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Inhomogeneous Relativistic Cosmology : nonperturbative models and spatial averaging of the Einstein equations

Pierre Mourier

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**Cosmologie inhomogène relativiste :
modèles non perturbatifs et moyennes
spatiales des équations d'Einstein**

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*From my point of view one cannot arrive, by way of theory,
at any at least somewhat reliable results in the field of cosmology,
if one makes no use of the principle of general relativity.*

Albert Einstein, 1949

Abstract

In the standard model of cosmology, the global dynamics of the Universe is modelled *via* a highly symmetric background spacetime with homogeneous and isotropic spatial sections. The coupling of the homogeneous fluid sources to the overall expansion is then determined by the Einstein equations of General Relativity. In addition, the formation of inhomogeneous matter structures is described either *via* a relativistic perturbation scheme assuming small deviations of all fields to the prescribed homogeneous background, or using Newtonian dynamics within the same expanding background, depending on the scale and epoch. However, the interpretation of observations within this model calls for an unexpectedly accelerated expansion requiring a poorly-understood ‘Dark Energy’ component, in addition to Dark Matter.

Inhomogeneous cosmology aims at relaxing the restrictions of these models on the geometry and sources while staying within the framework of General Relativity. It can allow, in particular, for an improved modelling of the formation of structures accounting for strong deviations from homogeneity in the matter distribution and the geometry. It can also study the dynamical consequences, or *backreaction* effects, of the development of such inhomogeneities on the expansion of larger scales. Such a backreaction may then reproduce, at least partially, the behaviours attributed to Dark Energy or Dark Matter.

During my PhD under the direction of Thomas Buchert, I have been working on several analytical aspects of general-relativistic inhomogeneous cosmology. I present below the results of collaborations in which I played a major role in the context of the PhD. I first focussed on the expression of a relativistic Lagrangian approximation scheme for the description of the local dynamics of structures up to a nonlinear regime in irrotational perfect barotropic fluids. I then considered the effective description of inhomogeneous fluids with vorticity and a general energy-momentum tensor in terms of two possible schemes of spatial averaging. These schemes are applicable to any choice of spatial hypersurfaces of averaging, providing for each choice a set of effective evolution equations, featuring several backreaction terms, for an averaging region comoving with the sources. This allows for a qualitative discussion of the dependence of the average equations and backreactions on the foliation choice. I also studied the rewriting of such averaging schemes and evolution equations under a unified and manifestly 4-covariant form. This latter result will allow for a more explicit investigation of foliation dependence.

Résumé

Le modèle standard de la cosmologie décrit l'expansion de l'Univers au moyen d'un espace-temps idéalisé. Celui-ci est supposé admettre des sections spatiales dont la géométrie et le contenu énergétique, couplés par les équations d'Einstein de la Relativité Générale, sont strictement homogènes et isotropes. Cette hypothèse de forte symétrie contraint la dynamique d'un tel modèle, qui peut alors être décrite entièrement par une fonction, un facteur d'échelle dépendant seulement du temps. Cette fonction est obtenue comme solution d'une équation différentielle ordinaire, dépendant d'un nombre restreint de paramètres à contraindre par l'observation. La description dans ce cadre des phénomènes physiques se produisant à différentes époques au cours de l'expansion s'en trouve considérablement simplifiée. Cela s'applique par exemple à la formation des noyaux atomiques simples, ou à l'émergence des galaxies et à leur organisation en structures de plus grande échelle.

Toutefois, pour rester compatible avec les observations, ce modèle requiert deux composantes de nature encore indéterminée, la Matière Noire et l'Énergie Noire. Toujours dans ce modèle, cette dernière provoquerait l'accélération de l'expansion, et pourrait correspondre à une composante énergétique exotique issue d'un champ (souvent scalaire) fondamental ou simplement être modélisée par une constante cosmologique. Dans les deux cas, se pose le problème de la surprenante coïncidence du début de la phase accélérée avec l'époque de la formation des structures de matière à grande échelle.

Le modèle standard de la cosmologie inclut une description de la mise en place et de l'évolution de ces structures. Selon l'échelle et l'époque considérées, elles sont modélisées soit par un schéma perturbatif relativiste avec de faibles déviations de toutes les grandeurs par rapport au modèle de fond homogène et isotrope, soit par une dynamique newtonienne au sein de ce même fond en expansion. Dans les deux cas, l'expansion aux plus grandes échelles est prescrite *a priori*, *via* ce fond, et ne peut être affectée par le développement d'inhomogénéités dans la distribution de matière. Pourtant, selon les principes de la Relativité Générale, ces inhomogénéités dans les sources doivent affecter la géométrie locale. La déformation associée peut être importante dans les régions très denses, ou à l'inverse particulièrement vides, apparaissant dans l'Univers tardif, pour lesquelles une description en termes de faibles fluctuations par rapport à une densité homogène est insuffisante. Ce changement dynamique dans la géométrie peut à son tour affecter l'expansion, par un phénomène de rétroaction (« *backreaction* »), contribuant éventuellement aux effets imputés à l'Énergie Noire ou à la Matière Noire.

Mon travail de thèse sous la direction de Thomas Buchert s'inscrit dans le cadre de la cosmologie inhomogène et relativiste, qui vise à décrire la formation de structures et les effets de rétroaction au sein de la Relativité Générale. J'ai ainsi été amené à contribuer de façon majeure à l'approfondissement et à l'extension de deux approches précédemment développées dans ce cadre pour la modélisation analytique de ces phénomènes.

La première est un schéma de perturbation relativiste lagrangien sur des champs de déformation de la métrique, développé dans le cadre d’un modèle de fluide sans pression ni vorticité [Buchert and Ostermann, 2012, Buchert et al., 2013, Alles et al., 2015, Al Roumi et al., 2017]. Il permet de décrire la formation des structures au-delà d’un régime linéaire grâce à l’évaluation non-perturbative de la densité, de la courbure et d’autres observables à partir du champ de déformation approché, selon un principe similaire à l’approximation de Zel’dovich pour la gravitation newtonienne. La seconde stratégie, développée pour des fluides parfaits irrotationnels avec ou sans pression [Buchert, 2000, 2001], consiste en l’écriture de moyennes spatiales pour des scalaires inhomogènes sur des domaines compacts d’espace qui suivent la propagation du fluide source. Ceci permet l’écriture d’équations-bilans sur la dynamique effective de tels domaines, pouvant être comparées aux équations de Friedmann qui seraient obtenues pour un fluide homogène. Les différences font apparaître explicitement des termes de rétroaction (cinématique, dynamique, ou de courbure). Cette analyse peut être appliquée à la première approche qui fournit alors un modèle d’évolution locale. Dans les deux cas, le formalisme 3+1 de la Relativité Générale décrivant l’espace-temps au moyen de sections spatiales évoluant dynamiquement est utilisé, avec le choix spécifique (permis par l’absence de vorticité) de sections spatiales orthogonales à la quadrivitesse du fluide en tout point.

Je décris ces deux approches plus en détail dans l’introduction ci-dessous (chapitre 1), après une description du formalisme 3+1 utilisé dans l’ensemble de cette thèse et, pour comparaison, un rappel détaillé du modèle de fond et du schéma de perturbations relativiste du modèle cosmologique standard. Les résultats de plusieurs travaux effectués au sein de ma thèse dans le cadre de plusieurs collaborations, et dont je suis l’un des principaux auteurs, font l’objet des chapitres suivants. Ces travaux sont basés sur les deux approches inhomogènes rappelées ci-dessus.

Je présente ainsi au chapitre 2 un travail conjoint avec Yongzhuang Li et David Wiltshire de l’University of Canterbury (Nouvelle-Zélande) et Thomas Buchert. Nous y montrons comment un schéma de perturbation lagrangien, inspiré de celui adapté aux fluides sans pression, peut encore être développé pour des fluides parfaits barotropes irrotationnels. Comme pour les fluides sans pression, ce schéma se base sur une réécriture des équations d’Einstein (sous forme 3+1) au moyen d’une base de 1-formes spatiales (ou *coframes*), qui sont ensuite perturbées au premier ordre par rapport à un fond homogène et isotrope. Une emphase particulière est mise sur la trace de la perturbation : nous écrivons son équation d’évolution linéarisée, examinons les limites newtonienne et à pression nulle, et donnons une solution analytique dans le cas d’une relation barotrope linéaire, applicable par exemple à un fluide radiatif. Nous étudions également les contraintes sur les conditions initiales ainsi que la dynamique de la partie sans trace. Une décomposition de cette dernière en parties électrique et magnétique est déterminée et peut permettre une modélisation d’ondes gravitationnelles se propageant dans le fluide. Enfin, nous illustrons sur un exemple simple l’utilisation de l’ensemble des composantes de la solution linéarisée sur les *coframes* en vue d’une évaluation non-perturbative de la densité au cours de la formation d’une structure, pour laquelle les gradients de pression sont supposés modéliser une dispersion

de vitesse.

Au chapitre 3, je décris les résultats de deux études conduites par Xavier Roy, Thomas Buchert et moi-même, l'une étant en cours de finalisation et l'autre publiée sous forme de lettre, afin de généraliser le schéma de moyennes spatiales de [Buchert, 2000, 2001]. Nous y présentons deux approches possibles pour moyenniser spatialement des scalaires sur un domaine compact d'espace suivant, là encore, le flot du fluide source des équations d'Einstein, applicables à un feuilletage quelconque de l'espace-temps en hypersurfaces spatiales. La source est cette fois un fluide quelconque, *a priori* non parfait et pouvant présenter de la vorticit . Le premier sch ma propos  se base sur l' l ment de volume riemannien des hypersurfaces et permet d'examiner l' volution de la g om trie de ces derni res. Nous le comparons   la litt rature, o  un sch ma similaire a  t  propos  plusieurs fois mais jamais appliqu    des domaines pr servant leur contenu de fluide au cours du temps. Le second sch ma, utilisant le volume propre des  l ments de fluide et se basant sur les variables cin matiques du fluide, permet une analyse plus intrins que de la dynamique de ce dernier (nous le qualifions donc d'« intrins que »). Nous obtenons pour ces deux sch mas les  quations d' volution moyennes pour le facteur d' chelle effectif (estimant la taille du domaine en fonction du temps) et l'expression des termes de r troaction associ s. Pour le sch ma intrins que en particulier, sur lequel nos deux  tudes insistent, nous pr sentons  galement des formes effectives plus compactes, quoique toujours aussi g n rales, de ces  quations. Nous mettons en exergue le concept d'un feuilletage par hypersurfaces   temps propre constant pour le fluide et la description lagrangienne associ e, et montrons comment une telle description simplifie encore davantage les  quations moyennes obtenues avec le sch ma intrins que. Un aspect de notre lettre est  galement d di    une discussion qualitative de l'influence d'un changement de feuilletage spatial sur les moyennes et les termes de r troaction.

J'expose ensuite au chapitre 4 le travail que j'ai effectu  en collaboration avec Asta Heinesen de l'University of Canterbury et Thomas Buchert, sur une r criture sous forme manifestement covariante de diff rents sch mas de moyenne spatiale scalaire. En nous inspirant de [Gasperini et al., 2009, 2010], nous  crivons une telle forme pour des moyennes gr ce   une fonction fen tre qui s lectionne les sections spatiales et le domaine compact sur lequel moyenniser. En l' crivant sous une forme g n ralis e, cette fonction nous permet  galement de s lectionner la mesure de volume, pour inclure notamment les deux sch mas de moyenne du chapitre 3 sous une forme unifi e, mais  galement des moyennes pond r es. Nous  crivons les  quations d' volution moyenn es sur un domaine et les termes de r troaction obtenus avec un repr sentant quelconque de la classe de fonctions fen tres consid r e. Nous retrouvons en particulier, sous une forme manifestement covariante, les r sultats du chapitre 3 pour le sch ma de moyenne intrins que et pour un domaine suivant la propagation du fluide.

Enfin, le chapitre 5 dresse un bilan des diff rents r sultats d taill s dans les chapitres pr c dents. J'y esquisse  galement les projets en cours et des perspectives de d veloppements futurs pour un approfondissement des r sultats pr sent s dans cette th se.

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Foreword

Cosmology is the investigation of the shape and dynamical behaviour of the Universe on the largest scales, in particular the observed overall expansion and its time dependence. It also studies the main energy-momentum sources in the Universe through time and their coupling to the global dynamics, as well as the progressive mutual decoupling of particles and the formation of the first atomic nuclei in the early Universe. As for the late Universe, another major focus of cosmology is the description of the structures in the large-scale distribution of matter (the so-called *Cosmic Web*) and their formation and dynamics in interaction with the global expansion.

The first approaches to a full mathematical cosmological model arose a century ago with General Relativity, which describes the coupling of spacetime geometry and sources, hence allowing a global study of the dynamics of the Universe depending on its energy contents. These first studies (*e.g.*, [de Sitter \[1917\]](#), [Einstein \[1917\]](#), [Friedmann \[1922\]](#), [Lemaître \[1927\]](#); see, *e.g.*, [Peebles \[1971\]](#), p.3-14 for a short review) were based on an idealized geometry assuming strict homogeneity and isotropy of space, a class of models which is still of major use in today's cosmology as a reference 'background' to the standard model of cosmology (see section 1.2 in the next chapter).

The picture of an eternal Universe that would be globally static on its largest scales, possibly limited to our own Galaxy, was rather widespread at the time and led Einstein to design a static model with a positive cosmological constant Λ counterweighting gravitational attraction. This conception was later disfavoured in view of the observation of other galaxies and of the discovery of their increasing distances to Earth indicated by their redshift in observations by Slipher (reviewed by Eddington in [\[Eddington, 1923\]](#)). This showed the expansion of the observable Universe, as analysed by Lemaître [\[Lemaître, 1927\]](#), and later by Hubble [\[Hubble, 1929\]](#), and led to the preference for expanding homogeneous models without an unnecessary cosmological constant.

The sources of such models were radiation, ordinary matter, and since the late 1930s Dark Matter as an additional matter source, dominant in mass proportions and behaving as baryons with respect to gravitation but seemingly decoupled from other interactions, deduced from observed motions within galaxies and galaxy clusters. The dynamics of such cosmological models (with or without Λ) together with the observation of a nearly blackbody and isotropic radiative background suggesting a very dense early Universe with coupled radiation and matter releasing

this radiation as it expands, established the idea that the Universe had a well-defined start in terms of an initial singularity nicknamed ‘Big Bang’.

In the 1980s and 1990s observations and predictions of the matter density and age of the Universe from homogeneous models without a cosmological constant started to appear incompatible (see, *e.g.*, [Efstathiou et al. \[1990\]](#) and references therein). The refined observational study of distant objects redshifts *via* supernovae in 1998 [[Riess et al., 1998](#), [Perlmutter et al., 1999](#)] within the assumption of homogeneous models indicated that only such models with an accelerated expansion could be compatible with observations. This required an energy source opposing gravitational collapse of more ordinary matter, which was named *Dark Energy* and has appeared to be compatible with the re-introduction of a positive cosmological constant in addition to the other sources. This raised, however, the so-called *Coincidence Problem*. The value of Λ as a parameter of General Relativity is fully unconstrained *a priori*, and it cannot be simply predicted, *e.g.*, as a vacuum energy from Quantum Field Theory as attempts in doing so famously lead to a prediction exceeding the observed value by 120 orders of magnitude (see, *e.g.*, [Carroll \[2001\]](#) and references therein). Thus, why should the energy associated to Λ be of the same order of magnitude (about 2/3 of the energy balance) as that of Dark and ordinary Matter today?

The importance of the contribution of a cosmological constant, or even the presence of acceleration, do depend on the cosmological model considered. Hence, alternative or complementary approaches to the cosmological constant (a nonzero Λ can still be included in most of these models) regarding the apparent acceleration of the expansion in the most widely-used models have also been suggested, sometimes simultaneously addressing the nature of Dark Matter or the gravitational dynamics attributed to it. This includes, for instance, models of Dark Energy as a fundamental ‘quintessence’ scalar field source (see, *e.g.*, [Wetterich \[1988\]](#), [Ratra and Peebles \[1988\]](#) and references therein), or departures from General Relativity usually modelled as modifications to the Einstein-Hilbert action and consequently to the Einstein equations (see for instance [Carroll et al. \[2005\]](#), [Nojiri and Odintsov \[2005\]](#), [Joyce et al. \[2016\]](#) and references therein).

Another possible approach, suggested by Ellis [[Ellis, 1984](#)], is to generalize the class of cosmological models considered, within General Relativity and without additional fundamental sources, to account for large local departures from homogeneity implied by the gravitational clustering of matter in the late Universe. A hint towards the relevance of this clustering for the large-scale dynamics could be seen in its approximate coincidence with the beginning of the accelerating phase of the expansion as deduced from homogeneous models [[Roukema et al., 2013](#)], another possible formulation of the Coincidence Problem.

In the standard model of cosmology, the formation of these large-scale matter structures (galaxy clusters, filaments, and walls, separated by voids) is usually modelled either as Newtonian self-gravitating fluid dynamics or as relativistic linear perturbations, in both cases over a given homogeneous expanding background model. Inhomogeneous cosmology aims at a relativistic description of the dynamics of these structures without too restrictive assumptions regarding

deviations from homogeneity, and ideally without a prescribed background. This allows in principle for a more realistic and more accurate modelling of structure formation within an expanding Universe.

Conversely, it can account as well for the consequences or *backreaction* effects of the development of such inhomogeneities onto the large-scale expansion, some of which possibly corresponding to the effects expected from Dark Energy [Buchert, 2000, 2008, Buchert and Räsänen, 2012]. Modifications to the dynamics of smaller-scale regions due to highly nonlinear structure formation within them can also be investigated, and may contribute in part to the behaviour attributed to Dark Matter. The different dynamics induced by large inhomogeneities may also help explaining some of the observational disagreements or ‘tensions’ with respect to the standard model of cosmology, most notably the discrepancy between values of the present-time expansion rate deduced within this model from local or from early Universe-based observations [Riess et al., 2016, 2019] (see Buchert et al. [2016] for a review of other such tensions).

During my PhD under the direction of Thomas Buchert, I have been working within this inhomogeneous relativistic approach to cosmology. I have been mainly focussing in particular on the description of the effective dynamics of regions of inhomogeneous space in terms of spatial averages and resulting backreaction terms as introduced in [Buchert, 2000, 2001], by investigating the extension of such schemes to general sources and general definitions of ‘space’ as a three-dimensional section of spacetime, and manifestly 4-covariant formulations of such approaches. As another important application of inhomogeneous cosmology, I also contributed to the development of a Lagrangian relativistic scheme for the approximate description of a mildly nonlinear regime of structure formation for perfect fluids with nonvanishing pressure and pressure gradients, building upon previous analyses for pressureless fluids [Buchert and Ostermann, 2012, Buchert et al., 2013, Alles et al., 2015, Al Roumi et al., 2017]. I worked on these projects as part of several collaborations. This led to several papers in joint first authorship and more minor contributions to other projects [Roukema, Mourier, Buchert, and Ostrowski, 2017a, Roukema, Ostrowski, Buchert, and Mourier, 2017b]; I will present only the former results here.

This thesis is structured as follows. In the introductory chapter 1, I recall the 3+1 (space + time) formalism of General Relativity used throughout this thesis as a key framework for relativistic cosmologies (section 1.1) and the main assumptions of the standard model of cosmology in terms of a homogeneous background model and perturbations thereof (sections 1.2 and 1.3). I also subsequently describe inhomogeneous cosmological frameworks of major interest for this thesis: the relativistic Zel’dovich approximation (section 1.4) and the spatial averaging procedures and associated backreactions for inhomogeneous irrotational fluids seen in their rest frames (section 1.5). I present in chapter 2 an extension of the relativistic Zel’dovich approximation to fluids with pressure. I then introduce in chapter 3 proposals for averaging schemes for arbitrary fluid sources within general three-dimensional spatial slices, together with a comparison with suggestions in the literature in this direction. In chapter 4 I show how such schemes can be written under a manifestly covariant form and further extended, before concluding in chapter 5.

Remarks on conventions and notations

The spacetime metric signature convention adopted here is $(-+++)$.

Units are taken such that the vacuum speed of light c is 1, but c will occasionally be recovered when insightful. The gravitational constant G is not adimensioned.

The exterior derivative for forms is denoted as d , and their exterior product as \wedge .

A coordinate system is often used; it is then noted (x^μ) and is associated to the coordinate vector basis (∂_{x^μ}) (also written (∂_μ)) and its dual exact 1-form basis (dx^μ) . Components of tensorial objects are implicitly expressed in these bases and at the point of coordinates x^μ , unless otherwise stated. Tensors, vectors or forms are usually written in bold font, with their components using the same symbol in plain font with appropriate indices; *e.g.*, the 4-velocity vector field is written as $\mathbf{u} = u^\mu \partial_\mu$. For vectors noted in this way, the associated metric-dual 1-form is underlined, *e.g.* for \mathbf{u} , $\underline{\mathbf{u}} = u_\mu dx^\mu$. In the Newtonian framework briefly used in section 1.4, three-dimensional vectors are rather topped with an arrow, *e.g.*, \vec{g} for the gravitational field, of components g^i .

Greek letters denote spacetime indices running in $\{0, 1, 2, 3\}$, while Latin letters stand for spatial indices, running in $\{1, 2, 3\}$. The time coordinate x^0 will be rather noted t , and accordingly the vector ∂_0 and 1-form dx^0 are rather written ∂_t and dt , respectively.

Partial derivatives with respect to the coordinate x^i are noted ∂_i , or ∂_{x^i} when there is a risk of ambiguity with another spatial coordinate set. Similarly, partial derivatives with respect to t in the coordinate set (t, x^i) are noted either ∂_t , or $\partial_t|_{x^i}$ when necessary.

The symbol \equiv or $:=$ expresses an equality by definition, and \propto indicates a proportionality.

Chapter 1

Introduction

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1.1 Spatial foliations and 3+1 picture of General Relativity

General Relativity describes the local coupling between the geometry of spacetime and energy sources. Particles subjected to gravitation only will follow geodesics in spacetime, and the theory thus fulfils its requirement of satisfying the *equivalence principle*: the trajectories of massive particles under the above conditions are independent from their mass.

In this theory, spacetime is modelled as a four-dimensional manifold \mathcal{M} endowed with a Lorentzian metric tensor \mathbf{g} , *i.e.*, a symmetric tensor field associated with a non-degenerate quadratic form of signature $(-, +, +, +)$ on each tangent space. In a chart, with a given coordinate basis (x^μ) with associated basis vectors ∂_{x^μ} and coordinate 1-forms dx^μ , the components of \mathbf{g} will be written as $g_{\mu\nu}$: $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$, and those of its inverse will be written as $g^{\mu\nu}$.

Still as part of the general-relativistic framework, \mathcal{M} is moreover equipped with the parallel transport defined from the Levi-Civita connection of \mathbf{g} , *i.e.*, there is no torsion and parallel transport preserves the metric. The corresponding covariant derivative operator ∇ is thus commutative for scalars and vanishes for \mathbf{g} : in coordinate components, $\nabla_{[\mu}\nabla_{\nu]}f = 0$ for any scalar f , and $\nabla_\rho g_{\mu\nu} = 0$. The components of the covariant derivative of a vector $\mathbf{A} = A^\mu \partial_{x^\mu}$ or of its metric-dual 1-form $A_\mu dx^\mu$, $A_\mu \equiv g_{\mu\nu} A^\nu$, $A^\mu = g^{\mu\nu} A_\nu$, read for instance

$$\nabla_\mu A^\nu = \partial_{x^\mu} A^\nu + \Gamma_{\mu\rho}^\nu A^\rho \quad ; \quad \nabla_\mu A_\nu = \partial_{x^\mu} A_\nu - \Gamma_{\mu\nu}^\rho A_\rho , \quad (1.1)$$

respectively, with the Levi-Civita connection coefficients or Christoffel symbols

$$\Gamma_{\mu\nu}^\rho \equiv \frac{1}{2} g^{\rho\sigma} (\partial_{x^\mu} g_{\nu\sigma} + \partial_{x^\nu} g_{\mu\sigma} - \partial_{x^\sigma} g_{\mu\nu}) \quad : \quad \Gamma_{[\mu\nu]}^\rho = 0 . \quad (1.2)$$

Commutation of covariant derivatives for vectors defines the Riemann tensor, of components $R^\mu{}_{\nu\rho\sigma}$, as a measure of spacetime curvature: for any vector $\mathbf{A} = A^\mu \partial_{x^\mu}$, following the sign conventions of Misner, Thorne, and Wheeler [1973],

$$\nabla_\mu \nabla_\nu A^\rho - \nabla_\nu \nabla_\mu A^\rho \equiv R^\rho{}_{\sigma\mu\nu} A^\sigma . \quad (1.3)$$

From it are also defined the (symmetric) Ricci tensor $R_{\mu\nu} dx^\mu \otimes dx^\nu$ and the Ricci scalar R by successive contractions, $R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}$, $R \equiv R^\mu{}_\mu$. A Levi-Civita connection and associated covariant derivatives, Riemann and Ricci tensors and Ricci scalar can similarly be defined in any dimension and for a Riemannian (positive-definite) metric, as will be used later for the Riemannian metrics induced by \mathbf{g} on three-dimensional spacelike sections of \mathcal{M} .

The coupling between spacetime geometry (metric and curvature) and the stress-energy tensor of the sources $T_{\mu\nu} dx^\mu \otimes dx^\nu$ in General Relativity is described by the *Einstein equation*:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} , \quad (1.4)$$

in components, where G is Newton's gravitational constant. This general form of the Einstein equation includes the constant parameter Λ known as the *cosmological constant*. The symmetric tensor of components $G_{\mu\nu} \equiv R_{\mu\nu} - (R/2) g_{\mu\nu}$ is the *Einstein tensor* and obeys a conservation equation: $\nabla_\mu G^{\mu\nu} = 0$. The Einstein equation thus automatically ensures the conservation of the (also symmetric) energy-momentum tensor appearing in its right-hand side: $\nabla_\mu T^{\mu\nu} = 0$.

The Einstein equation can be obtained from a variational principle, with the Einstein-Hilbert

action for the geometric part, $S_{EH} = (16\pi G)^{-1} \int_{\mathcal{M}} (R - 2\Lambda) \sqrt{g} d^4x$ (with the determinant $g \equiv |\det(g_{\mu\nu})|$), added to the action S_M for the matter sources described as fundamental fields. The energy-momentum tensor is then defined from the variation of S_M with respect to the metric tensor components. It may alternatively be defined under a phenomenological form in a fluid approximation for the sources. Although this approximation may not be fully accounting for the general-relativistic geometry at the scale of the fluid elements (which may themselves be somewhat ill-defined in a cosmological context) [Wiltshire, 2011, Coley and Wiltshire, 2017], it offers a tractable framework of wide use for the modelling of the late Universe and it will be conservatively adopted in the following.

Building a spatial foliation

A cosmological model is usually described in terms of a split of spacetime into a three dimensional space (which models the Universe at a given time) and the time evolution of its properties. While such a picture is *a priori* especially suited for a Newtonian framework with absolute and global space and time notions, this is still compatible with General Relativity. Selecting a set of three-dimensional ‘space’ slices parametrized by ‘time’ is still possible in this case, although the choice of such a set is far from unique, in contrast to the Newtonian case.

The description of the spacetime manifold and of the Einstein equations in terms of such a split corresponds to the 3+1 picture of General Relativity. This picture gives a natural view of the Einstein equations as evolving physical observables with time from given, constrained initial data, and is at the core of the investigations of existence and uniqueness of solutions to these equations as a Cauchy problem (most notably Choquet-Bruhat and Geroch [1969]). I will now recall an adapted framework for this picture (as detailed, *e.g.*, in [Misner, Thorne, and Wheeler, 1973], p.505-532; [Gourgoulhon, 2012], p.78-88; [Alcubierre, 2008], p.64-75) which is part of the ADM formalism [Arnowitt, Deser, and Misner, 1962] (the latter also aiming at a Hamiltonian formulation), and which is used throughout the present thesis¹.

On top of the general-relativistic assumptions, the spacetime manifold \mathcal{M} will be assumed to be globally hyperbolic, allowing for a foliation into Cauchy surfaces [Bernal and Sánchez, 2003], which can be used as the spatial slices. When described in a coordinate chart, the four coordinates (x^μ) will be split into the time coordinate $x^0 = t$ and the remaining, spatial coordinates x^i .

A foliation of \mathcal{M} into a family of spacelike hypersurfaces Σ_i , $\mathcal{M} = \cup_{i \in \mathbb{R}} \Sigma_i$, $\Sigma_i \cap \Sigma_j = \emptyset$ for $j \neq i$, can be chosen among all possible such foliations, denoting by \mathbf{n} their timelike, future-oriented, unit normal 4-vector. Each hypersurface corresponds to a three-dimensional ‘space’ slice, and can be seen at each point as the local rest frame of some arbitrarily chosen set of reference observers, with 4-velocity \mathbf{n} . The foliation can be characterized by a regular scalar function S strictly increasing along any future-pointing timelike line, and defined such that each spatial hypersurface is a level set of S . The dual 1-form $\underline{\mathbf{n}}$ to \mathbf{n} will thus be proportional

¹The remaining of this section is inspired by the introduction of the 3+1 formalism in the paper in preparation (subsections 2.1 and 3.1 therein) presented in chapter 3, developing it in more detail.

to the gradient of S : $\underline{\mathbf{n}} = -\alpha dS$ for some strictly positive scalar function α and with d the exterior derivative. Hence $d\underline{\mathbf{n}} = -\alpha^{-1} d\alpha \wedge \underline{\mathbf{n}}$, showing that \mathbf{n} has to be irrotational, which is equivalent to $\underline{\mathbf{n}} \wedge d\underline{\mathbf{n}} = 0$. For simplicity, the time coordinate t will be chosen as being a strictly increasing function of S (implying the reciprocal relation $S = S(t)$), and will be used to label the hypersurfaces. The spatial coordinates x^i , on the other hand, are kept arbitrary.

In such a spacetime coordinate basis, the components of \mathbf{n} and its dual $\underline{\mathbf{n}}$ are written, respectively:

$$n^\mu = \frac{1}{N} (1, -N^i) \quad ; \quad n_\mu = -N (1, 0, 0, 0) . \quad (1.5)$$

The strictly positive lapse function N determines the distance between consecutive slices at each point per unit of coordinate time. It also measures, through its spatial variations, the covariant acceleration $\mathbf{a}^{(n)}$ of components $a_\mu^{(n)} \equiv n^\nu \nabla_\nu n_\mu$ of the observers with 4-velocity \mathbf{n} (Gourgoulhon [2012], p.62; Alcubierre [2008], p.122):

$$a_i^{(n)} = \frac{\partial_{x^i} N}{N} \quad ; \quad a_0^{(n)} = N^i a_i^{(n)} , \quad (1.6)$$

where the second equation arises from the orthogonality of \mathbf{n} and its acceleration, $n^\mu a_\mu^{(n)} = 0$, itself a consequence of the unitarity of \mathbf{n} . The shift vector $\mathbf{N} = N^i \partial_{x^i}$ generates a spatial diffeomorphism that relates points in consecutive slices. Following the usual conventions in the literature cited hereabove, this lapse will from now on be associated to the coordinate functions defining the propagation of the local spatial coordinates between slices. In coordinate components, it describes the propagation of the local spatial coordinates between slices. By definition lapse and shift relate to \mathbf{n} and the time coordinate vector ∂_t as $\partial_t = N\mathbf{n} + \mathbf{N}$ (see Figure 1.1 for an illustration of the foliation and these vectors).

Specifying the lapse as a function, or from an evolution equation, selects the foliation and the time function used to label the slices. The shift characterizes a spacetime flow crossing each hypersurface exactly once, that can be (and is, in the present formalism) used to propagate the spatial coordinates.

3+1 split of the Einstein equations

Spacetime tensors are projected onto the hypersurfaces of the foliation by means of the \mathbf{n} -orthogonal spatial projection operator $h^\mu{}_\nu \partial_{x^\mu} \otimes dx^\nu$,

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu , \quad h^\mu{}_\nu n^\nu = 0 , \quad h^\mu{}_\sigma h^\sigma{}_\nu = h^\mu{}_\nu , \quad h^{\mu\nu} h_{\mu\nu} = 3 . \quad (1.7)$$

The restriction of the associated type-(0,2) tensor to the spatial slices defines the spatial Riemannian metric $\mathbf{h} \equiv h_{ij} dx^i \otimes dx^j$, with inverse $h^{ij} \partial_{x^i} \otimes \partial_{x^j}$. Given this operator and the normal

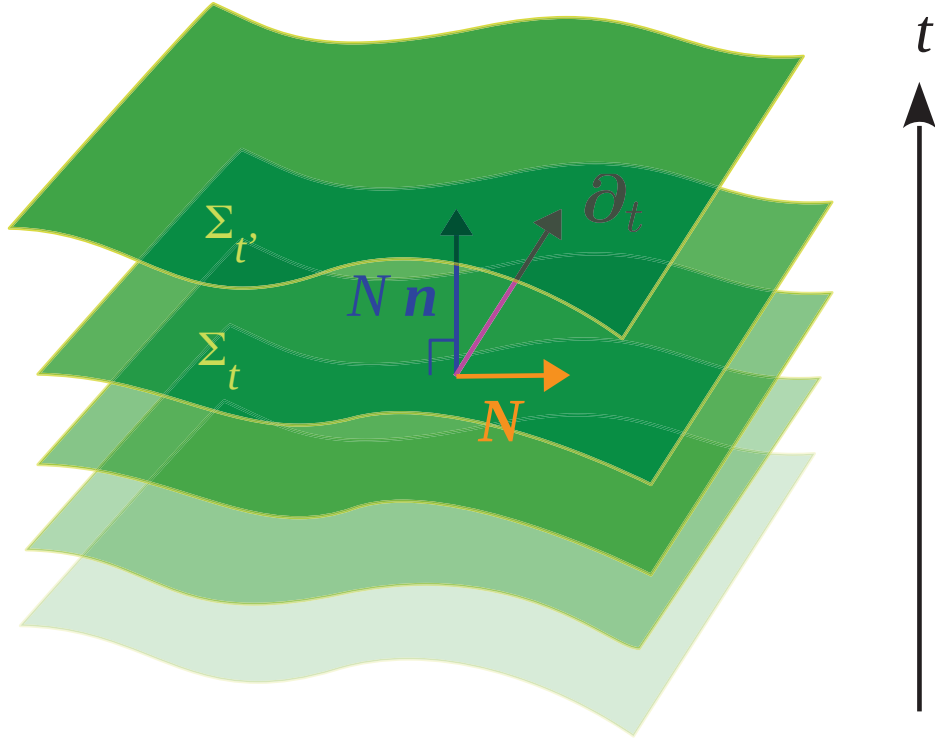


Figure 1.1: Diagrammatic representation of a foliation of a spacetime into spatial hypersurfaces Σ_t labelled by the time parameter t (here in 2 + 1 dimensions), and of the relation between the lapse N , the normal vector \mathbf{n} to the hypersurfaces, the tilt vector \mathbf{N} , and the time coordinate vector ∂_t .

vector \mathbf{n} , the four-dimensional line element can be decomposed into

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = - \left(N^2 - N^k N_k \right) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j , \quad (1.8)$$

with $N_i = h_{ij} N^j$.

The spatial metric \mathbf{h} characterizes the intrinsic geometry of the spatial hypersurfaces, with the (spatial) Ricci tensor $\mathcal{R}_{ij} dx^i \otimes dx^j$ of \mathbf{h} as their *intrinsic curvature*. In turn, the embedding of the slices within spacetime is characterized by their *extrinsic curvature*, with components $\mathcal{K}_{\mu\nu} \equiv -h^\rho{}_\mu h^\sigma{}_\nu \nabla_\rho n_\sigma$. From the orthogonality to \mathbf{n} in both indices by construction, the components of $\mathcal{K}_{\mu\nu}$ are fully determined by the spatial components \mathcal{K}_{ij} . The intrinsic and extrinsic curvatures

are related through decompositions of the spacetime Ricci tensor ([Gourgoulhon \[2012\]](#), p.51-53):

$$R + 2n^\mu n^\nu R_{\mu\nu} = \mathcal{R} + \mathcal{K}^2 - \mathcal{K}_{ij} \mathcal{K}^{ij} ; \quad (1.9)$$

$$h^\mu{}_{i} n^\nu R_{\mu\nu} = D_i \mathcal{K} - D_j \mathcal{K}^j{}_i ; \quad (1.10)$$

$$\begin{aligned} h^\mu{}_{i} h^\nu{}_{j} R_{\mu\nu} = & -\frac{1}{N} \partial_t|_{x^k} \mathcal{K}_{ij} + \frac{1}{N} N^k D_k \mathcal{K}_{ij} + \frac{1}{N} (\mathcal{K}_{ik} D_j N^k + \mathcal{K}_{kj} D_i N^k) \\ & - \frac{1}{N} D_i D_j N + \mathcal{R}_{ij} + \mathcal{K} \mathcal{K}_{ij} - 2 \mathcal{K}_{ik} \mathcal{K}^k{}_j , \end{aligned} \quad (1.11)$$

where D denotes the three-covariant derivative associated with the spatial metric h_{ij} and its Levi-Civita connection. The scalar intrinsic curvature \mathcal{R} (3-Ricci scalar) and scalar extrinsic curvature \mathcal{K} of the slices are given by the respective traces of the intrinsic and extrinsic curvatures, $\mathcal{R} \equiv h^{ij} \mathcal{R}_{ij}$, $\mathcal{K} \equiv \mathcal{K}^\mu{}_\mu = h^{ij} \mathcal{K}_{ij}$. The first two equations are the Gauss-Codazzi relations between all curvatures on a given hypersurface, while the last relation gives an evolution equation for \mathcal{K}_{ij} between successive slices.

The energy-momentum tensor can in turn be decomposed with respect to the spatial sections and their normal as

$$\begin{aligned} T_{\mu\nu} = & E n_\mu n_\nu + 2n_{(\mu} J_{\nu)} + S_{\mu\nu} , \quad (1.12) \\ \text{with } E \equiv & n^\mu n^\nu T_{\mu\nu} \quad , \quad J_\mu \equiv -h^\rho{}_\mu n^\sigma T_{\rho\sigma} \quad , \quad S_{\mu\nu} \equiv h^\rho{}_\mu h^\sigma{}_\nu T_{\rho\sigma} . \end{aligned}$$

Here, E is the energy density of the sources, J_μ (with $J_\mu n^\mu = 0$) their momentum density, and $S_{\mu\nu}$ (with $n^\mu S_{\mu\nu} = 0$) their symmetric stress density tensor, all as measured by observers with 4-velocity \mathbf{n} (*i.e.*, in the frames associated with the spatial slices). The isotropic pressure in these frames is given by $S/3$ with S the trace of the stress density tensor, $S \equiv g^{\mu\nu} S_{\mu\nu} = h^{ij} S_{ij}$.

The 3 + 1 form of the Einstein equation ([Arnowitz, Deser, and Misner \[1962\]](#), [Gourgoulhon \[2012\]](#) p.87, [Alcubierre \[2008\]](#) p.71-74), is then obtained from the above decompositions of the spacetime Ricci tensor and of the energy-momentum tensor, related through the Einstein equation including, here, a cosmological constant. Additionally, the time evolution of the spatial metric is related to the extrinsic curvature through the definition of the latter. This gives two (tensorial) evolution equations for h_{ij} and \mathcal{K}_{ij} as the dynamical variables characterizing the hypersurfaces and their embedding in spacetime:

$$\partial_t|_{x^k} h_{ij} = -2N \mathcal{K}_{ij} + D_j N_i + D_i N_j , \quad (1.13)$$

$$\begin{aligned} \partial_t|_{x^k} \mathcal{K}^i{}_j = & N \left(\mathcal{R}^i{}_j + \mathcal{K} \mathcal{K}^i{}_j + 4\pi G [(S - E) \delta^i{}_j - 2S^i{}_j] - \Lambda \delta^i{}_j \right) \\ & - D^i D_j N + N^k D_k \mathcal{K}^i{}_j + \mathcal{K}^i{}_k D_j N^k - \mathcal{K}^k{}_j D_k N^i , \end{aligned} \quad (1.14)$$

and two (one scalar and one vectorial) constraint equations on these quantities from the Gauss-

Codazzi relations, the energy constraint and the momentum constraint:

$$\mathcal{R} + \mathcal{K}^2 - \mathcal{K}^i_j \mathcal{K}^j_i = 16\pi G E + 2\Lambda ; \quad (1.15)$$

$$D_k \mathcal{K}^k_i - D_i \mathcal{K} = 8\pi G J_i . \quad (1.16)$$

In practice, these constraint equations only need to be ensured when setting initial conditions, as the above evolution equations propagate the constraints if initially satisfied; this gives a Cauchy problem on the variables h_{ij} , \mathcal{K}_{ij} from constrained initial conditions (Gourgoulhon [2012], p.88-89; Alcubierre [2008], p.74-75).

Most cosmological models are based on such a 3 + 1 split of spacetime. This is in particular the case of the simplest expanding model universes, where a preferred set of spatial hypersurfaces is characterized by a maximal symmetry condition.

1.2 Homogeneous and isotropic universes and the standard model of cosmology

1.2.1 The FLRW models and the Friedmann equations

The cosmological principle

The *Friedmann-Lemaître-Robertson-Walker* (thereafter FLRW) cosmological models are based on the assumption of the existence of a foliation of the spacetime manifold into *strictly homogeneous and isotropic* spatial sections. This relatively simple, idealized model universe is designed for the description of the largest spatial scales, beyond the so-called *homogeneity scale*. This scale of *statistical homogeneity* of matter distribution in the Universe is currently estimated from two-point correlation functions in the matter distribution to be about 70 Mpc/ h at present time (see, *e.g.*, Scrimgeour et al. [2012], Gonçalves et al. [2018], and references therein), where $h \simeq 0.7$ is a constant arising from the present-day expansion rate. Being of a statistical nature, such an homogeneity scale would not fully rule out the observed existence of a few matter structures and vacua larger than 400 Mpc/ h (*e.g.* Gott et al. [2005], Horváth et al. [2014], Kopylov and Kopylova [2002]), but would render them unlikely. In view of this and of other evaluation methods accounting for higher-order correlations (Sylos Labini et al. [2009], Wiegand et al. [2014]; see also Nesseris and Trashorras [2019] and references therein), the above homogeneity scale value and even the existence of such a scale remain under debate.

The assumption of the validity of the model above these scales is referred to as the *cosmological principle*. It is often presented as a consequence of the observation of approximate isotropy of the sky properties for an observer on Earth together with the *Copernican principle* stating that such an observer should be typical, *i.e.*, any observer should obtain a similar result. Provided one additionally assumes the isotropy property to be *exact*, at *all times*, in a smoothed picture beyond a certain scale, it will then hold around every point of each spatial section as a consequence of

the Copernican principle. Under these assumptions, the model universe then has to be spatially homogeneous at all times and a member of the FLRW class, with a shear-free, vorticity-free and geodesic perfect-fluid content (Ehlers [1961], Collins and Wainwright [1983], Ellis, Maartens, and MacCallum [2012] p.203).

Geometry of the FLRW models

The spatial sections of homogeneity of the FLRW models are three-dimensional isotropic Riemannian manifolds with constant (homogeneous) intrinsic curvature. Accordingly, they correspond to one of the three possible such manifolds, the unit 3-sphere \mathbb{S}^3 , the 3-dimensional Euclidean space \mathbb{R}^3 and the hyperbolic 3-space of unit ‘radius’ \mathbb{H}^3 , up to a global (homogeneous) conformal factor or *scale factor* for the metric. This scale factor can for instance be seen in the \mathbb{S}^3 case as defining a sphere with non-unit radius. The above three reference manifolds are respectively of constant positive, vanishing, and negative scalar curvature. By a suitable choice of coordinates r, θ, ϕ , the line element for each of these Riemannian manifolds may be written under the convenient unified form

$$ds_{\text{ref}}^2 = \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 (\sin \theta)^2 d\phi^2, \quad (1.17)$$

where the constant parameter k is taken as $+1$ for \mathbb{S}^3 , 0 for \mathbb{R}^3 and -1 for \mathbb{H}^3 . In each case, θ and ϕ are angular coordinates and r is a radial coordinate such that the constant- r surfaces remain 2-spheres of area $4\pi r^2$; they simply correspond to the spherical coordinates in the \mathbb{R}^3 case.

The strict isotropy assumption requires a vanishing spatial velocity in the spatial slices of homogeneity for any existing fluid source, *i.e.*, the sources must have a common 4-velocity \mathbf{u} orthogonal to these slices, $\mathbf{u} = \mathbf{n}$. A very natural choice for a coordinate basis for the description of the FLRW models is then set by the choice of constant time t on each spatial section of homogeneity (labelling these sections) arising from the 3+1 formalism of section 1.1, together with the requirement of spatial coordinates x^i comoving with the 4-velocity \mathbf{u} . The latter condition corresponds to a choice of vanishing shift associated with the coordinates (t, x^i) . The lapse $N = 1/(u^0)$ (in this coordinate system) that expresses in this framework the local rate of the proper time τ of the fluid sources with respect to the coordinate time t , $N = d\tau/dt$, must be a pure function of t due to the homogeneity assumption. This can also be seen as arising from the vanishing of the 4-acceleration of the fluid as a consequence of isotropy, since it corresponds to the acceleration of the normal vector to the slices with components given by (1.6). $N = N(t)$ implies that the spatial slices of homogeneity are also hypersurfaces of constant τ . Hence, the proper time is a valid and preferred choice for the hypersurface-labelling time coordinate t , which is then referred to as the *cosmic time*. In the corresponding coordinate basis $(t, x^i) = (\tau, x^i)$, the lapse is 1 and the components of the 4-velocity and its dual 1-form reduce to $u^\mu = (1, 0, 0, 0)$ and $u_\mu = (-1, 0, 0, 0)$, respectively.

The FLRW line element then reads in these coordinates

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + h_{ij} dx^i dx^j = -dt^2 + a(t)^2 ds_{\text{ref}}^2, \quad (1.18)$$

where $h_{\mu\nu}$ are the components of the metric induced on the spatial hypersurfaces. They are expressed as above in terms of the time-dependent scale factor $a(t) > 0$ ($a(t) = 0$ representing singularities) and the spatial line element ds_{ref}^2 given by one of the three reference Riemannian metrics mentioned above. By continuity, all spatial sections must correspond to the same reference Riemannian manifold up to their scale factor, *i.e.*, k is a constant. t is only defined up to an arbitrary translation, but if $a(t)$ reaches zero at a finite proper time value in the past (corresponding to an initial singularity or ‘Big Bang’), it is customary to set this value to $t = 0$.

For the dimensionless conventions $k = \pm 1$ (in the non-flat spatial sections cases) adopted here, the scale factor $a(t)$ becomes a dimensionful number measuring a characteristic size associated with the constant spatial curvature, *e.g.*, the radius of the spatial sections if they are 3-spheres. It can alternatively be made adimensional and be normalized to unity at a reference epoch by appropriately rescaling k and the coordinate r into dimensionful variables using the characteristic size at the reference epoch. In the flat spatial sections case ($k = 0$), $a(t)$ can simply be assumed to be dimensionless by interpreting r as having the dimension of a length. It is customary to do so in this case and to normalize the scale factor to unity at the current epoch t_0 , $a(t_0) = 1$.

Computing the spacetime Ricci tensor components in the coordinates (t, x^i) from the simple metric form (1.18) with ds_{ref}^2 given by (1.17) yields

$$R_{00} = -3\frac{\ddot{a}}{a} \quad ; \quad R_{0i} = R_{i0} = 0 \quad ; \quad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} \right) h_{ij} \quad , \quad (1.19)$$

for $i = 1, 2, 3$ and with the overdot notation $\dot{} \equiv \partial_t|_{x^i}$. The spacetime Ricci scalar thus reads

$$R \equiv g^{\mu\nu} R_{\mu\nu} = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) . \quad (1.20)$$

Similarly computing the Weyl tensor for this metric form shows that it is vanishing, *i.e.*, the FLRW metrics are conformally flat, and their Ricci tensor as given above fully characterize their spacetime curvature properties (*e.g.*, Ellis, Maartens, and MacCallum [2012], p.204). In the spatial sections, the intrinsic curvature components \mathcal{R}_{ij} and the scalar intrinsic curvature \mathcal{R} read, respectively:

$$\mathcal{R}_{ij} = \frac{2k}{a^2} h_{ij} \quad ; \quad \mathcal{R} = \frac{6k}{a^2} . \quad (1.21)$$

Coupling to the sources

The isotropy assumption prevents any momentum vector or trace-free (anisotropic) pressure contributions to the energy-momentum tensor. The latter thus has to take a perfect fluid form

[Ellis, Maartens, and MacCallum, 2012, p.203]:

$$T_{\mu\nu} = \epsilon u_\mu u_\nu + p h_{\mu\nu} , \quad (1.22)$$

where $\epsilon = \epsilon(t)$ and $p = p(t)$ are, respectively, the total energy density and isotropic pressure of the sources in their common rest frame, *i.e.*, as measured in the homogeneous spatial slices.

Using the expressions (1.19)–(1.20) for the spacetime Ricci tensor and scalar and the energy-momentum tensor form (1.22), the Einstein equations with a cosmological constant Λ then reduce in this simple framework to a set of ordinary differential equations for the main variable $a(t)$, the *Friedmann equations*:

$$3 \left(\frac{\dot{a}}{a} \right)^2 = 8\pi G \epsilon + \Lambda - \frac{3k}{a^2} ; \quad (1.23)$$

$$3 \frac{\ddot{a}}{a} = -4\pi G (\epsilon + 3p) + \Lambda . \quad (1.24)$$

In general, both right-hand sides above will be nonzero, making this system suitable as a simple model of an expanding Universe. For a non-exotic fluid with positive pressure, the acceleration equation (1.24) can only imply deceleration of the expansion for $\Lambda = 0$, and requires a positive and large enough cosmological constant for a positive acceleration. It also shows that this is required as well for Einstein's static FLRW model, with a specific Λ as a function of the energy sources, together with a specific positive spatial curvature as a consequence of the vanishing of both sides of the integral equation (1.23).

The above system can be complemented by the energy conservation law on $\epsilon(t)$, $p(t)$:

$$\dot{\epsilon} + 3 \frac{\dot{a}}{a} (\epsilon + p) = 0 . \quad (1.25)$$

This relation arises from the conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, projected onto the fluid 4-velocity. In this framework, it can also be directly deduced from a combination of (1.24) and the time derivative of (1.23), *i.e.* it is *not* an independent equation from the two Friedmann equations and it expresses a condition of compatibility between them. The other components of the energy-momentum tensor conservation equation, as obtained by projecting it orthogonally to the fluid 4-velocity, relate the fluid's 4-acceleration, of components $a^\mu = u^\nu \nabla_\nu u^\mu$, to the spatial pressure gradient,

$$a_\mu = -\frac{h_\mu{}^\nu \nabla_\nu p}{\epsilon + p} : \quad a_0 = 0 , \quad a_i = -\frac{\partial_{x^i} p}{\epsilon + p} . \quad (1.26)$$

This provides another derivation of the geodesic nature of the fluid in this setting where p is homogeneous in the spatial sections, $\partial_{x^i} p = 0$.

1.2.2 Specifying the model sources

Effective equations of state

The above system of only two independent equations (1.23)–(1.24) on the three variables $a(t)$, $\epsilon(t)$ and $p(t)$ is not closed, as expected given that the physical properties of the fluid have not been specified at this stage. This can be solved by specifying (at least partially) an equation of state for the fluid. It can be written in this case in term of a barotropic pressure–energy relation $p(\epsilon)$ at least within some range, since both ϵ and p are pure functions of time. The simplest case is a linear relation, $p = w\epsilon$ with $w = cst.$, for which the energy conservation equation (1.25) then provides the time evolution of the energy sources as $p \propto \epsilon \propto a(t)^{-3(1+w)}$. Such a linear relation is widely used as holding for all of the most common matter models. The latter are thus characterized by the corresponding value of w .

- $w = 0$ describes *dust*, *i.e.*, pressureless matter ($p = 0$), with $\epsilon \propto a(t)^{-3}$. In this case ϵ coincides with a rest mass density for the fluid (*i.e.*, the energy conservation equation reduces to a conserved rest mass current equation, $\nabla_\mu(\epsilon u^\mu) = 0$) and may as well be noted ϱ . This case can be used to model the non-interacting Dark Matter fluid, or even nonrelativistic baryons for which the pressure is much smaller than the mass energy density ϱ (that would read ϱc^2 in units where c is not adimensioned) and can thus be neglected. It is thus suited for the description of a cosmological era dominated by such nonrelativistic matter sources. In the case of flat spatial sections ($k = 0$) and a vanishing cosmological constant, the resulting pure dust FLRW model universe is known as the Einstein-de Sitter model and the scale factor time dependence can be deduced from (1.23) as $a(t) \propto t^{2/3}$.
- $w = 1/3$ describes a radiation fluid, with $\epsilon \propto a(t)^{-4}$. This can model a photon gas, or ultra-relativistic matter that may be coupled to radiation, and is thus suited for a cosmological era dominated by strongly coupled matter and radiation. From (1.23), in the simplest case $k = 0$ and $\Lambda = 0$, the scale factor of the pure radiation FLRW model universe evolves as $a(t) \propto \sqrt{t}$.
- $w = -1$ would correspond to a cosmological constant. The relation $p = -\epsilon$ implies that ϵ (assumed positive) and p (then negative) are constants of spacetime and would jointly contribute to both Friedmann equations exactly as a positive cosmological constant $8\pi G\epsilon$. This can be seen as an effective description of the Dark Energy component modelled by the cosmological constant Λ , effectively replacing the otherwise already present Λ contribution by a source $\epsilon = \Lambda/(8\pi G)$, $p = -\Lambda/(8\pi G)$. A pure Dark Energy FLRW model universe, for positive Λ , with no extra source and with $k = 0$, would have an exponentially growing scale factor with time, $a(t) \propto \exp(\sqrt{\Lambda/3}t)$, and would not reach a Big Bang singularity in a finite time to the past.
- $w = -1/3$ also corresponds to a negative pressure (or a negative energy density) and may similarly be used as an effective fluid description of the contribution of the spatial

curvature term $-3k/a^2$. The contribution of such a source to the first Friedmann equation (1.23) would indeed be a term $8\pi G\epsilon \propto a(t)^{-2}$ from (1.25), with a vanishing contribution to the acceleration equation (1.24). The latter shows that an empty FLRW model universe with $\Lambda = 0$, where an assumed nonzero curvature is the only source of the Friedmann equations, must have a linear scaling of the scale factor with time, $a(t) \propto t$. The first Friedmann equation (1.23) implies that the nonzero k has to be negative, corresponding to hyperbolic spatial sections. This picture corresponds to the so-called Milne model for a vacuum FLRW model with no cosmological constant. Inserting the Friedmann equations into the spacetime Ricci tensor expression (1.19) shows that it vanishes in this case, along with the Weyl tensor. Hence, the corresponding spacetime is simply the (Riemann-flat) Minkowski spacetime with an unusual slicing, with hyperbolic spatial hypersurfaces.

Other values of w are also sometimes encountered in the literature such as the “stiff fluid” model $w = 1$, implying an energy density scaling as a^{-6} and corresponding to the pressure–energy relation of a free fundamental scalar field described as a fluid source, or an unspecified value $-1 < w < -1/3$ as an alternative, dynamical model of Dark Energy as a fluid with negative pressure rather than a cosmological constant.

The Omega parameters

A more complete cosmological model can then be obtained from a sum of such sources, where the source i has an energy density ϵ_i and a linear partial equation of state parametrized by w_i : $\epsilon = \sum_i \epsilon_i$ and $p = \sum_i w_i \epsilon_i$, which would be valid over a range of cosmological eras. Considering these sources to be decoupled, each corresponding energy-momentum is separately conserved so that the time behaviour $\epsilon_i \propto a(t)^{-3(1+w_i)}$ still applies for each source i . The contributions from the cosmological constant and the spatial curvature can be replaced by effective sources within this decomposition as explained above, with $w = -1$ and an effective energy density $\epsilon_\Lambda = \Lambda/8\pi G$ for the cosmological constant and $w = -1/3$ and an effective energy density (which may be negative) $\epsilon_k = -3k/(8\pi G a^2)$ for curvature.

The first Friedmann equation (1.23) may then be rewritten as $3H^2 = \sum_i 8\pi G\epsilon_i$ with the Hubble parameter $H(t) \equiv \dot{a}/a$. In an expanding or collapsing phase ($\dot{a} \neq 0$) it can be divided by its left-hand side to define the dimensionless ‘Omega’ parameters expressing the relative contributions of each (real or effective) source to this energy balance:

$$\Omega_i \equiv \frac{8\pi G\epsilon_i}{3H^2} \quad ; \quad \sum_i \Omega_i = 1 . \quad (1.27)$$

The current observations appear to be compatible with the main physical sources (beyond the earliest epoch) being radiation (here noted r , and including photons and neutrinos, as well as the coupled photons+baryons radiation fluid at early times; $w_r = 1/3$), and baryonic matter and non-interacting ‘cold’ Dark Matter (CDM), both nonrelativistic at late times and here together

noted $m, \overline{w_m} = 0$. They will contribute to the energy balance along with the curvature k and the cosmological constant Λ : $1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k$. The matter parameter Ω_m may be split into the baryonic and cold Dark Matter components $\Omega_b, \Omega_{\text{DM}}$. (Note that in this cosmological context, ‘baryonic’ is taken in a broad sense that actually includes nonrelativistic leptons.) The present-day values $\Omega_i(t_0) \equiv \Omega_i^0$ of each of these contributions are key parameters for such a cosmological model.

Background dynamics in the standard model of cosmology

The widely accepted ‘ Λ CDM’ model (for the cosmological constant Λ and a cold Dark Matter component), or standard model of cosmology, considers the above matter sources as its physical content, and a cosmological constant $\Lambda > 0$. While allowing for deviations at all scales (see section 1.3 for the perturbative method for large scales), it is based on an assumed ‘background’ FLRW model as arising from the cosmological principle, with the same sources and Λ , and with a vanishing spatial curvature, $k = 0, \Omega_k = 0 \forall t$. The latter restriction appears to be compatible with the observational constraints on the present-day Omega parameters assuming a nearly FLRW model on large scales ($\Omega_k^0 = (0.7 \pm 1.9) \cdot 10^{-3}$ at 1σ confidence level in the *Planck* collaboration 2018 joint results [Planck Collaboration, 2018]).

The three remaining contributions to the first Friedmann equation (1.23) for the background, ϵ_r, ϵ_m and Λ , scale as a^{-4}, a^{-3} and a constant, respectively. This equation moreover shows that $\dot{a}/a \geq \sqrt{\Lambda/3} \forall t$, so that the model is indefinitely and unboundedly expanding, $a(t) \rightarrow +\infty$ for $t \rightarrow +\infty$, and $\dot{a}/a \geq cst/a^2$ for $cst > 0$, so that there is a Big Bang, $a(t) \rightarrow 0^+$ at a finite time that can be set to $t = 0^+$. The energy contribution of the matter sources is thus dominated by radiation at early times and by nonrelativistic matter at late times. The energy balance is ultimately dominated by Ω_Λ for $t \rightarrow +\infty$ where the scale factor will asymptotically undergo an exponential growth with t .

For a small enough Λ (as is observed, and as necessary for matter structures to be able to form), the model can be split in two main eras. The early era is dominated by radiation, with an approximate scale factor evolution $a(t) \propto \sqrt{t}$. The late era, once radiation has become negligible, is dominated by matter and (at even later times) the cosmological constant with $\Omega_m + \Omega_\Lambda \simeq 1$, *i.e.*, fully neglecting ϵ_r , $(\dot{a})^2 = H_0^2 \Omega_m^0/a + H_0^2 \Omega_\Lambda^0 a^2$ with the convention $a(t_0) = 1$ and with the present-day Hubble parameter $H_0 \equiv H(t_0)$. This evolution equation can be solved analytically to provide

$$a(t) = \sqrt[3]{\frac{\Omega_m^0}{\Omega_\Lambda^0}} \sinh^{2/3} \left(\frac{3}{2} H_0 \sqrt{\Omega_\Lambda^0} (t - t_1) \right), \quad (1.28)$$

for some reference time t_1 that may be tuned to match this solution with the end of the radiation-dominated era. The radiative era is often assumed to be preceded by an additional phase with fast, exponential expansion known as *inflation*, driven by a yet-unknown mechanism (for which many competing models exist) at play beyond the Planck energy scale. This can explain the

apparent near-thermal equilibrium of the observable Universe around the end of the radiative era, while all parts of it could otherwise not be causally connected at that time.

The *Planck* collaboration (2018 joint results) [Planck Collaboration, 2018] provides the following observational constraints on the present-day Omega parameters of the FLRW background of the Λ CDM model: $\Omega_m^0 = 0.3111 \pm 0.0056$, $\Omega_\Lambda^0 = 0.6889 \pm 0.0056$ (at 1σ confidence level). Ω_r^0 is smaller than 10^{-4} , depending on the number of still relativistic neutrinos (the contribution to Ω_r^0 from the energy density of photons alone being $\Omega_\gamma^0 \simeq 5.4 \cdot 10^{-5}$). This is thus consistent with a picture (for the assumed FLRW background) at present day of negligible radiation, and an energy balance dominated by the cosmological constant with a smaller but comparable contribution from Dark and baryonic Matter, making the above analytic solution (1.28) for $a(t)$ relevant around the present epoch. This matter contribution splits into Dark and baryonic Matter as $\Omega_{\text{DM}}^0 \simeq 0.261$ and $\Omega_b^0 \simeq 0.0490$, respectively, being dominated by the cold Dark Matter component.

The simplicity of the FLRW background model, with global dynamics fully determined by $a(t)$ as a single variable, makes it a very convenient setting for the description of a variety of physics phenomena in the early Universe within the standard model of cosmology. This reduces for instance the complexity of the study within a dynamical Universe of the phase transitions associated with the successive separations of the forces of the Standard Model of particle physics as energy density decreases, or of the formation of the lightest atomic nuclei (the Primordial Nucleosynthesis).

The Cosmic Microwave Background

After the formation of nuclei and in the transition period between the radiation- and the matter-dominated eras, the decoupling of ‘baryonic’ matter (including electrons) from radiation allows for the formation of atoms. This decoupling occurs as the decreasing density of the initial radiative photon-baryons mixture becomes low enough for the photons’ average time between scatterings to become of the order of the typical age of the Universe $1/H$ at that time. The collection of the typical last such scattering event at each point of space builds the *surface of last scattering*. Photons are freely emitted from this surface, building a mostly isotropic radiation feature that is observed today as the *Cosmic Microwave Background* (CMB), its typical wavelength being redshifted into the microwave range.

The CMB, and in particular the power spectrum of its fluctuations around a strictly isotropic radiation, is a fundamental tool for observational cosmological constraints. It is for instance the source of most of the constraints on the parameters of the Λ CDM model and some extensions thereof in the *Planck* collaboration results such as those cited hereabove. These fluctuations (see Figure 1.2) are coupled to the early inhomogeneities in matter density since the local baryonic density directly affects the local decoupling time. They may thus be used to construct initial data (at least on the observer’s past light cone) for the evolution of matter structures in the matter-dominated era. However, the description of such fluctuations and of their growth in

the first stages of matter structure formation clearly calls for going beyond the strictly FLRW ‘background’ model. This is usually performed with a perturbative scheme around an FLRW metric and associated homogeneous and isotropic sources.

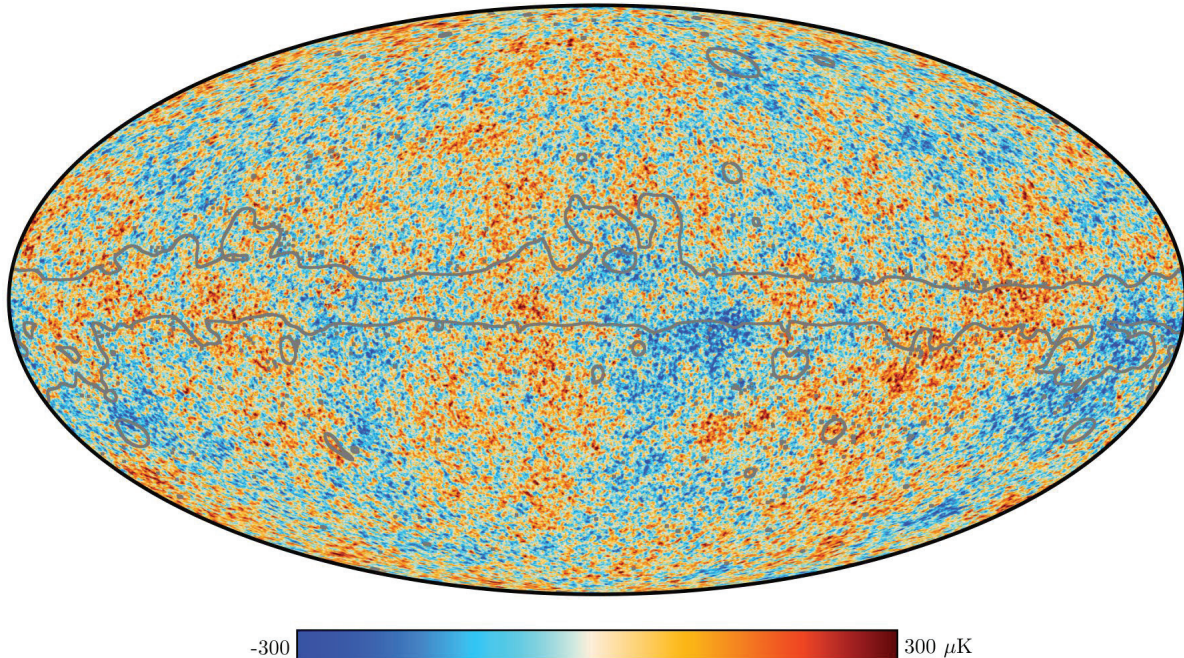


Figure 1.2: All-sky projected map from the *Planck* Collaboration 2018 results of the local fluctuations in the CMB temperature depending on the direction of observation. The isotropic component at about 2.7 K and the leading fluctuation (the dipole component with an amplitude of about 3.4 mK) have been removed. The areas delineated by the grey curves (mostly along the major axis of the map, corresponding to the Milky Way plane) are hidden by foregrounds and have been reconstructed. *Credit: ESA and the Planck Collaboration.*

1.3 Eulerian linear relativistic perturbation theory

Small deviations of a cosmological model from strict homogeneity and isotropy may be described in terms of a perturbative expansion of the metric and energy-momentum tensor components with respect to an FLRW reference model or ‘background’. Such an approach is commonly applied, within the standard model of cosmology, to the description of the early stages (linear regime) of the formation of matter structures, and to the largest scales in a coarse-grained picture where, *e.g.*, density or curvature perturbations are expected to be small. A relativistic framework (by contrast to a Newtonian model) is moreover needed for fluctuations over scales comparable to or larger than the causality horizon (as estimated by the Hubble length $c/H(t)$) near the onset of structure formation *i.e.* around the matter–radiation decoupling epoch, for

which the finite propagation speed of information cannot be ignored. These scales correspond to a typical present-day size of around 100 Mpc/ h . A relativistic model is also necessary for studying fluctuations in the radiation fluid.

1.3.1 Framework and perturbation variables

Metric form

This perturbation theory is based on a split of the metric \mathbf{g} into a (homogeneous) FLRW metric \mathbf{g}^H and a supposedly small deviation thereof, $g_{\mu\nu} = g_{\mu\nu}^H + \delta g_{\mu\nu}$. The background metric components are usually written under the form obtained from (1.18) after a change of time coordinate from the cosmic time t to the *conformal time* η . The latter is defined as a pure function of t from the FLRW scale factor by $d\eta = dt/a(t)$, amounting to setting a lapse $N = a(t)$ instead of 1, preserving the spatial hypersurfaces. The FLRW line element (1.18) then reads in these coordinates²:

$$ds_H^2 = a^2(\eta) [-d\eta^2 + \bar{g}_{ij} dx^i dx^j] , \quad (1.29)$$

in terms of the reference spatial metric with line element $\bar{g}_{ij} dx^i dx^j \equiv ds_{\text{ref}}^2$ from expression (1.17). In the Λ CDM model, the FLRW background is flat, *i.e.* the reference metric is Euclidean, and the spatial coordinates x^i are usually chosen as Cartesian coordinates, so that $\bar{g}_{ij} = \delta_{ij}$. The choice of η as the time coordinate thus makes the conformal relation between this FLRW metric and the Minkowski metric explicit, whence the name of this coordinate.

The perturbative approach at first order (Bardeen [1980], Kodama and Sasaki [1984], Stewart [1990], Mukhanov et al. [1992], Bruni et al. [1992], Malik and Wands [2009], Ellis, Maartens, and MacCallum [2012] p.249-262) assumes the existence of a corresponding coordinate system (η, x^i) on the model spacetime (the perturbed manifold) in which the line element can be split as

$$ds^2 = a^2(\eta) [-(1 + 2\phi) d\eta^2 + 2B_i d\eta dx^i + ((1 - 2\psi) \bar{g}_{ij} + G_{ij}) dx^i dx^j] , \quad (1.30)$$

where the space- and time-dependent perturbation variables ϕ , B_i , ψ , G_{ij} ($= G_{ji}$) are first-order quantities. Since it relies on an expansion of the metric components in coordinates that need not be directly related to the fluid content in general, and by analogy to the Newtonian perturbative frameworks, I will refer to this framework as a first-order (or linear) *Eulerian* perturbation theory, by contrast to a Lagrangian approach.

The variables B_i , G_{ij} may be further split into first-order scalar, vector and tensor parts:

$$B_i = \bar{D}_i B - S_i \quad ; \quad G_{ij} = 2 \bar{D}_i \bar{D}_j E + \bar{D}_i F_j + \bar{D}_j F_i + h_{ij} , \quad (1.31)$$

where \bar{D} denotes the spatial covariant derivative with respect to the reference metric $\bar{g}_{ij} dx^i \otimes dx^j$.

²Here and in the following, I will for ease of notation make use of the slightly improper form $a(\eta)$ to denote the scale factor at time $t = t(\eta)$, *i.e.* $a(t(\eta))$, even though it has in principle a different functional dependence on the variable η .

This operator reduces to a partial derivative when the reference metric is flat and written in Cartesian coordinates. It coincides at first order with the full spatial covariant derivative of the perturbed spacetime when applied to an already first-order variable, as above. The scalar perturbations are those built from the four functions ϕ , ψ , B and E . The vector parts arise from a divergence-free condition, $\bar{g}^{ij} \bar{D}_i S_j = \bar{g}^{ij} \bar{D}_i F_j = 0$ where \bar{g}^{ij} are the components of the inverse metric to $\bar{g}_{ij} dx^i \otimes dx^j$, in order to define the scalar parts uniquely. Similarly, the (symmetric) tensor part h_{ij} is defined as traceless and divergence-free: $h_{ij} = h_{ji}$, $\bar{g}^{ij} h_{ij} = 0$, $\bar{g}^{ij} \bar{D}_i h_{jk} = 0$.

Sources perturbation

The source fields in the energy-momentum tensor $T_{\mu\nu}$ are also similarly split into background (ϵ_H, p_H) and first-order quantities:

$$T_{\mu\nu} = \epsilon u^\mu u^\nu + p b_{\mu\nu} + \pi_{\mu\nu} ; \quad (1.32)$$

$$\epsilon = \epsilon_H + \delta\epsilon \quad ; \quad p = p_H + \delta p , \quad (1.33)$$

where u^μ is the 4-velocity of the sources corresponding to their energy frame [Ellis, Maartens, and MacCallum, 2012, p.91], *i.e.*, defined as an eigenvector of $T_{\mu\nu}$, so that the heat vector $q_\mu \equiv -u^\nu b^\sigma_\mu T_{\nu\sigma}$ vanishes, with $b_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ the components of the projector into the associated local rest frames. The anisotropic pressure tensor $\pi_{\mu\nu}$ is symmetric, fluid-orthogonal and traceless: $\pi_{\mu\nu} = \pi_{\nu\mu}$, $\pi^\mu_\mu = 0$, $u^\mu \pi_{\mu\nu} = 0$. It is part of the first-order variables as it vanishes in the isotropic background spacetime. Similarly to G_{ij} , it may be split into scalar, vector and tensor parts. The anisotropic pressure is however still vanishing at any order if the source is assumed to be a perfect fluid.

In order to have a consistent FLRW background model, the background sources $\epsilon_H(t)$ and $p_H(t)$ and the scale factor $a(t)$ are assumed to be coupled according to the Friedmann equations (1.23)–(1.24) (adding the subscript H to ϵ and p in these equations) for a given choice of the parameters k (in consistency with the choice of \bar{g}_{ij}) and Λ . The Friedmann equations for $a(t)$ may easily be rewritten in the time variable η instead of t to directly get the evolution equations for $a(\eta)$ depending on the background sources (giving for instance $a(\eta) \propto \eta^2$, with $\eta \propto t^{1/3}$, in the Einstein-de Sitter model $p_H = 0$, $\Lambda = 0$, $k = 0$).

The 4-velocity itself is also perturbed with respect to the FLRW field \mathbf{u}_H which is comoving with the reference spatial coordinates x^i , $u_H^\mu = (a^{-1}, 0, 0, 0)$ in the coordinate system (η, x^i) . It is thus split as $u^\mu = u_H^\mu + \delta u^\mu$ for a first-order δu^μ . In components, it then reads

$$u^\mu = a^{-1}(1 - \phi, V^i) , \quad (1.34)$$

introducing the independent first-order variable V^i , while the u^0 component is deduced from the first-order expansion of the normalization condition $g_{\mu\nu} u^\mu u^\nu = -1$ using the metric form (1.30). V^i can be split into a scalar and a vector part, $V^i = \bar{g}^{ij} \bar{D}_j v + v^i$ with $\bar{D}_i v^i = 0$.

Consistently splitting in this way all perturbation variables into scalar, vector and tensor parts (or modes) is of special interest in a first-order scheme since the evolution equations at this order will be linear by construction. Each mode will thus be decoupled from the others and can be studied separately. The uniqueness of such a split however requires specific boundary conditions (typically periodic conditions or requirements of damping at spatial infinity) or compact spatial sections (associated to an \mathbb{S}^3 reference spatial manifold, $k = +1$). Such conditions will provide a unique definition of each mode by ensuring that harmonic first-order variables (vanishing under the background spatial Laplacian operator $a^{-2} \bar{g}^{ij} \bar{D}_i \bar{D}_j$) vanish or have a vanishing gradient.

1.3.2 Gauge freedom

The gauge issue in Eulerian relativistic perturbation theory

Any coordinate system (η, x^i) in which the line element may be written under the perturbed form (1.30) with first-order perturbations is equally acceptable *a priori*. Each such choice may be interpreted as a mapping (specifically, a diffeomorphism) between the ‘perturbed’ (almost-FLRW) manifold and the assumed ‘background’ spacetime. The latter is a given reference manifold equipped with the background FLRW metric and an adapted coordinate system with conformal time and comoving (Cartesian in Λ CDM) spatial coordinates. The mapping, expressed in terms of the coordinate basis choice, associates to each event of the background manifold, with coordinates (η, x^i) , an event in the perturbed manifold which will be attributed the same coordinate values. Each such mapping singles out a ‘preferred’ spatial foliation (in the perturbed spacetime) as given by the level sets of the corresponding η coordinate, which are the images of the hypersurfaces of homogeneity in the background manifold. The value of a perturbed field at an event is then compared to the value of the corresponding background field at its image by the inverse map, *i.e.* at the background point with the same coordinates. This difference defines the perturbation variable for this field.

Another choice of mapping, and of the associated coordinate system, within the acceptable class, will change the perturbation variables $\phi, B, S_i, E, \delta\epsilon\dots$. As the mapping of the perturbed spacetime to the reference manifold is *a priori* arbitrary, this is considered a *gauge freedom* of the perturbation theory, and raises the issue of the *gauge dependence* of the perturbation variables.

From a given coordinate system $(x^\mu) = (\eta, x^i)$ in which the perturbed line element takes the form (1.30), any first-order coordinate change $x^\mu \mapsto \tilde{x}^\mu = x^\mu + \delta x^\mu(x^\nu)$ will preserve this first-order form. It will thus be another acceptable choice, building an associated new mapping, locally infinitesimally close to the previous one, and thus a new gauge. (Only infinitesimal transformations will be considered here, thus dismissing non-infinitesimal rotations and translations of the spatial coordinates which would also preserve the form of the perturbed metric, since the background coordinates themselves are defined up to such a transformation as well.) From the transformation of the metric components and their split, under such a gauge change, the scalar perturbations ϕ, B, ψ and E for instance will transform as [Ellis, Maartens, and MacCallum,

2012, p.253]:

$$\begin{aligned}\phi &\mapsto \tilde{\phi} = \phi - \frac{a'}{a}\delta\eta - (\delta\eta)' \quad ; \quad \psi \mapsto \tilde{\psi} = \psi + \frac{a'}{a}\delta\eta \\ B &\mapsto \tilde{B} = B + \delta\eta - \chi' \quad ; \quad E \mapsto \tilde{E} = E - \chi ,\end{aligned}\tag{1.35}$$

where the spatial coordinate change is also split into scalar and vector parts, $\delta x^i = \bar{g}^{ij}\bar{D}_j\chi + \chi^i$ with $\bar{D}_i\chi^i = 0$, and where a prime $'$ denotes a partial derivative with respect to η . The vector parts are also transformed (for instance $S_i \mapsto \tilde{S}_i = S_i + \bar{g}_{ij}\chi^{j'}$) while the tensor part h_{ij} of the metric perturbation remains invariant, $h_{ij} \mapsto \tilde{h}_{ij} = h_{ij}$.

Gauge-independent variables

In order to extract perturbations that are independent of the arbitrary mapping, one can build *gauge-invariant* quantities at a given order in the perturbation from the *a priori* gauge-dependent variables [Bardeen, 1980]. Gauge invariance to all orders is a very restricting requirement. This is due to the association of a given background event to an event in the perturbed spacetime that is different for each gauge. A gauge-invariance condition would then read as a comparison of the values of the perturbed field between distinct events (active transformation in the perturbed spacetime). For instance, the gauge transformation of a scalar φ under the coordinate change $x^\mu \mapsto \tilde{x}^\mu = x^\mu + \delta x^\mu(x^\nu)$ is the change between the two *functions* $\varphi(x^\mu)$ and $\tilde{\varphi}(x^\mu)$ (the fixed coordinate value x^μ mapping to a fixed point of the background manifold) where, as a scalar, $\tilde{\varphi}(\tilde{x}^\mu) = \varphi(x^\mu)$. Hence $\tilde{\varphi}(x^\mu) = \varphi(x^\mu - \delta x^\mu + o(\delta x^\mu)) = \varphi(x^\mu) - \delta x^\mu \partial_\mu \varphi + o(\delta x^\mu)$ and the gauge-invariance condition on φ is $\partial_\mu \varphi = 0$, leaving only constants as scalars that are gauge-invariant to any order.

However, gauge-invariance *to first order* is still possible for nontrivial quantities. Expressing the first-order results in the computation of physical observables in terms of such variables will then avoid interpretation issues with purely gauge-dependent terms. First-order gauge-invariant variables are usually obtained from combinations of first-order perturbations of tensor components (or scalars) [Bardeen, 1980].

As noted above, for instance, the ‘tensor part’ h_{ij} of the metric components first-order perturbation is such a first-order gauge invariant. From the transformation rules (1.35) of the ‘scalar parts’ of the first-order metric decomposition, ϕ , B , ψ and E , one can see that the following two combinations known as the *Bardeen potentials* are also gauge-invariant at first-order:

$$\Phi \equiv \phi - \frac{a'}{a}(E' - B) - (E'' - B') \quad ; \quad \Psi \equiv \psi + \frac{a'}{a}(E' - B).\tag{1.36}$$

No further independent such combination can be expected for the ‘scalar parts’ of the metric components due to the two ‘scalar part’ degrees of freedom in the coordinate change, $\delta\eta$ and χ , for four variables.

As a perturbation of a spacetime scalar with a nontrivial (time-dependent) background coun-

terpart, $\delta\epsilon$ is not gauge-invariant by itself at first order, transforming as $\delta\epsilon \mapsto \delta\epsilon - \epsilon'_H \delta\eta = \delta\epsilon + 3(a'/a)(\epsilon_H + p_H)\delta\eta$. In view of the transformation (1.35) of ψ , $\delta\epsilon - 3(a'/a)(\epsilon_H + p_H)\psi$ is thus a first-order gauge invariant based on $\delta\epsilon$, but there are various other possible such constructions for the energy density [Malik and Wands, 2009].

Rather than combinations based on components of tensor with different ranks, one may also define *covariant* variables that would vanish in the background, such as the (rescaled) fluid-orthogonal energy density variation as a vector, $a(\eta)(b_\mu{}^\nu \nabla_\nu \epsilon)/\epsilon$ [Ellis and Bruni, 1989], or its norm as a scalar. Such quantities being already of first order then ensures that they will remain invariant to this order under a gauge change [Stewart and Walker, 1974, Stewart, 1990].

Examples of gauge choices

Another way of dealing with the gauge freedom is to fix a gauge through more constraining requirements on the mapping to the background manifold or on the form taken by the perturbed metric components. Such requirements may need to be completed by further assumptions in order to eliminate residual gauge-dependent degrees of freedom. The following examples are some of the commonly used gauge choices:

- The *Poisson gauge* is set by the conditions $E = 0$, $B = 0$, and $S_i = 0$. In view of the transformation laws for these variables, such a condition can be achieved by a gauge change from any given other gauge (of parameters $E_o, B_o, S_{i,o}, \dots$) with $\delta\eta = E'_o - B_o$, $\chi = E_o$ and χ^i such that $\chi^{i'} = -\bar{g}^{ij}S_{j,o}$. Accordingly, this leaves some residual gauge freedom in the form of a time-independent pure vector spatial coordinate change, $\eta \mapsto \eta$, $x^i \mapsto x^i + \chi^i(x^k)$ with $\bar{D}_i \chi^i = 0$, associated with a time-independent relabelling of the spatial coordinates x^i [Malik and Wands, 2009]. The conditions on B and S_i imply a block-diagonal form for the metric, $g_{0i} = 0$, and correspond to a vanishing shift, associated to spatial coordinates x^i that propagate along the normal \mathbf{n} to the constant- η slices. The scalar parts of the metric perturbation are reduced to ϕ and ψ which coincide in this case with the Bardeen potentials (1.36), $\phi = \Phi$, $\psi = \Psi$. Writing down the Einstein equations at first order within this gauge results in several equations reminiscent of the Newtonian ones, including a Poisson-like equation for Φ as a weak-field equivalent of the Newtonian gravitational potential [Malik and Wands, 2009]. (Accordingly, this gauge may also be referred to as a Newtonian gauge or Newtonian slicing when only scalar perturbations are considered.) This also allows to relate both scalar parts of the metric perturbation through the fluid's anisotropic pressure, giving $\Phi = \Psi$ (leaving a single scalar-part variable) if $\pi_{\mu\nu} = 0$. This relation between both Bardeen potentials may also be obtained by applying the Einstein equation while already using only such gauge-invariant variables [Bardeen, 1980].
- The *synchronous gauge* is obtained by setting $\phi = 0$ and $B_i = \bar{D}_i B - S_i = 0$. The first condition implies that g_{00} reduces to $-a(\eta)^2$. It corresponds to setting the τ coordinate (defined by $d\tau = a(\eta) d\eta$ and corresponding to cosmic time in the background) to still be

interpreted in the perturbed manifold as the proper time measured along the normal to the constant- η slices from a given ‘initial’ spatial hypersurface [Kodama and Sasaki, 1984]. In this gauge, the metric is also block-diagonal ($g_{0i} = 0$). These choices allow for a simple form of the metric, the perturbations being only contained in the spatial components g_{ij} . There is residual gauge freedom in terms of free time-independent functions which may be used for a coordinate change, $\eta \mapsto \eta + a(\eta)^{-1} \widehat{\delta\eta}(x^k)$, $x^i \mapsto x^i + (\delta x^i)(x^k)$ (with here $\widehat{\delta\eta} = a \delta\eta$) [Malik and Wands, 2009]. The freedom on η corresponds to the arbitrary choice of the ‘initial’ hypersurface. It may be removed *via* additional requirements, such as a vanishing of the scalar velocity perturbation variable v on the initial hypersurface. The freedom on the spatial coordinates simply corresponds to their relabelling in a time-independent way.

- The *uniform curvature gauge* is defined by the requirements $E = 0$, $\psi = 0$ and $F_i = 0$ with no residual gauge freedom, leaving the tensor part h_{ij} as the only perturbation of the spatial metric components g_{ij} . The scalar intrinsic curvature \mathcal{R} of the constant- η slices being given in general by $a^2\mathcal{R} = 6k + 12k\psi + 4\bar{g}^{ij}\bar{D}_i\bar{D}_j\psi$ [Ellis, Maartens, and MacCallum, 2012, p.256], this gauge indeed corresponds to an absence of perturbation on this curvature which keeps the uniform value $\mathcal{R} = 6k/a^2$ of the background. For a flat background ($k = 0$), this gauge is also referred to as the *flat gauge* [Malik and Wands, 2009], with $\mathcal{R} = 0$ still in the perturbed spacetime.
- The *comoving orthogonal gauge* requires $B_i = \bar{D}_i B - S_i = 0$ and $V^i = \bar{g}^{ij}\bar{D}_j v + v^i = 0$. These two scalar and two vector conditions represent more requirements than the two scalar ($\delta\eta, \chi$) and one vector (χ^i) gauge degrees of freedom can allow to fulfill in general: the additional assumption of the vanishing of the gauge-invariant vector quantity $v_i - S_i$ [Kodama and Sasaki, 1984] must be made in order for this gauge to be defined. The gauge conditions correspond to the vanishing of the spatial 4-velocity components $u^i = a^{-1}V^i$, meaning that the spatial coordinates are comoving with the fluid flow, and of the spatial components of the velocity 1-form $u_i = a(B_i + \bar{g}_{ij}V^j)$, meaning that the constant- η slices are orthogonal to the fluid flow, $\mathbf{u} = \mathbf{n}$. The latter implies that the 4-velocity is hypersurface-forming and thus that it must be irrotational, which is expressed at first order by the condition $v_i - S_i = 0$. This gauge provides a natural association of both the spatial foliation and the spatial coordinates to the fluid content. The remaining gauge freedom is a time-independent relabelling of the spatial coordinates, $x^i \mapsto x^i + (\delta x^i)(x^k)$, and a relabelling of the spatial slices, $\eta \mapsto \eta + (\delta\eta)(\eta)$.

Yet further gauge conditions may be imposed *e.g.* by alternative specific requirements on how the constant- η hypersurfaces should be defined, such as hypersurfaces of constant energy density ($\delta\epsilon = 0$) or constant scalar extrinsic curvature. Additional conditions may then be set to fully determine the gauge, since the choice of the spatial coordinates otherwise remains free.

1.3.3 Limitations and alternatives to this formalism

The above formalism is *a priori* suited for the study of fluctuations in the radiation-dominated epoch, as well as the linear growth of structures in the early regime of their formation on large scales. Some of its aspects however impede its applicability beyond these regimes. In particular, the deviations from homogeneity of all variables (metric components, velocity field, density, and their spatial derivatives) need to be assumed small and controlled by a given first-order smallness parameter in order to be comparable (once properly adimensioned). Hence, it cannot be used to describe the dynamics at the scales of the largest walls and voids and below, beyond the early onset of structures after the last scattering epoch, when nonlinear inhomogeneities in density (and curvature, as metric derivatives coupled to density) start to form. Deviations from a homogeneous density even become nonperturbative at later times and smaller scales such as those of galaxy superclusters.

The perturbative scheme can be extended to second order (see, *e.g.*, [Malik and Wands \[2009\]](#)) and beyond, to encompass a wider range of perturbations, However, this requires new constructions of gauge-invariant quantities or extensions of the specific gauge choices for each order, and the scheme cannot be readily extended into a nonperturbative approach as would be needed for the dynamics at small scales. This may require going beyond the tight links to the assumed background from which the mapping and gauge-dependence notions originate. The need for prescribing a background also precludes a full investigation of possible ‘backreaction’ effects of the small scale dynamics to the global expansion behaviour at the largest scales, as the latter is given by the background evolution.

The dynamics in the matter-dominated era at scales below the causality horizon size of order $c/H(t)$, are often described in terms of Newtonian models instead. This offers a simpler framework for the investigation of potentially nonlinear dynamics, at scales and for fluid models for which at least special-relativistic effects are of limited relevance. One may use a perturbation scheme based on the Eulerian picture for a Newtonian expanding fluid model, where all variables (velocity field, density, ...) are expanded in terms of small deviations with respect to a prescribed homogeneous and isotropic Hubble flow (see, *e.g.*, [Peebles \[1980\]](#), p.47-68). Such an approach is reminiscent of the relativistic scheme presented in this section in terms of its variables, actually being inspired from a similar relativistic perturbation scheme [[Lifshitz, 1946](#)]. It has a similarly limited applicability (small density fluctuations being assumed), in addition to being restricted to nonrelativistic regimes.

The Newtonian formalism, however, also allows for relatively fast N-body-based numerical investigations of structure formation on top of a prescribed background expansion without restrictions on the growth of density contrasts or other observables. Such numerical schemes overall reproduce rather well the observed distribution of matter, at least above the scales of galaxy groups [[Angulo et al., 2012](#), [Alimi et al., 2012](#)].

On the analytic side, such evolutions may be described thanks to perturbation schemes based on the Newtonian dynamics of an expanding fluid in a *Lagrangian* picture instead, in particular

the Zel'dovich approximation and the underlying Lagrangian scheme [Zel'dovich, 1970]. The perturbation is encoded into the deformation field alone (giving the position of a particle in Eulerian space at a given time as a function of its Lagrangian — initial — position) as a deviation from that of the homogeneous and isotropic Hubble flow. Observables are then computed directly from the perturbed deformation field through exact first integrals: for these calculations, the approximation scheme is nonperturbative. Very large overdensities, for instance, are allowed, and will be reached near shell-crossings (corresponding to a degenerate deformation field).

The Zel'dovich approximation is of rather common use for the onset of initial conditions for Newtonian simulations. While much faster as mostly analytic, it also provides a rather good match to the matter dynamics and distributions obtained with N-body simulations until shell-crossings and multistreaming at small scales become too important [Buchert, 1996]. It may even be extended into these regimes *via* the addition of the contributions of several flows after shell-crossing, a modelling of velocity dispersion as an effective pressure term, and/or the expansion of the deformation field at second order or beyond.

A similar scheme may be implemented in a general-relativistic setting to handle nontrivial curvature contributions and special-relativistic effects, to simultaneously describe the largest scales and those which enter the nonlinear regime of density distributions. This *Relativistic Zel'dovich Approximation* or RZA is also based on the perturbation of a single deformation variable from which observables can be computed in a nonperturbative manner.

1.4 A relativistic Lagrangian scheme

1.4.1 Principle of Lagrangian approaches

Lagrangian picture in Newtonian dynamics and deformation field

In Newtonian fluid mechanics, a Lagrangian description keeps track of the fluid elements through their evolution, and expresses the dynamics in terms of the *deformation field* $\vec{f}(t, \vec{X})$. This vector corresponds to the Eulerian position \vec{x} at time t of the fluid element that was at spatial coordinate \vec{X} at some initial time, $\vec{f}(t_1, \vec{X}) = \vec{X}$. All fields are then expressed as functions of time and of this Lagrangian coordinate \vec{X} which builds a time-independent label for each fluid particle.

The time derivative d/dt along the fluid flow (total or Lagrangian time derivative) is defined in terms of Eulerian positions and the velocity field \vec{v} as $d/dt \equiv \partial_t|_{x^i} + v^i \partial_{x^i}$, featuring an advection term which accounts for the time-dependent Eulerian positions of the fluid elements. However, in the Lagrangian coordinates set (t, \vec{X}) , the total time derivative coincides by construction with the partial time derivative $\partial_t|_{X^i}$, and can be denoted by an overdot for short. The velocity and particle acceleration fields are then straightforwardly expressed from the deformation field as, respectively, $\vec{v} = \dot{\vec{f}}$ and $d\vec{v}/dt = \ddot{\vec{f}}$. Accordingly, unlike Eulerian descriptions, the velocity is not an independent field in a Lagrangian approach.

Considering a self-gravitating dust fluid, the Euler equation reduces to setting the acceleration

as equal to the gravitational field strength \vec{g} , $\ddot{\vec{f}} = \vec{g}$, as a result of assuming the equivalence of inertial and gravitational masses. \vec{g} is constrained as being an irrotational field, $\partial_{x^i} g^j - \partial_{x^j} g^i = 0$, and as satisfying the Poisson equation relating it to its source, the mass density field ϱ : $\partial_{x^i} g^i = -4\pi G\varrho$. The evolution of ϱ is given by the mass conservation equation, $d\varrho/dt + \varrho \partial_{x^i} v^i = 0$. This can be integrated analytically in terms of the initial conditions on ϱ , $\varrho_{\mathbf{i}}(\vec{X}) \equiv \varrho(t_{\mathbf{i}}, \vec{X}) = \varrho(t_{\mathbf{i}}, \vec{x})$, and of the deformation field,

$$\dot{\varrho} + \varrho \frac{{}^N J}{N J} = 0 \quad : \quad \varrho(t, \vec{X}) = \frac{\varrho_{\mathbf{i}}(\vec{X})}{{}^N J(t, \vec{X})} \quad , \quad (1.37)$$

where ${}^N J$ is the Jacobian of the coordinate transformation $\vec{X} \mapsto \vec{x}$ at a given time (the label N standing for ‘Newtonian’),

$${}^N J \equiv \det \left(\frac{\partial f^i}{\partial X^j} \right) \quad ; \quad {}^N J(t_{\mathbf{i}}, \vec{X}) = 1 \quad . \quad (1.38)$$

Hence, apart from $\varrho_{\mathbf{i}}(\vec{X})$ as the initial condition, ϱ , \vec{v} and \vec{g} are no longer independent fields but are expressed in terms of \vec{f} , while the Euler and rest mass conservation equations are automatically satisfied. The only remaining system of equations for the dynamics is thus given by the irrotationality condition and Poisson equation on the acceleration $\ddot{\vec{f}}$ (being equal to \vec{g}). These two equations can be re-expressed in terms of Lagrangian positions and the deformation field as follows, respectively, using the integral (1.37) for ϱ :

$$\epsilon_{klm} \left(\partial_{X^k} \ddot{f}^i \right) \left(\partial_{X^l} f^i \right) \left(\partial_{X^m} f^j \right) = 0 \quad \forall j \quad ; \quad \frac{1}{2} \epsilon_{ijk} \epsilon^{lmn} \left(\partial_{X^l} \ddot{f}^i \right) \left(\partial_{X^m} f^j \right) \left(\partial_{X^n} f^k \right) = -4\pi G \varrho_{\mathbf{i}} \quad , \quad (1.39)$$

with the totally antisymmetric Levi-Civita symbol ϵ_{ijk} , $\epsilon^{lmn} \equiv \delta_i^l \delta_j^m \delta_k^n \epsilon_{ijk}$.

As one can see, for given initial conditions, this Lagrangian system only depends on the single variable $\vec{f}(t, \vec{X})$ and its derivatives. More precisely, it only depends on the Jacobian matrix of the Lagrangian to Eulerian coordinates transformation, ${}^N J_j^i(t, \vec{X}) \equiv \partial_{X^j} f^i(t, \vec{X})$, and its time derivatives. It is then possible to consistently assume a perturbation of \vec{f} or ${}^N J_j^i$ as the *only unknown* as a small deviation with respect to an assumed solution.

The perturbed deformation field and nonperturbative evaluations

In a (Newtonian) cosmological context, the deformation field would typically be written as a deviation from the homogeneous and isotropic Hubble flow solution $\vec{f}_H(t, \vec{X}) = a(t)\vec{X}$:

$$\vec{f}(t, \vec{X}) \equiv a(t) (\vec{X} + \vec{P}(t, \vec{X})) \quad : \quad {}^N J_j^i = a(t) (\delta_j^i + \partial_{X^j} P^i) \quad , \quad (1.40)$$

with $a(t_{\mathbf{i}}) = 1$ and a perturbation vector field $\vec{P}(t, \vec{X})$ such that $\vec{P}(t_{\mathbf{i}}, \vec{X}) = \vec{0}$ and $|\partial_{X^j} P^i| \ll 1$ with also small time derivatives. $a(t)$ is set to follow the Friedmann equations (1.23)–(1.24)

for a dust fluid (with density $\varrho_H(t) = \varrho_{H\mathbf{i}}/a(t)^3$ for some initial homogeneous density $\varrho_{H\mathbf{i}}$ as a parameter) with $\Lambda = 0$ and k as a simple integration constant, which also hold for a Newtonian dust homogeneous and isotropic Hubble flow. A nonzero Λ could be included with a modification to the Poisson equation, which would be sourced by $\Lambda - 4\pi G\varrho$ instead of $-4\pi G\varrho$.

The system of evolution equations on ${}^N J_j^i$ (1.39) is then linearised (or developed to some further order) in the spatial derivatives of \vec{P} to get an approximate solution for the deformation field. At first order for instance, one can find the most general solution of the linearised evolution equations, and use it as the generic expression for \vec{f} for the following steps; it is also possible to restrict it to its growing mode, as done in the original Zel'dovich approximation [Zel'dovich, 1970]. However, observables that can be deduced from \vec{f} are handled in a nonperturbative way, as they are computed from this approximate \vec{f} with their exact, functional expression. In this Newtonian case, this applies most prominently to ϱ , which is computed according to (1.37) from the solution \vec{f} to the linearised version of (1.39), without any further approximation. This allows a strict mass conservation with this deformation field. Moreover, with this procedure, ϱ is allowed to enter a non-linear regime and have large departures from a homogeneous value, as is physically observed in a Universe where structures are already well developed.

The Relativistic Zel'dovich approximation (RZA), as introduced in [Buchert and Ostermann, 2012] (see also Kasai [1995], Matarrese and Terranova [1996] without the formulation of all equations in terms of a single variable), extends these ideas to the general-relativistic framework by writing the 3+1 Einstein equations in terms of a single variable. The perturbation scheme will similarly be set on this variable with respect to a FLRW background. The 3+1 equations are then linearised to get a first-order (or beyond) solution for the perturbation variable, which is used to compute in a nonperturbative way the observables from their functional expression in terms of this variable. Such observables include in this case the energy density, but also, for instance, spatial curvatures or metric distances.

1.4.2 A perturbation variable for the 3+1 Einstein equations

Geometric framework

The RZA framework presented here follows the conventions of Buchert et al. [2013], later applied and extended in [Alles et al., 2015, Al Roumi et al., 2017], as a follow-up to [Buchert and Ostermann, 2012] with a small variant in the decomposition of the metric. Following the assumptions of these papers, the model spacetime considered in the remainder of this section, globally hyperbolic and obeying the Einstein equations, is filled with a dust fluid of 4-velocity \mathbf{u} : $T^{\mu\nu} = \varrho u^\mu u^\nu$ with a vanishing pressure and a rest mass density ϱ associated with the conserved current ϱu^μ . The fluid is also assumed to be irrotational, *i.e.*, in terms of the dual 1-form $\underline{\mathbf{u}}$ to \mathbf{u} , $\underline{\mathbf{u}} \wedge d\underline{\mathbf{u}} = 0$. As seen in section 1.1, this is a necessary condition for the use of the fluid's 4-velocity \mathbf{u} as the normal vector \mathbf{n} to spatial hypersurfaces. The Frobenius theorem ensures that it is locally a sufficient condition, and it will be assumed here that such (automatically

space-like) hypersurfaces can be defined globally to build a spatial foliation and used within the 3+1 formalism.

The remainder of this section uses this 3+1 formalism as described in section 1.1, with the above choice of fluid-orthogonal hypersurfaces, labelled by the coordinate t with a choice of lapse as $N = 1$. The latter is allowed by the vanishing of the spatial variation (1.6) of the lapse, since the dust fluid flow is geodesic, $u^\mu \nabla_\mu u_\nu = 0$, and $\mathbf{n} = \mathbf{u}$. This also means that t is a proper time for the fluid, as the cosmic time of FLRW models (see section 1.2). A vanishing shift is also set, fully specifying the degrees of freedom of the 3+1 picture. The associated spatial coordinates X^i are comoving with the fluid flow and are thus the relativistic analogs of the Lagrangian positions in the Newtonian setting, hence the use of the same notation. The partial time derivative in the coordinate system (t, X^i) thus coincides with the proper time directional derivative along the fluid lines, $\partial_t|_{X^k} = u^\mu \partial_\mu$, and will be denoted hereafter by an overdot. In the coordinates (t, X^i) , the metric line element takes the form

$$ds^2 = -dt^2 + h_{ij} dX^i dX^j . \quad (1.41)$$

The spatial coframes

The objects that will play an analogous role to the spatial derivatives of the Newtonian deformation field (Jacobian matrix) in the RZA are the components η^a_i of spatial coframes $\boldsymbol{\eta}^a$, where $a = 1, 2, 3$ is a counting index. These coframes are 1-forms building a generalized (non-coordinate) basis of the forms on the spatial hypersurfaces, and they are defined as satisfying

$$\mathbf{h} = G_{ab} \boldsymbol{\eta}^a \otimes \boldsymbol{\eta}^b \quad , \quad \partial_t|_{X^k}(G_{ab}) = 0 , \quad (1.42)$$

or, in components on the (t, X^i) coordinate basis, with $\boldsymbol{\eta}^a \equiv \eta^a_i dX^i$,

$$g_{ij}(t, X^k) = h_{ij}(t, X^k) = G_{ab}(X^k) \eta^a_i(t, X^k) \eta^b_j(t, X^k) . \quad (1.43)$$

Here G_{ij} are the initial spatial metric coefficients, that is

$$G_{ab}(X^k) = \delta_a^i \delta_b^j g_{ij}(t_i, X^k) , \quad (1.44)$$

so that the spatial metric coefficients in the coframe basis G_{ab} are time-independent, and the coframes themselves contain all the information about the time evolution of the spatial metric. This is a generalization of the coframes that are more usually set to be orthonormal (Cartan coframes $\tilde{\boldsymbol{\eta}}^a$: $\mathbf{h} = \delta_{ab} \tilde{\boldsymbol{\eta}}^a \otimes \tilde{\boldsymbol{\eta}}^b$), as in [Buchert and Ostermann, 2012] for instance. Instead, the convention of Buchert et al. [2013] as used here allows for a simpler form of the RZA, closer to the Newtonian case. In particular, the coframes coefficients can be assumed (in compatibility with (1.44)) to be initially $\eta^a_i(t_i, X^k) = \delta^a_i$ as were $\partial_i f^a$ in the Newtonian case, without restriction of generality and in particular without constraining the initial spatial metric to be Euclidean.

The dual basis of the coframes are the frames $e_a \equiv e_a^i \partial_{X^i}$, satisfying

$$e_a^i \eta_j^a = \delta_j^i \ ; \ e_a^i \eta_i^b = \delta_a^b . \quad (1.45)$$

Introducing the determinant of the coframe coefficients matrix,

$$J \equiv \det(\eta_i^a) = \frac{1}{6} \epsilon_{abc} \epsilon^{ijk} \eta_i^a \eta_j^b \eta_k^c , \quad (1.46)$$

the coefficients of the frames can be expressed in terms of those of the coframes as

$$e_a^i = \frac{1}{2J} \epsilon_{abc} \epsilon^{ijk} \eta_j^b \eta_k^c . \quad (1.47)$$

As the inverse metric is simply expressed in terms of the frames and the inverse of the initial metric (G^{ab} satisfying $G^{ab} G_{bc} = \delta_c^a$), $g^{ij} = G^{ab} e_a^i e_b^j$, it can also be expanded in terms of the coframes and $G^{ab}(X^k)$.

The requirement (1.42) on the coframes allows for a freedom in their definition, in terms of a spacetime-dependent ‘rotation’ (isometry of the initial metric, reducing to a rotation if the latter is Euclidean), $\boldsymbol{\eta}^a \mapsto A^a_b(t, X^k) \boldsymbol{\eta}^b$ with $G_{ab}(X^k) A^a_c(t, X^k) A^b_d(t, X^k) = G_{cd}(X^k)$. This freedom can be used to set a symmetry condition on the evolution of the coframes:

$$G_{ab} \dot{\eta}^a_{[i} \eta^b_{j]} = 0 . \quad (1.48)$$

This condition will simplify further expressions and, together with the initial choice $\eta_i^a(t_{\mathbf{i}}, X^k) = \delta_i^a$, fully determines the coframes. It is moreover required in order for the coframes to be a relativistic equivalent of the Newtonian deformation field within this irrotational framework³.

Rewriting the 3+1 Einstein equations in terms of the coframes

Using the above symmetry condition, the spatial components of the mixed-indices extrinsic curvature can be expressed from (1.13) as

$$\mathcal{K}^i_j = -e_a^i \dot{\eta}^a_j = -\frac{1}{2J} \epsilon_{abc} \epsilon^{ikl} \dot{\eta}^a_j \eta_k^b \eta_l^c . \quad (1.49)$$

The opposite of its trace coincides in this fluid-orthogonal setting with the expansion scalar $\Theta \equiv \nabla_\mu u^\mu = \nabla_\mu n^\mu = -\mathcal{K}$ involved in the rest mass conservation equation, $\dot{\varrho} + \varrho\Theta = 0$. With the coframe choice (1.42), J relates to the determinants $G \equiv \det(G_{ab})$ and $g = |\det(g_{\mu\nu})|$ (which

³The *Minkowski restriction* [Buchert and Ostermann, 2012, Buchert et al., 2013] consists in assuming that G_{ab} can be taken as δ_{ab} for appropriate fluid-comoving coordinates X^i and that the coframes are exact spatial 1-forms, $\boldsymbol{\eta}^a = df^a$ within each spatial slice. The spatial metric is then Euclidean, $\mathbf{h} = \delta_{ab} df^a \otimes df^b$ (cf. (1.61)), with components δ_{ab} in the coordinates $x^a = f^a(t, X^k)$. The latter coordinates are then Eulerian coordinates in this flat space, and in these coordinates, the above symmetry condition reduces in this restriction to a vanishing rotational of the velocity field, $\partial f^{[a} / \partial x^{b]} = 0$.

in the present framework is also equal to $\det(h_{ij})$, as $J = \sqrt{g/G}$. Hence, using moreover (1.13) to relate $\dot{g} = h^{ij}\dot{h}_{ij}$ and $\Theta = -\mathcal{K}$, one has

$$\Theta = \frac{\partial_t \sqrt{g}}{\sqrt{g}} = \frac{\dot{J}}{J}. \quad (1.50)$$

As a consequence, the rest mass conservation equation can be exactly integrated as

$$\varrho(t, X^k) = \frac{\varrho_{\mathbf{i}}(X^k)}{J(t, X^k)} \quad ; \quad \varrho_{\mathbf{i}}(X^k) \equiv \varrho(t_{\mathbf{i}}, X^k). \quad (1.51)$$

This is a direct analog of the Newtonian integral with the formal substitution $\partial_{X^i} f^a \mapsto \eta^a_i$.

Using the above expressions of \mathcal{K}^i_j (from the 3+1 equation (1.13)) and ϱ , the remaining 3+1 Einstein equations (1.14)–(1.16) can be re-expressed in terms of the coframes:

$$\frac{1}{2} \epsilon_{abc} \epsilon^{ikl} \ddot{\eta}^a_j \eta^b_k \eta^c_l + \epsilon_{abc} \epsilon^{ikl} \dot{\eta}^a_j \dot{\eta}^b_k \eta^c_l = (4\pi G \varrho_{\mathbf{i}} + \Lambda J) \delta^i_j - J \mathcal{R}^i_j ; \quad (1.52)$$

$$\epsilon_{abc} \epsilon^{ikl} \dot{\eta}^a_i \dot{\eta}^b_k \eta^c_l = 16\pi G \varrho_{\mathbf{i}} + (2\Lambda - \mathcal{R}) J ; \quad (1.53)$$

$$D_i (\epsilon_{abc} \epsilon^{ikl} \dot{\eta}^a_j \eta^b_k \eta^c_l) = D_j (\epsilon_{abc} \epsilon^{ikl} \dot{\eta}^a_i \eta^b_k \eta^c_l) . \quad (1.54)$$

This system is complemented by the symmetry condition (1.48). One can explicitly add the (redundant) *Raychaudhuri equation* arising from the combination of the trace of the evolution equation (1.52) and the Hamilton constraint (1.53) which eliminates the spatial Ricci tensor:

$$\frac{1}{2} \epsilon_{abc} \epsilon^{ikl} \ddot{\eta}^a_i \eta^b_k \eta^c_l = \Lambda J - 4\pi G \varrho_{\mathbf{i}} . \quad (1.55)$$

Under this form, it is almost manifest that the above system of constraints and evolution equations only depends on the coefficients η^a_i as dynamical variables, in addition to constrained initial conditions. This dependency is only implicit for J , the spatial covariant derivatives and the spatial Ricci tensor, but in the same way, they can all be expressed as functions of the coframes alone.

This is indeed obvious for J from equation (1.46), whereas the covariant derivatives only involve connection coefficients that can be expressed in terms of the inverse metric and derivatives of the metric, thus in terms of the coframes and their derivatives. The spatial Ricci tensor is deduced (by contraction of the spatial Riemann tensor, of components \mathcal{R}^i_{jkl}) from the commutation of the spatial covariant derivatives of the coframes,

$$2 D_{[k} D_{l]} \eta^a_i = \mathcal{R}_{kli}{}^m \eta^a_m , \quad (1.56)$$

implying

$$\mathcal{R}_{ij} = 2 e_a{}^k D_{[j} D_{k]} \eta^a_i . \quad (1.57)$$

As the frames, the spatial covariant derivatives, and the rising of indices are all expressed in terms of the coframes, so are the spatial Ricci tensor with mixed indices \mathcal{R}^i_j and the Ricci scalar $\mathcal{R} = \mathcal{R}^i_i$. However, for compactness, these quantities will not usually be explicitly expressed in terms of the coframes in the following.

As the main variable for the RZA, the coframes can then be perturbed with respect to a reference (here FLRW) expression, similarly to the deformation field in the Newtonian case.

1.4.3 First-order expansion of the coframes and nonperturbative evaluations

A first-order solution for the coframes

Writing the coframes as a first-order deviation from their FLRW expression, one has

$$\eta^a_i \equiv a(t) \left(\delta^a_i + P^a_i(t, X^k) \right), \quad (1.58)$$

with $|P^a_i| \ll 1$ and $P^a_i(t_i, X^k) = 0$. The scale factor $a(t)$ is initially equal to 1 and follows the Friedmann equations (1.23)–(1.24) for a dust fluid with some homogeneous energy density $\varrho_H(t)$ evolving as $\varrho_H(t) = a(t)^{-3} \varrho_H(t_i)$, and freely specifiable parameters $\varrho_H(t_i)$, Λ and k .

The constraints and evolution equations (1.48), (1.52–1.55) can then be linearised in P^a_i to search for the general solution for the perturbed coframes at this order. Finding this general solution would amount to solving a linear system of coupled partial differential equations in the 9 components of P^a_i in the variables (t, X^i) .

Instead, following [Buchert and Ostermann, 2012, Buchert et al., 2013], this problem can be simplified by restricting the general solution to the case where all components of P^a_i have the same time evolution as its trace $P \equiv \delta_a^i P^a_i$, considering that this trace part already captures most aspects of the interactions with the matter sources and the development of inhomogeneities. Such an assumption is possible due to the direct coupling between spatial derivatives of P^a_i and its trace arising from the momentum constraint (1.54), linearised in P^a_i and $G_{ab} \simeq \delta_{ab}$ (for a small initial intrinsic curvature) and time-integrated as $\partial_j P^j_i = \partial_i P$ with $P^j_i \equiv \delta_a^j P^a_i$. This assumption of direct coupling of all components extends the ‘slaving’ principle of the (Newtonian) Zel’dovich approximation, where the velocity field is assumed to be proportional to the acceleration by a choice of initial conditions selecting only the growing mode. As in this Newtonian case, only the growing mode will be kept here in the solutions for the evolution of P itself, on top of the assumption on the trace-free components, assuming the other modes to be already absent in the initial conditions.

The evolution of the trace part is obtained from linearising the Raychaudhuri equation (1.55) and applying an additional time derivative, which gives, in terms of $H(t) = \dot{a}(t)/a(t)$:

$$\partial_t \left[a^3 \left(\ddot{P} + 2H(t)\dot{P} - 4\pi G \varrho_H(t)P \right) \right] = 0. \quad (1.59)$$

As it follows a third-order linear ordinary differential equation, the solution for P is made of three

modes, reducing to a sum of two independent modes after making use of the initial condition $P(t_i, X^k) = 0$. As said above, as part of the approximation, the decaying mode is assumed to vanish, so that P takes the separable growing-mode form $P(t, X^k) = \xi(t)\mathcal{P}(X^k)$ where \mathcal{P} is time-independent, otherwise arbitrary, and $\xi(t)$ satisfies

$$\begin{aligned} \ddot{\xi} + 2H(t)\dot{\xi} - 4\pi G_{\rho H}(t)(\xi + K) &= 0 ; \\ \xi(t_i) &= 0 ; \dot{\xi}(t_i) = 1 , \end{aligned} \quad (1.60)$$

with an arbitrary constant K indicating the absence of constraint on $\ddot{\xi}(t_i)$.

All components will thus take the form $P_i^a(t, X^k) = \xi(t)\mathcal{P}_i^a(X^k)$ within the above assumptions, giving the following prescription for the components of the first-order approximate coframes:

$$\eta_i^a(t, X^k) = a(t) [\delta_i^a + \xi(t)\mathcal{P}_i^a(X^k)] , \quad (1.61)$$

with $\xi(t)$ obeying (1.60). The time-independent functions \mathcal{P}_i^a are arbitrary apart from their need to comply with the constraints provided by (1.48) and (1.53)–(1.54) at initial time, at least to first order.

Nonperturbative estimates for the density and other observables

Finally, all observables shall be derived non-perturbatively from their exact functional expression in terms of the coframes, the latter being replaced by their prescribed value (1.61). The rest mass density for instance is computed according to (1.51), which ensures exact mass conservation, as in the Newtonian case. Introducing the following initial invariants, that are functions of the spatial position only:

$$\text{I}_i \equiv \frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\mathcal{P}_i^a\delta_k^b\delta_l^c = \mathcal{P} ; \quad (1.62)$$

$$\text{II}_i \equiv \frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\mathcal{P}_i^a\mathcal{P}_k^b\delta_l^c ; \quad (1.63)$$

$$\text{III}_i \equiv \frac{1}{6}\epsilon_{abc}\epsilon^{ikl}\mathcal{P}_i^a\mathcal{P}_k^b\mathcal{P}_l^c = \det(\mathcal{P}_i^a) , \quad (1.64)$$

one gets within the above prescription (1.61) for η_i^a :

$$\varrho = \frac{\varrho_i}{a^3(1 + \xi \text{I}_i + \xi^2 \text{II}_i + \xi^3 \text{III}_i)} . \quad (1.65)$$

This allows for large overdensities for a large enough $\xi(t)$ at fluid elements where the denominator becomes small.

The spatial curvature can be similarly evaluated from its coframe expression (1.57), and metric distances would be computed from the exact expression of the metric components $g_{ij} = a^2 G_{ab}(\delta_i^a + P_i^a)(\delta_j^b + P_j^b)$, keeping the quadratic terms in P_k^c . It should be noted, however, that

each quantity may in principle be evaluated from several different functional expressions (*e.g.*, the scalar curvature, or the density, can alternatively be computed from the Hamilton constraint (1.53)), giving different result since the coframes used only approximately solve the Einstein equations. Comparing such different results provides a test of the quality of the approximation; see, *e.g.*, Doroshkevich et al. [1973] for such a comparison for density in the Newtonian framework.

Extending the scheme

The RZA can be further extended by carrying the expansion of the coframes beyond the first order (see Alles et al. [2015]) or, within the first order, by relaxing the assumptions on the coupling of all components of the perturbation to its trace. For the latter generalization, following [Al Roumi et al., 2017], the traceless part $\Pi_j^i \equiv P_j^i - P \delta_j^i/3$ can be split into an electric part ${}^E\Pi_j^i$ and a magnetic part⁴ ${}^H\Pi_j^i$, $\Pi_j^i = {}^E\Pi_j^i + {}^H\Pi_j^i$. The electric part is required to satisfy the direct coupling to the trace arising from the linearised momentum constraint, $\partial_j {}^E\Pi_j^i = \partial_j \Pi_j^i = 2 \partial_i P/3$, by having the same time dependence as P as a longitudinal mode. It is thus the part already considered above. The magnetic part decouples from the trace and is transverse, $\partial_j {}^H\Pi_j^i = 0$. It contains additional degrees of freedom that are related to the description of gravitational waves, *via* the damped wave equation $a^2({}^H\ddot{\Pi}_j^i + 3H {}^H\dot{\Pi}_j^i) - \delta^{kl} \partial_k \partial_l {}^H\Pi_j^i = s_j^i$ for some time-independent source $s_j^i(X^k)$, and according to the detailed comparison of this field to the standard description of gravitational waves made explicit in [Al Roumi et al., 2017]. Other possible extensions can include the consideration of model fluids with nonzero pressure, to which part of this thesis has been dedicated (see chapter 2).

Despite relaxing the assumptions on the deviation of the energy density and other observables from homogeneity and thus probing further into the nonlinear regime of structure formation, the RZA still relies on the prescription of a background. It might be modified into an iterative scheme that updates the background at each time step to account for the non-FLRW metric evolution (see Roy and Buchert [2012] for a description of deviations out of an updating background, but without the use of the Lagrangian coframes as the main variable). This approach relies on the definition of an ‘effective’ scale factor and associated effective FLRW metric at each step, from an average of the inhomogeneous dynamics. Such descriptions in terms of averages are not sufficient by themselves (without, *e.g.*, associating them with approximations such as the RZA) to give a local model of the evolving spatial metric and matter distribution. As I will detail now, they provide, however, an efficient and background-free description of the effective dynamics on a given scale (including a comparison to FLRW on the largest scales) as being affected by the smaller-scale inhomogeneities.

⁴This split is not, in general, directly related to the electric and magnetic parts of the Weyl tensor. At first order in the RZA, the magnetic Weyl tensor (in the fluid frame), for instance, depends on all components of the perturbation field P_j^i (see Al Roumi et al. [2017] for details).

1.5 Spatial averaging and backreaction from inhomogeneities

1.5.1 Averaging procedure

This section presents the spatial averaging framework for scalars and associated description of the averaged dynamics of a spatial region introduced in [Buchert and Ehlers, 1997, Buchert, 2000] for dust fluids and generalized in [Buchert, 2001] for fluids with pressure. Following the framework of both references, the model general-relativistic universe under consideration is filled with an irrotational perfect fluid of 4-velocity \mathbf{u} , and is described in an adapted 3+1 picture.

As for the RZA in the previous section, for such an irrotational fluid a foliation of spacetime into fluid-orthogonal spatial hypersurfaces ($\mathbf{n} = \mathbf{u}$) is considered, and spatial coordinates X^i are chosen as being comoving with the fluid elements, so that the associated shift vanishes. As in [Buchert, 2001], without further assumptions, the time coordinate t is freely chosen among the possible labels that are constant on each spatial slice, so that the lapse N is only defined by the constraint (1.6) on its spatial variations (with $\mathbf{a}^{(n)}$ equal to the 4-acceleration $\mathbf{a} = a^\mu \partial_{x^\mu}$ of the fluid). In the coordinates (t, X^i) , the line element takes the following form:

$$ds^2 = -N^2 dt^2 + h_{ij} dX^i dX^j, \quad (1.66)$$

and the components of the 4-velocity \mathbf{u} and its dual read, respectively:

$$u^\mu = \left(\frac{1}{N}, 0, 0, 0 \right) \quad ; \quad u_\mu = (-N, 0, 0, 0). \quad (1.67)$$

Such a 3+1 setting may be seen as a background-free, nonperturbative analog of the comoving orthogonal gauge of Eulerian relativistic perturbation theory (*cf.* subsection 1.3.2).

Spatial averaging can then be performed over any given compact domain \mathcal{D} lying within a fluid-orthogonal spatial slice. This averaging domain is propagated between all slices *via* the fluid flow, *i.e.* it is defined as a Lagrangian domain, which follows the fluid through its evolution. This ensures the conservation of the collection of fluid elements it contains, and in particular of the total fluid rest mass within \mathcal{D} .

The Riemannian volume of the domain within a given spatial slice is given by

$$\mathcal{V}_{\mathcal{D}}(t) \equiv \int_{\mathcal{D}} \sqrt{h}(t, X^k) d^3 X, \quad (1.68)$$

where $h \equiv \det(h_{ij})$, *i.e.*, $\sqrt{h} d^3 X$ is the Riemannian spatial volume element in the fluid-orthogonal hypersurfaces. The spatial average of a scalar ψ over the domain at a given hypersurface is then defined as its volume average [Buchert and Ehlers, 1997, Buchert, 2000, 2001]:

$$\langle \psi \rangle_{\mathcal{D}}(t) \equiv \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}} \psi(t, X^k) \sqrt{h}(t, X^k) d^3 X, \quad (1.69)$$

the normalization ensuring that $\langle \psi \rangle_{\mathcal{D}} = \psi$ if ψ is a constant or is homogeneous on the slice. t being only used as a parameter, and the foliation being selected from a geometric requirement, this definition is coordinate-independent.

This operation is only well-defined for scalars. A similar procedure for higher-type tensors would additionally require a comparison of the values of a tensor at different events, hence between different tangent or cotangent spaces (or higher-type tensor spaces), raising the issue of the non-uniqueness of mappings between these different spaces. Accordingly, as in [Buchert, 2000, 2001], the problem of defining an averaging operation for tensors will not be addressed here; see, *e.g.*, Zalaletdinov [1992], Paranjape [2009], Korzyński [2010] and references therein for some proposals for such a formalism.

1.5.2 Effective dynamics of the region of averaging and backreactions

The overall evolution of the Lagrangian averaging domain \mathcal{D} may be characterized by an *effective scale factor* $a_{\mathcal{D}}(t)$ as an estimate of its typical ‘size’ averaged over all directions at a given time, defined from the volume at time t and the volume $\mathcal{V}_{\mathcal{D}_i}$ at initial time t_i as

$$a_{\mathcal{D}}(t) \equiv \sqrt[3]{\frac{V_{\mathcal{D}}(t)}{V_{\mathcal{D}_i}}} . \quad (1.70)$$

In a strictly homogeneous and isotropic geometry, this would reduce to the FLRW scale factor $a(t)$ (normalized as $a(t_i) = 1$) for any Lagrangian domain: differences in the evolution of $a_{\mathcal{D}}(t)$ with respect to such an FLRW scale factor will represent *backreaction* effects of inhomogeneities within the domain on the domain’s dynamics.

These effects will be first described below for a pressureless fluid, with energy-momentum tensor components $T_{\mu\nu} = \rho u_{\mu}u_{\nu}$, following [Buchert, 2000]. In this case, as for the RZA framework above (see subsection 1.4.2), t is chosen as a proper time for the dust fluid, reducing the lapse to 1 and the line element to the simpler form (1.41).

Commutation rule for dust

Backreaction effects arise as a consequence of the nonlinearity of the local dynamical equations, and of the lack of commutation of the spatial averaging operation and of the time derivative along the fluid flow $d/dt \equiv \partial_t|_{X^k}$ (simply reducing to the time derivative for a purely time-dependent function such as $\mathcal{V}_{\mathcal{D}}(t)$, $a_{\mathcal{D}}(t)$ or an average). The latter property is characterized by the following *commutation rule*, expressed in terms of the expansion scalar of the fluid $\Theta = \nabla_{\mu}u^{\mu}$ for any scalar ψ [Buchert and Ehlers, 1997, Buchert, 2000]:

$$\frac{d}{dt} \langle \psi \rangle_{\mathcal{D}} - \left\langle \frac{d\psi}{dt} \right\rangle_{\mathcal{D}} = \langle \Theta \psi \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}} . \quad (1.71)$$

Unless Θ or ψ is homogeneous on the $t = cst$ slice, the correlation function on the right-hand side is nonzero and there is non-commutation.

This rule can be shown by first considering the time evolution of the volume integral of a scalar ψ over the domain, $(d/dt)(\int_{\mathcal{D}} \psi \sqrt{h} d^3X)$. Since \mathcal{D} and the spatial coordinates X^i are comoving with the fluid flow, the (compact) domain of integration in the spatial coordinates space is time-independent for these spatial coordinates, and the integration and time derivative $d/dt = \partial_t|_{X^k}$ can be commuted. Further using the first equality in (1.50) obtained within the same framework, $\Theta = h^{-1/2} d(\sqrt{h})/dt$ since $g = h$, one gets

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{D}} \psi \sqrt{h} d^3X &= \int_{\mathcal{D}} \frac{d}{dt} (\psi \sqrt{h}) d^3X \\ &= \int_{\mathcal{D}} \left(\frac{d\psi}{dt} + \psi \Theta \right) \sqrt{h} d^3X . \end{aligned} \quad (1.72)$$

Applying this formula to $\psi = 1$ and dividing by $\mathcal{V}_{\mathcal{D}}$ gives the rates of evolution of the volume and the effective scale factor:

$$\frac{1}{\mathcal{V}_{\mathcal{D}}} \frac{d\mathcal{V}_{\mathcal{D}}}{dt} = \frac{3}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} = \langle \Theta \rangle_{\mathcal{D}} . \quad (1.73)$$

The combination of (1.72) for a given ψ and of (1.73) gives the commutation rule (1.71) for ψ from $d\langle \psi \rangle_{\mathcal{D}}/dt = \mathcal{V}_{\mathcal{D}}^{-1}(d/dt)(\int_{\mathcal{D}} \psi \sqrt{h} d^3X) - \langle \psi \rangle_{\mathcal{D}} \mathcal{V}_{\mathcal{D}}^{-1} d\mathcal{V}_{\mathcal{D}}/dt$.

Averaging the dust-source Einstein equations

The *expansion tensor* of the fluid is in general defined as the symmetrized fluid-orthogonal projected gradient of the 4-velocity, $\Theta_{\mu\nu} \equiv b^{\rho}_{(\mu} b^{\sigma}_{\nu)} \nabla_{\rho} u_{\sigma}$ with $b^{\mu}_{\nu} \equiv \delta^{\mu}_{\nu} + u^{\mu} u_{\nu}$ the components of the fluid-orthogonal projection operator. The antisymmetric part $b^{\rho}_{[\mu} b^{\sigma}_{\nu]} \nabla_{\rho} u_{\sigma}$ defines the *vorticity 2-form* components $\omega_{\mu\nu}$ and vanishes for the irrotational fluid considered here. The expansion tensor can be split into its trace which is the expansion scalar, and its traceless part defining the (symmetric) *shear tensor*:

$$\Theta_{\mu\nu} = \frac{1}{3} \Theta b_{\mu\nu} + \sigma_{\mu\nu} \quad ; \quad \Theta = \Theta^{\mu}_{\mu} \quad ; \quad \sigma^{\mu}_{\mu} = 0 \quad ; \quad u^{\mu} \sigma_{\mu\nu} = 0 . \quad (1.74)$$

The latter is associated with the positive-definite *shear scalar*, $\sigma^2 \equiv \sigma_{\mu\nu} \sigma^{\mu\nu} / 2$.

For the fluid-orthogonal foliation considered, $b^{\mu}_{\nu} = h^{\mu}_{\nu}$, and $\nabla_{\mu} u_{\nu} = \nabla_{\mu} n_{\nu}$, so that the opposite of the extrinsic curvature coincides with the expansion tensor of the fluid, $-\mathcal{K}_{ij} = \Theta_{ij}$ in spatial components while the other components vanish. In this situation, with $N^i = 0$ and

$N = 1$ and within the dust fluid assumption, the 3+1 Einstein equations (1.13)–(1.16) reduce to

$$\partial_t|_{X^k} h_{ij} = 2\Theta_{ij} ; \quad (1.75)$$

$$\partial_t|_{X^k} \Theta^i_j = -\Theta\Theta^i_j - \mathcal{R}^i_j + (4\pi G\rho + \Lambda)\delta^i_j ; \quad (1.76)$$

$$\frac{2}{3}\Theta^2 - 2\sigma^2 = 16\pi G\rho + 2\Lambda - \mathcal{R} ; \quad (1.77)$$

$$D_k\Theta^k_i = D_i\Theta . \quad (1.78)$$

The trace of (1.76) can be combined with the Hamilton constraint (1.77) to provide an additional scalar equation, the Raychaudhuri equation:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 = -4\pi G\rho + \Lambda . \quad (1.79)$$

This system can be complemented by the rest mass conservation equation arising from $\nabla_\mu T^{\mu\nu} = 0$ projected onto the fluid 4-velocity, giving yet another scalar equation:

$$\dot{\rho} + \rho\Theta = 0 . \quad (1.80)$$

The averaging operator defined hereabove can then be applied to the scalar equations (1.77), (1.79) and (1.80). Using the commutation rule (1.71) and the scale factor evolution rate expression (1.73), one gets the following evolution equations for the effective scale factor [Buchert, 2000]:

$$3\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = 8\pi G\langle\rho\rangle_{\mathcal{D}} + \Lambda - \frac{1}{2}\langle\mathcal{R}\rangle_{\mathcal{D}} - \frac{1}{2}\mathcal{Q}_{\mathcal{D}} ; \quad (1.81)$$

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G\langle\rho\rangle_{\mathcal{D}} + \Lambda + \mathcal{Q}_{\mathcal{D}} ; \quad (1.82)$$

$$\langle\rho\rangle_{\mathcal{D}}\dot{a}_{\mathcal{D}} + 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\langle\rho\rangle_{\mathcal{D}} = 0 , \quad (1.83)$$

introducing the *kinematical backreaction* term,

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left(\langle\Theta^2\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}^2\right) - 2\langle\sigma^2\rangle_{\mathcal{D}} . \quad (1.84)$$

The average rest mass density conservation equation (1.83) is equivalent to the conservation of the total fluid rest mass within the domain $\mathcal{M}_{\mathcal{D}} \equiv \int_{\mathcal{D}} \rho \sqrt{h} d^3X = \mathcal{V}_{\mathcal{D}} \langle\rho\rangle_{\mathcal{D}}$: $d\mathcal{M}_{\mathcal{D}}/dt = 0$, as a consequence of the fluid-comoving domain assumption.

Backreactions for a dust fluid

The system of averaged dynamical equations (1.81)–(1.83) exhibits several qualitative differences with respect to the Friedmann equations (1.23)–(1.25) for a strictly FLRW model universe.

One important difference is the dependence of each term (except Λ) on the averaging region, hence on the scale considered or on the position of the domain with respect to over- and underdense regions. Hence the dynamics of $a_{\mathcal{D}}$, as well as the contributions (and even the signs) of $\langle \mathcal{R} \rangle_{\mathcal{D}}$ and $\mathcal{Q}_{\mathcal{D}}$ can vary with the region of averaging.

These equations also feature the kinematical backreaction term $\mathcal{Q}_{\mathcal{D}}$ as an additional source to the dynamics of $a_{\mathcal{D}}$. This term compares two positive contributions: the variance of the expansion scalar and the average shear scalar. In an FLRW model, the former vanishes by homogeneity and the latter is zero due to isotropy, implying the absence of kinematical backreaction. $\mathcal{Q}_{\mathcal{D}}$ is thus a direct effect and a measure of inhomogeneity and anisotropy within the domain, with qualitative consequences on the effective dynamics of the set of fluid elements considered. The acceleration equation (1.82) shows that a positive backreaction for a given domain will contribute positively to its effective scale factor acceleration, thus contributing to the role played by a positive Λ or another description of an acceleration-inducing Dark Energy. Conversely, a negative backreaction for another domain choice will contribute as an additional mass to the deceleration, playing the role of a Dark Matter-like source. Such behaviours indeed arise in some inhomogeneous cosmological models (see, *e.g.*, Räsänen [2006] (in a Newtonian model), Buchert and Carfora [2008], Chuang et al. [2008], Wiegand and Buchert [2010]; and Buchert and Räsänen [2012] for a review).

The contribution from the averaged intrinsic spatial scalar curvature also differs from FLRW models. The local curvature is inhomogeneous, being coupled to the matter distribution, and its time evolution is not globally constrained: while a curvature $\mathcal{R}_H = 6k/a(t)^2$ is accounted for in the Friedmann equations, in general $\langle \mathcal{R} \rangle_{\mathcal{D}}$ will be domain-dependent and needs not evolve as $1/a_{\mathcal{D}}^2$. Its nontrivial evolution is directly coupled to $\mathcal{Q}_{\mathcal{D}}$. This can be seen by combining the time derivative of (1.81) with (1.82), using $\langle \varrho \rangle_{\mathcal{D}}(t) = \langle \varrho \rangle_{\mathcal{D}}(t_i)/a_{\mathcal{D}}(t)^3$ as an integral of (1.83). The result amounts to expressing the compatibility between the evolution equations (1.81) and (1.82) as the following *integrability condition*:

$$\dot{\mathcal{Q}}_{\mathcal{D}} + 6 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \mathcal{Q}_{\mathcal{D}} + \langle \dot{\mathcal{R}} \rangle_{\mathcal{D}} + 2 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \langle \mathcal{R} \rangle_{\mathcal{D}} = \frac{1}{a_{\mathcal{D}}^6} \frac{d}{dt} (a_{\mathcal{D}}^6 \mathcal{Q}_{\mathcal{D}}) + \frac{1}{a_{\mathcal{D}}^2} \frac{d}{dt} (a_{\mathcal{D}}^2 \langle \mathcal{R} \rangle_{\mathcal{D}}) = 0. \quad (1.85)$$

This differs from the FLRW models where the compatibility between the homogeneous Hamilton and Raychaudhuri equations (1.23)–(1.24) is equivalent to the energy conservation equation (1.25). Here, the three average equations (1.81)–(1.83) are independent.

The different dynamics of the average (as opposed to homogeneous) curvature can thus be seen as another backreaction effect of the inhomogeneities within \mathcal{D} on its evolution. This can be made explicit by rewriting the evolution equations for $a_{\mathcal{D}}$ (1.81), (1.82), and the integrability condition (1.85) combined with the average energy conservation equation (1.83), respectively,

under an effective non-dust Friedmannian form [Buchert, 2001, 2005]:

$$3 \left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 = 8\pi G \epsilon_{\text{eff}}^{\mathcal{D}} + \Lambda - \frac{3k_{\mathcal{D}}}{a_{\mathcal{D}}^2} ; \quad (1.86)$$

$$3 \frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G (\epsilon_{\text{eff}}^{\mathcal{D}} + 3p_{\text{eff}}^{\mathcal{D}}) + \Lambda ; \quad (1.87)$$

$$\dot{\rho}_{\text{eff}}^{\mathcal{D}} + 3 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} (\epsilon_{\text{eff}}^{\mathcal{D}} + p_{\text{eff}}^{\mathcal{D}}) = 0 , \quad (1.88)$$

where $k_{\mathcal{D}}$ is a domain-dependent constant parameter. This system is formally equivalent to the Friedmann equations and FLRW energy conservation law (1.23)–(1.25) with domain-dependent effective perfect fluid source terms $\epsilon_{\text{eff}}^{\mathcal{D}}(t)$ and $p_{\text{eff}}^{\mathcal{D}}(t)$, defined as follows:

$$\epsilon_{\text{eff}}^{\mathcal{D}} \equiv \langle \varrho \rangle_{\mathcal{D}} - \frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}} - \frac{1}{16\pi G} \mathcal{W}_{\mathcal{D}} ; \quad (1.89)$$

$$p_{\text{eff}}^{\mathcal{D}} \equiv -\frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}} + \frac{1}{48\pi G} \mathcal{W}_{\mathcal{D}} . \quad (1.90)$$

In this rewriting, the averaged curvature contribution has been explicitly split into a Friedmannian term decreasing as $1/a_{\mathcal{D}}^2$ and the deviation with respect to this behaviour as a *curvature backreaction* $\mathcal{W}_{\mathcal{D}}$, $\langle \mathcal{R} \rangle_{\mathcal{D}}(t) \equiv 6k_{\mathcal{D}}/a_{\mathcal{D}}(t)^2 + \mathcal{W}_{\mathcal{D}}(t)$, where $k_{\mathcal{D}}$ may for instance be set from initial conditions in the domain, $k_{\mathcal{D}} \equiv \langle \mathcal{R} \rangle_{\mathcal{D}}(t_{\text{i}})/6$.

Perfect fluids with pressure

In the more general case of an irrotational perfect fluid with nonzero, inhomogeneous pressure described in a fluid-orthogonal foliation (within the setting exposed in subsection 1.5.1), as considered in [Buchert, 2001], the acceleration of the fluid and the resulting inhomogeneous lapse (which cannot be set to 1) have to be accounted for.

In this setting, the lapse expresses the relative rates of the proper time τ measured by the fluid elements and the coordinate time t , along the fluid worldlines: $N = d\tau/dt$, where d/dt is still defined as the coordinate time derivative along the fluid flow $\partial_t|_{X^k}$. This difference of time rates affects the volume and effective scale factor expansion rates and the commutation rule for a given scalar ψ , which respectively become [Buchert, 2001]

$$\frac{1}{\mathcal{V}_{\mathcal{D}}} \frac{d\mathcal{V}_{\mathcal{D}}}{dt} = \frac{3}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} = \langle N\Theta \rangle_{\mathcal{D}} ; \quad (1.91)$$

$$\frac{d}{dt} \langle \psi \rangle_{\mathcal{D}} - \left\langle \frac{d\psi}{dt} \right\rangle_{\mathcal{D}} = \langle N\Theta \psi \rangle_{\mathcal{D}} - \langle N\Theta \rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}} . \quad (1.92)$$

In these equations, Θ is thus replaced by the rescaled rate $N\Theta$: since Θ expresses the local volume expansion rate per unit fluid proper time, $N\Theta = \Theta(d\tau/dt)$ is the local volume expansion rate per unit coordinate time t .

The evolution equations for the effective scale factor arise from the averages of the scalar parts of the 3+1 Einstein equations for a perfect fluid, subsequently making use of the above commutation rule and scale factor evolution rate. They take the following form for nonvanishing pressure (*cf.* Buchert [2001], with here the inclusion of a nonzero cosmological constant):

$$3 \left(\frac{1}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \right)^2 = 8\pi G \langle N^2 \epsilon \rangle_{\mathcal{D}} + \Lambda \langle N^2 \rangle_{\mathcal{D}} - \frac{1}{2} \langle N^2 \mathcal{R} \rangle_{\mathcal{D}} - \frac{1}{2} \tilde{\mathcal{Q}}_{\mathcal{D}} ; \quad (1.93)$$

$$\frac{3}{a_{\mathcal{D}}} \frac{d^2 a_{\mathcal{D}}}{dt^2} = -4\pi G \langle N^2 (\epsilon + 3p) \rangle_{\mathcal{D}} + \Lambda \langle N^2 \rangle_{\mathcal{D}} + \tilde{\mathcal{Q}}_{\mathcal{D}} + \tilde{\mathcal{P}}_{\mathcal{D}} , \quad (1.94)$$

with a rescaling of the sources, of Λ , and of \mathcal{R} by a factor N^2 , a modified kinematical backreaction $\tilde{\mathcal{Q}}_{\mathcal{D}}$, and an additional *dynamical backreaction* term $\tilde{\mathcal{P}}_{\mathcal{D}}$:

$$\tilde{\mathcal{Q}}_{\mathcal{D}} \equiv \frac{2}{3} \left(\langle N^2 \Theta^2 \rangle_{\mathcal{D}} - \langle N \Theta \rangle_{\mathcal{D}}^2 \right) - 2 \langle N^2 \sigma^2 \rangle_{\mathcal{D}} \quad ; \quad \tilde{\mathcal{P}}_{\mathcal{D}} \equiv \langle N^2 \nabla_{\mu} a^{\mu} \rangle_{\mathcal{D}} + \left\langle \Theta \frac{dN}{dt} \right\rangle_{\mathcal{D}} . \quad (1.95)$$

These equations are covariant, but individual terms (especially $d^2 a_{\mathcal{D}}/dt^2$ and $\tilde{\mathcal{P}}_{\mathcal{D}}$ due to its second term) depend on the choice of t as a parametrization of the hypersurfaces and need to be interpreted in direct relation to the chosen t in specific applications.

The combination of (1.94) with the time derivative of (1.93) gives the corresponding integrability condition,

$$\begin{aligned} \frac{d}{dt} \tilde{\mathcal{Q}}_{\mathcal{D}} + \frac{6}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \tilde{\mathcal{Q}}_{\mathcal{D}} + \frac{d}{dt} \langle N^2 \mathcal{R} \rangle_{\mathcal{D}} + \frac{2}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \langle N^2 \mathcal{R} \rangle_{\mathcal{D}} + \frac{4}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \tilde{\mathcal{P}}_{\mathcal{D}} - 2\Lambda \frac{d}{dt} \langle N^2 \rangle_{\mathcal{D}} \\ = 16\pi G \left(\frac{d}{dt} \langle N^2 \epsilon \rangle_{\mathcal{D}} + \frac{3}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \langle N^2 (\epsilon + p) \rangle_{\mathcal{D}} \right) . \end{aligned} \quad (1.96)$$

Similarly to the dust case, the above averaged equations may as well be rewritten under an effective Friedmannian form (see Buchert [2001], in the $\Lambda = 0$ case). They would now feature rescaled local energy sources in the effective sources, additional contributions of $\langle N^2 p \rangle_{\mathcal{D}}$ and of $\tilde{\mathcal{P}}_{\mathcal{D}}$ in the effective pressure term, and an inclusion in both effective sources of the deviation of the cosmological constant contribution $\Lambda \langle N^2 \rangle_{\mathcal{D}}(t)$ to (1.93)–(1.94) from an actual constant.

With the assumed perfect fluid form of the energy-momentum tensor, $T_{\mu\nu} = \epsilon u_{\mu} u_{\nu} + p b_{\mu\nu}$, the local energy conservation equation arising from $\nabla_{\mu} T^{\mu\nu}$ projected along the fluid 4-velocity reads $u^{\mu} \partial_{\mu} \epsilon + \Theta(\epsilon + p) = 0$. Rescaling it by N^2 before applying the averaging operation and the commutation rule (1.92) gives the averaged energy conservation equation as another expression for the right-hand side of the above integrability condition:

$$\frac{d}{dt} \langle N^2 \epsilon \rangle_{\mathcal{D}} + \frac{3}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \langle N^2 (\epsilon + p) \rangle_{\mathcal{D}} = \langle N \Theta \rangle_{\mathcal{D}} \langle N^2 p \rangle_{\mathcal{D}} - \langle \Theta N^3 p \rangle_{\mathcal{D}} + 2 \left\langle N \frac{dN}{dt} \epsilon \right\rangle_{\mathcal{D}} . \quad (1.97)$$

Applications and generalizations

The systems of averaged equations presented above may be seen as balance equations describing and comparing all contributions to the overall dynamics of a finite, inhomogeneous region, introducing measures of the dynamical effects of such an inhomogeneity under the form of backreaction terms. These systems, nevertheless, feature only averages and are based on the scalar parts of the 3+1 Einstein equations only. They are consequently not closed, even once a local equation of state is assumed for the sources, and do not keep track of the whole local information. They are not sufficient to solve for the local dynamics such as the formation of structures, and to determine for instance the amplitude of the backreactions as a function of time. This is to be achieved in combination with assumptions on a cosmological model, which would ideally be nonperturbative and/or background-free.

The relativistic Zel'dovich approximation may for instance be used to evolve initial conditions which can be set as fluctuations around a homogeneous matter distribution (*e.g.*, modelling the distribution near the last scattering epoch deduced from CMB observations), restricting the attention to dust fluids in the framework presented in section 1.4 or applying the generalization of the RZA to fluids with pressure which will be introduced in chapter 2. Alternatively, one may use as cosmological models exact inhomogeneous solutions to the Einstein equations (in particular the Lemaitre-Tolman-Bondi [Bondi, 1947] and Szekeres [Szekeres, 1975] models; see, *e.g.*, Bolejko et al. [2011], Sussman [2011, 2014] and references therein for applications to the study of kinematical backreaction), silent-Universe models (*e.g.*, Bolejko [2018]), patchings of exact solutions (*e.g.*, Bolejko and C el erier [2010], Lavinto et al. [2013] and references therein), or phenomenological models such as the Timescape model [Wiltshire, 2009, 2011] or multi-scale models [Buchert and Carfora, 2008, Wiegand and Buchert, 2010]. The recently emerging fully relativistic cosmological simulations [Bentivegna and Bruni, 2016, Mertens et al., 2016, Giblin et al., 2016, Macpherson et al., 2019], based on the integration of the 3+1 Einstein equations, are also promising frameworks for the investigation of backreaction effects⁵ in realistic inhomogeneous model universes.

The framework of this section can be further extended to spatial averaging in spatial foliations that are not required to be orthogonal to the 4-velocity of the sources. This allows for the description of more general fluid flows which may have a nonzero vorticity, and of non-perfect fluid energy-momentum contributions which in general create vorticity. A major part of this thesis has been dedicated to proposals for such an extension that preserve the main ideas of the above schemes (such as a fluid-comoving averaging domain). These results will be presented in chapters 3 and 4, the latter focussing on a manifestly 4-covariant writing of the averaged equations.

⁵The more widespread Newtonian simulations, on the other hand, are usually insufficient for this investigation due to the periodic simulation box used (a 3-torus topology) for which the Newtonian equivalent of the kinematical backreaction vanishes [Buchert and Ehlers, 1997]. Moreover, spatial curvature coupling dynamically to the general-relativistic backreaction should play an important role [Buchert and Carfora, 2008, Roukema et al., 2013] which is not accounted for by the Newtonian framework.

Chapter 2

A relativistic Lagrangian approximation for fluids with pressure

The RZA (Relativistic Zel’dovich Approximation), as introduced in [Buchert and Ostermann, 2012] and further refined in [Buchert et al., 2013, Alles et al., 2015, Al Roumi et al., 2017], recalled above in chapter 1, section 1.4, considers a model universe sourced by an irrotational dust fluid described in its spatial rest frames. This was also an underlying assumption in the relativistic Lagrangian approximation schemes suggested in [Kasai, 1995, Matarrese and Terranova, 1996]. This is a rather sensible assumption for the modelling of large cosmological scales in the late Universe. This era is dominated by matter with nonrelativistic velocities, so that its pressure is much smaller than the energy density associated with its rest mass, and non-negligible pressure gradients and vorticity are mostly restricted to small-scale collapsing or collapsed regions.

A more comprehensive model for structure formation would require the inclusion of a nonvanishing pressure. This would allow for the handling of earlier epochs where radiation is dominant or cannot be neglected. One would also be able to probe structure formation beyond the initial gravitational collapse stage and account for the appearance and stabilization of virialized matter structures such as galaxy clusters within the studied domain. This can impact the dynamics of the domain, especially when it already lies at the scales corresponding to these objects.

The main limitation of Lagrangian approaches is the occurrence of shell-crossings: fluid elements are followed through their evolution, but their worldlines can cross. This leads to a degeneracy in the definition of the 4-velocity field and the fluid flow at the event where this shell-crossing occurs. The degeneracy remains at later times as several crossing flows would need to be modelled, a phenomenon known in Newtonian contexts as multistreaming or velocity dispersion, and that still occurs in a general-relativistic framework. Hence in the corresponding region of spacetime, the single-fluid model and the associated Lagrangian picture are ill-defined and lead to singularities in matter density as well as in geometric quantities such as spatial curvature. A more involved multi-fluid or phase space approach would be more suitable for the description of these regimes.

Shell-crossings are naturally expected for irrotational dust models in collapsing regions. However, when pressure is not exactly zero (as for baryonic matter), pressure gradients will become very strong in a collapse and can oppose it, stabilizing the structure and avoiding shell-crossings.

Moreover, the velocity dispersion in Newtonian dynamics may be modelled in a first approximation by an effective pressure gradient term in the Euler equation, with increasing validity as more shell-crossings occur and the distribution of velocities becomes more isotropic (Binney and Tremaine [2008], Adler and Buchert [1999]; Buchert and Domínguez [2005] and references therein). The effective fluid described in this way will then itself better avoid shell-crossings due to the effective pressure gradient, allowing for a Lagrangian description for longer times while already modelling velocity dispersion and the formation of virialized objects. The validity of the extension of such an effective description of velocity dispersion to a general-relativistic setting would be an assumption. However, for nonrelativistic velocities as for Dark and baryonic Matter in the late Universe, it may be a reasonable approximation since the direct contribution of the effective pressure into the energy sources (which may bias the dynamics) will remain negligible with respect to the rest mass density. Hence, the main dynamical contribution will still be the effective pressure gradient as in the Newtonian framework.

Vorticity also plays a role on small scales and can contribute to the avoidance of shell-crossings by angular momentum conservation. I will not consider it in this chapter for simplicity, since a nonzero vorticity implies further major changes to the RZA framework, with the necessary use of more general spatial slices, tilted with respect to the fluid flow. This will be addressed in a more general setting and in relation to the averaging problem in the next chapters.

In this chapter, I will show how the relativistic Lagrangian framework of section 1.4 and the RZA may be modified and extended to account for irrotational inhomogeneous barotropic fluids⁶ described in a fluid-orthogonal foliation, and for the 4-acceleration and inhomogeneous lapse associated with their pressure gradients. This broadens the range of these formalisms to the radiation-dominated era and to small-scale baryonic matter dynamics. Under the above assumption of the validity of a general-relativistic modelling of velocity dispersion by an effective pressure, this will also be applicable to the regimes of structure formation where velocity dispersion plays an important stabilizing role. Numerical results are presented for illustrative toy-models of the latter situation, along with analytic solutions for linear pressure – energy density relations applicable to radiation. The recovery of the dust case and of Newtonian results in Lagrangian frameworks is considered. Building upon [Al Roumi et al., 2017] for dust, the

⁶The very assumption of the energy-momentum tensor being that of a perfect fluid with a local thermodynamic equation of state does imply a restriction on the possible classes of spacetime metrics. As discussed in detail in Krasiński et al. [1997], the sources computed from a given spacetime metric and the Einstein equations do not always allow for the definition of a specific entropy which complies with the Gibbs equation, and thus local thermodynamic equilibrium may not hold. A barotropic relation between the sources is, of course, even more restrictive in this sense. This should be kept in mind before applying the formalism presented below to a specific class of spacetime metrics. Note, however, that the Lagrangian approximation scheme below determines a time evolution for the metric, so that only the class of metrics for the initial data needs to be restricted when imposing physical assumptions on the local thermodynamic relations within the fluid.

behaviour of the traceless part of the perturbation field in fluids with pressure is also discussed, and a split into electric and magnetic parts is shown to still be possible despite a similar time behaviour for both the trace and the traceless parts of the perturbation being no longer possible. As for the dust case, the electric and magnetic components of the traceless part defined in this way are not directly related in general to the electric/magnetic split of the Weyl tensor.

These results, presented below, were obtained as part of a collaboration with Yongzhuang Li and David Wiltshire, University of Canterbury (New Zealand), and Thomas Buchert. The remaining of this chapter corresponds to the contents of our paper which has been recently published as [Li, Mourier, Buchert, and Wiltshire, 2018].

Lagrangian theory of structure formation in relativistic cosmology.

V. Irrotational fluids

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We extend the general relativistic Lagrangian perturbation theory, recently developed for the formation of cosmic structures in a dust continuum, to the case of model universes containing a single fluid with a single-valued analytic equation of state. Using a coframe-based perturbation approach, we investigate evolution equations for structure formation in pressure-supported irrotational fluids that generate their rest-frame spacetime foliation. We provide master equations to first order for the evolution of the trace and traceless parts of barotropic perturbations that evolve in the perturbed space, where the latter describes the propagation of gravitational waves in the fluid. We illustrate the trace evolution for a linear equation of state and for a model equation of state describing isotropic velocity dispersion, and we discuss differences to the dust matter model, to the Newtonian case, and to standard perturbation approaches.

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I. INTRODUCTION

Relativistic cosmological perturbation theory is based on evolving the Einstein equations with a global foliation of the spacetime metric, via the $3 + 1$ formalism [7, 41]. In the standard approach a spatially homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) geometry is assumed as the unperturbed global background spacetime, and Einstein’s equations are then solved to some order on this predefined background [45]. The standard approach is Eulerian in the sense that perturbations are represented and propagate on this background that corresponds, in the Newtonian limit, to Eulerian perturbation theory. In this latter case, a perturbation method for the density and velocity fields is used to solve the Euler–Poisson system of equations that governs the fluid evolution [10]. Cosmological structure formation in the nonperturbative regime is also generally modeled within the Newtonian framework.

An alternative approach to structure formation has also been developed, principally in the Newtonian regime, which is directly tied to fluid elements. It is consequently known as Lagrangian perturbation theory [8, 9, 11–17, 26, 27, 33, 52, 53, 56, 57, 71, 78] to distinguish it from the Eulerian approach based on coordinates on an assumed global background. The Lagrangian approach uses a single perturbation variable, the fluid’s deformation field. This gives it the advantage of also applying in the nonlinear regime, where Eulerian density perturbations are large. In recent years, Lagrangian perturbation theory has been generalized to general relativistic cosmologies with a dust continuum [L1, L2, L3, L4]; see also [37, 44, 50, 58, 59, 63, 64].

In the Newtonian regime, an extension of Lagrangian

perturbation theory to fluids with dynamic pressure was considered first in terms of isotropic pressure [6]. The resulting Lagrangian perturbation equations have been solved up to second order for a polytropic fluid [51, 67]. For third order perturbative solutions in Newtonian Lagrangian perturbation theory with pressure, see Ref. [66]. Models with isotropic pressure can also be considered as phenomenological models for the generally anisotropic pressure originating from the velocity dispersion of dust particles [47–49], by taking velocity moments of the collisionless Vlasov equation [24, 25]. For a sequence of modeling assumptions used in nonperturbative extensions of Lagrangian perturbation theory, see the summary [21].

In this paper we will extend relativistic Lagrangian perturbation theory for a dust matter model [L1, L2, L3, L4] to the case of irrotational perfect fluids, and also to cases that are relevant for the modeling of multistream regimes where the dust approximation breaks down. This will provide a framework not only to deal with a relativistic generalization of Newtonian Lagrangian perturbation theory with pressure at late epochs, but also to the fully relativistic situation of the early Universe.

A primary motivation for such an investigation is to establish a framework which is better suited to studies of the backreaction of inhomogeneities in cosmology as compared to standard perturbation theory. In particular, standard cosmological perturbation theory conventionally assumes that average cosmic evolution is *exactly* described by a solution to Einstein’s equations with a prescribed energy–momentum tensor on a global hypersurface irrespective of the scale of coarse-graining of the matter fields. No fundamental physical principle demands such an outcome [76].

The scalar averaging scheme introduced in [19, 20, 22, 30] is an example of an approach to backreaction of inhomogeneities in cosmology, in which the Einstein equations are assumed to hold on small scales, where they are well-tested, but not for the average cosmic evolu-

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tion on arbitrarily large spatial scales. This is a consequence of the fact that a generic averaging operation includes nonlocal fluctuation terms, and it should not be confused with modified gravity approaches which change the Einstein-Hilbert action. A variety of phenomenological interpretations of the Buchert scheme are possible [28, 29, 46, 61, 62, 72–75, 79], since additional ingredients are required to relate statistical quantities to physical observables determined from our own cosmological observations.

To date, no phenomenological approach to backreaction has fully utilized the general scalar averaging framework for perfect fluids [20]. In the timescape scenario [73–75], solutions to the corresponding system of averaged scalar equations have been given with matter and radiation [31] extending smoothly into the early radiation-dominated epoch in the early Universe. However, in deriving these solutions it was assumed that backreaction is insignificant before photon–electron decoupling, so that backreaction involving pressure terms was neglected.

Neglecting backreaction in the primordial plasma may seem to be a reasonable approximation for the evolution of the background universe to leading order, given that it is extremely close to being spatially homogeneous and isotropic at early times. However, backreaction can nonetheless make a significant difference when considering the growth of perturbations. In particular, even if the difference from the Friedmann equation is of order 10^{-5} as a fraction of energy density at decoupling, this is nonetheless of the same order as the density perturbations. A recent study of Cosmic Microwave Background (CMB) anisotropies in the timescape model found that neglecting such small differences in initial conditions at last scattering leads to systematic uncertainties of 8–13% for particular cosmological parameters at the present epoch [54]. This remark applies to the conservative assumption that the background universe does not already contain backreaction arising from earlier epochs that could be compatible with large-scale homogeneity and isotropy [28].

For these reasons, we desire a new approach to cosmological perturbation theory which is intrinsic to the fluid and not anchored to an embedding space. Relativistic Lagrangian perturbation theory represents a promising avenue, as it is intimately tied to physical particles. To proceed to a fully realistic theory will require important steps beyond those which we investigate in this paper. Such steps will include:

- An extension from one fluid to the many fluids pertinent to the early Universe, which requires considering a tilt between various fluid flow vectors and the normal to the spatial hypersurfaces;¹

- Identifying relevant physical scales and volume partitioning the model universe into regions whose average evolves by averaged dynamical equations, rather than by global Friedmann equations;
- Aiming at a background-free description. While perturbations are still formulated in the present paper as deviations from a fixed background cosmology, a general volume partitioning can be implemented without referring to a background [23, 72].

As a first step towards these goals, in the present paper we will firstly consider relativistic Lagrangian perturbation theory for the same system that was considered in Ref. [20], namely a single component perfect fluid with barotropic equation of state. We will also include an explicit cosmological constant term.

This paper is organized as follows. In Section II we employ a 3 + 1 formalism [7, 41] with Lagrangian spatial coordinates, presenting the general framework and foliation structure for a general irrotational matter model. We then restrict our attention to a barotropic fluid and discuss in detail the fluid variables and their equation of state. In this context, in Section III we introduce Cartan’s coframe formalism, proceeding with the relativistic Lagrangian perturbation approach. We develop the first-order Lagrangian scheme and derive master equations for the trace and trace-free parts of the perturbation field. In Section IV we apply the first-order Lagrangian scheme to particular matter models, allowing us to explicitly derive solutions for the trace part, and we illustrate and discuss the results. Particular solutions for the gravitoelectric traceless part are studied in Appendix A. We summarize our main results in Section V.

II. SPACETIME FOLIATION STRUCTURE AND 3+1 EINSTEIN EQUATIONS

In this paper we will consider a model universe containing a single irrotational fluid, so that a foliation of spacetime into flow-orthogonal hypersurfaces can be introduced.

A. Decomposition of Einstein’s equations for flow-orthogonal hypersurfaces

The irrotationality assumption on the fluid amounts to the existence of two scalar functions, N and t , such that the 1-form dual to the normalized 4-velocity vector u^μ of the fluid can be written as:²

$$u_\mu = -N \partial_\mu t \quad ; \quad N := (-\partial^\mu t \partial_\mu t)^{-1/2} . \quad (1)$$

matter components [70].)

² In the convention we use here, greek letters μ, ν, \dots are spacetime indices running from 0 to 3, while lowercase latin letters i, j, \dots are spatial indices running from 1 to 3. We use units in which $c = 1$, if not otherwise stated.

¹ Note that in the standard approach, the same FLRW frame is used for different matter components. (Even in this idealized case there are important differences to be respected for the different

The level sets of t then define flow-orthogonal hypersurfaces, labeled Σ_t , which foliate spacetime, with unit normal u^μ , $u^\mu u_\mu = -1$. We will now follow the 3 + 1 formalism [7, 41] and define our time coordinate as coinciding with this function t . In this case, $N(t, x^i)$ is the *lapse function*.

In addition, we choose the spatial coordinates to be spatial Lagrangian (or comoving) coordinates, denoted X^i , that are assumed to be constant along each flow line. In the set of coordinates $(X^\mu) = (t, X^i)$, the components of the fluid 4-velocity vector and its dual are then respectively:

$$u^\mu = \frac{1}{N}(1, 0, 0, 0) \quad ; \quad u_\mu = (-N, 0, 0, 0), \quad (2)$$

while the line element can be written as

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu = -N^2 dt^2 + g_{ij} dX^i dX^j. \quad (3)$$

Here, g_{ij} corresponds both to the spatial coefficients of the 4-metric $g_{\mu\nu}$ and to the components of the 3-metric that it induces on the hypersurfaces Σ_t . Introducing the projector onto Σ_t , $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, this 3-metric is indeed

$$h_{ij} := g_{\mu\nu} h^\mu{}_i h^\nu{}_j = g_{ij}. \quad (4)$$

The spatial metric and the lapse function N together encode the inhomogeneities. (We will later use the more elementary coframe coefficients instead of the 3-metric coefficients.) We use $\mathcal{R}^i{}_j$ to denote the Ricci tensor coefficients of this spatial metric, with \mathcal{R} the corresponding Ricci scalar.

Without loss of generality, the energy-momentum tensor of the fluid is given by

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu} + \pi_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu, \quad (5)$$

where $\pi_{\mu\nu}$ is an anisotropic pressure, with $\pi_{[\mu\nu]} = 0$, $u^\mu \pi_{\mu\nu} = 0$ and $\pi^\mu{}_\mu = 0$, and q_μ the heat flux, with $q_\mu u^\mu = 0$.

Introducing the expansion tensor (as minus the extrinsic curvature) of the hypersurfaces,

$$\Theta_{ij} := \nabla_\nu n_\mu h^\mu{}_i h^\nu{}_j = \frac{1}{2N} \partial_t g_{ij}, \quad (6)$$

Einstein's equations with a cosmological constant may be cast into a set of constraint and evolution equations. The constraint equations are the energy and momentum constraints:³

$$\begin{aligned} \mathcal{R} + \Theta^2 - \Theta^i{}_j \Theta^j{}_i &= 16\pi G \epsilon + 2\Lambda; \\ \Theta^i{}_{j|i} - \Theta_{|j} &= -8\pi G q_j. \end{aligned} \quad (7)$$

The propagation equations are given by

$$\begin{aligned} \Theta^i{}_j &= \frac{1}{2N} g^{ik} \partial_t g_{kj}; \\ N^{-1} \partial_t \Theta^i{}_j &= -\Theta \Theta^i{}_j - \mathcal{R}^i{}_j + \mathcal{A}^i{}_j \\ &\quad + 4\pi G [(\epsilon - p) \delta^i{}_j + 2\pi^i{}_j] + \Lambda \delta^i{}_j, \end{aligned} \quad (8)$$

where $a_\mu := u^\nu \nabla_\nu u_\mu = N^{-1} N_{||\mu}$ is the covariant acceleration of the fluid (with ∇ denoting the 4-covariant derivative), and $\mathcal{A}^i{}_j := a^i{}_{||j} + a^i a_j = N^{-1} N^i{}_{||j}$. Combining the trace of the second equation with the energy constraint yields the Raychaudhuri equation:

$$N^{-1} \partial_t \Theta = -\frac{1}{3} \Theta^2 - 2\sigma^2 - 4\pi G(\epsilon + 3p) + \mathcal{A} + \Lambda, \quad (9)$$

where $\mathcal{A} := \mathcal{A}^i{}_i = \nabla_\mu a^\mu = N^{-1} N^i{}_{||i}$.

With the spacetime described by the given metric, the energy-momentum conservation laws are expressed as follows:

$$\partial_t \epsilon + N \Theta (\epsilon + p) = -N (q^\mu{}_{||\mu} + 2q^\mu a_\mu + \sigma_{\mu\nu} \pi^{\mu\nu}); \quad (10)$$

$$\begin{aligned} (\epsilon + p) a_\mu + p_{||\mu} &= -(\pi_{\mu\nu}{}^{||\nu} + a^\nu \pi_{\mu\nu}) \\ &\quad - \left(\frac{4}{3} \Theta q_\mu + q^\nu \sigma_{\mu\nu} + u^\nu \nabla_\nu q_\mu - q^\nu a_\nu u_\mu \right). \end{aligned} \quad (11)$$

In what follows, we will specialize to the case of isotropic pressure, $\pi_{\mu\nu} = 0$, and vanishing heat flux, $q_\mu = 0$. Note that with these assumptions we do still allow for some nonperfect fluids, since p is not necessarily the local thermodynamic equilibrium pressure [38]. Such a restriction is required here since both extra terms in general create vorticity, which cannot be covered by the class of flow-orthogonal foliations considered in this work.

Let us illustrate this by considering more closely the irrotationality condition for a fluid with negligible heat flux, $q_\mu = 0$, to see how this condition constrains the equation of state and the anisotropic pressure. The vanishing of the vorticity 2-form implies vanishing of the antisymmetrized projected gradient of the acceleration, $a_{[\nu}{}_{||\mu]} = 0$, since $a_\mu = (\ln N)_{||\mu}$ from (1), being a consequence of the existence of the fluid-orthogonal foliation. From this, one obtains through (11) the following constraint on the energy-momentum components:

$$\begin{aligned} \epsilon_{||[\mu} p_{||\nu]} + (\epsilon + p)_{||[\mu} h^\rho{}_{\nu]} \nabla_\sigma \pi^\sigma{}_\rho \\ - (\epsilon + p) h^\rho{}_{[\mu} h^\sigma{}_{\nu]} \nabla_\rho \nabla_\tau \pi^\tau{}_\sigma = 0. \end{aligned} \quad (12)$$

Since $\nabla_\mu \pi^\mu{}_\nu = 0$ would imply the vanishing of the right hand sides of (10)–(11), an anisotropic pressure that does contribute to the dynamics will satisfy $\nabla_\mu \pi^\mu{}_\nu \neq 0$ and thus will not fulfill the above condition in general, producing a vortical flow. Conversely, a barotropic fluid flow with $\pi_{\mu\nu} = 0$ and an effective equation of state of the form $p = \beta(\epsilon)$, automatically satisfies the above constraint. Moreover, for such a fluid, (11) allows one to

³ The symbol $||$ denotes the covariant derivative with respect to the 3-metric h_{ij} . When applied to scalars it reduces to a partial derivative, denoted $|$, with respect to the Lagrangian coordinates, X^i .

write the acceleration as a flow–orthogonal projected gradient, and it will indeed obey the relativistic equivalent of the Kelvin–Helmholtz theorem, so that irrotationality will be preserved along the flow lines [32, 38].

B. Barotropic perfect fluid spacetimes

For the remainder of this paper we will only consider fluids with $q_\mu = 0$ and $\pi_{\mu\nu} = 0$. The energy–momentum tensor (5) then reduces to perfect fluid form:

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (13)$$

while its conservation equations (10)–(11) become, respectively

$$\partial_t \epsilon + N\Theta(\epsilon + p) = 0; \quad (14)$$

$$a_\mu = -\frac{p_{||\mu}}{\epsilon + p}. \quad (15)$$

As a further restriction we will assume that the fluid flow is *barotropic*, *i.e.*, we assume a local relation of the form $p = \beta(\epsilon)$ to effectively hold throughout the entire fluid,⁴ that we will henceforth call *the equation of state* or EoS. As noted earlier, such a relation will ensure that the flow remains irrotational. For such a fluid, setting some reference constant energy and rest mass density values ϵ_1, ϱ_1 , we may use the EoS to define a formal rest mass density $\varrho(\epsilon)$ and a related specific enthalpy $h(\epsilon)$ – as an injection energy per fluid element and unit formal rest mass [42] – respectively, by

$$\varrho := F(\epsilon) := \varrho_1 \exp \int_{\epsilon_1}^{\epsilon} \frac{dx}{x + \beta(x)}; \quad (16)$$

$$h(\epsilon) := \frac{\epsilon + \beta(\epsilon)}{F(\epsilon)} = \frac{\epsilon + p}{\varrho}. \quad (17)$$

The energy–momentum conservation equations (10) and (11) then, respectively, provide a conservation law for ϱ ,

$$\partial_t \varrho + N\Theta\varrho = 0, \quad (18)$$

and a relation between the specific enthalpy (17) and the lapse,

$$\frac{N_{||\mu}}{N} = a_\mu = -\frac{h_{||\mu}}{h} : (Nh)_{|i} = 0. \quad (19)$$

⁴ Considering the local dynamical solution for these variables, there is always a freedom of integration constant that depends on the Lagrangian coordinates, *i.e.*, on the particular fluid element. We assume here that the same relation holds for all fluid elements. Only this assumption makes the dynamical relation an apparent equation of state that is valid throughout the fluid flow. All related variables then also depend on this assumption, which is a restriction imposed on initial data.

By an appropriate choice of the hypersurface–labeling function t , the lapse can thus be rescaled so that [20, 38]

$$N = \frac{1}{h} = \frac{F(\epsilon)}{\epsilon + \beta(\epsilon)}. \quad (20)$$

If we assume that the fluid remains in thermodynamic equilibrium locally, and if it has a nonvanishing rest mass density, then this density will follow the same evolution law (18) as $\varrho = F(\epsilon)$, by rest mass conservation. This formal ϱ and the actual rest mass density will then coincide up to a possible different spatial dependence (*cf.*, footnote 4). These two quantities may be made equal by a suitable choice of initial conditions for the rest mass density or local thermodynamic equilibrium assumptions.⁵ This would then ensure the validity of the interpretation of ϱ and h as the physical rest mass density (or particle number density) and specific enthalpy of the fluid, respectively. We will not, however, make such assumptions in the following Section III, to keep its level of generality. This will allow us to consider the case of a zero rest mass fluid (for which $F(\epsilon) \neq 0$ and $h(\epsilon)$ are still well–defined), as well as that of a nonzero rest mass density with less constrained initial conditions. It will also allow us to consider the variable p as an effective pressure term – *e.g.*, modeling velocity dispersion – instead of the local thermodynamic equilibrium pressure. For the general treatment including these cases it will suffice to formally define ϱ and h from Equations (16)–(17) using the single barotropic assumption $p = \beta(\epsilon)$. We follow the notation of Ref. [20] here.

⁵ Let us take the local state of the fluid to belong to a thermodynamic Gibbs space admitting the equation of state $u(s, v)$, where s is the specific entropy, v is the specific volume and $u = \epsilon v$ is the specific internal energy. If we now assume that p is the local thermodynamic equilibrium pressure of the fluid, it can then be expressed as $p(s, v) = -\partial u / \partial v$. Provided that a specific equation of state does not render the above relations degenerate, then these relations may be inverted to provide $v(\epsilon, p)$. Within a barotropic flow satisfying $p = \beta(\epsilon)$, the actual rest mass density v^{-1} thus only depends on the energy density ϵ , which fully determines its initial conditions. From the conservation equations of both quantities, $\partial_t \epsilon / (\epsilon + \beta(\epsilon)) = -N\Theta = \partial_t (v^{-1}) / v^{-1}$, this dependency must be $v^{-1} = F(\epsilon)$, for Θ not identically vanishing, up to a constant prefactor which can be absorbed in the choice of ϱ_1 . Hence, in this case, $F(\epsilon)$ is indeed the rest mass density of the fluid with no further loss of generality. Also note that under the same assumptions, s is also a function of ϵ , preserved along the flow lines as the flow is adiabatic: $\partial_t s = 0 = (ds/d\epsilon) \partial_t \epsilon$, while $\partial_t \epsilon$ is not identically vanishing. The flow is thus *isentropic*, s being a constant s_1 that depends neither on time nor on the fluid element. The barotropic relation then corresponds to the equation $p(\epsilon, s)$ deduced from the thermodynamic equation of state, and taken at constant s , $\beta(\epsilon) = p(\epsilon, s = s_1)$ (see [32, 38, 39, 42, 43, 65]).

III. LAGRANGIAN PERTURBATION SCHEME

In this section we will introduce the coframe formalism to describe spacetime, which is a set of four deformation 1-form fields dual to a generally noncoordinate basis of vectors at every point of the manifold [35, 36, 77]. A general relativistic version of a coframe-based perturbative approach for an irrotational dust continuum has been proposed in Ref. [44], developed further in Ref. [50] and in final form, featuring only the coframes as the single perturbation variable in Ref. [L1].

A. Coframe formulation

Following [L2, L3, L4], we construct a set of three spatial coframes $\boldsymbol{\eta}^a$ such that the spatial metric can be rewritten in the form

$${}^{(3)}\mathbf{g} = G_{ab} \boldsymbol{\eta}^a \otimes \boldsymbol{\eta}^b \quad : \quad g_{ij} = G_{ab} \eta_i^a \eta_j^b. \quad (21)$$

Here $G_{ab}(\mathbf{X})$ is the Gram matrix that encodes all the initial spatial metric perturbations, $G_{ab}(\mathbf{X}) := \delta_a^i \delta_b^j G_{ij}(\mathbf{X})$, with the initial metric coefficients, $G_{ij}(\mathbf{X}) := g_{ij}(t_i, \mathbf{X})$. On the other hand we can also include the temporal component into the matrix and rewrite it as

$$\tilde{G}_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & G_{ab} \end{pmatrix}. \quad (22)$$

With this we introduce a full set of four spacetime coframes $\boldsymbol{\eta}^\alpha$ to describe the 4-metric ${}^{(4)}\mathbf{g}$:

$${}^{(4)}\mathbf{g} = \tilde{G}_{\alpha\beta} \boldsymbol{\eta}^\alpha \otimes \boldsymbol{\eta}^\beta, \quad (23)$$

by defining the coframe components as

$$\eta^0_\mu = (-N, 0, 0, 0) \quad ; \quad \eta^a_\mu = (0, \eta^a_i). \quad (24)$$

We now define the transformation between coordinate and noncoordinate bases as: $\mathcal{J} = \sqrt{-g}/\sqrt{-G} = \sqrt{-g}/\sqrt{G}$ (the signature adopted here being $(-1, 1, 1, 1)$), and using the notation $g := \det({}^{(4)}\mathbf{g})$, $\tilde{G} := \det(\tilde{G}_{\alpha\beta})$ and $G := \det(G_{ab})$. This corresponds to $\mathcal{J} = -\det(\eta^\alpha_\mu)$, or,

$$\begin{aligned} \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta} \boldsymbol{\eta}^\alpha \wedge \boldsymbol{\eta}^\beta \wedge \boldsymbol{\eta}^\gamma \wedge \boldsymbol{\eta}^\delta = \\ - \frac{1}{4!} \mathcal{J} \epsilon_{\mu\nu\rho\sigma} dX^\mu \wedge dX^\nu \wedge dX^\rho \wedge dX^\sigma. \end{aligned} \quad (25)$$

From Eq. (24), in terms of the spatial components of the coframes, \mathcal{J} becomes

$$\mathcal{J} = \frac{1}{3!} N \epsilon_{abc} \epsilon^{ijk} \eta_i^a \eta_j^b \eta_k^c = N \det(\eta_i^a), \quad (26)$$

while correspondingly, the dual vector basis can be described by the four frames $\mathbf{e}_\alpha = e_\alpha^\mu \partial/\partial X^\mu$:

$$\begin{aligned} e_\alpha^\mu \eta^\alpha_\nu &= \delta^\mu_\nu \quad ; \quad e_\alpha^\mu \eta^\beta_\mu = \delta_\alpha^\beta \quad ; \\ e_\alpha^\mu &= -\frac{1}{6\mathcal{J}} \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \eta^\beta_\nu \eta^\gamma_\rho \eta^\delta_\sigma \quad ; \\ e_a^i &= \frac{1}{2\mathcal{J}} N \epsilon_{abc} \epsilon^{ijk} \eta_j^b \eta_k^c \quad ; \\ e_0^\mu &= \frac{1}{N} (-1, 0, 0, 0) \quad ; \quad e_a^\mu = (0, e_a^i). \end{aligned} \quad (27)$$

With this choice, the evolution equations for \mathcal{J} and the expansion tensor coefficients Θ^i_j read:

$$\begin{aligned} \partial_t \mathcal{J} &= \frac{\partial_t N}{N} \mathcal{J} + \mathcal{J} N \Theta \quad ; \\ \Theta^i_j &= \frac{1}{2\mathcal{J}} \epsilon_{abc} \epsilon^{ikl} (\partial_t \eta^a_j) \eta^b_k \eta^c_l \quad ; \\ \frac{\partial_t \Theta^i_j}{N} &= -\Theta \Theta^i_j + \frac{1}{2\mathcal{J}} \epsilon_{abc} \epsilon^{ikl} \partial_t \left(\frac{1}{N} \partial_t \eta^a_j \right) \eta^b_k \eta^c_l \\ &\quad + \frac{1}{N\mathcal{J}} \epsilon_{abc} \epsilon^{ikl} (\partial_t \eta^a_j) (\partial_t \eta^b_k) \eta^c_l. \end{aligned} \quad (28)$$

From the constraint and evolution equations (7)–(9), together with the definition of \mathcal{J} and Eqs. (28), the Lagrange–Einstein system of an irrotational barotropic fluid model is cast into the following form:

$$G_{ab} \partial_t \eta^a_{[i} \eta^b_{j]} = 0 \quad ; \quad (29)$$

$$\begin{aligned} \frac{1}{2\mathcal{J}} \epsilon_{abc} \epsilon^{ikl} \partial_t \left(\frac{1}{N} (\partial_t \eta^a_j) \eta^b_k \eta^c_l \right) &= \mathcal{A}^i_j - \mathcal{R}^i_j \\ &\quad + [4\pi G(\epsilon - p) + \Lambda] \delta^i_j \quad ; \end{aligned} \quad (30)$$

$$\epsilon_{abc} \epsilon^{ijk} (\partial_t \eta^a_i) (\partial_t \eta^b_j) \eta^c_k = (16\pi G\epsilon + 2\Lambda - \mathcal{R}) N \mathcal{J} \quad ; \quad (31)$$

$$\left[\frac{1}{\mathcal{J}} \epsilon_{abc} \epsilon^{ikl} (\partial_t \eta^a_j) \eta^b_k \eta^c_l \right]_{\parallel i} = \left[\frac{1}{\mathcal{J}} \epsilon_{abc} \epsilon^{ikl} (\partial_t \eta^a_i) \eta^b_k \eta^c_l \right]_{\parallel j} \quad ; \quad (32)$$

$$p = \beta(\epsilon). \quad (33)$$

Equations (29)–(32) are not closed unless an EoS, here (33), is specified. Recall that the lapse appearing above can be replaced by its expression in terms of ϵ , $N = (\epsilon + \beta(\epsilon))^{-1} F(\epsilon)$. The evolution equation (30) may be split into a trace part, which we then combine with the energy constraint (31) to obtain the Raychaudhuri equation, and a traceless part, yielding respectively:

$$\frac{1}{2\mathcal{J}} \epsilon_{abc} \epsilon^{ikl} \partial_t \left(\frac{1}{N} \partial_t \eta^a_i \right) \eta^b_k \eta^c_l = \mathcal{A} - 4\pi G(\epsilon + 3p) + \Lambda \quad ; \quad (34)$$

$$\begin{aligned} \frac{1}{2\mathcal{J}} \left[\epsilon_{abc} \epsilon^{ikl} \partial_t \left(\frac{1}{N} (\partial_t \eta^a_j) \eta^b_k \eta^c_l \right) \right. \\ \left. - \frac{1}{3} \epsilon_{abc} \epsilon^{mkl} \partial_t \left(\frac{1}{N} (\partial_t \eta^a_m) \eta^b_k \eta^c_l \right) \delta^i_j \right] = \xi^i_j - \tau^i_j, \end{aligned} \quad (35)$$

where $\tau_j^i := \mathcal{R}_j^i - \frac{1}{3}\mathcal{R}\delta_j^i$ are the coefficients of the traceless part of the spatial Ricci tensor, and $\xi_j^i := \mathcal{A}_j^i - \frac{1}{3}\mathcal{A}\delta_j^i$.

The Lagrange–Einstein system, Eqs. (29)–(33), is closed and provides the components η^a_i of coframes, from which one can calculate the evolution of the perturbations. The system comprises 14 equations, where 9 equations describe the evolution for the coefficient functions of 3 spatial Cartan coframe fields, and the remaining 5 equations originate from the 4 constraints and the EoS defining the properties of the fluid.

B. Perturbation ansatz

1. Background

We will choose a spatially flat, homogeneous and isotropic model universe as the background spacetime, with the same barotropic EoS, and including a possible constant curvature term into the first-order perturbations, (*cf.*, e.g., [L3]). Accordingly, the spatial metric coefficients of the background will be $a^2(t)\delta_{ij}$, $a(t)$ being the background scale factor. We prescribe a homogeneous lapse $N_H(t)$ for this homogeneous and isotropic background, by setting its relation to the background energy density ϵ_H , formal rest mass density $\varrho_H := F(\epsilon_H)$ and pressure $p_H = \beta(\epsilon_H)$ as being the same relations as those for the inhomogeneous quantities,

$$N_H = \frac{\varrho_H}{\epsilon_H + p_H} = \frac{F(\epsilon_H)}{\epsilon_H + \beta(\epsilon_H)}. \quad (36)$$

We may then write the background line element as

$$ds_H^2 = -N_H^2(t)dt^2 + a^2(t)\delta_{ij}dX^i dX^j. \quad (37)$$

Note that the evolution of the background lapse function $N_H(t)$ will be given by its definition (36) and the EoS, making it time-dependent for $p_H \neq 0$. One should keep in mind that our choice of time coordinate t will consequently not coincide in general with the usual ‘cosmic time’ coordinate for the background, and will evolve at a different rate. The usual cosmic time \tilde{t} would rather be defined by $d\tilde{t} = N_H(t)dt$, so that the background line element (37) would take the usual Friedmannian form for homogeneous and isotropic model universes:⁶

$$ds_H^2 = -d\tilde{t}^2 + a^2[\tilde{t}]\delta_{ij}dX^i dX^j. \quad (38)$$

⁶ The notation $a[\tilde{t}]$ signifies that the scale factor still takes the same values, $a[\tilde{t}] := a(t)$, but has a different functional dependence on the alternative time coordinate.

With this time variable, the standard Friedmann equations would indeed be recovered:

$$\begin{aligned} 3\frac{\partial_t^2 a}{a} &= -4\pi G(\epsilon_H + 3p_H) + \Lambda; \\ 3\left(\frac{\partial_t a}{a}\right)^2 &= 8\pi G\epsilon_H + \Lambda; \\ \partial_t \epsilon_H + 3\frac{\partial_t a}{a}(\epsilon_H + p_H) &= 0. \end{aligned} \quad (39)$$

However, for consistency with the lapsed foliation used for the full inhomogeneous spacetime, in what follows we include the homogeneous lapse N_H into the background and use the coordinate t . In terms of this variable, the acceleration and Friedmann equations are respectively:

$$\begin{aligned} \frac{3}{N_H^2}\frac{\partial_t^2 a}{a} &= -4\pi G(\epsilon_H + 3p_H) + \Lambda + 3\frac{\partial_t a}{a}\frac{\partial_t N_H}{N_H^3}; \\ \frac{3}{N_H^2}\left(\frac{\partial_t a}{a}\right)^2 &= 8\pi G\epsilon_H + \Lambda, \end{aligned} \quad (40)$$

while the energy–momentum conservation equation is formally unchanged:

$$\partial_t \epsilon_H + 3\frac{\partial_t a}{a}(\epsilon_H + p_H) = 0. \quad (41)$$

2. Coframes decomposition

It is important to express the full set of equations in terms of a single perturbation variable, the coframes, so that the Lagrangian perturbation approach is well-defined. Although this is not made fully explicit in the Lagrange–Einstein system (29)–(33), it is implicitly the case as the Ricci tensor and covariant derivatives are functionals of the metric, and hence of the coframes, and ϵ , p , N and \mathcal{A}_j^i can be expressed in terms of the coframes and initial energy density data. The latter relations are obtained *via* the conservation equation (18) for $\varrho = F(\epsilon)$ and the evolution equation for $J := \mathcal{J}/N = \det(\eta^a_i)$ from the first equation in (28):

$$N\Theta = -\frac{\partial_t F(\epsilon)}{F(\epsilon)} = \frac{\partial_t J}{J}; \quad \epsilon = F^{-1}\left(\frac{F(\epsilon_i)}{J}\right), \quad (42)$$

where for any quantity A , A_i denotes the quantity at initial time t_i . Here $J_i = 1$ as a result of the choice of initial conditions for the coframes. The barotropic EoS and choice of N then allow us to determine p , N and $\mathcal{A}_j^i = N^{-1}N^{||i}_{||j}$, and to express these fields as functions of $J = \det(\eta^a_i)$.

We then follow the previous papers [L1]–[L4] and decompose the coframes into a FLRW coframe set and deviations thereof,

$$\eta^a = \eta^a_i dX^i = a(t)(\delta^a_i + P^a_i) dX^i. \quad (43)$$

At this nonperturbative level, the metric coefficients are then related to the deformation field by

$$g_{ij} = a^2(t) (G_{ij} + 2P_{(ij)} + G_{ab}P_a^i P_b^j), \quad (44)$$

where we have defined

$$P_j^i := \delta_a^i P_a^j; \quad P := P_k^k = \delta_a^k P_a^k; \quad P_{ij} := G_{ai} P_a^j. \quad (45)$$

Recall that the Gram matrix coefficients G_{ab} have been defined to encode the initial metric inhomogeneities, so that the coefficients P_a^i can be set to zero initially. Also recall that this coframe split is made with respect to a FLRW background with a nontrivial lapse included, and that the functional dependence of a , or of the deformation field, on the time coordinate t will be affected accordingly.

We then expand the deformation fields P_a^i into a perturbative sum, so that the coframes are given by:

$$\eta^a = a(t) \left(\delta_a^i + \sum_{m=1}^{\infty} P_a^i{}^{(m)} \right) dX^i, \quad (46)$$

where the m th-order deformation field coefficients $P_a^i{}^{(m)}$ are of order ε^m for some bookkeeping parameter $\varepsilon \ll 1$. In this paper we will only focus on first-order deformations.

3. Initial conditions

We will follow the steps of Refs. [L3, L4] to prescribe the initial data. The deformation field and its time-derivatives are given at some initial time t_i by:

$$\begin{aligned} P_a^i(t_i) &= 0; \\ (\partial_t P_a^i)(t_i) &=: U_a^i; \\ (\partial_t^2 P_a^i)(t_i) &=: W_a^i - 2H_i U_a^i, \end{aligned} \quad (47)$$

where $H := \partial_t a/a$ is the Hubble function. Hereafter, we will normalize the scale factor as $a_i = 1$. The six 1-form fields $\mathbf{U}^a = U_a^i dX^i$ and $\mathbf{W}^a = W_a^i dX^i$ are 1-form generalizations of the initial Newtonian peculiar-velocity and peculiar-acceleration gradient fields, respectively.

The Lagrange-Einstein system with its split into trace and traceless parts according to (29)–(35) then translates into constraints on the initial data:

$$U_{[ij]} = 0 \quad ; \quad W_{[ij]} = 0; \quad (48)$$

$$\begin{aligned} W - U \left(\frac{\partial_t N}{N} \right)_i &= 3H_i \left[\left(\frac{\partial_t N}{N} \right)_i - \left(\frac{\partial_t N_H}{N_H} \right)_i \right] \\ &+ \Lambda (N_i^2 - N_{H_i}^2) + N_i^2 \mathcal{A}_i \\ &- 4\pi G [(\epsilon_i + 3p_i)N_i^2 - (\epsilon_{H_i} + 3p_{H_i})N_{H_i}^2]; \end{aligned} \quad (49)$$

$$\begin{aligned} {}^{\text{tl}}W_a^j \delta_a^i + \left(H_i - \left(\frac{\partial_t N}{N} \right)_i \right) {}^{\text{tl}}U_a^j \delta_a^i \\ + U {}^{\text{tl}}U_a^j \delta_a^i - \left(U_a^k \delta_a^i U_b^j \delta_b^k - \frac{1}{3} U_a^l \delta_a^k U_b^l \delta_b^i \delta_j^i \right) \\ = N_i^2 (\xi_j^i(t_i) - \tau_j^i(t_i)); \end{aligned} \quad (50)$$

$$\begin{aligned} U^2 - U_a^i \delta_a^j U_j^b \delta_b^i + 4H_i U \\ = 16\pi G (\epsilon_i N_i^2 - \epsilon_{H_i} N_{H_i}^2) + 2\Lambda (N_i^2 - N_{H_i}^2) - \mathcal{R}_i N_i^2; \end{aligned} \quad (51)$$

$$(N_i^{-1} U_a^j \delta_a^i)_{||i} = (N_i^{-1} U)_{|j} + 2H_i (N_i^{-1})_{|j}; \quad (52)$$

$$p_i = \beta(\epsilon_i) \quad ; \quad p_{H_i} = \beta(\epsilon_{H_i}). \quad (53)$$

The abbreviations $U := U_a^k \delta_a^k$, $W := W_a^k \delta_a^k$, and ${}^{\text{tl}}W_a^i := W_a^i - (1/3)W\delta_a^i$, ${}^{\text{tl}}U_a^i := U_a^i - (1/3)U\delta_a^i$, are used for the trace and traceless parts, respectively.

C. First-order Lagrange-Einstein system

We now expand the above Lagrange-Einstein system and its initial conditions to first order⁷ in the only dynamical variable in this Lagrangian perturbation approach, namely the deformation field P_a^i . In what follows we omit the index ⁽¹⁾ for the first-order deformation field and the associated initial conditions U_{ij} , W_{ij} , but keep the index for the other variables, as functionals of P_a^i . We first need to express these functionals explicitly at first order.

1. Dependent variables at first order

In order to express the first-order Ricci tensor and scalar curvature in terms of the coframes, we expand the initial metric coefficients to first order as $G_{ij}(\mathbf{X}) = \delta_{ij} + G_{ij}^{(1)}(\mathbf{X})$ since they reduce to δ_{ij} at the unperturbed zero-order level. Introducing the first-order quantities $G^{(1)ij} := \delta^{ik} \delta^{jl} G_{kl}^{(1)}$, $P^{ij} := \delta^{ik} \delta^{jl} P_{kl}$ for the inverse metric, we can then substitute the metric and its inverse, truncated to first order,

$$g_{ij} = a^2 \left(\delta_{ij} + G_{ij}^{(1)} + 2P_{(ij)} \right); \quad (54)$$

$$g^{ij} = a^{-2} \left(\delta^{ij} - G^{(1)ij} - 2P^{(ij)} \right), \quad (55)$$

into the definitions of the spatial Christoffel symbols and of the spatial Ricci tensor. We thereby obtain:

$$\Gamma_{ij}^{k(1)} = \frac{1}{2} \delta^{kl} \left(G_{il|j}^{(1)} + G_{lj|i}^{(1)} - G_{ij|l}^{(1)} \right) \quad (56)$$

$$+ \delta^{kl} (P_{(il)|j} + P_{(lj)|i} - P_{(ij)|l});$$

$$\mathcal{R}_{ij}^{(1)} = \mathcal{R}_{ij} + P_{[j|k]|i}^k + P_{[j^k|k]|i} + P_{(ik)|j}{}^k - P_{(ij)}{}^k{}_{|k}; \quad (57)$$

$$\mathcal{R}^{(1)} = a^{-2} \mathcal{R} + 2a^{-2} \left(P^{ki}{}_{|i|k} - P^k{}_{|k} \right), \quad (58)$$

⁷ Note that initial data can be assumed, without loss of generality, to be first order.

where $\mathcal{R}_{ij} := G_{i[k]j}^{(1)k} + G_{[j|k]i}^{k(1)}$, and $\mathcal{R} := \delta^{ij}\mathcal{R}_{ij} = 2G_{[k|l]}^{l|k(1)}$ are the initial conditions for the curvature tensor coefficients and their trace, respectively.

An important difference from the dust case is that here the spatial Ricci scalar will in general not be constrained to evolve as $\mathcal{R}(\mathbf{X})a(t)^{-2}$ at first order, due to the contributions from the lapse in the momentum constraints. As will be shown below, these contributions give rise to a nonzero evolution for the (initially vanishing) second term ($P_{|i|k}^{ki} - P_{|k}^{lk}$), or equivalently a nonconserved scalar curvature, $\partial_t \mathcal{R}^{(1)} + 2H\mathcal{R}^{(1)} = a^{-2}\partial_t(a^2\mathcal{R}^{(1)}) \neq 0$, in contrast to the dust case.

Using the barotropic EoS and the corresponding solution (42) to the energy conservation equation (14), we can also expand ϵ , p , N and \mathcal{A}_j^i in terms of the first-order deformation field. We write $\epsilon_i := \epsilon_{H\mathbf{i}}(1 + \delta\epsilon_i)$ at first order, and expand $J^{-1} = a^{-3}\det(\delta_i^\alpha + P_i^\alpha)^{-1}$ at the same order as $a^{-3}(1 - P)$. The solution (42) for ϵ as a function of J can then be expanded to first order in the perturbation as

$$\begin{aligned} \epsilon &= F^{-1} \left(\frac{F(\epsilon_{H\mathbf{i}}) + F'(\epsilon_{H\mathbf{i}})\epsilon_{H\mathbf{i}}\delta\epsilon_i - F(\epsilon_{H\mathbf{i}})P}{a^3} \right) \\ &= F^{-1} \left(\frac{F(\epsilon_{H\mathbf{i}})}{a^3} \right) \\ &+ \left[\frac{1}{a^3}\epsilon_{H\mathbf{i}}F'(\epsilon_{H\mathbf{i}})\delta\epsilon_i - P\frac{F(\epsilon_{H\mathbf{i}})}{a^3} \right] (F^{-1})' \left(\frac{F(\epsilon_{H\mathbf{i}})}{a^3} \right). \end{aligned} \quad (59)$$

The energy-momentum conservation equation (42) still holds for background quantities, giving

$$F(\epsilon_H) = \frac{F(\epsilon_{H\mathbf{i}})}{a^3}. \quad (60)$$

This can be substituted into (59) to give

$$\epsilon = \epsilon_H \left[1 + \frac{F(\epsilon_H)}{\epsilon_H F'(\epsilon_H)} \left(\frac{\epsilon_{H\mathbf{i}}F'(\epsilon_{H\mathbf{i}})}{F(\epsilon_{H\mathbf{i}})}\delta\epsilon_i - P \right) \right]. \quad (61)$$

The further use of the definition of F , Eq. (16), allows us to simplify the above to

$$\epsilon = \epsilon_H \left[1 - \left(1 + \frac{p_H}{\epsilon_H} \right) \bar{P} \right], \quad (62)$$

which we have written for convenience in terms of a shifted deformation trace,

$$\bar{P} := P - \alpha_{H\mathbf{i}}\delta\epsilon_i, \quad (63)$$

where $\alpha_{H\mathbf{i}} := (\epsilon_{H\mathbf{i}} + \beta(\epsilon_{H\mathbf{i}}))^{-1}\epsilon_{H\mathbf{i}}$ is a constant, and $\delta\epsilon_i$ is the initial energy perturbation.

The pressure can in turn be expanded to first order as $p = \beta(\epsilon)$, yielding

$$p = p_H - \beta'(\epsilon_H)(\epsilon_H + p_H)\bar{P}. \quad (64)$$

Note that the factor $\beta'(\epsilon_H)$ corresponds to the (generally time-dependent) dimensionless ratio of the background speed of sound to speed of light squared, $\beta'(\epsilon_H) =: c_S^2(t)/c^2$, if p_H is the thermodynamic equilibrium pressure for the background fluid.

We then expand the lapse $N = (\epsilon + p)^{-1}F(\epsilon)$ as

$$N = N_H [1 + \beta'(\epsilon_H)\bar{P}] \quad (65)$$

at first order in the deformation field. At this order, one will then have (with $\partial_t P = \partial_t \bar{P}$):

$$\begin{aligned} \frac{\partial_t N}{N} &= \frac{\partial_t N_H}{N_H} + \beta'(\epsilon_H)\partial_t \bar{P} \\ &- 3H(\epsilon_H + \beta(\epsilon_H))\beta''(\epsilon_H)\bar{P}, \end{aligned} \quad (66)$$

with

$$\frac{\partial_t N_H}{N_H} = 3H\beta'(\epsilon_H). \quad (67)$$

This also allows one to obtain the first-order expression for $\mathcal{A}_j^i = N^{-1}N^{|i|}_{|j|}$:

$$\mathcal{A}_j^{i(1)} = a^{-2}\beta'(\epsilon_H)\delta^{ik}\bar{P}_{j|k}. \quad (68)$$

2. First-order system

Using the above expansions, the Lagrange-Einstein system (29)–(32) can be rewritten at first order in the deformation field as follows:

$$\partial_t P_{[ij]} = 0; \quad (69)$$

$$\begin{aligned} \partial_t^2 P_j^i + 3H[1 - \beta'(\epsilon_H)]\partial_t P_j^i \\ + H[1 - \beta'(\epsilon_H) - \mathcal{V}(t)]\partial_t \bar{P}\delta_j^i \\ = N_H^2 \mathcal{A}_j^{i(1)} - N_H^2 \left(\mathcal{R}_j^{i(1)} - \frac{\mathcal{V}(t)}{4}\mathcal{R}^{(1)}\delta_j^i \right); \end{aligned} \quad (70)$$

$$\partial_t (P_{j|i}^i - \bar{P}_{|j}) = -2H\beta'(\epsilon_H)\bar{P}_{|j}, \quad (71)$$

$$\begin{aligned} H\partial_t \bar{P} + 4\pi G [\epsilon_H + p_H - (2\epsilon_H + \tilde{\Lambda})\beta'(\epsilon_H)] N_H^2 \bar{P} \\ = -\frac{1}{4}N_H^2 \mathcal{R}^{(1)}, \end{aligned} \quad (72)$$

with $\partial_t P = \partial_t \bar{P}$, and where $\mathcal{A}_j^{i(1)}$, $\mathcal{R}_j^{i(1)} = a^{-2}\delta^{ik}\mathcal{R}_{kj}^{(1)}$ and $\mathcal{R}^{(1)}$ are expressed as functions of P_i^α according to the formulas given above, $\tilde{\Lambda} := \Lambda/(4\pi G)$, and we introduce the abbreviation

$$\begin{aligned} \mathcal{V}(t) &:= [\epsilon_H + p_H - (2\epsilon_H + \tilde{\Lambda})\beta'(\epsilon_H)]^{-1} \\ &\times \left\{ \epsilon_H + p_H - (3\epsilon_H - p_H + 2\tilde{\Lambda})\beta'(\epsilon_H) \right. \\ &\left. + (2\epsilon_H + \tilde{\Lambda})(\epsilon_H + p_H)\beta''(\epsilon_H) \right\}. \end{aligned} \quad (73)$$

Equation (70) has been obtained from the first-order expansion of the extrinsic curvature evolution equation (30) by combining it with the first-order energy constraint (72). The EoS (33) has already been used to expand ϵ , p and N in terms of the first-order deformation field.

D. First-order master equations

Following the approach of Ref. [L4] the above system can be reexpressed by decomposing the deformation fields into trace, trace-free symmetric and antisymmetric parts:

$$P^i_j = \frac{1}{3}P\delta^i_j + \Pi^i_j + \mathfrak{P}^i_j, \quad (74)$$

where $\Pi_{ij} = P_{(ij)} - \frac{1}{3}P\delta_{ij}$ and $\mathfrak{P}_{ij} = P_{[ij]}$.

We will now derive the governing equations for these parts, named *master equations*. For the trace part we use the new variable \bar{P} from Eq. (63). Accordingly, (69)–(70) become:

$$\partial_t \mathfrak{P}_{ij} = 0 : \quad \mathfrak{P}_{ij} = \mathfrak{P}_{ij}(t_i) = 0; \quad (75)$$

$$\begin{aligned} \partial_t^2 \bar{P} + 3H[2 - 2\beta'(\epsilon_H) - \mathcal{V}(t)]\partial_t \bar{P} \\ = N_H^2 \mathcal{A}^{(1)} - N_H^2 \left(1 - \frac{3}{4}\mathcal{V}(t)\right) \mathcal{R}^{(1)}; \end{aligned} \quad (76)$$

$$\partial_t^2 \Pi^i_j + 3H[1 - \beta'(\epsilon_H)]\partial_t \Pi^i_j = N_H^2 \left(\xi^i_j{}^{(1)} - \tau^i_j{}^{(1)}\right); \quad (77)$$

$$\partial_t \left(\Pi^i_{j|i} - \frac{2}{3}\bar{P}_{|j}\right) = -2H\beta'(\epsilon_H)\bar{P}_{|j}. \quad (78)$$

Once again the first-order quantities $\mathcal{A}^{(1)}$, $\xi^i_j{}^{(1)}$, $\mathcal{R}^{(1)}$ and $\tau^i_j{}^{(1)}$ are used as shorthand notations but are meant to be expressed in terms of the deformation field. These expressions are obtained from the results above, Eqs. (57), (58), (68), as follows:⁸

$$a^2 \mathcal{A}^{(1)} = \beta'(\epsilon_H) \Delta_0 \bar{P}; \quad (79)$$

$$a^2 \xi^i_j{}^{(1)} = \beta'(\epsilon_H) \left(\bar{P}^{|i}_{|j} - \frac{\delta^i_j}{3} \Delta_0 \bar{P}\right); \quad (80)$$

⁸ The expression given for $\tau^i_j{}^{(1)}$ makes use of the momentum constraints (78), which imply, through their spatial derivative, $\partial_t \Pi_{k[ij]}{}^{|k} = 0$, and thus $\Pi_{k[ij]}{}^{|k} = \Pi_{k[ij]}{}^{|k}(t_i) = 0$. Also note that since P and \bar{P} differ by an initial spatial function, we can express (79)–(82) in terms of either variable. Here we have adopted the most compact possibility, noting that the initial value of \bar{P} is nonzero, whereas (81) and (82) involve the initial curvature which is independent of the initial perturbation field.

$$a^2 \mathcal{R}^{(1)} = \mathcal{R} + 2 \left(\Pi^{ki}{}_{|k|i} - \frac{2}{3}P^{|k}_{|k}\right); \quad (81)$$

$$\begin{aligned} a^2 \tau^i_j{}^{(1)} = \mathcal{F}_j^i + 2\Pi^i{}_{k|j}{}^{|k} - \Pi^i_j{}^{|k}_{|k} \\ - \frac{1}{3} \left(2\Pi^k{}_{l|k}{}^{|l} \delta^i_j + P^{|i}_{|j} - \frac{1}{3}\Delta_0 P \delta^i_j\right), \end{aligned} \quad (82)$$

with $\mathcal{F}_j^i := \mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j = \tau^i_j{}^{(1)}(t_i)$, and with Δ_0 the coordinate Laplacian operator in the Lagrangian coordinates $\{X^i\}$, $\Delta_0 := \delta^{ij}\partial_i\partial_j$.

1. Master equation for the trace

Contracting the momentum constraints (78) with a spatial derivative ${}_{|j}$ yields the first-order evolution equation for the nontrivial part of the scalar curvature:

$$\begin{aligned} \partial_t \left(P^{ki}{}_{|k|i} - P^{|k}_{|k}\right) = \partial_t \left(\Pi^{ki}{}_{|k|i} - \frac{2}{3}\bar{P}^{|k}_{|k}\right) \\ = -2H\beta'(\epsilon_H)\Delta_0 \bar{P}. \end{aligned} \quad (83)$$

From the respective expressions (58), (79) for $\mathcal{R}^{(1)}$ and $\mathcal{A}^{(1)}$, this amounts to the following evolution for $\mathcal{R}^{(1)}$:

$$\begin{aligned} \partial_t \mathcal{R}^{(1)} + 2H\mathcal{R}^{(1)} = -4Ha^{-2}\beta'(\epsilon_H)\Delta_0 \bar{P} \\ = -4H\mathcal{A}^{(1)}, \end{aligned} \quad (84)$$

which unlike the case of dust does remain coupled to the dynamics of the inhomogeneous perturbation.

Combining this evolution equation with the linearized energy constraint (72) and its time-derivative one then obtains the *master equation for the evolution of the trace* (63) of the first-order deformation field:⁹

$$\begin{aligned} \partial_t^2 \bar{P} + 2H(1 - 3\beta'(\epsilon_H))\partial_t \bar{P} - \mathcal{W}(t)N_H^2 \bar{P} \\ = a^{-2}N_H^2 \beta'(\epsilon_H)\Delta_0 \bar{P}, \end{aligned} \quad (85)$$

where $p_H = \beta(\epsilon_H)$ and $N_H = F(\epsilon_H)/(\epsilon_H + p_H)$ still, and

$$\begin{aligned} \mathcal{W}(t) := 4\pi G[\epsilon_H + p_H - (2\epsilon_H + \tilde{\Lambda})\beta'(\epsilon_H)][4 - 3\mathcal{V}(t)] \\ = 4\pi G \left[\epsilon_H + p_H + (\epsilon_H - 3p_H + 2\tilde{\Lambda})\beta'(\epsilon_H)\right] \\ - 12\pi G(2\epsilon_H + \tilde{\Lambda})(\epsilon_H + p_H)\beta''(\epsilon_H). \end{aligned} \quad (86)$$

To avoid potential confusion, since the time coordinate t used in this paper has a different rate as compared to the conventional cosmic time, it will sometimes be convenient

⁹ This equation can also be derived by combining the energy constraint (72) with the trace (76) of the evolution equation to eliminate $\mathcal{R}^{(1)}$, or equivalently by directly expanding the Raychaudhuri equation (34) to first-order. In both cases, the master equation for the trace would then be recovered after replacing the first-order acceleration divergence $\mathcal{A}^{(1)}$ by its explicit expansion (79).

for further applications to use the (time–coordinate–independent) background scale factor a as the time variable instead. With this change of parametrization, the energy constraint (72) and the master equation for the trace (85) may be rewritten as follows:

$$a \frac{\partial \bar{P}}{\partial a} + \alpha_0 \bar{P} = -\frac{N_H^2}{4H^2} \mathcal{R}^{(1)} ; \quad (87)$$

$$\frac{\partial^2 \bar{P}}{\partial a^2} + \frac{\alpha_1}{a} \frac{\partial \bar{P}}{\partial a} - \frac{\alpha_2}{a^2} \bar{P} = \frac{\alpha_3}{a^4} \Delta_0 \bar{P}, \quad (88)$$

respectively, with time–dependent coefficients,

$$\begin{aligned} \alpha_0 &:= 4\pi G \frac{N_H^2}{H^2} \left[\epsilon_H + p_H - (2\epsilon_H + \tilde{\Lambda}) \beta'(\epsilon_H) \right] ; \\ \alpha_1 &:= \alpha_0 + 4\pi G \frac{N_H^2}{H^2} \left[\tilde{\Lambda} - 2p_H \right] ; \\ \alpha_2 &:= N_H^2 \mathcal{W}(t)/H^2 \quad ; \quad \alpha_3 := N_H^2 \beta'(\epsilon_H)/H^2 , \end{aligned} \quad (89)$$

where we recall that from the background Friedmann equation we have $H^2/N_H^2 = 4\pi G(2\epsilon_H + \tilde{\Lambda})/3$.

From Eq. (88) we can introduce a time–dependent background Jeans wave number $k_J(\epsilon_H)$ by¹⁰

$$k_J(\epsilon_H) := \frac{1}{c} \sqrt{\frac{\alpha_2}{\alpha_3}} = \frac{1}{c} \sqrt{\frac{\mathcal{W}(t)}{\beta'(\epsilon_H)}}, \quad (90)$$

provided that the term in the square root is positive. Pressure should be positive for sound waves to resist gravitational collapse, and the existence of the Jeans length is intimately related to the energy conditions satisfied by the matter field.

A remark is in order here. In general, one would expect the evolution of the inhomogeneous deformation to be affected by the local, inhomogeneous speed of sound and density, so that a nonperturbative Lagrangian realization would rather feature a local Jeans wave number $k_J(\epsilon)$ [21]. The dynamics in the presence of a significant density contrast will thus only be partially captured by the above first–order equation, where ϵ has been expanded in P_i^{α} and, accordingly, only zero–order background factors such as $k_J(\epsilon_H)$ survive in front of the first–order \bar{P} .

As in the dust case, the advantages of the Lagrangian approach are only fully realized *via* nonlinear extrapolation, e.g., by computing the energy density as a full nonlinear functional from the first–order deformation. This is part of the Relativistic Zel’dovich Approximation scheme, as defined for dust fluids in [L1]. As in the dust case and in contrast to standard Eulerian linear perturbation schemes, applying this procedure to compute the energy density out of the solution to first–order equations such as (85), will already capture part of the

nonlinear features. This is due to the nonlinear extrapolation and to the use of Lagrangian spatial coordinates which follow the fluid propagation in an exact manner. Further nonlinear effects of inhomogeneous pressure will, however, still be missed due to the absence of local Jeans length contributions in the equation used for \bar{P} , compared to what should appear in the nonperturbative evolution equation.

We will not go beyond this procedure in the present paper. Let us nonetheless suggest here a possible direction for improvement. It would require properly defining the local Jeans length in the relativistic context as a functional of the deformation. This quantity would then replace the background Jeans length in the trace master equation. The corresponding nonlinear master equation could then be solved in an iterative manner, by computing at each step the local Jeans length *via* functional extrapolation out of the previous estimate for the deformation field. Note that each step would also involve a search for the traceless part of the deformation, as all of its components would be required for the extrapolation.

The evolution equation (85) may be rewritten in an alternative form *via* a time–dependent rescaling of the variable $\bar{P} \mapsto \bar{P}/N_H(t)$. Using the variation rate (67) of the background lapse one finds the more transparent form:

$$\begin{aligned} \partial_t^2 \left(\frac{\bar{P}}{N_H} \right) + 2H \partial_t \left(\frac{\bar{P}}{N_H} \right) - 4\pi G(\epsilon_H + p_H) N_H^2 \left(\frac{\bar{P}}{N_H} \right) \\ = a^{-2} N_H^2 \beta'(\epsilon_H) \Delta_0 \left(\frac{\bar{P}}{N_H} \right) . \end{aligned} \quad (91)$$

• *Dust limit:* Setting $p_H = \beta(\epsilon_H) = 0$, we find $\mathcal{W}(t) = 4\pi G \epsilon_H = 4\pi G \varrho_H = 4\pi G \varrho_{H_1} a^{-3}$ and $N_H(t) = (\epsilon_H + p_H)^{-1} \varrho_H = 1$, and consequently both t –variable forms of the trace master equation, Eqs. (85) and (91), reduce to the dust deformation trace evolution equation of [L1]–[L4]. The trace master equation becomes:

$$\partial_t^2 P + 2H \partial_t P - 4\pi G \varrho_{H_1} a^{-3} P = -4\pi G \varrho_{H_1} a^{-3} \delta \epsilon_i . \quad (92)$$

With $N_H = 1$ the time variable used is the standard FLRW time coordinate $\tilde{t} = t$, so that the above time–derivatives coincide with those used in [L1]–[L4] (denoted there by overdots). Finally, as evaluating Eq. (92) at the initial time gives $W = -4\pi G \varrho_{H_1} \delta \epsilon_i$, its right hand side can always be rewritten as $W a^{-3}$, and the dust–case master equation for the trace (e.g., Eq. (41) of [L4]) is thus recovered.

• *Newtonian limit:* The Newtonian limit is obtained by the joint application of the *Minkowski Restriction* (MR) for the deformation field, as introduced for dust in [L1, L2], and of the $c \rightarrow \infty$ limit together with the assumption of a nonrelativistic pressure.

The latter two assumptions imply that the pressure is no longer a source of the gravitational field, as the energy density is then $\epsilon \simeq \varrho c^2 \gg p$ (where the constant c has been temporarily restored), so that all source terms

¹⁰ We include the factor c explicitly so that the dimensional content of this relation is clear. The right hand side of (86) must be divided by c^2 if units $c \neq 1$ are restored.

reduce to the contribution of ϱ . Note that ϱ can be considered as equal to the actual rest mass density in this limit. A further consequence of this is that the lapse becomes trivial, $N = \varrho c^2 / (\epsilon + p) \simeq 1$, consistent with its spatial variation rate, $N^{-1} N_{|i} = -(\epsilon + p)^{-1} p_{|i} \simeq -(\varrho c^2)^{-1} p_{|i} \rightarrow 0$ when $c \rightarrow \infty$, for any pressure spatial gradient. It is also the case for the (already homogeneous, but generally time-dependent) background lapse that $N_H \simeq 1$. Consequently, the fluid-orthogonal hypersurface time label t now coincides with the fluid's proper time τ (since $1 \simeq N = \partial_t \tau$) as well as with the standard background cosmic (proper) time \tilde{t} . All these notions thus consistently define a time reference that can be used as the Newtonian absolute time. We will denote the corresponding Lagrangian time-derivative operator by an overdot.

With $N = 1$ the line element (3) reduces to the one used in [L1, L2] for irrotational dust, and one can thus directly use the corresponding definition of the MR in this context.¹¹ This restriction amounts to assuming that the initial metric is Euclidean and that the spatial coframes are exact in the three-dimensional hypersurfaces, *i.e.*, that there exist spatial coordinates $x^a = f^a(X^i, t)$ such that $G_{ab} = \delta_{ab}$ and

$$\eta^a_{|i} = a(t) (\delta^a_{|i} + P^a_{|i}) = f^a_{|i}. \quad (93)$$

In any $t = \text{const}$ hypersurface, the spatial line element then reads $ds^2 = \delta_{ab} dx^a dx^b$. The coordinates x^a thus define Cartesian-type Eulerian coordinates in which the metric coefficients are manifestly Euclidean at each time, and they can be used to define a Newtonian spatial reference frame. Through its second equality, Eq. (93) also implies that the deformation 1-forms P^a are also exact and accordingly define a deformation vector \mathbf{P} , with components P^a ,

$$\mathbf{x} = a(t) [\mathbf{X} + \mathbf{P}(\mathbf{X}, t)] \quad , \quad P^a_{|i} := P^a_{|i}. \quad (94)$$

With these two assumptions the master equation (91) on the trace $P = \delta_a^i P^a_{|i}$ becomes an equation on the Lagrangian divergence $\nabla_0 \cdot \mathbf{P} := \delta_a^i P^a_{|i}$ of \mathbf{P} :

$$\begin{aligned} \nabla_0 \cdot \ddot{\mathbf{P}} + 2H \nabla_0 \cdot \dot{\mathbf{P}} - 4\pi G \varrho_H (\nabla_0 \cdot \mathbf{P} - \delta \varrho_i) \\ = a^{-2} \frac{d p_H}{d \varrho_H} \Delta_0 (\nabla_0 \cdot \mathbf{P} - \delta \varrho_i), \end{aligned} \quad (95)$$

¹¹ Note that the Minkowski Restriction introduced for the dust case is in principle independent of a possible $c \rightarrow \infty$ limit and can still otherwise be applied in a Minkowskian regime, as the name suggests. In the present case, when c is still finite, this procedure would need to be extended to the presence of pressure and consequently of an inhomogeneous lapse. We believe, however, that such an extension to this case would require a modification of the perturbation framework used so far in this paper, through the use of a spacetime foliation better adapted to this purpose, and we will consequently not attempt to provide such a generalization here.

with $\varrho_H = a^{-3} \varrho_{H\mathbf{i}}$ still, and $\varrho_i := \varrho_{H\mathbf{i}} (1 + \delta \varrho_i)$. Note that, although the pressure itself no longer contributes as a source of gravitation, its spatial gradient still produces an acceleration (as obviously expected in a Newtonian framework), which is why it still affects the dynamics of $\nabla_0 \cdot \mathbf{P}$ above through the sound speed squared factor $d p_H / d \varrho_H$ in front of its Laplacian.

The above Eq. (95) matches¹² the corresponding equation for the deformation vector obtained in the Newtonian Lagrangian framework, Eq. (24b) in [6] or Eq. (45) in [25] written for the longitudinal part of the deformation vector. By definition, this part obeys the same evolution equation as the Lagrangian divergence of the vector, as can be seen in the unnumbered equations involving that divergence before Eq. (24a) in [6]. Note that in this reference, the Laplacian term features a local sound speed squared (related to the local Jeans length) $d p / d \varrho$, but it is already noted there that it should actually be replaced by the background value for consistency with the first-order expansion, and it is indeed replaced by the corresponding background expression in [25].

2. Master equation for the traceless part

The first-order evolution of the traceless symmetric part Π^i_j is given by Eq. (77), with $\xi_j^{i(1)}$ and $\tau_j^{i(1)}$ replaced by their expressions (80) and (82), respectively. Eliminating the initial traceless curvature \mathcal{T}_j^i by evaluation of the evolution equation at the time corresponding to the initial condition (114), then first yields the following evolution equation for the traceless symmetric part:

$$\begin{aligned} \partial_t^2 \Pi^i_j + 3H [1 - \beta'(\epsilon_H)] \partial_t \Pi^i_j \\ + \frac{N_H^2}{a^2} \left(2 \Pi^i_{|k|j}{}^{|k} - \Pi^i_{|j|k}{}^{|k} - \frac{2}{3} \Pi^k_{|l|k}{}^{|l} \delta^i_j \right) \\ = \frac{N_H^2}{3a^2} \left([1 + 3\beta'(\epsilon_H)] \mathcal{D}^i_j \bar{P} - [1 + 3\beta'(\epsilon_{H\mathbf{i}})] \mathcal{D}^i_j \bar{P}_{\mathbf{i}} \right) \\ + \frac{N_H^2}{a^2 N_{H\mathbf{i}}^2} \left({}^{\text{tl}}W^i_j + H_{\mathbf{i}} [1 - 3\beta'(\epsilon_{H\mathbf{i}})] {}^{\text{tl}}U^i_j \right). \end{aligned} \quad (96)$$

Here $\bar{P}_{\mathbf{i}} = -\alpha_{H\mathbf{i}} \delta \epsilon_i$ due to the vanishing of the initial spatial perturbation field, and we have introduced the coordinate traceless spatial Hessian operator $\mathcal{D}^i_j := \delta^{ik} \partial_k \partial_j - (1/3) \delta^i_j \Delta_0$.

¹² Eq. (95) features additional contributions from the initial density perturbations $\delta \varrho_i$ as compared to the original Newtonian result obtained in [6]. These perturbations were actually neglected there, by assuming $\varrho_i = \varrho_{H\mathbf{i}}$, as is also assumed in Zel'dovich's original work for the dust case [78]. However, as is demonstrated in Appendix A of [6], such an assumption can be made without loss of generality in the Newtonian context within the first-order perturbation scheme in the deformation vector, through a suitable change of Lagrangian coordinates, making both approaches equivalent.

This equation still explicitly features the trace, but it can be fully expressed in terms of Π_j^i by making use of the momentum constraints (78). This can be achieved by rewriting (78) as

$$\frac{1}{N_H} \partial_t \Pi_j^i = \frac{2}{3} \partial_t \left(\frac{\bar{P}_{|j}}{N_H} \right). \quad (97)$$

A time-integration and spatial differentiation of this equation allows one to express $\mathcal{D}_j^i \bar{P}$ as

$$\frac{\mathcal{D}_j^i \bar{P}}{N_H} = \frac{\mathcal{D}_j^i \bar{P}_i}{N_{H_i}} + \frac{1}{2} \int_{t_i}^t \frac{\partial_t \left(3 \Pi_{j|k}^k \delta_j^i - \Pi_{l|k}^k \delta_j^i \right)}{N_H} dt'. \quad (98)$$

The pair of equations {(96),(98)} together comprise the *master equation for the traceless part*. When $p_H = 0$, one simply has $N_H(t) = 1$ and $\beta'(\epsilon_H) = 0$ so that this master equation reduces to the corresponding one in the dust case, Eq. (43) in [L4].

3. Master equations for free and scattered gravitational waves

Following the approach developed in [L3, L4], we can gain more insight into the evolution of Π_j^i by splitting the full master equation for the traceless variable into gravitoelectric and gravitomagnetic parts.

To this end, we first define a corresponding split of the initial conditions for the traceless variables:

$${}^{\text{tl}}U_j^i = {}^{\text{tl},E}U_j^i + {}^{\text{tl},H}U_j^i; \quad {}^{\text{tl}}W_j^i = {}^{\text{tl},E}W_j^i + {}^{\text{tl},H}W_j^i; \quad (99)$$

$${}^{\text{tl},H}U_{j|i} = 0; \quad {}^{\text{tl},H}W_{j|i} = 0; \quad (100)$$

$$2 \Delta_0 {}^{\text{tl},E}U_j^i + {}^{\text{tl},E}U_{l|k}^k \delta_j^i - 3 {}^{\text{tl},E}U_{k|j}^i \delta^k = 0; \quad (101)$$

$$2 \Delta_0 {}^{\text{tl},E}W_j^i + {}^{\text{tl},E}W_{l|k}^k \delta_j^i - 3 {}^{\text{tl},E}W_{k|j}^i \delta^k = 0. \quad (102)$$

These conditions can be jointly required because of the following geometric identity (taking its first two time-derivatives and evaluating them at the initial time):

$$(2 \Delta_0 \Pi_j^i + \Pi_{l|k}^k \delta_j^i - 3 \Pi_{k|j}^i \delta^k)_{|i} = 0. \quad (103)$$

This in turn is due to $\Pi_{[i|j]|k}^k = 0$, which is a consequence of the momentum constraints (see footnote 8).

We can then define the gravitoelectric and gravitomagnetic traceless parts, respectively, ${}^E\Pi_j^i$ and ${}^H\Pi_j^i$, from their vanishing initial values and their respective initial first time-derivatives ${}^{\text{tl},E}U_j^i$ and ${}^{\text{tl},H}U_j^i$, requiring them

to obey the following evolution equations:

$$\begin{aligned} & \partial_t^2 {}^H\Pi_j^i + 3H[1 - \beta'(\epsilon_H)] \partial_t {}^H\Pi_j^i - \frac{N_H^2}{a^2} \Delta_0 {}^H\Pi_j^i \\ &= \frac{N_H^2}{a^2 N_{H_i}^2} \left({}^{\text{tl},H}W_j^i + H_i [1 - 3\beta'(\epsilon_{H_i})] {}^{\text{tl},H}U_j^i \right); \quad (104) \\ & \partial_t^2 {}^E\Pi_j^i + 3H[1 - \beta'(\epsilon_H)] \partial_t {}^E\Pi_j^i + \frac{N_H^2}{3a^2} \Delta_0 {}^E\Pi_j^i \\ &= \frac{N_H^2}{3a^2} \left([1 + 3\beta'(\epsilon_H)] \mathcal{D}_j^i \bar{P} - [1 + 3\beta'(\epsilon_{H_i})] \mathcal{D}_j^i \bar{P}_i \right) \\ &+ \frac{N_H^2}{a^2 N_{H_i}^2} \left({}^{\text{tl},E}W_j^i + H_i [1 - 3\beta'(\epsilon_{H_i})] {}^{\text{tl},E}U_j^i \right). \quad (105) \end{aligned}$$

Equation (104) is the *master equation for free gravitational waves*, while Equation (105), after elimination of the coupling to the trace, is the *master equation for the gravitational wave part that is scattered at the fluid source*. We will discuss the coupling to the trace of this latter equation in more detail below.

The above evolution equations ensure that we indeed get a decomposition of the traceless deformation field obeying (96) at all times:

$$\Pi_j^i = {}^E\Pi_j^i + {}^H\Pi_j^i. \quad (106)$$

They will also propagate the initial constraints (99)–(102) that define the split of ${}^{\text{tl}}U_j^i$ and ${}^{\text{tl}}W_j^i$. This will ensure the preservation at all times of the divergence-free nature of free gravitational waves as well as the geometric identity on their scattered part, similar to the dust case (cf. [L3, L4]):

$${}^H\Pi_{j|i}^i = 0; \quad (107)$$

$$2 \Delta_0 {}^E\Pi_j^i + {}^E\Pi_{l|k}^k \delta_j^i - 3 {}^E\Pi_{k|j}^i \delta^k = 0. \quad (108)$$

The (also propagating) momentum constraints (97) split as follows:

$${}^H\Pi_{j|i}^i = 0; \quad \frac{1}{N_H} \partial_t {}^E\Pi_{j|i}^i = \frac{2}{3} \partial_t \left(\frac{\bar{P}_{|j}}{N_H} \right). \quad (109)$$

Observe that ${}^H\Pi_j^i$ decouples from the trace in both the momentum constraints and the evolution equation, while ${}^E\Pi_j^i$ remains coupled to the trace in both cases.

Alternatively, using a time integral of the momentum constraints,

$${}^E\Pi_{j|i}^i = \frac{2}{3} \int_{t_i}^t N_H \partial_t \left(\frac{\bar{P}_{|j}}{N_H} \right) dt', \quad (110)$$

the geometric constraint (108) on ${}^E\Pi_j^i$ can be expressed as follows:

$$\Delta_0 {}^E\Pi_j^i = \mathcal{D}_j^i \left(\int_{t_i}^t N_H \partial_t \left(\frac{\bar{P}}{N_H} \right) dt' \right). \quad (111)$$

This is to be compared to the dust-case relation, Eq. (51) in [L4], to which it reduces when $p_H = 0$ and accordingly

$N_H(t) = 1$: $\Delta_0^E \Pi_j^i = \mathcal{D}_j^i(\bar{P} - \bar{P}_i) = \mathcal{D}_j^i P$. Hence, in the presence of pressure, in contrast to the dust case, the gravitoelectric traceless part and the trace, although still tightly coupled, will in general have different time behaviors.

With the antisymmetric part vanishing at all times, the evolution equations for the trace and for the gravitoelectromagnetic split of the traceless symmetric part, coupled through the momentum constraints, characterize the behavior of the first-order Lagrangian deformation field for this general barotropic single fluid. These evolution equations have yet to be complemented by the set of initial constraints (48)–(53), to which we turn now.

E. First-order initial conditions

The constraints on the initial conditions for the deformation field, the density and the spatial curvature are expressed at the first-order level as follows:

$$U_{[ij]} = 0 \quad ; \quad W_{[ij]} = 0 ; \quad (112)$$

$$W - 6H_{\mathbf{i}} \beta'(\epsilon_{H\mathbf{i}}) U = -N_{H\mathbf{i}}^2 \alpha_{H\mathbf{i}} \left[\mathcal{W}(t_{\mathbf{i}}) \delta\epsilon_{\mathbf{i}} + \beta'(\epsilon_{H\mathbf{i}}) \Delta_0(\delta\epsilon_{\mathbf{i}}) \right] ; \quad (113)$$

$${}^{\text{tl}}W_j^i + H_{\mathbf{i}} \left[1 - 3\beta'(\epsilon_{H\mathbf{i}}) \right] {}^{\text{tl}}U_j^i = -N_{H\mathbf{i}}^2 \mathcal{S}_j^i - N_{H\mathbf{i}}^2 \alpha_{H\mathbf{i}} \beta'(\epsilon_{H\mathbf{i}}) \left[(\delta\epsilon_{\mathbf{i}})^i{}_{|j} - \frac{1}{3} \Delta_0(\delta\epsilon_{\mathbf{i}}) \delta_j^i \right] ; \quad (114)$$

$$H_{\mathbf{i}} U = -\frac{1}{4} \mathcal{R} N_{H\mathbf{i}}^2 + 4\pi G N_{H\mathbf{i}}^2 \alpha_{H\mathbf{i}} \delta\epsilon_{\mathbf{i}} \times \left[\epsilon_{H\mathbf{i}} + p_{H\mathbf{i}} - (2\epsilon_{H\mathbf{i}} + \tilde{\Lambda}) \beta'(\epsilon_{H\mathbf{i}}) \right] ; \quad (115)$$

$$U^i{}_{j|i} - U_{|j} = 2H_{\mathbf{i}} \alpha_{H\mathbf{i}} \beta'(\epsilon_{H\mathbf{i}}) (\delta\epsilon_{\mathbf{i}})_{|j} ; \quad (116)$$

$$p_{\mathbf{i}} = p_{H\mathbf{i}} + \epsilon_{H\mathbf{i}} \beta'(\epsilon_{H\mathbf{i}}) \delta\epsilon_{\mathbf{i}} \quad ; \quad p_{H\mathbf{i}} = \beta(\epsilon_{H\mathbf{i}}) . \quad (117)$$

This set of initial conditions can also be obtained by evaluating the linearized Lagrange–Einstein system at the initial time. It can be complemented by the requirements (99)–(102) which define the initial split into gravitoelectric and gravitomagnetic parts of the traceless deformation field.

Note that the above set keeps more variables coupled than the corresponding ones in [L4]. This is to be expected, since in the dust case a vanishing pressure and a constant lapse allowed for the elimination of ϵ and Λ between the first two constraints, leaving only a relation among U , W and \mathcal{R} . Here, we also have contributions from p , Λ (due to the lapse factor in the Λ term) and the nonvanishing $\mathcal{A}_{\mathbf{i}}^{(1)}$. Accordingly, the dependence on the initial energy density $\epsilon_{\mathbf{i}}$ and its spatial derivatives can no longer be explicitly removed in general. However, as in the dust case, the scalar constraints (113) and (115), together with the initial EoS (117), show that only two independent first-order initial conditions need to be

given for the scalar variables U , W , \mathcal{R} , $\epsilon_{\mathbf{i}}$, and $p_{\mathbf{i}}$. One could for instance only specify U and W as can be done in the dust case, fully determining the other scalar initial conditions. In contrast to the dust case, however, determining $\epsilon_{\mathbf{i}}$ in this situation would involve solving for the Laplacian differential equation (113).

IV. APPLICATION TO SPECIFIC EQUATIONS OF STATE

Concrete results can be obtained by looking at special cases of the barotropic EoS. In this section, we will first consider the family of linear relations between the pressure and the energy density. We then proceed to a special nonlinear polytropic EoS that allows one to model the isotropic part of a velocity dispersion field up to late epochs of nonlinear structure formation.

A. Case of a linear Equation of State: $p = w\epsilon$

In the previous section we have derived the evolution equations for the first-order deformation field, sourced by a general barotropic fluid. In this section we will consider as an example the simplest barotropic EoS, $p = \beta(\epsilon) = w\epsilon$ with w a constant parameter obeying the dominant energy condition, $-1 \leq w \leq 1$. In addition to the radiation fluid, with $w = 1/3$, other interesting cases include a “stiff fluid” corresponding to a free scalar field source, with $w = 1$, and a “curvature” or “string gas” equation of state, with $w = -1/3$. For this class of linear EoS we can readily apply the procedure suggested in [L3, L4] to find the relativistic Lagrangian first-order solutions.

The formal rest mass density $F(\epsilon)$ and the lapse are found to be as follows:

$$F(\epsilon) = \varrho_1 \left(\frac{\epsilon}{\epsilon_1} \right)^{1/(1+w)} \quad ; \quad N = \frac{\varrho_1}{\epsilon_1(1+w)} \left(\frac{\epsilon}{\epsilon_1} \right)^{-w/(1+w)} , \quad (118)$$

if $w \neq -1$. (The case $w = -1$ for a “vacuum energy equation of state” can be treated separately by the explicit cosmological term.)

The solution (42) of the energy conservation law then yields the energy density, and the lapse as deduced from (118), as the following respective functionals of the coframes, with $J = \det(\eta^a{}_i)$:

$$\epsilon = \epsilon_{\mathbf{i}} J^{-(1+w)} \quad ; \quad N = N_{\mathbf{i}} J^w . \quad (119)$$

Similar equations hold for the background spacetime,

$$\epsilon_H = \epsilon_{H\mathbf{i}} a^{-3(1+w)} \quad ; \quad N_H = N_{H\mathbf{i}} a^{3w} \quad ; \quad \frac{\partial_t N_H}{N_H} = 3wH . \quad (120)$$

Given the linear barotropic EoS, the pressure and background pressure are immediately deduced from the expression of the corresponding energy densities, and will share their functional dependencies.

1. First-order equations

With the linear EoS $\beta(\epsilon) = w\epsilon$, $\beta'(\epsilon_H)$ reduces to the constant value w , $\beta''(\epsilon_H)$ vanishes at all times, and $\alpha_{H\mathbf{i}} = (1+w)^{-1}$. Consistent with a first-order evaluation of the exact formulas above, the first-order expressions (62)–(66) for \bar{P} , ϵ , p , $F(\epsilon)$, N (and its rate of evolution) thus simplify to

$$\begin{aligned}\bar{P} &= P - (1+w)^{-1}\delta\epsilon_{\mathbf{i}} ; \\ \epsilon &= \epsilon_H \left[1 - (1+w)\bar{P} \right] ; \quad p = p_H - w(1+w)\epsilon_H\bar{P} ; \\ F(\epsilon) &= F(\epsilon_H) \left[1 - \bar{P} \right] ; \\ N &= N_H \left[1 + w\bar{P} \right] ; \quad \frac{\partial_t N}{N} = 3wH + w\partial_t\bar{P} .\end{aligned}\quad (121)$$

Eq. (73) reduces to

$$\mathcal{V}(t) = \frac{\epsilon_H(1-w)^2 - 2w\tilde{\Lambda}}{\epsilon_H(1-w) - w\tilde{\Lambda}}, \quad (122)$$

and the first-order Lagrange–Einstein system (72), (75)–(78) becomes:¹³

$$\begin{aligned}\partial_t \mathfrak{P}_{ij} &= 0 : \quad \mathfrak{P}_{ij} = \mathfrak{P}_{ij}(t_i) = 0 ; \\ \partial_t^2 \bar{P} + 3H \frac{\epsilon_H(1-w)^2 + 2w^2\tilde{\Lambda}}{\epsilon_H(1-w) - w\tilde{\Lambda}} \partial_t \bar{P} \\ &= N_{H\mathbf{i}}^2 a^{6w} \left[\mathcal{A}^{(1)} - \frac{\epsilon_H(1-w)(1+3w) + 2w\tilde{\Lambda}}{4\epsilon_H(1-w) - 4w\tilde{\Lambda}} \mathcal{R}^{(1)} \right] ;\end{aligned}\quad (123)$$

$$\partial_t^2 \Pi_j^i + 3H(1-w)\partial_t \Pi_j^i = N_{H\mathbf{i}}^2 a^{6w} \left(\xi_j^{i(1)} - \tau_j^{i(1)} \right) ; \quad (124)$$

$$\begin{aligned}H \partial_t \bar{P} + 4\pi G [\epsilon_H(1-w) - w\tilde{\Lambda}] N_{H\mathbf{i}}^2 a^{6w} \bar{P} \\ = -\frac{1}{4} N_{H\mathbf{i}}^2 a^{6w} \mathcal{R}^{(1)} ;\end{aligned}\quad (125)$$

$$\partial_t \left(\Pi_{j|i}^i - \frac{2}{3} \bar{P}_{j|} \right) = -2wH\bar{P} . \quad (126)$$

The acceleration gradient and its trace and traceless parts are expressed in terms of the deformation field at first order according to Eqs. (68), (79), and (80), yielding

$$\mathcal{A}_j^{i(1)} = a^{-2} w \bar{P}_{|j}^i ; \quad (127)$$

$$\mathcal{A}^{(1)} = a^{-2} w \Delta_0 \bar{P} ; \quad (128)$$

$$\xi_j^{i(1)} = a^{-2} w \mathcal{D}_j^i \bar{P} , \quad (129)$$

while the first-order expressions (58), (81), and (82) of the Ricci tensor and its trace/traceless split are formally unchanged. Since for the chosen EoS, $\mathcal{W}(t)$ yields

$$\begin{aligned}\mathcal{W}(t) &= 4\pi G \left[\epsilon_H(1-w)(1+3w) + 2w\tilde{\Lambda} \right] \\ &= 4\pi G \left[\epsilon_{H\mathbf{i}} a^{-3(1+w)}(1-w)(1+3w) + 2w\tilde{\Lambda} \right] ,\end{aligned}\quad (130)$$

the master equation (85) for the trace of the perturbation now reads:

$$\begin{aligned}\partial_t^2 \bar{P} + 2H(1-3w)\partial_t \bar{P} \\ - 4\pi G N_{H\mathbf{i}}^2 [\epsilon_{H\mathbf{i}}(1-w)(1+3w) a^{3(w-1)} + 2w\tilde{\Lambda} a^{6w}] \bar{P} \\ = w N_{H\mathbf{i}}^2 a^{6w-2} \Delta_0 \bar{P} .\end{aligned}\quad (131)$$

In turn, the master equation (96) for the traceless symmetric part of the deformation field becomes

$$\begin{aligned}\partial_t^2 \Pi_j^i + 3H(1-w)\partial_t \Pi_j^i \\ + N_{H\mathbf{i}}^2 a^{6w-2} \left\{ 2\Pi_{k|j}^i{}^{|k} - \Pi_{j|k}^i{}^{|k} - \frac{2}{3}\Pi_{l|k}^k{}^{|l} \delta_j^i \right. \\ \left. - \frac{1}{3}(1+3w)\mathcal{D}_j^i (\bar{P} - \bar{P}_{\mathbf{i}}) \right\} \\ = a^{6w-2} [{}^{\text{tl}}W_j^i + (1-3w)H_{\mathbf{i}}{}^{\text{tl}}U_j^i] ,\end{aligned}\quad (132)$$

with, from the momentum constraints (126),

$$a^{-3w} \mathcal{D}_j^i \bar{P} = \mathcal{D}_j^i \bar{P}_{\mathbf{i}} + \int_{t_i}^t \frac{\partial_t \left(3\Pi_{j|k}^k{}^{|i} - \Pi_{l|k}^k{}^{|l} \delta_j^i \right)}{2a^{3w}} dt' . \quad (133)$$

We can finally rewrite the set of initial conditions (112)–(117) for the linear EoS:

$$U_{[ij]} = 0 ; \quad W_{[ij]} = 0 ; \quad (134)$$

$$\begin{aligned}W - 6wH_{\mathbf{i}}U = -\frac{N_{H\mathbf{i}}^2}{1+w} \left(w\Delta_0(\delta\epsilon_{\mathbf{i}}) \right. \\ \left. + 4\pi G [\epsilon_{H\mathbf{i}}(1-w)(1+3w) + 2w\tilde{\Lambda}] \delta\epsilon_{\mathbf{i}} \right) ;\end{aligned}\quad (135)$$

$${}^{\text{tl}}W_j^i + (1-3w)H_{\mathbf{i}}{}^{\text{tl}}U_j^i = -N_{H\mathbf{i}}^2 \left[\mathcal{S}_j^i + \frac{w}{1+w} \mathcal{D}_j^i(\delta\epsilon_{\mathbf{i}}) \right] ; \quad (136)$$

$$H_{\mathbf{i}}U = -\frac{1}{4} \mathcal{R} N_{H\mathbf{i}}^2 + \frac{4\pi G N_{H\mathbf{i}}^2}{1+w} [\epsilon_{H\mathbf{i}}(1-w) - w\tilde{\Lambda}] \delta\epsilon_{\mathbf{i}} ; \quad (137)$$

$$U_{j|i}^i - U_{|j} = 2 \frac{w}{1+w} H_{\mathbf{i}}(\delta\epsilon_{\mathbf{i}})_{|j} ; \quad (138)$$

$$p_{\mathbf{i}} = p_{H\mathbf{i}} + w\epsilon_{H\mathbf{i}}\delta\epsilon_{\mathbf{i}} ; \quad p_{H\mathbf{i}} = w\epsilon_{H\mathbf{i}} . \quad (139)$$

2. Solutions for the trace of the deformation field

Similarly to [L1, L2], we will now further investigate the behavior of the trace P of the first-order deformation.

¹³ It is worth noting in the case when $\Lambda = 0$, $\mathcal{V}(t)$ simplifies further and reduces to the constant $1-w$, so that (123) becomes

$$\partial_t^2 \bar{P} + 3H(1-w)\partial_t \bar{P} = N_{H\mathbf{i}}^2 a^{6w} \left[\mathcal{A}^{(1)} - \frac{1+3w}{4} \mathcal{R}^{(1)} \right] .$$

For simplicity, we will restrict attention to the case of a vanishing cosmological constant, $\Lambda = 0$, as may be reasonably assumed during the radiation-dominated era. In this case Eqs. (87)–(89) reduce to

$$a \frac{\partial \bar{P}}{\partial a} + \frac{3}{2}(1-w)\bar{P} = \frac{-3}{32\pi G\epsilon_{H1}} a^{3(1+w)} \mathcal{R}^{(1)}; \quad (140)$$

$$\frac{\partial^2 \bar{P}}{\partial a^2} + \frac{\alpha_1}{a} \frac{\partial \bar{P}}{\partial a} - \frac{\alpha_2}{a^2} \bar{P} = \alpha_{3i} a^{3w-1} \Delta_0 \bar{P}, \quad (141)$$

with the constant parameters

$$\alpha_1 = \frac{3(1-3w)}{2}; \quad \alpha_2 = \frac{3(1-w)(3w+1)}{2}; \quad (142)$$

$$\alpha_{3i} = \frac{3w}{8\pi G\epsilon_{H1}}.$$

If $w > 0$ (implying $\alpha_{3i} > 0$), as we will assume in the following, then Eq. (141) is a second-order hyperbolic partial differential equation (PDE).¹⁴ This equation is formally analogous to the standard Eulerian propagation equations for a linearized density contrast [38, 55, 69] once those are reexpressed in terms of the variable a .¹⁵ In the Eulerian case, assuming global flat-space spatial coordinates, one can find the analytical general solution using a Fourier transformation. A discussion of the differences between the Eulerian and Lagrangian approaches has been given in [L4]. (See also the related discussion in [71].) Ref. [L4] also elucidated a procedure for finding general-relativistic Lagrangian first-order solutions for the deformation field in the dust case. We show here that this procedure can be readily extended to the presence of pressure and apply it to the determination of a Lagrangian solution for the trace part.¹⁶

First, we can use the formal identity of Eq. (141), written in Lagrangian coordinates on the nontrivial space-time manifold, with an equation written in Euclidean space. We can thus work within this flat space with its effective ‘Eulerian’ Cartesian spatial coordinates x^i and solve Equation (141) with $\Delta_0 \mapsto \delta^{ij} \partial_{x^i} \partial_{x^j}$ for the unknown $\bar{P}(a, \mathbf{x})$. On this space we can then apply an inverse Fourier transformation

$$\bar{P}(a, \mathbf{x}) = \iiint \bar{P}_k(a, \mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}, \quad (143)$$

¹⁴ It would be an elliptic PDE for $w < 0$ (i.e., $\alpha_{3i} < 0$), while for the parabolic case $w = 0$ (and consequently $\alpha_{3i} = 0$) it reduces, as expected, to the evolution equation for the dust case, with decoupled time and space variables.

¹⁵ Note that in terms of the conventional cosmic time \tilde{t} introduced in (38), Eq. (141) reduces to $\partial_{\tilde{t}}^2 \bar{P} + (2-3w)a^{-1} \partial_{\tilde{t}} a \partial_{\tilde{t}} \bar{P} - 4\pi G [(1-w)(1+3w)\epsilon_H + 2w\tilde{\Lambda}] \bar{P} = wa^{-2} \Delta_0 \bar{P}$. This is formally equivalent to the linearized Eulerian equation (3.2.17) of Ref. [69] in that the coefficients agree, but both the dependent and (spatial) independent variables differ.

¹⁶ A complementary picture of an equivalent procedure is shown in Appendix A 2 and applied to the search for a particular solution for the traceless part.

and thus get a second-order linear ordinary differential equation:

$$\frac{d^2 \bar{P}_k}{da^2} + \frac{\alpha_1}{a} \frac{d\bar{P}_k}{da} - (\alpha_2 a^{-2} - \alpha_{3i} k^2 a^{3w-1}) \bar{P}_k = 0, \quad (144)$$

where we have used $\mathbf{k}\cdot\mathbf{x} := \delta_{ij} k^i x^j$ and $k := (\delta_{ij} k^i k^j)^{1/2}$.

In this case the background Jeans wave number (90) satisfies

$$k_J(\epsilon_H)^2 = \frac{\alpha_2}{\alpha_{3i}} a^{-3(1+w)}$$

$$= 4\pi G\epsilon_{H1} \frac{(1-w)(3w+1)}{w a^{3(1+w)}}, \quad (145)$$

where we recall that $0 < w \leq 1$ is assumed. The behavior of the solution to Eq. (141) will then depend on the relative values of k and $a k_J(\epsilon_H)$.

One can first proceed by investigating the extreme cases, as is commonly done in the Eulerian analyses. When $k \ll a k_J(\epsilon_H)$, Eq. (144) may be solved as

$$\bar{P}_k = a^{1+3w} C_{k,1} + a^{\frac{3}{2}(w-1)} C_{k,2}, \quad (146)$$

where $C_{k,1(2)}$ are two functions of \mathbf{k} encoding the initial conditions. This corresponds, as expected, to the unstable regime since the term with coefficient $C_{k,1}$ is a growing mode.

In the opposite situation when $k \gg a k_J(\epsilon_H)$, the solution reads

$$\bar{P}_k = a^{\frac{9w-1}{4}} \left[J_{\hat{\nu}} \left(B a^{\frac{1+3w}{2}} k \right) C_{k,1} + Y_{\hat{\nu}} \left(B a^{\frac{1+3w}{2}} k \right) C_{k,2} \right];$$

$$B := \frac{2\sqrt{\alpha_{3i}}}{1+3w}; \quad \hat{\nu} := \frac{9w-1}{2+6w}, \quad (147)$$

with different \mathbf{k} -dependent coefficients $C_{k,1(2)}$, and where $J_{\nu}(x)$ and $Y_{\nu}(x)$ denote the Bessel functions of the first and second kind, respectively. This corresponds to a ‘stable’ regime of acoustic oscillations, although their amplitude will grow over time (as $a^{(3w-1)/2}$ for large a) for an unusual EoS with $w > 1/3$. The latter remark includes the ‘stiff fluid’ EoS $w = 1$, for which the above solution is exact at all times, since it corresponds to $k_J(\epsilon_H) = 0$.

From the expression (145) of $k_J(\epsilon_H)$, the noncomoving Jeans wave number $a k_J(\epsilon_H)$ decreases over time, so that even an initially unstable solution will eventually enter the stable regime. Such a solution will cross the threshold $k \simeq a k_J(\epsilon_H)$ and it may be useful to be able to describe this transition period as well.

As in the Newtonian case in the Eulerian approach, with different coefficients (see, e.g., [40]), the Bessel functions actually allow for an explicit solution of Eq. (144) for any mode at all times. The general solution is the same as (147) up to a change of the order of the Bessel functions:

$$\bar{P}_k = a^{\frac{9w-1}{4}} \left[J_{\nu} \left(B a^{\frac{1+3w}{2}} k \right) C_{k,1} + Y_{\nu} \left(B a^{\frac{1+3w}{2}} k \right) C_{k,2} \right];$$

$$B = \frac{2\sqrt{\alpha_{3i}}}{1+3w}; \quad \nu = \frac{5+3w}{2+6w}. \quad (148)$$

The integration constants $C_{k,1(2)}$ are derived from the initial conditions on \bar{P} and its time-derivative, $\bar{P}_i(\mathbf{X})$ and $U(\mathbf{X})$. To this end, one formally replaces these quantities by functions of the ‘Eulerian’ coordinates x^i on the Euclidean space, with the same functional dependence, $\bar{P}_i(\mathbf{x})$ and $U(\mathbf{x})$. One is then working on flat-space, and the respective Fourier transforms $\bar{P}_k(a = a_i = 1, \mathbf{k})$ and $(\partial_t \bar{P}_k)(a = 1, \mathbf{k}) = H_i(\partial_a \bar{P}_k)(a = 1, \mathbf{k})$ can be computed, from which $C_{k,1(2)}(\mathbf{k})$ are deduced. Knowing these, $\bar{P}(a, \mathbf{k})$ is expressed as the full solution given by Eq. (148) and its inverse Fourier transform (143) gives $\bar{P}(a, \mathbf{x})$ in Euclidean space.

Finally, one can formally replace the Eulerian spatial coordinates by the Lagrangian ones in $\bar{P}(a, \mathbf{x})$ while preserving the functional form. The resulting Lagrangian function $\bar{P}(a, \mathbf{X})$ then gives a solution to the evolution equation (141) in the nonconstant curvature spatial sections, thanks to the algebraic identity of this equation with its Euclidean space counterpart. It is now a Lagrangian solution, however, and must be interpreted as such: the coordinates X^i are comoving with the inhomogeneous fluid flow. They are local coordinates on the perturbed manifold; thus the solution $P(a, \mathbf{X})$ describes perturbations as they evolve in the perturbed space. This perturbed space is in general not isometric to Euclidean space. Note that the Fourier modes $\bar{P}(a, \mathbf{k})$ are only an intermediate resolution step as they only correspond to modes in the ancillary Euclidean space. As the inversion of the solution (148) does not allow for an explicit general analytic expression, it requires the specification of the initial conditions and will usually involve numerical integration with the given $C_{k,1(2)}(\mathbf{k})$ to realize this solution procedure.

B. Case of a polytropic Equation of State: $p = \kappa \varrho^\gamma$

As a second class of models we will now turn to the nonlinear case of polytropic equations of state.

1. Equation of state and resolution procedure

We consider the polytropic EoS, $p = \kappa \varrho^\gamma$, $\varrho = F(\epsilon)$, where κ is the polytropic constant, and $\gamma > 1$ the polytropic exponent. For such flows the pressure and the energy density obey the relation [38, 60]

$$\epsilon = \beta^{-1}(p) = \frac{1}{\gamma-1} p + A p^{1/\gamma} = \frac{1}{\gamma-1} \kappa \varrho^\gamma + A \kappa^{1/\gamma} \varrho, \quad (149)$$

where A is a constant parameter. We will assume in this section that the formal $\varrho = F(\epsilon)$ actually coincides with the rest mass density of the fluid, *e.g.*, via suitable initial conditions. For $A = 0$, we again obtain the (nondust) linear case $p = w\epsilon$ with $w := \gamma - 1 > 0$. In the following, we will instead consider the case $A \kappa^{1/\gamma} = 1$ (in particular

$A > 0$), corresponding to an EoS of the type of a non-relativistic adiabatic ideal gas, the energy density being the sum of the rest mass density and an internal energy density equal to $p/(\gamma - 1)$.

As a relevant example, we will focus on the case $\gamma = 5/3$, which is an exact solution for a locally isotropic distribution with velocity dispersion, derived from the relativistic kinetic theory of collisionless matter [34]. (See also [68] and references therein.) This EoS also coincides with the corresponding exact solution in Newtonian cosmology derived from kinetic theory [24, 25]. In these latter papers it is also shown that this particular EoS arises in the inhomogeneous case by closing the hierarchy of kinetic equations through truncation of the third and higher reduced moments. In the inhomogeneous case this law is, however, phenomenological, since there is a non-vanishing anisotropic part. Neglecting this part strictly results in shear-free motion confirming the exactness of the law in the homogeneous case.

The conservation law (18), combined with $p = \kappa \varrho^\gamma$, gives for the evolution of p :

$$\partial_t p + \gamma N \Theta p = 0 ; \quad \gamma = \frac{5}{3} . \quad (150)$$

The same relation holds within the background space-time, so that $p_H a^5 = p_{H_1} a_1^5$. The assumption of the background sources following the same EoS also gives, for $\gamma = 5/3$:

$$\begin{aligned} \epsilon_H &= \beta^{-1}(p_H) = \frac{3}{2} p_H + A p_H^{3/5} ; \\ \beta'(\epsilon_H) &= \frac{2}{3} \frac{5}{5 + 2 A p_H^{-2/5}} ; \\ \beta''(\epsilon_H) &= \frac{80 A p_H^{-7/5}}{9 \left(5 + 2 A p_H^{-2/5} \right)^3} . \end{aligned} \quad (151)$$

The procedure outlined in the last subsection for solving the trace master equation, Eq. (88), in terms of Fourier transformation within a set of coordinates formally equivalent to Eulerian spatial coordinates, is still applicable in this case. We can thus substitute (151) and (89) in the Eulerian coordinate analogue of (88), and solve the corresponding ordinary differential evolution equation for each Fourier mode. This has to be performed by numerical integration as the more complicated time-evolution of the coefficients prevents an explicit analytic solution. Once initial conditions are specified we can then numerically compute the inverse Fourier transform, and formally replace the (Eulerian) spatial coordinates by the Lagrangian coordinates X^i (see Section IV A 2) to obtain the solution for $\bar{P}(t, X^i)$.

2. Behavior of the first-order trace for a model overdense region

As an instructive toy model, we will now consider the evolution of an initial spherical Gaussian deformation:

$$-\bar{P}_i = \alpha_{H_i} \delta \epsilon_i = c_i \exp\left(-\frac{R^2}{2\sigma^2}\right), \quad (152)$$

where σ and c_i respectively define the characteristic scale and maximum amplitude of the initial perturbation, and $R := (\delta_{ij} X^i X^j)^{1/2}$ is a Lagrangian coordinate ‘radius’.¹⁷ We will take $c_i > 0$ and $c_i \ll 1$. The perturbation can then be seen to describe a small initial local overdensity, since the initial rest mass density contrast,

$$\delta_i := \frac{\rho_i}{\rho_{H_i}} - 1 = \frac{F(\epsilon_i)}{F(\epsilon_{H_i})} - 1 = \frac{F(\epsilon_{H_i}[1 + \delta\epsilon_i]) - F(\epsilon_{H_i})}{F(\epsilon_{H_i})}, \quad (153)$$

is well approximated by $\alpha_{H_i} \delta \epsilon_i = -\bar{P}_i$ for $c_i \ll 1$.

The actual value of the amplitude c_i is irrelevant for the evolution of \bar{P} itself, since it obeys a linear equation. However, it will matter for the nonlinear evaluation of any physical quantity such as ρ determined by the first-order solution for \bar{P} through the extrapolation procedure mentioned above from the Relativistic Zel’dovich Approximation. To best illustrate the effect of this procedure, we choose a rather large overdensity with the arbitrary amplitude $c_i = 10^{-3}$ at an initial time that corresponds to the epoch of last scattering. As we will see, this will let the unstable perturbations enter the mildly nonlinear regime (where $|\bar{P}| < 1$ but is of order 1) around the present epoch, *i.e.*, around $a = a_0 \simeq 1090$ since we set $a_i = 1$.

The other independent initial condition amounts to specifying the first time-derivative $(\partial_t \bar{P})(t_i)$. For this we simply consider an initially stationary deformation and set $(\partial_t \bar{P})(t_i) = U = 0$.

The present formalism focuses on the description of a single fluid source, as it allows for a description in terms of a single velocity field and a single EoS. We will consequently make the simplifying assumption of a model universe filled with a single-component matter fluid and a cosmological constant. The description of model universes with multicomponent fluids is beyond the scope of the present paper, and is left to future work. The background density parameters Ω_m , Ω_Λ for the matter component and the cosmological constant respectively, satisfy $\Omega_m + \Omega_\Lambda = 1$. We will take the present epoch value $\Omega_\Lambda^0 = 0.692$ in agreement with the best-fit Λ CDM parameters from the Planck Collaboration [5].

The background is also affected by the polytropic EoS (149) of the source fluid. As noted above, our polytrope is exact for the background and is parametrized by the arbitrary constant κ , or equivalently A as we set $A\kappa^{1/\gamma} = 1$. Specifying its value amounts to choosing the initial instability scale as determined by $k_J(\epsilon_{H_i})$. It also controls the ratio between pressure and rest mass density at a given time, and hence the deviation of the background from a dust–fluid Λ CDM model. The value we adopt for our examples below, $A p_{H_i}^{-2/5} = 3/2$, requires the background fluid pressure to be relativistic (and radiation-like) at the initial time, $p_{H_i} = \epsilon_{H_i}/3$, with $p_{H_i}/\rho_{H_i} = 2/3$. However, it subsequently quickly becomes negligible as $p_H/\rho_H \propto a^{-2}$, keeping the late-time dynamics of the background very close to that of the Λ CDM model. We choose to make the lengths R , σ dimensionless by setting the initial instability scale $k_J(\epsilon_{H_i})^{-1}$ (as derived from substituting (151) into (90) at the initial time) to be our length unit. Thus $\sigma < 1$ means that the scale of the initial perturbation is below the Jeans scale $k_J(\epsilon_{H_i})^{-1}$, and above it for $\sigma > 1$. For the value of A adopted in the present example and estimating ρ_{H_i} from Λ CDM background parameters [5], this length unit is approximately 98 kpc. This would correspond to a large background comoving initial overdensity size of $a_0 k_J(\epsilon_{H_i})^{-1} \simeq 107$ Mpc.¹⁸

Figs. 1–3 show the numerical results for \bar{P} with the procedure, initial conditions and parameters given above, for three different values of σ .

The first case, $\sigma = 10$ (Fig. 1), corresponds to a super-Jeans length, hence unstable, initial perturbation. Figs. 1(a),(b) show the numerical results for the evolution of the perturbation $-\bar{P}$ as a function of the scale factor at several values of R , and over the whole range of radii R for increasing values of a , respectively. As expected, this perturbation is unstable and remains so by growing at all times, the pressure gradient being insufficient to prevent the collapse of the structure. The evolution is similar to the dust case with the fast onset of a linear growth of the perturbation with a before a late-time slow down due to the presence of Λ .

The second case, $\sigma = 0.2$ (Fig. 2), illustrates the opposite situation of an initially sub-Jeans length perturbation. Figs. 2(a),(b) show the numerical solution for $-\bar{P}$ in this situation along the same reasoning as for Figs. 1(a),(b). At the early stage, the pressure gradi-

¹⁷ We have chosen the set of Lagrangian coordinates X^i such that the components of the spatial metric at initial time, G_{ij} , are approximately δ_{ij} (at leading order) in these coordinates. They can thus be considered as Cartesian-like coordinates, and R is thus a fluid-comoving radial coordinate. It does not, however, coincide with the spatial metric distance between the fluid elements of the respective Lagrangian coordinates (X^i) and $(0, 0, 0)$. (This is true irrespective of a possible normalization by $a(t)$ to make it a background comoving distance.)

¹⁸ Note that $k_J(\epsilon_{H_i})^{-1}$ defines an initial instability ‘scale’ only in terms of Lagrangian coordinates, *e.g.*, in terms of R . This means that the corresponding ‘background comoving’ distance, $a(t)k_J(\epsilon_{H_i})^{-1}$ evaluated at present time, does not coincide with the present-day physical size of an object that would initially have been of this scale, as such a size must be evaluated using the actual, deformed, spatial metric. (See previous footnote.) $a_0 k_J(\epsilon_{H_i})^{-1}$ may be seen as a rough estimate of this physical size, as obtained by fully neglecting the deformations $G_{ab}^{(1)}$, P_i^a , in the evaluation of the integrated spatial line element.

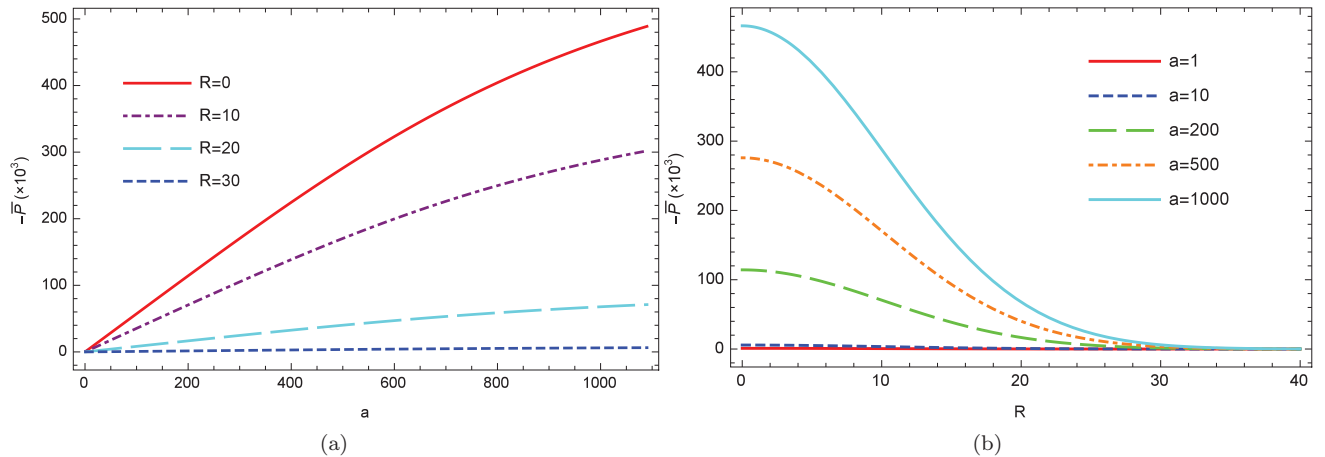


FIG. 1: Numerical solution for the first-order trace $-\bar{P}$ in Lagrangian space, for an initial spherical Gaussian overdensity with a peak amplitude of 10^{-3} at $R = 0$ and a standard deviation σ such that $k_J(\epsilon_{H1})\sigma = 10$. (a). Evolution of $-\bar{P}$ as a function of a for fixed values of the Lagrangian radius R . From top to bottom: $R = 0, 10, 20$ and 30 . (b). Spatial variation of $-\bar{P}$ with R , for several values of the background scale factor. From bottom to top: $a = 1, 10, 200, 500$ and 1000 . The perturbation strongly grows over time, corresponding to a collapsing structure.

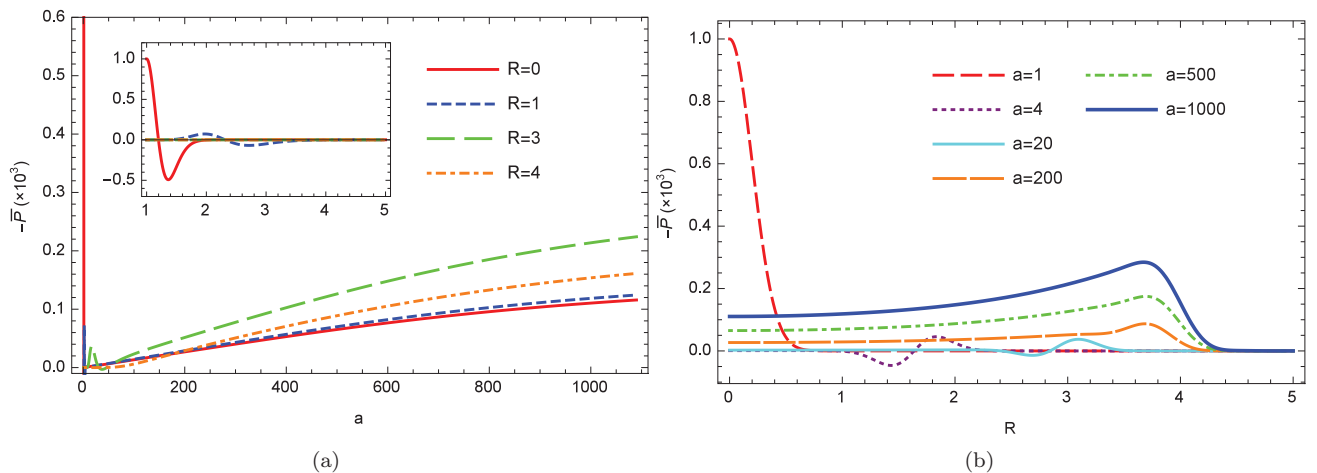


FIG. 2: Numerical solution for the first-order trace $-\bar{P}$ in Lagrangian space, for an initial spherical Gaussian overdensity with a peak amplitude of 10^{-3} at $R = 0$ and a standard deviation σ such that $k_J(\epsilon_{H1})\sigma = 0.2$. (a). Evolution of $-\bar{P}$ as a function of a at fixed distance R . From top to bottom at $a = 1000$: $R = 3, R = 4, R = 1$ and $R = 0$. The inset panel shows a detail of the early evolution (small values of a), where only the $R = 0$ (solid line) and $R = 1$ (dashed line) are visibly nonzero. (b). Spatial variation of $-\bar{P}$ with the Lagrangian radius, for several values of the background scale factor. The structure is first damped and spread out by the Lagrangian pressure gradient, before starting to grow back after the critical wave number $a k_J(\epsilon_H)$ has increased, as the perturbation enters the unstable regime.

ent dominates and opposes the gravitational collapse. The perturbation behaves as an acoustic wave and is damped as it propagates away from the initial peak at $R = 0$. However, the instability wave number $a k_J(\epsilon_H)$ quickly starts increasing over time (cf., Fig. 4). That is why around $a = 50$ to 100 the perturbation starts to grow as its typical wave number (estimated by $\sigma^{-1} = 5$) ends up below the critical value, with $a k_J(\epsilon_H) = 5$ for $a \simeq 94$, and it enters the unstable regime. The peak of this growing structure remains at a mostly stationary Lagrangian position, at $R \simeq 3.7$, while its increasing

amplitude still remains small and below the initial value $-\bar{P}(a = 1, R = 0) = 10^{-3}$ up to present time ($a \simeq 1090$).

For comparison we also consider the special case where the initial scale lies at the stability threshold, $\sigma = 1$. The evolution of the corresponding solution for $-\bar{P}$ with a at several radii is shown in Figs. 3(a),(b), with the latter highlighting the early evolution ($1 \leq a \leq 20$). Fig. 3(c) shows the spatial dependence of $-\bar{P}$ with R at some values of the scale factor. The behavior of the perturbation in this case is as expected intermediate, with an initial acoustic damping and propagation away from $R = 0$ sim-

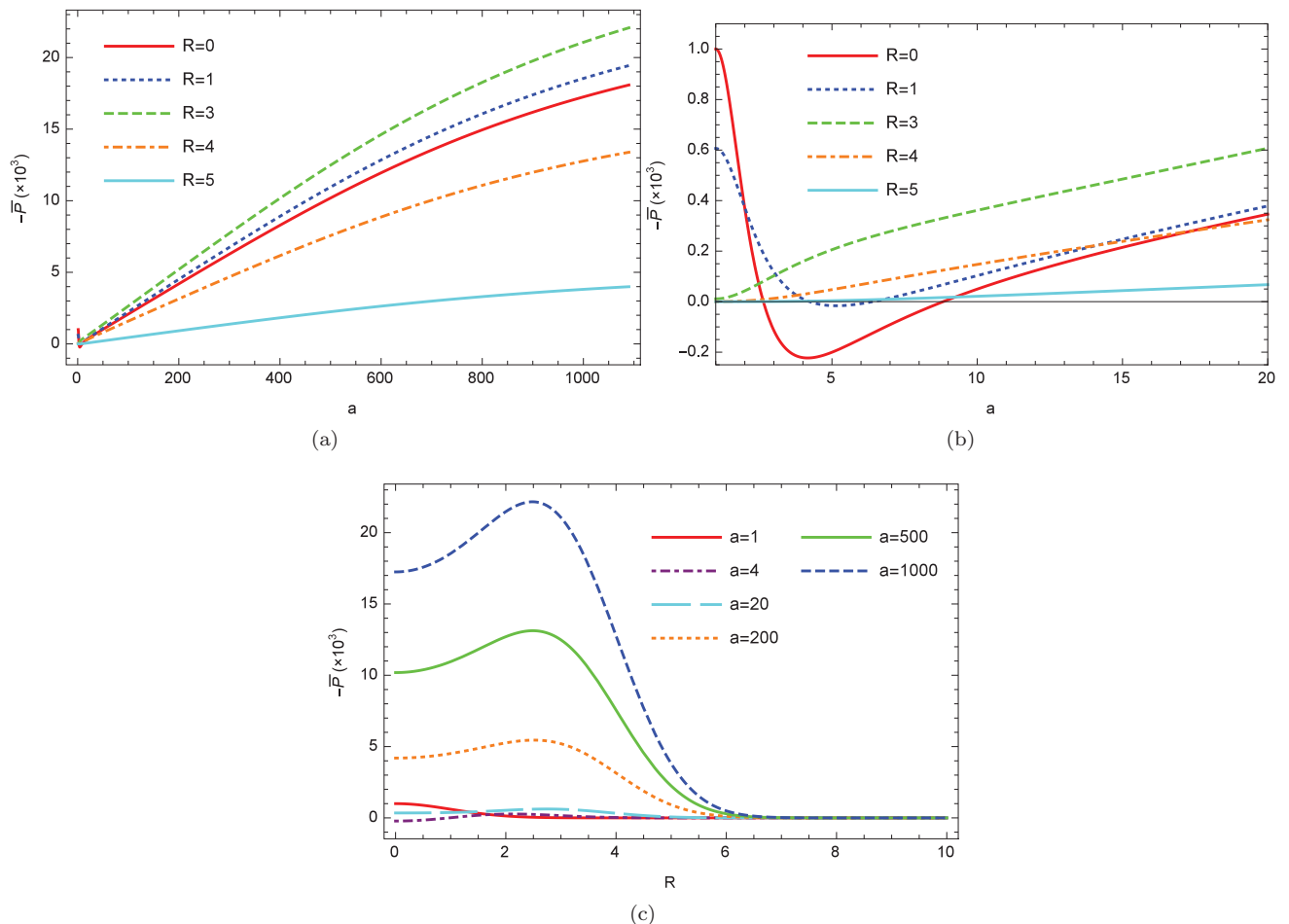


FIG. 3: Numerical solution for the first-order trace $-\bar{P}$ in Lagrangian space, for an initial spherical Gaussian overdensity with a peak amplitude of 10^{-3} at $R = 0$ and a standard deviation σ such that $k_J(\epsilon_{H1})\sigma = 1$. (a) and (b). Evolution of $-\bar{P}$ as a function of a at a given distance R , for late and early times, respectively. From top to bottom at $a = 1000$ for (a): $R = 3$, $R = 1$, $R = 0$, $R = 4$, $R = 5$; same order for (b) at $a = 20$. (c). Spatial variation of $-\bar{P}$ with R , for fixed values of the background scale factor. From top to bottom at $R = 0$: $a = 1000$, $a = 500$, $a = 200$, $a = 1$, $a = 20$, $a = 4$. The behavior is rather similar to the previous case of $k_J(\epsilon_{H1})\sigma = 0.2$; as expected, the unstable regime is, however, reached sooner, and the perturbation then grows similarly to the case of $k_J(\epsilon_{H1})\sigma = 10$, up to much above its initial amplitude.

ilarly to the $\sigma = 0.2$ case, but more rapidly entering an unstable regime, after $a \simeq 5$. The amplitude of the perturbation then starts growing with a dust-like behavior up to beyond 20 times its initial value at present time, with a shifted peak as in the $\sigma = 0.2$ case, that stays around $R \simeq 2.5$.

3. Evaluating the nonlinear density contrast

As we recalled above, even the first-order Lagrangian perturbation scheme allows one to probe part of the nonlinear regime in the evaluation of observable quantities. This involves extrapolating these observables as exact, nonlinear functionals of the deformation field, the latter being evaluated as a solution to its first-order evolution equations and constraints.

Adopting this procedure for the rest mass density we evaluate it as the exact integral to the rest mass conservation equation (18):

$$\varrho = \frac{\varrho_i}{J} \quad ; \quad J = \det(\eta^a_i) = a^3 \det(\delta^a_i + P^a_i), \quad (154)$$

where P^a_i are the components of the deformation field. The density contrast δ is then deduced from the above:

$$\delta := \frac{\varrho - \varrho_H}{\varrho_H} = \frac{\varrho_i}{\varrho_{H1} a^{-3} J} - 1 \quad ; \quad a^{-3} J = \det(\delta^a_i + P^a_i), \quad (155)$$

and it is evaluated by replacing P^a_i by the first-order solution.

Using the polytropic EoS and the parameters adopted

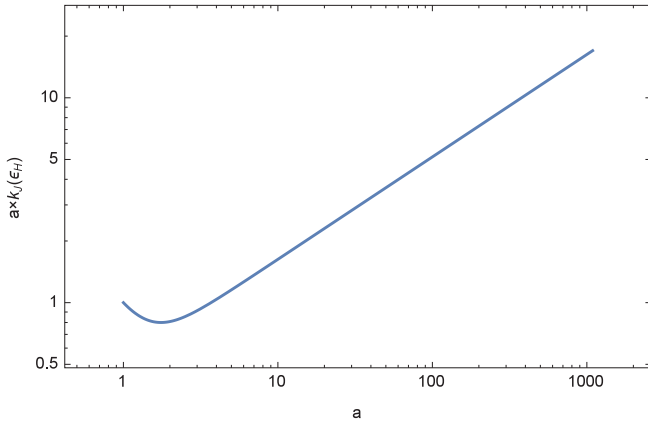


FIG. 4: Evolution of the instability wave number $a k_J(\epsilon_H)$ with the scale factor a for the polytropic EoS considered here, with the unit of length convention $k_J(\epsilon_{H1}) = 1$. As this wave number only depends on the background by construction, this result applies to all examples considered in this Subsection IV B. After a small initial dip, $a k_J(\epsilon_H)$ exceeds its initial value around $a \simeq 4$ and enters the increasing power law regime $a k_J(\epsilon_H) \propto \sqrt{a}$ (valid as long as $(\Omega_\Lambda/\Omega_m)(a/a_1)^{-2} \ll 1$, which is satisfied up to the present epoch) as expected from the large a expansion of its expression for the present polytropic EoS.

here, the lapse may then be computed from

$$N = \frac{\varrho}{\epsilon + p} = \frac{\varrho}{\varrho + \frac{\gamma}{\gamma-1} \kappa \varrho^\gamma} = \frac{1}{1 + \frac{5}{3}(1 + \delta)^{2/3} a^{-2}}, \quad (156)$$

with δ expressed from the deformation field as above. This formula shows that the lapse is 1 in pressure-free (here empty) regions ($\delta = -1$) and decreases with increasing density contrast at a given time. The deviation $(1 - N)$ rapidly decreases over time as $\propto a^{-2}$, with late time values of order 10^{-6} (when $a \simeq 1000$), as long as δ remains at most of order unity.

We will now illustrate this process for the density contrast with two examples using the same polytropic EoS as above. Note that this evaluation requires the knowledge of all components of the deformation field, including the traceless part. We specify procedures in Appendix A to obtain a particular (gravitoelectric) solution for the first-order traceless part from the initial conditions for the trace in specific cases. These procedures have been used to determine a consistent solution for the full deformation field in the examples below. We have also made use of the fact that the initial density $\varrho_{\mathbf{i}} = F(\epsilon_{H\mathbf{i}} [1 + \delta\epsilon_{\mathbf{i}}])$ is well approximated by $F(\epsilon_{H\mathbf{i}})(1 + \alpha_{H\mathbf{i}} \delta\epsilon_{\mathbf{i}}) = \varrho_{H\mathbf{i}}(1 - \bar{P}_{\mathbf{i}})$ for a small, still linear, initial density perturbation (with $\alpha_{H\mathbf{i}} = 3/4$ for the chosen EoS parameters) for the evaluation of δ .

a. Localized overdensity:

Let us first retain the ‘spherical’ initial overdensity example studied thus far in this section, with the initial conditions for the trace given by (152), with $c_{\mathbf{i}} = 10^{-3}$, and $U = 0$. The first-order solution for the trace in this situation has been determined above, and is comple-

mented by a gravitoelectric solution for the first-order traceless part through the use of the procedure given in Appendix A 2 that directly applies to this case. The determinant J is then computed from this solution as in Appendix A 4, giving δ from Eq. (155).

Note that when all components of the deformation field are very small, *i.e.*, when it lies fully in the linear regime, then the extrapolated δ remains quantitatively close to $-\bar{P}$, which corresponds to its expansion at first order in the deformation field. This is the case in the initially stable or marginally stable cases $\sigma = 0.2$ and $\sigma = 1$, where the initial acoustic damping of the perturbation keeps its amplitude small up to the present time despite the late-time growth. In both of these cases, the resulting density contrast indeed remains indistinguishable from the value of $-\bar{P}$ already depicted above (Figs. 2–3).

We will consequently focus from now on on the case $\sigma = 10$, where the unstable deformation reaches into the mildly nonlinear regime before the present time, as can be seen for the trace (whose amplitude reaches about 0.5 at the present epoch).

Figs. 5(a),(b) show the result of the nonlinear evaluation of the density contrast in this situation, as a function of a at given radii R , and as a function of the radius at several moments in its evolution, respectively. Although the general behavior is roughly similar to that of $-\bar{P}$ (*cf.* Fig. 1), nonlinear effects are visible in the amplified growth of δ at late times near $R = 0$, with a maximal overdensity reaching about 0.7 at present.

This nonlinear deviation of the density contrast functional with respect to its first-order estimate $-\bar{P}$ is made explicit by the direct comparison of the peak ($R = 0$) amplitude evolution of δ and $-\bar{P}$ as a function of the background scale factor in Fig. 6(a). The spatial dependence on R of both quantities at late times, compared in Fig. 6(b) at $a = 1000$, is also visibly affected by the amplified growth of the density contrast where \bar{P} is no longer small, *i.e.*, around $R = 0$.

b. Lagrangian monochromatic wave:

The second toy model we consider is that of a single Lagrangian monochromatic wave deformation. The choices of background parameters and the length unit ($k_J(\epsilon_{H\mathbf{i}}) = 1$) are unchanged. The initial perturbation is now chosen to be

$$-\bar{P}_{\mathbf{i}} = c_{\mathbf{i}} \cos(KX) \quad ; \quad U = 0, \quad (157)$$

where we will again take $c_{\mathbf{i}} = 10^{-3}$ as an initial amplitude. This situation corresponds to an initially stationary monochromatic wave in the given Lagrangian coordinate set,¹⁹ $-\bar{P}_{\mathbf{i}} = c_{\mathbf{i}} \cos(\delta_{ij} K^i X^j + \phi_0)$ with $\phi_0 = 0$ and a

¹⁹ Similarly to the interpretation of R for the previous example, it is important to keep in mind that the perturbation we are considering here only has a sinusoidal dependence in the chosen Lagrangian coordinates X^i . It would have a different functional dependence in terms of actual physical (metric) spatial distance

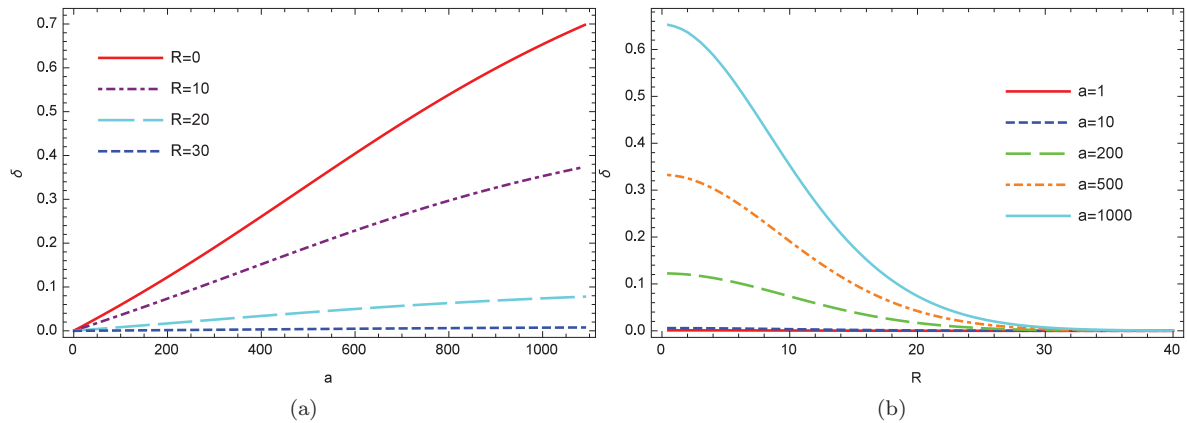


FIG. 5: Numerical evaluation of the nonlinear density contrast δ as extrapolated from the first-order Lagrangian perturbation, where the initial $-\bar{P}$ is the same spherical Gaussian field as for Fig. 1, with peak value of 10^{-3} and $k_J(\epsilon_{Hi})\sigma = 10$. (a). Evolution of δ with the background scale factor at fixed distances R . From top to bottom: $R = 0, 10, 20$ and 30 . (b). Spatial variation of δ with the Lagrangian radius, for given values of a . From bottom to top: $a = 1, 10, 200, 500$ and 1000 . The overall behavior of δ is similar to the results of Fig. 1 for the first-order $-\bar{P}$ in the same situation, but the extrapolated density contrast grows faster at late times near the $R = 0$ maximal overdensity. Additional nonlinear effects concerning the comparison with a standard perturbation approach, not studied here, could also be revealed by using instead as the x -axis for (b) the actual spatial metric distance to the $R = 0$ fluid element (as an ‘Eulerian radius’), altering the spatial dependence. (See the discussion in Section IV B 4.)

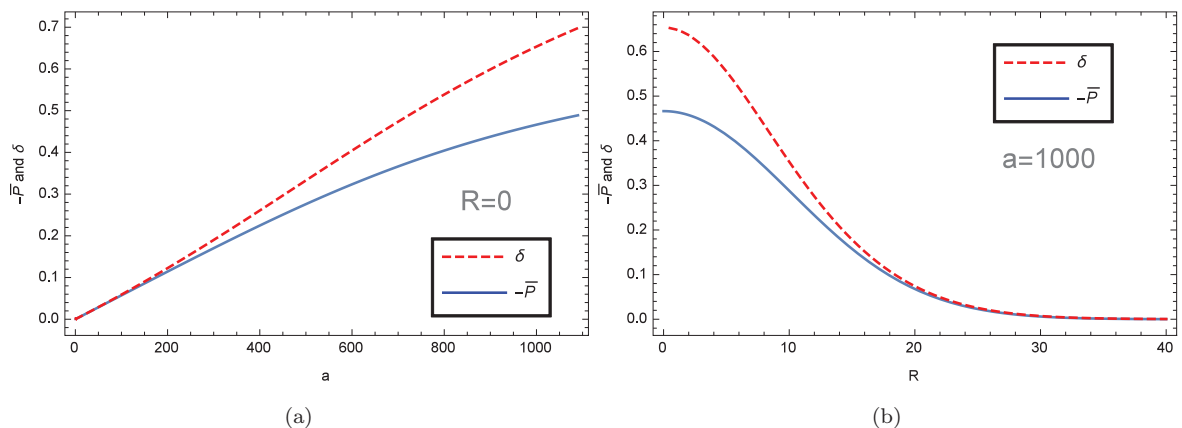


FIG. 6: Comparison of the extrapolated nonlinear density contrast δ (dashed line) with the first-order solution for the sign-inverted deformation trace $-\bar{P}$ (solid line) within the same setting as Figs. 1 and 5. (a). Comparison of the evolution of both quantities as a function of a at the centre of the overdensity ($R = 0$). (b). Comparison of the spatial variation of both quantities with R at a late time ($a = 1000$). In this situation, the perturbation grows large enough to enter the nonlinear regime and to render the time evolution and spatial behavior of the extrapolated δ clearly deviating from those of $-\bar{P}$.

Lagrangian wave vector \mathbf{K} along the first coordinate X , with components $K^i = (K, 0, 0)$.

between two points on a given hypersurface $t = \text{const.}$ One expects for instance, at a given late time t and along a given spatial geodesic line, the distance between the successive perturbation nodes at $KX = -\pi/2$ and $KX = \pi/2$ (surrounding a collapsing overdensity) to be shorter than the distance between the nodes at $KX = \pi/2$ and $KX = 3\pi/2$ (surrounding an expanding underdensity), despite all nodes being equally separated in terms of the Lagrangian coordinate X .

The first-order trace solution then remains in this monochromatic mode form in the Lagrangian coordinates at all times, $\bar{P} = \hat{P}_{\mathbf{K}}(t) \cos(KX)$. The amplitude $\hat{P}_{\mathbf{K}}(t)$ evolves according to the ordinary differential equation (A2) which is solved by numerical integration for a given wave number K . A gravitoelectric solution for the traceless part is then determined along the lines of Appendix A 1, where the relevant amplitude $\hat{Q}_{\mathbf{K}}(t)$ is again numerically evaluated, knowing $\hat{P}_{\mathbf{K}}(t)$, through its defining time integral formula (A3). From these, one can calculate the density contrast in the same way as in the previous ex-

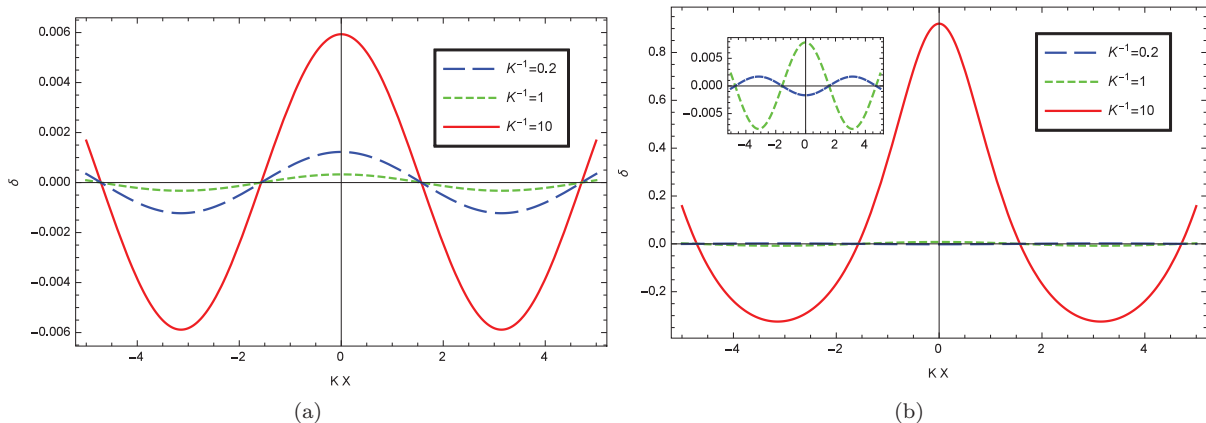


FIG. 7: Numerical evaluation of the nonlinear density contrast δ as extrapolated from the first-order Lagrangian perturbation. The first-order deformation trace is taken as a plane-wave in Lagrangian coordinates of wave vector \mathbf{K} (of norm K) along the X coordinate, $-\bar{P} \propto \cos(KX)$, of initial amplitude 10^{-3} . The result is shown at a given time as a function of KX for three possible values of K , which is expressed in units $k_J(\epsilon_{H1}) = 1$. (a). At $a = 10$, for $K = 0.1$ ($K^{-1} = 10$), $K = 5$ ($K^{-1} = 0.2$) and $K = 1$ by order of decreasing amplitude. (b). At $a = 1000$, for $K = 0.1$ ($K^{-1} = 10$), $K = 1$ and $K = 5$ ($K^{-1} = 0.2$) by order of decreasing amplitude. The side panel displays the (otherwise barely visible) latter two curves on a different vertical scale. The most unstable perturbation, for $K^{-1} = 10$, displays a non-sinusoidal asymmetric shape at late times as it reaches the mildly nonlinear regime. This shape would be further nonlinearly modified, *via* a different x -axis dependence, if this axis were expressed alternatively in terms of an Eulerian-type, regularly spaced (in terms of spatial metric distances), x coordinate.

ample, with the determinant J evaluated as detailed in Appendix A 4.

Here we again study three cases distinguished by their wave number in direct analogy to the previous example, with K^{-1} playing the role of the characteristic length σ . We accordingly choose $K^{-1} = 0.2$, $K^{-1} = 1$ and $K^{-1} = 10$, which at the initial time are stable, marginally stable and unstable, respectively. The corresponding spatial dependence of δ as a function of KX for the three wave number choices is shown at an early time ($a = 10$) in Fig. 7(a), and at a late time ($a = 1000$) in Fig. 7(b).

In this situation, in the first two cases the components of the deformation field again remain small at all times, due to initial acoustic oscillations, and the density contrast thus follows the sinusoidal shape of $-\bar{P}$ at all times. This is also the case for the unstable mode $K^{-1} = 10$ at $a = 10$ when it is still in the linear regime. At $a = 1000$, however, this mode clearly deviates from this behavior as its amplitude is no longer linear. In particular, an asymmetry develops between the under- and overdensity magnitudes as the latter is sharply amplified by the nonlinear evolution of δ .

4. Discussion

In both examples above, the Lagrangian scheme and the proposed extrapolation procedure exhibit nonlinear effects on the overdensity for unstable perturbations when they become large enough. The amplitude of large overdensities in these examples is clearly underestimated when they are approximated by the first-order expression

$-\bar{P}$ instead of using the nonlinear extrapolation for δ .

An even higher initial overdensity amplitude could actually lead to a vanishing determinant $a^{-3}J$ at the maximum overdensity at a late enough time, implying $\varrho \rightarrow \infty$ with deformation coefficients still of order 1. This situation corresponds to a shell-crossing, beyond which the first-order Lagrangian scheme is no longer valid.

The presence of pressure can delay its occurrence by damping the perturbation. An improvement of the perturbative scheme to account for further local nonlinear effects in the dynamical evolution, e.g., allowing for a nonlinear coefficient to define the Jeans length is needed, however, to fully circumvent this problem. Velocity dispersion effects may in principle allow us to model the multistream regime, and the stabilization of structure formation in the form of virialization, which may help to avoid shell-crossings [18, 25].

We emphasize that the current Lagrangian perturbation scheme already contains another effect of nonlinear structure evolution, which lies in the exact propagation of the spatial coordinates used along the fluid flow lines. This is analogous to the inclusion of quadratic convection terms within linear Lagrangian time derivatives in the Newtonian framework.²⁰

²⁰ In addition to the time derivatives being taken at different fixed spatial coordinates, a difference also comes from the spatial derivative operators, such as the Laplacian Δ_0 appearing in the trace master equation (85), being expressed in terms of Lagrangian coordinates and thus differing from the corresponding Eulerian operators. (See [6] for the explicit transformation in the Newtonian case.)

Let us suggest a procedure that would be required to make these effects explicit also in the relativistic context; its concrete application is beyond the scope of this paper.

Eulerian-like coordinates could first be recovered, at least along a given spatial geodesic direction, by labeling points at equal intervals of spatial metric distances. This would involve solving for the initial metric components G_{ab} such that their Ricci tensor is consistent with the initial conditions (114)–(115) for given initial deformation field data, and then functionally evaluating and integrating the line element as given by (A18) from the first-order solution for P^a_i . The resulting length, as a function of a Lagrangian coordinate, could then be used as an estimate of the Eulerian coordinate distance. Finally, this relation would have to be numerically inverted so that a given Lagrangian function obtained through the Relativistic Zel'dovich Approximation, such as $\varrho(X^i)$, could be expressed as a function of the Eulerian coordinate x along the chosen line.

A different functional dependence on this spatial distance (which may be normalized by $a(t)$ to become a background comoving distance), as compared to the fluid-comoving coordinates X^i , would thus include nonlinear effects of the fluid-propagation-dependent coordinate transformation.

Recall, however, that a three-dimensional family of Eulerian observers generally does not exist in a relativistic (intrinsic) description. Strictly, a coordinate transformation to Eulerian space can only be conducted after the Minkowski Restriction of the relativistic solution has been executed.

V. CONCLUSION

In this paper we have generalized the Lagrangian perturbation approach to the nonlinear evolution of inhomogeneous general relativistic model universes containing a single irrotational fluid obeying a general barotropic relation.

By choosing a suitable set of coframes, we obtained the master partial differential equations for the evolution of the trace and traceless parts of the first-order deformation field that reduce to the corresponding equations in the dust case. The trace part also matches the Newtonian limit of the corresponding Lagrangian perturbation problem.

We discussed the procedure proposed in previous papers of how to find the solution for perturbations that propagate in the perturbed space, and applied this procedure to specific toy models, illustrating the mildly nonlinear evolution of the density contrast. We also discussed the limits of a first-order Lagrangian scheme, and we proposed ideas for a nonperturbative generalization, which is needed especially in application to cases where the pressure term is taken to model multistreaming beyond the mildly nonlinear regime.

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Appendix A: Examples of solutions for the gravitoelectric traceless part

In this paper we will not attempt to find the general solution of Equations (96)–(97) for the traceless part. We will, however, discuss a procedure for finding one possible solution for suitably chosen traceless-part initial conditions. For any barotropic EoS, this yields one example of a full gravitoelectric solution for all components of the deformation field P^a_i . It can then be substituted into exact nonlinear formulae to extrapolate functionals of the coframes such as metric distances or the rest mass density.

To find such an example solution, we will focus on the gravitoelectric part which is directly coupled to the trace, and accordingly we set the gravitomagnetic part to zero.

1. Case of a Lagrangian monochromatic wave

Let us first assume that the first-order trace solution can be written as a single monochromatic wave mode in the given set of Lagrangian spatial coordinates X^i :

$$\bar{P}(t, X^i) = \varphi(\mathbf{K} \cdot \mathbf{X}) \hat{P}_{\mathbf{K}}(t), \quad (\text{A1})$$

for some constant Lagrangian wave vector \mathbf{K} , where $\mathbf{K} \cdot \mathbf{X} := \delta_{ij} K^i X^j$, and $\varphi(\mathbf{K} \cdot \mathbf{X}) = \cos(\mathbf{K} \cdot \mathbf{X} + \phi_0)$, with constant phase ϕ_0 . This form is a solution of the first-order trace master equation, if and only if $\hat{P}_{\mathbf{K}}(t)$ is a solution of the ordinary differential equation

$$\begin{aligned} \frac{d^2}{dt^2} \hat{P}_{\mathbf{K}} + 2H(1 - 3\beta'(\epsilon_H)) \frac{d}{dt} \hat{P}_{\mathbf{K}} - \mathcal{W}(t) N_H^2 \hat{P}_{\mathbf{K}} \\ = -a^{-2} N_H^2 \beta'(\epsilon_H) K^2 \hat{P}_{\mathbf{K}}, \end{aligned} \quad (\text{A2})$$

with $K := (\delta_{ij} K^i K^j)^{1/2}$. Then $P = \bar{P} - \bar{P}_i = \varphi(\mathbf{K} \cdot \mathbf{X}) (\hat{P}_{\mathbf{K}}(t) - \hat{P}_{\mathbf{K}}(t_i))$.

Setting

$$\begin{aligned}\hat{Q}_{\mathbf{K}}(t) &:= \int_{t_i}^t N_H(t') \partial_t \left(\frac{\hat{P}_{\mathbf{K}}}{N_H} \right) (t') dt' \\ &= \hat{P}_{\mathbf{K}}(t) - \hat{P}_{\mathbf{K}}(t_i) - 3 \int_{t_i}^t H(t') \beta'(\epsilon_H)(t') \hat{P}_{\mathbf{K}}(t') dt',\end{aligned}\quad (\text{A3})$$

the time integral of the momentum constraints (97) is

$$\Pi_{j|i}^i = \frac{2}{3} \hat{Q}_{\mathbf{K}}(t) K_j \varphi'(\mathbf{K} \cdot \mathbf{X}). \quad (\text{A4})$$

We now take Π_j^i to be a purely longitudinal mode and get the following solution to the momentum constraints (with $K_j := \delta_{jl} K^l$):

$$\Pi_j^i = \left(\frac{K^i K_j}{K^2} - \frac{1}{3} \delta_j^i \right) \hat{Q}_{\mathbf{K}}(t) \varphi(\mathbf{K} \cdot \mathbf{X}) \quad (\text{A5})$$

$$= \left(\frac{K^i K_j}{K^2} - \frac{1}{3} \delta_j^i \right) \left(\frac{\hat{Q}_{\mathbf{K}}(t)}{\hat{P}_{\mathbf{K}}(t) - \hat{P}_{\mathbf{K}}(t_i)} \right) P(t, X^i). \quad (\text{A6})$$

Substituting this form into the master equation (96) shows that it is consistently a solution of both equations for the traceless part. It is straightforward to show from the above formula that $2 \Delta_0 \Pi_j^i + \Pi_{l|k}^k \delta_j^i - 3 \Pi_{k|j}^i = 0$, *i.e.*, this Π_j^i obeys the defining relation (108) for the gravitoelectric part and evolves according to (105). This solution is thus a pure gravitoelectric one, amounting to setting the gravitomagnetic part to zero by the choice of vanishing gravitomagnetic traceless part of the initial deformation: $\Pi_j^i = {}^E \Pi_j^i$.

Choosing this solution amounts to specifying the following (gravitoelectric) initial conditions:

$${}^{\text{tl}}U_j^i = \left(\frac{K^i K_j}{K^2} - \frac{1}{3} \delta_j^i \right) \left(U + 3H_i \beta'(\epsilon_{H_i}) \alpha_{H_i} \delta_{\epsilon_i} \right); \quad (\text{A7})$$

$$\begin{aligned}{}^{\text{tl}}W_j^i &= \left(\frac{K^i K_j}{K^2} - \frac{1}{3} \delta_j^i \right) \left(W + 3H_i \beta'(\epsilon_{H_i}) U \right. \\ &\quad \left. + 3 [\partial_t (H \beta'(\epsilon_H))(t_i) + 2 H_i^2 \beta'(\epsilon_{H_i})] \alpha_{H_i} \delta_{\epsilon_i} \right).\end{aligned}\quad (\text{A8})$$

This is compatible with the set of constraints on the initial conditions given in Section III E, in particular the initial momentum constraints (116) and Eq. (114), provided that the latter is used to specify the traceless part of the initial first-order Ricci tensor \mathcal{S}_j^i .

The corresponding full perturbation field $P_j^i = \Pi_j^i + \frac{1}{3} \delta_j^i P$ then reads:

$$\begin{aligned}P_j^i &= \frac{K^i K_j}{K^2} \left(\frac{\hat{Q}_{\mathbf{K}}(t)}{\hat{P}_{\mathbf{K}}(t) - \hat{P}_{\mathbf{K}}(t_i)} \right) P \\ &\quad + \frac{1}{3} \delta_j^i \left(1 - \frac{\hat{Q}_{\mathbf{K}}(t)}{\hat{P}_{\mathbf{K}}(t) - \hat{P}_{\mathbf{K}}(t_i)} \right) P.\end{aligned}\quad (\text{A9})$$

Note that the corresponding deformation 1-forms $\mathbf{P}^a = \delta_k^a P_i^k dX^i$ are not exact due to the different time evolution of the trace and gravitoelectric traceless parts. This contrasts with the dust case where a purely gravitoelectric perturbation would lead to integrable coframes [L4], so that only the non-flat initial metric would prevent one obtaining an Euclidean spatial metric at all times in that situation.

By linearity of the equations, a solution for Π_j^i can also be obtained when the trace is a finite sum of such monochromatic waves, or the sum of the two time-evolution modes solutions of the evolution equation (A2) for a given wave vector \mathbf{K} , simply by summing the corresponding solutions as given by (A5).

2. Case of a spatially localized solution

We assume here either that the spatial slices are globally diffeomorphic to the Euclidean space \mathbb{R}^3 , *i.e.*, that they can be covered by a single chart, or that the deformation field can be assumed to vanish outside a given chart. In either case it suffices to work within the Euclidean space spanned by the spatial coordinates in a given chart.

Let us now consider a spatially localized solution for the trace, *e.g.*, a local overdensity evolving from an initial Gaussian perturbation in terms of the given set of spatial Lagrangian coordinates, as studied in the numerical examples of Section IV. More specifically, we require the solution for the trace to always be a square-integrable function of the spatial coordinates in the chart, so that its Fourier transform in these coordinates can be performed and inverted. We can thus write:

$$\bar{P}(t, X^i) = \iiint e^{-i\mathbf{K} \cdot \mathbf{X}} \hat{P}(t, \mathbf{K}) d^3 \mathbf{K}, \quad (\text{A10})$$

where $\hat{P}(t, \mathbf{K})$ is a solution of the evolution equation (A2) at fixed \mathbf{K} , with the initial conditions set by the forward Fourier transform in the chart coordinates:

$$\hat{P}(t_i, \mathbf{K}) = -\frac{1}{(2\pi)^3} \alpha_{H_1} \iiint e^{i\mathbf{K} \cdot \mathbf{X}} \delta_{\epsilon_1}(\mathbf{X}) d^3 X; \quad (\text{A11})$$

$$(\partial_t \hat{P})(t_i, \mathbf{K}) = \frac{1}{(2\pi)^3} \iiint e^{i\mathbf{K} \cdot \mathbf{X}} U(\mathbf{X}) d^3 X. \quad (\text{A12})$$

Note that the above approach represents an alternative and complementary formulation of the method of solution presented in [L4] which formally replaces the Lagrangian coordinates by ‘Eulerian’ ones. In the present paper it is applied in Sections IV A 2 and IV B. The reformulation suggested here allows us to be more explicit about the required assumptions, as well as expressing the coordinate components of tensors such as Π_j^i in a more convenient form. In both formulations, the use of plane-wave modes and flat-space Fourier transformations is sufficient since the Lagrangian first-order master equations

to be solved only involve the metric-independent coordinate spatial derivatives ${}_{|i}$ and Laplacian $\Delta_0 = {}_{|i|j} \delta^{ij}$ as spatial derivative operators.

By linearity of the equations, a solution for the (gravitoelectric) traceless part is obtained by summation of the plane wave solutions for all Fourier modes:

$$\begin{aligned} \Pi_j^i = {}^E\Pi_j^i &= \iiint e^{-i\mathbf{K}\cdot\mathbf{X}} \frac{K^i K_j}{K^2} \hat{Q}(t, \mathbf{K}) d^3\mathbf{K} \\ &\quad - \frac{1}{3} \delta_j^i \iiint e^{-i\mathbf{K}\cdot\mathbf{X}} \hat{Q}(t, \mathbf{K}) d^3\mathbf{K}, \end{aligned} \quad (\text{A13})$$

with

$$\hat{Q}(t, \mathbf{K}) := \int_{t_i}^t N_H(t') \partial_t \left(\frac{\hat{P}(t, \mathbf{K})}{N_H(t)} \right) (t') dt'. \quad (\text{A14})$$

Using this solution again implies a specific choice of initial conditions for the traceless deformation field (in particular taking it to be gravitoelectric) and for the traceless part of the spatial Ricci tensor.

In the case of spherically symmetric initial conditions in the chart coordinates, *i.e.*, when $\delta\epsilon_i(X^i)$ and $U(X^i)$ only depend on $R := (\delta_{ij} X^i X^j)^{1/2}$, their Fourier transform will also depend only on K . From the evolution equation (A2), this feature is preserved over time, so that one can write $\hat{P}(t, \mathbf{K})$ as $\hat{P}(t, K)$ and consequently $\hat{Q}(t, \mathbf{K})$ as $\hat{Q}(t, K)$ and $\bar{P}(t, X^i)$ as $\bar{P}(t, R)$. The above solution for Π_j^i can then be computed as

$$\Pi_j^i = \left(\frac{X^i X_j}{R^2} - \frac{1}{3} \delta_j^i \right) q(t, R), \quad (\text{A15})$$

with $X_j := \delta_{jk} X^k$ and

$$\begin{aligned} q(t, R) &:= \frac{4\pi}{R} \int_0^\infty K \sin(RK) \hat{Q}(t, K) dK \\ &\quad - \frac{4\pi}{R^3} \int_0^\infty \left(\frac{\sin(RK)}{K} - R \cos(RK) \right) \hat{Q}(t, K) dK. \end{aligned} \quad (\text{A16})$$

3. Time integral of the gravitoelectric evolution equation

The above procedure gives a way of obtaining a traceless part consistent with the momentum constraints and evolution equations in particular situations, and when only initial conditions on the trace part (or on the energy density) are explicitly specified. Alternatively, and still focusing on a purely gravitoelectric traceless part, a solution can be derived from the gravitoelectric traceless evolution equation (105), if the trace part and the (gravitoelectric) traceless initial conditions are known. It can be achieved by rewriting this evolution equation as fol-

lows:

$$\begin{aligned} \partial_t \left(\frac{a^3}{N_H} \partial_t {}^E\Pi_j^i \right) &= -\frac{aN_H}{3} \mathcal{D}_j^i \left(\int_{t_i}^t N_H \partial_t \left(\frac{\bar{P}}{N_H} \right) dt' \right) \\ &\quad + \frac{aN_H}{3} \left([1 + 3\beta'(\epsilon_H)] \mathcal{D}_j^i \bar{P} - [1 + 3\beta'(\epsilon_{H_i})] \mathcal{D}_j^i \bar{P}_i \right) \\ &\quad + \frac{aN_H}{N_H^2} \left({}^{\text{tl},E}W_j^i + H_i [1 - 3\beta'(\epsilon_{H_i})] {}^{\text{tl},E}U_j^i \right), \end{aligned} \quad (\text{A17})$$

after replacing $\Delta_0 {}^E\Pi_j^i$ by its integral expression (111) in terms of \bar{P} . It can be readily time-integrated twice to give ${}^E\Pi_j^i$. This yields the full Π_j^i if the initial conditions are chosen such that the gravitomagnetic part vanishes.

In contrast to the previous subsections, this procedure can be applied in general, allowing the gravitoelectric initial conditions for the traceless part to be freely set. However, this requires the initial conditions ${}^{\text{tl}}U_j^i = {}^{\text{tl},E}U_j^i$ and ${}^{\text{tl}}W_j^i = {}^{\text{tl},E}W_j^i$ to be explicitly specified. While the trace parts relate to the energy density and spatial scalar curvature, the tracefree parts are related to properties of the gravitational wave components at the initial time. The latter have to be set in such a way as to fulfill the momentum constraints and their time derivative at the initial time, as well as the geometric constraints (101)–(102) for the gravitoelectric parts.

4. On the evaluation of physical quantities

From given solutions for the trace and traceless parts, the full deformation field is straightforwardly obtained as $P_j^i = \Pi_j^i + (1/3)P \delta_j^i$, with $P = \bar{P} - \bar{P}_i$. This expression can then be inserted into the Lagrangian functional expressions for various physical quantities in terms of the deformation field. They can then be directly evaluated without any further linearization. This extrapolation is a crucial part of the Relativistic Zel'dovich Approximation as defined in [L1], and it generally requires the knowledge of all components of the deformation field.

One would for instance directly compute a spatial distance from the line element

$$ds^2 = a(t)^2 G_{ab} (\delta^a_i + P^a_i) (\delta^b_j + P^b_j) dX^i dX^j, \quad (\text{A18})$$

where knowledge of $G_{ab}(X^k)$ is also required. In turn, the rest mass density (with initial conditions set in such a way that it does coincide with $\varrho = F(\epsilon)$) would be computed as

$$\varrho = \frac{\varrho_i}{J} = \frac{\varrho_{H_i} (1 + \alpha_{H_i} \delta\epsilon_i)}{a^3 \det(\delta^a_i + P^a_i)}. \quad (\text{A19})$$

For the evaluation of the latter, note that in the case of a monochromatic wave (with one or both time-evolution modes), the deformation field components can be written as follows:

$$P_j^i = \lambda_1 \frac{K^i K_j}{K^2} + \lambda_2 \delta_j^i, \quad (\text{A20})$$

and similarly in the case of a localized spherically symmetric perturbation,

$$P_j^i = \lambda_1 \frac{X^i X_j}{K^2} + \lambda_2 \delta_j^i. \quad (\text{A21})$$

The coefficients $\lambda_1(t, X^k)$, $\lambda_2(t, X^k)$ for the monochromatic case are directly deduced from (A9) or from a sum of two such solutions, while in the localized spherically symmetric case, $\lambda_1(t, X^k) = q(t, R)$ and $\lambda_2(t, X^k) = (P(t, R) - q(t, R))/3$. The determinant of the spatial coframe coefficients, from which ϱ is evaluated, is then expressed in both cases by

$$J = a^3 (1 + \lambda_2)^2 (1 + \lambda_1 + \lambda_2), \quad (\text{A22})$$

leading to an infinite rest mass density (from shell-crossing) whenever $\lambda_2 \rightarrow -1$ or $\lambda_1 + \lambda_2 \rightarrow -1$.

Such an extrapolation procedure provides the exact metrical distances, density and other physical properties as produced by the deformation field at a given order. In particular, this gives powerful approximations for the Ricci and Weyl curvatures that are not available in standard perturbation theory. It is, however, clear that the resulting expressions are approximations that must be controlled.

We can further combine the exact functionals for a given deformation with exact averages of Einstein's equations. An example was given in [L2] that also showed that the resulting prescription can even lead to exact results. For example, the combination of the first-order Lagrangian dust model with exact averages led to an exact formula for the kinematical backreaction within a class of averaged Lemaître–Tolman–Bondi solutions [L2].

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Chapter 3

Averaging and backreactions in general foliations

The RZA (Relativistic Zel'dovich Approximation) including fundamental or effective pressure sources detailed in the previous chapter can be used as one of the possible models for the local dynamics in an inhomogeneous Universe. In this case it provides a nonperturbative approximation for the evolution of the matter or radiation energy density from given initial conditions, able to probe from the very early up to rather late regimes of structure formation, as well as scales where dust models are insufficient. It may then be used in conjunction with the spatial averaging procedure and averaged dynamical equations of Buchert [2001], recalled above in chapter 1 (section 1.5), for an investigation of the backreaction consequences of the emergence and stabilization of structures on the overall dynamics of a given set of matter fluid elements, including possible contributions to Dark Matter-like effects at small scales.

As mentioned earlier, a natural further extension to this scheme would be the inclusion of vorticity in the fluid, which can have non-negligible contributions to the dynamics of collapsing regions. One would thus consider a model fluid of 4-velocity \mathbf{u} with a nonzero vorticity 2-form $\boldsymbol{\omega}$, of components $\omega_{\mu\nu} \equiv b^\rho_{[\mu} b^\sigma_{\nu]} \nabla_\rho u_\sigma$ with $b^\rho_\mu = \delta^\rho_\mu + u^\rho u_\mu$. In terms of the dual 1-form $\underline{\mathbf{u}}$ to \mathbf{u} , this implies $\underline{\mathbf{u}} \wedge d\underline{\mathbf{u}} \neq 0$, *i.e.*, \mathbf{u} does not form orthogonal hypersurfaces. A more general foliation than the fluid-orthogonal ones considered so far has thus to be used in this case, allowing for a nonzero, varying tilt between the fluid 4-velocity and the normal to the hypersurfaces.

While adapting the RZA to non-fluid-orthogonal foliations as required to account for vorticity will be a future project, a major part of my PhD has been dedicated to the generalization of the spatial averaging scheme of Buchert [2000, 2001] to such less restrictive possibilities of building the spatial slices of averaging for the study of backreaction in inhomogeneous, possibly rotational fluids.

Beyond accounting for vorticity contributions, considering a freely specifiable 3+1 spatial foliation has other advantages. As no more restrictions need to be made on the fluid's kinematics, its energy-momentum tensor can also be taken under the most general form. A non-

barotropic isotropic pressure, an anisotropic pressure, and a heat vector can be included, while they will all generally create vorticity (see [Ellis, Maartens, and MacCallum \[2012\]](#) p.124-125, or [Christopherson et al. \[2009\]](#) for perturbative results). It also implies that, while still focussing on the frames and worldlines of a particular fluid component (with 4-velocity \mathbf{u}), other components with distinct 4-velocities can be included, despite them generally adding non-perfect fluid contributions to the total energy-momentum tensor as decomposed with respect to \mathbf{u} [[Maartens et al., 1999](#), [Delgado Gaspar et al., 2019](#)].

Moreover, such a generality allows for the choice of a foliation of specific physical interest, which may not be fluid-orthogonal even for an irrotational fluid, in each concrete application of the formalism to a cosmological model. This can include, for instance, hypersurfaces with constant scalar extrinsic curvature. Another choice of interest is a foliation by constant-fluid proper time slices, which can always be built for a nonsingular fluid flow and provides a natural interpretation of its ‘time’ parameter. I will further investigate this option below.

Relativistic numerical simulations for cosmology may also adopt a non-fluid-orthogonal foliation for an improved numerical stability, since the formation of shell-crossings in the fluid would otherwise lead to coordinate singularities in the associated 3+1 scheme. The analysis of resulting average dynamics and backreactions then requires an averaging scheme applicable to the chosen hypersurfaces. This is the situation considered in the recent work of [Macpherson et al. \[Macpherson et al., 2019\]](#) where the foliation and time parametrization are selected by setting an evolution equation on the lapse along the normal vector \mathbf{n} as⁷ $n^\mu \partial_\mu N = -N\mathcal{K}/3$.

Accounting for any possible spatial foliation choice can also allow for the determination of how the effective dynamics and the backreaction terms found in the fluid-orthogonal framework of [Buchert \[2000, 2001\]](#) would change for another foliation, since spatial averages are by definition taken over a subset of the spatial slices considered. This question has been raised in the literature where it is sometimes designated as a possible ‘gauge dependence’ (*e.g.*, [Larena \[2009\]](#), [Brown et al. \[2009\]](#)). This name refers to the Eulerian perturbation theories around an FLRW background (see subsection 1.3.2) where a different mapping to the background spacetime’s slices of homogeneity leads to the consideration of a different foliation in the perturbed manifold. =In the background-free frameworks considered here, changes in averaged quantities between different foliation choices are rather referred to as *foliation dependence*.

Several proposals for such a generalization of the framework of [Buchert \[2000, 2001\]](#) have already been exposed in the literature ([Kasai et al. \[2006\]](#), [Tanaka and Futamase \[2007\]](#); [Larena \[2009\]](#); [Brown et al. \[2009\]](#); [Gasperini et al. \[2009, 2010\]](#); [Räsänen \[2010\]](#); [Beltrán Jiménez et al. \[2014\]](#); [Smirnov \[2014\]](#)). However, none of these works considers a non-global averaging domain propagating along the fluid flow lines and preserving its fluid elements contents over time, directly affecting the effective dynamics. The consequences of such different propagation choices as well as other characteristics of the formalism of each of these works will be discussed in more detail

⁷This choice would give a parametrization by the conformal time η in an FLRW (Friedmann-Lemaître-Robertson-Walker) model.

below.

I will present in this chapter the results of the joint work with Xavier Roy and Thomas Buchert on averages of scalars in arbitrary spatial foliations. This corresponds to two studies: a paper in the last stages of preparation, and a Letter recently published as [Buchert, Mourier, and Roy, 2018]. The latter has been the object of an additional CQG+ Comment for originality⁸ at the journal's request.

In the first study, constituting the first part of this chapter below, we introduce two possible averaging schemes for such a general framework⁹, and the resulting effective regional dynamics for a compact averaging domain comoving with the fluid flow. The latter can have vorticity and acceleration and has no restriction on its stress-energy tensor. Both schemes are mostly similar, and both reduce to the scheme of Buchert [2000, 2001] for a fluid-orthogonal foliation, but, when the hypersurfaces are tilted with respect to the fluid flow, these schemes are distinguishable from each other by their volume measures and the local variables on which they focus. The first one is identical to those used in the literature proposals mentioned above apart from the domain propagation: it focusses on the geometry of the slices themselves, in the region delineated by the averaging domain, by using the Riemannian volume measure from the induced metric on the slices. After its description and application to averaging the 3+1 Einstein equations, we compare it with the previous works and analyse their specificities. The second averaging scheme is more intrinsic to the fluid flow due to using its proper volume as a volume measure, while still integrating over the (fluid-comoving) domain lying within the arbitrary spatial slices. It focusses on the kinematic variables of the fluid. Providing more insight into the effective dynamics of the fluid content, this scheme is also well suited to foliations at constant fluid proper time. We discuss its application to this particular choice, as well as its restriction to the fluid-orthogonal subcase and to a Newtonian framework.

This is followed as a second part to this chapter by the results of our Letter. We summarize there the main findings of our general scalar averaging framework and we discuss further their implications, focussing on the fluid-intrinsic scheme and its application to constant– fluid proper time foliations. We address in particular the explicit dependence of the average evolution equations and of the backreaction terms on the choice of foliation. We argue for a small possible such dependence in a cosmological context, thanks to the intrinsic approach, despite the existence of opposite claims such as in [Adamek et al., 2019] within extrinsic averaging schemes.

⁸<https://cqgplus.com/2018/11/28/space-is-the-place/>

⁹Although the averaged dynamical equations derived in this study arise from the local Einstein equations, the definition of the averaging operator and formalism itself only relies on the assumption of a semi-Riemannian manifold, and thus holds within any corresponding metric gravity theory. It may thus also be applied to such alternative theories of gravity, such as those discussed as alternative explanations to the apparent dynamical effects attributed to Dark Matter or Dark Energy.

On Average Properties of Inhomogeneous Fluids in General Relativity III: General Fluid Cosmologies

T. Buchert · P. Mourier · X. Roy

Abstract We investigate effective equations governing the volume expansion of spatially averaged portions of inhomogeneous cosmologies in spacetimes filled with an arbitrary fluid. This work is a follow-up to previous studies focused on irrotational dust models (Paper I) and irrotational perfect fluids (Paper II) in flow-orthogonal foliations of spacetime. It complements them by considering arbitrary foliations (hence arbitrary lapse and shift) and by allowing for a tilted fluid flow with vorticity. As for the first studies, the propagation of the spatial averaging domain is chosen to follow the congruence of the fluid, which avoids unphysical dependencies in the averaged system that is obtained. We present two different averaging schemes and corresponding systems of averaged evolution equations providing generalizations of Papers I and II. The first one retains the averaging operator used in several other generalizations found in the literature. We extensively discuss relations to these formalisms and pinpoint limitations, in particular in relation to averaging domain rest mass conservation. The alternative averaging scheme that we subsequently introduce follows the spirit of Papers I and II and focuses on the fluid flow and the associated $1 + 3$ threading congruence, used jointly with the $3 + 1$ foliation that builds the surfaces of averaging. This results in compact averaged equations with a minimal number of cosmological backreaction terms. We highlight that this system becomes especially transparent when applied to a natural class of proper time foliations.

Keywords relativistic cosmology · foliations · Lagrangian description · cosmological backreaction

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1 Introduction

A viable cosmological model provides an effective evolution history of the inhomogeneous Universe. The procedure of spatially averaging the scalar characteristics of an inhomogeneous model universe yields a system of Friedmann-type equations with an effective energy-momentum tensor, featuring so-called backreaction terms (see [15, 16], respectively referred to as Paper I and II hereafter). These additional terms contribute to and may potentially replace the dark constituents of the Universe that have to be postulated as fundamental sources in the standard model of cosmology [14, 18]. For recent reviews and references, we direct the attention of the reader to [33, 19, 29, 45, 53, 28, 22].

Extensions of this averaging framework have been investigated, concentrating on general foliations of spacetime within the $3+1$ formalism, to include a possible shift vector and a tilted fluid 4-velocity with vorticity [46, 10]. Some misinterpretations and drawbacks can be identified in these papers, and we are going to point them out in specially dedicated sections on the comparison with results in the literature. A four-dimensional averaging procedure has also been proposed [37, 38] in order to provide an explicit 4-covariant expression of the backreaction terms and to relate these to gauge-invariant variables.

We describe in this paper a unified and general framework within the $3+1$ formalism, leaving its four degrees of freedom (lapse and shift vector) unspecified and allowing for a tilted and vortical fluid flow. We shall emphasize (i) the use of an averaging domain comoving with the $1+3$ threading congruence of the fluid, and (ii) the Lagrangian point of view, that has been employed previously, without averaging, for fluids with vorticity [6] and pressure [5]. The present general investigation is also useful to relax some restricting assumptions of Papers I and II, to better understand the relation to Newtonian averaged cosmologies [23], and to extend the range of applicability of the effective equations.

The averaged system that we derive furnishes a background-free approach to relativistic cosmologies. It can alternatively be interpreted as a *general background cosmology* with a ‘background’ that is not fixed *a priori* [45], but interacts with the formation of structures. Fluctuations can then be investigated with respect to the physical average,¹ abandoning standard perturbative frameworks where fluctuations are referred to a fixed reference background and thus eliminating the need to consider gauge transformations.

This paper is organized as follows. Section 2 gives a comprehensive outline of the $3+1$ framework and the general fluid content we consider. We here also introduce the Lagrangian description, the relevance of which we shall emphasize in what follows. We introduce in section 3 an averaging framework similar to one commonly used in relativistic cosmological modeling (named here *fluid-extrinsic approach*), but with emphasis on a comoving evolution of the averaging domain. We derive the corresponding averaged evolution equations for the domain and comment on the resulting backreaction terms. We close this section with a detailed discussion of existing results in the literature. Section 4 opens a new perspective on the averaging problem by proposing a *fluid-intrinsic approach* that is inspired by a $1+3$ threading of spacetime and that focusses on the fluid’s metric and volume

¹First results on a corresponding perturbation scheme that makes structures evolve on such a physical background have been communicated [57].

forms rather than the metric and volume forms of the hypersurfaces. This allows for a compact formulation of the effective equations governing hypersurface averages of fluid properties, and it agrees in spirit with what has been presented in Papers I and II. We conclude in Section 5 after a discussion of various subcases of interest in order to illustrate our fluid-intrinsic approach and to prepare applications.

2 Foliation of spacetime and decomposition of the fluid

This section sets the definitions and notations for the $3+1$ foliation of spacetime and for the decomposition of the fluid flow and of its energy-momentum tensor (see, e.g., [4, 51, 60, 1, 39] for more details). The *comoving* and *Lagrangian* pictures are then introduced as natural possible coordinate descriptions adapted to the fluid flow.

2.1 Description of the geometry

Our spacetime model is a globally hyperbolic four-dimensional manifold, endowed with the pseudo-Riemannian metric tensor \mathbf{g} and described by a local system of coordinates $x^\mu = (t, x^i)$.²

We foliate this manifold into a family of spacelike hypersurfaces, and we denote by \mathbf{n} their timelike, future-oriented, unit normal 4-vector. The foliation can be characterized by a regular scalar function S strictly increasing along each flow line, and defined such that each spatial hypersurface is a level set of S . For simplicity, we choose the time coordinate t as being a strictly increasing function of S (implying the reciprocal relation $S = S(t)$), and use it to label the hypersurfaces. The spatial coordinates x^i , on the other hand, are kept arbitrary.

In such a spacetime coordinate basis, the components of \mathbf{n} are written:

$$n^\mu = \frac{1}{N} (1, -N^i), \quad (2.1)$$

and the components of its non-exact dual form $\underline{\mathbf{n}}$ read:

$$n_\mu = -N (1, 0). \quad (2.2)$$

The positive lapse function N determines how far consecutive slices are from each other at each point, while the shift vector \mathbf{N} generates a spatial diffeomorphism that relates pairs of points between the slices. Following the usual conventions of a $3+1$ formalism, we here associate this lapse to the coordinate functions defining the propagation of the local spatial coordinates between slices. By definition we have:

$$\partial_t = N\mathbf{n} + \mathbf{N}. \quad (2.3)$$

²Greek letters are assigned to spacetime indices, they run in $\{0, 1, 2, 3\}$, and Latin letters refer to space indices, running in $\{1, 2, 3\}$. The signature of the metric is taken as $(-+++)$, and units are such that $c = 1$. The coordinate system x^μ is associated to the coordinate basis $\{\partial_{x^\mu}\} := \{\partial_t, \partial_{x^i}\}$ and its dual exact basis $\{dx^\mu\} := \{dt, dx^i\}$. Unless otherwise specified, components of tensorial objects should be understood as expressed in these bases, with arguments (t, x^i) .

We shall keep the lapse and shift unspecified for the derivation of the averaged system, thereby preserving the four degrees of freedom of the foliation. We shall, however, introduce in subsection 2.4 convenient foliations and coordinate choices that may be adopted for the description of the system (these amount to setting the shift, or both the lapse and the shift).

Spacetime tensors are projected onto the hypersurfaces of the foliation by means of the operator $\mathbf{h} = h_{\alpha\beta} dx^\alpha \otimes dx^\beta$,

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad h_{\alpha\mu} n^\alpha = 0, \quad h^\mu{}_\alpha h^\alpha{}_\nu = h^\mu{}_\nu, \quad h^{\alpha\beta} h_{\alpha\beta} = 3, \quad (2.4)$$

whose restriction on the spatial slices defines the spatial Riemannian metric h_{ij} , with inverse h^{ij} . Given this operator and the normal vector \mathbf{n} , the four-dimensional line element can be decomposed into

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\left(N^2 - N^k N_k\right) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j. \quad (2.5)$$

Note that the lapse N also measures, through its spatial variations, the acceleration of the frames associated with \mathbf{n} :

$$a_\mu^{(n)} := n^\alpha \nabla_\alpha n_\mu = \frac{N_{||\mu}}{N}, \quad (2.6)$$

where ∇_α denotes the four-covariant derivative, and $||$ the three-covariant derivative associated with the spatial metric h_{ij} .

2.2 Description of the fluid

We consider in this work a model universe sourced by a single general fluid, the flow of which is described by a unit timelike vector \mathbf{u} , tilted with respect to the normal \mathbf{n} of the foliation.

2.2.1 Decomposition of the 4-velocity

The fluid 4-velocity vector \mathbf{u} can be decomposed in all generality into

$$\mathbf{u} = \gamma (\mathbf{n} + \mathbf{v}), \quad (2.7)$$

$$\text{with } n_\alpha v^\alpha = 0, \quad \gamma = -n_\alpha u^\alpha = \frac{1}{\sqrt{1 - v^\alpha v_\alpha}}, \quad (2.8)$$

where \mathbf{v} (hereafter *Eulerian velocity*) is the spatial velocity of the fluid relative to the *normal frames*, which are defined as being at rest within the hypersurfaces and transported along the normal \mathbf{n} . The vector \mathbf{v} identifies the direction and magnitude of the above-mentioned tilt. The magnitude is equivalently measured by the Lorentz factor γ or by the tilt angle ϕ , defined as $\phi := \text{arcosh}(\gamma)$ [44, 38]. For a vanishing tilt, $\mathbf{u} = \mathbf{n}$, we have $\mathbf{v} = \mathbf{0}$, $\gamma = 1$, and $\phi = 0$.

Introducing the spatial *coordinate velocity* of the fluid,

$$\mathbf{V} = \frac{d\mathbf{x}}{dt}, \quad \text{with } n_\alpha V^\alpha = 0, \quad (2.9)$$

where \mathbf{x} is the spatial position of a fluid element in the coordinate system (t, x^i) and d/dt is the derivative with respect to t along the fluid flow lines, we can write the Eulerian velocity as (see, e.g., [60, 1, 39]):

$$\mathbf{v} = \frac{1}{N} (\mathbf{N} + \mathbf{V}) . \quad (2.10)$$

Equation (2.7) can then be reformulated in the general form:

$$\mathbf{u} = \frac{\gamma}{N} (N\mathbf{n} + \mathbf{N} + \mathbf{V}) , \quad (2.11)$$

with $\frac{\gamma}{N} = \frac{1}{\sqrt{N^2 - (N^\alpha + V^\alpha)(N_\alpha + V_\alpha)}} .$

In contrast to the Eulerian velocity \mathbf{v} which is covariantly defined, the coordinate velocity \mathbf{V} depends on the way the spatial coordinates propagate between neighboring hypersurfaces; hence it depends on the shift. For instance for a coordinate system comoving with the fluid, which corresponds to a specific shift, we have $\mathbf{V} = \mathbf{0}$, while for a vanishing tilt, we have $\mathbf{V} = -\mathbf{N}$, whatever shift is chosen. We represent in figure 1 the different vector fields introduced thus far.

Note that a foliation orthogonal to the fluid, where $\mathbf{n} := \mathbf{u}$ and $\mathbf{v} = \mathbf{0}$ (as considered in Papers I and II), is only possible for a fluid flow with no vorticity. Even for irrotational fluids, introducing a tilt allows us to keep the freedom in the construction of the spatial hypersurfaces.

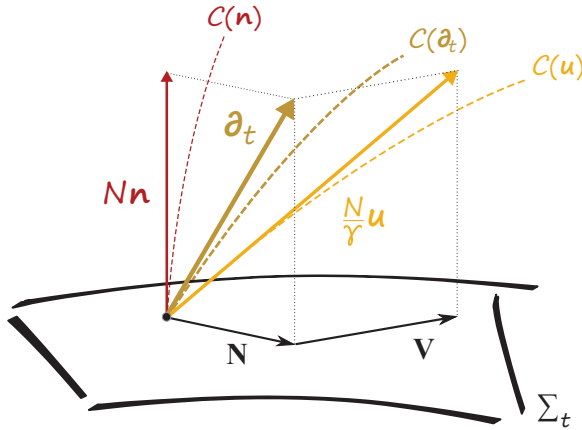


Fig. 1 Representation of the different vector fields at hand, on a spatial hypersurface Σ_t . \mathbf{n} is the vector normal to the hypersurface and it transports the normal frames; ∂_t is the time-vector of the coordinate basis, tangent to the integral curves $C(\partial_t)$ (with $x^i = \text{const.}$); and \mathbf{u} is the 4-velocity of the fluid, tangent to the congruence $C(\mathbf{u})$. The deviations between \mathbf{n} and ∂_t , on the one hand, and between ∂_t and \mathbf{u} , on the other hand, are identified respectively by \mathbf{N} and \mathbf{V} . The tilt between \mathbf{u} and \mathbf{n} is given by $\mathbf{v} = (\mathbf{N} + \mathbf{V})/N$. (Note that although \mathbf{N} and \mathbf{V} are tangent to Σ_t , we represent them at the heads of the other vectors for simplicity.) For a coordinate system comoving with the fluid, we have $\mathbf{V} = \mathbf{0}$ and $\partial_t = (N/\gamma) \mathbf{u}$. Even though the coordinate velocity vanishes in this situation, the fluid can still experience a spatial motion within the hypersurface, given by \mathbf{v} , and the shift would be set to $\mathbf{N} = N\mathbf{v}$. Alternatively, in the case of a fluid flow orthogonal to the hypersurfaces, we would have $\mathbf{u} = \mathbf{n}$, and hence $\mathbf{V} = -\mathbf{N}$ for any shift.

The components of \mathbf{u} and its dual are obtained by noticing that any spatial vector χ can be extended to a four-dimensional vector by writing:

$$\chi^\mu = (\chi^0, \chi^i), \quad \text{with } \chi^0 = 0. \quad (2.12)$$

The components of its dual 1-form are then deduced from the property $n^\alpha \chi_\alpha = 0$ along with expression (2.1):

$$\chi_\mu = (\chi_0, \chi_i), \quad \text{with } \chi_0 = N^k \chi_k. \quad (2.13)$$

Applying (2.12) and (2.13) to the shift vector and the coordinate velocity, we obtain from (2.11) the component expressions:

$$u^\mu = \frac{\gamma}{N} (1, V^i), \quad u_\mu = \frac{\gamma}{N} (-N^2 + N^k (N_k + V_k), N_i + V_i). \quad (2.14)$$

2.2.2 Kinematic variables and acceleration

Let us introduce the operator $\mathbf{b} = b_{\alpha\beta} dx^\alpha \otimes dx^\beta$ that projects tensors onto the local rest frames of the fluid orthogonal to \mathbf{u} :

$$b_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad b_{\alpha\mu} u^\alpha = 0, \quad b^\mu{}_\alpha b^\alpha{}_\nu = b^\mu{}_\nu, \quad b^{\alpha\beta} b_{\alpha\beta} = 3. \quad (2.15)$$

The projectors \mathbf{b} and \mathbf{h} usually differ because of the tilt of \mathbf{u} with respect to the normal \mathbf{n} of the slices. From relations (2.15), we can decompose the 4-covariant derivative of the 1-form \underline{u} into the 4-acceleration and the kinematic parts of the fluid [31] as follows:

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \frac{1}{3} \Theta b_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \quad (2.16)$$

$$\text{with } a_\mu := u^\alpha \nabla_\alpha u_\mu, \quad \Theta := \nabla_\alpha u^\alpha,$$

$$\text{and } \sigma_{\mu\nu} := b^\alpha{}_\mu b^\beta{}_\nu \nabla_{(\alpha} u_{\beta)} - \frac{1}{3} \Theta b_{\mu\nu}, \quad \omega_{\mu\nu} := b^\alpha{}_\mu b^\beta{}_\nu \nabla_{[\alpha} u_{\beta]}, \quad (2.17)$$

where the round and square brackets respectively imply symmetrization and anti-symmetrization over the indices enclosed. \mathbf{a} is the acceleration of the fluid, Θ its expansion rate, σ its shear tensor, and ω is its vorticity tensor.³

Recall that the rest frames of the fluid are not hypersurface-forming if ω does not vanish.

³The shear, vorticity and acceleration of the fluid, as seen in the normal frames, can be derived from the projections onto the three-surfaces of the proper shear σ , proper vorticity ω and proper acceleration \mathbf{a} , respectively. For instance, the second would read $h^\alpha{}_\mu h^\beta{}_\nu \omega_{\alpha\beta} = h^\alpha{}_\mu h^\beta{}_\nu b^\delta{}_\alpha b^\xi{}_\beta \nabla_{[\delta} u_{\xi]}$, which differs from $h^\alpha{}_\mu h^\beta{}_\nu \nabla_{[\alpha} u_{\beta]}$ when \mathbf{a} is not null.

2.2.3 Stress-energy tensor and conservation laws

The stress-energy tensor of the fluid can be decomposed with respect to the fluid rest frames as follows:

$$T_{\mu\nu} = \epsilon u_\mu u_\nu + 2q_{(\mu} u_{\nu)} + p b_{\mu\nu} + \pi_{\mu\nu}, \quad (2.18)$$

with $\epsilon := u^\alpha u^\beta T_{\alpha\beta}$, $q_\mu := -b^\alpha{}_\mu u^\beta T_{\alpha\beta}$, $p b_{\mu\nu} + \pi_{\mu\nu} := b^\alpha{}_\mu b^\beta{}_\nu T_{\alpha\beta}$, $b^{\mu\nu} \pi_{\mu\nu} = 0$.

ϵ denotes the energy density of the fluid in its rest frame, q_μ the spatial heat vector, p the isotropic pressure, and $\pi_{\mu\nu}$ the spatial and traceless anisotropic stress. Alternatively, it can be decomposed with respect to the normal frames as

$$T_{\mu\nu} = E n_\mu n_\nu + 2n_{(\mu} J_{\nu)} + S_{\mu\nu}, \quad (2.19)$$

with $E := n^\alpha n^\beta T_{\alpha\beta}$, $J_\mu := -h^\alpha{}_\mu n^\beta T_{\alpha\beta}$, $S_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta}$,

where E is the energy density of the fluid, J_μ its momentum density, and $S_{\mu\nu}$ its stress density, all as measured in the normal frames. The isotropic part of $S_{\mu\nu}$ is given by the trace $S := g^{\alpha\beta} S_{\alpha\beta}$. This last decomposition will be used in section 3 for the derivation of the averaged equations. Using expression (2.7), we can relate the scalar quantities of both decompositions as

$$E = \gamma^2 \epsilon + (\gamma^2 - 1)p + 2\gamma v^\alpha q_\alpha + v^\alpha v^\beta \pi_{\alpha\beta}, \quad (2.20)$$

$$S = (\gamma^2 - 1)\epsilon + (\gamma^2 + 2)p + 2\gamma v^\alpha q_\alpha + v^\alpha v^\beta \pi_{\alpha\beta}. \quad (2.21)$$

From the property $\nabla_\beta T^{\alpha\beta} = 0$ along with relations (2.18) and (2.17), we derive the energy conservation law:

$$u_\alpha \nabla_\beta T^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \dot{\epsilon} + \Theta(\epsilon + p) = -a_\alpha q^\alpha - \nabla_\alpha q^\alpha - \pi^{\alpha\beta} \sigma_{\alpha\beta}, \quad (2.22)$$

and the momentum conservation law:

$$\begin{aligned} & b_{\mu\alpha} \nabla_\beta T^{\alpha\beta} = 0 \\ \Leftrightarrow \quad a_\mu &= -\frac{1}{\epsilon + p} \left(b^\alpha{}_\mu \nabla_\alpha p + b_{\mu\alpha} \dot{q}^\alpha + \frac{4}{3} \Theta q_\mu + q^\alpha (\sigma_{\alpha\mu} + \omega_{\alpha\mu}) + b_{\mu\alpha} \nabla_\beta \pi^{\alpha\beta} \right), \end{aligned} \quad (2.23)$$

where the overdot is defined below in subsection 2.3. These relations can be complemented by the conservation of the rest mass density ρ of the fluid in its rest frame:

$$\nabla_\alpha (\rho u^\alpha) = 0, \quad \text{or equivalently,} \quad \dot{\rho} + \Theta \rho = 0. \quad (2.24)$$

2.3 Time derivatives and their relations

The existence of two different times (the coordinate time t and the fluid proper time τ) and of three timelike congruences (see figure 1) leads to several possible definitions of time derivatives. Those of interest for the present work are:

- the *covariant derivative* along the fluid flow lines, denoted by an overdot; for any tensor field \mathbf{F} , we have $\dot{\mathbf{F}} := u^\alpha \nabla_\alpha \mathbf{F}$;

- the comoving derivative along the fluid flow lines and according to the proper time τ , or Lagrangian derivative, denoted by $d/d\tau$;
- the comoving derivative d/dt along the fluid flow lines and according to the coordinate time t ;
- the partial coordinate time derivative along the vector ∂_t , *i.e.* along the integral curves of constant x^i , denoted by $\partial_t|_{x^i}$.

The last three derivatives are related by:

$$\frac{dF^{\mu\nu\dots}_{\alpha\beta\dots}}{dt} = \frac{\partial F^{\mu\nu\dots}_{\alpha\beta\dots}}{\partial t}\Big|_{X^i} = \frac{\partial F^{\mu\nu\dots}_{\alpha\beta\dots}}{\partial t}\Big|_{x^i} + V^i \frac{\partial F^{\mu\nu\dots}_{\alpha\beta\dots}}{\partial x^i}, \quad (2.25)$$

$$\frac{dF^{\mu\nu\dots}_{\alpha\beta\dots}}{d\tau} = \frac{\gamma}{N} \frac{dF^{\mu\nu\dots}_{\alpha\beta\dots}}{dt}, \quad (2.26)$$

for any tensor field $\mathbf{F} = F^{\mu\nu\dots}_{\alpha\beta\dots} \partial_\mu \otimes \partial_\nu \otimes \dots \otimes \mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta \otimes \dots$. For a scalar field ψ , the first two derivatives are identical: $\dot{\psi} = u^\alpha \partial_\alpha \psi = d\psi/d\tau$.

Proof Let us consider the components $F^{\mu\nu\dots}_{\alpha\beta\dots}$ of a tensor field \mathbf{F} in the coordinate basis associated with (t, x^i) (see footnote 2). For notational ease, we drop in what follows the indices and write $F := F^{\mu\nu\dots}_{\alpha\beta\dots}$. The total coordinate-time derivative of F along any timelike curve \mathcal{C} can be decomposed in terms of the coordinate partial derivatives as

$$\frac{dF}{dt}\Big|_{\mathcal{C}} = \frac{\partial F}{\partial t}\Big|_{x^i} + \frac{\partial F}{\partial x^i} \frac{dx^i}{dt}\Big|_{\mathcal{C}}. \quad (2.27)$$

Considering the variation along the congruence $\mathcal{C}(\mathbf{u})$ of the fluid, and therefore making use of definition (3.7), we obtain

$$\frac{dF}{dt} := \frac{dF}{dt}\Big|_{\mathcal{C}(\mathbf{u})} = \frac{\partial F}{\partial t}\Big|_{x^i} + \frac{\partial F}{\partial x^i} V^i. \quad (2.28)$$

Moreover, for the Lagrangian coordinates X^i , by definition constant along the fluid flow lines, we have $(dX^i/dt)|_{\mathcal{C}(\mathbf{u})} = 0$, and hence $dF/dt = \partial_t|_{X^i} F$, which concludes the proof of (2.25).

The total derivative of F with respect to the proper time τ of the fluid along the congruence $\mathcal{C}(\mathbf{u})$ satisfies

$$\frac{dF}{d\tau} := \frac{dF}{d\tau}\Big|_{\mathcal{C}(\mathbf{u})} = \frac{dt}{d\tau}\Big|_{\mathcal{C}(\mathbf{u})} \frac{dF}{dt}\Big|_{\mathcal{C}(\mathbf{u})}. \quad (2.29)$$

From the definition of \mathbf{u} and its component expression (2.14), we have $(dt/d\tau)|_{\mathcal{C}(\mathbf{u})} = u^0 = \gamma/N$, and thus

$$\frac{dF}{d\tau} = \frac{\gamma}{N} \frac{dF}{dt}, \quad (2.30)$$

which proves (2.26). Reformulating the right-hand side by means of (2.28), and using again the component expression of \mathbf{u} finally yields:

$$\frac{dF}{d\tau} = u^0 \frac{\partial F}{\partial t}\Big|_{x^i} + u^i \frac{\partial F}{\partial x^i} = u^\alpha \partial_{x^\alpha} F, \quad (2.31)$$

hence $d/d\tau = u^\alpha \partial_\alpha$. This operator coincides with the overdot, $\dot{} = u^\alpha \nabla_\alpha$, when applied to a scalar variable. \square

2.4 Comoving and Lagrangian descriptions

2.4.1 Comoving description

For any given foliation, the shift vector can be chosen in such a way that the spatial components (2.14) of \mathbf{u} vanish: by setting $\mathbf{N} = N\mathbf{v}$, given relation (2.10), we have $\mathbf{V} = \mathbf{0}$. This choice corresponds to spatial coordinates propagating along the fluid flow lines, i.e. to *comoving* (or *Lagrangian*) spatial coordinates. We will refer to the use of these spatial coordinates as a *comoving description* of the fluid, and denote them by X^i . Note that a comoving description is a “weak” form of a *Lagrangian description* (as introduced below) in that no constraints are set on the time coordinate t .

In the coordinates (t, X^i) of the comoving description, the components (2.14) of the fluid velocity read:

$$u^\mu = \frac{\gamma}{N} (1, 0), \quad u_\mu = \left(-\frac{N}{\gamma}, \gamma v_i \right), \quad (2.32)$$

while the line element (2.5) reduces to

$$ds^2 = -\frac{N^2}{\gamma^2} dt^2 + 2Nv_i dt dX^i + h_{ij} dX^i dX^j. \quad (2.33)$$

The components of the acceleration and kinematic quantities simplify as follows. From the anti-symmetric part of (2.16) we can write in any coordinate system:

$$\omega_{\mu\nu} = u_{[\mu} a_{\nu]} + \nabla_{[\mu} u_{\nu]} = u_{[\mu} a_{\nu]} + \partial_{[\mu} u_{\nu]}. \quad (2.34)$$

In comoving coordinates, the $(0, i)$ components of this expression vanish, given that $\omega_{\alpha i} u^\alpha = 0$. Combining this property with $a_0 = 0$, from $a_\alpha u^\alpha = 0$, we can thus write the spatial components of the acceleration as

$$a_i = \frac{\gamma}{N} \left(\frac{d}{dt} u_i + \frac{\gamma}{N} \partial_i \left(\frac{N}{\gamma} \right) \right), \quad (2.35)$$

where we also used $u_0 = -N/\gamma$ and $\partial_t = d/dt$. Inserting (2.35) back into the (i, j) components of (2.34) yields the non-vanishing components of the vorticity:

$$\omega_{ij} = \frac{\gamma}{N} u_{[i} \frac{d}{dt} u_{j]} + \frac{N}{\gamma} \partial_{[i} \left(\frac{\gamma}{N} u_{j]} \right). \quad (2.36)$$

The expansion tensor can be related to the Lie derivative $\mathcal{L}_{\mathbf{u}} \mathbf{b}$ of the projector \mathbf{b} along the fluid flow in any coordinates according to

$$(\mathcal{L}_{\mathbf{u}} \mathbf{b})_{\mu\nu} = u^\alpha \nabla_\alpha b_{\mu\nu} + b_{\alpha\nu} \nabla_\mu u^\alpha + b_{\mu\alpha} \nabla_\nu u^\alpha = 2u_{(\mu} a_{\nu)} + 2\nabla_{(\mu} u_{\nu)} = 2\Theta_{\mu\nu}, \quad (2.37)$$

where we have used the symmetric part of (2.16) for the last equality. The covariant derivatives of the second expression can be equivalently replaced by partial derivatives. This provides the non-vanishing comoving-coordinates components of the expansion tensor as $\Theta_{ij} = (\mathcal{L}_{\mathbf{u}} \mathbf{b})_{ij} / 2 = u^0 \partial_0 b_{ij} / 2$, and hence

$$\Theta_{ij} = \frac{1}{2} \frac{\gamma}{N} \frac{d}{dt} b_{ij}. \quad (2.38)$$

The trace and traceless parts are deduced from the above. For convenience, we express them in terms of a representative length ℓ in the fluid rest frames, defined by $\hat{\ell}/\ell := \Theta/3$ [31]:

$$\Theta = \frac{1}{2} \frac{\gamma}{N} b^{ij} \frac{d}{dt} b_{ij} = \frac{3}{\ell} \frac{\gamma}{N} \frac{d\ell}{dt}; \quad \sigma_{ij} = \frac{1}{2} \frac{\gamma}{N} \ell^2 \frac{d}{dt} (\ell^{-2} b_{ij}). \quad (2.39)$$

2.4.2 Lagrangian description

An appropriate choice of foliation can allow for the hypersurfaces to be labelled by a proper time τ of the fluid [34,35]. Such a construction identifies a class of foliations which we call *fluid proper time foliations*. It is realized by level sets of the fluid proper time τ , as defined from its comoving coordinate-time evolution rate $d\tau/dt = N/\gamma$ (see section 2.3) and an initial spacelike hypersurface Γ (parametrized by an equation $t = t_\Gamma(X^i)$) on which it takes a given constant value $\tau_{\mathbf{i}}$,⁴

$$\tau(t, X^i) := \tau_{\mathbf{i}} + \int_{t_\Gamma(X^i)}^t \frac{N(\hat{t}, X^i)}{\gamma(\hat{t}, X^i)} d\hat{t}. \quad (2.40)$$

The hypersurface labelled by a given value τ can equivalently be defined as the image at time τ of Γ by the flow operator defined from the unitary vector field \mathbf{u} .

The fluid proper time foliations set the normal vector \mathbf{n} , and determine the lapse N up to a time-dependent factor. The fluid proper time can then be used as the time parameter t labelling these hypersurfaces, $t := \tau$, fully determining the lapse and tying it to the Lorentz factor, $N = \gamma$. Note that for such foliations, the hypersurfaces cannot be fluid-orthogonal, namely a tilt must be present, except in the case of irrotational geodesic flows (*e.g.* irrotational dust) [31]. In general, such a tilt may be expected to grow with time and become large and highly inhomogeneous on the slices. This may even imply in some cases that not all slices remain everywhere spacelike; hence, when using such a foliation, we will implicitly restrict our attention to the part of spacetime where the hypersurfaces do remain spatial, if necessary. Within this class of foliation and lapse choice, the additional requirement of using comoving spatial coordinates defines a *comoving*

⁴The proper time is not uniquely defined *a priori*, but it is fully determined by the choice of an initial Cauchy surface to build one of its level sets [35]. Another proper time function τ' , taking the constant value $\tau'_{\mathbf{i}}$ on another initial hypersurface Γ' , would differ from τ by a function φ constant along the fluid flow lines, $\tau'(t, X^i) = \tau(t, X^i) + \varphi(X^i)$. This relation follows by writing

$$\tau' := \tau'_{\mathbf{i}} + \int_{t_{\Gamma'}(X^i)}^t \frac{N(\hat{t}, X^i)}{\gamma(\hat{t}, X^i)} d\hat{t},$$

with Γ' parametrized by $t = t_{\Gamma'}(X^i)$, yielding

$$\varphi(X^i) = \tau'_{\mathbf{i}} - \tau_{\mathbf{i}} + \int_{t_{\Gamma'}(X^i)}^{t_\Gamma(X^i)} \frac{N(\hat{t}, X^i)}{\gamma(\hat{t}, X^i)} d\hat{t}.$$

The expressions defining τ and τ' are here given in terms of comoving coordinates. They could alternatively be written covariantly, by setting the value of $\tau - \tau_{\mathbf{i}}$ (resp. $\tau' - \tau'_{\mathbf{i}}$) at a given spacetime event as the total length of the unique fluid flow line joining this event to the hypersurface Γ (resp. Γ'). The properties of both proper times and their relation through φ of course hold in this description.

and *synchronous* picture which we call the *Lagrangian description* of the fluid (see Asada and Kasai [6] and Asada [5], inspired by Friedrich [36]).

In the coordinates (τ, X^i) of the Lagrangian description, the components (2.14) of the 4-velocity and its dual read:

$$u^\mu = (1, 0), \quad u_\mu = (-1, \gamma v_i), \quad (2.41)$$

while the line element (2.5) takes the form:

$$ds^2 = -dt^2 + 2\gamma v_i dX^i dt + h_{ij} dX^i dX^j. \quad (2.42)$$

The Lagrangian condition $u^\mu = \delta^\mu_0$, as introduced in [36], is therefore equivalent to setting simultaneously $N^i = Nv^i$ and $N = \gamma$. It implies $g_{00} = -1$ or, equivalently, $N^2 - N^k N_k = 1$. In this description, as a special case of a comoving description (with the additional requirement of $N = \gamma$), the spatial components of the fluid acceleration reduce to

$$a_i = \frac{d}{d\tau} u_i, \quad (2.43)$$

and those of the kinematic variables become:

$$\begin{aligned} \Theta_{ij} &= \frac{1}{2} \frac{d}{d\tau} b_{ij}; & \Theta &= \frac{1}{2} b^{kl} \frac{d}{d\tau} b_{kl} = \frac{3}{\ell} \frac{d\ell}{d\tau}; & \sigma_{ij} &= \frac{1}{2} \ell^2 \frac{d}{d\tau} (\ell^{-2} b_{ij}); \\ \omega_{ij} &= u_{[i} \frac{d}{d\tau} u_{j]} + \partial_{[i} u_{j]}. \end{aligned} \quad (2.44)$$

In the following derivations of the extrinsic and intrinsic averaging schemes, we will keep the lapse and shift unspecified, thereby considering a general description and preserving the four degrees of freedom of the foliation. The formulation of the averaged system in the Lagrangian description will be discussed later on as a particularly insightful special case within the intrinsic scheme.

3 Rest mass-preserving scalar averaging: fluid-extrinsic approach

In this section we recall the 3 + 1 formulation of Einstein's equations with respect to the hypersurfaces of normal \mathbf{n} , we formalize spatial averaging over a compact domain that lies within the spatial hypersurfaces and that follows the fluid flow. We then derive the commutation rule and averaged equations for the scalar parts of Einstein's equations and we discuss some properties of the resulting backreaction terms and their relation to boundary terms. At the end of the section, we compare our approach and its results to previous proposals of generalizations of the framework of Papers I and II that can be found in the literature, and we discuss in detail the differences and pinpoint limitations.

3.1 Dynamical equations

The 3 + 1 foliation of Einstein's equations [4, 51, 60, 1, 39], with the cosmological constant Λ included, comprises the following evolution equations:

$$\partial_t|_{x^i} h_{ij} = -2N\mathcal{K}_{ij} + N_{i||j} + N_{j||i}, \quad (3.1)$$

$$\begin{aligned} \partial_t|_{x^i} \mathcal{K}^i_j &= N \left(\mathcal{R}^i_j + \mathcal{K}\mathcal{K}^i_j + 4\pi G [(S - E)\delta^i_j - 2S^i_j] - \Lambda\delta^i_j \right) \\ &\quad - N^{||i}_{||j} + N^k \mathcal{K}^i_{j||k} + \mathcal{K}^i_k N^k_{||j} - \mathcal{K}^k_j N^i_{||k}, \end{aligned} \quad (3.2)$$

together with the momentum and energy constraints:

$$\mathcal{K}^k_{i||k} - \mathcal{K}_{||i} = 8\pi G J_i, \quad (3.3)$$

$$\mathcal{R} + \mathcal{K}^2 - \mathcal{K}^i_j \mathcal{K}^j_i = 16\pi G E + 2\Lambda. \quad (3.4)$$

\mathcal{R}_{ij} and $\mathcal{K}_{ij} := -h^\alpha_i h^\beta_j \nabla_\alpha n_\beta$ are the components of the 3-Ricci tensor and the extrinsic curvature of the hypersurfaces, respectively. $\mathcal{R} := h^{ij} \mathcal{R}_{ij}$ and $\mathcal{K} := h^{ij} \mathcal{K}_{ij}$ are their respective traces.

In Appendix A we give the evolution equations for h_{ij} and \mathcal{K}^i_j along the congruence of the fluid, using the derivative d/dt instead of $\partial_t|_{x^i}$, and we specify their expressions in the comoving and Lagrangian descriptions.

3.2 Fluid-extrinsic scalar averaging

3.2.1 Comoving-to-reference map

We introduce a set of Lagrangian (or comoving) spatial coordinates $\mathbf{X} = \{X^i\}$. The comoving coordinates of each fluid element remain constant along its flow line, as opposed in general to its arbitrary reference spatial coordinates $\mathbf{x} = \{x^i\}$. This arises from the different directions between the threading congruence of the fluid ($t, X^i = \text{const.}$), given by \mathbf{u} , and the arbitrary coordinate congruence ($t, x^i = \text{const.}$), given by ∂_t . The two sets of spatial coordinates \mathbf{x} and \mathbf{X} are related by a one-parametric family of diffeomorphisms⁵ parametrized by the coordinate time t ,

$$\begin{aligned} \Phi_t : \mathcal{D}_{\mathbf{X}} &\rightarrow \mathcal{D}_{\mathbf{x}} = \Phi_t(\mathcal{D}_{\mathbf{X}}), \\ \mathbf{X} &\mapsto \mathbf{x} = \Phi_t(\mathbf{X}) := \mathbf{f}(t, \mathbf{X}), \end{aligned} \quad (3.5)$$

$$\text{with } \mathbf{f}(t_1, \mathbf{X}) = \mathbf{X}, \quad \text{and } J(t, \mathbf{X}) := \det \frac{\partial \mathbf{f}(t, \mathbf{X})}{\partial \mathbf{X}}, \quad (3.6)$$

where \mathcal{D} refers to a compact domain lying within the hypersurfaces and transported along the congruence of the fluid flow (hereafter *comoving domain*). This specific transport ensures that the domain encloses the same collection of fluid elements at all times (an important feature to which we shall come back in the discussion). We denote the set of spatial coordinate values corresponding to this collection at a given time t by $\mathcal{D}_{\mathbf{x}}(t)$, or $\mathcal{D}_{\mathbf{x}}$ for short, in the reference coordinates, and by $\mathcal{D}_{\mathbf{X}}$ (by definition time-independent) in the comoving coordinates. The maps Φ_t define on each constant- t hypersurface a coordinate transformation between \mathbf{x} and \mathbf{X} .

From (3.5) we reformulate the coordinate velocity (2.9) as

$$\mathbf{V} = \frac{d\mathbf{x}}{dt} = \frac{d}{dt} \mathbf{f}(t, \mathbf{X}) = \partial_t|_{X^i} \mathbf{f}(t, \mathbf{X}), \quad (3.7)$$

⁵Note that we assume throughout the regularity of the fluid flow implied by the existence of congruences and invertible maps (diffeomorphisms), which excludes the description of caustics that may occur for particular matter models.

where along the direction of the derivative $\partial_t|_{X^i}$, given by the fluid flow lines, the comoving spatial coordinates X^i are kept fixed. Using (3.6) together with (3.7) we have the identity:

$$\partial_t|_{X^i} J = J \partial_i V^i. \quad (3.8)$$

We note that the Newtonian tools developed for the Lagrangian description of structure formation in cosmology can be applied to this diffeomorphism without difficulty (see [13], [32] and references therein).

3.2.2 Volume of a domain and its comoving time-evolution

The Riemannian volume of the spatial domain \mathcal{D} is given by

$$\mathcal{V}_{\mathcal{D}}(t) := \int_{\mathcal{D}_{\mathbf{x}}} n^\mu d\sigma_\mu = \int_{\mathcal{D}_{\mathbf{x}}} \sqrt{h(t, x^i)} d^3x, \quad (3.9)$$

where h is the determinant of the spatial metric, $h := \det(h_{ij})$, and $d\sigma_\mu$ is the oriented spatial volume element, $d\sigma_\mu := -n_\mu \sqrt{h} d^3x$. We seek the coordinate-time variation of (3.9) along the fluid flow lines, namely we search for the expression of

$$\frac{d}{dt} \int_{\mathcal{D}_{\mathbf{x}}} \sqrt{h(t, x^i)} d^3x. \quad (3.10)$$

The operators d/dt and $\int_{\mathcal{D}_{\mathbf{x}}} \cdot d^3x$ do not commute in general since the endpoints of the integral, determined by the spatial region $\mathcal{D}_{\mathbf{x}}$, depend themselves on time. The fluid is moving with respect to the coordinate system (t, x^i) , and the domain of integration is attached to the fluid⁶ (see figure 2). We need to reformulate the integrand to get rid of this time-dependence.

To this aim, we consider the family of maps $\Phi_t = \mathbf{f}(t, \cdot)$ introduced above to change the coordinates from x^i to X^i . We have:

$$x^i = f^i(t, \mathbf{X}), \quad d^3x = \det\left(\frac{\partial f(t, \mathbf{X})}{\partial \mathbf{X}}\right) d^3X = J(t, \mathbf{X}) d^3X, \quad (3.11)$$

while the region of integration transforms as $\mathcal{D}_{\mathbf{x}} \rightarrow \mathcal{D}_{\mathbf{X}} = \Phi_t^{-1}(\mathcal{D}_{\mathbf{x}})$. Inserting (3.11) into (3.9), we get:

$$\mathcal{V}_{\mathcal{D}}(t) = \int_{\mathcal{D}_{\mathbf{X}}} \sqrt{h(t, f^i(t, \mathbf{X}))} J(t, \mathbf{X}) d^3X. \quad (3.12)$$

The invariance of the volume element $\sqrt{h(t, x^i)} d^3x$ (here integrated over the same collection of fluid elements) with respect to changes of spatial coordinates appears here by noticing that $\sqrt{h(t, f^i(t, \mathbf{X}))} J(t, \mathbf{X})$ above corresponds to the square root of the determinant of the components in the coordinate system (t, \mathbf{X}) of the spatial metric \mathbf{h} . Obviously, the fluid is at rest in this coordinate system, allowing for the commutation of $d/dt = \partial_t|_{X^i}$ and $\int_{\mathcal{D}_{\mathbf{X}}} \cdot d^3X$.⁷ We can now write:

$$\frac{d}{dt} \mathcal{V}_{\mathcal{D}} = \int_{\mathcal{D}_{\mathbf{X}}} \frac{d}{dt} \left(\sqrt{h(t, f^i(t, \mathbf{X}))} J \right) d^3X, \quad (3.13)$$

⁶For the same reason, the operators $d/d\tau$ and $\partial_t|_{x^i}$ do not commute either with $\int_{\mathcal{D}_{\mathbf{x}}} \cdot d^3x$.

⁷Note that, in contrast to the operator d/dt , the operator $d/d\tau$ does not commute in general with $\int_{\mathcal{D}_{\mathbf{X}}} \cdot d^3X$, since $d/d\tau = (\gamma/N) d/dt$ depends on the spatial coordinates.

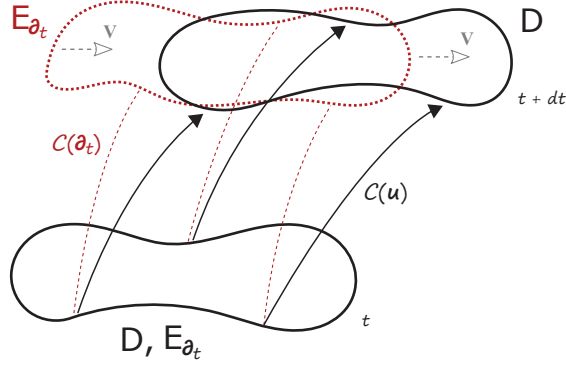


Fig. 2 Representation of the motion of a compact domain \mathcal{D} between neighboring hypersurfaces. \mathcal{D} is transported along the congruence of the fluid $\mathcal{C}(\mathbf{u})$, with $X^i = \text{const.}$, and contains by construction the same collection of fluid elements throughout its evolution. We introduce in this figure another compact domain, \mathcal{E}_{∂_t} , carried along the congruence $\mathcal{C}(\partial_t)$, with $x^i = \text{const.}$, that coincides with \mathcal{D} at time t . \mathcal{E}_{∂_t} encloses the same collection of fluid elements as \mathcal{D} at that time. At $t + dt$, the two domains do not coincide anymore as the fluid undergoes a spatial motion of velocity \mathbf{V} in the coordinate system (t, x^i) (hence d/dt and $\int_{\mathcal{D}_x} \cdot d^3x$ do not commute). This motion induces a flux of fluid particles across the boundary of \mathcal{E}_{∂_t} . In the comoving and Lagrangian descriptions, the congruences $\mathcal{C}(\partial_t)$ and $\mathcal{C}(\mathbf{u})$ are identical and this flux does not occur. A similar distinction would have to be made between \mathcal{D} and a domain transported along the flow of the hypersurfaces normal vector \mathbf{n} , with a flux of fluid particles across the boundaries of the latter, except in the absence of tilt.

and, transforming the coordinates back to x^i with the help of Φ_t^{-1} , we obtain:

$$\frac{d}{dt} \mathcal{V}_{\mathcal{D}} = \int_{\mathcal{D}_x} \frac{d}{dt} (J\sqrt{h}) J^{-1} d^3x = \int_{\mathcal{D}_x} \left(\frac{d}{dt} \sqrt{h} + \sqrt{h} J^{-1} \frac{d}{dt} J \right) d^3x. \quad (3.14)$$

Using the relations (2.25) and (3.8), this implies:

$$\begin{aligned} \frac{d}{dt} \mathcal{V}_{\mathcal{D}} &= \int_{\mathcal{D}_x} \left(\partial_t|_{x^i} \sqrt{h} + V^k \partial_k \sqrt{h} + \partial_k V^k \sqrt{h} \right) d^3x \\ &= \int_{\mathcal{D}_x} \left(\frac{1}{2} h^{ij} \partial_t|_{x^i} h_{ij} + \frac{1}{2} h^{ij} V^k \partial_k h_{ij} + \partial_k V^k \right) \sqrt{h} d^3x. \end{aligned} \quad (3.15)$$

From the trace of the evolution equation (3.1) and noticing that

$$\frac{1}{2} h^{ij} \partial_k h_{ij} V^k + \partial_k V^k = V^k{}_{||k}, \quad (3.16)$$

we finally end up with the expression of the coordinate-time comoving variation of the Riemannian volume (see Appendix A for an alternative derivation using instead the 3 + 1 evolution equations along the congruence of the fluid):

$$\begin{aligned} \frac{d}{dt} \mathcal{V}_{\mathcal{D}} &= \int_{\mathcal{D}_x} \left(-N\mathcal{K} + N^i{}_{||i} + V^i{}_{||i} \right) \sqrt{h} d^3x \\ &= \int_{\mathcal{D}_x} \left(-N\mathcal{K} + (Nv^i)_{||i} \right) \sqrt{h} d^3x, \end{aligned} \quad (3.17)$$

where we used relation (2.10) for the last equality.

3.2.3 Averaging and commutation rule

We define the *extrinsic spatial hypersurface volume average* of any scalar ψ on a compact comoving domain \mathcal{D} as

$$\langle \psi \rangle_{\mathcal{D}}(t) := \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}} \psi n^{\mu} d\sigma_{\mu} = \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}_{\mathbf{x}}} \psi(t, x^i) \sqrt{h(t, x^i)} d^3x. \quad (3.18)$$

Applying this definition on (3.17), we can write the rate of change of $\mathcal{V}_{\mathcal{D}}$ as

$$\frac{1}{\mathcal{V}_{\mathcal{D}}} \frac{d}{dt} \mathcal{V}_{\mathcal{D}} = \left\langle -N\mathcal{K} + (Nv^i)_{||i} \right\rangle_{\mathcal{D}}, \quad (3.19)$$

and express the comoving coordinate-time derivative of the averaged scalar ψ in the form:

$$\frac{d}{dt} \langle \psi \rangle_{\mathcal{D}} = - \left\langle -N\mathcal{K} + (Nv^i)_{||i} \right\rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}} + \frac{1}{\mathcal{V}_{\mathcal{D}}} \frac{d}{dt} \int_{\mathcal{D}_{\mathbf{x}}} \psi(t, x^i) \sqrt{h(t, x^i)} d^3x. \quad (3.20)$$

The second term on the right-hand side is evaluated by following the same procedure as above: we perform a coordinate change by means of the maps Φ_t ,

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{D}_{\mathbf{x}}} \psi(t, x^i) \sqrt{h(t, x^i)} d^3x &= \frac{d}{dt} \int_{\mathcal{D}_{\mathbf{X}}} \psi(t, f^i(t, \mathbf{X})) \sqrt{h(t, f^i(t, \mathbf{X}))} J(t, \mathbf{X}) d^3X \\ &= \int_{\mathcal{D}_{\mathbf{X}}} \frac{d}{dt} \left(\psi(t, f^i(t, \mathbf{X})) \sqrt{h(t, f^i(t, \mathbf{X}))} J(t, \mathbf{X}) \right) d^3X, \end{aligned}$$

and, transforming back to the reference coordinates, expanding the integrand, and using once again the definition (3.18), we end up with

$$\frac{1}{\mathcal{V}_{\mathcal{D}}} \frac{d}{dt} \int_{\mathcal{D}_{\mathbf{x}}} \psi \sqrt{h} d^3x = \left\langle \frac{d}{dt} \psi \right\rangle_{\mathcal{D}} + \left\langle \left(-N\mathcal{K} + (Nv^i)_{||i} \right) \psi \right\rangle_{\mathcal{D}}. \quad (3.21)$$

Plugging this equation into (3.20), we finally obtain the commutation rule for extrinsic averages over a spatial *comoving domain*. We formulate this new result in the form of a lemma.

Lemma 1 (Commutation rule for extrinsic volume averages)

The commutation rule between spatial averaging on a compact domain \mathcal{D} , lying within a t -constant hypersurface and comoving with the fluid, and comoving differentiation with respect to the coordinate time reads, for any $3+1$ foliation of spacetime and for any scalar ψ :

$$\frac{d}{dt} \langle \psi \rangle_{\mathcal{D}} = \left\langle \frac{d}{dt} \psi \right\rangle_{\mathcal{D}} - \left\langle -N\mathcal{K} + (Nv^i)_{||i} \right\rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}} + \left\langle \left(-N\mathcal{K} + (Nv^i)_{||i} \right) \psi \right\rangle_{\mathcal{D}}. \quad (3.22)$$

This commutation rule is independent of the shift vector, and hence is independent of the propagation of the spatial coordinates. This feature is inherited from the coordinate-independent definition of the propagation of the domain of averaging obtained by requiring it to be comoving with the fluid.

Note that, as shown in Appendix B (Eq.(B.2) therein), the local terms appearing in the volume rate of change (3.19) can be equivalently expressed in terms of the lapse, tilt and fluid expansion rate as $-N\mathcal{K} + (Nv^i)_{||i} = (N/\gamma)\Theta - \gamma^{-1}d\gamma/dt$. The commutation rule can thus alternatively be written under the following form for any scalar ψ :

$$\frac{d}{dt}\langle\psi\rangle_{\mathcal{D}} = \left\langle\frac{d}{dt}\psi\right\rangle_{\mathcal{D}} - \left\langle\frac{N}{\gamma}\Theta - \frac{1}{\gamma}\frac{d\gamma}{dt}\right\rangle_{\mathcal{D}}\langle\psi\rangle_{\mathcal{D}} + \left\langle\left(\frac{N}{\gamma}\Theta - \frac{1}{\gamma}\frac{d\gamma}{dt}\right)\psi\right\rangle_{\mathcal{D}}, \quad (3.23)$$

which will be useful when applied to fluid rest frame variables such as ϵ or ϱ .

3.3 Conservation of the fluid rest mass

We introduce the conserved fluid rest mass flux vector \mathbf{M} as

$$M^\mu := \varrho u^\mu, \quad \nabla_\mu M^\mu = 0, \quad (3.24)$$

from the (conserved) rest mass density ϱ . The rest mass of the fluid within the domain \mathcal{D} is given by the flow of \mathbf{M} through \mathcal{D} :

$$M_{\mathcal{D}} := \int_{\mathcal{D}} M^\mu d\sigma_\mu = \int_{\mathcal{D}} -\varrho u^\mu n_\mu \sqrt{h} d^3x = \mathcal{V}_{\mathcal{D}} \langle\gamma\varrho\rangle_{\mathcal{D}}, \quad (3.25)$$

with the oriented spatial volume element $d\sigma_\mu = -n_\mu \sqrt{h} d^3x$, and where we used $-u^\mu n_\mu = \gamma$, Eq. (2.8).

The conservation of this rest mass can be seen by integrating the conservation equation of \mathbf{M} over the spacetime tube \mathcal{T} swept by the domain \mathcal{D} between two hypersurfaces at times t_1 and $t_2 > t_1$:

$$0 = \int_{\mathcal{T}} \nabla_\mu M^\mu \sqrt{g} d^4x = \oint_{\partial\mathcal{T}} M^\mu d\eta_\mu, \quad (3.26)$$

where $d\eta_\mu$ is the outward-oriented volume element on the boundary $\partial\mathcal{T}$ of \mathcal{T} . Introducing the timelike part \mathcal{A} of $\partial\mathcal{T}$, with \mathbf{A} its outward-oriented unit normal vector and $dV_{\mathcal{A}}$ its volume 3-form, we rewrite the above as:

$$\begin{aligned} 0 &= \int_{\mathcal{D}_{t_2}} \gamma\varrho\sqrt{h} d^3x - \int_{\mathcal{D}_{t_1}} \gamma\varrho\sqrt{h} d^3x + \int_{\mathcal{A}} M^\mu A_\mu dV_{\mathcal{A}} \\ &= M_{\mathcal{D}_{t_2}} - M_{\mathcal{D}_{t_1}} + \int_{\mathcal{A}} M^\mu A_\mu dV_{\mathcal{A}}. \end{aligned} \quad (3.27)$$

The last term cancels out precisely because the domain propagates along the fluid flow lines so that the normal vector \mathbf{A} is orthogonal to \mathbf{u} everywhere on the boundary \mathcal{A} . We therefore end up with the conservation of the rest mass within \mathcal{D} : $M_{\mathcal{D}_{t_2}} = M_{\mathcal{D}_{t_1}}$.

Alternatively, one can make use of the local continuity equation (2.24) for ϱ , equivalent to the conservation of M (3.24), rewritten in terms of a coordinate-time derivative:

$$\frac{d}{dt}\varrho + \frac{N}{\gamma}\Theta\varrho = 0. \quad (3.28)$$

Applying the commutation rule expressed in terms of Θ , Eq. (3.23), and the corresponding form of the volume expansion rate, $\mathcal{V}_{\mathcal{D}}^{-1}d\mathcal{V}_{\mathcal{D}}/dt = \langle(N/\gamma)\Theta - \gamma^{-1}d\gamma/dt\rangle_{\mathcal{D}}$, to the average of the above local continuity equation multiplied by γ then gives $d(\mathcal{V}_{\mathcal{D}}\langle\gamma\varrho\rangle_{\mathcal{D}})/dt = 0$, recovering the conservation of $M_{\mathcal{D}}$.

3.4 Averaged inhomogeneous cosmologies in the fluid-extrinsic approach

We introduced in the previous sections a scalar averaging procedure on a compact spatial domain comoving with the fluid. We derived the corresponding commutation rule and showed the preservation of the total fluid rest mass within the comoving domain. Both hold for any foliation of spacetime. By means of this formalism, and from the Einstein equations given in subsection 2.2, we now give an (under-determined) set of scalar balance equations describing the effective dynamics of spatially averaged comoving and compact regions of inhomogeneous cosmologies.

3.4.1 Averaged evolution equations

Following the original proposal of [23] (used in Papers I and II), we define the *effective scale factor* $a_{\mathcal{D}}$ of the comoving domain \mathcal{D} as

$$a_{\mathcal{D}}(t) := \left(\frac{\mathcal{V}_{\mathcal{D}}(t)}{\mathcal{V}_{\mathcal{D}_i}} \right)^{1/3}, \quad (3.29)$$

where \mathcal{D}_i refers to the domain at the initial time t_i . The volume expansion rate (3.19) then gives:

$$\frac{1}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} = \frac{1}{3} \left\langle -NK + (Nv^i)_{||i} \right\rangle_{\mathcal{D}}. \quad (3.30)$$

From the average of the trace of $N \times$ (3.2) and that of $N^2 \times$ (3.4), and upon using the commutation rule (3.22) along with relation (3.30), we obtain the effective evolution equations for an inhomogeneous model universe in the fluid-extrinsic averaging procedure, that we formulate in the form of a *Theorem*.

Theorem 1.a (Extrinsically averaged evolution equations)

The evolution equations for the effective scale factor of a compact spatial domain \mathcal{D} comoving with a general fluid read, for any 3 + 1 foliation of spacetime:

$$3 \frac{1}{a_{\mathcal{D}}} \frac{d^2 a_{\mathcal{D}}}{dt^2} = -4\pi G \left\langle N^2 (\epsilon + 3p) \right\rangle_{\mathcal{D}} + \left\langle N^2 \right\rangle_{\mathcal{D}} \Lambda + \mathcal{Q}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}} + \frac{1}{2} \mathcal{T}_{\mathcal{D}}, \quad (3.31)$$

$$3 \left(\frac{1}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \right)^2 = 8\pi G \left\langle N^2 \epsilon \right\rangle_{\mathcal{D}} + \left\langle N^2 \right\rangle_{\mathcal{D}} \Lambda - \frac{1}{2} \left\langle N^2 \mathcal{R} \right\rangle_{\mathcal{D}} - \frac{1}{2} \mathcal{Q}_{\mathcal{D}} - \frac{1}{2} \mathcal{T}_{\mathcal{D}}, \quad (3.32)$$

with $\mathcal{Q}_{\mathcal{D}}$, $\mathcal{P}_{\mathcal{D}}$ and $\mathcal{T}_{\mathcal{D}}$ respectively the kinematical backreaction, the dynamical backreaction, and the stress-energy backreaction, defined as

$$\mathcal{Q}_{\mathcal{D}} := \left\langle N^2 \left(\mathcal{K}^2 - \mathcal{K}_{ij} \mathcal{K}^{ij} \right) \right\rangle_{\mathcal{D}} - \frac{2}{3} \left\langle -N\mathcal{K} + (Nv^i)_{||i} \right\rangle_{\mathcal{D}}^2, \quad (3.33)$$

$$\begin{aligned} \mathcal{P}_{\mathcal{D}} := & \left\langle \left((Nv^i)_{||i} \right)^2 + \frac{d}{dt} \left((Nv^i)_{||i} \right) - 2N\mathcal{K} (Nv^i)_{||i} - N^2 v^i \mathcal{K}_{||i} \right\rangle_{\mathcal{D}} \\ & + \left\langle NN_{||i}^i - \mathcal{K} \frac{dN}{dt} \right\rangle_{\mathcal{D}}, \end{aligned} \quad (3.34)$$

$$\mathcal{T}_{\mathcal{D}} := -16\pi G \left\langle N^2 \left((\gamma^2 - 1)(\epsilon + p) + 2\gamma v^\alpha q_\alpha + v^\alpha v^\beta \pi_{\alpha\beta} \right) \right\rangle_{\mathcal{D}}. \quad (3.35)$$

Remarks to Theorem 1.a: Care should be taken in the interpretation of the system ((3.31),(3.32)). These equations are globally invariant under the remaining coordinate freedoms, that is, (i) under any change of the spatial coordinates, or (ii) under a change of the time coordinate of the form $t \mapsto T(t)$ with $dT/dt > 0$ and of the lapse as $N \mapsto N' = N (dT/dt)^{-1}$ (which corresponds to a re-parametrization of the hypersurfaces). However, individual terms, as well as each equation side taken separately, are invariant under the former transformation only. A time change as above would rescale most terms, such as $\mathcal{Q}_{\mathcal{D}}$, $\mathcal{T}_{\mathcal{D}}$ or $3((1/a_{\mathcal{D}}) da_{\mathcal{D}}/dt)^2$, by the time-dependent factor $(dT/dt)^{-2}$ (strictly preserving their sign). The terms $\mathcal{P}_{\mathcal{D}}$ and $(3/a_{\mathcal{D}}) d^2 a_{\mathcal{D}}/dt^2$ would undergo an affine transformation, with this same rescaling plus an additional term (the same for both, thus preserving the equation globally) proportional to $(da_{\mathcal{D}}/dt) (d^2 T/dt^2)$, so that even their sign can be arbitrarily changed in a time-dependent manner.

Accordingly, depending on what t represents, the left-hand sides of equations ((3.31),(3.32)) may not follow an interpretation similar to the corresponding $3(\dot{a}/a)^2$ and $3\ddot{a}/a$ of the standard Friedmann equations. These are unambiguously expressed as derivatives with respect to the common proper of the comoving fluid.⁸ Without a well-specified choice for t , conclusions may only be drawn on quantities that are invariant under the change of time coordinate expressed above. Such invariants include the sign of the contribution of each term except the dynamical backreaction and scale factor acceleration terms, as we shall, *e.g.*, discuss for the stress-energy backreaction in section 3.5.2, or effective dimensionless “ Ω ” parameters that may be defined for a non-stationary $a_{\mathcal{D}}$ ($da_{\mathcal{D}}/dt \neq 0$) by dividing each term of Eq. (3.32) by $3[(1/a_{\mathcal{D}}) da_{\mathcal{D}}/dt]^2$. The generality of Theorem 1.a allows us to choose the most suited definition for t in any specific application.

The Friedmannian interpretation of t and its derivatives can be recovered for some choices that are applicable to general settings. This is the case for instance for the synchronous and the Lagrangian descriptions, which involve a choice of foliation (see section 4.4 for an example). One could also choose t within any foliation such that it coincides with the proper time along some given timelike

⁸Note that one could in the same way parametrize the Friedmann model by a different time coordinate while staying within the homogeneous foliation, and similarly get rescaled terms and an arbitrarily altered acceleration term (see, *e.g.*, the system of equations (20) in Paper II [16] or the system of equations (40) in [47]). The usual form of the Friedmann equations removes this freedom by choosing the proper time as the most natural time parameter in this situation. As, additionally, the spatial coordinates generally used in this framework are comoving with the fluid content, this picture corresponds to what we termed in this work a Lagrangian description.

wordline, for instance taken to model the worldline of an observer on Earth. Once a specification of the time label is performed, each term of the above equations, including the acceleration term $(3/a_{\mathcal{D}}) d^2 a_{\mathcal{D}}/dt^2$ or its sign, can be interpreted in direct relation to the physical meaning of the chosen t .

3.4.2 Integrability and energy balance conditions

We proceed by deriving the *integrability condition* for the system of equations of Theorem 1.a, which provides the relation that has to hold for (3.32) to be the integral of (3.31). This condition is obtained by taking the comoving coordinate-time derivative of (3.32), and by inserting the set of equations (3.31) and (3.32) back into the result.⁹ Complementing this condition by the average of the energy conservation equation, we write the second part of the above *Theorem*:

Theorem 1.b (Integrability and energy balance conditions)

A necessary condition of integrability of equation (3.31) to yield equation (3.32) is given by the relation:

$$\begin{aligned} \frac{d}{dt} \mathcal{Q}_{\mathcal{D}} + \frac{6}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \mathcal{Q}_{\mathcal{D}} + \frac{d}{dt} \langle N^2 \mathcal{R} \rangle_{\mathcal{D}} + \frac{2}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \langle N^2 \mathcal{R} \rangle_{\mathcal{D}} + \frac{d}{dt} \mathcal{T}_{\mathcal{D}} + \frac{4}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} (\mathcal{T}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}}) \\ = 16\pi G \left(\frac{d}{dt} \langle N^2 \epsilon \rangle_{\mathcal{D}} + \frac{3}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \langle N^2 (\epsilon + p) \rangle_{\mathcal{D}} \right) + 2\Lambda \frac{d}{dt} \langle N^2 \rangle_{\mathcal{D}}, \end{aligned} \quad (3.36)$$

where the source part on the right-hand side satisfies the averaged energy conservation law:

$$\begin{aligned} \frac{d}{dt} \langle N^2 \epsilon \rangle_{\mathcal{D}} + \frac{3}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \langle N^2 (\epsilon + p) \rangle_{\mathcal{D}} = \left\langle \frac{N}{\gamma} \Theta \right\rangle_{\mathcal{D}} \langle N^2 p \rangle_{\mathcal{D}} - \left\langle \frac{N}{\gamma} \Theta N^2 p \right\rangle_{\mathcal{D}} - \left\langle \frac{1}{\gamma} \frac{d\gamma}{dt} \right\rangle_{\mathcal{D}} \langle N^2 p \rangle_{\mathcal{D}} \\ + \left\langle \left(2 \frac{1}{N} \frac{dN}{dt} - \frac{1}{\gamma} \frac{d\gamma}{dt} \right) N^2 \epsilon \right\rangle_{\mathcal{D}} - \left\langle \frac{N^3}{\gamma} \left(q^\alpha a_\alpha + \nabla_\alpha q^\alpha + \pi^{\alpha\beta} \sigma_{\alpha\beta} \right) \right\rangle_{\mathcal{D}}. \end{aligned} \quad (3.37)$$

This conservation law can be complemented by the conservation of the fluid rest mass, $dM_{\mathcal{D}}/dt = 0$, which may be rewritten as follows:

$$\frac{d}{dt} \langle \gamma \varrho \rangle_{\mathcal{D}} + \frac{3}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \langle \gamma \varrho \rangle_{\mathcal{D}} = 0. \quad (3.38)$$

Proof The local energy conservation law (2.22) implies:

$$\frac{d}{dt} (N^2 \epsilon) + \frac{N}{\gamma} \Theta (N^2 (\epsilon + p)) = 2 \frac{1}{N} \frac{dN}{dt} N^2 \epsilon - \frac{N^3}{\gamma} \left(q^\alpha a_\alpha + \nabla_\alpha q^\alpha + \pi^{\alpha\beta} \sigma_{\alpha\beta} \right). \quad (3.39)$$

Relation (3.37) is then recovered by averaging the local equation (3.39) and applying the commutation rule expressed in terms of Θ , Eq.(3.23). \square

We present as *Corollary 1* in Appendix B an equivalent formulation of the system of equations of Theorem 1, focussing explicitly on the kinematic and dynamical variables of the fluid rather than the geometric properties of the hypersurfaces (such as their intrinsic and extrinsic curvatures).

⁹Alternatively, we can derive the integrability condition directly from the Einstein equations. For this we note that we can derive the evolution equations for the square of the trace-free part of the extrinsic curvature and the evolution equation for the 3-scalar curvature using (3.2) and (3.4). Averaging these equations and combining them, we also obtain the integrability condition.

The system of equations of this theorem could also be rewritten in more compact ways, as we shall illustrate for similar equations obtained within an alternative averaging approach in section 4. We will keep it under the current form, as it is already sufficient to discuss important properties and relations to the literature, to which we turn now.

3.5 Discussion

We summarize in the first part of this subsection the framework of our study. We then discuss the backreaction terms that were defined, investigate boundary effects and boundary-free global domains, and finally discuss relations to the literature for global and general domains successively.

3.5.1 Summary

We have worked with three sets of independent worldlines: the normal congruence along \mathbf{n} , everywhere orthogonal to the hypersurfaces of constant coordinate time t , the congruence of the *coordinate frames* along ∂_t , and the threading congruence of the *comoving frames* (or, equivalently, the fluid rest frames) along \mathbf{u} . The deviations between \mathbf{n} and ∂_t , on the one hand, and between \mathbf{n} and \mathbf{u} , on the other hand, are identified respectively by the vector fields \mathbf{N} and \mathbf{v} , while that between ∂_t and \mathbf{u} is pinpointed by \mathbf{V} (see figure 1).

This general configuration allows for a fluid flow with vorticity and tilted with respect to the normal of the three-surfaces, and for an arbitrary propagation of the spatial coordinates. Also, the lapse function is left unspecified, preserving the freedom in the construction of the spatial slices.

We have considered a compact spatial domain \mathcal{D} , lying within the hypersurfaces and transported along the fluid flow lines, thus enclosing by construction the same collection of fluid elements throughout the evolution. In the generic situation, this domain undergoes a spatial motion in the coordinate system (t, x^i) , since the integral curves of ∂_t and \mathbf{u} do not coincide (see figure 2).

Within this framework, we have established the general commutation rule (formula (3.22)) between spatial averaging and differentiation with respect to the coordinate time along the fluid flow lines. We have then derived in *Theorem 1* a set of scalar equations describing the regional dynamics of spatially averaged portions of an inhomogeneous fluid. The results obtained hold for a general fluid and for a general foliation of spacetime and, in particular, are independent of the propagation of the spatial coordinates. In such a general foliation, however, we have stressed the risk of too hastily interpreting these results, in particular of interpreting the time acceleration term in the same way as the proper-time acceleration term \ddot{a}/a of the standard Friedmann equations: its meaning strongly depends on the interpretation of the chosen time parameter t itself. We have also highlighted the Lagrangian foliation and coordinates choice as a transparent setting that allows us to recover the common interpretation.

3.5.2 Comments on the backreaction terms

The kinematical backreaction $\mathcal{Q}_{\mathcal{D}}$ (3.33) and the dynamical backreaction $\mathcal{P}_{\mathcal{D}}$ (3.34) generalize the expressions given in Paper II. The emphasis is set here on the geometric variables of the foliation $\{\mathcal{K}, \mathcal{K}_{ij}, \text{etc.}\}$, rather than on the kinematic variables of the fluid $\{\Theta, \Theta_{ij}, \text{etc.}\}$ (see Appendix B for a formulation in terms of the latter). These two sets of variables are identical in the fluid-orthogonal approach of Paper II, but they differ in the present framework. Differences with the setup of Paper II can be made explicit in the kinematical backreaction term by reformulating it as

$$\mathcal{Q}_{\mathcal{D}} = \frac{2}{3} \left(\left\langle N^2 \mathcal{K}^2 \right\rangle_{\mathcal{D}} - \left\langle -N\mathcal{K} + (Nv^i)_{||i} \right\rangle_{\mathcal{D}}^2 \right) - 2 \left\langle N^2 \mathcal{K}_{\text{tl}}^2 \right\rangle_{\mathcal{D}}, \quad (3.40)$$

where the traceless part of the extrinsic curvature defines the shear scalar of the normal congruence, $\mathcal{K}_{\text{tl}}^2 := \frac{1}{2} (\mathcal{K}_{ij} - \frac{1}{3} \mathcal{K} h_{ij}) (\mathcal{K}^{ij} - \frac{1}{3} \mathcal{K} h^{ij})$, and its trace \mathcal{K} gives (up to a sign change) the expansion rate of the normal congruence. This formulation is reminiscent of Paper II. However, it is no longer expressed in terms of kinematic variables, and it highlights an additional contribution $(Nv^i)_{||i}$ from the Eulerian velocity (or, equivalently, the tilt). We can also notice additional terms due to the Eulerian velocity in the expression of the dynamical backreaction (3.34).

If the fluid is vorticity-free, we can choose a fluid-orthogonal foliation, namely we can set $\mathbf{n} = \mathbf{u}$ as in Paper II and, thus, have $\mathbf{v} = \mathbf{0}$ and $\gamma = 1$. In this configuration the geometric and kinematic variables coincide, and we recover the expressions of $\mathcal{Q}_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{D}}$ given in Paper II. As this setting also implies the vanishing of the stress-energy backreaction $\mathcal{T}_{\mathcal{D}}$, we formally get back the same set of evolution equations for the effective scale factor (up to the additional inclusion of the cosmological constant contribution). This could have been expected, but notice that here, in contrast to Paper II, we allow for a non-vanishing shift vector and a non-perfect fluid. As already discussed, and as for the commutation rule (3.22), the shift does not contribute because local evolutions are regarded along the fluid flow lines, and the spatial domain of averaging is comoving with the fluid; the shift vector plays no dynamical role locally and on average. However, even though nonperfect-fluid effects are not formally present in the evolution equations for the effective scale factor, they still influence the dynamics through the local and average evolution of the energy density (see equations (2.22) and (3.37)).

In addition to contributing to the kinematical and dynamical backreaction terms, the tilt also yields the additional backreaction term $\mathcal{T}_{\mathcal{D}}$, which we named *stress-energy backreaction*, and which can be interpreted in the following ways.

Firstly, it measures the difference between the fluid's energy as seen in the normal frames and its rest frames. In this sense, it is thus (up to an overall negative factor) an average measure of the kinetic energy of the fluid in the normal frames. Indeed, using relation (2.20) we can write

$$(\gamma^2 - 1)(\epsilon + p) + 2\gamma v^\alpha q_\alpha + v^\alpha v^\beta \pi_{\alpha\beta} = E - \epsilon = T_{\mu\nu} n^\mu n^\nu - T_{\mu\nu} u^\mu u^\nu, \quad (3.41)$$

so that

$$\mathcal{T}_{\mathcal{D}} = -16\pi G \left\langle N^2 (E - \epsilon) \right\rangle_{\mathcal{D}} = -16\pi G \left\langle N^2 (T_{\mu\nu} n^\mu n^\nu - T_{\mu\nu} u^\mu u^\nu) \right\rangle_{\mathcal{D}}. \quad (3.42)$$

Secondly, it also expresses the difference in isotropic pressure as seen in both frames, since combining relations (2.20) and (2.21) gives

$$E - \epsilon = S - 3p = T_{\mu\nu}h^{\mu\nu} - T_{\mu\nu}b^{\mu\nu} . \quad (3.43)$$

This backreaction term has been introduced to express the dynamics of the averaging domain as sourced by averages of scalar dynamical quantities of the fluid as seen in its rest frames, ϵ and p (recall equations (3.31)–(3.32)), rather than the quantities measured in the normal frames, E and S , since only the former correspond to intrinsic thermodynamical quantities of the fluid that are directly described by its equation of state.

Thirdly, as will be shown in subsection 3.5.3, it corresponds to the ‘bulk’ tilt contribution in that it survives for a boundary-free domain, while the tilt contributions to $\mathcal{Q}_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{D}}$ are boundary terms.

The last expression in equation (3.42) shows that our stress-energy backreaction corresponds (up to a numerical factor) to the ‘fluid corrections’ terms introduced by Brown *et al.* in [10], while the first form (3.35) is sufficient to identify it with the (unnamed and slightly more general) $\langle F \rangle$ term appearing in Räsänen’s equations in [54], and to see that it reduces to the ‘tilt effects’ noticed by Gasperini *et al.* in [38] in the particular case of a perfect fluid, still up to numerical factors.

The sign of $\mathcal{T}_{\mathcal{D}}$ will usually be constrained and will remain negative, consistently with the interpretation of $-\mathcal{T}_{\mathcal{D}}$ as a measure of kinetic energy, so that this backreaction will contribute as a deceleration term to the effective acceleration equation (3.31). This constraint is expressed by the following *Proposition*.

Proposition 1 (sign of the stress-energy backreaction)

If the matter stress-energy tensor satisfies the Null Energy Condition, then

$$(\gamma^2 - 1)(\epsilon + p) + v^\mu v^\nu \pi_{\mu\nu} \geq 0 , \quad (3.44)$$

and the following assumptions on the heat vector \mathbf{q} separately impose $\mathcal{T}_{\mathcal{D}} \leq 0$:

- (i) *a vanishing heat vector, $\mathbf{q} = \mathbf{0}$ (this includes the case of a perfect fluid, for which the constant sign of the corresponding ‘tilt effects’ was already noticed in [38]); or,*
- (ii) *a preferred mutual spatial orientation between \mathbf{v} and (the projection onto the hypersurfaces of) \mathbf{q} ensuring $N^2 \gamma q_\mu v^\mu \geq 0$, locally or on average; or,*
- (iii) *on the contrary and more realistically, a variable orientation of the heat vector de-correlated from that of \mathbf{v} and from the value of the lapse N and Lorentz factor γ , so that the variable-sign term $N^2 \gamma q_\mu v^\mu$ is averaged out while the other terms all add up positively: $\left| \langle N^2 \gamma q_\mu v^\mu \rangle_{\mathcal{D}} \right| \ll \langle (\gamma^2 - 1)(\epsilon + p) + v^\mu v^\nu \pi_{\mu\nu} \rangle_{\mathcal{D}}$.*

Proof Noting that $(b^\mu_\nu v^\nu)(b_{\mu\rho} v^\rho) = b_{\mu\nu} v^\mu v^\nu = \gamma^2 - 1 = \gamma^2 v^2$, one can define two future-pointing null vectors \mathbf{k}_+ , \mathbf{k}_- as $k^\mu_\pm := \gamma v u^\mu \mp b^\mu_\nu v^\nu = \gamma v u^\mu \pm b^\mu_\nu v^\nu$. The projections of the stress-energy tensor onto these vectors yield:

$$T_{\mu\nu} k^\mu_\pm k^\nu_\pm = (\gamma^2 - 1)(\epsilon + p) + \pi_{\mu\nu} v^\mu v^\nu \mp 2\gamma v q_\mu v^\mu . \quad (3.45)$$

According to the Null Energy Condition (which we recall is a condition of positiveness of the projection $T_{\mu\nu} k^\mu k^\nu$ for any future-oriented null vector \mathbf{k}), both projections are positive, hence

$$(\gamma^2 - 1)(\epsilon + p) + \pi_{\mu\nu} v^\mu v^\nu \geq 2\gamma v |q_\mu v^\mu| \geq 0 . \quad (3.46)$$

Recalling that $\mathcal{T}_{\mathcal{D}} = -16\pi G \langle N^2 ((\gamma^2 - 1)(\epsilon + p) + \pi_{\mu\nu} v^\mu v^\nu + 2\gamma q_\mu v^\mu) \rangle_{\mathcal{D}}$ (equation (3.35)), and since $2\gamma v < 2\gamma$, even the (stronger) first inequality is insufficient to conclude on the sign of $\mathcal{T}_{\mathcal{D}}$ without further assumptions on \mathbf{q} . This was to be expected since the same reasoning could be applied similarly after interchanging the roles played by \mathbf{u} and \mathbf{n} (that is, using the normal-frame decomposition of the stress-energy tensor, which replaces for instance \mathbf{q} by \mathbf{J} , and using the null vectors $k_{\pm}^{\mu} := \gamma v n^{\mu} \mp h^{\mu}_{\nu} u^{\nu}$ instead of k_{\pm}^{μ}), which exchanges $\mathcal{T}_{\mathcal{D}}$ and $-\mathcal{T}_{\mathcal{D}}$. This symmetry in the roles played by \mathbf{u} and \mathbf{n} is broken by the possibility of constraining \mathbf{q} , which is an intrinsic property of the fluid, through physical assumptions (e.g. assuming a perfect fluid), while this is not possible for the foliation-dependent vector \mathbf{J} . \square

Note that the same results hold under any of the other standard (Weak, Strong, Dominant) Energy Conditions as they all imply the Null Energy Condition [40, 66].

3.5.3 Boundary terms and global averages

As previously illustrated (see figure 2), the spatial motion of \mathcal{D} in the coordinate system (t, x^i) induces a flux of fluid elements with velocity \mathbf{V} across the boundary of the domain $\mathcal{E}_{\partial t}$, coinciding at some instant with \mathcal{D} and transported along the congruence of ∂t . In the same line of thoughts, there also exists a flux of fluid elements with velocity $\mathbf{N} + \mathbf{V} = N\mathbf{v}$ across the boundary of the domain $\mathcal{E}_{\mathbf{n}}$, coinciding with \mathcal{D} at some instant and carried along the normal congruence.

The first boundary effect is related to the choice of the spatial coordinates, and it can be made to vanish by adopting a comoving picture. The second one is generated by the tilt, that is, the deviation of the fluid 4-velocity with respect to \mathbf{n} , that translates into a tilted motion of the comoving domain boundaries with respect to the normal of the slices. It will be present in general unless the foliation is fluid-orthogonal, a foliation choice which is not possible if the fluid has non-vanishing vorticity. It is this second effect that impacts on the time variation of the Riemannian volume, as one can see upon writing expression (3.17) as

$$\frac{d}{dt} \mathcal{V}_{\mathcal{D}} = \int_{\mathcal{D}_{\mathbf{x}}} -N\mathcal{K}\sqrt{h} d^3x + \int_{\mathcal{D}_{\mathbf{x}}} (Nv^i)_{||i} \sqrt{h} d^3x = \int_{\mathcal{D}_{\mathbf{x}}} -N\mathcal{K}\sqrt{h} d^3x + \oint_{\partial\mathcal{D}_{\mathbf{x}}} Nv^i \varkappa_i d\varsigma, \quad (3.47)$$

where we have used Gauss' theorem for the second equality. Above, \varkappa is the outward-pointing unit normal vector of the boundary $\partial\mathcal{D}$, whose surface element is denoted by $d\varsigma$. This rewriting allows to clearly see how the tilt, as measured by \mathbf{v} , contributes as a boundary term to the evolution of the domain's volume.

Similar tilt-related boundary terms affect the commutation rule (3.22) and the evolution equations of the effective scale factor (3.31)–(3.32). They arise from the averages of covariant spatial three-divergences, which are boundary terms as implied by Gauss' theorem:

$$\langle A^i{}_{||i} \rangle_{\mathcal{D}} = \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}_{\mathbf{x}}} A^i{}_{||i} \sqrt{h} d^3x = \frac{1}{\mathcal{V}_{\mathcal{D}}} \oint_{\partial\mathcal{D}_{\mathbf{x}}} A^i \varkappa_i d\varsigma, \quad (3.48)$$

for any spatial vector field \mathbf{A} . These effects cannot be neglected in general; for a given fluid flow, their contribution entirely depends on the way the slices are constructed, which locally affects the amplitudes of the lapse and the tilt, and on the choice of the domain of interest (locally defining a specific boundary).

As an example, let us consider the commutation rule (3.22). Successively applying (3.48) to $\mathbf{A} = N\mathbf{v} = \mathbf{N} + \mathbf{V}$ and $\mathbf{A} = \psi N\mathbf{v}$, we can rewrite it for any scalar ψ under the following forms:

$$\begin{aligned} \frac{d}{dt} \langle \psi \rangle_{\mathcal{D}} &= \left\langle \frac{d}{dt} \psi \right\rangle_{\mathcal{D}} + \langle NK \rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}} - \left\langle \left(NK - (Nv^i)_{||i} \right) \psi \right\rangle_{\mathcal{D}} \\ &\quad - \langle \psi \rangle_{\mathcal{D}} \frac{1}{\mathcal{V}_{\mathcal{D}}} \oint_{\partial \mathcal{D}_{\mathbf{x}}} N v^i \varkappa_i d\varsigma \end{aligned} \quad (3.49)$$

$$\begin{aligned} &= \langle N n^\mu \partial_\mu \psi \rangle_{\mathcal{D}} + \langle NK \rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}} - \langle NK \psi \rangle_{\mathcal{D}} \\ &\quad + \frac{1}{\mathcal{V}_{\mathcal{D}}} \oint_{\partial \mathcal{D}_{\mathbf{x}}} \psi N v^i \varkappa_i d\varsigma - \langle \psi \rangle_{\mathcal{D}} \frac{1}{\mathcal{V}_{\mathcal{D}}} \oint_{\partial \mathcal{D}_{\mathbf{x}}} N v^i \varkappa_i d\varsigma, \end{aligned} \quad (3.50)$$

where the second expression makes use of the total coordinate-time derivative with respect to \mathbf{n} , instead of \mathbf{u} (as in the first expression), replacing $\frac{d}{dt}$ by $N n^\mu \partial_\mu$.

For simplicity, we do not make the boundary contributions explicit in the evolution equations for $a_{\mathcal{D}}$, although this could be done in the same manner. Instead, we illustrate their effect by comparing the set of averaged equations in the generic case to a restricted situation where all boundary terms cancel out. We consider to this aim the case of topologically closed spatial sections (that is, we assume that the hypersurfaces are compact three-dimensional manifolds without boundaries), and we extend the averaging domain to the full compact boundary-free hypersurface, which we denote by Σ . From (3.47), the evolution of the domain volume becomes in this case:

$$\frac{1}{\mathcal{V}_{\Sigma}} \frac{d\mathcal{V}_{\Sigma}}{dt} = - \langle NK \rangle_{\Sigma}, \quad (3.51)$$

so that the scale factor here satisfies $(da_{\Sigma}/dt)/a_{\Sigma} = - \langle NK \rangle_{\Sigma}/3$. Then, from (3.50), the *commutation rule for a global boundary-free averaging domain* can be written under the following equivalent forms:

$$\begin{aligned} \frac{d}{dt} \langle \psi \rangle_{\Sigma} &= \left\langle \frac{d}{dt} \psi \right\rangle_{\Sigma} + \langle NK \rangle_{\Sigma} \langle \psi \rangle_{\Sigma} - \left\langle \left(NK - (Nv^i)_{||i} \right) \psi \right\rangle_{\Sigma}; \\ \frac{d}{dt} \langle \psi \rangle_{\Sigma} &= \langle N n^\mu \partial_\mu \psi \rangle_{\Sigma} + \langle NK \rangle_{\Sigma} \langle \psi \rangle_{\Sigma} - \langle NK \psi \rangle_{\Sigma}, \end{aligned} \quad (3.52)$$

for any scalar ψ .

Applying Theorem 1.a to a global domain on topologically closed hypersurfaces ($\mathcal{D} = \Sigma$), we infer that the system of evolution equations (3.31)–(3.32) for the effective scale factor remains formally unchanged as written, while the global backreaction terms reduce to the following:

$$\mathcal{Q}_{\Sigma} = \left\langle N^2 \left(\mathcal{K}^2 - \mathcal{K}_{ij} \mathcal{K}^{ij} \right) \right\rangle_{\Sigma} - \frac{2}{3} \langle NK \rangle_{\Sigma}^2; \quad (3.53)$$

$$\mathcal{P}_{\Sigma} = - \langle NK n^\mu \partial_\mu N \rangle_{\Sigma} - \left\langle N^{||i} N_{||i} \right\rangle_{\Sigma}; \quad (3.54)$$

$$\mathcal{T}_{\Sigma} = - 16\pi G \left\langle N^2 \left((\gamma^2 - 1)(\epsilon + p) + 2\gamma v^\alpha q_\alpha + v^\alpha v^\beta \pi_{\alpha\beta} \right) \right\rangle_{\Sigma}, \quad (3.55)$$

thanks to the vanishing of the averages of spatial divergences (which are boundary terms) on Σ . In particular, for the calculation of the expression of \mathcal{P}_{Σ} from the

general $\mathcal{P}_{\mathcal{D}}$ (3.34), successive uses of this property provide the following equivalent expressions:

$$\mathcal{P}_{\Sigma} = \left\langle NN^{\parallel i} - \mathcal{K} \frac{dN}{dt} \right\rangle_{\Sigma} - \left\langle N(\mathcal{K} N v^i)_{\parallel i} \right\rangle_{\Sigma} ; \quad (3.56)$$

$$\mathcal{P}_{\Sigma} = - \left\langle N^{\parallel i} N_{\parallel i} + \mathcal{K} \frac{dN}{dt} \right\rangle_{\Sigma} + \left\langle \mathcal{K} N v^i N_{\parallel i} \right\rangle_{\Sigma} . \quad (3.57)$$

The backreaction formulae (3.53)–(3.55) can be compared with the expressions in the general case, (3.33)–(3.35): the differences are the boundary contributions to the backreactions, erased when $\mathcal{D} = \Sigma$. These include all explicit contributions of the tilt vector v to the kinematical and dynamical backreactions, which have disappeared in the above expressions (3.53)–(3.54). The alternative expressions (3.56)–(3.57) for the dynamical backreaction when $\mathcal{D} = \Sigma$ show, nevertheless, that the tilt vector still manifests itself through the difference between coordinate-time evolutions along the fluid flow $\frac{d}{dt}$ and along the hypersurface-orthogonal flow $N n^{\mu} \partial_{\mu}$, here regarding the lapse N . Moreover, the existence of a tilt still influences the dynamics of the effective scale factor through the stress-energy backreaction, which is unchanged whether the domain has boundaries or not. Indeed, the stress-energy backreaction is not a boundary effect but instead a manifestation of, e.g., the *local* difference between the rest frame energy of the fluid and its energy as measured in the normal frames.

The integrability condition and the averaged energy conservation law for an average performed over a closed hypersurface are, respectively, deduced from relations (3.36) and (3.37) without change. The same terms are involved, since no explicit three-divergence term appears in these two expressions. However, the backreactions appearing in the integrability condition should again be replaced by their simplified expressions above.

3.5.4 Relations to the literature: global averages

The averaged equations and the commutation rule that we obtained in the particular case $\mathcal{D} = \Sigma$ are equivalent to those derived by Räsänen in [54],¹⁰ where all averages were taken on the whole boundary-free hypersurface (which was not assumed to be topologically closed and compact; instead, the existence of the averages was implied by an assumption of statistical homogeneity of the spatial hypersurfaces). The above average equations for the $\mathcal{D} = \Sigma$ case are also identical to those obtained by Tanaka & Futamase in [64] (following from [43] and supplementing their equations with the contributions of the cosmological constant), while the commutation rule was not explicitly given in these papers. Periodic boundary conditions were assumed, so that the situation considered was equivalent to a global averaging over hypersurfaces with a closed 3-torus topology. The vanishing

¹⁰This is not obvious at first glance, due to a different choice of the scalars that have been averaged, i.e. in contrast to our case the averaged quantities in [54] do not involve the factor N^2 . Hence, the averaged equations do not appear identical to those obtained in the present work, and to see that they are equivalent the use of the corresponding local equations is necessary. The notations also differ (mostly because the description adopted in [54] is explicitly 4-covariant); one should take care in particular of the fact that in [54] the notation ∂_t is used for the coordinate-time covariant derivative along \mathbf{n} (i.e. $N n^{\mu} \nabla_{\mu}$ in the notations of the present work), rather than for the coordinate-time partial derivative $\partial_t|_{x^i}$.

shift considered in these papers does not affect the results since, as seen above, this vector does not contribute to the local and average dynamics.

One also recovers the same averaged equations and commutation rule as in subsection 3.5.3 above by restricting in the same way the expressions obtained by Brown *et al.* in [10] to the compact boundary-free domain case (whereas it is not the case for the results of Larena in [46] due to the different choice of scale factor). More surprisingly, the averaged and commutation relations derived by Gasperini *et al.* in [38] (or by Smirnov in [61] within the same formalism) remain formally similar to the equations we get in our boundary-free $\mathcal{D} = \Sigma$ case hereabove, even when applied to a general domain. This originates from the different propagation of the averaging domain, which in [38,61] is chosen to be along the flow of \mathbf{n} ; accordingly, the natural time derivative in their approach is $Nn^\mu\partial_\mu$ (in the notations of the present work). This similitude (or, equivalently, the fact that the averaged equations and commutation rule of [38,61] are formally unchanged by restricting them to the case $\mathcal{D} = \Sigma$) indeed shows that boundary terms only occur when the domain's boundaries follow a tilted flow with respect to the normal to the hypersurfaces in which the domain is embedded. There is no such tilt in the domain propagation in [38,61], hence boundary terms are absent, despite the non-vanishing local tilt vector between the fluid and normal flows. As in our case, this local tilt still influences the dynamics *via* the difference in energy density and pressure between the local frames orthogonal to each of these flows.

3.5.5 Relations to the literature: transport of the averaging domain

In the more generic case of an averaging domain not covering the whole hypersurface, its time propagation needs to be specified. Three choices in particular, determined by the three congruences we introduced (see figure 1), may appear as ‘natural’ definitions of the transport of the averaging domain.

The first choice is to assume a domain evolving along the congruence of the coordinate frames ∂_t . This is the situation implicitly considered by Larena [46] and Brown *et al.* [10] (see also the respective applications of these papers in [65] and [11,12]). Such a construction picks up two important issues: first, given a particular choice of shift, the vectors ∂_t and \mathbf{u} will not be collinear in general, hence there will be a flow of fluid elements across the domain boundary. This calls the physical relevance of the averaged system into question as the domain will not encompass the same collection of fluid elements throughout its evolution, i.e. it will not conserve its rest mass content. Second, for the same spacetime and the same foliation, the location of the domain at a given time will depend on the choice of the shift vector, as it determines the direction of ∂_t . This leads to an unphysical dependency of the averaged system (hence, of all spatial average properties) on the choice of the spatial coordinates and on the way they propagate.

The second choice is to assume a spatial domain evolving along the integral curves of the normal frames \mathbf{n} . This is the configuration considered by Gasperini *et al.* [38] (see also the follow-up paper [49]). Their averaging formalism, as introduced in [37], is based on the construction of a spacetime window function characterizing the averaging domain to be considered, and is written in manifestly 4-covariant form. While this formalism is suitable for a freely specifiable propagation of the domain boundaries, the averaged system of equations derived in [38], both in

4-covariant and 3 + 1 forms, has assumed a transport along \mathbf{n} (see equation (3.2) therein).¹¹

This choice of propagation was also the one adopted by Smirnov [61] and Beltrán Jiménez *et al.* [7]. In these papers, \mathbf{n} is assumed to be geodesic and to correspond to the 4-velocity of an irrotational non-interacting dust contribution to the stress-energy tensor, in contrast to [38] where this normal vector was freely specifiable. The formalism of Smirnov is otherwise close to that of Gasperini *et al.* [38], from which it is directly inspired, with both 4-covariant and 3 + 1 forms of the averaged equations. Beltrán Jiménez *et al.* [7] consider a 3 + 1 description, with a vanishing shift and a trivial lapse ($N = 1$) but still tilted fluid flows, and their domain actually follows both ∂_t and \mathbf{n} as the vanishing shift makes these two directions identical.

The choice of a domain transport along \mathbf{n} leads to formally simpler averaged equations in terms of the geometric variables of the foliation due to the vanishing boundary terms (see subsection 3.5.4). It also makes the propagation of the averaging domain independent of the propagation of the spatial coordinates, but this propagation becomes instead dependent on the choice of the foliation which defines the vector \mathbf{n} . One could argue that such a dependence is inherently present in any spatial averaging scheme, since the domain of averaging lies by definition within the hypersurfaces built from the foliation. However, the dependence we refer to can be understood from a spacetime perspective: by changing the foliation, and hence the vector \mathbf{n} , the four-dimensional tube spanned by the domain transported along this vector will not remain the same (see figure 3). We also notice that the second drawback mentioned previously for an evolution along ∂_t also holds for a transport along \mathbf{n} : the rest mass of the fluid within the domain will generically not be conserved, as the particle content of the domain will be altered during its evolution.

Two similar generalization schemes have been suggested by Räsänen in [54] (see also the application [55]), and by Kasai *et al.* in [43] followed by Tanaka & Futamase in [64], where such issues related to the propagation of the domain boundaries are avoided. However, in both cases this requires specific choices of the averaging domain that restrict the scope to large scales and to a class of foliations where the assumptions made in these papers can hold. Räsänen [54] derives the averaged equations in a 4-covariant form for a domain covering the whole hypersurfaces, thus without the need for specifying its propagation. The convergence of the averages for such an infinite domain is ensured by the assumption of statistical homogeneity to hold in these hypersurfaces. In turn, the system of averaged equations obtained by Tanaka & Futamase in [64] (slightly generalizing that of [43]) requires a domain and foliation where periodic boundary conditions can be assumed. A system of averaged equations is given in a background-independent scheme as a preliminary step in [43, 64]. However, the emphasis is subsequently

¹¹Accordingly, and in contrast to a statement of [38], the resulting averaged system of equations, as expressed in 3 + 1 form, is *not* identical to that of Brown *et al.* [10] for a non-vanishing shift, as in this latter study the domain is transported along ∂_t . This becomes true if a vanishing shift is chosen, due to the proportionality of \mathbf{n} and ∂_t in this case. As correctly stated, however, the averaged system of equations in 3 + 1 form of [38] becomes identical to that of Paper II for an irrotational perfect fluid if the fluid rest frames are used to generate the spatial hypersurfaces. This is indeed expected as in this case $\mathbf{n} = \mathbf{u}$, hence the domain has the same (fluid-comoving) evolution as in Paper II.

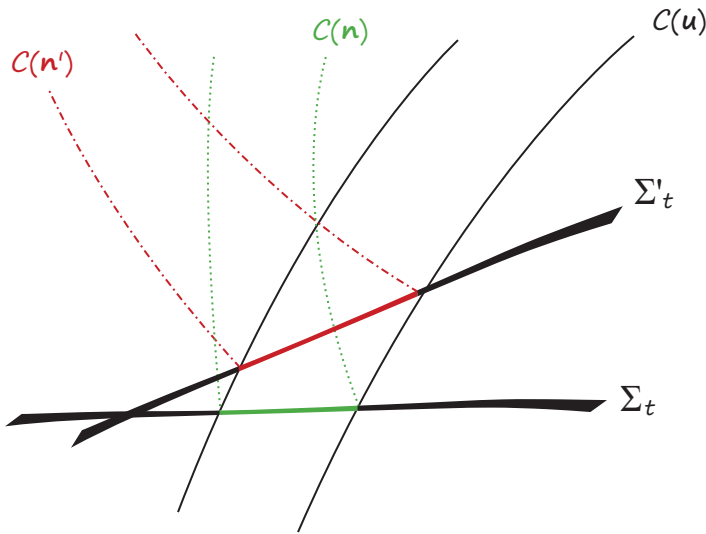


Fig. 3 We here illustrate the situation where the propagation of the averaging domain is chosen so as to follow the normal of the hypersurfaces at stake. For the foliation of slices Σ_t , the domain locus is described by the associated normal congruence $\mathcal{C}(n)$ (green dotted lines). For another foliation of slices Σ'_t , it is described by the normal congruence $\mathcal{C}(n')$ (red dash-dotted lines), which differs in general from $\mathcal{C}(n)$. The spatial domain selected in this way at time t by each foliation is represented by the continuous-line colored section of the corresponding hypersurface, Σ_t or Σ'_t . Choosing a domain transport along the normal of the hypersurfaces constructs different four-dimensional tubes, corresponding to different physical systems, for different foliations. It will also imply a flow of fluid elements across the domain boundary in general.

put on linear perturbation theory at a Friedmannian background, on which the main conclusions are based. Accordingly, no or negligible contributions from backreaction are found in this setting, which is expected due to the nonlinear and background-free nature of backreaction.¹² The transport of the averaging domain is not specified; this does not affect the results due to the vanishing of any boundary term. Comparing with [38] and in view of the discussion above in subsection 3.5.4, we conclude that the results obtained in both latter schemes [54, 64] would remain valid in a general foliation, and for any domain, provided it is required that its boundaries propagate along \mathbf{n} (which would also be a propagation along ∂_t in [64] in view of the vanishing shift vector choice) in order to prevent the occurrence of extra boundary terms. A wider applicability of the schemes would thus be recovered, but the drawbacks highlighted above for such a propagation would also be retained.

The third choice, which we adopt in the present work, is that of a domain comoving with the fluid. As its boundaries follow the fluid flow \mathbf{u} , the averaging domain always sweeps out the same four-dimensional tube of spacetime, whatever the choice of the foliation and spatial coordinates. This option also ensures, by def-

¹²We emphasize that mixing background-dependent applications with a background-free framework may imply strong restrictions, e.g. the small backreaction found by Russ *et al.* [59] in second-order perturbation theory at a Friedmannian background must in reality vanish due to the geometric constraints imposed (see the comments in Paper I [15], Sect. 3.4.).

inition, that the domain encloses the same collection of fluid elements throughout its evolution, which in turn implies the conservation of the fluid rest mass within \mathcal{D} . Choosing such a domain propagation therefore avoids all of the drawbacks mentioned above. It should be noted, however, that the advantage of rest mass conservation within the spatial domain would not hold, in general, for the averaged description of a model universe filled with several fluids. A multi-fluid approach would require to pick up and follow one preferred fluid congruence, preserving the corresponding rest mass only, while allowing the others to flow across the domain boundaries, see e.g. [30]. However, the rest mass within the domain could be conserved simultaneously for every fluid only by assuming that the 4-velocities of all fluids coincide, at least at the domain boundary,¹³ or that the spatial domain is extended to the whole hypersurface. In the present work we consider a cosmological model sourced by a single fluid, which should satisfactorily account for the description of the main cosmological epochs largely dominated by a particular fluid (radiation or dust).

3.5.6 Relations to the literature: comparison of the final averaged equations

Most authors cited in the above discussion base their studies either on a direct 3 + 1 formulation of the evolution and averaged equations, or on a formulation using explicitly 4-covariant terms from which a 3 + 1 form is explicitly deduced. This allows for a rather direct comparison with the formalism and results presented so far in this paper (section 3.4).¹⁴

All of the corresponding systems of 3 + 1 averaged equations are manifestly different from the one we obtain in subsection 3.4 due to the different propagation of the averaging domain. However, we notice a formal similarity between the commutation rule (3.22) and the system of dynamical equations for the effective scale factor (3.31)–(3.32) we present, and those of Brown *et al.* [10]. The tilt vector pondered by the lapse $N\mathbf{v}$ appearing in several terms in the commutation rule and backreaction formulas would there be formally replaced by the shift vector \mathbf{N} , both representing the deviation of the vector flow followed by the domain (respectively \mathbf{u} and ∂_t) to the normal to the slices \mathbf{n} in the corresponding framework. Similarly, the time derivative d/dt along \mathbf{u} would be replaced by the time derivative $\partial_t|_{x^i}$ along ∂_t . This allows to easily see that both systems of equations become equivalent in the special case of a comoving description (within which $N\mathbf{v} = \mathbf{N}$ and $d/dt = \partial_t|_{x^i}$), as expected since in this case the spatial coordinates are chosen in such a way that both domains follow the same flow $\partial_t \propto \mathbf{u}$. Despite the same domain propagation choice, the averaged equations of Larena [46] remain different from the former even in a comoving picture due to a different notion

¹³The averaged equations are in general defined for arbitrary domains. If an assumption is adopted that distinct fluid congruences coincide or “average out” on the boundary, the arbitrariness of the domain choice has to be given up.

¹⁴The averaged energy conservation equation and the integrability condition (see subsection 3.4.2 above) are not always considered. The 3 + 1 approach of Beltrán Jiménez *et al.* [7] differs from the one used here in that it does neither include lapse nor shift, while Tanaka & Futamase [64] consider a nontrivial lapse along with a vanishing shift. In the approach of Räsänen [54], the formulation is only given in explicitly 4-covariant terms; also in this case can a 3 + 1 formulation be readily deduced, for comparison with the above averaged equations, upon making a coordinate choice including the appropriate time t .

of effective scale factor.¹⁵ Finally, as already discussed, the choice of a domain propagating along the normal to the slices (or in the last two cases, the use of global assumptions on the domain that erase boundary terms, yielding the same evolution) made by Gasperini *et al.* [38], Beltrán Jiménez *et al.* [7], Smirnov [61], Tanaka & Futamase [64] and Räsänen [54] would require to take either global averages or fluid-orthogonal hypersurfaces (when possible) in each case to make the averaged equations of these papers equivalent to those derived in the above section 3.4.

The reader may find a complete comparison of the averaging formalisms discussed above in Appendix C and synthetic tables therein.

4 Rest mass-preserving scalar averaging: fluid-intrinsic approach

In this section we propose an alternative averaging procedure aimed at characterizing average properties that are fully intrinsic to the fluid. We start with presenting the motivations for this approach.

4.1 Motivation for a fluid-intrinsic averaging procedure

In the previous section we learned that most of the literature on the generalization of spatially averaged cosmologies for arbitrary foliations abandons the intrinsic fluid averaging approach that was a primary element of Papers I and II. Instead, the averaging procedures considered were built from averaging domains evolving along the normal congruence of the hypersurfaces of arbitrary foliations. We pointed out that this choice inherits problems with regards to the foliation-dependent evolution of the domain,¹⁶ and especially the non-conservation of the rest mass of the averaging domain in general situations. These problems are avoided for our choice of a comoving domain of averaging, i.e. of a domain transported along the fluid congruence.

The approach we presented in section 3 complies, however, with the definition of the averaging operation, and with the set of foliation-related local variables explicitly appearing in the equations, adopted in the aforementioned literature

¹⁵Such additional differences with the results of [46] arise from a definition of the effective scale factor in this latter study that makes its evolution different from that of the cubic root of the domain's volume. Since the aim of an averaging framework is to investigate the regional dynamics of comoving domains lying within spatial hypersurfaces, it appears to be more appropriate to define the scale factor from their volume. The reader may refer to [65] for a comparison of the different averaged energy constraints obtained for different choices of $a_{\mathcal{D}}$, and for an analysis of the backreaction effects obtained for each choice in a Friedmann-Lemaître-Robertson-Walker (FLRW) model perturbed up to second order. Note, however, that in these studies the domain also follows the congruence of the coordinate frames along ∂_t , implying the drawbacks already highlighted in section 3.5.5.

¹⁶We emphasize that the averaged dynamics and the definitions of backreaction terms in this approach involve the extrinsic curvature, which depends on derivatives of the normal vector. Even if the tilt measuring the deviation of the 4-velocity with respect to the normal is small (the Lorentz factor is close to unity), its derivatives can be large. This may lead to a strong foliation dependence of the averaged variables and backreaction terms that is to be considered irrelevant for a cosmological model, since in such an approach these quantities only characterize properties of a family of extrinsic observers.

(although some ‘mixed’ fluid and foliation scalars such as $h^{\mu\nu}\nabla_\mu u_\nu$ have also been used by Larena [46]). This *extrinsic* approach could be employed to measure the deviations from the dynamics of a homogeneous-isotropic model universe in a geometric way, since it focuses on averages of foliation-dependent scalars characterizing the hypersurfaces such as the respective traces of the extrinsic and intrinsic curvatures. We argue, however, that intrinsic properties of the fluid content such as those measured by the rest frame kinematic quantities Θ , σ^2 and ω^2 , defined in subsection 2.2.2, are relevant for the characterization of an effective cosmological model. It is not only a philosophical question to consider as a viable cosmology the evolution of averaged fluids formulated in its own variables, rather than looking at averages ‘from outside’ that mostly focus on the study of geometrical properties of the hypersurfaces. This risks invoking a quasi-Newtonian understanding of a moving fluid with respect to some fiducial external spacetime.

Having said this, the reader may point out that focusing on the properties of the fluid congruence is more reminiscent of a 1 + 3 (*threading*) point of view. Indeed, we employ in this work a 1 + 3 threading formalism, but jointly with a 3 + 1 foliation, simply because hypersurfaces are needed for the averaging operation. Going as far as possible toward a fluid-intrinsic description avoids an excessive foliation-dependence of the variables considered. However, this goal will encounter limitations, since the rest frames of a vortical fluid are not hypersurface-forming. A fully intrinsic construction of effective cosmologies will thus in general require other choices. The foliation at constant fluid proper time, as part of the Lagrangian description (see subsection 2.4.2) allows for a spatial averaging over hypersurfaces that are built from the fluid flow itself. Another possibility that is opened with the intrinsic approach would be to characterize hypersurfaces statistically. This strategy will be discussed in subsection 5.3.3.

As a first step toward an intrinsic approach, we present in Appendix B a re-expression of the extrinsic evolution equations (3.31)–(3.32) in terms of the fluid’s intrinsic variables. This provides more insight into the contributions of these quantities to the averaged dynamics, in particular the influence of the vorticity can be better understood, but it also raises additional contributions from the tilt factor γ . In the following, we shall go another route heading toward an intrinsic fluid point of view. For this aim we introduce a slightly different generalization of the fluid-orthogonal averaging formalism of Papers I and II that will also allow us to derive a more compact form of averaged cosmologies. We first motivate this route by contemplating on the conservation of the rest mass of the fluid.

4.1.1 Regional rest mass conservation

We have shown in subsection 3.3 that the total fluid rest mass within the domain \mathcal{D} , $M_{\mathcal{D}} = \int_{\mathcal{D}} M^\mu d\sigma_\mu$, with $M^\mu = \rho u^\mu$ the conserved rest mass flux vector, is preserved in time ($dM_{\mathcal{D}}/dt = 0$) as a consequence of the domain’s fluid-comoving propagation. We have also shown that $M_{\mathcal{D}}$ can be expressed in terms of the volume and averaging operator introduced by (3.18) as follows:

$$M_{\mathcal{D}} = \int_{\mathcal{D}} \gamma \rho \sqrt{h} d^3x = \mathcal{V}_{\mathcal{D}} \langle \gamma \rho \rangle_{\mathcal{D}}. \quad (4.1)$$

The relevant scalar to be integrated over the spatial domain is therefore $\gamma \rho$, rather than the rest mass density ρ as it could have been expected. Unless the foliation

is fluid-orthogonal ($\gamma = 1$), the quantity $\int_{\mathcal{D}} \varrho \sqrt{h} d^3x = \mathcal{V}_{\mathcal{D}} \langle \varrho \rangle_{\mathcal{D}}$ is not the fluid rest mass within \mathcal{D} and accordingly is not conserved. Indeed, using the continuity equation (3.28) for ϱ as well as the commutation rule (3.23) and the associated volume evolution rate expression, we have

$$\frac{d}{dt} \left(\mathcal{V}_{\mathcal{D}} \langle \varrho \rangle_{\mathcal{D}} \right) = -\mathcal{V}_{\mathcal{D}} \left\langle \frac{1}{\gamma} \frac{d\gamma}{dt} \varrho \right\rangle_{\mathcal{D}}. \quad (4.2)$$

The need to account for the factor γ is a consequence of the conserved ϱ being a rest mass density of the fluid in its local rest frames. It is thus a density with respect to the measure of proper volume of the fluid elements, while $\gamma\varrho$ is the corresponding density with respect to the (Lorentz-contracted) normal frames volume measure $\sqrt{h} d^3x$ used in the definition of the extrinsic volume averaging operator $\langle \cdot \rangle_{\mathcal{D}}$.

The total fluid rest mass within the domain is alternatively obtained by integrating the rest mass density per unit of fluid proper volume, ϱ , with the corresponding fluid rest frames volume element, $\sqrt{b} d^3x$ with $b := \det(b_{ij})$: given the relation between the determinants b and h ,

$$\begin{aligned} b &= \det(g_{ij} + u_i u_j) = \det(h_{ij} + u_i u_j) = h \det(\delta^i_j + h^{ik} u_k u_j) \\ &= h (1 + h^{ij} u_i u_j) = h (1 + h^{\mu\nu} u_\mu u_\nu) = h \gamma^2, \end{aligned} \quad (4.3)$$

we have $\sqrt{b} d^3x = \gamma \sqrt{h} d^3x$, and therefore

$$M_{\mathcal{D}} = \int_{\mathcal{D}} \varrho \sqrt{b} d^3x. \quad (4.4)$$

The rest mass of the fluid $M_{\mathcal{D}}$ is thus more naturally defined in terms of the proper volume measure $\sqrt{b} d^3x$.

Note that the two covariant¹⁷ volume measures $\sqrt{h} d^3x$ and $\sqrt{b} d^3x$ coincide in the case of a flow-orthogonal foliation (possible for an irrotational fluid), which is the situation considered in Papers I and II. A degeneracy between both volumes is present in these papers, while they are distinct for any other choice of foliation. This is similar to the difference between hypersurface-orthogonal and fluid-comoving propagation choices for the averaging domain, that emerges outside the fluid-orthogonal foliation framework of Papers I and II where both choices can be made simultaneously. We have argued above that once this distinction needs to be done, preserving the comoving character of the domain propagation is the relevant choice for a physical description of average properties of a regional subset of the fluid. Here we also notice that keeping a volume measure that corresponds to a proper volume for the fluid appears to be the most suited to describe the integrated contribution of variables that are primarily defined from the fluid's rest frames, as, e.g. for the expression of the mass within the domain from the rest mass density ϱ .

We shall accordingly introduce a new volume for the domain and a new averaging operator based on the fluid proper volume element. It will allow us to

¹⁷As $\sqrt{h(t, x^k)} d^3x$, the fluid-orthogonal volume 3-form $\sqrt{b(t, x^k)} d^3x$ is also invariant under a change of spatial coordinates, as can be checked either directly or by rewriting it as $\gamma(t, x^k) \sqrt{h(t, x^k)} d^3x$, $\gamma = -n^\mu u_\mu$ being a 4-scalar. It reads in particular $\sqrt{b(t, x^i)} d^3x = \sqrt{b(t, f^i(t, \mathbf{X}))} J(t, X^i) d^3X$ in comoving spatial coordinates X^i , with $b(t, f^i(t, \mathbf{X})) J(t, X^i)^2$ being the determinant of the spatial components of the fluid rest frame projector \mathbf{b} in the comoving coordinate system (t, X^i) .

define these notions intrinsically from the source content, leaving only the integration itself as based on the foliation choice since the spatial integration domain lies within a hypersurface.¹⁸ We will also recover the expected relation between mass and averaged rest mass density.

4.1.2 Intrinsic averaging operator

We consider as before a compact domain \mathcal{D} transported along the fluid flow lines and contained within hypersurfaces of normal the unit timelike vector field \mathbf{n} . Instead of using the hypersurface Riemannian domain volume $\mathcal{V}_{\mathcal{D}}^h$, where the superscript h is used for clarity, we introduce the proper volume of the fluid elements within \mathcal{D} :

$$\mathcal{V}_{\mathcal{D}}^b(t) := \int_{\mathcal{D}} u^\mu d\sigma_\mu = \int_{\mathcal{D}} \gamma(t, x^i) \sqrt{h(t, x^i)} d^3x = \int_{\mathcal{D}} \sqrt{b(t, x^i)} d^3x, \quad (4.5)$$

and we define the *fluid-intrinsic average* over \mathcal{D} of any scalar ψ as¹⁹

$$\begin{aligned} \langle \psi \rangle_{\mathcal{D}}^b &:= \frac{1}{\mathcal{V}_{\mathcal{D}}^b} \int_{\mathcal{D}} \psi u^\mu d\sigma_\mu = \frac{1}{\mathcal{V}_{\mathcal{D}}^b} \int_{\mathcal{D}} \psi(t, x^i) \gamma(t, x^i) \sqrt{h(t, x^i)} d^3x \\ &= \frac{1}{\mathcal{V}_{\mathcal{D}}^b} \int_{\mathcal{D}} \psi(t, x^i) \sqrt{b(t, x^i)} d^3x. \end{aligned} \quad (4.6)$$

In other words, we make use of the restriction to the slices of the Hodge dual $\star \mathbf{u}$ of the 1-form \mathbf{u} as the volume 3-form, rather than the volume 3-form similarly built from \mathbf{n} that was used for the extrinsic definitions of section 3.

Similarly to the extrinsic hypersurface averager of section 3, we recover from (4.5) and (4.6) the volume and averager of Papers I and II when considering a foliation orthogonal to an irrotational fluid flow. The two averaging schemes can be formally related as

$$\mathcal{V}_{\mathcal{D}}^b = \mathcal{V}_{\mathcal{D}}^h \langle \gamma \rangle_{\mathcal{D}}^h; \quad \langle \psi \rangle_{\mathcal{D}}^b = \frac{\langle \gamma \psi \rangle_{\mathcal{D}}^h}{\langle \gamma \rangle_{\mathcal{D}}^h}, \quad (4.7)$$

for any scalar ψ , where we label the extrinsic averaging operator used throughout section 3 with a superscript h for a more explicit distinction. This shows the identity of both operators in the absence of tilt ($\gamma = 1$), and their approximate identity in the case of a small tilt, i.e. of non-relativistic Eulerian velocities of the fluid in the chosen foliation ($\gamma \simeq 1$).

¹⁸In the following we shall emphasize the choice of a proper time foliation that, in particular, forms hypersurfaces that are themselves defined intrinsically from the fluid (up to the choice of an initial hypersurface).

¹⁹Note that the intrinsic averager (4.6) can be obtained in the framework of Gasperini *et al.* [38] by rewriting their window function, $W_\Omega = n^\mu \nabla_\mu (\mathcal{H}(A(x^\alpha) - A_0)) \mathcal{H}(r_0 - B(x^\alpha))$, as $W_\Omega = u^\mu \nabla_\mu (\mathcal{H}(A(x^\alpha) - A_0)) \mathcal{H}(r_0 - B(x^\alpha))$. Considering in addition the constraint $u^\mu \partial_\mu B = 0$, which defines a comoving domain propagation, yields the same averaged system that we are going to derive in the present section. (See [41] for a detailed analysis.)

4.2 Intrinsic effective dynamics of general fluids seen in general foliations

4.2.1 Fluid-intrinsic volume measure and averager: time evolution

Starting with the reformulation of the extrinsic averaging scheme in Appendix B, we obtain from the volume evolution rate (B.3), the commutation rule (B.4), and the above relations between both schemes (4.7), the evolution rate of the fluid-intrinsic volume:

$$\frac{1}{\mathcal{V}_{\mathcal{D}}^b} \frac{d}{dt} \mathcal{V}_{\mathcal{D}}^b = \left\langle \frac{N}{\gamma} \Theta \right\rangle_{\mathcal{D}}^b = \langle \tilde{\Theta} \rangle_{\mathcal{D}}^b, \quad (4.8)$$

where we have introduced the rescaled scalar expansion rate $\tilde{\Theta} := (N/\gamma)\Theta$. Since $N/\gamma = d\tau/dt$, $\tilde{\Theta}$ can be seen as the fluid's local expansion rate with respect to the coordinate time t , while Θ expresses this rate with respect to the proper time τ .

This evolution rate formula can alternatively be derived in the same way as it was done for the hypersurface Riemannian domain volume in subsection 3.2.2, changing the spatial coordinates to comoving ones in the integral to commute integration and comoving coordinate-time derivative. Using the invariance of the fluid rest frame volume form with respect to such a spatial diffeomorphism, we then get the above result through the second equality of relation (2.39) holding in comoving coordinate systems.

Both methods can be equally used to obtain a new commutation rule for the intrinsic averager, which we now express in the form of a *Lemma*:

Lemma 2 (commutation rule for fluid-intrinsic volume averages)

The commutation rule between fluid-intrinsic averaging on a compact domain \mathcal{D} , lying within the t -constant hypersurfaces and comoving with the fluid, and comoving differentiation with respect to the coordinate time reads, for any $3 + 1$ foliation of spacetime and for any scalar ψ :

$$\frac{d}{dt} \langle \psi \rangle_{\mathcal{D}}^b = \left\langle \frac{d}{dt} \psi \right\rangle_{\mathcal{D}}^b - \langle \tilde{\Theta} \rangle_{\mathcal{D}}^b \langle \psi \rangle_{\mathcal{D}}^b + \langle \tilde{\Theta} \psi \rangle_{\mathcal{D}}^b. \quad (4.9)$$

This simple relation is independent of the shift due to the spatial coordinate-independent definitions of the domain propagation and averaging procedure; it only depends on the lapse and the tilt through the threading lapse factor N/γ in $\tilde{\Theta}$, rescaling the proper time evolutions to coordinate-time evolutions.

Within this fluid-intrinsic averaging scheme, the averaged rest mass density takes the expected form:

$$\langle \varrho \rangle_{\mathcal{D}}^b = \frac{M_{\mathcal{D}}}{\mathcal{V}_{\mathcal{D}}^b}. \quad (4.10)$$

From the commutation rule (4.9) and the continuity equation $d\varrho/dt + \tilde{\Theta}\varrho = 0$, we obtain $d(\mathcal{V}_{\mathcal{D}}^b \langle \varrho \rangle_{\mathcal{D}}^b)/dt = 0$, which shows again the preservation of the domain rest mass $M_{\mathcal{D}}$.

4.2.2 Averaged evolution equations

We define the effective scale factor of the fluid body via the intrinsic domain volume:

$$a_{\mathcal{D}}^b(t) := \left(\frac{\mathcal{V}_{\mathcal{D}}^b(t)}{\mathcal{V}_{\mathcal{D}_i}^b} \right)^{1/3}, \quad (4.11)$$

so that its rate of change yields the averaged expansion rate as seen in coordinate time t :

$$H_{\mathcal{D}}^b := \frac{1}{a_{\mathcal{D}}^b} \frac{da_{\mathcal{D}}^b}{dt} = \frac{1}{3} \left\langle \tilde{\Theta} \right\rangle_{\mathcal{D}}^b. \quad (4.12)$$

Equivalently, the rate of change of the fluid scale factor can be defined as

$$H_{\mathcal{D}}^b := \left\langle \frac{1}{\ell} \frac{d\ell}{dt} \right\rangle_{\mathcal{D}}^b, \quad (4.13)$$

with ℓ being the representative length lying in the rest frames of the fluid, and satisfying

$$\frac{\dot{\ell}}{\ell} = \frac{1}{3} \Theta, \quad \text{or} \quad \frac{1}{\ell} \frac{d\ell}{dt} = \frac{1}{3} \tilde{\Theta}. \quad (4.14)$$

In other words, ℓ denotes the spatial isotropic deviation of two neighbouring fluid elements.²⁰

Instead of using the Einstein equations projected along \mathbf{n} , yielding equations (3.2) and (3.4) (expressed in terms of the intrinsic and extrinsic curvature of the hypersurfaces), we here express the local dynamics of the fluid directly through the Raychaudhuri equation:

$$\dot{\Theta} = -\frac{1}{3} \Theta^2 - 2\sigma^2 + 2\omega^2 + \nabla_{\mu} a^{\mu} - 4\pi G(\epsilon + 3p) + \Lambda, \quad (4.15)$$

obtained from a projection of the Einstein equations along \mathbf{u} ; it relates rest frame kinematic and dynamical scalars of the fluid, thus being relevant for the present fluid-focussed approach. It can be complemented by an analogue in terms of fluid-intrinsic quantities of the foliation-related energy constraint (3.4) by defining a ‘fluid rest frame 3–curvature’ scalar \mathcal{R} from the 4–Ricci tensor $R_{\mu\nu}$ and scalar R , following Ellis et al. [34], as follows:

$$\mathcal{R} := \nabla_{\mu} u^{\nu} \nabla_{\nu} u^{\mu} - \nabla_{\mu} u^{\mu} \nabla_{\nu} u^{\nu} + R + 2 R_{\mu\nu} u^{\mu} u^{\nu}. \quad (4.16)$$

Noting that the covariant derivatives above can be equivalently replaced by their projections orthogonal to \mathbf{u} ($\nabla_{\rho} u^{\sigma} \mapsto b_{\rho}^{\kappa} b^{\sigma}_{\tau} \nabla_{\kappa} u^{\tau}$), the scalar Gauss equation [1,

²⁰The difference to the averager used in section 3 can be made explicit by introducing l as the counterpart of ℓ :

$$\frac{1}{l} \frac{dl}{dt} = \frac{1}{3} \left(\tilde{\Theta} - \frac{1}{\gamma} \frac{d\gamma}{dt} \right) = \frac{1}{\ell} \frac{d\ell}{dt} - \frac{1}{3} \frac{1}{\gamma} \frac{d\gamma}{dt}.$$

We thus have $l^3 = \ell^3/\gamma$, i.e. l is a length (cubic root of a volume) associated with the volume contraction of ℓ by the Lorentz factor γ (lengths are contracted by γ in one spatial direction and are not affected in the other orthogonal two, implying a factor $\gamma^{1/3}$ for the isotropically averaged length contraction).

39] applied to the \mathbf{u} -orthogonal hypersurfaces when those exist (i.e. for vanishing vorticity) shows that \mathcal{R} corresponds in this case to the scalar intrinsic curvature of these hypersurfaces. For non-zero vorticity, such hypersurfaces cannot be built, and \mathcal{R} is not transparently interpreted as a scalar curvature.²¹ It should be kept in mind that it does not in general correspond to the intrinsic curvature \mathcal{R} of the \mathbf{n} -orthogonal hypersurfaces in which the domain \mathcal{D} is embedded.

Inserting the trace of the Einstein equations and their projection along \mathbf{u} in the definition (4.16) of \mathcal{R} allows to relate it to the fluid's rest frame energy density within a constraint equation where the covariant derivative of \mathbf{u} has been decomposed into its kinematic parts:

$$\frac{2}{3}\Theta^2 - 2\sigma^2 + 2\omega^2 + \mathcal{R} = 16\pi G\epsilon + 2\Lambda. \quad (4.17)$$

Analogously to what has been done in section 3 within the extrinsic averaging scheme, we can now apply the fluid-intrinsic averager to equations (4.17) and (4.15) multiplied by $(N/\gamma)^2$ and use expression (4.8) for the evolution rate of $\mathcal{V}_{\mathcal{D}}^b$ as well as the commutation rule (4.9) to obtain the effective evolution equations of the scale factor $a_{\mathcal{D}}^b$. We formulate them in the following *Theorem* in terms of rescaled variables defined similarly to $\tilde{\Theta}$: rescaled kinematic variables, $\tilde{\sigma}^2 := (N/\gamma)^2\sigma^2$ and $\tilde{\omega}^2 := (N/\gamma)^2\omega^2$, dynamical variables, $\tilde{\epsilon} := (N/\gamma)^2\epsilon$ and $\tilde{p} := (N/\gamma)^2p$, acceleration 4-divergence, $\tilde{\mathcal{A}} := (N/\gamma)^2\mathcal{A}$ with $\mathcal{A} := \nabla_{\mu}a^{\mu}$, and fluid 3-curvature, $\tilde{\mathcal{R}} := (N/\gamma)^2\mathcal{R}$.

Theorem 2.a (fluid-intrinsically averaged evolution equations)

The evolution equations for the effective scale factor of the fluid body within a compact and comoving regional spatial domain \mathcal{D} of an inhomogeneous general fluid, and for any 3 + 1 foliation of spacetime, read:

$$3 \frac{1}{a_{\mathcal{D}}^b} \frac{d^2 a_{\mathcal{D}}^b}{dt^2} = -4\pi G \langle \tilde{\epsilon} + 3\tilde{p} \rangle_{\mathcal{D}}^b + \tilde{\Lambda}_{\mathcal{D}}^b + \tilde{\mathcal{Q}}_{\mathcal{D}}^b + \tilde{\mathcal{P}}_{\mathcal{D}}^b; \quad (4.18)$$

$$3 (H_{\mathcal{D}}^b)^2 = 8\pi G \langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + \tilde{\Lambda}_{\mathcal{D}}^b - \frac{1}{2} \tilde{\mathcal{Q}}_{\mathcal{D}}^b - \frac{1}{2} \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}}^b, \quad (4.19)$$

with a time- and scale-dependent contribution from the cosmological constant,

$$\tilde{\Lambda}_{\mathcal{D}}^b := \Lambda \left\langle \frac{N^2}{\gamma^2} \right\rangle_{\mathcal{D}}^b, \quad (4.20)$$

²¹However, \mathcal{R} can indeed arise as the 3-Ricci scalar associated to a ‘ \mathbf{u} -orthogonal spatial Riemann-like’ tensor which can be built from the \mathbf{u} -orthogonal spatial covariant derivative operator (defined for tensors fully orthogonal to \mathbf{u} as the projection through \mathbf{b} of their covariant 4-derivative on every component) as well as from its spacetime embedding [50, 34, 56]. Although for non-vanishing vorticity this Riemann-like tensor does not possess all the symmetry properties of a true Riemann tensor and the way of defining such a spatial curvature tensor is not unique, \mathcal{R} may accordingly be seen as the scalar part of local 3-curvature at the \mathbf{u} -orthogonal subspace of the tangent space at each spacetime point. Boersma and Dray introduce so-called parametric manifolds to define this quantity as the curvature of the parametric submanifold [8]. Alternatively, we may see it simply as a definition through equation (4.17).

and with $\tilde{Q}_{\mathcal{D}}^b$ and $\tilde{\mathcal{P}}_{\mathcal{D}}^b$ denoting the intrinsic kinematical and dynamical backreaction terms, respectively, as seen in the t -hypersurfaces. They are defined as follows:

$$\tilde{Q}_{\mathcal{D}}^b := \frac{2}{3} \left\langle \left(\tilde{\Theta} - \langle \tilde{\Theta} \rangle_{\mathcal{D}}^b \right)^2 \right\rangle_{\mathcal{D}}^b - 2 \langle \tilde{\sigma}^2 \rangle_{\mathcal{D}}^b + 2 \langle \tilde{\omega}^2 \rangle_{\mathcal{D}}^b ; \quad (4.21)$$

$$\tilde{\mathcal{P}}_{\mathcal{D}}^b := \langle \tilde{\mathcal{A}} \rangle_{\mathcal{D}}^b + \left\langle \tilde{\Theta} \frac{\gamma}{N} \frac{d}{dt} \left(\frac{N}{\gamma} \right) \right\rangle_{\mathcal{D}}^b . \quad (4.22)$$

As for Theorem 1.a, the left-hand side of equation (4.18) above should not be directly interpreted as a time-acceleration of the scale factor, unless a framework such as the Lagrangian picture, that we develop below, is adopted (compare the discussion and proof in subsection 3.4.1).

Note also that the backreaction terms introduced above do not correspond in general to the terms $Q_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{D}}$ appearing in the extrinsic averaging scheme. They do coincide, however, in case of a fluid-orthogonal foliation as can be seen by direct comparison with the definitions (3.33)–(3.34) of $Q_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{D}}$, and by noting that in this case $\mathcal{K}_{ij} = \Theta_{ij}$, $\omega^2 = 0$, and (through relation (2.6) between lapse and acceleration of the normal frames), $\tilde{\mathcal{A}} = NN^{\parallel i}$.

The above system of averaged equations can alternatively be derived (through relations (4.7) between both averaging schemes) from the analogous relations for the effective scale factor $a_{\mathcal{D}}$ of the extrinsic averaging scheme, provided the latter relations are re-expressed in terms of the fluid rest frame local kinematic and dynamical variables, as exposed in Appendix B. The use of the local dynamical equations (4.17) and (4.15) is still required in the process since the local quantities to be averaged differ between both schemes by a factor γ .

4.2.3 Integrability and energy balance conditions

As for the extrinsic averaging formalism (see subsection 3.4.2), a condition of integrability of the system of averaged equations (4.18)–(4.19) can be obtained by applying the Lagrangian coordinate-time derivative $\frac{d}{dt}$ to the averaged constraint equation (4.17) and inserting $2(a_{\mathcal{D}}^b)^{-1} (da_{\mathcal{D}}^b/dt) \times ((4.18) - (4.19))$ into the result. The averaged fluid source terms appearing in the resulting condition are themselves constrained by the local energy balance equation (2.22), which can be rescaled by a factor $(N/\gamma)^3$ to yield:

$$\frac{d}{dt} \tilde{\epsilon} + \tilde{\Theta} (\tilde{\epsilon} + \tilde{p}) = 2 \tilde{\epsilon} \frac{\gamma}{N} \frac{d}{dt} \left(\frac{N}{\gamma} \right) - \frac{N^3}{\gamma^3} (q^\mu a_\mu + \nabla_\mu q^\mu + \pi^{\mu\nu} \sigma_{\mu\nu}) . \quad (4.23)$$

Applying to it the intrinsic averager, the commutation rule (4.9) yields an evolution equation for $\langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b$, which we express along with the integrability condition in the second part of the *Theorem*:

Theorem 2.b (integrability and energy balance conditions to 2.a)

A necessary condition of integrability of equation (4.18) to yield equation (4.19) is given by:

$$\begin{aligned} \frac{d}{dt} \tilde{Q}_{\mathcal{D}}^b + 6H_{\mathcal{D}}^b \tilde{Q}_{\mathcal{D}}^b + \frac{d}{dt} \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}}^b + 2H_{\mathcal{D}}^b \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}}^b + 4H_{\mathcal{D}}^b \tilde{\mathcal{P}}_{\mathcal{D}}^b \\ = 16\pi G \left(\frac{d}{dt} \langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + 3H_{\mathcal{D}}^b \langle \tilde{\epsilon} + \tilde{p} \rangle_{\mathcal{D}}^b \right) + 2 \frac{d}{dt} \tilde{\Lambda}_{\mathcal{D}}^b, \end{aligned} \quad (4.24)$$

where the first terms on the right-hand side obey an averaged energy balance equation:

$$\begin{aligned} \frac{d}{dt} \langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + 3H_{\mathcal{D}}^b \langle \tilde{\epsilon} + \tilde{p} \rangle_{\mathcal{D}}^b = \langle \tilde{\Theta} \rangle_{\mathcal{D}}^b \langle \tilde{p} \rangle_{\mathcal{D}}^b - \langle \tilde{\Theta} \tilde{p} \rangle_{\mathcal{D}}^b \\ - \left\langle \frac{N^3}{\gamma^3} (\nabla_{\mu} q^{\mu} + q^{\mu} a_{\mu} + \pi^{\mu\nu} \sigma_{\mu\nu}) \right\rangle_{\mathcal{D}}^b + 2 \left\langle \tilde{\epsilon} \frac{\gamma}{N} \frac{d}{dt} \left(\frac{N}{\gamma} \right) \right\rangle_{\mathcal{D}}^b. \end{aligned} \quad (4.25)$$

This balance equation can be supplemented by the rest mass conservation law $dM_{\mathcal{D}}/dt = 0$, which can be equivalently expressed in terms of the averaged rest mass density $\langle \varrho \rangle_{\mathcal{D}}^b = M_{\mathcal{D}}/\mathcal{V}_{\mathcal{D}}^b$:

$$\frac{d}{dt} \langle \varrho \rangle_{\mathcal{D}}^b + 3H_{\mathcal{D}}^b \langle \varrho \rangle_{\mathcal{D}}^b = 0. \quad (4.26)$$

4.3 Effective forms of the fluid-intrinsic cosmological equations

We now introduce effective forms of the fluid-intrinsically averaged equations providing compact expressions that are suitable for applications.

4.3.1 Effective Friedmannian form

Following the suggestion in Paper II, the set of equations given in Theorem 2, which features deviations from the standard Friedmann equations, can be seen as a (scale-dependent) Friedmannian dynamics sourced by an *effective* energy-momentum tensor. The corresponding effective homogeneous energy density and pressure are defined as:

$$\epsilon_{\text{eff}}^b := \langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b - \frac{1}{16\pi G} \tilde{Q}_{\mathcal{D}}^b - \frac{1}{16\pi G} \tilde{\mathcal{W}}_{\mathcal{D}}^b + \frac{1}{8\pi G} \tilde{\mathcal{L}}_{\mathcal{D}}^b; \quad (4.27)$$

$$p_{\text{eff}}^b := \langle \tilde{p} \rangle_{\mathcal{D}}^b - \frac{1}{16\pi G} \tilde{Q}_{\mathcal{D}}^b + \frac{1}{48\pi G} \tilde{\mathcal{W}}_{\mathcal{D}}^b - \frac{1}{8\pi G} \tilde{\mathcal{L}}_{\mathcal{D}}^b - \frac{1}{12\pi G} \tilde{\mathcal{P}}_{\mathcal{D}}^b, \quad (4.28)$$

where we have introduced the backreaction terms $\tilde{\mathcal{W}}_{\mathcal{D}}^b$, for the deviation of the averaged fluid 3-curvature $\langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}}^b$ from a constant-curvature behaviour, and $\tilde{\mathcal{L}}_{\mathcal{D}}^b$ for the deviation from the cosmological constant Λ :²²

$$\tilde{\mathcal{W}}_{\mathcal{D}}^b := \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}}^b - 6 \frac{k_{\mathcal{D}}}{(a_{\mathcal{D}}^b)^2}; \quad \tilde{\mathcal{L}}^b := \tilde{\Lambda}^b - \Lambda. \quad (4.29)$$

²²In the standard cosmological model it is assumed that the cosmological constant Λ models Dark Energy; the averaged equations show that we then also have to account for *Dark Energy backreaction* $\mathcal{L}_{\mathcal{D}}^b$ in cases where $N \neq \gamma$, cf. (4.20).

$k_{\mathcal{D}}$ is a domain-dependent arbitrary constant, which can for instance be defined as $k_{\mathcal{D}} = (a_{\mathcal{D}}^b)^2(t_i) \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}}^b(t_i) / 6$ to feature the initial scalar curvature. Equations (4.18)–(4.19) can then be written as Friedmann-like equations for the effective sources and the effective Hubble function $H_{\mathcal{D}}^b$, summarized in the following *Corollary to Theorem 2*:

Corollary 2.a (Effective Friedmannian form)

The set of cosmological evolution equations of Theorem 2 can be written in Friedmannian form for the effective sources (4.27) and (4.28):

$$3 \frac{1}{a_{\mathcal{D}}^b} \frac{d^2 a_{\mathcal{D}}^b}{dt^2} = -4\pi G (\epsilon_{\text{eff}}^b + 3p_{\text{eff}}^b) + \Lambda, \quad (4.30)$$

$$3 (H_{\mathcal{D}}^b)^2 = 8\pi G \epsilon_{\text{eff}}^b - 3 \frac{k_{\mathcal{D}}}{(a_{\mathcal{D}}^b)^2} + \Lambda; \quad (4.31)$$

while the integrability condition (4.24) reduces to the effective conservation equation:

$$\frac{d}{dt} \epsilon_{\text{eff}}^b + 3 H_{\mathcal{D}}^b (\epsilon_{\text{eff}}^b + p_{\text{eff}}^b) = 0. \quad (4.32)$$

4.3.2 Effective scalar field form

Looking at the effective sources (4.27) and (4.28), we appreciate that the kinematical backreaction term $-\tilde{\mathcal{Q}}_{\mathcal{D}}^b/(16\pi G)$ individually obeys an effective *stiff equation of state*, i.e., its contributions $p_{\mathcal{Q}}^b$ and $\epsilon_{\mathcal{Q}}^b$ to the effective pressure and energy density (respectively) obey $p_{\mathcal{Q}}^b = \epsilon_{\mathcal{Q}}^b$, while the curvature deviation term $-\tilde{\mathcal{W}}_{\mathcal{D}}^b/(16\pi G)$ individually obeys an effective *curvature equation of state*, $p_{\mathcal{W}}^b = -\epsilon_{\mathcal{W}}^b/3$ (with similar notations), and the Dark Energy backreaction term $\tilde{\mathcal{L}}_{\mathcal{D}}^b/(8\pi G)$ an effective *Dark Energy equation of state*, $p_{\mathcal{L}}^b = -\epsilon_{\mathcal{L}}^b$. The dynamical backreaction term $-\tilde{\mathcal{P}}_{\mathcal{D}}^b/(12\pi G)$ arises as an additional effective geometric pressure. This consideration motivates the introduction of a scalar field language, since a free scalar field in the fluid analogy also obeys a stiff equation of state, and the scalar field potential also features a different sign for the effective potential in the expressions for the energy density and the pressure. With this analogy the backreaction terms (by definition only time-dependent, as spatial averages) can be represented by an effective homogeneous scalar field, the *morphon field*, as introduced in [24]. The resulting Friedmann-like equations are sourced in this description by the following effective homogeneous energy density and pressure:²³

$$\epsilon_{\text{eff}}^b =: \langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + \epsilon_{\text{eff}}^{\Phi, b}; \quad p_{\text{eff}}^b = \langle \tilde{p} \rangle_{\mathcal{D}}^b + p_{\text{eff}}^{\Phi, b}. \quad (4.33)$$

with the morphon variables (for the simplest choice of a scalar field fluid analogy),

$$\epsilon_{\text{eff}}^{\Phi, b} := \frac{1}{2} \left(\frac{d}{dt} \tilde{\Phi}_{\mathcal{D}} \right)^2 + U_{\text{eff}}^b(\tilde{\Phi}_{\mathcal{D}}); \quad p_{\text{eff}}^{\Phi, b} := \frac{1}{2} \left(\frac{d}{dt} \tilde{\Phi}_{\mathcal{D}} \right)^2 - U_{\text{eff}}^b(\tilde{\Phi}_{\mathcal{D}}). \quad (4.34)$$

²³In the paper introducing the morphon field [24], the possibility of phantom energies has been discussed too, which in this effective picture does not violate energy conditions. We have omitted this possible parametrization here.

The morphon field is therefore defined in terms of the backreaction terms as:

$$24\pi G \left(\frac{d}{dt} \tilde{\Phi}_{\mathcal{D}} \right)^2 := -3\tilde{Q}_{\mathcal{D}}^b - 2\tilde{\mathcal{P}}_{\mathcal{D}}^b - \tilde{W}_{\mathcal{D}}^b ; \quad (4.35)$$

$$24\pi G U_{\text{eff}}^b(\tilde{\Phi}_{\mathcal{D}}) := 3\tilde{\mathcal{L}}_{\mathcal{D}}^b + \tilde{\mathcal{P}}_{\mathcal{D}}^b - \tilde{W}_{\mathcal{D}}^b . \quad (4.36)$$

We summarize the resulting equations in the following *Corollary*:

Corollary 2.b (Effective Friedmannian form with effective scalar field)

The set of cosmological evolution equations of Theorem 2 can be written in Friedmannian form for the averaged energy sources and effective scalar field energies:²⁴

$$3 \frac{1}{a_{\mathcal{D}}^b} \frac{d^2 a_{\mathcal{D}}^b}{dt^2} = -4\pi G \left(\langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + \epsilon_{\text{eff}}^{\Phi, b} + 3 \left(\langle \tilde{p} \rangle_{\mathcal{D}}^b + p_{\text{eff}}^{\Phi, b} \right) \right) + \Lambda . \quad (4.37)$$

$$3 (H_{\mathcal{D}}^b)^2 = 8\pi G \left(\langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + \epsilon_{\text{eff}}^{\Phi, b} \right) - 3 \frac{k_{\mathcal{D}}}{(a_{\mathcal{D}}^b)^2} + \Lambda ; \quad (4.38)$$

The integrability condition (4.24), written in terms of the deviation fields $\tilde{W}_{\mathcal{D}}^b$ and $\tilde{\mathcal{L}}_{\mathcal{D}}^b$ (cf. (4.29),

$$\begin{aligned} \frac{d}{dt} \tilde{Q}_{\mathcal{D}}^b + 6H_{\mathcal{D}}^b \tilde{Q}_{\mathcal{D}}^b + \frac{d}{dt} \tilde{W}_{\mathcal{D}} + 2H_{\mathcal{D}}^b \tilde{W}_{\mathcal{D}} + 4H_{\mathcal{D}}^b \tilde{\mathcal{P}}_{\mathcal{D}}^b - 2 \frac{d}{dt} \tilde{\mathcal{L}}_{\mathcal{D}}^b \\ = 16\pi G \left(\frac{d}{dt} \langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + 3H_{\mathcal{D}}^b \langle \tilde{\epsilon} + \tilde{p} \rangle_{\mathcal{D}}^b \right) , \end{aligned} \quad (4.39)$$

is mapped to a conservation law for the effective homogeneous scalar field energies, equivalent to an effective Klein Gordon operator, applied to $\tilde{\Phi}_{\mathcal{D}}$:

$$\frac{d}{dt} \epsilon_{\text{eff}}^{b-\Phi} + 3H_{\mathcal{D}}^b \left(\epsilon_{\text{eff}}^{b-\Phi} + p_{\text{eff}}^{b-\Phi} \right) + \mathfrak{S}_{\mathcal{D}}^b = 0 , \quad (4.40)$$

$$\text{i.e.} \quad \frac{d\tilde{\Phi}_{\mathcal{D}}}{dt} \left(\frac{d^2 \tilde{\Phi}_{\mathcal{D}}}{dt^2} + 3H_{\mathcal{D}}^b \frac{d\tilde{\Phi}_{\mathcal{D}}}{dt} + \frac{\partial U_{\text{eff}}^b(\tilde{\Phi}_{\mathcal{D}})}{\partial \tilde{\Phi}_{\mathcal{D}}} \right) + \mathfrak{S}_{\mathcal{D}}^b = 0 , \quad (4.41)$$

balanced by the averaged conservation law for the sources (cf. (4.25)):

$$\begin{aligned} \mathfrak{S}_{\mathcal{D}}^b := \frac{d}{dt} \langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + 3H_{\mathcal{D}}^b \left(\langle \tilde{\epsilon} \rangle_{\mathcal{D}}^b + \langle \tilde{p} \rangle_{\mathcal{D}}^b \right) &= \langle \tilde{\Theta} \rangle_{\mathcal{D}}^b \langle \tilde{p} \rangle_{\mathcal{D}}^b - \langle \tilde{\Theta} \tilde{p} \rangle_{\mathcal{D}}^b \\ &- \left\langle \frac{N^3}{\gamma^3} (\nabla_{\mu} q^{\mu} + q^{\mu} a_{\mu} + \pi^{\mu\nu} \sigma_{\mu\nu}) \right\rangle_{\mathcal{D}}^b + 2 \left\langle \tilde{\epsilon} \frac{\gamma}{N} \frac{d}{dt} \left(\frac{N}{\gamma} \right) \right\rangle_{\mathcal{D}}^b , \end{aligned} \quad (4.42)$$

so that in total the conservation law for the total effective energy densities (4.32) holds.

²⁴The language of a given effective scalar field theory can be freely specified. We may think of other scalar field theories, e.g. non-minimally coupled, especially, if we set the scalar field analogy within an extrinsic averaging formalism, where another dictionary could be a better choice. In this line, the analogy—here set up for fluid-intrinsic averaging—could have interesting implications for the relation of different scalar field theories with different foliation choices. By construction, the scalar field obtained here obeys the evolution equations of a homogeneous scalar field, being built from pure functions of t . One may, however, define it first (following the above procedure) as a function of the time t of a given foliation, and then consider this field in another foliation choice, where it will in general be inhomogeneous. In this way, the scalar field would acquire a nonvanishing spatial gradient and would so allow for a comparison with phenomenological inhomogeneous scalar fields employed in standard perturbation theory.

4.4 Lagrangian effective forms

The effective averaged equations derived from the fluid-intrinsic approach can be further simplified by moving to the Lagrangian picture, where the rescaled variables $(\tilde{\Theta}, \tilde{\epsilon}, \dots)$ reduce to the original variables $(\Theta, \epsilon, \dots)$ since $N = \gamma$. We recall that the Lagrangian picture requires both a foliation choice of hypersurfaces at constant proper time τ and the natural adapted spacetime coordinate choice (τ, X^i) . We shall list below arguments why we consider this choice as the most adapted one, both to the geometric structure and to cosmological applications. The choice of fluid-comoving spatial coordinates X^i actually remains optional in the following, as we have seen that the average equations do not depend on the shift.

Within this picture, the commutation rule (4.9) and scale factor evolution rate (4.12) become respectively:

$$\langle \psi \rangle_{\mathcal{D}}^{b \cdot} = \langle \dot{\psi} \rangle_{\mathcal{D}}^b - \langle \Theta \rangle_{\mathcal{D}}^b \langle \psi \rangle_{\mathcal{D}}^b + \langle \Theta \psi \rangle_{\mathcal{D}}^b \quad ; \quad \frac{\dot{a}_{\mathcal{D}}^b}{a_{\mathcal{D}}^b} = \frac{1}{3} \langle \Theta \rangle_{\mathcal{D}}^b, \quad (4.43)$$

where the operators $\dot{}$ and d/dt are here equivalent for scalars.

4.4.1 Lagrangian effective cosmological equations

We summarize the Lagrangian formulation of the averaged cosmological equations in the following *Corollary*. (Note that the property $N = \gamma$ of the constant- τ foliation choice makes the Dark Energy backreaction $\tilde{\mathcal{L}}_{\mathcal{D}}^b$ vanish.)

Corollary 3.a (Lagrangian effective cosmological equations)

The evolution equations for the proper volume scale factor $a_{\mathcal{D}}^b$ (4.18)–(4.19) for a choice of constant fluid proper time foliation parametrized by $t = \tau$, read:

$$3 \frac{\ddot{a}_{\mathcal{D}}^b}{a_{\mathcal{D}}^b} = -4\pi G \langle \epsilon + 3p \rangle_{\mathcal{D}}^b + \Lambda + \mathcal{Q}_{\mathcal{D}}^b + \mathcal{P}_{\mathcal{D}}^b; \quad (4.44)$$

$$3 (H_{\mathcal{D}}^b)^2 = 8\pi G \langle \epsilon \rangle_{\mathcal{D}}^b + \Lambda - \frac{1}{2} \mathcal{Q}_{\mathcal{D}}^b - \frac{1}{2} \langle \mathcal{R} \rangle_{\mathcal{D}}^b, \quad (4.45)$$

with $H_{\mathcal{D}}^b = \dot{a}_{\mathcal{D}}^b/a_{\mathcal{D}}^b$ and the backreaction terms reduced to

$$\mathcal{Q}_{\mathcal{D}}^b = \frac{2}{3} \left\langle \left(\Theta - \langle \Theta \rangle_{\mathcal{D}}^b \right)^2 \right\rangle_{\mathcal{D}}^b - 2 \langle \sigma^2 \rangle_{\mathcal{D}}^b + 2 \langle \omega^2 \rangle_{\mathcal{D}}^b; \quad (4.46)$$

$$\mathcal{P}_{\mathcal{D}}^b = \langle \mathcal{A} \rangle_{\mathcal{D}}^b. \quad (4.47)$$

The corresponding integrability condition (c.f. equation (4.24)) now becomes:

$$\dot{\mathcal{Q}}_{\mathcal{D}}^b + 6H_{\mathcal{D}}^b \mathcal{Q}_{\mathcal{D}}^b + \langle \mathcal{R} \rangle_{\mathcal{D}}^{b \cdot} + 2H_{\mathcal{D}}^b \langle \mathcal{R} \rangle_{\mathcal{D}}^b + 4H_{\mathcal{D}}^b \mathcal{P}_{\mathcal{D}}^b = 16\pi G \left(\langle \epsilon \rangle_{\mathcal{D}}^{b \cdot} + 3H_{\mathcal{D}}^b \langle \epsilon + p \rangle_{\mathcal{D}}^b \right), \quad (4.48)$$

with the right-hand side satisfying the averaged energy conservation equation (4.25) under the following simpler form:

$$\langle \epsilon \rangle_{\mathcal{D}}^{b \cdot} + 3H_{\mathcal{D}}^b \langle \epsilon + p \rangle_{\mathcal{D}}^b = \langle \Theta \rangle_{\mathcal{D}}^b \langle p \rangle_{\mathcal{D}}^b - \langle \Theta p \rangle_{\mathcal{D}}^b - \langle \nabla_{\mu} q^{\mu} + q^{\mu} a_{\mu} + \pi^{\mu\nu} \sigma_{\mu\nu} \rangle_{\mathcal{D}}^b. \quad (4.49)$$

4.4.2 Effective Friedmannian and Lagrangian form

Combining the above form with the effective Friedmannian form provides the most compact writing of the averaged cosmological equations, summarized in the following second part of the *Corollary*:

Corollary 3.b (Compact form of Lagrangian cosmologies)

In the Lagrangian picture (implying $N = \gamma$, where all fields are expressed in terms of fluid-intrinsic coordinates (τ, X^i)), the effective Friedmann equations (4.31)–(4.32), reduce to the following form:

$$3 \frac{\ddot{a}_{\mathcal{D}}^b}{a_{\mathcal{D}}^b} = -4\pi G(\epsilon_{\text{eff}}^b + 3p_{\text{eff}}^b) + \Lambda ; \quad (4.50)$$

$$3(H_{\mathcal{D}}^b)^2 = 8\pi G \epsilon_{\text{eff}}^b - 3 \frac{k_{\mathcal{D}}}{(a_{\mathcal{D}}^b)^2} + \Lambda ; \quad (4.51)$$

$$\dot{\epsilon}_{\text{eff}}^b + 3H_{\mathcal{D}}^b (\epsilon_{\text{eff}}^b + p_{\text{eff}}^b) = 0 , \quad (4.52)$$

with $H_{\mathcal{D}}^b = \dot{a}_{\mathcal{D}}^b/a_{\mathcal{D}}^b$. The effective energy density ϵ_{eff}^b and effective pressure p_{eff}^b , as defined in (4.27) and (4.28), are here simplified to the following expressions:

$$\epsilon_{\text{eff}}^b = \langle \epsilon \rangle_{\mathcal{D}}^b - \frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}}^b - \frac{1}{16\pi G} \mathcal{W}_{\mathcal{D}}^b ; \quad (4.53)$$

$$p_{\text{eff}}^b = \langle p \rangle_{\mathcal{D}}^b - \frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}}^b + \frac{1}{48\pi G} \mathcal{W}_{\mathcal{D}}^b - \frac{1}{12\pi G} \mathcal{P}_{\mathcal{D}}^b , \quad (4.54)$$

with $\mathcal{Q}_{\mathcal{D}}^b$ and $\mathcal{P}_{\mathcal{D}}^b$ as given by (4.46) and (4.47), and with the curvature deviation term $\tilde{\mathcal{W}}_{\mathcal{D}}^b$ reduced to $\mathcal{W}_{\mathcal{D}}^b = \langle \mathcal{R} \rangle_{\mathcal{D}}^b - 6k_{\mathcal{D}}/(a_{\mathcal{D}}^b)^2$.

In the Lagrangian picture the scalar field analogy can be interpreted with the help of a morphon field in a simplified form.

Corollary 3.c (Compact form of Lagrangian cosmologies with morphon)

The effective Friedmann equations (4.51)–(4.52) can be interpreted as being sourced by homogeneous morphon energy densities by reformulating the backreaction terms in (4.53) and (4.54):

$$\epsilon_{\text{eff}}^b = \langle \epsilon \rangle_{\mathcal{D}}^b + \epsilon_{\text{eff}}^{\Phi, b} \quad ; \quad \epsilon_{\text{eff}}^{\Phi, b} := \frac{1}{2} \dot{\Phi}_{\mathcal{D}}^2 + U_{\text{eff}}^b(\Phi_{\mathcal{D}}) \quad (4.55)$$

$$p_{\text{eff}}^b = \langle p \rangle_{\mathcal{D}}^b + p_{\text{eff}}^{\Phi, b} \quad ; \quad p_{\text{eff}}^{\Phi, b} := \frac{1}{2} \dot{\Phi}_{\mathcal{D}}^2 - U_{\text{eff}}^b(\Phi_{\mathcal{D}}) , \quad (4.56)$$

with the simplified morphonic dictionary:

$$24\pi G \dot{\Phi}_{\mathcal{D}}^2 := -3\mathcal{Q}_{\mathcal{D}}^b - 2\mathcal{P}_{\mathcal{D}}^b - \mathcal{W}_{\mathcal{D}}^b ; \quad (4.57)$$

$$24\pi G U_{\text{eff}}^b(\Phi_{\mathcal{D}}) := \mathcal{P}_{\mathcal{D}}^b - \mathcal{W}_{\mathcal{D}}^b . \quad (4.58)$$

The conservation law (4.52) couples the conservation law for the material sources (4.49) to an effective Klein-Gordon operator applied to $\Phi_{\mathcal{D}}$:

$$\dot{\Phi} \left(\ddot{\Phi}_{\mathcal{D}} + 3H_{\mathcal{D}}^b \dot{\Phi}_{\mathcal{D}} + \frac{\partial U_{\text{eff}}^b(\Phi_{\mathcal{D}})}{\partial \Phi_{\mathcal{D}}} \right) + \mathfrak{S}_{\mathcal{D}}^b = 0 ; \quad (4.59)$$

with $\mathfrak{S}_{\mathcal{D}}^b$ here reduced to

$$\mathfrak{S}_{\mathcal{D}}^b = \langle \epsilon \rangle_{\mathcal{D}}^b + 3H_{\mathcal{D}}^b \langle \epsilon + p \rangle_{\mathcal{D}}^b = \langle \Theta \rangle_{\mathcal{D}}^b \langle p \rangle_{\mathcal{D}}^b - \langle \Theta p \rangle_{\mathcal{D}}^b - \langle \nabla_{\mu} q^{\mu} + q^{\mu} a_{\mu} + \pi^{\mu\nu} \sigma_{\mu\nu} \rangle_{\mathcal{D}}^b .$$

We note the important property that $\mathfrak{S}_{\mathcal{D}}^b$ only vanishes in general in the case of dust matter, separating the individually satisfied conservation law from a Klein-Gordon equation for $\Phi_{\mathcal{D}}$. In more general cases, $\mathfrak{S}_{\mathcal{D}}^b$ is non-zero. As an example it was pointed out in Paper II that this property implies that the spatially averaged inhomogeneous radiation fluid does not follow the volume expansion law of the homogeneous-isotropic radiation-dominated cosmos.

Useful characteristics for cosmological models such as dimensionless effective cosmological ‘parameters’ can be defined along the lines explained in [18] (sect. 2.4).

5 Discussion and Concluding Remarks

In this article we have distinguished *fluid-extrinsic* (section 3) and *fluid-intrinsic* (section 4) averaging procedures of the scalar parts of the Einstein equations within a general 3 + 1 spacetime split and for a general fluid whose congruence is allowed to be tilted with respect to the normal congruence of the spacelike hypersurfaces. While the former applies to all previous investigations in the literature, which we have compared in detail with our proposal of fluid-extrinsic averaging—formulated, however, for comoving domains of averaging²⁵—, the latter forms our new proposal of constructing effective cosmological equations.

5.1 Recovering the results of Paper I and Paper II

Fluid-intrinsic scalar averaging generalizes, in form and spirit, the previously suggested cosmological equations restricted to flow-orthogonal foliations of spacetime: Paper I [15] for irrotational dust (case I below), and Paper II [16] for irrotational perfect fluids (case II-C below). Within the fluid-intrinsic picture, it is straightforward to recover these subcases. We additionally get a Lagrangian picture for irrotational perfect fluids with pressure (case II-L below).

- I (irrotational, non-tilted dust in Lagrangian form): We set $\omega = 0$, $p = 0$, $q^{\mu} = 0$, $\pi_{\mu\nu} = 0$, and $\epsilon = \varrho$, and a fluid-orthogonal foliation. The coordinate time t is then already the proper time τ , cf. (2.40), and we can start from the Lagrangian form of Corollary 3. Moreover, $a_{\mathcal{D}}^b = a_{\mathcal{D}}$ and extrinsic and intrinsic averaging operators become equivalent, $\langle \dots \rangle_{\mathcal{D}}^b = \langle \dots \rangle_{\mathcal{D}}$, for this fluid-orthogonal situation. The index b becomes redundant for all expressions, and we directly recover the cosmological equations of Paper I.
- II-C (irrotational, non-tilted perfect fluids in comoving form): In Paper II the choice $\gamma = 1$ with a non-constant lapse function N was adopted, hence, this does not correspond to a Lagrangian picture. In other words, τ does not reduce to the

²⁵Recall that we reformulated the extrinsic averaging approach with this property of the averaging operator in order to avoid a number of drawbacks that may arise by not requiring this constraint.

coordinate time t , cf. (2.40). To recover the same form we have to use the equations of *Corollary 2*, in which we can select a fluid-orthogonal foliation. We can then omit the index b for the same reasons as in case I. Setting $\gamma = 1$, $\omega = 0$, $q^\mu = 0$, $\pi_{\mu\nu} = 0$ and $\Lambda = 0$ in these equations, we recover the equations of Paper II (were Λ was not considered).

II-L (irrotational, non-tilted perfect fluids in Lagrangian form): We can set $\omega = 0$, $q^\mu = 0$, $\pi_{\mu\nu} = 0$ and consider a fluid proper time foliation, with $t = \tau$ (hence the equations of *Corollary 3* can be used). For nonvanishing pressure gradients in the fluid local rest frames, this foliation is not fluid orthogonal, $\gamma > 1$, and we get different, simpler averaged equations with respect to Paper II, with an intrinsic averaging operator that is distinct from the extrinsic one.

5.2 Recovering the Newtonian form of the general cosmological equations

The compact form of the cosmological equations of *Corollary 3* enjoys a straightforward transformation to the corresponding equations that arise in Newtonian Cosmology [23]. We formulate this ‘Newtonian limit’ in terms of a restriction of the fluid deformations to integrable fluid deformations, according to the *Minkowski Restriction* as defined and executed for various variables in the series of papers [27, 26, 2, 3, 47], see especially [2, 3]. As in [47], we will combine it with a nonrelativistic limit in the special-relativistic sense, $c \rightarrow \infty$.

The cosmological equations presented in this paper do not explicitly refer to a particular spatial metric, they only depend functionally on a spatial metric. To explain the notion of *integrability* we write the spatial metric in terms of three Cartan co-frame (1-form) fields, η^a , where $a = 1, 2, 3$ counts the spatial co-frames,

$$\mathbf{h} = \delta_{ab} \eta^a \otimes \eta^b. \quad (5.1)$$

The *Minkowski Restriction*, denoted by “ \rightarrow ”, if applied to the spatial co-frames, restricts the general 1-forms to exact forms: $\eta^a \rightarrow df^a$. We consider here a fluid proper time foliation, and in this subsection we consider all tensor fields restricted to the three-dimensional spatial hypersurfaces, with the associated *spatial* exterior derivative. f^a defines an embedding into Euclidean space (a becomes a coordinate index of Eulerian coordinates $x^a = f^a(t, X^i)$, where f^a defines a diffeomorphism); the proper time τ reduces to the coordinate time t that then labels flat space sections. In an exact basis dX^i , associated with Lagrangian spatial coordinates X^i , the coefficients of the Cartan co-frames reduce to the Newtonian deformation matrix $f^a_{|i}$ in Lagrangian coordinates, $\eta^a = \eta^a_i dX^i \rightarrow df^a = f^a_{|i} dX^i$, where a vertical slash denotes derivative with respect to Lagrangian coordinates. The Riemannian spatial metric reduces to the Euclidean metric up to the coordinate transformation from Lagrangian to Eulerian coordinates, $g_{ij} dX^i \otimes dX^j = \delta_{ab} f^a_{|i} f^b_{|j} dX^i \otimes dX^j = \delta_{ij} dx^i \otimes dx^j$. The 3 + 1 Einstein equations then reduce to the Newtonian equations in Lagrangian form (for an explicit demonstration in the case of an irrotational dust matter model see [27] for the Einstein equations and [13, 32] for the Newtonian equations in Lagrangian form, admitting non-zero vorticity.)

We illustrate this transformation to the Newtonian equations for the case of a *rotational dust matter model*. We apply the *Minkowski Restriction* to this context

in a fluid proper time foliation, and we additionally assume that the relative velocities (compared to c) between fluid-comoving and normal observers and the associated special-relativistic effects of differences of clock rates are negligible. In other words, we can neglect the local tilt (not its variations) between the fluid rest frames and the Euclidean constant- τ hypersurfaces considered, $\gamma \simeq 1$, and the intrinsic averages (as the extrinsic ones) reduce to being expressed in terms of Euclidean volume integrals. Writing these Euclidean averages without the unnecessary label b , we obtain the following set of cosmological equations from *Corollary 3*:

$$3(H_{\mathcal{D}}^b)^2 = 8\pi G \epsilon_{\text{eff}}^N - 3\frac{k_{\mathcal{D}}}{a_{\mathcal{D}}^2} + \Lambda; \quad (5.2)$$

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G(\epsilon_{\text{eff}}^N + 3p_{\text{eff}}^N) + \Lambda; \quad (5.3)$$

$$\dot{\epsilon}_{\text{eff}}^N + 3H_{\mathcal{D}}(\epsilon_{\text{eff}}^N + p_{\text{eff}}^N) = 0, \quad (5.4)$$

with $H_{\mathcal{D}}^b = \dot{a}_{\mathcal{D}}/a_{\mathcal{D}}$, and with the effective sources

$$\epsilon_{\text{eff}}^N = \langle \varrho \rangle_{\mathcal{D}} - \frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}}^N - \frac{1}{16\pi G} \mathcal{W}_{\mathcal{D}}^N; \quad (5.5)$$

$$p_{\text{eff}}^N = -\frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}}^N + \frac{1}{48\pi G} \mathcal{W}_{\mathcal{D}}^N. \quad (5.6)$$

The kinematical backreaction reduce to

$$\mathcal{Q}_{\mathcal{D}}^N = \frac{2}{3} \left\langle \left(\Theta - \langle \Theta \rangle_{\mathcal{D}} \right)^2 \right\rangle_{\mathcal{D}} - 2 \langle \sigma^2 \rangle_{\mathcal{D}} + 2 \langle \omega^2 \rangle_{\mathcal{D}}; \quad (5.7)$$

where Θ , σ and ω are the scalar kinematic variables of the (now integrable) expansion and vorticity tensors, *i.e.*, related to the kinematic invariants of the velocity gradient field $d\dot{f}^i = (\partial \dot{f}^a / \partial x^i) dx^i$ with coefficients expressed in the Eulerian co-ordinate basis dx^i .

In Newtonian theory, $k_{\mathcal{D}}$ is a constant of integration (not associated with a constant curvature term), and the ‘‘curvature deviation’’ $\mathcal{W}_{\mathcal{D}}^N$ also loses its interpretation as a Riemannian curvature. It is defined through (5.4) which splits into two equations. Firstly, the energy conservation law (4.49) reduces for dust to the continuity equation for the average density:

$$\langle \varrho \rangle_{\mathcal{D}}^{\cdot} + 3H_{\mathcal{D}} \langle \varrho \rangle_{\mathcal{D}} = 0. \quad (5.8)$$

Secondly, the Newtonian form of the integrability condition, and hence the definition of $\mathcal{W}_{\mathcal{D}}^N$ follows from integrating the remaining part of (5.4), which is

$$\dot{\mathcal{Q}}_{\mathcal{D}}^N + 6H_{\mathcal{D}} \mathcal{Q}_{\mathcal{D}}^N + \mathcal{W}_{\mathcal{D}}^N + 2H_{\mathcal{D}} \mathcal{W}_{\mathcal{D}}^N = 0, \quad (5.9)$$

i.e. (cf. [18], sect. 2.3):

$$\frac{1}{2} \left(\mathcal{Q}_{\mathcal{D}}^N + \mathcal{W}_{\mathcal{D}}^N \right) = -\frac{1}{a_{\mathcal{D}}^2} \int_{t_i}^t dt' \mathcal{Q}_{\mathcal{D}}^N \frac{d}{dt'} a_{\mathcal{D}}(t'), \quad (5.10)$$

defining $\mathcal{W}_{\mathcal{D}}^N$ through $\mathcal{Q}_{\mathcal{D}}^N$ and the time-history of $\mathcal{Q}_{\mathcal{D}}^N$.²⁶

²⁶In Newtonian Cosmology we have to abandon the background-free character of general relativity: in order to obtain unique solutions, we have to introduce a background in terms of

5.3 Concluding remarks on the fluid-intrinsic and Lagrangian approaches

5.3.1 Interest of the fluid-intrinsic averaged equations

Corollary 2 shows that in the fluid-intrinsic approach tilt effects are no longer an issue, and the *stress-energy backreaction* disappears. This is different from all approaches presented in the literature, put into perspective in subsection 3.5. Tilt effects may, however, be important for the observational interpretation, since the observer may be tilted with respect to the cosmic fluid. For effective cosmologies we advocate the fluid-intrinsic point of view, focussing on the effective evolution of the *model universe*, and eliminating wherever possible observer-specific issues. It is then an entirely different question, well-separated from the model universe, how the variables of these cosmologies are related to observables, which is a question related to light-cone averages, not considered in this work.

5.3.2 Interest of the Lagrangian picture choice

Corollary 3 shows, in addition to *Corollary 2*, that a constant fluid proper time foliation choice, parametrized by $t = \tau$ (implying $N = \gamma$), makes the *Dark Energy backreaction* $\mathcal{L}_{\mathcal{D}}^b$ disappear and removes the need to account for a difference of time rates in the dynamical backreaction and in the rescaling of all variables to be averaged. Despite these simplifications, we emphasize that the above set of compact cosmological equations holds for a general fluid. The difference between *Corollary 2* (presenting the cosmological equations for general fluids *and* general foliations), and *Corollary 3* (making the fluid proper time foliation choice) may serve for a discussion of the robustness of the averaged equations with respect to a foliation choice.

We here summarize the advantages of the use of the Lagrangian picture in combination with the fluid-intrinsic averaging framework for cosmological backreaction:

- (i) It links the foliation itself to the fluid, in a way alternative to the fluid-orthogonal choice, but in contrast to the latter it comes with a unique time-normalization and holds for any fluid;
- (ii) It allows for a simple and compact form of the cosmological equations, removing the need for rescalings and extra terms due to the different clock rates;
- (iii) The corresponding choice of time is formally unique up to a constant along each flow line (see, however, our remarks on effective times below), and of clear physical interpretation. The associated time derivatives, in particular the scale factor expansion and acceleration rates, are well-defined as proper rates for the fluid elements;
- (iv) It reduces without change to the usual (fluid-orthogonal) approach for irrotational dust and for homogeneous fluids with pressure (FLRW);

a linear reference velocity field, $V_i = H_{ij}x_j$, with homogeneous expansion, shear and vorticity, $H_{ij} =: \Theta_H(t)/3 + \Sigma_{ij}(t) + \Omega_{ij}(t)$. Deviations thereof are to be bound to a 3-torus topology. As a result of the integrability of the Newtonian variables on flat space sections, $\mathcal{Q}_{\mathcal{D}}^N$ (which does not depend on the background variables) can be written in terms of full divergences of vector fields. Hence, $\mathcal{Q}_{\mathcal{D}}^N$ has to vanish on the no-boundary 3-torus, see [23] and the recent discussion [20].

- (v) It also allows simple, transparent, Newtonian-like formulas for the kinematic variables (especially the components of the tensorial variables), and it reduces to the Newtonian averaged equations for spatially flat space sections (Minkowski Restriction) and nonrelativistic velocities.

A possible important drawback however has to be checked in individual cases: the so-defined foliation may become strongly tilted with respect to the fluid, and might even not always be spacelike in the entire spacetime, if acceleration and/or vorticity induce an infinite tilt after a finite time. The averaging formalism would become ill-defined in the spacetime regions where the hypersurfaces are no longer spacelike. As for the comoving domain moreover, for several fluid components, the advantages can only be preserved in general for one fluid, considered as preferred, from which the proper time will be defined.

5.3.3 Limitations and outlook

The results of this work are *general* in various respects, culminating in compact forms of effective cosmologies by arguing in favour of a Lagrangian description. However, there are issues that are worth to be addressed, and we highlight some of them in what follows.

The issue of closure The presented sets of averaged equations and compact cosmological equations are not closed. This known issue is obvious from the very approach of performing averages of only the scalar parts of the Einstein equations. It is also obvious since a balance equation on averages will not allow to reconstruct the inhomogeneous metric (similarly to, in Newtonian contexts, the virial relations not allowing for the reconstruction of the orbits in phase space). We do not enter the issue of averaging or smoothing tensorial variables here, but we emphasize that even averaging further scalar equations would result in a hierarchy of equations that would not close (similar as the hierarchy of moment equations in kinetic theory). As in the standard Friedmannian framework, where closure conditions have to be imposed in terms of *equations of state* determining fluid properties, closure conditions may here be represented as *effective equations of state* in the effective Friedmannian and Lagrangian forms. These effective relations encode inhomogeneous properties and evolution details of the fluid and, hence, they are dynamical and not simply derivable from thermodynamical properties. Closure conditions can be studied in terms of exact scaling solutions [18, 24, 58], global assumptions on model universes [17], exact solutions of the Einstein equations, *e.g.* [9, 63], or generic but approximate models for inhomogeneities, *e.g.* [26].

Statistical hypersurfaces of averaging The framework of Papers I and II allows for averaging on fluid-comoving domains and on hypersurfaces formed by the fluid itself, but only in cases where the fluid is irrotational and non-tilted. We proposed here a way of dealing with rotational and tilted cases by introducing a fluid-intrinsic averaging procedure that reduces to the standard Riemannian volume average for cases of irrotational fluids in their fluid-orthogonal foliations, and we suggested the fluid proper time foliations as a possible way of still building the hypersurfaces from the fluid. Alternatively, we can take a statistical point of view by investigating hypersurfaces of ‘statistically averaged’ geometries, a notion

that has yet to be formalized. The example of vorticity may illustrate the physical idea behind such a concept: if we view vorticity as arising on small scales only while expecting that, by going to larger scales, it will become unimportant, we may wonder whether vorticity ‘averages out’ (in a statistical sense) on some scale of averaging. On scales larger than this, a potential flow is expected, and the fluid can be described as hypersurface-forming, while ‘averaged-out’ scales may still feature a statistical ‘dressed’ [21] contribution from vorticity. The idea of viewing averaged equations as providing a definition of ‘statistical hypersurfaces of averaging’ has been advocated (*e.g.* [54]) and, in Newtonian theory, assumptions such as homogeneous-isotropic turbulence have been advanced to construct statistical averages [52].

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A 3 + 1 evolution equations along the congruence of the fluid

The evolution equations for h_{ij} and \mathcal{K}^i_j along the congruence of the fluid are obtained from expressions (3.1) and (3.2), by relating the derivative $\partial_t|_{x^i}$ to d/dt with the help of (2.25). They read:

$$\frac{d}{dt}h_{ij} = -2N\mathcal{K}_{ij} + N_{i||j} + N_{j||i} + V^k\partial_k h_{ij}, \quad (\text{A.1})$$

$$\begin{aligned} \frac{d}{dt}\mathcal{K}^i_j &= N \left(\mathcal{R}^i_j + \mathcal{K}\mathcal{K}^i_j + 4\pi G [(S-E)\delta^i_j - 2S^i_j] - \Lambda\delta^i_j \right) \\ &\quad - N^i_{||i} + N^k\mathcal{K}^i_{j||k} + \mathcal{K}^i_k N^k_{||j} - \mathcal{K}^k_j N^i_{||k} + V^k\partial_k\mathcal{K}^i_j. \end{aligned} \quad (\text{A.2})$$

Comoving coordinate system. In the comoving picture, as described in subsection 2.4, we have $\mathbf{N} = N\mathbf{v}$, or equivalently $\mathbf{V} = \mathbf{0}$. Equations (A.1) and (A.2) hence read:

$$\frac{d}{dt}h_{ij} = -2N\mathcal{K}_{ij} + N(v_{i||j} + v_{j||i}) + v_i N_{||j} + v_j N_{||i}, \quad (\text{A.3})$$

$$\begin{aligned} \frac{d}{dt}\mathcal{K}^i_j &= N \left(\mathcal{R}^i_j + \mathcal{K}\mathcal{K}^i_j + 4\pi G [(S-E)\delta^i_j - 2S^i_j] - \Lambda\delta^i_j \right) - N^i_{||j} \\ &\quad + Nv^k\mathcal{K}^i_{j||k} + N\mathcal{K}^i_k v^k_{||j} + \mathcal{K}^i_k v^k N_{||j} - N\mathcal{K}^k_j v^i_{||k} - \mathcal{K}^k_j v^i N_{||k}. \end{aligned} \quad (\text{A.4})$$

Comoving coordinate system and constant-fluid proper time slicing. In the Lagrangian picture, also described in subsection 2.4, we have $\mathbf{N} = N\mathbf{v}$ (or equivalently $\mathbf{V} = \mathbf{0}$), and $N = \gamma$ as a consequence of the slicing and the additional parametrization choice $t = \tau$. Expressions (A.1) and (A.2) accordingly read:

$$\frac{d}{d\tau}h_{ij} = -2\gamma\mathcal{K}_{ij} + \gamma(v_{i||j} + v_{j||i}) + v_i\gamma_{||j} + v_j\gamma_{||i}, \quad (\text{A.5})$$

$$\begin{aligned} \frac{d}{d\tau}\mathcal{K}^i_j &= \gamma \left(\mathcal{R}^i_j + \mathcal{K}\mathcal{K}^i_j + 4\pi G [(S-E)\delta^i_j - 2S^i_j] - \Lambda\delta^i_j \right) - D^i D_j \gamma \\ &\quad + \gamma v^k\mathcal{K}^i_{j||k} + \gamma\mathcal{K}^i_k v^k_{||j} + \mathcal{K}^i_k v^k\gamma_{||j} - \gamma\mathcal{K}^k_j v^i_{||k} - \mathcal{K}^k_j v^i\gamma_{||k}, \end{aligned} \quad (\text{A.6})$$

where we have used the equality between the proper time and coordinate time derivatives along the fluid flow.

The equations of evolution along the fluid flow in general coordinates and slicing allow an alternative derivation of the coordinate time derivative of the Riemannian volume,

$$\frac{d}{dt}\mathcal{V}_D = \int_{\mathcal{D}_x} \left(-N\mathcal{K} + (Nv^i)_{||i} \right) \sqrt{h} d^3x, \quad (\text{A.7})$$

by restarting from (3.14) and expanding its integrand as

$$\frac{d}{dt}\mathcal{V}_D = \int_{\mathcal{D}_x} \left(\frac{1}{2}h^{ij}\frac{d}{dt}h_{ij} + J^{-1}\frac{d}{dt}J \right) \sqrt{h} d^3x. \quad (\text{A.8})$$

The trace of (A.1) can then be used together with (3.8) to obtain:

$$\frac{d}{dt}\mathcal{V}_D = \int_{\mathcal{D}_x} \left(-N\mathcal{K} + N^i_{||i} + \frac{1}{2}h^{ij}V^k\partial_k h_{ij} + \partial_k V^k \right) \sqrt{h} d^3x. \quad (\text{A.9})$$

This expression then allows to catch up with the end of the derivation given in section 3.2.2, so that a similar use of relations (3.16) and (2.10) again gives the evolution of the volume (A.7).

B Extrinsic averaging procedure in fluid-intrinsic kinematic variables

The system of equations for extrinsic averages on \mathcal{D} derived in section 3 is mostly expressed in terms of geometric variables of the \mathbf{n} -orthogonal hypersurfaces, such as their extrinsic curvature. We here suggest an alternative formulation of the same equations focusing instead on the intrinsic, rest-frame kinematic quantities of the fluid (see subsection 2.2.2), which do not depend on the foliation choice.

We can first rewrite the volume expansion rate expression (3.19) and commutation rule (3.22) in terms of the intrinsic expansion rate of the fluid Θ by re-expressing the three-divergence of Nv as

$$(Nv^i)_{||i} = Nv^i_{||i} + v^i N_{||i} = N\nabla_\mu v^\mu, \quad (\text{B.1})$$

where we have employed (2.6) for the last equality. Noticing that $\mathcal{K} = -\nabla_\mu n^\mu$ and making use of expression (2.7), we get²⁷

$$-N\mathcal{K} + (Nv^i)_{||i} = \frac{N}{\gamma}\Theta - \frac{1}{\gamma}\frac{d\gamma}{dt} = \tilde{\Theta} - \frac{1}{\gamma}\frac{d\gamma}{dt} := \tilde{\Theta}^T, \quad (\text{B.2})$$

where we have defined a *tilted and scaled expansion rate* $\tilde{\Theta}^T$ out of the scaled rate $\tilde{\Theta} = (N/\gamma)\Theta$. This allows to recast the volume expansion rate and the commutation rule, respectively, into the following expressions:

$$3H_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\frac{dV_{\mathcal{D}}}{dt} = \langle \tilde{\Theta}^T \rangle_{\mathcal{D}}; \quad (\text{B.3})$$

$$\frac{d}{dt}\langle \psi \rangle_{\mathcal{D}} = \left\langle \frac{d}{dt}\psi \right\rangle_{\mathcal{D}} - \langle \tilde{\Theta}^T \rangle_{\mathcal{D}}\langle \psi \rangle_{\mathcal{D}} + \langle \tilde{\Theta}^T \psi \rangle_{\mathcal{D}}. \quad (\text{B.4})$$

We notice that, even for the general configuration we are investigating (see figure 1), the commutation rule, as well as the domain volume expansion rate, can be cast into a simple form with respect to the fluid quantities, although this extrinsic averaging framework requires the explicit contribution from the evolving tilt.

The use of the Raychaudhuri equation (4.15) and the energy constraint (4.17) (instead of the scalar parts of the extrinsic 3 + 1 Einstein equations (3.2)–(3.4)), together with this alternative form of the commutation rule, allows a rewriting of the effective evolution equations for $a_{\mathcal{D}}$. This yields the following equivalent formulation of *Theorem 1*, in terms of rescaled fluid-intrinsic kinematic and dynamical variables, $\tilde{\sigma}^2 = (N^2/\gamma^2)\sigma^2$, $\tilde{\omega}^2 = (N^2/\gamma^2)\omega^2$, $\tilde{\mathcal{R}} = (N^2/\gamma^2)\mathcal{R}$, $\tilde{\epsilon} = (N^2/\gamma^2)\epsilon$, $\tilde{p} = (N^2/\gamma^2)p$, and $\tilde{\mathcal{A}} = (N^2/\gamma^2)\mathcal{A}$ (with $\mathcal{A} = \nabla_\mu a^\mu$), as well as $\tilde{\Lambda} := (N^2/\gamma^2)\Lambda$.

Corollary 1.a (extrinsically averaged evolution equations in fluid variables)

The effective averaged evolution equations for $a_{\mathcal{D}}$ can be written under the following form:

$$3\left(\frac{1}{a_{\mathcal{D}}}\frac{da_{\mathcal{D}}}{dt}\right)^2 = 8\pi G\langle \tilde{\epsilon} \rangle_{\mathcal{D}} + \langle \tilde{\Lambda} \rangle_{\mathcal{D}} - \frac{1}{2}\langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}} - \frac{1}{2}\mathcal{Q}_{\mathcal{D}}^T; \quad (\text{B.5})$$

$$\frac{3}{a_{\mathcal{D}}}\frac{d^2a_{\mathcal{D}}}{dt^2} = -4\pi G\langle (\tilde{\epsilon} + 3\tilde{p}) \rangle_{\mathcal{D}} + \langle \tilde{\Lambda} \rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}^T + \mathcal{P}_{\mathcal{D}}^T, \quad (\text{B.6})$$

²⁷The factor N/γ in expressions (B.3)–(B.4) adjusts the clock rate between the proper time of the fluid and the coordinate time. This can also be seen upon writing:

$$\frac{N}{\gamma}\Theta - \frac{1}{\gamma}\frac{d\gamma}{dt} = \frac{N}{\gamma}\left(\Theta - \frac{1}{\gamma}\frac{d\gamma}{d\tau}\right) = \frac{d\tau}{dt}\left(\Theta - \frac{1}{\gamma}\frac{d\gamma}{d\tau}\right),$$

where we have used the relation (2.26) between d/dt and $d/d\tau$. The extra tilt term $-\gamma^{-1}d\gamma/dt$ can be understood as the effect of the evolving mutual tilt between the hypersurfaces embedding \mathcal{D} and the fluid flow. This affects the volume measure and the evolution of the volume due to the fluid's intrinsic expansion.

with alternative, ‘tilted’ kinematical and dynamical backreactions, reading respectively:

$$\mathcal{Q}_{\mathcal{D}}^{\mathbb{T}} := \frac{2}{3} \left[\left\langle (\tilde{\Theta}^T)^2 \right\rangle_{\mathcal{D}} - \langle \tilde{\Theta}^T \rangle_{\mathcal{D}}^2 \right] - 2 \langle \tilde{\sigma}^2 \rangle_{\mathcal{D}} + 2 \langle \tilde{\omega}^2 \rangle_{\mathcal{D}} + \frac{2}{3} \left\langle 2 \tilde{\Theta}^T \frac{1}{\gamma} \frac{d\gamma}{dt} + \left(\frac{1}{\gamma} \frac{d\gamma}{dt} \right)^2 \right\rangle_{\mathcal{D}} ; \quad (\text{B.7})$$

$$\mathcal{P}_{\mathcal{D}}^{\mathbb{T}} := \langle \tilde{\mathcal{A}} \rangle_{\mathcal{D}} + \left\langle \frac{\gamma}{N} \frac{d}{dt} \left(\frac{N}{\gamma} \right) \tilde{\Theta}^T \right\rangle_{\mathcal{D}} - \left\langle 2 \tilde{\Theta}^T \frac{1}{\gamma} \frac{d\gamma}{dt} + \left(\frac{1}{\gamma} \frac{d\gamma}{dt} \right)^2 + \frac{N}{\gamma} \frac{d}{dt} \left(\frac{\gamma}{N} \frac{1}{\gamma} \frac{d\gamma}{dt} \right) \right\rangle_{\mathcal{D}} . \quad (\text{B.8})$$

We recall that, as in Theorem 1.a, the left-hand sides in the above equations should be seen as derivatives with respect to the chosen parameter t , and be interpreted according to the physical meaning of the latter. In particular, the term $3a_{\mathcal{D}}^{-1} d^2 a_{\mathcal{D}}/dt^2$ in equation (B.6) is the proper time scale factor acceleration in a Lagrangian picture, but not in general.

Under this form, only two backreaction terms appear, $\mathcal{Q}_{\mathcal{D}}^{\mathbb{T}}$ and $\mathcal{P}_{\mathcal{D}}^{\mathbb{T}}$, as the tilt only contributes under these combinations. These backreaction terms will not in general be directly related to the $\mathcal{Q}_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{D}}$ appearing in Theorem 1.a, as they do not collect the same local terms in their expression. They do coincide, however, for a fluid-orthogonal foliation ($\mathcal{Q}_{\mathcal{D}}^{\mathbb{T}} = \mathcal{Q}_{\mathcal{D}}$, $\mathcal{P}_{\mathcal{D}}^{\mathbb{T}} = \mathcal{P}_{\mathcal{D}}$, while $\mathcal{T}_{\mathcal{D}} = 0$).

Note that there is no explicit non-perfect fluid contribution to these effective evolution equations for the scale factor $a_{\mathcal{D}}$, although the non-perfect fluid components of the energy-momentum tensor do have an influence on the dynamics *via* the local (and average, see below) evolution of ϵ and p .

As before, this set of equations goes along with an integrability condition, and must be complemented by the evolution equation for the averaged energy density and pressure.

Corollary 1.b (integrability and energy balance conditions to Corollary 1.a)

The corresponding integrability condition reads:

$$\begin{aligned} & \frac{d}{dt} \mathcal{Q}_{\mathcal{D}}^{\mathbb{T}} + 6H_{\mathcal{D}} \mathcal{Q}_{\mathcal{D}}^{\mathbb{T}} + \frac{d}{dt} \langle \tilde{\mathcal{H}} \rangle_{\mathcal{D}} + 2H_{\mathcal{D}} \langle \tilde{\mathcal{H}} \rangle_{\mathcal{D}} + 4H_{\mathcal{D}} \mathcal{P}_{\mathcal{D}}^{\mathbb{T}} \\ & = 16\pi G \left(\frac{d}{dt} \langle \tilde{\epsilon} \rangle_{\mathcal{D}} + 3H_{\mathcal{D}} \langle (\tilde{\epsilon} + \tilde{p}) \rangle_{\mathcal{D}} \right) + 2 \frac{d}{dt} \langle \tilde{\mathcal{A}} \rangle_{\mathcal{D}} , \end{aligned} \quad (\text{B.9})$$

while the associated averaged conservation equation for the scaled energy density $\tilde{\epsilon}$ and pressure \tilde{p} becomes:

$$\begin{aligned} & \frac{d}{dt} \langle \tilde{\epsilon} \rangle_{\mathcal{D}} + 3H_{\mathcal{D}} \langle \tilde{\epsilon} + \tilde{p} \rangle_{\mathcal{D}} = \left(\langle \tilde{\Theta}^T \rangle_{\mathcal{D}} \langle \tilde{p} \rangle_{\mathcal{D}} - \langle \tilde{\Theta}^T \tilde{p} \rangle_{\mathcal{D}} \right) - \left\langle \frac{1}{\gamma} \frac{d\gamma}{dt} (\tilde{\epsilon} + \tilde{p}) \right\rangle_{\mathcal{D}} \\ & - \left\langle \frac{N^3}{\gamma^3} \left(a_{\mu} q^{\mu} + \nabla_{\mu} q^{\mu} + \pi^{\mu\nu} \sigma_{\mu\nu} \right) \right\rangle_{\mathcal{D}} + 2 \left\langle \tilde{\epsilon} \frac{\gamma}{N} \frac{d}{dt} \left(\frac{N}{\gamma} \right) \right\rangle_{\mathcal{D}} . \end{aligned} \quad (\text{B.10})$$

(The expression $\langle \tilde{\Theta}^T \rangle_{\mathcal{D}} \langle \tilde{p} \rangle_{\mathcal{D}} - \langle \tilde{\Theta}^T \tilde{p} \rangle_{\mathcal{D}}$, can also be written as $\langle d\tilde{p}/dt \rangle_{\mathcal{D}} - d\langle \tilde{p} \rangle_{\mathcal{D}}/dt$.)

We add a technical remark. In the fluid-intrinsic point of view we can borrow one element from the 1+3 formalism to foliate spacetime, the so-called *spacetime threading*, although spatial volume averaging only makes sense on hypersurfaces. We recall that in the 1+3 decomposition, the four-dimensional line element reads (see, e.g., [42], sect.10):

$$ds^2 = -\mathcal{M}^2 dt^2 + 2\mathcal{M}^2 \mathcal{M}_i dt dX^i + (b_{ij} - \mathcal{M}^2 \mathcal{M}_i \mathcal{M}_j) dX^i dX^j . \quad (\text{B.11})$$

with \mathcal{M} the threading lapse and $\underline{\mathcal{M}}$ the threading shift, which relate to the lapse and Eulerian velocity as follows:

$$\mathcal{M} := \frac{N}{\gamma} ; \quad \underline{\mathcal{M}} := \frac{\gamma^2}{N} \left(\underline{v} + v^k v_k \underline{n} \right) : \mathcal{M} \mathcal{M}_{\mu} = \gamma(0, v_i) = (0, u_i) . \quad (\text{B.12})$$

In the Lagrangian description we have:

$$\mathcal{M} = 1 ; \quad \underline{\mathcal{M}} = \gamma \left(\underline{v} + v^k v_k \underline{n} \right) : \mathcal{M}_{\mu} = \gamma(0, v_i) = (0, u_i) . \quad (\text{B.13})$$

Note that in the most compact form of the general averaged equations, we only deal with appearances of $N/\gamma = \mathcal{M}$.

C Summarized literature comparison

We present in Table 1 a synthetic comparison of the various formalisms used in the existing generalization proposals of the system of averaged scalar equations of Papers I and II to general foliations.

In this table we express all notations in terms of those used in this work to make comparisons easier. In particular, when considering the 4-scalar expressions of [38, 54, 61], we define the lapse N as $(-\partial_\mu T \partial^\mu T)^{-1/2}$, where T is the scalar function which labels the hypersurfaces. This quantity (noted Γ in [54]) plays a role analogous to the 3 + 1 lapse as $n_\mu = -N \partial_\mu T$, and it indeed coincides with the usual lapse if the 4-scalar formalism is split into a 3 + 1 description with the natural choice of T as the time coordinate.

Reference	Domain boundary flow (Mass-preserving ?)	Fluid content	Formulation	Foliation vector \mathbf{n}
Tanaka & Futamase [64]	Equivalent to having \mathbf{n} , and ∂_t (No) ^a	General $T_{\mu\nu}$	3 + 1 with $\mathbf{N} = \mathbf{0}$	General
Larena [46]	Implicitly, ∂_t (No)	One perfect fluid	3 + 1	General
Brown <i>et al.</i> [10]	Implicitly, ∂_t (No)	Sum of general fluids	3 + 1	General
Gasperini <i>et al.</i> [38]	\mathbf{n} (No)	One perfect fluid	Both (mostly) ^b 4-scalar and 3 + 1	General
Räsänen [54]	Global domain (Yes) ^c	General $T_{\mu\nu}$	4-scalar	General
Beltrán Jiménez <i>et al.</i> [7]	$\mathbf{n}, = \partial_t$ (No)	General $T_{\mu\nu}$ with a dust part ^d	3 + 1 with $\mathbf{N} = \mathbf{0}$ and $N = 1$	Geodesic (dust velocity)
Smirnov [61]	\mathbf{n} (No)	General $T_{\mu\nu}$ plus a dust part ^d	Both 4-scalar and 3 + 1	Geodesic (dust velocity)
Section 3 of this work	\mathbf{u} (Yes)	One general fluid	3 + 1	General

^a In [64], boundary terms are removed by an (*a priori* background-dependent) assumption of periodic boundary conditions on the large enough but still compact domain. As discussed in section 3.5.4 above, this implies equivalent results to the more general case (not considered in [64]) of a compact domain propagating along \mathbf{n} , at the expense of the mass preservation. As the shift vector is chosen to be zero in [64], this would also amount to a propagation along ∂_t .

^b Formally, the boundaries of the domain are assumed to be determined by some scalar function, in which case the averages and the equations are truly covariant; but the authors mention the difficulty of finding such a scalar on physical grounds which may constrain one to choose a function of the coordinates instead of a scalar, hence inducing deviations from general covariance in the averages. The follow-up paper [49] makes these deviations explicit at second order in perturbation theory.

^c The equations would formally still hold without change if a regional domain propagating along \mathbf{n} were considered instead. However, it would not be mass-preserving in this case.

^d In both cases (Beltrán Jiménez *et al.* [7] and Smirnov [61]) it is assumed that there are ‘natural’ observers corresponding to some irrotational dust as part of the fluid content of the model universe, not interacting with the rest, so that it is moreover geodesic, and the corresponding geodesic irrotational normalized velocity field is used as the normal vector \mathbf{n} to build the hypersurfaces. In [7] it is assumed to represent the baryonic and Dark Matter on large scales and hence is a well-defined part of $T_{\mu\nu}$ (whereas the remaining parts can account for other fluids such as radiation or Dark Energy, or for effects due to a departure from General Relativity). In [61], it can either be some component intrinsically contained within $T_{\mu\nu}$, or some ‘test observers’ that are added to the fluid content with an assumed negligible source contribution.

Table 1 Summary of the main differences between the various generalizations of the averaged scalar 3 + 1 Einstein equations to general foliations suggested in the literature. This table is split into three parts respectively focussing on the setup, the equations presented (and corresponding effective Hubble parameter considered), and the terms they feature.

Reference	Effective Hubble parameter ($\times 3$)	Evolution of scale factor	Integrability condition	Averaged energy balance
Tanaka & Futamase [64]	$\frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \langle -NK \rangle$	Yes	No	No
Larena [46]	$\langle Nh^{\mu\nu} \nabla_{\mu} u_{\nu} \rangle \neq \frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt}$ ^e	Yes	Yes	Yes
Brown <i>et al.</i> [10]	$\frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \langle -NK + N^i{}_{ i} \rangle$ ^f	Yes	No	No
Gasperini <i>et al.</i> [38]	$\frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \langle -NK \rangle$	Yes	No	Only in case $\mathbf{n} = \mathbf{u}$
Räsänen [54]	$\frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \langle -NK \rangle$	Yes	No	Yes
Beltrán Jiménez <i>et al.</i> [7]	$\frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \langle -\mathcal{K} \rangle$ ($N = 1$)	Yes	Yes	Yes
Smirnov [61]	$\frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \langle -NK \rangle$	Yes	No	No
Section 3 of this work	$\frac{1}{\mathcal{V}_{\mathcal{D}}} \frac{d\mathcal{V}_{\mathcal{D}}}{dt} = \langle -NK + (Nv^i)_{ i} \rangle_{\mathcal{D}}$	Yes	Yes	Yes

^e The application paper [65] introduces instead five possible definitions of the effective Hubble parameter H in order to compare them, and derives the averaged energy constraint for each of them. The first four of these definitions are respectively: $3H = \dot{\mathcal{V}}/\mathcal{V} = \langle -NK + N^i{}_{||i} \rangle$ (which becomes simply $\langle -NK \rangle$ later in the paper as the shift is set to zero); $3H = \langle -\mathcal{K} \rangle$; $3H = \langle Nh^{\mu\nu} \nabla_{\mu} u_{\nu} \rangle$; $3H = \langle h^{\mu\nu} \nabla_{\mu} u_{\nu} \rangle$; where all averages are taken over a domain on the \mathbf{n} -orthogonal hypersurfaces. The last proposal for $3H$ consists in averaging simply the intrinsic fluid expansion rate Θ (without any lapse factor) over a domain on \mathbf{u} -orthogonal hypersurfaces, in case \mathbf{u} is irrotational. In all of the \mathbf{n} -orthogonal cases, the domain still implicitly evolves along ∂_t , whereas in the last case the averaged (dust) equation is recalled from Paper I, meaning that in this case the domain must be assumed to be fluid-comoving.

^f In the first application [11], the average of $h^{\mu\nu} \nabla_{\mu} u_{\nu}$ is also considered, while the second application [12] focuses on the average of Θ ; however, in both cases, the corresponding averaged equations are not explicated.

Table 1 Summary of the main differences between the various generalizations of the averaged scalar 3 + 1 Einstein equations to general foliations suggested in the literature. (continued)

Reference	Dynamical variables (from $T_{\mu\nu}$)	Kinematic variables	Inclusion of Λ	Explicitly identified backreaction terms	Main local time derivative
Tanaka & Futamase [64]	Normal-frame	Normal-frame (<i>i.e.</i> , built from the extrinsic curvature)	Yes	None	$\partial_t _{x^i}$ ($= Nn^\mu\partial_\mu$)
Larena [46]	Intrinsic ^g	Mixed (<i>e.g.</i> , $h^{\mu\nu}\nabla_\mu u_\nu$) ^g	Yes	\mathcal{Q} , \mathcal{P} , and three more ^h	$\partial_t _{x^i}$
Brown <i>et al.</i> [10]	Intrinsic (for each fluid)	Normal-frame	Yes	\mathcal{Q} , \mathcal{P} , and one \mathcal{T} term per fluid	$\partial_t _{x^i}$
Gasperini <i>et al.</i> [38]	Normal-frame ⁱ	Normal-frame	No	None	$Nn^\mu\partial_\mu$
Räsänen [54]	Either normal-frame or in a general frame ^j	Normal-frame	Implicit (can be included in $T_{\mu\nu}$)	Only \mathcal{Q}	$Nn^\mu\partial_\mu$
Beltrán Jiménez <i>et al.</i> [7]	Normal-frame	Normal-frame	Implicit (can be included in $T_{\mu\nu}$)	Only \mathcal{Q} ; there $\mathcal{P} = 0$	$\partial_t _{x^i}$ ($= n^\mu\partial_\mu$)
Smirnov [61]	Normal-frame (plus T^μ_μ)	Normal-frame	Implicit (can be included in $T_{\mu\nu}$)	Only \mathcal{Q} , not in all equations	$Nn^\mu\partial_\mu$
Section 3 of this work	Intrinsic	Normal-frame	Yes	\mathcal{Q}_D , \mathcal{P}_D , \mathcal{T}_D	$\frac{d}{dt} = \frac{N}{\gamma}u^\mu\partial_\mu$

^g However, in contrast to other papers, the averages of the intrinsic dynamical quantities alone (multiplied by N^2) do not appear explicitly: the dynamical variables appearing in the averaged equations are actually averages of the local normal-frame variables as expressed in terms of the local intrinsic ones through the tilt.

^h In the application paper [65], where the averaged energy constraint is derived for five proposals of effective Hubble parameter choices (see footnote ^d above), the kinematic variables appearing in the equations are the best-suited for each case: based on the normal frames in the first two cases, mixed in the third and fourth cases, and intrinsic in the last case. The backreaction terms introduced there also depend on the Hubble parameter choice and can be either only \mathcal{Q} , \mathcal{Q} and \mathcal{P} , or \mathcal{Q} and another backreaction denoted \mathcal{L}_D , with a different expression for \mathcal{Q} in each case.

ⁱ However, in the $3+1$ form of the equations, the intrinsic dynamical variables ϵ and p are used instead, which allows for an explicit separation of the difference to the average of the normal-frame variables, corresponding to the ‘stress-energy backreaction’ of the present work.

^j Two forms of the equations are given, with an explicit separation of the contribution of the dynamical variables as seen either in the normal frames, or in an independent, general frame.

Table 1 Summary of the main differences between the various generalizations of the averaged scalar $3+1$ Einstein equations to general foliations suggested in the literature. (continued)

Additional specificities of some of the papers compared here:

- Räsänen [54]: The velocity field \mathbf{u} that is introduced in addition to \mathbf{n} is fully general and is not related either to \mathbf{n} nor to the content of the model universe (it could be chosen to be a fluid's velocity as in the present approach, but this would be a restriction of generality). It is supposed to represent the normalized velocity field of the observers. In the application paper [55], this field is restricted to be everywhere very close to \mathbf{n} (and so has a small vorticity), whereas \mathbf{n} is assumed to be chosen such that it builds hypersurfaces of statistical homogeneity and isotropy. These restrictions are already both suggested in the original paper [54] but the equations are kept general.
- Beltrán Jiménez *et al.* [7]: The main objective of using a general $T_{\mu\nu}$ in this paper is to account for theories beyond General Relativity whose differences are transferred into $T_{\mu\nu}$ as effective terms. Note also that this paper features an additional average equation giving the evolution of the averaged shear scalar $\partial_t \langle \sigma^2 \rangle$, as well as the corresponding local equation; these equations are absent from the other papers quoted in this appendix, including the present work (the reason being that the resulting system is still not closed by adding this equation; work about looking deeper into the hierarchy of equations is in preparation).
- Smirnov [61]: not only the choice of hypersurfaces (or of \mathbf{n}) and the choice of the time that parametrizes them are specific in this paper, but this is also the case for the domain, although this is not explicit in the equations and it could as well be any domain evolving along this specific \mathbf{n} . Indeed, the domain is there chosen as a 'sphere' in some \mathbf{n} -comoving coordinates Z^i on the hypersurfaces, as defined by $H_{ij} Z^i Z^j \leq r_0$ for some $r_0 > 0$ and with H_{ij} the components of the spatial metric in these coordinates. This choice was a response to the series of papers of Gasperini *et al.* and Marozzi [37, 38, 49] to show how it is possible to determine the boundary of the domain *via* a scalar function (here in the sense that the Z^i are fixed *a priori* without link to the actual spacetime coordinate choices).
- In the present work, we also introduce, in section 4, a different averaging formalism that measures scalar quantities and volume in the local rest frames of the fluid, even if they are integrated over a domain lying in the not necessarily fluid-orthogonal hypersurfaces. We then obtain the corresponding commutation rule and averaged dynamics under rather simple forms as expressed in terms of the *intrinsic* dynamical quantities of the fluid (for instance, the effective Hubble parameter, still defined as 1/3 of the volume expansion rate, can be simply expressed as the average of $\frac{N}{\gamma} \Theta$) and only two backreactions, kinematical and dynamical, distinct in general from the terms $\mathcal{Q}_{\mathcal{D}}$, $\mathcal{P}_{\mathcal{D}}$ of section 3. This formalism and this system of equations clearly differ from the literature compared in this Appendix (including our section 3, although it otherwise follows the same setup), due to the different volume and averaging operator definition.

D Erratum

We wish to point out a small mistake in Paper II [16] (repeated in the appendix of [18] after Equations (A23) and (A28) therein). For this we recall that the spatial components of the acceleration, $a_i = N_{||i}/N$, the 4-divergence $\mathcal{A} := \nabla_\mu a^\mu = a^i_{||i} + a^i a_i$, and its expression in terms of the lapse N or the injection energy per fluid element h (related to the relativistic enthalpy), respectively,

$$\mathcal{A} = \left(\frac{N_{||i}}{N} \right)_{||i} + \frac{N_{||i} N_{||i}}{N^2} = \frac{N_{||i}}{N} = h \left(\frac{1}{h} \right)_{||i} = -\frac{h_{||i}}{h} + 2 \frac{h_{||i} h_{||i}}{h^2} ,$$

are correctly written, but the first equality in Equation (10a) of Paper II (and of its review in [18]) is incorrect, $\mathcal{A} \neq (N_{||i}/N)_{||i}$, due to an omission of the $a^i a_i$ contribution to \mathcal{A} here.

There is also an imprecise statement: in Paper II, in footnote 3, it is stated that for scalars the operator $||$ amounts to a partial derivative. This statement is only true for spatial components (for a scalar, $||_i := \partial_i$, but $||_0 = N^i \partial_i \neq \partial_t$; $||_0$ was identically zero for scalars due to the vanishing shift in Paper II).

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LETTER TO THE EDITOR

Cosmological backreaction and its dependence on spacetime foliation

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Abstract. The subject of cosmological backreaction in General Relativity is often approached by coordinate-dependent and metric-based analyses. We present in this letter an averaging formalism for the scalar parts of Einstein’s equations that is coordinate-independent and only functionally depends on a metric. This formalism is applicable to general $3 + 1$ foliations of spacetime for an arbitrary fluid with tilted flow. We clarify the dependence on spacetime foliation and argue that this dependence is weak in cosmological settings. We also introduce a new set of averaged equations that feature a global cosmological time despite the generality of the setting, and we put the statistical nature of effective cosmologies into perspective.

Keywords: cosmology—foliations—Lagrangian description—backreaction

1. Context

Cosmology deals with models for the evolution of the Universe and, within General Relativity, entails the question of how to split the 4-dimensional spacetime into a 3-dimensional space evolving in time. This question can be formally answered by a $1 + 3$ threading along a preferred timelike congruence (see, e.g. [16, 23]), or by a $3 + 1$ slicing (foliation) into a family of spacelike hypersurfaces (see, e.g. [2, 18]). Both decompositions introduce four degrees of freedom, which are given in terms of a lapse function and a shift vector (or one-form). We shall consider the threading and slicing approaches jointly to formalize configurations where the fluid content is described by a 4-velocity tilted with respect to the hypersurface normal. *A priori*, only in special cases does the slicing keep the proper time of the fluid elements synchronous.

In standard cosmology one commonly idealizes the geometry of the Universe by a homogeneous-isotropic background metric with constant spatial curvature. In the case of the so-called concordance or Λ CDM model (“Cold Dark Matter with dark energy modeled by the cosmological constant Λ ”), the metric form features a global time t labeling Euclidean spatial sections that admit global coordinates x^i , with a global

rescaling factor $a(t)$,¹ ${}^4\mathbf{g}^{\text{hom}} = -\mathbf{d}t^2 + a^2(t)\delta_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j$. It is known that other choices of slicing change Friedmann's equations which determine the only gravitational degree of freedom, $a(t)$. See, e.g. [5]. The scale factor may even become space-dependent, for instance for a general slicing lapse. The problem of dependence on spacetime foliation therefore also exists in standard cosmology, where it is solved by choosing a preferred (proper time) foliation anchored to the Cosmic Microwave Background (CMB) rest frame.

Nonlinear structure formation in cosmology is most often investigated within the Newtonian framework of self-gravitating fluids. Efforts to describe relativistic models of inhomogeneities often rely on metric forms that are designed to be 'close to' the homogeneous-isotropic metric form above. The idea is to describe 'small' perturbations, which is a sensible assumption for metric perturbations, since they are indeed very small except in the vicinity of strong field astrophysical objects [22].² In the *longitudinal gauge*, the lapse function and the spatial 3-metric are referred to a 'perturbed Newtonian setting', with an assumed metric form for the physical spacetime,

$${}^4\mathbf{g}^{\text{pert}} = -N^2\mathbf{d}t^2 + g_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j, \quad (1)$$

where the slicing lapse N and the 3-metric coefficients g_{ij} of a family of spacelike hypersurfaces $t = \text{const.}$ are written as (here for scalar perturbations only):³

$$N^2 \equiv 1 + 2\Phi(t, x^i) \quad ; \quad g_{ij} \equiv a^2(t)[1 - 2\Psi(t, x^i)]\gamma_{ij}. \quad (2)$$

We note that the extrinsic curvature has no trace-free part, i.e. for a fluid 4-velocity parallel to the normal congruence $N^{-1}\partial_t$ the above metric describes homogeneous solutions in cosmologically relevant cases [12, 13, 14].⁴

Metric forms that are designed to stay 'close to' a homogeneous solution are also used to address the backreaction problem by devising simulations that include relativistic corrections. As an example we read in [1] (see also references therein) that 'the backreaction from structure can differ by many orders of magnitude depending upon the slicing of spacetime one chooses to average over'. We shall confront this statement with a covariant and background-free result about averaged dynamics that allows us to discuss the foliation dependence of backreaction without the need to consider gauge transformations.

¹ We adopt the conventions that Greek indices are assigned to spacetime indices running in $\{0, 1, 2, 3\}$, and Latin indices refer to space indices, running in $\{1, 2, 3\}$. The signature of the metric is taken as $(- + + +)$, and the units are such that $c = 1$.

² However, the derivatives of the metric can be large. Even for metric perturbations of order 10^{-6} , curvature perturbations can be of order unity and therefore out of reach in this setting [9]. Green and Wald [19] have modified earlier statements of [22] emphasizing that curvature can be large. (Their statement of trace-free backreaction, however, has no physical justification [10].)

³ Here, $a(t)$ denotes the same scale factor as in the homogeneous-isotropic case, which follows by setting $0 = \Phi = \Psi$; γ_{ij} denotes a constant curvature metric that is commonly considered to be flat, $\gamma_{ij} = \delta_{ij}$.

⁴ It is commonly assumed that the 4-velocity is tilted with respect to the normal congruence, but that spatial velocities are non-relativistic, i.e. that the Lorentz factor γ is close to 1. Our remark implies that by replacing the approximate sign by an equality sign the fluid has to be *shear-free* in the metric form $\{(1) \text{ and } (2)\}$ and, hence, *homogeneous* in cosmologically relevant cases [12, 13, 14].

We finally emphasize that cosmological backreaction can only be present if the average spatial curvature, and hence the large-scale average of cosmological variables, are allowed to evolve. Schemes that suppress average curvature evolution by, e.g. employing periodic boundary conditions as in Newtonian models [7], cannot describe global backreaction, but only backreaction in the interior of an assumed background model, i.e. ‘cosmic variance’.⁵

2. Explicit foliation dependence of backreaction

Cosmological backreaction is the study of inhomogeneity effects on the global evolution of the model universe. This involves averaging strategies which can for instance be unambiguously defined on the basis of volume averages of scalars. For *irrotational dust* and *irrotational perfect fluids* the answer has been given in terms of volume averaged scalar parts of the Einstein equations in [4, 5, 6]. This yielded cosmological equations of Friedmannian form for an effective energy-momentum tensor including averages of (extrinsic and intrinsic) curvature invariants of geometrical inhomogeneities in fluid-orthogonal spatial domains. These results are background-free, they depend on the averaging domain (e.g. on spatial scale), and they imply a dependence on the metric only via the morphology of the domain and the volume element of integration. As we shall discuss, this implicit dependence on the metric can be exploited for a statistical interpretation of the effective cosmological equations.

In a forthcoming investigation we derive the scalar-averaged equations for arbitrary $3 + 1$ foliations with general tilted fluid flow [11]. There, we discuss in detail relations to other works where such generalizations are offered. These earlier proposals focus on an *extrinsic approach*, i.e. they perform averages of the geometrical variables as seen by hypersurface observers. As we also discuss in [11], this approach inherits problems such as the non-conservation of the number of fluid elements within the averaging domain as it evolves.

We present in this letter the general scalar-averaged equations derived from an *intrinsic approach*, therefore following the spirit of the original works [4, 5]. Specifically, we perform averages of the fluid variables as seen by fluid observers. We consider an arbitrary spatial foliation which can be tilted with respect to the fluid congruence; this is necessary for a general flow as a fluid-orthogonal foliation is impossible as soon as the fluid has nonzero vorticity [17]. Accordingly, *local* spacelike projections can be performed onto the local tangent spaces of the hypersurfaces of the foliation along their normal \mathbf{n} , with $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, or onto the rest frames of the fluid elements along their 4-velocity \mathbf{u} , with $b_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. These projectors define two covariant volume measures on the tangent spaces of the hypersurfaces: $\sqrt{\det(h_{ij})}d^3x$ and $\sqrt{\det(b_{ij})}d^3x = \gamma\sqrt{\det(h_{ij})}d^3x$, with x^i arbitrary local spatial coordinates, and γ the Lorentz factor given by the fluid

⁵ Theoretical foundations of the cosmological backreaction effect via structure-emerging average spatial curvature may be found in [4, 6]. (See also illustrations within a class of background-free simulations in [3].)

spatial velocity \mathbf{v} , as a measure of the local tilt between \mathbf{n} and \mathbf{u} , as follows:

$$\gamma = \frac{1}{\sqrt{1 - v^\alpha v_\alpha}} \quad ; \quad u^\mu = \gamma(n^\mu + v^\mu) \quad ; \quad n^\alpha v_\alpha = 0 . \quad (3)$$

We associate accordingly to the same averaging domain \mathcal{D} lying in the hypersurfaces two different volumes: the Riemannian volume $V_{\mathcal{D}}^h \equiv \int_{\mathcal{D}} \sqrt{\det(h_{ij})} d^3x$, and the fluid *proper volume*, $V_{\mathcal{D}}^b \equiv \int_{\mathcal{D}} \sqrt{\det(b_{ij})} d^3x$. The former appears on average Lorentz-contracted with respect to the latter: introducing the *proper volume averager*, defined for any scalar φ as $\langle \varphi \rangle_{\mathcal{D}}^b \equiv 1/V_{\mathcal{D}}^b \int_{\mathcal{D}} \varphi \sqrt{\det(b_{ij})} d^3x$, we have $V_{\mathcal{D}}^h = V_{\mathcal{D}}^b \langle 1/\gamma \rangle^b$, which shows identity in the absence of tilt, i.e. when $\gamma = 1$. The integral is here again performed over a domain lying within the hypersurfaces of normal \mathbf{n} . As we shall only consider proper volume averages in the following, we shall omit the index b for notational ease.

We apply the averaging operator to the scalar parts of the Einstein equations over a compact domain \mathcal{D} lying within the hypersurfaces. Following [4, 5], \mathcal{D} is chosen to be a *comoving domain*, i.e. it is transported along the fluid congruence, which ensures the absence of matter flow across its boundaries and the preservation of its total rest mass. From this procedure we obtain the following *expansion* and *acceleration laws*, together with their *integrability condition*, for rescaled kinematic fluid variables (the squared rates of expansion, shear and vorticity, $\tilde{\Theta}^2 \equiv M^2 \Theta^2$, $\tilde{\sigma}^2 \equiv M^2 \sigma^2$, $\tilde{\omega}^2 \equiv M^2 \omega^2$), energy density and pressure ($\tilde{\epsilon} \equiv M^2 \epsilon$, $\tilde{p} \equiv M^2 p$), divergence of the fluid's 4-acceleration a^μ ($\tilde{\mathcal{A}} \equiv M^2 \mathcal{A}$, with $\mathcal{A} \equiv \nabla_\mu a^\mu$), and fluid 3-curvature ($\tilde{\mathcal{R}} \equiv M^2 \mathcal{R}$):⁶

$$\begin{aligned} 3 \frac{1}{a_{\mathcal{D}}} \frac{d^2 a_{\mathcal{D}}}{dt^2} &= -4\pi G \left\langle \tilde{\epsilon} + 3\tilde{p} \right\rangle_{\mathcal{D}} + \tilde{\Lambda}_{\mathcal{D}} + \tilde{\mathcal{Q}}_{\mathcal{D}} + \tilde{\mathcal{P}}_{\mathcal{D}} ; \\ 3 \left(\frac{1}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \right)^2 &= 8\pi G \left\langle \tilde{\epsilon} \right\rangle_{\mathcal{D}} + \tilde{\Lambda}_{\mathcal{D}} - \frac{1}{2} \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}} - \frac{1}{2} \tilde{\mathcal{Q}}_{\mathcal{D}} ; \\ \frac{d}{dt} \tilde{\mathcal{Q}}_{\mathcal{D}} + 6H_{\mathcal{D}} \tilde{\mathcal{Q}}_{\mathcal{D}} + \frac{d}{dt} \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}} + 2H_{\mathcal{D}} \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}} + 4H_{\mathcal{D}} \tilde{\mathcal{P}}_{\mathcal{D}} \\ &= 16\pi G \left(\frac{d}{dt} \left\langle \tilde{\epsilon} \right\rangle_{\mathcal{D}} + 3H_{\mathcal{D}} \left\langle \tilde{\epsilon} + \tilde{p} \right\rangle_{\mathcal{D}} \right) + 2 \frac{d}{dt} \tilde{\Lambda}_{\mathcal{D}} . \end{aligned} \quad (4)$$

The first terms on the right-hand side of the last equation also obey an averaged energy balance equation sourced by the non-perfect-fluid parts of the energy-momentum tensor. We observe a time- and domain-dependent contribution from the cosmological constant, $\tilde{\Lambda}_{\mathcal{D}} \equiv \Lambda \langle N^2/\gamma^2 \rangle_{\mathcal{D}}$, and two terms $\tilde{\mathcal{Q}}_{\mathcal{D}}$ and $\tilde{\mathcal{P}}_{\mathcal{D}}$ denoting the intrinsic *kinematical* and *dynamical backreaction* terms, respectively. These are defined in terms of the rescaled

⁶ We defined $M \equiv N/\gamma$ (the *threading lapse* in a 1 + 3 threading of spacetime). The hypersurfaces are parametrized by a monotonic scalar function t . From it we can define the *comoving time-derivative* d/dt as the derivative with respect to t along the fluid flow lines, and the *effective Hubble rate* $H_{\mathcal{D}} \equiv (da_{\mathcal{D}}/dt)/a_{\mathcal{D}}$ for the *volume scale factor* $a_{\mathcal{D}} \equiv (V_{\mathcal{D}}/V_{\mathcal{D}i})^{1/3}$. Θ , $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are, respectively, the trace, the symmetric traceless part, and the antisymmetric part of the projected 4-velocity gradient, $b^\alpha_\mu b^\beta_\nu \nabla_\alpha u_\beta$. $\sigma^2 := (1/2)\sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\omega^2 := (1/2)\omega_{\mu\nu}\omega^{\mu\nu}$ define the rates of shear and vorticity. The 'fluid 3-curvature' \mathcal{R} is defined from the energy constraint in the fluid rest frames, $\mathcal{R} \equiv -(2/3)\Theta^2 + 2\sigma^2 - 2\omega^2 + 16\pi G\epsilon + 2\Lambda$ (see [15]), and reduces to the 3-Ricci scalar of the fluid-orthogonal hypersurfaces for vanishing vorticity.

fluid variables as follows:

$$\begin{aligned}\tilde{Q}_{\mathcal{D}} &\equiv \frac{2}{3} \left\langle \left(\tilde{\Theta} - \langle \tilde{\Theta} \rangle_{\mathcal{D}} \right)^2 \right\rangle_{\mathcal{D}} - 2 \langle \tilde{\sigma}^2 \rangle_{\mathcal{D}} + 2 \langle \tilde{\omega}^2 \rangle_{\mathcal{D}} ; \\ \tilde{\mathcal{P}}_{\mathcal{D}} &\equiv \langle \tilde{\mathcal{A}} \rangle_{\mathcal{D}} + \left\langle \tilde{\Theta} \frac{\gamma}{N} \frac{d}{dt} \left(\frac{N}{\gamma} \right) \right\rangle_{\mathcal{D}} .\end{aligned}\quad (5)$$

The *dynamical backreaction* thus consists of an acceleration 4–divergence and of a contribution that captures the *rate of desynchronization of the clocks*, with the proper time τ of the fluid obeying $d\tau/dt = N/\gamma = M$. By defining an effective diagonal energy-momentum tensor with the following effective sources:⁷

$$\begin{aligned}\epsilon_{\text{eff}} &\equiv \langle \tilde{\epsilon} \rangle_{\mathcal{D}} - \frac{\tilde{Q}_{\mathcal{D}}}{16\pi G} - \frac{\tilde{\mathcal{W}}_{\mathcal{D}}}{16\pi G} + \frac{\tilde{\mathcal{L}}_{\mathcal{D}}}{8\pi G} ; \\ p_{\text{eff}} &\equiv \langle \tilde{p} \rangle_{\mathcal{D}} - \frac{\tilde{Q}_{\mathcal{D}}}{16\pi G} + \frac{\tilde{\mathcal{W}}_{\mathcal{D}}}{48\pi G} - \frac{\tilde{\mathcal{L}}_{\mathcal{D}}}{8\pi G} - \frac{\tilde{\mathcal{P}}_{\mathcal{D}}}{12\pi G} ,\end{aligned}\quad (6)$$

the set of effective cosmological equations can be cast into ‘Friedmannian form’:

$$\begin{aligned}3 \left(\frac{1}{a_{\mathcal{D}}} \frac{da_{\mathcal{D}}}{dt} \right)^2 &= 8\pi G \epsilon_{\text{eff}} - 3 \frac{k_{\mathcal{D}}}{(a_{\mathcal{D}})^2} + \Lambda ; \\ 3 \frac{1}{a_{\mathcal{D}}} \frac{d^2 a_{\mathcal{D}}}{dt^2} &= -4\pi G (\epsilon_{\text{eff}} + 3 p_{\text{eff}}) + \Lambda ; \\ \frac{d}{dt} \epsilon_{\text{eff}} + 3 H_{\mathcal{D}} (\epsilon_{\text{eff}} + p_{\text{eff}}) &= 0 ,\end{aligned}\quad (7)$$

where the last equation, the *effective energy conservation law*, is equivalent to the *integrability condition*. The set of equations (7) needs a closure condition, e.g. an *effective equation of state* that relates ϵ_{eff} , p_{eff} and $a_{\mathcal{D}}$.

3. Effective cosmological equations in the fluid proper time foliation

Starting from an arbitrary Cauchy hypersurface, one can globally construct a $3 + 1$ foliation the slices of which are obtained by transporting the initial hypersurface through the (general) 4–velocity \mathbf{u} of the fluid. Each hypersurface of this foliation corresponds to a constant value of proper time τ , measured along the fluid world lines and being set to $\tau_{\mathbf{i}} \equiv t_{\mathbf{i}}$ on the initial slice. The proper time τ can thus be used to label the hypersurfaces, defining a global time parameter. The same construction can be performed from any choice of the initial Cauchy hypersurface, identifying what we call the class of *fluid proper time foliations*. (See also [16], chapter 4.1.)

Such a construction sets the normal vector \mathbf{n} and the lapse N , which in this case equals the Lorentz factor: $N = \gamma$. A natural choice for the shift vector \mathbf{N} would be

⁷ We have defined new backreaction variables: $\tilde{\mathcal{W}}_{\mathcal{D}}$ for the deviation of the averaged fluid 3–curvature from a constant-curvature behaviour, $\tilde{\mathcal{W}}_{\mathcal{D}} \equiv \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}} - 6k_{\mathcal{D}}/(a_{\mathcal{D}})^2$, and $\tilde{\mathcal{L}}_{\mathcal{D}}$ for the deviation from the cosmological constant Λ , $\tilde{\mathcal{L}}_{\mathcal{D}} \equiv \tilde{\Lambda}_{\mathcal{D}} - \Lambda$. $k_{\mathcal{D}}$ is an *a priori* domain-dependent arbitrary constant which can be set to $k_{\mathcal{D}} \equiv (a_{\mathcal{D}})^2(t_{\mathbf{i}}) \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}}(t_{\mathbf{i}})/6$. In the standard cosmological model it is assumed that the cosmological constant Λ models Dark Energy; the averaged equations show that we then also have to account for *Dark Energy backreaction* $\tilde{\mathcal{L}}_{\mathcal{D}}$ in cases where $N \neq \gamma$.

$\mathbf{N} = N\mathbf{v}$ (for which $N = \gamma$ implies $N^2 - N^\mu N_\mu = 1$), identifying the points on each hypersurface that correspond to the same fluid element. However, the choice of a shift does neither affect the definition of our averaging formalism nor the resulting averaged equations. Apart from the case of irrotational dust, the hypersurfaces of a fluid proper time foliation cannot be fluid-orthogonal, namely a tilt must be present. As we shall see, this choice carries a number of advantages in the context of the averaging problem.

Within a fluid proper time foliation, the general *volume expansion* and *acceleration laws* for the fluid scale factor $a_{\mathcal{D}}$ (together with their *integrability condition*), (4), reduce to the following effective cosmological equations:

$$\begin{aligned} 3 \left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 &= 8\pi G \epsilon_{\text{eff}} - 3 \frac{k_{\mathcal{D}}}{(a_{\mathcal{D}})^2} + \Lambda \quad ; \quad 3 \frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G (\epsilon_{\text{eff}} + 3 p_{\text{eff}}) + \Lambda \ ; \\ \dot{\epsilon}_{\text{eff}} + 3 H_{\mathcal{D}} (\epsilon_{\text{eff}} + p_{\text{eff}}) &= 0 \ . \end{aligned} \quad (8)$$

The overdot denotes the covariant derivative with respect to proper time. The effective energy density ϵ_{eff} and effective pressure p_{eff} , as defined in (6), become

$$\begin{aligned} \epsilon_{\text{eff}} &= \langle \epsilon \rangle_{\mathcal{D}} - \frac{\mathcal{Q}_{\mathcal{D}}}{16\pi G} - \frac{\mathcal{W}_{\mathcal{D}}}{16\pi G} \ ; \\ p_{\text{eff}} &= \langle p \rangle_{\mathcal{D}} - \frac{\mathcal{Q}_{\mathcal{D}}}{16\pi G} + \frac{\mathcal{W}_{\mathcal{D}}}{48\pi G} - \frac{\mathcal{P}_{\mathcal{D}}}{12\pi G} \ , \end{aligned} \quad (9)$$

with $\mathcal{Q}_{\mathcal{D}}$ as given by (5) with non-rescaled variables (since here $M = 1$), and where the dynamical backreaction reduces to $\mathcal{P}_{\mathcal{D}} = \langle \mathcal{A} \rangle_{\mathcal{D}}$, removing the contribution from clock desynchronization. The cosmological constant deviation $\tilde{\mathcal{L}}_{\mathcal{D}}$ vanishes, and the curvature deviation term $\tilde{\mathcal{W}}_{\mathcal{D}}$ reduces to $\mathcal{W}_{\mathcal{D}} = \langle \mathcal{R} \rangle_{\mathcal{D}} - 6k_{\mathcal{D}}/(a_{\mathcal{D}})^2$.

We emphasize that the above system and the corresponding proper time foliation choices are covariantly defined, i.e. are coordinate-independent [20]. For concrete calculations of local variables, a specific set of coordinates may then be chosen depending on the problem being investigated. For instance, for the formation of structure in relativistic Lagrangian perturbation theory [8], an appropriate set can be constructed as follows. First, as for the hypersurfaces labeled in terms of proper time, we can introduce *spatial labels* X^i to identify each fluid element in the general threading congruence defined by \mathbf{u} , which can always be relabeled in this covariant framework. (The spatial labels X^i provide the same identification of points as the shift vector choice $\mathbf{N} = N\mathbf{v}$.) Second, for any given foliation, these labels may be used as a set of spatial coordinates propagating along the fluid flow lines. These are *comoving* (or *Lagrangian*) spatial coordinates, where the spatial coordinate velocity (hence the spatial components of u^μ) vanish. We name this choice *comoving description* of the fluid, in conformity with the literature. This description is a ‘weak’ form of a *Lagrangian description* of the fluid where in addition τ is used as the time-coordinate. The coordinate assignment (X^i, τ) provides $u^\mu = (1, 0, 0, 0)$. This defines *Lagrangian observers* who in the standard model are called *fundamental observers*.

4. Conclusion and Discussion

Looking at the set of equations (4) and their backreaction terms (5) we appreciate that the explicit foliation dependence is solely given in terms of the threading lapse $M = N/\gamma$. In the *fluid proper time foliation* we have $M = 1$, which does not differ significantly from the value of the threading lapse in the metric form {(1) and (2)} for the usual assumptions $N = 1 + \varepsilon$; $|\varepsilon| \ll 1$ and $\gamma = 1 + \zeta$; $\zeta \ll 1$. The remaining foliation dependence of the amount of backreaction arises in the realization of the averaged model, when integration of local variables is performed over specific hypersurfaces that are not fully determined by N/γ due to the degeneracy of this ratio.

Let us now narrow down the class of relevant foliations, focussing on matter-dominated model universes. We think of a cosmological coarse-graining that smoothes over scales where vorticity, velocity dispersion and pressure play a role. In view of observations one can then reasonably assume the existence of a class of foliations where the hypersurfaces reflect statistical homogeneity and isotropy and in which the motions of all coarse-grained fluid elements are non-relativistic, i.e. $\gamma \simeq 1$, thus identifying a class relevant to cosmology (see also the related discussion in [24]). This implies that the tilt is negligible on these scales, $u^\mu \simeq n^\mu$, and, in view of the negligible pressure gradients over the coarse-graining scale, that the lapse function can be set to $N \simeq 1$.⁸ Overall this estimates M to be close to a Lagrangian description, $M \simeq 1$, while the domain of integration selected by the hypersurfaces is bound to small variations in spacetime, since these hypersurfaces are constrained to remain almost orthogonal to \mathbf{u} everywhere. Thus, these conditions imply only small variations of the large-scale backreaction terms (of the order of the deviations of the lapse and the Lorentz factor from 1) under a change of cosmological spacetime foliation. Explicit bounds on such variations will be investigated in a forthcoming paper [21]. These covariantly defined requirements cannot be reproduced in a coordinate-dependent setting such as that used in [1]. The variations can of course be larger when going beyond this restricted class of foliations that are favoured on cosmological scales, as it would, e.g. be needed for evaluating backreaction on smaller scales. These scales, where tilt, vorticity and pressure gradients matter, can be treated as well within the general framework introduced in this letter.

We emphasize that the lapse and the Lorentz factor only depend on the normal vector flow, and not on its derivatives, allowing for strong constraints on variations of the backreaction with the foliation when the normal vector itself is constrained. In our formalism, the kinematical backreaction does not involve the extrinsic curvature, which depends on derivatives of the normal vector. It features instead derivatives of the relative velocities of the fluid elements (such as Θ). These foliation-independent scalars can be large despite velocities themselves being small (cf., footnote 2), allowing for large backreaction. We remark in this context that the fact that $M \simeq 1$ in the metric form {(1) and (2)}, together with the smallness assumptions made, does not mean that the

⁸ Another issue arises if we also consider the coarse-graining of ‘time’ that may accumulate an effective lapse during differing histories of voids and clusters, cf. the ‘timescape scenario’ [25].

estimates of backreaction in paper [1] fall within our conclusions about the small impact of the foliation choice. These authors employ an extrinsic averaging formalism where dependencies on *derivatives* of the normal vector \mathbf{n} (and, thus, on derivatives of N and γ) are introduced in the backreaction terms *via* the dependence on the extrinsic curvature of the hypersurfaces. This may lead to unphysical foliation dependence of backreaction, just because the variables to be averaged are defined from the hypersurfaces themselves, i.e. they characterize the properties of a family of extrinsic observers. (We consider this additional foliation dependence ‘unphysical’, since such observers only exist as a mathematical abstraction.)⁹

The *fluid proper time foliation* and its representation in terms of the *Lagrangian description* appear to be natural choices for an effective cosmology. These settings should not be disqualified in favour of a choice of foliation extrinsic to the fluid by emphasizing the need to avoid singularities. For example, evolving a dust matter model implies the development of shell-crossing, as discussed in [8], as a manifestation of the breakdown of the dust approximation. Improving the matter model may or may not avoid these or other (e.g. black hole) singularities.

A possible shortcoming of the proper time foliations relates to the spacelike character of the corresponding hypersurfaces generated from the evolution of a single fluid. While such foliations are always well-defined under the assumption that the fluid flow contains no singularity, one has to guarantee that the hypersurfaces, generated from the initial spacelike slice, remain spacelike for all times considered. This will hold at least locally in general and globally for an irrotational dust model with a fluid-orthogonal initial hypersurface (since the whole foliation will then be fluid-orthogonal). The construction of a proper time foliation is based on the choice of an initial Cauchy hypersurface, which has to be specified; it may be best anchored to the last scattering surface at the CMB epoch. These aspects have to be judged within specific applications.

The proper time choice can also be criticized because it requires following the details of inhomogeneities developing in the fluid. This latter view originates, however, from looking at a single realization of the fluid’s evolution and a single inhomogeneous metric. What the averaged equations embody goes beyond the picture obtained from a single realization of the metric. Changing the metric will change the morphology of the averaging domain and the volume element, but we are entitled to implement the cosmological model through a statistical ensemble of realizations. With this statistical interpretation of the averaged equations, the effective cosmological equations no longer trace individual metric variations as suggested by a one-metric-based picture. In this context, a further important question for the definition of statistical hypersurfaces will be whether the tilt, depending on physics on smaller scales, would average out to provide an effective flow-orthogonal foliation on cosmological scales. Follow-up work is dedicated

⁹ We also remark that if backreaction happens to be zero in one foliation (e.g. if subjected to a 3-torus constraint on a flat space section [7]), and if it is represented by a small number in its numerical realization, a still small but nonzero backreaction parameter in another foliation could suggest a *ratio* of several orders of magnitude, even if both estimates were in reality comparable.

to explicitly implementing these statistical aspects.

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Chapter 4

A manifestly 4-covariant form for averages and the averaged equations

The two averaging frameworks introduced in the previous chapter for a general choice of spatial slices are both volume-based but use different volume measures. They have been defined covariantly, as integrals based on scalars and covariant volume measures on a given three-dimensional domain selected in spacetime *via* a geometric construction. For the sake of transparency they have been written in the form obtained within a specific coordinate set adapted to the foliation, following the usual 3+1 framework. Nevertheless, as I will show below, building upon the formalism suggested in [Gasperini et al., 2009, 2010], a manifestly covariant form of the same volumes and averages definitions can also be written, and allows for a combination of both frameworks under the same form.

Spatial volumes and volume-averages of scalars are defined in [Gasperini et al., 2009, 2010] in terms of integrals over the whole spacetime manifold \mathcal{M} , with the spatial domain of averaging \mathcal{D} selected by a scalar window function $W_{\mathcal{D}}$: $I(\psi) \equiv \int_{\mathcal{M}} \sqrt{g} \psi W_{\mathcal{D}} d^4x$ for any scalar ψ defines the volume as $I(1)$ and the average of a scalar ψ as $I(\psi)/I(1)$. With the conventions of Gasperini et al. [2010], the window function is then defined as $W_{\mathcal{D}}(x^\nu) = n^\mu \nabla_\mu (H(A(x^\nu) - A_0)) H(B_0 - B(x^\nu))$ with H the Heaviside step function. In this writing, A is a ‘time’ scalar function whose level sets define the spatial hypersurfaces of averaging, the constant parameter A_0 labelling the current slice (the slice at time $A = A_0$), while the scalar function B and the constant B_0 select the spatial boundary of the averaging domain at all time (a point of coordinates x^μ is in the averaging domain at time $A = A(x^\mu)$ if, and only if, $B(x^\mu) \leq B_0$).

With a choice of 3+1 spacetime coordinates adapted to the foliation by constant- A slices, *i.e.*, choosing a time coordinate t as a function of A , the first (extrinsic) averaging framework presented in chapter 3 is recovered. The above definition is thus simply a manifestly 4-covariant form of this same averager. The only difference lies in the domain propagation remaining unspecified in the definition of Gasperini et al. [2010]. For the derivation of the averaged dynamical equations in the latter work, a propagation of the domain along the normal to the slices is assumed, rather

than an averaging domain comoving with the fluid flow.

The main body of this chapter below stems from the recently published work [Heinesen, Mourier, and Buchert, 2019] as a result of a collaboration with Asta Heinesen (University of Canterbury, New Zealand) and Thomas Buchert. In this study, we show how to generalize the above window function to encompass both of the averaging frameworks introduced above in chapter 3 as well as any similar spatial averaging procedure defined from another volume measure. This provides in particular a manifestly covariant form of the second, fluid-intrinsic averaging operator of chapter 3. Averages with a different weight than volume (such as mass-weighted averages) are also included in the class of window functions considered. Giving some examples of possible choices for the window function, we discuss the range of applicability of the general form and the possibility of its extension to averages over light cones (such as the formalisms introduced in [Gasperini et al., 2011]), which would be useful for an application to observations. We then derive the manifestly 4-covariant form of the averaged 3+1 Einstein equations and of the corresponding backreaction terms with such a general window function. In these results, the propagation of the domain between slices remains freely specifiable, but a special emphasis is still set on a domain comoving with a physical fluid flow.

The question of the quantitative dependence of averages and backreactions on the foliation choice, requiring an averaging formalism that allows for general spatial foliations, has not been fully addressed in the analysis given in chapter 3 of such a formalism. This was limited to a mostly qualitative discussion, due to the selection of the spatial hypersurfaces remaining implicit in the 3+1 form of the definition of these averages. The manifestly covariant form discussed in the present chapter, in which the foliation is explicitly selected by the scalar A , however, is well suited to the more explicit investigation of foliation dependence of averaging-based quantities. As of the writing of this thesis, I am pursuing the collaboration with Asta Heinesen on this question as a follow-up project to the study exposed in this chapter.

On the covariance of scalar averaging and backreaction in relativistic inhomogeneous cosmology

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Abstract. We introduce a generalization of the 4-dimensional averaging window function of Gasperini, Marozzi and Veneziano (2010) that may prove useful for a number of applications. The covariant nature of spatial scalar averaging schemes to address the averaging problem in relativistic cosmology is an important property that is implied by construction, but usually remains implicit. We employ here the approach of Gasperini et al. for two reasons. First, the formalism and its generalization presented here are manifestly covariant. Second, the formalism is convenient for disentangling the dependencies on foliation, volume measure, and boundaries in the averaged expressions entering in scalar averaging schemes. These properties will prove handy for simplifying expressions, but also for investigating extremal foliations and for comparing averaged properties of different foliations directly. The proposed generalization of the window function allows for choosing the most appropriate averaging scheme for the physical problem at hand, and for distinguishing between the role of the foliation itself and the role of the volume measure in averaged dynamic equations. We also show that one particular window function obtained from this generalized class results in an averaging scheme corresponding to that of a recent investigation by Buchert, Mourier and Roy (2018) and, as a byproduct, we explicitly show that the general equations for backreaction derived therein are covariant.

Keywords: general relativity—foliations—Lagrangian description—backreaction

1. Introduction

Cosmology is the discipline of describing overall dynamic properties of the Universe in a spatially and/or statistically averaged sense. For a cosmology founded on general relativistic principles, this aim is hard to obtain for at least two reasons:

- (i) In general relativity a global and canonical notion of time is not in general expected to exist. There is no unique and general way of extending the eigentime of a world line to a global time parameter at each point in space-time. Thus, global dynamics

is not easily defined since a natural ‘laboratory frame’ is missing. A cosmological model would usually describe congruences of fundamental observers following source fluid flows, and would naturally attempt to build global frames based on such a family of observers. However, the identification of observer congruences in our space-time, that ‘at present day’ involves a complicated hierarchy of structure, is a difficult task. Moreover, a congruence of fluid-comoving observers does not build global rest frames in presence of vorticity (expected to appear on small scales), so that alternative definitions of observers-based spatial sections may be required.

- (ii) Averages and statistical descriptions are not naturally formulated within general relativity. Tensor quantities are intrinsic to the tangent-space in which they live; while there are ways of mapping tensor quantities between tangent-spaces, such mappings are not unique. Furthermore, point particles as matter sources are not compatible with the formulation of general relativity. Projecting such a particle picture into a continuous space-time setting may for instance involve an extension to a curved manifold of the Newtonian procedure of coarse-graining particles in phase space by filtering the Klimontovich density and of forming appropriate moments. For these reasons statistical matter descriptions are highly involved in general relativity.

The standard paradigm of cosmology relies on pre-assuming a statistical geometry and a corresponding matter description (disentangled from curvature degrees of freedom). Assuming also decoupling of scales, approximate large-scale statistical homogeneity and isotropy is used as a motivation for taking the Friedman-Lemaître-Robertson-Walker (FLRW) class of metrics as an idealization for the average properties of the Universe on the largest scales. However, the FLRW class of metrics assumes local isotropy which results in a homogeneous geometry on all scales, not only on the largest scales. The assumption that the FLRW geometries match the average properties does not follow from first principles.

In the field of inhomogeneous cosmology we are interested in studying the failure of the FLRW idealization as an accurate description of geometry on the largest scales, meaning the failure of it to describe the average dynamics of inhomogeneities propagating on all scales and the motions of test particles through them. In general relativity geometry and matter couple *locally*. This core feature is missed by any large-scale description that neglects structure on small scales and only deals with coupling between an assumed large-scale geometry and averaged matter sources.

The usual approaches to describe structure on cosmological scales involve weak field approximations around a homogeneous background. However, typical weak field argumentation in cosmology has limitations. It is assumed that there is a *global* FLRW background metric around which the weak field is to be taken everywhere; clearly local potentials associated with most structures in our Universe are weak; the question in cosmology is what an appropriate *background* is for such a weak field limit [26]. Moreover, even if metric perturbations are small with respect to a global FLRW metric

over a spatial section of the Universe, their derivatives can be non-perturbative. This is the case for non-linear density fields (which are present at nested scales in our present-day Universe), in which case second-order derivatives of the potentials are necessarily non-linear (see, e.g. [6]). In such cases, the expansion of the Einstein equation into a zero-order FLRW part and a first-order part breaks down, and from first principles we would not expect the FLRW field equations to be satisfied as independent equations decoupled from the dynamics of structures.

Here, we shall focus on quantifications of the non-linear backreaction of smaller scales on the large scale evolution that involves averaging of ‘local’ quantities. We shall focus only on averaging schemes for space-time scalars as done in [2, 3], and later generalized by many authors (see, e.g. the reviews [9, 10] and references therein). We note that the fundamental problems in describing averaged cosmological dynamics as outlined in (i) and (ii) are not fully addressed in this form of averaging. In particular, the assumption of a ‘local’ fluid description, where fluid elements are implicitly coarse-grained by neglecting their internal curvature degrees of freedom, is built into the Buchert equations [2, 3] (see, e.g. [27]). However, we do not assume an averaged homogeneous and isotropic fluid as a source for a large-scale statistical geometry: geometry and matter couple at the fluid resolution scale. The average behaviour is formulated directly from the physics at this ‘local’ scale, and inhomogeneities at local scales appear explicitly in the resulting generalizations of the Friedmann equations, reflecting the non-commutativity of averaging and evolution in time.

In this work we introduce a 4-dimensional averaging window function that generalizes the window function presented in [13, 14] for integration over hypersurfaces. There are multiple purposes in doing so. First, we shall often be interested in a fluid-intrinsic averaging operation (when a fundamental fluid exists in our space-time); such intrinsic formulation will in general not be compatible with the class of window functions considered in [13, 14]. Second, the generalized scheme allows for maximal freedom in the choices of averaging domain and volume measure, while still being compact and easy to interpret. Covariance is built explicitly into the averaging scheme, guaranteeing that any generalization of the Buchert scheme formulated from this will be coordinate-independent by construction. Third, the introduction of the new window function has applications for further investigations on extremal foliations and on the dependence of averaged quantities on the foliation. Such studies are beyond the scope of this paper, but will be considered in a future paper [16].

We are solely concerned with *covariance* here; we do not consider gauge-invariance as defined in standard model perturbation theory.‡ In standard model perturbation theory the fields of interest are perturbation degrees of freedom of the space-time metric defined relative to a background metric. These fields are defined in terms of components of the metric and the background metric and *do not* transform as tensors in the differential geometry definition of a tensor, i.e. they are not covariant. This

‡ We emphasize the focus of this paper on covariant variables only, in distinction to [13] where both covariance and standard model perturbation theory gauge invariance are discussed.

includes the Bardeen variables, which are ‘gauge-invariant’ in this context, i.e. they are invariant under first-order changes of the diffeomorphism between the background manifold and the physical space-time manifold, but they are not 4–scalars.

We emphasize that there is no reference to a background space-time in the context of this paper, and that we use the conventional general relativistic wording throughout. When referring to scalar degrees of freedom we mean quantities that do not transform under arbitrary coordinate transformations. When we refer to ‘gauge’ degrees of freedom in this paper, this will be in the broad sense of the word, i.e. as redundant degrees of freedom in the parameterization of a physical system.

This paper is organized as follows. In section 2 we introduce the averaging scheme as formulated in terms of a covariant window function. We discuss the interpretation of the generalized adapted volume measure entering this scheme and we give examples of relevant subcases. In section 3 we discuss the commutation rule for such an averaging operation and apply it to the conservation of regional rest mass. The averaged Einstein equations for a general fundamental fluid source are derived in section 4 for a general window function, expressed in such a way that boundary terms vanish by construction, except for the average energy conservation law. We consider domains propagated along the fluid world lines as a special case that allow for a more transparent interpretation of the averaged equations. We conclude in section 5.

2. The averaging scheme

We now introduce the averaging scheme used to quantify averaged dynamics in this paper. This averaging formalism is a direct generalization of that presented in [14], the difference being that we allow for an arbitrary volume measure on the selected hypersurfaces. We discuss the interpretation of the generalized volume measure, and highlight several relevant subcases of the averaging scheme in relation to the existing literature.

2.1. The window function

Following [13, 14] we consider scalar functions integrated over space-time domains that are selected out of the space-time 4–manifold \mathcal{M} by appropriate choices of window functions. In the context of this paper we shall consider window functions that single out compact regions of 3–dimensional spatial hypersurfaces. Averaging over 3–dimensional hypersurfaces is natural when we want to describe the evolution of averaged properties of spatial sections of the Universe.

Here we shall consider a slightly broader class of $3 + 1$ window functions than in [13, 14], to allow for arbitrary positive volume measures on the hypersurface of integration. Hence, we do not restrict ourselves to having the volume measure coincide with the adapted volume measure in the frame of the foliation. Such a more general volume measure is natural in several settings, some of which we shall investigate below.

This furthermore allows us to make explicit which properties of the averaged expressions are related to the foliation and which are related to the volume measure. When investigating foliation dependence [16] the separation of these contributions will be useful.

We shall consider the broad class of window functions

$$W_{A,A_0,B,B_0,\mathbf{V}} = -V^\mu \nabla_\mu (\mathcal{H}(A_0 - A)) \mathcal{H}(B_0 - B) = (V^\mu \nabla_\mu A) \delta(A_0 - A) \mathcal{H}(B_0 - B), \quad (1)$$

where A is a scalar with time-like gradient that determines the spatial foliation of integration (with hypersurfaces $A = \text{const.}$) and B is a scalar with space-like (or possibly null) gradient that is used to bound the averaging domain. A_0 and B_0 are constants that respectively select a specific hypersurface of the foliation ($A = A_0$) and the domain's spatial boundary ($B = B_0$). \mathbf{V} is an arbitrary time-like vector field, that need not be normalized, and that will in general not be normal to the hypersurfaces defined by A . \mathcal{H} is the unit step function; we use the convention $\mathcal{H}(0) = 1$ throughout. We shall call A the hypersurface scalar, B the boundary scalar, and \mathbf{V} the volume measure vector. We shall drop the subscripts denoting the dependencies of W in the following.

This form of the window function generalizes that of [14] through the freedom of choice of the volume measure vector, which in [14] is restricted to being the unit normal vector \mathbf{n} to the hypersurfaces defined by A . \mathbf{V} determines the volume measure on the hypersurfaces defined by A . This corresponds to considering the usual oriented volume element

$$dV^\lambda = -n^\lambda \frac{\sqrt{g}}{6} n^\mu \epsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \quad ; \quad n_\mu = \frac{-\nabla_\mu A}{(-g^{\nu\sigma} \nabla_\nu A \nabla_\sigma A)^{1/2}}, \quad (2)$$

(where $g \equiv -\det(g_{\mu\nu})$, and ϵ is the Levi-Civita symbol) projected along the vector \mathbf{V} . Thus, the integration measure that we use on the surfaces defined by constant A is

$$d\mathcal{V} \equiv V_\mu dV^\mu. \quad (3)$$

We can think of $V_\mu dV^\mu$ as the flux of \mathbf{V} through the infinitesimal volume dV^μ .

If \mathbf{V} is taken to be the normal vector \mathbf{n} to the $A = \text{const.}$ hypersurfaces, we simply recover the Riemannian volume measure of the hypersurfaces, $d\mathcal{V} = n_\mu dV^\mu$. Alternatively, we may take the volume measure vector \mathbf{V} to be a 4-velocity field \mathbf{u} of physical interest, in general tilted with respect to the normal \mathbf{n} . In this case, the integration measure defined in (3) becomes

$$\begin{aligned} d\mathcal{V} &\equiv u_\mu dV^\mu = -u_\mu n^\mu \frac{\sqrt{g}}{6} n^\lambda \epsilon_{\lambda\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= \gamma \frac{\sqrt{g}}{6} n^\lambda \epsilon_{\lambda\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= \gamma \frac{\sqrt{g}}{6} (-\nabla_\nu A \nabla^\nu A)^{-1/2} \epsilon_{ijk} d\bar{x}^i \wedge d\bar{x}^j \wedge d\bar{x}^k = \gamma n_\mu dV^\mu, \end{aligned} \quad (4)$$

where $\bar{x}^\mu = (A, \bar{x}^i)$ is an adapted coordinate system to the foliation of A , and where $\gamma \equiv -\mathbf{u} \cdot \mathbf{n}$ is the tilt, or Lorentz factor, between the normal of the hypersurfaces and the 4-velocity \mathbf{u} . The infinitesimal volume element $d\mathcal{V}$ measures the local proper volume (around $A = A_0$) of the fluid element defined by the infinitesimal fluid flow tube

that intersects the hypersurface $\{A = A_0\}$ at the points of the time coordinate (in the \bar{x}^μ basis) $A = A_0$ and of the spatial coordinates spanning the range $[\bar{x}^i, \bar{x}^i + d\bar{x}^i]$. The Riemannian volume measure $n_\mu dV^\mu$ of this fluid element as it intersects the hypersurface $\{A = A_0\}$, is its volume measure in the frame defined by \mathbf{n} , and it is thus Lorentz-contracted with respect to $d\mathcal{V}$. Hence, the choice $\mathbf{V} = \mathbf{u}$ introduces a local proper volume measure of the fluid as the Riemannian volume measure multiplied by the local Lorentz factor γ .

2.2. Averages of scalars

We define the integral over a scalar S over the space-time domain $\{A = A_0, B \leq B_0\}$ singled out by the window function W as follows:

$$I_W(S) \equiv \int_{\mathcal{M}} d^4x \sqrt{g} S W, \quad (5)$$

and we define the average of a scalar S as

$$\langle S \rangle_W \equiv \frac{\int_{\mathcal{M}} d^4x \sqrt{g} S W}{\int_{\mathcal{M}} d^4x \sqrt{g} W} = \frac{I_W(S)}{\mathcal{V}}, \quad (6)$$

where $\mathcal{V} \equiv I_W(1)$ is the volume of the domain as measured by $d\mathcal{V}$. The functional dependencies of $I_W(S)$ and $\langle S \rangle_W$ on the variables of W are kept implicit for ease of notation, and we shall also drop the window function index W in what follows.

2.3. Examples of window functions

We now present several possible choices for the window function, adapted to specific descriptions.

2.3.1. Riemannian averages: As discussed above, the choice $\mathbf{V} = \mathbf{n}$ implies integration with respect to the Riemannian volume element of the hypersurfaces determined by A in the definitions (5)–(6) for integration and averages. This choice corresponds to the averaging formalisms that are often used in the literature for general foliations, in addition to specific (not always covariantly defined) conditions on the propagation of the domain boundary (see a comprehensive list of such general foliation extensions of [2, 3] in the literature comparison investigated in [8]). This is the choice made in [14], where the propagation of the domain is in principle kept general, but is specified as following the normal vector, $\mathbf{n} \cdot \nabla B = 0$, when derivation of averaged Einstein equations is considered.

2.3.2. Lagrangian window functions: One can also use the integration measure arising from $\mathbf{V} = \mathbf{u}$, where \mathbf{u} is the generator of flow lines of a physical fluid, together with the requirement of a domain propagating along the fluid flow, $\mathbf{u} \cdot \nabla B = 0$. We do not at this point specify the time function A . We call such a choice a *Lagrangian* window function, since the spatial domain is comoving with the fluid, and the volume measure is defined as the proper volume measure of the fluid elements.

The proper volume element of the fluid (4) and the associated volume and averages as defined by (6) are equivalent to those of [7], here derived from a manifestly covariant window function. This explicitly shows that all results derived from the integration of scalars with this choice of volume element in [7] are covariant, as well as the former results of [2, 3] obtained with the same volume element in the case of a fluid-orthogonal foliation ($\mathbf{V} = \mathbf{u} = \mathbf{n}$).

2.3.3. Mass-weighted averages: Consider a fluid with 4-velocity \mathbf{u} and with an associated conserved local rest mass current \mathbf{M} ,

$$M^\mu = \rho u^\mu \quad ; \quad \nabla_\mu M^\mu = 0, \quad (7)$$

where ρ is the rest mass density. We can define a mass-weighted Lagrangian average by choosing $V^\mu = M^\mu$ in (1) and $\mathbf{u} \cdot \nabla B = 0$. This mass-weighted average corresponds to that formulated for irrotational dust in fluid-orthogonal foliations in [20], but here expressed in the explicitly covariant formalism and extended to arbitrary fluids and foliations.

2.3.4. Other weighted averages: As illustrated by the previous example, the freedom of choice of \mathbf{V} allows for any weighting of the averages. One may thus use the window function (1) to define, e.g., averages weighted by curvature, or by other functions related to curvature degrees of freedom in the spirit of the ‘q-average’ of Sussman [21, 22]§, writing the corresponding window function under a manifestly covariant form.

2.3.5. Extensions to light cone averages: One may choose a boundary scalar with null gradient such that $\{B = B_0\}$ defines the past light cone of a given event, as studied in [15] in the case $\mathbf{V} = \mathbf{n}$. Integrals and averages are then taken over the spatial region defined by the interior of the light cone at time $A = A_0$.

Because \mathbf{V} is not constrained to be the unit normal vector to the $A = \text{const.}$ hypersurfaces, the formalism can also be straightforwardly extended to averaging over past light cones by choosing A as the appropriate scalar with light-like gradient and \mathbf{V} as a fixed time-like vector, e.g. the 4-velocity \mathbf{u} of a fluid source. One might then also replace B by a scalar of time-like gradient; another averaging operator discussed in [15] is recovered in this case if \mathbf{V} is taken as the normalized gradient of B . For either a space-like or a time-like ∇B , such a window function would then select a bounded part of the past light cone of a given event. The variations of integrals or averages with respect to A_0 then provide information on *drift* effects as this event changes, while

§ Note that the ‘q-average’ is constructed for the specific metrics of the Lemaître-Tolman-Bondi and Szekeres models by introducing a weighting in the average that is defined from metric degrees of freedom in a particular coordinate system. It is therefore not formulated in a manifestly covariant way. However, we may simply extend the definition of the weighting to any other coordinate system, by requiring the weighing to be invariant under the change of coordinates. With such an extension the weighting function is per construction a 4-scalar, and the ‘q-average’ becomes covariant.

the description of time evolution along a fixed past light cone would instead require an analysis of variations with respect to B_0 .

3. The Buchert-Ehlers commutation rule

We now give a generalization of the commutation rule [5], [2–4, 12], and the corresponding manifestly covariant version [14]. We focus on different possible rewritings of the commutation rule, which can prove useful for interpretation and for compactness of averaged equations. We then apply it to a Lagrangian window function and to the evolution of the fluid rest mass within the integration domain.

3.1. General formulation

The essential insight of scalar averaging schemes is that time-derivatives and averaging operations do not commute in general. The commutation rule for the integral can be derived by differentiating the expression for $I(S)$ in the form (5) with respect to A_0 :

$$\begin{aligned}
I(S)' &= \int_{\mathcal{M}} d^4x \sqrt{g} S V^\nu \nabla_\nu A \left(\frac{\partial}{\partial A_0} \delta(A_0 - A) \right) \mathcal{H}(B_0 - B) \\
&= \int_{\mathcal{M}} d^4x \sqrt{g} S V^\nu \nabla_\nu A \left(-\frac{\partial}{\partial A} \delta(A_0 - A) \right) \mathcal{H}(B_0 - B) \\
&= \int_{\mathcal{M}} d^4x \sqrt{g} S V^\nu \nabla_\nu A \left(-\frac{Z^\mu}{Z^\nu \nabla_\nu A} \nabla_\mu \delta(A_0 - A) \right) \mathcal{H}(B_0 - B) \\
&= \int_{\mathcal{M}} d^4x \sqrt{g} W \frac{\nabla_\mu \left(S Z^\mu \frac{V^\kappa \nabla_\kappa A}{Z^\sigma \nabla_\sigma A} \mathcal{H}(B_0 - B) \right)}{V^\nu \nabla_\nu A} \\
&= I \left(\frac{Z^\mu \nabla_\mu S}{Z^\sigma \nabla_\sigma A} \right) + I \left(\frac{S \nabla_\mu \left(\frac{Z^\mu V^\kappa \nabla_\kappa A}{Z^\sigma \nabla_\sigma A} \right)}{V^\nu \nabla_\nu A} \right) - I \left(\frac{S Z^\mu \nabla_\mu B \delta(B_0 - B)}{Z^\sigma \nabla_\sigma A} \right), \quad (8)
\end{aligned}$$

with the notation $' \equiv \partial/\partial A_0$, and where \mathbf{Z} is an arbitrary vector field obeying $\mathbf{Z} \cdot \nabla A \neq 0$ everywhere. The third line of (8) follows from $\mathbf{Z} \cdot \nabla (\delta(A_0 - A)) = (\mathbf{Z} \cdot \nabla A) \partial_A (\delta(A_0 - A))$, and the fourth line follows from partial integration, with the convention $\mathcal{H}(0) = 1$ implying $\mathcal{H}(x)\delta(x) = \delta(x)$.

\mathbf{Z} represents the freedom of the direction in which we define local time derivatives with respect to A . Non-commutativity is given by the failure of the boundary to be parallel-transported along $\mathbf{Z}/(\mathbf{Z} \cdot \nabla A)$ and by the change of volume measure along the flow lines of $\mathbf{Z}/(\mathbf{Z} \cdot \nabla A)$. We denote the first term of (8) the evolution term, the second term the expansion term, and the third term the boundary term.

The full result (8) is not dependent on \mathbf{Z} , but different choices of \mathbf{Z} allow us to trade between the three terms in (8). For instance, we can make the boundary terms disappear by choosing \mathbf{Z} such that $\mathbf{Z} \cdot \nabla B = 0$,^{||} i.e., the boundary term contribution does not appear if the direction chosen for time derivation follows the propagation of

^{||} Taking \mathbf{Z} to be time-like or null automatically ensures $\mathbf{Z} \cdot \nabla A \neq 0$ if ∇A is time-like.

the boundary. Similarly, we might make the evolution term vanish by choosing a \mathbf{Z} such that $\mathbf{Z} \cdot \nabla S = 0$. ¶ The rate of evolution of the volume $I(1)$ and the commutation rule for the average follow from (8) and are given respectively by

$$\frac{I(1)'}{I(1)} = \left\langle \frac{\nabla_\mu \left(Z^\mu \frac{V^\kappa \nabla_\kappa A}{Z^\sigma \nabla_\sigma A} \right)}{V^\nu \nabla_\nu A} \right\rangle - \left\langle \frac{Z^\mu \nabla_\mu B \delta(B_0 - B)}{Z^\sigma \nabla_\sigma A} \right\rangle ; \quad (9)$$

$$\begin{aligned} \langle S \rangle' &= \frac{I(S)'}{I(1)} - \langle S \rangle \frac{I(1)'}{I(1)} = \\ &\left\langle \frac{Z^\mu \nabla_\mu S}{Z^\sigma \nabla_\sigma A} \right\rangle + \left\langle \frac{(S - \langle S \rangle) \nabla_\mu \left(Z^\mu \frac{V^\kappa \nabla_\kappa A}{Z^\sigma \nabla_\sigma A} \right)}{V^\nu \nabla_\nu A} \right\rangle - \left\langle \frac{(S - \langle S \rangle) Z^\mu \nabla_\mu B \delta(B_0 - B)}{Z^\sigma \nabla_\sigma A} \right\rangle. \end{aligned} \quad (10)$$

Again, we might trade between the three terms in (10) by changing \mathbf{Z} , e.g., we can still make the third term vanish by choosing \mathbf{Z} to be a time-like vector field comoving with the spatial boundaries of the domain.

When it is possible to choose a time-like \mathbf{Z} such that $\nabla_\mu \left(Z^\mu \frac{V^\kappa \nabla_\kappa A}{Z^\sigma \nabla_\sigma A} \right) = 0$, and $Z^\mu \nabla_\mu B = 0$ simultaneously, there is a sense in which time-derivative and the averaging operation commute in (8) and (10): in this case it is possible to construct flow lines along which the only contribution to the change of $\langle S \rangle$ is the change of S itself. This is the case for a mass-weighted window function (see section 2.3.3). In this case, $\mathbf{Z} = \mathbf{u}$ satisfies the above requirements, so that the commutation rule (10) reduces to

$$\langle S \rangle' = \left\langle \frac{u^\mu \nabla_\mu S}{u^\sigma \nabla_\sigma A} \right\rangle. \quad (11)$$

Hence, there is commutation of this particular averaging operation and time-derivative along the flow lines of \mathbf{u} , generalizing this result obtained for irrotational dust in the fluid-orthogonal foliation [20]. This commutation is, however, obtained at the expense of a more complicated definition required for a physical volume (and associated scale factor). In this setting, the ‘volume’ $I(1)$ actually corresponds to a total rest mass within the integration domain, as described in section 3.3. Thus, as noticed in [20], defining a physical volume would require to compensate for the weighting by ϱ , e.g. by considering $I(1/\varrho)$.

We may choose \mathbf{Z} to be the most convenient vector field for simplifying the commutation rules, or may choose it from a geometric motivation as, e.g. in [14], where \mathbf{Z} is chosen to coincide with the normal to the hypersurfaces. Alternatively, one may choose a physical vector field for \mathbf{Z} , e.g. $\mathbf{Z} = \mathbf{u}$, where \mathbf{u} is the 4-velocity of a physical fluid of interest. In this formulation the terms in (8) and (10) can be interpreted in terms of evolution along physical flow lines of a fluid and its expansion.

¶ Note, however, that if $\nabla S \propto \nabla A$, then this choice is not possible, and the evolution term cannot be put to zero.

3.2. Application to the case of a Lagrangian window function

Let us consider a Lagrangian window function as defined in section 2.3.2. Writing the commutation rule (8) with $\mathbf{Z} = \mathbf{u}$ we have in this case

$$I(S)' = I\left(\frac{u^\mu \nabla_\mu S}{u^\sigma \nabla_\sigma A}\right) + I\left(\frac{S \nabla_\mu u^\mu}{u^\sigma \nabla_\sigma A}\right) \quad ; \quad I(1)' = I\left(\frac{\nabla_\mu u^\mu}{u^\sigma \nabla_\sigma A}\right), \quad (12)$$

where the first contribution comes from the change of S along the flow lines of \mathbf{u} , and the second contribution from the expansion $\nabla_\mu u^\mu$ of the fluid. Note the normalization $u^\sigma \nabla_\sigma A$, which is a change of measure between the proper time parameter τ of the fluid and the foliation parameter A along each fluid flow line. Hence, this normalization reduces to unity if and only if A is a proper time of \mathbf{u} .

The analogous commutation rule for the average (10) yields

$$\langle S \rangle' = \left\langle \frac{u^\mu \nabla_\mu S}{u^\sigma \nabla_\sigma A} \right\rangle + \left\langle \frac{(S - \langle S \rangle) \nabla_\mu u^\mu}{u^\sigma \nabla_\sigma A} \right\rangle. \quad (13)$$

There are at least two natural ways of choosing A in the Lagrangian spirit of formulating the window function. In cases where \mathbf{u} is irrotational, it is then proportional to the gradient of a scalar α , and we can choose A to define a foliation in the rest frame of the fluid (i.e. fluid-orthogonal hypersurfaces) by $A = \alpha$. An alternative natural choice of A is a proper time parameter τ of \mathbf{u} [7,8]. This has the advantage of being always possible, even if \mathbf{u} has vorticity, and of providing a clear physical interpretation of A as the time parameter in evolution equations for average quantities. However, the time-like nature of $\nabla\tau$ can in general not be guaranteed. Note that the above conditions define classes of foliation scalars, i.e. further specifications are required to determine them uniquely.⁺ A choice of proper time foliation can be simultaneously fluid-orthogonal only when the fluid is irrotational and geodesic.*

3.3. Total rest mass of the averaging domain

Consider a conserved local rest mass current $M^\mu = \rho u^\mu$ as in (7). We can define a total rest mass within the domain at $A = A_0$ as

$$M(A_0) \equiv \int_{\mathcal{M}} d^4x \sqrt{g} M^\mu \nabla_\mu (\mathcal{H}(A - A_0)) \mathcal{H}(B_0 - B), \quad (14)$$

⁺ The *proper time foliation* $A = \tau$ is only specified up to an additive function β obeying $\mathbf{u} \cdot \nabla\beta = 0$. The *fluid frame foliation* $A = \alpha$ is only specified up to a reparametrization, $A = f(\alpha)$, for any non-decreasing function f of α . This freedom can be denoted a gauge freedom, since it can be viewed as a time reparametrization within the original foliation itself. See Appendix A for further details on gauge freedom in the labeling of hypersurfaces.

* A fluid-orthogonal foliation implies that $\mathbf{u} = \mathbf{n} = -N\nabla A$ with the lapse $N = (-\nabla A \cdot \nabla A)^{-1/2}$. The vorticity of \mathbf{u} thus has to vanish, which is part of Frobenius' theorem. It also implies that the 4-acceleration \mathbf{a} of the fluid relates to the lapse variations as $a^\mu = N^{-1} b^{\mu\nu} \nabla_\nu N$ [7,23], with \mathbf{b} the fluid-orthogonal projector. If A is additionally required to be a proper time function for the fluid, $\mathbf{u} \cdot \nabla A = 1$, then $N = 1$ everywhere and $\mathbf{a} = 0$. This shows that the fluid flow must also be geodesic.

i.e., as $I(1)$ for a window function with $V^\mu = M^\mu$ (e.g. the mass-weighted window function, see section 2.3.3). Applying (8) gives the evolution of $M(A_0)$ which, due to the local conservation of M^μ , reduces to a single boundary term

$$M(A_0)' = - \int_{\mathcal{M}} d^4x \sqrt{g} M^\mu \nabla_\mu B \mathcal{H}(A - A_0) \delta(B_0 - B), \quad (15)$$

i.e. the evolution of mass is given by the flux of the mass current M^μ out of the averaging domain. Thus, $M(A_0)$ is constant in A_0 when the domain is comoving with the fluid elements, $\mathbf{u} \cdot \nabla B = 0$. For such a comoving integration domain, $M = M(A_0)$ (for any A_0), as defined by (14), corresponds to the total conserved rest mass of the fluid within the domain. In this case, the additional requirement $\mathbf{V} = \mathbf{u}$ sets a Lagrangian window function (as defined in section 2.3.2). The conserved total rest mass within the domain then takes the natural form $M = I(\varrho)$. For other volume measures, in general, $I(\varrho)$ would not correspond to the rest mass within the domain and would not be conserved, due to a weighting or due to the volume not being measured in the fluid's local rest frames. (For instance, for the hypersurfaces Riemannian volume measure, $\mathbf{V} = \mathbf{n}$, and still for a comoving domain, the integrated rest mass would have to be written $M = I(\gamma\varrho)$ with $\gamma = -\mathbf{n} \cdot \mathbf{u}$.) A Lagrangian window function $\{\mathbf{V} = \mathbf{u}, \mathbf{u} \cdot \nabla B = 0\}$ thus appears as a particularly natural choice to follow and characterize a given collection of fluid elements, if a preferred fluid frame with an associated rest mass current is present in the model universe. We shall focus again in section 4.3.1 on domains that follow the propagation of the fluid—hence preserving the associated rest mass—as a subcase of particular interest of more general averaged evolution equations, to which we turn now.

4. The averaged Einstein equations

The general averaging formalism and the commutation rule are applied below to scalar projections of the Einstein equations. The resulting system of averaged evolution equations allows for a covariant definition of *cosmological backreaction terms*. We shall then explicitly provide the simpler form taken by these equations for a domain that follows the fluid world lines, and we discuss the natural choices $\mathbf{V} = \mathbf{n}$ and $\mathbf{V} = \mathbf{u}$.

4.1. Local variables and relations

In this subsection we consider an averaging domain defined by a time-like propagation of its boundary. We thus assume that a unit time-like propagation vector field \mathbf{P} can be defined such that it satisfies $\mathbf{P} \cdot \nabla B = 0$, at least on the domain's boundary $\{B = B_0\}$. Applying the commutation rules (8)–(10) with the choice $\mathbf{Z} = \mathbf{P}$ will then ensure the vanishing of the boundary terms in these equations.

Kinematic variables may then be defined for this vector field by decomposing its gradient with respect to \mathbf{P} and its null-space as follows, using the orthogonal projector \mathbf{k} with components $k_{\mu\nu} = g_{\mu\nu} + P_\mu P_\nu$:

$$\nabla_\mu P_\nu = -P_\mu a_\nu^P + \frac{1}{3} \Theta_P k_{\mu\nu} + \sigma_{\mu\nu}^P + \omega_{\mu\nu}^P;$$

$$\begin{aligned}
a_\mu^P &= P^\nu \nabla_\nu P_\mu ; \quad \Theta_P = k^{\mu\nu} \nabla_\mu P_\nu ; \quad \sigma_{\mu\nu}^P = k_{(\mu}^\alpha k_{\nu)}^\beta \nabla_\alpha P_\beta - \frac{1}{3} \Theta_P k_{\mu\nu} ; \quad \omega_{\mu\nu}^P = k_{[\mu}^\alpha k_{\nu]}^\beta \nabla_\alpha P_\beta ; \\
\sigma_P^2 &= \frac{1}{2} \sigma_{\mu\nu}^P \sigma^{P,\mu\nu} ; \quad \omega_P^2 = \frac{1}{2} \omega_{\mu\nu}^P \omega^{P,\mu\nu} .
\end{aligned} \tag{16}$$

Assuming the presence of a preferred non-singular fluid flow as a source, with 4-velocity \mathbf{u} , the (fully general) energy-momentum tensor is naturally decomposed with respect to \mathbf{u} and its null-space:

$$\begin{aligned}
T_{\mu\nu} &= \epsilon u_\mu u_\nu + 2 q_{(\mu} u_{\nu)} + p b_{\mu\nu} + \pi_{\mu\nu} ; \\
\epsilon &\equiv u^\mu u^\nu T_{\mu\nu} ; \quad q_\mu \equiv -b_\mu^\alpha u^\beta T_{\alpha\beta} ; \quad p \equiv \frac{1}{3} b^{\mu\nu} T_{\mu\nu} ; \quad \pi_{\mu\nu} \equiv b_\mu^\alpha b_\nu^\beta T_{\alpha\beta} - p b_{\mu\nu} ,
\end{aligned} \tag{17}$$

where \mathbf{b} is the projector onto the fluid's rest frames, with components $b_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. It may alternatively be decomposed using \mathbf{P} . In particular, one can define the energy density E_P and pressure $S_P/3$, in the frames defined by \mathbf{P} , from, respectively:

$$E_P \equiv P^\mu P^\nu T_{\mu\nu} ; \quad S_P = k^{\mu\nu} T_{\mu\nu} . \tag{18}$$

These variables are related to the fluid rest frame energy density ϵ , pressure p , and to the non-perfect fluid contributions *via*

$$E_P - \epsilon = \frac{1}{2} [E_P + S_P - (\epsilon + 3p)] = (\epsilon + p) [(u^\mu P_\mu)^2 - 1] + 2 (u^\mu P_\mu) (P^\nu q_\nu) + \pi_{\mu\nu} P^\mu P^\nu . \tag{19}$$

The following Raychaudhuri equation for \mathbf{P} is then obtained by combining the Einstein equation projected twice along \mathbf{P} , and its trace:

$$P^\mu \nabla_\mu \Theta_P = -\frac{1}{3} \Theta_P^2 - 2\sigma_P^2 + 2\omega_P^2 + \nabla^\mu a_\mu^P - 4\pi G (E_P + S_P) + \Lambda . \tag{20}$$

We define an effective scalar 3-curvature for the null-space of \mathbf{P} (which is not hypersurface-forming if $\omega_P^2 \neq 0$) as follows:

$$\mathcal{R}_P \equiv \nabla_\mu P^\nu \nabla_\nu P^\mu - \nabla_\mu P^\mu \nabla_\nu P^\nu + R + 2R_{\mu\nu} P^\mu P^\nu . \tag{21}$$

This definition of effective 3-curvature reduces to the scalar 3-curvature of the \mathbf{P} -orthogonal hypersurfaces when they exist (i.e., for $\omega_P^2 = 0$, by Frobenius' theorem). Such a generalization of the hypersurface-based notion is not unique; we here follow a similar definition as that of, e.g. [11]. This convention implies the following relation in the form of an energy constraint:

$$\frac{2}{3} \Theta_P^2 = -\mathcal{R}_P + 2\sigma_P^2 - 2\omega_P^2 + 16\pi G E_P + 2\Lambda . \tag{22}$$

4.2. Averaged evolution equations

We use the general window function (1) and define an effective 'scale factor' a as $a = (I(1)/I(1)_i)^{1/3}$, where the subscript \mathbf{i} denotes a value on some initial hypersurface $A = A_i$.

As noted for the example of the mass-weighted average [20], it should be kept in mind that this definition is only relevant as a scale factor if it can be interpreted as a typical length derived from a volume, i.e. only when the choice of integration measure defined by \mathbf{V} allows for the interpretation of $I(1)$ as a volume. Another definition

of ‘scale factor’ that does relate it to a physical volume (e.g. to $I(1/\varrho)$ in the case of the mass-weighted average) may otherwise be more appropriate. It should also be noted, that the effective ‘scale factor’ a in general does not have an interpretation in terms of mean redshift of null bundles (the averaging scheme presented in this paper is too general to make a direct link to statistical light propagation). However, when $I(1)$ does measure a volume, and under the assumptions that (i) the frame of averaging is associated with statistical homogeneity and isotropy, that (ii) structures are slowly evolving (allowing null-rays to probe the statistical homogeneity scale), and that (iii) typical emitters and observers of light are reasonably close to being in the averaging frame, a might be interpreted as the inverse of a ‘statistical redshift’ averaged over many observers and emitters [19]. More generally, only assuming a choice of window function such that $I(1)$ measures a physical volume, a should merely be interpreted as an effective length scale of an averaging region defined in a given foliation.

Averaging the above equations (22) and (21) with the averaging definition (6), and making use of the volume evolution rate (9) and the commutation rule (10) with the choice $\mathbf{Z} = \mathbf{P}$, implying $\mathbf{Z} \cdot \nabla B = 0$, yields the following evolution equations for a :

$$3 \left(\frac{a'}{a} \right)^2 = 8\pi G \left\langle \frac{\epsilon}{(P^\mu \nabla_\mu A)^2} \right\rangle + \Lambda \left\langle \frac{1}{(P^\mu \nabla_\mu A)^2} \right\rangle - \frac{1}{2} \left\langle \frac{\mathcal{R}_P}{(P^\mu \nabla_\mu A)^2} \right\rangle - \frac{1}{2} \mathcal{Q} - \frac{1}{2} \mathcal{T}; \quad (23)$$

$$3 \frac{a''}{a} = -4\pi G \left\langle \frac{\epsilon + 3p}{(P^\mu \nabla_\mu A)^2} \right\rangle + \Lambda \left\langle \frac{1}{(P^\mu \nabla_\mu A)^2} \right\rangle + \mathcal{Q} + \mathcal{P} + \frac{1}{2} \mathcal{T}. \quad (24)$$

These equations feature three backreaction terms, a *kinematical backreaction* \mathcal{Q} , a *dynamical backreaction* \mathcal{P} , and an *energy-momentum backreaction* \mathcal{T} that captures the difference of the energy densities as measured in two different frames (see [8]). These backreaction terms are defined as follows:

$$\begin{aligned} \mathcal{Q} &\equiv \frac{2}{3} \left[\left\langle \frac{\Theta_P^2}{(P^\rho \nabla_\rho A)^2} \right\rangle - \left\langle \frac{\Theta_P + \Gamma_P^{-1} P^\mu \nabla_\mu \Gamma_P}{P^\rho \nabla_\rho A} \right\rangle^2 \right] - \left\langle \frac{2\sigma_P^2}{(P^\mu \nabla_\mu A)^2} \right\rangle + \left\langle \frac{2\omega_P^2}{(P^\mu \nabla_\mu A)^2} \right\rangle; \\ \mathcal{P} &\equiv \left\langle \frac{\nabla^\mu a_\mu^P}{(P^\mu \nabla_\mu A)^2} \right\rangle + \left\langle \frac{\Theta_P}{(P^\rho \nabla_\rho A)^2} \left(2 \frac{P^\mu \nabla_\mu \Gamma_P}{\Gamma_P} - \frac{P^\mu \nabla_\mu (P^\nu \nabla_\nu A)}{P^\sigma \nabla_\sigma A} \right) \right\rangle \\ &\quad + \left\langle \frac{\Gamma_P^{-1} P^\mu \nabla_\mu (P^\nu \nabla_\nu \Gamma_P)}{(P^\mu \nabla_\mu A)^2} \right\rangle - \left\langle \frac{\Gamma_P^{-1} P^\mu \nabla_\mu \Gamma_P}{(P^\rho \nabla_\rho A)^2} \frac{P^\nu \nabla_\nu (P^\kappa \nabla_\kappa A)}{P^\sigma \nabla_\sigma A} \right\rangle; \\ \mathcal{T} &= -16\pi G \left\langle \frac{E_P - \epsilon}{(P^\mu \nabla_\mu A)^2} \right\rangle, \end{aligned} \quad (25)$$

with the energy difference $E_P - \epsilon$ given by (19), and with the ratio of ‘Lorentz factors’ $\Gamma_P \equiv (V^\mu \nabla_\mu A)/(P^\nu \nabla_\nu A) = (-V^\mu n_\mu)/(-P^\nu n_\nu)$, $-V^\mu n_\mu$ being a Lorentz factor when \mathbf{V} is normalized.

From the requirement of (23) being the integral of (24) we get the integrability condition:

$$\begin{aligned} \mathcal{Q}' + 6 \frac{a'}{a} \mathcal{Q} + \left\langle \frac{\mathcal{R}_P}{(P^\sigma \nabla_\sigma A)^2} \right\rangle' + 2 \frac{a'}{a} \left\langle \frac{\mathcal{R}_P}{(P^\sigma \nabla_\sigma A)^2} \right\rangle + \mathcal{T}' + 4 \frac{a'}{a} \mathcal{T} + 4 \frac{a'}{a} \mathcal{P} \\ = 16\pi G \left(\left\langle \frac{\epsilon}{(P^\sigma \nabla_\sigma A)^2} \right\rangle' + 3 \frac{a'}{a} \left\langle \frac{\epsilon + p}{(P^\sigma \nabla_\sigma A)^2} \right\rangle \right) + 2\Lambda \langle (P^\sigma \nabla_\sigma A)^{-2} \rangle'. \end{aligned} \quad (26)$$

Defining the kinematic variables of the fluid from the decomposition of the 4-velocity gradient,

$$\begin{aligned}\nabla_\mu u_\nu &= -u_\mu a_\nu + \frac{1}{3}\Theta b_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}; \\ a_\mu &= u^\nu \nabla_\nu u_\mu; \Theta = b^{\mu\nu} \nabla_\mu u_\nu; \sigma_{\mu\nu} = b^\alpha_{(\mu} b^\beta_{\nu)} \nabla_\alpha u_\beta - \frac{1}{3}\Theta b_{\mu\nu}; \omega_{\mu\nu} = b^\alpha_{[\mu} b^\beta_{\nu]} \nabla_\alpha u_\beta; \\ \sigma^2 &= \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}; \omega^2 = \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu},\end{aligned}\tag{27}$$

we can express the energy-momentum conservation equation projected onto the fluid frame as follows:

$$-u^\mu \nabla_\nu T^\nu_\mu = u^\mu \nabla_\mu \epsilon + \Theta(\epsilon + p) + a^\mu q_\mu + \nabla_\mu q^\mu + \pi_{\mu\nu} \sigma^{\mu\nu} = 0.\tag{28}$$

One can then divide this relation by $(P^\mu \nabla_\mu A)^2$, take the average and apply the commutation rule (8) with $\mathbf{Z} = \mathbf{u}$. This yields the average energy conservation law satisfied by the right-hand side of (26):

$$\begin{aligned}\left\langle \frac{\epsilon}{(P^\sigma \nabla_\sigma A)^2} \right\rangle' + 3\frac{a'}{a} \left\langle \frac{\epsilon + p}{(P^\sigma \nabla_\sigma A)^2} \right\rangle &= -\left\langle \frac{\Theta}{\dot{A}} \frac{p}{(P^\sigma \nabla_\sigma A)^2} \right\rangle + \left\langle \frac{\Theta}{\dot{A}} \right\rangle \left\langle \frac{p}{(P^\sigma \nabla_\sigma A)^2} \right\rangle \\ &+ \left\langle \frac{\dot{\Gamma}/\Gamma}{\dot{A}} - \frac{(u^\mu \nabla_\mu B) \delta(B_0 - B)}{\dot{A}} \right\rangle \left\langle \frac{p}{(P^\sigma \nabla_\sigma A)^2} \right\rangle - \left\langle \frac{\epsilon}{(P^\sigma \nabla_\sigma A)^2} \frac{(u^\mu \nabla_\mu B) \delta(B_0 - B)}{\dot{A}} \right\rangle \\ &+ \left\langle \frac{\epsilon}{(P^\sigma \nabla_\sigma A)^2} \frac{2(\dot{\Gamma}_P/\Gamma_P) - (\dot{\Gamma}/\Gamma) - 2(\dot{A}/\dot{A})}{\dot{A}} \right\rangle - \left\langle \frac{a_\mu q^\mu + \nabla_\mu q^\mu + \pi_{\mu\nu} \sigma^{\mu\nu}}{\dot{A} (P^\sigma \nabla_\sigma A)^2} \right\rangle,\end{aligned}\tag{29}$$

with $\Gamma \equiv (V^\mu \nabla_\mu A)/(u^\nu \nabla_\nu A) = (-V^\mu n_\mu)/\gamma$, and using the shorthand notation \dot{S} for the proper-time covariant derivative along \mathbf{u} of a scalar S , $\dot{S} \equiv u^\mu \nabla_\mu S$. This average conservation equation features two boundary terms that provide the variations in volume and average energy density due to the flux of fluid elements across the domain's boundary if $u^\mu \nabla_\mu B \neq 0$.

The above system of averaged equations (23,24,26,29) is covariant since it only features explicitly covariant terms. The form of these equations is moreover globally preserved under a change of the parametrization of the foliation (using a non-decreasing function of A instead of A , preserving the set of hypersurfaces), but the individual terms they contain are not. This is no different from the time-parameter dependence of the expansion and acceleration terms of the Friedmann equations in homogeneous and isotropic cosmologies. This freedom of relabeling the hypersurfaces is important to keep in mind when interpreting averaged evolution equations: as for any parametric equations, e.g. acceleration terms (as second derivatives with respect to a parameter) can be tuned in any desirable way, including the change of sign, by an appropriate change of the parameter. This is discussed in more detail in the specific context of the above averaged equations in Appendix A. This interpretation issue is simply solved by the choice of a time label with a clear physical meaning for the hypersurfaces. Such a choice can be made specifically for the physical model considered, or from more general conditions, such as taking τ itself as the parameter A when working within a foliation at constant fluid proper time τ (see the related remarks that conclude section 3.2).

This general set of averaged equations is naturally expressed in terms of geometric variables such as the extrinsic curvature or the intrinsic scalar 3–curvature of the $A = \text{const.}$ hypersurfaces for a domain propagation along the normal vector field, i.e., for $\mathbf{P} = \mathbf{n}$. In this case, and for $\mathbf{V} = \mathbf{n}$ (i.e. for Riemannian averages), this system corresponds to the averaged system derived in [14], with the addition of the integrability condition and the general form of the averaged energy conservation law.

For a general propagation vector \mathbf{P} , the explicit contribution of the geometric variables in the above equations can also be recovered by an alternative writing. It can be done by splitting \mathbf{P} into a component along \mathbf{n} and a component orthogonal to \mathbf{n} , $\mathbf{P} = \gamma_P(\mathbf{n} + \mathbf{v}_P)$ with $\gamma_P = -\mathbf{P} \cdot \mathbf{n}$ and $\mathbf{n} \cdot \mathbf{v}_P = 0$. The contributions from the decomposition of the gradient of \mathbf{P} to the averaged equations can then be expressed in terms of the extrinsic curvature of the hypersurface, e.g. by applying the following split in the commutation rule:

$$\frac{\nabla_\mu \left(P^\mu \frac{V^\rho \nabla_\rho A}{P^\sigma \nabla_\sigma A} \right)}{V^\nu \nabla_\nu A} = \frac{\Theta_P + \Gamma_P^{-1} P^\mu \nabla_\mu \Gamma_P}{P^\rho \nabla_\rho A} = -N\mathcal{K} + N \frac{\nabla_\mu (V^\nu n_\nu v_P^\mu)}{V^\rho n_\rho} + \frac{N n^\mu \nabla_\mu (V^\nu n_\nu)}{V^\rho n_\rho},$$

with the lapse function $N \equiv (\nabla^\mu A \nabla_\mu A)^{-1/2}$ and the trace of the extrinsic curvature $\mathcal{K} \equiv -\nabla_\mu n^\mu$. The set of equations using this decomposition will then simplify when using the Riemannian volume measure of the hypersurfaces, $\mathbf{V} = \mathbf{n}$. In the comoving domain case, $\mathbf{P} = \mathbf{u}$, this returns one of the sets of equations obtained in [8] when geometric variables–based expressions for the spatial Riemannian volume measure and a domain comoving with the fluid flow are considered.

4.3. Examples of applications

4.3.1. Comoving domains: We now specify the above results to the case of a domain comoving with the fluid, i.e. for which $\mathbf{u} \cdot \nabla B = 0$. One can thus take $\mathbf{P} = \mathbf{u}$. The adapted local Raychaudhuri equation (20) and energy constraint (22) are then expressed in terms of rest frame variables of the fluid:

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 2\sigma^2 + 2\omega^2 + \nabla_\mu a^\mu - 4\pi G(\epsilon + 3p) + \Lambda; \quad (30)$$

$$\frac{2}{3}\Theta^2 = -\mathcal{R} + 2\sigma^2 - 2\omega^2 + 16\pi G\epsilon + 2\Lambda, \quad (31)$$

with the effective scalar 3–curvature of the rest frames of \mathbf{u} [11],

$$\mathcal{R} \equiv \nabla_\mu u^\nu \nabla_\nu u^\mu - \nabla_\mu u^\mu \nabla_\nu u^\nu + R + 2R_{\mu\nu} u^\mu u^\nu. \quad (32)$$

The corresponding evolution equations for the effective ‘scale factor’ a (which may still not be the most appropriate definition in cases where $I(1)$ is not interpreted as a volume) are then written as follows:

$$3 \left(\frac{a'}{a} \right)^2 = 8\pi G \left\langle \frac{\epsilon}{\dot{A}^2} \right\rangle + \Lambda \left\langle \frac{1}{\dot{A}^2} \right\rangle - \frac{1}{2} \left\langle \frac{\mathcal{R}}{\dot{A}^2} \right\rangle - \frac{1}{2} \mathcal{Q}; \quad (33)$$

$$3 \frac{a''}{a} = -4\pi G \left\langle \frac{\epsilon + 3p}{\dot{A}^2} \right\rangle + \Lambda \left\langle \frac{1}{\dot{A}^2} \right\rangle + \mathcal{Q} + \mathcal{P}. \quad (34)$$

The energy-momentum backreaction vanishes since $\mathbf{P} = \mathbf{u}$, and the kinematical and dynamical backreaction terms reduce to the following:

$$\mathcal{Q} \equiv \frac{2}{3} \left(\left\langle \frac{\Theta^2}{\dot{A}^2} \right\rangle - \left\langle \frac{\Theta + \dot{\Gamma}/\Gamma}{\dot{A}^2} \right\rangle^2 \right) - 2 \left\langle \frac{\sigma^2}{\dot{A}^2} \right\rangle + 2 \left\langle \frac{\omega^2}{\dot{A}^2} \right\rangle ; \quad (35)$$

$$\mathcal{P} \equiv \left\langle \frac{\nabla_\mu a^\mu}{\dot{A}^2} \right\rangle + \left\langle \frac{\Theta}{\dot{A}^2} \left(2 \frac{\dot{\Gamma}}{\Gamma} - \frac{\ddot{A}}{\dot{A}} \right) \right\rangle + \left\langle \frac{\ddot{\Gamma}/\Gamma}{\dot{A}^2} \right\rangle - \left\langle \frac{(\ddot{A}/\dot{A})(\dot{\Gamma}/\Gamma)}{\dot{A}^2} \right\rangle . \quad (36)$$

The integrability condition (26) now becomes

$$\begin{aligned} \mathcal{Q}' + 6 \frac{a'}{a} \mathcal{Q} + \left\langle \frac{\mathcal{R}}{\dot{A}^2} \right\rangle' + 2 \frac{a'}{a} \left\langle \frac{\mathcal{R}}{\dot{A}^2} \right\rangle + 4 \frac{a'}{a} \mathcal{P} \\ = 16\pi G \left(\left\langle \frac{\epsilon}{\dot{A}^2} \right\rangle' + 3 \frac{a'}{a} \left\langle \frac{\epsilon + p}{\dot{A}^2} \right\rangle \right) + 2\Lambda \left\langle \frac{1}{\dot{A}^2} \right\rangle' , \end{aligned} \quad (37)$$

where the right-hand side obeys the averaged energy conservation law (29) that reduces to

$$\begin{aligned} \left\langle \frac{\epsilon}{\dot{A}^2} \right\rangle' + 3 \frac{a'}{a} \left\langle \frac{\epsilon + p}{\dot{A}^2} \right\rangle = - \left\langle \frac{\Theta}{\dot{A}} \frac{p}{\dot{A}^2} \right\rangle + \left\langle \frac{\Theta + \dot{\Gamma}/\Gamma}{\dot{A}} \right\rangle \left\langle \frac{p}{\dot{A}^2} \right\rangle \\ + \left\langle \frac{\epsilon}{\dot{A}^2} \left(\frac{\dot{\Gamma}/\Gamma - 2\ddot{A}/\dot{A}}{\dot{A}} \right) \right\rangle - \left\langle \frac{q^\mu a_\mu + \nabla_\mu q^\mu + \pi_{\mu\nu} \sigma^{\mu\nu}}{\dot{A}^3} \right\rangle . \end{aligned} \quad (38)$$

Remark: The requirement $\mathbf{u} \cdot \nabla B = 0$ in the choice of the window function corresponds to the definition of an averaging domain that follows the fluid flow. It thus ensures by construction the preservation over time of the collection of fluid elements to be averaged, in particular preserving their total rest mass (as shown in section 3.3) when it can be defined.

4.3.2. Lagrangian window function: The above equations for a comoving domain, $\mathbf{u} \cdot \nabla B = 0$, simplify further when in addition the fluid proper volume measure is used, $\mathbf{V} = \mathbf{u}$, yielding a Lagrangian window function. This corresponds to setting $\Gamma = 1$ in equations (33)–(38) above, dropping all terms that depend on its evolution. The system of averaged equations in the framework corresponding to the Lagrangian window function in [7, 8] is thus recovered, under an equivalent, here manifestly covariant form. As discussed in the above references, it becomes particularly transparent in a foliation by hypersurfaces of constant fluid proper time, $A = \tau$.

Remark: The Lagrangian window function choice, based on a preferred fluid 4-velocity field, is especially adapted to analyzing average properties within single-fluid cosmological models. This could apply, e.g. to the description of a dark matter-dominated late Universe within a dust model, or to the radiation-dominated era within a model of a pressure-supported fluid. It can as well be used in a model involving

several non-comoving fluids, e.g. to describe a mixture of dark matter and radiation with different 4-velocities. In this case, it would require choosing one of the fluids to be followed through its evolution and to define a proper volume measure. The total energy-momentum tensor would then have to be decomposed with respect to the corresponding frame, in which contributions from the other fluids will generally appear in the form of non-perfect fluid terms [24].

4.3.3. Riemannian volume averages: As discussed at the end of section 4.2, the choice of a Riemannian volume measure, $\mathbf{V} = \mathbf{n}$, is the most adapted for analyzing averaged geometric properties of the hypersurfaces themselves, e.g. by providing expressions of the averaged equations in terms of the extrinsic curvature of the hypersurfaces. This is expected since the scale factor and averages are then based on the intrinsic spatial volume form of the hypersurfaces. The evolution equations for the scale factor with such a choice and for a comoving domain, $\mathbf{u} \cdot \nabla B = 0$, may be obtained from equations (33)–(38) by setting $\Gamma = 1/\gamma$. This gives a manifestly covariant system of equations equivalent to that given in Appendix B of [8], also expressed in terms of the rest frame fluid variables. Recovering the dependence in the geometric variables such as the trace of extrinsic curvature then requires rewriting these local quantities along the lines suggested at the end of section 4.2.

Remark: The choice of a Riemannian volume measure, $\mathbf{V} = \mathbf{n}$, is especially suited for studying the behaviour of hypersurfaces defined from geometric conditions, such as the Constant Mean Curvature requirement, which is frequently used in general relativity.

The averaged equations for this volume measure take their simplest form for a propagation of the domain along the normal vector \mathbf{n} ($\mathbf{n} \cdot \nabla B = 0$). The evolution equations for such a choice of propagation of the domain can be directly obtained in terms of the geometric variables from the general equations of section 4.2, recovering the framework and results of [14]. However, a geometric propagation of the domain ($\mathbf{n} \cdot \nabla B = 0$) will in general imply a flow of fluid elements (with a 4-velocity \mathbf{u}) across the domain boundary. Preservation of fluid elements could be recovered with additional assumptions; for instance, for an irrotational fluid model with averaging defined in the corresponding global fluid rest frames, with $\mathbf{n} = \mathbf{u}$. In a more general cosmological setting, one may assume on large scales that vorticity effects may be neglected, at least near the domain boundary, allowing for a foliation where a propagation of the domain boundary along the normal vector would approximate a comoving propagation ($\mathbf{u} \cdot \nabla B = 0$). One may also assume a choice of hypersurfaces where statistical homogeneity holds for all observables, effectively leaving the evolution equations defined over such a choice of hypersurfaces invariant under the increase of scale of the domain B_0 above a suitable homogeneity scale cut-off. This would then allow for a computation of averages over a global range ($B_0 \rightarrow +\infty$), effectively eliminating the need for distinguishing the possible propagations of the domain boundary for this choice; see [19] for an investigation of this framework.

4.3.4. *Light propagation:* As discussed in section 2.3.5, an alternative choice for the domain boundary would be that of binding it to the past light cone of a given event by choosing the appropriate scalar B with light-like gradient, covering the evolution of the average properties of spatial sections in the interior of this light cone.

Alternatively, one might consider the case where A has light-like gradient such that $A = A_0$ singles out a null surface that might be associated with the light cone of an observer, and where B has time-like or space-like gradient (e.g. ∇B being proportional to an irrotational fluid 4-velocity \mathbf{u}). Variation of average properties with respect to emitting times of the sources along a given cone then requires a variation of the parameter B_0 , while the above results for the dependence in A_0 would provide insight on drift effects as the observer changes. These situations have been investigated in detail with similar covariant averaging schemes in [15] (see also the application in an adapted coordinate system [1]).

Remark: Averaging domains defined from the light cone are natural candidates for relating the averaging formalism discussed in this paper to observations. It is important to keep in mind that the formalism presented in this paper is general, allowing for averaging over hypersurfaces of arbitrary globally hyperbolic space-times. In particular, the average equations only implicitly depend on the metric of space-time. While we consider this being an advantage, as it allows to express average properties independently of a specific form of the space-time metric, it implies the need for further specifications and assumptions in order to connect the general result to observations. For example, assumptions must be made in order to interpret averaged quantities defined over spatial hypersurfaces in terms of (averaged) energy, flux, etc., of photon bundles emitted by matter sources and absorbed by specified classes of observers. Such an interpretation may become more natural if the formalism is specified to light cone averaging [15], but further assumptions would still be needed in order to close the system of averaged equations (e.g. by specifying a model for the inhomogeneous metric [1]), and to relate the obtained averages to observational results that are usually based on idealizing assumptions on the geometry. It is beyond the scope of this paper to go into details about the difficult task of establishing connections between averaged cosmological evolution equations and (statistical) observations of selected observers. For papers addressing the link between the averaging formalism and its observational interpretation, see e.g. [18,19] (with a covariant formalism for global spatial averages in the second case), and [17,25] for local and bi-local investigations.

5. Conclusion

Covariance is a requirement for any physical theory, and a cornerstone in the formulation of general relativity. In this paper we have investigated scalar covariant formulations of global dynamics relevant for the description of backreaction effects in cosmology. We have considered a generalized window function, allowing for arbitrary foliation, spatial boundary, and volume measure.

We provided an explicitly covariant form for the commutation rule and for the spatially averaged scalar parts of Einstein’s equations, with the associated integrability condition, using this general window function. The absence of restrictions imposed on the energy-momentum tensor of the fluid sources allows us to apply these schemes to the early Universe as well as to the matter-dominated later stages, and they cover all spatial scales down to which the fluid approximation can be considered as valid. Backreaction terms are introduced from these equations, and are thus also expressed under a manifestly covariant form. We then applied these results to the physically relevant subcase of a comoving domain.

We have given a procedure for providing several possible decompositions of the commutation rule and the resulting averaged equations. This allows us, for example, to get rid of boundary terms, or to keep them as transparent boundary flux terms, for any choice of domain propagation. We have discussed the effect on averaged equations of a relabeling of the hypersurfaces in a given foliation, and we have stressed the importance of being able to physically interpret the chosen label.

The formalism used in this paper provides a unifying framework encompassing various scalar averaging schemes that have been suggested or could be used for the description of averaged properties of cosmological models. It can be straightforwardly adapted to a given specific scheme by suitably choosing the window function. Several examples of such possible applications were given. In particular, we have shown that the manifestly covariant averaging scheme used in this work reduces to the averaging scheme considered in [8] for a so-called Lagrangian window function, providing covariant formulas for the latter scheme. The explicit selection of the foliation by a scalar function in the scheme used in this work also makes it suitable for the forthcoming investigation of foliation dependence of averaged expressions [16], and it may be helpful for other related considerations.

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Appendix A. Freedom of hypersurfaces labeling

We here investigate in more detail the consequences of a change of the hypersurfaces label A (without change of the hypersurfaces) for the terms appearing in the evolution equations for the effective scale factor a .

Any transformation of the form

$$A \mapsto f(A), \quad (\text{A.1})$$

where f is a strictly monotonically increasing function, is a transformation of the foliation of A onto itself (i.e. the same set of hypersurfaces is considered, with a different parametrization), since

$$n_\mu = -\frac{\nabla_\mu f(A)}{\sqrt{-\nabla_\nu f(A)\nabla^\nu f(A)}} = -\frac{\nabla_\mu A}{\sqrt{-\nabla_\nu A\nabla^\nu A}}. \quad (\text{A.2})$$

The class of transformations (A.1) is thus a gauge of the foliation.

This seemingly innocent parametrization freedom can cause issues if we are naively evaluating averaged quantities without paying attention to the interpretation on what the time label A represents in the equations. As an example, the interpretation of the Friedmann equations under their usual form relies on the fact that their time parameter has a transparent meaning as the eigentime of ideal fundamental observers.

Let us consider an integrand

$$SW = -SV^\mu \nabla_\mu (\mathcal{H}(A_0 - A)) \mathcal{H}(B_0 - B), \quad (\text{A.3})$$

where the vector $SV^\mu \mathcal{H}(B_0 - B)$ is invariant under reparametrizations (A.1) of A . (This is for instance the case if S , \mathbf{V} and B, B_0 are independent of A or only depend on it *via* the normal vector \mathbf{n} .) Under such a reparametrization, the integral $I(S) = I(S)_{A,A_0}$ (recovering provisionally an explicit indication of the dependence in A and A_0 of the window function) becomes

$$I(S)_{A,A_0} \mapsto I(S)_{f(A),f(A_0)} = I(S)_{A,A_0}, \quad (\text{A.4})$$

where we have used that

$$\mathcal{H}(f(A_0) - f(A)) = \mathcal{H}(A_0 - A), \quad (\text{A.5})$$

for strictly increasing functions f . Such an integral thus only depends on the chosen foliation and the selected slice, but not on the parametrization, and we can remove the subscript notation A, A_0 in the following.

Derivatives with respect to the parameter transform as

$$\frac{\partial I(S)}{\partial A_0} \mapsto \frac{\partial I(S)}{\partial (f(A_0))} = \frac{1}{f'(A_0)} \frac{\partial I(S)}{\partial A_0}, \quad (\text{A.6})$$

while second derivatives become

$$\frac{\partial^2 I(S)}{\partial A_0^2} \mapsto \frac{\partial^2 I(S)}{\partial (f(A_0))^2} = \frac{1}{f'(A_0)^2} \frac{\partial^2 I(S)}{\partial A_0^2} - \frac{f''(A_0)}{f'(A_0)^3} \frac{\partial I(S)}{\partial A_0}. \quad (\text{A.7})$$

We can therefore tune first derivatives by any positive rescaling $f'(A_0)$ through the transformations (A.1), while second derivatives may even be canceled or change sign,

since $f''(A_0)$ is not constrained in its sign. The above results similarly apply to the average $\langle S \rangle$ and its derivatives with respect to A_0 .

We conclude that, without a physical interpretation of the hypersurface label A , statements about the magnitude of first-order derivatives (A.6), as well as *any* statements (about magnitude or sign) about second-order derivatives (A.7), are degenerate with the choice of A . This applies for instance to the left-hand sides of the averaged dynamical equations (23)–(24), or (33)–(34), that are proportional to $(\partial I(1)/\partial A_0)^2$ and $\partial^2 I(1)/\partial A_0^2$, assuming that \mathbf{V} , B and B_0 are defined independently of A or only depend on it *via* the normal vector \mathbf{n} .

Under the same assumption, the conclusions about parametrization-dependence also hold for the terms on the right-hand sides of (23)–(24). Most of them can be written as $\langle S/(P^\sigma \nabla_\sigma A)^2 \rangle$ with a scalar S that is unchanged under the reparametrization (A.1), even when it depends on A , such as Γ_P , and would thus rescale by a factor $f'(A_0)^2$, as does $(\partial I(1)/\partial A_0)^2$. The only exception is the combination of terms $\langle -(\Theta_P + \Gamma_P^{-1} P^\mu \nabla_\mu \Gamma_P) P^\nu \nabla_\nu (P^\sigma \nabla_\sigma A) (P^\rho \nabla_\rho A)^{-3} \rangle$ appearing in \mathcal{P} in (24), which would transform as

$$\left\langle -\frac{(\Theta_P + \Gamma_P^{-1} P^\mu \nabla_\mu \Gamma_P) P^\nu \nabla_\nu (P^\sigma \nabla_\sigma A)}{(P^\rho \nabla_\rho A)^3} \right\rangle \mapsto \frac{1}{f'(A_0)^2} \left\langle -\frac{(\Theta_P + \Gamma_P^{-1} P^\mu \nabla_\mu \Gamma_P) P^\nu \nabla_\nu (P^\sigma \nabla_\sigma A)}{(P^\rho \nabla_\rho A)^3} \right\rangle - \frac{f''(A_0)}{f'(A_0)^3} \frac{\partial I(1)}{\partial A_0}, \quad (\text{A.8})$$

i.e. in the same way as $\partial^2 I(1)/\partial A_0^2$. These identical transformations of both sides of the averaged evolution equations ensure the preservation of the form of these equations under a reparametrization. The same remarks hold for the equations (33)–(34) with $\mathbf{P} = \mathbf{u}$ in this case.

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Chapter 5

Conclusion and perspectives

This PhD under the direction of Thomas Buchert has been a rewarding opportunity to work on several analytical aspects of the formation of matter structures in the Universe and its consequences, within the framework of General Relativity. I have studied approximate models of the local dynamics as well as effective, averaged descriptions of the evolution of a set of fluid elements, focussing for the latter on the possible use of various definitions of ‘space’ slices in spacetime and of local measures of volume. For both aspects, a highlight has been the extension of the validity range of existing schemes to less restrictive fluid models, improving in particular the modelling of small scales and of collapsing objects in their late formation stages.

In the introduction of this thesis (chapter 1), I first recalled the key notion of foliations of spacetime into sets of spacelike hypersurfaces and the corresponding 3+1 formulation of General Relativity. I then exposed the main framework of the standard model of cosmology. The corresponding background models with homogeneous and isotropic spatial hypersurfaces provide a simple framework which can be fully described in terms of a single function of time, the scale factor $a(t)$. They only rely on a small set of parameters as degrees of freedom to be determined. The observational constraints on these models, however, require them to undergo an accelerated expansion and to be dominated by poorly-understood energy components, Dark Matter (with dynamical effects that are also detectable at smaller scales) and Dark Energy. The perturbation theory built in the standard cosmological model accounts for small perturbations in the matter distribution out of the homogeneous background, while it cannot be applied to the non-linear stages of formation of inhomogeneous structures due to this smallness assumption. It also introduces a notion of gauge freedom in terms of the mapping to the assumed background model.

In the last two sections of the introduction I presented two frameworks which aim at better accounting for the emergence of nonlinear inhomogeneities and their coupling to the global dynamics. The Relativistic Zel’dovich Approximation (RZA), designed for irrotational pressureless model fluids, is an efficient Lagrangian scheme for the nonperturbative estimate of the local energy density and other inhomogeneous observables such as curvature, out of perturbative solutions for a set of spatial coframes. The effects of such local inhomogeneities on the overall

dynamics of larger scales are modelled by the second framework in terms of spatial averages, effective scale factors, and backreaction terms. The latter represent the deviation from the behaviour of homogeneous models and may account for some of the effects attributed to Dark Energy and Dark Matter, depending on the averaging scale. These two formalisms formed the basis of the investigations of the following chapters.

In chapter 2, I exposed a Lagrangian approximation scheme building upon the RZA to model structure formation in accelerated irrotational fluids with a barotropic isotropic pressure contribution. The evolution of the trace of the perturbation variable was determined, with an analytic solution in the case of a linear barotropic relation. Procedures for finding solutions for the traceless part were also discussed, which could be split into electric and magnetic parts modelling a propagation of gravitational waves in the fluid. Trace and traceless parts are both required for the nonperturbative evaluation of quantities of interest such as the rest mass density. This was illustrated by a numerical application example in terms of a toy-model for a collapsing structure in which pressure gradients modelled multistreaming.

I then presented in chapter 3 two averaging procedures that can be applied to any definition of the spatial foliation and to a model fluid with vorticity and a general energy-momentum tensor. The commutation rule and the evolution equations for the relevant effective scale factor were derived for both procedures for an averaging domain comoving with the fluid flow, with a discussion of the backreaction terms that arise in each case. The first approach focusses mostly on the evolution of geometric properties of the hypersurfaces despite following a set of fluid elements moving through these slices. The second formalism brings more physical insight into the dynamics of the fluid through the use of a different volume measure, based on the proper volume of the fluid, and through a focus on variables arising from the fluid rest frames. This strategy strongly restricts the dependence of the resulting averages and backreaction terms on the freedom of choice of the foliation. It is especially suited to a foliation by hypersurfaces of constant proper time of the fluid, which exist even in presence of vorticity. With this latter choice, the whole averaging procedure is built from the physical fluid flow, including the spatial slices, and the averaged dynamical equations and the backreactions take an even more transparent form. Due to their generality, both frameworks of this chapter will be applicable to a range of cosmological models and scales for the study of the dynamical contribution of backreactions.

Finally, I showed in chapter 4 how both of the above averaging procedures could be written under a manifestly 4-covariant form. The freedoms in the window function that defines volumes and averages can be used to select any spatial foliation and averaging domain, including the specification of the domain propagation, but also any volume measure. Weighted averages are also possible at the expense of the simple form of an effective scale factor which could still be interpreted as a typical length scale. The manifestly covariant commutation rule, averaged dynamical equations and backreaction expressions were also derived for a generic member of this class of window functions, recovering in particular the results of the previous chapter for a fluid-comoving domain and a measure based on the proper volume for the fluid.

The explicit selection of the foliation by the window function in the manifestly 4-covariant form of the averaging framework allows for a direct comparison of integrals or averages over a domain between two foliation choices. The dependence of these quantities, including backreactions, on the foliation choice can then be studied and bounded explicitly. A variational analysis in terms of infinitesimal foliation changes can also be performed to search for foliations that extremize a specific integral (such as the volume) or average. I am pursuing these studies as part of an ongoing project in collaboration with Asta Heinesen.

I have also initiated an application of the fluid-intrinsic averaging framework of chapter 3 to the analysis of general-relativistic cosmological simulations using non-fluid-orthogonal foliations. This project is a collaboration with Hayley Macpherson (Monash University, Australia), which we will continue in the near future, for a new analysis of the recent results of [Macpherson et al. \[2019\]](#) in terms of intrinsic kinematic variables of the matter flow and the associated kinematical backreaction.

I am additionally planning as a future project a rewriting of the RZA that would hold in any spatial foliation, allowing in this way for the handling of vorticity as well as non-perfect fluid contributions. As already mentioned, this would be a natural further extension beyond the inclusion of isotropic pressure and would improve the modelling of small scales. I would moreover benefit from the experience with the general framework needed already used for the study of general averages in chapter 3.

Another possible future project would be the application of this latter general averaging framework to specific cosmological models and fluid sources for the investigation of backreactions in these cases. One example of particular interest could be the use of the two vector fields at play (the 4-velocity of some source \mathbf{u} and the normal vector to the slices \mathbf{n}), with a relative tilt, to model the early Universe era with two main fluid components, Dark Matter and radiation, having two different velocities associated with \mathbf{u} and \mathbf{n} , respectively. For this, the radiation fluid would be assumed to remain irrotational. This follows a suggestion of David Wiltshire with whom a collaboration would be possible.

While continuing on these projects, I will also turn after the end of this PhD to the study of other consequences of General Relativity, especially gravitational waves, through the opportunity of a postdoctoral position at the Max Planck Institut für Gravitationsphysik in Hannover. In this context, the electric and magnetic split of traceless components of linearized spatial coframes introduced within the RZA framework may bring an additional tool of interest for the modelling of gravitational waves as they propagate through matter.

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