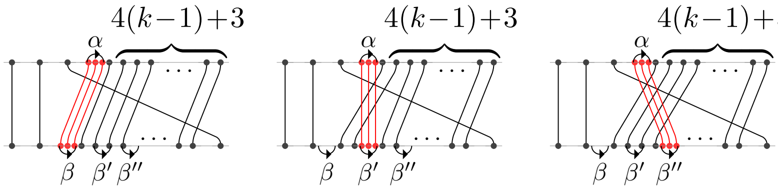
- *if* $2\ell = 4k$



$(\{2\ell+1\}, 1, s)$ $\qquad$ $(\{2\ell+1\}, 1, -s)$ $\qquad$ $(\{2\ell+1\}, s)$

The first two permutations are the ones with invariant $(\{2\ell+1\}, 1, \pm s)$ *constructed by theorem 176. The invariant of the third one are justified by applying proposition 168 on* α, β' *and* β'' *since* α *is part of a even cycle of length* $2k$ *and* β'' *and* β'*' are consecutive arc of another even cycle of length* $2k$ *by proposition 175 (second line third case of figure 7.16).*

- *if* $2\ell = 4k+2$*:*



$(\{2\ell+1\}, 1, s)$ $\qquad$ $(\{2\ell+1\}, 1, -s)$ $\qquad$ $(\{2\ell+1\}, s)$

Once again the first two permutations are the one with invariant $(\lambda, r, \pm s)$ constructed by theorem 176 . The invariant of the third one are justified by applying proposition 168 on α, β' and β''.

No other permutation produced by theorem 176 are exceptional since by appendix C there are only two exceptional permutations starting with $\sigma(1) = 1$ and $\sigma(2) = 2$ those are id_n and id'_n. We just solved the case of id'_n and id_n is not I_2X.

We now consider the case (λ, r) contains even cycles. For that purpose, proposition 173 is not sufficient since it does not allow one to add pairs of even cycles of differing lengths. Recall that by theorem 137 (and more generaly the classification theorem) that even cycles must be in even number in a permutation, thus they can only be added in at least pairs.

The following proposition complements proposition 173 and makes it possible to add pair of even cycles of differing lengths.

Proposition 179 (Adding cycles 2)**.** *let σ be a permutation with cycle invariant (λ, r) and let $p > p'$, we call $\sigma_i(C_{2p} \bigcup C_{2p'})$ the permutation obtained by : (see also figure 7.19)*

1. *replacing the ith edge of σ by the cross permutation C_{2p}*
2. *replacing the first parallel edge of C_{2p} by the cross permutation $C_{2p'}$*
3. *and finally inserting a double-edge within α the leftmost top arc of $C_{2p'}$ and β the leftmost bottom arc of C_{2p}.*

The cycle invariant (λ', r) of $\sigma_i(C_{2p} \bigcup C_{2p'})$ verifies $\lambda' = \lambda \bigcup \{2(p+1), 2(p'+1)\}$:

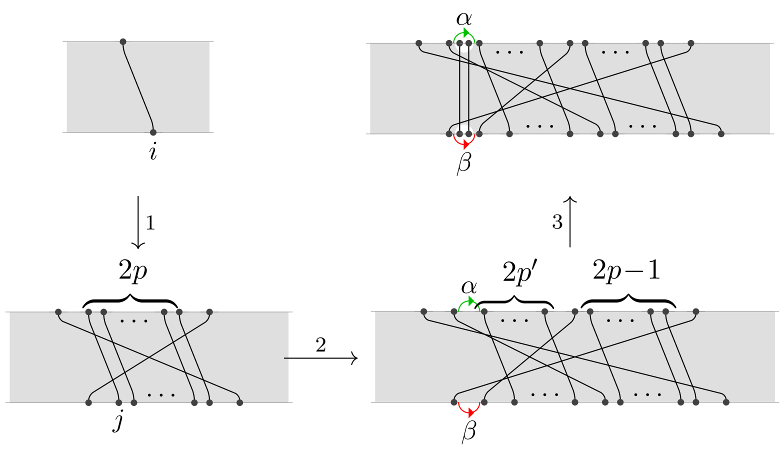


Figure 7.19: The scheme to add two different even cycles to a permutation.

Proof. The proof follows from the diagram below.

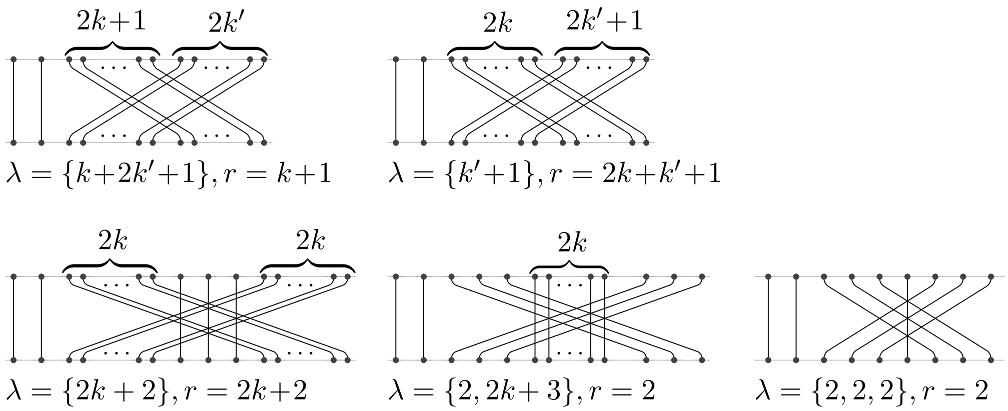


Figure 7.20: Two families of I_2X-permutations with their respective cycle invariant.

$$
\begin{tikzcd}
\sigma,\ (\lambda, r) \arrow[d, "1"] & \sigma_i(C_{2p} \bigcup C_{2p'}),\ (\lambda \bigcup \{2p+2, 2p'+2\}, r) \\
\sigma' = \sigma_i(C_{2p}),\ (\lambda \bigcup \{2p+1\}, r) \arrow[r, "2"] & s'' = \sigma'_j(C_{2p'}),\ (\lambda \bigcup \{2p+1, 2p'+1\}, r) \arrow[u, "3"']
\end{tikzcd}
$$

The modifications of the cycle invariants are justified in step 1 and 2 by the proposition 173 and in step 3 by the double-edge insertion proposition 157 since $\sigma_i(C_{2p} \bigcup C_{2p'}) = \sigma''|_{2,\alpha,\beta}$ and α and β are two arcs of two different cycles of length $2p+1$ and $2p'+1$ respectively. $\square$

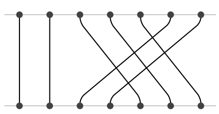
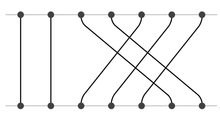
The second theorem handles the case σ has even cycles. Fortunately since the sign is 0 in this case there are no need two produce two permutations with opposite sign. However there are more base permutations to consider so the proof is not much shorter.

Proposition 180 (Base permutations 2). *For every $k, k' \geq 0$,*

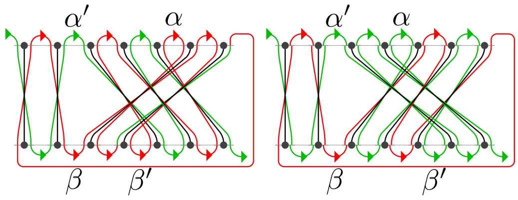
The permutations $X_{2k+1,2k'}$ and, $X_{2k,2k'+1}$ (described in the first line of figure 7.16 have cycle invariant $(\lambda = \{k+2k'+1\}, r = k+1)$ and $(\lambda = \{k'+1\}, r = 2k+k'+1)$ respectively.

The permutations $X_{2k,3,2k}$, $X_{3,2k,3}$ and $X_{2,1,3}$ (described in the second line of figure 7.16) have cycle invariant $(\lambda = \{2k+2\}, r = 2k+2)$, $(\lambda = \{2, 2k+3\}, r = 2)$ and $(\lambda = \{2, 2, 2\}, r = 2)$ respectively.

Proof. By induction on k, k'. In order to highlight the structure of the cycle invariant we start the base cases $k, k' \geq 1$. The base cases for the first line are respectively :

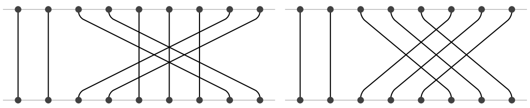
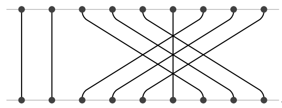
 and .

And they have cycle invariant (λ, r) : $(\{4\}, 2)$, $(\{2\}, 4)$ as shown just below :

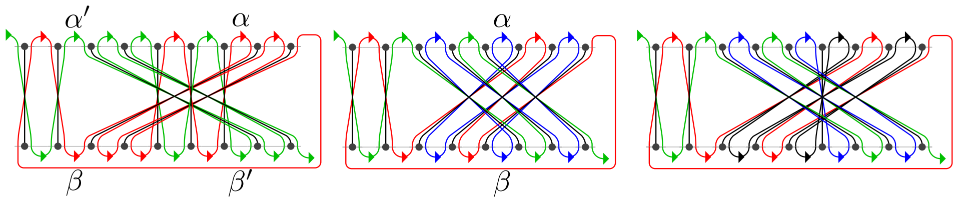


Then the statement follows by induction from the insertion a double-edge within α and β or within α' and β' (the resulting change of the cycle invariant are described in proposition 157)

The base cases for the second family are respectively (for the second we make k start at 0 since the strcuture of the cycle invariant is just as explicit here) :

 and .

And they have cycle invariant (λ, r) : $(\{4\}, 4)$, $(\{2, 3\}, 2)$, $(\{2, 2, 2\}, 2)$ as shown below :



Then the statement follows by induction from the insertion of a pair of double-edges within α and β and within α' and β' for the first permutation. For the second permutation the double-edge must be inserted within α and β (the resulting change of the cycle invariant are described in proposition 157). $\square$

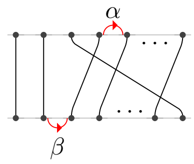
Theorem 181. *Let* $(\lambda, r, 0)$ *be an invariant with even cycle, then there exists a shift-irreducible family with invariant* $(\lambda, r, 0)$.

Proof. We can always consider that the size of the permutation is at least 10, for smaller size the result can be obtained by automatic search.

Let (λ, r, s) be an invariant with even cycles. Following the proof sketch of the introduction, we fist construct a base permutation σ^0 with invariant $(\lambda' \subseteq \lambda, r)$ then add cycles to obtain a I_2X-permutation σ_1 with invariant $(\lambda, r, 0)$.

- If the rank is odd and there is at least one odd cycle $2\ell + 1$, then the base permutation is either $X_{1,2\ell}$, $X_{2,1,2\ell}$ ot $X_{2,r-3}$ as in theorem 176.
- if the rank is odd and there are no odd cycles :
  - If the rank $r = 1$, then there are two base cases: If λ has 2 even cycles of the same length 2ℓ, the base permutation σ^0 with invariant $(\lambda' = \{2\ell, 2\ell\}, 1)$ for any $\ell \geq 1$ is $X_{1,4(\ell-1)+3}$ according to lemma 175 (second line, first case).

If λ does not have two even cycles of the same length, the base permutation σ^0 with invariant $(\lambda' = \{2\ell, 2\ell' + 2\ell\}, 1)$ $\ell', \ell \geq 1$ is obtained in two steps. First we take, as above, $\sigma'^0 = X_{1,4(\ell-1)+3}$, it has invariant $(\lambda' = \{2\ell, 2\ell\}, 1)$. Then we choose the two arcs α and β as below:

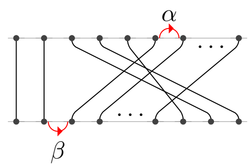
$X_{1,4(\ell-1)+3} =$ 

α and β are in the same cycle of length 2ℓ, thus by the double-edge insertion proposition 157 $\sigma'^0|_{2\ell',\alpha,\beta}$ the permutation resulting from the insertion of ℓ double-edges within α and β has invariant $(\lambda' = \{2\ell, 2\ell' + 2\ell\}, 1)$.

- If the rank $r = 3$, then there are two base permutations:

  If λ has 2 even cycle of the same length 2ℓ, the base permutation σ^0 with invariant $(\lambda' = \{2\ell, 2\ell\}, 3)$ for any $\ell \geq 1$ is $X_{2,1,4(\ell-1)+3}$ according to lemma 175 (second line, second case).

  If λ does not have two even cycles of the same length, the base permutation σ^0 with invariant $(\lambda' = \{2\ell, 2\ell' + 2\ell\}, 1)$ $\ell', \ell \geq 1$ is obtained in two steps. First we take, as above, $\sigma'^0 = X_{2,1,4(\ell-1)+3}$, it has invariant $(\lambda' = \{2\ell, 2\ell\}, 3)$. Then we choose the two arcs α and β as below:

  $X_{2,1,4(\ell-1)+3} =$ 

  α and β are in the same cycle of length 2ℓ, thus by the double-edge insertion proposition 157 $\sigma'^0|_{2\ell',\alpha,\beta}$ the permutation resulting from the insertion of ℓ double-edges within α and β has invariant $(\lambda' = \{2\ell, 2\ell' + 2\ell\}, 3)$.

- If the rank $r > 5$, then the base permutation σ^0 with invariant $(\lambda' = \varnothing, r)$ is exactly $X_{2,r-3}$ according to proposition 175 (first line, third case of figure 7.16 since r is odd implies $r - 3$ is even).

• If the rank r is even and the longest even cycle of λ has length $2\ell + r$, $\ell \geq 1$ then σ^0 is $X_{2r-1,2\ell}$ and has invariant $(\lambda = \{2\ell + r\}, r)$ by proposition 180 (first line, first case).

• If the rank r is even and the longest even cycle of λ has length 2ℓ, $r > 2\ell \geq 1$ then σ^0 is $X_{r-2\ell,4\ell-1}$ and has invariant $(\lambda = \{2\ell\}, r)$ by proposition 180 (first line, second case).

• If the rank r is even and the longest even cycle of λ has also length r :

  - If $r > 2$ then σ^0 is $X_{r-2,3,r-2}$ and has invariant $(\lambda = \{r\}, r)$ by proposition 180 (second line, first case).

– If $r = 2$ and there is a odd cycle of length $2\ell' + 1$, $\ell' \geq 1$ in λ then σ^0 is $X_{3,2\ell'-1,3}$ and has invariant $(\lambda = \{2, 2\ell' + 1\}, 2)$ by proposition 180 (second line, second case).
– If $r = 2$ and there are no odd cycle (thus every cycle has length two) in λ then σ^0 is $X_{2,1,3}$ and has invariant $(\lambda = \{2, 2, 2\}, 2)$ by proposition 180 (second line, third case).

□

7.7 The induction

Let us list a few lemma before beginning the induction.

Lemma 182. *Let σ be a permutation and let c be a $(2k, 2r)$-coloring such that either the edge $e_1 = (\sigma^{-1}(1), 1)$ or $e_2 = (n, \sigma(n))$ are grayed. Let τ be the corresponding reduction and let S be a sequence. Define $\tau' = S(\tau)$.*

Then $(\sigma', c') = B(S)(\sigma, c)$ has the following property: the gray edge e_1 is $(\sigma'^{-1}(1), 1)$ or the gray edge e_2 is $(n, \sigma'(n))$ in (σ', c').

Proof. Let us do the case for e_1, the case for e_2 is identical

Let (Π_b, Π_t) be a consistent labelling for τ then the gray edge e_1 of σ, c is inserted within the arcs with label r_0^{rk} and $b \in \Sigma_b$ (since $e_1 = (\sigma'^{-1}(1), 1)$. By theorem 151 the image of a consistent labelling is a consistent labelling thus $\Pi'_t = S(\Pi_t)$ verifies $\Pi_t'^{-1}(r_0^{rk}) = 1$ by definition. Since the labelling is compatible with the boosted dynamics (cf proposition 153) the top endpoint of the gray edge e_1 of (σ', c') is still within r_0^{rk} and thus inserted within the top arc with position 1 in (τ', Π'_b). □

In other words, the leftmost top endpoint of the edge $(\sigma^{-1}(1), 1)$ and the rightmost bottom endpoint of the edge $(n, \sigma(n))$ are fixed by the dynamics. It had already be proved many times but this proof is a good illustration of how we will employ the labelling and the boosted dynamics.

Lemma 183 ($d(\sigma)$ for σ of type X)**.** *Let σ be a standard permutation with invariant (λ, r, s) of type $X(r, i)$ and let $\tau = d(\sigma)$. Then τ has cycle invariant $(\lambda \setminus \{i\}, r+i-1)$ and type $H(i, r)$. Moreover for any consistent labelling (Π_b, Π_t) of τ we have $\sigma_1 = R(\sigma) = \tau|_{1,t_0^{rk},b_{i-1}^{rk}}$.*

Proof. Let σ and τ be as in the lemma. Then τ has indeed cycle invariant $(\lambda \setminus \{i\}, r + i - 1)$ and type $H(i, r)$, by proposition 159.

let (Π_b, Π_t) be a consistent labelling for τ and define $\beta = \tau^{-1}(n) - 1$ the bottom arc to the left of the edge $(\tau^{-1}(n), n)$. Then clearly $\sigma_1 = R(\sigma)$ is $\tau_{1,1,\beta}$, moreover $\Pi_b(\beta) = b_{i-1}^{rk}$ since τ has type $H(i, r)$ and therefore the top part of the rank (connecting the top left corner to the top right corner) has length i.

Thus we also have $\sigma_1 = \tau_{1,t_0^{rk},b_{i-1}^{rk}}$ for $\tau, (\Pi_b, \Pi_t)$. See figure 7.21 □

Lemma 184 ($d(\sigma)$ for σ of type H)**.** *Let σ be a standard permutation with invariant (λ, r, s) of type $H(i, r_2)$ and let $\tau = d(\sigma)$. Then τ has cycle invariant $(\lambda \bigcup \{i-1\}, r_2 - 1)$ and type $X(r_2 - 1, i - 1)$. Moreover there exists a consistent labelling (Π_b, Π_t) of τ such that we have $\sigma_1 = R(\sigma) = \tau|_{1,t_0^{rk},b_{i-2,i-1,1}}$.*

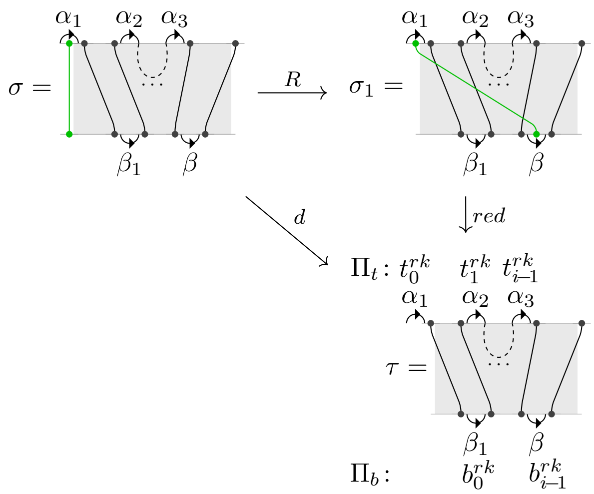


Figure 7.21: The case σ has type $X(r,i)$ of lemma 183. We have $\sigma_1 = \tau_{1,\alpha_1=1,\beta} = \tau_{1,t_0^{rk},b_{i-1}^{rk}}$

Proof. Let σ and τ be as in the lemma. Then τ has indeed cycle invariant $(\lambda \bigcup \{i-1\}, r_2 - 1)$ and type $H(i-1, r_2-1)$, by proposition 159.

Let (Π_b, Π_t) be a consistent labelling for τ such that the label of the first top arc of the principal cycle (i.e the arc with position $\alpha = \sigma(1)+1$) is $t_{1,i-1,1}$. Such a labelling exists since the principal cycle has length $i-1$.

Then consider $\beta' = \Pi^{-1}(b_{2,3,1})$ and $\beta = \Pi^{-1}(b_{3,3,1})$ the last two consecutives bottom arcs (in this order) of the principal cycle. Thus $\beta = \tau^{-1}(n-1)-1$ the bottom arc to the left of the edge $(\tau^{-1}(n-1), n-1)$ (τ has size $n-1$). See figure 7.22 bottom.

Clearly $\sigma_1 = R(\sigma)$ is $\tau_{1,1,\beta} = \tau_{1,t_0^{rk},b_{i-2,i-1,1}}$. See figure 7.22 top: right and middle.
□

Remark 185. *Note that if σ has invariant (λ, r, s) and type $H(r_1, r_2)$ then $r_1 + r_2 = r+1$.*

Let us proceed with the induction. The statements 1 to 7 are true at small size < 10.

Inductive case: By induction we suppose that statements 1 to 7 are true at size up to $n-1$, let us prove that they are true at size n:

7.7.1 Statement 1: Every non exceptional class has a shift-irreducible family

Let C be a non exceptional class with invariant (λ, r, s) and $\sigma \in C$ a standard permutation with $\sigma(2) = 2$. Let c be the $(n-2,2)$-coloring in which the edge $e = (n, \sigma(n))$ of σ is grayed and τ the corresponding reduction. Clearly τ is irreducible since $\tau(1) = 1$ (and $\tau(2) = 2$).

We distinguish two cases : τ is not in an exceptional class or it is.

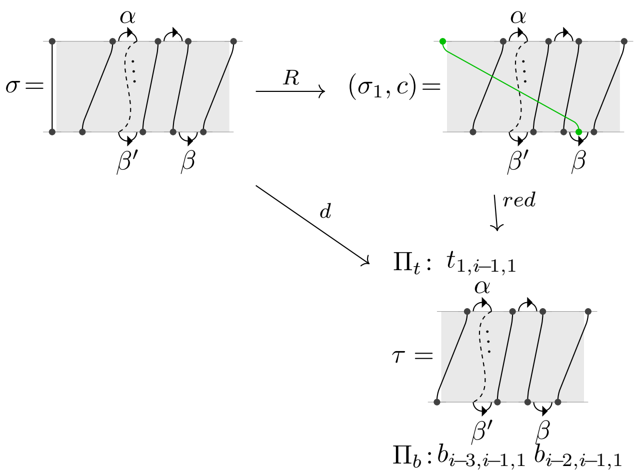


Figure 7.22: The case σ has type $H(i, r_2)$ of lemma 184. We have $\sigma_1 = \tau_{1,\alpha_1=1,\beta} = \tau_{1,t_0^{rk},b_{i-1,i-1,1}}$

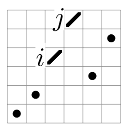
- **Suppose that τ is not in an exceptional class.** By induction (proposition 141 statement 6) there exists a I_2X permutation τ' in the class of τ. Thus there exists S such that $S(\tau) = \tau'$.

Let $B(S)$ be the boosted sequence of S, we have $B(S)(\sigma, c) = (\sigma', c')$ with c' having the following property: the gray edge e is $(n, \sigma'(n))$ by lemma 182.

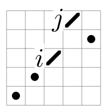
Thus the class C contains a permutation σ' such that removing the last edge $(n, \sigma'(n))$ gives a I_2X-permutation τ'. By the characterisation proposition for shift-irreducible permutations (proposition 163) a simple case study on the value of $\sigma'(n)$ shows that the standard family of σ' is shift-irreducible.

If τ is in an exceptional class then it is either in Id'_n or Id_n.

- **Suppose $\tau \in \mathrm{Id}'_{n-1}$.** By the characterisation theorem of Id' (appendix C) since $\tau(1) = 1$ and $\tau(2) = 2$, we must have $\tau = L^2(\mathrm{id}'_{n-1})$. But then, the standard family of σ is shift-irreducible by the characterisation proposition 163 since σ has the following form :

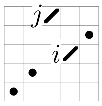
$\sigma =$  $i, j \geq 0$

- **Suppose $\tau \in \mathrm{Id}_n$.** By the characterisation theorem of Id (appendix C) since $\tau(1) = 1$ and $\tau(2) = 2$, we must have $\tau = \mathrm{id}_{n-1}$, thus σ has the form:

$\sigma =$  $i > 0, j > 0$

We must have $i > 0$ since otherwise $\sigma = L^2(\mathrm{id}'_n)$ and $j > 0$ since otherwise $\sigma = \mathrm{id}_n$, in both cases σ would be in an exceptional class which is false by hypothesis.

Now the permutation $\sigma' = R^{-1}L^iR^{-j}L^j(\sigma)$ has the form:

$\sigma' =$ 

$i, j > 0.$

and the standard family of σ' is shift-irreducible by the characterisation proposition 163.

7.7.2 Statement 2: Proof of theorem 137

$*$ Let C be a class with invariant (λ, r), we show that the list $\lambda \bigcup \{r\}$ has an even number of even parts.

Let σ be a standard permutation with $\sigma(2) = 2$, then $\tau = d(\sigma)$ is irreducible for it is standard and has invariant (λ', r').

By induction hypothesis the list $\lambda' \bigcup \{r'\}$ has an even number of even parts. Moreover, by inspection of table 4.2 it is easy to see that the parity of the number of even parts is preserved when adding an edge $(i, 1)$ to a permutation for every i. Thus since $\sigma = add_1(\tau)$ the list $\lambda \bigcup \{r\}$ must contain an even number of even parts

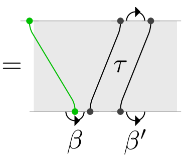
7.7.2.0.1 $*$ Let $\sigma \in C$ be a permutation with cycle invariant (λ, r) we must prove that :

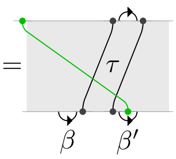
$$\overline{A}(\sigma) = \begin{cases} \pm 2^{\frac{n+\ell}{2}} & \text{If the number of even parts of the list } \lambda \bigcup \{r\} \text{ is } 0. \\ 0 & \text{Otherwise.} \end{cases}$$

Where ℓ is the number of parts in λ.

We distinguish two cases :

- **Suppose $\lambda = \varnothing$.** We must prove that $\overline{A}(C) = \pm 2^{\frac{n}{2}}$ since there are no even cycles and $\ell = 0$. **Proof idea:** We apply proposition 166 to two permutations σ and σ' of C and conclude by induction.

$\sigma = \tau|_{1,1,\beta} =$ 

$\sigma' = \tau|_{1,1,\beta'} =$ 

More formaly, the proof proceeds from the two following lemma :

Lemma. *Let C be as above, There exists τ with $\overline{A}(\tau) = \pm 2^{\frac{n}{2}}$ and two consecutive bottom arcs β, β' such that $\tau|_{1,1,\beta}$ and $\tau|_{1,1,\beta'}$ are in C.*

and

Lemma. *Let τ be a permutation and let β, β' be two consecutives bottom arcs such that $\tau|_{1,1,\beta}$ and $\tau|_{1,1,\beta'}$ are in the same class C then $\overline{A}(C) = \overline{A}(\tau)$.*

Clearly the two lemmas put together prove the statement.

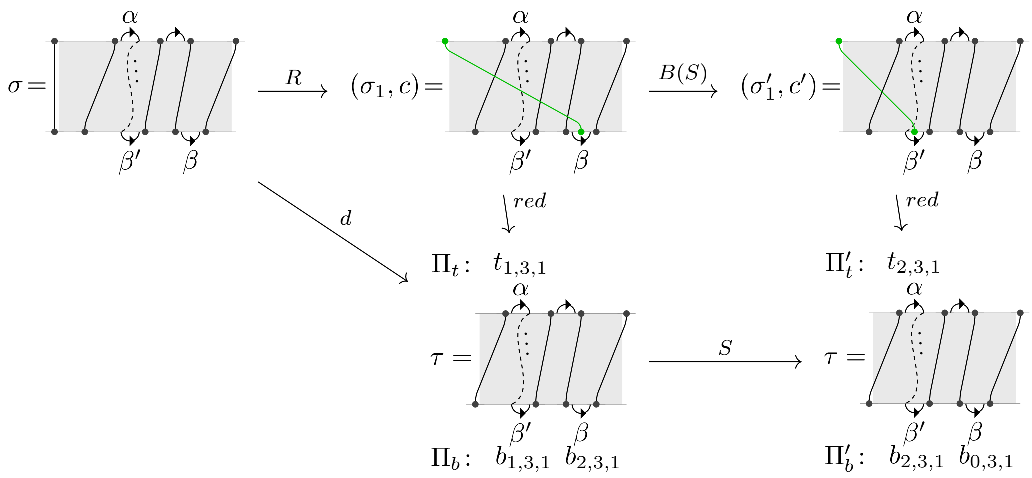


Figure 7.23: The construction of the proof of the lemma. We have $\sigma_1 = \tau_{1,1,\beta'}$, $\sigma_1' = \tau_{1,1,\beta}$ and $\sigma_1 = B(S)(\sigma_1)$ moreover $\overline{A}(\tau) = \pm 2^{\frac{n}{2}}$.

Proof of the first lemma. Let St be a shift-irreducible family of C and let $\sigma \in St$ the unique permutation with type $H(4, r-4+1)$ (it exists by proposition 158). let $\tau, (\Pi_b, \Pi_t)$ and $\sigma_1 = \tau|_{1,t_0^{rk},b_{2,3,1}}$ be as in lemma 184. Then τ has cycle invariant $(\{3\}, r-4)$, type $X(r-4,3)$ and is irreducible since the family is shift-irreducible (proposition 162). Moreover by induction it has $\overline{A}(d(\sigma)) = \pm 2^{\frac{n-1+1}{2}}$. Let us say wlog $\overline{A}(d(\sigma)) = 2^{\frac{n}{2}}$.

Consider $\beta' = \Pi^{-1}(b_{1,3,1})$ and $\beta = \Pi^{-1}(b_{2,3,1})$ the last two consecutives bottom arcs of the principal cycle. Thus $\beta = \tau^{-1}(n-1) - 1$ is the bottom arc to the left of edge $(\tau^{-1}(n-1), n-1)$. See figure 7.23 bottom: middle.

By induction we apply the corollary 155 of the monodromy theorem 154(which is true by induction hypothesis). Thus there exists a loop S on τ such that $\Pi_b' = S(\Pi_b)$ verifies $\Pi_b'(\beta') = b_{2,3,1}$ since the cycle is odd (has length 3).

Let c be $(n-2,2)$-coloring of σ_1 in which the edge $e = (\sigma_1^{-1}(1), 1)$ is grayed (by choice of c the reduction is τ). Thus $e \in (r_0^{rk}, b_{2,3,1})$ in $\tau, (\Pi_b, \Pi_t)$. See figure 7.23 top: right and middle.

By lemma 182 and since the labelling is compatible with the boosted dynamics (theorem 153) $(s_1', c') = B(S)(\sigma_1, c)$ verifies $\sigma_1' = \tau_{1,1,\beta'}$. Indeed the gray edge $e \in (r_0^{rk}, b_{2,3,1})$ in $S(\tau, (\Pi_b, \Pi_t)) = \tau(\Pi_b', \Pi_t')$ i.e. the edge e is inserted within the top arc with position 1 and bottom arc with position $\beta' = \Pi_b'^{-1}(b_{2,3,1})$. See figure 7.23 top and bottom right.

The permutations constructed σ_1', σ_1 and τ prove the lemma. $\square$

Proof of the second lemma. Let us apply proposition 166 to the permutations $\tau|_{1,1,\beta}$, $\tau|_{1,1,\beta'}$ and τ.

$$\overline{A}(\tau|_{1,1,\beta}) + \overline{A}(\tau|_{1,1,\beta'}) = 2\overline{A}(\tau)$$

We know that

$$\overline{A}(\tau|_{1,1,\beta}) = \overline{A}(\tau|_{1,1,\beta'})$$

since they are in the same class.

Thus

$$\overline{A}(C) = \overline{A}(\tau|_{1,1,\beta}) = \overline{A}(\tau)$$

$\square$

Let us now consider the second case.

- **Suppose** $\lambda \neq \varnothing$. **Proof idea:** Since $\lambda \neq \varnothing$ there must be a shift-irreducible standard permutation σ of C of type $X(r,i)$. Then the result follows from an application of proposition 172 and lemma 183.

Let σ be a standard shift-irreducible permutation of type $X(r,i)$, and let τ and $\sigma_1 = \tau|_{1,t_0^{rk},b_{i-1}^{rk}}$ be as in lemma 183. Then τ is irreducible since σ is shift-irreducible (proposition 162) with invariant $(\lambda \setminus \{i\}, r' = r+i-1)$ and type $H(i,r)$.

Let us choose $j = r' - i + 1 = r$ then $r' - j = i - 1$. According to lemma 170 we have

$$\overline{A}(\sigma_1 = \tau_{1,t_0^{rk},b_{r-j}^{rk}}) = \begin{cases} 0, \text{ if } j \equiv 0 \mod 2 \\ 2\overline{A}(\tau) \text{ Otherwise.} \end{cases} \tag{7.32}$$

Clearly if $j = r$ is even $\overline{A}(\sigma_1) = 0$. However since r is even and the list $\lambda \bigcup \{r\}$ contains even parts.

If $j = r$ is odd and λ does contain some even cycles then the list $\lambda \setminus \{i\} \bigcup \{r'\}$ also contains even parts and by induction $\overline{A}(\tau) = 0$, thus $\overline{A}(R(\sigma)) = 2\overline{A}(\tau) = 0$.

If $j = r$ is odd and λ does not contain even cycles then the list $\lambda \setminus \{i\} \bigcup \{r'\}$ does not contains even parts either, thus by induction $\overline{A}(\tau) = \pm 2^{\frac{n-1+\ell}{2}}$ (where ℓ is the number of parts of $\lambda \setminus \{i\}$) thus

$$\overline{A}(\sigma_1) = 2 \cdot \pm 2^{\frac{n-1+\ell}{2}} = \pm 2^{\frac{n+\ell+1}{2}} = \pm 2^{\frac{n+\ell'}{2}}$$

where $\ell' = \ell + 1$ is the number of parts of λ.

Note that we have proven the following lemma (that will be used in the section just below)

Lemma 186. *Let σ be a shift-irreducible permutation with invariant $(\lambda \neq \varnothing, r, s)$ of type $X(r,i)$, let $\tau = d(\sigma)$ and suppose the list $\lambda \bigcup \{r\}$ contains no even cycle. Then τ is irreducible and has invariant $(\lambda \setminus \{i\}, r' = r+i-1, s)$.*

7.7.3 Statement 3, Existence: For every valid invariant (λ, r, s) there exists a permutation with invariant (λ, r, s).

The statement is a direct consequence of theorems 176 and 176 where we constructed a I_2X-permutation for every valid (λ, r, s).

Note that we can indeed apply theorem 176 since we have just proved theorem 137 up to n (cf warning remark 177 and 169).

7.7.4 Statement 4, Proposition 139: First step of the labelling method

We already outlined the proof in the proof overview section.

Let σ_a and σ_b with invariant (λ, r, s). We shall find σ_1 and σ'_1 two standard shift-irreducible permutations of the same type (X or H) connected to σ and σ', such that $\tau = d(\sigma_1)$ and $\tau' = d(\sigma'_1)$ have the same invariant (λ', r', s').

Then the proposition 139 is proven by taking σ_1 and σ'_1 as in lemma 183 or lemma 184. Indeed if σ has type $X(r_i)$ by lemma 183 $\sigma_1 = R(\sigma) = \tau|_{1,t_0^{rk},b_{i-1}^{rk}}$. Let c be the $(n-2,2)$-coloring of σ_1 where the edge $e = (\sigma_1^{-1}(1), 1)$ is grayed. Then the reduction of (σ_1, c) is τ. The same reasoning apply to σ' which gives a (σ'_1, c') with reduction τ'.

The case $H(i, r_2)$ is similar with the use of lemma 184 instead.

Case 1: Suppose $\lambda \neq \varnothing$ and $\lambda \bigcup \{r\}$ has no even parts. Let $i \in \lambda$, let σ, σ' be two standard shift-irreducible permutation of type $X(r, i)$ in the class of σ_a and σ_b respectively (they exist by proposition 158).

By lemma 186 $\tau = d(\sigma)$ and $\tau' = d(\sigma')$ are irreducible and have the same invariant. $\square$

Case 2: Suppose $\lambda \neq \varnothing$ and λ has at least two even cycles (equivalently $\lambda \bigcup \{r\}$ has at least four even parts). Let $i \in \lambda$, let σ, σ' be two standard shift-irreducible permutations of type $X(r, i)$ in the class of σ_a and σ_b respectively (they exist by proposition 158).

By lemma 183 τ and τ have cycle invariant $(\lambda \setminus \{i\}, r + i - 1)$, thus the list $\lambda \setminus \{i\} \bigcup \{r + i - 1\}$ must still contains even cycles and by theorem 137 τ and τ have sign 0. Thus they have the same invariant. Moreover they are irreducible since σ and σ' are shift-irreducible (proposition 136). $\square$

Case 3: Suppose $r \geq 4$. Let σ, σ' be two standard shift-irreducible permutations of type $H(3, r-2)$ in the class of σ_a and σ_b respectively (they exist by proposition 158 since $r \geq 4$).

By lemma 184 τ and τ have cycle invariant $(\lambda \bigcup \{2\}, r - 3)$. Thus the list $\lambda \bigcup \{2\} \bigcup \{r - 3\}$ contains even cycles and by theorem 137 τ and τ have sign 0. Therefore they have the same invariant. Moreover they are irreducible since σ and σ' are shift-irreducible (proposition 136). $\square$

Case 4: Suppose $\lambda = 2k$ and $r = 2$. We will make use of proposition 172. We first prove the following lemma:

Lemma 187. *Let σ be a standard permutation with invariant $(\lambda, r, 0)$ of type $X(r, i)$. Let $e = (\sigma^{-1}(n), n)$ and $e' = (\sigma^{-1}(n)-1, \sigma(\sigma^{-1}(n)-1))$ and define $j = \sigma(\sigma^{-1}(n)-1)$ then $\sigma_{next} = L^{n-j+1}(\sigma)$ is of type $X(r, i)$.*

Moreover $\overline{A}(d(\sigma_{next})) = -\overline{A}(d(\sigma))$.

Proof. Clearly σ_{next} is of type $X(r, i)$ (see figure 7.24).

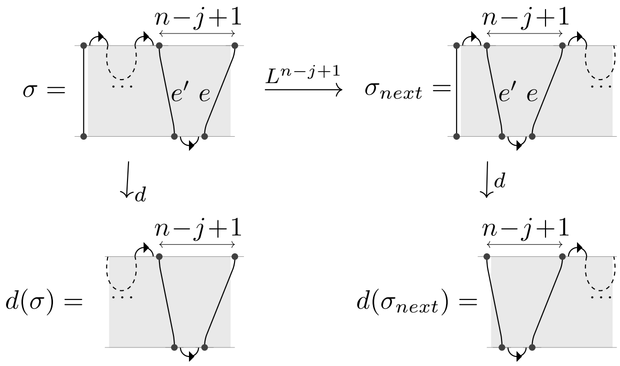


Figure 7.24: The construction of the proof of the lemma 187. Both σ and σ_{next} have type $X(r,i)$ and $\overline{A}(d(\sigma)) = -\overline{A}(d(\sigma_{next}))$.

For the second part of the statement we apply proposition 172 to $d(\sigma_{next})$, $d(\sigma)$ and σ. we have $\overline{A}(d(\sigma_{next})) + \overline{A}(d(\sigma)) = \overline{A}(\sigma)$. Thus $\overline{A}(d(\sigma_{next})) = -\overline{A}(d(\sigma))$ since $\overline{A}(\sigma) = 0$. □

Let σ, σ' be two standard shift-irreducible permutation of type $X(r, 2k)$ in the class of σ_a and σ_b respectively (they exist by proposition 158). Applying lemma 187 to σ and σ' we obtain respectively σ_{next} and σ'_{next} of type $X(r, 2k)$ such that $d(\sigma)$ and $d(\sigma_{next})$ as well as $d(\sigma')$ and $d(\sigma'_{next})$ have opposite sign.

Thus choosing $\tau = d(\sigma)$ or $\tau = d(\sigma_{next})$ and $\tau' = d(\sigma')$ or $\tau' = d(\sigma'_{next})$, τ and τ' have the same cycle invariant by lemma 183 and the same sign invariant. They are also irreducible since σ or σ_{next} and σ' or σ'_{next} are shift-irreducible (proposition 136). □

The four cases overlap somewhat, however they do cover all possibilities. Indeed 'no cycles' is covered by case 3, 'no even parts and some cycles' by case 1, 'at least four even parts' by case 2, 'exactly two even parts' by case 3 (for $r \geq 4$) and case 4 (for $r = 2$). Recall that there are always an even number of even parts in $\lambda \bigcup \{r\}$ by theorem 137 so every single possible invariant is handled.

We must now justify that the τ obtained are not in an exceptional class. First note that in case 1,2 and 4 the rank of τ is strictly more than one and in case 3 τ has a cycle of length 2 thus $\tau \notin \mathrm{Id}'_{n-1}$ since the cycle invariants are not compatible. Likewise in case 3 $\tau \notin \mathrm{Id}_{n-1}$ since the cycle invariants are not compatible.

It thus remains the case 1,2,4 with a rank < 4. Consider the following lemma (proven in the appendix C).

Lemma 188. *Let C be a non-exceptional class and let St be a standard family. Then at most one $\sigma \in St$ has $d(\sigma) \in \mathrm{Id}_{n-1}$.*

Since the rank is small, there must be either many cycles or a cycle of large length (since n is at least 10 in the induction). If we are unlucky and the permutation σ or σ' of type $X(r,i)$ has $d(\sigma) \in Id_{n-1}$ or $d(\sigma') \in Id_{n-1}$ we know that it is the only such one in the standard family and we can choose another of type $X(r,i)$ (since there are im_i permutation of type $X(r,i)$ by proposition 158).

7.7.5 Statement 5, Completeness: Every pair of permutations (σ, σ') with invariant (λ, r, s) are connected.

The completeness statement is demonstrated by the labelling method. (Refer to section 5)

- Let σ_a and σ_b be two irreducible permutation with invariant (λ, r, s). By proposition 139 there exists (σ_1, c) and (σ'_1, c') connected to σ_a and σ_b respectively with the following property:

  c and c' are the $(2n-2,2)$-coloring of σ_1 and σ'_1 where the edge $e = (\sigma_1^{-1}(1), 1)$ and $e' = (\sigma_1'^{-1}(1), 1)$ are grayed respectively and τ and τ' the reduction of (σ_1, c) and (σ'_1, c') are irreducible, have the same invariant (λ', r', s') and are not in exceptional classes.

  Moreover by the proof of proposition 139 $\sigma = R^{-1}(\sigma_1)$ and $\sigma' = R^{-1}(\sigma'_1)$ are standard and have the same type $X(r, i)$ or $H(i, r_2)$.

  If they have type $X(r, i)$ then by lemma 183 there exists a consistent labelling (Π_b, Π_t) of τ and (Π'_b, Π'_t) of τ' such that $\sigma_1 = R(\sigma) = \tau|_{1, t_0^{rk}, b_{i-1}^{rk}}$ and $\sigma'_1 = \tau'|_{1, t_0^{rk}, b_{i-1}^{rk}}$. Define $t = t_0^{rk}$ and $b = b_{i-1}^{rk}$, then $e \in (t, b)$ in $(\tau, (\Pi_b, \Pi_t))$ and $e' \in (t, b)$ in $(\tau', (\Pi'_b, \Pi'_t))$.

  If they have type $H(i, r_2)$ then by lemma 184 there exists a consistent labelling (Π_b, Π_t) of τ and (Π'_b, Π'_t) of τ' such that $\sigma_1 = \tau|_{1, t_0^{rk}, b_{i-2,i-1,1}}$ and $\sigma'_1 = \tau'|_{1, t_0^{rk}, b_{i-2,i-1,1}}$. Define $t = t_0^{rk}$ and $b = b_{i-2,i-1,1}$ then $e \in (t, b)$ in $(\tau, (\Pi_b, \Pi_t))$ and $e' \in (t, b)$ in $(\tau', (\Pi'_b, \Pi'_t))$. Moreover $\lambda' = \lambda \bigcup i-1$ (lemma 184 again).

- Since τ and τ' are irreducible and have the same invariant they are in the same class by the classification theorem (which is true by induction hypothesis). Therefore there exists S such that $\tau' = S(\tau)$.

- Since there exists a boosted dynamics and the labelling is compatible with the boosted dynamics (theorem 153) there exists $B(S)$ such that the reduction of $(\sigma_2, c_2) = B(S)(\sigma_1, c)$ is τ' and the gray edge e of (σ_2, c_2) is inserted within the arcs with labels t and b of $\tau', (\Pi''_b, \Pi''_t) = S(\tau, (\Pi_b, \Pi_t))$. Moreover by theorem 151 (Π''_b, Π''_t) is a consistent labelling.

- **First case:** $b = b_{i-1}^{rk}$. Both (Π''_b, Π''_t) and (Π'_b, Π'_t) are consistent labellings of τ' and the labels of the rank are fixed by definition, thus we have $\beta = \Pi_b''^{-1}(b_{i-1}^{rk}) = \Pi_b'^{-1}(b_{i-1}^{rk})$ (and $1 = \alpha = \Pi_t''^{-1}(t_0^{rk}) = \Pi_t'^{-1}(t_0^{rk})$). Therefore the gray edge e of (s_2, c_2) and e' of (s'_1, c'_1) are both inserted within the arcs with position $\alpha = 1$ and β. Thus $(s_2, c_2) = (s'_1, c'_1)$.

  Second case: $b = b_{i-2,i-1,1}$. Since (Π'_b, Π'_t) is consistent the arc with position $\beta = \Pi_b'^{-1}(b_{i-2,i-1,1})$ is an arc of a cycle of length $i-1$ (More precisely it is the last bottom arc of the principal cycle by lemma 184) . Since (Π''_b, Π''_t) is also consistent $\Pi''_b(\beta) = b_{c,i-1,d}$ for some $0 \leq c < i-1$ and $1 \leq d \leq m_{i-1}$ where m_{i-1} is the multiplicity of $i-1$ in λ'.

  If λ' has no even cycle or at least two even cycles, then by corollary 155 of theorem 154 (which is true by induction hypothesis) there exists a loop

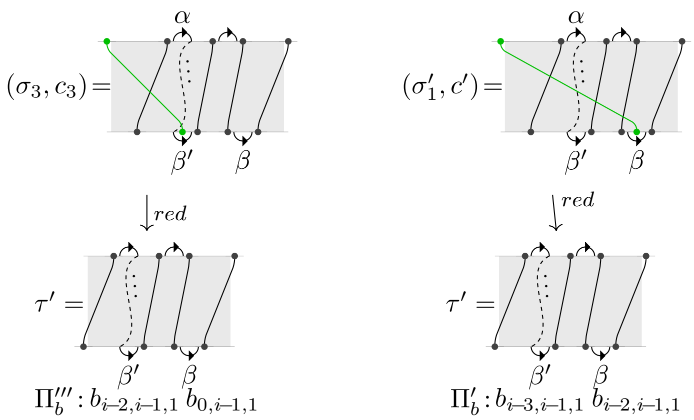


Figure 7.25: (σ_3, s'_1) are both permutation with invariant $(\lambda, r, s \neq 0)$ and τ' has sign invariant 0. However proposition 166 applied to σ_3, s'_1 and τ' implies that σ_3 should have invariant $-s$, This is contradictory and thus the case $\Pi'''_b(\beta) = b_{0,i-1,1}$ cannot happen.

S_1 of τ such that $\Pi'''_b = S_2(\Pi''_b)$ verifies $\Pi'''_b(\beta) = b_{i-2,i-1,1}$. Let $B(S_1)$ be the boosted sequence of S_1 then the gray edge e of $(s_3, c_3) = B(S_1)(s_2, c_2)$ is inserted within the arcs with position $\alpha = 1$ and β therefore $(s_3, c_3) = (s'_1, c'_1)$.

If λ' has exactly one even cycle: the cycle of length $i - 1$. In this case $d = 1$ and $\Pi''_b(\beta) = b_{c,i-1,1}$ then by corollary 155 of theorem 154 (which is true by induction hypothesis) there exists a loop S_1 of τ such that $\Pi'''_b = S_2(\Pi''_b)$ verifies $\Pi'''_b(\beta) = b_{i-2,i-1,1}$ if $i - 2 - c \equiv 0 \mod (2)$ and $\Pi'''_b(\beta) = b_{0,i-1,1}$ otherwise. In the first case we conclude as in the previous paragraph.

Let us show that $\Pi'''_b(\beta) = b_{0,i-1,1}$ cannot happen. Refer also to figure 7.25.

We constructed $\sigma'_1 = \tau'|_{1,t_0^{rk},b_{i-2,i-1,1}}$ and (Π'_b, Π'_t) by lemma 184 therefore, as said above, β is the last arc of the principal cycle of τ' (to the left of the edge $(\tau'^{-1}(n-1), n-1)$) and $\lambda = \lambda' \setminus \{i-1\}$, thus the list $\lambda \bigcup \{r\}$ has no even parts and $s \neq 0$.

If $\Pi'''_b(\beta) = b_{0,i-1,1}$, let $\beta' = \Pi'''^{-1}_b(b_{i-2,i-1,1})$ then β' and β are consecutive (in this order) since Π'''_b is a consistent labelling and $i-2$ and 0 are consecutive (modulo $i-1$) indices. Moreover $(s_3, c_3) = B(S_1)(s_2, c_2)$ verifies $\sigma_3 = \tau'|_{1,1,\beta'}$ since the gray edge e of (s_3, c_3) is inserted within the arcs with label t^{rk_0} and $b_{i-2,i-1,1}$ of $\tau', (\Pi'''_b, \Pi'''_t)$.

However, we have $\overline{A}(s_3) = \overline{A}(s_1) = \pm 2^{\frac{n+\ell}{2}}$ since both have invariant (λ, r, s) and $\overline{A}(s_1) = \pm 2^{\frac{n+\ell}{2}}$ by theorem 137 moreover $\overline{A}(\tau) = 0$ since λ' has an even cycle. Let us apply proposition 166 to s_3, s'_1 and τ'.

$$\overline{A}(s_3) + \overline{A}(s_1) = 2 \cdot \pm 2^{\frac{n+\ell}{2}} = 0$$

this is contradictory thus $\Pi'''_b(\beta) = b_{0,i-1,1}$ cannot happen.

The classification theorem (theorem 194) is a consequence of statements 3 and 5.

7.7.6 Statement 6, Proposition 141: Every non exceptional class contains a I_2X-permutation.

The statement is a direct consequence of theorems 176 and 176 where we constructed a I_2X permutation for every valid (λ, r, s) and of the classification theorem 194 which says that there is only one non-exceptional class per invariant.

7.7.7 Statement 7: the 2-point monodromy theorem 154

Before tackling the monodromy theorem, let us consider the following problem: we have two standard permutations σ and σ' of type $X(r,i)$ and a labelling (Π_b, Π_t) of σ. The labels of the principal cycle of σ are $t_{c,i,j}, \dots, t_{c+i-1 \mod i,i,j}$ in that order.

We wish to find a sequence S' such that $S'((\Pi_b, \Pi_t))$ is a labelling of σ' and the labels of the principal cycle of σ' are also $t_{c,i,j}, \dots, t_{c+i-1 \mod i,i,j}$ in that order.

The following proposition tells us that it is possible if $\tau = d(\sigma)$ and $\tau' = d(\sigma')$ are in the same class.

Proposition 189. *Let σ and σ' be two standard permutations with invariant (λ, r, s) and type $X(r,i)$. Let (Π_b, Π_t) be a consistent labelling of σ, and let $(\alpha_1, \dots, \alpha_i)$ and $(\alpha'_1, \dots, \alpha'_i)$ be the arcs (in that order) of the principal cycle of σ and σ' respectively. Then*

$$\Pi_t(\alpha_1) = t_{c,i,j}, \dots, \Pi_t(\alpha_i) = t_{c+i-1 \mod i,i,j} \text{ for some } c \text{ and } j .$$

Let $\tau = d(\sigma)$ and $\tau' = d(\sigma')$ and suppose there exists S such that $\tau' = S(\tau)$. Finally let $\sigma', (\Pi'_b, \Pi'_t) = R^{-1}B(S)R(\sigma, (\Pi_b, \Pi_t))$. We have

$$\Pi'_t(\alpha'_1) = t_{c,i,j}, \dots, \Pi'_t(\alpha'_i) = t_{c+i-1 \mod i,i,j}.$$

Proof. Let $\tau, (\Pi_b^\ell, \Pi_t^\ell)$, $\sigma_1 = R(\sigma) = \tau|_{1,t_0^{rk},b_{i-1}^{rk}}$ and $\tau', (\Pi_b^{\ell_1}, \Pi_t^{\ell_1})$, $\sigma_1 = R(\sigma') = \tau'|_{1,t_0^{rk},b_{i-1}^{rk}}$ be as in lemma 183.

Let c and c' be the $(n-2,2)$-coloring of σ_1 and σ'_1 such that the corresponding reductions are τ and τ' (i.e $e \in (t_0^{rk}, b_{i-1}^{rk})$ in $\tau, (\Pi_b^\ell, \Pi_t^\ell)$ likewise for e').

Let $B(S)$ be the boosted sequence of S then $B(S)(\sigma_1, c) = (\sigma'_1, c')$ since both e and e' are inserted within t_0^{rk}, b_{i-1}^{rk} in $\tau', (\Pi_b'^\ell, \Pi_t'^\ell) = S(\tau, (\Pi_b^\ell, \Pi_t^\ell))$ and $\tau', (\Pi_b^{\ell_1}, \Pi_t'^{\ell_1})$ respectively, and the labels of the rank are fixed. (This part of the proof follows that of statement 5).

For the following, refer to figure 7.26

In $\sigma, \Pi_t(\alpha_1) = t_{c,i,j}, \dots, \Pi_t(\alpha_i) = t_{c+i-1 \mod i,i,j}$ however in $\tau, \Pi_t^\ell(\alpha_1) = t_0^{rk}, \dots, \Pi_t^\ell(\alpha_i) = t_{i-1}^{rk}$. Thus the labels of the principal cycle of σ are attached to the ith first labels of the rank of τ and they will move along with them in the boosted sequence.

In $\tau', (\Pi_b'^\ell, \Pi_t'^\ell)$ we have $\Pi_t'^\ell(\alpha'_1) = t_0^{rk}, \dots, \Pi_t'^\ell(\alpha'_i) = t_{i-1}^{rk}$ since the labels of the rank are fixed, thus for $\sigma', (\Pi'_b, \Pi'_t) = R^{-1}B(S)R(\, s(\Pi_b, \Pi_t))$, we must also have $\Pi'_t(\alpha'_1) = t_{c,i,j}, \dots, \Pi'_t(\alpha'_i) = t_{c+i-1 \mod i,i,j}$.

□

We now prove the monodromy theorem 154. Let C be a class with invariant (λ, r, s), if we establish the theorem for a given permutation with a consistent labelling $\sigma, (\Pi_b, \Pi_t)$ then by conjugation the theorem is valid for any $\sigma' \in C$ and any consistent labelling of σ' . (refer in the labelling method section 5 to the discussion below definition 101 for more details)

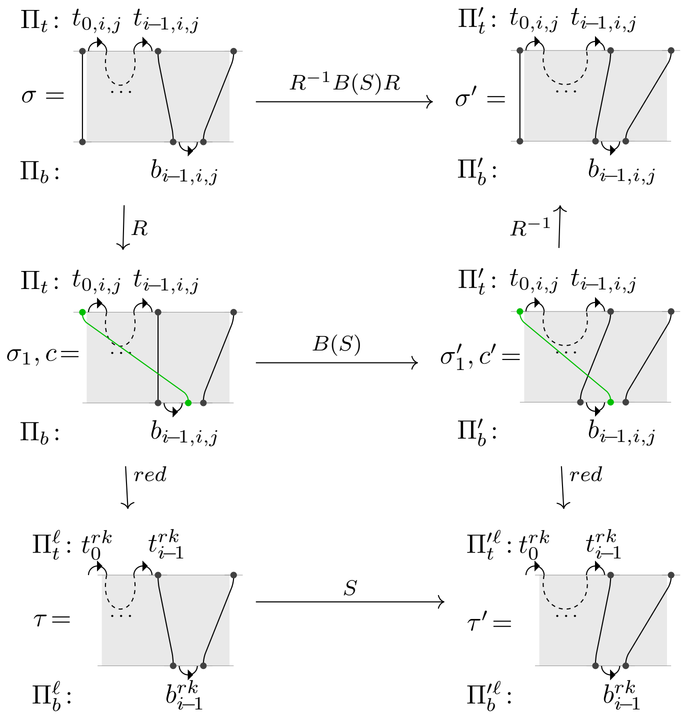


Figure 7.26: σ and σ' are standard of type $X(r,i)$. The labels of the principal cycles of σ are send to the arcs of the principal cycle of σ' by the sequence $R^{-1}B(S)R$. Indeed they are attached to the ith first labels of the rank of τ and the labels of the rank are fixed. Thus they are also attached to ith first the labels of the rank of τ' which are the arcs of the principal cycle of σ'.
In the figure, we choose $\Pi_t(\alpha) = t_{0,i,j}$ instead of $t_{c,i,j}$ for some c for space-saving purpose.

The proof of every case will follow the same pattern: we start with a standard permutation σ of type $X(r,i)$ with a consistent labelling (Π_b, Π_t), the top labels of the principal cycle are $t_1, \dots, t_i \in \Sigma_t$ in that order. The objective is to find a loop S such that the top labels of the principal cycle becomes $t'_1, \dots, t'_i$ in that order.

For that purpose, we construct a standard permutation σ' of type $X(r,i)$ with the following properties:

- $\sigma', (\Pi'_b, \Pi'_b) = S_1(\sigma, (\Pi_b, \Pi_t))$ for some S_1 and the labels of the principal cycle of σ' are $t'_1, \dots, t'_i$ in that order.
- $\tau' = d(\sigma)$ and $\tau = d(\sigma)$ are in the same class, thus there is a sequence S_2 such that $\tau = S_2(\tau')$.

Then we conclude by applying proposition 189 to σ' and σ, thus $S = R^{-1}B(S_2)RS_1$.

Note that σ' is always another permutation of the standard family of σ so $S_1 = L^k$ for some k.

- **Let us demonstrate by the statement on the cycle 1-shift. In this case λ has no even cycles or at least 2 even cycles** Let $i \in \lambda$, let σ a

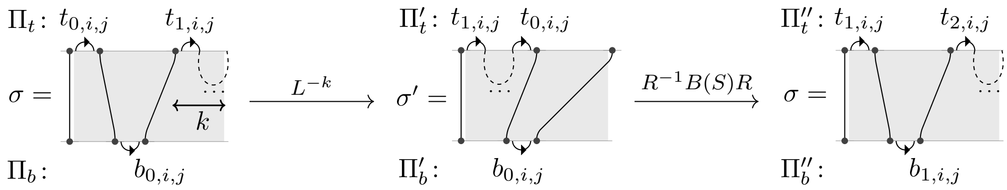


Figure 7.27: The case of the cycle 1-shift. We apply proposition 189 on σ' and σ.

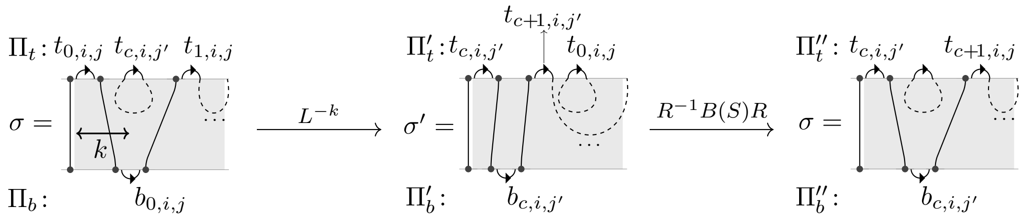


Figure 7.28: The case of the cycle jump. We apply proposition 189 on σ' and σ.

standard shift-irreducible permutation of type $X(r,i)$ and let (Π_b, Π_t) be a choosen consistent labelling of σ. Let $\alpha_1, \ldots, \alpha_i$ be the top arcs of the principal cycle, we have $\Pi_t(\alpha_1) = t_{0,i,j}$ and $\Pi_t(\alpha_2) = t_{1,i,j}, \ldots, \Pi_t(\alpha_i) = t_{i-1,i,j}$.

We construct a loop S of σ such that $(\Pi''_b, \Pi''_t) = S(\Pi_b, \Pi_t)$ verifies $\Pi''_t(\alpha_1) = t_{1,i,j}$ and $\Pi''_t(\alpha_2) = t_{2,i,j} \ldots \Pi''_t(\alpha_i) = t_{0,i,j}$.

Let $k = \alpha_2 - 2$ then $\sigma', (\Pi'_b, \Pi'_t) = L^{-k}(\sigma, (\Pi_b, \Pi_t))$ is a standard shift-irreducible permutation of type $X(r,i)$. Let $\alpha'_1, \ldots, \alpha'_i$ be the arc of the princicpal cycle, we have $\Pi'_t(\alpha'_1) = t_{1,i,j}, \Pi'_t(\alpha'_2) = t_{2,i,j}, \ldots, \Pi'_t(\alpha'_i) = t_{0,i,j}$. See figure 7.27 left and middle.

By the proof of proposition 139 case 1 and 2, $\tau = d(\sigma)$ and $\tau' = d(\sigma')$ are irreducible and have the same invariant, thus there is S such that $S(\tau') = \tau$ by the classification theorem. Therefore by proposition 189 let us define $\sigma, (\Pi'_b, \Pi'_t) = R^{-1}B(S)R(\sigma', (\Pi'_b, \Pi'_t))$, we have $\Pi''_t(\alpha_1) = t_{1,i,j}, \Pi''_t(\alpha_2) = t_{2,i,j}, \ldots, \Pi''_t(\alpha_i) = t_{0,i,j}$ as expected. See figure 7.27 right.

• **Let us demonstrate the statement on the cycle jump. In this case λ has at least two cycles of length i** Let $i \in \lambda$, let σ a standard shift-irreducible permutation of type $X(r,i)$ and let (Π_b, Π_t) be a choosen consistent labelling of σ. Let $\alpha_1, \ldots, \alpha_i$ be the top arcs of the principal cycle, we have $\Pi_t(\alpha_1) = t_{0,i,j}$ and $\Pi_t(\alpha_2) = t_{1,i,j} \ldots \alpha_i = t_{i-1,i,j}$. Let α' be an arc of another cycle of length i, $\Pi_t(\alpha') = t_{c,i,j'}$ for some c, j'.

We construct a loop S of σ such that $(\Pi''_b, \Pi''_t) = S(\Pi_b, \Pi_t)$ verifies $\Pi''_t(\alpha_1) = t_{c,i,j'}$ and $\Pi''_t(\alpha_2) = t_{c+1 \mod i,i,j}, \ldots, \Pi''_t(\alpha_i) = t_{c+i-1 \mod i,i,j}$

Let $k = \alpha' - 2$ then $\sigma', (\Pi'_b, \Pi'_t) = L^{-k}(\sigma, (\Pi_b, \Pi_t))$ is a standard shift-irreducible permutation of type $X(r,i)$. let $\alpha'_1, \ldots, \alpha'_i$ be the arc of the princicpal cycle, we have $\Pi'_t(\alpha'_1) = t_{c,i,j'}, \Pi'_t(\alpha'_2) = t_{c+1 \mod i,i,j'}, \ldots, \Pi'_t(\alpha'_i) = t_{c+i-1 \mod i,i,j'}$. See figure 7.28 left and middle.

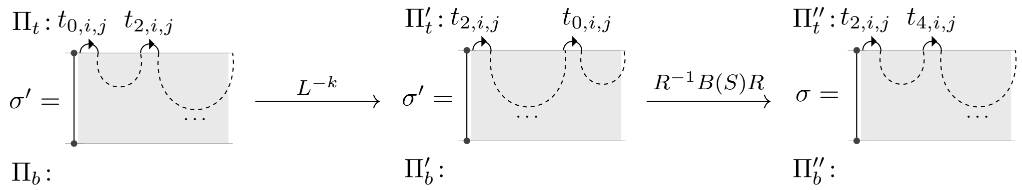


Figure 7.29: The case of the cycle 2-shift. We apply proposition 189 on σ' and σ.

By the proof of proposition 139 case 1 and 2, $\tau = d(\sigma)$ and $\tau' = d(\sigma')$ are irreducible and have the same invariant, thus there is S such that $S(\tau') = \tau$ by the classification theorem. Therefore, by proposition 189 let us define $\sigma, (\Pi'_b, \Pi'_t) = R^{-1}B(S)R(\sigma', (\Pi'_b, \Pi'_t))$, we have $\Pi''_t(\alpha_1) = t_{c,i,j'}, \Pi''_t(\alpha_2) = t_{c+1 \mod i,i,j'}, \dots, \Pi''_t(\alpha_i) = t_{c+i-1 modi,i,j'}$ as expected. See figure 7.28 right.

- **Let us demonstrate the statement on the cycle 2-shift. In this case λ has at exactly one cycle of even length, the cycle of length i.** Let σ a standard shift-irreducible permutation of type $X(r,i)$ and let (Π_b, Π_t) be a choosen consistent labelling of σ. Let $\alpha_1, \dots, \alpha_i$ be the top arcs of the principal cycle, we have $\Pi''_t(\alpha_1) = t_{0,i,j}$ and $\Pi''_t(\alpha_2) = t_{1,i,j}, \dots, \Pi''_t(\alpha_i) = t_{i-1,i,j}$.

We construct a loop S of σ such that $(\Pi''_b, \Pi''_t) = S(\Pi_b, \Pi_t)$ verifies $\Pi_t(\alpha_1) = t_{2,i,j}$ and $\Pi_t(\alpha_2) = t_{3,i,j} \dots \alpha_i = t_{1 \mod i,i,j}$.

Let $k = \alpha_3 - 2$ then $\sigma', (\Pi'_b, \Pi'_t) = L^{-k}(\sigma, (\Pi_b, \Pi_t))$ is a standard shift-irreducible permutation of type $X(r,i)$. Let $\alpha'_1, \dots, \alpha'_i$ be the arc of the princicpal cycle, we have $\Pi'_t(\alpha'_1) = t_{2,i,j}, \Pi'_t(\alpha'_2) = t_{3,i,j}, \dots, \Pi'_t(\alpha'_i) = t_{1,i,j}$. See figure 7.29 left and middle.

We must establish that $\tau = d(\sigma)$ and $\tau' = d(\sigma')$ are irreducible and have the same invariant. For the irreducibility and the cycle invariant, this follows from lemma 183 and the fact that σ and σ' are shift irreducible.

The sign invariant is slightly more complicated, we employ lemma 187 as we already did in case 4 of the proof of proposition 139. In a few words, define $\sigma'' = \sigma'_{next}$ as in the lemma, then we have $\sigma''_{next} = \sigma$ thus $\overline{A}(\sigma) = -\overline{A}(\sigma'') = \overline{A}(\sigma')$.

Thus there is S such that $S(\tau') = \tau$ by the classification theorem. Then by proposition 189 let $\sigma, (\Pi'_b, \Pi'_t) = R^{-1}B(S)R(\sigma', (\Pi'_b, \Pi'_t))$, we have $\Pi''_t(\alpha_1) = t_{3,i,j}, \Pi''_t(\alpha'_2) = t_{4,i,j}, \dots, \Pi''_t(\alpha_i) = t_{1,i,j}$ as expected. See figure 7.29 right. □

This complete the proof of theorem 154.

Part III

Future work

Chapter 8

The Rauzy dynamics $\mathcal{M}_n$

We introduced the Rauzy dynamics for matching $\mathcal{M}_n$ early on in section 3.1.1 (and chapter 1 for the palindrome counting motivation) but delayed its study until now. The classification of the Rauzy classes for matchings is established by induction with the labelling method described in section 5.

The proof is much easier than for the permutation case (section 7) mainly due to the fact that we only have to deal with one invariant: The cycle invariant $\lambda(m)$ (and irreducibility). There will still be some difficulties inherent to the matching case (in particular the lack of easily obtained standard family) but, overall, the complexity is much less than for the permutation case.

We will not carry out the classification, however the work has already been done and will be published in a subsequent article.

For ease of reference, we reproduce the definition of the dynamics $\mathcal{M}_n$ below,

$\mathcal{M}_n$: The space of configuration is $\mathfrak{M}_n^{\mathrm{irr}}$, irreducible n-arc matchings (a matching $m \in \mathfrak{M}_n$ is irreducible if there are no k such that $m(\{1, \ldots, k\}) = \{2n-k+1, \ldots, 2n\}$). There are two generators, L and R, with $a_L(m)$ being the index paired to 1, $a_R(m)$ the index paired to $2n$, $\alpha_{L,i} = \gamma_{L,i,2n}$, and $\alpha_{R,i} = \gamma_{R,i,2n}$ (a and α are as in Definition 12). See Figure 8.1.

As for the case of permutation, the reason for restricting to irreducible matching comes from the fact that the operators L and R only act on the extern-most block in case of a reducible matching (see figure 8.2).

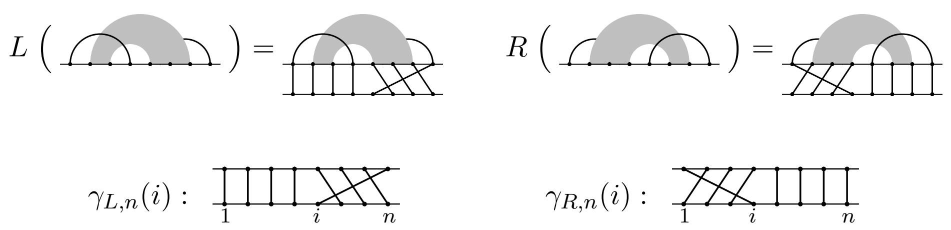


Figure 8.1: Top: the $\mathcal{M}_n$ case. Bottom: The permutations $\gamma_{L,i,2n}$ and $\gamma_{R,i,2n}$

8.1 Definition of the invariant

In this section, we present the only invariant of the Rauzy dynamics for matching: the cycle invariant $\lambda(m)$. The construction is quite similar to the permutation case and if σ is taken as a matching m then $\lambda(m)$ is in bijection with $\lambda'(\sigma)$ the cycle invariant for the extended dynamics. The bijective map is

$$\begin{aligned} f\colon \lambda(m) &\to \lambda'(\sigma) \\ \lambda_i &\mapsto \lambda_i/2. \end{aligned}$$

Let m be a matching, identified with its diagram. An arc of m is a pair (i, j), for $j = m(i)$, where $i < j$ thus i and j denote respectively the left and rigth end of the arc in the diagram. Perform the following manipulations on the diagram: (1) replace each arc with a pair of concentric arcs; more precisely, replace each arc endpoint, say i, by a black and a white endpoint, i_b and i_w (the black on the left), then introduce the arcs (i_b, j_w) and (i_w, j_b). (2) connect by an arc the points i_w and $(i+1)_b$, for $i = 1, \dots, 2n-1$, on the bottom the diagram; (3) connect by a long arc the rightmost and leftmost endpoints, $2n_w$ and 1_b. Call this arc the "-1 mark".

The resulting structure is composed of a number of closed cycles. If it is a cycle that goes through the -1 mark, we call it the *principal cycle*. Define the length of a closed path as the number bottom arcs (connecting a white endpoint to a black endpoint) in the path. These numbers are always positive integers (for $n > 1$ and irreducible matching). Note that, contrary to the permutation case, these lengths are not even in general. Call $\lambda(m) = \{\lambda_i\}$, the collection of lengths of the cycles, the *cycle invariant* of m. Define $\ell(m)$ as the number of cycles in m. See Figure 8.3, for an example.

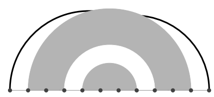
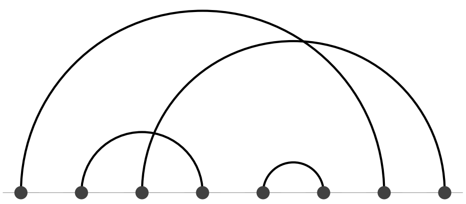


Figure 8.2: A reducible matching. It is clear that the operators L and R will leave the inner block unchanged.



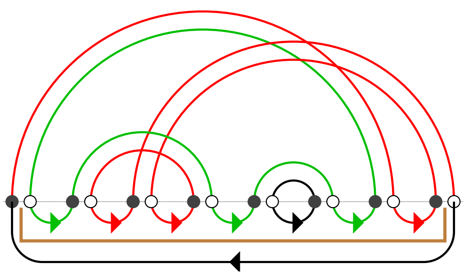


Figure 8.3: Left: an irreducible matching, $m = [\,(17)(24)(38)(56)\,]$. Right: the construction of the cycle invariant. Different cycles are in different colour. The length of a cycle or path, defined as the number of bottom arcs, is thus 3 for red and green, and 1 for black. As a result, in this example $\lambda(m) = (3, 3, 1)$. Counting the path-length of a cycle (i.e. the number of matching arcs) gives the same result except for the principal cycle where one has to soustract 1 for the -1 mark arc.

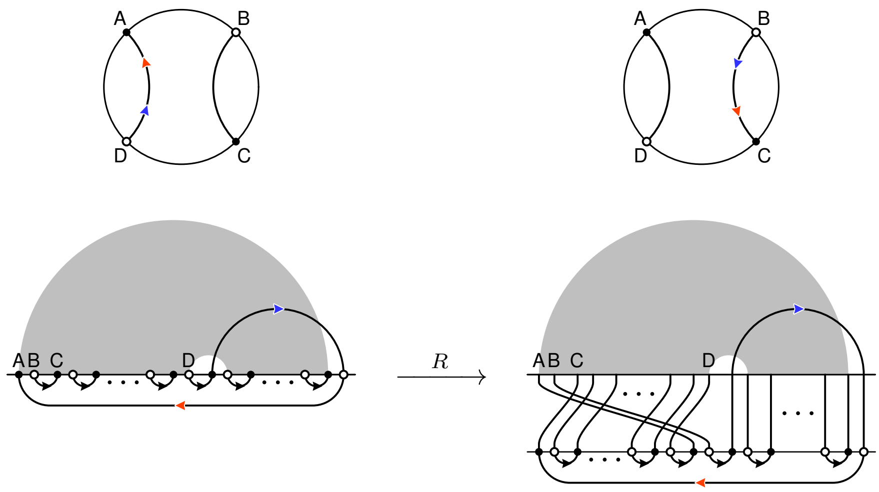


Figure 8.4: Invariance of $\lambda(m)$ in the $\mathcal{M}_n$ dynamics illustrated for the operator R. The case of operator L is deduced by symmetry.

Note that this quantity does coincide with the ordinary path-length of the corresponding paths (as opposed to the permutation case): The path-length of a cycle of length k is k, unless it goes through the -1 mark, in which case it is $k+1$. This justifies the name of "-1 mark" for the corresponding arc in the construction of the cycle invariant.

It is easily seen that

$$\sum_i \lambda_i = 2n - 1 \tag{8.1}$$

And we call, once again, this formula the *dimension formula.*

We have

Proposition 190. *The quantity λ is invariant in the $\mathcal{M}$ dynamics.*

Proof. The proof is strictly similar to the proof at the beginning of section 4.1.1 for the case of permutation. Again the idea is that the operators of the dynamics perform local modifications on portions of paths, without changing their lengths. For all cycles except for at most two of them this is completely evident: each arc of the cycles is deformed by (say) the permutation $\gamma_{L,2n}(i)$ in a way which can be retracted without changing the topology of the connections.

For the paths passing from fours special endpoints (marked as A, B, C and D in the figure), the invariance of the cycle structure holds through a more subtle mechanism, involving the exchange of two arcs, and, for what concerns path-lengths, by a crucial use of the -1 mark. The way in which the topology of connections among these four points is modified by the permutation $\gamma_{L,2n}(i)$ is hardly explained in words, but is evident from Figure 8.4. This proves that the list λ is invariant.

8.2 Cycle of length two and concentric arcs

As we noted at the beginning of section 8.1, cycles of length 1 in a permutation correspond to cycles of length 2 in the associated matching. Thus, for consistency

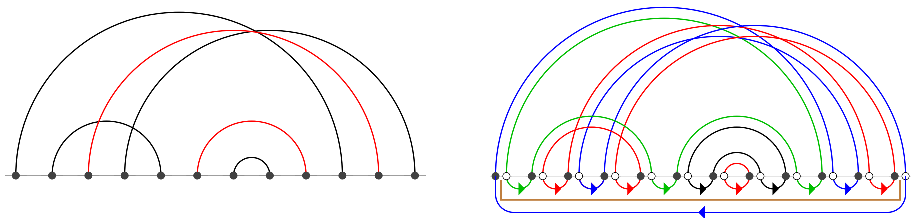


Figure 8.5: Left: a matching m with a concentric arc and the special concentric arc. Right: In this example $\lambda(m) = (3, 3, 2, 2, 1)$. $\text{prim}(m)$ is represented in figure 8.3.

with the permutation case (cf section 3.1.3.3 and appendix B), cycle of length 2 in matchings should also have an especially simple behaviour.

Definition 191. *For a matching m in the $\mathcal{M}_n$ dynamics, we say that the arc $(i, m(i))$ is a concentric arc is $(i+1, m(i)-1)$ is also an arc of the matching and it is a special concentric arc is $m(1) = i-1$ and $m(2n) = m(i)+1$.*

Concentric arcs (special or not) are associated to cycles of length two. As for the permutation case, call a matching primitif if it has no concentric arcs (or cycles of length 2), and define $\text{prim}(m)$, the *primitive of* m, as the configuration obtained by removing the concentric (special or not) arcs. See figure 8.5.

Then we have

Proposition 192. *$m \sim m'$ iff $\text{prim}(m) \sim \text{prim}(m')$ and $|m| = |m'|$.*

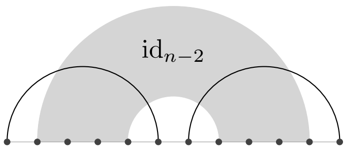
The proof of this proposition is a simple adaptation of the proof in appendix B to the case of matchings. Thus we will not prove it.

8.3 The exceptional class

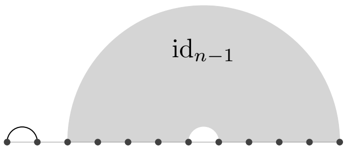
As mentioned in section 3.1.4, our classifications theorems have sometimes exceptions: the set of invariant characterize all classes except for a few that we call *exceptional classes*. The $\mathcal{M}_n$ dynamics has only one exceptional class, called Id_n^{match} due to its structural similarity with the class Id_n of the $\mathcal{S}_n$ dynamics.

There are three interesting class representatives of Id_n^{match}, id_n^{match}, $\text{id}_n^{match,1}$ and $\text{id}_n^{match,2}$. Those are defined as:

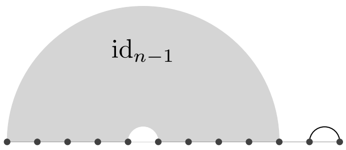
$$\mathrm{id}_n^{match} = ((1,n),(2,n{+}2),(3,(n{+}3),\ldots,(n{-}1,n{+}(n{-}1),(n{+}1,2n))$$

$\mathrm{id}_n^{match} =$ 

$$\mathrm{id}_n^{match,1} = ((1,2),(3,3{+}n{-}1),(4,4+(n{+}{-}1),\ldots,(n{+}1,n{+}1{+}(n{-}1))$$

$\mathrm{id}_n^{match,1} =$ 

$$\mathrm{id}_n^{match,2} = ((1,n),(2,n{+}1),(3,(n{+}2),\ldots,(n{-}1,n{+}(n{-}2),(2n{-}1,2n))$$

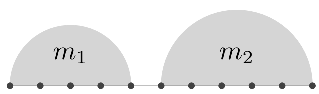
$\mathrm{id}_n^{match,2} =$ 

A convenient notation for writing matching such as $\mathrm{id}^{match,1}$ and $\mathrm{id}^{match,2}$ is the *concatenation*:

Notation 7. *We say that a matching m of size n is the concatenation of two matchings m_1 of size k and m_2 of size $n-k$ and we note $m = m_1|m_2$ if*

$$\forall i \leq 2k,\ m(i) = m_1(i) \text{ and } \forall i > 2k,\ m(i) = m_2(i-2k)+2k.$$

In the diagram representation, m has the form:

$m =$ 

We emphasize here the points of interest for the proof of classification theorem:

Lemma 193. *There are exactly two matchings of Id_n^{match} of the form $m = \mathrm{id}_k|m'$ where m' is a matching. Those matchings are $\mathrm{id}_n^{match,1} = \mathrm{id}_1|id_{n-1}$ and $\mathrm{id}_n^{match,2} = \mathrm{id}_{n-1}|\mathrm{id}_1$*

The cycle invariant of Id_n^{match} depends on the size mod 2 and is found in the table 8.1

	n even	n odd
(λ) of Id_n^{match}	$\{n-1,n-1,1\}$	$\{2(n-1),1\}$

Table 8.1: Cycle invariant of the exceptional class Id_n^{match}.

8.4 The classification theorem

Theorem 194. *Two matching m and m' are in the same class iff they have the same number of concentric arcs, and* $\mathrm{prim}(m)$, $\mathrm{prim}(m')$, *are in the same class.*

A matching m is primitive iff $\lambda(m)$ has no parts of size 2. Besides the exceptional class Id^{match}, *which has cycle invariants described in Table 8.1, the number of primitive classes with cycle invariant λ (no $\lambda_i = 2$) depends on the number ℓ of elements in the list λ, and is, for $n \geq 9$,*

one, *if* $\ell \equiv n+1 \mod 2$ *and* $n - \ell \geq -1$.

zero, *otherwise.*

$p = (\{1,3,6\},\{2,9\},\{4,5,7,8\}) \in \mathcal{P}_9$

Set partition diagram representation

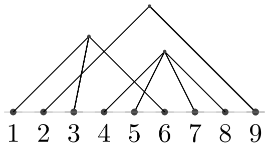


Figure 9.1: A set partition diagram representations of a set partition.

Chapter 9

More Rauzy-type dynamics

In this section, we present dynamics for which the classification proof can be achieve with labelling method. As said before, the invariant set that we identify for those dynamics are not necessarily complete. This absence of completeness is the main obstacle to a proof of the classification theorem for those dynamics.

As mentionned in the beginning of part II, starting from this section, we do not carry out the proof of the classification of the dynamics we consider. We merely provide the invariants of the dynamics that we found, and in case we gathered enough numerical data to make a good guess, we indicate whether the set of invariants should be complete.

9.1 The involution dynamics for set partition $\mathfrak{I}_n^{\mathcal{P}}$

We introduce in this section a generalisation of the involution dynamics to set partitions. Let $\mathcal{P}_n$ denote the set of set partitions of $[n]$.

We represent a set partition $p = (S_1, \ldots, S_k)$ over $[n]$ as k hyperarcs $(h_i)_i$ in the upper half plane connecting n points on the real line in the following way: the hyperarc h_i connects together all the points of the subset S_i (see figure 9.1). We call this representation a *set partition diagram representation.*

As for the involution dynamics on matching, we study two main dynamics $\mathfrak{I}_n^{\mathcal{P}}$ and $\mathfrak{I}_n^{\mathcal{P},ex}$.

$\mathfrak{I}_n^{\mathcal{P}}$ **:** The space of configuration is $\mathcal{P}_n$, the set of set partition of $[n]$. For every k-hyperarc $h = \{i_1, \ldots, i_k\}$ of a set partition p, there are $2k-3$ generators, $(A^I_{i_j,i_{j+1}})_{0<j<k}$ and $(A^I_{i_j,i_{j+2}})_{0<j<k-1}$, with $\alpha_{a_{A^I_{i,j}(p)}} = \gamma_{A^I,n}(q,r)$ where $q,r = i_j, i_{j+1}$ for some $0<j<k$ or $q,r = i_j, i_{j+2}$ for some $0<j<k-1$. See Figure 9.2, top and middle.

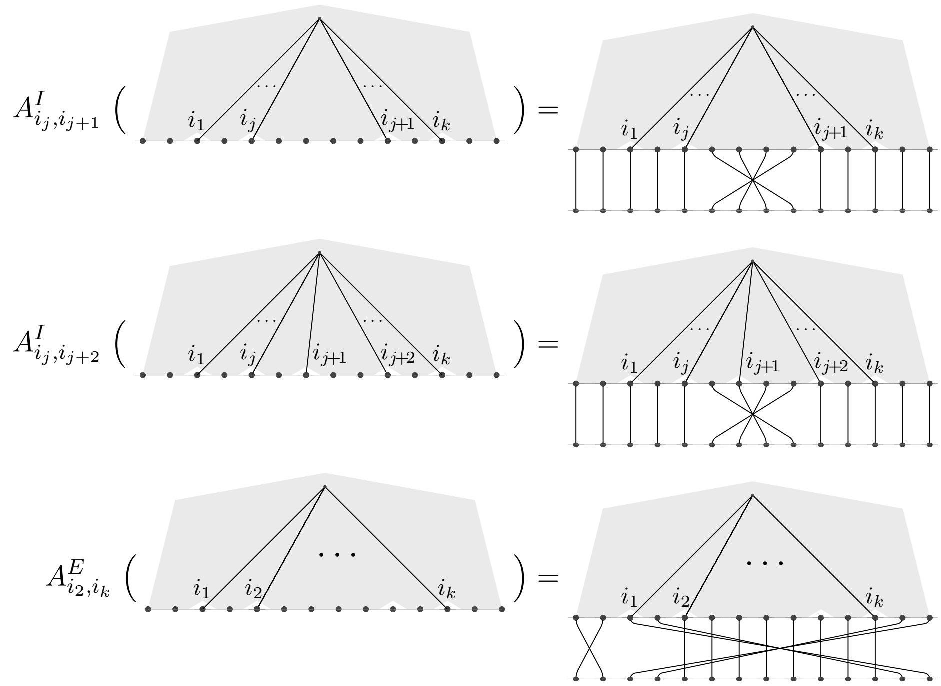


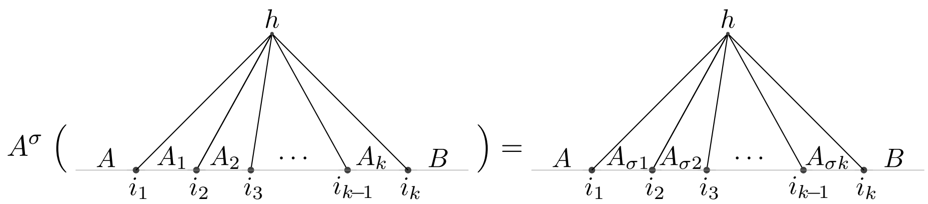
Figure 9.2: Top and middle: the $\mathfrak{I}_n^{\mathcal{P}}$ case in diagram representation. Bottom: in the $\mathfrak{I}_n^{\mathcal{P},ex}$ case we have the same operators as in the $\mathfrak{I}_n^{\mathcal{P}}$ case plus the the three operators (A^E_*) for each hyperarc. We chose to represent only the case $A^E_{i_2,i_k}$.

$\mathfrak{I}_n^{\mathcal{P},ex}$: The space of configuration is $\mathcal{P}_n$, the set of set partition of $[n]$. For every k-hyperarc $h = \{i_1, \ldots, i_k\}$ of a set partition, we now have $2k$ generators, the $(A^I_{i_j,i_{j+1}})_{0<j<k}$ and $(A^I_{i_j,i_{j+2}})_{0<j<k-1}$ as above, and the three operators $A^E_{i_1,i_k}, A^E_{i_2,i_k}, A^E_{i_1j,i_{k-1}}$, with $\alpha_{a_{A^E_{q,r}(p)}} = \gamma_{A^E,n}(q,r)$ where $q,r = i_1, i_k$ or $q,r = i_2, i_k$ or $q,r = i_1, i_{k-1}$. See Figure 9.2, bottom.

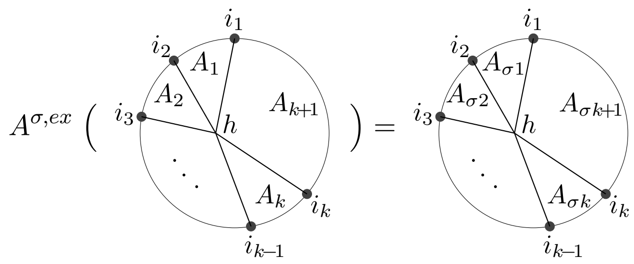
In the definition, we chose a set of operators that was minimal. The combination of those operators on a given hyperarc yields a more practical set of operators:

Proposition 195. *Let $p \in \mathcal{P}_n$ and let $h = \{i_1, \ldots, i_k\}$, $i_j < i_{j+1}$ for all $0 < j < k$ be a hyperarc. $\forall\ 0 < j < k$, let A_j be the endpoints of p in the interior of the arcs i_j, i_{j+1} of the hyperarc h. Finally, let A and B be the endpoints of p to the left and right of the hyperarc h respectively.*

Then in the $\mathfrak{I}^{\mathcal{P}}$ dynamics, for any $\sigma \in \mathfrak{S}_k$ there exists an operators A^σ such that:



Likewise in the $\mathfrak{I}^{\mathcal{P},ex}$ dynamics, for any $\sigma \in \mathfrak{S}_k$ we have:



Proof. It is clear that the operator $A^I_{i_j,i_{j+2}}$ exchanges A_{i_j} with $A_{i_{j+1}}$ and reverses them. Thus the sequence $A^I_{i_j,i_{j+1}}A^I_{i_{j+1},i_{j+2}}A^I_{i_j,i_{j+2}}$ exchanges A_{i_j} with $A_{i_{j+1}}$. From there, since transpositions generate the symmetric group, there exists a sequence of operators such that the ordered list $(A_i)_i$ becomes the ordered list $(A_\sigma(i))_i$. We call such sequence A^σ.

The proof for the $\mathfrak{I}^{\mathcal{P},ex}$ dynamics is similar. □

9.1.1 Definition of the invariant

As can be expected, the invariant of the involution dynamics for partitions is also a multigraph $\mathfrak{G}^{\mathcal{P}}(p)$.

Let p be a partition, identified with its diagram and let $(h_i)_{1\leq i\leq n}$ be the hyperarcs of p. We say that two hyperarcs $h_i = (i_1, \dots, i_k)$ and $h_j = (j_1, \dots, j_{k'})$ are adjacent if there exists p, r, such that$|i_q - j_r| = 1$, and we define the adjacence number of h_i, h_j to be the number of time the two hyperarcs are adjacent.

Construct the multigraph $\mathfrak{G}(m)$ in the following way: The vertex set $V(\mathfrak{G}(m))$ is $(h_i)_i$. There are two labelled vertices, the left vertex $h_1 = (1, i_2, \dots, i_k)$ and the right vertex $a_\ell = (j_1, \dots, j_{k'-1}, 2n)$. Two vertices h_i, h_j are connected by a multi-edge if and only if they are adjacent and the degree of the multi-edge is the adjacence number of h_i, h_j. Loops can arise on vertex $h_i = (m_1, \dots, m_{k''}$, if there exists j such that $m_{j+1} = m_j + 1$.

Thus a vertex associated to a k-hyperarc has degree $2k$ (loops count for 2) since every endpoint of an hyperarc is adjacent to two (possibly equal) hyperarcs, except the left and right vertices that have degree $2k - 1$ and $2k' - 1$ respectively since one of their endpoints is on the left and right extermities of the partition respectively. Confer to figure 9.3 top for an example of the construction.

We have

Proposition 196. *The multigraph $\mathfrak{G}^{\mathcal{P}}(\cdot)$ with its two labelled vertices is invariant in the $\mathfrak{I}^{\mathcal{P}}$ dynamics.*

The proof follows that of proposition 103 and is omitted.

As for the matching case, connect by an edge the left and right vertices of $\mathfrak{G}^{\mathcal{P}}(p)$ and unlabel them. Call $\mathfrak{G}^{\mathcal{P},ex}(p)$ the resulting multigraph (see figure 9.3 bottom).

Then we have

Proposition 197. *The multigraph $\mathfrak{G}^{\mathcal{P},ex}(\cdot)$ is invariant in the $\mathfrak{I}^{\mathcal{P},ex}$ dynamics.*

This proof also follows the lines of proposition 104 and is thus omitted.

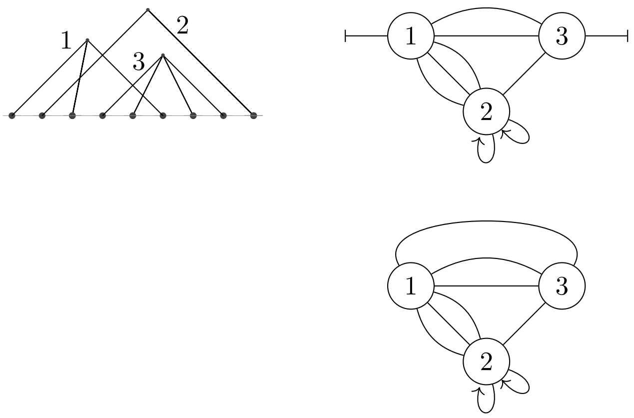


Figure 9.3: Top left: the partition $p = \big(\{1,3,6\},\{2,9\},\{4,5,7,8\}\big)$ in its diagram representation with the hyperarcs labelled from 1 to n. Top right: the construction of the graph invariant $\mathfrak{G}^{\mathcal{P}}(p)$.
Bottom right: The construction of the graph invariant $\mathfrak{G}^{\mathcal{P},ex}(p)$.

We do no know whether the multigraph invariant is a complete invariant set for either the $\mathfrak{I}^{\mathcal{P}}$ dynamics or the $\mathfrak{I}^{\mathcal{P},ex}$ dynamics

9.2 A Rauzy-type dynamics on chord diagram $\mathcal{D}_n$

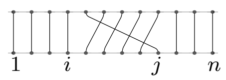
We define in this section $\mathcal{D}_n$, a Rauzy-type dynamics on matchings over $[2n]$ somewhat similar to the Rauzy dynamics in that it preserves the cycle invariant (in its chord diagram definition).

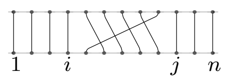
Let us define a special set of permutations (in cycle notation)

$$\gamma_{A,n}(i,j) = (1)(2)\dots(i-2)(j\ j-1\ \cdots\ i-1)(j+1)\dots(n)\,; \tag{9.1a}$$

$$\gamma_{B,n}(i,j) = (1)(2)\cdots(i)(i+1\ i+2\ \cdots\ j-1)(j)\dots(n)\,; \tag{9.1b}$$

I.e in a picture:

$\gamma_{A,n}(i,j)$: 

$\gamma_{B,n}(i,j)$: 

The definition of the dynamics $\mathcal{D}_n$ is a bit longer than usual:

$\mathcal{D}_n$ **:** The space of configuration is $\mathfrak{M}_n$, the n-arc matchings. There are n generators $(M^k)_k$, one per arc. Let m be a matching and $a_k = (i,j)$ $(i<j)$ be an arc. Let $a^1 = (i+1, m(i+1))$ be the arc to the right of the left endpoint of a_i. Define:

$$M_1^k = \begin{cases} \gamma_{A,2n}\,(i+1, m(i+1)) & \text{if } m(i+1) > i+1 \\ \gamma_{B,2n}\,(m(i+1), i+1) & \text{if } m(i+1) < i+1. \end{cases}$$

Let $m' = M_1^k(m)$ then in m' the arc a_k is now $a_k = (i' = M_1^k(i), j' = M_1^k(j))$. Define $j'_{mod} = j' \,(\mathrm{mod}\ n) + 1$ and let $a^2 = (j'_{mod}, m(j'_{mod}))$ be the arc to the

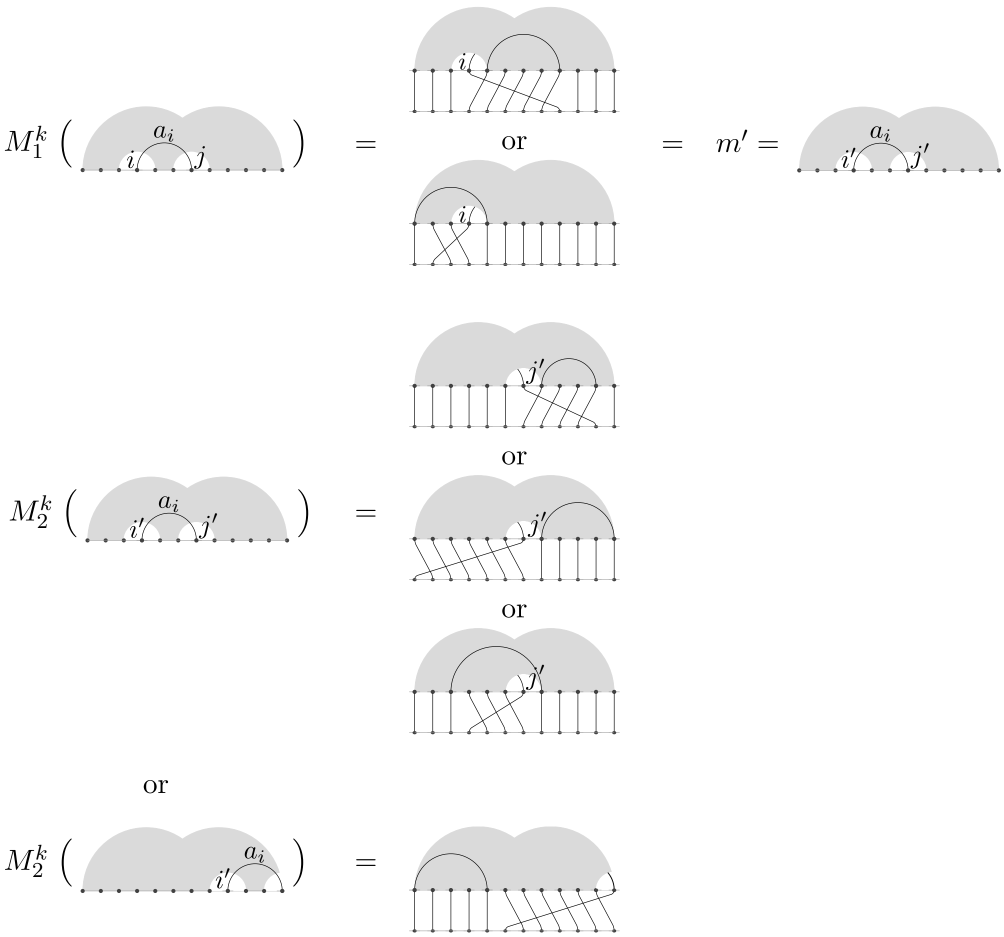


Figure 9.4: The $\mathcal{D}_n$ dynamics in diagram representation. The diagram representation is not the most convenient way to represent the action of the dynamics. A more fitting representation is illustrated in figure 9.6 using the chord diagram representation.

right of the endpoint j' of a_k. As can be seen from the definition of a^2 if a_k ends in n then the arc to the right of it is $(1, m(1))$, thus the dynamics is better visualized in the chord diagram representation (see below definition 198 and figure 9.6). Define

$$M_2^k = \begin{cases} \gamma_{A,2n}\left(j'_{mod}, m(j'_{mod})\right) & \text{if } m'(j'_{mod}) > j'_{mod} \text{ and } j'_{mod} \neq 1 \text{ and } m(j'_{mod}) \neq n \\ \gamma_{B,2n}\left(0, j'_{mod}+1\right)) & \text{if } m'(j'_{mod}) > j'_{mod} \text{ and } m(j'_{mod}) = n \\ \gamma_{B,2n}\left(m(j'_{mod}), j'_{mod}\right) & \text{if } m'(j'_{mod}) < j'_{mod} \\ \gamma_{b,2n}\left(m(j'_{mod}), 2n+1\right)) & \text{if } m'(j'_{mod}) > j'_{mod} \text{ and } j'_{mod} = 1 \\ \end{cases}$$

Then we have $M^k = M_2^k \circ M_1^k$. See figure 9.4.

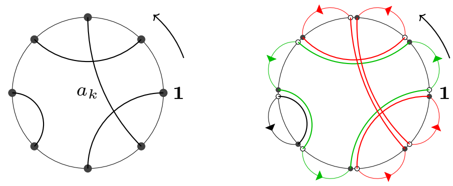


Figure 9.5: Left: a matching, $m = [\,(17)(24)(38)(56)\,]$. Right: the construction of the cycle invariant. Different cycles are in different colour. The length of a cycle, defined as the number of outisde arcs, is thus 3 for red, 4 for the green, and 1 for black. As a result, in this example $\lambda^c(m) = (4, 3, 1)$. One can compare with figure 8.3 in section 8.1 where the construction of the cycle invariant for the same matching gives $\lambda(m) = \{3, 3, 1\}$: the red cycle (which is the principal cycle) is shorter by 1.

9.2.1 Cycle invariant on chord diagram representation

The chord diagram representation of matchings was introduced in section 6. In this section we construct the cycle invariant on chord diagrams which is an invariant for the $\mathcal{D}_n$ dynamics.

Let m be a matching $[2n]$, identified with its chord diagram. Perform the following manipulations on the chord diagram: (1) replace each arc with a pair of parallel arcs; more precisely, replace each arc endpoint, say i, by a black and a white endpoint, i_b and i_w (the black first in the anticlockwise direction), then introduce the arcs (i_b, j_w) and (i_w, j_b) in the conformal disk. (2) connect by an arc, on the outside of the disk, the points i_w and $(i \,(\mathrm{mod}\ 2n) + 1)_b$, for $i = 1, \ldots, 2n$.

The resulting structure is composed of a number of closed cycles. Define the length of a closed cycle as the number of outside arcs (connecting a white endpoint to a black endpoint) in the cycle. $\lambda^c(m) = \{\lambda_i\}$, the collection of lengths of the cycles, will be called the *cycle invariant of chord diagram* of m. Define $\ell(m)$ as the number of cycles in m. See Figure 9.5, for an example.

The cycle invariant of chord diagram $\lambda^c(m)$ and the classical cycle invariant $\lambda(m)$ differ only on the length of the principal cycle c_p (in the chord diagram the principal cycle is the cycle that goes through the outside arc connecting the points $2n_w$ and 1_b): in $\lambda^c(m)$ the principal cycle is longer by one.

Now that we have given the construction of the cycle invariant, it becomes possible to state a much clearer definition of the dynamics using the the chord diagram representation.

Definition 198. *Let m be a matching and a_k be an arc. Construct the cycle invariant in the chord diagram. Then $M^k(m)$ is the matching where both the endpoints of the arc a_k have moved along the cycle invariant once, in the anticlockwise direction. See figure 9.6.*

We have

Proposition 199. *The quantity λ^c is invariant in the $\mathcal{D}$ dynamics.*

Proof. The operators $(M^k)_k$ of the dynamics, perform local modifications on portions of paths, without changing their lengths. For all cycles except for at most

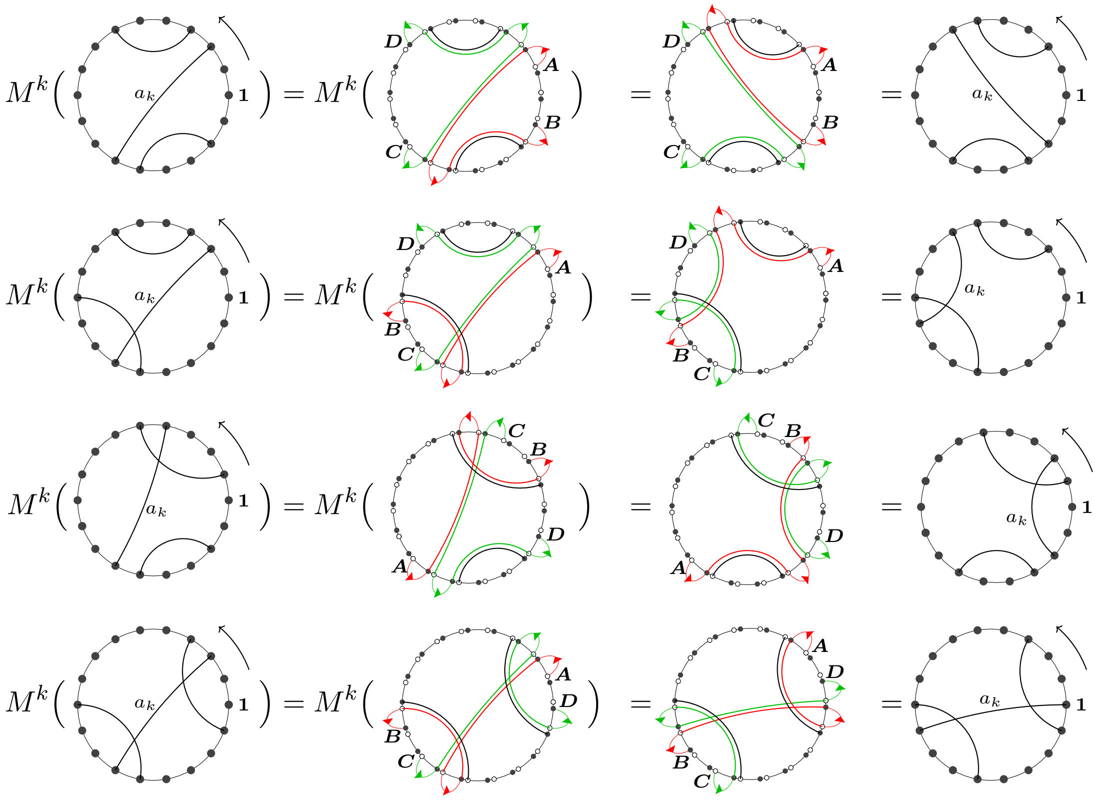


Figure 9.6: The $\mathcal{D}_n$ dynamics in the chord diagram representation and the invariance of $\lambda^c(m)$. The four cases have been represented:
$m(i+1) > i+1$ and $m'(j'_{mod}) > j'_{mod}$ and $m(i+1) > i+1$ and $m'(j'_{mod}) < j'_{mod}$
$m(i+1) < i+1$ and $m'(j'_{mod}) > j'_{mod}$ and $m(i+1) < i+1$ and $m'(j'_{mod}) < j'_{mod}$.
The other cases of the definition do not occur since they are a byproduct of the maching representation.

two of them, this is completely evident: each arc of the cycles is deformed by (say) the permutation $\gamma_{B,2n}(j'_{mod}, m(j'_{mod})) \circ \gamma_{A,2n}(i+1, m(i+1))$ in a way which can be retracted without changing the topology of the connections, this in turns certifying that the length of the involved cycle does not change.

For the paths passing from fours special endpoints (marked as A, B, C and D in the figure 9.6), the invariance of the cycle structure holds through a more subtle mechanism: The way in which the topology of connections among these four points is modified by the operator M^k is hardly explained in words, but is evident from Figure 9.6. This proves that the list λ^c is invariant. $\square$

We do no know whether the $\{\lambda^c\}$ is a complete invariant set for the $\mathcal{D}$ dynamics

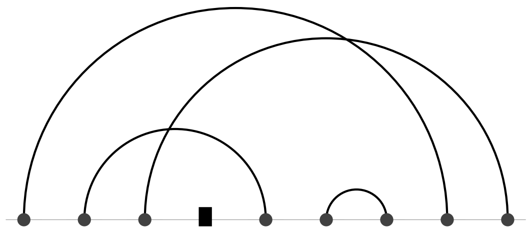


Figure 9.7: A generalized permutation over [9].

9.3 The Rauzy dynamics for generalized permutation $\mathcal{L}_n$

The Rauzy dynamics on generalized permutation was introduced in [Lan08] and [BL09] to classify the connected components of the strata of the moduli space of quadratic differentials. In essence, [BL09] proved that the Rauzy classes of the extended Rauzy dynamics on generalized permutation were in one-to-one correspondence with the connected component of the strata of the moduli space of quadratic differentials. Thus a classification theorem for one provides a classification theorem for the other.

Presently, the only proof of this classification theorem uses tools from translation surfaces and works with the strata of the moduli space of quadratic differential. It is likely that the labelling method developed in this thesis could yield a combinatorial proof working with the Rauzy dynamics on generalized permutations.

Let $\mathfrak{L}_n$ denote the set of generalized permutations over $[2n+1]$. A generalized permutation over $[2n+1]$ is a matching with n arc and one fixed point, that we name the mark. As can be expected, we represent generalized permutations over $[2n+1]$ as n arcs in the upper half plane, connecting paiwise $2n$ points on the real line and one point standing alone. See figure 9.7.

The definition of irreducibility for generalized permutations stated in [BL09] is much longer than for matching.

Definition 200. *We say that a generalized permutation σ is reducible if there exists $A, B, C, D \subset [2n+1]$ not all empty such that σ has the form:*

$$\underline{B \bigcup D \;\; *** \;\; A \bigcup B' \blacksquare \, A' \bigcup C' \quad *** \quad C' \bigcup D'}$$

with $A' = \sigma(A), B' = \sigma(B), C' = \sigma(C),\ D' = \sigma(D)$ and $\sigma(***) = ***$. Moreover, one of the following statements holds

- *None of the four cluster is empty.*
- *Only one cluster is empty and it is next to the mark.*
- *Exactly two clusters are empty and they are either both next to the mark or both to the extermities.*

Note that the standard definition of reducibility for matching is recovered in the case that two clusters are empty and are next to the mark.

Let $\mathcal{L}_n^{irr}$ denote the set of irreducible generalized permutations over $[2n+1]$ such that the mark is not on a side. Given a generalized permutation σ, let $\overline{M}(\sigma)$ denote

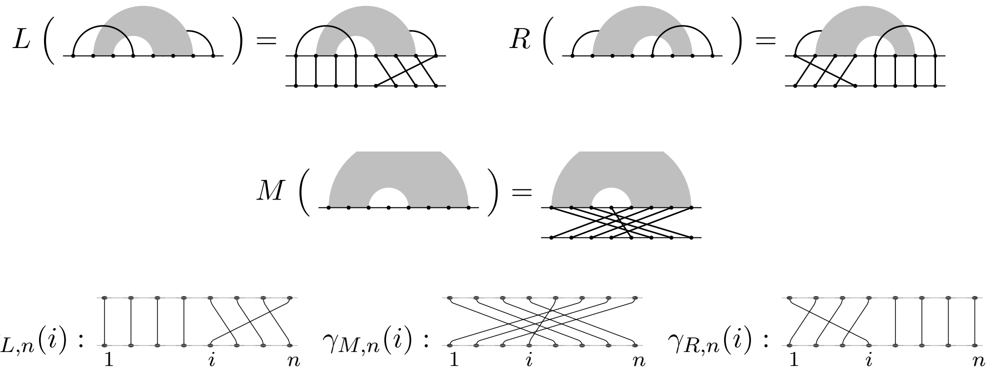


Figure 9.8: Top: the $\mathcal{L}_n$ case. Middle: In the case $\mathcal{L}_n^{ex}$ we have the same operators as in the case $\mathcal{L}_n$ plus the operator M. Bottom: The permutations $\gamma_{L,i,n}$, $\gamma_{M,i,n}$ and $\gamma_{R,i,n}$

the position of its mark. The two monoid dynamics we define in this section are the Rauzy dynamics for generalized permutation $\mathcal{L}$ and the extended Rauzy dynamics for generalized permutation $\mathcal{L}^{ex}$:

$\mathcal{L}_n$ **:** The space of configuration is $\mathfrak{L}_n^{irr}$. There are two generators, L and R, with $a_L(\sigma)$ being the index paired to 1, $a_R(\sigma)$ the index paired to $2n+1$, $\alpha_{L,i} = \gamma_{L,2n+1}(i)$, and $\alpha_{R,i} = \gamma_{R,2n+1}(i)$ (a and α are as in Definition 12). See Figure 9.8.

However the operator L is not defined on σ if $\sigma(2n+1) > \overline{M}(\sigma)$ and $\forall j > \overline{M}(\sigma), \sigma(j) < \overline{M}(\sigma)$. In other words, L is not defined if the arc ending in $2n+1$ has both its endpoints to the right of the mark while any other arc has at most one endpoint to the right of the mark.

Likewise the operator R is not defined on σ, if $\sigma(1) < \overline{M}(\sigma)$ and $\forall j < \overline{M}(\sigma), \sigma(j) > \overline{M}(\sigma)$. In other words, R is not defined if the arc ending in 1 has both its endpoints to the left of the mark while any other arc has at most one endpoint to the left of the mark.

$\mathcal{L}_n^{ex}$ **:** The space of configuration is $\mathfrak{L}_n^{irr}$. There are three operators L, R and M, with $a_M(\sigma)$ being the index of $\overline{M}(\sigma)$ and $\alpha_{M,i} = \gamma_{M,2n+2-i,2n+1}$

It is clear from the definition of the dynamics that the Cayley digraphs of the Rauzy classes of $\mathcal{L}_n$ (once the mark is removed from the generalized permutations) are subdigraphs of the Cayley graphs of the Rauzy classes of $\mathcal{M}_n$, the relationship between $\mathcal{M}_n$ and $\mathcal{L}_n^{ex}$ is less obvious.

9.3.0.1 Definition of the invariant

The cycle invariant for $\mathcal{L}_n$ and $\mathcal{L}_n^{ex}$ is inherited from $\mathcal{M}_n$: let σ be a generalized permutation and let m be the matching obtained when removing the mark of σ. We apply the construction of the cycle invariant to m and then reinsert the mark. In the counting of the arcs to determine the length of the cycles, we require that the arc below the mark counts for 0. See figure 9.9.

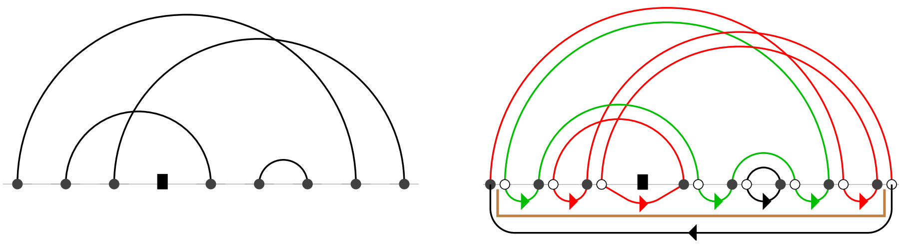


Figure 9.9: Left: a generalized permutation, $\sigma = [(17)(25)(39)(4)(67)]$. Right: the construction of the cycle invariant. Different cycles are in different colour. The length of a cycle, defined as the number of bottom arcs (except for the arc below the mark that count for 0), is thus 2 for the rank in red, 3 for green, and 1 for black. As a result, in this example $(\lambda(\sigma) = (3, 1), r = 2)$ and $\lambda^{ex}(\sigma) = (3, 2, 1)$.

Call *rank* the cycle containing the arc below the mark and let r be its length (as defined above, its length will be the number of its bottom arcs minus 1) . Define $\lambda(\sigma)$ to be the collection of the lengths of the cycles (save for the rank) in σ. Finally let $\lambda^{ex}(\sigma) = \lambda(\sigma) \bigcup \{r\}$

Then we have

Proposition 201. *The pair (λ, r) is invariant in the $\mathcal{L}$ dynamics.*

Proof. The proof of the invariance of the quantity $\lambda \bigcup \{r\}$ follows the proof of proposition 190. The invariance of r alone is more subtle. In the construction of the cycle invariant every endpoint i is doubled into a black point i_b and a white point i_w and the endpoints of two bottom arcs are connected to i_b and i_w respectively, thus the endpoint i is part of two cycles (possibly equal) the one going through (the arc connected to) i_b and the other going through (the arc connected to) i_w. The same remark can be made about the mark if we make a small change to the construction of the invariant as detailed below:

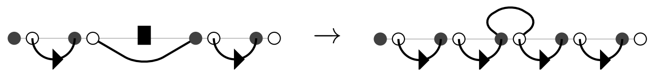


Figure 9.10: The mark and the long arc below it (counting for 0 in the invariant) are replaced with a black and white points connected above by a small loop. Two bottom arcs (both counting for 0 in the invariant) are also added.

Moreover, if the endpoint is not an endpoint of a pivot (i.e. an endpoint of the arcs $(1, \sigma(1))$ or $(\sigma(2n + 1), 2n + 1)$), the cycles associated to it do not change after the application of either operators.

In the case of a R operator (the case for L is symmetrical), the situation is described in figure 8.4 (see proposition 190). For any endpoint (other than the endpoints of the pivots) except the endpoint 2 and $\sigma(2n + 1) - 1$ the statement is clear: each arc of the cycles is deformed by the permutation $\gamma_{R,2n+1}(\sigma(2n + 1))$ in a way which can be retracted without changing the topology of the connections. For the endpoint 2 the left arc goes from B to C in both before and after the application of the operator R likewise for the endpoint $\sigma(2n + 1) - 1$ the right arc goes from

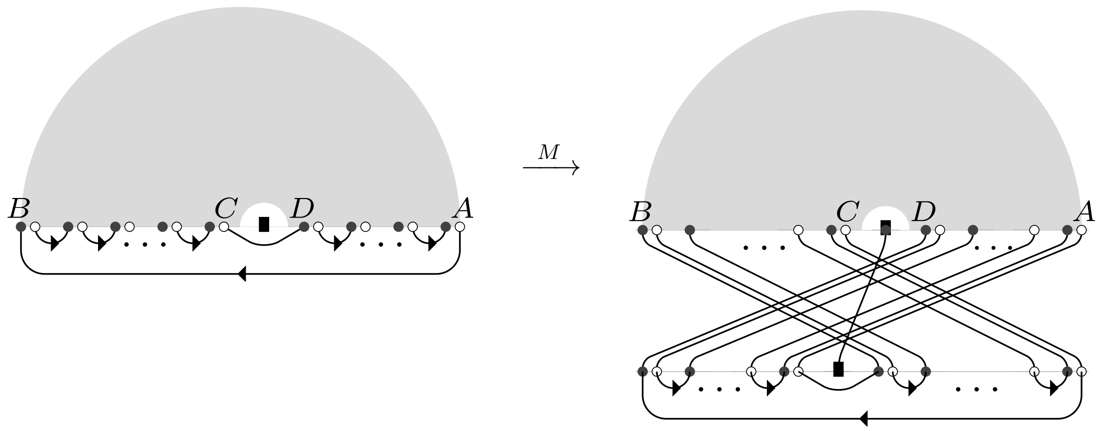


Figure 9.11: Invariance of $\lambda^{ex}(\sigma)$ in $\mathcal{L}_n$, illustrated for the operator M.

D to A in both cases. Thus in those cases as well the two cycles associated to the endpoints do not change.

The mark cannot be an endpoint of a pivot since it can never be on either end side of the generalized permutation. Indeed the operators L and R become undefined before this can happen. Thus the rank is invariant since the cycle below the mark can never change. □

and

Proposition 202. *The quantity λ^{ex} is invariant in the $\mathcal{L}^{ex}$ dynamics.*

Proof. The invariance of λ^{ex} is clear for the operators R and L, thus it remains to be proven for M. As always, M performs local modifications on portions of paths, without changing their lengths. For all cycles except for at most two of them, this is completely evident: each arc of the cycles is deformed by the permutation $\gamma_{M,2n+1}(2n+2-i)$ in a way which can be retracted without changing the topology of the connections, this in turns certifying that the length of the involved cycle does not change.

For the paths passing from fours special endpoints (marked as A, B, C and D in the figure), the invariance of the cycle structure holds through a more subtle mechanism, involving the exchange of two arcs: the long bottom arc (connecting $2n+1$ to 1) and the arc below the mark. Both arc count for 0 in the length of the cycle so the length of the two paths involved (A to B and C to D) do not change. The way in which the topology of connections among these four points is modified by the permutation $\gamma_{M,2n+1}(2n+2-i)$ is shown in Figure 9.11. This proves that the list $\lambda^{ex} = \lambda \cup \{r\}$ is invariant. □

9.3.0.2 The classification theorems

The following theorem was proven in [BL09]:

Theorem 203. *The (extended) Rauzy classes of $\mathcal{L}^{ex}$ are in one-to-one correspondence with the connected component of the strata of the moduli space of quadratic differentials.*

Thus we can reformulate the classification theorem for quadratic differentials in [Lan08] into a classification theorem for the dynamics $\mathcal{L}^{ex}$. Before that, we need to clarify the difference of terminology between this thesis and the works [Lan08] and [BL09]:

An extended Rauzy class with invariant $\lambda^{ex} = \{\lambda_1, \dots, \lambda_\ell\}$ corresponds to a connected component of the stratum $\mathcal{Q}(\lambda_1 - 2, \dots, \lambda_\ell - 2)$. Furthermore we have the formula

$$4g - 4 = 2n - 2 - 2\ell$$

between the genus g, the size of a generalized permutation $2n+1$ and it number of cycles ℓ.

Unlike the previous classification theorems, we state this one as a function of the genus rather than $2n+1$ the size of the generalized permutations. For the $\mathcal{L}^{ex}$ dynamics we have:

Theorem 204. *For $g \geq 5$, define the following three families of invariants:*

$$\begin{array}{lll}
\mathcal{F}_2 = \{ \quad \lambda^{ex} = \{4(g-(k+1)), 4(k+1)\} & | & 0 \leq k \leq g-2\} \\
\mathcal{F}_3 = \{ \quad \lambda^{ex} = \{4(g-(k+1)), 2k+3, 2k+3\} & | & 0 \leq k \leq g-1\} \\
\mathcal{F}_4 = \{ \quad \lambda^{ex} = \{2(g-k)-1, 2(g-k)-1, 2k+3, 2k+3\} & | & -1 \leq k \leq g-2\}
\end{array}$$

Then the number of classes with invariant λ^{ex} are:

two, *if λ^{ex} is in $\mathcal{F}_i$ for $i = 2, 3, 4$ with one class being hyperelliptic, the other no.*

one, *otherwise.*

The classification theorem for small $g \leq 4$ can be found in [Lan08].

For the $\mathcal{L}$ dynamics, the classification theorem comes from the work [Boi12] where it was shown that, as could be expected, every extended Rauzy class with invariant $\lambda^{ex} = \{\lambda_1^{i_1}, \dots, \lambda_k^{i_k}\}$ was partitioned into k Rauzy classes with respective invariant $(\lambda = \lambda^{ex} \smallsetminus \{\lambda_j\}, \lambda_j)$.

To conclude this presentation of the Rauzy dynamics for generalized permutations, let us compare $\mathcal{L}_n$ and $\mathcal{M}_n$ ($\mathcal{M}_n$ and $\mathcal{L}^{ex}$ are not comparable since they do not have the same operators). As mentioned above, removing the mark of a generalized permutation σ over $[2n+1]$ yields a matching m over $[2n]$. In term of cycle invariant, if σ has cycle invariant (λ, r) then m has cycle invariant $(\lambda \bigcup \{r+1\})$ since the cycle of length r contains the mark and the arc below the mark counts for 0 in the cycle invariant for $\mathcal{L}_n$ while it counts for 1 for $\mathcal{M}_n$.

Thus, the Cayley graph of Rauzy class for $\mathcal{M}_n$ with invariant $\lambda = \{\lambda_1^{i_1}, \dots, \lambda_k^{i_k}\}$ contains as subgraphs the k Cayley digraphs of the Rauzy classes for $\mathcal{L}_n$ with respective invariant $(\lambda \smallsetminus \{\lambda_j\}, \lambda_j - 1)$, with some exceptions in cases of hyperelliptic classes.

Given the situation it is not clear that the classification theorem for $\mathcal{M}_n$ can be obtained easily from the classification theorem for $\mathcal{L}_n$.

Appendix A

Representation of Cayley graphs

This short appendix deals with a graphical convention on the representation of the dynamics.

Given a group dynamics with $h+k$ generators $\{\iota_1, \dots, \iota_h, \kappa_1, \dots, \kappa_k\}$, of which h involutive and k non-involutive, a *Cayley (di)graph* can be constructed, with one connected component per equivalence class. Vertices v_x are associated to configurations x. The edges are labeled by a generator. They are unoriented in the case of involutive generators, and oriented otherwise. We have the unoriented edge (v_x, v_y) with label ι_a if $\iota_a x = y$, and the oriented edge (v_x, v_y) with label κ_a if $\kappa_a x = y$. As a result, each vertex has overall degree $h + 2k$.

For dynamics with two (non-involutive) generators, such as $\mathcal{S}_n$ and $\mathcal{M}_n$, the Cayley graph has thus degree 4. Even for relatively small classes, this makes a too complex structure for being visualised in an effective way. In this section we introduce a notation that, through an embedding on a two-dimensional grid, allows to omit to draw a fraction of the edges with no loss of information. As a result of this pruning, some of the Cayley graphs of classes are reduced to regular intelligible structures, this being in particular the case for exceptional classes, discussed in Appendix C. From this point on, we concentrate on the case of non-involutive generators.

Clearly, such an enhancement cannot be made for graphs in general, and we shall use the specialties coming from the fact that these graphs are Cayley graphs of a group action.

The iterated application of a single generator partitions a Cayley graph into cyclic orbits. Let us embed our graph in dimension $d \geq 2$, and choose a direction θ_a, in the sphere of dimension $d-1$, for each generator κ_a. If we take care of representing a directed edge with label κ_a as oriented in the direction θ_a, then one edge per orbit can be omitted without loss of information: an orbit of length ℓ is turned into a linear chain with ℓ vertices and $\ell - 1$ collinear oriented edges, and it will be understood without ambiguities that the image of the last vertex of the chain under the operator is the first vertex of the chain. In fact, edges can be omitted completely, and only the angles θ_a be specified, at the condition that no other vertices, besides those in the orbit, lay on the line in $\mathbb{R}^d$ associated to the orbit. If $d = k$, the directions κ_a can be taken to constitute the canonical basis without loss of generality.

The most useful application of this strategy is when there are two generators, and thus the graph is conveniently represented on a plane. In this special case, we will use edges with a single- or double-arrow, and in red or blue, for the two generators,

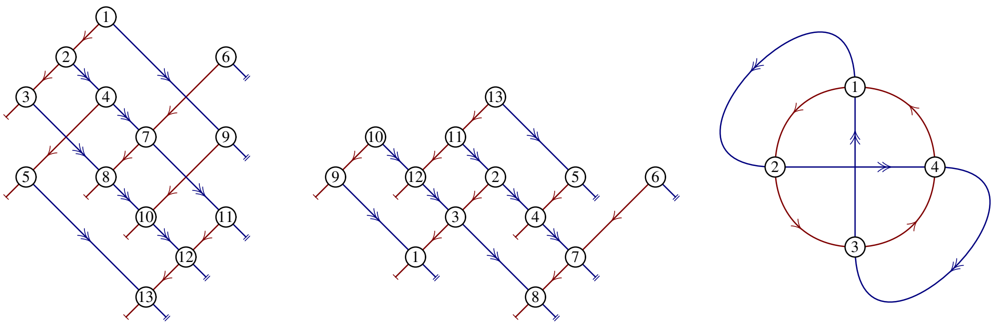


Figure A.1: Left: Straight representation of a digraph. This representation is *not* planar. Middle: a planar straight representation of the same digraph. Right: a digraph which is not straight: the four vertices are in the same red and blue orbit, and cannot be simultaneously collinear along x and along y, on two distinct directions and in the given cyclic orderings, while staying distinct.

and add a small tag on the last vertex of any chain, for further enhancing the visualisation.

See Figure A.1, left, for an example of representation of such a digraph.

We say that a Cayley graph admitting a representation with these properties is *straight*, and if it admits a representation with these properties, and such that the edges do not cross, it is *straight planar*.

It is easy to see that if a graph is straight, it also allows a straight representation in which all vertices are distinct vertices of $\mathbb{Z}^2$, and different orbits are in different rows/columns of the grid, thus straight digraphs are also representable as finite subsets of $\mathbb{Z}^2$, and a list of rational numbers (one per generator) associated to the slopes.

It is also easy to see that not all digraphs are straight. See Figure A.1, right, for a simple counter-example.

Appendix B

Non-primitive classes

In this appendix we discuss how the classification theorem on primitive classes implies, with a small amount of further reasoning, a classification theorem for all classes, including non-primitive ones. More generally, we prove how even the full structure of the Cayley graph of a non-primitive class can be evinced from the Cayley graph of the associated primitive class. This result is announced in Section 3.1.3.3.

Recall that we announced, in Corollary 23, that the map $\mathrm{prim}(\,\cdot\,)$ is a homomorphism for the dynamics, so that, as a result of the analysis performed below, if $\sigma \in C$, we can naturally define the *primitive* $C' = \mathrm{prim}(C)$ as the class containing $\mathrm{prim}(\sigma)$ for any $\sigma \in C$.

We start by discussing the most basic example of irreducible non-primitive class, namely the set of classes T_n, of size $n \geq 2$, such that $\mathrm{prim}(T_n) = \mathrm{Id}_2$. [1]

We claim that the class T_n has cardinality $\binom{n}{2}$, and that the Cayley graph admits an especially simple straight planar representation, illustrated in Figure B.1, namely a triangular portion of the square grid.

More precisely, w.r.t. the coordinates a and b shown in figure, we claim that the configurations with coordinate (a, b) are composed of three bundles, and have a special descent, if $a + b < n$, and are composed of two bundles, and have no special descent, if $a + b = n$, as shown in Figure B.2. These facts are easily verified.

[1] This is the smallest example among irreducible configurations, as if $\mathrm{prim}(\sigma) = \mathrm{id}_1$ and $\sigma \neq \mathrm{id}_1$, then σ is reducible.

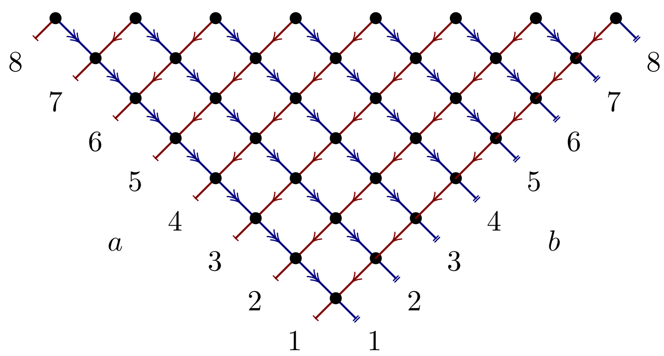


Figure B.1: The straight representation of the Cayley graph of the class T_n. Here $n = 9$.

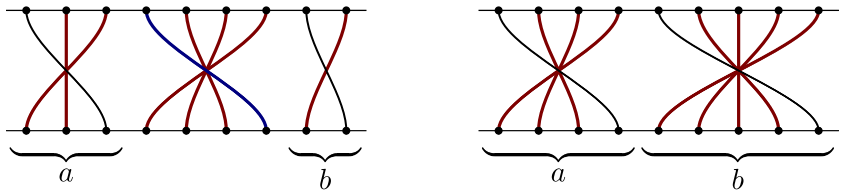
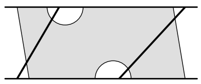
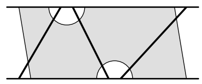


Figure B.2: Left, the configuration with coordinate $(a, b) = (3, 2)$ in T_9, composed of three bundles of crossing edges, and having a special descent. Right, the configuration with coordinate $(a, b) = (4, 5)$ in T_9, composed of two bundles of crossing edges, and having no special descent.

For a permutation σ of size n, with no special descent, call $\tilde{\sigma}$ the permutation of size $n + 1$ in which the special descent has been added:

$\sigma =$  $\tilde{\sigma} =$ 

Now we want to illustrate how the structure of classes T_n is sufficiently general to describe the local structure of non-primitive classes C in terms of the one of the class $C' = \mathrm{prim}(C)$. Let n and n' be the sizes of classes C and C'. Let $\sigma \in C$, without a special descent, and $\tau = \mathrm{prim}(\sigma)$. The configuration σ thus consists of n' bundles of crossing edges. Let $\boldsymbol{m} = (m_1, \dots, m_{n'})$ be the cardinalities of these bundles (so that $m_i \geq 1$ and $m_1 + \cdots + m_{n'} = n$). In particular, we single out the multiplicities of the two pivots, i.e., $\boldsymbol{m} = (m_1, \dots, m_{\tau^{-1}(n')}, \dots) =: (m_L, \dots, m_R, \dots)$. The datum of the pair $(\tau, \boldsymbol{m})$ is of course equivalent to σ. The following properties are easily verified:

- For $i < m_L$, $L^i\sigma$, gives the permutation associated to $(\tilde{\tau}, (m_L{-}i, \dots, i, m_R, \dots))$. Similarly, for $i < m_R$, $R^i\sigma$ gives $(\tilde{\tau}, (m_L, \dots, i, m_R - i, \dots))$.
- $L^{m_L}\sigma$ gives $(L\tau, (\dots, m_L, m_R, \dots))$.
- Similarly, $R^{m_R}\sigma$ gives $(R\tau, (m_L, \dots, m_R, \dots))$.

As a corollary, if $m_L \geq 2$ we have that $R^{-1}L\sigma$ gives $(\tau, (m_L - 1, \dots, m_R + 1, \dots))$, and if $m_R \geq 2$ then $L^{-1}R\sigma$ gives $(\tau, (m_L + 1, \dots, m_R - 1, \dots))$, which implies

$$(\tau, (m_L, \dots, m_R, \dots)) \sim (\tau, (m_L{+}c, \dots, m_R{-}c, \dots)) \qquad \forall \quad -m_L{+}1 \leq c \leq m_R{-}1\,. \tag{B.1}$$

More generally, combining the remarks above, we have

$$\begin{aligned}(\tau, (m_L, \dots, m_R, \dots)) &\sim (\tilde{\tau}, (m_L - a, \dots, a + b, m_R - b, \dots)) \\ &\qquad \forall\ a \leq m_L - 1,\ b \leq m_R - 1,\ a + b \geq 0 \\ &\sim (L\tau, (\dots, m_L + c, m_R - c, \dots)) \\ &\qquad \forall\ -m_L + 1 \leq c \leq m_R - 1 \\ &\sim (R\tau, (m_L + c, \dots, m_R - c, \dots)) \\ &\qquad \forall\ -m_L + 1 \leq c \leq m_R - 1\,.\end{aligned} \tag{B.2}$$

In other words, a portion of the Cayley graph for C, containing $\sigma = (\tau, (m_L, \dots, m_R, \dots))$, is isomorphic to a large patch of the class $T_{m_L+m_R}$ represented in Figure B.1, with the identification of parameters $(a, b) = (m_L, m_R)$, up to an important difference. For the class T_k in itself, according to the notation introduced in Appendix A, when we exit the straight-planar representation of T_k from one boundary, we come back to a vertex on another boundary of the same class. The portion of T_n within C, instead, is such that when we exit from the boundary of its planar representation, we enter into a different copy $T_{k'}$, and associated to a different set of omitted parameters $(\tau', \boldsymbol{m'} \setminus \{m'_L, m'_R\})$.

It is a fortunate coincidence that the simple straight representation of the class T_k presented above has the orbits cut exactly at the position at which the different copies of T_k's are patched together.

Proof of Proposition 22. As another application of Lemma 49, we see that

$$(\tau, (m_1, m_2, \dots, m_n)) \sim (\tau, (m'_1, m'_2, \dots, m'_n))$$

whenever $\sum_i m_i = \sum_i m'_i$, and all m_i, m'_i are strictly positive. This implies the statement for σ having no special descent. A configuration σ' with a special descent is always at alternating distance 1 from a σ with no special descent, this allowing to complete the reasoning. $\square$

The clarification of the Cayley graph of C in terms of the Cayley graph of C' has a corollary on the size of these graphs. Let C be a class of size $n + k$, with k descents. Then C' is a class of size n. Let $\tau \in C'$. Consider the number of configurations $\sigma \in C$ which can be written as $\sigma = (\tau, (m_L, \dots, m_R, \dots))$ (if σ has no special descent) or as $\sigma = (\tilde{\tau}, (m_L, \dots, m_{\text{pivot}}, m_R, \dots))$ (if σ has a special descent). Identify $(\tau, (m_L, \dots, m_R, \dots)) \equiv (\tilde{\tau}, (m_L, \dots, 0, m_R, \dots))$, and consider the list $(m_L - 1, m_2 - 1, \dots, m_{\text{pivot}}, m_R - 1, \dots, m_n - 1) =: (k_1, \dots, k_{n+1})$. At fixed $\tau \in C'$, each list with $k_i \geq 0$ and $\sum_i k_i = k$ is obtained from a single $\sigma \in C$, and lists not satisfying the properties above are never obtained. As a result, we have

Proposition 205. *With notations as above,*

$$|C| = \binom{n+k}{n} |C'| \,. \tag{B.3}$$

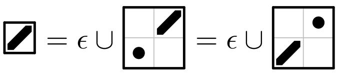
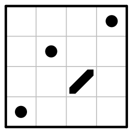
A version of this proposition, for the case of the extended dynamcis, has been first established by Delecroix in [Del13, Thm. 2.4]. His derivation is quite different in spirit, as it results from the analysis of a complicated general formula for $|C|$, instead of deriving directly the weaker (and easier) result on $|C|/|C'|$.

Appendix C

Exceptional classes for $\mathcal{S}_n$

In this appendix we describe the structure of the Cayley graph of the classes Id_n and Id'_n, which we have called *exceptional classes* in the main text. This structure is so rigid that, for any permutation σ, it is 'easily' tested if $\sigma \mathrm{Id}_n$, $\sigma \in \mathrm{Id}'_n$ or none of the two (by 'easily' we mean, in particular, that this can be tested in a time $\sim n$, instead of the naïve upper bound $\sim 2^n$ based on the cardinality of these classes). This result is of crucial importance for our classification theorem, as the latter is based on a result of the form "a list of invariants for the dynamics is complete". Our main invariants are the cycle (with the rank) and the sign. We have painfully proven that these invariants distinguish non-exceptional classes, however at generic n each of the two exceptional classes have the same cycle and sign invariant of one non-exceptional class, so that the complete list of invariants must include the boolean 'exceptionality' invariant, i.e. the outcome of the forementioned membership testing algorithm.

In this appendix, when using a matrix representation of configurations, it is useful to adopt the following notation: The symbol ϵ denotes the 0×0 empty matrix. The symbol  denotes a square block in a matrix (of any size ≥ 0), filled with an identity matrix. A diagram, containing these special symbols and the ordinary bullets used through the rest of the appendix, describes the set of all configurations that could be obtained by specifying the sizes of the identity blocks. In such a syntax, we can write equations of the like

id :=  ; id′ :=  . (C.1)

The sets id and id′ contain one element per size, id_n and id'_n, for $n \geq 0$ and $n \geq 3$ respectively.

The two exceptional classes Id_n and Id'_n contain the configurations id_n and id'_n, respectively (and are primitive for $n \geq 4$ and $n \geq 5$, respectively).

C.1 Classes Id_n

The structure of the classes Id_n is summarised by the following relation:

$$\mathrm{Id} := \bigcup_n \mathrm{Id}_n = \left(\bigcup_{k \geq 1} (X^k_{RL} \cup X^k_{LR} \cup X^k_{LL} \cup X^k_{RR}) \right) \cup \mathrm{id} \tag{C.2}$$

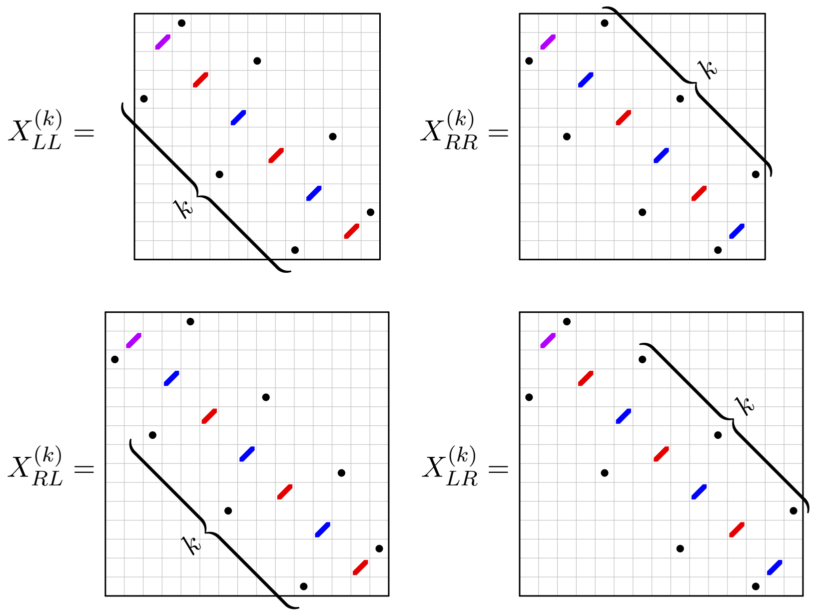


Figure C.1: Description of the structure of configurations in the identity class Id_n, besides the configuration id_n.

where the configurations $X^k_{\cdot\cdot}$ are defined as in figure C.1 (discard colours for the moment).

In other words, we claim that the configurations in classes Id_n are partitioned into five disjoint sets: the identity configurations id_n, and those contained in the four sets $X^k_{\cdot\cdot}$, with $\cdot = R, L$ and total size n. More in detail, the configuration $\sigma_{RL}^{n;(i_1,j_1,\dots,i_k,j_k)} := R^{j_k+1}\cdots L^{i_2+1}R^{j_1+1}L^{i_1+1}\mathrm{id}_n$ is in X^k_{RL} whenever $i_1 + j_1 + i_2 + \cdots + j_k = n - 2k - 2 + \delta$, with $\delta \geq 0$, and it is represented exactly as in figure C.1, bottom-left, with red blocks having size i_1, i_2, $\dots$, i_k, (from bottom-right to top-left), blue blocks having size j_1, j_2, $\dots$, j_k (still from bottom-right to top-left), and the violet box having size δ. The other three sets have similar definitions.

We also claim that the class Id_n has a straight planar representation (in the sense of Appendix A) consisting of a complete binary tree of height $n-2$ (and, in particular, $|\mathrm{Id}_n| = 2^{n-1} - 1$). An illustration of this fact for Id_6 is presented in Figure C.2. The root configuration of the tree is id_n. All the vertices of the tree can be described in terms of the unique path that reaches them starting from the root. When we have a vertex such that the corresponding path starts with $i_1 + 1$ left-steps, followed by $j_1 + 1$ right steps, followed by $i_2 + 1$ left-steps, and so on, and finally terminating with $j_k + 1$ right step, the corresponding configuration is the forementioned $\sigma_{RL}^{n;(i_1,j_1,\dots,i_k,j_k)}$. The other three cases are described analogously.

In order to see why this correspondence holds, we have to verify that the action of R and L on these configurations is consistent with this straight representation.

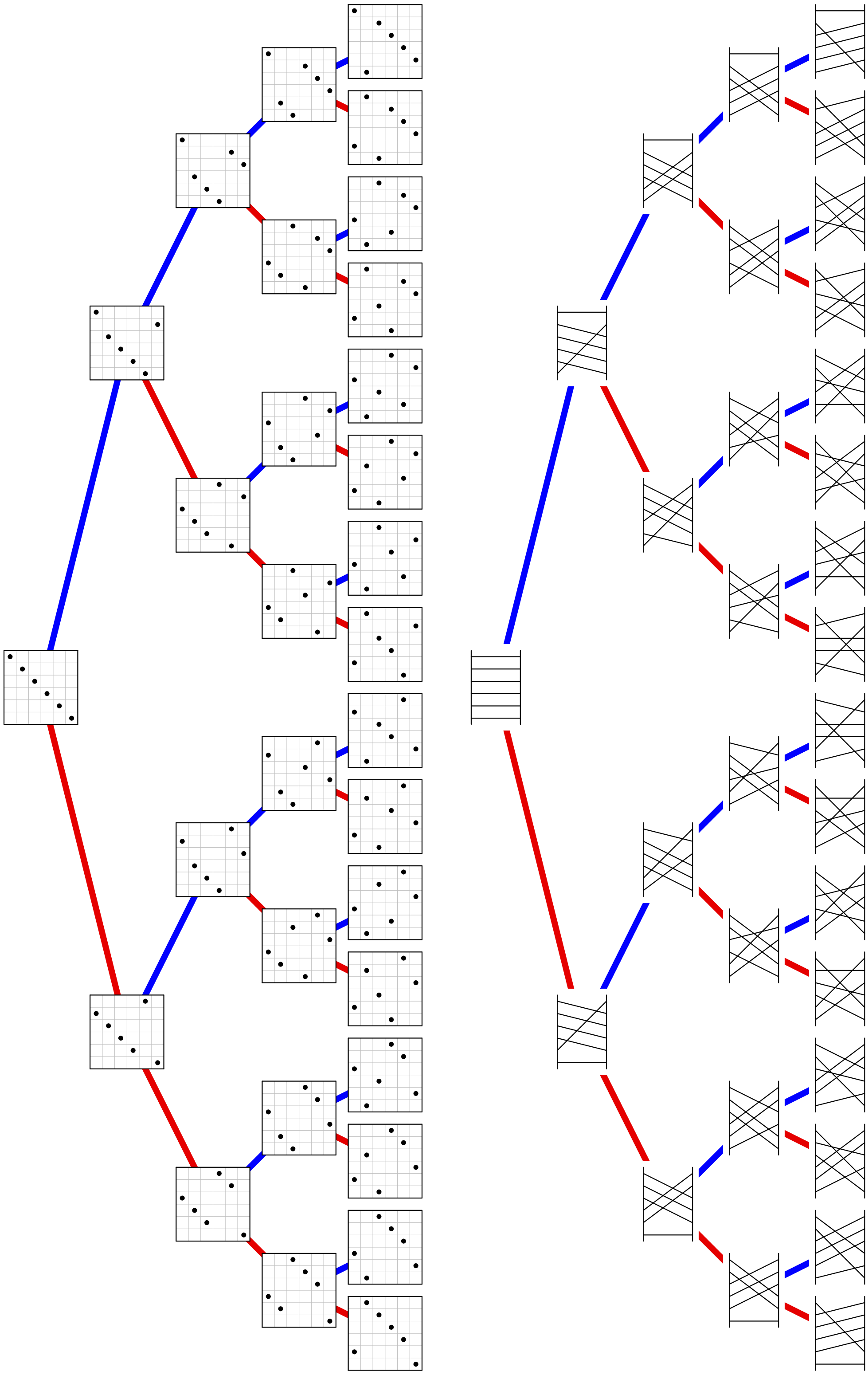


Figure C.2: The Cayley graph of Id_6, with configurations in matrix and diagram representation. The image shall be observed rotated by 90 degrees (i.e., the root of the tree is on top). Red and blue edges correspond to actions of L and R respectively, and are omitted with the same rules of straight representations.

The relations implied by the straight representation as a binary tree are

$$L\,\sigma_{RL}^{n;(i_1,j_1,\dots,i_k,j_k)} = \begin{cases} \sigma_{LL}^{n;(i_1,j_1,\dots,i_k,j_k,1)} & \delta \geq 1 \\ \sigma_{RL}^{n;(i_1,j_1,\dots,i_k,j_k)} & \delta = 0 \end{cases} \tag{C.3a}$$

$$R\,\sigma_{RL}^{n;(i_1,j_1,\dots,i_k,j_k)} = \begin{cases} \sigma_{RL}^{n;(i_1,j_1,\dots,i_k,j_k+1)} & \delta \geq 1 \\ \sigma_{LL}^{n;(i_1,j_1,\dots,i_k)} & \delta = 0 \end{cases} \tag{C.3b}$$

Similarly we have

$$L\,\sigma_{LL}^{n;(i_1,j_1,\dots,i_k)} = \begin{cases} \sigma_{LL}^{n;(i_1,j_1,\dots,i_k+1)} & \delta \geq 1 \\ \sigma_{RL}^{n;(i_1,j_1,\dots,j_{k-1})} & \delta = 0 \end{cases} \tag{C.4a}$$

$$R\,\sigma_{LL}^{n;(i_1,j_1,\dots,i_k)} = \begin{cases} \sigma_{RL}^{n;(i_1,j_1,\dots,i_k,1)} & \delta \geq 1 \\ \sigma_{LL}^{n;(i_1,j_1,\dots,i_k)} & \delta = 0 \end{cases} \tag{C.4b}$$

(the other two cases are treated analogously). Recalling that the graphical description of the dynamics, in matrix representation, is given by figure 3.2, we find that these equations are verified, as is illustrated in Figure C.3 for equations (C.4) (the other cases are treated analogously)

The characterisation of Id_n has a couple of interesting consequences. A first one is the fact that Id_n has a unique standard family, containing id_n (this comes from the direct inspection of figure C.1). The second one is the following

Corollary 206. *We can decide in linear time if* $\sigma \in \mathrm{Id}_n$.

Proof. It is easily seen that we can check in linear time if σ has one of the four structures of figure C.1. In fact, for each of the two sets $\mathrm{id} \cup X_{RL} \cup X_{LL}$ and $\mathrm{id} \cup X_{LR} \cup X_{RR}$, the test takes $\sim 2n$ accesses to the matrix, if successful, and less than this if unsuccessful.

Another way of seeing this is to use a linear-time standardisation algorithm, and the uniqueness of the standard family. However, the naïve standardisation algorithm is quadratic in time.

At this point, it is specially easy to prove the Lemma 188, needed in Section 7.7.5. This lemma states that, for St a standard family within a non-exceptional class C at size $n+1$, there is at most one element σ in St such that, when removing the $\sigma(1) = 1$ entry, the resulting permutation at size n is in Id_n.

Proof of Lemma 188. This is equivalent to say that there are no pairs of permutations $\sigma_1, \sigma_2 \in \mathrm{Id}_n$ which allow for a block decomposition

$$\sigma_1 = \begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array} \qquad\qquad \sigma_2 = \begin{array}{|c|}\hline B \\ \hline A \\ \hline \end{array} \tag{C.5}$$

If the block A has ℓ rows, we say that σ_2 is the result of shifting σ_1 by ℓ.

Clearly, at the light of the structure of configurations that we have presented (refer in particular to Figure C.1), this pattern is incompatible with σ_1 or σ_2 being id_n (as a non-trivial shift produces a configuration which is not even irreducible), so we have excluded the cases in which, still with reference to the figure, we have only one violet block, and the number of black points is at least 3, for $X_{LL}^{(k)}$ and $X_{RR}^{(k)}$, and at least 4, for $X_{LR}^{(k)}$ and $X_{RL}^{(k)}$. Note that the black points are the positions in the grid which are south-west or north-east extremal (i.e., positions $(i, j) \in \sigma$ such that there is no $(i', j') \in \sigma$ with $i' < i$ and $j' < j$, or the analogous statement with $i' > i$ and $j' > j$). Let us call *number of records*, $\rho(\sigma)$, this parameter. Thus we have that configurations in $X_{LL}^{(k)}$ and $X_{RR}^{(k)}$ have $\rho = 2k + 1$, and configurations in $X_{LR}^{(k)}$ and $X_{RL}^{(k)}$ have $\rho = 2k + 2$.

Now, if we perform a shift within one block of consecutive ascents, it is easily seen, by investigation of the sub-configuration at the right of the entry of the new configuration in the bottom-most row, or the one at the left of the entry of the new configuration in the top-most row, that the resulting structure is incompatible with the structure of Id. The same reasoning apply if we perform the shift at the beginning/end of a non-empty diagonal block, which is not the one at the bottom-right/top-left. On the other side, if we perform a shift in any other configuration, we have a new configuration in which ρ has strictly decreased. As σ_2 is a non-trivial shift of σ_1, and σ_1 is a non-trivial shift of σ_2, we can thus conclude. $\square$

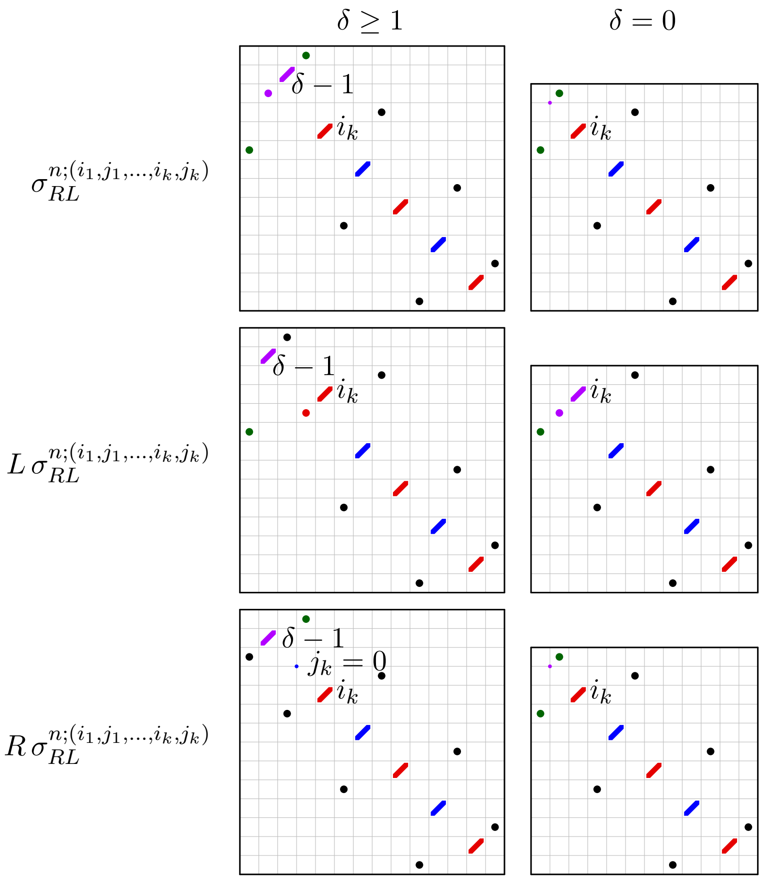


Figure C.3: Action of the dynamics on a configuration in Id_n. We analyse separately the cases of $\delta \geq 1$ (in this case we can single out one point in the top-right identity block) and $\delta = 0$ (in this case we can omit the top-right identity block). In all the four cases we obtain a configuration in one of the four forms considered in figure C.1, and all the sizes of the identity blocks, except for a finite number of blocks on the top-left corner, are left unchanged, this allowing for the analysis of a generic configuration.

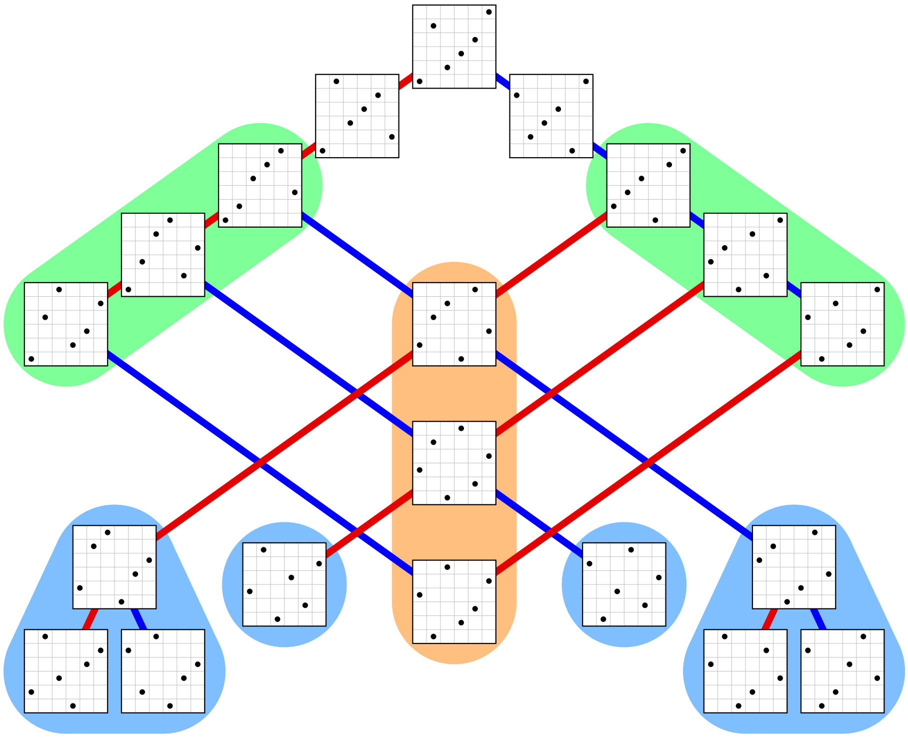


Figure C.4: The Cayley graph of Id'_6, with configurations in matrix representation. Red and blue edges correspond to actions of L and R respectively, and are omitted with the same rules of straight representations. Green, orange and cyan blocks of configurations correspond to configurations with different structure, illustrated in the text. In particular, the cyan blocks are isomorphic to classes Id_k, except for the action of L^{-1} (or R^{-1}, depending on the position w.r.t. the vertical axis) on the root of the binary tree, and the action of L (resp. R) on the left-most leaf of the tree (resp. right-most).

C.2 Classes Id'_n

This class is somewhat more complicated than Id_n, however, within its exponential cardinality, the vast majority of configurations and transitions follow the same basic mechanism of Id_n (and are arranged into complete binary trees), the new ingredients being confined to a number, linear in n, of special configurations, with a simple structure (i.e., at generic n, described by a finite number of bullets and identity blocks).

We use again a straight representation of the Cayley graph. This could be represented in a planar way (with some edges being very stretched), but we will instead use a non-planar representation, illustrated in Figure C.4 for the case of Id'_6.

Figure C.5 illustrates the general structure of the class Id'_n. We observe four families of configurations:

- The three configurations id', $L\,\mathrm{id}'$ and $R\,\mathrm{id}'$, which are on the top part of our straight representation of the Cayley graph.

- Two linear families of configurations, related one another by the symmetry of

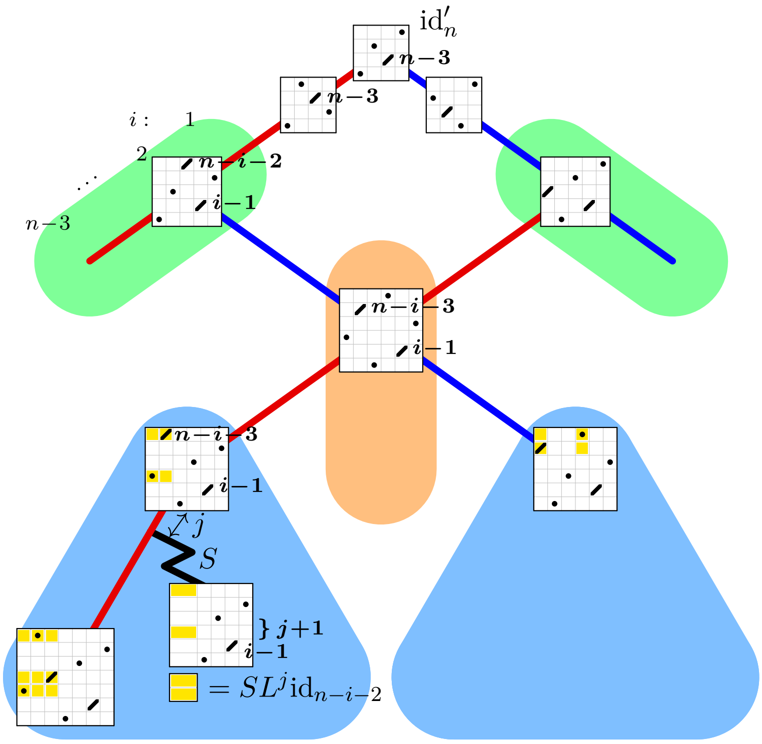


Figure C.5: A schematisation of the Cayley graph of Id'_n, with configurations in matrix representation. Only a few branches are represented (the full structure is evinced from comparison with Figure C.4). The configuration on the bottom-left corner has been added for clarity, in order to illustrate its 'anomalous behaviour' under the application of L, w.r.t. the copy of Id_{n-i-2} in which it is contained.

reflection along the diagonal, and denoted in green in the figures C.4 and C.5. An index $1 \le i \le n-3$ is associated to the size of one of the two diagonal blocks.

- One linear family of configurations, which are symmetric, and denoted in orange in the figures. Again, an index $1 \le i \le n-3$ is associated to the size of one of the two diagonal blocks.

- Two linear families, related one another by the symmetry, of subgraphs of the Cayley graph, isomorphic to the graph of Id_{n-i-2} except for one transition (interpret Id_0 as an empty graph). These are denoted by the cyan triangles in the figures. In figure C.5, the restriction to the yellow blocks of these configurations coincides with the corresponding configurations in Id_{n-i-2}.

At the light of the results of the previous section on the structure of Id, it is easy to verify that this Cayley graph indeed describes the dynamics on this class.

Corollary 207. Id'_n *has size* $2^{n-2} + n - 2$.

Proof. We have the three configurations id', $L\,\mathrm{id}'$ and $R\,\mathrm{id}'$, then $2(n-3)$ 'green' configurations (w.r.t. the colours of Figure C.4), $n-3$ 'orange' ones, and $2\sum_{i=1}^{n-3}(2^{i-1} -$

1) $= 2^{n-2} - 2n + 4$ for the 'cyan' ones. Collecting all the summands gives the statement.

Corollary 208. *We can decide in linear time if $\sigma \in \mathrm{Id}'_n$.*

Proof. All configurations in Id'_n have either a structure described by a finite number of bullets and identity blocks, within the finite list of Figure C.5, or a structure of this form, plus a block which is in the class Id_k for some k. With an argument similar to the one of Corollary 206, and in light of the latter, we can thus conclude.

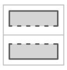
C.3 Synthetic presentation of classes Id_n and Id'_n

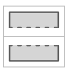
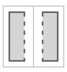
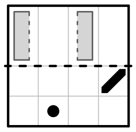
Instead of using the explicit descriptions of figures C.1 and C.5, we can describe the sets Id and Id′ in a synthetic recursive way. To this aim we need a few definitions.

Definition 209. *A configuration $\sigma \in \mathrm{Id}_n$ is* of type (L, j) *if $\sigma(n) = j > \sigma^{-1}(1)$, and it is* of type (R, j) *if $\sigma(n) < j = \sigma^{-1}(1)$. We denote by $\mathrm{Id}_n^{(L,j)}$ and $\mathrm{Id}_n^{(R,j)}$ the corresponding sets.*

Proposition 210. *The configurations in Id_n are the disjoint union of id_n, configurations of type (L, j) for $2 \le j \le n-1$, and configurations of type (R, j) for $2 \le j \le n-1$. The set $\mathrm{Id}_n^{(R,j)}$ consists of the transpose of matrices in set $\mathrm{Id}_n^{(L,j)}$.*

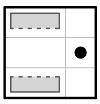
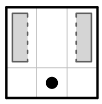
For compactness of the following expressions, we do the identification $\mathrm{id} \equiv \mathrm{Id}_n^{(L,1)} \equiv \mathrm{Id}_n^{(R,1)}$, and we introduce $\mathrm{Id}_n(L) = \cup_{j \ge 1} \mathrm{Id}_n^{(L,1)}$, and similarly for $\mathrm{Id}_n(R)$.

Define also recursively the family of square matrices divided into two rectangular blocks  as

 =  ^{T} $= \epsilon \cup$  (C.6)

Then the set Id defined as

$$\mathrm{Id} = \mathrm{Id}(L) \cup \mathrm{Id}(R)\,; \qquad \mathrm{Id}(L) \cap \mathrm{Id}(R) = \mathrm{id}\,; \tag{C.7}$$

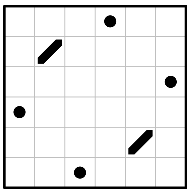
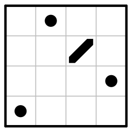
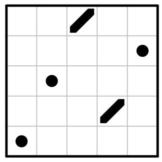
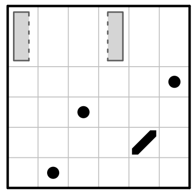
$\mathrm{Id}(L) :=$  ; $\mathrm{Id}(R) :=$  ; (C.8)

consists of configurations of size ≤ 3, which are not primitive, and the union of Id_n for $n \ge 4$.

Similarly, the set Id′ defined as follows

$$\mathrm{Id}' = \mathrm{Id}'(L) \cup \mathrm{Id}'(R) \cup \mathrm{Id}'(T)\,; \qquad \mathrm{Id}'(R) = \left(\mathrm{Id}'(L)\right)^T; \tag{C.9}$$

$$(\mathrm{Id}'(L) \cup \mathrm{Id}'(R)) \cap \mathrm{Id}'(T) = \varnothing\,; \qquad \mathrm{Id}'(L) \cap \mathrm{Id}'(R) = \mathrm{id}'\,; \tag{C.10}$$

Id$'(T) =$  ; Id$'(L) =$  $\cup$  $\cup$  ; (C.11)

consists of configurations of size 3 and 4, which are not primitive, and the union of Id'_n for $n \geq 5$.

Note that, with respect to the notations of Section C.1, we have $\mathrm{Id}(L) = \mathrm{id} \cup X_{RL} \cup X_{LL}$ and $\mathrm{Id}(R) = \mathrm{id} \cup X_{LR} \cup X_{RR}$, and with respect to the notations of Section C.2 we have that $\mathrm{Id}'(T)$ contains the orange configurations, and $\mathrm{Id}'(L)$, $\mathrm{Id}'(R)$ both contain id', and for the rest contain each half of the configurations which are not symmetric w.r.t. reflection along the diagonal.

Appendix D

Some facts on graph automorphimsms

In this appendix we establish a lemma which is crucial in solving the 2-point monodromy problem for the dynamics $\mathfrak{I}_n$ on matchings, studied in Chapter 6. However this is the monodromy problem in the general case when study the full set of Rauzy classes.

As mentioned in the introduction of chapter 6, we can work with a closed subset of Rauzy classes for which the monodromy problem is much easier and still recover the classification of the full theorem thanks to the result of section 6.6.

Nevertheless this appendix provides the means to directly carry out the labelling method in the general case.

Let $G = (V, E)$ be a graph. Call $\vec{E}$ the set of oriented edges of G (in particular $|\vec{E}| = 2|E|$). A permutation $\phi \in \mathfrak{S}_{\vec{E}}$ is an automorphism if it preserves the adjacency structure of the graph. In this case, it induces permutations in $\mathfrak{S}_E$, and in $\mathfrak{S}_V$, which we still denote with the same symbol.

As each permutation, an automorphism is invertible and has a finite order. For any subgraph $H \subseteq G$, we define the *order of H w.r.t. ϕ* as the smallest positive integer $\ell = \ell(H, \phi)$ such that $\phi^\ell H = H$.

Definition 211 (Cycle-type of a vertex)**.** *Let $v \in V(G)$, of degree d, and $\phi \in$ $\mathrm{Aut}(G)$. The map $\phi^{\ell(v,\phi)}$ induces a permutation $\sigma \in \mathfrak{S}_d$ on the d oriented edges incident to v. This permutation has some cycle-type λ. We say that v* has cycle-type λ w.r.t. ϕ.

We say that a graph is a *pointed quartic graph* (shortened to p.q.g.) if it is connected, it has two vertices of degree 1, labeled v_L and v_R, and called the *roots*, and all other vertices, which are unlabeled, of degree 4. We define $\mathrm{Aut}(G)$ as the set of automorphisms of G, as a completely unlabeled graph (i.e., including also those which exchange v_L and v_R, if there are any), and $\mathrm{Aut}_+(G) \subseteq \mathrm{Aut}(G)$ the subgroup of the automorphisms which leaves v_L and v_R fixed. Of course, either $\mathrm{Aut}(G) \equiv \mathrm{Aut}_+(G)$ or $\mathrm{Aut}(G)$ consists of two cosets, the one connected to the identity being $\mathrm{Aut}_+(G)$.

Lemma 212. *Let G be a pointed quartic graph with at least 3 vertices and no triple edges. Let $\phi \in \mathrm{Aut}_+(G)$. Then there exists a vertex of degree 4 with cycle-type different from* (4) *and* (3, 1)*, and whose order is a power of* 3.

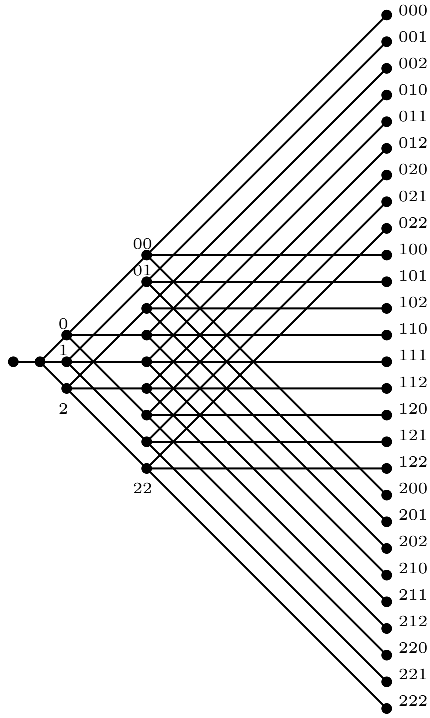


Figure D.1: Complete ternary tree of depth $d = 3$. The indices of some of the vertices are shown. The automorphism acts by translating one step down, vertically, in a cyclic way.

Proof. Observe that, if there are loops in the graph, because of the degree constraint, these must be isolated, and the corresponding vertex must have cycle-type in the list $\{(2,2),(2,1,1)\}$. Similarly, if there are double edges, these must come in linear chains (i.e., vertices u_0, u_1, ..., u_k, with $k \geq 1$, are such that u_i is connected to u_{i+1} with a double edge, and no other double edges are incident to the u_j's). As a result, u_0 and u_k have cycle-type in the same list above.

Call u_L and v_R the two roots of G. Call u the only neighbour of u_L. As $|V(G)| \geq 3$, $u \neq v_R$ and the edge (u_L, u) has order 1. So u has order $1 = 3^0$, and, unless it has cycle-type $(3,1)$, we are done. So we investigate this case, and declare $C = \{u\}$ to be the 'current orbit' of ϕ.

If the graph is not simple at C (i.e., if there are loops or double-edges incident on C), from the reasoning above we are done. So we restrict the attention to the case in which the graph is simple at C. In this case, the three other neighbours of u must be distinct and isomorphic, so in particular none of them is v_R. Without loss of generality, we can call them u_0, u_1 and u_2, and we can state that $\phi u_i = u_{i+1}$, where "+1" is intended in $\mathbb{Z}_3$ (we use $\mathbb{Z}_p$ as a shortcut for $\mathbb{Z}/p\mathbb{Z}$).

Now we perform similar reasonings, while proceeding by induction. Assume that the neighbourhood of radius $d+1$ of the vertex u_L has the form of a complete ternary tree of depth d, in which:

- vertices at distance $k+1$ from u_L are labeled with strings $\omega \in \{0,1,2\}^k$, identified to elements in $\mathbb{Z}_{3^k}$; they have order 3^k w.r.t. ϕ, and are all isomorphic;
- besides (u_L, u), the edges are exactly the pairs $(u_\omega, u_{x\omega})$, with $x \in \{0,1,2\}$,

$\omega \in \{0,1,2\}^k$ and $0 \le k \le d-1$, have order 3^{k+1} w.r.t. ϕ, and are all isomorphic;

- the image under ϕ of u_ω is $u_{\omega+1}$, and the image of $(u_\omega, u_{x\omega})$ is $(u_{\omega+1}, u_{x\omega+1})$, where "+1" is in the group $\mathbb{Z}_{3^k}$ pertinent to the label;
- the 'current orbit' is the set $C = \{u_\omega\}_{|\omega|=d}$.

In this case, we claim that either the current orbit is composed of vertices of cycle-type in the list $\{(2,2),(2,1,1),(1,1,1,1)\}$, or the neighbourhood of radius $d+2$ of u_L is the tree of depth $d+1$. This claim is clearly equivalent to our statement, as in the first case we have found the seeked orbit, and the second case cannot be iterated infinitely many times, as we have a finite graph in which u_L and v_R are connected.

Clearly, the vertices in the current orbit cannot be of type (4). And, as we said above, the graph must be simple at the current orbit, in order to have cycle-type $(3,1)$. If all the 3^{d+1} neighbours of the vertices at depth d are distinct, and distinct from the ones which have been already disclosed, we can label them as in the description of the tree above, at depth $d+1$. Also, up to renaming the vertices, the only way in which ϕ can act on these vertices, which is compatible with the fact that the vertices at depth d are of type $(3,1)$, is by adding $+1$ to the index, modulo 3^{d+1} (and these vertices would constitute the new current orbit). So, in the case of distinct vertices we would have verified the second part of the claim.

Thus, we need to exclude any other possibility, namely that C has cycle-type $(3,1)$, and one of the two following events occurs: (1) that some neighbour of a u_ω is a $u_{\omega'}$, for $\omega, \omega' \in \mathbb{Z}_{3^d}$; (2) that some of the neighbours are not distinct.

We start with case (1), which is rather easy. As all the u_ω's are of cycle type $(3,1)$ we know that all of the 3^{d+1} edges going out of the u_ω's and not in the tree are isomorphic. So, all these edges would be between vertices at depth d in the tree, which is obviously inconsistent (e.g., because v_R would be disconnected, or because we cannot pair 3^{d+1} half-edges as this is an odd number).

Now we pass to case (2), while remebering that the graph is simple at C. Say that two (distinct) vertices at depth d are connected to the same neighbour, and this new vertex is thus at depth $d+1$. say that these two vertices have indices $\omega' 0 \omega$ and $\omega'' 1 \omega$ (the cases with 0 or 1 replaced by 2 is completely analogous). Let $|\omega| = k$ and $|\omega'| = |\omega''| = d-k-1$. For future convenience, let us call $v_{0\omega}$ this neighbour. Thus we have (here $u \sim v$ means that u and v are adjacent in the graph)

$$u_{\omega' 0 \omega}, u_{\omega'' 1 \omega} \sim v_{0\omega} \,.$$

Define $s = \omega'' 1 - \omega' 0 \in \mathbb{Z}_{3^{d-k}}$. Then, from the way in which ϕ acts on vertices at depth d, we know that, defining the set $W = \{\omega' 0 + as \pmod{3^{d-k}}\}_{a \in \mathbb{Z}}$ (this is a coset in $\mathbb{Z}_{3^d}$), we have $u_{\omega''\omega} \sim v_{0\omega}$ for all $\omega'' \in W$. As the degree of $v_{0\omega}$ is at most 4, all cosets have cardinality a power of 3, and W has cardinality at least 2, it must be $|W| = 3$, i.e. $s = 3^{d-k-1}$ or $2 \cdot 3^{d-k-1}$. From the explicit form of s, we know that its last digit is non-zero, i.e. we deduce that $k = d-1$, i.e. that $\omega \in \{0,1,2\}^{d-1}$, and

$$u_{0\omega}, u_{1\omega}, u_{2\omega} \sim v_{0\omega} \,.$$

As we have already observed, the 3^{d+1} edges going out of the u_ω's are isomorphic, so in particular there exists vertices $v_{1\omega}$ and $v_{2\omega}$ such that

$$u_{x\omega} \sim v_{y\omega} \qquad \forall\, x, y \in \{0,1,2\} \,,$$

and in fact the relation above, if valid for *one* word $\omega \in \{0,1,2\}^{d-1}$, must be valid for *all* words $\omega \in \{0,1,2\}^{d-1}$, and all the $v_{y\omega}$'s are isomorphic.

For obvious reasons, the cycle-type of the v's cannot be neither (4), nor $(2,2)$. It may seem, at first, that the cases $(1,1,1,1)$, $(2,1,1)$ and $(3,1)$ are still viable. However, the fact that the edges $\{(u_{x\omega}, v_{y\omega})\}$ are all isomorphic easily allows to exclude all these cases, except possibly for $(3,1)$, and only in the configuration such that the forementioned edges are those in the cycle of length 3 for $\sigma(v_{y\omega}, \phi)$. In this case, the v's would be all isomorphic, and have order 3^d w.r.t. ϕ. However, also this possibility shall be excluded: if both the u_ω's and the v_ω's, for $|\omega| = d$, are of order 3^d, and there are no triple edges in the graph, (and recalling that ϕ acts as an increment by 1 on the indices both of u and of v vertices), we would get that the edges between them are of order 3^d, which is a contradiction, as we know from the tree-structure of vertices up to distance $d+1$ imply that they are of order 3^{d+1}. As a result, the only viable possibility is that the neighbours of the u_ω's are all distinct, i.e. the case which we have already analysed above, with a positive result.

So, from the fact that the graph is finite and connected, we have concluded that there must exist a value $d \geq 0$ such that we have an orbit of vertices of cardinality 3^d and cycle-type in the list $\{(2,2),(2,1,1),(1,1,1,1)\}$, that these vertices are all and only the vertices at distance $d+1$ from u_L, and that the neighbourhood of radius $d+1$ of u_L is the complete tree of depth d depicted above. This is a slight refinement of the claim in the lemma. $\square$

At this point, we have to establish that a certain manipulation performed at the class of vertices identified by the lemma above has a number of special properties that, in turns, will make our induction work, for what concerns the monodromy problem for the dynamics $\mathfrak{I}_n$ described in Section 6.3. We start with a few definitions, and then provide a further lemma.

We call a *family of automorphisms* $A = \{A(G)\}_{G \text{ p.q.g.}}$ as the datum of one subgroup $A(G) \subseteq \mathrm{Aut}(G)$ for each pointed quartic graph G. Similarly as for Aut, we call $A_+(G)$ the restriction of $A(G)$ to automorphisms which not exchange the two roots.

Let G be a pointed quartic graph, let $\phi \in \mathrm{Aut}(G)$, and let C be a class of cycle-type in the list $\{(2,2),(2,1,1),(1,1,1,1)\}$. Say $|C| = m$. Let us call $C = \{v_1, \ldots, v_m\}$ (with the convention $\phi(v_m) = v_{m+1}$, and $m+1$ intended modulo m), and $\{e_{i,a}\}_{1 \leq i \leq m; 1 \leq a \leq 4}$ the $4m$ oriented edges incident on the v_i's, with the convention that $\phi(e_{i,a}) = e_{i+1,a}$ for $i < m$, and $\phi(e_{m,a}) = e_{1,a'}$ with (a,a') in the list $\{(1,2),(2,1),(3,4),(4,3)\}$, or $a = a'$ (this depends on the cycle type of C). We say that a pairing P of the $4m$ edges is *consistent with* ϕ if the pairs in P are the set $\{(e_{i,a_1}, e_{i,a_2}), (e_{i,a_3}, e_{i,a_4})\}_{1 \leq i \leq m}$, with:

- $\{(a_1,a_2),(a_3,a_4)\}$ being any pairing of $\{1,2,3,4\}$, if the cycle-type is $(1,1,1,1)$;
- $\{(a_1,a_2),(a_3,a_4)\}$ being the pairing $\{(1,2),(3,4)\}$, if the cycle-type is $(2,1,1)$ or $(2,2)$.

This condition is equivalent to say that the image under ϕ of two paired edges consists of two paired edges.

For a given triple (G,C,P) as above, we define the map $\kappa_0 : G \to G_0$ as the map that replaces each pair of edges in P with one edge, and drops the vertices of C. The graph G_0 must consist of one connected component containing v_L and v_R, plus possibly other components composed only of vertices of degree 4. We call $\kappa_1 : G \to G_1$ the composition of κ_0, and the restriction to the connected component containing the roots.

If $|C| = 1$, and G_0 is not connected, say that $P = \{(1,2),(3,4)\}$, and the concatenation of the pair $(e_{1,1}, e_{1,2})$ of P is in G_1, and the concatenation of $(e_{1,3}, e_{1,4})$ is not. Define $\kappa_2 : G \to G_2$ as the graph obtained by *not* concatenating $(e_{1,3}, e_{1,4})$, dropping v_1 and the whole of G_1, and, naming v_L and v_R the endpoints of $e_{1,3}$ and $e_{1,4}$ that were previously incident on v_1.

Say that $\phi_1 \in \mathrm{Aut}(G_1)$ is *compatible with* ϕ if the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \big\downarrow{\scriptstyle \kappa_1} & & \big\downarrow{\scriptstyle \kappa_1} \\ G_1 & \xrightarrow{\phi_1} & G_1 \end{array} \tag{D.1}$$

commutes. When κ_2 and G_2 are defined, say analogously that $\phi_2 \in \mathrm{Aut}(G_2)$ is *compatible with* ϕ if the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \big\downarrow{\scriptstyle \kappa_2} & & \big\downarrow{\scriptstyle \kappa_2} \\ G_2 & \xrightarrow{\phi_2} & G_2 \end{array} \tag{D.2}$$

commutes. Note that both G_1 and G_2 are pointed quartic graphs.

A crucial definition is the following:

Definition 213 (Consistent family of automorphisms)**.** *A family of automorphisms* $\{A(G)\}_{G \text{ p.q.g.}}$ *is* consistent *if*

- *If there exists $\iota \in \mathrm{Aut}(G)$ exchanging the two roots, then there exists $\iota' \in A(G)$ exchanging the two roots.*
- *For e_1, e_2 double-edges of G, $\mathfrak{S}_{\{e_1,e_2\}} \subseteq A_+(G)$.*
- *For a triple (G, C, P) as above, if there exists $\phi_1 \in A_+(G_1)$, then there exists $\phi \in A_+(G)$ such that C and P are compatible*[1] *with ϕ, and the diagram (D.1) commutes.*
- *For a triple (G, C, P) as above, if κ_2 and G_2 are defined, and there exists $\phi_2 \in A(G_2)$, then there exists $\phi \in A_+(G)$ such that C and P are compatible with ϕ, and the diagram (D.2) commutes.*

Please remark the subtle interplay between A and A_+ in the definition.

Definition 214. *Let G be a pointed quartic graph, and e_1, e_2 two oriented edges of G, which are isomorphic w.r.t. the action of* $\mathrm{Aut}_+(G)$*. Let* $A_+(G) \subseteq \mathrm{Aut}_+(G)$*. We say that e_1 and e_2 are* connected by A_+ *if there exists $\phi \in A_+(G)$ such that $\phi(e_1) = e_2$. We say that G is* 2-edge connected by A_+ *if all of its pairs of isomorphic oriented edges are connected by A_+.*

[1] I.e., C is an orbit of ϕ of cycle-type $\{(2,2),(2,1,1),(1,1,1,1)\}$, and P is a compatible pairing.

We shall prove the following

Theorem 215. *Let $\{A(G)\}_{G \text{ p.q.g.}}$ be a consistent family of automorphisms. Then all graphs G are 2-edge connected by A_+.*

Proof. If e and e' are double-edges, we conclude immediately from the second defining property of consistent family of automorphisms.

For the other cases, we proceed by induction on $|V(G)|$. If G consists of just the two root vertices and the edge that connects them, $\mathrm{Aut}(G)$ is the identity and there is nothing to prove. Now, assume that the theorem holds for graphs with at most $n-1$ vertices, and say that G has n vertices, and the oriented edges e and e' are isomorphic as in the definition. This implies in particular that there exists $\phi \in \mathrm{Aut}_+(G)$ such that $\phi(e) = e'$, and we shall analyse the result of the map κ_1 (and, if defined, κ_2) on some datum (G, C, P), with C an orbit of ϕ, and P compatible with ϕ.

Several situations may occur, of which only one is delicate:

- If there exists an orbit C of cycle-type $(1,1,1,1)$, all three pairings P are compatible with ϕ, and at least one of them is such that κ_1 produces a subgraph G_1 containing both e and e'.[2] In this case there must exist $\phi_1 \in A(G_1)$ such that $\phi_1(e) = e'$, and, from the third defining property of compatible family of automorphisms, we are done.

- If we have a class C in which the cycle-type is either $(2,2)$ or $(2,1,1)$, it is still possible that, by accident, the unique consistent pairing P is such that e and e' are contained in G_1, and are not concatenated by κ_1, so we can conclude as above.

- If there exists a class C of cardinality 1, and we are not in the case above (so that the cycle-type is either $(2,2)$ or $(2,1,1)$, and there is a unique viable pairing, say $P = \{(1,2),(3,4)\}$), and e and e' (or their image under κ_1) are not both in G_1, by isomorphism of e and e' we conclude that they are both contained in G_2, so we conclude by the fourth defining property of compatible family, where we use an automorphism $\phi_2 \in A_+(G_2)$ if $e_{1,3}$ and $e_{1,4}$ are not exchanged by ϕ, and an automorphism ϕ_2 in the coset of $A(G_2)$ not containing the identity otherwise.

- If none of the above apply, we need a more subtle argument, that we produce below.

Let us consider the class C, of cardinality 3^d, whose existence is implied by Lemma 212 (which applies as, at this stage of the induction, we have at least one vertex of degree 4). As we are not in one of the cases above, this means that the cycle-type of C is either $(2,2)$ or $(2,1,1)$, and that $d > 0$.

We shall prove that there exists a $\psi \in A_+(G)$ by proving that there exist automorphisms ψ_2 and ψ_1, both in $A_+(G)$, which, for what concerns the orbit of e and

[2] With some precisions: if e and e' are adjacent to the vertices of C, we refer here to the corresponding edges resulting from the application of κ_1, which are uniquely identified, except when e and e' are incident on the same vertex of C and these edges are paired, but, from the choice of three different pairing that we have in this case, it is easily seen that there exists one pairing avoiding such a situation.

e' under ϕ, are such that ψ_2 behaves as ϕ^{3^d}, and ψ_1 behaves as ϕ^2. Then, as $A_+(G)$ is a group, it suffices to consider $\psi = \psi_2(\psi^1)^{-(3^d-1)/2}$, which is in $A_+(G)$ and is such that $\psi(e) = e'$.

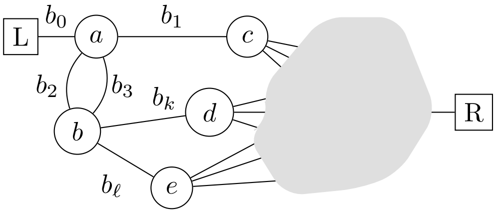
The automorphism ϕ^2 is such that C is still an orbit of order 3^d, however its cycle-type is the cycle-type of $\sigma(C, \phi^2) = \sigma(C, \phi)^2$, thus it must be $(1, 1, 1, 1)$, there exists a $\phi_1 \in A_+(G_1)$, and from the third defining property of consistent family of automorphisms we can conclude that there exists a $\psi_1 \in A(G)$, behaving as ϕ^2 for what concerns the restriction to G_1,

On the other side, ϕ^{3^d} splits C into orbits of cardinality 1. For a whatever consistent pairing P, the associated ϕ_1, and possibly also ϕ_2, exist in the corresponding groups A, depending if $\kappa_0(G)$ is connected or not. In the first case, e and e' are in G_1, but we have a $\phi_1 \in A_+(G_1)$, and we conclude immediately, by setting $\psi_2 = \phi_1$. In the second case, we have only proven the existence of an automorphism $\phi_2 \in A(G_2)$ on the graph G_2. In this case we shall have exactly 3^d components isomorphic to G_2, connected by the action of ϕ, and one component containing the roots, and e and e' are in some copies G_2 (may be the same one or two different ones). Let us set ψ_2 as the action of ϕ_2 on each copy of G_2, transported by the action of ϕ, and let ψ_2 act as the identity on the component containing the roots. Yet again we have that ψ_2 behaves as ϕ^{3^d} on all the components isomorphic to G_2, which thus contain both e and e'. This allows us to conclude. $\square$

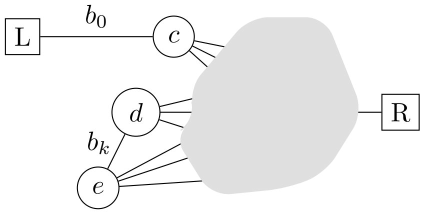
Finally, let us state a couple of rather simple facts on the automorphism of pointed quartic graphs, in presence of double-edges, which are used in the proof of the involution dynamics on the subset $\mathfrak{M}_n^{\mathrm{d}}$ of matchings, discussed in Chapter 6.

We need to establish two facts, one for the case in which there is a one-edge cut at distance 1 from L, that separates L from R, and one for when this is not the case. We start with this second case.

Let us call G the graph at size n shown in the figure below:



and G_1 the graph at size $n-2$ shown in the figure below:



Then we have

Proposition 216. *For G and G_1 defined as above,* $\mathrm{Aut}_+(G) \simeq (\mathrm{Aut}_+(G_1) \times \mathfrak{S}_{b_2,b_3})|_{\pm b_k \textit{ fixed}}$, *that is, the restriction of the group* $\mathrm{Aut}_+(G_1) \times \mathfrak{S}_{b_2,b_3}$ *to maps that* *send b_k to either b_k or its reverse, as oriented edges. For any automorphism (ϕ, σ) on the RHS, we can construct the associated automorphism ψ on the LHS, by letting ψ act as ϕ on the part of graph G common to G_1, as σ on the edges b_2 and b_3, and as the identity on the edges b_0 and b_1. Furthermore, it acts as the identity on the edges b_ℓ and b_k of G, if $\phi(b_k) = b_k$ on G_1, while it interchanges the edges b_ℓ and b_k of G, if $\phi(b_k) = -b_k$ on G_1.*

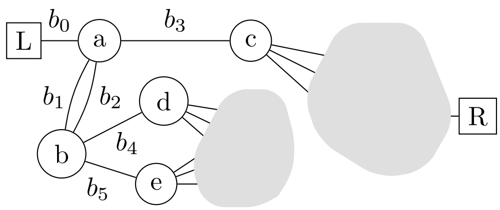
Proof. It is clear that the construction of an automorphism ψ from a pair (ϕ, σ) gives indeed an element in $Aut_+(G)$, and is injective. What is left to prove is the surjectivity. At this aim we need to investigate the neighbourhood of the vertex L, in order to determine that:

- a must be a fixed vertex, because it is the only one adjacent to L.
- b_3 must be a fixed oriented edge, because it is the only one incident on a, which does not connect to L and is not a double-edge.
- Similarly, the subgraph consisting of b_1 and b_2 must be left fixed by any automorphism.
- as a result of this, also the vertex b must be fixed.

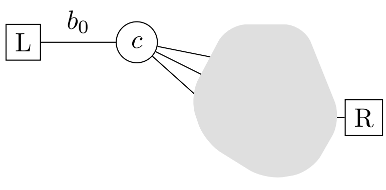
Thus, the conditions for ψ being an automorphism, namely of preserving the adjacency structure on G, consist exactly of the conditions on the adjacency structure on G_1, plus the forementioned list of conditions for the neighbourhood of the vertex L. As this coincides with the explicit construction on pairs (ϕ, σ), we have proven surjectivity. ☐

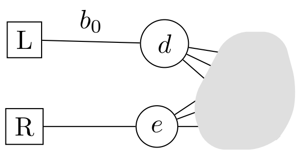
Let us continue with the case in which we have an edge-cut.

Let us call G the graph at size n shown in the figure below:



and G_1 and G_2 the graphs at size m and $n - m - 2$ shown (from left to right) in the figure below:





Then we have

Proposition 217. *For G, G_1 and G_2 defined as above,* $\mathrm{Aut}_+(G) \simeq \mathrm{Aut}_+(G_1) \times \mathrm{Aut}(G_2) \times \mathfrak{S}_{b_2,b_3}$. *For any automorphism (ϕ_1, ϕ_2, σ) on the RHS, we can construct the associated automorphism ψ on the LHS, by letting ψ act as ϕ_1 and ϕ_2 on the* *part of graph G common to G_1 and G_2, respectively, as σ on the edges b_2 and b_1, and as the identity on the edges b_0 and b_3.*

Furthermore, it acts as the identity on the edges b_ℓ and b_k of G, if ϕ_2 is in the coset of $\mathrm{Aut}(G_2)$ that sends L to L and R to R, while it interchanges the edges b_ℓ and b_k of G, if ϕ_2 is in the coset that interchanges L and R.

The proof is similar to that of the previous proposition.

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