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Gamma positivity in enumerative combinatorics

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Résumé

La positivité gamma d'une suite combinatoire unifie à la fois l'unimodalité et la symétrie de cette suite. Trouver des nouvelles familles d'objets dont les polynômes énumératifs ont une positivité gamma est un défi et un sujet important en combinatoire et géométrie. Il a attiré beaucoup d'attention ces derniers temps en raison de la conjecture de Gal, qui affirme que le γ -vecteur a des coefficients positifs pour n'importe quel polytope simple. Souvent, le h -polynôme pour les polytopes simpliciaux de signification combinatoire peut être donné en tant que fonction génératrice sur un ensemble d'objets combinatoires apparentés par rapport à une statistique telle que le nombre des descentes, dont les polynômes énumératifs sur les permutations sont des polynômes Eulériens.

Ce travail traite des propriétés gamma de plusieurs famille de polynômes énumératifs de permutations comme les polynômes Eulériens et les polynômes de Narayana. Cette thèse contient quatre chapitres. L'introduction donne quelques notions de base et des résultats généraux sur le développement gamma de polynômes.

Chapitre 1 vise à généraliser la formule de Stembridge reliant les polynômes de pic et les polynômes Eulériens en utilisant des statistiques circulaires et en employant la méthode de la fraction continue. Cette approche était inspirée par le fait bien connu que les polynômes Eulériens peuvent être interprétés à la fois comme des polynômes de descentes et de l'excédance de permutations. Nous prouvons également des formules similaires pour les polynômes d'excédance à permutations restreintes telles que les dérangements et les permutations à motifs exclus.

Chapitre 2 traite des permutations qui évitent un motif unique de longueur trois. On sait que ces permutations sont énumérées par les nombres de Catalan. Nous étudions une sorte de (q, t) -nombres de Catalan définis par des fractions continues et fournissons plusieurs nouvelles interprétations ainsi que leurs développements gamma correspondants en utilisant des permutations évitant certains motifs. Nous donnons également une caractérisation complète de certains (-1) -phénomène pour chaque sous-ensemble de permutations évitant un motif de longueur trois, et discutons de leurs q -analogues.

Chapitre 3 étudie une classe de permutations introduite par Flajolet et Françon pour donner une interprétation combinatoire des coefficients de Taylor des fonctions elliptiques de Jacobi. En généralisant deux fractions continues de Rogers et de Stieltjes, nous donnons des interprétations combinatoires des coefficients de Taylor correspondants ainsi que leurs coefficients gamma.

Chapitre 4 est à étudier le tableau de différences associé aux polynômes énumérative de permutations colorée par rapport aux nombres de points fixes. Ceci conduit à une généralisation des résultats d'Eriksen–Freij–Wästlund et de Liese et Remmel sur les k -excédances en permutations. De plus, en considérant les r -dérangements dans le contexte du tableau de différences, nous obtenons un λ -analogue des nombres de r -dérangements dus à Wang–Miska–Mező.

Mots-clés: statistiques de permutation, descente, excédance, pic, inversion, inversion admissible, tableau des différences d'Euler, polynômes Eulériens, γ -positivité, (q, t) -nombre de Catalan, permutations à motifs exclus, dérangement, permutation alternée, (-1) -phénomène, action modifiée de Foata-Strehl, fraction continue, fonction elliptique de Jacobi, double permutation.

Abstract

The gamma positivity of a combinatorial sequence unifies both unimodality and symmetry. Finding new family of objects whose enumerative sequences have gamma positivity is a challenge and important topic in recent years. It has received considerable attention in recent times because of Gal's conjecture, which asserts that the γ -vector has nonnegative entries for any flag simple polytope. Often times, the h -polynomial for simplicial polytopes of combinatorial signification can be given as a generating function over a related set of combinatorial objects with respect to some statistic like the descent numbers, whose enumerative polynomials on permutations are Eulerian polynomials.

This work deals with the gamma properties of several enumerative polynomials of permutation such as Eulerian polynomials and Narayana polynomials. This thesis contains four chapters. The introduction gives some backgrounds and general results on gamma expansion of polynomials.

Chapter 1 aims to generalize Stembridge's formula relating peak polynomials and Eulerian polynomials by using their cycle analogues and using continued fraction method. This approach was motivated by the well-known fact that Eulerian polynomials can be interpreted as both descent and excedance polynomials of permutations. We also prove similar formulae for excedance polynomials of restricted permutations such as derangements and permutations avoiding certain patterns.

Chapter 2 discusses permutations that avoid single pattern of length three. These permutations are known to be enumerated by Catalan numbers. We define a kind of (q, t) -Catalan numbers using continued fractions and provide several new interpretations along with their corresponding gamma expansions using pattern avoiding permutations. We also give a complete characterization of certain (-1) -phenomenon for each subset of permutations avoiding pattern of length three, and discuss their q -analogues.

Chapter 3 studies a class of permutations introduced by Flajolet and Françon to give a combinatorial interpretation for the Taylor coefficients of Jacobian elliptic functions. By considering two more general continued fractions of Rogers and Stieltjes, we provide combinatorial interpretations of the corresponding Taylor coefficients as well as their gamma coefficients.

Chapter 4 is to study the difference table associated with enumerative polynomials of colored permutations with respect to the numbers of fixed points. This leads to a generalization of Eriksen–Freij–Wästlund's result and Liese and Remmel's result on k -excedances in permutations. Moreover, by considering r -derangements in the context of the difference table, we obtain a λ -analogue of r -derangement numbers due to Wang–Miska–Mező.

Keywords: permutation statistic, descent, excedance, peak, inversion, admissible inversion, Euler's difference table, Eulerian polynomial, gamma-positivity, (q, t) -Catalan number, permutation pattern avoidance, derangement, alternating permutation, (-1) -

phenomenon, Modified Foata-Strehl action, continued fraction, Jacobi elliptic function, doubled permutation.

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Contents

Introduction	1
1 Eulerian polynomials and excedance statistics	13
1.1 Introduction	13
1.2 Background and preliminaries	16
1.2.1 Permutation statistics and two bijections	16
1.2.2 The bijection Φ	17
1.2.3 The bijection Ψ	18
1.2.4 The star variation	19
1.2.5 Laguerre histories as permutation encodings	21
1.2.6 Françon-Viennot bijection	22
1.2.7 Restricted Françon-Viennot bijection	23
1.2.8 Foata-Zeilberger bijection	23
1.2.9 Pattern avoidances and 2-Motzkin paths	23
1.3 Main results	26
1.4 Proofs using group actions	32
1.4.1 Proof of Theorem 1.15	32
1.4.2 Proof of Theorem 1.25	36
1.4.3 Proof of Theorems 1.31 and 1.32	39
1.5 Proofs via continued fractions	40
1.5.1 Some combinatorial continued fractions	40
1.5.2 Proof of Theorems 1.15–1.28	42
1.5.3 Proof of Theorems 1.29–1.30	47
2 Gamma positivity in restricted permutations	49
2.1 Introduction	49
2.2 Definitions and preliminaries	52
2.2.1 Permutation statistics and a proof of Theorem 2.4	52
2.2.2 Other combinatorial interpretations of $C_n(q)$	54
2.3 Proofs of Theorems 2.1, 2.2 and 2.3	56
2.4 A variant of q -Narayana polynomials	62
2.5 Characterization for avoiding a pattern of length three	63
2.5.1 The 231-avoiding des-case and its q -analogues	63
2.5.2 Other des-cases avoiding one pattern of length three and their q -analogues	65
2.5.3 Other exc-cases avoiding one pattern of length three	71
2.6 Two cases avoiding two patterns of length four	72
2.6.1 The case of (2413,3142)-avoiding alternating permutations.	73
2.6.2 The case of (1342,2431)-avoiding alternating permutations.	76
2.7 Final remarks	77

3	Elliptic Functions and Gamma Positivity	79
3.1	Introduction	79
3.2	Main results	80
3.3	Definitions and Preliminaries	84
3.4	Proof of Theorem 3.1	85
3.5	Proof of Theorem 3.2	86
	3.5.1 Analytic method	86
	3.5.2 Group action method	87
3.6	Proof of Theorem 3.4	88
3.7	Proof of Theorem 3.5	90
3.8	Proof of Theorem 3.7	91
3.9	Proof of Theorem 3.8	92
3.10	Proof of Theorem 3.9	94
3.11	Interpret $\text{cn}(z, x)$ in terms of alternating permutations	96
4	λ-Euler's difference table for colored permutations	99
4.1	Introduction	99
4.2	Definitions and main results	100
4.3	Proof of Theorem 4.1	105
4.4	Proof of Theorem 4.4	107
4.5	Proof of Theorem 4.5	115
4.6	Proof of Theorem 4.6	115
	4.6.1 First Proof	116
	4.6.2 Second Proof	116
4.7	Generating functions and further recurrence relations	117
4.8	Proof of Theorem 4.7	119
4.9	Combinatorial proof of recurrence relation (4.46)	120
4.10	Final remarks	121
	Bibliography	123
	Index	131

Indeed, the so-called Waring's formula [42] reads:

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j}.$$

Let $\gamma_{n,j} = (-1)^j \frac{n}{n-j} \binom{n-j}{j}$, then

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j} (xy)^j (x+y)^{n-2j}.$$

Since $a_j = a_{n-j}$ (symmetry), we have

$$\begin{aligned} a_j x^j + a_{n-j} x^{n-j} &= a_j x^j (1 + x^{n-2j}) \\ &= a_j x^j \sum_k \gamma_{n-2j,k} x^k (x+1)^{n-2j-2k} \\ &= a_j \sum_k \gamma_{n-2j,k} x^{k+j} (x+1)^{n-2(j+k)}. \end{aligned}$$

If $h(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1+x)^{n-2k}$, we call $\{\gamma_k\}_{k=0}^{\lfloor n/2 \rfloor}$ the γ -vector of h . Since the binomial coefficients $\{\binom{n}{k}\}_{k=0}^n$ for fixed n are unimodal, having a nonnegative γ -vector leads to the unimodality of $\{a_k\}_{k=0}^n$ directly. We say that h is γ -positive if the γ -vector of h is nonnegative.

Let Γ_+^n denotes the convex cone of polynomials that have nonnegative coefficients when expanded in B_n . It is easy to see that

$$\Gamma_+^m \cdot \Gamma_+^n := \{fg : f \in \Gamma_+^m \text{ and } g \in \Gamma_+^n\} \subseteq \Gamma_+^{m+n}. \quad (1)$$

Remark 0.1. Suppose that a polynomial $h(x) = \sum_{k=0}^n a_k x^k \in \mathfrak{A}_+[x]$ has a nonnegative and symmetric coefficients with center of symmetry $n/2$, if all its zeros are real, then the negative zeros can be paired into reciprocal pairs,

$$h(x) = Ax^k \prod_{i=1}^{\ell} (x + \theta_i)(x + 1/\theta_i) = Ax^k \prod_{i=1}^{\ell} ((1+x)^2 + (\theta_i + 1/\theta_i - 2)x),$$

where $A > 0$. Since x and $(1+x)^2 + (\theta_i + 1/\theta_i - 2)x$ are in Γ_+^1 and Γ_+^2 , from (1) we see that $h(x)$ is γ -positive.

The classical *Eulerian polynomials* $A_n(t)$ are defined by

$$\sum_{k \geq 0} (k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}} \quad (2)$$

for integers $n \geq 0$. These polynomials were introduced by Euler [37] in the 18th century when he looked for an expression for the *alternating* sum $\sum_{1 \leq i \leq k} i^n (-1)^i$. The research about the Eulerian polynomials plays an important role in numerous arithmetical and combinatorial studies [48, 51, 59]. For instance, from (2) we can derive the equivalent definition of the Eulerian polynomials

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{zt} - te^z}. \quad (3)$$

We give the first few values of Eulerian polynomials as follows,

$$A_n(t) = \begin{cases} 1, & \text{if } n = 1, \\ 1 + t, & \text{if } n = 2, \\ 1 + 4t + t^2, & \text{if } n = 3, \\ 1 + 11t + 11t^2 + t^3, & \text{if } n = 4, \\ 1 + 26t + 66t^2 + 26t^3 + t^4, & \text{if } n = 5. \end{cases} \quad (4)$$

A *permutation* of $[n] := \{1, 2, \dots, n\}$ is a bijection $\pi : [n] \rightarrow [n]$. Let \mathfrak{S}_n denote the set of all permutations of $[n]$. A *statistic* on \mathfrak{S}_n is just a function $st : \mathfrak{S}_n \rightarrow \mathbb{N}$. The study of permutations of sets and wide variety of interesting statistics of permutations is the richest topic in enumerative combinatorics and its history can at least go back to more than a century ago due to André [2] and MacMahon [76, 77].

A permutation $\pi = \pi(1)\pi(2) \cdots \pi(n) \in \mathfrak{S}_n$ is an *alternating* (resp. *falling alternating*) permutation if $\pi(1) < \pi(2)$, $\pi(2) > \pi(3)$, $\pi(3) < \pi(4)$, etc. (resp. $\pi(1) > \pi(2)$, $\pi(2) < \pi(3)$, $\pi(3) > \pi(4)$, etc.). We denote by \mathbb{A}_n the set of alternating permutations of length n . The *complement* transformation

$$\pi \mapsto \pi^c = (n + 1 - \pi(1))(n + 1 - \pi(2)) \cdots (n + 1 - \pi(n))$$

on the permutations $\pi \in \mathbb{A}_n$ shows that the number of alternating permutations is equal to the number of falling alternating permutations. We define the *Euler numbers*, denoted by E_n , to be the number of alternating (resp. *falling alternating*) permutations on $[n]$. Note that the sequence is called Euler number because Euler is the first to consider the generating function (6), see [100].

n	E_n	alternating permutations
1	1	1
2	1	12
3	2	132, 231
4	5	1324, 1423, 2314, 2413, 3412

Figure 1: The alternating permutations and Euler numbers for $1 \leq n \leq 4$.

At the end of the 19th century, André [2] gave the following generation function of alternating permutations in \mathbb{A}_n .

Theorem 0.2. *We have*

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Note that since $\sec x$ (resp. $\tan x$) is an even function (resp. odd function), from Theorem 0.2,

$$\sec x = 1 + \frac{x^2}{2!} + 5 \frac{x^4}{4!} + 61 \frac{x^6}{6!} + 1385 \frac{x^8}{8!} + \cdots + E_{2n} \frac{x^{2n}}{(2n)!} + \cdots, \quad (5)$$

$$\tan x = x + 2 \frac{x^3}{3!} + 16 \frac{x^5}{5!} + 272 \frac{x^7}{7!} + \cdots + E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} + \cdots, \quad (6)$$

then the even (resp. odd) Euler numbers are also called *secant* (resp. *tangent*) numbers.

In the early 20th century MacMahon studied four fundamental permutation statistics, the descent number, the excedance number, the inversion number and the major index.

For each permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$, an integer $i \in [n-1]$ is a descent (resp. excedance) of π if $\pi(i) > \pi(i+1)$ (resp. $\pi(i) > i$). Denote $\text{DES}(\pi)$ (resp. $\text{EXC}(\pi)$) the set of descents (resp. excedances) and denote $\text{des } \pi$ (resp. $\text{exc } \pi$) the number of descents (resp. excedances) of π . MacMahon [76, 77] showed that the descent number and excedance number on permutations are equidistributed, i.e.,

$$\sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} = \sum_{\pi \in \mathfrak{S}_n} t^{\text{exc } \pi}.$$

\mathfrak{S}_3	des	exc	inv	maj
123	0	1	0	0
132	1	1	1	2
213	1	1	1	1
231	1	2	2	2
312	1	1	2	1
321	2	1	3	3

Figure 2: Four fundamental statistics for \mathfrak{S}_3 .

Riordan [87] later discovered the following combinatorial interpretation for the Eulerian polynomials

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi}. \quad (7)$$

With the above interpretation, each statistic equidistributed with the descent number (or excedance number) on permutations is usually called *Eulerian statistic*. The well known *Foata's first fundamental transformation* [46] provides the bijection to show the equidistribution of the decent number and the excedance number on permutations (or more generally on words).

An *inversion* of a permutation $\pi \in \mathfrak{S}_n$ is a pair $(\pi(i), \pi(j))$ such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. Denote $\text{INV}(\pi)$ the set of inversions and $\text{inv } \pi$ the number of the inversions of π . The set \mathfrak{S}_n has a group structure by composition. The resulting group is called the symmetric group, or the type A Coxeter group of order n . A well-known result in the type A Coxeter group is that the length of an element equals its *inversion number* [8]. Rodrigues [88] showed the the distribution of the inversion statistic,

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{inv } \pi} = [n]_q!,$$

where

$$[n]_q! := [1]_q [2]_q \cdots [n]_q$$

and

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

MacMahon [75, 76] defined the *major index* by

$$\text{maj } \pi = \sum_{i \in \text{DES}(\pi)} i,$$

and showed that the inversion number and the major index are identically distributed on permutations, i.e.,

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{maj } \pi} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{inv } \pi}.$$

In memory of MacMahon, any permutation statistic equidistributed with the inversion number (or major index) is called a *Mahonian statistic*. The well known *Foata's second fundamental transformation* [47] provides the bijection to prove the equidistribution of the inversion number and the major index on permutations (or more generally on words).

With (4), the first few terms of Eulerian polynomials $A_n(t)$ have the following expressions,

$$A_n(t) = \begin{cases} 1, & \text{if } n = 1, \\ 1 + t, & \text{if } n = 2, \\ (1 + t)^2 + 2t, & \text{if } n = 3, \\ (1 + t)^3 + 8t(1 + t), & \text{if } n = 4, \\ (1 + t)^4 + 22t(1 + t)^2 + 16t^2, & \text{if } n = 5. \end{cases}$$

Let $\pi \in \mathfrak{S}_n$ and by convention $\pi(0) = \pi(n + 1) = \infty$, we call $\pi(i)$ is a *double descent* of π if $\pi(i - 1) > \pi(i) > \pi(i + 1)$, denote by $\text{dd } \pi$ the number of double descents of π . A classical result due to Foata and Schützenberger [51] states that Eulerian polynomials have the following γ -expansion.

Theorem 0.3 (Foata–Schützenberger).

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^A t^k (1+t)^{n-1-2k}, \quad (8)$$

where

$$\gamma_{n,k}^A = \#\{\pi \in \mathfrak{S}_n : \text{dd } \pi = 0, \text{des } \pi = k\}. \quad (9)$$

Note that Foata and Strehl's celebrated valley-hopping [52] was a neat combinatorial argument that lead to (8) (see also [83, Chapter 4] for a nice exposition and the references therein). Moreover, we can take $t = -1$ in (8) and recover the following combinatorial interpretation of a classical identity involving the odd Euler number E_{2n+1} [37]:

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{des } \pi} = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{exc } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n & \text{if } n \text{ is odd.} \end{cases} \quad (10)$$

Recall that a permutation $\sigma \in \mathfrak{S}_n$ is a *derangement* if it has no fixed points, i.e., $\sigma(i) \neq i$ for all $i \in [n]$. Let \mathfrak{D}_n be the set of *derangements* in \mathfrak{S}_n . A parallel result for the even Euler number E_{2n} was first given by Roselle [90]:

$$\sum_{\pi \in \mathfrak{D}_n^*} (-1)^{\text{des } \pi} = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\text{exc } \pi} = \begin{cases} (-1)^{\frac{n}{2}} E_n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (11)$$

where \mathfrak{D}_n^* denotes the set of *coderangements* of length n , see Definition 2.3.

Recall [6] it is common to consider permutations graphically. For a permutation $\pi = \pi(1) \cdots \pi(n)$, its *plot* consists of the the points $(i, \pi(i))$ in the Euclidean plane, for $i = 1, \dots, n$.

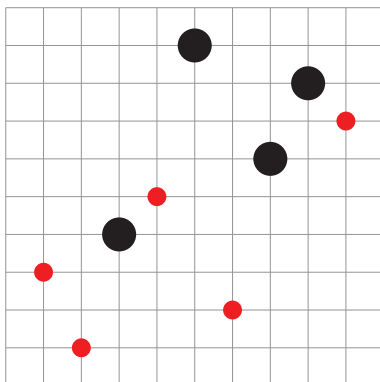


Figure 3: The plot of permutation **314592687** with a **1423** pattern marked with black dots

Given two permutations $\pi \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$, $k \leq n$, we say that π *contains the pattern* p if there exists a set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ of π is order-isomorphic to p . From the graphical perspective, π contains p if erasing zero or more points from the plot of π and then rescaling the axes appropriately leads to the plot of p . For example, 314592687 contains 1423 because the subsequence 4968 is order-isomorphic to 1423, see Figure 3.

If π does not contain p , we say that π *avoids* p . The set of permutations of length n that avoid patterns p_1, p_2, \dots, p_m is denoted as $\mathfrak{S}_n(p_1, p_2, \dots, p_m)$. For example, 314592687 avoids 3241 and 4231 since it has no subsequence ordered-isomorphic to 3241 and 4231.

Given two permutations σ and τ with lengths k and ℓ respectively, define the *direct sum* $\sigma \oplus \tau$ the permutation of length $k + \ell$ consisting of σ followed by a shifted copy of τ :

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } i \leq k, \\ k + \tau(i - k) & \text{if } k + 1 \leq i \leq k + \ell. \end{cases}$$

The *skew sum* $\sigma \ominus \tau$ is defined analogously. See Figure 4 for the illustration.

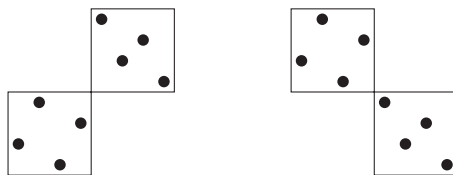


Figure 4: The direct sum **2413** \oplus **4231**, the skew sum **2413** \ominus **4231**.

The study of patterns in permutations and words has a long history [63]. MacMahon [75] (resp. Knuth [64]) showed that the size of $\mathfrak{S}_n(123)$ (resp. $\mathfrak{S}_n(231)$) is the Catalan number. Simion and Schmidt [98] were the first to study the relationship among the different permutation patterns and gave a bijection between $\mathfrak{S}_n(132)$ and $\mathfrak{S}_n(123)$. Over the past decades many mathematicians have paid a lot of attention to pattern avoidance in permutations.

Chapter 1

The integer i ($2 \leq i \leq n - 1$) is a *peak* of $\pi = \pi(1)\pi(2)\cdots\pi(n)$ if $\pi(i - 1) < \pi(i) > \pi(i + 1)$, denote $\text{peak}' \pi$ the number of peaks of π , The peak number played a key role in

Stembridge's theory of enriched P-partitions, see [103]. Define the peak polynomials

$$P_n^{\text{peak}}(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\text{peak}' \pi}.$$

The following equivalent form of (8) was first given by Stembridge [103, Remark 4.8],

$$A_n(t) = \left(\frac{1+t}{2}\right)^{n-1} P_n^{\text{peak}}\left(\frac{4t}{(1+t)^2}\right). \quad (12)$$

Let

$$P_n^{(\text{peak}, \text{des})}(x, t) := \sum_{\pi \in \mathfrak{S}_n} x^{\text{peak}' \pi} t^{\text{des } \pi}.$$

Recently Zhuang [112, Theorem 4.2] proved the following result, which reduces to (12) when $x = 1$,

$$A_n(t) = \left(\frac{1+xt}{1+x}\right)^{n-1} P_n^{(\text{peak}, \text{des})}\left(\frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt}\right). \quad (13)$$

We prove several new formulas expressing polynomials counting permutations by various excedance statistics in terms of Eulerian polynomials, which are analogous to Zhuang's formulas. Our methods include permutations enumeration technics involving continued fractions and cycle version of modified Foata-Strehl action. Moreover, we prove similar formulae for restricted permutations such as derangements and permutations avoiding certain patterns.

The contents of this chapter are available online in [arXiv:1908.01084](https://arxiv.org/abs/1908.01084), 1–41.

Chapter 2

Similar with the combinatorial interpretation of Eulerian polynomials, t -Catalan numbers (or Narayana polynomials) (cf. [83, Section 2.3]) are defined by

$$N_n(t) = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des } \pi} = \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k} t^k. \quad (14)$$

There is an γ -expansion (cf. [83, Theorem 4.2]) for the t -Catalan numbers

$$N_n(t) := \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des } \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^N t^k (1+t)^{n-1-2k}, \quad (15)$$

where

$$\gamma_{n,k}^N = \#\{\pi \in \mathfrak{S}_n(231) : \text{dd } \pi = 0, \text{des } \pi = k\}. \quad (16)$$

The Clarke-Steingrímsson-Zeng bijection [24] linking des based statistics with exc based ones is crucial for our ensuing derivation. It is the composition, say Φ , of two bijections between \mathfrak{S}_n and the set of certain weighted two colored Motzkin paths of length n . One bijection is due to Françon and Viennot [54], the other is due to Foata and Zeilberger [53]. See [24] for a direct description of Φ and further details. The following equidistribution result reveals further properties of Φ and is equivalent to [95, Theorem 8] modulo one

application of the inverse map: $\pi \mapsto \pi^{-1}$. For any $n \geq 1$, there is a bijection Φ on \mathfrak{S}_n such that

$$(\text{des, fmax, 31-2, 2-31, MAD}) \pi = (\text{exc, fix, cros, nest, inv}) \Phi(\pi) \quad \text{for all } \pi \in \mathfrak{S}_n.$$

Shin and Zeng [95] deduced the continued fraction expansion for the quint-variate generating function of \mathfrak{S}_n with respect to the above statistics. We first prove several new interpretations of a kind of (q, t) -Catalan numbers along with their corresponding γ -expansions using pattern avoiding permutations. Secondly, we give a complete characterization of certain (-1) -phenomenon for each subset of permutations avoiding a single pattern of length three, and discuss their q -analogues utilizing the newly obtained q - γ -expansions. Moreover, we enumerate the alternating permutations avoiding simultaneously two patterns, namely $(2413, 3142)$ and $(1342, 2431)$, of length four, and consider such (-1) -phenomenon for these two subsets as well.

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Chapter 3

The Jacobi elliptic function $\text{sn}(z, x)$ is defined by the inverse of an elliptic integral, i.e.,

$$\text{sn}(z, x) = y \quad \text{iff} \quad z = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}.$$

where $x \in (0, 1)$ is a fixed *modulus*.

The other two Jacobi elliptic functions are defined by

$$\begin{aligned} \text{cn}(z, x) &:= \sqrt{1 - \text{sn}^2(z, x)}, \\ \text{dn}(z, x) &:= \sqrt{1 - x^2 \text{sn}^2(z, x)}. \end{aligned}$$

A variety of problems related to Jacobi elliptic functions have been extensively studied in mathematical physics, combinatorics and number theory ([25, 26, 31, 32, 43, 44, 80, 89, 104, 106]). When $x = 0$ or $x = 1$, the Jacobi elliptic functions degenerate into trigonometric or hyperbolic functions, to be more precise,

$$\begin{aligned} \text{sn}(z, 0) &= \sin z, \quad \text{cn}(z, 0) = \cos z, \quad \text{dn}(z, 0) = 1, \\ \text{sn}(z, 1) &= \tanh z, \quad \text{cn}(z, 1) = \text{dn}(z, 1) = \text{sech } z. \end{aligned}$$

Furthermore, the Jacobi elliptic functions relate the classic Euler numbers,

$$-\text{isn}(iz, 1) + \text{cn}(iz, 1) = \tan z + \sec z = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!},$$

where $i = \sqrt{-1}$.

The three Jacobi elliptic functions also satisfy the following differential system (see [11]):

$$\begin{cases} \frac{d}{dz} \text{sn}(z, x) = \text{cn}(z, x) \text{dn}(z, x), \\ \frac{d}{dz} \text{cn}(z, x) = -\text{sn}(z, x) \text{dn}(z, x), \\ \frac{d}{dz} \text{dn}(z, x) = -x^2 \text{sn}(z, x) \text{cn}(z, x), \end{cases}$$

with the initial condition,

$$\operatorname{sn}(0, x) = 0, \quad \operatorname{cn}(0, x) = 1, \quad \operatorname{dn}(0, x) = 1.$$

With the above relations, the Taylor series coefficients of the Jacobian elliptic function are well studied by several authors, see [92, 31, 32]. The first few terms of Taylor series expansions of these Jacobian elliptic functions are given as follows:

$$\operatorname{sn}(z, x) = z - (1 + x^2) \frac{z^3}{3!} + (1 + 14x^2 + x^4) \frac{z^5}{5!} - (1 + 135x^2 + 135x^4 + x^6) + \dots, \quad (17)$$

$$\operatorname{cn}(z, x) = 1 - \frac{z^2}{2!} + (1 + 4x^2) \frac{z^4}{4!} - (1 + 44x^2 + 16x^4) \frac{z^6}{6!} + \dots, \quad (18)$$

$$\operatorname{dn}(z, x) = 1 - x^2 \frac{z^2}{2!} + x^2(4 + x^2) \frac{z^4}{4!} - x^2(16 + 44x^2 + x^4) \frac{z^6}{6!} + \dots. \quad (19)$$

Rogers and Stieltjes [89, 104] considered the following Laplace-Borel transforms of sn and cn :

$$S_1(z, x) = \int_0^\infty e^{-t} \operatorname{sn}(zt, x) dt \quad \text{and} \quad C_0(z, x) = \int_0^\infty e^{-t} \operatorname{cn}(zt, x) dt,$$

i.e., the series obtained from (17) and (18) by replacing $z^n/n!$ by z^n , have the following continued fractions forms,

$$S_1(z, x) = \frac{z}{1 + (1 + x^2)z^2 - \frac{z}{1 + (1 + x^2)3^2z^2 - \frac{z}{1 + (1 + x^2)5^2z^2 - \dots}}},$$

$$C_0(z, x) = \frac{1}{1 + z^2 - \frac{1}{1 + (3^2 + 2^2x^2)z^2 - \frac{1}{1 + (5^2 + 4^2x^2)z^2 - \dots}}}.$$

According to [44], the combinatorial interpretation of the coefficients of

$$(-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (\text{resp. } (-1)^n \frac{z^{2n}}{(2n)!})$$

in the Taylor series expansions of Jacobi elliptic functions $\operatorname{sn}(z, x)$ (resp. $\operatorname{cn}(z, x)$) were first considered by Schützenberger. Afterwards, a series of paper [106, 43, 31, 44] answered the problem by providing interpretations via different methods. In particular, Flajolet-Françon [44] gave an interpretation of the elliptic functions as generating functions of *doubled permutations*, see Definition 3.1.

By generalizing the continued fraction expansions of Rogers and Stieltjes, we give the combinatorial interpretation of the Taylor coefficients of the generalized Jacobian elliptic functions. The second goal of this chapter is to study the expansion of the Taylor coefficients of the Jacobian elliptic functions, which implies the symmetric and unimodal property of the Taylor coefficients of the Jacobian elliptic functions. The main tools are the combinatorial theory of continued fractions due to Flajolet and bijections due to Françon-Viennot [54], Foata-Zeilberger [53] and Clarke-Steingrímsson-Zeng [24].

Chapter 4

Let l be a fixed positive integer and $C_l \wr \mathfrak{S}_n$ be the wreath product of the cyclic group C_l of order l by the symmetric group \mathfrak{S}_n of order n . The group $C_l \wr \mathfrak{S}_n$ is also known as the *colored permutation group*. In the case $l = 1$ (resp. $l = 2$) the wreath product $C_l \wr \mathfrak{S}_n$ is the symmetric group \mathfrak{S}_n (resp. the group of the signed permutations or the *Type B Coxeter group* B_n). Various statistics on colored permutation groups have been studied in the literature and several q -analogs of colored Eulerian polynomials have been proposed [38, 39, 41, 49, 50, 70, 71, 102]. Euler [33] studied the difference table $(g_n^m)_{0 \leq m \leq n}$, where the coefficients are defined by $g_n^n = n!$ and

$$g_n^m = g_n^{m+1} - g_{n-1}^m, \quad (20)$$

for $0 \leq m \leq n - 1$. The first terms of these coefficients for g_n^m are given in Tables 1.

$n \setminus m$	0	1	2	3	4	5
0	1					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
5	44	53	64	78	96	5!

Table 1: Values of g_n^m for $0 \leq m \leq n \leq 5$.

Dumont and Randrianarivony [33] studied the combinatorial interpretation of g_n^m in the symmetric group \mathfrak{S}_n , which consists of permutations of $[n] = \{1, \dots, n\}$. In particular, they showed that the sequence $\{g_n^0\}_{n \geq 0}$ is the number of derangements, i.e., the fixed point free permutations in \mathfrak{S}_n . Wang–Miska–Mező [107] introduced the r -derangement number, which counts the derangements of $[n]$ with the first r elements appear in distinct cycles. In ordinary rook theory, rook placements can be associated with permutations of set $[n]$, see [61]. Briggs and Remmel [16] generalized the rook theory linked with the elements of $C_\ell \wr \mathfrak{S}_n$. Rakotondrajao [84] developed further combinatorial interpretations of g_n^m in terms of k -successions, Liese and Remmel [69] used the rook theory to obtain a series of recurrence formula for the number of permutations with m k -successions. The reader is referred to [33, 84, 85, 40, 23, 35, 69, 20], where several generalizations of Euler’s difference table with combinatorial meanings were studied.

We continue the research about Euler’s difference table. In 2008 Faliharimalala and Zeng [40] studied the Euler’s difference table $(g_{\ell,n}^m)_{0 \leq m \leq n}$ for $C_\ell \wr \mathfrak{S}_n$,

$$\begin{cases} g_{\ell,n}^n = \ell^n n! & (m = n); \\ g_{\ell,n}^m = g_{\ell,n}^{m+1} - g_{\ell,n-1}^m & (0 \leq m \leq n - 1). \end{cases} \quad (21)$$

In 2009 Eriksen–Freij–Wästlund [35] studied the following λ -Euler’s difference table $(g_n^m(\lambda))_{0 \leq m \leq n}$,

$$\begin{cases} g_n^m(\lambda) = n! & (m = n); \\ g_n^m(\lambda) = g_n^{m+1}(\lambda) + (\lambda - 1)g_{n-1}^m(\lambda) & (0 \leq m \leq n - 1), \end{cases} \quad (22)$$

and interpreted $g_n^m(\lambda)$ when λ is a non-negative interger. Motivated by the above works, we studied the combinatorial interpretation of λ -analogue of Euler’s difference table $g_{\ell,n}^m(\lambda)$ when λ is a variable. Using Briggs and Remmel’s rook theory for $C_\ell \wr \mathfrak{S}_n$, we interpret the

coefficients of polynomial $g_{\ell,n}^m(\lambda)$. Moreover, by relating with the sequences arising from the difference table, we obtain λ -analogue of Wang–Miska–Mező’s r -derangement numbers for colored permutations.

The contents of this chapter are published in **Electron. J. Combin.** **25 (2018)**, no. **4**, Paper 4.25, 27 pp.

Chapter 1

Eulerian polynomials and excedance statistics¹

1.1 Introduction

The Eulerian polynomials $A_n(t)$ can be defined through the continued fraction expansion [104]

$$\sum_{n \geq 0} A_n(t) z^n = 1/1 - 1 \cdot z/1 - t \cdot z/1 - 2 \cdot z/1 - 2t \cdot z/1 - 3 \cdot z/1 - 3t \cdot z/1 - \dots \quad (1.1)$$

For an n -permutation $\sigma := \sigma(1)\sigma(2) \cdots \sigma(n)$ of the word $1 \dots n$, an index i ($1 \leq i \leq n-1$) is a *descent* (resp. *excedance*) of σ if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$). It is well-known [51, 83] that

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma}, \quad (1.2)$$

where \mathfrak{S}_n is the set of n -permutations and $\text{des } \sigma$ (resp. $\text{exc } \sigma$) denotes the number of descents (resp. excedances) of σ . The value $\sigma(i)$ ($2 \leq i \leq n-1$) is a *peak* of σ if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ and the *peak polynomials* are defined by

$$P_n^{\text{peak}}(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{\text{peak}' \sigma} \quad (1.3)$$

where $\text{peak}' \sigma$ denotes the number of peaks of σ . The peak polynomials are related to the Eulerian polynomials by Stembridge's identity [103, Remark 4.8], see also [13, 112],

$$A_n(t) = \left(\frac{1+t}{2} \right)^{n-1} P_n^{\text{peak}} \left(\frac{4t}{(1+t)^2} \right), \quad (1.4)$$

which can be used to compute the peak polynomials. Obviously Eq. (1.4) is equivalent to the so-called γ -expansion of Eulerian polynomials

$$A_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^{2k+1-n} \gamma_{n,k} t^k (1+t)^{n-1-2k}, \quad (1.5)$$

¹The contents of this chapter are available online in [arXiv:1908.01084](https://arxiv.org/abs/1908.01084), 1–41, see [60].

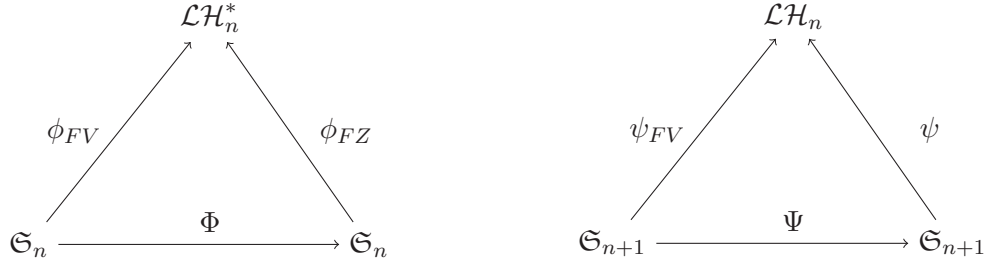


Figure 1.1: Bijections $\Phi = \phi_{FZ}^{-1} \circ \phi_{FV}$ and $\psi = \psi_{FV} \circ \Psi^{-1}$.

where $\gamma_{n,k}$ is the number of n -permutations with k peaks. In the form of (1.5) it is not difficult to see that Stembridge's formula (1.4) is actually equivalent to a formula of Foata and Schützenberger [51, Théorème 5.6] via Brändén's modified Foata-Strehl action (cf. [13]). In the last two decades, many people studied the refinements of Stembridge's identity, see Brändén [13], Petersen [82], Shin and Zeng [96, 97], Zhuang [112], Athanasiadis [3] and the references therein. In particular, Zhuang [112] has proved several identities expressing polynomials counting permutations by various descent statistics in terms of Eulerian polynomials, extending results of Stembridge, Petersen, and Brändén.

By contracting the continued fraction (1.1) starting from the first and second lines (see Lemma 1.39), respectively, we derive the two J-type continued fraction formulae (cf. [43])

$$\sum_{n \geq 0} A_{n+1}(t)z^n = \frac{1}{1 - (1+t) \cdot z - \frac{1 \cdot 2 \cdot t \cdot z^2}{1 - 2(1+t) \cdot z - \frac{2 \cdot 3 \cdot t \cdot z^2}{1 - 3(1+t) \cdot z - \frac{3 \cdot 4 \cdot t \cdot z^2}{1 - \dots}}}} \quad (1.6)$$

and

$$\sum_{n \geq 0} A_n(t)z^n = \frac{1}{1 - (1+0 \cdot t) \cdot z - \frac{1^2 \cdot z^2}{1 - (2+1 \cdot t) \cdot z - \frac{2^2 \cdot t \cdot z^2}{1 - (3+2 \cdot t) \cdot z - \frac{3^2 \cdot t \cdot z^2}{1 - \dots}}}}. \quad (1.7)$$

In view of Flajolet's combinatorial interpretation in terms of weighted Motzkin paths for generic J-type continued fraction expansions [43], Françon-Viennot's bijection ψ_{FV} (resp. its restricted version ϕ_{FV}) between permutations and *Laguerre histories* provides a bijective proof of (1.6) (resp. (1.7)), while Foata-Zeilberger's bijection ϕ_{FZ} [53] gives a bijective proof of (1.7). More precisely, Françon-Viennot [54] set up a bijection (and its restricted version) from permutations to *Laguerre histories* using *linear statistics of permutation*, while Foata-Zeilberger's bijection [53] uses *cyclic statistics of permutations*. Clarke-Steingrímsson-Zeng [24] gave a direct bijection Φ on permutations converting statistic des into exc on permutations, and linking the restricted Françon-Viennot's bijection ϕ_{FV} to Foata-Zeilberger bijection ϕ_{FZ} , see Figure 1. As a variation of Φ , Shin and Zeng [96] constructed a bijection Ψ on permutations to derive a cycle version of linear statistics on permutations, which are obtained via Françon-Viennot bijection ψ_{FV} . One of our main results (cf. Theorem 1.6) shows that a direct description of the bijection $\psi := \psi_{FV} \circ \Psi^{-1}$

from \mathfrak{S}_{n+1} to \mathcal{LH}_n is straightforward. It turns out that ψ is connected to a recent bijection of Yan, Zhou and Lin [110], see Theorem 1.31 and Figure 1.4.

A permutation σ is called *231-avoiding permutation* if there is no triple of indices $i < j < k$ such that $\sigma(k) < \sigma(i) < \sigma(j)$. The Narayana polynomials are defined by

$$N_n(t) = \sum_{\sigma \in \mathfrak{S}_n(231)} t^{\text{des } \sigma},$$

where $\mathfrak{S}_n(231)$ is the set of 231-avoiding permutations in \mathfrak{S}_n . It is well known that Narayana polynomial is γ -positive and have the expansion [83, Chapter 4]:

$$N_n(t) = \sum_{k=0}^{n/2} \tilde{\gamma}_{n,j} t^j (1+t)^{n-1-2j}, \quad (1.8)$$

where $\tilde{\gamma}_{n,j} = |\{\sigma \in \mathfrak{S}_n(231) : \text{des}(\sigma) = \text{peak}(\sigma) = j\}|$. As for Eulerian polynomials, by contraction, from

$$\sum_{n \geq 0} N_n(t) z^n = 1/1 - z/1 - t \cdot z/1 - z/1 - t \cdot z/1 - \dots \quad (1.9)$$

we derive immediately the following continued fractions

$$\sum_{n \geq 0} N_{n+1}(t) z^n = \frac{1}{1 - (1+t) \cdot z - \frac{t \cdot z^2}{1 - (1+t) \cdot z - \frac{t \cdot z^2}{1 - \dots}}} \quad (1.10)$$

and

$$\sum_{n \geq 0} N_n(t) z^n = \frac{1}{1 - z - \frac{t \cdot z^2}{1 - (1+t) \cdot z - \frac{t \cdot z^2}{1 - (1+t) \cdot z - \dots}}} \quad (1.11)$$

Note that $N_n(1)$ is the n -th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

In this chapter we shall prove generalizations of Stembridge's formula using excedance statistics by further exploiting the continued fraction technique in [95, 96, 97]. Indeed, since the observation (cf. [96]) that the gamma-positive formula of Eulerian polynomials (1.5) is equal to the Jacobi-Rogers polynomial corresponding to (1.6), it becomes clear that Flajolet-Viennot's combinatorial theory of formal continued fractions could shed more lights on this topic. Our main tool is the combinatorial theory of continued fractions due to Flajolet [43] and bijections due to Françon-Viennot, Foata-Zeilberger between permutations and *Laguarre histories*, see [54, 53, 43, 24, 95]. As in [96] this approach uses both linear and cycle statistics on permutations. There are several well-known q -Narayana polynomials in the literature; see [56] and the references therein. As a follow-up to [56], we shall give more results on q -Narayana polynomials using pattern avoiding permutations.

The rest of this chapter is organized as follows: in Section 2 we introduce the work in [95, 96, 97], and construct a bijection ψ , which is an analogue of Foata-Zeilberger's bijection from \mathfrak{S}_{n+1} to \mathcal{LH}_n and related to a recent bijection of Yan-Zhou-Lin [110]; in Section 3 we present three classes of analogues of (1.4) using excedance statistics for permutations, derangements and pattern avoiding permutations as well as a type B analogue for Eulerian polynomials. In Section 4 we prove Theorems 1.15, 1.25, 1.31 and 1.32. using variations of modified Foata-Strehl action on permutations or Laguerre histories. In Section 5 we prove the remaining theorems by exploiting the continued fraction technique.

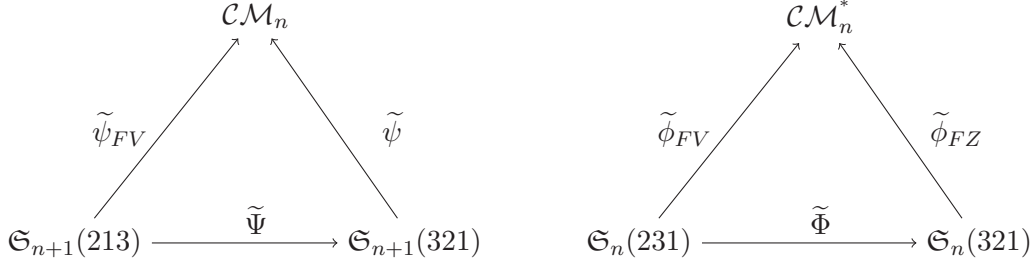


Figure 1.2: Bijections $\tilde{\Psi} = \tilde{\psi}^{-1} \circ \tilde{\psi}_{FV}$ and $\tilde{\Phi} = \tilde{\phi}_{FZ}^{-1} \circ \tilde{\phi}_{FV}$

1.2 Background and preliminaries

1.2.1 Permutation statistics and two bijections

For $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$ with convention $0-0$, i.e., $\sigma(0) = \sigma(n+1) = 0$, a value $\sigma(i)$ ($1 \leq i \leq n$) is called

- a *peak* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) > \sigma(i+1)$;
- a *valley* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double ascent* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double descent* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) > \sigma(i+1)$.

The set of Peaks (resp. Valleys, double ascents, double descents) of σ is denoted by

$$\text{Pk}\sigma \quad (\text{resp. Val}\sigma, \text{Da}\sigma, \text{Dd}\sigma).$$

Let $\text{peak } \sigma$ (resp. $\text{valley } \sigma$, $\text{da } \sigma$, $\text{dd } \sigma$) be the number of peaks (resp. valleys, double ascents, double descents) of σ . For $i \in [n] := \{1, \dots, n\}$, we introduce the following statistics:

$$\begin{aligned} (31-2)_i \sigma &= \#\{j : 1 < j < i \text{ and } \sigma(j) < \sigma(i) < \sigma(j-1)\} \\ (2-31)_i \sigma &= \#\{j : i < j < n \text{ and } \sigma(j+1) < \sigma(i) < \sigma(j)\} \\ (2-13)_i \sigma &= \#\{j : i < j < n \text{ and } \sigma(j) < \sigma(i) < \sigma(j+1)\} \\ (13-2)_i \sigma &= \#\{j : 1 < j < i \text{ and } \sigma(j-1) < \sigma(i) < \sigma(j)\} \end{aligned} \tag{1.12}$$

and define (see (1.55)):

$$(31-2) = \sum_{i=1}^n (31-2)_i, \quad (2-31) = \sum_{i=1}^n (2-31)_i, \quad (2-13) = \sum_{i=1}^n (2-13)_i, \quad (13-2) = \sum_{i=1}^n (13-2)_i.$$

Now, we consider $\sigma \in \mathfrak{S}_n$ as a bijection $i \mapsto \sigma(i)$ for $i \in [n]$, a value $x = \sigma(i)$ is called

- a *cyclic peak* if $i = \sigma^{-1}(x) < x$ and $x > \sigma(x)$;
- a *cyclic valley* if $i = \sigma^{-1}(x) > x$ and $x < \sigma(x)$;
- a *double excedance* if $i = \sigma^{-1}(x) < x$ and $x < \sigma(x)$;
- a *double drop* if $i = \sigma^{-1}(x) > x$ and $x > \sigma(x)$;
- a *fixed point* if $x = \sigma(x)$.

We say that $i \in [n-1]$ is an *ascent* of σ if $\sigma(i) < \sigma(i+1)$ and that $i \in [n]$ is a *drop* of σ if $\sigma(i) < i$. Let Cpk (resp. Cval , Cda , Cdd , Fix , Drop) be the set of *cyclic peaks* (resp. *cyclic valleys*, *double excedances*, *double drops*, *fixed points*, *drops*) and denote the corresponding cardinality by cpk (resp. cvalley , cda , cdd , fix , drop). Moreover, define

$$\text{wex } \sigma = \#\{i : i \leq \sigma(i)\} = \text{exc } \sigma + \text{fix } \sigma \quad (1.13)$$

$$\text{cros}_i \sigma = \#\{j : j < i < \sigma(j) < \sigma(i) \text{ or } \sigma(i) < \sigma(j) \leq i < j\}, \quad (1.14)$$

$$\text{nest}_i \sigma = \#\{j : j < i < \sigma(i) < \sigma(j) \text{ or } \sigma(j) < \sigma(i) \leq i < j\}. \quad (1.15)$$

Let $\text{cros}^1 = \sum_{i=1}^n \text{cros}_i$ and $\text{icr } \sigma = \text{cros } \sigma^{-1}$. Define $\text{nest}^2 = \sum_{i=1}^n \text{nest}_i$ and $\text{ine } \sigma = \text{nest } \sigma^{-1}$.

Remark 1.1. Although we introduced ine above, it is really the same statistic as nest . Namely, we have $\text{ine } \pi = \text{nest } \pi$ for any permutation π . When $\text{fix } \pi = 0$ this should be clear from definition. In general, it will suffice to observe that for any i such that $\pi(i) = i$, there are as many $j < i$ with $\pi(j) > i$ as $k > i$ with $\pi(k) < i$.

A pair of integers (i, j) is an *inversion* of $\sigma \in \mathfrak{S}_n$ if $i < j$ and $\sigma(i) > \sigma(j)$, and $\sigma(i)$ (resp. $\sigma(j)$) is called *inversion top* (resp. *bottom*). Let $\text{inv } \sigma$ be the inversion number of σ .

For $\sigma \in \mathfrak{S}_n$ with convention $0-\infty$, i.e., $\sigma(0) = 0$ and $\sigma(n+1) = \infty$, let Lpk (resp. Lval , Lda , Ldd) be the set of *peaks* (resp. *valleys*, *double ascents* and *double descents*) and denote the corresponding cardinality by lpeak (resp. lvalley , lda and ldd). For $i \in [n]$, the value $\sigma(i)$ is called a *left-to-right maximum* (resp. *right-to-left minimum*) if $\sigma(i) = \max\{\sigma(1), \sigma(2), \dots, \sigma(i)\}$ (resp. $\sigma(i) = \min\{\sigma(i), \dots, \sigma(n-1), \sigma(n)\}$). Similarly, we define *left-to-right minimum* (resp. *right-to-left maximum*).

A double ascent $\sigma(i)$ ($i = 1, \dots, n$) is called a *foremaximum* (resp. *afterminimum*) of σ if it is at the same time a left-to-right maximum (resp. right-to-left minimum). Denote the number of foremaxima (resp. afterminima) of σ by $\text{fmax } \sigma$ (resp. $\text{amin } \sigma$). Note that for the peak number peak' in (1.3) we have following identities :

$$\text{peak}' = \text{valley} = \text{peak} - 1 \quad \text{and} \quad \text{lvalley} = \text{lpeak}. \quad (1.16)$$

Now we recall two bijections Φ and Ψ due to Clarke et al. [24] and Shin-Zeng [96], respectively.

1.2.2 The bijection Φ

Let $\sigma = \sigma(1) \dots \sigma(n) \in \mathfrak{S}_n$, an *inversion top number* (resp. *inversion bottom number*) of a letter $x := \sigma(i)$ in the word σ is the number of occurrences of inversions of form (i, j) (resp. (j, i)). A letter $\sigma(i)$ is a *descent top* (resp. *descent bottom*) if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i-1) > \sigma(i)$). Given a permutation σ , we first construct two biwords, $\binom{f}{f'}$ and $\binom{g}{g'}$, where f (resp. g) is the subword of descent bottoms (resp. nondescent bottoms) in σ ordered increasingly, and f' (resp. g') is the permutation of descent tops (resp. nondescent tops) in σ such that the inversion bottom (resp. top) number of each letter $x := \sigma(i)$ in f' (resp. g') is $(2-31)_x \sigma$, and then form the biword $w = \binom{f \quad g}{f' \quad g'}$ by concatenating f and g , and f' and g' , respectively. Rearranging the columns of w , so that the bottom row is in increasing order, we obtain the permutation $\tau = \Phi(\sigma)$ as the top row of the rearranged bi-word.

The following result can be found in [96, Theorem 12] and its proof.

¹Our definition of cros corresponds to icr in [56].

²Our definition of nest corresponds to ine in [56].

Lemma 1.1 (Shin-Zeng). *For $\sigma \in \mathfrak{S}_n$, we have*

$$\begin{aligned} & (2\text{-}31, 31\text{-}2, \text{des}, \text{asc}, \text{lda} - \text{fmax}, \text{ldd}, \text{lvalley}, \text{lpeak}, \text{fmax})\sigma \\ &= (\text{nest}, \text{icr}, \text{drop}, \text{exc} + \text{fix}, \text{cda}, \text{cdd}, \text{cvalley}, \text{cpk}, \text{fix})\Phi(\sigma) \\ &= (\text{nest}, \text{cros}, \text{exc}, \text{drop} + \text{fix}, \text{cdd}, \text{cda}, \text{cvalley}, \text{cpk}, \text{fix})(\Phi(\sigma))^{-1}, \end{aligned} \quad (1.17)$$

$$(\text{Lval}, \text{Lpk}, \text{Lda}, \text{Ldd})\sigma = (\text{Cval}, \text{Cpk}, \text{Cda} \cup \text{Fix}, \text{Cdd})\Phi(\sigma), \quad (1.18)$$

$$(2\text{-}31)_i\sigma = \text{nest}_i\Phi(\sigma) \quad \forall i = 1, \dots, n. \quad (1.19)$$

1.2.3 The bijection Ψ

Given a permutation $\sigma \in \mathfrak{S}_n$, let

$$\hat{\sigma} = \begin{pmatrix} 1 & 2 & \dots & n & n+1 \\ \sigma(1)+1 & \sigma(2)+1 & \dots & \sigma(n)+1 & 1 \end{pmatrix}, \quad (1.20)$$

and $\tau := \Phi(\hat{\sigma}) \in \mathfrak{S}_{n+1}$. Since the last element of $\hat{\sigma}$ is 1, the first element of τ should be $n+1$. Define the bijection $\Psi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by

$$\Psi(\sigma) := \tau(2) \dots \tau(n+1) \in \mathfrak{S}_n. \quad (1.21)$$

Example 1.2. *If $\sigma = 4\ 1\ 2\ 7\ 9\ 6\ 5\ 8\ 3$, then $\hat{\sigma} = 5\ 2\ 3\ 8\ 10\ 7\ 6\ 9\ 4\ 1$, and reading from left to right, we obtain the corresponding numbers $(2\text{-}31)_i : 1, 1, 1, 2, 0, 1, 1, 0, 0, 0$ for $i = 5, 2, \dots, 1$, and*

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 6 & 7 \\ 4 & 9 & 5 & 7 & 10 \end{pmatrix}, \quad \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} 3 & 5 & 8 & 9 & 10 \\ 2 & 3 & 8 & 6 & 1 \end{pmatrix}.$$

Hence

$$w = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 6 & 7 & 3 & 5 & 8 & 9 & 10 \\ 4 & 9 & 5 & 7 & 10 & 2 & 3 & 8 & 6 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 3 & 5 & 1 & 4 & 9 & 6 & 8 & 2 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}.$$

Thus $\tau = \Phi(\hat{\sigma}) = 10\ 3\ 5\ 1\ 4\ 9\ 6\ 8\ 2\ 7$, and $\Psi(\sigma) = \tau(2) \dots \tau(10) = 3\ 5\ 1\ 4\ 9\ 6\ 8\ 2\ 7$.

Lemma 1.3. *For $i \in [n]$, we have*

$$(2\text{-}31)_{i+1}\hat{\sigma} = \begin{cases} (2\text{-}13)_i\sigma + 1 & \text{if } i+1 \in \text{Lval}\hat{\sigma} \cup \text{Lda}\hat{\sigma}; \\ (2\text{-}13)_i\sigma & \text{if } i+1 \in \text{Lpk}\hat{\sigma} \cup \text{Ldd}\hat{\sigma}. \end{cases}$$

Proof. An increasing (resp. decreasing) run of σ is a maximum consecutive increasing (resp. decreasing) subsequence $R := \sigma(i)\sigma(i+1) \dots \sigma(j)$ of σ such that $\sigma(i-1) > \sigma(i)$ and $\sigma(j) > \sigma(j+1)$ with $1 \leq i \leq j \leq n$. For any $i \in [n]$, as $\hat{\sigma}(n+1) = 1$, there is a unique way to write

$$\hat{\sigma} = \begin{cases} w_1(i+1)u_1d_2 \dots u_{k-1}d_k & \text{if } i+1 \in \text{Lval}\hat{\sigma} \cup \text{Lda}\hat{\sigma}; \\ w_1(i+1)d_1u_2d_2 \dots u_kd_k & \text{if } i+1 \in \text{Lpk}\hat{\sigma} \cup \text{Ldd}\hat{\sigma}, \end{cases}$$

where u_i (resp. d_i) is an increasing (resp. decreasing) run, and $(i+1)u_1$ (resp. $(i+1)d_1$) is an increasing (resp. decreasing) sequence. We say that a run R covers i if i is bounded by $\max(R)$ and $\min(R)$. It is not hard to show that

$$\#\{i \geq 2 : u_i \text{ covers } i+1\} = \begin{cases} \#\{i \geq 2 : d_i \text{ covers } i+1\} + 1 & \text{if } i+1 \in \text{Lval}\hat{\sigma} \cup \text{Lda}\hat{\sigma}; \\ \#\{i \geq 2 : d_i \text{ covers } i+1\} & \text{if } i+1 \in \text{Lpk}\hat{\sigma} \cup \text{Ldd}\hat{\sigma}. \end{cases}$$

Since $(2-13)_i$ (resp. $(2-31)_i$) is the number of increasing (resp. decreasing) runs covering i to the right of i , we are done. \square

We use the aforementioned statistics to define variant boundary conditions. Given a permutation $\sigma \in \mathfrak{S}_n$ with convention $\infty - 0$, the number of corresponding peaks, valleys, double ascents, and double descents of permutation $\sigma \in \mathfrak{S}_n$ is denoted by $\text{rpeak } \sigma$, $\text{rvalley } \sigma$, $\text{rda } \sigma$, $\text{rdd } \sigma$ respectively. A double descent $\sigma(i)$ is called a *aftermaximum* (resp. *foreminimum*) of σ if it is at the same time a right-to-left maximum (resp. left-to-right minimum). Denote the number of aftermaxima (resp. foreminimum) of σ by $\text{amax } \sigma$ (resp. $\text{fmin } \sigma$).

For $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$, we define two permutations σ^c and σ^r by $\sigma^c(i) = n + 1 - \sigma(i)$ and $\sigma^r(i) = \sigma(n + 1 - i)$ for $i \in [n]$. It is not difficult to verify the following properties

$$(2-31, 31-2, \text{des}, \text{lda} - \text{fmax}, \text{ldd}, \text{lvalley}, \text{fmax}) \sigma \quad (1.22)$$

$$=(13-2, 2-13, \text{asc}, \text{rdd} - \text{amax}, \text{rda}, \text{rvalley}, \text{amax}) \sigma^r \quad (1.23)$$

$$=(31-2, 2-31, \text{des}, \text{lda} - \text{amin}, \text{ldd}, \text{lpeak}, \text{amin}) \sigma^{r \cdot c} \quad (1.24)$$

$$=(2-13, 13-2, \text{asc}, \text{rdd} - \text{fmin}, \text{rda}, \text{rvalley}, \text{fmin}) \sigma^{r \cdot c \cdot r}, \quad (1.25)$$

where $\sigma^{r \cdot c} = (\sigma^r)^c$ and $\sigma^{r \cdot c \cdot r} = (\sigma^{r \cdot c})^r = (\sigma^r)^{c \cdot r}$.

1.2.4 The star variation

For $\sigma = \sigma(1)\cdots\sigma(n) \in \mathfrak{S}_n$, we define its *star companion* σ^* as a permutation of $\{0, \dots, n\}$ by

$$\sigma^* = \begin{pmatrix} 0 & 1 & 2 & \dots & n \\ n & \sigma(1) - 1 & \sigma(2) - 1 & \dots & \sigma(n) - 1 \end{pmatrix}. \quad (1.26)$$

We define the following sets of cyclic star statistics for σ :

$$\text{Cpk}^* \sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) < i > \sigma^*(i)\}, \quad (1.27)$$

$$\text{Cval}^* \sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) > i < \sigma^*(i)\}, \quad (1.28)$$

$$\text{Cda}^* \sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) < i < \sigma^*(i)\}, \quad (1.29)$$

$$\text{Cdd}^* \sigma = \{i \in [n-1] : (\sigma^*)^{-1}(i) > i > \sigma^*(i)\}, \quad (1.30)$$

$$\text{Fix}^* \sigma = \{i \in [n-1] : i = \sigma^*(i)\}, \quad (1.31)$$

$$\text{Wex}^* \sigma = \{i \in [n-1] : i \leq \sigma^*(i)\} (= \text{exc } \sigma), \quad (1.32)$$

$$\text{Drop}^* \sigma = \{i \in [n] : i > \sigma^*(i)\}. \quad (1.33)$$

The corresponding cardinalities are denoted by cpk^* , cvalley^* , cda^* , cdd^* , fix^* , wex^* and drop^* , respectively. By (1.27), (1.30) and (1.33), we have $\text{drop}^* - 1 = \text{cdd}^* + \text{cpk}^*$. Let $\text{cyc } \sigma$ be the number of cycles of σ and $\text{cyc}^* \sigma := \text{cyc } \sigma^*$. For example, for $\sigma = 3762154$, we have $\sigma^* = 72651043$, which has two cycles $1 \rightarrow 2 \rightarrow 6 \rightarrow 4 \rightarrow 1$ and $7 \rightarrow 3 \rightarrow 5 \rightarrow 0 \rightarrow 7$. Thus $\text{cyc}^* \sigma = 2$.

Theorem 1.4. *For $\sigma \in \mathfrak{S}_n$, we have*

$$(\text{Val}, \text{Pk} \setminus \{n\}, \text{Da}, \text{Dd})\sigma = (\text{Cval}^*, \text{Cpk}^*, \text{Cda}^* \cup \text{Fix}^*, \text{Cdd}^*)\Psi(\sigma) \quad (1.34)$$

and

$$((2-13)_i, (31-2)_i)\sigma = (\text{nest}_i, \text{cross}_i)\Psi(\sigma) \quad \text{for } i \in [n]. \quad (1.35)$$

Proof. We first recall the construction of Ψ : if $\sigma \in \mathfrak{S}_n$, let $\hat{\sigma} = (\sigma(1)+1) \dots (\sigma(n)+1)(1) \in \mathfrak{S}_{n+1}$ and $\tau = \Phi(\hat{\sigma})$ (with $\tau(1) = n+1$), then $\Psi(\sigma) := \tau(2) \dots \tau(n+1) \in \mathfrak{S}_n$, i.e. $\Psi(\sigma)(i) = \tau(i+1)$ for $i \in [n]$. As $(\Psi(\sigma))^*(i) = \Psi(\sigma)(i) - 1$, we have

$$\tau(i+1) = (\Psi(\sigma))^*(i) + 1. \quad (1.36)$$

For $S \subset \mathbb{N}$, let $S+1 := \{s \in S : s+1\}$. Given $\sigma \in \mathfrak{S}_n$, it is easy to check that

$$((\text{Val } \sigma + 1) \cup \{1\}, \text{Pk } \sigma + 1, \text{Da } \sigma + 1, \text{Dd } \sigma + 1) = (\text{Lval}, \text{Lpk}, \text{Lda}, \text{Ldd})\hat{\sigma}. \quad (1.37)$$

For $2 \leq i \leq n$, $i \in \text{Cval } \tau$ means $i < \tau(i)$ and $i < \tau^{-1}(i)$, namely,

$$i < \Psi(\sigma)(i-1) \text{ and } i < (\Psi(\sigma))^{-1}(i) + 1,$$

therefore,

$$i-1 \in \text{Cval}^*(\Psi(\sigma)) \iff i < \Psi(\sigma)(i-1) \text{ and } i-1 < (\Psi(\sigma))^{-1}(i).$$

Thus

$$(\text{Cval}^* \Psi(\sigma) + 1) \cup \{1\} = \text{Cval } \Phi(\hat{\sigma}) \quad (1.38)$$

In the same vein, we have

$$\begin{aligned} \text{Cpk}^* \Psi(\sigma) + 1 &= \text{Cpk } \Phi(\hat{\sigma}) \setminus \{n+1\} \\ \text{Cda}^* \Psi(\sigma) \cup \text{Fix}^* \Psi(\sigma) + 1 &= \text{Cda } \Phi(\hat{\sigma}) \cup \text{Fix } \Phi(\hat{\sigma}) \\ \text{Cdd}^* \Psi(\sigma) + 1 &= \text{Cdd } \Phi(\hat{\sigma}). \end{aligned} \quad (1.39)$$

Comparing (1.37) and (1.38)-(1.39) and using (1.18) we derive (1.34).

As $\text{nest}_i \tau = (2-31)_i \hat{\sigma}$ by (1.19), to prove $\text{nest}_i \Psi(\sigma) = (2-13)_i \sigma$, it is sufficient to show that

$$\text{(nest)}_{i+1} \tau = \begin{cases} (\text{nest})_i \Psi(\sigma) + 1 & \text{if } i+1 \in \text{Cval } \tau \cup \text{Cda } \tau \cup \text{Fix } \tau; \\ (\text{nest})_i \Psi(\sigma) & \text{if } i+1 \in \text{Cpk } \tau \cup \text{Cdd } \tau. \end{cases} \quad (1.40)$$

By (1.18), comparing (1.40) with Lemma 1.3, we are done. We omit the proof of $\text{cros}_i \Psi(\sigma) = (31-2)_i \sigma$ for it is similar and left to the reader.

Now we are ready to prove (1.40). Note that the index $\text{nest}_i \sigma$ ($i \in [n]$) can be characterized using σ^* as follows:

$$\text{nest}_i \sigma = \#\{j \in [n] : j < i \leq \sigma^*(i) < \sigma^*(j) \text{ or } \sigma^*(j) < \sigma^*(i) < i < j\}. \quad (1.41)$$

We consider three cases of $i+1$.

(i) if $i+1 < \tau(i+1)$, then $i < (\Psi(\sigma))^*(i)$. By (1.36), we have

$$\#\{j \in [n] : j+1 < i+1 < \tau(i+1) < \tau(j+1)\} = \#\{j \in [n] : j < i \leq (\Psi(\sigma))^*(i) < (\Psi(\sigma))^*(j)\},$$

$$\#\{j \in [n] : j+1 > i+1 \geq \tau(i+1) > \tau(j+1)\} = \#\{j \in [n] : j > i > (\Psi(\sigma))^*(i) > (\Psi(\sigma))^*(j)\}.$$

Since $\tau(1) = n+1$, we have $\text{nest}_{i+1} \tau = \text{nest}_i \Psi(\sigma) + 1$ by using (1.15) (resp. (1.41)) to compute $\text{nest}_{i+1} \tau$ (resp. $\text{nest}_i \Psi(\sigma)$).

(ii) if $i+1 > \tau(i+1)$, then $i > (\Psi(\sigma))^*(i)$. Similarly we get $\text{nest}_{i+1} \tau = \text{nest}_i \Psi(\sigma)$.

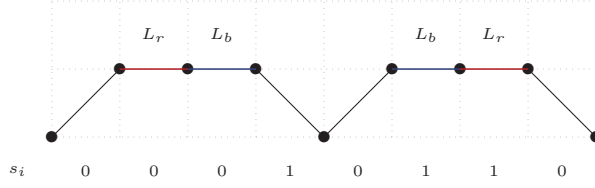


Figure 1.3: $(\mathbf{s}, \mathbf{p}) = ((U, L_r, L_b, D, U, L_b, L_r, D), (0, 0, 0, 1, 0, 1, 1, 0)) \in \mathcal{LH}_8$

(iii) if $i + 1 = \tau(i + 1)$, then $i = (\Psi(\sigma))^*(i)$. For $i \in \text{Fix } \tau$, it is easy to see that

$$\#\{j \in [n + 1] : j > i > \tau(j)\} = \#\{j \in [n + 1] : j < i < \tau(j)\}.$$

Then by (1.36), we have

$$\begin{aligned} & \#\{j \in [n] : j + 1 > i + 1 > \tau(j + 1)\} \\ &= \#\{j \in [n] : j + 1 < i + 1 < \tau(j + 1)\} + 1 \quad (\text{as } \tau(1) = n + 1) \\ &= \#\{j \in [n] : j < i < (\Psi(\sigma))^*(j)\} + 1. \end{aligned}$$

Then, we have $\text{nest}_{i+1} \tau = \text{nest}_i \Psi(\sigma) + 1$ by using (1.15) (resp. (1.41)) to compute $\text{nest}_{i+1} \tau$ (resp. $\text{nest}_i \Psi(\sigma)$).

□

Since $\text{asc} = \text{valley} + \text{da}$, $\text{des} = \text{peak} + \text{dd} - 1$, $\text{wex}^* = \text{cvalley}^* + \text{cda}^* + \text{fix}^*$, $\text{drop}^* - 1 = \text{cdd}^* + \text{cpk}^*$, we get the following result in [96, Theorem 12].

Corollary 1.5 (Shin-Zeng). *For $\sigma \in \mathfrak{S}_n$ we have*

$$\begin{aligned} & (2\text{-}13, 31\text{-}2, \text{des}, \text{asc}, \text{da}, \text{dd}, \text{valley})\sigma \\ &= (\text{nest}, \text{cros}, \text{drop}^* - 1, \text{wex}^*, \text{cda}^* + \text{fix}^*, \text{cdd}^*, \text{cvalley}^*)\Psi(\sigma). \end{aligned} \quad (1.42)$$

1.2.5 Laguerre histories as permutation encodings

A *2-Motzkin path* of length n is a word $\mathbf{s} := s_1 \dots s_n$ on the alphabet $\{U, D, L_r, L_b\}$ such that $|s_1 \dots s_n|_U = |s_1 \dots s_i|_D$ and

$$h_i := |s_1 \dots s_i|_U - |s_1 \dots s_i|_D \geq 0 \quad (i = 1, \dots, n), \quad (1.43)$$

where $|s_1 \dots s_i|_U$ is the number of letters U in the word $s_1 \dots s_i$. By (1.10) we see that the number of 2-Motzkin paths of length n is the Catalan number C_{n+1} .

A *Laguerre history* (resp. *restricted Laguerre history*) of length n is a pair (\mathbf{s}, \mathbf{p}) , where \mathbf{s} is a 2-Motzkin path $s_1 \dots s_n$ and $\mathbf{p} = (p_1, \dots, p_n)$ with $0 \leq p_i \leq h_{i-1}$ (resp. $0 \leq p_i \leq h_{i-1} - 1$ if $s_i = L_b$ or D) and $h_0 = 0$. Let \mathcal{LH}_n (resp. \mathcal{LH}_n^*) be the set of Laguerre histories (resp. restricted Laguerre histories) of length n . There are several well-known bijections between \mathfrak{S}_n and \mathcal{LH}_n^* and \mathcal{LH}_{n-1} , see [5, 7, 28, 24, 29, 36] and references therein.

1.2.6 Françon-Viennot bijection

We recall a version of Françon and Viennot's bijection $\psi_{FV} : \mathfrak{S}_{n+1} \rightarrow \mathcal{LH}_n$. Given $\sigma \in \mathfrak{S}_{n+1}$, the Laguerre history (\mathbf{s}, \mathbf{p}) is defined as follows:

$$s_i = \begin{cases} U & \text{if } i \in \text{Val}\sigma \\ D & \text{if } i \in \text{Pk}\sigma \\ L_r & \text{if } i \in \text{Da}\sigma \\ L_b & \text{if } i \in \text{Dd}\sigma \end{cases} \quad (1.44)$$

and $p_i = (2-13)_i \sigma$ for $i = 1, \dots, n$. Given $\sigma = 4\ 1\ 2\ 7\ 9\ 6\ 5\ 8\ 3$, see Figure 1 for $\psi_{FV}(\sigma)$.

Theorem 1.6. *The mapping $\psi := \psi_{FV} \circ \Psi^{-1}$ is a bijection from \mathfrak{S}_{n+1} to \mathcal{LH}_n . If $\psi(\sigma) = (\mathbf{s}, \mathbf{p})$ with $\sigma \in \mathfrak{S}_{n+1}$, then, for $i = 1, \dots, n$,*

$$s_i = \begin{cases} U & \text{if } i \in \text{Cval}^* \sigma; \\ D & \text{if } i \in \text{Cpk}^* \sigma; \\ L_r \text{ (resp. } L_b) & \text{if } i \in \text{Cda}^* \sigma \cup \text{Fix}^* \sigma \text{ (resp. } \text{Cdd}^* \sigma), \end{cases} \quad (1.45)$$

with $p_i = \text{nest}_i \sigma$.

Proof. This follows from Theorem 1.4 by comparing (1.45) with (1.44), see the commutative diagram in Figure 1.1. \square

Corollary 1.7. *The two sextuple statistics*

$$(\text{nest}, \text{cros}, \text{exc}, \text{cdd}^*, \text{cda}^* + \text{fix}^*, \text{cpk}^*) \quad \text{and} \quad (2-13, 31-2, \text{des}, \text{da}, \text{dd}, \text{peak} - 1)$$

are equidistributed on \mathfrak{S}_n .

Proof. For $\sigma \in \mathfrak{S}_n$, let $\tau = \psi^{-1} \circ \Theta \circ \psi_{FV}(\sigma)$, where Θ is the involution on \mathcal{LH}_n defined by

$$\forall (\mathbf{s}, \mathbf{p}) \in \mathcal{LH}_n, \quad \Theta(\mathbf{s}, \mathbf{p}) = \begin{cases} (s_i, p_i) & \text{if } s_i = U \text{ or } D \\ (L_r, p_i) & \text{if } s_i = L_b \\ (L_b, p_i) & \text{if } s_i = L_r. \end{cases} \quad (1.46)$$

It follows from (1.44), (1.45) and (1.46) that, $\forall i \in [n]$,

$$(\text{Val}, \text{Pk}, \text{Dd}, \text{Da}, (2-13)_i)\sigma = (\text{Cval}^*, \text{Cpk}^*, \text{Cda}^* \cup \text{Fix}^*, \text{Cdd}^*, \text{nest}_i)\tau.$$

Let $\psi_{FV}(\sigma) = (\mathbf{s}, \mathbf{p})$ and $\psi(\tau) = (\mathbf{s}', \mathbf{p}')$. Then $h_i(\mathbf{s}, \mathbf{p}) = h_i(\mathbf{s}', \mathbf{p}')$ for all $i \in [n]$. It is not difficult to prove by induction that

$$(2-13)_i \sigma + (31-2)_i \sigma = h_{i-1}(\mathbf{s}, \mathbf{p}) \quad (1.47)$$

$$\text{nest}_i \sigma + \text{cros}_i \sigma = h_{i-1}(\mathbf{s}', \mathbf{p}'). \quad (1.48)$$

Thus we have $(31-2)_i \sigma = \text{cros}_i \tau$. As $\text{exc} = \text{wex}^* = \text{cvalley}^* + \text{cda}^* + \text{fix}^*$, $\text{des} = \text{valley} + \text{dd}$, $\text{cpk}^* = \text{cvalley}^*$, and $\text{valley} = \text{peak} - 1$, we complete the proof. \square

For $k \in [n]$ we define the two subsets of \mathfrak{S}_n :

$$\mathcal{DD}_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{des}(\sigma) = k, \text{dd}(\sigma) = 0\} \quad (1.49)$$

$$\text{DE}_{n,k}^* := \{\sigma \in \mathfrak{S}_n : \text{exc}(\sigma) = k, \text{cda}^*(\sigma) + \text{fix}^*(\sigma) = 0\}. \quad (1.50)$$

As (cf. [95, Eq. (40)])

$$\text{inv} = \text{exc} + 2 \text{nest} + \text{cros}, \quad (1.51)$$

we derive the following result from Corollary 1.7.

Corollary 1.8. *We have*

$$\sum_{\sigma \in \mathcal{DD}_{n,k}} q^{2(2-13)+(31-2)} = \sum_{\sigma \in \text{DE}_{n,k}^*} q^{\text{inv}(\sigma) - \text{exc}(\sigma)}. \quad (1.52)$$

We recall two bijections ϕ_{FZ} and ϕ_{FV} from \mathfrak{S}_n to \mathcal{LH}_n^* .

1.2.7 Restricted Françon-Viennot bijection

We recall a restricted version of Françon and Viennot's bijection $\phi_{FV} : \mathfrak{S}_n \rightarrow \mathcal{LH}_n^*$. Given $\sigma \in \mathfrak{S}_n$, the Laguerre history (\mathbf{s}, \mathbf{p}) is defined as follows:

$$s_i = \begin{cases} U & \text{if } i \in \text{Lval}\sigma \\ D & \text{if } i \in \text{Lpk}\sigma \\ L_r & \text{if } i \in \text{Lda}\sigma \\ L_b & \text{if } i \in \text{Ldd}\sigma \end{cases} \quad (1.53)$$

and $p_i = (2-31)_i \sigma$ for $i = 1, \dots, n$.

1.2.8 Foata-Zeilberger bijection

This bijection ϕ_{FZ} encodes permutations using cyclic statistics. Given $\sigma \in \mathfrak{S}_n$, $\phi_{FZ} : \mathfrak{S}_n \rightarrow \mathcal{LH}_n^*$ is for $i = 1, \dots, n$,

$$s_i = \begin{cases} U & \text{if } i \in \text{Cval}\sigma; \\ D & \text{if } i \in \text{Cpk}\sigma; \\ L_r & \text{if } i \in \text{Cda}\sigma \cup \text{Fix}\sigma; \\ L_b & \text{if } i \in \text{Cdd}\sigma, \end{cases} \quad (1.54)$$

with $p_i = \text{nest}_i \sigma$. By (1.18) and (1.19), we can build a comutative diagram, see the right diagram of Figure 1.1.

1.2.9 Pattern avoidances and 2-Motzkin paths

Given two permutations $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$, we say that σ *contains the pattern* τ if there exists a set of indices $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that the subsequence $\sigma(i_1)\sigma(i_2) \cdots \sigma(i_k)$ of σ is order-isomorphic to τ . Otherwise, σ is said to *avoid* τ . For example, the permutation 15324 contains the pattern 321 and avoids the pattern 231. The set of permutations of length n that avoid patterns $\tau_1, \tau_2, \dots, \tau_m$ is denoted as $\mathfrak{S}_n(\tau_1, \tau_2, \dots, \tau_m)$.

Moreover we shall consider the so-called *vincular patterns* [4]. The number of occurrences of vincular patterns 31-2, 2-31, 2-13 and 13-2 in $\pi \in \mathfrak{S}_n$ are defined (cf. (1.12)) by

$$\begin{aligned} (31-2) \pi &= \#\{(i, j) : i+1 < j \leq n \text{ and } \pi(i+1) < \pi(j) < \pi(i)\}, \\ (2-31) \pi &= \#\{(i, j) : j < i < n \text{ and } \pi(i+1) < \pi(j) < \pi(i)\}, \\ (2-13) \pi &= \#\{(i, j) : j < i < n \text{ and } \pi(i) < \pi(j) < \pi(i+1)\}, \\ (13-2) \pi &= \#\{(i, j) : i+1 < j \leq n \text{ and } \pi(i) < \pi(j) < \pi(i+1)\}. \end{aligned} \quad (1.55)$$

Similarly, we use $\mathfrak{S}_n(31-2)$ to denote the set of permutations of length n that avoid the vincular pattern 31-2, etc. In order to apply Laguerre history to count pattern-avoiding permutations, we will need the following results in [56, Lemma 2.8 and 2.9].

Lemma 1.9. [56, Lemma 2.8] *For any $n \geq 1$, we have*

$$\mathfrak{S}_n(2-13) = \mathfrak{S}_n(213), \quad \mathfrak{S}_n(31-2) = \mathfrak{S}_n(312), \quad (1.56)$$

$$\mathfrak{S}_n(13-2) = \mathfrak{S}_n(132), \quad \mathfrak{S}_n(2-31) = \mathfrak{S}_n(231). \quad (1.57)$$

Proof. By definition we have $\mathfrak{S}_n(213) \subset \mathfrak{S}_n(2-13)$, Conversely, if $\pi \notin \mathfrak{S}_n(213)$, then π has the pattern 213, that is, there exists $i, j, k \in [n]$ such that $k < i < j$, $\pi(i) < \pi(k) < \pi(j)$, then there must be some i' , $i \leq i' < j$, and $\pi(i') < \pi(k) < \pi(i'+1)$, then $\pi \notin \mathfrak{S}_n(2-13)$.

The first equality in Lemma 1.9 was already observed by Claesson [22, Lemma 2]. The proofs of the remaining three equalities are essentially the same and thus omitted. \square

Lemma 1.10. [56, Lemma 2.9]

(i) *A permutation $\pi \in \mathfrak{S}_n$ belongs to $\mathfrak{S}_n(321)$ if and only if $\text{nest } \pi = 0$*

(ii) *The mapping Φ has the property that $\Phi(\mathfrak{S}_n(231)) = \mathfrak{S}_n(321)$.*

Proof. (i) If $\pi \in \mathfrak{S}_n$ with pattern 321, there are indices i, j and k such that $\pi(i) > \pi(j) > \pi(k)$ with $1 \leq i < j < k \leq n$. There are two cases to consider:

- if $\pi(j) < j$, then $\pi(k) < \pi(j) < j < k$ form a nesting of π ;
- if $\pi(j) \geq j$, then $i < j \leq \pi(j) < \pi(i)$ form a nesting of π .

Therefore in either case, we have $\text{nest } \pi > 0$. Suppose that $\pi \in \mathfrak{S}_n$ with $\text{nest } \pi > 0$. Then there is a pair $(i, j) \in [n] \times [n]$ such that $i < j \leq \pi(j) < \pi(i)$ or $\pi(j) < \pi(i) < i < j$. There are also two cases:

- if $j \leq \pi(j)$, as $\pi(i) > j$, there is a $k > j$ such that $\pi(k) < j$, thus $\pi(i) > \pi(j) > \pi(k)$;
- if $\pi(j) < j$, as $\pi(j) < i$, there is a $k < i$ such that $\pi(k) > i$, thus $\pi(k) > \pi(i) > \pi(j)$.

Therefore in either case, we have $\pi \notin \mathfrak{S}_n(321)$.

(ii) By Lemmas 1.1 and 1.9 we have $\pi \in \mathfrak{S}_n(231)$ if and only if $\text{ine } \Phi(\pi) = \text{nest } \Phi(\pi) = 0$. Thus, it follows from (i) that $\pi \in \mathfrak{S}_n(231)$ if and only if $\Phi(\pi) \in \mathfrak{S}_n(321)$. \square

We use \mathcal{CM}_n to denote the set of 2-Motzkin paths of length n and \mathcal{CM}_n^* to denote its subset that is composed of 2-Motzkin paths without L_b -step at level zero, i.e., if $h_{i-1} = 0$, then $s_i \neq L_b$.

Let $\tilde{\phi}_{FV}$, $\tilde{\phi}_{FZ}$, $\tilde{\psi}_{FV}$ and $\tilde{\psi}$ be the restriction of ϕ_{FV} , ϕ_{FZ} , ψ_{FV} and ψ on the sets $\mathfrak{S}_n(231)$, $\mathfrak{S}_n(321)$, $\mathfrak{S}_{n+1}(213)$ and $\mathfrak{S}_{n+1}(321)$, respectively.

Theorem 1.11. *We have*

1. The mapping $\tilde{\phi}_{FV}$ is a bijection from $\mathfrak{S}_n(231)$ to \mathcal{CM}_n^* .
2. The mapping $\tilde{\phi}_{FZ}$ is a bijection from $\mathfrak{S}_n(321)$ to \mathcal{CM}_n^* .
3. The mapping $\tilde{\psi}_{FV}$ is a bijection from $\mathfrak{S}_{n+1}(213)$ to \mathcal{CM}_n .
4. The mapping $\tilde{\psi}$ is a bijection from $\mathfrak{S}_{n+1}(321)$ to \mathcal{CM}_n .

Proof. We just prove (1) and leave the others to the reader. If $\sigma_1, \sigma_2 \in \mathfrak{S}_n(231)$, let $\phi_{FV}(\sigma_i) = (\mathbf{s}_i, \mathbf{p}_i)$ for $i = 1, 2$. By definition we have $(2-31)\sigma_1 = (2-31)\sigma_2 = 0$, which implies that $\mathbf{p}_1 = \mathbf{p}_2 = (0, 0, \dots, 0)$; as ϕ_{FV} is a bijection, we derive that $\mathbf{s}_1 \neq \mathbf{s}_2$. Hence, the mapping ϕ_{FV} is an injection from $\mathfrak{S}_n(231)$ to \mathcal{CM}_n^* . Noticing that the generating function $\sum_{n \geq 0} |\mathcal{CM}_n^*| z^n$ has the continued fraction expansion (1.11) with $t = 1$, we derive that $|\mathfrak{S}_n(231)| = |\mathcal{CM}_n^*| = C_n$. Thus, the mapping $\tilde{\phi}_{FV}$ is a bijection. \square

Theorem 1.12. *Let $\tilde{\Phi}$ be the restriction of Φ on $\mathfrak{S}_n(231)$. Then $\tilde{\Phi}$ is a bijection from $\mathfrak{S}_n(231)$ to $\mathfrak{S}_n(321)$. Moreover, for $\sigma \in \mathfrak{S}_n(231)$, we have*

$$\begin{aligned} & (31-2, \text{des}, \text{asc}, \text{lda} - \text{fmax}, \text{ldd}, \text{lvalley}, \text{lpeak}, \text{fmax}) \sigma \\ &= (\text{icr}, \text{drop}, \text{exc} + \text{fix}, \text{cda}, \text{cdd}, \text{cvalley}, \text{cpk}, \text{fix}) \tilde{\Phi}(\sigma) \\ &= (\text{cros}, \text{exc}, \text{drop} + \text{fix}, \text{cdd}, \text{cda}, \text{cvalley}, \text{cpk}, \text{fix}) (\tilde{\Phi}(\sigma))^{-1} \end{aligned} \quad (1.58)$$

Proof. For $\sigma \in \mathfrak{S}_n(231)$, we have $(2-31)_i = 0$ for $i \in [n]$. So the inversion bottom (resp. top) number of each letter in f' (resp. g') equals 0. Let $\tau = \tilde{\Phi}(\sigma)$. By definition of Φ (cf. Section 1.2.2) the letters in f' (resp. g') are in increasing order. It is not hard to verify that $\text{nest}_i(\tau) = 0$ for each $i \in [n]$. By Lemma 1.10, we derive that $\tau \in \mathfrak{S}_n(321)$. For $\sigma_1, \sigma_2 \in \mathfrak{S}_n(231)$, since Φ is a bijection, we have $\tilde{\Phi}(\sigma_1) \neq \tilde{\Phi}(\sigma_2)$. And $|\mathfrak{S}_n(231)| = |\mathfrak{S}_n(321)| = C_n$, so $\tilde{\Phi}$ is a bijection from $\mathfrak{S}_n(231)$ to $\mathfrak{S}_n(321)$. Finally, the equidistribution (1.58) follows from Lemma 1.1. \square

Theorem 1.13. *Let $\tilde{\Psi}$ be the restriction of Ψ on $\mathfrak{S}_n(213)$. Then $\tilde{\Psi}$ is a bijection from $\mathfrak{S}_n(213)$ to $\mathfrak{S}_n(321)$.*

Proof. If $\sigma \in \mathfrak{S}_n(213)$, then $(2-13)_i \sigma = 0$ for $i \in [n]$. Thus, $(2-31)_1 \hat{\sigma} = 0$, and by Lemma 1.3, $(2-31)_{i+1} \hat{\sigma} = 1$ if $i+1$ is a nondescent top and $(2-31)_{i+1} \hat{\sigma} = 0$ otherwise. By definition of Φ , we construct two biwords, $\binom{f}{f'}$ and $\binom{g}{g'}$, where f (resp. g) is the subword of descent bottoms (resp. nondescent bottoms) in $\hat{\sigma}$ ordered increasingly, and f' (resp. g') is the permutation of descent tops (resp. nondescent tops) in $\hat{\sigma}$ such that the letters (resp. except 1 at the end) in f' (resp. g') are in increasing order.

Let $\tau = \tilde{\Psi}(\hat{\sigma})$. It is not hard to verify that $\text{nest}_i(\tau) = 1$ if $i \in g' \setminus \{1\}$ and $\text{nest}_i(\tau) = 0$ otherwise. Thus, by (1.40), we have $\text{nest}(\tilde{\Psi}(\hat{\sigma})) = 0$. By Lemma 1.10, $\tilde{\Psi}(\hat{\sigma}) \in \mathfrak{S}_n(321)$. For $\sigma_1, \sigma_2 \in \mathfrak{S}_n(213)$, since Ψ is a bijection, we have $\tilde{\Psi}(\sigma_1) \neq \tilde{\Psi}(\sigma_2)$. And $|\mathfrak{S}_n(213)| = |\mathfrak{S}_n(321)| = C_n$, so $\tilde{\Psi}$ is a bijection from $\mathfrak{S}_n(213)$ to $\mathfrak{S}_n(321)$. \square

Example 1.14. *If $\sigma = 1\ 6\ 8\ 9\ 7\ 2\ 5\ 3\ 4$, then $\hat{\sigma} = 2\ 7\ 9\ 10\ 8\ 3\ 6\ 4\ 5\ 1$, and reading from left to right, we obtain the corresponding numbers $(2-31)_i : 1, 1, 1, 0, 0, 1, 0, 1, 0, 0$ for $i = 2, 7, \dots, 1$, and*

$$\binom{f}{f'} = \begin{pmatrix} 1 & 3 & 4 & 8 \\ 5 & 6 & 8 & 10 \end{pmatrix}, \quad \binom{g}{g'} = \begin{pmatrix} 2 & 5 & 6 & 7 & 9 & 10 \\ 2 & 3 & 4 & 7 & 9 & 1 \end{pmatrix}.$$

Hence

$$w = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 & 8 & 2 & 5 & 6 & 7 & 9 & 10 \\ 5 & 6 & 8 & 10 & 2 & 3 & 4 & 7 & 9 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 2 & 5 & 6 & 1 & 3 & 7 & 4 & 9 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}.$$

Thus $\tau = \Phi(\hat{\sigma}) = 10\ 2\ 5\ 6\ 1\ 3\ 7\ 4\ 9\ 8$, and $\tilde{\Psi}(\sigma) = \tau(2)\dots\tau(10) = 2\ 5\ 6\ 1\ 3\ 7\ 4\ 9\ 8$.

Combining Theorems 1.11, 1.12, 1.13 and Figure 1.1 we obtain the diagrams in Figure 1.2.

1.3 Main results

For a finite set of permutations Ω and m statistics $\text{stat}_1, \dots, \text{stat}_m$ on Ω , we define the generating polynomial

$$P^{(\text{stat}_1, \dots, \text{stat}_m)}(\Omega; t_1, \dots, t_m) := \sum_{\sigma \in \Omega} t_1^{\text{stat}_1 \sigma} \dots t_m^{\text{stat}_m \sigma}. \quad (1.59)$$

Theorem 1.15. For $n \geq 1$,

$$\begin{aligned} & P^{(\text{nest}, \text{cros}, \text{exc}, \text{fix})}(\mathfrak{S}_n; p, q, tq, r) \\ &= \left(\frac{1+xt}{1+x} \right)^n P^{(\text{nest}, \text{cros}, \text{cpk}, \text{exc}, \text{fix})} \left(\mathfrak{S}_n; p, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{q(x+t)}{1+xt}, \frac{(1+x)r}{1+xt} \right), \end{aligned} \quad (1.60)$$

equivalently,

$$\begin{aligned} & P^{(\text{nest}, \text{cros}, \text{cpk}, \text{exc}, \text{fix})}(\mathfrak{S}_n; p, q, x, qt, r) \\ &= \left(\frac{1+u}{1+uv} \right)^n P^{(\text{nest}, \text{cros}, \text{exc}, \text{fix})} \left(\mathfrak{S}_n; p, q, qv, \frac{(1+uv)r}{1+u} \right), \end{aligned} \quad (1.61)$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

Remark 1.2. Cooper et al. [27, Theorem 11] have recently proved the $p = q = 1$ case of (1.60) by applying Sun and Wang's CMFS action [105].

Applying Lemma 1.10 and Theorem 1.15 with $p = 0$, we obtain the following result.

Corollary 1.16. For $n \geq 1$,

$$\begin{aligned} & P^{(\text{cros}, \text{exc}, \text{fix})}(\mathfrak{S}_n(321); q, tq, r) \\ &= \left(\frac{1+xt}{1+x} \right)^n P^{(\text{cros}, \text{cpk}, \text{exc}, \text{fix})} \left(\mathfrak{S}_n(321); q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{q(x+t)}{1+xt}, \frac{(1+x)r}{1+xt} \right), \end{aligned} \quad (1.62)$$

equivalently,

$$\begin{aligned} & P^{(\text{cros}, \text{cpk}, \text{exc}, \text{fix})}(\mathfrak{S}_n(321); q, x, qt, r) \\ &= \left(\frac{1+u}{1+uv} \right)^n P^{(\text{cros}, \text{exc}, \text{fix})} \left(\mathfrak{S}_n(321); q, qv, \frac{(1+uv)r}{1+u} \right), \end{aligned} \quad (1.63)$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

We define the polynomial

$$A_n(p, q, t) := \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest } \sigma} q^{\text{cross } \sigma} t^{\text{exc } \sigma}. \quad (1.64)$$

The following is a generalization of Stembridge's identity (1.4).

Theorem 1.17. *For $n \geq 1$, we have*

$$A_n(p, q, t) = \left(\frac{1+xt}{1+x} \right)^{n-1} P^{(\text{nest, cross, cpk}^*, \text{exc})} \left(\mathfrak{S}_n; p, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \quad (1.65)$$

equivalently,

$$P^{(\text{nest, cross, cpk}^*, \text{exc})}(\mathfrak{S}_n; p, q, x, t) = \left(\frac{1+u}{1+uv} \right)^{n-1} A_n(p, q, v), \quad (1.66)$$

$$\text{where } u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t} \text{ and } v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}.$$

By Corollary 1.7, we obtain the following linear generalization of Stembridge's identity.

Corollary 1.18. *For $n \geq 1$, we have*

$$A_n(p, q, t) = \left(\frac{1+xt}{1+x} \right)^{n-1} P^{(2-13, 31-2, \text{peak}-1, \text{des})} \left(\mathfrak{S}_n; p, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \quad (1.67)$$

equivalently,

$$P^{(2-13, 31-2, \text{peak}-1, \text{des})}(\mathfrak{S}_n; p, q, x, t) = \left(\frac{1+u}{1+uv} \right)^{n-1} A_n(p, q, v), \quad (1.68)$$

$$\text{where } u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t} \text{ and } v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}.$$

Remark 1.3. When $x = 1$ or $p = q = 1$ we recover two special cases of (1.65) due to Brändén's result [13, Eq (5.1)] and Zhuang [112, Theorem 4.2], respectively.

With Lemma 1.9 and (1.16), let $p = 0$ (resp. $q = 0$) in Corollary 1.18, we obtain the following corollary.

Corollary 1.19. *For all positive integers n and each triple statistic*

$$(\tau, \text{stat}_1, \text{stat}_2) \in \{(213, 31-2, \text{valley}), (312, 2-13, \text{valley})\},$$

we have

$$\begin{aligned} & P^{(\text{stat}_1, \text{des})}(\mathfrak{S}_n(\tau); q, t) \\ &= \left(\frac{1+xt}{1+x} \right)^{n-1} P^{(\text{stat}_1, \text{stat}_2, \text{des})} \left(\mathfrak{S}_n(\tau); q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \end{aligned} \quad (1.69)$$

equivalently,

$$P^{(\text{stat}_1, \text{stat}_2, \text{des})}(\mathfrak{S}_n(\tau); q, x, t) = \left(\frac{1+u}{1+uv} \right)^{n-1} P^{(\text{stat}_1, \text{des})}(\mathfrak{S}_n(\tau); q, v), \quad (1.70)$$

$$\text{where } u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t} \text{ and } v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}.$$

Remark 1.4. When $x = 1$, (1.69) reduces to [56, Eqs. (1.5) and (1.6)]. When $(\tau, \text{stat}_1, \text{stat}_2) = (213, 31-2, \text{valley})$ and $q = 1$, (1.69) reduces to [112, Corollary 5.3].

From (1.51) and (1.67) we derive the following result, which is an extension of Shin and Zeng [97, Theorem 1].

Corollary 1.20. For $n \geq 1$,

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv } \sigma - \text{exc } \sigma} t^{\text{exc } \sigma} \\ &= \left(\frac{1+xt}{1+x} \right)^{n-1} P^{(2-13, 31-2, \text{peak } -1, \text{des})} \left(\mathfrak{S}_n; q^2, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right). \end{aligned} \quad (1.71)$$

Define the cycle-refinement of the Eulerian polynomial $A_n(t)$ by

$$A_n^{(\text{cyc}^* - \text{fix}^*, \text{exc})}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{(\text{cyc}^* - \text{fix}^*) \sigma} t^{\text{exc } \sigma},$$

we obtain a cyclic analogue of Zhuang's formula [112, Theorem 4.2].

Theorem 1.21. For $n \geq 1$, we have

$$\begin{aligned} & A_n^{(\text{cyc}^* - \text{fix}^*, \text{exc})}(q, t) \\ &= \left(\frac{1+xt}{1+x} \right)^{n-1} P^{(\text{cyc}^* - \text{fix}^*, \text{cpk}^*, \text{exc})} \left(\mathfrak{S}_n; q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \end{aligned} \quad (1.72)$$

equivalently,

$$P^{(\text{cyc}^* - \text{fix}^*, \text{cpk}^*, \text{exc})}(\mathfrak{S}_n; q, x, t) = \left(\frac{1+u}{1+uv} \right)^{n-1} A_n(q, v), \quad (1.73)$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

Let $p = q = 1$ in (1.65) or $q = 1$ in (1.72), we get the following corollary.

Corollary 1.22. For $n \geq 1$, we have

$$A_n(t) = \left(\frac{1+xt}{1+x} \right)^{n-1} P^{(\text{cpk}^*, \text{exc})} \left(\mathfrak{S}_n; \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \quad (1.74)$$

equivalently,

$$P^{(\text{cpk}^*, \text{exc})}(\mathfrak{S}_n; x, t) = \left(\frac{1+u}{1+uv} \right)^{n-1} A_n(v), \quad (1.75)$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

Recall that a permutation $\sigma \in \mathfrak{S}_n$ is a derangement if it has no fixed points, i.e., $\sigma(i) \neq i$ for all $i \in [n]$. Let

$$D_n^{(\text{stat}_1, \text{stat}_2)}(q, t) := \sum_{\sigma \in \mathfrak{D}_n} q^{\text{stat}_1 \sigma} t^{\text{stat}_2 \sigma},$$

where \mathfrak{D}_n is the set of *derangements* in \mathfrak{S}_n .

Taking $(p, q, tq, r) = (q, 1, t, 0)$ (resp. $(p, q, tq, r) = (q^2, q, tq, 0)$) in Theorem 1.15 and by (1.51), we obtain the following corollary.

Corollary 1.23. *For all positive integers n and for each statistic $\text{stat} \in \{\text{nest}, \text{inv}\}$,*

$$D_n^{(\text{stat}, \text{exc})}(q, t) = \left(\frac{1+xt}{1+x} \right)^n P^{(\text{stat}, \text{cpk}, \text{exc})} \left(\mathfrak{D}_n; q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \quad (1.76)$$

equivalently,

$$P^{(\text{stat}, \text{cpk}, \text{exc})}(\mathfrak{D}_n; q, x, t) = \left(\frac{1+u}{1+uv} \right)^n D_n^{(\text{stat}, \text{exc})}(q, v), \quad (1.77)$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

By (1.51) and Lemma 1.10, the $r = 0$ case of (1.62) yields the following result in parallel with Corollary 1.23, which generalizes Lin's identity [72, Theorem 1.4].

Corollary 1.24. *For $n \geq 1$,*

$$\begin{aligned} & P^{(\text{inv}, \text{exc})}(\mathfrak{D}_n(321); q, t) \\ &= \left(\frac{1+xt}{1+x} \right)^n P^{(\text{inv}, \text{cpk}, \text{exc})} \left(\mathfrak{D}_n(321); q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \end{aligned} \quad (1.78)$$

equivalently,

$$P^{(\text{inv}, \text{cpk}, \text{exc})}(\mathfrak{D}_n(321); q, x, t) = \left(\frac{1+u}{1+uv} \right)^n P^{(\text{inv}, \text{exc})}(\mathfrak{D}_n(321); q, v), \quad (1.79)$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

Moreover, we have the the following formula.

Theorem 1.25. *For all positive integers n ,*

$$D_n^{(\text{cyc}, \text{exc})}(q, t) = \left(\frac{1+xt}{1+x} \right)^n P^{(\text{cyc}, \text{cpk}, \text{exc})} \left(\mathfrak{D}_n; q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \quad (1.80)$$

equivalently,

$$P^{(\text{cyc}, \text{cpk}, \text{exc})}(\mathfrak{D}_n; q, x, t) = \left(\frac{1+u}{1+uv} \right)^n D_n^{(\text{cyc}, \text{exc})}(q, v), \quad (1.81)$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

Consider the generalized q -Narayana polynomials $N_n(t, q, r)$ defined by

$$N_n(t, q, r) := \sum_{\sigma \in \mathfrak{S}_n(321)} t^{\text{exc } \sigma} q^{\text{inv } \sigma} r^{\text{fix } \sigma}. \quad (1.82)$$

In particular, we have

$$N_n(t/q, q, 1) = \sum_{\sigma \in \mathfrak{S}_n(321)} t^{\text{exc } \sigma} q^{\text{inv } \sigma - \text{exc } \sigma} \quad (1.83)$$

$$N_n(t, q, t) = \sum_{\sigma \in \mathfrak{S}_n(321)} t^{\text{wex } \sigma} q^{\text{inv } \sigma}. \quad (1.84)$$

Fu et al. [56] gave more interpretations of $N_n(t/q, q, 1)$ and $N_n(t, q, t)$ in terms of n -permutation patterns. We further prove the following interpretations by using the $(n-1)$ -permutation patterns.

Table 1.1: Five choices of $(\tau, \text{stat}_1, \text{stat}_2, \text{stat}_3)$

#	τ	stat_1	stat_2	stat_3
1	321	exc	inv	fix
2	231	des	des +31-2	fmax
3	132	asc	asc +2-13	amax
4	312	des	des +2-31	amin
5	213	asc	asc +13-2	fmin

Theorem 1.26. For $n \geq 1$, the following identities hold

$$N_n(t/q, q, 1) = \sum_{\sigma \in \mathfrak{S}_{n-1}(\tau)} t^{\text{stat}_1 \sigma} q^{\text{stat}_2 \sigma} (1+t)^{\text{stat}_3 \sigma}, \quad (1.85)$$

$$N_n(t, q, t) = t^n \sum_{\sigma \in \mathfrak{S}_{n-1}(\tau)} (q^2/t)^{\text{stat}_1 \sigma} q^{\text{stat}_2 \sigma} (1+q^2/t)^{\text{stat}_3 \sigma}, \quad (1.86)$$

where five choices for the quadruples $(\tau, \text{stat}_1, \text{stat}_2, \text{stat}_3)$ are listed in Table 1.1.

For $0 \leq k \leq n$, define the sets

$$\overline{\mathfrak{S}}_{n,k}(321) = \{\sigma \in \mathfrak{S}_n(321) : \text{exc } \sigma = k, \text{ cda } \sigma = 0\}, \quad (1.87)$$

$$\overline{\mathfrak{S}}_{n,k}(213) = \{\sigma \in \mathfrak{S}_n(213) : \text{asc } \sigma = k, \text{ rda } \sigma = 0\}, \quad (1.88)$$

$$\overline{\mathfrak{S}}_{n,k}(312) = \{\sigma \in \mathfrak{S}_n(312) : \text{des } \sigma = k, \text{ ldd } \sigma = 0\}, \quad (1.89)$$

$$\overline{\mathfrak{S}}_{n,k}(132) = \{\sigma \in \mathfrak{S}_n(132) : \text{asc } \sigma = k, \text{ rda } \sigma = 0\}, \quad (1.90)$$

$$\overline{\mathfrak{S}}_{n,k}(231) = \{\sigma \in \mathfrak{S}_n(231) : \text{des } \sigma = k, \text{ ldd } \sigma = 0\}, \quad (1.91)$$

and $\overline{\mathfrak{S}}_n(321) = \cup_{k=0}^n \overline{\mathfrak{S}}_{n,k}(321)$.

Theorem 1.27. For $n \geq 1$, the following q -analogue of (1.8) holds

$$N_n(t/q, q, 1) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n-1,k}(q) t^k (1+t)^{n-1-2k}, \quad (1.92)$$

where

$$\gamma_{n-1,k}(q) = \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k}(321)} q^{\text{inv } \pi} \quad (1.93)$$

$$= \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k}(231)} q^{(31-2) \pi + \text{des } \pi} = \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k}(312)} q^{(2-31) \pi + \text{des } \pi} \quad (1.94)$$

$$= \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k}(132)} q^{(2-13) \pi + \text{asc } \pi} = \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k}(213)} q^{(13-2) \pi + \text{asc } \pi}. \quad (1.95)$$

Theorem 1.28. For $n \geq 1$, the following q -analogue of (1.8) holds

$$N_n(t, q, t) = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \tilde{\gamma}_{n-1,k-1}(q) t^k (1+t/q)^{n+1-2k}, \quad (1.96)$$

where

$$\tilde{\gamma}_{n-1,k-1}(q) = \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k}(321)} q^{n-1+\text{inv } \pi - \text{exc } \pi} \quad (1.97)$$

$$= \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k-1}(231)} q^{n-1+(31-2)\pi} = \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k-1}(312)} q^{n-1+(2-31)\pi} \quad (1.98)$$

$$= \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k-1}(132)} q^{n-1+(2-13)\pi} = \sum_{\pi \in \overline{\mathfrak{S}}_{n-1,k-1}(213)} q^{n-1+(13-2)\pi}. \quad (1.99)$$

Remark 1.5. Other interpretations for $\gamma_{n-1,k}(q)$ and $\tilde{\gamma}_{n-1,k-1}(q)$ are given in [73, 72, 56].

Let \mathcal{B}_n be the set of permutations σ of $\{\pm 1, \dots, \pm n\}$ with $\sigma(-i) = -\sigma(i)$ for every $i \in [n]$. As Steingrímsson [102, Definition 3], we define the *excedance* of $\sigma \in \mathcal{B}_n$ by $i <_f \sigma(i)$ for $i \in [n]$, in the *friends order* $<_f$ of $\{\pm 1, \dots, \pm n\}$:

$$1 <_f -1 <_f 2 <_f -2 <_f \dots <_f n <_f -n,$$

and denote the number of excedances of $\sigma \in \mathcal{B}_n$ by $\text{exc}_B(\sigma)$. As Brenti [15] we say that $i \in [0, n-1]$ is a *B-descent* of σ if $\sigma(i) > \sigma(i+1)$ in the natural order $<$ of $\{\pm 1, \dots, \pm n\}$:

$$-n < \dots < -2 < -1 < 1 < 2 < \dots < n,$$

where $\sigma(0) = 0$. Denote the number of *B-descents* of σ by $\text{des}_B(\sigma)$.

Define the sign type B Eulerian polynomial as follows,

$$B_n(y, t) = \sum_{\sigma \in \mathcal{B}_n} y^{\text{neg } \sigma} t^{\text{exc}_B \sigma}.$$

Theorem 1.29. *We have*

$$B_n(y, t) = \sum_{\sigma \in \mathcal{B}_n} y^{\text{neg } \sigma} t^{\text{des}_B \sigma}. \quad (1.100)$$

We can write the sign type B Eulerian polynomials in terms of cyclic peak analogue of Eulerian polynomials $P^{(\text{cpk}, \text{exc})}(\mathfrak{S}_n; x, t)$.

Theorem 1.30. *For $n \geq 1$,*

$$B_n(y, t) = (1 + yt)^n P^{(\text{cpk}, \text{exc})} \left(\mathfrak{S}_n; \frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt} \right), \quad (1.101)$$

equivalently,

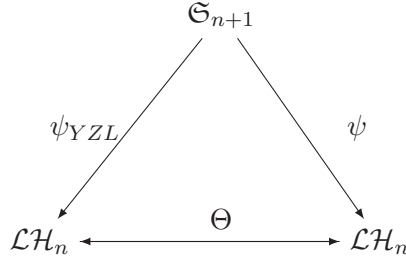
$$P^{(\text{cpk}, \text{exc})}(\mathfrak{S}_n; y, t) = \frac{1}{(1+uv)^n} B_n(u, v), \quad (1.102)$$

where $u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t}$ and $v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}$.

Yan, Zhou and Lin [110] constructed a bijection $\psi_{YZL} : \mathfrak{S}_{n+1} \rightarrow \mathcal{L}_n$. The next result shows that their bijection is actually almost equal to our ψ .

Theorem 1.31. *The bijection ψ_{YZL} is related to ψ as follows*

$$\Theta \circ \psi_{YZL} = \psi. \quad (1.103)$$

Figure 1.4: Involution Θ connecting ψ_{YZL} and ψ .

For $k \in [n]$ we define the subset of \mathfrak{S}_n :

$$\text{SDE}_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{exc}(\sigma) = k, \text{scda} = 0\}. \quad (1.104)$$

As $\text{exc} = \text{scval} + \text{scda}$, we have

$$\text{SDE}_{n,k} = \{\sigma \in \mathfrak{S}_n : \text{scval}(\sigma) = k, \text{scda}(\sigma) = 0\}.$$

Theorem 1.32. *We have*

$$\sum_{\sigma \in \text{SDE}_{n,k}} q^{\text{inv}(\sigma) - \text{exc}(\sigma)} = \sum_{\sigma \in \mathcal{DD}_{n,k}} q^{2(31-2) + (2-13)}. \quad (1.105)$$

Comparing (1.104) and (1.50) we derive $\text{DE}_{n,k}^* = \text{SDE}_{n,k}$ and the following result of Yan-Zhou-Lin [110].

Corollary 1.33 (Yan-Zhou-Lin).

$$\sum_{\sigma \in \text{SDE}_{n,k}} q^{\text{inv}(\sigma) - \text{exc}(\sigma)} = \sum_{\sigma \in \mathcal{DD}_{n,k}} q^{2(2-13) + (31-2)}. \quad (1.106)$$

1.4 Proofs using group actions

1.4.1 Proof of Theorem 1.15

We use a variant of Foata-Zeilberger's bijection $\phi_{FZ} : \mathfrak{S}_n \rightarrow \mathcal{LH}_n^*$ (cf. (1.54)). Given $\sigma \in \mathfrak{S}_n$, we construct the restricted Laguerre history $\phi'_{FZ} : \mathfrak{S}_n \rightarrow \mathcal{L}_n^*$ as follows. For $i = 1, \dots, n$,

$$s_i = \begin{cases} U & \text{if } i \in \text{Cval } \sigma; \\ D & \text{if } i \in \text{Cpk } \sigma; \\ L_r & \text{if } i \in \text{Cdd } \sigma \cup \text{Fix } \sigma; \\ L_b & \text{if } i \in \text{Cda } \sigma, \end{cases} \quad (1.107)$$

with $p_i = \text{nest}_i \sigma$.

First, define

$$L_r^*((\mathbf{s}, \mathbf{p})) := \{i \in [n] : s_i = L_r \text{ and } p_i = h_{i-1}\},$$

we give the following lemma.

Lemma 1.34. *If $\phi'_{FZ}(\sigma) = (\mathbf{s}, \mathbf{p}) \in \mathcal{L}_n^*$, we have*

$$\text{fix } \sigma = \#L_r^*((\mathbf{s}, \mathbf{p})), \quad (1.108)$$

$$\text{exc } \sigma = \#L_b(\mathbf{s}) + \#U(\mathbf{s}), \quad (1.109)$$

$$\text{nest } \sigma = \sum_{i=1}^n p_i, \quad (1.110)$$

$$\text{exc } \sigma + \text{cros } \sigma + \text{nest } \sigma = \sum_{i=1}^n h_{i-1}, \quad (1.111)$$

Proof. From the construction of ϕ'_{FZ} , it is easy to see (1.109)-(1.110). Define

$$\text{exc}_i \sigma = \begin{cases} 1, & \text{if } \sigma(i) > i \\ 0, & \text{if } \sigma(i) \leq i. \end{cases}$$

By inductions on $i \in [n]$ we have

$$\text{exc}_i \sigma + \text{nest}_i \sigma + \text{cros}_i \sigma = \begin{cases} h_{i-1} + 1, & \text{if } s_i = U; \\ h_{i-1}, & \text{if } s_i = L_r; \\ h_{i-1} - 1, & \text{if } s_i = D; \\ h_{i-1}, & \text{if } s_i = L_b, \end{cases}$$

which implies (1.111) immediately. For $i \in \text{Fix } \sigma$, we have $\text{exc}_i \sigma + \text{cros}_i \sigma = 0$, then $p_i = \text{nest}_i \sigma = h_{i-1}$ and $s_i = L_r$. This proves (1.108). \square

Yan-Zhou-Lin [110] introduced a group action on \mathcal{LH}_n in the spirit of the Foata–Strehl action on permutations. For $(\mathbf{s}, \mathbf{p}) \in \mathcal{LH}_n$ and $i \in [n]$, if $s_i = L_b$ or L_r , then φ_i be the mapping on \mathcal{LH}_n such that $\varphi_i((\mathbf{s}, \mathbf{p})) = ((\mathbf{s}', \mathbf{p}))$, where \mathbf{s}' is the 2-Motzkin path obtained from \mathbf{s} by changing the color of the i -th step, namely $L_b \leftrightarrow L_r$. Otherwise, define $\varphi_i((\mathbf{s}, \mathbf{p})) = (\mathbf{s}, \mathbf{p})$. We use another similar \mathbb{Z}_2^n -action on \mathcal{LH}_n^* . Let $i \in [n]$ and $(\mathbf{s}, \mathbf{p}) \in \mathcal{LH}_n^*$. Define the *restricted group action* φ'_i as follows,

$$\varphi'_i((\mathbf{s}, \mathbf{p})) = \begin{cases} (\mathbf{s}, \mathbf{p}), & \text{if } i \in L_r^*((\mathbf{s}, \mathbf{p})) \\ \varphi_i((\mathbf{s}, \mathbf{p})), & \text{otherwise.} \end{cases}$$

For any subset $S \subseteq [n]$ define the mapping $\varphi'_S : \mathcal{LH}_n^* \rightarrow \mathcal{LH}_n^*$ by $\varphi'_S((\mathbf{s}, \mathbf{p})) = \prod_{i \in S} \varphi'_i((\mathbf{s}, \mathbf{p}))$. Hence the group \mathbb{Z}_2^n acts on \mathcal{LH}_n^* via the function φ'_S , which is a generalization of the group action on \mathcal{CM}_n^* in [72]. We see that for $i \in [n]$, p_i and h_{i-1} are invariant under the group action. This action divides the set \mathcal{LH}_n^* into disjoint orbits and each orbit has a unique restricted Laguerre history which has all its level steps not included in $L_r^*((\mathbf{s}, \mathbf{p}))$ labelled by L_b . It is easy to see that the restricted group action preserves the elements in $L_r^*((\mathbf{s}, \mathbf{p}))$ in each orbit of \mathcal{L}_n^* .

For $0 \leq j \leq n$ and $1 \leq k \leq n$, let us introduce

$$\mathcal{R}_{n,j} = \{(\mathbf{s}, \mathbf{p}) \in \mathcal{LH}_n^* \mid \#L_r^*((\mathbf{s}, \mathbf{p})) = j\}.$$

Let $\text{Orb}((\mathbf{s}, \mathbf{p})) := \{\varphi'_S((\mathbf{s}, \mathbf{p})) \mid S \subseteq [n]\}$ be the orbit of $(\mathbf{s}', \mathbf{p}) \in \mathcal{R}_{n,j}$ under the restricted group action. First we prove the following identity.

Lemma 1.35. *For $(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}$, we have*

$$\begin{aligned} & (1+x)^{\#L_b(\mathbf{s})+\#L_r(\mathbf{s})-\#L_r^*((\mathbf{s}, \mathbf{p}))} \sum_{(\mathbf{s}', \mathbf{p}) \in \text{Orb}(\mathbf{s}, \mathbf{p})} t^{\#L_b(\mathbf{s}')+\#U(\mathbf{s}')} \\ &= \sum_{(\mathbf{s}', \mathbf{p}) \in \text{Orb}(\mathbf{s}, \mathbf{p})} (1+xt)^{\#L_r(\mathbf{s}')-\#L_r^*((\mathbf{s}', \mathbf{p}))} (x+t)^{\#L_b(\mathbf{s}')} t^{\#U(\mathbf{s}')}. \end{aligned} \quad (1.112)$$

Proof. Let $L(\mathbf{s}) := \{i \in [n] : s_i = L_b \text{ or } L_r\} \setminus L_r^*((\mathbf{s}, \mathbf{p}))$ with cardinality $\ell = \#L_r(\mathbf{s}) + \#L_b(\mathbf{s}) - \#L_r^*((\mathbf{s}, \mathbf{p}))$. As $\#U(\mathbf{s}') = \#U(\mathbf{s})$ for $(\mathbf{s}', \mathbf{p}) \in \text{Orb}(\mathbf{s}, \mathbf{p})$, the above identity is equivalent to

$$(1+x)^\ell \sum_{S \subset L(\mathbf{s})} t^{|S|} = \sum_{S \subset L(\mathbf{s})} (1+xt)^{|S|} (x+t)^{\ell-|S|},$$

namely, $(1+x)^\ell \cdot (1+t)^\ell = (1+xt+x+t)^\ell$. \square

We are ready to prove Theorem 3.1. For $(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}$, since $2\#U(\mathbf{s}) + \#L_r(\mathbf{s}) - \#L_r^*((\mathbf{s}, \mathbf{p})) + \#L_b(\mathbf{s}) + j = n$ and Eq. (1.112), we have

$$\begin{aligned} & \left(\sum_{(\mathbf{s}', \mathbf{p}) \in \text{Orb}(\mathbf{s}, \mathbf{p})} t^{\#L_b(\mathbf{s}') + \#U(\mathbf{s}')} \right) (1+x)^{n-j-2\#U(\mathbf{s})} \\ &= \sum_{(\mathbf{s}', \mathbf{p}) \in \text{Orb}(\mathbf{s}, \mathbf{p})} (1+xt)^{n-j-2\#U(\mathbf{s}') - \#L_b(\mathbf{s}')} (x+t)^{\#L_b(\mathbf{s}')} t^{\#U(\mathbf{s}')}, \end{aligned} \quad (1.113)$$

dividing both sides by $(1+x)^{n-j-2\#U(\mathbf{s})}$ for the above equation,

$$\sum_{(\mathbf{s}', \mathbf{p}) \in \text{Orb}(\mathbf{s}, \mathbf{p})} t^{\#U(\mathbf{s}') + \#L_b(\mathbf{s}')} = \sum_{(\mathbf{s}', \mathbf{p}) \in \text{Orb}(\mathbf{s}, \mathbf{p})} \frac{(1+xt)^{n-j-2\#U(\mathbf{s}') - \#L_b(\mathbf{s}')} (x+t)^{\#L_b(\mathbf{s}')} t^{\#U(\mathbf{s}')}}{(1+x)^{n-j-2\#U(\mathbf{s})}}.$$

Then, summing over all the orbits leads to the restricted Laguerre histories in $\mathcal{R}_{n,j}$,

$$\sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}} t^{\#L_b(\mathbf{s}) + \#U(\mathbf{s})} = \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}} \frac{(1+xt)^{n-j-2\#U(\mathbf{s}) - \#L_b(\mathbf{s})} (x+t)^{\#L_b(\mathbf{s})} t^{\#U(\mathbf{s})}}{(1+x)^{n-j-2\#U(\mathbf{s})}}.$$

Since $\sum_i p_i$ and $\sum_{i=1}^n h_{i-1}$ is invariant under the group action,

$$\begin{aligned} & \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}} \left(p^{\sum_i p_i} q^{\sum_{i=1}^n h_{i-1} - \sum_i p_i} \right) t^{\#L_b(\mathbf{s}) + \#U(\mathbf{s})} \\ &= \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}} \left(p^{\sum_i p_i} q^{\sum_{i=1}^n h_{i-1} - \sum_i p_i} \right) \frac{(1+xt)^{n-j-2\#U(\mathbf{s}) - \#L_b(\mathbf{s})} (x+t)^{\#L_b(\mathbf{s})} t^{\#U(\mathbf{s})}}{(1+x)^{n-j-2\#U(\mathbf{s})}}. \end{aligned} \quad (1.114)$$

Applying the bijection $\phi'_{FZ^{-1}}$ to the above identity, with (1.108)–(1.111) we obtain

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_{n,j}} \left(p^{\text{nest } \sigma} q^{\text{cros } \sigma + \text{exc } \sigma} \right) t^{\text{exc } \sigma} \\ &= \sum_{\sigma \in \mathfrak{S}_{n,j}} \left(p^{\text{nest } \sigma} q^{\text{cros } \sigma + \text{exc } \sigma} \right) \frac{(1+xt)^{n-j-\text{exc } \sigma - \text{cpk } \sigma} (x+t)^{\text{exc } \sigma - \text{cpk } \sigma} t^{\text{cpk } \sigma}}{(1+x)^{n-j-2\text{cpk } \sigma}}, \end{aligned} \quad (1.115)$$

where $\mathfrak{S}_{n,j}$ is the set of permutations in \mathfrak{S}_n with j fixed points. It is easy to see (1.115) is equal to

$$\left(\frac{1+xt}{1+x} \right)^{n-j} P^{(\text{nest}, \text{cros}, \text{cpk}, \text{exc})} \left(\mathfrak{S}_{n,j}; p, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{q(x+t)}{1+xt} \right),$$

which is (1.60). By using the substitution $u = \frac{(1+x^2)t}{(x+t)(1+xt)}$ and $v = \frac{x+t}{1+xt}$ as in (1.60), we obtain (1.61) immediately. We complete the proof.

Remark 1.6. When $x = 1$ in (1.114), we have

$$\begin{aligned} & \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}} \left(p^{\sum_i p_i} q^{\sum_{i=1}^n h_{i-1} - \sum_i p_i} \right) t^{\#L_b(\mathbf{s}) + \#U(\mathbf{s})} \\ &= \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}} \left(p^{\sum_i p_i} q^{\sum_{i=1}^n h_{i-1} - \sum_i p_i} \right) \frac{(1+t)^{n-j-2\#U(\mathbf{s})} t^{\#U(\mathbf{s})}}{2^{n-j-2\#U(\mathbf{s})}}. \end{aligned} \quad (1.116)$$

Define

$$\mathcal{O}_{n,k,j} = \{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j} : s \text{ has no } L_b\text{-step and } k \text{ } U\text{-steps}\}.$$

By the restricted group action on $\mathcal{R}_{n,j}$, it is easy to see that there are $2^{n-j-2\#U(\mathbf{s})}$ elements in each orbit, then we have

$$2^{n-2k-j} |\mathcal{O}_{n,k,j}| = |\{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j} : \#U(\mathbf{s}) = k\}|.$$

Then (1.116) is equivalent to

$$\begin{aligned} & \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{R}_{n,j}} \left(p^{\sum_i p_i} q^{\sum_{i=1}^n h_{i-1} - \sum_i p_i} \right) t^{\#L_b(\mathbf{s}) + \#U(\mathbf{s})} \\ &= \sum_{k=0} \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{O}_{n,j,k}} \left(p^{\sum_i p_i} q^{\sum_{i=1}^n h_{i-1} - \sum_i p_i} \right) (1+t)^{n-j-2k} t^k, \end{aligned} \quad (1.117)$$

applying the bijection ϕ_{FZ}^{-1} to the above identity, with (1.108)–(1.111) we obtain Shin and Zeng's result, see [96, Theorem 8],

$$\begin{aligned} & \sum_{\pi \in \mathfrak{S}_n} (tq)^{\text{exc } \pi} p^{\text{nest } \pi} q^{\text{cros } \pi} r^{\text{fix } \pi} \\ &= \sum_{j=0}^n r^j \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} \left(\sum_{\sigma \in \mathfrak{S}_{n,k,j}} p^{\text{nest } \sigma} q^{\text{cros } \sigma + \text{exc } \sigma} \right) t^k (1+t)^{n-j-2k}, \end{aligned} \quad (1.118)$$

where

$$\mathfrak{S}_{n,k,j} = \{\sigma \in \mathfrak{S}_n, \text{cpk } \sigma = k, \text{fix } \sigma = j, \text{cda } \sigma = 0\}.$$

Taking $p = 0$ in (1.118), by (1.51) and Lemma 1.10, we obtain the following result due to Lin, see [72, Theorem 2.4],

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_n(321)} t^{\text{exc } \sigma} q^{\text{inv } \sigma} r^{\text{fix } \sigma} \\ &= \sum_{j=0}^n r^j \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} \left(\sum_{\sigma \in \mathfrak{S}_{n,k,j}(321)} q^{\text{inv } \sigma} \right) t^k (1+t)^{n-j-2k}, \end{aligned} \quad (1.119)$$

where

$$\mathfrak{S}_{n,k,j}(321) := \{\sigma \in \mathfrak{S}_n(321) : \text{fix } \sigma = j, \text{exc } \sigma = k, \text{cda } \sigma = 0\}.$$

1.4.2 Proof of Theorem 1.25

Let $\sigma \in \mathfrak{S}_n$ with convention $0-\infty$. For any $x \in [n]$, the x -factorization of σ reads $\sigma = w_1 w_2 x w_3 w_4$, where w_2 (resp. w_3) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all smaller than x . Following Foata and Strehl [52] we define the action φ_x by

$$\varphi_x(\sigma) = w_1 w_3 x w_2 w_4.$$

Note that if x is a double ascent (resp. double descent), then $w_3 = \emptyset$ (resp. $w_2 = \emptyset$), and if x is a valley then $w_2 = w_3 = \emptyset$. For instance, if $x = 5$ and $\sigma = 26471583 \in \mathfrak{S}_7$, then $w_1 = 2647, w_2 = 1, w_3 = \emptyset$ and $w_4 = 83$. Thus $\varphi_5(\sigma) = 26475183$. Clearly, φ_x is an involution acting on \mathfrak{S}_n and it is not hard to see that φ_x and φ_y commute for all $x, y \in [n]$. Brändén [13] modified the map φ_x to be

$$\varphi'_x(\sigma) := \begin{cases} \varphi_x(\sigma), & \text{if } x \text{ is not a peak of } \sigma; \\ \sigma, & \text{if } x \text{ is a peak of } \sigma. \end{cases}$$

It is clear that φ'_x is involution and commutes with φ'_y for $x \neq y$. For any subset $S \subseteq [n]$ we can then define the map $\varphi'_S : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by

$$\varphi'_S(\sigma) = \prod_{x \in S} \varphi'_x(\sigma).$$

Hence the group \mathbb{Z}_2^n acts on \mathfrak{S}_n via the functions $\varphi'_S, S \subseteq [n]$. This action is called the *Modified Foata–Strehl action* (MFS-action for short).

The idea of MFS-action is due to Foata–Strehl, which was later considered by Shapiro, Woan, and Getu [93]. Brändén [13] developed the MFS-action much more extensively and Sun–Wang [105] gave a cyclic analogue of the MFS-action, whose results in [13, 74, 105] motivated much of the work in this section.

Recall that the cycle structure of a permutation $\sigma \in \mathfrak{S}_n$ can be written as a disjoint union of its distinct cycles C_1, C_2, \dots, C_k . Define the standard cycle representation of σ by

- writing the largest element of each cycle first,
- arranging the cycles in increasing order according to their largest elements.

Let $\sigma = C_1 C_2 \dots C_k$, define $o(\sigma)$ to be the permutation obtained from σ by erasing the parentheses of cycles. For example, for $\sigma = 26471583 \in \mathfrak{D}_8$, the standard cycle representation is $(6512)(8347)$, then $o(\sigma) = 65128347$. In this section, we consider the statistics of $o(\sigma)$ with the convention $0-\infty$.

From the above map $o : \mathfrak{S}_n \mapsto \mathfrak{S}_n$, we observe the following result directly.

Lemma 1.36. *For $\sigma = C_1 C_2 \dots C_k \in \mathfrak{D}_n$, we have*

$$\begin{aligned} \text{cvalley } \sigma &= \text{lvalley } o(\sigma) = \text{lpeak } o(\sigma) = \text{cpk } \sigma, & \text{lda } o(\sigma) &= \text{exc } \sigma - \text{cpk } \sigma, \\ \text{ldd } o(\sigma) &= n - \text{cpk } \sigma - \text{exc } \sigma, & \text{lda } o(\sigma) + \text{ldd } o(\sigma) &= n - 2 \text{cpk } \sigma. \end{aligned}$$

Proof. The first two identities are easily seen by the definitions of σ and $o(\sigma)$. For the third identity,

$$\begin{aligned} \text{lodd } o(\sigma) &= n - (\text{lpeak } o(\sigma) + \text{lvalley } o(\sigma) + \text{lda } o(\sigma)) \\ &= n - (\text{cpk } \sigma + \text{cvalley } \sigma + \text{exc } \sigma - \text{cvalley } \sigma) \\ &= n - \text{cpk } \sigma - \text{exc } \sigma. \end{aligned}$$

With the second and third identities, the fourth identity can be derived directly. □

For $\sigma \in \mathfrak{D}_n$, define the map $\tau_x^c : \mathfrak{D}_n \mapsto \mathfrak{D}_n$ by

$$\tau_x^c(\sigma) := o^{-1}(\varphi'_x(o(\sigma))).$$

It is easy to see that τ_x^c is involution and commutes with τ_y^c for $x \neq y$. For any subset $S \subseteq [n]$, we define $\tau_S^c : \mathfrak{D}_n \rightarrow \mathfrak{D}_n$ by

$$\tau_S^c(\sigma) = \prod_{x \in S} \tau_x^c(\sigma).$$

Sun and Wang [105] defined the group action of \mathbb{Z}_2^n on \mathfrak{D}_n via the involutions τ_S^c over all $S \subseteq [n]$; this group action is called the *cyclic modified Foata–Strehl action*, abbreviated *CMFS-action*, see Figure 1.5 for an illustration. For any permutation $\sigma \in \mathfrak{S}_n$, let $\text{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the *orbit of σ under the CMFS-action*.

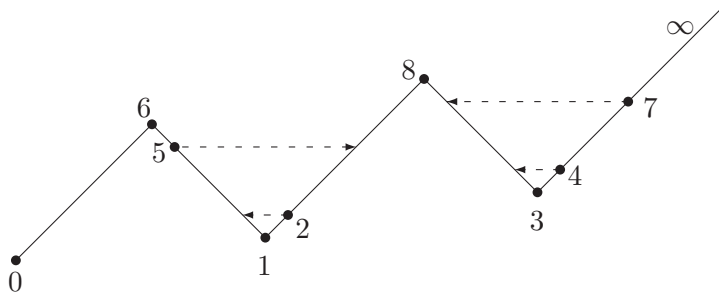


Figure 1.5: CMFS-actions on $(6512)(8347)$

Remark 1.7. The CMFS-action divides the set \mathfrak{D}_n into disjoint orbits. Moreover, for $\sigma \in \mathfrak{D}_n$, x is a double drop (resp. double excedance) of σ if and only if x is a double excedance (resp. double drop) of $\tau_x^c(\sigma)$. A double drop (resp. double excedance) x of σ remains a double drop (resp. double excedance) of $\tau_y^c(\sigma)$ for any $y \neq x$. Hence, there is a unique permutation in each orbit which has no double excedance. Let $\check{\sigma}$ be this unique element in $\text{Orb}(\sigma)$, and for any other $\sigma' \in \text{Orb}(\sigma)$, it can be obtained from $\check{\sigma}$ by repeatedly applying τ_x^c for some double drop x of $\check{\sigma}$. Each time this happens, exc increases by 1 and cdd decreases by 1. Thus by Lemma 1.36, we have

$$\sum_{\sigma \in \text{Orb } \sigma} t^{\text{exc } \sigma} = t^{\text{exc } \check{\sigma}} (1+t)^{\text{cdd } \check{\sigma}} = t^{\text{cpk } \check{\sigma}} (1+t)^{n-2 \text{cpk } \check{\sigma}}.$$

We obtain gamma expansion of derangement polynomials immediately by summing over all the orbits that form \mathfrak{D}_n .

For any $\Pi \subseteq \mathfrak{S}_n$ let

$$A^{(\text{exc}, \text{cyc})}(\Pi; w, t) := \sum_{\sigma \in \Pi} w^{\text{cyc } \sigma} t^{\text{exc } \sigma}.$$

The set Π is *invariant* under the CMFS-action if $\tau_S^\xi(\sigma) \in \Pi$ for any $\sigma \in \Pi$ and any $S \subseteq [n]$.

Then we will give a generalized version of Theorem 1.25.

Theorem 1.37. *If $\Pi \subseteq \mathfrak{D}_n$ is invariant under the CMFS-action, then*

$$A^{(\text{cyc}, \text{exc})}(\Pi; w, t) = \left(\frac{1+xt}{1+x} \right)^n P^{(\text{cyc}, \text{cpk}, \text{exc})} \left(\Pi; w, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt} \right), \quad (1.120)$$

equivalently,

$$P^{(\text{cyc}, \text{cpk}, \text{exc})}(\Pi; x, t, w) = \left(\frac{1+u}{1+uv} \right)^{n+1} A^{(\text{cyc}, \text{exc})}(\Pi; w, v),$$

where $u = \frac{1+t^2-2xt-(1-t)\sqrt{(1+t)^2-4xt}}{2(1-x)t}$ and $v = \frac{(1+t)^2-2xt-(1+t)\sqrt{(1+t)^2-4xt}}{2xt}$.

Let σ be a derangement of $[n]$. First we prove the following identity.

Lemma 1.38. *We have*

$$(1+x)^{\text{cda } \sigma + \text{cdd } \sigma} \sum_{\sigma' \in \text{Orb}(\sigma)} t^{\text{exc } \sigma'} = \sum_{\sigma' \in \text{Orb}(\sigma)} (1+xt)^{\text{cdd } \sigma'} (x+t)^{\text{cda } \sigma' + \text{cvalley } \sigma'}. \quad (1.121)$$

Proof. Let $J(\sigma) := \{i \in [n] : \sigma(i) \text{ is a double excedance or double drop}\}$ be the set of indices of double excedances and double drops of σ with cardinality $j = \text{cda } \sigma + \text{cdd } \sigma$. As $\text{exc } \sigma' = \text{cda } \sigma' + \text{cvalley } \sigma'$ and $\text{cvalley } \sigma' = \text{cvalley } \sigma$, the above identity is equivalent to

$$(1+x)^j \sum_{S \subset J(\sigma)} t^{|S|} = \sum_{S \subset J(\sigma)} (1+xt)^{|S|} (x+t)^{j-|S|},$$

namely, $(1+x)^j \cdot (1+t)^j = (1+xt+x+t)^j$. \square

We are ready to prove the Theorem 1.37. With Lemma 1.36 and Eq. (1.121), we have

$$\left(\sum_{\sigma' \in \text{Orb}(\sigma)} t^{\text{exc } \sigma'} \right) (1+x)^{n-2 \text{cpk } \sigma'} = \sum_{\sigma' \in \text{Orb}(\sigma)} (1+xt)^{n-\text{exc } \sigma' - \text{cpk } \sigma'} (x+t)^{\text{exc } \sigma' - \text{cpk } \sigma'} t^{\text{cpk } \sigma'},$$

dividing both sides by $(1+x)^{n-2 \text{cpk } \sigma'}$,

$$\sum_{\sigma' \in \text{Orb}(\sigma)} t^{\text{exc } \sigma'} = \sum_{\sigma' \in \text{Orb}(\sigma)} \frac{(1+xt)^{n-\text{exc } \sigma' - \text{cpk } \sigma'} (x+t)^{\text{exc } \sigma' - \text{cpk } \sigma'} t^{\text{cpk } \sigma'}}{(1+x)^{n-2 \text{cpk } \sigma'}}.$$

Then, summing over all the orbits leads to the permutations in Π ,

$$\sum_{\sigma \in \Pi} t^{\text{exc } \sigma} = \sum_{\sigma \in \Pi} \frac{(1+xt)^{n-\text{exc } \sigma - \text{cpk } \sigma} (x+t)^{\text{exc } \sigma - \text{cpk } \sigma} t^{\text{cpk } \sigma}}{(1+x)^{n-2 \text{cpk } \sigma}},$$

For $\sigma' \in \text{Orb}(\sigma)$, first we have $\text{cyc}(\sigma') = \text{cyc}(\sigma)$. From the definition of $o(\sigma)$, we have $\text{cyc}(\sigma)$ is equal to the number of left-to-right maximum of $o(\sigma)$. It is easy to see that the

number of left-to-right maximum is invariant under MFS-action. Thus we have $\text{cyc}(\sigma')$ is invariant for $\sigma' = \text{Orb}(\sigma)$.

Then we have

$$\begin{aligned} \sum_{\sigma \in \Pi} t^{\text{exc } \sigma} w^{\text{cyc } \sigma} &= \sum_{\sigma \in \Pi} \frac{(1+xt)^{n-\text{exc } \sigma - \text{cpk } \sigma} (x+t)^{\text{exc } \sigma - \text{cpk } \sigma} t^{\text{cpk } \sigma}}{(1+x)^{n-2\text{cpk } \sigma}} w^{\text{cyc } \sigma}, \\ &= \left(\frac{1+xt}{1+x} \right)^n P^{(\text{cpk}, \text{exc}, \text{cyc})} \left(\Pi; \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt}, w \right). \end{aligned}$$

We complete the proof.

Remark 1.8. Recently, using the joint distribution of the cyclic valley number and excedance number statistics Cooper, Jones and Zhuang [27] have generalized the formula of Stembridge by applying Sun and Wang's CMFS action. In particular they also obtained the $w = 1$ case of Theorem 1.37.

1.4.3 Proof of Theorems 1.31 and 1.32

Proof of Theorem 1.31. For $\sigma \in \mathfrak{S}_{n+1}$, an index $i \in [n]$ is called

- a *shifted cycle valley* if $i < \sigma(i)$ and $i+1 \leq \sigma^{-1}(i+1)$;
- a *shifted cycle peak* if $i \geq \sigma(i)$ and $i+1 > \sigma^{-1}(i+1)$;
- a *shifted double excedance* if $i < \sigma(i)$ and $i+1 > \sigma^{-1}(i+1)$;
- a *shifted no double excedance* if $i \geq \sigma(i)$ and $i+1 \leq \sigma^{-1}(i+1)$.

Denote by Scval , Scpeak , Scda and Scdn , the sets of shifted cycle valleys, shifted cycle peaks, shifted double excedance and shifted no double excedance. It is easy to see that the shifted statistics are almost equal to our star statistics, i.e.,

$$(\text{Scval}, \text{Scpeak}, \text{Scdn}, \text{Scda}) = (\text{Cval}^*, \text{Cpk}^*, \text{Cdd}^*, \text{Cda}^* \cup \text{Fix}^*). \quad (1.122)$$

We just prove $\text{Scval} = \text{Cval}^*$. As $\text{Cval}^*(\sigma) = \{i : i+1 < \sigma(i), i < \sigma^{-1}(i+1)\}$, it is sufficient to show that $\text{Scval} \subset \text{Cval}^*$. If $\sigma \in \mathfrak{S}_n$ and $i \in \text{Scval}(\sigma)$, then $i < \sigma(i)$ and $i+1 \leq \sigma^{-1}(i+1)$. Suppose $i+1 = \sigma(i)$ then $\sigma^{-1}(i+1) = i$, which contradicts the second inequality. So $i+1 < \sigma(i)$, and $i \in \text{Cval}^*$. We can reformulate the bijection ψ_{YZL} in [110] as follows

$$s_i = \begin{cases} U & \text{if } i \in \text{Cval}^* \sigma; \\ D & \text{if } i \in \text{Cpk}^* \sigma; \\ L_r \text{ (resp. } L_b) & \text{if } i \in \text{Cdd}^* \sigma \text{ (resp. } \text{Cda}^* \cup \text{Fix}^* \sigma), \end{cases} \quad (1.123)$$

with $p_i = \text{nest}_i \sigma$. Comparing (1.123) with (1.45) and by (1.122), we see that the only difference between the Laguerre histories obtained by applying the bijections ψ_{YZL} and ψ is the color of their level steps. \square

Proof of Theorem 1.32. For $\sigma \in \mathfrak{S}_n$, let $\sigma^r := \sigma(n) \cdots \sigma(2)\sigma(1)$ be the reverse of σ . By (1.42),

$$[2(31-2) + 2-13]\sigma = [2(2-13) + 31-2]\sigma^r = (2 \text{ nest} + \text{cross})\Psi(\sigma^r).$$

$\sigma \in \mathcal{DD}_{4,k}$	σ	σ^r	$\Psi(\sigma) \in \text{SDE}_{4,k}$	$(31-2)\sigma$	$(2-13)\sigma$	$\text{inv } \Psi(\sigma)$	$\text{exc } \Psi(\sigma)$
$k = 0$	1324	4231	1423	0	1	2	1
	1423	3241	1432	1	0	3	1
	2314	4132	4123	0	2	3	1
$k = 1$	2413	3142	4132	1	1	4	1
	3412	2143	3214	1	0	3	1
	2134	4312	3124	0	1	2	1
	3124	4213	4213	1	1	4	1
	4123	3214	4231	2	0	5	1

Figure 1.6: Illustration of Ψ on $\mathcal{DD}_{4,k}$ with their statistics.

By (1.51), we derive

$$[2(31-2) + (2-13)]\sigma = (\text{inv} - \text{exc})\Psi(\sigma^r). \quad (1.124)$$

Besides, by (1.42) and (1.122) we have

$$\begin{aligned} (\text{des}, \text{dd})\sigma &= (\text{asc}, \text{da})\sigma^r \\ &= (\text{wex}^*, \text{cda}^* + \text{fix}^*)\Psi(\sigma^r) \\ &= (\text{cvalley}^* + \text{cda}^* + \text{fix}^*, \text{cda}^* + \text{fix}^*)\Psi(\sigma^r) \\ &= (\text{scval} + \text{scda}, \text{scda})\Psi(\sigma^r). \end{aligned}$$

Hence, when $\text{dd}(\sigma) = \text{scda } \Psi(\sigma) = 0$, from (1.42) we derive that $\sigma \in \mathcal{DD}_{n,k}$ if and only if $\rho(\sigma) \in \text{SDE}_{n,k}$. With (1.124) this implies (1.106). \square

1.5 Proofs via continued fractions

1.5.1 Some combinatorial continued fractions

We first recall a standard contraction formula for continued fractions, see [95, Eq. (44)].

Lemma 1.39 (Contraction formula). *The following contraction formulae hold*

$$\begin{aligned} \frac{1}{1 - \frac{\alpha_1 z}{1 - \frac{\alpha_2 z}{1 - \frac{\alpha_3 z}{1 - \frac{\alpha_4 z}{1 - \dots}}}}} &= \frac{1}{1 - \alpha_1 z - \frac{\alpha_1 \alpha_2 z^2}{1 - (\alpha_2 + \alpha_3)z - \frac{\alpha_3 \alpha_4 z^2}{1 - \dots}}} \\ &= 1 + \frac{\alpha_1 z}{1 - (\alpha_1 + \alpha_2)z - \frac{\alpha_2 \alpha_3 z^2}{1 - (\alpha_3 + \alpha_4)z - \frac{\alpha_4 \alpha_5 z^2}{1 - \dots}}}. \end{aligned}$$

The following four combinatorial continued fraction expansions are proved by Shin and Zeng [96]. Let

$$A_n(p, q, t, u, v, w) := \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest}} \sigma q^{\text{cross}} \sigma t^{\text{exc}} \sigma u^{\text{cdd}^*} \sigma v^{\text{cda}^* + \text{fix}^*} \sigma w^{\text{cpk}^*} \sigma \quad (1.125)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des}} \sigma u^{\text{da}} \sigma v^{\text{dd}} \sigma w^{\text{peak}} \sigma^{-1}, \quad (1.126)$$

where the equality of the two enumerative polynomials follows from Lemma (1.7).

Lemma 1.40. [96, Eq. (28)] *We have*

$$\begin{aligned} & \sum_{n \geq 0} A_{n+1}(p, q, t, u, v, w) z^n \\ &= \frac{1}{1 - (u + tv)[1]_{p,q}z - \frac{[1]_{p,q}[2]_{p,q}twz^2}{1 - (u + tv)[2]_{p,q}z - \frac{[2]_{p,q}[3]_{p,q}twz^2}{\dots}}}, \end{aligned} \quad (1.127)$$

where $[n]_{p,q} = (p^n - q^n)/(p - q)$.

Let

$$B_n(p, q, t, u, v, w, y) := \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest}} \sigma q^{\text{cross}} \sigma t^{\text{exc}} \sigma u^{\text{cdd}} \sigma v^{\text{cda}} \sigma w^{\text{cvalley}} \sigma y^{\text{fix}} \sigma. \quad (1.128)$$

Lemma 1.41. [96, Eq. (34)] *We have*

$$1 + \sum_{n=1}^{\infty} B_n(p, q, t, u, v, w, y) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{1 - b_2 z - \frac{a_2 c_3 z^2}{\dots}}}}, \quad (1.129)$$

where $a_h = tw[h + 1]_{p,q}$, $b_h = yp^h + (qu + tv)[h]_{p,q}$ and $c_h = [h]_{p,q}$ for $h \geq 0$.

Let

$$C_n(q, t, u, v, w) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{cyc}^*} \sigma \text{-fix}^* \sigma t^{\text{wex}^*} \sigma u^{\text{cda}^* + \text{fix}^*} \sigma v^{\text{cdd}^*} \sigma w^{\text{cvalley}^*} \sigma, \quad (1.130)$$

Lemma 1.42. [96, Eq. (50)] *We have*

$$\begin{aligned} & \sum_{n \geq 0} C_{n+1}(q, t, u, v, w) z^n \\ &= \frac{1}{1 - (tu + v)z - \frac{1(q + 1)twz^2}{1 - 2(tu + v)z - \frac{2(q + 2)twz^2}{1 - 3(tu + v)z - \frac{3(q + 3)twz^2}{\dots}}}}}. \end{aligned} \quad (1.131)$$

Let

$$D_n(q, t, u, v, w) := \sum_{\sigma \in \mathfrak{D}_n} q^{\text{cyc}} \sigma t^{\text{exc}} \sigma u^{\text{cda}} \sigma v^{\text{cdd}} \sigma w^{\text{cvalley}} \sigma. \quad (1.132)$$

Lemma 1.43. [96, Eq. (41)] We have

$$1 + \sum_{n=1}^{\infty} D_n(q, t, u, v, w) z^n = \frac{1}{1 - 0(tu + v)z - \frac{1(q+0)twz^2}{1 - 1(tu + v)z - \frac{2(q+1)twz^2}{1 - 2(tu + v)z - \frac{3(q+2)twz^2}{\ddots}}}}. \quad (1.133)$$

1.5.2 Proof of Theorems 1.15–1.28

Proof of Theorem 1.15. In view of Lemma 1.41, let $(p, q, t, u, v, w, y) = (p, q, t, 1, 1, w, y)$, then $a_h = tw[h+1]_{p,q}$, $b_0 = y$, $b_h = yp^h + (q+t)[h]_{p,q}$, $c_h = [h]_{p,q}$, we have

$$B_n(p, q, t, 1, 1, w, y) = \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest } \sigma} q^{\text{cros } \sigma} t^{\text{exc } \sigma} w^{\text{cvalley } \sigma} y^{\text{fix } \sigma},$$

and

$$\sum_{n=0}^{\infty} B_n(p, q, t, 1, 1, w, y) z^n = \frac{1}{1 - yz - \frac{tw[1]_{p,q}^2 z^2}{1 - (yp + (q+t)[1]_{p,q})z - \frac{tw[2]_{p,q}^2 z^2}{1 - (yp^2 + (q+t)[2]_{p,q})z - \frac{tw[3]_{p,q}^2 z^2}{\ddots}}}}. \quad (1.134)$$

The right-hand side of Eq. (1.60) is

$$\sum_{n \geq 0} P^{(\text{nest, cros, cpk, exc, fix})} \left(\mathfrak{S}_n; p, q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{q(x+t)}{1+xt}, \frac{(1+x)r}{1+xt} \right) \left(\frac{(1+xt)z}{1+x} \right)^n.$$

By transforming $\frac{q(x+t)}{1+xt}$, $\frac{(1+x)^2 t}{(x+t)(1+xt)}$, $\frac{(1+xt)z}{1+x}$ and $\frac{(1+x)r}{1+xt}$ to t , w , z and y in (1.134), respectively, and observing (1.129), we obtain the equation corresponds to the continue fraction expansion of $S_n(p, q, tq, r)$ immediately. \square

Proof of Theorem 1.17. In view of (1.127), we have

$$A_n(p, q, t, 1, 1, w) = \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest } \sigma} q^{\text{cros } \sigma} t^{\text{exc } \sigma} w^{\text{cpk}^* \sigma}.$$

It follows that

$$\begin{aligned}
& \sum_{n=0}^{\infty} A_{n+1}(p, q, t, 1, 1, w) z^n \\
= & \frac{1}{1 - [1]_{p,q}(t+1)z - \frac{[1]_{p,q}[2]_{p,q}twz^2}{1 - [2]_{p,q}(t+1)z - \frac{[2]_{p,q}[3]_{p,q}twz^2}{1 - [3]_{p,q}(t+1)z - \frac{[3]_{p,q}[4]_{p,q}twz^2}{\ddots}}}} \quad (1.135)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{n=0}^{\infty} A_{n+1}(p, q, t) z^n \\
= & \frac{1}{1 - [1]_{p,q}(t+1)z - \frac{[1]_{p,q}[2]_{p,q}tz^2}{1 - [2]_{p,q}(t+1)z - \frac{[2]_{p,q}[3]_{p,q}tz^2}{1 - [3]_{p,q}(t+1)z - \frac{[3]_{p,q}[4]_{p,q}tz^2}{\ddots}}}} \quad (1.136)
\end{aligned}$$

The generating function of the right side of Eq. (1.65) is

$$\sum_{n \geq 0} P^{(\text{nest}, \text{cros}, \text{cpk}^*, \text{exc})} \left(\mathfrak{S}_{n+1}; p, q, \frac{(1+w)^2 t}{(w+t)(1+wt)}, \frac{w+t}{1+wt} \right) \left(\frac{(1+wt)z}{1+w} \right)^n.$$

Substituting (t, w, z) by $(\frac{w+t}{1+wt}, \frac{(1+w)^2 t}{(w+t)(1+wt)}, \frac{z(1+wt)}{1+w})$ in (1.135), the right-hand-side becomes that of (1.136) immediately. Thus their left-hand-sides are also equal. \square

Proof of Theorem 1.21. In view of Eq. (1.130), the generating function of the left side of Eq. (1.72) is

$$\begin{aligned}
& \sum_{n \geq 0} A_{n+1}^{(\text{cyc}^* - \text{fix}^*, \text{exc})}(q, t) z^n \\
= & \frac{1}{1 - 1(t+1)z - \frac{1(q+1)tz^2}{1 - 2(t+1)z - \frac{2(q+2)tz^2}{1 - 3(t+1)z - \frac{3(q+3)tz^2}{\ddots}}}} \quad (1.137)
\end{aligned}$$

In view of Eq. (1.130), we have

$$C_n(q, t, 1, 1, w) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{cyc}^* \sigma - \text{fix}^* \sigma} w^{\text{cvalley}^* \sigma} t^{\text{exc} \sigma},$$

and

$$\begin{aligned} & \sum_{n \geq 0} C_{n+1}(q, t, 1, 1, w) z^n \\ &= \frac{1}{1 - 1(t+1)z - \frac{1(q+1)twz^2}{1 - 2(t+1)z - \frac{2(q+2)twz^2}{1 - 3(t+1)z - \frac{3(q+3)twz^2}{\ddots}}}}}. \end{aligned} \quad (1.138)$$

The generating function of the right side of Eq. (1.72) is

$$\sum_{n \geq 0} P^{(\text{cyc}^* - \text{fix}^*, \text{cvalley}^*, \text{exc})} \left(\mathfrak{S}_{n+1}; q, \frac{(1+w)^2 t}{(w+t)(1+wt)}, \frac{w+t}{1+wt} \right) \left(\frac{(1+wt)z}{1+w} \right)^n.$$

By transforming $\frac{w+t}{1+wt}$, $\frac{(1+w)^2 t}{(w+t)(1+wt)}$ and $\frac{z(1+wt)}{(1+w)}$ to t , w and z in (1.138), respectively, we obtain (1.137) immediately. By the above argument, we see that the two sides of Eq. (1.72) have the same continue fraction. \square

Proof of Theorem 1.25. When $\text{stat} = \text{cyc}$, in view of Eq. (1.133), the generating function of the left side of Eq. (1.80) is

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} D_n^{(\text{cyc}, \text{exc})}(q, t) z^n \\ &= \frac{1}{1 - 0(t+1)z - \frac{1(q+0)tz^2}{1 - 1(t+1)z - \frac{2(q+1)tz^2}{1 - 2(t+1)z - \frac{3(q+2)tz^2}{\ddots}}}}}. \end{aligned} \quad (1.139)$$

In view of Eq. (1.133), we have

$$D_n(q, t, 1, 1, w) = \sum_{\sigma \in \mathfrak{D}_n} q^{\text{cyc } \sigma} w^{\text{cpk } \sigma} t^{\text{exc } \sigma},$$

and

$$1 + \sum_{n=1}^{\infty} D_n(q, t, 1, 1, w) z^n = \frac{1}{1 - \frac{1(q+0)twz^2}{1 - 1(t+1)z - \frac{2(q+1)twz^2}{1 - 2(t+1)z - \frac{3(q+2)twz^2}{\ddots}}}}}. \quad (1.140)$$

The generating function of the right side of Eq. (1.80) is

$$\sum_{n \geq 0} P^{(\text{cyc}, \text{cpk}, \text{exc})} \left(\mathfrak{D}_n; q, \frac{(1+w)^2 t}{(w+t)(1+wt)}, \frac{w+t}{1+wt} \right) \left(\frac{(1+wt)z}{1+w} \right)^n.$$

By transforming $\frac{w+t}{1+wt}$, $\frac{(1+w)^2t}{(w+t)(1+wt)}$ and $\frac{z(1+wt)}{(1+w)}$ to t , w and z in (1.140), respectively, we obtain (1.139) immediately. By the above argument, we see that the two sides of Eq. (1.80) for $\text{stat} = \text{cyc}$ have the same continue fraction. \square

Proof of Theorem 2.1. Recall [21, Theorem 7.2] that

$$\sum_{n=0}^{\infty} N_n(t, q, r) z^n = \frac{1}{1 - rz - \frac{tqz^2}{1 - (1+t)qz - \frac{tq^3z^2}{1 - (1+t)q^2z - \frac{tq^5z^2}{\ddots}}}}. \quad (1.141)$$

Writing $\sum_{n=1}^{\infty} N_{n-1}(t, q, r) z^n = z \sum_{n=0}^{\infty} N_n(t, q, r) z^n$ we have

$$1 + \sum_{n \geq 1} N_{n-1}(t, q, 1+t) z^n = 1 + \frac{z}{1 - (1+t)z - \frac{tqz^2}{1 - q(t+1)z - \frac{tq^3z^2}{1 - \dots}}}, \quad (1.142)$$

which is $\sum_{n \geq 0} N_n(t/q, q, 1) z^n$ by applying contraction formula (cf. Lemma 1.39). Thus

$$N_n(t/q, q, 1) = N_{n-1}(t, q, 1+t). \quad (1.143)$$

Since $\text{nest } \pi^{-1} = \text{nest } \pi$ (cf. Remark 1.1) and $\text{wex } \pi^{-1} = n - \text{exc } \pi$ for any $\pi \in \mathfrak{S}_n$, by Lemma 1.10, we deduce that $\pi \in \mathfrak{S}_n(321)$ if and only if $\pi^{-1} \in \mathfrak{S}_n(321)$. Using the inverse mapping $\pi \mapsto \pi^{-1}$ on $\mathfrak{S}_n(321)$ we can rewrite $N_n(t, q, t) := \sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{wex } \pi} q^{\text{inv } \pi}$ as follows:

$$N_n(t, q, t) = t^n \sum_{\pi \in \mathfrak{S}_n(321)} (1/t)^{\text{exc } \pi} q^{\text{inv } \pi} = t^n N_n(q/t, q, 1).$$

It follows from (1.143) that

$$N_n(t, q, t) = t^n N_{n-1}(q^2/t, q, 1 + q^2/t). \quad (1.144)$$

In view of (1.82) identities (1.143) and (1.144) provide the first interpretation in Table 1. Other interpretations in Table 1 can be derived from the equidistribution results in (1.17) and (1.22)–(1.25) \square

Proof of Theorem 1.27. By Lemma 1.10, (1.51) and (1.128), we have

$$B_n(0, q, tq, 1, 0, 1, 1) = \sum_{\sigma \in \overline{\mathfrak{S}_n(321)}} q^{\text{inv } \sigma} t^{\text{exc } \sigma}. \quad (1.145)$$

It follows from Lemma 1.41 that

$$\sum_{n=0}^{\infty} \sum_{\sigma \in \overline{\mathfrak{S}_n(321)}} q^{\text{inv } \sigma} t^{\text{exc } \sigma} z^n = \frac{1}{1 - z - \frac{tqz^2}{1 - qz - \frac{tq^3z^2}{1 - q^2z - \frac{tq^5z^2}{\ddots}}}}. \quad (1.146)$$

On the other hand, the generating function of the right-hand side of Eq. (1.92) is

$$\begin{aligned} & 1 + z \sum_{n \geq 1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\sum_{\sigma \in \overline{\mathfrak{S}}_{n-1,k}(321)} q^{\text{inv}(\sigma)} \right) t^k (1+t)^{n-1-2k} z^{n-1} \\ &= 1 + z \sum_{n \geq 0} \sum_{\sigma \in \overline{\mathfrak{S}}_n(321)} q^{\text{inv}(\sigma)} \left(\frac{t}{(1+t)^2} \right)^{\text{exc } \sigma} ((1+t)z)^n, \end{aligned} \quad (1.147)$$

which, combining with (1.146), is equal to

$$1 + \frac{z}{1 - (1+t)z - \frac{tqz^2}{1 - (1+t)qz - \frac{tq^3z^2}{1 - (1+t)q^2z - \frac{tq^5z^2}{\ddots}}}} \quad (1.148)$$

$$= \frac{1}{1 - z - \frac{tz^2}{1 - (q+t)z - \frac{tq^2z^2}{1 - (q+t)qz - \frac{tq^4z^2}{\dots}}}}, \quad (1.149)$$

which is equal to $\sum_{n \geq 0} N_n(t/q, q, 1)z^n$ by (1.141). Other interpretations can be obtained by the equidistribution results of (1.17) and (1.22)–(1.25). \square

Proof of Theorem 1.28. The generating function of the right side of (1.96) can be written as

$$\begin{aligned} & 1 + z \sum_{n \geq 1} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(\sum_{\sigma \in \overline{\mathfrak{S}}_{n-1,k-1}(321)} q^{n-1+\text{inv } \sigma - \text{exc } \sigma} \right) t^k (1+t/q)^{n+1-2k} z^{n-1} \\ &= 1 + zt \sum_{n \geq 0} \left(\sum_{\sigma \in \overline{\mathfrak{S}}_n(321)} q^{\text{inv } \sigma - \text{exc } \sigma} \left(\frac{tq^2}{(q+t)^2} \right)^{\text{exc } \sigma} \right) ((q+t)z)^n. \end{aligned} \quad (1.150)$$

By using the second claim of Lemma 1.10, (1.51) and (1.128), we have

$$B_n(0, q, t, 1, 0, 1, 1) = \sum_{\sigma \in \overline{\mathfrak{S}}_n(321)} q^{\text{inv } \sigma - \text{exc } \sigma} t^{\text{exc } \sigma},$$

Lemma 1.41 implies that

$$\sum_{n=0}^{\infty} \sum_{\sigma \in \overline{\mathfrak{S}}_n(321)} q^{\text{inv } \sigma - \text{exc } \sigma} t^{\text{exc } \sigma} z^n = \frac{1}{1 - z - \frac{tz^2}{1 - qz - \frac{tq^2z^2}{1 - q^2z - \frac{tq^4z^2}{\ddots}}}}.$$

Making the substitution $z \mapsto (q+t)z$ and $t \mapsto tq^2/(q+t)^2$ in the above equation, we obtain

$$1 + \frac{zt}{1 - (q+t)z - \frac{tq^2z^2}{1 - (q+t)qz - \frac{tq^4z^2}{1 - (q+t)q^2z - \frac{tq^6z^2}{\ddots}}}} \quad (1.151)$$

$$= \frac{1}{1 - tz - \frac{tqz^2}{1 - (1+t)qz - \frac{tq^3z^2}{1 - (1+t)q^2z - \frac{tq^5z^2}{\dots}}}}, \quad (1.152)$$

which is equal to $\sum_{n \geq 0} N_n(t, q, t)z^n$ by (1.141). Other interpretations can be obtained by the equidistribution results of (1.17) and (1.22)–(1.25). \square

1.5.3 Proof of Theorems 1.29–1.30

Recall the *color order* $<_c$ of $\{\pm 1, \dots, \pm n\}$:

$$-1 <_c -2 <_c \dots <_c -n <_c 1 <_c 2 <_c \dots <_c n,$$

and define the following statistics:

$$\begin{aligned} \text{fix } \sigma &= \#\{i \in [n] : i = \sigma(i)\}, \\ \text{exc}_A \sigma &= \#\{i \in [n] : i <_c \sigma(i)\}, \\ \text{wex}_A \sigma &= \#\{i \in [n] : i \leq_c \sigma(i)\} = \text{exc}_A \sigma + \text{fix } \sigma, \\ \text{wex}_C \sigma &= \#\{i \in [n] : i \leq |\sigma(i)| \text{ and } \sigma(i) < 0\}, \\ \text{neg } \sigma &= \#\{i \in [n] : \sigma_i < 0\}. \end{aligned}$$

For convenience, we use the following compact notation for the J-type continued fraction

$$J[z; b_n, \lambda_n] = \frac{1}{1 - b_0z - \frac{\lambda_1z^2}{1 - b_1z - \frac{\lambda_2z^2}{1 - b_2z - \frac{\lambda_3z^2}{1 - \dots}}}}. \quad (1.153)$$

Let

$$F_n(t, w, r, y) = \sum_{\sigma \in \mathcal{B}_n} t^{\text{wex}_A \sigma} w^{\text{wex}_C \sigma} r^{\text{fix } \sigma} y^{\text{neg } \sigma}.$$

Lemma 1.44. [97, Lemma 16] For $n \geq 1$,

$$\sum_{n \geq 0} F_n(t, w, r, y)z^n = J[z; b_n, \lambda_n], \quad (1.154)$$

where the coefficients λ_n and b_n are given by

$$\begin{aligned} \lambda_n &= n(t + wy)(1 + y) \quad (n \geq 1) \\ b_n &= n(1 + y) + t(r + n) + wy(n + 1) \quad (n \geq 0). \end{aligned}$$

We need the following lemma, see [111, Lemma 12] and [57, p. 307].

Lemma 1.45. *Suppose that two sequences $\{P_n\}_n$ and $\{Q_n\}_n$ satisfy the equation*

$$\sum_{n \geq 0} P_n \frac{z^n}{n!} = e^{\alpha z} \sum_{n \geq 0} Q_n \frac{z^n}{n!}.$$

If $\sum_{n \geq 0} Q_n z^n = J[z; b_n, \lambda_n]$, then $\sum_{n \geq 0} P_n z^n = J[z; b_n + \alpha, \lambda_n]$.

Proof of Theorem 1.29. Since $\text{exc} = (\text{wex}_A - \text{fix}) + \text{wex}_C$, see [97, Eq. (4.5)], we have

$$F_n(t, t, 1/t, y) = \sum_{\sigma \in \mathcal{B}_n} t^{\text{exc}_B \sigma} y^{\text{neg } \sigma} = B_n(y, t),$$

and formula (1.154) becomes

$$\sum_{n \geq 0} B_n(y, t) z^n = J[z; b_n, \lambda_n], \quad (1.155)$$

where $b_n = (n+1)(1+yt) + n(t+y)$ and $\lambda_n = (1+y)^2 t n^2$.

On the other hand, let

$$\widehat{B}_n(y, t) := \sum_{\sigma \in \mathcal{B}_n} y^{\text{neg } \sigma} t^{\text{des}_B \sigma}. \quad (1.156)$$

Then Brenti [15, Theorem 3.4] proved

$$\sum_{n \geq 0} \widehat{B}_n(y, t) \frac{z^n}{n!} = \frac{(1-t)e^{z(1-t)}}{1 - te^{z(1-t)}(1+y)} = e^{y(t-1)z} S((1+y)z; t) \quad (1.157)$$

with $S(z; t) = \frac{(1-t)e^{z(1-t)}}{1 - te^{z(1-t)}}$, which is the exponential generating function of Eulerian polynomials $A_n(t)$. By Lemma 1.45 we derive from (1.7) and (1.157) that $\sum_{n \geq 0} \widehat{B}_n(y, t) z^n$ has the same continued fraction (1.155). \square

Proof of Theorem 1.30. By (1.128), we have

$$P^{(\text{cpk}, \text{exc})}(\mathfrak{S}_n; w, t) = B_n(1, 1, t, 1, 1, w, 1).$$

Specializing (p, q, t, u, v, w, y) in Lemma 1.41 yields

$$\sum_{n=0}^{\infty} P^{(\text{cpk}, \text{exc})}(\mathfrak{S}_n; w, t) z^n = J[z; b_n, \lambda_n],$$

where $b_n = 1 + (1+t)n$ and $\lambda_n = twn^2$. It follows that the series

$$\sum_{n \geq 0} P^{(\text{cpk}, \text{exc})} \left(\mathfrak{S}_n; \frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt} \right) ((1+yt)z)^n$$

has the same continued fraction expansion for $\sum_{n \geq 0} B_n(y, t) z^n$ in (1.155). \square

Chapter 2

Gamma positivity in restricted permutations ¹

2.1 Introduction

One of the simple and fertile characterizations of *Catalan numbers* $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the following Stieltjes continued fraction expansion (cf. [12, 43])

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1}{1 - \frac{z}{1 - \frac{z}{\ddots}}}$$

In this chapter we define the (q, t) -Catalan numbers $C_n(t, q)$ as the Taylor coefficients in the following continued fraction expansion

$$\sum_{n=0}^{\infty} C_n(t, q) z^n = \frac{1}{1 - \frac{z}{1 - \frac{tz}{1 - \frac{\ddots}{1 - \frac{q^{k-1}z}{1 - \frac{tq^{k-1}z}{\ddots}}}}}}. \quad (2.1)$$

When we take $q = 1$ in (2.1), the right-hand side reduces to the continued fraction expansion for the generating function of the *Narayana polynomials* $C_n(t, 1)$, see [12], and for $t = 1$ we recover the classical q -Catalan numbers of Carlitz-Riordan [17]. Blanco and Petersen [9] considered a related (q, t) -analog of Catalan numbers $\text{Dyck}(n; t, q)$. Indeed, by comparing the continued fraction (2.1) with that in [9, Proposition 2.6], we have

$$\text{Dyck}(n; t, q) = C_n(tq, q^2). \quad (2.2)$$

The combinatorial constructions behind the proof of Theorem 2.1, one of our main results, originated from the fundamental work of Flajolet [43] for the lattice path interpretation of the formal continued fractions, and two bijections between sets of certain weighted

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Table 2.1: Ten choices for (τ, stat)

#	τ	stat	#	τ	stat
1	231	13-2	6	132	2-31
2	231	ai*	7	231	31-2
3	312	2-13	8	312	2-31
4	312	ai	9	213	13-2
5	213	31-2	10	132	2-13

Motzkin paths and permutations due to Françon-Viennot [54], and Foata-Zeilberger [53], see also [24, 95, 96].

The first goal of this chapter is to establish new combinatorial interpretations for $C_n(t, q)$, as well as their corresponding γ -expansions, using pattern avoiding permutations, which we define now. Denote by \mathfrak{S}_n the set of permutations of length n . Given two permutations $\pi \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$, $k \leq n$, we say that π *avoids the pattern* p if there does not exist a set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ of π is order-isomorphic to p . For example, the permutation 15324 avoids 231. The set of permutations of length n that avoid patterns p_1, p_2, \dots, p_m is denoted as $\mathfrak{S}_n(p_1, p_2, \dots, p_m)$.

A polynomial $f(x) = \sum_{i=0}^n a_i x^i \in \mathfrak{R}[x]$ is called *palindromic* if $a_i = a_{n-i}$ for $0 \leq i \leq n$. Clearly $\{x^i(1+x)^{n-2i}\}_{i=0}^{\lfloor n/2 \rfloor}$ forms a basis for the vector space of all palindromic polynomials of degree no greater than n . We call the unique expansion

$$f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$$

the γ -*expansion* for $f(x)$. Actually most of the expansions derived in this chapter are q - γ -*expansions*, in the sense that the coefficients are polynomials in q , and when we let $q = 1$, we recover the original γ -expansions.

Now we give the first three main results of this chapter, with the definitions of permutation statistics, permutation patterns and restricted permutations sets postponed to the next section.

Theorem 2.1. *The (q, t) -Catalan numbers $C_n(t, q)$ have the following ten interpretations*

$$C_n(t, q) = \sum_{\pi \in \mathfrak{S}_n(\tau)} t^{\text{des } \pi} q^{\text{stat } \pi},$$

with τ being a pattern of length 3, and stat being a permutation statistic. Ten choices for the pair (τ, stat) are listed in Table 2.1.

The last four interpretations for the γ -coefficients given in the next theorem correspond in a natural way to the interpretations labelled as entries 1, 3, 5, 6 in Table 2.1.

Theorem 2.2. *For $n \geq 1$, the following γ -expansions formula holds true*

$$C_n(t, q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) t^k (1+t)^{n-1-2k}, \quad (2.3)$$

where

$$\gamma_{n,k}(q) = \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(321)} q^{\text{inv } \pi - \text{exc } \pi} \quad (2.4)$$

$$= \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(213)} q^{(31-2) \pi} = \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(312)} q^{(2-13) \pi} \quad (2.5)$$

$$= \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(132)} q^{(2-31) \pi} = \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(231)} q^{(13-2) \pi}. \quad (2.6)$$

Remark 2.1. Eq. (2.3) with interpretation (2.4) is due to Lin and Fu [73]. Moreover, Blanco and Petersen [9] also obtained a γ -expansion formula for $C_n(tq, q^2)$, which should yield another interpretation for the γ -coefficients.

We will also prove the following variation of Theorems 2.1 and 2.2.

Theorem 2.3. *We have*

$$\sum_{\pi \in \mathfrak{S}_n(213)} t^{\text{des } \pi} q^{\text{ai } \pi} = \sum_{\pi \in \mathfrak{S}_n(132)} t^{\text{des } \pi} q^{\text{ai}^* \pi} \quad (2.7)$$

$$= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(213)} q^{\text{ai } \pi} \right) t^k (1+t)^{n-1-2k} \quad (2.8)$$

$$= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(132)} q^{\text{ai}^* \pi} \right) t^k (1+t)^{n-1-2k}. \quad (2.9)$$

Our second goal is to derive new examples of the following (-1) -phenomenon: for certain combinatorial generating functions for a set of permutations or *derangements*, substituting -1 for one of the variables gives an associated generating function over *alternating permutations* in the set. A permutation is said to be alternating (or up-down) if it starts with an ascent and then descents and ascents come in turn. (This has been called *reverse alternating* in Stanley's survey [100] and some of the other literature, but we stick with this convention throughout the chapter.) We denote by \mathbb{A}_n the set of alternating permutations of length n , and by $\mathbb{A}_n(p_1, p_2, \dots, p_m)$ the set of alternating permutations of length n that avoid patterns p_1, p_2, \dots, p_m .

The rest of the chapter is organized as follows. In Section 2.2, we give most of the definitions and provide the previously known results from the literature, which will be used to prove Theorems 2.1, 2.2 and 2.3 in Section 2.3. We consider a variation involving the *weak excedance* in Section 2.4. Next in Section 2.5, we completely determine the existence of (-1) -phenomena for $\mathfrak{S}_n(\tau)$, where τ runs through all permutations in \mathfrak{S}_3 . For example, we have the following q -version of the (-1) -phenomenon on $\mathfrak{S}_n(321)$ concerning exc. Recall [9] that Carlitz's q -Catalan numbers $C_n(q)$ are defined by

$$C_n(q) := C_n(q, q^2). \quad (2.10)$$

Using the Dyck path interpretation for $C_n(t, q)$ in (2.1) (see [43]), we see that $C_n(q)$ is a

polynomial of degree $\binom{n}{2}$. For instance,

$$\begin{aligned} C_0(q) &= C_1(q) = 1, \\ C_2(q) &= q + 1, \\ C_3(q) &= q^3 + q^2 + 2q + 1, \\ C_4(q) &= q^6 + q^5 + 2q^4 + 3q^3 + 3q^2 + 3q + 1. \end{aligned}$$

Theorem 2.4. For any $n \geq 1$,

$$C_n(-1, q) = \sum_{\pi \in \mathfrak{S}_n(321)} (-1)^{\text{exc } \pi} q^{\text{inv } \pi - \text{exc } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \quad (2.11)$$

$$\sum_{\pi \in \mathfrak{D}_n(321)} (-1)^{\text{exc } \pi} q^{\text{inv } \pi} = \begin{cases} (-q)^{\frac{n}{2}} C_{\frac{n}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \quad (2.12)$$

Motivated by Lewis' work [65, 66, 67, 68], many authors [10, 19, 109, 108, 81] have studied pattern avoidance on alternating permutations, especially the Wilf-equivalence problem for patterns of length four. As for alternating permutations that avoid two patterns of length four simultaneously, our results in Section 2.6 concerning $\mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n(1342, 2431)$ appear to be new. We close with some remarks to motivate further study along this line.

2.2 Definitions and preliminaries

2.2.1 Permutation statistics and a proof of Theorem 2.4

In this chapter, we define the variant labels for some notations and the nomenclature of various permutation statistics of Chapter 1, other statistics use the same labels as mentioned before.

Definition 2.1. Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ be a permutation, assume $\pi(n+1) = n+1$. A *descent* (resp. an *ascent*) in π is a triple $(i, \pi(i), \pi(i+1))$ such that $i \in [n]$ and $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$). Here $\pi(i)$ is called the *descent top* (resp. the *ascent bottom*) and $\pi(i+1)$ is called the *descent bottom* (resp. the *ascent top*). An *excedance* (resp. a *nonexcedance*) in π is a pair $(i, \pi(i))$ such that $i \in [n]$ and $\pi(i) > i$ (resp. $\pi(i) \leq i$). Here i is called the *excedance bottom* (resp. the *nonexcedance top*) and $\pi(i)$ is called the *excedance top* (resp. the *nonexcedance bottom*). The *number of descents* (resp. the *number of ascents*) in π is denoted by $\text{des } \pi$ (resp. $\text{asc } \pi$), and the *number of excedances* in π is denoted by $\text{exc } \pi$.

Next, we introduce four statistics involving three consecutive letters in π . Define

$$\begin{aligned} \tilde{\mathfrak{S}}_{n,k}(213) &:= \{\pi \in \mathfrak{S}_n(213) : \text{dd}^* \pi = 0, \text{des } \pi = k\}, \\ \tilde{\mathfrak{S}}_{n,k}(312) &:= \{\pi \in \mathfrak{S}_n(312) : \text{dd}^* \pi = 0, \text{des } \pi = k\}, \\ \tilde{\mathfrak{S}}_{n,k}(132) &:= \{\pi \in \mathfrak{S}_n(132) : \text{dd } \pi = 0, \text{des } \pi = k\}, \\ \tilde{\mathfrak{S}}_{n,k}(231) &:= \{\pi \in \mathfrak{S}_n(231) : \text{dd } \pi = 0, \text{des } \pi = k\}. \end{aligned}$$

Definition 2.2. The statistic MAD is defined by

$$\text{MAD } \pi := \text{des } \pi + (31-2) \pi + 2 \cdot (2-31) \pi$$

Definition 2.3. [95, Definition 7] A permutation π is called *coderangement* if $\text{fmax } \pi = 0$. Let \mathfrak{D}_n^* be the set of all coderangements in \mathfrak{S}_n .

Shin and Zeng introduced the above linear model of derangements. As suggested by its name, the set of coderangements is equinumerous with the set of derangements.

For the rest of this subsection, we collect all the lemmas that will be useful in later sections, and prove Theorem 2.4.

The Clarke-Steingrímsson-Zeng bijection [24] linking descent based statistics with excedance based ones is crucial for our ensuing derivation. It is the composition, say Φ , of two bijections between \mathfrak{S}_n and the set of certain weighted two colored Motzkin paths of length n . One bijection is due to Françon and Viennot [54], the other is due to Foata and Zeilberger [53]. See [24] for a direct description of Φ and further details. The following equidistribution result reveals further properties of Φ and is equivalent to [95, Theorem 8] modulo one application of the inverse map: $\pi \mapsto \pi^{-1}$.

Lemma 2.5 (Shin-Zeng). *For any $n \geq 1$, there is a bijection Φ on \mathfrak{S}_n such that*

$$(\text{des}, \text{fmax}, 31\text{-}2, 2\text{-}31, \text{MAD}) \pi = (\text{exc}, \text{fix}, \text{cros}, \text{nest}, \text{inv}) \Phi(\pi) \quad \text{for all } \pi \in \mathfrak{S}_n.$$

Shin and the fourth author [95] deduced the continued fraction expansion for the quintivariate generating function of \mathfrak{S}_n with respect to the above statistics.

Lemma 2.6 (Shin-Zeng). *Let*

$$\begin{aligned} A_n(x, y, q, p, s) &:= \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} y^{\text{fmax } \pi} q^{(31\text{-}2) \pi} p^{(2\text{-}31) \pi} s^{\text{MAD } \pi} \\ &= \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc } \pi} y^{\text{fix } \pi} q^{\text{cros } \pi} p^{\text{nest } \pi} s^{\text{inv } \pi}. \end{aligned} \quad (2.13)$$

Then we have

$$1 + \sum_{n=1}^{\infty} A_n(x, y, q, p, s) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\ddots}}}, \quad (2.14)$$

where, for $h \geq 0$,

$$a_h = s^{2h+1} [h+1]_{q,ps}, \quad b_h = yp^h s^{2h} + (x+q)s^h [h]_{q,ps},$$

and

$$c_h = x[h]_{q,ps}, \quad [h]_{u,v} := (u^h - v^h)/(u - v).$$

In order to make (2.13) suitable for the pattern-avoiding subsets, we have to invoke the following two lemmas.

Lemma 2.7. *For any $n \geq 1$,*

$$\begin{aligned} \mathfrak{S}_n(2\text{-}13) &= \mathfrak{S}_n(213), \quad \mathfrak{S}_n(31\text{-}2) = \mathfrak{S}_n(312), \\ \mathfrak{S}_n(13\text{-}2) &= \mathfrak{S}_n(132), \quad \mathfrak{S}_n(2\text{-}31) = \mathfrak{S}_n(231). \end{aligned}$$

Proof. See the proof of Lemma 1.9 in Chapter 1. □

Lemma 2.8. (i) A permutation $\pi \in \mathfrak{S}_n$ belongs to $\mathfrak{S}_n(321)$ if and only if $\text{nest } \pi = 0$. (ii) The mapping Φ has the property that $\Phi(\mathfrak{S}_n(231)) = \mathfrak{S}_n(321)$.

Proof. See the proof of Lemma 1.10 in Chapter 1. \square

By the second claim of Lemma 2.8, the special case of Lemma 2.6 where $p = 0$, $q = 1$ yields a result of Cheng et al. [21, Theorem 7.3].

Lemma 2.9 (Cheng et al.). *We have*

$$\sum_{n=0}^{\infty} \left(\sum_{\pi \in \mathfrak{S}_n(321)} q^{\text{inv } \pi} t^{\text{exc } \pi} y^{\text{fix } \pi} \right) z^n = \frac{1}{1 - yz - \frac{tqz^2}{1 - (1+t)qz - \frac{tq^3z^2}{1 - (1+t)q^2z - \frac{tq^5z^2}{\ddots}}}}. \quad (2.15)$$

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Taking $(t, y) = (-1/q, 1)$ in (2.15), we have by applying the contraction formula

$$1 + \sum_{n=1}^{\infty} \left(\sum_{\pi \in \mathfrak{S}_n(321)} (-1)^{\text{exc } \pi} q^{\text{inv } \pi - \text{exc } \pi} \right) z^n = 1 + \frac{z}{1 + \frac{qz^2}{1 + \frac{q^3z^2}{1 + \frac{q^5z^2}{\ddots}}}}.$$

We derive (2.11) by comparing this with (2.1).

In the same vein, by setting $(t, y) = (-1, 0)$ in (2.15), we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{\pi \in \mathfrak{D}_n(321)} (-1)^{\text{exc } \pi} q^{\text{inv } \pi} \right) z^n = \frac{1}{1 - \frac{(-q)z^2}{1 - \frac{(-q^3)z^2}{1 - \frac{(-q^5)z^2}{\ddots}}}}.$$

Comparing with (2.1), we readily get (2.12). \square

2.2.2 Other combinatorial interpretations of $C_n(q)$

We can derive several pattern avoiding interpretations for our q -Catalan numbers $C_n(q)$ from γ -expansions proved in [73, 72]. Let

$$\widehat{\mathfrak{S}}_{n,k}(321) := \{\pi \in \mathfrak{S}_n(321) : \text{exc } \pi = k \text{ and if } i < \pi(i), \text{ then } i+1 \text{ is a nonexcedance bottom}\}.$$

According to this definition, for any $\pi \in \widehat{\mathfrak{S}}_{n,k}(321)$, each occurrence of an excedance uniquely leads to an occurrence of a nonexcedance. So when n is odd, the maximum for $\text{exc } \pi$ is achieved at $k = \frac{n-1}{2}$, and in this case, the “if” condition becomes “if and only if.” More precisely, take any $\pi \in \widehat{\mathfrak{S}}_{n, \frac{n-1}{2}}(321)$, we have for $1 \leq i \leq n-1$ that the following are equivalent:

- $i < \pi(i)$,
- $i + 1$ is a nonexcedance bottom, and
- $\pi(i) - 1$ is a nonexcedance top.

This analysis shows that $\widehat{\mathfrak{S}}_{2n+1,n}(321)$ is exactly the set described in [101, Exercice 145], and therefore it is enumerated by C_n .

Interestingly, we find yet another two q - γ -expansions in Lin's work [72, Theorems 1.2 and 1.4].

Lemma 2.10 (Lin). *For any $n \geq 1$,*

$$\sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{wex } \pi} q^{\text{inv } \pi} = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(\sum_{\pi \in \text{NDW}_{n,k}(321)} q^{\text{inv } \pi} \right) t^k (1 + t/q)^{n+1-2k}, \quad (2.16)$$

$$\sum_{\pi \in \mathfrak{D}_n(321)} t^{\text{exc } \pi} q^{\text{inv } \pi} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{\pi \in \text{NDE}_{n,k}(321)} q^{\text{inv } \pi} \right) t^k (1 + t)^{n-2k}, \quad (2.17)$$

where

$$\text{NDW}_{n,k}(321) := \{ \pi \in \mathfrak{S}_n(321) : \text{wex } \pi = k, \text{ no } i \text{ such that } \pi(i+1) \geq i+1, \\ i \geq \pi^{-1}(i) \},$$

$$\text{NDE}_{n,k}(321) := \{ \pi \in \mathfrak{D}_n(321) : \text{exc } \pi = k, \text{ no } i \text{ such that } \pi^{-1}(i) < i < \pi(i) \}.$$

Now we can give the following three alternative interpretations for $C_n(q)$.

Proposition 2.11. *For any $n \geq 1$,*

$$\begin{aligned} C_n(q^2) &= q^{-2n} \sum_{\pi \in \widehat{\mathfrak{S}}_{2n+1,n}(321)} q^{\text{inv } \pi} = q^{-2n} \sum_{\pi \in \text{NDW}_{2n+1,n+1}(321)} q^{\text{inv } \pi} \\ &= q^{-n} \sum_{\pi \in \text{NDE}_{2n,n}(321)} q^{\text{inv } \pi}. \end{aligned}$$

Proof. The first equality involving $\widehat{\mathfrak{S}}_{2n+1,n}(321)$ follows from putting $t = -1$ in (2.3) and comparing (2.4) with (2.11). The second interpretation involving $\text{NDW}_{2n+1,n+1}(321)$ is a result of taking $t = -q$ in (2.16), replacing $\text{wex } \pi$ by $2n+1 - \text{exc } \pi$, and comparing the result with (2.11). The last one follows from (2.17) (setting $t = -1$) and (2.12) similarly. \square

Remark 2.2. Two remarks on Proposition 2.11 are in order. First, as a by-product we note that $\text{inv } \pi$ is even for any $\pi \in \widehat{\mathfrak{S}}_{2n+1,n}(321)$ (resp. $\pi \in \text{NDW}_{2n+1,n+1}(321)$), and $\text{inv } \pi$ has the parity of n for any $\pi \in \text{NDE}_{2n,n}(321)$. A direct combinatorial explanation of this might be interesting. On the other hand, from a bijective point of view, we note that the second equality above is a natural result of the inverse map $\pi \mapsto \pi^{-1}$, while a bijection deducing the third equality is possible via the two colored Motzkin path [72, 73]. We leave the details as exercises for motivated readers. Moreover, we note in passing that $|\text{NDE}_{2n,n}(321)| = C_n$ is equivalent to Exercise 151 in [101].

2.3 Proofs of Theorems 2.1, 2.2 and 2.3

In order to prove Theorem 2.1, we begin with a crucial lemma, which follows from [96, Eq. (39)].

Lemma 2.12 (Shin-Zeng). *The following four polynomials are equal*

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} p^{(2-13) \pi} q^{(31-2) \pi} &= \sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} p^{(31-2) \pi} q^{(2-13) \pi} \\ &= \sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} p^{(2-31) \pi} q^{(31-2) \pi} = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} p^{(31-2) \pi} q^{(2-31) \pi}. \end{aligned}$$

Proof. Indeed, equation (39) in [96] reads:

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} p^{(2-13) \pi} q^{(31-2) \pi} \right) z^n &= \sum_{n=0}^{\infty} \left(\sum_{\pi \in \mathfrak{S}_n} t^{\text{des } \pi} p^{(2-31) \pi} q^{(31-2) \pi} \right) z^n \\ &= \frac{1}{1 - \frac{c_1 z}{1 - \frac{c_2 z}{1 - \frac{c_3 z}{\ddots}}}} \end{aligned}$$

with $c_{2i} = t[i]_{p,q}$ and $c_{2i-1} = [i]_{p,q}$ for $i \geq 1$, where two misprints in [96] are corrected. The continued fraction shows clearly that the generating function is symmetric in p and q . \square

Next we give the definition of the statistic *admissible inversion*, which was first introduced by Shareshian and Wachs [94].

Definition 2.4. Let $\pi = \pi(1)\pi(2) \cdots \pi(n)$ be a permutation of \mathfrak{S}_n and $\pi(0) = \pi(n+1) = 0$. An *admissible inversion* of π is an inversion pair $(\pi(i), \pi(j))$, i.e., $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$, satisfying either of the following conditions:

- $\pi(j) < \pi(j+1)$ or
- there is some l such that $i < l < j$ and $\pi(j) > \pi(l)$.

If we apply reverse-complement to this definition, we get the following version, which was also used in [74, Definition 1].

Definition 2.5. Let $\pi = \pi(1)\pi(2) \cdots \pi(n)$ be a permutation of \mathfrak{S}_n and $\pi(0) = \pi(n+1) = n+1$. A *star admissible inversion* of π is a pair $(\pi(i), \pi(j))$ such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$, satisfying either of the following conditions:

- $\pi(i-1) < \pi(i)$ or
- there is some l such that $i < l < j$ and $\pi(i) < \pi(l)$.

Let $\text{ai } \pi$ and $\text{ai}^* \pi$ be the numbers of admissible inversions and star admissible inversions of $\pi \in \mathfrak{S}_n$, respectively. For example, if $\pi = 231$, then $\text{ai } \pi = 0$ while $\text{ai}^* \pi = 2$.

Lemma 2.13. *We have*

$$\text{ai } \pi = (2-13) \pi, \text{ if } \pi \in \mathfrak{S}_n(312), \tag{2.18}$$

$$\text{ai}^* \pi = (13-2) \pi, \text{ if } \pi \in \mathfrak{S}_n(231). \tag{2.19}$$

Proof. By Definition 2.4, an inversion pair $(\pi(i), \pi(j))$ of a permutation $\pi \in \mathfrak{S}_n$ is admissible if and only if either of the following conditions holds

- the triple $(\pi(i), \pi(j), \pi(j+1))$ forms a pattern 2-13 or 3-12;
- the triple $(\pi(i), \pi(l), \pi(j))$ with $i < l < j$ forms a pattern 312.

Thus, if $\pi \in \mathfrak{S}_n(312)$, the permutation π avoids both 312 and 3-12. This proves (2.18). The proof of (2.19) is similar. \square

We are now in position to prove Theorem 2.1. Our starting point is the following interpretation of $C_n(t, q)$ using pattern avoiding permutations first appeared in [73]

$$C_n(t, q) = \sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{exc } \pi} q^{\text{inv } \pi - \text{exc } \pi}. \quad (2.20)$$

We note that equality (2.20) follows also from Cheng et al. [21, Theorem 7.3] (see Lemma 2.9) by applying a standard contraction formula (see Lemma 1.39). For the reader's convenience, we are using a graph (see Figure 2.1) on the vertices $0, 1, \dots, 10$, where label 0 represents the interpretation in (2.20) while the remaining labels correspond to the ten interpretations in Table 2.1 of Theorem 2.1, and an edge between two vertices represents the two interpretations will be proven to be equivalent. Thus we break down the proof of Theorem 2.1 as showing equivalences represented by all the edges in Fig 2.1.

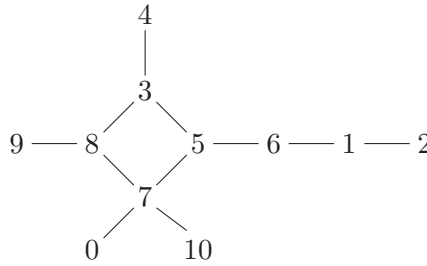


Figure 2.1: The proof flowchart for Theorem 2.1

Proof of Theorem 2.1.

- 0—7: We make the substitutions $(x, y, q, p, s) = (t/q, 1, 1, 0, q)$ in (2.13), then apply Lemma 2.8 and the definition of MAD to obtain

$$\sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{exc } \pi} q^{\text{inv } \pi - \text{exc } \pi} = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des } \pi} q^{(31-2) \pi}.$$

- 1—2, 3—4: These two follow directly from Lemma 2.13.
- 5—3—7—8: With Lemma 2.7 in mind, we set $p = 0$ in Lemma 2.12 to get

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_n(213)} t^{\text{des } \pi} q^{(31-2) \pi} &= \sum_{\pi \in \mathfrak{S}_n(312)} t^{\text{des } \pi} q^{(2-13) \pi} = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des } \pi} q^{(31-2) \pi} \\ &= \sum_{\pi \in \mathfrak{S}_n(312)} t^{\text{des } \pi} q^{(2-31) \pi}. \end{aligned}$$

- 5—6: The reverse-complement transformation $\pi \mapsto \pi^{rc}$ provides us with

$$\sum_{\pi \in \mathfrak{S}_n(213)} t^{\text{des } \pi} q^{(31-2) \pi} = \sum_{\pi \in \mathfrak{S}_n(132)} t^{\text{des } \pi} q^{(2-31) \pi}.$$

- 8—9, 7—10, 6—1: Recall that $C_n(t, q)$ is palindromic in t , then we apply the reverse map $\pi \mapsto \pi^r$ to get

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_n(312)} t^{\text{des } \pi} q^{(2-31) \pi} &= \sum_{\pi \in \mathfrak{S}_n(312)} t^{n-1-\text{des } \pi} q^{(2-31) \pi} = \sum_{\pi \in \mathfrak{S}_n(213)} t^{\text{des } \pi} q^{(13-2) \pi}, \\ \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des } \pi} q^{(31-2) \pi} &= \sum_{\pi \in \mathfrak{S}_n(231)} t^{n-1-\text{des } \pi} q^{(31-2) \pi} = \sum_{\pi \in \mathfrak{S}_n(132)} t^{\text{des } \pi} q^{(2-13) \pi}, \\ \sum_{\pi \in \mathfrak{S}_n(132)} t^{\text{des } \pi} q^{(2-31) \pi} &= \sum_{\pi \in \mathfrak{S}_n(132)} t^{n-1-\text{des } \pi} q^{(2-31) \pi} = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des } \pi} q^{(13-2) \pi}. \end{aligned}$$

By gathering all the equalities above, we complete the proof. \square

The proofs of Theorems 2.2 and 2.3 build on several lemmas.

Definition 2.6 (MFS-action). Let $\pi \in \mathfrak{S}_n$ with boundary condition $\pi(0) = \pi(n+1) = 0$, for any $x \in [n]$, the x -factorization of π reads $\pi = w_1 w_2 x w_3 w_4$, where w_2 (resp. w_3) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all larger than x . Following Foata and Strehl [52] we define the action φ_x by

$$\varphi_x(\pi) = w_1 w_3 x w_2 w_4.$$

Note that if x is a double ascent (resp. double descent), then $w_2 = \emptyset$ (resp. $w_3 = \emptyset$), and if x is a peak then $w_2 = w_3 = \emptyset$. For instance, if $x = 3$ and $\pi = 28531746 \in \mathfrak{S}_7$, then $w_1 = 2, w_2 = 85, w_3 = \emptyset$ and $w_4 = 1746$. Thus $\varphi_x(\pi) = 23851746$. Clearly, φ_x is an involution acting on \mathfrak{S}_n and it is not hard to see that φ_x and φ_y commute for all $x, y \in [n]$. Brändén [13] modified the map φ_x to be

$$\varphi'_x(\pi) := \begin{cases} \varphi_x(\pi), & \text{if } x \text{ is not a valley of } \pi; \\ \pi, & \text{if } x \text{ is a valley of } \pi. \end{cases}$$

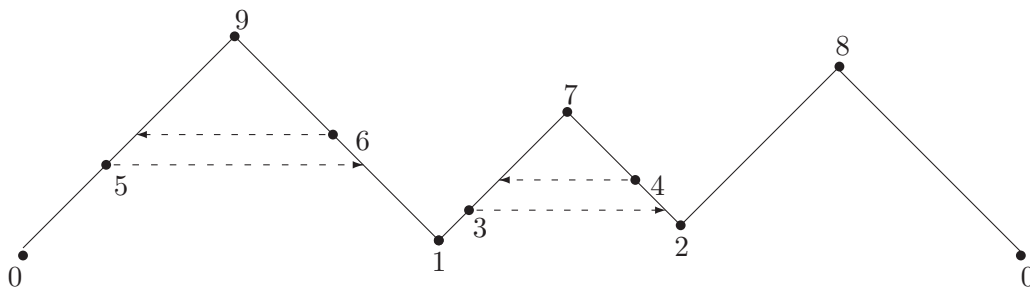


Figure 2.2: MFS-actions on 596137428 (recall $\pi(0) = \pi(10) = 0$)

See Figure 2.2 for illustration, where exchanging w_2 and w_3 in the x -factorisation is equivalent to moving x from a double ascent to a double descent or vice versa. Note that the boundary condition does matter. Take the permutation 596137428 in Fig 3.2 as an example. If $\pi(0) = 10$ instead, then 5 becomes a valley and will be fixed by φ'_5 .

It is clear that φ'_x 's are involutions and commute. For any subset $S \subseteq [n]$ we can then define the map $\varphi'_S : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by

$$\varphi'_S(\pi) = \prod_{x \in S} \varphi'_x(\pi).$$

Hence the group \mathbb{Z}_2^n acts on \mathfrak{S}_n via the functions φ'_S , $S \subseteq [n]$. This action will be called the *Modified Foata–Strehl action* (*MFS-action* for short). For any permutation $\pi \in \mathfrak{S}_n$, let $\text{Orb}(\pi) = \{g(\pi) : g \in \mathbb{Z}_2^n\}$ be the *orbit of π under the MFS-action*.

Remark 2.3. If we change the boundary condition to $\pi(0) = \pi(n+1) = n+1$, and in the x -factorization, we take w_2 (resp. w_3) to be the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all **smaller** than x , then we have the version used by Lin-Zeng in [74]. Note that for $321 \in \mathfrak{S}_3(231)$, $\varphi'_2(321) = 231 \notin \mathfrak{S}_3(231)$ using our version, while $\varphi'_2(321) = 312 \in \mathfrak{S}_3(231)$ using Lin-Zeng's version. Thus we need both versions in this chapter, just as we have defined both ai and ai^* . More precisely, when patterns $\{231, 132, 2-31, 13-2\}$ are concerned, we use Lin-Zeng's version, while for patterns $\{213, 312, 2-13, 31-2\}$ we use our current version.

Lemma 2.14. *The statistic ai is constant on any orbit under the MFS-action.*

Proof. Let $\pi \in \mathfrak{S}_n$, we aim to show that for each $x \in [n]$, we have $\text{ai } \pi = \text{ai } \varphi'_x(\pi)$. We discuss by three cases, according to the type of x . If x is a peak or a valley of π , then $\varphi'_x(\pi) = \pi$ and the result is true. If x is a double descent of π , then the x -factorization of π is $\pi = w_1 w_2 x w_3 w_4$ with w_3 being the empty word, and there are no admissible inversions of π formed by x and one letter in w_2 . As $\varphi'_x(\pi) = w_1 w_3 x w_2 w_4$, there are no inversions of $\varphi'_x(\pi)$ between w_2 and x . Let $(\pi(i), \pi(j)) \notin \{(y, x) : y \text{ is a letter in } w_2\}$ be a pair in π such that $i < j$. We claim that $(\pi(i), \pi(j))$ is an admissible inversion of π if and only if it is an admissible inversion of $\varphi'_x(\pi)$, from which the result follows for this case. Finally, suppose x is a double ascent of π . Recall that φ'_x is an involution, and x is a double ascent of π if and only if x is a double descent of $\varphi'_x(\pi)$, so this case follows as well.

Now we prove the claim from last paragraph. For a word w , we write $a \in w$ if a is a letter in w . To check the claim, we must consider cases depending on whether $\pi(i)$ and $\pi(j)$ belong to w_1 , w_2 , x , or w_4 . We will show only the case $\pi(i) \in w_2 x$, $\pi(j) \in w_4$, other cases are similar. If $(\pi(i), \pi(j))$ is an admissible inversion of π , then $\pi(i) > \pi(j) < \pi(j+1)$ or $\pi(i) > \pi(j) > \pi(l)$ for some $i < l < j$. Since φ'_x does not change the relative order between the letters other than x , we see that for the first case and the second case with $\pi(l) \neq x$, $(\pi(i), \pi(j))$ remains an admissible inversion of $\varphi'_x(\pi)$. For the second case with $\pi(l) = x$, we denote x' the first letter of w_4 . Then $x' < x < \pi(j) < \pi(i)$. Now in view of the triple $(\pi(i), x', \pi(j))$ in $\varphi'_x(\pi)$, we deduce that $(\pi(i), \pi(j))$ is an admissible inversion of $\varphi'_x(\pi)$. To show that, if $(\pi(i), \pi(j))$ is an admissible inversion of $\varphi'_x(\pi)$ then $(\pi(i), \pi(j))$ is an admissible inversion of π , is similar and we omit it. This finishes the proof of our claim. \square

Lemma 2.15. *The statistics (2-31), (13-2), (2-13) and (31-2) are constant on any orbit under the MFS-action.*

Proof. For $\pi \in \mathfrak{S}_n$, when $\pi(0) = \pi(n+1) = n+1$, the cases (2-31) and (13-2) were proved by Brändén [13, Theorem 5.1]. For the case (2-13) (see Fig. 2.3), let $\pi(0) = \pi(n+1) = 0$,

and note that

$$\begin{aligned}
(2-13) \quad \pi &= \#\{(i, j) : 1 \leq i < j < n, \pi(j) < \pi(i) < \pi(j+1)\} \\
&= \#\{(i, j, k) : 1 \leq i < j < k \leq n, \pi(j) < \pi(i) < \pi(k), \pi(j) \text{ is a valley,} \\
&\quad \pi(k) \text{ is a peak, } \pi(l) \text{ is neither a valley nor a peak, for } j < l < k\}.
\end{aligned}$$

Here the first equality is by definition. To see the second equality, suppose we are given a pair of indices (i, j) that satisfies the conditions in the first line. Starting with j , search to the left looking for the largest valley, say $\pi(j')$, then $i < j' \leq j$. Starting with $j+1$, search to the right looking for the smallest peak, say $\pi(k)$, then $j+1 \leq k \leq n$. Now clearly (i, j', k) forms a triple that is counted by the second and third lines above. Conversely, given such a triple, we can uniquely find a pair that is counted by the first line. Hence these two sets are in bijection. Finally, the number of these triples is invariant under the action since $\pi(j)$ and $\pi(k)$ cannot move and neither can $\pi(i)$ hop over the valley $\pi(j)$. A similar argument leads to the case (31-2). \square

Lemma 2.16. *The MFS-action φ'_S is closed on the subsets $\mathfrak{S}_n(\tau)$, for $\tau = 213, 312, 132, 231$.*

Proof. This follows directly from Lemmas 2.7 and 2.15. \square

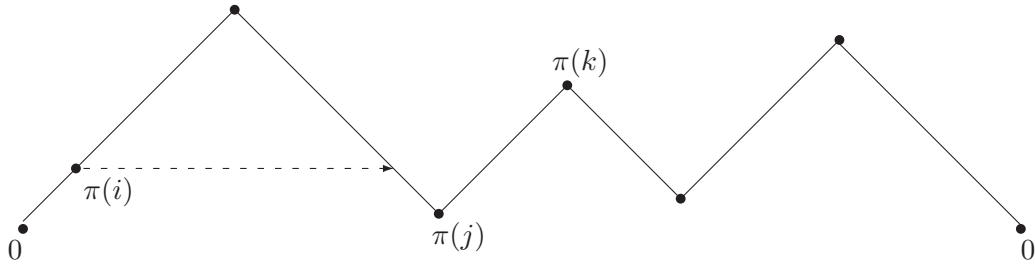


Figure 2.3: MFS-actions on pattern avoidance 2-13

Proof of Theorem 2.2. By Lemma 2.15, the statistics tracked by the power of q remain constant inside each orbit under the MFS-action. We first prove the 213-avoiding case in (2.5). The MFS-action divides the set \mathfrak{S}_n into disjoint orbits. Moreover, for $\pi \in \mathfrak{S}_n$, x is a double descent (resp. double ascent) of π if and only if x is a double ascent (resp. double descent) of $\varphi'_x(\pi)$. A double descent (resp. double ascent) x of π remains a double descent (resp. double ascent) of $\varphi'_y(\pi)$ for any $y \neq x$. Hence, there is a unique permutation in each orbit which has no double descent. Graphically, this is the permutation all of whose active dots hop to the left. Now, let $\bar{\pi}$ be this unique element in $\text{Orb}(\pi)$, then $\text{da}^* \bar{\pi} = n - \text{peak}^* \bar{\pi} - \text{valley}^* \bar{\pi}$ and $\text{des} \bar{\pi} = \text{peak}^* \bar{\pi} - 1 = \text{valley}^* \bar{\pi}$. And for any other $\pi' \in \text{Orb}(\pi)$, it can be obtained from $\bar{\pi}$ by repeatedly applying φ'_x for some double ascent x of $\bar{\pi}$. Each time this happens, des increases by 1 and da decreases by 1. Thus

$$\sum_{\sigma \in \text{Orb} \pi} q^{(31-2) \sigma} t^{\text{des} \sigma} = q^{(31-2) \bar{\pi}} t^{\text{des} \bar{\pi}} (1+t)^{\text{da} \bar{\pi}} = q^{(31-2) \bar{\pi}} t^{\text{des} \bar{\pi}} (1+t)^{n-2 \text{des} \bar{\pi}-1}.$$

We obtain the 213-avoiding case (2.5) immediately from Lemma 2.16 and by summing over all the orbits that form $\mathfrak{S}_n(213)$. The proofs of the remaining three cases are quite analogous, although one should keep in mind that for the 231-avoiding and 132-avoiding cases in (2.6), we use Lin-Zeng's version of MFS-action and the initial condition $\pi(0) = \pi(n+1) = n+1$ instead. \square

Proof of Theorem 2.3. Clearly the reverse-complement transformation $\pi \mapsto \pi^{rc}$ satisfies $(\text{des}, 213, \text{ai}) \pi = (\text{des}, 132, \text{ai}^*) \pi^{rc}$, which yields (2.7) directly. With Lemma 2.14 and Lemma 2.16, we obtain (2.8) and (2.9) via the MFS-action in a similar fashion as in the proof of Theorem 2.2. \square

Lemma 2.17. *All inversions in a down-up permutation of odd length are admissible. Moreover, for $\pi \in \tilde{\mathfrak{S}}_{2n+1,n}(213)$, we have*

$$\text{ai } \pi + \text{ai } \pi^r = 2n^2 + n, \quad (2.21)$$

$$\text{ai } \pi = 2 \cdot (3-12) \pi, \quad (2.22)$$

$$\text{ai } \pi^r = (31-2) \pi. \quad (2.23)$$

Proof. Assume $\pi \in \mathfrak{S}_{2n+1}$ is a down-up permutation, and $(\pi(i), \pi(j))$ is an inversion pair. If $j = 2n + 1$ or $\pi(j)$ is a peak, then $\pi(j - 1) < \pi(j)$. Otherwise $\pi(j)$ must be a valley and $j < 2n + 1$, then we see $\pi(j) < \pi(j + 1)$. In both cases, we see that $(\pi(i), \pi(j))$ is indeed admissible. So we have $\text{inv } \pi = \text{ai } \pi$.

Now suppose $\pi \in \tilde{\mathfrak{S}}_{2n+1,n}(213)$, the initial condition $\pi(2n + 2) = 0$ and the restriction of having no double descents force $\pi(2n) < \pi(2n + 1)$. Since π has exactly n descents and no double descents, its descents and ascents must alternate. Moreover, if $2n + 1$ is not the first letter of π then it will cause a pattern of 213. In summary, π must be a down-up permutation with the first letter being $2n + 1$. Also note that the reversal of a down-up permutation of odd length is still a down-up permutation. Therefore we have

$$\text{ai } \pi + \text{ai } \pi^r = \text{inv } \pi + \text{inv } \pi^r = \binom{2n + 1}{2},$$

where the last equality relies on the simple fact that any pair of letters in π is either an inversion pair for π , or an inversion pair for π^r . This proves (2.21).

To get (2.22), we construct a 1-to-2 map from triples that form 3-12 patterns in π , to inversion pairs of π . Suppose $(\pi(i), \pi(j), \pi(j + 1))$ is such a triple, then it is mapped to two inversion pairs, namely $(\pi(i), \pi(j))$ and $(\pi(i), \pi(j + 1))$. This map is well-defined. Since π is down-up, each inversion pair can only be the image under this map for at most one triple. Now it will suffice to show that all inversion pairs in π arise in this way. Suppose $(\pi(i), \pi(j))$ is an inversion pair. If $\pi(j)$ is a peak, then $(\pi(i), \pi(j - 1), \pi(j))$ forms a 3-12 pattern. Otherwise $\pi(j)$ must be a valley, so $\pi(j) < \pi(j + 1)$. Now if $\pi(j + 1) > \pi(i)$, $(\pi(i), \pi(j), \pi(j + 1))$ will form a 213 pattern which should be avoided by π . So we must have $\pi(j + 1) < \pi(i)$ and $(\pi(i), \pi(j), \pi(j + 1))$ forms a 3-12 pattern.

Finally for (2.23), simply note that if $\pi \in \mathfrak{S}_n(213)$, then $\pi^r \in \mathfrak{S}_n(312)$. Then by Lemma 2.13 we get $\text{ai } \pi^r = (2-13) \pi^r = (31-2) \pi$ and the proof is completed. \square

With the above lemma we obtain another combinatorial interpretation of $C_n(q)$.

Proposition 2.18. *For any $n \geq 1$,*

$$C_n(q) = \sum_{\pi \in \tilde{\mathfrak{S}}_{2n+1,n}(213)} q^{n^2 - (3-12) \pi}$$

Proof. Using the #5 interpretation for $C_n(t, q)$ and making the same argument as we did in the proof of Theorem 2.2 concerning the MFS-action, we have

$$C_{2n+1}(t, q) = \sum_{\pi \in \tilde{\mathfrak{S}}_{2n+1}(213)} t^{\text{des } \pi} q^{(31-2) \pi} = \sum_{k=0}^n \left(\sum_{\pi \in \tilde{\mathfrak{S}}_{2n+1,k}(213)} q^{(31-2) \pi} \right) t^k (1 + t)^{2n-2k}.$$

We take $t = -1$ in the above equation and apply (2.11) for the left-hand side to get

$$C_{2n+1}(-1, q) = (-q)^n C_n(q^2).$$

For the right-hand side, only the term with $k = n$ remains. Equating this with the left-hand side and use Lemma 2.17 to get the desired expression for $C_n(q)$. \square

2.4 A variant of q -Narayana polynomials

Let $W_n(t, q) := \sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{wex } \pi} q^{\text{inv } \pi}$, then combining Lemma 2.8 (i) and Lemma 2.19 below gives us $t^n W_n(t^{-1}, 1) = C_n(t, 1)$. Therefore $W_n(t, q)$ can be viewed as a variant of q -Narayana polynomials. We explore its various interpretations and q - γ -expansions in this section.

Definition 2.7. For $\pi \in \mathfrak{S}_n$, a value $x = \pi(i)$ ($i \in [n]$) is called

- a cyclic valley if $i = \pi^{-1}(x) > x$ and $x < \pi(x)$;
- a double excedance if $i = \pi^{-1}(x) < x$ and $x < \pi(x)$;
- a drop if $x = \pi(i) < i$.

Let cvalley (resp. cda , drop) denote the number of cyclic valleys (resp. double excedances, drops) in π . The following result is due to Shin-Zeng [96, Theorem 5].

Lemma 2.19 (Shin-Zeng). *There is a bijection Υ on \mathfrak{S}_n such that for all $\pi \in \mathfrak{S}_n$,*

$$\begin{aligned} & (\text{ine}, \text{icr}, \text{drop}, \text{cda}, \text{cdd}, \text{cvalley}, \text{fix}) \pi \\ &= (2\text{-}31, 31\text{-}2, \text{des}, \text{lda} - \text{fmax}, \text{ldd}, \text{lvalley}, \text{fmax}) \Upsilon(\pi), \end{aligned}$$

where the linear statistics on the right-hand side are defined with the convention $\pi(0) = 0$ and $\pi(n+1) = n+1$ for $\pi \in \mathfrak{S}_n$.

Theorem 2.20. *we have*

$$\begin{aligned} W_n(t, q) &= \sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{wex } \pi} q^{\text{inv } \pi} \\ &= t^n \sum_{\pi \in \mathfrak{S}_n(231)} (q/t)^{\text{des } \pi} q^{(31\text{-}2) \pi} = t^n \sum_{\pi \in \mathfrak{S}_n(231)} (q/t)^{\text{des } \pi} q^{(13\text{-}2) \pi} \\ &= t^n \sum_{\pi \in \mathfrak{S}_n(231)} (q/t)^{\text{des } \pi} q^{\text{ai}^* \pi} = t^n \sum_{\pi \in \mathfrak{S}_n(312)} (q/t)^{\text{des } \pi} q^{(2\text{-}31) \pi} \\ &= t^n \sum_{\pi \in \mathfrak{S}_n(312)} (q/t)^{\text{des } \pi} q^{(2\text{-}13) \pi} = t^n \sum_{\pi \in \mathfrak{S}_n(312)} (q/t)^{\text{des } \pi} q^{\text{ai } \pi} \\ &= t^n \sum_{\pi \in \mathfrak{S}_n(213)} (q/t)^{\text{des } \pi} q^{(31\text{-}2) \pi} = t^n \sum_{\pi \in \mathfrak{S}_n(213)} (q/t)^{\text{des } \pi} q^{(13\text{-}2) \pi} \\ &= t^n \sum_{\pi \in \mathfrak{S}_n(132)} (q/t)^{\text{des } \pi} q^{(2\text{-}31) \pi} = t^n \sum_{\pi \in \mathfrak{S}_n(132)} (q/t)^{\text{des } \pi} q^{(2\text{-}13) \pi}. \end{aligned}$$

Proof. Since $\text{drop } \pi = n - \text{wex } \pi$ and $\text{inv } \pi = n - \text{wex } \pi + \text{cros } \pi + 2 \text{nest } \pi$ ([95, Eq. (40)]), we have

$$\sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{wex } \pi} q^{\text{inv } \pi} = \sum_{\pi \in \mathfrak{S}_n(321)} t^{n - \text{drop } \pi} q^{\text{inv } \pi} = t^n \sum_{\pi \in \mathfrak{S}_n(321)} (q/t)^{\text{drop } \pi} q^{\text{cros } \pi}.$$

By Theorem 2.1 and Lemma 2.19, we have

$$\begin{aligned} t^n \sum_{\pi \in \mathfrak{S}_n(321)} (q/t)^{\text{drop } \pi} q^{\text{cross } \pi} &= t^n \sum_{\pi \in \mathfrak{S}_n(231)} (q/t)^{\text{des } \pi} q^{(31-2) \pi} \\ &= t^n \sum_{\pi \in \mathfrak{S}_n(231)} (q/t)^{\text{des } \pi} q^{(13-2) \pi}, \end{aligned}$$

and the remaining equalities follow similarly. \square

We can now derive another q - γ -expansion for the joint distribution of wex and inv over $\mathfrak{S}_n(321)$.

Theorem 2.21. *For any $n \geq 1$,*

$$\sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{wex } \pi} q^{\text{inv } \pi} = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(q^{n-k} \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k-1}(231)} q^{(13-2) \pi} \right) t^k (1+t/q)^{n+1-2k}. \quad (2.24)$$

Proof. By Theorems 2.20 and 2.2,

$$t^n \sum_{\pi \in \mathfrak{S}_n(231)} (q/t)^{\text{des } \pi} q^{(13-2) \pi} = t^n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(231)} q^{(13-2) \pi} \right) (q/t)^k (1+q/t)^{n-1-2k}.$$

For the right-hand side of above equation, by shifting k to $k-1$, we get (2.24). \square

In a similar fashion, the combination of Theorems 2.20 and 2.2 yields another three expressions. Now comparing (2.24) with (2.16) gives us the following five different interpretations for the same q - γ -coefficients in (2.24).

Corollary 2.22. *The following holds*

$$\begin{aligned} \sum_{\pi \in \text{NDW}_{n,k}(321)} q^{\text{inv } \pi} &= q^{n-k} \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k-1}(231)} q^{(13-2) \pi} = q^{n-k} \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k-1}(132)} q^{(2-31) \pi} \\ &= q^{n-k} \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k-1}(312)} q^{(2-13) \pi} = q^{n-k} \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k-1}(213)} q^{(31-2) \pi}. \end{aligned}$$

2.5 Avoiding one pattern of length three: a complete characterization

In this section, we completely determine the existence of (-1) -phenomena for $\mathfrak{S}_n(\tau)$, with respect to des (in subsections 5.1 and 5.2) and exc (in subsection 5.3), where τ runs through all permutations in \mathfrak{S}_3 . The reader can refer to Tables 2.2–2.4 for quick access to the results.

2.5.1 The 231-avoiding des -case and its q -analogues

The 231-avoiding alternating permutations were first enumerated by Mansour [79]:

$$|\mathbb{A}_{2n+1}(231)| = |\mathbb{A}_{2n}(231)| = C_n, \quad \text{for } n \geq 0. \quad (2.25)$$

A bijective proof of this fact with further implications was given by Lewis [65]. Indeed, combining (2.25) with exercises 146, 147, 149 and 150 in [101], utilizing the reverse map, the complement map, as well as the reverse complement map, one get the complete enumerations for all alternating permutations avoiding a single pattern of length three (see Table 2.2).

Recall the *standardization* of a word w with n distinct ordered letters, denoted as $\text{st}(w)$, is the unique permutation in \mathfrak{S}_n that is order isomorphic to w . We say a word w_1 is superior to another word w_2 and denote as $w_1 > w_2$, if for any two letters $l_1 \in w_1, l_2 \in w_2$, we always have $l_1 > l_2$. The following decomposition is crucial for deriving q -analogues of the (-1) -phenomenon on pattern-avoiding subsets of the coderangements.

Lemma 2.23. *Let $P_0(t, q) = Q_0(t, q) = R_1(t, q) = 1, P_1(t, q) = Q_1(t, q) = 0$, and for $n \geq 2$,*

$$\begin{aligned} P_n(t, q) &:= \sum_{\pi \in \mathfrak{D}_n^*(231)} t^{\text{des } \pi} q^{(13-2) \pi}, \\ Q_n(t, q) &:= \sum_{\pi \in \mathfrak{D}_n^*(132)} t^{\text{des } \pi} q^{(2-31) \pi}, \\ R_n(t, q) &:= \sum_{\pi \in \mathfrak{D}_n^*(213)} t^{\text{des } \pi} q^{(31-2) \pi}. \end{aligned}$$

Then for $n \geq 2$,

$$P_n(t, q) = \sum_{m=0}^{n-2} tq^{n-m-1} P_m(t, q) C_{n-m-1}(t, q), \quad (2.26)$$

$$Q_n(t, q) = \sum_{m=0}^{n-2} tq^m Q_m(t, q) C_{n-m-1}(t, q), \quad (2.27)$$

$$R_n(t, q) = \sum_{m=1}^{n-1} tq^{n-m-1} R_m(t, q) C_{n-m-1}(t, q). \quad (2.28)$$

Proof. The key observation is that, $\pi \in \mathfrak{D}_n^*(231)$ if and only if $\pi = \pi^{(1)} n \pi^{(2)}$, for some subwords $\pi^{(1)}$ and $\pi^{(2)} \neq \emptyset$, with $\pi^{(2)} > \pi^{(1)}$. In addition, $\pi^{(1)} \in \mathfrak{D}_m^*(231)$ and $\text{st}(\pi^{(2)}) \in \mathfrak{S}_{n-m-1}(231)$, for some $m, 0 \leq m \leq n-2$. Indeed, if $\pi \in \mathfrak{D}_n^*(231)$, since n must be a left-to-right maximum of π , it cannot be at the end of π , so $\pi^{(2)} \neq \emptyset$. And since $\pi^{(2)}$ is preceded by n , it contains no left-to-right maximum, therefore the coderangement restriction does not affect $\pi^{(2)}$ at all, but we do need it to avoid 231 as a subword of π . The condition $\pi^{(2)} > \pi^{(1)}$ is to guarantee that xny does not form a 231 pattern for any $x \in \pi^{(1)}$ and $y \in \pi^{(2)}$. The above discussion shows the “only if” part of the claim, the “if” part should be clear as well.

Now that the claimed decomposition is justified, we use the appropriate 231-avoiding interpretation for $C_{n-m-1}(t, q)$ taken from Theorem 2.1 and examine the change of des and $(13-2)$ during this decomposition. This should give us (2.26), the proofs of (2.27) and (2.28) are similar and thus omitted. \square

Now we can derive the following q -analogues for the (-1) -phenomenon on $\mathfrak{S}_n(231)$ concerning des , which parallels Theorem 2.4 nicely.

Theorem 2.24. For any $n \geq 1$,

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_n(231)} (-1)^{\text{des } \pi} q^{(31-2)\pi} &= \sum_{\pi \in \mathfrak{S}_n(231)} (-1)^{\text{des } \pi} q^{(13-2)\pi} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \end{aligned} \quad (2.29)$$

$$\sum_{\pi \in \mathfrak{D}_n^*(231)} (-q)^{\text{des } \pi} q^{(31-2)\pi} = \begin{cases} (-q)^{\frac{n}{2}} C_{\frac{n}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (2.30)$$

$$\sum_{\pi \in \mathfrak{D}_n^*(231)} (-1)^{\text{des } \pi} q^{(13-2)\pi} = \begin{cases} (-1)^{\frac{n}{2}} C_{\frac{n}{2}}^*(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (2.31)$$

where $C_n^*(q) := \sum_{\pi \in \mathbb{A}_{2n}(132)} q^{(2-31)\pi}$.

Proof. All we need to do to prove (2.29) (resp. (2.30)) is take $t = -1$ (resp. $(x, y, q, p, s) = (-1, 0, 1, 0, q)$) in Theorem 2.1 (resp. (2.13)), then apply Theorem 2.4. Next for (2.31), with the decomposition (2.26) in mind, we note that $P_{2n+1}(-1, q) = 0$ follows from induction on $P_m(-1, q)$ and using (2.11) for

$C_{n-m-1}(-1, q)$. In the same vein, the even $2n$ case reduces to proving the following identity:

$$C_n^*(q) = \sum_{m=0}^{n-1} q^{3n-3m-2} C_m^*(q) C_{n-m-1}(q^2). \quad (2.32)$$

Combining Proposition 2.11 and Corollary 2.22, we get the desired interpretation that meshes well with that of $C_m^*(q)$:

$$q^{n-m-1} C_{n-m-1}(q^2) = \sum_{\pi \in \mathbb{A}_{2n-2m-1}(132)} q^{(2-31)\pi}.$$

Next we plug this back to (2.32) and decompose permutations in $\mathbb{A}_{2n}(132)$ similarly as in the proof of (2.26) to complete the proof. \square

The first few values for $C_n^*(q)$ are:

$$\begin{aligned} C_0^*(q) &= C_1^*(q) = 1, \\ C_2^*(q) &= 2q, \\ C_3^*(q) &= 3q^2 + 2q^4, \\ C_4^*(q) &= 4q^3 + 6q^5 + 2q^7 + 2q^9, \\ C_5^*(q) &= 5q^4 + 12q^6 + 9q^8 + 8q^{10} + 4q^{12} + 2q^{14} + 2q^{16}. \end{aligned}$$

2.5.2 Other des-cases avoiding one pattern of length three and their q -analogues

In a search for results analogous to Theorems 2.24 and 2.4, we consider all the remaining subsets that avoid one pattern of length three, and summarize the results in Tables 2.3 and 2.4. The (-1) -evaluations for the non- \star entries with missing parity are understood to

Table 2.2: The enumeration of $\mathbb{A}_n(\tau)$, for $n \geq 3$ odd and even.

τ	123	132	213	231	312	321
$\mathbb{A}_{2n+1}(\tau)$	C_{n+1}	C_n	C_{n+1}	C_n	C_{n+1}	C_{n+1}
$\mathbb{A}_{2n}(\tau)$	C_n	C_n	C_n	C_n	C_n	C_{n+1}

Table 2.3: The (-1) -evaluation over $\mathfrak{S}_n(\tau)$ and $\mathfrak{D}_n^*(\tau)$ with respect to des.

des \ τ	123	132	213	231	312	321
\mathfrak{S}_{2n+1}	\star	$(-1)^n C_n$	$(-1)^n C_n$	$(-1)^n C_n$	$(-1)^n C_n$	\star
\mathfrak{D}_{2n}^*	\star	$(-1)^n C_n$	$(-1)^n C_{n-1}$	$(-1)^n C_n$	\star	\star

vanish. For instance, $\sum_{\pi \in \mathfrak{S}_{2n}(132)} (-1)^{\text{des } \pi} = 0$. A “ \star ” means there is no such phenomenon in this case. Take the top-left \star in Table 2.3 for example, we put it there to indicate that $\sum_{\pi \in \mathfrak{S}_{2n}(123)} (-1)^{\text{des } \pi}$ does not always vanish, or $\sum_{\pi \in \mathfrak{S}_{2n+1}(123)} (-1)^{\text{des } \pi}$ does not seem to be a recognizable sequence. For all the des-cases, we actually obtain the stronger q -versions. We begin by proving three useful lemmas.

Lemma 2.25. *For any $n \geq 1$,*

$$\sum_{\pi \in \mathfrak{D}_n^*(213)} t^{\text{des } \pi} q^{(13-2) \pi} = t \sum_{\pi \in \mathfrak{S}_{n-1}(213)} t^{\text{des } \pi} q^{(13-2) \pi}.$$

Proof. It is easy to see from the definition of \mathfrak{D}_n^* that $\pi \in \mathfrak{D}_n^*(213)$ if and only if $\pi = n\pi'$ with $\pi' \in \mathfrak{S}_{n-1}(213)$. Moreover, we note that $\text{des } \pi = 1 + \text{des } \pi'$ and $(13-2) \pi = (13-2) \pi'$. Summing over all the $\pi \in \mathfrak{D}_n^*(213)$ completes the proof. \square

Lemma 2.26. *For any $n \geq 1$ and $\pi \in \mathfrak{S}_n$,*

$$\text{des } \pi + (31-2) \pi + 1 = \text{fl } \pi + (13-2) \pi, \quad (2.33)$$

where $\text{fl } \pi = \pi(1)$ is the first letter of π .

Proof. We use induction on n . The $n = 1$ case holds trivially. Assume (2.33) is true for any permutation with length less than n . Let $\pi \in \mathfrak{S}_n$ and i be such that $\pi(i) = n$. If $i \in \{1, n\}$, the statement is easily checked. Otherwise we assume $2 \leq i \leq n-1$ and let $\pi' = \pi(1) \cdots \pi(i-1)\pi(i+1) \cdots \pi(n)$.

- If $\pi(i-1) < \pi(i+1)$, then $\text{des } \pi = \text{des } \pi' + 1$, $\text{fl } \pi = \text{fl } \pi'$, and

$$(13-2) \pi - (13-2) \pi' = (31-2) \pi - (31-2) \pi' + 1,$$

where we only need to check the contributions for 13-2 and 31-2 coming from the triple with n playing the role of 3.

- $\pi(i-1) > \pi(i+1)$, then $\text{des } \pi = \text{des } \pi'$, $\text{fl } \pi = \text{fl } \pi'$, and

$$(13-2) \pi - (13-2) \pi' = (31-2) \pi - (31-2) \pi'.$$

In both cases, we see that (2.33) holds for n as well. \square

Lemma 2.27. *For any $n \geq 2$,*

$$\sum_{\pi \in \mathfrak{D}_n^*(132)} t^{\text{des } \pi} q^{\text{fl } \pi} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{\pi \in \overline{\mathfrak{D}}_{n,k}^*(132)} q^{\text{fl } \pi} \right) t^k (1+t)^{n-2k}, \quad (2.34)$$

where $\overline{\mathfrak{D}}_{n,k}^*(132) := \{\pi \in \mathfrak{D}_n^*(132) : \text{dd}^* \pi = 1, \text{des } \pi = k\}$.

Proof. Since pattern 132 is concerned here, per Remark 2.3, we shall use Lin-Zeng's dual version of the MFS-action φ_x . In addition, we modify it differently in the following way. This new variant of MFS-action is denoted as $\overline{\varphi}_x$.

$$\overline{\varphi}_x(\pi) := \begin{cases} \pi, & \text{if } x \text{ is a valley, a peak, or a left-to-right maximum of } \pi; \\ \varphi_x(\pi), & \text{otherwise.} \end{cases}$$

We state without proving the following facts about $\overline{\varphi}_x$, all of which can be verified similarly as for φ'_x .

- $\overline{\varphi}_x$'s are involutions and commute;
- the map $\overline{\varphi}_x$ is closed on $\mathfrak{D}_n^*(132)$;
- for any $\pi \in \mathfrak{D}_n^*(132)$ and each $x \in [n]$, $\text{fl } \pi = \text{fl } \overline{\varphi}_x(\pi)$.

Let $\pi \in \mathfrak{D}_n^*(132)$. The above facts, together with a similar argument about the orbits under this new MFS-action, tell us that there is a unique permutation in $\text{Orb}(\pi)$ which has exactly one double descent at the first letter (this is due to the definition of coderangements \mathfrak{D}^* and the convention that $\pi(0) = \pi(n) = n+1$). Now, let $\bar{\pi}$ be this unique element in $\text{Orb}(\pi)$, then $\text{da } \bar{\pi} = n-1 - \text{peak } \bar{\pi} - \text{valley } \bar{\pi}$ and $\text{des } \bar{\pi} = \text{peak } \bar{\pi} + 1 = \text{valley } \bar{\pi}$. Thus

$$\sum_{\sigma \in \text{Orb } \pi} t^{\text{des } \sigma} q^{\text{fl } \sigma} = q^{\text{fl } \bar{\pi}} t^{\text{des } \bar{\pi}} (1+t)^{\text{da } \bar{\pi}} = q^{\text{fl } \bar{\pi}} t^{\text{des } \bar{\pi}} (1+t)^{n-2 \text{des } \bar{\pi}}.$$

Summing over all the orbits establishes (2.34). □

Now we are ready to present the q -analogues for all the remaining entries shown in Table 2.3.

Theorem 2.28. For any $n \geq 1$,

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_n(132)} (-1)^{\text{des } \pi} q^{(2-31) \pi} &= \sum_{\pi \in \mathfrak{S}_n(132)} (-1)^{\text{des } \pi} q^{(2-13) \pi} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \end{aligned} \quad (2.35)$$

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_n(213)} (-1)^{\text{des } \pi} q^{(31-2) \pi} &= \sum_{\pi \in \mathfrak{S}_n(213)} (-1)^{\text{des } \pi} q^{(13-2) \pi} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \end{aligned} \quad (2.36)$$

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_n(312)} (-1)^{\text{des } \pi} q^{(2-31) \pi} &= \sum_{\pi \in \mathfrak{S}_n(312)} (-1)^{\text{des } \pi} q^{(2-13) \pi} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \end{aligned} \quad (2.37)$$

$$\sum_{\pi \in \mathfrak{D}_n^*(132)} (-1)^{\text{des } \pi} q^{(2-31) \pi} = \begin{cases} (-1)^{\frac{n}{2}} \widehat{C}_{\frac{n}{2}}(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (2.38)$$

$$\sum_{\pi \in \mathfrak{D}_n^*(132)} (-q)^{\text{des } \pi} q^{(31-2) \pi} = \begin{cases} (-q)^{\frac{n}{2}} \overline{C}_{\frac{n}{2}}(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (2.39)$$

$$\sum_{\pi \in \mathfrak{D}_n^*(213)} (-1)^{\text{des } \pi} q^{(13-2) \pi} = \begin{cases} (-1)^{\frac{n}{2}} q^{\frac{n-2}{2}} C_{\frac{n-2}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (2.40)$$

$$\sum_{\pi \in \mathfrak{D}_n^*(213)} (-q)^{\text{des } \pi} q^{(31-2) \pi} = \begin{cases} (-1)^{\frac{n}{2}} q^{\frac{3n-4}{2}} C_{\frac{n-2}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (2.41)$$

where

$$\widehat{C}_n(q) := \sum_{\pi \in \mathbb{A}_{2n}(231)} q^{(13-2) \pi} \quad \text{and} \quad \overline{C}_n(q) := \sum_{\pi \in \mathbb{A}_{2n}(231)} q^{(2-13) \pi}.$$

Proof. (2.35)–(2.37) follow directly by taking $t = -1$ in Theorem 2.1 and applying (2.11). The proof of (2.38) parallels that of (2.31), only that we use the decomposition in (2.27) this time. To prove (2.39), we first note that

$$\begin{aligned} \sum_{\pi \in \mathfrak{D}_n^*(132)} (-q)^{\text{des } \pi} q^{(31-2) \pi} &= \sum_{\pi \in \mathfrak{D}_n^*(132)} (-1)^{\text{des } \pi} q^{\text{des } \pi + (31-2) \pi} \\ &\stackrel{(2.33)}{=} \sum_{\pi \in \mathfrak{D}_n^*(132)} (-1)^{\text{des } \pi} q^{\text{fl } \pi - 1}, \end{aligned}$$

which gives directly the odd $2n + 1$ case in view of the expansion (2.34). For the even $2n$ case, we compute using (2.34) again that

$$\sum_{\pi \in \mathfrak{D}_{2n}^*(132)} (-1)^{\text{des } \pi} q^{\text{fl } \pi - 1} = (-1)^n \sum_{\pi \in \overline{\mathfrak{D}}_{2n,n}^*(132)} q^{\text{fl } \pi - 1} \stackrel{(2.33)}{=} (-q)^n \sum_{\pi \in \overline{\mathfrak{D}}_{2n,n}^*(132)} q^{(31-2) \pi}.$$

Moreover, we note that $\pi \in \overline{\mathfrak{D}}_{2n,n}^*(132)$ if and only if $\pi^r \in \mathbb{A}_{2n}(231)$, which implies (2.39).

Finally, (2.40) follows from (2.36) and Lemma 2.25. In view of the similarity between (2.40) and (2.41), it is a straightforward calculation basing on identity (2.33) and that $\text{fl } \pi = n$ for any $\pi \in \mathfrak{D}_n^*(213)$, as pointed out in the proof of Lemma 2.25. \square

The first few values for $\widehat{C}_n(q)$ and $\overline{C}_n(q)$ are:

$$\begin{aligned}\widehat{C}_0(q) &= \widehat{C}_1(q) = 1, \\ \widehat{C}_2(q) &= q + q^2, \\ \widehat{C}_3(q) &= q^2 + q^3 + q^4 + q^5 + q^6, \\ \widehat{C}_4(q) &= q^3 + q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + 2q^9 + q^{10} + q^{11} + q^{12}, \\ \widehat{C}_5(q) &= q^4 + q^5 + 3q^6 + 3q^7 + 4q^8 + 3q^9 + 5q^{10} + 3q^{11} + 4q^{12} + 3q^{13} + 3q^{14} \\ &\quad + 2q^{15} + 2q^{16} + 2q^{17} + q^{18} + q^{19} + q^{20}, \\ \overline{C}_0(q) &= \overline{C}_1(q) = 1, \\ \overline{C}_2(q) &= 1 + q, \\ \overline{C}_3(q) &= 1 + 2q + 2q^2, \\ \overline{C}_4(q) &= 1 + 3q + 5q^2 + 5q^3, \\ \overline{C}_5(q) &= 1 + 4q + 9q^2 + 14q^3 + 14q^4, \\ \overline{C}_6(q) &= 1 + 5q + 14q^2 + 28q^3 + 42q^4 + 42q^5.\end{aligned}$$

The q -Catalan numbers $\overline{C}_n(q)$ merit some further investigation for their own sake. First we utilize (2.33) again to get another interpretation for $\overline{C}_n(q)$:

$$\sum_{\pi \in \mathbb{A}_{2n}(231)} q^{(2-13)\pi} = \sum_{\pi \in \mathbb{A}_{2n}(231)} q^{(31-2)\pi^r} \stackrel{(2.33)}{=} q^{-n-1} \sum_{\pi \in \mathbb{A}_{2n}(231)} q^{\text{fl } \pi^r}.$$

Definition 2.8. Let $\overline{C}_n(q) = q^{-n-1} \sum_{\pi \in \mathbb{A}_{2n}(231)} q^{\text{fl } \pi^r} := \sum_{k=0}^{n-1} a_{n,k} q^k$, where

$$\mathfrak{a}_{n,k} = \{\pi \in \mathbb{A}_{2n}(231) : \text{fl } \pi^r = n + k + 1\} \text{ and } a_{n,k} = |\mathfrak{a}_{n,k}|.$$

The first few examples are :

$$\begin{aligned}\mathfrak{a}_{1,0} &= \{12\}; \\ \mathfrak{a}_{2,0} &= \{1423\} \quad \text{and} \quad \mathfrak{a}_{2,1} = \{1324\}; \\ \mathfrak{a}_{3,0} &= \{162534\}, \quad \mathfrak{a}_{3,1} = \{162435, 132645\} \quad \text{and} \quad \mathfrak{a}_{3,2} = \{132546, 152436\}.\end{aligned}$$

Recall that the ballot numbers $f(n, k)$ satisfy (see [1, 18]) the recurrence relation

$$f(n, k) = f(n, k-1) + f(n-1, k), \quad (n, k \geq 0), \quad (2.42)$$

where $f(n, k) = 0$ if $n < k$ and $f(0, 0) = 1$, and have the explicit formula

$$f(n, k) = \frac{n-k+1}{n+1} \binom{n+k}{k}, \quad (n \geq k \geq 0).$$

With the initial values $a_{1,0} = a_{2,0} = a_{2,1} = 1$, and comparing (2.42) and (2.43), we establish the following connection.

Proposition 2.29. For $0 \leq k \leq n-1$,

$$a_{n,k} = f(n-1, k) = \frac{n-k}{n} \binom{n-1+k}{k}.$$

Proof. For $n, k \geq 0$ let $a_{0,0} = 1$ and $a_{n,k} = 0$ if $k \geq n$ or $k < 0$. It suffices to prove the following recurrence relation for $a_{n,k}$:

$$a_{n+1,k} = a_{n+1,k-1} + a_{n,k}. \quad (2.43)$$

First note two useful facts for any $\pi \in \mathbb{A}_{2n}(231)$.

- a) $\text{fl } \pi = 1$, since otherwise $(\pi(1), \pi(2), 1)$ will form a 231 pattern.
- b) $\pi(1) < \pi(3) < \dots < \pi(2n-1)$, i.e., the valleys of π form an increasing subsequence.

Due to fact a), we can assume $\text{fl } \pi^r = \pi(2n) > 1$. Now we decompose $\mathfrak{a}_{n,k}$ as the union of two disjoint subsets:

$$\begin{aligned} \mathfrak{a}_{n,k}^p &:= \{\pi \in \mathfrak{a}_{n,k} : \pi(2n) - 1 \text{ is a peak}\}, \\ \mathfrak{a}_{n,k}^v &:= \{\pi \in \mathfrak{a}_{n,k} : \pi(2n) - 1 \text{ is a valley}\}. \end{aligned}$$

We proceed to show that $|\mathfrak{a}_{n+1,k}^p| = |\mathfrak{a}_{n+1,k-1}|$ and $|\mathfrak{a}_{n+1,k}^v| = |\mathfrak{a}_{n,k}|$ via two bijections $\alpha : \mathfrak{a}_{n+1,k}^p \rightarrow \mathfrak{a}_{n+1,k-1}$ and $\beta : \mathfrak{a}_{n+1,k}^v \rightarrow \mathfrak{a}_{n,k}$, and thus proving (2.43).

The first map α is relatively easier. For any $\pi \in \mathfrak{a}_{n+1,k}^p$, we get its image $\alpha(\pi)$ by switching the position of two peaks $\pi(2n)$ and $\pi(2n) - 1$. A moment of reflection should reveal that $\alpha : \mathfrak{a}_{n+1,k}^p \rightarrow \mathfrak{a}_{n+1,k-1}$ is indeed well-defined and bijective.

We have to lay some ground work for the second map $\beta : \mathfrak{a}_{n+1,k}^v \rightarrow \mathfrak{a}_{n,k}$. The key observation is on the last three letters. We claim that for any $\pi \in \mathfrak{a}_{n+1,k}^v$,

$$\pi(2n) = \pi(2n+2) + 1, \quad \pi(2n+1) = \pi(2n+2) - 1. \quad (2.44)$$

First we see $2n+2 \neq \pi(2n+2)$, since otherwise $2n+1 = \pi(2n+2) - 1$ cannot be a valley. So $2n+2$ must be a non-terminal peak. Now notice that $2n+1$ cannot appear to the left of $2n+2$, otherwise it will cause a 231 pattern. It must also be a peak, since there are no other letters larger than it except for $2n+2$. If $2n+1 = \pi(2n+2)$ is the last peak, then $2n$ being a valley forces $(\pi(2n), \pi(2n+1), \pi(2n+2)) = (2n+2, 2n, 2n+1)$, which means (2.44) holds true. Otherwise $2n+1$ is a non-terminal peak and we consider $2n$ next. This deduction must end in finitely many steps since the total number of peaks is n (and finite). At this ending moment we find some m as the last peak, and $2n+2, 2n+1, \dots, m+1$ are all peaks decreasingly ordered to its left, then $m-1$ being a valley, together with fact b) force us to have (2.44) again. So the claim is proved.

The definitions and validity of β and its inverse become transparent, in view of (2.44).

β : For $\pi \in \mathfrak{a}_{n+1,k}^v$, delete $\pi(2n+1)$ and $\pi(2n+2)$, then decrease the remaining letters larger than $\pi(2n+2)$ by 2.

β^{-1} : For $\sigma \in \mathfrak{a}_{n,k}$, increase the letters no less than $\sigma(2n)$ by 2, and append two letters $\sigma(2n)$ and $\sigma(2n) + 1$ to the right of σ , in that order.

The proof ends here and we give the following example for illustration. □

Table 2.4: The (-1) -evaluation over $\mathfrak{S}_n(\tau)$ and $\mathfrak{D}_n(\tau)$ with respect to exc.

exc $\setminus \tau$	123	132	213	231	312	321
\mathfrak{S}_{2n+1}	*	$(-1)^n C_n$	$(-1)^n C_n$	*	*	$(-1)^n C_n$
\mathfrak{D}_{2n}	Conjecture 2.32	$(-1)^n C_n$	$(-1)^n C_n$	*	*	$(-1)^n C_n$

Example 2.30. The two bijections $\alpha : \mathfrak{a}_{n+1,k}^p \rightarrow \mathfrak{a}_{n+1,k-1}$ and $\beta : \mathfrak{a}_{n+1,k}^v \rightarrow \mathfrak{a}_{n,k}$ for the case of $n = 3$ are shown below.

$$\begin{array}{l}
\mathfrak{a}_{4,3}^p \left\{ \begin{array}{l} 13254768 \\ 13274658 \\ 15243768 \\ 17243658 \\ 17263548 \end{array} \right\} \xrightarrow{\alpha} \left\{ \begin{array}{l} 13254867 \\ 13284657 \\ 15243867 \\ 18243657 \\ 18263547 \end{array} \right\} \mathfrak{a}_{4,2} \\
\mathfrak{a}_{4,2}^p \left\{ \begin{array}{l} 13284657 \\ 18243657 \\ 18263547 \end{array} \right\} \xrightarrow{\alpha} \left\{ \begin{array}{l} 13284756 \\ 18243756 \\ 18273546 \end{array} \right\} \mathfrak{a}_{4,1} \\
\mathfrak{a}_{4,1}^p \left\{ 18273546 \right\} \xrightarrow{\alpha} \left\{ 18273645 \right\} \mathfrak{a}_{4,0} \\
\mathfrak{a}_{4,2}^v \left\{ \begin{array}{l} 13254867 \\ 15243867 \end{array} \right\} \xrightarrow{\beta} \left\{ \begin{array}{l} 132546 \\ 152436 \end{array} \right\} \mathfrak{a}_{3,2} \\
\mathfrak{a}_{4,1}^v \left\{ \begin{array}{l} 13284756 \\ 18243756 \end{array} \right\} \xrightarrow{\beta} \left\{ \begin{array}{l} 132645 \\ 162435 \end{array} \right\} \mathfrak{a}_{3,1} \\
\mathfrak{a}_{4,0}^v \left\{ 18273645 \right\} \xrightarrow{\beta} \left\{ 162534 \right\} \mathfrak{a}_{3,0}
\end{array}$$

2.5.3 Other exc-cases avoiding one pattern of length three

In this subsection we present the parallel (-1) -phenomena with respect to exc, note the differences when one compares Table 2.4 with Table 2.3. Unfortunately we have not found any q -analogues at this moment.

Theorem 2.31. For any $n \geq 1$,

$$\sum_{\pi \in \mathfrak{S}_n(213)} (-1)^{\text{exc } \pi} = \sum_{\pi \in \mathfrak{S}_n(132)} (-1)^{\text{exc } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases} \quad (2.45)$$

$$\sum_{\pi \in \mathfrak{D}_n(213)} (-1)^{\text{exc } \pi} = \sum_{\pi \in \mathfrak{D}_n(132)} (-1)^{\text{exc } \pi} = \begin{cases} (-1)^{\frac{n}{2}} C_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \quad (2.46)$$

Proof. We first apply the $q = 1$ case of Theorem 2.4, and the following identity due to Elizalde [34] to derive the second equalities in both (2.45) and (2.46).

$$\sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{exc } \pi} y^{\text{fix } \pi} = \sum_{\pi \in \mathfrak{S}_n(132)} t^{\text{exc } \pi} y^{\text{fix } \pi}, \quad \text{for } n \geq 1. \quad (2.47)$$

Next we observe the following facts, which can be easily checked.

$$\begin{aligned}
\pi \in \mathfrak{S}_n(132) &\Leftrightarrow \pi^{rc} \in \mathfrak{S}_n(213), \\
\text{exc}(\pi) &= n - \text{exc}(\pi^{rc}) - \text{fix}(\pi), \quad \text{fix}(\pi) = \text{fix}(\pi^{rc}).
\end{aligned}$$

Consequently we have

$$\sum_{\pi \in \mathfrak{S}_n(132)} t^{\text{exc } \pi} y^{\text{fix } \pi} = t^n \sum_{\pi \in \mathfrak{S}_n(213)} t^{-\text{exc } \pi - \text{fix } \pi} y^{\text{fix } \pi}. \quad (2.48)$$

Plugging in $t = -1, y = 0$ gives us directly the first equality in (2.46). Finally, taking $t = y = -1$ in (2.48), (2.47) and $t = -1, q = 1$ in (2.24) leads to:

$$(-1)^n \sum_{\pi \in \mathfrak{S}_n(213)} (-1)^{\text{exc } \pi} = \sum_{\pi \in \mathfrak{S}_n(321)} (-1)^{\text{wex } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases}$$

which is exactly the first equality in (2.45). \square

The only non- \star entry in Table 2.4 that is not covered by Theorems 2.4 or 2.31 is still a conjecture.

Conjecture 2.32. *For any $n \geq 1$, the polynomials $G_n(t) := \sum_{\pi \in \mathfrak{D}_n(123)} t^{\text{exc } \pi}$ have nonnegative coefficients in their γ -expansions. Moreover, there is a sequence $\{F_n\}_{n \geq 1}$ of positive integers such that*

$$G_n(-1) = \sum_{\pi \in \mathfrak{D}_n(123)} (-1)^{\text{exc } \pi} = \begin{cases} (-1)^{\frac{n}{2}} F_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

We note that neither $\{G_n(1)\}_{n \geq 1}$ nor $\{F_n\}_{n \geq 1}$ is registered in the OEIS. The first values are given by $G_n(1) = 0, 1, 2, 7, 20, 66, 218, 725, \dots$ and $F_n = 1, 7, 58, 545, 5570, \dots$. For the first few $n \geq 1$, we have

$$\begin{aligned} G_1(t) &= 0, G_2(t) = t, \\ G_3(t) &= t + t^2 = t(1 + t), \quad G_4(t) = 7t^2, \\ G_5(t) &= 10t^2 + 10t^3 = 10t^2(1 + t), \\ G_6(t) &= 2t^2 + 62t^3 + 2t^4 = 2t^2(1 + t)^2 + 58t^3, \\ G_7(t) &= 109t^3 + 109t^4 = 109t^3(1 + t), \\ G_8(t) &= 45t^3 + 635t^4 + 45t^5 = 45t^3(1 + t)^2 + 545t^4, \\ G_9(t) &= 5t^3 + 1264t^4 + 1264t^5 + 5t^6 = 5t^3(1 + t)^3 + 1249t^4(1 + t), \\ G_{10}(t) &= 769t^4 + 7108t^5 + 769t^6 = 769t^4(1 + t)^2 + 5570t^5. \end{aligned}$$

The symmetry of $G_n(t)$ follows from the map $\pi \mapsto \pi^{rc}$, which is stable on $\mathfrak{S}_n(123)$ and $\mathfrak{D}_n(123)$, and satisfies $\text{exc}(\pi) = n - \text{exc}(\pi^{rc}) - \text{fix}(\pi)$. Thus, if $\pi \in \mathfrak{D}_n(123)$, we obtain the symmetry.

2.6 Two cases avoiding two patterns of length four

We first enumerate $\mathbb{A}_n(2413, 3142)$ and $\mathbb{A}_n(1342, 2431)$, then put these results in the context of (-1) -evaluations of the descent polynomials over $\mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n(1342, 2431)$.

Letting $q = 1$ in the last interpretation (2.6) of Theorem 2.2 we derive immediately the following γ -expansion for Narayana polynomials (see also [83, Chapter 4]).

$$C_n(t, 1) = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des } \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^N t^k (1+t)^{n-1-2k}, \quad (2.49)$$

where $\gamma_{n,k}^N := \gamma_{n,k}(1)$. The following two γ -expansions (2.50) and (2.51), which were obtained recently by Fu-Lin-Zeng [55] and Lin[72], respectively, will be crucial in our (-1) -evaluations.

$$S_n(t) := \sum_{\pi \in \mathfrak{S}_n(2413,3142)} t^{\text{des } \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^S t^k (1+t)^{n-1-2k}, \quad (2.50)$$

$$Y_n(t) := \sum_{\pi \in \mathfrak{S}_n(1342,2431)} t^{\text{des } \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^Y t^k (1+t)^{n-1-2k}, \quad (2.51)$$

where

$$\gamma_{n,k}^S = \#\{\pi \in \mathfrak{S}_n(2413,3142) : \text{dd } \pi = 0, \text{des } \pi = k\}, \quad (2.52)$$

$$\gamma_{n,k}^Y = \#\{\pi \in \mathfrak{S}_n(1342,2431) : \text{dd } \pi = 0, \text{des } \pi = k\}. \quad (2.53)$$

It follows that

$$|\mathbb{A}_n(2413,3142)| = \gamma_{n, \lfloor \frac{n-1}{2} \rfloor}^S, \quad |\mathbb{A}_n(1342,2431)| = \gamma_{n, \lfloor \frac{n-1}{2} \rfloor}^Y. \quad (2.54)$$

Recall the γ -coefficients in the expansions (2.49), (2.50)–(2.51). For $* = N, S, Y$, let

$$\Gamma_*(x, z) := \sum_{n=1}^{\infty} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^* x^k z^n$$

be the generating functions for $\gamma_{n,k}^N, \gamma_{n,k}^S$ and $\gamma_{n,k}^Y$, respectively. We need the following two algebraic equations for $\Gamma_S(x, z)$ and $\Gamma_Y(x, z)$, which were first derived by Lin [72].

$$\Gamma_S = z + z\Gamma_S + xz\Gamma_S^2 + x\Gamma_S^3, \quad (2.55)$$

$$\Gamma_Y = z + z\Gamma_Y + 2xz\Gamma_N\Gamma_Y + x\Gamma_N^2(\Gamma_Y - z). \quad (2.56)$$

2.6.1 The case of (2413,3142)-avoiding alternating permutations.

Theorem 2.33. *Let $r_n := |\mathbb{A}_{2n+1}(2413,3142)|$, $n \geq 0$, $R(x) := \sum_{n=1}^{\infty} r_n x^n$, then*

$$R(x) = x(R(x) + 1)^2 + x(R(x) + 1)^3. \quad (2.57)$$

Consequently, $r_0 = 1$ and for $n \geq 1$,

$$r_n = \frac{2}{n} \sum_{i=0}^{n-1} 2^i \binom{2n}{i} \binom{n}{i+1}. \quad (2.58)$$

Proof. First, (2.54) gives us $r_n = \gamma_{2n+1,n}^S$. Therefore, in order to get a recurrence relation for r_n , we should extract the coefficient of z^{2n+1} in (2.55) and then compare the coefficients of x^n from both sides. This gives us, for $n \geq 1$,

$$r_n = [x^{n-1}] ([z^{2n}] \Gamma_S^2(x, z)) + [x^{n-1}] ([z^{2n+1}] \Gamma_S^3(x, z)).$$

Now we take a closer look at $[z^{2n}] \Gamma_S^2(x, z)$.

$$[z^{2n}] \Gamma_S^2(x, z) = \sum_{m=1}^{2n-1} \left(\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \gamma_{m,j}^S x^j \right) \cdot \left(\sum_{k=0}^{\lfloor \frac{2n-m-1}{2} \rfloor} \gamma_{2n-m,k}^S x^k \right),$$

So for each term in this summation, the power of x is

$$j + k \leq \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{2n-m-1}{2} \right\rfloor \leq n-1.$$

Hence we get contributions for x^{n-1} only from odd m 's, with j and k both achieving their maxima. Similar analysis applies to the term involving Γ_S^3 and the details are omitted. All these amount to

$$r_n = \sum_{m=0}^{n-1} r_m r_{n-m-1} + \sum_{m,l=0}^{n-1} r_m r_l r_{n-m-l-1}.$$

In terms of the generating function $R(x)$, we obtain (2.57). Next we rewrite (2.57) as

$$x = \frac{R}{(R+1)^2(R+2)}, \quad (2.59)$$

which is ripe for applying the Lagrange inversion (cf. [45]). A straightforward computation leads to (2.58) and completes the proof. \square

Theorem 2.34. *Let $t_n := |\mathbb{A}_{2n}(2413, 3142)|$, $n \geq 1$, $T(x) := \sum_{n=1}^{\infty} t_n x^n$, then*

$$\frac{1}{2}R(x) = \frac{1}{2}R(x) \cdot T(x) + T(x). \quad (2.60)$$

Consequently, $t_1 = 1$ and for $n \geq 2$,

$$t_n = \frac{4}{n-1} \sum_{i=0}^{n-2} 2^i \binom{2n-1}{i} \binom{n-1}{i+1}. \quad (2.61)$$

Proof. While it might be possible to establish (2.60) algebraically from (2.55), we present a combinatorial argument, showing both sides generate the same set of permutations.

The first thing to notice is that for an alternating permutation $\pi \in \mathbb{A}_{2n+1}(2413, 3142)$, $n \geq 1$, its reverse $\pi^r \neq \pi$ is also in $\mathbb{A}_{2n+1}(2413, 3142)$. This implies that r_n is even for $n \geq 1$. Moreover, we call a permutation $\pi \in \mathfrak{S}_n$, $n \geq 2$ *normal* if 1 appears to the left of n . For example, there are three normal permutations in \mathfrak{S}_3 : 213, 123, 132. Now we see that exactly one permutation in the pair $\{\pi, \pi^r\}$ is normal, and consequently the number of normal permutations in $\mathbb{A}_{2n+1}(2413, 3142)$ is $r_n/2$. Therefore the left-hand side of (2.60) generates all normal, alternating, and (2413, 3142)-avoiding permutations of odd length larger than 1. Next we show that the right-hand side does precisely the same. The following claimed decomposition, whose proof given separately, is the key ingredient. Recall two classical operations, the *direct sum* “ \oplus ” and the *skew sum* “ \ominus ”. If $\pi = \pi^{(1)}\pi^{(2)}$ with $\pi^{(1)} < \pi^{(2)}$, then we write $\pi = \pi^{(1)} \oplus \text{st}(\pi^{(2)})$. Similarly, if $\pi = \pi^{(1)}\pi^{(2)}$ with $\pi^{(1)} > \pi^{(2)}$, then we write $\pi = \text{st}(\pi^{(1)}) \ominus \pi^{(2)}$. For instance, we have $12354 = 123 \oplus 21$ and $34521 = 123 \ominus 21$.

Claim 2.35. *Let π be a normal, alternating, and (2413, 3142)-avoiding permutation of odd length larger than 1, then there exists a unique pair of permutations $(\pi^{(1)}, \pi^{(2)})$, such that*

1. $\pi = \pi^{(1)} \oplus \pi^{(2)}$,
2. either $\pi^{(1)} = 1$ or $\pi^{(1)}$ is of odd length and non-normal, alternating and (2413, 3142)-avoiding,

3. $\pi^{(2)}$ is of even length (≥ 2) and (2413, 3142)-avoiding, its reverse is alternating.

In view of the claim above, $\frac{1}{2}R(x) \cdot T(x)$ accounts for the cases when $\pi^{(1)}$ is of length 3 or longer, while $T(x)$ corresponds to the case when $\pi^{(1)} = 1$. Now since the above decomposition using \oplus is unique, we get (2.60).

Applying (2.59), we can rewrite (2.60) as

$$T = \frac{R}{R+2} = x(R+1)^2.$$

This form is suitable for the more general Lagrange-Bürmann formula (cf. [45]), and we get for $n \geq 2$,

$$\begin{aligned} t_n &= [x^{n-1}](R+1)^2 = \frac{1}{n-1}[R^{n-2}] (2(R+1)(R+1)^{2n-2}(R+2)^{n-1}) \\ &= \frac{2}{n-1} \sum_{i=0}^{n-2} 2^{n-1-i} \binom{n-1}{i} \binom{2n-1}{n-2-i} \\ &= \frac{4}{n-1} \sum_{i=0}^{n-2} 2^i \binom{n-1}{i+1} \binom{2n-1}{i}. \end{aligned}$$

The proof is now completed. \square

Proof of Claim 2.35. A permutation avoids both 2413 and 3142 if and only if it is *separable* (cf. [63, page 57]), which means it can be decomposed as either $\pi = \pi^{(1)} \oplus \pi^{(2)}$ or $\pi = \pi^{(1)} \ominus \pi^{(2)}$. Now π being normal excludes the case of $\pi = \pi^{(1)} \ominus \pi^{(2)}$. Such \oplus -decomposition may not be unique. To make it unique as claimed, we always take the decomposition where $\pi^{(1)}$ is shortest in length. This also means $\pi^{(1)}$ itself cannot be \oplus -decomposed further. Therefore $\pi^{(1)} = 1$ or $\pi^{(1)}$ can be \ominus -decomposed and thus non-normal. Being subwords of π , $\pi^{(1)}$ and $\pi^{(2)}$ should avoid 2413 and 3142 as well. The remaining restrictions on $\pi^{(1)}$ and $\pi^{(2)}$ can be verified easily. \square

Remark 2.4. In view of the similarity in the expressions for r_n and t_n , we can unify them as the following formula:

$$|\mathbb{A}_n(2413, 3142)| = \frac{2^{n-2m}}{m} \sum_{i=0}^{m-1} 2^i \binom{m}{i+1} \binom{n-1}{i}, \text{ where } m = \left\lfloor \frac{n-1}{2} \right\rfloor, \text{ and } n \geq 3.$$

Moreover, the two sequences $\{r_n\}_{n \geq 0}$ and $\{t_n\}_{n \geq 1}$ have been cataloged in the OEIS (see [oeis:A027307](#) and [oeis:A032349](#)), and were considered, for instance, by Deutsch et al. [30] as enumerating certain type of lattice paths. Then a natural question would be to find a bijection between these two combinatorial models.

Now we turn to the (-1) -evaluation for $S_n(t)$, which is a direct result of (2.50) and (2.54).

Theorem 2.36. *For any $n \geq 1$, the following holds*

$$S_n(-1) = \sum_{\pi \in \mathfrak{S}_n(2413, 3142)} (-1)^{\text{des } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} r_{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases} \quad (2.62)$$

2.6.2 The case of (1342,2431)–avoiding alternating permutations.

Theorem 2.37. Let $u_n := |\mathbb{A}_{2n+1}(1342, 2431)|$ and $U(x) := \sum_{n=0}^{\infty} u_n x^n$, then

$$U(x) = \frac{\sqrt{1-4x}}{\sqrt{1-4x}-2x} = \frac{1}{1 - \frac{2x}{1 - \frac{2x}{1 - \frac{x}{1 - \frac{x}{\ddots}}}}}}. \quad (2.63)$$

Proof. We only sketch the proof since it is quite analogous to that of Theorem 2.33. We use (2.56) in a similar way as we use (2.55) in the proof of (2.57), i.e., we extract the coefficients of z^{2n+1} from both sides and then compare the coefficients of x^n . This leads to the following recurrence relation that involves the Catalan number C_n , since we have already shown that $\gamma_{2n+1, n}^N = |\mathbb{A}_{2n+1}(231)| = C_n$. For $n \geq 1$, we have:

$$\begin{aligned} u_n &= 2 \sum_{m=0}^{n-1} u_m C_{n-1-m} + \sum_{m=1}^{n-1} u_m \sum_{l=0}^{n-m-1} C_l C_{n-m-l-1} \\ &= 2 \sum_{m=0}^{n-1} u_m C_{n-1-m} + \sum_{m=1}^{n-1} u_m C_{n-m}. \end{aligned}$$

In terms of generating functions, this means

$$2(U(x) - 1) = 2xU(x)C(x) + (U(x) - 1)C(x),$$

where

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

is the generating function for the Catalan numbers. We plug in $C(x)$ and solve for $U(x)$ to finish the proof. \square

Remark 2.5. Interestingly, our result above seems to be the first combinatorial interpretation for u_n , and the sequence $\{u_n\}_{n \geq 0}$ is also on OEIS (see [oeis:A084868](#)). Although a single sum formula for u_n can be derived from (2.63) by using standard method, we prefer to give a multiple sum formula as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} u_n x^n &= \frac{1}{1 - \frac{2x}{\sqrt{1-4x}}} = \sum_{m=0}^{\infty} \left(\frac{2x}{\sqrt{1-4x}} \right)^m \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\infty} 2 \binom{2k}{k} x^{k+1} \right)^m. \end{aligned}$$

Thus we obtain, for $n \geq 1$,

$$u_n = \sum_{m=1}^n 2^m \sum_{k_1 + \dots + k_m = n-m} \prod_{i=1}^m \binom{2k_i}{k_i}. \quad (2.64)$$

The above formula shows that u_n is a multiple of 4 when $n \geq 2$.

With the aid of (2.51) and (2.54), we obtain

Theorem 2.38. *For any $n \geq 1$, the following holds*

$$Y_n(-1) = \sum_{\pi \in \mathfrak{S}_n(1342, 2413)} (-1)^{\text{des } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} u_{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases} \quad (2.65)$$

We end this section by noting that both $\mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n(1342, 2431)$ exhibit (-1) -phenomenon only for the entire set of permutations, but not for the subset of coderangements. This should not come as a surprise in view of the reversal relations between the two patterns that we avoid, namely $(2413)^r = 3142$, $(1342)^r = 2431$, and the fact that the definition of coderangements is incompatible with the reverse map. Other subsets of \mathfrak{S}_n instead of \mathfrak{D}_n^* should be examined to hunt for the other half of the (-1) -phenomenon.

2.7 Final remarks

It would be interesting to give direct combinatorial proofs of the (-1) -phenomena of this chapter. The expansions we have in Theorems 2.2 and 2.3, Lemma 2.27 are all natural, in the sense that the statistics (powers of q) appear in the γ -coefficients on the expansion side, are the same as those that appear on the left-hand side, the avoiding patterns are also the same. And we prove them uniformly using the MFS-action and its variation. Each interpretation listed in Theorem 2.1 (resp. Theorem 2.20) should have a q - γ -expansion in theory. Namely, once we have an expansion for one of them, the others all share this expansion. But expansions derived this way are unnatural (for instance, (2.24) is unnatural). So now the question is, do the other ones that we are missing in Theorem 2.2 (to be precise, in Tabel 2.1) have natural expansions? It seems the MFS-action cannot help anymore.

It would be appealing to establish a multivariate generating function (in the spirit of Shin-Zeng's Lemma 2.6) that specializes to the $(2413, 3142)$ -avoiding permutations or $(1342, 2413)$ -avoiding permutations, and consequently giving us q -analogues of (2.62) or (2.65).

Another direction to extend the results presented here is to place \mathfrak{S}_n in the broader context of Coxeter groups, and consider the so-called Narayana polynomials of types B and D (see [3, Theorems 2.32 and 2.33]). This approach was shown fruitful for permutations in a recent work of Eu, Fu, Hsu and Liao [36].

Finally, in a different context, some (-1) -phenomenon have been generalized to the deeper *cyclic sieving phenomenon* (CSP), see [86, 91]. It would be interesting to see whether there are any CSP-analogue for our (-1) -phenomenon.

Chapter 3

Elliptic Functions, Continued Fractions and Gamma Positivity

3.1 Introduction

The Jacobi elliptic function $\operatorname{sn}(z, x)$ is defined by the inverse of an elliptic integral, i.e.,

$$\operatorname{sn}(z, x) = y \text{ iff } z = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}},$$

where $x \in (0, 1)$ is fixed *modulus*. The other two Jacobi elliptic functions are defined by

$$\begin{aligned} \operatorname{cn}(z, x) &:= \sqrt{1 - \operatorname{sn}^2(z, x)}, \\ \operatorname{dn}(z, x) &:= \sqrt{1 - x^2 \operatorname{sn}^2(z, x)}. \end{aligned}$$

The first few terms of Taylor series expansions of these Jacobian elliptic functions are given [44],

$$\operatorname{sn}(z, x) = z - (1+x^2)\frac{z^3}{3!} + (1+14x^2+x^4)\frac{z^5}{5!} - (1+135x^2+135x^4+x^6)z^7 + \dots, \quad (3.1)$$

$$\operatorname{cn}(z, x) = 1 - \frac{z^2}{2!} + (1+4x^2)\frac{z^4}{4!} - (1+44x^2+16x^4)\frac{z^6}{6!} + \dots, \quad (3.2)$$

$$\operatorname{dn}(z, x) = 1 - x^2\frac{z^2}{2!} + x^2(4+x^2)\frac{z^4}{4!} - x^2(16+44x^2+x^4)\frac{z^6}{6!} + \dots. \quad (3.3)$$

Rogers and Stieltjes [89, 104] considered the following Laplace-Borel transforms of sn and cn :

$$S_1(z, x) = \int_0^\infty e^{-t} \operatorname{sn}(zt, x) dt \text{ and } C_0(z, x) = \int_0^\infty e^{-t} \operatorname{cn}(zt, x) dt,$$

i.e., the series obtained from (3.1) and (3.2) by replacing $z^n/n!$ by z^n , have the following continued fractions expansions,

$$S_1(z, x) = \frac{z}{1 + (1+x^2)z^2 - \frac{z}{1 + (1+x^2)3^2z^2 - \frac{3 \cdot 4^2 \cdot 5 \cdot x^2 z^4}{1 + (1+x^2)5^2z^2 - \dots}}}, \quad (3.4)$$

$$C_0(z, x) = \frac{1}{1 + z^2 - \frac{1^2 \cdot 2^2 \cdot x^2 z^4}{1 + (3^2 + 2^2 x^2)z^2 - \frac{3^2 \cdot 4^2 \cdot x^2 z^4}{1 + (5^2 + 4^2 x^2)z^2 - \dots}}}. \quad (3.5)$$

According to [44], the question of the possible combinatorial significance of the coefficients of

$$(-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (\text{resp. } (-1)^n \frac{z^{2n}}{(2n)!})$$

in the Taylor Series expansions of Jacobi elliptic functions $\text{sn}(z, x)$ (resp. $\text{cn}(z, x)$) was first raised by Schützenberger. The first combinatorial interpretation was given by Viennot [106], and is expressed in terms of so-called Jacobi permutations. Flajolet [43] has shown the coefficients of $\text{cn}(z, x)$ to count classes of alternating (up-and-down) permutations based on the parity of peaks. Dumont [31] finally discovered some further relations between these functions and the cycle structure of permutations. Flajolet-Françon [44] gave an interpretation of the elliptic functions as generating functions of *double permutations*.

A polynomial $f(x) = \sum_i a_i x^i \in \mathfrak{R}[x]$ is called γ -positive if $f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$ for $n \in \mathbb{N}$ and nonnegative reals $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor n/2 \rfloor}$. The notion of γ -positivity appeared first in the work of Foata and Schützenberger [51], a recent survey on γ -positivity in combinatorics and geometry was given by Athanasiadis [3]. In a series of papers Shin and Zeng [95, 96, 97] exploited the combinatorial theory of continued fractions to derive various γ -positivity results.

In this chapter, we study a class of permutations introduced by Flajolet and Françon to give combinatorial interpretations for the Taylor coefficients of Jacobian elliptic functions. By considering two more general continued fractions of Rogers and Stieltjes, we provide combinatorial interpretations of the corresponding Taylor coefficients as well as their gamma coefficients.

3.2 Main results

In this chapter, we define the variant labels for some notations and the nomenclature of various permutation statistics of pervious chapters, other statistics use the same labels as mentioned before.

For $\pi \in \mathfrak{S}_n$, consider the *ordinal type* any entry $\pi(i)$ ($i \in [n]$) can be classified into four categories (cf. Subsection 1.2.1), we give the following definition.

Definition 3.1. A permutation is a *Doubled Permutation* iff for all i , elements (i.e. values) $2i+1$ and $2i+2$ have the same ordinal type. The set of doubled permutations is denoted by \mathcal{DP}_n .

Define $J_{2n+1}(p, q, x, u, v, w)$ the coefficients in the following continued fraction expansion,

$$\begin{aligned} & \sum_{n \geq 0} (-1)^n J_{2n+1}(p, q, x, u, v, w) z^{2n+1} \\ &= \frac{z}{1 + (u^2 + x^2 v^2) [1]_{p,q}^2 z^2 - \frac{[1]_{p,q} [2]_{p,q}^2 [3]_{p,q} x^2 w^2 z^4}{1 + (u^2 + x^2 v^2) [3]_{p,q}^2 z^2 - \frac{[3]_{p,q} [4]_{p,q}^2 [5]_{p,q} x^2 w^2 z^4}{\dots}}}, \quad (3.6) \end{aligned}$$

where $[n]_{p,q} = (p^n - q^n)/(p - q)$. Then we give the following combinatorial interpretation of the coefficients in the continued fraction expansions (3.6).

Theorem 3.1. *We have*

$$J_{2n+1}(p, q, x, u, v, w) := \sum_{\pi \in \mathcal{DP}_{2n+1}} p^{(2-13)\pi} q^{(31-2)\pi} x^{\text{des } \pi} u^{\text{da } \pi} v^{\text{dd } \pi} w^{\text{valley } \pi} \quad (3.7)$$

Remark 3.1. Let $J_{2n+1}(x) := J_{2n+1}(1, 1, x, 1, 1, 1)$, Eq. (3.7) reduces to Flajolet-Françon's result [44]. In other words,

$$\text{sn}(z, x) := \sum_{n \geq 0} (-1)^n J_{2n+1}(x) \frac{z^{2n+1}}{(2n+1)!}.$$

Define

$$\mathcal{DP}_{2n+1,2k} := \{\pi \in \mathcal{DP}_{2n+1}, \text{dd } \pi = 0, \text{des } \pi = 2k\}.$$

We have the double analogue of Eulerian polynomials expansion formula as follows.

Theorem 3.2.

$$J_{2n+1}(p, q, x, u, v, w) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{2n+1,2k}(p, q) (xw)^{2k} (u^2 + v^2 x^2)^{n-2k}, \quad (3.8)$$

where

$$a_{2n+1,2k}(p, q) = \sum_{\pi \in \mathcal{DP}_{2n+1,2k}} p^{(2-13)\pi} q^{(31-2)\pi}. \quad (3.9)$$

Moreover, for all $0 \leq k \leq \lfloor n/2 \rfloor$, the following divisibility holds

$$(p+q)^{2k} \mid a_{2n+1,2k}(p, q). \quad (3.10)$$

In particular, the coefficients in the Taylor series expansion of Jacobi elliptic function $\text{sn}(z, x)$ have the following γ -expansions.

Corollary 3.3. *For all $n \geq 1$, we have*

$$J_{2n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} |\mathcal{DP}_{2n+1,2i}| x^{2i} (1+x^2)^{n-2i}. \quad (3.11)$$

The second goal of this chapter is to explore the coefficients in the Taylor series expansion of Jacobi elliptic function $\text{cn}(z, x)$ by generalizing the continued fractions of (3.5).

For $\pi \in \mathfrak{S}_n$, consider the *cyclic ordinal type* any entry $\pi(i)$ ($i \in [n]$) can be classified into four categories such as cyclic peak, cyclic valley, double excedance and fixed point, double drop (cf. Subsection 1.2.1) we give the following definition.

Definition 3.2. A permutation π is said to be a *Cyclic Doubled Permutation of the First Kind* iff for all i , elements (i.e. values) $2i+2$ and $2i+3$ in π have the same cyclic ordinal type. The set of first cyclic doubled permutations is denoted by \mathcal{FCDP}_n . A permutation π is said to be a *Cyclic Doubled Permutation of the Second Kind* iff for all i , elements (i.e. values) $2i+1$ and $2i+2$ in π have the same cyclic ordinal type. The set of second cyclic doubled permutations is denoted by \mathcal{SCDP}_n .

The Clarke-Steingrímsson-Zeng [24, 96] constructed a bijection Φ from des based statistics with exc based ones for $\pi \in \mathfrak{S}_n$, we show that the bijection Φ has following property from $\sigma \in \mathcal{DP}_{2n}$ to $\Phi(\sigma) \in \mathcal{SCDP}_{2n}$.

Theorem 3.4. *For $n \geq 1$, there exists a bijection $\Phi : \mathcal{DP}_{2n} \mapsto \mathcal{SCDP}_{2n}$ such that*

$$\begin{aligned} & (2\text{-}31, 31\text{-}2, \text{des}, \text{lda} - \text{fmax}, \text{ldd}, \text{lvalley}, \text{fmax}) \sigma \\ & = (\text{nest}, \text{icr}, \text{drop}, \text{cda}, \text{cdd}, \text{cvalley}, \text{fix}) \Phi(\sigma). \end{aligned} \quad (3.12)$$

Define $J_{2n}(p, q, x, u, v, w, y)$ the coefficients in the following continued fraction expansion,

$$\sum_{n \geq 0} (-1)^n J_{2n}(p, q, x, u, v, w, y) z^{2n} = \frac{1}{1 + b_0 z^2 - \frac{\lambda_1 z^4}{1 + b_1 z^2 - \frac{\lambda_2 z^4}{1 + b_2 z^2 \dots}}}, \quad (3.13)$$

where $b_n = (p^{2n}y + qu[2n]_{p,q})^2 + x^2v^2[2n]_{p,q}^2$ and $\lambda_n = [2n-1]_{p,q}^2[2n]_{p,q}^2x^2w^2$. Then we give the following combinatorial interpretation of the coefficients in the continued fraction expansions (3.13).

Theorem 3.5. *We have*

$$J_{2n}(p, q, x, u, v, w, y) = \sum_{\sigma \in \mathcal{SCDP}_{2n}} p^{\text{nest}} \pi q^{\text{icr}} \pi x^{\text{drop}} \pi u^{\text{cda}} \pi v^{\text{cdd}} \pi w^{\text{cvalley}} \pi y^{\text{fix}} \pi \quad (3.14)$$

$$= \sum_{\pi \in \mathcal{DP}_{2n}} p^{(2\text{-}31)} \pi q^{(31\text{-}2)} \pi x^{\text{des}} \pi u^{\text{lda}} \pi^{-\text{fmax}} \pi v^{\text{ldd}} \pi w^{\text{lvalley}} \pi y^{\text{fmax}} \pi \quad (3.15)$$

$$= \sum_{\sigma \in \mathcal{SCDP}_{2n}} p^{\text{nest}} \pi q^{\text{cros}} \pi x^{\text{exc}} \pi u^{\text{cdd}} \pi v^{\text{cda}} \pi w^{\text{cvalley}} \pi y^{\text{fix}} \pi. \quad (3.16)$$

Remark 3.2. Let $J_{2n}(x) := J_{2n}(1, 1, x, 1, 1, 1, 1)$, Eq. (3.15) reduces to Flajolet-Françon's result [44]. In other words,

$$\text{cn}(z, x) := 1 + \sum_{n \geq 1} (-1)^n J_{2n}(x) \frac{z^{2n}}{(2n)!}.$$

By taking $y = 0$ in the Theorem 3.5, we consider the derangement analogue of doubled permutations. Define $D_{2n}(p, q, x, u, v, w)$ the coefficients in the following continued fraction expansion

$$\begin{aligned} & \sum_{n \geq 0} (-1)^n D_{2n}(p, q, x, u, v, w) z^{2n} \\ & = \frac{1}{1 - \frac{[1]_{p,q}^2 [2]_{p,q}^2 x^2 w^2 z^4}{1 + [2]_{p,q}^2 (q^2 u^2 + x^2 v^2) z^2 - \frac{[3]_{p,q}^2 [4]_{p,q}^2 x^2 w^2 z^4}{\dots}}}. \end{aligned} \quad (3.17)$$

By Theorem 3.5, we give the following combinatorial interpretation of the coefficients in the continued fraction expansions (3.17).

Corollary 3.6. *We have*

$$D_{2n}(p, q, x, u, v, w) := \sum_{\pi \in \mathcal{DD}_{2n}^*} p^{(2-31)\pi} q^{(31-2)\pi} x^{\text{des } \pi} u^{\text{lda } \pi} v^{\text{ldd } \pi} w^{\text{lvalley } \pi} \quad (3.18)$$

$$= \sum_{\pi \in \mathcal{DD}_{2n}} p^{\text{nest } \pi} q^{\text{icr } \pi} x^{\text{drop } \pi} u^{\text{cda } \pi} v^{\text{cdd } \pi} w^{\text{cvalley } \pi} \quad (3.19)$$

$$= \sum_{\pi \in \mathcal{DD}_{2n}} p^{\text{nest } \pi} q^{\text{cros } \pi} x^{\text{exc } \pi} u^{\text{cdd } \pi} v^{\text{cda } \pi} w^{\text{cvalley } \pi}. \quad (3.20)$$

Where $\mathcal{DD}_{2n} := \{\pi \in \mathcal{SCDP}_{2n}, \text{fix } \pi = 0\}$ and $\mathcal{DD}_{2n}^* = \{\pi \in \mathcal{DP}_{2n}, \text{fmax } \pi = 0\}$.

Define $D_{2n}^{\text{cyc}}(\beta, x, u, v, w)$ the coefficients in the following continued fraction expansion

$$1 + \sum_{n \geq 1} (-1)^n D_{2n}^{\text{cyc}}(\beta, x, u, v, w) z^{2n} = \frac{1}{1 + b_0 z^2 - \frac{\lambda_1 x^2 w^2 z^4}{1 + b_1 z^2 - \frac{\lambda_2 x^2 w^2 z^4}{\dots}}}, \quad (3.21)$$

where, for $k \geq 0$,

$$b_k = (2k)^2(x^2 u^2 + v^2), \quad \text{and} \quad \lambda_{k+1} = (2k+1)(2k+2)(\beta+2k)(\beta+2k+1).$$

Then we give the following combinatorial interpretation of the coefficients in the continued fraction expansions (3.21).

Theorem 3.7. *We have*

$$D_{2n}^{\text{cyc}}(\beta, x, u, v, w) := \sum_{\pi \in \mathcal{DD}_{2n}} \beta^{\text{cyc } \pi} x^{\text{exc } \pi} u^{\text{cda } \pi} v^{\text{cdd } \pi} w^{\text{cvalley } \pi},$$

where $\text{cyc } \pi$ denote the number of its cycles for any permutation $\pi \in \mathcal{DD}_{2n}$.

Define the double Derangement polynomial,

$$D_{2n}(x) := \sum_{\pi \in \mathcal{DD}_{2n}} x^{\text{exc } \pi}.$$

We give the first few terms of $D_{2n}(x)$ as follows,

$$D_2(x) = 0,$$

$$D_4(x) = 4x^2,$$

$$D_6(x) = 16x^4 + 16x^2,$$

$$D_8(x) = 64x^6 + 720x^4 + 64x^2,$$

$$D_{10}(x) = 256x^8 + 14720x^6 + 14720x^4 + 256x^2,$$

$$D_{12}(x) = 1024x^{10} + 253696x^8 + 1111360x^6 + 253696x^4 + 1024x^2.$$

By observing above equations, we have the following q -analogue of double Derangement polynomials expansion formula.

Theorem 3.8. *For all positive integers n and for each statistic $\text{stat} \in \{\text{cyc}, \text{inv}, \text{nest}\}$,*

$$\sum_{\pi \in \mathcal{DD}_{2n}} q^{\text{stat } \pi} x^{\text{exc } \pi} = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{2n, 2k}(q) x^{2k} (1+x^2)^{n-2k}, \quad (3.22)$$

where

$$b_{2n,2k}(q) = \sum_{w \in \mathcal{DD}_{2n,2k}} q^{\text{stat } \pi}, \quad (3.23)$$

and $\mathcal{DD}_{2n,2k}$ consists of all elements of \mathcal{DD}_{2n} with exactly $2k$ excedances and no double excedance.

The third goal of this chapter is to give another combinatorial interpretation of the coefficients in the Taylor series expansion of Jacobi elliptic functions $\text{sn}(z, x)$ and $\text{cn}(z, x)$ by using the fundamental transformation.

Theorem 3.9. *The coefficient of*

$$(-1)^n \frac{z^{2n+1}}{(2n+1)!} x^{2k}$$

in $\text{sn}(z, x)$ counts the number of (even) first cyclic doubled permutations over $2n+2$ terminated by 1 having $2k+1$ excedances. The coefficient of

$$(-1)^n \frac{z^{2n}}{(2n)!} x^{2k}$$

in $\text{cn}(z, x)$ counts the number of (even) second cyclic doubled permutations over $2n$ having $2k$ excedances.

Remark 3.3. In this chapter, we prove above theorem by using fundamental transformation of Foata-Schützenberger, see [51, p. 13]. The bijection of Clarke-Steingrímsson-Zeng [24] can be used to give another proof.

3.3 Definitions and Preliminaries

A Motzkin path of length n in the plan $\mathbb{N} \times \mathbb{N}$ is a sequence of points (s_0, \dots, s_n) , where $s_0 = (0, 0)$, $s_i - s_{i-1} = (1, 0), (1, \pm 1)$ and $s_n = (n, 0)$. Each step (s_{i-1}, s_i) is called East (resp. North-East, South-East) if $s_i - s_{i-1} = (1, 0)$ (resp. $s_i - s_{i-1} = (1, 1)$, $s_i - s_{i-1} = (1, -1)$). The *height* of the step (s_{i-1}, s_i) is the ordinate of s_{i-1} .

Given a Motzkin path γ , for convenience we consider two types of horizontal steps, either *blue* or *red*, the set of *2-Motzkin path* of length $n \geq 1$ is denoted by \mathcal{CM}_n . Denoting the North-East step by a , the East blue step by b , the East red step by b' and the South-East step by c , see Figure 3.1 for a *2-Motzkin path*. If we weight each East blue (resp. East red, North-East, South-East) step of height i by b_i (resp. b'_i, a_i and c_i), the weight of γ is defined by the product of its step weights and denoted by $w(\gamma)$. Then,

$$\sum_{n=0}^{\infty} \sum_{\gamma \in \mathcal{CM}_n} w(\gamma) z^n = \frac{1}{1 - (b_0 + b'_0)z - \frac{a_0 c_1 z^2}{1 - (b_1 + b'_1)z - \frac{a_1 c_2 z^2}{\ddots}}}. \quad (3.24)$$

A *2-Motzkin path* is a *doubled path* if the step at odd position is always followed by a step of the same type. See Figure 3.1 for a doubled path γ , whose weight is $w(\gamma) = a_0 a_1 b_2 b_2 c_2 c_1 a_0 a_1 c_2 c_1 b'_0 b'_0$. Grouping steps 2 by 2 in a doubled path of length $2n$ yields a *2-Motzkin path* of length n , then we obtain the following lemma.

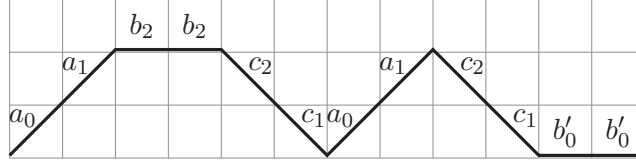


Figure 3.1: The doubled path γ can be written as the word $aabbccaacccb$.

Lemma 3.10. *If DM_{2n} is the set of doubled paths of length $2n$, then*

$$\sum_{n=0}^{\infty} \sum_{\gamma \in \text{DM}_{2n}} w(\gamma) z^n = \frac{1}{1 - (b_0^2 + b'_0)^2 z - \frac{a_0 a_1 c_2 c_1 z^2}{1 - (b_2^2 + b'_2)^2 z - \frac{a_2 a_3 c_4 c_3 z^2}{\ddots}}}, \quad (3.25)$$

A *doubled Laguerre history* of length $2n$ denoted by \mathfrak{DL}_{2n} is a couple $(\gamma, (p_1, \dots, p_{2n}))$ such that γ is a doubled path of length $2n$ and (p_1, \dots, p_{2n}) is a sequence satisfying $0 \leq p_i \leq v(s_{i-1}, s_i)$, where $v(s_{i-1}, s_i) = k$ if $s_{i-1} = (i-1, k)$.

3.4 Proof of Theorem 3.1

For $i \in [2n+1]$, define $(31-2)_k \sigma$, $(2-31)_k \sigma$ and $(2-13)_k \sigma$ for $\sigma \in \mathcal{DP}_{2n+1}^*$ by

$$\begin{aligned} (31-2)_k \sigma &= \#\{i : i+1 < j \text{ and } \sigma(i+1) < \sigma(j) = k < \sigma(i)\}, \\ (2-31)_k \sigma &= \#\{i : j < i-1 \text{ and } \sigma(i) < \sigma(j) = k < \sigma(i-1)\}, \\ (2-13)_k \sigma &= \#\{i : j < i-1 \text{ and } \sigma(i-1) < \sigma(j) = k < \sigma(i)\}. \end{aligned}$$

The numbers $l_k = (31-2)_k \sigma$ (resp. $r_k = (2-31)_k \sigma$) are called the *left embracing numbers* (resp. *right embracing numbers*) of $k \in [2n]$ in σ .

For any $\sigma \in \mathcal{DP}_{2n+1}$, the doubled Laguerre history $(s_0, \dots, s_{2n}, p_1, \dots, p_{2n})$ is constructed as follows: let $s_0 = (0, 0)$ and for $i = 1, \dots, 2n$,

- the step (s_{i-1}, s_i) is North-East if i is a valley,
- the step (s_{i-1}, s_i) is South-East if i is a peak,
- the step (s_{i-1}, s_i) is East blue if i is a double ascent,
- the step (s_{i-1}, s_i) is East red if i is a double descent.

While $p_i = (2-13)_i \sigma$ for $i = 1, \dots, 2n$. If h_i is the height of (s_{i-1}, s_i) , i.e., $s_{i-1} = (i-1, h_i)$, then $(2-13)_i \sigma + (31-2)_i \sigma = h_i$. Since $\sigma(0) = \sigma(2n+2) = 0$, so $2n+1$ must be a peak and valley $\sigma = \text{peak } \sigma - 1$. Thus $(s_0, \dots, s_{2n}, p_1, \dots, p_{2n})$ is a doubled Laguerre history of length $2n$ and

$$w(\sigma) = x^{\text{ER } \gamma + \text{NE } \gamma} u^{\text{EB } \gamma} v^{\text{ER } \gamma} w^{\text{NE } \gamma} \prod_{i=1}^{2n} p^i q^{h_i - p_i},$$

where $\text{NE } \gamma$ (resp. $\text{EB } \gamma$, $\text{ER } \gamma$) is the number of North-East steps (resp. East blue steps, East red steps) of γ . Therefore,

$$J_{2n+1}(p, q, x, u, v, w) = \sum_{\gamma \in \text{DM}_{2n}} x^{\text{ER } \gamma + \text{NE } \gamma} u^{\text{EB } \gamma} v^{\text{ER } \gamma} w^{\text{NE } \gamma} \prod_{i=1}^{2n} [h_i + 1]_{p, q}, \quad (3.26)$$

where $[n]_{p,q} = (p^n - q^n)/(p - q)$. Given a doubled path γ , the weight of each step at height k is created by

$$a_k := xw[k+1]_{p,q}, \quad b_k := u[k+1]_{p,q}, \quad b'_k := xv[k+1]_{p,q}, \quad c_k := [k+1]_{p,q}, \quad (3.27)$$

if the step is North-East, East blue, East red and South-East, respectively, and the weight of γ is defined to be the product of the step weights. Summing over all the doubled paths of length $2n$ with the rules (3.27), we have

$$J_{2n+1}(p, q, x, u, v, w) = \sum_{\gamma \in \text{DM}_{2n}} w(\gamma). \quad (3.28)$$

With Lemma 3.10 and above equation, $J_{2n+1}(p, q, x, u, v, w)$ are the coefficients in the following continued fraction expansion,

$$\begin{aligned} & \sum_{n \geq 0} J_{2n+1}(p, q, x, u, v, w) z^n \\ = & \frac{1}{1 - (u^2 + x^2v^2)[1]_{p,q}^2 z - \frac{[1]_{p,q}[2]_{p,q}^2 [3]_{p,q} x^2 w^2 z^2}{1 - (u^2 + x^2v^2)[3]_{p,q}^2 z - \frac{[3]_{p,q}[4]_{p,q}^2 [5]_{p,q} x^2 w^2 z^2}{\dots}}}, \end{aligned} \quad (3.29)$$

by transforming z to $-z^2$ and multiplying both sides by z , we obtain (3.7) immediately.

3.5 Proof of Theorem 3.2

In this section, we give a proof using continued fraction and a proof using Modified Foata-Strehl action.

3.5.1 Analytic method

In view of (3.29), for $0 \leq k \leq \lfloor n/2 \rfloor$, let $a_{2n+1,2k}(p, q, x, u, v)$ be the coefficient of w^{2k} in $J_{2n+1}(p, q, x, u, v, w)$, i.e.,

$$J_{2n+1}(p, q, x, u, v, w) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{2n+1,2k}(p, q, x, u, v) w^{2k}. \quad (3.30)$$

Transforming z and w to $\frac{z}{(u^2+x^2v^2)}$ and $\frac{w(u^2+x^2v^2)}{x}$ in (3.29), respectively, we have

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{a_{2n+1,2k}(p, q, x, u, v)}{x^{2k}(u^2 + x^2v^2)^{n-2k}} w^{2k} z^n \\ = & \frac{1}{1 - [1]_{p,q}^2 z - \frac{[1]_{p,q}[2]_{p,q}^2 [3]_{p,q} w^2 z^2}{1 - [3]_{p,q}^2 z - \frac{[3]_{p,q}[4]_{p,q}^2 [5]_{p,q} w^2 z^2}{1 - [5]_{p,q}^2 z - \frac{[5]_{p,q}[6]_{p,q}^2 [7]_{p,q} w^2 z^2}{\dots}}}}. \end{aligned} \quad (3.31)$$

Since the continued fraction expansion of (3.31) is free of variables x , u , and v , the coefficient of $w^{2k}z^n$ of the above equation is a polynomial in p and q with nonnegative integral coefficients. Denoted the coefficient of (3.31) by

$$P_{2n+1,2k}(p, q) := \frac{a_{2n+1,2k}(p, q, x, u, v)}{x^{2k}(u^2 + x^2v^2)^{n-2k}}. \tag{3.32}$$

On the other hand, taking $(p, q, x, u, v, w) = (p, q, 1, 1, 0, w)$ in (3.30), the continued fraction (3.29) becomes the right-hand side of (3.31) immediately. With the definition of $a_{2n+1,2k}$ in (3.9), we see that

$$P_{2n+1,2k}(p, q) = a_{2n+1,2k}(p, q, 1, 1, 0) = a_{2n+1,2k}(p, q). \tag{3.33}$$

Combining (3.30), (3.31) (3.32) and (3.33), this proves (3.8). Finally, since $(p + q)^2 \mid [2n - 1]_{p,q}[2n]_{p,q}^2[2n + 1]_{p,q}$ for all $n \geq 1$, in the right-hand side of (3.31), each w^2 appears with a factor $(p + q)^2$, and the polynomial $P_{2n+1,2k}(p, q)$ is divisible by $(p + q)^{2k}$. This completes the proof.

3.5.2 Group action method

Definition 3.3 (MFS-action). Let $\pi \in \mathfrak{S}_n$ with boundary condition $\pi(0) = \pi(n + 1) = 0$, for any $a \in [n]$, the a -factorization of π reads $\pi = w_1w_2aw_3w_4$, where w_2 (resp. w_3) is the maximal contiguous subword immediately to the left (resp. right) of a whose letters are all larger than a . Following Foata and Strehl [52] we define the action φ_a by

$$\varphi_a(\pi) = w_1w_3aw_2w_4.$$

Note that if a is a double ascent (resp. double descent), then $w_2 = \emptyset$ (resp. $w_3 = \emptyset$), and if a is a peak then $w_2 = w_3 = \emptyset$. For instance, if $a = 3$ and $\pi = 28531746 \in \mathfrak{S}_7$, then $w_1 = 2, w_2 = 85, w_3 = \emptyset$ and $w_4 = 1746$. Thus $\varphi_a(\pi) = 23851746$. Clearly, φ_a is an involution acting on \mathfrak{S}_n and it is not hard to see that φ_a and φ_b commute for all $a, b \in [n]$. Brändén [13] modified the map φ_a to be

$$\varphi'_a(\pi) := \begin{cases} \varphi_a(\pi), & \text{if } a \text{ is not a valley of } \pi; \\ \pi, & \text{if } a \text{ is a valley of } \pi. \end{cases}$$

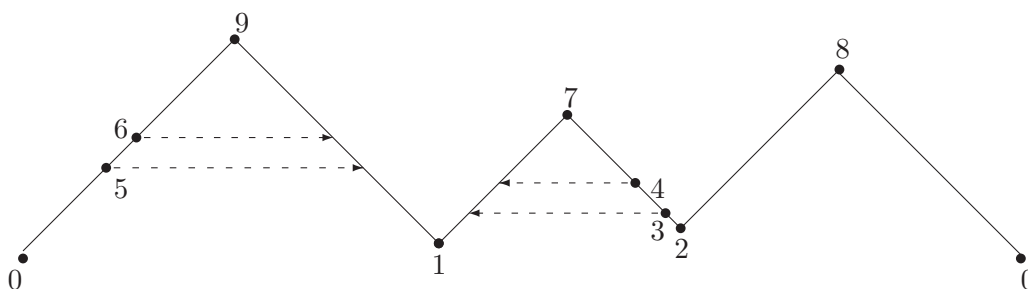


Figure 3.2: MFS-actions on 569174328 (recall $\pi(0) = \pi(10) = 0$)

See Figure 3.2 for illustration, where exchanging w_2 and w_3 in the a -factorisation is equivalent to move a from a double ascent to a double descent or vice versa. Note that the boundary condition does matter. Take the permutation 569173428 in Figure 3.2 as an example. If $\pi(0) = 10$ instead, then 5 becomes a valley and will be fixed by φ'_5 .

It is clear that φ'_a 's are involutions and commute. For any subset $S \subseteq [n]$ we can then define the map $\varphi'_S : \mathcal{DP}_{2n+1} \rightarrow \mathcal{DP}_{2n+1}$ by

$$\varphi'_S(\pi) = \prod_{a \in S} \varphi'_{2a-1}(\pi) \varphi'_{2a}(\pi).$$

Note that $\varphi'_{2n+1}(\pi) = \pi$ and the concatenation of $\varphi'_{2a-1}(\pi) \varphi'_{2a}(\pi)$ is closed for $\pi \in \mathcal{DP}_{2n}$. Hence the group \mathbb{Z}_2^n acts on \mathcal{DP}_{2n+1} via the functions φ'_S , $S \subseteq [n]$. This action will be called the *Modified Foata–Strehl action* (*MFS-action* for short).

Proof of Corollary 3.3. For any permutation $\pi \in \mathcal{DP}_{2n+1}$, let $\text{Orb}(\pi) = \{g(\pi) : g \in \mathbb{Z}_2^n\}$ be the orbit of π under the MFS-action. The MFS-action divides the set \mathcal{DP}_{2n+1} into disjoint orbits. Moreover, for $\pi \in \mathcal{DP}_{2n+1}$, $2a-1$ and $2a$ are double descents (resp. double ascents) of π if and only if $2a-1$ and $2a$ are double ascents (resp. double descents) of $\varphi'_{2a-1}(\pi) \varphi'_{2a}(\pi)$. Double descents (resp. double ascents) $2a-1$ and $2a$ of π remains a double descent (resp. double ascent) of $\varphi'_{2b-1}(\pi) \varphi'_{2b}(\pi)$ for any $b \neq a$. Hence, there is a unique permutation in each orbit which has no double descent. Let $\bar{\pi}$ be this unique element in $\text{Orb}(\pi)$, then $\text{da } \bar{\pi} = 2n+1 - \text{peak } \bar{\pi} - \text{valley } \bar{\pi}$ and $\text{des } \bar{\pi} = \text{peak } \bar{\pi} - 1 = \text{valley } \bar{\pi}$. And for any other $\pi' \in \text{Orb}(\pi)$, it can be obtained from $\bar{\pi}$ by repeatedly applying φ'_{2a-1} and φ'_{2a} for some double ascents $2a-1$ and $2a$ of $\bar{\pi}$. Once $\varphi'_{2a-1} \varphi'_{2a}$ is used, $\frac{\text{des}}{2}$ increases by 1 and $\frac{\text{da}}{2}$ decreases by 1. Thus

$$\sum_{\sigma \in \text{Orb } \pi} x^{\frac{\text{des } \sigma}{2}} = x^{\frac{\text{des } \bar{\pi}}{2}} (1+x)^{\frac{\text{da}^* \bar{\pi}}{2}} = x^{\frac{\text{des } \bar{\pi}}{2}} (1+x)^{n - \text{des } \bar{\pi}},$$

by summing over all the orbits that compose together to form \mathcal{DP}_{2n+1} , we obtain

$$\sum_{\pi \in \mathcal{DP}_{2n+1}} x^{\frac{\text{des}}{2}} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} |DP_{2n+1, 2i}| x^i (1+x)^{n-2i},$$

by transforming x to x^2 , (3.11) is derived immediately. \square

3.6 Proof of Theorem 3.4

The Clarke–Steingrímsson–Zeng [24, 96] constructed a bijection Φ from des based statistics with exc based ones for $\pi \in \mathfrak{S}_n$, to be specific, for $n \geq 1$ and $\sigma \in \mathfrak{S}_n$,

$$\begin{aligned} & (2\text{-}31, 31\text{-}2, \text{des}, \text{lda} - \text{fmax}, \text{ldd}, \text{lvalley}, \text{fmax}) \sigma \\ & = (\text{nest}, \text{icr}, \text{drop}, \text{cda}, \text{cdd}, \text{cvalley}, \text{fix}) \Phi(\sigma). \end{aligned} \quad (3.34)$$

In this section, we first recall the bijection Φ and show that the above bijection also has above properties from $\sigma \in \mathcal{DP}_{2n}$ to $\Phi(\sigma) \in \mathcal{SCDP}_{2n}$.

Recall (1.14) and (1.15), let $\sigma \in \mathcal{SCDP}_{2n}$, the refinements of inverse crossing and inverse nesting are defined by

$$\begin{aligned} \text{icr}_k \sigma &= \#\{i \in [2n] : (i < k \leq \sigma(i) < \sigma(k)) \vee (i > k > \sigma(i) > \sigma(k))\}, \\ \text{ine}_k \sigma &= \#\{i \in [2n] : (i < k \leq \sigma(k) < \sigma(i)) \vee (i > k > \sigma(k) > \sigma(i))\}. \end{aligned}$$

It is clear that $\text{icr } \sigma = \sum_{k=1}^{2n} \text{icr}_k \sigma$ and $\text{ine } \sigma = \sum_{k=1}^{2n} \text{ine}_k \sigma$.

For $\sigma = \sigma(1)\sigma(2) \cdots \sigma(2n) \in \mathcal{DP}_{2n}$, let $\sigma(0) = 0$ and $\sigma(2n+1) = 2n+1$. For $1 \leq i \leq 2n$, the letter $\sigma(i)$ (resp. $\sigma(i+1)$) is called a *descent top* (resp. *descent bottom*)

of σ if $\sigma(i) > \sigma(i+1)$; the letter $\sigma(i)$ (resp. $\sigma(i+1)$) is called a *nondescent top* (resp. *nondescent bottom*) of σ if $\sigma(i) < \sigma(i+1)$. We define *inversion top number* (resp. *inversion bottom number*) of a letter i in the permutation σ by the number of appearances of inversion form (i, j) (resp. (j, i)) in σ .

The permutation $\tau = \Phi(\sigma)$ is constructed by

$$(2-31)_k \sigma = \text{ine}_k \tau, \quad \text{for } k = 1, \dots, 2n.$$

Given a permutation σ , for two biwords, $\begin{pmatrix} f \\ f' \end{pmatrix}$ and $\begin{pmatrix} g \\ g' \end{pmatrix}$, by concatenating f and g , and f' and g' , respectively, we obtain the biword $\tau' = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$.

- Define the word f by the subword of descent bottoms of σ in increasing order.
- Define the word g by the subword of nondescent bottoms of σ in increasing order.
- Define the word f' by the permutation on descent tops in σ , where the inversion bottom number of each letter a in f' is equal to the right embracing number of a in σ .
- Define the word g' by the permutation on nondescent tops in σ , where the inversion top number of each letter b in g' is equal to the right embracing number of b in σ .

Then we rearrange the columns of τ' such that the bottom row is in increasing order, the top row of the rearranged bi-word leads to the desired permutation $\tau = \Phi(\sigma)$.

Example 3.11. Let $\sigma = 8\ 2\ 3\ 4\ 10\ 6\ 7\ 5\ 9\ 1 \in \mathcal{DP}_{10}$, with right embracing numbers $2, 1, 1, 1, 0, 2, 1, 1, 0, 0$. Then

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 9 & 7 & 10 & 8 \end{pmatrix}, \quad \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} 3 & 4 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 6 & 5 & 1 \end{pmatrix},$$

$$\begin{aligned} \tau' &= \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 6 & 3 & 4 & 7 & 8 & 9 & 10 \\ 9 & 7 & 10 & 8 & 2 & 3 & 4 & 6 & 5 & 1 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 10 & 3 & 4 & 7 & 9 & 8 & 2 & 6 & 1 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}. \end{aligned}$$

and thus $\Phi(\sigma) = \tau = 10\ 3\ 4\ 7\ 9\ 8\ 2\ 6\ 1\ 5 \in \mathcal{SCDP}_{10}$.

Proof of Theorem 3.4. Given $\sigma \in \mathcal{DP}_{2n}$, let $\tau = \Phi(\sigma)$. From the construction process every column $\begin{pmatrix} i \\ j \end{pmatrix}$ in $\begin{pmatrix} f \\ f' \end{pmatrix}$ (resp. $\begin{pmatrix} g \\ g' \end{pmatrix}$) satisfies $i < j$ (resp. $i \geq j$).

- Each letter $\sigma(i)$ appears in f' is a descent top $\sigma(i)$ in σ and also a drop $\sigma(i)$ in τ , in other words,

$$\sigma(i) > \sigma(i+1) \quad (1 \leq i \leq 2n) \quad \iff \sigma(i) > \tau(\sigma(i)). \quad (3.35)$$

Therefore, $\text{des } \sigma = \text{drop } \tau$.

- By the construction, $(2-31)\sigma$ is equal to the sum of the right embracing numbers in σ and the sum of the inversions in the words f' and g' . For an inversion pair (i, j) in f' (resp. g'), we have $\tau(i) < \tau(j) < j < i$ (resp. $j < i \leq \tau(i) < \tau(j)$), then the pair (i, j) is a inverse nesting in τ . Thus $(2-31)_k \sigma = \text{ine}_k \tau, \quad \forall k \in [2n]$.

- Similarly, $(31-2)_k\sigma = \text{icr}_k \tau$, $\forall k \in [2n]$.
- If k is a foremaximum in σ , then k is a nondescent bottom and a nondescent top, so k appears in g and g' . From the construction, $(31-2)_k\sigma = 0$. Therefore, the column $\binom{k}{k}$ occurs in $\binom{g}{g'}$, i.e., $\tau(k) = k$. Conversely, if $\tau(k) = k$, then the column $\binom{k}{k}$ occurs in $\binom{g}{g'}$ and $(31-2)_k\sigma = 0$, which yields that k is a foremaximum in σ . Thus $\text{fmax } \sigma = \text{fix } \tau$.
- If k is a double ascent of σ and not a foremaximum of σ , then k is a nondescent bottom and nondescent top. So k appears in g and g' . From the construction, $(31-2)_k\sigma > 0$. Therefore, $\binom{g}{g'}$ does not contain the column $\binom{k}{k}$, i.e., $\tau^{-1}(k) < k < \tau(k)$, which implies that k is a double excedance of τ . Conversely, if $\tau^{-1}(k) < k < \tau(k)$, then the column k occurs in g and g' and $(31-2)_k\sigma > 0$. It follows that k is a double ascent of σ and not a foremaximum of σ . Thus $(\text{lda} - \text{fmax}) \sigma = \text{cda } \tau$.
- If k is a double descent of σ , then k is a descent bottom and descent top. So k occurs in f and f' , then $\tau^{-1}(k) > k > \tau(k)$, which implies that k is a double drop of τ . Conversely, if $\tau^{-1}(k) > k > \tau(k)$, then the column k occurs in f and f' . It follows that k is a double descent of σ . Thus $\text{ldd } \sigma = \text{cdd } \tau$.
- If k is a valley of σ , then k is a descent bottom and nondescent top. So k occurs in f and g' , then $\tau^{-1}(k) > k < \tau(k)$, which implies that k is a cyclic valley of τ . Conversely, if $\tau^{-1}(k) > k < \tau(k)$, then the column k occurs in f and g' . It follows that k is a valley of σ . Thus $\text{lvalley } \sigma = \text{cvalley } \tau$.
- If k is a peak of σ , then k is a nondescent bottom and descent top. So k occurs in f' and g , then $\tau^{-1}(k) < k > \tau(k)$, which implies that k is a cyclic peak of τ . Conversely, if $\tau^{-1}(k) < k > \tau(k)$, then the column k occurs in f' and g . It follows that k is a peak of σ . Thus $\text{lpeak } \sigma = \text{cpk } \tau$.

With the Definitions 3.1 and 3.2, then $\tau \in \mathcal{SCDP}_{2n}$ and the statistics above satisfy (3.12). This completes the proof. \square

3.7 Proof of Theorem 3.5

Using Foata-Zeilberger's bijection $\Psi_{FZ} : \mathcal{SCDP}_{2n} \rightarrow \mathcal{D}\mathfrak{H}_{2n}$, the doubled Laguerre history $(s_0, \dots, s_{2n}, p_1, \dots, p_{2n})$ is constructed as follows, let $s_0 = (0, 0)$,

- the step (s_{i-1}, s_i) is North-East if i is a cyclic valley,
- the step (s_{i-1}, s_i) is South-East if i is a cyclic peak,
- the step (s_{i-1}, s_i) is East blue if i is a double excedance (or fixed point),
- the step (s_{i-1}, s_i) is East red if i is a double drop.

and $p_i = \text{nest}_i \sigma$ for $i = 1, \dots, 2n$. It is easy to see that

$$\text{nest}_i \sigma + \text{icr}_i \sigma = \begin{cases} h_i, & \text{if } (s_{i-1}, s_i) \text{ is North-East;} \\ h_i - 1, & \text{if } (s_{i-1}, s_i) \text{ is South-East;} \\ h_i, & \text{if } (s_{i-1}, s_i) \text{ is East blue;} \\ h_i - 1, & \text{if } (s_{i-1}, s_i) \text{ is East red.} \end{cases}$$

Then $(s_0, \dots, s_{2n}, p_1, \dots, p_{2n})$ is a doubled Laguerre history of length $2n$ and

$$w(\sigma) = x^{\text{ER}\gamma + \text{NE}\gamma} u^{\text{EB}\gamma} v^{\text{ER}\gamma} w^{\text{NE}\gamma} y^{\text{EB}^*\gamma} q^{\text{NE}\gamma + \text{EB}\gamma} \prod_{i=1}^{2n} p^{p_i} q^{h_i - 1 - p_i},$$

where $\text{NE}\gamma$ (resp. $\text{EB}\gamma$, $\text{ER}\gamma$, $\text{EB}^*\gamma$) is the number of North-East steps (East blue steps, and East red steps and East blue steps whose height is equal to p_i) of γ . Given a doubled path γ , the weight of each step at height k is created by using the following rules:

$$a_k := xw[k+1]_{p,q}, \quad b_k := yp^k + qu[k]_{p,q}, \quad b'_k := xv[k]_{p,q}, \quad c_k := [k]_{p,q}, \quad (3.36)$$

if the step is North-East, East Blue, East Red and South-East, respectively, and the weight of γ is defined by the product of the step weights. Summing over all the doubled paths of length $2n$ with the rules (3.36), we have

$$J_{2n}(p, q, x, u, v, w, y) = \sum_{\gamma \in \text{DM}_{2n}} w(\gamma). \quad (3.37)$$

With Lemma 3.10 and above equation, $J_{2n}(p, q, x, u, v, w, y)$ are the coefficients in the following continued fraction expansion,

$$\begin{aligned} & \sum_{n \geq 0} J_{2n}(p, q, x, u, v, w, y) z^n \\ = & \frac{1}{1 - y^2 z - \frac{[1]_{p,q}^2 [2]_{p,q}^2 x^2 w^2 z^2}{1 - ((qu[2]_{p,q} + p^2 y)^2 + x^2 v^2 [2]_{p,q}^2) z - \frac{[3]_{p,q}^2 [4]_{p,q}^2 x^2 w^2 z^2}{1 - \dots}}}}, \end{aligned} \quad (3.38)$$

by transforming z to $-z^2$, we obtain (3.13) immediately.

Claim: for $\pi \in \mathcal{SCDP}_{2n}$, we have

$$\begin{aligned} & (\text{nest, icr, drop, cda, cdd, cvalley, fix})\pi \\ = & (\text{nest, cros, exc, cdd, cda, cvalley, fix})\pi^{-1}. \end{aligned} \quad (3.39)$$

With Eqs. (3.12) and (3.39), the other two interpretations (3.15) and (3.16) is obtained.

Take $y = 0$ in (3.38), we have

$$\begin{aligned} & \sum_{n \geq 0} D_{2n}(p, q, x, u, v, w) z^n \\ = & \frac{1}{1 - \frac{[1]_{p,q}^2 [2]_{p,q}^2 x^2 w^2 z^2}{1 - ((q^2 u^2 + x^2 v^2) [2]_{p,q}^2) z - \frac{[3]_{p,q}^2 [4]_{p,q}^2 x^2 w^2 z^2}{1 - ((q^2 u^2 + x^2 v^2) [4]_{p,q}^2) z \dots}}}}, \end{aligned} \quad (3.40)$$

which is equivalent to (3.17) by transforming z to $-z^2$.

3.8 Proof of Theorem 3.7

Let

$$D_{2n}^{\text{cyc}}(\beta, x, u, v, w) := \sum_{\sigma \in \mathcal{D}_{2n}} \beta^{\text{cyc}\sigma} x^{\text{exc}\sigma} u^{\text{cda}\sigma} v^{\text{cdd}\sigma} w^{\text{cvalley}\sigma},$$

in view of [95, Eq. 40], given a doubled path γ , the weight is created by using the following rules:

$$b_k + b'_k := k(xu + v) \quad \text{and} \quad a_k c_{k+1} := (k+1)(\beta + k)xw, \quad (3.41)$$

where a_k (resp. b_k , b'_k and c_k) is the weight of North-East (resp. East blue, East red and South-East) step at height k . The weight of γ is defined by the product of the step weights. Summing over all the doubled paths of length $2n$ with the rules (3.41), we have

$$D_{2n}^{\text{cyc}}(\beta, x, u, v, w) = \sum_{\gamma \in \text{DM}_{2n}} w(\gamma).$$

With Lemma 3.10 and above equation, $D_{2n}^{\text{cyc}}(p, q, x, u, v, w, y)$ are the coefficients in the following continued fraction expansion,

$$\begin{aligned} & 1 + \sum_{n \geq 1} D_{2n}^{\text{cyc}}(\beta, x, u, v, w) z^n \\ = & \frac{1}{1 - 0(x^2 u^2 + v^2)z - \frac{2\beta(\beta + 1)x^2 w^2 z^2}{1 - 2^2(x^2 u^2 + v^2)z - \frac{3(\beta + 2)4(\beta + 3)x^2 w^2 z^2}{\dots}}}, \end{aligned} \quad (3.42)$$

which is equivalent to (3.21) by transforming z to $(-z)^2$.

3.9 Proof of Theorem 3.8

In this section, we give the proof by considering three cases.

1. When $\text{stat} = \text{nest}$, considering the generating function of the left side of Eq. (3.22),

$$D_{2n}(q, 1, x, 1, 1, 1) = \sum_{\pi \in \mathcal{DD}_{2n}} q^{\text{nest } \pi} x^{\text{exc } \pi}.$$

In view of (3.40), we have

$$1 + \sum_{n \geq 1} D_{2n}(q, 1, x, 1, 1, 1) z^n = \frac{1}{1 - \frac{[1]_q^2 [2]_q^2 x^2 z^2}{1 - [2]_q^2 (x^2 + 1)z - \frac{[3]_q^2 [4]_q^2 x^2 z^2}{\dots}}}. \quad (3.43)$$

Then the generating function of the right side of Eq. (3.22) is equal to

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/2 \rfloor} b_{2n, 2k}(q) x^{2k} (1 + x^2)^{n-2k} z^n \\ = & \sum_{n \geq 0} \sum_{\pi \in \mathcal{D}_{2n, 2k}} q^{\text{nest } \pi} \left(\frac{x}{1 + x^2}\right)^{2k} ((1 + x^2)z)^n \\ = & \sum_{n \geq 0} D_{2n}(q, 1, \frac{x}{1 + x^2}, 1, 0, 1) ((1 + x^2)z)^n. \end{aligned} \quad (3.44)$$

By Eq. (3.40),

$$\sum_{n \geq 0} D_{2n}(q, 1, x, 1, 0, 1)z^n = \frac{1}{1 - \frac{[1]_q^2 [2]_q^2 x^2 z^2}{1 - [2]_q^2 z - \frac{[3]_q^2 [4]_q^2 x^2 z^2}{\dots}}}. \quad (3.45)$$

By transforming $\frac{x}{1+x^2}$ and $(1+x^2)z$ into x and z , respectively, then Eq. (3.44) is equivalent to (3.43) immediately. This completes the proof.

2. When $\text{stat} = \text{inv}$, invoking the known result (see [95, Eq. (40)]),

$$\text{inv} = \text{drop} + \text{cros} + 2 \text{nest},$$

by using a inverse bijection, we have

$$\text{inv} = \text{exc} + \text{icr} + 2 \text{ine}.$$

Considering the generating function of the left side of Eq. (3.22),

$$D_{2n}(q^2, q, qx, 1, 1, 1) = \sum_{\pi \in \mathcal{DD}_{2n}} q^{\text{inv } \pi} x^{\text{exc } \pi}.$$

In view of (3.40), we have

$$1 + \sum_{n \geq 1} D_{2n}(q^2, q, qx, 1, 1, 1)z^n = \frac{1}{1 - \frac{[1]_{q^2, q}^2 [2]_{q^2, q}^2 x^2 q^2 z^2}{1 - [2]_{q^2, q}^2 (x^2 q^2 + q^2)z - \frac{[3]_{q^2, q}^2 [4]_{q^2, q}^2 x^2 q^2 z^2}{\dots}}}. \quad (3.46)$$

Then the generating function of the right side of Eq. (3.22) is equal to

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/2 \rfloor} b_{2n, 2k}(q) x^{2k} (1+x^2)^{n-2k} z^n \\ &= \sum_{n \geq 0} \sum_{\pi \in \mathcal{D}_{2n, 2k}} q^{\text{inv } \pi} \left(\frac{x}{1+x^2}\right)^{2k} ((1+x^2)z)^n \\ &= \sum_{n \geq 0} D_{2n}(q^2, q, \frac{qx}{1+x^2}, 1, 0, 1) ((1+x^2)z)^n. \end{aligned} \quad (3.47)$$

By Eq. (3.40), we have

$$\sum_{n \geq 0} D_{2n}(q^2, q, qx, 1, 0, 1)z^n = \frac{1}{1 - \frac{[1]_{q^2, q}^2 [2]_{q^2, q}^2 x^2 z^2}{1 - q^2 [2]_{q^2, q}^2 z - \frac{[3]_{q^2, q}^2 [4]_{q^2, q}^2 x^2 z^2}{\dots}}}. \quad (3.48)$$

By transforming $\frac{x}{1+x^2}$ and $(1+x^2)z$ into x and z , respectively, then Eq. (3.47) is equivalent to (3.46) immediately. This completes the proof.

3. When $\text{stat} = \text{cyc}$, considering the generating function of the left side of Eq. (3.22),

$$C_{2n}(q, x, 1, 1, 1) =: \sum_{\pi \in \mathcal{DD}_{2n}} q^{\text{cyc } \pi} x^{\text{exc } \pi}.$$

In view of (3.42), we have

$$1 + \sum_{n \geq 1} C_{2n}(q, x, 1, 1, 1) z^n = \frac{1}{1 - \frac{2q(q+1)x^2 z^2}{1 - 2^2(x^2+1)z - \frac{3(q+2)4(q+3)x^2 z^2}{\dots}}}. \quad (3.49)$$

Then the generating function of the right side of Eq. (3.22) is equal to

$$\begin{aligned} & 1 + \sum_{n \geq 1} \sum_{k=0}^{\lfloor n/2 \rfloor} b_{2n, 2k}(q) x^{2k} (1+x^2)^{n-2k} z^n \\ &= 1 + \sum_{n \geq 1} \sum_{\pi \in \mathcal{D}_{2n, 2k}} q^{\text{cyc } \pi} \left(\frac{x}{1+x^2}\right)^{2k} ((1+x^2)z)^n \\ &= 1 + \sum_{n \geq 1} C_{2n}\left(q, \frac{x}{1+x^2}, 0, 1, 1\right) ((1+x^2)z)^n. \end{aligned} \quad (3.50)$$

By Eq. (3.42),

$$1 + \sum_{n \geq 1} C_{2n}(q, x, 0, 1, 1) z^n = \frac{1}{1 - \frac{2q(q+1)x^2 z^2}{1 - 2^2 z - \frac{3(q+2)4(q+3)x^2 z^2}{\dots}}}. \quad (3.51)$$

By transforming $\frac{x}{1+x^2}$ and $(1+x^2)z$ into x and z , respectively, then Eq. (3.50) is equivalent to (3.49) immediately. This completes the proof.

3.10 Proof of Theorem 3.9

The fundamental bijection of Foata-Schützenberger linking des statistics with drop is crucial for our ensuing derivation, see [51] and [99].

Recall that the cycle structure of a permutation $\pi \in \mathfrak{S}_n$ can be written as a disjoint union of its distinct cycles C_1, C_2, \dots, C_k , i.e., $\pi = C_1 C_2 \cdots C_k$. Define the standard cycle representation of π in following two steps.

1. Writing the largest element of each cycle first,
2. Arranging the cycles in increasing order according to their largest elements.

For example, for $\pi = 26471583 \in \mathfrak{S}_8$, the standard cycle representation is $(6512)(8347)$.

For $\pi \in \mathfrak{S}_n$, define $\Psi(\pi)$ in two steps.

1. Inserting a left parenthesis in $\pi = \pi(1)\pi(2) \cdots \pi(n)$ before each *left-to-right maximum*, i.e., an element a_i such that $a_i > a_j$ for every $j < i$.
2. Then inserting a closing parenthesis before each internal left parenthesis and at the end.

$\sigma \in \mathcal{DP}_5$	$\tau = \Psi_1(\sigma)$	$\tau^{-1} \in \mathcal{FCDP}_6$	des σ	da* σ	dd* σ	valley* σ
			drop $\tau - 1$	cda $\tau + \text{fix } \tau$	cdd τ	cvalley $\tau - 1$
			exc $\tau^{-1} - 1$	cdd $\tau^{-1} + \text{fix } \tau^{-1}$	cda τ^{-1}	cvalley $\tau^{-1} - 1$
12345	623451	623451	0	4	0	0
31425	645231	645231	2	0	0	2
32415	654321	654321	2	0	0	2
41325	645321	654231	2	0	0	2
42315	654231	645321	2	0	0	2
31524	645213	546231	2	0	0	2
32514	654312	564321	2	0	0	2
51324	645312	564231	2	0	0	2
52314	654213	546321	2	0	0	2
41523	654123	456321	2	0	0	2
42513	645132	465231	2	0	0	2
51423	654132	465321	2	0	0	2
52413	645123	456231	2	0	0	2
12543	623145	423561	2	2	2	0
34521	612453	236451	2	2	2	0
54321	612345	234561	4	0	4	0

Figure 3.3: Illustration of Ψ_1 on doubled permutations on [5] with their statistics

$\sigma \in \mathcal{DP}_5$	$\tau = \Psi_2(\sigma')$	$\tau^{-1} \in \mathcal{SCDP}_4$	des σ	da* σ	dd* σ	valley* σ
			drop τ	cda $\tau + \text{fix } \tau$	cdd τ	cvalley τ
			exc τ^{-1}	cdd $\tau^{-1} + \text{fix } \tau^{-1}$	cda τ^{-1}	cvalley τ^{-1}
12345	1234	1234	0	4	0	0
31425	3412	3412	2	0	0	2
32415	4321	4321	2	0	0	2
41325	3421	4312	2	0	0	2
42315	4312	3421	2	0	0	2

Figure 3.4: Illustration of Ψ_2 on doubled permutations on [5] with their statistics

Conversely, define $\Psi^{-1}(\pi)$ to be the permutation obtained from π by writing in standard form and erasing the parentheses. Then the above map $\Psi : \pi \mapsto \pi$ is a bijection from \mathfrak{S}_n to itself, known as the *fundamental bijection*.

Remark 3.4. For $\tau = \Psi(\sigma)$, it is easy to observe the following properties.

- If the value i is a double ascent of σ , the value i is a double excedance or fixed point of τ .
- If the value i is a double descent of σ , the value i is a double drop of τ .
- If the value i is a peak of σ , the value i is a cyclic peak of τ .
- If the value i is a valley of σ , the value i is a cyclic valley of τ .

Proof of Theorem 3.9. For Jacobi elliptic function $\text{sn}(z, x)$, recall [44, Theorem 1] the Taylor coefficients count the number (odd) doubled permutations over $2n + 1$ having $2k$ descents. We define a new bijection Ψ_1 on \mathcal{DP}_{2n+1} . Given a permutation $\sigma = \sigma(1) \dots \sigma(2n + 1) \in \mathcal{DP}_{2n+1}$, let $\hat{\sigma} = (\sigma(1) + 1) \dots (\sigma(2n + 1) + 1)(1)$, where elements $2i + 2$ and $2i + 3$ are of the same ordinal type. Consider the permutation $\tau = \Psi(\hat{\sigma})$.

- The bijection Ψ satisfies that the elements $2i + 2$ and $2i + 3$ in τ are of the same cyclic ordinal type.

- Since the last element $\hat{\sigma}(2n + 2)$ of $\hat{\sigma}$ is 1, the final element of last cycle of τ should be 1, i.e., $\tau(1) = 2n + 2$, by using a classical inverse bijection,

$$\tau^{-1} := \tau^{-1}(1)\tau^{-1}(2) \cdots \tau^{-1}(n).$$

Then the last element $\tau^{-1}(2n + 2)$ of τ^{-1} should be 1.

Then $\tau := \Psi_1(\sigma) = \Psi(\hat{\sigma}) \in \mathcal{FCDP}_{2n+2}$, and

$$\text{des } \sigma = \text{des } \hat{\sigma} - 1 = \text{drop } \tau - 1 = \text{exc } \tau^{-1} - 1.$$

Then we see that the Taylor coefficients count the number (even) first cyclic doubled permutations over $2n + 2$ terminated by 1 and having $2k + 1$ excedances. We give an example for the bijection Ψ_1 on doubled permutations on [5], see Figure 3.3.

For Jacobi elliptic function $\text{cn}(z, x)$, recall [44, Theorem 1] the Taylor coefficients count the number (odd) doubled permutations over $2n + 1$ terminated by $2n + 1$ and having $2k$ descents. Let $\sigma := \sigma(1) \dots \sigma(2n)(2n + 1)$, the elements $2i + 1$ and $2i + 2$ are of the same ordinal type. Using the fundamental bijection Ψ to $\sigma' := \sigma(1) \dots \sigma(2n)$, consider the permutation $\tau' := \Psi_2(\sigma) = \Psi(\sigma')$, where elements $2i + 1$ and $2i + 2$ in τ' are of the same cyclic ordinal type. Then $\tau' \in \mathcal{SCDP}_{2n}$, and

$$\text{des } \sigma = \text{des } \sigma' = \text{drop } \tau' = \text{exc } \tau'^{-1}.$$

Then we see that the Taylor coefficients count the number (even) second cyclic doubled permutations over $2n$ having $2k$ excedances. We give an example for the bijection Ψ_2 on doubled permutations on [5], see Figure 3.4. \square

3.11 Interpret $\text{cn}(z, x)$ in terms of alternating permutations

A permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ is *alternating* (resp. *falling alternating*) permutation if $\sigma_1 < \sigma_2, \sigma_2 > \sigma_3, \sigma_3 < \sigma_4$, etc. (resp. $\sigma_1 > \sigma_2, \sigma_2 < \sigma_3, \sigma_3 > \sigma_4$, etc.). Let \mathbb{A}_n^* (resp. \mathbb{A}_n) be the set of alternating (resp. *falling alternating*) permutations on $[n]$. Let $\text{evalley } \pi$ and $\text{opeak } \pi$ denote the number of even valleys and odd peaks of π .

A *Dyck path* is a Motzkin path without horizontal step. Let Dyck_{2n} denote the set of Dyck paths of length $2n$. Then, it is well known (see [43]) that

$$1 + \sum_{n \geq 1} \sum_{\gamma \in \text{Dyck}_{2n}} w(\gamma)z^{2n} = \frac{1}{1 - \frac{a_0 c_1 z^2}{1 - \frac{a_1 c_2 z^2}{1 - \frac{a_2 c_3 z^2}{\ddots}}}}. \tag{3.52}$$

Using Lemma 1.39 in (3.13), we obtain that $J_{2n}(p, q, x) := J_{2n}(p, q, x, 1, 1, 1, 1)$ are the coefficients of the following continued fractions,

$$\sum_{n=0}^{\infty} (-1)^n J_{2n}(p, q, x) z^{2n} = \frac{1}{1 + \frac{[1]_{p,q}^2 z^2}{1 + \frac{[2]_{p,q}^2 (xz)^2}{1 + \frac{[3]_{p,q}^2 z^2}{1 + \frac{[4]_{p,q}^2 (xz)^2}{\ddots}}}}}. \tag{3.53}$$

Remark 3.5. When $p = q = x = 1$, (3.53) reduces to the continued fraction of classic secant numbers E_{2n} , see [95, Eq. (9)]. In 2012, Kim [62] tried to find a formula for $J_{2n}(p, q, 1)$.

Observing the continued fractions (3.53), we give another combinatorial interpretation of $J_{2n}(p, q, x)$.

Theorem 3.12. *For $n \geq 1$, we have*

$$J_{2n}(p, q, x) = \sum_{\pi \in \mathbb{A}_{2n}} p^{(2-31)^\pi} q^{(31-2)^\pi} x^{\text{valley } \pi + \text{peak } \pi}. \quad (3.54)$$

Proof. For any $\sigma = \sigma_1 \sigma_2 \dots \sigma_{2n} \in \mathbb{A}_{2n}$, let $\sigma(2n+1) = 2n+1$, the corresponding Dyck path diagram $(s_0, \dots, s_{2n}, p_1, \dots, p_{2n})$ is constructed as follows: let $s_0 = (0, 0)$ and for $i = 1, \dots, 2n$,

- the step (s_{i-1}, s_i) is North-East if i is a valley,
- the step (s_{i-1}, s_i) is South-East if i is a peak.

While $p_i = (2-31)_i \sigma$ for $i = 1, \dots, 2n$. If h_i is the height of (s_{i-1}, s_i) , i.e., $s_{i-1} = (i-1, h_i)$, Since 1 is the valley then $(31-2)_1 \sigma + (2-31)_1 \sigma = 0 = h_1$, and for $i > 1$,

$$\begin{aligned} & (31-2)_i \sigma + (2-31)_i \sigma \\ = & \begin{cases} (31-2)_{i-1} \sigma + (2-31)_{i-1} \sigma + 1 & \text{if } i-1 \text{ is a valley and } i \text{ is a valley,} \\ (31-2)_{i-1} \sigma + (2-31)_{i-1} \sigma - 1 & \text{if } i-1 \text{ is a peak and } i \text{ is a peak.} \\ (31-2)_{i-1} \sigma + (2-31)_{i-1} \sigma & \text{if } i-1 \text{ is a valley and } i \text{ is a peak,} \\ (31-2)_{i-1} \sigma + (2-31)_{i-1} \sigma & \text{if } i-1 \text{ is a peak and } i \text{ is a valley,} \end{cases} \end{aligned}$$

by induction we have

$$(31-2)_i \sigma + (2-31)_i \sigma = \begin{cases} h_i & \text{if } i \text{ is a valley,} \\ h_i - 1 & \text{if } i \text{ is a peak.} \end{cases}$$

Therefore,

$$w(\sigma) = x^{\text{ENE } \gamma + \text{OSE } \gamma} q^{\text{NE } \gamma} \prod_{i=1}^{2n} p^{p_i} q^{h_i - 1 - p_i},$$

where $\text{NE } \gamma$, $\text{SE } \gamma$, ENE and $\text{OSE } \gamma$ are the number of North-East steps, South-East steps, North-East steps at even positions (s_{2i-1}, s_{2i}) ($1 \leq i \leq n-1$) (corresponding height h_{2i} is odd) and South-East steps at odd positions $(s_{2i'}, s_{2i'+1})$ ($1 \leq i' \leq n-1$) (corresponding height $h_{2i'+1}$ is even) of γ , respectively. For example, $\sigma = 645231$ gives the path (NE, ENE, OSE, ENE, OSE, SE) and the weight $(1, tp, tq, tp, tq, 1)$. Therefore,

$$J_{2n}(p, q, x) = \sum_{\gamma \in \text{Dyck}_{2n}} x^{\text{ENE } \gamma + \text{OSE } \gamma} q^{\text{NE } \gamma} \prod_{i=1}^{2n} [h_i]_{p,q}, \quad (3.55)$$

where $[n]_{p,q} = (p^n - q^n)/(p - q)$. Given a Dyck path γ , the weight of each step is created by using the following rules:

$$a_{2k} := [2k+1]_{p,q}, \quad a_{2k+1} := [2k+2]_{p,q} x, \quad c_{2k} := [2k]_{p,q} x, \quad c_{2k+1} := [2k+1]_{p,q}, \quad (3.56)$$

if the step is North-East at height $2k$, North-East at height $2k + 1$, South-East at height $2k$, and South-East at height $2k + 1$, respectively, and the weight of γ is defined to be the product of the step weights. Summing over all the doubled paths of length $2n$ with the rules (3.56), we have

$$J_{2n}(p, q, x) = \sum_{\gamma \in \text{Dyck}_{2n}} w(\gamma). \quad (3.57)$$

With (3.52) and above equation, $J_{2n}(p, q, x)$ are the coefficients in the following continued fraction expansion,

$$\sum_{n=0}^{\infty} J_{2n}(p, q, x) z^{2n} = \frac{1}{1 - \frac{[1]_{p,q}^2 z^2}{1 - \frac{[2]_{p,q}^2 (xz)^2}{1 - \frac{[3]_{p,q}^2 z^2}{1 - \frac{[4]_{p,q}^2 (xz)^2}{\ddots}}}}}, \quad (3.58)$$

by transforming z^2 to $-z^2$, we obtain (3.53) immediately. □

Remark 3.6. For a Dyck path $\gamma \in \text{Dyck}_{2n}$, a North-East step at positions (s_{2i-1}, s_{2i}) ($1 \leq i \leq n-1$) is matched by some South-East step at $(s_{2i'}, s_{2i'+1})$ ($1 \leq i' \leq n-1$), i.e., the number of North-East steps at odd height is equal to the number of South-East steps at even height. In the construction from the alternating permutation to Dyck path of above proof, we see that the number of even valleys is equal to the number of odd peaks for $\pi \in \mathbb{A}_{2n}$. Therefore, when $p = q = 1$, (3.54) reduces to Flajolet's result [43, Theorem 4].

Chapter 4

λ -Euler's difference table for colored permutations ¹

4.1 Introduction

Euler [33] studied the difference table $(g_n^m)_{0 \leq m \leq n}$, where the coefficients are defined by $g_n^n = n!$ and

$$g_n^m = g_n^{m+1} - g_{n-1}^m, \quad (4.1)$$

for $0 \leq m \leq n-1$. Dumont and Randrianarivony [33] studied the combinatorial interpretation of g_n^m in the symmetric group S_n , which consists of permutations of $[n] = \{1, \dots, n\}$. In particular, they showed that the sequence $\{g_n^0\}_{n \geq 0}$ is the number of derangements, i.e., the fixed point free permutations in S_n . Then Rakotondrajao [84] developed further combinatorial interpretations. The reader is referred to [33, 84, 85, 40, 23, 35, 69, 20], where several generalizations of Euler's difference table with combinatorial meanings were studied.

Definition 4.1. For fixed integer $\ell \geq 1$, we define λ -Euler's difference table $(g_{\ell,n}^m(\lambda))_{0 \leq m \leq n}$ for $C_\ell \wr S_n$, where the coefficients are defined by

$$\begin{cases} g_{\ell,n}^n(\lambda) = \ell^n n! & (m = n); \\ g_{\ell,n}^m(\lambda) = g_{\ell,n}^{m+1}(\lambda) + (\lambda - 1)g_{\ell,n-1}^m(\lambda) & (0 \leq m \leq n - 1). \end{cases} \quad (4.2)$$

From the above definition, it is easy to see the coefficients $g_{\ell,n}^m(\lambda)$ are polynomials in λ . Faliharimalala and Zeng [40] studied the combinatorial interpretation of $g_{\ell,n}^m(0)$ in terms of k -circular successions in $C_\ell \wr S_n$. Eriksen et al. [35] gave a combinatorial interpretation for the coefficients $g_{1,n}^m(\lambda)$ by assuming that λ is a non-negative integer. They showed that $g_{1,n}^m(\lambda)$ count the number of permutations of $[n]$ such that fixed points on the last $n - k$ positions may be colored in any one of λ colors. Liese and Remmel [69] interpreted the coefficients of polynomial $g_{1,n}^m(\lambda)$ by counting certain rook placements in the $[n] \times [n]$ board.

It is not hard to see that the coefficient $g_{\ell,n}^m(\lambda)$ is divisible by $\ell^m m!$. This prompted us to introduce $d_{\ell,n}^m(\lambda) = g_{\ell,n}^m(\lambda) / \ell^m m!$. Then we derive the following allied array $(d_{\ell,n}^m(\lambda))_{0 \leq m \leq n}$ from (4.1).

¹The contents of this chapter are published in **Electron. J. Combin.** **25** (2018), no. 4, Paper 4.25, 27 pp, see [58].

Definition 4.2. For fixed integer $\ell \geq 1$, the coefficients of the λ -difference table $(d_{\ell,n}^m(\lambda))_{0 \leq m \leq n}$ are defined by

$$\begin{cases} d_{\ell,n}^n(\lambda) = 1 & (m = n); \\ d_{\ell,n}^m(\lambda) = \ell(m+1)d_{\ell,n}^{m+1}(\lambda) + (\lambda-1)d_{\ell,n-1}^m(\lambda) & (0 \leq m \leq n-1). \end{cases} \quad (4.3)$$

The first terms of these coefficients for $\ell = 1, 2$ are given in Tables 4.1 and 4.2.

$n \setminus m$	0	1	2	3	4
0	1				
1	λ	1			
2	$\lambda^2 + 1$	$\lambda + 1$	1		
3	$\lambda^3 + 3\lambda + 2$	$\lambda^2 + 2\lambda + 3$	$\lambda + 2$	1	
4	$\lambda^4 + 6\lambda^2 + 8\lambda + 9$	$\lambda^3 + 3\lambda^2 + 9\lambda + 11$	$\lambda^2 + 4\lambda + 7$	$\lambda + 3$	1

Table 4.1: Values of $d_{\ell,n}^m(\lambda)$ for $0 \leq m \leq n \leq 4$ and $\ell = 1$.

$n \setminus m$	0	1	2	3	4
0	1				
1	$\lambda + 1$	1			
2	$\lambda^2 + 2\lambda + 5$	$\lambda + 3$	1		
3	$\lambda^3 + 3\lambda^2 + 15\lambda + 29$	$\lambda^2 + 6\lambda + 17$	$\lambda + 5$	1	
4	$\lambda^4 + 4\lambda^3 + 30\lambda^2 + 116\lambda + 233$	$\lambda^3 + 9\lambda^2 + 51\lambda + 131$	$\lambda^2 + 10\lambda + 37$	$\lambda + 7$	1

Table 4.2: Values of $d_{\ell,n}^m(\lambda)$ for $0 \leq m \leq n \leq 4$ and $\ell = 2$.

Two combinatorial interpretations of $d_{\ell,n}^m(0)$ were given in [40]. When λ is a non-negative integer, Eriksen et al. [35] gave a combinatorial interpretation for the coefficients $d_{1,n}^m(\lambda)$ in the symmetric group. Wang et al. [107] introduced the r -derangement number, which counts the derangements of $[n]$ with the first r elements appear in distinct cycles.

Motivated by [40, 35, 69, 107], we study the combinatorial interpretation of $g_{\ell,n}^m(\lambda)$ and $d_{\ell,n}^m(\lambda)$ in the colored symmetric group $G_{\ell,n}$, i.e., the wreath product of a cyclic group and a symmetric group. The chapter is organized as follows. In Sections 3 and 4, we interpret the polynomial $g_{\ell,n}^m(\lambda)$ and the coefficients in $g_{\ell,n}^m(\lambda)$, respectively. In Sections 5 and 6, we prove the linear combinatorial interpretation and cyclic combinatorial interpretation of $d_{\ell,n}^m(\lambda)$, respectively. In Section 7, we obtain the generating functions and recurrence relations of $d_{\ell,n}^m(\lambda)$. In Section 8, we generalize r -derangement number by relating with $d_{\ell,n}^m(\lambda)$. In Section 9, we give a combinatorial proof of recurrence relation of $d_{\ell,n}^m(\lambda)$.

4.2 Definitions and main results

For positive integers $\ell, n \geq 1$, the group of *colored permutations* of n elements with ℓ colors is the wreath product $G_{\ell,n} := C_\ell \wr S_n = C_\ell^n \rtimes S_n$, where C_ℓ is the ℓ -cyclic group generated by $\zeta = e^{2i\pi/\ell}$ ($\zeta^\ell = 1$). From definition, it is obvious to see the elements in $G_{\ell,n}$ are pairs $(\epsilon, \sigma) \in C_\ell^n \times S_n$.

And $G_{\ell,n}$ can also be seen as a permutation group on the colored set:

$$\Sigma_{\ell,n} := C_\ell \times [n] = \{\zeta^j i \mid i \in [n], 0 \leq j \leq \ell - 1\}.$$

Clearly there are $\ell^n n!$ signed permutations in the group $G_{\ell,n}$. For more details, see [38].

A signed permutation $\pi \in G_{\ell,n}$ can be written in two-line form. For example, if $\pi = (\epsilon, \sigma) \in G_{4,11}$, where $\epsilon = (1, \zeta^3, 1, \zeta, 1, 1, \zeta^2, \zeta, 1, \zeta, 1)$ and

$$\sigma = 7 \ 5 \ 3 \ 1 \ 2 \ 6 \ 8 \ 9 \ 4 \ 10 \ 11,$$

we write

$$\pi = \left(\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \zeta^2 7 & 5 & 3 & 1 & \zeta^3 2 & 6 & \zeta 8 & 9 & \zeta 4 & \zeta 10 & 11 \end{array} \right).$$

To be convenient, we write j bars over i instead of $\zeta^j i$. Thus, we rewrite the above permutation in linear form as $\pi = \bar{7} \ 5 \ 3 \ 1 \ \bar{2} \ 6 \ \bar{8} \ 9 \ \bar{4} \ \bar{10} \ 11$, or in disjoint cyclic form as

$$\pi = (1, \bar{7}, \bar{8}, 9, \bar{4}) (\bar{2}, 5) (3) (6) (\bar{10}) (11).$$

That is, when using disjoint cyclic notation to determine the image of a number, we ignore the sign on that number and only consider the sign on the number to which it is mapped. Thus, in the above example, we ignore the sign ζ^2 on the 7 and 7 maps to $\zeta 8$ since the sign on 8 is ζ . Moreover, let $[m+1, n]$ denote the interval $\{m+1, \dots, n\}$, and we give the following conventions:

- i) If $\pi = (\epsilon, \sigma) \in G_{\ell,n}$, we define $|\pi| = \sigma$ and $\text{sign}_\pi(i) = \epsilon_i$ for $i \in [n]$. For example, if $\pi = 1 \ \bar{4} \ 3 \ \bar{2}$ then $\epsilon = (1, \zeta^2, 1, \zeta)$ and $\text{sign}_\pi(4) = \zeta$.
- ii) For $i \in [n]$ and $j \in \{0, 1, \dots, \ell-1\}$, we define $\zeta^j i + k = \zeta^j(i+k)$ for $0 \leq k \leq n-i$, and $\zeta^j i - k = \zeta^j(i-k)$ for $0 \leq k \leq i$. For example, $\bar{2} + 1 = \bar{3}$ in $G_{4,11}$.
- iii) We define the total ordering on $\Sigma_{\ell,n}$ as follows. For $i, j \in \{0, \dots, \ell-1\}$ and $a, b \in [n]$,

$$\zeta^i a < \zeta^j b \iff i > j \quad \text{or} \quad i = j \text{ and } a < b.$$

In $G_{\ell,n}$, Faliharimalala and Zeng [40] introduced the k -successions as follows.

Definition 4.3. Given a permutation $\pi \in G_{\ell,n}$ and an integer $0 \leq k \leq n-1$, $\pi(i)$ is a k -succession at position $i \in [n-k]$ if $\pi(i) = i+k$. In particular, the 0-succession is also called fixed point.

Note that the above k -succession $\pi(i)$ needs to be uncolored, that is, $\text{sign}_\pi(\pi(i)) = 1$.

To obtain the combinatorial interpretation of $g_{\ell,n}^m(\lambda)$, we introduce the following definition.

Definition 4.4. For any integer $0 \leq k \leq n-1$, let $SUC_k(\pi)$ denote the set of k -successions in $\pi \in G_{\ell,n}$, i.e.,

$$SUC_k(\pi) = \{\pi(i) | \pi(i) = i+k, i \in [n-k], \pi \in G_{\ell,n}\}.$$

For integer $0 \leq m \leq n$, we define the statistic $\text{suc}_{>m}^{(k)}(\pi)$ is the number of k -successions included in $[m+1, n]$ for $\pi \in G_{\ell,n}$, i.e.,

$$\text{suc}_{>m}^{(k)}(\pi) = \#\{\pi(i) \in [m+1, n] | \pi(i) \in SUC_k(\pi)\}.$$

In particular, for $\pi \in G_{\ell,n}$, by taking $k=0$ and $k=m$, $\text{suc}_{>m}^{(k)}$ is the number of fixed points and m -successions concerning $\pi \in G_{\ell,n}$, respectively, which are included in $[m+1, n]$.

For example, when $\pi \in G_{4,11}$, if $\pi = 5 \bar{3} 1 \bar{\bar{2}} \bar{6} \bar{8} 9 \bar{4} \bar{10} 11 \bar{7}$ and $\pi' = 3 1 \bar{\bar{2}} \bar{6} \bar{8} 9 \bar{4} \bar{10} 11 \bar{7} \bar{5}$, we have $SUC_1(\pi) = SUC_2(\pi') = \{3, 6, 11\}$ and $\text{suc}_{>4}^{(1)}(\pi) = \text{suc}_{>4}^{(2)}(\pi') = 2$.

Theorem 4.1. *For fixed integers ℓ, k, m and n , let $\ell \geq 1$ and $0 \leq k \leq m \leq n$, we have*

$$g_{\ell,n}^m(\lambda) = \sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)}. \tag{4.4}$$

Remark 4.1. We recover Faliharimalala and Zeng's result [40, Theorem 3] about the combinatorial interpretation of $g_{\ell,n}^m(0)$ in $G_{\ell,n}$. And we prove Theorem 4.1 in Section 4.3.

We give an example to illustrate the above theorem. For $\ell = 2, n = 2$ and $m = 1$, the permutations in $G_{2,2}$ are

$$1 \bar{2}, \bar{1} 2, 1 \bar{2}, \bar{1} \bar{2}, 2 \bar{1}, \bar{2} 1, 2 \bar{1}, \bar{2} \bar{1}.$$

For $k = 0$, $\sum_{\pi \in G_{2,2}} \lambda^{\text{suc}_{>1}^{(0)}(\pi)} = 2\lambda + 6$. For $k = 1$, $\sum_{\pi \in G_{2,2}} \lambda^{\text{suc}_{>1}^{(1)}(\pi)} = 2\lambda + 6$.

For $n, m, s \geq 0$, Rakotondrajao [85] also studied the number of permutations in S_n having exactly s m -successions. Similarly, we define that $c_{\ell,n,s}^m$ is the number of permutations $\pi \in G_{\ell,n}$ having s m -successions. In other words,

$$c_{\ell,n,s}^m = |\{\pi \in G_{\ell,n} \mid |SUC_m(\pi)| = s\}|, \quad \text{for } n, s, m \geq 0.$$

With Theorem 4.1 and above definition, we state an expression of $g_{\ell,n}^m(\lambda)$ as follows.

Corollary 4.2. *For $\ell \geq 1, 0 \leq m \leq n$ and $0 \leq s \leq n - m$, we have*

$$g_{\ell,n}^m(\lambda) = \sum_{s \geq 0} c_{\ell,n,s}^m \lambda^s. \tag{4.5}$$

Remark 4.2. With the equations (4.2) and (4.5), we obtain that

$$c_{\ell,n,s}^{m+1} = c_{\ell,n,s}^m + c_{\ell,n-1,s}^m - c_{\ell,n-1,s-1}^m,$$

which is the result of [40, Theorem 4].

To show the combinatorial interpretations and recursions of $c_{\ell,n,s}^m$, we review the generalized rook theory model in [16].

Let B_n^ℓ be the $n \times \ell n$ array of squares, we label the n columns from left to right by $1, 2, \dots, n$ and the ℓn rows from bottom to top by

$$1, \zeta 1, \dots, \zeta^{\ell-1} 1, 2, \zeta 2, \dots, \zeta^{\ell-1} 2, \dots, n, \zeta n, \dots, \zeta^{\ell-1} n,$$

respectively. For instance, the board B_n^3 is pictured in Figure 4.1. The square in the column labeled with i and the row labeled with $\zeta^r j$ is denoted by $(i, \zeta^r j)$. Each such square is called a *cell* and the rows labeled by $j, \zeta j, \dots, \zeta^{\ell-1} j$ are called *level* j .

Given a board $B \subseteq B_n^\ell$, we let $R_{k,n}^\ell(B)$ denote the set of k element subsets \mathbb{P} of B such that no two elements lie in the same level or column for non-negative integers k . We call the subset \mathbb{P} a placement of non-attacking ℓ -rooks in B . Since the cells in the placement are considered to contain ℓ -rooks, we define the k th ℓ -rook number of B by $r_{k,n}^\ell(B) = |R_{k,n}^\ell(B)|$.

Given a permutation $\pi \in G_{\ell,n}$, we can identify π with a placement \mathbb{P}_π of n ℓ -rooks in B_n^ℓ . In other word, $\mathbb{P}_\pi = \{(i, \zeta^r j) : \pi(i) = \zeta^r j \text{ for } 1 \leq i \leq n\}$, then we define the k th ℓ -hit

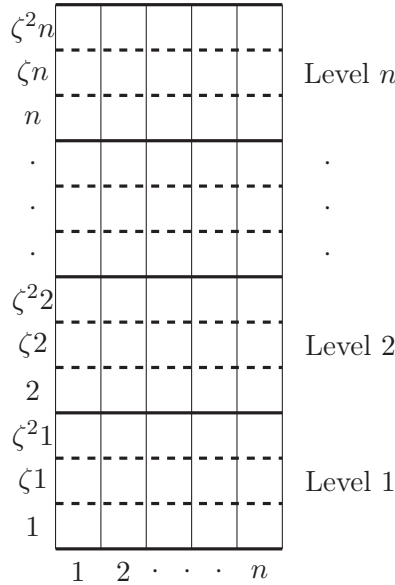


Figure 4.1: A board B_n^3 .

number of B denoted by $h_{k,n}^\ell(B)$, which is the number of $\pi \in G_{\ell,n}$ such that the placement \mathbb{P}_π intersects the board B in exactly k cells, i.e.,

$$h_{k,n}^\ell(B) = |\{\mathbb{P}_\pi | \pi \in G_{\ell,n} \text{ and } |\mathbb{P}_\pi \cap B| = k\}|.$$

Briggs and Remmel [16, Theorem 1] found the following relationship between the ℓ -hit numbers and the ℓ -rook numbers.

Theorem 4.3 (Briggs-Remmel). *Let B be a board contained in B_n^ℓ . Then*

$$\sum_{k=0}^n h_{k,n}^\ell(B)x^k = \sum_{k=0}^n r_{k,n}^\ell(B)\ell^{n-k}(n-k)!(x-1)^k.$$

By interpreting $c_{\ell,n,s}^m$ in terms of ℓ -hit numbers for a certain board, we obtain the following formula.

Theorem 4.4. *For $\ell, n \geq 1$, $0 \leq m \leq n$ and $s \geq 0$, we have*

$$c_{\ell,n,s}^m = \sum_{t=s}^{n-m} (-1)^{t-s} \ell^{n-t} (n-t)! \binom{t}{s} \binom{n-m}{t}. \tag{4.6}$$

Remark 4.3. When $\ell = 1$, (4.6) reduce to the result of [69, Theorem 2.2]. And we prove Theorem 4.4 in Section 4.4.

To make our arguments of interpreting the coefficients $d_{\ell,n}^m(\lambda)$ clear, we only consider the case $k = 0$ in statistics $\text{suc}_{>m}^{(k)}(\pi)$. We define $FIX(\pi) := SUC_0(\pi)$, which denote the set of fixed points in $\pi \in G_{\ell,n}$, i.e., $FIX(\pi) = \{\pi(i) | \pi(i) = i, i \in [n], \pi \in G_{\ell,n}\}$. Define $\text{fix}_{>m}(\pi) := \text{suc}_{>m}^{(0)}(\pi)$, i.e.,

$$\text{fix}_{>m}(\pi) := \#\{\pi(i) \in [m+1, n] | \pi(i) \in FIX(\pi)\}.$$

For example, when $\pi \in G_{4,11}$, if $\pi = \bar{7} \ 5 \ 3 \ 1 \ \bar{2} \ 6 \ \bar{8} \ 9 \ \bar{4} \ \bar{10} \ 11$, we have $FIX(\pi) = \{3, 6, 11\}$ and $\text{fix}_{>4}(\pi) = 2$.

To give the linear interpretation of $d_{\ell,n}^m(\lambda)$, we give the following definition.

Definition 4.5. For $0 \leq m \leq n$, a permutation $\pi = (\epsilon, \sigma) \in G_{\ell,n}$ is called an m -decreasing permutation if satisfies the following conditions:

- i) $\text{sign}_{\pi}(\pi(i)) = 1(i \in [m])$;
- ii) $\pi(1) > \pi(2) > \cdots > \pi(m)$.

Let $L_{\ell,n}^m$ be the set of m -decreasing permutations in $G_{\ell,n}$. For example, when $\ell = 2, n = 3$ and $m = 2$,

$$L_{2,3}^2 = \{21\bar{3}, 21\bar{3}, 31\bar{2}, 31\bar{2}, 32\bar{1}, 32\bar{1}\}, \quad \text{and} \quad \sum_{\pi \in L_{2,3}^2} \lambda^{\text{fix}_{>2}(\pi)} = \lambda + 5.$$

Theorem 4.5. For $0 \leq m \leq n$, we have

$$d_{\ell,n}^m(\lambda) = \sum_{\pi \in L_{\ell,n}^m} \lambda^{\text{fix}_{>m}(\pi)}.$$

Remark 4.4. When $\lambda = 0$, Theorems 4.5 reduce to the result of [40, Theorem 10], we prove above theorem in Section 4.5.

To give the cyclic interpretation of $d_{\ell,n}^m(\lambda)$, we give the following definition.

Definition 4.6. For $0 \leq m \leq n$, a permutation $\pi = (\epsilon, \sigma) \in G_{\ell,n}$ is called m -separated permutation if satisfies the following conditions:

- i) $\text{sign}_{\pi}(i) = 1(i \in [m])$;
- ii) the first m elements belong into distinct cycles.

Let $C_{\ell,n}^m$ be the set of m -separated permutations in $G_{\ell,n}$. For example, when $\ell = 2, n = 3$ and $m = 2$,

$$C_{2,3}^2 = \{(13)(2), (1\bar{3})(2), (1)(2\bar{3}), (1)(2\bar{3}), (1)(2)(3), (1)(2)(\bar{3})\},$$

and

$$\sum_{\pi \in C_{2,3}^2} \lambda^{\text{fix}_{>2}(\pi)} = \lambda + 5.$$

Theorem 4.6. For $0 \leq m \leq n$, we have

$$d_{\ell,n}^m(\lambda) = \sum_{\pi \in C_{\ell,n}^m} \lambda^{\text{fix}_{>m}(\pi)}.$$

Remark 4.5. When $\lambda = 0$, Theorems 4.6 reduce to the result of [40, Theorem 12], we prove above theorem in Section 4.6.

To generalize the definition of r -derangement number, we give the following definition.

Definition 4.7. For $0 \leq m \leq n$, a permutation $\pi \in G_{\ell, n}$ is called m -fixed point-free colored permutation if satisfies the following conditions:

- i) For $i \in [m]$, let $\pi(i) \in [m+1, n]$ and $\text{sign}_{\pi}(i) = \text{sign}_{\pi}(\pi(i)) = 1$;
- ii) no two elements of $[m]$ are in the same cycle.

Let $F_{\ell, n+m}^m$ be the set of m -fixed point-free colored permutations in $G_{\ell, n+2m}$, we define

$$f_{\ell, n+m}^m(\lambda) = \sum_{\pi \in F_{\ell, n+m}^m} \lambda^{\text{fix}_{>m}(\pi)}. \quad (4.7)$$

For example, when $\ell = 2$, $n = 1$ and $m = 1$,

$$F_{2,2}^1 = \{(12)(3), (12)(\bar{3}), (13)(2), (13)(\bar{2}), (123), (12\bar{3}), (132), (13\bar{2})\}$$

and

$$f_{2,2}^1(\lambda) = 2\lambda + 6.$$

Remark 4.6. When $(\ell, \lambda) = (1, 0)$, the equation (4.7) reduce to the sum over $\{\pi \in F_{\ell, n+m}^m \mid \text{fix}_{>m}(\pi) = 0\}$, then the polynomial $f_{\ell, n+m}^m(\lambda)$ reduces to the r -derangement number, see [107, Definition 1]. By the above definition, we generalize the generating functions and recurrence relations of Wang et al. [107].

By observing the above definitions, we prove the following combinatorial relation between the $f_{\ell, n+m}^m(\lambda)$ and $d_{\ell, n+m}^m(\lambda)$ in Section 4.8.

Theorem 4.7. For $\ell \geq 1$ and $m, n \geq 0$, we have

$$f_{\ell, n+m}^m(\lambda) = \frac{(n+m)!}{n!} d_{\ell, n+m}^m(\lambda). \quad (4.8)$$

4.3 Proof of Theorem 4.1

In the section, to prove Theorem 4.1, we prove the following equations,

$$\left\{ \begin{array}{l} \sum_{\pi \in G_{\ell, n}} \lambda^{\text{suc}_{>n}^{(k)}(\pi)} = \ell^n n! \quad (m = n); \\ \sum_{\pi \in G_{\ell, n}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} = \sum_{\pi \in G_{\ell, n}} \lambda^{\text{suc}_{>m+1}^{(k)}(\pi)} + (\lambda - 1) \sum_{\pi \in G_{\ell, n-1}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} \quad (0 \leq m \leq n-1). \end{array} \right. \quad (4.9)$$

Lemma 4.8. For any integer k such that $0 \leq k \leq m$ and $0 \leq m \leq n$, there holds

$$\sum_{\substack{\pi \in G_{\ell, n} \\ m+1 \in \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} = \lambda \sum_{\pi \in G_{\ell, n-1}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)}. \quad (4.10)$$

Proof. Let us define the bijection $\psi: G_{\ell,n} \mapsto G_{\ell,n-1}$. For $\pi \in G_{\ell,n}$, we delete the $m+1$ at position $m+1-k$ and define the $\psi(\pi) = \widehat{\pi}_1 \widehat{\pi}_2 \dots \widehat{\pi}_{m-k} \widehat{\pi}_{m-k+2} \dots \widehat{\pi}_{n-1} \in G_{\ell,n-1}$ where

$$\widehat{\pi}_i = \begin{cases} \pi_i, & \text{if } |\pi|_i < m+1; \\ \pi_i - 1, & \text{if } |\pi|_i > m+1. \end{cases}$$

Conversely, starting from $\psi(\pi) = \widehat{\pi}_1 \widehat{\pi}_2 \dots \widehat{\pi}_{m-k} \widehat{\pi}_{m-k+2} \dots \widehat{\pi}_n \in G_{\ell,n-1}$, we define $\pi = \pi_1 \pi_2 \dots \pi_n \in G_{\ell,n}$ where

$$\pi_i = \begin{cases} \widehat{\pi}_i, & \text{if } |\widehat{\pi}|_i < m+1; \\ \widehat{\pi}_i + 1, & \text{if } |\widehat{\pi}|_i \geq m+1. \end{cases}$$

Then we put $m+1$ at the position $m+1-k$, from the map, we can easily see $\text{suc}_{>m}^{(k)}(\pi) = \text{suc}_{>m}^{(k)}(\psi(\pi)) + 1$. \square

For example $\ell = 4, n = 9, m = 4, k = 1, \pi = \bar{7} \ 3 \ 4 \ 5 \ \bar{2} \ \bar{1} \ 8 \ 9 \ \bar{6}$, $\psi(\pi) = \bar{6} \ 3 \ 4 \ \bar{2} \ \bar{1} \ 7 \ 8 \ \bar{5}$, and $\text{suc}_{>4}^{(1)}(\pi) = \text{suc}_{>4}^{(1)}(\psi(\pi)) + 1$.

Lemma 4.9. *For $0 \leq m \leq n$, there holds*

$$\sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m+1}^{(k)}(\pi)}. \quad (4.11)$$

Proof. It follows similar arguments as in the proof of Lemma 4.8. \square

Proof of Theorem 4.1. First we check the initial condition in (4.9), when $m = n$, $\text{suc}_{>n}^{(k)}(\pi) = 0$, $\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{>n}^{(k)}(\pi)} = \ell^n n!$.

We start to prove the recurrence in (4.9). Then, by considering the following equation,

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \notin \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} + \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)}. \quad (4.12)$$

Because for $\pi \in G_{\ell,n}$ with $m+1 \notin \text{SUC}_k(\pi)$, we have $\text{suc}_{>m}^{(k)}(\pi) = \text{suc}_{>m+1}^{(k)}(\pi)$, then (4.12) is equivalent to

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \notin \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m+1}^{(k)}(\pi)} + \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)}. \quad (4.13)$$

By equations (4.10) and (4.13), we obtain that

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \notin \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m+1}^{(k)}(\pi)} + \lambda \sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)}. \quad (4.14)$$

By combining the equations (4.11) and (4.14), we obtain

$$\begin{aligned} & \sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} \\ &= (\lambda - 1) \sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{suc}_{>m}^{(k)}(\pi)} + \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \notin \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m+1}^{(k)}(\pi)} + \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in \text{SUC}_k(\pi)}} \lambda^{\text{suc}_{>m+1}^{(k)}(\pi)}. \end{aligned}$$

4				
4				
3				
3				
2				
2				
1				
1				
	1	2	3	4

Figure 4.2: The board $B_{4,1}^2$ corresponds to the shaded cells.

With (4.12), it is easy to see that the above equation is equivalent to the recurrence relation in (4.9), this completes the proof of Theorem 4.1. \square

Remark 4.7. Since why $g_{\ell,n}^m(\lambda)$ is independent from k ($0 \leq k \leq m$) in the above proof is not mentioned, we state an argument as follows. By considering the bijection d which transforms $\pi = \pi_1\pi_2\pi_3 \cdots \pi_n$ into $d(\pi) = \pi' = \pi_2\pi_3 \cdots \pi_n\pi_1$. It is easy to see that the k -successions of π are in $[m+1, n]$ if and only if the $(k+1)$ -successions of π' are in $[m+1, n]$. Hence, let the composition of j times of d is denoted by d^j , the application of $d^{k_2-k_1}$ permits to transfer the k_1 -successions to k_2 -successions if $k_1 < k_2$. In particular if we apply d^m to a permutation whose fixed points are in $[m+1, n]$, then we obtain a permutation whose m -successions are in $[m+1, n]$ and vice versa.

4.4 Proof of Theorem 4.4

In this section, first we prove the following expressions of $c_{\ell,n,s}^m$ in Theorem 4.4,

$$c_{\ell,n,s}^m = \sum_{t=s}^{n-m} (-1)^{t-s} \ell^{n-t} (n-t)! \binom{t}{s} \binom{n-m}{t}. \quad (4.15)$$

Then we derive several recurrence relations of $c_{\ell,n,s}^m$.

Proof of Theorem 4.4. First, we give the combinatorial interpretation of $c_{\ell,n,s}^m$ as follows. Let $B_{n,m}^\ell$ be the board contained in B_n^ℓ consisting of the cells $(1, 1+m), (2, 2+m), (3, 3+m), \dots, (n-m, n)$. For example, the board $B_{4,1}^2$ is pictured in Figure 4.2. Then the number of $\pi \in G_{\ell,n}$ with s m -successions is the s -th ℓ -hit number of $B_{n,m}^\ell$, i.e.,

$$c_{\ell,n,s}^m = h_{s,n}^\ell(B_{n,m}^\ell). \quad (4.16)$$

With the definitions of $B_{n,m}^\ell$, we have $r_{s,n}^\ell(B_{n,m}^\ell) = \binom{n-m}{s}$. By Theorem 4.3,

$$\begin{aligned} g_{\ell,n}^m(\lambda) &= \sum_{s=0}^n c_{\ell,n,s}^m \lambda^s = \sum_{s=0}^n h_{s,n}^\ell(B_{n,m}^\ell) \lambda^s \\ &= \sum_{s=0}^n r_{s,n}^\ell(B_{n,m}^\ell) \ell^{n-s} (n-s)! (\lambda-1)^s \\ &= \sum_{s=0}^n \binom{n-m}{s} \ell^{n-s} (n-s)! (\lambda-1)^s. \end{aligned} \quad (4.17)$$

Equating the coefficients of λ^s yields (4.15) immediately. \square

Remark 4.8. We also obtain the above expression (4.17) of $g_{\ell,n}^m(\lambda)$ by generating function, see Proposition 4.21.

Let $s = n - m$ in (4.15), we obtain the following corollary.

Corollary 4.10. For all $\ell \geq 1$ and $n \geq m \geq 0$,

$$c_{\ell,n,n-m}^m = \ell^m m!$$

Next we show the recurrence relations of $c_{\ell,n,s}^m$ in colored symmetric group.

Proposition 4.11. For all $\ell \geq 1, n \geq 2, 0 \leq m < n$, and $s \geq 1$,

$$c_{\ell,n,s}^m = (\ell(n-s-1) + (\ell-1))c_{\ell,n-1,s}^m + \ell(s+1)c_{\ell,n-1,s+1}^m + c_{\ell,n-1,s-1}^m. \quad (4.18)$$

Proof. Let us consider the map from $\pi = \pi_1 \dots \pi_{n-1} \in G_{\ell,n-1}$ to $\bar{\pi} \in G_{\ell,n}$ such that $\bar{\pi}$ has s m -successions, we consider the following three cases.

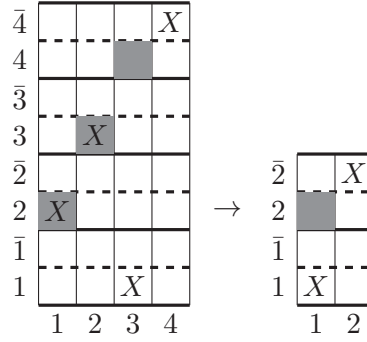
- If $\pi \in G_{\ell,n-1}$ has s m -successions.
 1. Let $\bar{\pi} = \pi_1 \pi_{n-m-1}(\zeta^j n) \pi_{n-m+1} \dots \pi_{n-1} \pi_{n-m}$ and $1 \leq j \leq \ell-1$, the number of m -successions of $\bar{\pi}$ and π are the same, so there are $(\ell-1)c_{\ell,n-1,s}^m$ permutations in this case.
 2. Let $\bar{\pi} = \pi_1 \pi_{i-1}(\zeta^j n) \pi_{i+1} \dots \pi_{n-1} \pi_i$, where $i \neq n-m$ and i is a position without m -successions, the number of m -successions of $\bar{\pi}$ and π are the same. Since we have $n-s-1$ choices for position i and $0 \leq j \leq \ell-1$, there are $\ell(n-s-1)c_{\ell,n-1,s}^m$ permutations in this case.
- If $\pi \in G_{\ell,n-1}$ has $s+1$ m -successions. Let $\bar{\pi} = \pi_1 \pi_{i-1}(\zeta^j n) \pi_{i+1} \dots \pi_{n-1} \pi_i$, where i is a position with m -succession, the number of m -successions of $\bar{\pi}$ is the number of m -successions of π minus one. Since we have $s+1$ choices for position i and $0 \leq j \leq \ell-1$, there are $\ell(s+1)c_{\ell,n-1,s+1}^m$ permutations in this case.
- If $\pi \in G_{\ell,n-1}$ has $s-1$ m -successions. Let $\bar{\pi} = \pi_1 \pi_{n-m-1}(\zeta^j n) \pi_{n-m+1} \dots \pi_{n-1} \pi_{n-m}$ and $j = 0$, the number of m -successions of $\bar{\pi}$ is the number of m -successions of π plus one, so there are $c_{\ell,n-1,s-1}^m$ permutations in this case.

\square

Proposition 4.12. For all $\ell, n \geq 1, 0 \leq m < n$, and $s \geq 0$,

$$c_{\ell,n,s}^m = \binom{n-m}{s} c_{\ell,n-s,0}^m. \quad (4.19)$$

Proof. Note that in (4.16), $c_{\ell,n,s}^m$ is the number of placements of n non-attacking ℓ -rooks in B_n^ℓ that intersect $B_{n,m}^\ell$ in exactly s squares. By removing the level $i+m$ and column i of these ℓ -rooks which lie in the cell $(i, i+m)$ ($1 \leq i \leq n-m$), we obtain these placements of $n-s$ non-attacking ℓ -rooks in B_{n-s}^ℓ that intersect $B_{n-s,m}^\ell$ in exactly 0 squares, which is counted by $c_{\ell,n-s,0}^m$. The process is pictured in Figure 4.3. \square

Figure 4.3: From Board B_4^2 to Board B_2^2 .

Remark 4.9. Faliharimalala and Zeng [40, Lemma 14] proved the above (4.19) directly by interpreting $c_{\ell,n,s}^m$ as the number of permutation in $G_{\ell,n}$ with s m -successions. However, we give a trivial proof by interpreting $c_{\ell,n,s}^m$ as the number of placements of n non-attacking ℓ -rooks in B_n^ℓ that intersect $B_{n,m}^\ell$ in exactly s squares.

Proposition 4.13. For all $\ell \geq 1$, $n \geq 2$ and $0 \leq m < n$,

$$c_{\ell,n,0}^m = (\ell n - 1)c_{\ell,n-1,0}^m + \ell(n - m - 1)c_{\ell,n-2,0}^m. \quad (4.20)$$

Proof. Let us consider the map from $\pi = \pi_1 \dots \pi_n \in G_{\ell,n}$ to $\pi' \in G_{\ell,n-1}$, starting from π without m -successions, we define

$$\pi' = \begin{cases} \pi_1 \dots \pi_{i-1} \pi_n \pi_{i+1} \dots \pi_{n-1}, & \text{if } \pi_i = \zeta^j n (0 \leq j \leq \ell - 1) \text{ for } 1 \leq i < n; \\ \pi_1 \dots \pi_{n-1}, & \text{if } \pi_n = \zeta^j n (0 \leq j \leq \ell - 1). \end{cases}$$

1. $\pi' \in G_{\ell,n-1}$ has no m -successions.

Either $\pi_n = \zeta^j n (0 \leq j \leq \ell - 1)$ or $\pi_i = \zeta^j n (1 \leq i < n, 0 \leq j \leq \ell - 1)$ and $\pi_n \neq i + m$, π' has no m -successions. Conversely, for $\pi' \in G_{\ell,n-1}$ without m -successions, by inserting $\zeta^j n (0 \leq j \leq \ell - 1)$ into π' in every position except putting n in to position $n - m$, we obtain the permutation in $G_{\ell,n}$ without m -successions. Since $\zeta^j n (0 \leq j \leq \ell - 1)$ can be in any position except $\pi_{n-m} = n$, there are $(\ell n - 1)c_{\ell,n-1,0}^m$ permutations.

2. $\pi' \in G_{\ell,n-1}$ has 1 m -succession.

When $\pi_i = \zeta^j n (1 \leq i \leq n - 1 - m, 0 \leq j \leq \ell - 1)$ and $\pi_n = i + m$, π' has 1 m -succession, then the 1 m -succession of π' corresponds to the ℓ -rook $(i, i + m)$ of the rook placement in the board B_{n-1}^ℓ . For the rook placement corresponds to π' in B_{n-1}^ℓ , removing the column i and level $i + m$ from the board B_{n-1}^ℓ , we obtain the rook placement in B_{n-2}^ℓ without intersecting $B_{n-2,m}^\ell$, which corresponding to the permutation denoted by $\tilde{\pi} \in G_{\ell,n-2}$, and $\tilde{\pi}$ has no m -successions.

Conversely, let $\tilde{\pi}$ be a permutation in $G_{\ell,n-2}$ without m -successions, we obtain $\pi \in G_{\ell,n}$ in two steps.

Step 1. For $1 \leq i \leq n - 1 - m$, by adding the column i and level $i + m$ to the boards B_{n-2}^ℓ , we choose $(i, i + m)$ as the new ℓ -rook and take the same rook placement corresponds to $\tilde{\pi}$, then we obtain the rook placement in B_{n-1}^ℓ corresponding to $\pi' \in G_{\ell,n-1}$ with 1 m -succession.

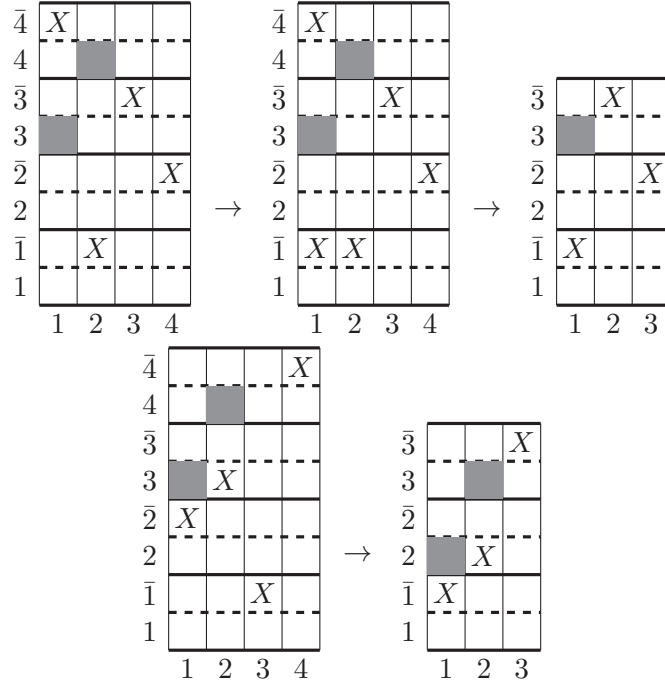


Figure 4.4: Reducing rook placements from $B_{4,2}^2$ by the rook position in level 1 .

Step 2. Adding the column n and level n in the board B_{n-1}^ℓ , by taking away the ℓ -rook $(i, i+m)$ and putting ℓ -rooks at $(i, \zeta^j n) (0 \leq j \leq \ell - 1)$ and $(n, i+m)$, we obtain the rook placement without intersecting $B_{n,m}^\ell$, which corresponds to the permutation $\pi \in G_{\ell,n}$ without m -successions. Since $1 \leq i \leq n - 1 - m$ and $0 \leq j \leq \ell - 1$, there are $\ell(n - 1 - m)c_{\ell,n-2,0}^m$ permutations.

□

Remark 4.10. When $m = 0$, we define that

$$D_n^\ell := c_{\ell,n,0}^0, \tag{4.21}$$

which counts the number of derangements in $G_{\ell,n}$. It is easy to see (4.20) reduce to

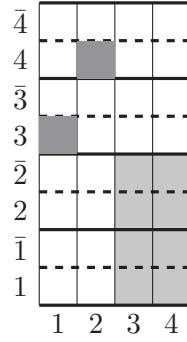
$$D_n^\ell = (\ell n - 1)D_{n-1}^\ell + \ell(n - 1)D_{n-2}^\ell.$$

Proposition 4.14. For all $\ell \geq 1$, $n \geq 2$, and $1 \leq m < n$,

$$c_{\ell,n,0}^m = \ell m c_{\ell,n-1,0}^{m-1} + \ell(n - m)c_{\ell,n-1,0}^m. \tag{4.22}$$

Proof. We prove the above equation by considering level 1 in the rook placement corresponding to the permutation $\pi \in G_{\ell,n}$ without m -successions.

1. When the ℓ -rook of level 1 is in column $i (1 \leq i \leq n - m)$, if the ℓ -rook of level $i + m$ lies at the position $(k, \zeta^j(i + m)) (k \neq i \text{ and } 0 \leq j \leq \ell - 1)$, by adding a ℓ -rook at the position $(k, \zeta^j 1)$ in the level 1, we obtain a placement of $n + 1$ ℓ -rooks without intersecting $B_{n,m}^\ell$. Then removing column i and level $i + m$ will result in a placement

Figure 4.5: Board $B_{4,2}^2$ with some lightly shaded cells.

of $n-1$ non-attacking ℓ -rooks without intersecting $B_{n-1,m}^\ell$. Since the rook placement corresponding to π has ℓ different positions in level 1 and column $1 \leq i \leq n-m$, thus there are $\ell(n-m)c_{\ell,n-1,0}^m$ placements. The process is illustrated in top of the Figure 4.4.

2. When the ℓ -rook of level 1 is in column i ($n-m < i \leq n$), by removing column i and level 1, we obtain a placement of $n-1$ non-attacking ℓ -rooks without intersecting $B_{n-1,m-1}^\ell$. Thus there are $\ell m c_{\ell,n-1,0}^{m-1}$ placements. The process is illustrated in the bottom of Figure 4.4.

□

Proposition 4.15. For all $\ell, n \geq 1$ and $0 \leq m < n$,

$$c_{\ell,n,0}^m = \ell^m m! \sum_{r=0}^m \binom{m}{r} \binom{n-m}{m-r} c_{\ell,n-m,0}^{m-r}. \quad (4.23)$$

Proof. To obtain a placement of n non-attacking ℓ -rooks without intersecting $B_{n,m}^\ell$, starting from the lightly shaded cells in the lower right corner of the board in Figure 4.5, we see that 0 to m ℓ -rooks can be placed in this area. Suppose that we choose r levels in this area, there are $\binom{m}{r}$ ways. Since there should be m ℓ -rooks in the last m columns, we choose $m-r$ ℓ -rooks above the lower right corner of the board. Thus we choose $m-r$ levels from the $n-m$ levels, there are $\binom{n-m}{m-r}$ ways.

After picking the m levels that contain the ℓ -rooks in the last m columns, there are $\ell^m m!$ ways to place the ℓ -rooks in the last m columns.

Let P denote the non-attacking rook placement in the last m columns with r ℓ -rooks falling in the lightly shaded area, we extend P to a non-attacking rook placement Q with n ℓ -rooks, where there is no intersection with $B_{n,m}^\ell$. By removing the levels and columns of rook placement P , we obtain the non-attacking rook placement without intersecting $B_{n-m,m-r}^\ell$, which is counted by $c_{\ell,n-m,0}^{m-r}$. Summing over all possible values of r yields the desired result. □

Proposition 4.16. For all $\ell \geq 1$, $n \geq 2$ and $0 \leq m < n$,

$$c_{\ell,n,0}^m = \ell c_{\ell,n,1}^{m+1} + (\ell m + \ell - 1) c_{\ell,n-1,0}^m. \quad (4.24)$$

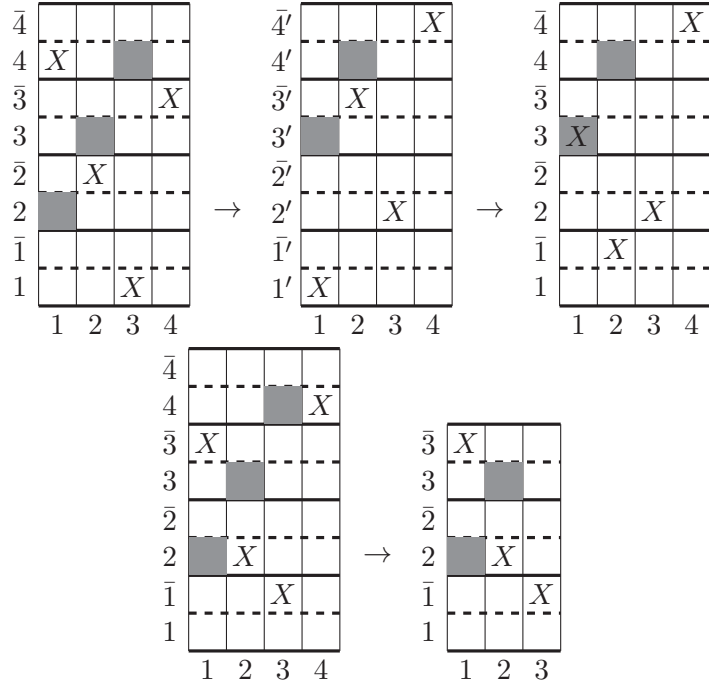


Figure 4.6: Reducing rook placements from $B_{4,1}^2$ by the rook position in top level .

Proof. Let us consider the rook position of level n in the rook placement which corresponding to the permutation $\pi \in G_{\ell,n}$ without m -successions.

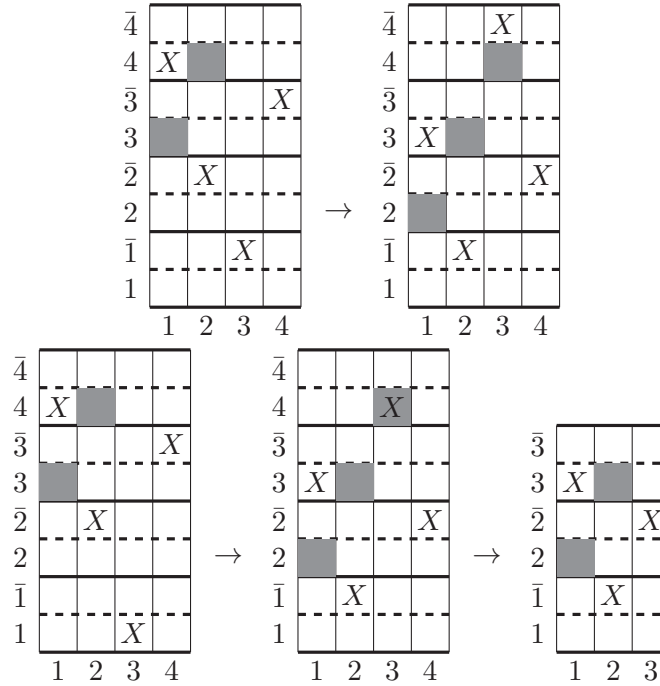
1. When the ℓ -rook of level n is in column $i(1 \leq i \leq n - m - 1)$. If the ℓ -rook of column i is in row n , we keep it unchanged. If the ℓ -rook of column i is in row $\zeta^j n(1 \leq j \leq \ell - 1)$, we exchange the row $\zeta^j n$ with row n . Then we move the level n to the bottom level of the board, which is denoted by level $1'$, other levels are increased by one such as level $2', \dots, \text{level } n'$. By exchanging the level $1'$ and level $(i + m + 1)'$, we obtain a non-attacking rook placement that intersect $B_{n,m+1}^\ell$ one rook $(i, i + m + 1)$. Since the ℓ -rook can be in the row $\zeta^j n(0 \leq j \leq \ell - 1)$, there are $\ell c_{n,1}^{m+1}$ permutations in this case. This process is shown in top of Figure 4.6.
2. When the ℓ -rook of level n is in column $i(n - m \leq i \leq n)$, the ℓ -rook can be in the position $(i, \zeta^j n)(n - m \leq i \leq n, 0 \leq j \leq \ell - 1)$, since π has no m -successions, the ℓ -rook can not be in the square $(n - m, n)$, so there are $\ell m + \ell - 1$ choices in level n . Removing the level n and column i will result in a non-attacking rook placement without intersecting $B_{n-1,m}^\ell$. Then there are $(\ell m + \ell - 1)B_{n-1,m}^\ell$ permutations in this kind. This process is shown in bottom of Figure 4.6.

□

Proposition 4.17. For $\ell \geq 1, n \geq 2$ and $1 \leq m < n$,

$$c_{\ell,n,0}^m = c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1}. \tag{4.25}$$

Proof. Let us consider the non-attacking rook placement corresponding to the permutation $\pi \in G_{\ell,n}$ without m -successions. We move the bottom level to the top level and all other levels reduced by one, which is shown in Figure 4.7.

Figure 4.7: Moving the bottom level to top level in $B_{4,2}^2$.

1. When the bottom ℓ -rook is not in the position $(n - m + 1, 1)$, the process is shown in the top of Figure 4.7. After the movement of ℓ -rooks in the board, we obtain the non-attacking rook placement without intersecting $B_{n,m-1}^\ell$. Thus there are $c_{\ell,n,0}^{m-1}$ permutations in this case.
2. When the bottom ℓ -rook is in the position $(n - m + 1, 1)$, the process is shown in the bottom of Figure 4.7. The resulting rook placement intersect $B_{n,m-1}^\ell$ in the position $(n - m + 1, n)$. By removing the column $n - m + 1$ and level n , we get the non-attacking rook placement without intersecting $B_{n-1,m-1}^\ell$. Thus there are $c_{\ell,n-1,0}^{m-1}$ permutations in this case.

□

Proposition 4.18. For $\ell, n \geq 1$ and $0 \leq m < n$,

$$c_{\ell,n,0}^m = \sum_{r=0}^m \binom{m}{r} D_{n-m+r}^\ell. \quad (4.26)$$

Proof. We prove this theorem by inductions on m . If $m = 0$, we have $c_{\ell,n,0}^0 = D_n^\ell$ by equation (4.21). Suppose that $c_{\ell,n,0}^i = \sum_{r=0}^i \binom{i}{r} D_{n-i+r}^\ell$ is satisfied for $i \leq m - 1$, then

$$c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1} = \sum_{r=0}^{m-1} \binom{m-1}{r} D_{n-m+1+r}^\ell + \sum_{r=0}^{m-1} \binom{m-1}{r} D_{n-m+r}^\ell. \quad (4.27)$$

By separating out the $m - 1$ term of the first sum and the 0 term of the second sum in

(4.27), which is equivalent to

$$\binom{m-1}{m-1} D_n^\ell + \sum_{r=0}^{m-2} \binom{m-1}{r} D_{n-m+1+r}^\ell + \binom{m-1}{0} D_{n-m}^\ell + \sum_{r=1}^{m-1} \binom{m-1}{r} D_{n-m+r}^\ell.$$

By transforming r to $r-1$ in the first sum and using $\binom{m-1}{r-1} + \binom{m-1}{r} = \binom{m}{r}$, we have

$$c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1} = \sum_{r=0}^m \binom{m}{r} D_{n-m+r}^\ell.$$

With the recurrence (4.25),

$$c_{\ell,n,0}^m = c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1},$$

we obtain

$$c_{\ell,n,0}^m = \sum_{r=0}^m \binom{m}{r} D_{n-m+r}^\ell.$$

The proof is thus completed. \square

With equations (4.19) and (4.26), we give the relation between $c_{\ell,n,s}^m$ and D_n^ℓ directly.

Corollary 4.19. *For $\ell, n \geq 1$, $0 \leq m < n$ and $s \geq 0$,*

$$c_{\ell,n,s}^m = \binom{n-m}{s} \sum_{r=0}^m \binom{m}{r} D_{n-s-m+r}^\ell.$$

By observing the coefficients of polynomial $g_{\ell,n}^m(\lambda)$, we find $c_{\ell,n,s}^m$ decreases as s increases.

Proposition 4.20. *For $\ell, n \geq 1$, $1 \leq m < n$ and $s \geq 1$,*

$$c_{\ell,n,s-1}^m \geq c_{\ell,n,s}^m. \quad (4.28)$$

Proof. With recursion (4.19) and (4.22), we have

$$\begin{aligned} & c_{\ell,n,s-1}^m - c_{\ell,n,s}^m \\ &= \binom{n-m}{s-1} c_{\ell,n-s+1,0}^m - \binom{n-m}{s} c_{\ell,n-s,0}^m \\ &= (\ell(n-m-s+1) \binom{n-m}{s-1} - \binom{n-m}{s}) c_{\ell,n-s,0}^m + \ell m \binom{n-m}{s-1} c_{\ell,n-s,0}^{m-1}. \end{aligned}$$

Since

$$\ell(n-m-s+1) \binom{n-m}{s-1} - \binom{n-m}{s} = \frac{(n-m)!}{(n-m-s)!s!} (\ell s - 1),$$

we obtain (4.28) immediately. \square

Remark 4.11. In particular, when $m = 0$, we have the similar result for $c_{\ell,n,s}^0$. By using $D_n^\ell = \ell n D_{n-1}^\ell + (-1)^n$ [40, Equation (2.8)] and similar arguments above, we have $c_{\ell,n,s-1}^0 \geq c_{\ell,n,s}^0$ for $2 \leq s \leq n-2$ and $n \geq 3$.

Remark 4.12. When $\ell = 1$, the above expressions and relations of $c_{\ell,n,s}^m$ in this section reduce to Liese and Remmel's results [69, Sections 2 and 3].

4.5 Proof of Theorem 4.5

In this section, to prove Theorem 4.5, we prove the following equations,

$$\sum_{\pi \in L_{\ell,n}^m} \lambda^{\text{fix}_{>m}(\pi)} = \sum_{\pi \in G_{\ell,n}} \lambda^{\text{fix}_{>m}(\pi)} / \ell^m m!, \quad \text{for } 0 \leq m \leq n. \quad (4.29)$$

Proof of Theorem 4.5. For $1 \leq k \leq n$, if $\pi = \pi(1)\pi(2)\dots\pi(k-1)\pi(k)\pi(k+1)\dots\pi(n) \in G_{\ell,n}$, let $T(\pi)$ be the vector that record the numbers of the last $n-k$ positions in π , i.e., $T(\pi) = (\pi(k+1), \pi(k+2), \dots, \pi(n))$. For example, if $n = 12$, $k = 4$, $\pi = \bar{9} \ 5 \ \bar{4} \ 1 \ 3 \ \bar{\bar{8}} \ 2 \ 6 \ \bar{7} \ 10 \ \bar{12} \ 11 \in G_{4,12}$, then $T(\pi) = (3, \bar{\bar{8}}, \dots, 11)$. We define the relation \sim on $G_{\ell,n}$ by

$$\pi \sim \pi' \Leftrightarrow T(\pi) = T(\pi'),$$

it is easy to see this is an equivalence relation. Let us consider the map $\delta : (\eta, \pi) \rightarrow \delta(\eta, \pi)$ from $G_{\ell,m} \times G_{\ell,n}$ to $G_{\ell,n}$, where $G_{\ell,m}$ can be seen as a permutation group of colored set $C_\ell \times \{|\pi|(1), |\pi|(2), \dots, |\pi|(m)\}$. Define the permutation $\delta(\eta, \pi)$ such that $\delta(\eta, \pi)(i) = \eta(i)$ ($i \leq m$), and $\delta(\eta, \pi)(i) = \pi(i)$ ($i > m$). For example, if $\pi = \bar{9} \ 5 \ \bar{4} \ 1 \ 3 \ \bar{\bar{8}} \ 2 \ 6 \ \bar{7} \ 10 \ \bar{12} \ 11 \in G_{4,12}$, and $\eta = 5 \ \bar{4} \ 1 \ \bar{9} \in G_{4,4'}$, then

$$\delta(\eta, \pi) = 5 \ \bar{4} \ 1 \ \bar{9} \ 3 \ \bar{\bar{8}} \ 2 \ 6 \ \bar{7} \ 10 \ \bar{12} \ 11.$$

So the equivalence class of $\pi \in G_{\ell,n}$ is $\{\delta(\eta, \pi) | \eta \in G_{\ell,m}\}$, it's easy to see the cardinality of each equivalence class is $\ell^m m!$, choosing the representative of the equivalence class $\delta(\iota, \pi)$ such that

$$\text{sign}(|\iota|(i)) = 1 \quad \text{and} \quad \iota(1) > \iota(2) \cdots > \iota(m).$$

Since the fix points of π and $\delta(\iota, \pi)$ on $[m+1, n]$ keep unaltered. By Theorem 4.1, we obtain that the number of equivalence class is $g_{\ell,n}^m(\lambda) / \ell^m m!$, which yields the equation (4.29). \square

Remark 4.13. As in the proof of Theorem 4.1, we can also prove the recurrence relations of (4.3) by constructing bijections directly, the proof is left to the interested reader.

4.6 Proof of Theorem 4.6

In this section, we give two proofs of Theorem 4.6. In the first proof, we give a bijection from $C_{\ell,n}^m$ to $L_{\ell,n}^m$, that is,

$$\sum_{\pi \in L_{\ell,n}^m} \lambda^{\text{fix}_{>m}(\pi)} = \sum_{\pi \in C_{\ell,n}^m} \lambda^{\text{fix}_{>m}(\pi)} \quad \text{for } 0 \leq m \leq n. \quad (4.30)$$

In the second proof, we prove this cyclic result by constructing a equivalence relation on $G_{\ell,n}$, that is,

$$\sum_{\pi \in C_{\ell,n}^m} \lambda^{\text{fix}_{>m}(\pi)} = \sum_{\pi \in G_{\ell,n}} \lambda^{\text{fix}_{>m}(\pi)} / \ell^m m!, \quad \text{for } 0 \leq m \leq n. \quad (4.31)$$

4.6.1 First Proof

we will give a bijection $\rho : \pi \rightarrow \pi'$ from $C_{\ell,n}^m$ to $L_{\ell,n}^m$ such that $\text{fix}_{>m}(\pi) = \text{fix}_{>m}(\pi')$. First we give the map $|\pi| \rightarrow |\pi'|$ and then construct the sign transformation.

- Let $|\pi'| = |\pi'|(1), \dots, |\pi'|(m), \dots, |\pi'|(n)|$, where $|\pi'|(1), \dots, |\pi'|(m)$ in decreasing rearrangement of $|\pi|(1), \dots, |\pi|(m)$ and $|\pi'|(m+i) = |\pi|(m+i)$ ($1 \leq i \leq n-m$). Conversely, we give the reverse map by $\pi' \rightarrow \pi$ from $L_{\ell,n}^m$ to $C_{\ell,n}^m$. For $\pi' \in L_{\ell,n}^m$, we define $P := \{|\pi'|(i), i \in [m]\}$. let

$$(|\pi'|^{-s}(i), \dots, |\pi'|^{-2}(i), |\pi'|^{-1}(i), i)$$

be the cycle of $|\pi|$ containing i ($i \in [m]$), where s is the least non-negative number such that $|\pi'|^{-s}(i) \in P$ and if $|\pi'|(j) = i$ ($j \in [n]$), then $|\pi'|^{-1}(i) := j$. And setting $|\pi'|^0(i) = i$, that is, if $i \in P \cap [m]$, then $s = 0$, and i is a fixed point of $|\pi|$. The other cycles keep in accordance with $|\pi'|$.

- We define the sign transformation as follows. Since each element $i \in [m]$ in π and $\pi(i)$ ($i \in [m]$) in π' are uncolored, we exchange the sign of $|\pi|(i) \in [m]$ in π and $i \in [m]$ in π' . In other words,

$$\text{sign}_{\pi}(i) = \text{sign}_{\pi'}(|\pi|(i)) = 1 \quad \text{and} \quad \text{sign}_{\pi}(|\pi|(i)) = \text{sign}_{\pi'}(i), \quad i \in [m].$$

The signs of other elements remain unchanged, i.e.,

$$\text{sign}_{\pi'}(i) = \text{sign}_{\pi}(i), \quad i \notin [m] \cup \{|\pi|(i) | i \in [m]\}.$$

For example: For $\ell = 4, n = 12, m = 4$, $\pi = (1 \bar{9}) (2 \bar{7}) (3 \bar{5}) (4) (6 \bar{8}) (10) (11 \bar{\bar{12}}) \in C_{4,12}^4$,

$$\text{sign}_{\pi'}(1) = \text{sign}_{\pi}(|\pi|(1)) = \zeta^2, \text{sign}_{\pi'}(2) = \text{sign}_{\pi}(|\pi|(2)) = \zeta,$$

we have

$$\pi' = 9 \bar{7} 5 \bar{4} 3 \bar{8} \bar{2} 6 \bar{1} 10 \bar{\bar{12}} 11 \in L_{4,12}^4 \quad \text{and} \quad \text{fix}_{>4}(\pi) = \text{fix}_{>4}(\pi') = 1.$$

4.6.2 Second Proof

We decompose a permutation $\pi \in G_{\ell,n}$ as a product of disjoint cycles. For each $i \in [m]$, we define $\omega_{\pi}(i) = \pi(i)\pi^2(i) \dots \pi^{s-1}(i)$ where $s \geq 1$ is the least integer such that $|\pi|^s(i) \in [m]$. Obviously $\omega_{\pi}(i) = \emptyset$ if $s = 1$. Let $\Omega(\pi)$ be the product of cycles of π which have no common elements with $\{\zeta^j i | i \in [m], 0 \leq j \leq \ell - 1\}$, let $\pi_m \in G_{\ell,m}$ be the permutation obtained from π by deleting elements in $\omega_{\pi}(i)$ and the cycles in $\Omega(\pi)$ for $i \in [m]$.

For example, if $\ell = 4, n = 12, m = 4$ and $\pi = (\bar{1} \bar{9} \bar{7} \bar{2} \bar{5} \bar{3} \bar{4}) (6 \bar{8}) (10) (11 \bar{\bar{12}})$, then $\pi_4 = (\bar{1} \bar{2} \bar{3} \bar{4})$ and

$$\omega_{\pi}(1) = 9\bar{7}, \quad \omega_{\pi}(2) = 5, \quad \omega_{\pi}(3) = \emptyset, \quad \omega_{\pi}(4) = \emptyset, \quad \Omega(\pi) = (6 \bar{8}) (10) (11 \bar{\bar{12}}).$$

Setting $E(\pi) = (\omega_{\pi}(1), \omega_{\pi}(2), \dots, \omega_{\pi}(k), \Omega(\pi))$, we define the relation \sim on $G_{\ell,n}$ by

$$\pi_1 \sim \pi_2 \Leftrightarrow E(\pi_1) = E(\pi_2),$$

it is easy to see that this is an equivalence relation. Then we define the mapping $\theta : (\tau, \pi) \mapsto \theta(\tau, \pi)$ from $G_{\ell, m} \times G_{\ell, n}$ to $G_{\ell, n}$. We obtain the permutation $\theta(\tau, \pi)$ by inserting the elements $\omega_i(\pi)$ after the elements $\zeta^j i (i \in [m], 0 \leq j \leq \ell - 1)$ of τ and adding the cycles of $\Omega(\pi)$.

For example, if $\pi = (\bar{1} \ 9 \ \bar{7} \ 2 \ 5 \ \bar{3} \ 4) (6 \ \bar{8}) (10) (11 \ \bar{\bar{12}})$ and $\tau = (1 \ \bar{2}) (\bar{3}) (4)$ then

$$\theta(\tau, \pi) = (1 \ 9 \ \bar{7} \ \bar{2} \ 5) (\bar{3}) (4) (6 \ \bar{8}) (10) (11 \ \bar{\bar{12}}).$$

Obviously $\{\theta(\tau, \pi) | \tau \in G_{\ell, m}\}$ is the equivalence class of $\pi \in G_{\ell, n}$. From the construction of $\theta(\tau, \pi)$, for $\tau \in G_{\ell, m}$ and $\pi \in G_{\ell, n}^m$, we have $\theta(\tau, \pi) \sim \pi$. Conversely, if $\pi' \sim \pi$, then $\pi' = \theta(\pi'_m, \pi)$, and if $\theta(\tau, \pi) = \theta(\tau', \pi) = \pi'$ for $\tau, \tau' \in G_{\ell, m}$, then $\tau = \tau' = \pi'_m$. Hence the cardinality of each equivalence class is $\ell^m m!$. Let η be the identity permutation of $G_{\ell, m}$, then we choose $\theta(\eta, \pi)$ as the representative of each equivalence class $\{\theta(\tau, \pi) | \tau \in G_{\ell, m}\}$, that is, $\theta(\eta, \pi)$ represents the the permutation $\pi \in G_{\ell, n}$ where $\text{sign}_\pi(i) = 1 (i \in [m])$ with the first m elements belong into distinct cycles. It is obvious to see $\text{fix}_{>m}(\pi) = \text{fix}_{>m}(\theta(\eta, \pi))$. By Theorem 4.1, the number of equivalence classes is $g_{\ell, n}^m(\lambda) / \ell^m m!$, as desired.

4.7 Generating functions and further recurrence relations

In this section, by using the recurrence relation (4.2), we obtain the generating functions and further recurrence relations of $g_{\ell, n}^m(\lambda)$ and $d_{\ell, n}^m(\lambda)$.

Proposition 4.21. *For $m \geq 0$ we have the following identities:*

$$g_{\ell, n+m}^m(\lambda) = \sum_{i=0}^n (\lambda - 1)^{n-i} \binom{n}{i} \ell^{m+i} (m+i)!, \quad (4.32)$$

$$\sum_{n \geq 0} g_{\ell, n+m}^m(\lambda) \frac{u^n}{n!} = \frac{\ell^m m! \exp((\lambda - 1)u)}{(1 - \ell u)^{m+1}}, \quad (4.33)$$

$$\sum_{m, n \geq 0} g_{\ell, n+m}^m(\lambda) \frac{x^m}{m!} \frac{u^n}{n!} = \frac{\exp((\lambda - 1)u)}{1 - \ell x - \ell u}. \quad (4.34)$$

Proof. For any function $f(k) (k \geq 0) : \mathbb{Z}[\lambda] \rightarrow \mathbb{C}[\lambda]$, we define the operator

$$\Delta f(n)(\lambda) = f(n)(\lambda) + (\lambda - 1)f(n-1)(\lambda).$$

By inductions on $N \geq 0$, we have

$$\Delta^N f(n)(\lambda) = \sum_{i=0}^N (\lambda - 1)^i \binom{N}{i} f(n-i)(\lambda) = \sum_{i=0}^N (\lambda - 1)^{N-i} \binom{N}{i} f(n-N+i)(\lambda). \quad (4.35)$$

If $f(n)(\lambda) = g_{\ell, n}^n(\lambda)$, thus $g_{\ell, n+m}^{n+m-i}(\lambda) = \Delta^i f(n+m)(\lambda)$ for $i \geq 0$. From (4.35), we obtain

$$g_{\ell, n+m}^m(\lambda) = \Delta^n f(n+m)(\lambda) = \sum_{i=0}^n (\lambda - 1)^{n-i} \binom{n}{i} \ell^{m+i} (m+i)!. \quad (4.36)$$

For the above identity, multiplying both sides by $u^n/n!$ and summing over $n \geq 0$, we obtain

$$\sum_{n \geq 0} g_{\ell, n+m}^m(\lambda) \frac{u^n}{n!} = \ell^m m! \sum_{n, i \geq 0} (\lambda - 1)^{n-i} \binom{m+i}{i} \frac{\ell^i u^n}{(n-i)!}.$$

By shifting n to $n+i$, we have

$$\sum_{n \geq 0} g_{\ell, n+m}^m(\lambda) \frac{u^n}{n!} = \ell^m m! \left(\sum_{n \geq 0} (\lambda - 1)^n \frac{u^n}{n!} \right) \cdot \left(\sum_{i \geq 0} \binom{m+i}{i} (\ell u)^i \right).$$

Clearly the above equation implies (4.33) immediately. Finally multiplying both sides of (4.33) by $x^m/m!$ and summing over $m \geq 0$ yields (4.34). \square

Remark 4.14. Setting $m = 0$ in (4.32), we obtain

$$d_{\ell, n}^0(\lambda) = g_{\ell, n}^0(\lambda) = n! \sum_{i=0}^n \frac{(\lambda - 1)^i}{i!} \ell^{n-i}, \quad (4.37)$$

which implies immediately the following recurrence relation,

$$d_{\ell, n}^0(\lambda) = \ell n d_{\ell, n-1}^0(\lambda) + (\lambda - 1)^n \quad (n \geq 1). \quad (4.38)$$

Proposition 4.22. For $\ell \geq 1$ and $0 \leq m \leq n - 2$ we have

$$g_{\ell, n}^m(\lambda) = (\ell n + \lambda - 1) g_{\ell, n-1}^m(\lambda) - \ell(n - m - 1)(\lambda - 1) g_{\ell, n-2}^m(\lambda) \quad (n \geq 2); \quad (4.39)$$

$$g_{\ell, n}^m(\lambda) = \ell(n - m) g_{\ell, n-1}^m(\lambda) + \ell m g_{\ell, n-1}^{m-1}(\lambda) \quad (m \geq 1, n \geq 1); \quad (4.40)$$

$$g_{\ell, n}^m(\lambda) = \ell n g_{\ell, n-1}^m(\lambda) + \ell m(\lambda - 1) g_{\ell, n-2}^{m-1}(\lambda) \quad (m \geq 1, n \geq 2), \quad (4.41)$$

where $g_{\ell, 0}^0(\lambda) = 1$, $g_{\ell, 1}^0(\lambda) = \lambda + \ell - 1$ and $g_{\ell, 1}^1(\lambda) = \ell$.

Proof. Let $F(u)$ denote the left-hand side of (4.33). By using the differentiation of $F(u)$ and (4.33), we obtain

$$(1 - \ell u)F'(u) = [\ell(m + 1) + (\lambda - 1)(1 - \ell u)]F(u). \quad (4.42)$$

By equating the coefficients of $u^n/n!$ in (4.42), we have

$$g_{\ell, n+m+1}^m(\lambda) = [\ell(m + n + 1) + \lambda - 1] g_{\ell, n+m}^m(\lambda) - \ell n(\lambda - 1) g_{\ell, n+m-1}^m(\lambda),$$

shifting $n + m + 1$ to n yields (4.39) immediately.

Then multiplying both sides of (4.33) by $1 - \ell u$, we have

$$(1 - \ell u) \sum_{n \geq 0} g_{\ell, n+m}^m(\lambda) \frac{u^n}{n!} = \frac{\ell^m m! \exp((\lambda - 1)u)}{(1 - \ell u)^m} = \ell m \sum_{n \geq 0} g_{\ell, n+m-1}^{m-1}(\lambda) \frac{u^n}{n!}. \quad (4.43)$$

By equating the coefficients of $u^n/n!$, we have

$$g_{\ell, n+m}^m(\lambda) - \ell n g_{\ell, n+m-1}^m(\lambda) = \ell m g_{\ell, n+m-1}^{m-1}(\lambda), \quad (4.44)$$

shifting $n + m$ to n yields (4.40).

Finally, from (4.40) and (4.2), we have

$$\begin{aligned} g_{\ell, n}^m(\lambda) &= \ell n g_{\ell, n-1}^m(\lambda) - \ell m (g_{\ell, n-1}^m(\lambda) - g_{\ell, n-1}^{m-1}(\lambda)) \\ &= \ell n g_{\ell, n-1}^m(\lambda) + \ell m(\lambda - 1) g_{\ell, n-2}^{m-1}(\lambda), \end{aligned}$$

which yields (4.41), the proof is completed. \square

With the above Proposition 4.22, we derive the following propositions immediately.

Proposition 4.23. *For $\ell \geq 1$ and $0 \leq m \leq n - 2$ we have*

$$d_{\ell,n}^m(\lambda) = (\ell n + \lambda - 1)d_{\ell,n-1}^m(\lambda) - \ell(n - m - 1)(\lambda - 1)d_{\ell,n-2}^m(\lambda) \quad (n \geq 2); \quad (4.45)$$

$$d_{\ell,n}^m(\lambda) = \ell(n - m)d_{\ell,n-1}^m(\lambda) + d_{\ell,n-1}^{m-1}(\lambda) \quad (m \geq 1, n \geq 1); \quad (4.46)$$

$$d_{\ell,n}^m(\lambda) - (\lambda - 1)d_{\ell,n-2}^{m-1}(\lambda) = \ell n d_{\ell,n-1}^m(\lambda) \quad (m \geq 1, n \geq 2), \quad (4.47)$$

where $d_{\ell,0}^0(\lambda) = 1$, $d_{\ell,1}^0(\lambda) = \lambda + \ell - 1$ and $d_{\ell,1}^1(\lambda) = 1$.

Proof. With Proposition 4.22, we can get these equations (4.45), (4.46) and (4.47) directly. \square

Remark 4.15. Setting $\ell = 1$, (4.3), (4.45), and (4.47) reduce to the result of Eriksen et al. [35, Propositions 8.1, 8.3 and 8.2]. In this case, (4.33) and (4.34) recover the result of Rakotondrajao [84, Theorem 6.7 and Theorem 6.8]. Setting $\lambda = 0$, Propositions 4.21, 4.22 and 4.23 reduce to the result of Faliharimalala and Zeng [40, Propositions 17, 18, and 19].

4.8 Proof of Theorem 4.7

In this section, to prove Theorem 4.7, we prove the following equation,

$$\sum_{\pi \in F_{\ell,n+m}^m} \lambda^{\text{fix}_{>m}(\pi)} = \frac{(n+m)!}{n!} \sum_{\pi \in C_{\ell,n+m}^m} \lambda^{\text{fix}_{>m}(\pi)} \quad \text{for } m, n \geq 0. \quad (4.48)$$

And with the generating functions of $d_{\ell,n}^m(\lambda)$, we obtain the generating functions and recurrence relations of $f_{\ell,n}^m(\lambda)$.

Proof of Theorem 4.7. For $\ell \geq 1$, we construct such a permutation $\pi \in F_{\ell,n+m}^m$ in following way, see Definition 4.7.

Starting from the set $[n+2m]$, we take m elements from the set $[m+1, 2m]$ as the image of $[1, m]$, which is labeled as $\pi(i) (i \in [m])$. Clearly there are $\binom{n+m}{m} m!$ ways to choose. Let $i' (i \in [m])$ represent the two element set $\{i, \pi(i)\}$, and let $i' (i \in [m+1, n+m])$ denote the remaining element $[n+2m] \setminus \{i, \pi(i)\}$. Let π' denote the permutation on the colored set $C_\ell \times \{1', 2', 3', \dots, (m+n)'\}$ such that $\text{sign}_{\pi'}(i') = 1 (i' \in [m])$ and $i' (i' \in [m])$ belong into distinct cycles, by transforming the i' into $\{i, \pi(i)\}$, we obtain the desired permutation in $F_{\ell,n+m}^m$ and vice versa. From this construction, we have $\text{fix}_{>m}(\pi) = \text{fix}_{>m}(\pi')$. This completes the proof. \square

Theorem 4.24. *For $\ell \geq 1$ and $0 \leq m \leq n$, we have*

$$\sum_{n \geq 0} f_{\ell,n}^m(\lambda) \frac{u^n}{n!} = \frac{u^m \exp(\lambda - 1)u}{(1 - \ell u)^{m+1}}. \quad (4.49)$$

Proof. According to the generating function (4.33) of $g_{\ell,n}^m(\lambda)$, it is clear to see that

$$\sum_{n \geq 0} d_{\ell,n+m}^m(\lambda) \frac{u^n}{n!} = \frac{\exp(\lambda - 1)u}{(1 - \ell u)^{m+1}}.$$

For the above identity, multiplying both sides by u^m , we obtain

$$\sum_{n \geq 0} \frac{(n+m)!}{n!} d_{\ell, n+m}^m(\lambda) \frac{u^{n+m}}{(n+m)!} = \frac{u^m \exp(\lambda-1)u}{(1-\ell u)^{m+1}}.$$

With Theorem 4.7,

$$\sum_{n \geq 0} f_{\ell, n+m}^m \frac{u^{n+m}}{(n+m)!} = \frac{u^m \exp(\lambda-1)u}{(1-\ell u)^{m+1}},$$

which is (4.49) by shifting $n+m$ to n . \square

With (4.3), (4.45), (4.46), (4.47) and Theorem 4.7, we obtain the following corollary.

Corollary 4.25. *For $\ell \geq 1, 1 \leq m \leq n-2$, we have*

$$(n-m+1)f_{\ell, n}^{m-1}(\lambda) = \ell m f_{\ell, n}^m(\lambda) + (\lambda-1)n f_{\ell, n-1}^{m-1}(\lambda); \quad (4.50)$$

$$(n-m)f_{\ell, n}^m(\lambda) = n(\ell n-1+\lambda)f_{\ell, n-1}^m(\lambda) - \ell(\lambda-1)n(n-1)f_{\ell, n-2}^m(\lambda); \quad (4.51)$$

$$f_{\ell, n}^m(\lambda) = \ell n f_{\ell, n-1}^m(\lambda) + n f_{\ell, n-1}^{m-1}(\lambda); \quad (4.52)$$

$$(n-m)f_{\ell, n}^m(\lambda) = (\lambda-1)n(n-1)f_{\ell, n-2}^{m-1}(\lambda) + \ell n^2 f_{\ell, n-1}^m(\lambda); \quad (4.53)$$

where $f_{\ell, 0}^0(\lambda) = 1$, $f_{\ell, 1}^0(\lambda) = \lambda + \ell - 1$ and $f_{\ell, 1}^1(\lambda) = 1$.

With (4.51) and (4.52), we have the following corollary.

Corollary 4.26. *For $\ell \geq 1$ and $1 \leq m \leq n-2$, we have*

$$f_{\ell, n}^m(\lambda) = m f_{\ell, n-1}^{m-1}(\lambda) - \ell(\lambda-1)(n-1)f_{\ell, n-2}^m(\lambda) + (\ell m + \ell n - 1 + \lambda)f_{\ell, n-1}^m(\lambda). \quad (4.54)$$

Remark 4.16. When $(\ell, \lambda) = (1, 0)$, (4.49) and (4.54) reduce to the results of [107, Theorem 3 and Theorem 2].

4.9 Combinatorial proof of recurrence relation (4.46)

In this section, we give the combinatorial proof of recurrence (4.46), that is,

$$\sum_{\pi \in C_{\ell, n}^m} \lambda^{\text{fix} > m(\pi)} = \ell(n-m) \sum_{\pi \in C_{\ell, n-1}^m} \lambda^{\text{fix} > m(\pi)} + \sum_{\pi \in C_{\ell, n-1}^{m-1}} \lambda^{\text{fix} > m-1(\pi)}, \quad (4.55)$$

other recurrences (4.45) and (4.47) can be proved in similar ways.

Lemma 4.27. *For $0 \leq m \leq n$,*

$$\sum_{\substack{\pi \in C_{\ell, n}^m \\ m \in \text{FIX}(\pi)}} \lambda^{\text{fix} > m(\pi)} = \sum_{\pi \in C_{\ell, n-1}^{m-1}} \lambda^{\text{fix} > m-1(\pi)}. \quad (4.56)$$

Proof. It follows similar arguments as in the proof of Lemma 4.8. \square

Lemma 4.28. *For $0 \leq m \leq n$,*

$$\sum_{\substack{\pi \in C_{\ell, n}^m \\ m \notin \text{FIX}(\pi)}} \lambda^{\text{fix} > m(\pi)} = \ell(n-m) \sum_{\pi \in C_{\ell, n-1}^m} \lambda^{\text{fix} > m(\pi)}. \quad (4.57)$$

Proof. Let us consider the map $\chi : \pi \rightarrow (\epsilon, \beta, \pi')$ from $C_{\ell, n}^m \cap \{\pi \in G_{\ell, n} \mid m \notin \text{FIX}(\pi)\}$ to $C_\ell \times [n - m] \times C_{\ell, n-1}^m$ such that $\text{fix}_{>m}(\pi) = \text{fix}_{>m}(\pi')$.

For $\pi \in C_{\ell, n}^m \cap \{\pi \in G_{\ell, n} \mid m \notin \text{FIX}(\pi)\}$, we decompose π as the product of disjoint cycles. Let $\pi(m) = \beta$, it is easy to see $|\beta| \in [m + 1, n]$ and $\text{sign}_\pi(\beta) = \epsilon$.

For the element $i \in \pi$, we delete the element β and define the element $i' \in [n - 1]$ in π' by

$$i' = \begin{cases} i, & \text{if } |i| < |\beta|; \\ i - 1, & \text{if } |i| > |\beta|. \end{cases}$$

Conversely, starting from $(\epsilon, \beta, \pi') \in C_\ell \times [n - k] \times C_{\ell, n-1}^m$, for the element $i' \in \pi'$, we define the element $i \in [n]$ in π by

$$i = \begin{cases} i', & \text{if } |i'| < |\beta|; \\ i' + 1, & \text{if } |i'| \geq |\beta|, \end{cases}$$

and let $\pi(m) = (\epsilon, \beta)$. □

For example, let $\ell = 4, n = 9, k = 4$, if $\pi = (1 \bar{7}) (2 \bar{5}) (3 \bar{8}) (4 \bar{9}) (6) \in C_{4,9}^4$,

$$\epsilon = \zeta, \beta = 9, \quad \pi' = (1 \bar{7}) (2 \bar{5}) (3 \bar{8}) (4) (6) \in C_{4,8}^4 \quad \text{and} \quad \text{fix}_{>4}(\pi) = \text{fix}_{>4}(\pi').$$

for $\pi = (1 \bar{7}) (2) (3 \bar{8}) (4 \bar{5}) (6) (9) \in C_{4,9}^4$,

$$\epsilon = \zeta^2, \beta = 5, \quad \pi' = (1 \bar{6}) (2) (3 \bar{7}) (4) (5) (8) \in C_{4,8}^4 \quad \text{and} \quad \text{fix}_{>4}(\pi) = \text{fix}_{>4}(\pi').$$

Proof of equation (4.55). By considering the following equation,

$$\sum_{\pi \in C_{\ell, n}^m} \lambda^{\text{fix}_{>m}(\pi)} = \sum_{\substack{\pi \in C_{\ell, n}^m \\ m \notin \text{FIX}(\pi)}} \lambda^{\text{fix}_{>m}(\pi)} + \sum_{\substack{\pi \in C_{\ell, n}^m \\ m \in \text{FIX}(\pi)}} \lambda^{\text{fix}_{>m}(\pi)}, \quad (4.58)$$

by Lemma 4.27, the (4.58) is equivalent to

$$\sum_{\pi \in C_{\ell, n}^m} \lambda^{\text{fix}_{>m}(\pi)} = \sum_{\substack{\pi \in C_{\ell, n}^m \\ m \notin \text{FIX}(\pi)}} \lambda^{\text{fix}_{>m}(\pi)} + \sum_{\pi \in C_{\ell, n-1}^{m-1}} \lambda^{\text{fix}_{>m-1}(\pi)}. \quad (4.59)$$

By Lemma 4.28, we obtain (4.55) immediately. This completes the proof. □

4.10 Final remarks

Faliharimalala and Zeng [41, eq. (1.2)] studied the wreath product analogue of Euler's q -difference table $\{g_{\ell, n}^m(q)\}_{0 \leq m \leq n}$ as follows.

Definition 4.8 (Faliharimalala-Zeng). For fixed integer $\ell \geq 1$, the coefficients of Euler's q -difference table $(g_{\ell, n}^m(q))_{0 \leq m \leq n}$ for $C_\ell \wr S_n$ are defined by

$$\begin{cases} g_{\ell, n}^n(q) = [\ell]_q [2\ell]_q \cdots [n\ell]_q, & (m = n); \\ g_{\ell, n}^m(q) = g_{\ell, n}^{m+1}(q) - q^{\ell(n-m-1)} g_{\ell, n-1}^m(q) & (0 \leq m \leq n - 1). \end{cases} \quad (4.60)$$

Faliharimalala and Zeng found a combinatorial interpretation of $(g_{\ell, n}^m(q))_{0 \leq m \leq n}$ by introducing a new Mahonian statistic $fmaf$ on the wreath products. So the natural question is to find a q - λ -Euler's difference table for λ -Euler's difference table in Definition 4.1, it seems the statistic $fmaf$ cannot help directly.

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Index

- $g_{\ell,n}^m(\lambda)$, 99
 (13-2), 16
 (2-13), 16
 (2-31), 16
 (31-2), 16
 (\mathbf{s}, \mathbf{p}) , 21
 $(d_{\ell,n}^m(\lambda))_{0 \leq m \leq n}$, 100
 $(g_n^m(\lambda))_{0 \leq m \leq n}$, 10
 $(g_n^m)_{0 \leq m \leq n}$, 10
 $(g_{\ell,n}^m)_{0 \leq m \leq n}$, 10
 $(g_{\ell,n}^m(\lambda))_{0 \leq m \leq n}$, 99
 $(g_{\ell,n}^m(q))_{0 \leq m \leq n}$, 121
 (q, t) -Catalan number, 49, 50
 $<_c$, 47
 $A_n(p, q, t)$, 27
 $A_n(p, q, t, u, v, w)$, 40
 $A_n(t)$, 2, 3, 13
 $A_n(x, y, q, p, s)$, 53
 $A_n^{(\text{cyc}^* - \text{fix}^*, \text{exc})}(q, t)$, 28
 $B_n(p, q, t, u, v, w, y)$, 41
 $B_n(y, t)$, 31
 $B_{n,m}^\ell$, 107
 $C_n^*(q)$, 65
 $C_0(z, x)$, 9, 80
 $C_n(q, t, u, v, w)$, 41
 $C_n(t, q)$, 49
 $C_{\ell,n}^m$, 104
 $D_n(q, t, u, v, w)$, 41
 $D_n^{(\text{stat}_1, \text{stat}_2)}(q, t)$, 28
 D_n^ℓ , 110
 $D_{2n}(p, q, x, u, v, w)$, 82
 $D_{2n}(x)$, 83
 $D_{2n}^{\text{cyc}}(\beta, x, u, v, w)$, 83
 FIX , 103
 $F_n(t, w, r, y)$, 47
 $F_{\ell,n+m}^m$, 105
 $G_n(t)$, 72
 $J[z; b_n, \lambda_n]$, 47
 $J_{2n+1}(p, q, x, u, v, w)$, 80
 $J_{2n+1}(x)$, 81
 $J_{2n}(p, q, x)$, 96
 $J_{2n}(p, q, x, u, v, w, y)$, 82
 $J_{2n}(x)$, 82
 $L_r^*((\mathbf{s}, \mathbf{p}))$, 32
 $L_{\ell,n}^m$, 104
 $N_n(t)$, 7, 15
 $N_n(t, q, r)$, 29
 $P^{(\text{stat}_1, \dots, \text{stat}_m)}(\Omega; t_1, \dots, t_m)$, 26
 $P_n(t, q)$, 64
 $P_n^{\text{peak}}(x)$, 7
 $P_n^{(\text{peak}, \text{des})}(x, t)$, 7
 $Q_n(t, q)$, 64
 $R(x)$, 73
 $R_n(t, q)$, 64
 $SUC_k(\pi)$, 101
 $S_1(z, x)$, 9, 79
 $S_n(t)$, 73
 $U(x)$, 76
 $W_n(t, q)$, 62
 $Y_n(t)$, 73
 \mathbb{A}_n , 3
 Cda^* , 19
 Cda , 17
 Cdd^* , 19
 Cdd , 17
 Cpk^* , 19
 Cpk , 17
 Cval^* , 19
 Cval , 17
 DD_{2n}^* , 83
 DD_{2n} , 83
 $\mathfrak{D}\mathfrak{H}_{2n}$, 85
 \mathcal{DP}_n , 80
 \mathfrak{D}_n^* , 5, 53
 \mathfrak{D}_n , 5
 Da , 16
 Dd , 16
 DES , 4

- Drop, 17
 EXC, 4
 \mathcal{FCDP}_n , 81
 Fix*, 19
 Fix, 17
 INV, 4
 \mathcal{LH}_n^* , 21
 \mathcal{LH}_n , 21
 Lda, 17
 Ldd, 17
 Lpk, 17
 Lval, 17
 $\text{Orb}((\mathbf{s}, \mathbf{p}))$, 33
 $\text{Orb}(\sigma)$, 37, 59, 88
 Pk, 16
 \mathcal{SCDP}_n , 81
 $\mathfrak{S}_{n,j}$, 34
 $\mathfrak{S}_{n,k,j}$, 35
 $\mathfrak{S}_{n,k,j}(321)$, 35
 Scpeak, 39
 Scval, 39
 Scda, 39
 Scdn, 39
 Val, 16
 13-2, 24
 ai π , 56
 ai* π , 56
 amax, 19
 amin, 17
 2-13, 24
 2-31, 24
 31-2, 24
 cda, 17
 cda*, 19
 cdd, 17
 cdd*, 19
 cpk, 17
 cpk*, 19
 cvalley, 17
 cvalley*, 19
 cyc, 19
 cyc*, 19
 da, 16
 dd, 16
 des, 4
 $\text{des}_B(\sigma)$, 31
 drop, 17
 drop*, 19
l-hit number, 103
l-rook number, 102
 $NDE_{n,k}(321)$, 55
 $NDW_{n,k}(321)$, 55
 exc, 4
 $\text{exc}_B(\sigma)$, 31
 fix, 17, 47
 fix*, 19
 fl, 66
 fmax, 8, 17
 fmin, 19
 $\gamma_{n,k}(q)$, 30, 50
 $\gamma_{n,k}^A$, 5
 $\gamma_{n,k}^N$, 7, 72
 $\gamma_{n,k}^S$, 73
 $\gamma_{n,k}^Y$, 73
 γ -positive, 2, 15, 80
 icr, 17, 88
 ine, 17, 88
 inv, 4
 λ -Euler's difference table, 10, 99
 lda, 17
 ldd, 17
 lpeak, 17
 lvalley, 17
 maj, 4
 $\mathbb{A}_{2n}(231)$, 63
 $\mathcal{DD}_{n,k}$, 23
 $\mathcal{DP}_{2n+1,2k}$, 81
 $\mathcal{O}_{n,k,j}$, 35
 $\mathcal{R}_{n,j}$, 33
 \mathcal{CM}_n^* , 24
 \mathcal{CM}_n , 24, 84
 $\mathfrak{a}_{n,k}^p$, 70
 $\mathfrak{a}_{n,k}^v$, 70
 $\mathfrak{a}_{n,k}$, 69
 $\text{DE}_{n,k}^*$, 23
 $\text{SDE}_{n,k}$, 32
 Wex*, 19
 Drop*, 19
 $\text{Dyck}(n; t, q)$, 49
 MAD, 8, 52
 cros, 8, 17, 53
 exc_A , 47
 neg, 47
 nest, 8, 17, 53
 wex_A , 47
 wex_C , 47
 wex, 17

- $\overline{C}_n(q)$, 68
 $\overline{\mathfrak{S}}_{n,k}(132)$, 30
 $\overline{\mathfrak{S}}_{n,k}(213)$, 30
 $\overline{\mathfrak{S}}_{n,k}(231)$, 30
 $\overline{\mathfrak{S}}_{n,k}(312)$, 30
 $\overline{\mathfrak{S}}_{n,k}(321)$, 30
 $\overline{\varphi}_x$, 67
 $\overline{\mathfrak{D}}_{n,k}^*(132)$, 67
 $\overline{\mathfrak{S}}_n(321)$, 30
 peak, 16
 peak', 6
 ϕ'_{FZ} , 32
 ϕ_{FV} , 23
 ϕ_{FZ} , 23
 ψ , 22
 ψ_{FV} , 22
 ψ_{YZL} , 31
 DM_{2n} , 85
 $\text{sn}(z, x)$, 8, 79
 valley, 16
 wex^* , 19
 $\tilde{\Phi}$, 25
 $\tilde{\Psi}$, 25
 $\tilde{\phi}_{FV}$, 25
 $\tilde{\phi}_{FZ}$, 25
 $\tilde{\psi}$, 25
 $\tilde{\psi}_{FV}$, 25
 $\{F_n\}_{n \geq 1}$, 72
 $\{P_n\}_n$, 48
 $\{Q_n\}_n$, 48
 $a_{2n+1,2k}(p, q)$, 81
 $c_{\ell,n,s}^m$, 102
 $d_{\ell,n}^m(\lambda)$, 100
 g_n^m , 10, 99
 $g_n^m(\lambda)$, 10
 $g_{\ell,n}^m$, 10
 $h_{k,n}^\ell(B)$, 103
 k -succession, 101
 q -Narayana polynomials, 29
 r -derangement number, 10, 100
 $r_{k,n}^\ell(B)$, 102
 $\text{fix}_{>m}$, 103
 $\text{cn}(z, x)$, 8, 79
 $\text{dn}(z, x)$, 8, 79
 $\text{suc}_{>m}^{(k)}(\pi)$, 101
 2-Motzkin path, 21, 24, 84
 231-avoiding permutation, 7, 15, 23
 Admissible inversion, 56
 Alternating permutation, 51, 96
 Catalan number, 15, 49
 Coderrangement, 5, 53
 Colored permutation group, 10, 100
 Crossing number, 17
 Cyclic Doubled Permutation, 81, 84
 Derangement, 5, 10, 28, 99
 Descent, 4
 Descent number, 4
 Doubled Laguerre history, 85, 90
 Doubled path, 84
 Doubled permutation, 9, 80
 Euler number, 3, 4, 8
 Euler's difference table, 10, 99
 Eulerian polynomial, 2, 7, 13
 Eulerian statistic, 4
 Excedance, 4
 Excedance number, 4
 Falling alternating permutation, 3, 96
 Foata-Zeilberger bijection, 14, 23, 32, 90
 Françon-Viennot bijection, 22, 23, 50, 53
 Group action on derangement, 37
 Group action on doubled permutation, 88
 Group action on Laguerre history, 33
 Inverse crossing number, 17, 88
 Inverse nesting number, 17, 88
 Inversion, 4
 Inversion number, 4
 Jacobi elliptic function, 8, 79
 Laguerre history, 21–23
 Log-concave, 1
 Mahonian statistic, 5
 Major index, 4
 Narayana polynomial, 7, 15
 Nesting number, 17
 Pattern avoidance, 6, 23, 50
 Peak polynomial, 7, 13
 Permutation, 3
 Real rooted, 1
 Restricted Laguerre history, 21, 32
 Secant number, 4
 Sign type B Eulerian polynomial, 31

Star admissible inversion, 56

Statistic, 3

Symmetric, 1

Tangent number, 4

Unimodal, 1

Vincular pattern, 24