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Sébastien Gontard

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## THÈSE

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### **Courbures de métriques invariantes dans les variétés complexes non compactes**

### **Curvatures of metrics in non-compact complex manifolds**

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Curvatures of invariant metrics in noncompact  
complex manifolds

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October 7, 2019

## Résumé

Nous étudions les relations entre des propriétés géométriques et des propriétés métriques dans les domaines de  $\mathbb{C}^n$ . Plus précisément, étant donné un domaine pseudoconvexe borné  $D$  à bord  $\partial D$  lisse, nous nous intéressons au comportement asymptotique des courbures bisectionnelles holomorphes de métriques de Kähler invariantes, la métrique de Bergman et la métrique de Kähler-Einstein, en un point  $p$  de  $\partial D$ .

Lorsque  $p$  est un point de stricte pseudoconvexité,  $\partial D$  ressemble localement au bord d'une boule et les courbures de  $D$  sont asymptotiquement proches des courbures de cette boule à mesure que l'on se rapproche de  $p$ . Ce phénomène est également vrai lorsque la fonction de squeezing de  $D$  tend vers 1 en  $p$ .

Si  $p$  est un point de faible pseudoconvexité de type fini,  $\partial D$  ressemble localement à un domaine polynomial "modèle", et l'on s'attend à ce que les courbures de  $D$  s'approchent des courbures de ce "modèle". Il est donc naturel d'étudier les courbures des métriques de Bergman et de Kähler-Einstein dans ces domaines. Dans certains de ceux-ci (les domaines de Thullen et les domaines tubes dans  $\mathbb{C}^2$ ), les courbures bisectionnelles holomorphes des métriques suscitées sont pincées négativement. Ces résultats permettent de prouver que si  $D \subset \mathbb{C}^2$  est convexe ou Reinhardt complet et si un modèle en  $p$  est soit un domaine de Thullen soit un domaine tube, alors les courbures bisectionnelles holomorphes de  $D$  sont pincées négativement dans un voisinage conique de  $p$ .

## Abstract

We study the relationships between geometric and metric properties in domains of  $\mathbb{C}^n$ . Specifically, given a bounded pseudoconvex domain  $D$  with smooth boundary  $\partial D$ , we study the asymptotic behaviour of the holomorphic bisectional curvatures of invariant Kähler metrics, namely the Bergman metric and the Kähler-Einstein metric, at a point  $p$  of  $\partial D$ .

If  $p$  is a strict pseudoconvex point,  $\partial D$  locally looks like the boundary of a ball and the curvatures of  $D$  are asymptotically close to the curvatures of the ball as we approach  $p$ . This phenomenon also holds when the squeezing function of  $D$  tends to 1 at  $p$ .

If  $p$  is a weakly pseudoconvex boundary point of finite type in the sense of D'Angelo,  $\partial D$  locally looks like a polynomial model domain, and we expect that the curvatures of  $D$  approach curvatures of this model. It is therefore natural to study the curvatures of the Bergman and the Kähler-Einstein metrics in such domains. In some of these (namely

Thullen domains and tube domains in  $\mathbb{C}^2$ ), the holomorphic bisectional curvatures of the metrics mentioned above are negatively pinched. These results enable to prove that if  $D \subset \mathbb{C}^2$  is convex or complete Reinhardt and if a model at  $p$  is either a Thullen domain or a tube domain, then the holomorphic bisectional curvatures of  $D$  are negatively pinched in a conical neighbourhood of  $p$ .

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# Introduction

Given a bounded domain  $D$  in  $\mathbb{C}^n$  or a complex manifold with boundary it is natural to study the relationships between quantities containing information about the complex geometry of  $D$ . For instance one can ask whether the curvature of the boundary of  $D$  is related to the curvature of objects defined on  $D$ , such as invariant Kähler metrics. A Kähler metric on  $D$  associates with every point of  $D$  a way to measure angles between two directions, and the metric is called invariant if it respects the symmetries of the domain. If the metric is smooth, we may study its curvatures, which roughly measure the variations of the metric with respect to the point. We expect that the behaviour of the curvatures of the metric is influenced by the geometry of the boundary  $\partial D$  of  $D$ , at least when we look at points of  $D$  close to  $\partial D$ . As an example, we may hope that if  $\partial D$  “looks like the boundary of a ball  $B$ ” near a given point  $q$  of the boundary, then  $D$  is metrically “curved like  $B$ ” when we look at points in  $D$  near  $q$ .

The discussion can be formalised as follows. Let  $n \in \mathbb{N} \setminus \{0, 1\}$  be an integer, let  $D \subset \mathbb{C}^n$  be a bounded domain with boundary of class  $\mathcal{C}^\infty$ . We also assume that  $D$  is pseudoconvex and  $\partial D$  is of finite type in the sense of D’Angelo (see Definitions 1.5 and 1.11). Let  $[g_{i\bar{j}}]$  be either the Kähler-Einstein metric of  $D$  with Ricci curvature  $-(n+1)$  or the Bergman metric of  $D$  (see Section 1.4). We study the following question:

## **Question 1. Negative pinching**

*Does there exist a neighbourhood  $V$  of  $\partial D$  such that the holomorphic bisectional curvatures of the metric  $[g_{i\bar{j}}]$  are negatively pinched on  $D \cap V$ ?*

The first studies of the boundary behaviour of the curvatures of these metrics goes back to the late 1970. The answer to Question 1 is known for strictly pseudoconvex domains. Indeed, P. Klembeck proved in [39] that the Riemannian sectional curvatures (so in particular the holomorphic bisectional curvatures) of the Bergman metric near the boundary of a bounded strictly pseudoconvex domain with smooth boundary behave like

the ones of the Bergman metric on the ball and S.-Y. Cheng and S.-T. Yau proved in [12] a similar result for the holomorphic sectional curvatures of the Kähler-Einstein metric. Let  $q \in \partial D$ . It is natural to expect that the behaviour of the curvatures of the metric  $[g_{i\bar{j}}]$  in a neighbourhood of  $q$  depends only on the Cauchy-Riemann (CR) geometry of  $\partial D$  at  $q$ . Some results support this idea. For instance K.T. Kim and J. Yu proved in [37] that if  $q \in \partial D$  is a strictly pseudoconvex boundary point of  $D$ , then there exists a neighbourhood  $V$  of  $q$  such that the holomorphic sectional curvatures of the Bergman metric of  $D$  are negatively pinched on  $D \cap U$ . This result can be extended to holomorphic bisectonal curvatures, see Section 6 in [36]. Regarding the Kähler-Einstein metric, J. Bland proved a local version of the result of S.-Y. Cheng and S.-T. Yau if  $n \geq 6$  (see [4]).

If  $\partial D$  is not strictly pseudoconvex at  $q$ , the situation becomes more complicated and curvatures conditions on  $\partial D$  are necessary. The notion of finite type is a natural generalisation of strict pseudoconvexity in the pseudoconvex setting (see [14]). G. Herbort constructed an example of a bounded pseudoconvex domain with smooth boundary of finite type for which the holomorphic sectional curvatures of the Bergman metric are not bounded from below, see the conclusion for a discussion on the subject. This answers negatively Question 1 for a general pseudoconvex domain of finite type. However a domain  $D$  of finite type at  $q \in \partial D$  can be osculated by a model domain, in local coordinates near  $q$ , whose study should be simpler. Thus the natural question is:

**Question 2. Local behaviour**

*Does the metric of  $D$  behave like the metric of the corresponding model near  $q$ ?*

The answer is affirmative for the Bergman metric for a large class of domains in  $\mathbb{C}^n$  when dealing with holomorphic sectional curvatures (see [7]). A pinching of the holomorphic sectional curvatures of the Bergman metric in the associated model near  $q$  yields to a pinching of the same quantities for the domain in a neighbourhood of  $q$ . As an example, S. Fu proved that the local model at every boundary point of a bounded pseudoconvex complete Reinhardt domain of finite type  $D \subset \mathbb{C}^2$  is either a Thullen domain or tube domain (see [22]). Using estimates of the holomorphic sectional curvatures of the Bergman metric in Thullen domains obtained by K. Azukawa and M. Suzuki in [2] he proved that the holomorphic sectional curvatures of the Bergman metric of  $D$  are negatively pinched in a neighbourhood of  $\partial D$ .

In the case of the Kähler-Einstein metric the answer is not known, essentially for the

following two reasons: lack of localisation results for the Kähler-Einstein metric near weakly pseudoconvex boundary points, and lack of knowledge about the behaviour of the Kähler-Einstein metric in model domains. Indeed, the only model domains for which estimates of the holomorphic bisectional curvatures of the Kähler-Einstein metric are known are the Thullen domains, studied by J.S. Bland in [5].

Our objective has been to provide with estimates of the holomorphic bisectional curvatures of the Kähler-Einstein metric and the Bergman metric in some model domains, and to deduce estimates in some classes of domains in  $\mathbb{C}^n$ . More precisely, in Chapter 2 we study the Kähler-Einstein metric at boundary points at which the domain looks like a ball, namely strictly pseudoconvex boundary points and points at which the squeezing function of the domain tends to 1 (see [57]). We prove:

**Theorem 1.** *Let  $n \geq 2$ , let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$ , and let  $q \in \partial D$ . If  $q$  is a strictly pseudoconvex point of  $\partial D$  or if the squeezing function of  $D$  tends to 1 at  $q$  then,*

$$\sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left( \text{Bis}(g_{KE}^D)(z; v, w) + \left( 1 + \left( \frac{|\langle v, w \rangle_{g_{KE}^D, z}|}{\langle v, v \rangle_{g_{KE}^D, z} \langle w, w \rangle_{g_{KE}^D, z}} \right)^2 \right) \right) \xrightarrow{z \rightarrow q} 0.$$

Here,  $\text{Bis}(g_{KE}^D)(z; v, w)$  (respectively  $\langle v, w \rangle_{g_{KE}^D, z}$ ) denotes the holomorphic bisectional curvature (respectively the Hermitian scalar product) of the Kähler-Einstein metric of Ricci curvature  $-(n+1)$ , computed at point  $z$  and between vectors  $v$  and  $w$ . Notice that under the assumptions of Theorem 1, a model domain might be the unit ball  $\mathbb{B}$  in  $\mathbb{C}^n$  whose bisectional curvatures for the Kähler-Einstein metric of Ricci curvature  $-(n+1)$  are precisely  $-\left( 1 + \left( \frac{|\langle v, w \rangle_{g_{KE}^{\mathbb{B}}, z}|}{\langle v, v \rangle_{g_{KE}^{\mathbb{B}}, z} \langle w, w \rangle_{g_{KE}^{\mathbb{B}}, z}} \right)^2 \right)$  for every  $z \in \mathbb{B}$  and  $v, w \in \mathbb{C}^n \setminus \{0\}$  (hence the idea of "looking like a ball"). See Theorems 2.4, 2.5 and 2.6 for more detailed results.

In Chapter 3 we study the Kähler-Einstein metric in pseudoconvex domains in  $\mathbb{C}^2$ . We first focus on tube domains  $T'_p = \{z \in \mathbb{C}^2, \text{Re}(z_1) + \text{Re}(z_2)^{2p} < 0\}$  with  $p \in \mathbb{N}$  and prove that the holomorphic bisectional curvatures of  $T'_p$  are negatively pinched in certain "approach regions" of the weakly pseudoconvex point  $(0, 0)$ . More precisely:

**Theorem 2.** *There exist positive constants  $0 < c \leq C$  and  $0 < \alpha < 1$  such that the*

following holds for every  $z \in T'_p \cap \left( \left\{ \frac{\operatorname{Re}(z_2)^{2p}}{-\operatorname{Re}(z_1)} \leq \alpha \right\} \cup \left\{ 1 - \alpha \leq \frac{\operatorname{Re}(z_2)^{2p}}{-\operatorname{Re}(z_1)} < 1 \right\} \right)$ :

$$\forall v, w \in \mathbb{C}^2 \setminus \{0\}, \quad -C \leq \operatorname{Bis} \left( g_{KE}^{T'_p} \right) (z; v, w) \leq -c.$$

Using Theorem 2 and the study of the Kähler-Einstein metric in Thullen domains we prove the following (see Theorem 3.3):

**Theorem 3.** *Let  $D \subset \mathbb{C}^2$  be a bounded convex domain with boundary of class  $\mathcal{C}^\infty$ . Let  $q \in \partial D$  be of finite type and such that a local model at  $q$  is either a Thullen domain or a tube domain. Then there exists a neighbourhood  $U$  of  $q$  such that for every non tangential cone  $\Lambda$  with vertex at  $q$ ,  $\Lambda \cap U \subset D$ , there exist positive constants  $0 < c \leq C$  such that:*

$$\forall z \in \Lambda \cap U, \quad \forall v, w \in \mathbb{C}^2 \setminus \{0\}, \quad -C \leq \operatorname{Bis} \left( g_{KE}^D \right) (z; v, w) \leq -c.$$

In an attempt to prove a more general version of Theorem 3 we study the Kähler-Einstein metric in domains  $D'_H = \{z \in \mathbb{C}^2 / \operatorname{Re}(z_1) + H(z_2) < 0\}$  where  $H$  is a real-valued homogeneous convex polynomial. We prove the following partial result (see Theorem 3.4 for a more detailed statement):

**Theorem 4.** *Let  $p \in \mathbb{N}^*$ . Let  $H$  be a real-valued homogeneous polynomial function of degree  $2p$  which is subharmonic but not harmonic. Assume that there exists a complete Kähler-Einstein metric with Ricci curvature  $-3$  on  $D'_H$  induced by a potential  $g$  (see Section 1.4). Set  $K := \frac{2p+1}{3}$ . Then:*

$$\max_{v, w \in \mathbb{C}^n \setminus \{0\}} \operatorname{Bis} \left( g_{KE}^{D'_H} \right) ((-1, 0); v, w) = \max \left\{ -\frac{1}{K}, \frac{-3 + \frac{1}{K} + \frac{p-1}{pK} \frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0)}{2} \right\},$$

$$\min_{v, w \in \mathbb{C}^n \setminus \{0\}} \operatorname{Bis} \left( g_{KE}^{D'_H} \right) ((-1, 0); v, w) = \min \left\{ -3 + \frac{1}{K}, \frac{-3 - \frac{1}{K} - \frac{p-1}{pK} \frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0)}{2} \right\}.$$

Consequently, to prove Theorem 3 for all bounded convex domains of finite type in  $\mathbb{C}^2$  it would be sufficient to prove the inequality  $(p-1) \frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0) < 2p^2$ .

In Chapter 4 we study the Bergman metric. We first prove that the holomorphic bisectional curvatures in tube domains  $T'_p$  introduced above and in Thullen domains  $E_p := \{z \in \mathbb{C}^2 / |z_1|^2 + |z_2|^{2p} < 1\}$ ,  $p \in \mathbb{N}^*$ , are negatively pinched:

**Theorem 5.** *Let  $p \in \mathbb{N}^*$  be an integer. Then*

$$-\infty < \min_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis} \left( g_B^{E_p} \right) (0; v, w), \quad \max_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis} \left( g_B^{E_p} \right) (0; v, w) < 0,$$

$$-\infty < \min_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis} \left( g_B^{T'_p} \right) ((-1, 0); v, w), \quad \max_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis} \left( g_B^{T'_p} \right) ((-1, 0); v, w) < 0.$$

See Theorem 4.3 and also Proposition 4.7 for a pinching of the holomorphic bisectonal curvatures of the Bergman metric in Thullen domains  $E_p$  with  $p \in [1, +\infty[$ . As a consequence of Theorem 5 and of a standard localisation result, we prove (see Theorem 4.4):

**Theorem 6.** *Let  $D \subset \mathbb{C}^2$  be a bounded pseudoconvex complete Reinhardt domain with boundary of class  $\mathcal{C}^\infty$  and of finite type. Then there exist a neighbourhood  $U$  of  $\partial D$  and two constants  $0 < c < C$  such that  $-C \leq \text{Bis} \left( g_B^D \right) \leq -c$  on  $D \cap U \times (\mathbb{C}^2 \setminus \{0\})^2$ .*

# Chapter 1

## Geometric and metric properties of domains

### Abstract

In this chapter we introduce the general notions and notations needed in the rest of this thesis, and we provide with examples that highlight the relationships between the geometric and the metric notions of curvatures. Section 1.1 contains mostly basic notations. In Section 1.2 we recall the basic material needed about the geometry of domains in  $\mathbb{C}^n$ . In Section 1.3 we describe these notions in what we call "model" domains in  $\mathbb{C}^2$ . In Section 1.4 we recall the basic material needed regarding the Kähler metrics and their curvatures in domains of  $\mathbb{C}^n$  and we define Kähler-Einstein metrics and the Bergman metric. In Section 1.5 we give examples of Kähler metrics in certain classes of domains and in particular we give examples of domains of infinite type with vanishing holomorphic bisectional curvature for the Kähler-Einstein metric.

### 1.1 Notions and notations

Throughout this thesis we use Einstein notation when there is no possible confusion.

In this Section, we fix two non zero integers  $n$  and  $m$ .

We denote by  $\mathcal{M}_n(\mathbb{C})$  the set of square matrices of size  $n$ , with complex coefficients. In this set, we denote by  $0$  the null matrix and by  $I$  the identity matrix. The coefficient in position  $(i, j) \in \{1, \dots, n\}^2$  of a matrix  $M$  is noted  $M_{i\bar{j}}$  and we also note  $M = [M_{i\bar{j}}]$ .

Let  $A = [A_{i\bar{j}}], B = [B_{i\bar{j}}] \in \mathcal{M}_n(\mathbb{C}), v = [v_i] \in \mathbb{C}^n, w = [w_j] \in \mathbb{C}^n$  (here and from now

on we abusively identify vectors with column matrices).

If  $A$  is invertible, we note  $[A^{i\bar{j}}] = A^{-1}$ . It is characterised by the relations  $A^{i\bar{k}}A_{k\bar{j}} = A_{i\bar{k}}A^{k\bar{j}} = 1$  if  $i = j$ , 0 otherwise. Especially,  $Tr(A^{-1}B) = A^{i\bar{j}}B_{j\bar{i}}$ , where  $Tr$  denotes the trace function. We denote by  $Det(A)$  the determinant of  $A$ . To simplify notations we simply write  $Det(A_{i\bar{j}})$  instead of  $Det([A_{i\bar{j}}])$  and proceed likewise with the trace function. We denote by  ${}^tA$  the transpose matrix of  $A$ , and by  $\bar{A}$  its conjugate. They are respectively characterised by the relations  $({}^tA)_{i\bar{j}} = A_{j\bar{i}}$  and  $(\bar{A})_{i\bar{j}} = \overline{A_{i\bar{j}}}$  for every integer  $1 \leq i, j \leq n$ .

We denote by  $\mathcal{H}_n := \{A \in \mathcal{M}_n(\mathbb{C}) / {}^tA = \bar{A}\}$  the space of Hermitian matrices of order  $n$ . If  $A \in \mathcal{H}_n$  we note  $\langle v, w \rangle_A := A_{i\bar{j}}v_i\bar{w}_j$ . Recall that  $\langle v, v \rangle_A \in \mathbb{R}$ .

If  $A, B \in \mathcal{H}_n$ , we define the relation  $B \geq A$ , respectively  $B > A$ , if and only if the inequality  $\langle v, v \rangle_B \geq \langle v, v \rangle_A$  holds for every  $v \in \mathbb{C}^n \setminus \{0\}$ , respectively  $\langle v, v \rangle_B > \langle v, v \rangle_A$  for every  $v \in \mathbb{C}^n \setminus \{0\}$ . We note  $\mathcal{H}_n^+ := \{M \in \mathcal{H}_n / M \geq 0\}$  and  $\mathcal{H}_n^{++} := \{M \in \mathcal{H}_n / M > 0\}$ . If  $A \in \mathcal{H}_n^+$ , we note  $|v|_A := \sqrt{\langle v, v \rangle_A}$ .

We will need the following classical fact that we do not prove:

- Proposition 1.1.** *1. Let  $A \in \mathcal{H}_n^+$ . Then there exists  $R \in \mathcal{H}_n^+$  such that  $R^2 = A$ . The matrix  $R$  is called a square root of  $A$ .*  
*2. Let  $A \in \mathcal{H}_n^+$ . Then  $0 \leq A \leq Tr(A)I$ .*  
*3. Let  $A \in \mathcal{H}_n^{++}$ . Then there exist  $0 < \lambda \leq \Lambda$  such that  $\lambda I \leq A \leq \Lambda I$ .*

We work with the topology induced by the Euclidean norm given by  $|z|^2 := |z_1|^2 + \dots + |z_n|^2$  for every  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ . For  $p \in \mathbb{C}^n$  and  $R > 0$ , we denote by  $B(p, R) := \{z \in \mathbb{C}^n / |z - p| < R\}$  and  $S(p, R) := \{z \in \mathbb{C}^n / |z - p| = R\}$  the open ball and the sphere centered at point  $p$  and of radius  $R$ . In the special case  $p = 0$  and  $R = 1$  we also denote  $B(0, 1)$  by  $\Delta$  if  $n = 1$  and by  $\mathbb{B}$  if  $n \geq 2$ . The boundary of a set  $U \subset \mathbb{C}^n$  (with respect to the topology induced by the Euclidean norm) is denoted by  $\partial U$ , its closure by  $\bar{U}$ .

We recall the notion of Hausdorff convergence of sets. The Hausdorff distance between two sets  $A, B \subset \mathbb{C}^n$  is defined by:

$$d_H(A, B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |b - a| \right\}.$$

The space of non empty compact sets of  $\mathbb{C}^n$  equipped with the distance  $d_H$  is a complete space. Let  $(D_\nu)_{\nu \in \mathbb{N} \cup \{\infty\}}$  be a family of non empty domains in  $\mathbb{C}^n$ . We say that



the sequence  $(D_\nu)_{\nu \in \mathbb{N}}$  converges to  $D_\infty$  in the local Hausdorff topology if it satisfies  $\lim_{\nu \rightarrow \infty} d_H(\partial D_\nu \cap K, \partial D_\infty \cap K) = 0$  for every compact set  $K \subset \mathbb{C}^n$ . If for every  $\nu \in \mathbb{N} \cup \{\infty\}$   $D_\nu$  is convex, it is equivalent to  $\lim_{\nu \rightarrow \infty} d_H(\overline{D_\nu} \cap K, \overline{D_\infty} \cap K) = 0$  for every compact set  $K \subset \mathbb{C}^n$ .

Let  $U \subset \mathbb{C}^n$  be an open set and  $s \in \mathbb{N}$  be an integer. We denote by  $\mathcal{C}^s(U, \mathbb{C})$  the set of complex valued functions that are  $s$  times differentiable on  $U$ . We note  $\mathcal{C}^\infty(U, \mathbb{C}) := \bigcap_{s \in \mathbb{N}} \mathcal{C}^s(U, \mathbb{C})$ . Also, for  $s \in \mathbb{N} \cup \{\infty\}$  we note  $\mathcal{C}^s(U) := \{f \in \mathcal{C}^s(U, \mathbb{C}), f \text{ is real valued}\}$  and  $\mathcal{C}^\omega(U, \mathbb{C})$  the subset of  $\mathcal{C}^\infty(U)$  of real analytic functions in  $U$ . For every  $\alpha \in [0, 1]$ , we denote by  $\mathcal{C}^{s+\alpha}(U)$  the subset of functions in  $\mathcal{C}^s(U)$  such that all the partial derivatives of order  $s$  are Hölder with exponent  $\alpha$ . For  $f \in \mathcal{C}^1(U, \mathbb{C})$  and  $1 \leq j \leq n$ , we denote by  $f_j := \frac{\partial f}{\partial z_j} = \frac{1}{2} \left( \frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right)$  and  $\overline{f_j} := \frac{\partial f}{\partial \overline{z_j}} = \frac{1}{2} \left( \frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right)$ . Note that for every  $f \in \mathcal{C}^1(U)$  and every integer  $1 \leq j \leq n$  one has  $f_j = \overline{\overline{f_j}}$ .

Let  $U \subset \mathbb{C}^n, V \subset \mathbb{C}^m$  be two open sets and  $s \in \mathbb{N} \cup \{\infty, \omega\}$ . We denote by  $\mathcal{C}^s(U, V)$  the set of maps  $f = (f_1, \dots, f_m)$  having values in  $V$  and such that for every integer  $1 \leq i \leq m$  we have  $f_i \in \mathcal{C}^s(U, \mathbb{C})$ . When  $m \geq 2$  we use the notation  $\frac{\partial}{\partial z_j}$  to only denote the complex differentiation so that there is no confusion between coordinate functions and partial derivatives. Given  $f \in \mathcal{C}^1(U, V)$  and  $z \in U$  we denote by  $\partial_z f$  the  $\mathbb{C}$ -linear map defined by  $\partial_z f(v) := \left[ \sum_{1 \leq j \leq n} \frac{\partial f_i}{\partial z_j}(z) v_j \right] \in \mathbb{C}^m$  for every vector  $v \in \mathbb{C}^n$ .

Recall that a map  $f \in \mathcal{C}^1(U, V)$  is holomorphic in  $U$  if it satisfies the Cauchy-Riemann equations

$$\forall 1 \leq j \leq n, \quad \forall 1 \leq i \leq m, \quad \frac{\partial f_i}{\partial \overline{z_j}} = 0 \text{ in } U,$$

and is called a biholomorphic map between  $U$  and  $V$  if  $f$  is holomorphic in  $U$ , bijective from  $U$  to  $V$  and its inverse  $f^{-1}$  is holomorphic in  $V$ . We denote by  $\mathcal{H}(U, V)$  (respectively  $\mathcal{B}(U, V)$ ) the set of holomorphic maps (respectively biholomorphic maps) between  $U$  and  $V$ , and simply note  $\text{Aut}(U) = \mathcal{B}(U, U)$ . Recall that  $\mathcal{B}(U, V) = \emptyset$  if  $n \neq m$ . We say that  $U$  is homogeneous if for every  $z, z' \in U$  there exists a biholomorphic map  $\Phi \in \text{Aut}(U)$  such that  $\Phi(z) = z'$ .

If  $f \in \mathcal{C}^1(U, V)$  and  $z \in U$ , we denote by  $\text{Jac}_{\mathbb{C}}(f)(z) := \left[ \frac{\partial f_i}{\partial z_j}(z) \right]$  its complex Jacobian at point  $z$ .

## 1.2 Geometric properties of domains in $\mathbb{C}^n$

Let  $U \subset \mathbb{C}^n$  be an open set,  $f \in \mathcal{C}^2(U)$  (that is  $f$  is real valued),  $z \in U$  and  $v \in \mathbb{C}^n$ . The following objects are well defined:

- The complex differential, respectively the real differential, of  $f$  at point  $z$  and vector  $v$  is

$$\partial_z f(v) := \sum_{1 \leq i \leq n} f_i(z) v_i, \text{ respectively } d_z f(v) := 2\text{Re}(\partial_z f(v)).$$

- The complex Levi form, respectively the real Levi form, of  $f$  at point  $z$  and vector  $v$  is

$$\mathcal{L}^{\mathbb{C}}(f, z, v) := \langle v, v \rangle_{[f_{i\bar{j}}(z)]} = \sum_{1 \leq i, j \leq n} f_{i\bar{j}}(z) v_i \bar{v}_j,$$

respectively

$$\mathcal{L}^{\mathbb{R}}(f, z, v) := \text{Re} \left( \sum_{1 \leq i, j \leq n} f_{ij}(z) v_i v_j \right) + \mathcal{L}^{\mathbb{C}}(f, z, v).$$

With these notations the Taylor expansion of  $f$  at order 2 at point  $z \in U$  takes the following form:

$$f(z+v) = f(z) + \text{Re} \left( 2 \sum_{1 \leq i \leq n} f_i(z) v_i + \sum_{1 \leq i, j \leq n} f_{ij}(z) v_i v_j \right) + \mathcal{L}^{\mathbb{C}}(f, z, v) + o_{v \rightarrow 0}(|v|^2). \quad (1.1)$$

The function  $f$  is plurisubharmonic, respectively strictly plurisubharmonic, convex, strictly convex, at  $z$  if it satisfies the inequality  $\mathcal{L}^{\mathbb{C}}(f, z, v) \geq 0$ , respectively  $\mathcal{L}^{\mathbb{C}}(f, z, v) > 0$ ,  $\mathcal{L}^{\mathbb{R}}(f, z, v) \geq 0$ ,  $\mathcal{L}^{\mathbb{R}}(f, z, v) > 0$  for every vector  $v \in \mathbb{C}^n \setminus \{0\}$ . We say that  $f$  is plurisubharmonic (respectively strictly plurisubharmonic, convex, strictly convex) in  $U$  if it is plurisubharmonic (respectively strictly plurisubharmonic, convex, strictly convex) at every point of  $U$ . Since the equality  $2\mathcal{L}^{\mathbb{C}}(f, z, v) = \mathcal{L}^{\mathbb{R}}(f, z, v) + \mathcal{L}^{\mathbb{R}}(f, z, iv)$  holds for every vector  $v \in \mathbb{C}^n$  one easily sees that if  $f$  is (strictly) convex at  $z$  then  $f$  is (strictly) plurisubharmonic at  $z$  (this idea is also used in Proposition 1.6).

We will need the following fact that we do not prove:

**Proposition 1.2.** *Let  $U \subset \mathbb{C}^n$  be an open bounded set and let  $f \in \mathcal{C}^2(\bar{U})$  be a strictly plurisubharmonic function. Then there exist constants  $0 < \lambda \leq \Lambda$  such that  $\lambda I \leq [f_{i\bar{j}}] \leq \Lambda I$  on  $\bar{U}$ .*

Most of the geometric notions that we are about to introduce translate into conditions on the real differential and the complex Levi form of a given function, namely a

defining function of a given domain with smooth boundary. We recall the definition of the smoothness of the boundary of a domain.

**Definition 1.3.** *Let  $D \subset \mathbb{C}^n$  be a domain and let  $s \in \mathbb{N}^* \cup \{\infty, \omega\}$  be an integer and let  $U \subset \mathbb{C}^n$  be an open set such that  $\partial D \cap U \neq \emptyset$ . We say that  $\partial D \cap U$  is of class  $\mathcal{C}^s$  if there exists a function  $\rho \in \mathcal{C}^s(U)$  satisfying the following conditions:*

- $D \cap U = \{\rho < 0\}$ ,
- $\partial D \cap U = \{\rho = 0\}$ ,
- $\forall z \in \partial D, \quad d_z \rho \neq 0$ .

The function  $\rho$  is called a defining function for  $\partial D \cap U$ . For such a defining function we define the complex tangent space, respectively the real tangent space, of  $\partial D$  at  $z \in \partial D \cap U$  by

$$T_z^{\mathbb{C}} \partial D := \{v \in \mathbb{C}^n / \partial_z f(v) = 0\}, \text{ respectively } T_z^{\mathbb{R}} \partial D := \{v \in \mathbb{C}^n / d_z f(v) = 0\}.$$

If  $z \in \partial D$ , we say that  $\partial D$  is of class  $\mathcal{C}^s$  in a neighbourhood of  $z$  if there exists an open set  $U \subset \mathbb{C}^n$  containing  $z$  such that  $\partial D \cap U$  is of class  $\mathcal{C}^s$ . We say that  $\partial D$  is of class  $\mathcal{C}^s$  if  $\partial D \subset U$  in the above definition and in this case  $\rho$  is called a defining function for  $\partial D$ .

We make two remarks about these definitions. We use the notations of Definition 1.3. First, we stress out that most of the results of this thesis are formulated under the global assumption that  $\partial D$  is of class  $\mathcal{C}^s$ , however for some technical results or proofs we need to restrict ourselves to a piece of  $\partial D$  and thus work locally.

Second, if  $\rho, \rho' \in \mathcal{C}^s(U)$  are two defining functions for  $\partial D \cap U$ , then  $\frac{\rho'}{\rho}$  is well defined on  $U$  and  $\frac{\rho'}{\rho} \in \mathcal{C}^{s-1}(U, \mathbb{R}_+^*)$  (see for instance Lemma 8.3. in [44]). In particular, the real, respectively the complex, tangent space of  $\partial D$  at a boundary point  $z \in \partial D \cap U$  does not depend on the defining function (provided  $s \geq 2$ ). This property also tells us that a defining function of a domain with smooth boundary is essentially unique. In fact, the smoothness of the boundary of the domain is equivalent to the smoothness of its signed distance function, more precisely we have the following:

**Proposition 1.4.** *(See Appendix 14.6. of [27]) Let  $s \in \mathbb{N} \cup \{\infty, \omega\}$ ,  $s \neq 0, 1$  be an integer*

and  $D \subset \mathbb{C}^n$  be a domain with boundary of class  $\mathcal{C}^s$ . The function

$$d^D : \mathbb{C}^n \longrightarrow \mathbb{R}$$

$$z \longmapsto \begin{cases} -d(z, \partial D) & \text{if } z \in D, \\ d(z, \partial D) & \text{otherwise} \end{cases}$$

is called the signed distance function of  $\partial D$ . It is a defining function of class  $\mathcal{C}^s$  for  $\partial D$ .

For the rest of this section, we fix a number  $s \in (\mathbb{N} \setminus \{0, 1\}) \cup \{\infty, \omega\}$ , a domain  $D \subset \mathbb{C}^n$  with boundary of class  $\mathcal{C}^s$ . We also fix an open set  $U$  such that  $\partial D \cap U \neq \emptyset$  and  $\rho \in \mathcal{C}^s(U)$  a defining function for  $\partial D \cap U$ . We recall the definition of pseudoconvexity:

**Definition 1.5.** • We say that  $\partial D$  is strictly pseudoconvex at point  $z \in \partial D \cap U$  if it satisfies the following property:

$$\forall v \in T_z^{\mathbb{C}} \partial D \setminus \{0\}, \quad \mathcal{L}^{\mathbb{C}}(\rho, z, v) > 0. \quad (1.2)$$

- In case that  $\partial D$  is of class  $\mathcal{C}^s$  (here  $U \supset \partial D$ ), we say that  $D$  is a strictly pseudoconvex domain if it is strictly pseudoconvex at every boundary point.
- Let  $D' \subset \mathbb{C}^n$  be a domain. We say that  $D'$  is pseudoconvex if there exists a sequence  $(D^{(\nu)})_{\nu \in \mathbb{N}}$  of subsets of  $\mathbb{C}^n$  such that for every integer  $\nu \in \mathbb{N}$ ,  $D^{(\nu)}$  is a bounded strictly pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$  and satisfies  $\overline{D^{(\nu)}} \subset D^{(\nu+1)}$  and  $\bigcup_{\nu \in \mathbb{N}} D^{(\nu)} = D'$ . Such sequence  $(D^{(\nu)})_{\nu \in \mathbb{N}}$  is called an exhaustion of  $D'$ .

If  $D$  is pseudoconvex and  $z \in \partial D \cap U$ , one can check that

$$\forall v \in T_z^{\mathbb{C}} \partial D, \quad \mathcal{L}^{\mathbb{C}}(\rho, z, v) \geq 0. \quad (1.3)$$

Moreover if  $\partial D$  is of class  $\mathcal{C}^2$  and the inequality (1.3) holds for every  $z \in \partial D$  then  $D$  is a pseudoconvex domain. We say that  $D$  is weakly pseudoconvex at  $z$ , or  $\partial D$  is weakly pseudoconvex at  $z$ , if it satisfies (1.3) and there exists a vector  $v \in T_z^{\mathbb{C}} \partial D \setminus \{0\}$  such that  $\mathcal{L}^{\mathbb{C}}(\rho, z, v) = 0$ .

As for the definitions of real and complex tangent spaces, one can check that the (strict) pseudoconvexity of  $\partial D$  at a given point  $z \in \partial D$  actually does not depend on the defining function.

One can think of the pseudoconvexity as a local analogue of the convexity in the complex sense. Indeed, if the domain  $D$  is convex in the usual geometric sense, that is  $D$  contains

a segment if it contains its endpoints, then one can verify that it satisfies the following at every point  $z \in \partial D \cap U$ :

$$\forall v \in T_z^{\mathbb{R}} \partial D \setminus \{0\}, \quad \mathcal{L}^{\mathbb{R}}(\rho, z, v) \geq 0. \quad (1.4)$$

Thus we see that property (1.3) is just the complex analogue of (1.4).

We can also use Property (1.4) to give a first important class of examples of pseudoconvex domains:

**Proposition 1.6.** *Let  $z \in \partial D \cap U$ . If the property (1.4) holds at  $z$  then  $\partial D$  is pseudoconvex at  $z$ . Especially, if  $D$  is a convex domain and  $\partial D$  is of class  $\mathcal{C}^s$  then  $D$  is a pseudoconvex domain.*

*Proof of Proposition 1.6.* Let  $z \in \partial D \cap U$  such that the property (1.4) holds at  $z$  and let  $v \in T_z^{\mathbb{C}} \partial D$ . Since  $T_z^{\mathbb{C}} \partial D \subset T_z^{\mathbb{R}} \partial D$  one has  $\mathcal{L}^{\mathbb{R}}(\rho, z, v) \geq 0$ . Moreover one notices that  $\partial_z \rho(iv) = i \partial_z \rho(v) = 0$ , hence  $iv \in T_z^{\mathbb{C}} \partial D \subset T_z^{\mathbb{R}} \partial D$  so that  $\mathcal{L}^{\mathbb{R}}(\rho, z, iv) \geq 0$ . Thus  $\frac{\mathcal{L}^{\mathbb{R}}(\rho, z, v) + \mathcal{L}^{\mathbb{R}}(\rho, z, iv)}{2} \geq 0$ , that is  $\mathcal{L}^{\mathbb{C}}(\rho, z, v) \geq 0$  by definition of the real Levi form. Hence  $\partial D$  is pseudoconvex at  $z$ .  $\square$

For similar reasons, it can be checked that if  $\partial D$  is strictly convex at  $z$ , in the sense that the inequality in property (1.4) holds at  $z$  and is strict, then  $\partial D$  is strictly pseudoconvex at  $z$ .

Strictly pseudoconvex boundary points satisfy nice properties. Namely:

**Proposition 1.7** ([42]). *1. Let  $z \in \partial D \cap U$ . If  $\partial D$  is strictly pseudoconvex at  $z$ , then there exists an open set  $V \subset \mathbb{C}^n$  such that  $z \in V \subset U$  and a defining function  $\rho' \in \mathcal{C}^s(U)$  for  $\partial D$  such that  $[\rho'_{i\bar{j}}] > 0$  on  $V$ .*

*2. If  $s \neq \omega$  and  $D$  is a bounded strictly pseudoconvex domain (here we assume  $U \supset \partial D$ ), then there exists an open set  $V \subset \mathbb{C}^n$  containing  $\overline{D}$  and a defining function  $\rho' \in \mathcal{C}^s(V)$  which is strictly plurisubharmonic on  $V$ .*

**Remark 1.8.** *In the case  $s = \omega$ , the proof of part 2. of Proposition 1.7 still provides with a defining function  $\rho' \in \mathcal{C}^\infty(V)$  for  $\partial D$  which is strictly plurisubharmonic on  $V$  and of class  $\mathcal{C}^\omega$  in a neighbourhood of  $\partial D$ .*

If one removes the strict pseudoconvexity from the hypothesis of part 2. of Proposition 1.7 the conclusion fails in the general case. For instance, K.Diederich and J.E.Fornaess

constructed a bounded pseudoconvex domain in  $\mathbb{C}^2$  with boundary of class  $\mathcal{C}^\infty$  which does not admit a plurisubharmonic defining function of class  $\mathcal{C}^3$  (see Proposition 1 and Theorem 5 of [18]).

An interesting consequence of Proposition 1.7 is that strictly pseudoconvex domains are locally biholomorphic to convex domains. Namely:

**Proposition 1.9** (See lemma 3.2.2. in [42]). *Let  $z \in \partial D$  be a strictly pseudoconvex boundary point for  $\partial D$ . Then there exists a domain  $V \subset \mathbb{C}^n$  containing  $z$ , a domain  $W \subset \mathbb{C}^n$  and a biholomorphic map  $\Phi \in \mathcal{B}(V, W)$  such that  $\Phi(V \cap D)$  is convex.*

The conclusion of Proposition 1.9 fails if one replaces the strict pseudoconvexity of  $\partial D$  at  $z$  with the weak pseudoconvexity. In [9] S. Calamai proved that the domain

$$D := \left\{ \operatorname{Re}(z_1) + \frac{|z_1|^2}{5} + |z_1 z_2|^2 + |z_2|^8 + \frac{15}{7} |z_2|^2 \operatorname{Re}(z_2^6) + 10 |z_2|^{10} < 0 \right\}$$

is a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^\omega$ . Moreover the only weakly pseudoconvex boundary point is the origin, and there does not exist a neighbourhood of the origin  $V$  such that  $V \cap D$  is biholomorphic to a convex domain. This example is inspired from the more famous example of J.J. Kohn and L. Nirenberg  $\{ \operatorname{Re}(z_1) + |z_2|^8 + \frac{15}{7} |z_2|^2 \operatorname{Re}(z_2^6) < 0 \}$  (see [41]).

The pseudoconvexity is a local property of the boundary of a domain. For the purpose of this thesis we only need the following weaker statement:

**Proposition 1.10.** *Assume that  $\overline{D} \subset U$ . Let  $U' \subset \mathbb{C}^n$  be a domain, let  $\Phi \in \mathcal{B}(U, U')$  and set  $D' := \Phi(D)$ . Let  $z \in \partial D$ ,  $z' \in \partial D'$  such that  $\Phi(z) = z'$ . Then  $\partial D$  is pseudoconvex (respectively strictly pseudoconvex) at  $z$  if and only if  $\partial D'$  is pseudoconvex (respectively strictly pseudoconvex) at  $z'$ .*

*Proof of Proposition 1.10.* Set  $\rho' := \rho \circ \Phi^{-1}$ . Then the hypothesis on  $\rho$  and  $\Phi$  imply that  $\rho'$  is a defining function for  $\partial D'$  of class  $\mathcal{C}^s$ , and the chain rule yields:

$$\partial_{z'} \rho' = \partial_{z'} \rho \circ (\partial_z \Phi)^{-1}, \quad (1.5)$$

$$\mathcal{L}^{\mathbb{C}}(\rho', z', \cdot) = \mathcal{L}^{\mathbb{C}}(\rho, z, (\partial_z \Phi)^{-1}(\cdot)), \quad (1.6)$$

( $\partial_z \Phi$  is invertible because  $\Phi$  is a diffeomorphism).

Let  $v' \in T_{z'}^{\mathbb{C}} \partial D'$ . Then relation (1.5) implies that  $(\partial_z \Phi)^{-1}(v') \in T_z^{\mathbb{C}} \partial D$ , so that by pseudoconvexity of  $\partial D$  at  $z$  we deduce  $\mathcal{L}^{\mathbb{C}}(\rho, z, (\partial_z \Phi)^{-1}(v')) \geq 0$ . Using relation (1.6)

we deduce that  $\mathcal{L}^{\mathbb{C}}(\rho', z', v') \geq 0$ , hence  $\partial D'$  is pseudoconvex at  $z'$ . By replacing the signs  $\geq$  into  $>$  we obtain the result for the statement in the case of strict pseudoconvexity.  $\square$

We recall three notions of finite 1-type in the sense of D'Angelo. These notions somehow measure the best order of tangency that a holomorphic curve, respectively a holomorphic regular curve, respectively a complex line, can have with the boundary of a domain at a given point. We refer to [14] for more details.

Let  $V \subset \mathbb{C}$  be a domain containing the origin and let  $f \in \mathcal{C}^{\infty}(V, \mathbb{C})$  and  $m \in \mathbb{N}^*$ . We say that  $f$  has order of vanishing  $m$  at 0 if all the partial derivatives of  $f - f(0)$  up to order  $m - 1$  vanish at 0 and there exists a partial derivative of  $f$  of order  $m$  that does not vanish at 0, and we denote by  $\nu(f) = m$  the order of vanishing of  $f$  at 0. If all the partial derivatives of  $f - f(0)$  vanish at 0 we say that  $f$  vanishes to infinite order at 0, and set  $\nu(f) = \infty$ . If  $f = (f_1, \dots, f_n) \in \mathcal{C}^{\infty}(V, \mathbb{C}^n)$  we define  $\nu(f) := \min_{1 \leq i \leq n} \{\nu(f_i)\}$ . From now on, we assume that either  $s = \infty$  or  $s = \omega$ . Observe that if  $z \in \partial D \cap U$ ,  $V \subset \mathbb{C}$  is a domain containing the origin and  $f \in \mathcal{C}^{\infty}(V, \mathbb{C}^n)$  satisfies  $f(0) = z$ , then the function  $\rho \circ f$  is well defined in a neighbourhood of 0 so that  $\nu(\rho \circ f)$  is well defined.

**Definition 1.11.** *For a point  $z \in \partial D$ , let  $\mathcal{G}_z := \{f \in \mathcal{H}(\Delta, \mathbb{C}^n), f(0) = z\}$ ,  $\mathcal{G}_z^{\mathcal{R}} := \{f \in \mathcal{G}_z, f'(0) \neq 0\}$  and  $\mathcal{G}_z^{\mathcal{L}} := \{w \in \Delta \mapsto z + wv, v \in \mathbb{C}^n\}$ . We say that  $\partial D$  has finite variety type at  $z$  if the set  $V_z := \left\{ \frac{\nu(\rho \circ f)}{\nu(f)}, f \in \mathcal{G}_z \right\}$  is bounded. In that case, the supremum of  $V_z$  is called the variety type of  $\partial D$  at  $z$ . If  $V_z$  is unbounded, we say that  $\partial D$  has infinite variety type at  $z$ . Likewise we define the notions of finite and infinite regular type, respectively line type, by replacing  $\mathcal{G}_z$  with  $\mathcal{G}_z^{\mathcal{R}}$ , respectively with  $\mathcal{G}_z^{\mathcal{L}}$  in the definition of  $V_z$ . We say that a domain is of finite type if every boundary point has finite variety type.*

For the same reasons as for the pseudoconvexity, one can check that these notions do not depend on the defining function. Also, if the variety type (respectively the regular type or the line type) at some given boundary point of  $D$  is finite, then it is an integer. Moreover if the domain is pseudoconvex at a boundary point at which the variety type (respectively the regular type or the line type) is finite, then the variety type (respectively the regular type, the line type) at this point is an even integer and all the types at this point are equal to 2 if and only if the point is a strictly pseudoconvex boundary point. Because of the inclusions  $\mathcal{G}_z^{\mathcal{L}} \subset \mathcal{G}_z^{\mathcal{R}} \subset \mathcal{G}_z$  we always have at a given point  $z \in \partial D$

$$\text{Line type} \leq \text{Regular type} \leq \text{Variety type}.$$

In "good" cases some of the above inequalities are equalities (for the definition of a Reinhardt domain see the introduction of Chapter 4):

**Theorem 1.12.** *Let  $D \subset \mathbb{C}^n$  be a domain with boundary of class  $\mathcal{C}^\infty$  and let  $z \in \partial D$ .*

- *(see Theorem 1.1 in [47]) If  $D$  is convex, then the variety type at  $z$  is equal to the line type at  $z$ .*
- *(see Theorem 4 in [23]) If  $D$  is a pseudoconvex Reinhardt domain, then the variety type at  $z$  is equal to the regular type at  $z$ .*

In general, the line type and the regular type at a boundary point of a Reinhardt domain may not be equal. For instance, in the Reinhardt domain  $D := \{z \in \mathbb{C}^2, \text{Log } |z_1| + \text{Log } |z_2| + (\text{Log } |z_1| - \text{Log } |z_2|)^4 < 0\}$ , the regular type at  $(1, 1)$  is 4 whereas the line type is 2.

Like the pseudoconvexity, the type is invariant under local biholomorphisms. For the purpose of this thesis, we only need the following:

**Proposition 1.13.** *Assume that  $\overline{D} \subset U$ . Let  $U' \subset \mathbb{C}^n$  be a domain, let  $\Phi \in \mathcal{B}(U, U')$  and set  $D' := \Phi(D)$ . Let  $z \in \partial D$ ,  $z' \in \partial D'$  such that  $\Phi(z) = z'$ . Then  $\partial D$  is of finite type at  $z$  if and only if  $\partial D'$  is of finite type at  $z'$ . In that case, the type of  $\partial D$  at  $z$  is equal to the type of  $\partial D'$  at  $z'$ .*

*Proof of Proposition 1.13.* We sketch the proof for the regular type. Recall that the type does not depend of the choice of a defining function we choose to compute it. By working with  $\rho' := \rho \circ \Phi^{-1}$  as a defining function for  $\partial D'$  (see the proof of Proposition 1.10 for details) we see that  $V_z = V_{z'}$ , hence the result.  $\square$

### 1.3 Model domains in $\mathbb{C}^2$

Let  $D \subset \mathbb{C}^2$  be a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$  and let  $q \in \partial D$  be a point of finite (variety) type  $2p$  where  $p \in \mathbb{N}^*$ . Then there exists a real-valued homogeneous polynomial of degree  $2p$   $H$  which is subharmonic but not harmonic, an open set  $U \subset \mathbb{C}^2$  containing  $q$  and a biholomorphic map  $\Phi : U \rightarrow \Phi(U)$  such that  $\Phi(q) = 0$  and  $\Phi(D \cap U) = M'_H \cap \Phi(U)$  where

$$M'_H := \left\{ z \in \mathbb{C}^2 / \text{Re}(z_1) + H(z_2) + \underset{z \rightarrow 0}{O}(|z_2|^{2p+1} + |z_1||z|) < 0 \right\}.$$



The domain

$$D'_H := \{z \in \mathbb{C}^2 / \operatorname{Re}(z_1) + H(z_2) < 0\}$$

is a local model for  $\partial D$  at point  $q$ .

**Remark 1.14.** 1. *This construction might be generalised in higher dimension using the notion multitype and in this case the polynomial  $H$  is homogenous with weight depending on the multitype (see [10]).*

2. *If  $D \subset \mathbb{C}^2$  is a convex domain of finite type there exists a real-valued homogeneous convex polynomial of degree  $2p$   $H$  which is not harmonic, an open set  $U \subset \mathbb{C}^2$  containing  $q$  and an affine map  $\Phi \in \operatorname{Aut}(\mathbb{C}^2)$  such that  $\Phi(q) = 0$  and  $\Phi(D \cap U) = M'_H \cap \Phi(U)$  where  $M'_H$  is as above. In this case we call the associated domain  $D'_H$  a local model for  $\partial D$  at  $q$ . This is due to the fact for convex domains the multitype is given by the linear multitype (see [59] for more details).*

A defining function for  $D'_H$  is  $\rho(z) := \operatorname{Re}(z_1) + H(z_2)$ , where  $z = (z_1, z_2) \in \mathbb{C}^2$ . Clearly  $\rho \in \mathcal{C}^\omega(\mathbb{C}^2)$  so that  $D'_H$  is a pseudoconvex domain with real analytic boundary. When  $p = 1$  the domain  $D'_H$  is strictly pseudoconvex and biholomorphic to the unit ball in  $\mathbb{C}^2$ . We work with  $p \geq 2$  for the rest of this section. We have:

$$\forall z \in \mathbb{C}^2, \quad [\rho_i(z)] = \begin{bmatrix} \frac{1}{2} \\ \frac{\partial H}{\partial z}(z_2) \end{bmatrix}, \quad [\rho_{i\bar{j}}(z)] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial^2 H}{\partial z \partial \bar{z}}(z_2) \end{bmatrix},$$

so that a boundary point  $z \in \partial D'_H$  is a strictly pseudoconvex boundary point if and only if  $\frac{\partial^2 H}{\partial z \partial \bar{z}}(z_2) > 0$ . Especially the set  $i\mathbb{R} \times \{0\}$  consists of weakly pseudoconvex boundary points for  $\partial D'_H$ . Let us compute the variety type at such points. The following proposition is classical. We refer to [6] for a more general study.

**Proposition 1.15.** *Let  $a \in \mathbb{R}$ . The variety type of  $\partial D'_H$  at  $(ia, 0)$  is equal to  $2p$ .*

*Proof.* Proof of Proposition 1.15 Let  $f = (f_1, f_2) \in \mathcal{G}_0$ . Observe that if  $f_2 = 0$  then clearly  $\frac{\nu(\rho \circ f)}{\nu(f)} = 1$  and if  $f_1 = 0$  then clearly  $\frac{\nu(\rho \circ f)}{\nu(f)} = 2p$ . Assume that  $f_1, f_2 \neq 0$ . Let  $j, k \in \mathbb{N}^*$  and  $b, c \in \mathbb{C} \setminus \{0\}$  such that  $f(z) = \left( ia + bz^j + \underset{|z| \rightarrow 0}{o}(|z|^j), cz^k + \underset{|z| \rightarrow 0}{o}(|z|^k) \right)$ . Then  $\rho \circ f(z) = \operatorname{Re} \left( bz^j + \underset{|z| \rightarrow 0}{o}(|z|^j) \right) + H \left( cz^k + \underset{|z| \rightarrow 0}{o}(|z|^k) \right)$ . Since  $H$  is homogeneous of degree  $2p$ , the degree of leading term in the Taylor expansion of  $H \left( cz^k + \underset{|z| \rightarrow 0}{o}(|z|^k) \right)$  is

$2pk$ . If  $j < 2pk$  then  $\nu(\rho \circ f) = j$  so that  $\frac{\nu(\rho \circ f)}{\nu(f)} < 2p$ . If  $j > 2pk$  then  $\nu(\rho \circ f) = 2pk$  so that  $\frac{\nu(\rho \circ f)}{\nu(f)} = 2p$ . If  $j = 2pk$ , the non harmonicity of  $H$  ensures that the map  $z \mapsto H\left(cz^k + \underset{|z| \rightarrow 0}{o}\left(|z|^k\right)\right) + \operatorname{Re}(bz^{2pk})$  is not identically zero, so that  $\nu(\rho \circ f) = 2pk$  hence  $\frac{\nu(\rho \circ f)}{\nu(f)} = 2p$ . Therefore the variety type at  $(ia, 0)$  is equal to  $2p$  as the supremum of 1 and  $2p$ .  $\square$

Let us describe the automorphism group of  $D'_H$ . First observe that the families of affine maps

$$s : z \mapsto (z_1, -z_2), \quad \tau_t : z \mapsto z + it, \quad d_\lambda : z \mapsto \left(\lambda z_1, \lambda^{\frac{1}{2p}} z_2\right),$$

where  $t \in \mathbb{R} \times \{0\}$  and  $\lambda \in \mathbb{R}_+^*$  belong to  $\operatorname{Aut}(D'_H)$ . To complete the description of  $\operatorname{Aut}(D'_H)$ , we use the work of K. Oeljeklaus (see Theorem 1.3 in [50]). Our hypotheses (that  $H$  is subharmonic and  $p > 1$ ) leave us with the following three cases, which are exhaustives and exclusives:

1.  $H$  is invariant by rotation, that is there exists a positive number  $a$  such that  $H(z) = a|z|^{2p}$  for every complex number  $z$ . In that case we have:

$$\operatorname{Aut}(D'_H) = \{s, \tau_t, t \in \mathbb{R} \times \{0\}, d_\lambda, \lambda \in \mathbb{R}_+^*, r_\theta : z \mapsto (z_1, e^{i\theta} z_2), \theta \in \mathbb{R}\}.$$

2.  $H$  is invariant by translation, that is there exists a positive number  $a$  and a real number  $\alpha$  such that  $H(z) = a \operatorname{Re}(e^{i\alpha} z)^{2p}$  for every complex number  $z$ . In that case we have:

$$\operatorname{Aut}(D'_H) = \{s, \tau_{(t_1, t_2)}, (t_1, e^{i\alpha} t_2) \in \mathbb{R}^2, d_\lambda, \lambda \in \mathbb{R}_+^*, r_\theta : z \mapsto (z_1, e^{i\theta} z_2), \theta \in \mathbb{R}\}.$$

3.  $H$  is neither invariant by rotation, nor invariant by translation. In that case we have:

$$\operatorname{Aut}(D'_H) = \{s, \tau_t, t \in \mathbb{R} \times \{0\}, d_\lambda, \lambda \in \mathbb{R}_+^*\}.$$

The first type of domains are biholomorphic to  $\{\operatorname{Re}(z_1) + |z_2|^{2p} < 0\}$ . The set of boundary points of  $D'_H$  that are of type  $2p$  is  $(1, 0) + i\mathbb{R} \times \partial\Delta$ . Analogously to the case of the ball, the map  $z \mapsto \left(\frac{1+z_1}{1-z_1}, \frac{2^{\frac{1}{p}} z_2}{(1-z_1)^{\frac{1}{p}}}\right)$  is a biholomorphism between  $\{\operatorname{Re}(z_1) + |z_2|^{2p} < 0\}$  and  $E_p := \{|z_1|^2 + |z_2|^{2p} < 1\}$ .  $E_p$  is called the Thullen domain of type  $2p$ . By extension we call  $\{\operatorname{Re}(z_1) + |z_2|^{2p} < 0\}$  the "unbounded representation of  $E_p$ ".

The second type of domains are biholomorphic to  $T'_p := \{Re(z_1) + Re(z_2)^{2p} < 0\}$ .  $T'_p$  is called a tube domain. The set of boundary points of  $D'_H$  that are of type  $2p$  is  $(1, 0) + i\mathbb{R}^2$ . The tube domain  $T'_p$  is also biholomorphic to a bounded domain (because the subset  $T'_p \cap \mathbb{R}^2$  contains no real line), but there is no "nice" bounded representation as in the case of Thullen domains and most of the study on tube domains is done on unbounded representations such as  $T'_p$ .

Observe that both Thullen domains (even in their unbounded representation) and tube domains are convex. The following Proposition states that convex polynomial models are either "pinched between Thullen domain" or equal to a tube domain:

**Proposition 1.16.** *Assume that  $D'_H$  is a convex domain. If the restriction of  $H$  to the unit circle  $\partial\Delta$  is positive then there exists positive constants  $0 < c \leq C$  such that*

$$\{Re(z_1) + C|z_2|^{2p} < 0\} \subset D_H \subset \{Re(z_1) + c|z_2|^{2p} < 0\}.$$

*Otherwise,  $D'_H$  is biholomorphic to a tube domain. More precisely there exist a constant  $a \in \mathbb{C} \setminus \{0\}$  such that for every  $z \in \mathbb{C}$  one has  $H(z) = Re(az)^{2p}$ .*

In [17, Lemma 3.3.] the authors give an analytic proof of Proposition 1.16. The following proof is more geometric:

*Proof of Proposition 1.16.* Observe that the restriction of  $H$  to  $\partial\Delta$  is a continuous non negative function, thus it achieves its minimum and its maximum, that we denote by  $c$  and  $C$  respectively. If  $H$  does not vanish on  $\partial\Delta$ , then  $0 < c \leq C$ . Since  $H$  is homogenous of degree  $2p$  we have for every  $z \in \mathbb{C}^*$  one has  $c|z|^{2p} \leq H(z) = |z|^{2p} H\left(\frac{z}{|z|}\right) \leq C|z|^{2p}$ . If  $z = 0$  the previous inequalities still hold, hence the first result.

Assume that the restriction of  $H$  to  $\partial\Delta$  vanishes, and let  $u \in \partial\Delta$  such that  $H(u) = 0$ . Since  $H$  is invariant by the symmetry  $s$ , we also have  $H(-u) = 0$ . We prove that  $H$  does not vanish on  $\partial\Delta \setminus \{-u, u\}$  by contradiction. If there existed  $v \in \partial\Delta \setminus \{-u, u\}$  such that  $H(v) = 0$ , then for every number  $0 \leq t \leq 1$  we would have  $0 \leq H(tu + (1-t)v) \leq tH(u) + (1-t)H(v) = 0$  by convexity of  $H$ . Thus  $H(tu + (1-t)v) = 0$  and since  $H$  is homogeneous and  $tu + (1-t)v \neq 0$  we would deduce that  $H$  is identically equal to 0 on the angular sector of edge 0 and delimited by vectors  $u$  and  $v$ . We proceed likewise to deduce that  $H$  would be identically equal to 0 on the angular sector of edge 0 and delimited by vectors  $-u$  and  $v$ . Therefore  $H$  would be identically equal to 0 on the half circle centered at the origin delimited by points  $u$  and  $-u$  and containing  $v$ . Since  $H$  is

an even function, it would be identically equal to 0 on the whole unit circle. Since  $H$  is homogeneous,  $H$  would be identically equal to 0 on  $\mathbb{C}$ , hence the contradiction.

Up to a rotation we may assume that  $u = i$  so that for every real number  $y_2$  we have  $H(iy_2) = 0$ . Since  $D'_H$  is clearly invariant under the translations of the form  $\tau_{(y_1, 0)}$  where  $y_1 \in \mathbb{R}$ , the conclusion holds if we prove the following:

$$\forall (x_1, x_2) \in \mathbb{R}^2 \cap D'_H, \forall y_2 \in \mathbb{R}, \quad (x_1, x_2 + iy_2) \in D'_H.$$

Let  $(x_1, x_2) \in \mathbb{R}^2 \cap D'_H$  and  $y_2 \in \mathbb{R}$ . If  $x_2 = 0$  then clearly  $H(x_2 + iy_2) = H(iy_2) = 0$  thus  $(x_1, x_2 + iy_2) \in D'_H$ . Assume that  $x_2 \neq 0$ . Since  $D_H$  is an open set, there exists a positive constant  $R > 0$  such that  $B((x_1, x_2), R) \subset D'_H$ . Let  $\alpha_2 \in \left[0, \frac{R}{1 + \left|\frac{x_1}{x_2}\right|}\right]$  and let  $\alpha_1 := \frac{x_1}{x_2}\alpha_2$ . Then we have  $(x_1 + \alpha_1, x_2 + \alpha_2) \in B((x_1, x_2), R) \subset D'_H$ . Set  $t := 1 + \frac{\alpha_2}{x_2}$ . Then we have  $(x_1 + \alpha_1, x_2 + \alpha_2) + t((x_1, x_2 + iy_2) - (x_1 + \alpha_1, x_2 + \alpha_2)) = (0, iy_2) \in D'_H$ . Consequently we obtain  $(x_1, x_2 + iy_2) \in [(x_1 + \alpha_1, x_2 + \alpha_2), (0, iy_2)]$ , so in particular  $(x_1, x_2 + iy_2) \in D'_H$  by convexity of  $D'_H$ . Hence the result.  $\square$

To conclude this section, let us mention the following class of examples: let  $a, b \in \mathbb{R}$  and let  $H_{a,b}(z) := az^3\bar{z} + b|z|^4 + az\bar{z}^3$  for  $z \in \mathbb{C}$ . Then for every  $z \in \mathbb{C}$  we have  $\frac{\partial^2 H_{a,b}}{\partial z \partial \bar{z}}(z) = 3az^2 + 4b|z|^2 + 3a\bar{z}^2$  and  $\frac{\partial^2 H_{a,b}}{\partial z^2}(z) = 6a|z|^2 + 2b\bar{z}^2$ . Thus  $H$  is non negative if and only if  $b \geq 2|a|$ ,  $H$  is subharmonic if and only if  $b \geq \frac{3}{2}|a|$  and  $H$  is convex if and only if  $b \geq \frac{3}{\sqrt{2}}|a|$  (the convexity of  $H_{a,b}$  is equivalent to the condition  $\frac{\partial^2 H_{a,b}}{\partial z \partial \bar{z}} \geq \left|\frac{\partial^2 H_{a,b}}{\partial z^2}\right|$ , see for instance Lemma 4.2. in [17]). In particular we can construct pseudoconvex model domains  $D'_H$  such that  $H$  is negative in some directions (by taking  $a, b$  such that  $0 < \frac{3}{2}|a| \leq b < 2|a|$ ).

## 1.4 Kähler metrics of domains in $\mathbb{C}^n$

From now on we assume that the reader is familiar with Riemannian geometry. For the rest of this section we fix a domain  $D \subset \mathbb{C}^n$ .

### Kähler metrics and their curvatures

**Definition 1.17.** A Kähler metric on  $D$  is an element of  $\mathcal{C}(D, \mathcal{H}_n^{++})$ , that is a matrix  $[g_{i\bar{j}}]$  with continuous coefficients in  $D$  and such that for every  $z \in D$ ,  $[g_{i\bar{j}}(z)] \in \mathcal{H}_n^{++}$ .

In particular the quantity  $\langle v, w \rangle_{g,z} := \langle v, w \rangle_{[g_{i\bar{j}}(z)]}$  is well defined for every  $v, w \in \mathbb{C}^n$ . If we think of  $D$  as an open subset of  $\mathbb{R}^{2n}$  using the isomorphism

$$\begin{aligned} Can : \quad \mathbb{R}^{2n} &\longrightarrow \mathbb{C}^n \\ (x_1, \dots, x_n, y_1, \dots, y_n) &\longmapsto z := (x_1 + iy_1, \dots, x_n + iy_n), \end{aligned}$$

then the matrix  $\begin{bmatrix} [Re(g_{i\bar{j}})] & [Im(g_{i\bar{j}})] \\ [-Im(g_{i\bar{j}})] & [Re(g_{i\bar{j}})] \end{bmatrix} \in M_{2n}(\mathbb{R})$  defines a Riemannian metric on  $D$ . In particular we say that the Kähler metric is complete on  $D$  if the Riemannian metric induced by it is complete on  $D$ .

Given a Kähler metric  $[g_{i\bar{j}}]$  on  $D$  and a function  $f \in \mathcal{C}^2(D)$  we define the following:

- The gradient of  $f$  with respect to  $[g_{i\bar{j}}]$ :  $\nabla_g f := \overline{[g^{i\bar{j}}]} [f_{\bar{i}}]$ .
- The laplacian of  $f$  with respect to  $[g_{i\bar{j}}]$ :  $\Delta_g f := Tr([g^{i\bar{j}}] [f_{i\bar{j}}]) = \sum_{1 \leq i, j \leq n} g^{i\bar{j}} f_{j\bar{i}}$ .

A Kähler metric  $[g_{i\bar{j}}]$  is induced by a function  $u \in \mathcal{C}^2(D)$ , called a Kähler potential for the metric, if it satisfies the equality  $[u_{i\bar{j}}] = [g_{i\bar{j}}]$  in  $D$ .

In the case of the Euclidean metric, that is  $[g_{i\bar{j}}] = I$  on  $D$ , we drop the  $g$  subscripts and simply write  $\langle v, w \rangle, \nabla f, \Delta f$  etc.

## Curvatures of Kähler metrics

Here we assume that  $g_{i\bar{j}} \in \mathcal{C}^2(D, \mathbb{C})$  for every integer  $1 \leq i, j \leq n$ . Fix a point  $z \in D$ . The curvature coefficients of  $[g_{i\bar{j}}]$  at  $z$  are given by the following:

$$\forall 1 \leq i, j, k, l \leq n, \quad R_{i\bar{j}k\bar{l}}(g)(z) := \left( -g_{i\bar{j}k\bar{l}} + \sum_{1 \leq \alpha, \beta \leq n} g_{ik\bar{\alpha}} g^{\alpha\bar{\beta}} g_{\beta\bar{j}l} \right) (z). \quad (1.7)$$

They satisfy the relations

$$\forall 1 \leq i, j, k, l \leq n, \quad R_{i\bar{j}k\bar{l}}(g)(z) = R_{k\bar{l}i\bar{j}}(g)(z) = \overline{R_{j\bar{i}l\bar{k}}(g)}(z) \quad (1.8)$$

Given two vectors  $v, w \in \mathbb{C}^n \setminus \{0\}$ , the holomorphic bisectional curvature of  $[g_{i\bar{j}}]$  at point  $z$  between vectors  $v$  and  $w$  is defined by the following:

$$Bis(g)(z; v, w) = \frac{\sum_{1 \leq i, j, k, l \leq n} R_{i\bar{j}k\bar{l}}(g)(z) v_i \bar{v}_j w_k \bar{w}_l}{|v|_{g,z}^2 |w|_{g,z}^2}. \quad (1.9)$$

The holomorphic sectional curvature of the metric at  $z$  and at vector  $v \in \mathbb{C}^n \setminus \{0\}$  is  $H(g)(z; v) := Bis(g)(z; v, v)$ .

From relation (1.9) it is clear that the holomorphic bisectional curvature between two vectors actually does not depend on the length of the vectors at which it is computed, namely we have:

$$\forall v, w \in \mathbb{C}^n \setminus \{0\}, \quad \forall \lambda, \mu > 0, \quad \text{Bis}(g)(z; \lambda v, \lambda w) = \text{Bis}(g)(z; v, w). \quad (1.10)$$

The Ricci form of  $[g_{i\bar{j}}]$  at  $z$  is defined by  $[\text{Ric}(g)_{i\bar{j}}(z)] := \left[ -\text{Log}(\text{Det}(g_{p\bar{q}}))_{i\bar{j}}(z) \right]$ . By definition  $[\text{Ric}(g)_{i\bar{j}}(z)] \in \mathcal{H}_n$ . We say that the metric has Ricci curvature bounded from below (respectively from above) on  $D$  if there exists a constant  $\lambda \in \mathbb{R}$  such that one has  $\lambda [g_{i\bar{j}}] \leq [\text{Ric}(g)_{i\bar{j}}]$  (respectively  $[\text{Ric}(g)_{i\bar{j}}] \leq \lambda [g_{i\bar{j}}]$ ) on  $D$ . Finally we say that the metric  $[g_{i\bar{j}}]$  is Einstein if there exists  $\lambda \in \mathbb{R}$  such that  $[\text{Ric}(g)_{i\bar{j}}] = \lambda [g_{i\bar{j}}]$  on  $D$ . The function  $\text{Ric}(g) := -\text{Log}(\text{Det}(g_{p\bar{q}}))$  is a potential for the Ricci form of  $[g_{i\bar{j}}]$  on  $D$ .

The Ricci form and the curvature coefficients are related by the following formulas on  $D$  (which follow from the differentiation of  $\text{Log}(\text{Det}(g_{p\bar{q}}))$ ):

$$\forall 1 \leq i, j \leq n, \quad \text{Ric}(g)_{i\bar{j}}(z) = \left( \sum_{1 \leq k, l \leq n} R_{i\bar{j}k\bar{l}}(g) g^{l\bar{k}} \right) (z). \quad (1.11)$$

**Remark 1.18.** Let  $(e_1, \dots, e_n)$  be the canonical basis of  $\mathbb{C}^n$ , let  $R$  be a square root of  $[g_{i\bar{j}}(z)]$  and set  $e'_i := {}^t R^{-1} e_i$  for every integer  $1 \leq i \leq n$ . Then:

$$\forall v \in \mathbb{C}^n \setminus \{0\}, \quad \sum_{1 \leq j \leq n} \text{Bis}(g)(z; v, e'_j) = \langle v, v \rangle_{[\text{Ric}(g)_{i\bar{j}}(z)]}.$$

## Kähler-Einstein metrics and the Bergman metric

### Kähler-Einstein metrics

**Definition 1.19.** A Kähler-Einstein metric on  $D$  is a Kähler metric  $[g_{i\bar{j}}]$  on  $D$  with coefficients of class  $\mathcal{C}^2$  that is Einstein, that is there exists  $\lambda \in \mathbb{R}$  such that  $[\text{Ric}(g)_{i\bar{j}}] = \lambda [g_{i\bar{j}}]$  on  $D$ . In this case the number  $\lambda$  is referred as the Ricci curvature of  $[g_{i\bar{j}}]$ .

**Remark 1.20.** • It is equivalent to the fact that the Riemannian metric induced by  $[g_{i\bar{j}}]$  on  $D$  (seen as a real open subset of  $\mathbb{R}^{2n}$ ) is Einstein.

• Let  $[g_{i\bar{j}}]$  be a Kähler metric on  $D$  with coefficients of class  $\mathcal{C}^2$ . For every positive number  $\rho > 0$  the Ricci form of the Kähler metric  $[\tilde{g}_{i\bar{j}}] := \rho [g_{i\bar{j}}]$  satisfies  $[\text{Ric}(\tilde{g})_{i\bar{j}}] = [\text{Ric}(g)_{i\bar{j}}]$  so that  $[\tilde{g}_{i\bar{j}}]$  is Kähler-Einstein with Ricci curvature equal to  $\frac{\lambda}{\rho}$  if and only if  $[g_{i\bar{j}}]$  is Kähler-Einstein with Ricci curvature equal to  $\lambda$ . Therefore only the sign of the Ricci

curvature matters. We emphasize that in this thesis all the Kähler-Einstein metrics have Ricci curvature either equal to  $-(n+1)$  or equal to  $-1$  (the latter case only occurs when  $D \in \{\mathbb{B}, \Delta^n\}$  and the metric is the Bergman metric).

Kähler-Einstein metrics of prescribed negative Ricci curvature are invariant metrics in the following sense. Let  $D, D' \subset \mathbb{C}^n$  be two domains that are biholomorphic and let  $\Phi \in \mathcal{B}(D, D')$ . Assume that there exists a complete Kähler-Einstein metric  $[g_{i\bar{j}}]$  on  $D$ , there exists a complete Kähler-Einstein metric  $[g'_{i\bar{j}}]$  on  $D'$ , and that these metrics have the same negative Ricci curvature. Then they satisfy

$$\forall z \in D, \quad \forall v \in \mathbb{C}^n, \quad |v|_{g,z} = |\partial_z \Phi(v)|_{g', \Phi(z)}. \quad (1.12)$$

This invariance property of the Kähler-Einstein metric of negative Ricci curvature follows from the Yau-Schwarz lemma (see [48, 55]). This invariance property has two important consequences. Let  $\lambda > 0$ . If there exists a complete Kähler-Einstein metric on  $D$  of Ricci curvature  $-\lambda$ , then this metric is unique (simply take  $D' = D$ ,  $\Phi = Id$  and use the relation (1.12)). Additionally if  $D' \subset \mathbb{C}^n$  is another domain possessing a complete Kähler-Einstein metric with Ricci curvature  $-\lambda$  and if there exists a biholomorphic map  $\Phi \in \mathcal{B}(D, D')$  then the following holds

$$\forall z \in D, \quad \forall v, w \in \mathbb{C}^n \setminus \{0\}, \quad \text{Bis}(g)(z; v, w) = \text{Bis}(g')(\Phi(z); \partial_z \Phi(v), \partial_z \Phi(w)). \quad (1.13)$$

If a Kähler metric  $[g_{i\bar{j}}]$  is induced by a potential  $g$ , then it is Kähler-Einstein with Ricci curvature equal to  $\lambda$  if and only if there exists a function  $F \in \mathcal{C}^2(D)$  satisfying  $[F_{i\bar{j}}] = 0$  on  $D$  and such that  $g$  solves the Monge-Ampère equation  $\text{Det}(g_{i\bar{j}}) = e^{-\lambda g + F}$  on  $D$  (this directly comes from the definition of  $\text{Ric}(g)$ ). This observation is one of the keys to prove the existence of Kähler-Einstein metrics in pseudoconvex domains in  $\mathbb{C}^n$ . The construction of complete Kähler-Einstein metrics in pseudoconvex domains in  $\mathbb{C}^n$  is discussed in more details in Chapter 2.

**Remark 1.21.** *Let  $D, D' \subset \mathbb{C}^n$  be two domains that are biholomorphic and let  $\Phi \in \mathcal{B}(D, D')$ . Assume that there exists a complete Kähler-Einstein metric induced by a potential  $g$  on  $D$  and there exists a complete Kähler-Einstein metric induced by a potential  $g'$  on  $D'$ , and that these metrics have the same negative Ricci curvature. Then they satisfy*

$$[(g' \circ \Phi)_{i\bar{j}}] = [g_{i\bar{j}}] \quad \text{on } D. \quad (1.14)$$

## The Bergman pseudo-metric

We implicitly work with the Lebesgue measure on  $\mathbb{C}^n$  that we denote by  $\mu$ . We use standard notations from the theory of  $L^2$  spaces. Consider the Bergman space

$$\mathcal{H}^2(D) := \{f \in \mathcal{H}(D, \mathbb{C}) / \|f\|_{L^2} < +\infty\}.$$

It is a subspace of the Hilbert space  $(L^2(D), \|\cdot\|_{L^2})$ . It follows from the Cauchy formula that for every compact subset  $K \subset D$  there exists a positive constant  $C$  such that  $\sup_{z \in K} |f(z)| \leq C \|f\|_{L^2}$  for every function  $f \in \mathcal{H}^2(D)$ . This property implies that  $(\mathcal{H}^2(D), \|\cdot\|_{L^2})$  is a Hilbert space and also that for every  $z \in D$  the evaluation map at  $z$  is a continuous linear form on  $\mathcal{H}^2(D)$ , thus there exists a unique function  $\eta_z \in \mathcal{H}^2(D)$  such that

$$\forall f \in \mathcal{H}^2(D), \quad f(z) = \int_D f(w) \overline{\eta_z(w)} d\mu(w). \quad (1.15)$$

The Bergman kernel of  $D$  is defined by  $K(z, w) := \overline{\eta_z(w)}$  for every  $z, w \in D$ . It is not hard to check that  $K(z, w) = \overline{K(w, z)}$  and thus for every number  $w \in D$  we have  $K(\cdot, w) \in \mathcal{H}^2(D)$ . The Bergman kernel satisfies the following transformation formula (which follows from the theorem of change of variables):

**Proposition 1.22** (Proposition 1.4.12 in [42]). *Let  $D, D' \subset \mathbb{C}^n$  be two domains and let  $\Phi \in \mathcal{B}(D, D')$ . Let  $K^D$  be the Bergman kernel for  $D$  and  $K^{D'}$  be the Bergman kernel for  $D'$ . Then for every  $z, w \in D$  we have:*

$$K^D(z, w) = \text{Det}(\text{Jac}^{\mathbb{C}}(\Phi)(z)) K^{D'}(\Phi(z), \Phi(w)) \overline{\text{Det}(\text{Jac}^{\mathbb{C}}(\Phi)(w))} \quad (1.16)$$

Property (1.15) applied to  $K(\cdot, z)$  gives  $K(z, z) = \|K(\cdot, z)\|_{L^2(D)}^2 \geq 0$  for every number  $z \in D$ . Assume that  $K(z, z) > 0$  for every  $z \in D$  (we say that the Bergman kernel of  $D$  is positive). Then the function  $g$  defined by  $g(z) := \text{Log}(K(z, z))$  is well defined on  $D$  and is called the Bergman potential of  $D$ .

**Definition 1.23.** *If the Bergman kernel of  $D$  is positive and the Bergman potential of  $D$  is strictly plurisubharmonic in  $D$ , it defines a Kähler potential in  $D$ . In this case the metric  $[g_{i\bar{j}}]$  is called the Bergman metric of  $D$ .*

It follows from relation (1.16) that the Bergman metric is an invariant metric. Consequently it satisfies the properties described by relations (1.12), (1.13) and (1.14).

As already mentionned, the Bergman kernel of a given domain  $D$  may not be positive.



Even if it is, the potential  $g$  it induces may not be strictly plurisubharmonic on  $D$  and in this case the Bergman metric of  $D$  may still not be complete. Nonetheless it is the case for all the domains we work with in this thesis:

**Theorem 1.24** (see [1, 51]). *If  $D$  satisfies one of the following, its Bergman kernel is positive and induces a complete Kähler metric:*

- $D \subset \mathbb{C}^n$  is a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^1$ .
- $D = D'_H$  is a polynomial domain introduced in Section 1.3 and  $H$  is non negative on  $\mathbb{C}$ .

For further examples of domains with complete Bergman metric, see [1, 11].

## 1.5 Examples of Kähler metrics in domains

### 1.5.1 Strictly pseudoconvex domains

Fix  $s \in \mathbb{N} \setminus \{0, 1\} \cup \{\infty\}$  and a bounded strictly pseudoconvex domain  $D \subset \mathbb{C}^n$  with boundary of class  $\mathcal{C}^s$ . It is easy to construct complete Kähler metrics on  $D$ . Moreover these metrics enjoy a nice curvature behaviour near the boundary of the domain. This construction is the starting point to construct complete Kähler-Einstein metrics in such domains, as we detail in Chapter 2.

**Proposition 1.25.** *Let  $\rho'$  be as in point 2. of Proposition 1.7. Then  $g := -\text{Log}(-\rho')$  defines a complete Kähler potential of class  $\mathcal{C}^s$  on  $D$ , and the following formulas hold in  $D$ :*

$$(-\rho') [g_{i\bar{j}}] = [\rho'_{i\bar{j}}] + \left[ \frac{\rho'_i \rho'_{\bar{j}}}{-\rho'} \right], \quad (1.17)$$

$$[g^{i\bar{j}}] = (-\rho') [\rho'^{i\bar{j}}] - (-\rho') \frac{[\rho'^{i\bar{j}}][\rho'_i \rho'_{\bar{j}}][\rho'^{i\bar{j}}]}{-\rho' + |\nabla_{\rho'} \rho'|_{\rho'}^2}. \quad (1.18)$$

*Proof of Proposition 1.25.* The function  $g$  is well defined and of class  $\mathcal{C}^s$  in  $D$  by construction, and formula (1.17) directly comes from the chain rule. Let  $R$  be a square root of  $[\rho'_{i\bar{j}}]$  (cf Proposition 1.1). Observe that  $R$  is invertible because  $\text{Det}(R)^2 = \text{Det}([\rho'_{i\bar{j}}]) \neq 0$ . Set  $B := R^{-1} \left[ \frac{\rho'_i \rho'_{\bar{j}}}{-\rho'} \right] R^{-1}$  and  $A := (-\rho') R^{-1} [g_{i\bar{j}}] R^{-1} = I + B$ . Since the rank of  $B$  is 1, we have  $B^2 = \text{Tr}(B) B$ . Since  $\text{Tr}(B) = \frac{|\nabla_{\rho'} \rho'|_{\rho'}^2}{-\rho'} \geq 0 > -1$ , we can compute the

following:

$$A \left( I - \frac{B}{1 + \text{Tr}(B)} \right) = I + \left( \frac{-1}{1 + \text{Tr}(B)} + 1 - \frac{\text{Tr}(B)}{1 + \text{Tr}(B)} \right) B = I.$$

Consequently  $A$  is invertible and its inverse is  $I - \frac{B}{1 + \text{Tr}(B)} = I - R^{-1} \frac{[\rho'_i \rho'_{\bar{j}}]}{-\rho' + |\nabla_{\rho'} \rho'|_{\rho'}^2} R^{-1}$ .

Therefore we obtain the formula (1.18):

$$[g^{i\bar{j}}] = (-\rho') R^{-1} A^{-1} R^{-1} = (-\rho') [\rho'^{i\bar{j}}] - (-\rho') \frac{[\rho'^{i\bar{j}}] [\rho'_i \rho'_{\bar{j}}] [\rho'^{i\bar{j}}]}{-\rho' + |\nabla_{\rho'} \rho'|_{\rho'}^2}.$$

We admit the completeness of the distance induced by  $[-\text{Log}(-\rho')_{i\bar{j}}]$ .  $\square$

The ball  $\mathbb{B}$  is a special case of strictly pseudoconvex pseudoconvex domain with real analytic boundary. The metric induced by the potential  $g(z) := -\text{Log}(1 - |z|^2)$  is Kähler-Einstein with Ricci curvature equal to  $-(n + 1)$ , its curvature coefficients satisfy  $R_{i\bar{j}k\bar{l}}(g) = -(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$  for every  $1 \leq i, j, k, l \leq n$  and consequently its holomorphic bisectional curvature at point  $z$  between vectors  $v$  and  $w$  is given by  $\text{Bis}(g)(z; v, w) = -1 - \frac{|\langle v, w \rangle_{g,z}|^2}{|v|_{g,z}^2 |w|_{g,z}^2}$ . The metric constructed in Proposition 1.25 (here  $s \geq 4$ ) has holomorphic bisectional curvatures asymptotically close the holomorphic bisectional curvatures of the ball in the following sense. Computation of curvature coefficients yields

$$\begin{aligned} R_{i\bar{j}k\bar{l}}(g) &= -(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) \\ &+ \frac{1}{-\rho'} \left( R_{i\bar{j}k\bar{l}}(\rho') - \frac{1}{|\nabla_{\rho'} \rho'|_{\rho'}^2 - \rho'} (\rho'_{ik} - \rho'_{ik\bar{p}} \rho'^{p\bar{q}} \rho'_q) (\rho'_{\bar{j}\bar{l}} - \rho'_{\bar{p}\bar{l}} \rho'^{p\bar{q}} \rho'_{q\bar{l}}) \right), \end{aligned}$$

for every integer  $1 \leq i, j, k, l \leq n$ . In particular one has  $\rho' (R_{i\bar{j}k\bar{l}}(g) + (g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})) \in \mathcal{C}(\bar{D})$ . Using this and relation (1.17) above one easily obtains

$$\lim_{z \rightarrow \partial D} \sup_{v, w \in \mathbb{C}^n \setminus \{0\}} \left( \text{Bis}(g)(z; v, w) + \left( 1 + \left( \frac{|\langle v, w \rangle_{g,z}|}{\langle v, v \rangle_{g,z} \langle w, w \rangle_{g,z}} \right)^2 \right) \right) = 0. \quad (1.19)$$

This phenomenon indicates that the holomorphic bisectional curvatures of  $[g_{i\bar{j}}]$  are asymptotically the same as the holomorphic bisectional curvatures of the unit ball (hence the idea that  $D$  "looks like a ball" for the metric  $[g_{i\bar{j}}]$ ).

The asymptotic behaviour (1.19) may be seen as a result of the outstanding work of P.F. Klembeck in [39]. In the same paper the author also proved that if  $\partial D$  is of class

$\mathcal{C}^\infty$  then the Riemannian sectional curvatures of the Bergman metric of  $D$  tend to the Riemannian sectional curvatures of the Bergman metric of the ball when approaching  $\partial D$ . This relies on the following asymptotic expansion of the Bergman kernel obtained by C.L. Fefferman (see [19]):

$$\forall z \in D, \quad K(z, z) = \frac{\Phi(z)}{(-\rho'(z))^{n+1}} + \tilde{\Phi}(z) \text{Log}(-\rho'(z)), \quad (1.20)$$

where  $\Phi, \tilde{\Phi} \in \mathcal{C}^\infty(\overline{D})$  and  $\Phi(z) \neq 0$  for every  $z \in \partial D$ . In the case of the unit ball with  $\rho'(z) = |z|^2 - 1$ ,  $\Phi$  is a positive constant and  $\tilde{\Phi} = 0$ . In the general case the Ramadanov conjecture states that the vanishing of  $\tilde{\Phi}$  to infinite order on  $\partial D$  implies that  $\partial D$  is locally biholomorphic to  $\partial \mathbb{B}$  (see [53]). Notice that the terms  $\Phi$  and  $\tilde{\Phi}$  in the asymptotic expansion (1.20) contain invariant related to the CR geometry of  $\partial D$  (see the results in [13] by S. Curry and P. Ebenfelt and in [28] by R. Graham).

**Remark 1.26.** *Notice that in the case of bounded pseudoconvex domains with boundary of class  $\mathcal{C}^\infty$  and of finite type or in the case of model domains there is no general construction of a complete Kähler metric induced by a defining function. Moreover the boundary behaviour of the Bergman metric and of the Kähler-Einstein metric and of their curvatures is understood in very few cases. We provide with a study of these metrics in certain classes of pseudoconvex domains of finite type in Chapters 3,4 and 5.*

### 1.5.2 Convex domains of infinite type

The polydisc  $\Delta^n$  is a bounded convex domain. The set  $\{z \in \overline{\Delta^n}, \exists! 1 \leq i \leq n, |z_i| = 1\}$  consists of boundary point of  $\Delta^n$  of infinite type. The Bergman metric  $g_B$  of  $\Delta^n$  is Einstein with Ricci curvature equal to  $-1$ . Since  $\Delta^n$  is a homogeneous domain the computation of the metric and its curvatures at the origin is enough to determine the boundary behaviour of these quantities. Computations give  $H(g_B)(0; v) \leq -\frac{n+1}{2}$ ,  $Bis(g_B)(0; v, w) \leq 0$  for every  $z \in \Delta^n$  and  $v, w \in \mathbb{C}^2 \setminus \{0\}$ . Moreover one has  $Bis(g_B)(0; (1, 0, \dots, 0), (0, \dots, 0, 1)) = 0$  so the holomorphic bisectional curvatures vanish for certain pair of vectors.

The vanishing of the holomorphic bisectional curvatures metric is not specific to the Bergman metric. Actually, P. Yang proved the following striking result:

**Theorem 1.27** (See [54]). *The polydisc  $\Delta^n$  does not admit any Kähler metric with negatively pinched holomorphic bisectional curvatures.*

Theorem 1.27 seems to indicate that the finiteness of the type is a necessary condition for the existence of a complete Kähler metric with negatively pinched holomorphic bisectional curvatures. The non smoothness of the boundary of  $\Delta^n$  is misleading and one might think that the lack of smoothness causes for the vanishing of holomorphic bisectional curvatures. This is not the case. Indeed it follows from [8] that a bounded convex domain with smooth boundary in  $\mathbb{C}^n$  and with a point of infinite type does not admit a complete Kähler metric with negatively pinched holomorphic bisectional curvatures in a neighbourhood of its boundary. See Remark 3.31 for details.

### 1.5.3 A remark

The unit ball and the polydisc are bounded pseudoconvex domains for which the Bergman metric is Einstein. For a general bounded pseudoconvex domain these two metrics are different, which explains that we study them separately. In fact, S.-T. Yau conjectured in [56, Problem 44] that if the Bergman metric of a bounded pseudoconvex domain is Einstein, then the domain is homogeneous. For strictly pseudoconvex domains with smooth boundary the conjecture has been verified by X. Huang and M. Xiao:

**Theorem 1.28** (see [31]). *Let  $D \subset \mathbb{C}^n$  be a bounded strictly pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$ . If the Bergman metric of  $D$  is Einstein, then  $D$  is biholomorphic to the unit ball.*

In  $\mathbb{C}^2$ , the conclusion of Theorem 1.28 also holds if  $D$  is a pseudoconvex Reinhardt domain with boundary of class  $\mathcal{C}^\infty$  and of finite type (see [24, 49]). The general case of the conjecture is still open.

# Chapter 2

## The Kähler-Einstein metric in pseudoconvex domain: local behaviour at "ball like" boundary points

### Abstract

We briefly overview of the construction of the Kähler-Einstein metric in pseudoconvex domains. Then we establish a local regularity result for the Kähler-Einstein potential at strictly pseudoconvex boundary points and deduce that the holomorphic bisectional curvatures are negatively pinched in a neighbourhood of such point. Finally we prove that the same curvature behaviour holds at boundary points at which the squeezing function tends to one.

### Introduction

Kähler-Einstein metrics are by definition smooth Kähler metrics of constant Ricci curvature. In the case of bounded pseudoconvex domains, it follows from general results on non-negatively curved complete Riemannian manifolds that the Ricci curvature is necessarily negative (see the discussion below Equation (2.3.) in page 518 of [12]). The very first study of the existence of complete Kähler-Einstein metrics in bounded pseudoconvex domains is due to Cheng and Yau (see [12]). The authors first constructed complete Kähler-Einstein metrics in a given bounded strictly pseudoconvex domain by perturbing a "reference" complete Kähler metric. More precisely, they started with a bounded strictly pseudoconvex domain  $D$  with boundary of class  $\mathcal{C}^s$  ( $s$  being either an

integer such that  $s \geq 4$  or  $s \in \{\infty, \omega\}$ ), and a complete Kähler potential  $g \in \mathcal{C}^s(D)$  on  $D$  (for instance the one constructed in Proposition 1.25). They observed that if there exists a complete Kähler-Einstein potential on  $D$  with Ricci curvature equal to  $-(n+1)$  then the difference between this Kähler-Einstein potential and the potential  $g$ , that we denote by  $u$ , must satisfy the Monge-Ampère equation

$$\text{Det}(g_{i\bar{j}} + u_{i\bar{j}}) = e^{(n+1)u+F} \text{Det}(g_{i\bar{j}}), \quad (2.1)$$

where  $F := \text{Ric}(g) + (n+1)g$ , and conversly if there exists a function  $u \in \mathcal{C}^4(D)$  such that  $g' := g + u$  is a complete Kähler potential on  $D$  and  $u$  satisfies Equation (2.1) then  $g'$  satisfies the Monge-Ampère equation

$$\text{Det}(g'_{i\bar{j}}) = e^{(n+1)g'}, \quad (2.2)$$

hence  $g'$  is a complete Kähler-Einstein potential. They proved that if  $s \geq 7$  then there exists a function  $u$  such that  $g + u$  is a complete Kähler-Einstein potential. Actually they proved the more general following results:

**Theorem 2.1** (Modified versions of Theorem 4.4. and Corollary 4.5. in [12]). *Let  $s \geq 7$  be an integer, let  $D \subset \mathbb{C}^n$  be a bounded strictly pseudoconvex domain with boundary of class  $\mathcal{C}^s$ .*

1. (Construction as a perturbation) *Let  $\varphi \in \mathcal{C}^s(\overline{D})$  be a strictly plurisubharmonic defining function for  $\partial D$ . Set  $g := -\text{Log}(-\varphi)$ . Let  $\alpha \in ]0, 1[$  and let  $F \in \mathcal{C}^{s-2+\alpha}(\overline{D})$ . Then there exists a unique function  $u \in \mathcal{C}^{s+\alpha}(\overline{D})$  such that  $u$  satisfies condition*

$$\exists 0 < c \leq C \mid c [g_{i\bar{j}}] \leq [g_{i\bar{j}} + u_{i\bar{j}}] \leq C [g_{i\bar{j}}] \text{ on } D \quad (2.3)$$

*and solves the Monge-Ampère Equation (2.1) on  $D$ .*

2. (Implicit construction) *There exists a unique strictly plurisubharmonic function  $g \in \mathcal{C}^\omega(D)$  solving the Monge-Ampère equation (2.2) and satisfying the boundary condition*

$$g = +\infty \text{ on } \partial D. \quad (2.4)$$

They extend the existence of complete Kähler-Einstein potentials to bounded pseudoconvex domains using the fact that every bounded pseudoconvex domain can be exhausted by bounded strictly pseudoconvex domains with smooth boundary. Namely, they proved the following:

**Theorem 2.2** (Modified version of Theorem 7.5. in [12]). *Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^2$ . Then there exists a unique complete Kähler-Einstein potential  $g \in \mathcal{C}^\omega(D)$  satisfying Equation (2.2) with boundary condition (2.4).*

Later on Mok and Yau proved that the same result holds without any assumption on the regularity of the boundary, thus obtaining a characterisation of bounded pseudoconvex domains (see the Main Theorem in [48]). The existence of the complete Kähler-Einstein potential of factor  $-(n+1)$  has been extended to certain classes of unbounded domains, such as tube domains having a convex base that does not contain lines (see Corollary 4.6. in [12] and Proposition 3.1. in [32]).

In [12] the authors also studied the boundary behaviour of the Kähler-Einstein potential. To do so they start with a Kähler potential of the form  $-\text{Log}(-\varphi)$  that is "close" to the Kähler-Einstein potential near  $\partial D$  in the sense that the function  $\text{Ric}(g) + (n+1)g$  vanishes to some order on  $\partial D$  and proved that the perturbation  $u$  described above vanish to some order on  $\partial D$ , which implies that the curvatures of the Kähler-Einstein metric are asymptotically "close" to the curvatures of the metric induced by  $-\text{Log}(-\varphi)$ . They obtain the following:

**Theorem 2.3** (Modified version of Corollary 6.6. in [12]). *Let  $D \subset \mathbb{C}^n$  be a bounded strictly pseudoconvex domain with boundary of class  $\mathcal{C}^s$ ,  $s$  being an integer such that  $s \geq \max(2n+9, 3n+6)$ . Let  $g$  be the Kähler-Einstein potential constructed in point 2. of Theorem 2.1. Then there exists an open set  $U \subset \mathbb{C}^n$  containing  $\partial D$  such that for every  $\delta \in \left[0, \frac{1}{2}\right]$ , we have  $e^{-g} \in \mathcal{C}^{n+1+\delta}(\overline{\Omega \cap U})$ . Moreover the following curvature behaviour holds at every point  $q \in \partial D$ :*

$$\sup_{v \in \mathbb{C}^n \setminus \{0\}} (H(g)(z; v) + 2) \xrightarrow{z \rightarrow q} 0. \quad (2.5)$$

Using more algebraic considerations, Lee and Melrose completely describe the singularity of  $e^{-g}$  in the case of bounded strictly pseudoconvex domains with boundary of class  $\mathcal{C}^\infty$ , proving that the optimal regularity is  $e^{-g} \in \mathcal{C}^{n+1+\delta}(\overline{D})$  for every number  $\delta \in [0, 1[$ , except if  $D$  is biholomorphic to  $\mathbb{B}$  (in which case  $e^{-g} \in \mathcal{C}^\omega(\overline{D})$ ). See [45] for more details.

In this chapter we improve Theorem 2.3 in two ways. First we prove that the regularity result in Theorem 2.3 is local, more precisely:

**Theorem 2.4.** *Let  $D \subset \mathbb{C}^n$ ,  $n \geq 2$ , and  $q \in \partial D$ . Assume that there exists a neighbourhood of  $q$  on which  $\partial D$  is strictly pseudoconvex and of class  $\mathcal{C}^s$  with  $s \geq \max(2n+9, 3n+6)$ . Moreover, assume that  $D$  carries a complete Kähler-Einstein metric induced by a function  $g$  that satisfies conditions (2.2) and (2.4). Then there exists an open set  $U \subset \mathbb{C}^n$  containing  $q$  such that for every  $\delta \in \left[0, \frac{1}{2}\right]$ , we have:*

$$e^{-g} \in \mathcal{C}^{n+1+\delta}(\overline{D \cap U}).$$

Second we extend the asymptotic behaviour (2.5) to the holomorphic bisectional curvatures. Namely we prove:

**Theorem 2.5.** *Let  $D \subset \mathbb{C}^n$ ,  $n \geq 2$ , and  $q \in \partial D$ . Assume that there exists a neighbourhood of  $q$  on which  $\partial D$  is strictly pseudoconvex and of class  $\mathcal{C}^s$ ,  $s \geq \max(2n+9, 3n+6)$ . Moreover, assume that  $D$  carries a complete Kähler-Einstein metric induced by a function  $g$  that satisfies conditions (2.2) and (2.4). Then,*

$$\sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left( \text{Bis}(g)(z; v, w) + \left( 1 + \frac{|\langle v, w \rangle_{g,z}|^2}{|v|_{g,z}^2 |w|_{g,z}^2} \right) \right) \xrightarrow{z \rightarrow q} 0. \quad (2.6)$$

Theorem 2.5 leaves open the cases of strictly pseudoconvex boundary points at which the regularity of the boundary of the domain is not good enough. The following Theorem partially cover these cases:

**Theorem 2.6.** *Let  $D \subset \mathbb{C}^n$  be a pseudoconvex domain,  $n \geq 2$ , and  $q \in \partial D$ . Assume that the squeezing function of  $D$  tends to one at  $q$ . Moreover, assume that  $D$  carries a complete Kähler-Einstein metric induced by a function  $g$  solving equation (2.2) with condition (2.4) on  $D$ . Then,*

$$\sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left( \text{Bis}(g)(z; v, w) + \left( 1 + \frac{|\langle v, w \rangle_{g,z}|^2}{|v|_{g,z}^2 |w|_{g,z}^2} \right) \right) \xrightarrow{z \rightarrow q} 0.$$

In comparison with Theorems 2.4 and 2.5, Theorem 2.6 requires neither regularity assumptions on the boundary of the domain nor the strict pseudoconvexity at  $q$ , but gives no boundary regularity for the Kähler-Einstein potential.

We can apply Theorem 2.6 at  $\mathcal{C}^2$  strictly pseudoconvex boundary points of a domain admitting a Stein neighbourhood basis, at  $\mathcal{C}^2$  strictly convex boundary points of bounded domains, but also at every boundary point of the Fornaess-Wold domain, which is convex but not strictly pseudoconvex and has a boundary of class  $\mathcal{C}^2$  (see [21, 33, 38]).



This Chapter is organised as follows. In Section 2.1 we give more details about the construction of the Kähler-Einstein metric in bounded pseudoconvex domains. This section has an introductory purpose and contains no new result. In Section 2.2 we study the local behaviour of the complete Kähler-Einstein metric at strictly pseudoconvex boundary points and prove Theorem 2.4 and Theorem 2.5. In Section 2.3 we provide material regarding the squeezing function and prove Theorem 2.6.

## 2.1 Construction of Kähler-Einstein metrics in bounded pseudoconvex domains in $\mathbb{C}^n$

We recall the construction of the Kähler-Einstein metric on bounded pseudoconvex domains done in [12, 48]. The ideas developed here will help understanding Sections 2.2 and 2.3.

First we describe the construction when  $D \subset \mathbb{C}^n$  is a bounded strictly pseudoconvex domain with boundary of class  $\mathcal{C}^s$  with  $s \in \mathbb{N}$ ,  $s \geq 7$ . Let  $\rho \in \mathcal{C}^s(\overline{D})$  be a strictly plurisubharmonic defining function for  $\partial D$ . Then  $g := -\text{Log}(-\rho)$  is a complete Kähler potential on  $D$  (see Proposition 1.25). Moreover from formula (1.17) one derives  $\text{Det}(g_{i\bar{j}}) = e^{(n+1)g} e^{-F}$  where  $F := -\text{Log}\left(\left(-\rho + |\nabla_{\rho}\rho|^2_{\rho}\right) \text{Det}(\rho_{i\bar{j}})\right) = \text{Ric}(g) - (-(n+1)g) \in \mathcal{C}^{s-2}(\overline{D})$ . Thus in general  $g$  is not a Kähler-Einstein potential. However the function  $F$  which measures the defect of  $g$  to be a Kähler-Einstein potential has bounded partial derivatives up to the order  $s-2 \geq 2$ . A more detailed analysis of  $g$  indicates that  $D$  has *bounded geometry of order  $s-2$*  when equipped with the metric  $[g_{i\bar{j}}]$ . We refer to Definition 1.1 in [12] for a more precise statement.

Since  $g$  naturally compares to a Kähler-Einstein potential with Ricci curvature equal to  $-(n+1)$  we seek for a "perturbation"  $u \in \mathcal{C}^2(D)$  such that  $g+u$  is a Kähler-Einstein potential with Ricci curvature equal to  $-(n+1)$ .  $[g_{i\bar{j}} + u_{i\bar{j}}]$  satisfies the Kähler-Einstein condition if  $u$  verifies the relation (2.1). We want  $[g_{i\bar{j}} + u_{i\bar{j}}]$  to be a complete Kähler metric on  $D$ . Because of the bounded geometry and of Equation (2.1) it is relevant to impose that the background Kähler metric  $[g_{i\bar{j}}]$  and the perturbed metric  $[g_{i\bar{j}} + u_{i\bar{j}}]$  are equivalent, which translate into condition (2.3).

To solve Equation (2.1) with condition (2.3) S.-Y. Cheng and S.-T. Yau used a continuity

method. They proved that the set

$$I := \{t \in [0, 1] / \exists u \text{ solving } \text{Det}(g_{i\bar{j}} + u_{i\bar{j}}) = e^{(n+1)u+tF} \text{Det}(g_{i\bar{j}}) \text{ and satisfying (2.3)}\}$$

is open and closed. Since  $0 \in I$  one easily deduces  $I = [0, 1]$  by connectedness, and in particular  $1 \in I$ . To prove the openness and closedness of  $I$  one has to impose more regularity on the function  $u$ . The bounded geometry of  $D$  for the metric  $[g_{i\bar{j}}]$  and an *a priori* analysis of Equation (2.1) suggest to introduce a family of Hölder-like spaces that fits to the problem, the so-called Hölder-Cheng-Yau spaces (for a precise definition see the bottom of page 515 in [12]). In these spaces the continuity method works and S.-Y. Cheng and S.-T. Yau proved the existence of a solution  $u$  to Equation (2.1) satisfying condition (2.3). They also observed that this implies the existence of a complete Kähler potential  $g' \in \mathcal{C}^\omega(D)$  which satisfies Equation (2.2) and boundary condition (2.4) on  $D$ . They proved that in both cases the Kähler-Einstein potential constructed is unique. We refer to Theorem 4.4, Corollary 4.5 and Proposition 5.5 in [12] for much stronger statements.

Now assume that  $D$  is a bounded weakly pseudoconvex domain with boundary of class  $\mathcal{C}^2$ . Let  $(D_\nu)_{\nu \in \mathbb{N}}$  be an exhaustion of  $D$  by bounded strictly pseudoconvex domains with boundary of class  $\mathcal{C}^\infty$  (see Definition 1.5). For every  $\nu \in \mathbb{N}$  let  $g^{(\nu)} \in \mathcal{C}^\omega(D_\nu)$  be the complete Kähler-Einstein potential solution to Equation (2.2) with boundary condition (2.4) on  $D_\nu$ . In [12] the authors proved that for every integer  $\nu'$  the sequence  $(g_{|D_{\nu'}}^{(\nu)})_{\nu \geq \nu'}$  is a decreasing sequence of functions, so that there exists a function  $g$  defined on  $D$  satisfying  $\lim_{\nu \rightarrow \infty} g^{(\nu)}(z) = g(z)$  for every  $z \in D$  and condition (2.4) on  $D$ . Then they proved that for integer  $s$  and every compact set  $K \subset D$  the sequence  $(g_{|K}^{(\nu)})_{\nu \geq \nu_K}$  is bounded in  $\mathcal{C}^s(K)$  where  $\nu_K$  is an integer such that  $K \subset D_\nu$  for every  $\nu \geq \nu_K$ . Hence by Ascoli theorem there exists a subsequence  $(g_{|K}^{(\nu_k)})_{k \in \mathbb{N}}$  and a function  $g' \in \mathcal{C}^s(K)$  such that  $(g_{|K}^{(\nu_k)})_{k \in \mathbb{N}}$  converges to  $g'$  in  $\mathcal{C}^s(K)$ . By uniqueness of the pointwise limit, one has  $g = g'$ . Consequently  $g \in \mathcal{C}^\infty(D)$  and satisfies the Monge-Ampère equation (2.2) so that  $g \in \mathcal{C}^\omega(D)$  and is a Kähler-Einstein potential on  $D$ . This ends the construction of the Kähler-Einstein metric in bounded weakly pseudoconvex domains with boundary of class  $\mathcal{C}^2$ .

It should be noted that the uniqueness of the complete Kähler-Einstein metric implies that the function  $g$  constructed above does not depend on the exhaustion  $(D_\nu)_{\nu \in \mathbb{N}}$ . Moreover, for every integer  $s$  and every compact set  $K \subset D$ , the sequence  $(g_{|K}^{(\nu)})_{\nu \geq \nu_K}$  converges

to  $g$  in  $\mathcal{C}^s(K)$  because every subsequential  $\mathcal{C}^s(K)$ -limit of  $\left(g|_K^{(\nu)}\right)_{\nu \geq \nu_K}$  is equal to  $g$ . In particular we retain the following result that we will use in Section 2.3:

**Lemma 2.7.** *Let  $D_\infty \subset \mathbb{C}^n$  be a bounded pseudoconvex domain. Let  $(D_\nu)_{\nu \in \mathbb{N}}$  be an exhaustion of  $D_\infty$ . For every  $\nu \in \mathbb{N} \cup \{\infty\}$ , let  $g^{(\nu)} \in \mathcal{C}^\omega(D_\nu)$  be the solution of Equation (2.2) with boundary condition (2.4) on  $D_\nu$ . Then the following holds for every compact set  $K \subset D$ :*

$$\sup_{z \in K} \sup_{v, w \in \mathbb{C}^n \setminus \{0\}} \left| \langle v, w \rangle_{g^{(\nu)}, z} - \langle v, w \rangle_{g^{(\infty)}, z} \right| \xrightarrow{\nu \rightarrow \infty} 0,$$

$$\sup_{z \in K} \sup_{v, w \in \mathbb{C}^n \setminus \{0\}} \left| \text{Bis}(g^{(\nu)})(z; v, w) - \text{Bis}(g^{(\infty)})(z; v, w) \right| \xrightarrow{\nu \rightarrow \infty} 0.$$

## 2.2 The Kähler-Einstein metric at strictly pseudoconvex boundary points

This Section may be seen as a localisation of the results obtained in Section 6 in [12] completed with a study of the behaviour of the holomorphic bisectional curvature of the Kähler-Einstein metric. To obtain the local boundary regularity of the Kähler-Einstein potential we compare it with a "nice" family of local asymptotical Kähler-Einstein potentials. The construction of this family is inspired by the construction of asymptotically Kähler-Einstein potential originally explained by C. Fefferman in [20]. We explain it in more details in Subsection 2.2.1. In Subsection 2.2.2 we the Kähler-Einstein potential with the asymptotically Kähler-Einstein potential constructed in Subsection 2.2.1. In Subsection 2.2.3 we prove Theorems 2.4 and 2.5.

### 2.2.1 Local asymptotically Kähler-Einstein metrics

In the construction of local asymptotically Kähler-Einstein metrics, we need the following technical result:

**Lemma 2.8.** *Let  $D \subset \mathbb{C}^n$  be a domain, let  $n \geq 2$  be an integer, and let  $q \in \partial D$ . Assume that there exists a neighbourhood of  $q$  on which  $\partial D$  is of class  $\mathcal{C}^1$ . Let  $V \subset \mathbb{C}^n$  be an open set containing  $q$ , let  $\psi \in \mathcal{C}^1(V)$  be a defining function for  $\partial D \cap V$ . Let  $U \subset \bar{U} \subset V$  be a bounded open set containing  $q$ . Then, there exists a constant  $\epsilon > 0$  such that  $\inf_{\bar{U} \cap \{|\psi| \leq \epsilon\}} |\nabla \psi| > 0$ .*

*Proof of Lemma 2.8.* We argue by contradiction. Assume that there exists a sequence  $(z_i)_{i \in \mathbb{N}} \in \bar{U}^{\mathbb{N}}$  such that  $\lim_{i \rightarrow +\infty} \psi(z_i) = \lim_{i \rightarrow +\infty} |\nabla \psi|_{z_i} = 0$ . Since  $\bar{U}$  is compact, we can assume, up to extracting a subsequence, that  $(z_i)_{i \in \mathbb{N}}$  converges in  $\bar{U}$ . Denote by  $z$  its limit. By continuity of  $\psi$  at  $z$ , the condition  $\lim_{i \rightarrow +\infty} \psi(z_i) = 0$  implies  $\psi(z) = 0$ , which means that  $z \in \partial D \cap \bar{U} \subset \partial D \cap V$ . On the one hand, it implies that  $|\nabla \psi|_z > 0$  because  $\psi$  is a defining function for  $\partial D \cap V$ . On the other hand, the continuity of the function  $|\nabla \psi|$  at  $z$  implies that  $|\nabla \psi|_z = \lim_{i \rightarrow +\infty} |\nabla \psi|_{z_i} = 0$ . Hence the contradiction.  $\square$

Let  $V$  be an open set. Let  $s \geq 2$  be an integer. If  $\psi \in \mathcal{C}^s(V)$ , its Fefferman functional is defined by  $J(\psi) := (-1)^n \text{Det}(M(\psi))$  where

$$M(\psi) := \begin{bmatrix} \psi & [\psi_{\bar{j}}] \\ {}^t[\psi_i] & [\psi_{i\bar{j}}] \end{bmatrix}.$$

Then  $J(\psi) \in \mathcal{C}^{s-2}(V)$ . We observe that

$$J(\psi) = \psi^{n+1} \text{Det}[-\text{Log}(\psi)_{i\bar{j}}] \quad \text{on } \{\psi > 0\},$$

and that the function

$$F := -\text{Log}(J(\psi)) = \text{Ric}(-\text{Log}(\psi)) - (n+1)\text{Log}(\psi)$$

is well defined on  $\{\psi > 0\} \cap \{\text{Det}(-\text{Log}(\psi)_{i\bar{j}}) > 0\}$ . Especially, if  $[-\text{Log}(\psi)_{i\bar{j}}] > 0$ ,  $F$  is well defined and measures the defect of  $-\text{Log}(\psi)$  to be the potential of a Kähler-Einstein metric: the metric  $[-\text{Log}(\psi)_{i\bar{j}}]$  is Kähler-Einstein if and only if  $J(\psi) = 1$ .

Let  $D \subset \mathbb{C}^n$  be a domain and  $q \in \partial D$ . Assume that there exists a neighbourhood  $V$  of  $q$  such that  $\partial D \cap V$  is strictly pseudoconvex and of class  $\mathcal{C}^s$  with  $s \geq 2n+4$ . Without loss of generality, we may assume that  $V$  is a bounded domain. We describe Fefferman's iterating process in  $V$ .

Let  $\varphi \in \mathcal{C}^s(V)$  be a strictly plurisubharmonic defining function for  $\partial D \cap V$ . Let  $U_0 := \{J(-\varphi) > 0\}$ . Since  $\varphi \in \mathcal{C}^2(V)$  and  $J(-\varphi) > 0$  on  $\partial D \cap V$ , the set  $U_0$  contains  $\partial D \cap V$  and is open. Consider the following constructions on  $U_0$ :

$$\varphi^{(1)} := \frac{\varphi}{J(-\varphi)^{\frac{1}{n+1}}} \quad \text{and, for } 2 \leq l \leq n+1, \quad \varphi^{(l)} := \varphi^{(l-1)} \left( 1 + \frac{1 - J(-\varphi^{(l-1)})}{l(n+2-l)} \right).$$

Then, for every  $1 \leq l \leq n+1$ ,  $\varphi^{(l)}$  is well defined on  $U_0$  and  $\varphi^{(l)} \in \mathcal{C}^{s-2l}(U_0)$ . Moreover, according to the computations done by C. Fefferman in [20], we have  $\frac{J(-\varphi^{(l)}) - 1}{(-\varphi)^l} \in$

$\mathcal{C}^{s-2l-2}(U_0)$ . This ensures that the sets  $U_l := \left\{ |1 - J(-\varphi^{(l)})| < \frac{1}{2} \right\}$  are open and contain  $\partial D \cap V$  for every integer  $1 \leq l \leq n+1$ . Consequently, there exist two constants  $0 < r, R$  such that the set  $U := \left( \bigcap_{l=0}^{n+1} U_l \right) \cap ((B(q, R) \cap \partial D) + B(0, r))$  is open, contains  $q$ , satisfies  $\bar{U} \subset V$ , and on which every  $\varphi^{(l)}$  is a  $\mathcal{C}^{s-2l}$  defining function for  $\partial D \cap U$ . Then according to Lemma 2.8, we can assume (by taking smaller  $r$  and  $R$  if necessary) that  $\min_{1 \leq l \leq n+1} \inf_{z \in \bar{U}} |\nabla \varphi^{(l)}|_z > 0$  and also  $\inf_{z \in \bar{U}} |\nabla \varphi|_z > 0$ .

Since  $\partial D \cap V$  is strictly pseudoconvex, we can (by changing  $\varphi^{(l)}$  to  $\varphi^{(l)}(1 + t\varphi^{(l)})$  with  $t > 0$  small and taking smaller  $r$  and  $R$  if necessary) assume that each  $\varphi^{(l)}$  is strictly plurisubharmonic on  $\bar{U}$ .

Finally, the above construction gives, for every  $1 \leq l \leq n+1$ :

$$\begin{aligned} \frac{\text{Log}(J(-\varphi^{(l)}))}{(-\varphi)^l} &= \frac{\text{Log}(1 + (J(-\varphi^{(l)}) - 1))}{(-\varphi)^l} \\ &= \frac{J(-\varphi^{(l)}) - 1}{(-\varphi)^l} \left( 1 + \sum_{m=1}^{+\infty} \frac{(-1)^m}{m+1} (J(-\varphi^{(l)}) - 1)^m \right) \in \mathcal{C}^{s-2l-2}(\bar{U}). \end{aligned}$$

Let us summarize all these facts:

**Proposition 2.9.** *Let  $D \subset \mathbb{C}^n$  be a domain and let  $q \in \partial D$ . Assume that there exists a neighbourhood  $V$  of  $q$  such that  $\partial D \cap V$  is strictly pseudoconvex and of class  $\mathcal{C}^s$  with  $s \geq 2n+4$ . Then there exists a bounded domain  $U$  containing  $q$ , and a collection of functions  $(\varphi^{(l)})_{1 \leq l \leq n+1}$  satisfying, for every  $1 \leq l \leq n+1$ :*

1.  $\varphi^{(l)} \in \mathcal{C}^{s-2l}(\bar{U})$ ,
2.  $D \cap \bar{U} = \{\varphi^{(l)} < 0\} \cap \bar{U}$ ,
3.  $\inf_{z \in \bar{U}} |\nabla \varphi^{(l)}|_z > 0$ ,
4.  $\varphi^{(l)}$  is strictly plurisubharmonic on  $\bar{U}$ ,
5.  $|1 - J(-\varphi^{(l)})| \leq \frac{1}{2}$  on  $\bar{U}$ ,
6.  $\frac{J(-\varphi^{(l)}) - 1}{(-\varphi)^l} \in \mathcal{C}^{s-2l-2}(\bar{U})$ ,
7.  $\frac{\varphi^{(l)}}{\varphi} \in \mathcal{C}^{s-2l}(\bar{U})$  and is positive on  $\bar{U}$ ,
8.  $\frac{\text{Log}(J(-\varphi^{(l)}))}{(-\varphi)^l} \in \mathcal{C}^{s-2l-2}(\bar{U})$ .

Moreover, we have  $\inf_{z \in \bar{U}} |\nabla \varphi|_z > 0$ .

**Remark 2.10.** • Especially, conditions (1) to (4) imply that for every integer  $1 \leq l \leq n+1$ , the function  $\varphi^{(l)}$  is a strictly plurisubharmonic defining function of  $\partial D \cap U$  of class  $\mathcal{C}^{s-2l}$ .

• If  $s \geq 3n+5$ , then all the functions  $\varphi^{(l)}$ ,  $\frac{J(-\varphi^{(l)}) - 1}{(-\varphi)^l}$ ,  $\frac{\varphi^{(l)}}{\varphi}$  and  $\frac{\text{Log}(J(-\varphi^{(l)}))}{(-\varphi)^l}$  belong to  $\mathcal{C}^{n+1}(\bar{U})$ . If  $s \geq 3n+6$ , then all the aforementioned functions belong to  $\mathcal{C}^{n+2}(\bar{U}) \subset \bigcap_{0 \leq \delta \leq 1} \mathcal{C}^{n+1+\delta}(\bar{U})$ .

• The metrics  $\left[-\text{Log}(-\varphi^{(l)})_{i\bar{j}}\right]$  are called “asymptotically Kähler-Einstein” on  $\partial D \cap \bar{U}$ , since they satisfy the condition  $J(-\varphi^{(l)})(z) \xrightarrow{z \rightarrow \partial D \cap \bar{U}} 1$  (recall that  $\left[-\text{Log}(-\varphi^{(l)})_{i\bar{j}}\right]$  is Kähler-Einstein on  $D \cap \bar{U}$  if and only if  $J(-\varphi^{(l)}) = 1$  on  $D \cap \bar{U}$ ).

## 2.2.2 Local boundary regularity

In this subsection, we fix an integer  $n \geq 2$ , a domain  $D \subset \mathbb{C}^n$  and a point  $q \in \partial D$ . We assume that  $D$  satisfies the hypothesis of Theorem 2.4. Namely, there exists a complete Kähler-Einstein metric induced by a potential  $w' \in \mathcal{C}^\omega(D)$  that satisfies conditions (2.2) and (2.4), and there exists a neighbourhood  $V$  of  $q$  such that  $\partial D \cap V$  is strictly pseudoconvex and of class  $\mathcal{C}^s$  with  $s \geq \max(2n+9, 3n+6)$ . Thus, we can apply Proposition 2.9, and use the same notations introduced therein.

One of the main ideas to prove Theorem 2.4 is to compare the complete Kähler-Einstein metric  $\left[w'_{i\bar{j}}\right]$  to the asymptotically Kähler-Einstein metrics induced by the functions  $(\varphi^{(l)})_{1 \leq l \leq n+1}$  as follows. Let  $1 \leq l \leq n+1$ , and set

$$\eta := \frac{\varphi^{(l)}}{\varphi}, \quad w := -\text{Log}(-\eta\varphi), \quad F := -\text{Log}(J(-\eta\varphi)).$$

Then, according to points (5) – (8) of Proposition 2.9,  $\eta \in \mathcal{C}^{s-2l}(\bar{U})$ ,  $w \in \mathcal{C}^{s-2l}(D \cap U)$ ,  $F \in \mathcal{C}^{s-2l-2}(\bar{U})$ ,  $\frac{F}{(-\varphi)^l} \in \mathcal{C}^{s-2l-2}(\bar{U})$  and  $w$  and  $F$  are related on  $D \cap U$  by the equation (2.2) Let  $u := w' - w$ . Then, on  $D \cap U$ ,  $u$  solves the Monge-Ampère equation (2.1) (with the function  $g$  replaced by the function  $w$ ). Since  $w'$  is real analytic in  $D$  and  $w \in \mathcal{C}^{s-2l}(D \cap U)$ , then  $u \in \mathcal{C}^{s-2l}(D \cap U)$ .

So, for each integer  $1 \leq l \leq n+1$ , we have an asymptotically Kähler-Einstein metric  $\left[w_{i\bar{j}}\right]$  on  $\partial D \cap U$ , for which the defect of being Kähler-Einstein is encoded in the function  $F$ , and we study the difference between this metric and the Kähler-Einstein metric  $\left[w'_{i\bar{j}}\right]$

on  $D \cap \overline{U}$ . More precisely, we study the boundary regularity of the difference of their potentials, namely the function  $u$ .

Whether global (see [12]) or local (see [4]), the study of the boundary behaviour of  $u$  relies on its gradient estimate, which relies on the comparison between the metrics  $[w'_{i\bar{j}}]$  and  $[w_{i\bar{j}}]$  (see condition (2.8)). The gradient estimate enables to deduce the boundary behaviour of  $u$ , and then the boundary behaviour of the higher order derivatives of  $u$  by use of Schauder theory. All these estimates depend on the regularity of the gradient of  $\frac{F}{(-\varphi)^l}$ , for which we have the following result:

**Proposition 2.11.** *Under the hypothesis of Theorem 2.4, and with the notations introduced at the beginning of Subsection 3, we have  $\frac{|\nabla_w F|_w^2}{(-\varphi)^{2l-1}} \in \mathcal{C}^{s-2l-3}(\overline{D \cap U})$ . In particular, there exists a positive constant  $c_{\nabla}$ , such that the following holds on  $D \cap U$ :*

$$|\nabla_w F|_w^2 \leq c_{\nabla} (-\varphi)^{2l-1}. \quad (2.7)$$

*Proof of Proposition 2.11.* Let  $1 \leq i, j \leq n$ . Then, according to point (8) of Proposition 2.9,  $\frac{F_i}{(-\varphi)^{l-1}} = l \frac{F \varphi_i}{(-\varphi)^l} + \varphi \left( \frac{F}{(-\varphi)^l} \right)_i \in \mathcal{C}^{s-2l-3}(\overline{U})$ , and according to equation (1.18) as well as point (7) of Proposition 2.9,

$$\frac{w^{i\bar{j}}}{-\varphi} = \frac{\psi w^{i\bar{j}}}{\varphi - \psi} = \frac{\psi}{\varphi} \left( \psi^{i\bar{j}} + \frac{([\psi^{i\bar{j}}] [\psi_i \psi_{\bar{j}}] [\psi^{i\bar{j}}])_{ij}}{-\psi + |\nabla_{\psi} \psi|_{\psi}^2} \right) \in \mathcal{C}^{s-2l-2}(\overline{D \cap U}),$$

where  $\psi := \varphi^{(l)}$ .

$$\text{Hence } \frac{|\nabla_w F|_w^2}{(-\varphi)^{2l-1}} = \frac{w^{i\bar{j}}}{-\varphi} \frac{F_j}{(-\varphi)^{l-1}} \frac{F_{\bar{i}}}{(-\varphi)^{l-1}} \in \mathcal{C}^{s-2l-3}(\overline{D \cap U}). \quad \square$$

We improve the gradient estimate obtained in [4] by using the computations of [12] in a different way. Then we proceed exactly as in [4] to obtain the estimates of the other derivatives of  $u$ .

**Proposition 2.12.** *Under the hypothesis of Theorem 2.4, and with the notations introduced in Subsection 2.2.1 and in Proposition 2.11, for every  $\gamma \in ]0; \min(2n + 1, 2l - 1)[$ , there exist positive constants  $c$  and  $\epsilon$  such that  $|\nabla_w u|_w^2 \leq c (-\varphi)^{\gamma}$  on  $D \cap U \cap \{|\varphi| < \epsilon\}$ .*

**Remark 2.13.** • *Proposition 2.12 improves the results obtained in [4] in the sense that  $\partial D \cap U$  is not required to be “nice”.*

- *Proposition 2.12 is a local version of Proposition 6.4 in [12].*
- *The proof of Proposition 2.12 will use the fact that  $|\nabla_w u|_w^2 \in \mathcal{C}^2(D \cap U)$  and is bounded*

from above, which is true as long as  $s \geq 2n + 5$  (see page 297 of [4] for further details).

- It will also use the fact that Lemma II in [4] actually works for  $\mathcal{C}^2$  functions that are bounded below (see Lemma 2.14 for a version that fits to our situation).

*Proof of Proposition 2.12.* The strategy of the proof of Proposition 6.4 in [12] is first to show that there exists  $\delta_0 > 0$  such that for every  $0 < \alpha < n$ ,  $0 \leq \beta < n + 1$  and  $0 < \delta \leq \delta_0$  satisfying  $\alpha + \beta + \delta \leq 2l - 1$ , there exist positive constants  $\epsilon$  and  $c$  such that the following inequality holds on  $D \cap \{|\varphi| \leq \epsilon\}$ :

$$\Delta_{w'} \left( \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha \right) > \frac{n + 1 + n\beta - \beta^2}{2} \left( \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha \right),$$

and then to apply the generalized maximum principle and choose suitable constants  $\alpha$  and  $\beta$  to get the conclusion.

In our case, we wish to follow the same strategy when we restrict our considerations to  $D \cap U$ .

We focus our attention on explaining the necessary modifications in the proof of Proposition 6.4 in [12], keeping in mind that we look for local estimates in a neighbourhood of  $\partial D \cap U$ . For that purpose, we first explain the dependence of the constants  $c_1, \dots, c_9$  with respect to the local data in order to obtain conditions (2.14) and (2.15). Then we use formulas (2.14) and (2.15) to complete the proof. For each constant, we refer precisely to the condition in [12] where it is defined.

In the sequel,  $0 < \alpha < n$  and  $0 \leq \beta < n + 1$ .

- We apply the first Proposition of page 297 in [4] to derive the existence of positive constants  $\epsilon$  and  $\delta_0$  such that we have the following on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$[w'_{i\bar{j}}] = \left( 1 + O\left((- \varphi)^{\delta_0}\right) \right) [w_{i\bar{j}}], \quad (2.8)$$

which means that there exists a positive constant  $c'_1$  such that:

$$\left( 1 - c'_1 (-\varphi)^{\delta_0} \right) [w_{i\bar{j}}] \leq [w'_{i\bar{j}}] \leq \left( 1 + c'_1 (-\varphi)^{\delta_0} \right) [w_{i\bar{j}}]. \quad (2.9)$$

Hence by inverting it we obtain:

$$\left( 1 + c'_1 (-\varphi)^{\delta_0} \right)^{-1} [w^{i\bar{j}}] \leq [w'^{i\bar{j}}] \leq \left( 1 - c'_1 (-\varphi)^{\delta_0} \right)^{-1} [w^{i\bar{j}}].$$

Since  $\frac{1}{1-x} = 1 + \frac{x}{1-x} \leq 1 + 2x$  if  $x \in [0, \frac{1}{2}]$ , we have  $\frac{1}{1 - c'_1 (-\varphi)^{\delta_0}} \leq 1 + 2c'_1 (-\varphi)^{\delta_0}$  on the set  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$  whenever  $\epsilon \leq \left( \frac{1}{2c'_1} \right)^{\frac{1}{\delta_0}}$ .



Moreover, since  $\frac{1}{1+x} \geq 1-x \geq 1-2x$  for every  $x \in [0, \frac{1}{2}]$ , we also have  $\frac{1}{1+c'_1(-\varphi)^{\delta_0}} \geq 1-2c'_1(-\varphi)^{\delta_0}$  on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ . Thus, there exist positive constants  $\epsilon$  and  $c_1$  such that we have, on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$\left(1 - c_1(-\varphi)^{\delta_0}\right) \left[w^{i\bar{j}}\right] \leq \left[w'^{i\bar{j}}\right] \leq \left(1 + c_1(-\varphi)^{\delta_0}\right) \left[w^{i\bar{j}}\right].$$

We also take  $\epsilon \leq 1$  so that for every  $\delta \geq 0$  we have  $|\varphi|^\delta \leq 1$ . Consequently, we deduce the existence of constants  $\epsilon \in ]0, 1]$ ,  $\delta_0, c_1 > 0$  such that for every  $0 \leq \delta \leq \delta_0$ , we have the following on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$\left(1 - c_1|\varphi|^\delta\right) \left[w^{i\bar{j}}\right] \leq \left[w'^{i\bar{j}}\right] \leq \left(1 + c_1|\varphi|^\delta\right) \left[w^{i\bar{j}}\right]. \quad (2.10)$$

This is the same as condition (6.18) in [12], except that it holds in a neighbourhood of  $\partial D \cap \bar{U}$  in our situation (in [12], due to the global assumption of strict pseudoconvexity of  $\partial D$ , the inequalities in (2.10) are valid in a neighbourhood of  $\partial D$ ).

From now on, we let  $\delta \in ]0, \delta_0]$ .

- The constant  $c_2$  (see condition (6.19)) depends only on  $c_1$ .
- The constant  $c_3$  (see conditions (6.22) and (6.23)) depends only on  $c_1$ . Especially we have the following on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$1 - c_3(-\varphi)^\delta \leq \frac{|\nabla_{w'}\varphi|_{w'}^2}{\varphi^2} \leq 1 + c_3(-\varphi)^\delta.$$

In our situation we also assume that  $\epsilon \leq \left(\frac{1}{2c_3}\right)^{\frac{1}{\delta}}$ , so that we have the following on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$\frac{1}{2} \leq 1 - c_3(-\varphi)^\delta \leq \frac{|\nabla_{w'}\varphi|_{w'}^2}{\varphi^2} \leq 1 + c_3(-\varphi)^\delta. \quad (2.11)$$

- Set  $c_4 := 2nc_3$  (see condition (6.24)).
- According to inequality (6.25), we have, on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$-\Delta_{w'}(-\varphi)^\alpha \geq \alpha(-\varphi)^\alpha \left[ (n-\alpha) \frac{|\nabla_{w'}\varphi|_{w'}^2}{\varphi^2} - c_4(-\varphi)^\delta \right].$$

If we assume that  $\epsilon < \left(\frac{n-\alpha}{5c_4}\right)^{\frac{1}{\delta}}$ , then we derive the inequality  $\frac{(n-\alpha)}{2}|\nabla_{w'}\varphi|_{w'}^2 - c_4(-\varphi)^{\delta+2} > 0$  on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ , which leads to the following:

$$-\Delta_{w'}(-\varphi)^\alpha > \frac{\alpha(n-\alpha)}{2} \frac{|\nabla_{w'}\varphi|_{w'}^2}{\varphi^2} (-\varphi)^\alpha. \quad (2.12)$$

This is the same as inequality (6.26) in [12], but with  $c_5 = 0$ .

- Set  $c_6 := \beta c_4 + c_2$  (see condition (6.28)).
- The constant  $c_7$  depends only on  $c_6$  (see condition (6.29)).
- The constant  $c_8$  depends only on  $c_3$  and  $c_7$  (see condition (6.30)).
- If  $\epsilon < \left(\frac{n+1+n\beta-\beta^2}{2c_8}\right)^{\frac{1}{\delta}}$ , then we have, on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$\frac{n+1+n\beta-\beta^2}{2} - c_8(-\varphi)^\delta > 0,$$

so that in our case inequality (6.31) becomes the following:

$$\Delta_{w'} \left( \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} \right) > \frac{n+1+n\beta-\beta^2}{2} \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - |\nabla_w F|_w^2 (-\varphi)^{-(\delta+\beta)}. \quad (2.13)$$

Combining (2.12) and (2.13), we obtain, on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$  and for every  $c > 0$ :

$$\begin{aligned} \Delta_{w'} \left( \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha \right) &> \frac{n+1+n\beta-\beta^2}{2} \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - |\nabla_w F|_w^2 (-\varphi)^{-(\delta+\beta)} \\ &\quad + c \frac{\alpha(n-\alpha)}{2} \frac{|\nabla_{w'} \varphi|_{w'}^2}{\varphi^2} (-\varphi)^\alpha. \end{aligned}$$

This is exactly the same as inequality (6.31) in [12], but with  $c_9 = 0$ .

- Using condition (2.7) (Proposition 2.11), we observe that  $|\nabla_w F|_w^2 \leq c_\nabla (-\varphi)^{\alpha+\delta+\beta}$  whenever  $|\varphi| \leq 1$  and  $\alpha + \delta + \beta \leq 2l - 1$ . Therefore, according to (2.11), the following holds on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$\begin{aligned} -|\nabla_w F|_w^2 (-\varphi)^{-(\delta+\beta)} + c \frac{\alpha(n-\alpha)}{2} \frac{|\nabla_{w'} \varphi|_{w'}^2}{\varphi^2} (-\varphi)^\alpha &\geq (-\varphi)^\alpha \left( -c_\nabla + c \frac{\alpha(n-\alpha)}{2\varphi^2} \frac{|\nabla_{w'} \varphi|_{w'}^2}{\varphi^2} \right) \\ &\geq \left( -c_\nabla + c \frac{\alpha(n-\alpha)}{4} \right) (-\varphi)^\alpha. \end{aligned}$$

In particular if we take  $c > \frac{4c_\nabla}{\alpha(n-\alpha)}$  the right-hand side is non-negative. This is exactly what is derived from relation (6.32) in [12] (see the explanation below relation (6.33) in [12]), except that in our case it holds on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ .

For short, we have proved that there exists  $\delta_0 > 0$  such that for every  $0 < \alpha < n$ ,  $0 \leq \beta < n+1$  and  $0 < \delta \leq \delta_0$  satisfying  $\alpha + \beta + \delta \leq 2l - 1$ , there exist  $\epsilon \in ]0, 1]$  and  $c > 0$  such that the following inequalities hold on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ :

$$\Delta_{w'} \left( \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha \right) > 0, \quad (2.14)$$

$$\Delta_{w'} \left( \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha \right) > \frac{n+1+n\beta-\beta^2}{2} \left( \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha \right). \quad (2.15)$$

Inequality (2.14) implies that the function  $f := \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha$  cannot achieve its maximum on  $D \cap \bar{U} \cap \{|\varphi| \leq \epsilon\}$ , provided it is bounded from above on the set  $D_\epsilon := D \cap U \cap \{|\varphi| < \epsilon\}$ . Hence we can find a sequence  $(z_i)_{i \in \mathbb{N}} \in D_\epsilon^\mathbb{N}$  such that  $\lim_{i \rightarrow +\infty} f(z'_i) = \sup_{D_\epsilon} f$  and  $d_{w'}(z_i, \partial D_\epsilon) \xrightarrow{z \rightarrow +\infty} +\infty$ . Note that this implies that there exists a positive number  $R$  and an integer  $i_0 \in \mathbb{N}$  such that for every  $i \geq i_0$  we have  $d_{w'}(z_i, \partial D_\epsilon) \geq R$ .

The last step to conclude is to apply the local maximum principle due to J. Bland (see Lemma II in [4]) and use inequation (2.15). For completeness, we recall the local maximum principle in a version that fits our situation:

**Lemma 2.14.** *Let  $D \subset \mathbb{C}^n$  be a domain. Assume that there exists a Kähler-Einstein metric induced by a potential  $w'$  on  $D$ . Let  $D \subset D$  be a domain. Let  $f \in \mathcal{C}^2(D)$  bounded from above. If there exists a sequence  $(z_i)_{i \in \mathbb{N}} \in D^\mathbb{N}$  such that  $\lim_{i \rightarrow +\infty} f(z'_i) = \sup_D f$  and there exists  $R > 0$  such that for every integer  $i$ ,  $d_{w'}(z_i, \partial D) \geq R$ , then there exists another sequence  $(z'_i)_{i \in \mathbb{N}} \in D^\mathbb{N}$  such that*

$$\lim_{i \rightarrow +\infty} f(z'_i) = \sup_D f, \quad \limsup_{i \rightarrow +\infty} \Delta_{w'} f(z'_i) \leq 0.$$

We apply Lemma 2.14 to  $f = \frac{|\nabla_w u|_w^2}{(-\varphi)^\beta} - c(-\varphi)^\alpha$  with  $D = D_\epsilon$  and choose the suitable constants  $\alpha, \beta, \delta$  to conclude. We may argue as follows.

1. If  $2n + 1 \leq 2l - 1$ , we first apply Lemma 2.14 with  $\beta = 0$ ,  $\alpha = n - \frac{\delta}{4}$  and  $\delta \in ]0, \min(\delta_0, 4n)[$  to deduce the existence of constants  $\epsilon \in ]0, 1]$  and  $c > 0$  for which we have  $|\nabla_w u|_w^2 - c(-\varphi)^{n - \frac{\delta}{4}} \leq 0$  on  $D \cap D_\epsilon$ . Since  $(-\varphi) < \epsilon \leq 1$  on  $D \cap D_\epsilon$ , this directly implies:  $|\nabla_w u|_w^2 - c(-\varphi)^{n - \frac{\delta}{2}} \leq 0$  on  $D \cap D_\epsilon$ .

2. Hence we may apply Lemma 2.14 with  $\alpha = \beta = n - \frac{\delta}{2}$  and  $\delta \in ]0, \min(\delta_0, 2n)[$  to deduce the existence of constants  $\epsilon \in ]0, 1]$  and  $c > 0$  for which  $\frac{|\nabla_w u|_w^2}{(-\varphi)^{n - \frac{\delta}{2}}} - c(-\varphi)^{n - \frac{\delta}{2}} \leq 0$  on  $D \cap D_\epsilon$ . Again, since  $(-\varphi) < \epsilon \leq 1$  on  $D \cap D_\epsilon$ , this directly implies:  $|\nabla_w u|_w^2 - c(-\varphi)^{n+1 - \frac{\delta}{2}} \leq 0$  on  $D \cap D_\epsilon$ .

3. Hence we may apply once more Lemma 2.14 with  $\beta = \alpha + 1 = n + 1 - \frac{\delta}{2}$  and  $\delta \in ]0, \min(\delta_0, 2n)[$  to deduce the existence of  $c, \epsilon > 0$  for which  $\frac{|\nabla_w u|_w^2}{(-\varphi)^{n+1 - \frac{\delta}{2}}} - c(-\varphi)^{n - \frac{\delta}{2}} \leq 0$  on  $D \cap D_\epsilon$ . Finally, we directly deduce that  $|\nabla_w u|_w^2 \leq c(-\varphi)^{2n+1 - \delta}$  on  $D \cap D_\epsilon$ .

4. If  $2l - 1 < 2n + 1$ , we can proceed likewise: first taking  $\beta = 0, \alpha = \min\left(n, l - \frac{1}{2}\right) - \frac{\delta}{8}$  with  $\delta \in ]0, \min(\delta_0, 8 \min(n, l - \frac{1}{2}))[$ , then considering  $\alpha = \beta = \min\left(n, l - \frac{1}{2}\right) - \frac{\delta}{4}$  with  $\delta \in ]0, \min(\delta_0, 4 \min(n, l - \frac{1}{2}))[$ , and finally taking  $\alpha = \beta = l - \frac{1}{2} - \frac{\delta}{2}$  with  $\delta \in ]0, \min(\delta_0, 2l - 1)[$ .

In both cases, we obtain the desired conclusion by letting  $\delta$  tend to 0. Hence the result.  $\square$

In the rest of this Section, we use Proposition 2.12 first to derive the estimates of  $u$  of orders 0 (Proposition 2.16), second to derive estimates of higher order (Proposition 2.18), and finally to obtain a regularity result for  $\varphi e^{-u}$  (Proposition 2.19).

To obtain the estimates of  $u$  of order 0 we use the gradient estimate we obtain in Proposition 2.12. To do so we need the following result which gives a comparison of the gradient of  $u$  with respect to the Euclidean metric and its gradient with respect to another Kähler metric:

**Lemma 2.15.** *Let  $D \subset \mathbb{C}^n$  be a domain, and  $q \in \partial D$ . Assume that there exists a neighbourhood of  $q$  on which  $\partial D$  is strictly pseudoconvex and of class  $\mathcal{C}^2$ . Let  $V \subset \mathbb{C}^n$  be a bounded domain containing  $q$ ,  $\psi \in \mathcal{C}^2(V)$  be a strictly plurisubharmonic defining function for  $\partial D \cap V$ . Let  $g := -\text{Log}(-\psi)$ . Then for every bounded open set  $U \subset \bar{U} \subset V$  there exist  $0 < \lambda \leq \Lambda$  such that the following inequalities hold on  $\bar{D} \cap \bar{U}$ :*

$$\lambda \frac{\psi^2}{-\psi + |\nabla_\psi \psi|_\psi^2} I \leq [g^{i\bar{j}}] \leq \Lambda (-\psi) I. \quad (2.16)$$

*Proof of Lemma 2.15.* We use formula (1.18) and notations of Proposition 1.25 with  $U$  replaced with  $D \cap U$ . We also use the notations introduced in the proof of Proposition 1.25. According to Proposition 1.1, we have  $\frac{B}{1 + \text{Tr}(B)} \in \mathcal{H}_n^+$ , hence

$$0 \leq \frac{B}{1 + \text{Tr}(B)} \leq \frac{\text{Tr}(B)}{1 + \text{Tr}(B)} I.$$

Since  $A^{-1} = I - \frac{B}{1 + \text{Tr}(B)}$ , we deduce

$$\frac{1}{1 + \text{Tr}(B)} I = \left(1 - \frac{\text{Tr}(B)}{1 + \text{Tr}(B)}\right) I \leq A^{-1} \leq I.$$

Since  $-\psi > 0$ , we deduce the following:

$$\frac{-\psi}{-\psi + |\nabla_\psi \psi|_\psi^2} I \leq \frac{1}{-\psi} R [g^{i\bar{j}}] R \leq I,$$

$$\frac{\psi^2}{-\psi + |\nabla_\psi \psi|_\psi^2} [\psi^{i\bar{j}}] \leq [g^{i\bar{j}}] \leq (-\psi) [\psi^{i\bar{j}}].$$

Moreover, since  $[\psi^{i\bar{j}}]$  is continuous on the compact set  $\overline{U}$ , there exist  $0 < \lambda \leq \Lambda$  such that  $\lambda I \leq [\psi^{i\bar{j}}] \leq \Lambda I$  on  $\overline{U}$  (see Proposition 1.2). Hence:

$$\lambda \frac{\psi^2}{-\psi + |\nabla_\psi \psi|_\psi^2} I \leq [g^{i\bar{j}}] \leq \Lambda (-\psi) I.$$

□

**Proposition 2.16.** *Under the hypothesis and notations of Proposition 2.12, we have:*

1. *For every  $\gamma \in ]0, \min(2n+1, 2l-1)[$ , there exist positive constants  $\epsilon$  and  $c$  such that  $|\nabla u| \leq c(-\varphi)^{\frac{\gamma}{2}-1}$  on the set  $D \cap U \cap \{|\varphi| < \epsilon\}$ . In particular, if  $\gamma > 2$ , one has  $u \in \mathcal{C}^1(\overline{D \cap U})$ .*
2. *For every  $z \in \partial D \cap U$ ,  $|\nabla e^{-w'}|_z \neq 0$ .*
3. *For every  $\gamma \in ]0, \min(2n+1, 2l-1)[$  there exist positive constants  $c$  and  $\epsilon$  such that  $|u| \leq c(-\varphi)^{\frac{\gamma}{2}}$  on  $D \cap U \cap \{|\varphi| < \epsilon\}$ .*

**Remark 2.17.** • *Observe that relation (2.8) already gives an estimate of  $u$ . Indeed, by applying  $\text{Log} \circ \text{Det}$  on both sides, using equation (2.1), and simplifying both sides, we may successively obtain, on  $D \cap \overline{U} \cap \{|\varphi| \leq \epsilon\}$ :*

$$e^{(n+1)u+F} \text{Det} [w_{i\bar{j}}] = \left(1 + O\left(|\varphi|^{\delta_0}\right)\right)^n \text{Det} [w_{i\bar{j}}],$$

$$u = \frac{n}{n+1} \text{Log} \left(1 + O\left(|\varphi|^{\delta_0}\right)\right) - \frac{F}{n+1}.$$

*Thus, part (3) of Proposition 2.16 only improves the exponent  $\delta_0$ .*

• *Part (3) of Proposition 2.16 is exactly as in [4], the only difference being that we have it for every  $\gamma \in ]0, \min(2n+1, 2l-1)[$ . We prove it a slightly different way by first proving part (1) of Proposition 2.16.*

*Proof of Proposition 2.16.* 1. We apply Proposition 2.12, and use Lemma 2.15 with  $\psi = \eta\varphi$ ,  $g = w$  and  $U$  replaced with  $U \cap \{|\varphi| < \epsilon\}$ . With notations of Proposition 2.12 and Lemma 2.15, we have  $\frac{\epsilon}{\lambda} > 0$ . Moreover we know that  $-\psi + |\nabla_\psi \psi|_\psi^2, \left(\frac{1}{\eta}\right)^2 \in \mathcal{C}(\overline{U})$  and are positive functions. Hence they are bounded from above, so that there exist positive

constants  $M_1, M_2$  such that  $-\psi + |\nabla_\psi \psi|_\psi^2 \leq M_1$  and  $\left(\frac{1}{\eta}\right)^2 \leq M_2$  on  $\bar{U}$ . Thus, we have the following on  $D \cap \bar{U}$ :

$$\begin{aligned} |\nabla u|^2 &\leq \frac{1}{\lambda} \frac{-\psi + |\nabla_\psi \psi|_\psi^2}{\psi^2} |\nabla_g u|_g^2 \leq \frac{c}{\lambda} (-\psi + |\nabla_\psi \psi|_\psi^2) \left(\frac{\varphi}{\psi}\right)^2 (-\varphi)^{\gamma-2}, \\ &= \frac{c}{\lambda} (-\psi + |\nabla_\psi \psi|_\psi^2) \left(\frac{1}{\eta}\right)^2 (-\varphi)^{\gamma-2}, \\ &\leq \frac{c}{\lambda} M_1 M_2 (-\varphi)^{\gamma-2}. \end{aligned}$$

Therefore we obtain the conclusion by setting  $c' = \sqrt{\frac{c}{\lambda} M_1 M_2}$ . Especially, if  $\gamma > 2$ , then all the derivatives of  $u$  of order 1 extend continuously to  $\overline{D \cap U}$  (and equal 0 on  $\partial D \cap \bar{U}$ ), hence  $u \in \mathcal{C}^1(\overline{D \cap U})$ .

2. To prove part (2) of Proposition 2.16, we let  $l = n + 1$ . Then by construction  $e^{-w'} = -\varphi^{(n+1)} e^{-u}$ . Moreover, according to point (1) of Proposition 2.9 and to point (1) of Proposition 2.16, we have  $\varphi^{(n+1)}, u \in \mathcal{C}^1(\overline{D \cap U})$ . Thus  $e^{-w'} \in \mathcal{C}^1(\overline{D \cap U})$  so that we can differentiate in  $D \cap U$  and let  $z$  tend to any point in  $\partial D \cap U$  to deduce

$$\lim_{z \rightarrow \partial D \cap U} \left| \nabla e^{-w'} \right|_z = \lim_{z \rightarrow \partial D \cap U} \left| \nabla \varphi^{(n+1)} \right|_z \neq 0,$$

because of points (2), (3) of Proposition 2.9.

3. Fix  $\gamma \in ]0, \min(2n + 1, 2l - 1)[$ .

Let  $z \in U \cap \{|\varphi| < \epsilon\}$ . Let  $z_0 \in \partial D \cap U \cap \{|\varphi| < \epsilon\}$  such that  $d(z, \partial D) = |z - z_0| =: s$ .

Set  $\vec{v} := z - z_0$ . Define the following function:

$$\begin{aligned} f : [0, 1] &\longrightarrow \mathbb{R} \\ t &\longmapsto u(z_0 + t\vec{v}). \end{aligned}$$

According to point (1) of Proposition 2.16 we have  $f \in \mathcal{C}^1([0, s])$ . Moreover, by the Cauchy-Schwarz inequality we have  $|f'(t)| \leq |\nabla u|_{z_0 + t\vec{v}} |\vec{v}| = s |\nabla u|_{z_0 + t\vec{v}}$ . From point (1) of Remark 2.17 we also have  $u(z_0) = 0$ . Using the fundamental theorem of calculus

we deduce:

$$\begin{aligned}
|u(z)| &= |f(1) - f(0)| = \left| \int_0^1 f'(t) dt \right|, \\
&\leq s \int_0^1 |\nabla u|_{z_0+t\vec{v}} dt, \\
&\leq cs \int_0^1 (-\varphi(z_0 + t\vec{v}))^{\frac{\gamma}{2}-1} dt, \\
&\leq \frac{cs}{\inf_{[0,1]} h'(t)} \int_0^1 h'(t) (h(t))^{\frac{\gamma}{2}-1} dt, \\
&= \frac{2cs}{\gamma \inf_{[0,1]} h'(t)} \int_0^1 \left( h^{\frac{\gamma}{2}} \right)'(t) dt, \\
&= \frac{2cs}{\gamma \inf_{[0,1]} h'(t)} (-\varphi(z))^{\frac{\gamma}{2}}, \\
&\leq \frac{2cs}{\gamma \inf_{[0,1]} h'(t)},
\end{aligned}$$

where  $h := -\varphi(z_0 + \cdot \vec{v}) \in \mathcal{C}^1([0, 1])$ . According to point (3) of Proposition 2.9 we have  $\inf_{[0,1]} h' > 0$ . Hence the result. □

Proposition 2.18 is exactly as in [4], the only difference being that we have the estimates for every  $\gamma \in ]0, \min(2n + 1, 2l - 1)[$ .

**Proposition 2.18.** *Under the hypothesis and notations of Proposition 2.12, we have: for every  $\gamma \in ]0; \min(2n + 1, 2l - 1)[$ , there exist positive constants  $\epsilon$  and  $c$  such that for every integer  $0 \leq p \leq s - 2l$ , the following holds on  $D \cap U \cap \{|\varphi| < \epsilon\}$ :*

$$|D^p u|_w \leq c |\varphi|^{\frac{\gamma}{2}},$$

where  $|D^p u|_w$  is the length of the  $p$ -th covariant derivative of  $u$  with respect to  $[w_{i\bar{j}}]$ .

*Proof of Proposition 2.18.* We fix  $\gamma \in ]0, \min(2n + 1, 2l - 1)[$  and follow line by line the proof at the beginning of page 300 in [4], the only thing that changes being the range in which  $\gamma$  can be chosen. Namely, we apply  $\text{Log} \circ \text{Det}$  to equation (2.2) to obtain the following partial differential equation of second order:

$$(n + 1)u + F = h_{i\bar{j}} u_{j\bar{i}}, \tag{2.17}$$

where  $[h_{i\bar{j}}] := \left[ \int_0^1 (w + tu)^{i\bar{j}} dt \right] \in \mathcal{C}^{s-2l-2}(D \cap U \cap \{|\varphi| < \epsilon\}, \mathcal{H}_n^{++})$ . We use Equation (2.9) with  $\delta = 0$  to obtain  $(1 - c'_1) [w_{i\bar{j}}] \leq [(w + u)_{i\bar{j}}] \leq (1 + c'_1) [w_{i\bar{j}}]$  which implies

$(1 - tc'_1) [w_{i\bar{j}}] \leq [(w + tu)_{i\bar{j}}] \leq (1 + tc'_1) [w_{i\bar{j}}]$  for every  $0 \leq t \leq 1$ . By inverting these inequalities and integrating between  $t = 0$  and  $t = 1$  we deduce the existence of constants  $\epsilon, c > 0$  such that we have, on  $D \cap U \cap \{|\varphi| < \epsilon\}$ :

$$\frac{1}{c} [w^{i\bar{j}}] \leq [h_{i\bar{j}}] \leq c [w^{i\bar{j}}].$$

Moreover we have  $u \in \mathcal{C}^{s-2l}(D \cap U \cap \{|\varphi| < \epsilon\})$  and according to Proposition 2.9 we also have  $F, \frac{F}{(-\varphi)^l} \in \mathcal{C}^{s-2l-2}(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$ . We conclude by applying Schauder theory.  $\square$

In particular, we deduce the following, exactly as was done in [4]:

**Proposition 2.19.** *Under the notations and hypothesis of Proposition 2.12, for every number  $\gamma \in ]0, \min(2n + 1, 2l - 1)[$  and for every  $0 \leq \delta < \frac{\gamma}{2} - \lfloor \frac{\gamma}{2} \rfloor$  (where  $\lfloor \frac{\gamma}{2} \rfloor$  denotes the integral part of  $\frac{\gamma}{2}$ ), we have:  $u, e^{-u} \in \mathcal{C}^{\lfloor \frac{\gamma}{2} \rfloor + \delta}(\overline{D \cap U})$ . Moreover, if  $\gamma > 2$ , we have:  $\varphi e^{-u} \in \mathcal{C}^{\lfloor \frac{\gamma}{2} \rfloor + 1 + \delta}(\overline{D \cap U})$ .*

*Proof of Proposition 2.19.* This is exactly as in [4] (or [12] for a global version). Observe that since  $s - 2l \geq 3n + 6 - 2(n + 1) \geq n + 2 \geq \frac{\gamma}{2}$ ,  $u \in \mathcal{C}^{n+2}(D \cap U)$  and  $\varphi \in \mathcal{C}^{n+2}(\overline{U})$  (see Proposition 2.9), it is enough to prove the existence of a positive constant  $\epsilon$  such that for every  $0 \leq \delta < \frac{\gamma}{2} - \lfloor \frac{\gamma}{2} \rfloor$ , one has  $u \in \mathcal{C}^{\lfloor \frac{\gamma}{2} \rfloor + \delta}(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$  and  $\varphi e^{-u} \in \mathcal{C}^{\lfloor \frac{\gamma}{2} \rfloor + 1 + \delta}(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$ .

Let  $\gamma \in ]0, \min(2n + 1, 2l - 1)[$ . According to Proposition 2.18, there exist positive constants  $\epsilon$  and  $c$  such that for every integer  $0 \leq p \leq s - 2l$ , the following holds on  $D \cap U \cap \{|\varphi| < \epsilon\}$ :

$$|D^p u|_w \leq c |\varphi|^{\frac{\gamma}{2}}.$$

Moreover, according to Lemma 2.15, there exist positive constants  $\lambda \leq \Lambda$  such that the following holds on  $D \cap U$ :

$$\lambda \left( \frac{-\psi}{-\varphi} \right) \frac{-\psi}{-\psi + |\nabla_{\psi} \psi|_{\psi}^2} I \leq \left[ \frac{w^{i\bar{j}}}{-\varphi} \right] \leq \Lambda \left( \frac{-\psi}{-\varphi} \right) I.$$

Since  $\left( \frac{-\psi}{-\varphi} \right) \in \mathcal{C}(\overline{U})$  is a positive function (see Proposition 2.9) and  $\overline{U}$  is a compact set, we deduce that there exist positive constants  $M$  and  $M'$  such that the following holds on  $D \cap U$ :

$$\lambda M \frac{-\psi}{-\psi + |\nabla_{\psi} \psi|_{\psi}^2} I \leq \left[ \frac{w^{i\bar{j}}}{-\varphi} \right] \leq \Lambda M' I.$$



Together with the expression of  $|D^p u|_w$  in terms of the derivatives of  $u$  and of  $w$ , this implies the existence of positive constants  $\epsilon$  and  $c$  such that for every integer  $0 \leq p \leq s-2l$  and every multi-index  $(i_1, j_1, \dots, i_n, j_n) \in \mathbb{N}^{2n}$  satisfying  $\sum_{k=1}^n (i_k + j_k) \leq p$ , the following holds on  $D \cap U \cap \{|\varphi| < \epsilon\}$ :

$$|u_{i_1 \bar{j}_1 \dots i_n \bar{j}_n}|, \left| (e^{-u})_{i_1 \bar{j}_1 \dots i_n \bar{j}_n} \right| \leq c |\varphi|^{\frac{\gamma}{2}-p}.$$

- Let  $p = \lfloor \frac{\gamma}{2} \rfloor$ . Then the derivatives of  $u$  of order  $p$  extend continuously to  $D \cap U \cap \{|\varphi| < \epsilon\}$  (and are equal to 0 on  $\partial D \cap \bar{U}$ ), and these extensions are Hölder of exponent  $\delta$  for every  $0 \leq \delta < \frac{\gamma}{2} - \lfloor \frac{\gamma}{2} \rfloor$ . This gives the desired regularity of  $u$  and  $e^{-u}$ .
- According to the chain rule and the regularity of  $\varphi$  and  $e^{-u}$ , we have the existence of a constant  $c > 0$  such that the following holds on  $D \cap U \cap \{|\varphi| < \epsilon\}$ :

$$\left| (\varphi e^{-u})_{i_1 \bar{j}_1 \dots i_n \bar{j}_n} - \varphi_{i_1 \bar{j}_1 \dots i_n \bar{j}_n} e^{-u} \right| \leq c |\varphi|^{\frac{\gamma}{2}-(p-1)}.$$

Since  $\frac{\gamma}{2} > 1$  we also have  $\varphi_{i_1 \bar{j}_1 \dots i_n \bar{j}_n} e^{-u} \in \mathcal{C}^1(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$  hence  $\varphi_{i_1 \bar{j}_1 \dots i_n \bar{j}_n} e^{-u} \in \mathcal{C}^\delta(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$  for every  $0 \leq \delta < 1$ . Let  $p = \lfloor \frac{\gamma}{2} \rfloor + 1$ . Then the derivatives of  $\varphi e^{-u}$  of order  $p$  extend continuously to  $D \cap U \cap \{|\varphi| < \epsilon\}$  and these extensions are Hölder of exponent  $\delta$  for every  $0 \leq \delta < \frac{\gamma}{2} - \lfloor \frac{\gamma}{2} \rfloor$ . This gives the desired regularity of  $\varphi e^{-u}$ .  $\square$

### 2.2.3 Proof of Theorems 2.4 and 2.5

We deduce Theorem 2.4 by using Proposition 2.19:

*Proof of Theorem 2.4.* We take  $l = n + 1$ . Then, according to Proposition 2.11, the range of  $\gamma$  is  $]0, 2n + 1[$ . Let  $\alpha \in ]0, 1[$  and take  $\gamma := 2n + \alpha$  so that  $\lfloor \frac{\gamma}{2} \rfloor = n$ . We apply Proposition 2.19 to obtain  $\varphi e^{-u} \in \mathcal{C}^{n+1+\delta}(\overline{D \cap U})$  for every  $0 \leq \delta < \frac{\alpha}{2}$ . Since  $s - 2(n + 1) \geq n + 2$ , then  $\frac{\varphi^{(n+1)}}{\varphi} \in \mathcal{C}^{n+2}(\bar{U})$  by point (7) of Proposition 2.9. We directly deduce that  $-w' = \varphi^{(n+1)} e^{-u} = \left( \frac{\varphi^{(n+1)}}{\varphi} \right) \varphi e^{-u} \in \mathcal{C}^{n+1+\delta}(\overline{D \cap U})$ . This holds for every  $0 \leq \delta < \frac{\alpha}{2} < \frac{1}{2}$ , hence the result.  $\square$

We can also prove Theorem 2.5:

*Proof of Theorem 2.5.* In this proof for a Kähler metric  $[h_{i\bar{j}}]$  of class  $\mathcal{C}^2$  and two vectors  $v, w \in \mathbb{C}^n \setminus \{0\}$  we note  $Bis(h)(v, w)$  instead of  $Bis(h)(\cdot; v, w)$  the holomorphic bisectonal curvatures of the metric between  $v$  and  $w$ .

The curvature coefficients of  $[g_{i\bar{j}}]$  satisfy the following relation which follows from the definition by direct calculations:

$$\begin{aligned}
R_{i\bar{j}k\bar{l}}(g) &= -(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) \\
&+ \frac{1}{-\psi} \left( R_{i\bar{j}k\bar{l}}(\psi) - \frac{1}{|\nabla\psi|_{\psi}^2 - \psi} \underbrace{(\psi_{ik} - \psi_{ik\bar{p}}\psi^{\bar{p}q}\psi_q)}_{\psi_{,ik}:=} \underbrace{(\psi_{j\bar{l}} - \psi_{j\bar{p}}\psi^{\bar{p}q}\psi_{q\bar{j}\bar{l}})}_{\psi_{,j\bar{l}}:=} \right). \tag{2.18}
\end{aligned}$$

Using the definition of the holomorphic bisectional curvatures (1.9) and relation (2.18) we obtain the following on  $D \cap U$  for every  $v, w \in \mathbb{C}^n \setminus \{0\}$ :

$$\begin{aligned}
Bis(g)(v, w) &= - \underbrace{\left( 1 + \frac{|\langle v, w \rangle_g|^2}{|v|_g^2 |w|_g^2} \right)}_{=:T_1(v,w)} \\
&+ \underbrace{\frac{1}{-\psi} \frac{|v|_{\psi}^2 |w|_{\psi}^2}{|v|_g^2 |w|_g^2} Bis(\psi)(v, w)}_{=:T_2(v,w)} \\
&- \underbrace{\frac{1}{-\psi} \frac{1}{|\nabla_{\psi}\psi|_{\psi}^2 - \psi} \frac{\psi_{,ik}\psi_{,j\bar{l}}v_i\bar{v}_j\bar{w}_k\bar{w}_l}{|v|_g^2 |w|_g^2}}_{=:T_3(v,w)}.
\end{aligned}$$

Using the proof of Proposition 2.19 with  $\frac{\gamma}{2} = n + \delta \geq 2 + \delta$  for some fixed  $0 < \delta < \frac{1}{2}$  we have the existence of positive constants  $c, \epsilon > 0$  such that the following holds on  $D \cap U \cap \{|\varphi| < \epsilon\}$  for every  $1 \leq i, j, k, l \leq n$ :

$$|\psi_{i\bar{j}k\bar{l}}| \leq c |\varphi|^{-1+\delta},$$

and we also have  $\psi \in \mathcal{C}^3(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$ .

The rest of the proof consists of estimating  $|T_2(v, w)|$  and  $|T_3(v, w)|$ . This will directly follow from formulas (2.19) and (2.20).

- Using the notations of Proposition 1.25 and of the proof of Proposition 1.25, we have  $0 \leq B$ , hence  $I \leq A$ , hence  $[\psi_{i\bar{j}}] =: R^2 \leq RAR = (-\psi) [g_{i\bar{j}}]$ .

This means that for every  $v \in \mathbb{C}^n$ , the following holds on  $D \cap U$ :

$$|v|_{\psi}^2 \leq (-\psi) |v|_g^2. \tag{2.19}$$

- Since  $\overline{D \cap U \cap \{|\varphi| < \epsilon\}}$  is compact and  $\psi \in \mathcal{C}^2(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$ , we also have the

existence of a positive constant  $0 < \lambda_-$  such that the following inequality holds on  $D \cap U \cap \{|\varphi| < \epsilon\}$ :

$$\lambda_- I \leq [\psi_{i\bar{j}}]. \quad (2.20)$$

We complete the proof as follows. According to inequality (2.19), we have the following on  $D \cap U$  for every vectors  $v, w \in \mathbb{C}^n \setminus \{0\}$ :

$$\frac{1}{-\psi} \frac{|v|_\psi^2 |w|_\psi^2}{|v|_g^2 |w|_g^2} \leq -\psi = -\varphi e^{-u}.$$

Moreover, there exists a constant  $c > 0$  such that for all  $1 \leq i, j, k, l \leq n$  we have  $|R_{i\bar{j}k\bar{l}}(\psi)| \leq c|\varphi|^{\delta-1}$  on  $D \cap U \cap \{|\varphi| < \epsilon\}$ . Hence there exists a positive constant  $c > 0$  such that  $\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |T_2(v, w)| \leq c|\varphi|^\delta$  on  $D \cap U \cap \{|\varphi| < \epsilon\}$ .

Likewise, using inequalities (2.19) and (2.20) we obtain, on  $D \cap U$  and for every  $v, w \in \mathbb{C}^n \setminus \{0\}$ :

$$-\frac{1}{-\psi} \frac{1}{|\nabla_\psi \psi|_\psi^2} \frac{1}{-\psi} \frac{1}{|v|_g^2 |w|_g^2} \leq \frac{-\psi}{|\nabla_\psi \psi|_\psi^2 - \psi \lambda_-^2} \frac{1}{-\psi} = \frac{-\varphi}{|\nabla_\psi \psi|_\psi^2 - \psi \lambda_-^2} \frac{e^{-u}}{-\psi}.$$

Note that (up to taking a smaller positive  $\epsilon$ )  $|\nabla_\psi \psi|_\psi^2 - \psi \in \mathcal{C}(\overline{D \cap U \cap \{|\varphi| < \epsilon\}})$  and is a positive function according to point (2) of Proposition 2.16. Moreover, there exists a constant  $c > 0$  such that for all  $1 \leq i, j, k, l \leq n$  we have  $|\psi_{,ik} \psi_{,\bar{j}\bar{l}}| \leq c$  on  $D \cap U \cap \{|\varphi| < \epsilon\}$ . Hence there exists a positive constant  $c > 0$  such that  $\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |T_3(v, w)| \leq c|\varphi|$  on  $D \cap U \cap \{|\varphi| < \epsilon\}$ .

Using the triangle inequality, we deduce the existence of positive constants  $\epsilon, c > 0$  such that the following inequality holds on  $D \cap U \cap \{|\varphi| < \epsilon\}$ :

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |Bis_g(v, w) + T_1(v, w)| \leq c|\varphi|^\delta.$$

We obtain the result since  $\lim_{z \rightarrow \partial D \cap U} \varphi(z) = 0$  and  $\delta > 0$ . □

## 2.3 Behaviour of the Kähler-Einstein curvatures when the squeezing function tends to one

### 2.3.1 The squeezing function

We recall the definition of the squeezing function of a domain.

**Definition 2.20.** Let  $D \subset \mathbb{C}^n$  be a domain. For  $z \in D$ , let

$$\mathcal{F}_z^D := \{f \in \mathcal{H}(D, \mathbb{B}) \mid f \text{ is injective and } f(z) = 0\}.$$

The squeezing function of  $D$  at point  $z \in D$  is defined by  $s^D(z) := \sup\{r > 0 / \exists f \in \mathcal{F}_z^D, B(0, r) \subset f(D)\}$  if  $\mathcal{F}_z^D \neq \emptyset$ , and 0 otherwise.

We say that  $D$  satisfies a uniform squeezing property if  $\inf_{z \in D} s^D(z) > 0$ . More precisely for  $a \in ]0, 1]$  we say that  $D$  satisfies the  $a$ -squeezing property if  $s^D(z) \geq a$  for every  $z \in D$ .

**Remark 2.21.** Let  $D \subset \mathbb{C}^n$  be a domain and let  $z \in D$  such that  $s^D(z) > 0$ . It was observed in [15] that the supremum in the definition of  $s^D(z)$  is achieved.

Let  $D \subset \mathbb{C}^n$ . It is clear that if there exists a point  $z \in D$  such that  $s^D(z) = 1$  then  $D$  is biholomorphic to  $\mathbb{B}$ . Assume that there exists  $q \in \partial D$  such that  $\lim_{z \rightarrow q} s^D(z) = 1$ . For every sequence  $(z^{(\nu)})_{\nu \in \mathbb{N}} \in D^{\mathbb{N}}$  converging to  $q$  there exists a sequence of holomorphic injective maps  $(f^{(\nu)})_{\nu \in \mathbb{N}} \in \mathcal{H}(D, \mathbb{B})^{\mathbb{N}}$  and  $(r^{(\nu)})_{\nu \in \mathbb{N}} \in ]0, 1]^{\mathbb{N}}$  such that  $B(0, r^{(\nu)})_{\nu \in \mathbb{N}} \subset f^{(\nu)}(D) \subset \mathbb{B}$  and  $\lim_{\nu \rightarrow \infty} r^{(\nu)} = 1$ . This implies that the sequence of sets  $(f^{(\nu)}(D))_{\nu \in \mathbb{N}}$  converges to  $\mathbb{B}$  in the local Hausdorff topology. This is why we think of  $q$  as a "ball-like" boundary point.

Domains satisfying a uniform squeezing property enjoy many interesting properties. For instance, if  $D$  satisfies a uniform squeezing property, then  $D$  is pseudoconvex. Moreover there exists a complete Kähler-Einstein metric on  $D$ . Also, the Bergman kernel of  $D$  induces a complete metric, and the Bergman metric and the Kähler-Einstein metric of  $D$  are equivalent on  $D$ . We refer to [15, 38, 57] for proofs of these statements and other properties regarding domains satisfying a uniform squeezing property.

### 2.3.2 Proof of Theorem 2.6

In the rest of this subsection, every domain possesses a unique complete Kähler-Einstein potential which is solution to Equation (2.2) with condition (2.4) and we only consider this potential. Moreover, given a domain  $D$  with complete Kähler-Einstein potential  $g$  solving Equation (2.2) with condition (2.4), we use the notations  $\langle \cdot, \cdot \rangle^D, |\cdot|_z^D, Bis^D$  instead of the previous notations  $\langle \cdot, \cdot \rangle_g, |\cdot|_g, Bis(g)$  to avoid confusions.

We prove Theorem 2.6:

*Proof of Theorem 2.6.* Let  $(z^{(\nu)})_{\nu \in \mathbb{N}} \in D^{\mathbb{N}}$  such that  $\lim_{\nu \rightarrow \infty} z^{(\nu)} = q$ . For  $\nu \in \mathbb{N}$  let  $f^{(\nu)} \in \mathcal{F}_{z^{(\nu)}}^D$  such that  $B(0, s^D(z^{(\nu)})) \subset f^{(\nu)}(D)$ , let  $g^{(\nu)} := (1 - \frac{1}{2^{\nu+1}}) f^{(\nu)}$  and set  $D_\nu :=$

$g^{(\nu)}(D)$ . Since  $g^{(\nu)}$  is a biholomorphic mapping from the pseudoconvex domain  $D$  to  $D_\nu$ ,  $D_\nu$  is a bounded pseudoconvex domain. By construction of  $g^{(\nu)}$ , for every integer  $\nu \in \mathbb{N}$  we have  $\overline{D_\nu} \subset \mathbb{B}$ . Moreover we have  $\lim_{\nu \rightarrow \infty} s^D(z^{(\nu)}) = 1$  hence up to taking a subsequence we may assume that  $\overline{D_\nu} \subset D_{\nu+1}$ .

Let  $\nu \in \mathbb{N}$ . Since  $D_\nu$  is a bounded pseudoconvex domain, there exists an exhaustion of  $D_\nu$  by strictly pseudoconvex domains with smooth boundary, so that according to Lemma 2.7 there exists a strictly pseudoconvex domain  $S_\nu$  with boundary of class  $\mathcal{C}^\infty$  that satisfies  $\overline{D_{\nu-1}} \subset S_\nu \subset D_\nu$  and

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} \left| \left( \frac{|\langle v, w \rangle_0^{D_\nu}|}{|v|_0^{D_\nu} |w|_0^{D_\nu}} \right)^2 - \left( \frac{|\langle v, w \rangle_0^{S_\nu}|}{|v|_0^{S_\nu} |w|_0^{S_\nu}} \right)^2 \right| \leq \frac{1}{2^\nu},$$

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |Bis^{D_\nu}(0; v, w) - Bis^{S_\nu}(0; v, w)| \leq \frac{1}{2^\nu}.$$

For every  $v, w \in \mathbb{C}^n \setminus \{0\}$  set  $v^{(\nu)} := \partial_{z^{(\nu)}} g^{(\nu)}(v)$  and  $w^{(\nu)} := \partial_{z^{(\nu)}} g^{(\nu)}(w)$ . Since each  $g^{(\nu)}$  is holomorphic and injective, the linear map  $\partial_{z^{(\nu)}} g^{(\nu)}$  is invertible, hence  $v^{(\nu)}, w^{(\nu)} \neq 0$  and:

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} \left| \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{D_\nu}|}{|v^{(\nu)}|_0^{D_\nu} |w^{(\nu)}|_0^{D_\nu}} \right)^2 - \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{S_\nu}|}{|v^{(\nu)}|_0^{S_\nu} |w^{(\nu)}|_0^{S_\nu}} \right)^2 \right| \leq \frac{1}{2^\nu}, \quad (2.21)$$

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |Bis^{D_\nu}(0; v^{(\nu)}, w^{(\nu)}) - Bis^{S_\nu}(0; v^{(\nu)}, w^{(\nu)})| \leq \frac{1}{2^\nu}. \quad (2.22)$$

Because of the property  $\overline{D_\nu} \subset S_{\nu+1} \subset D_{\nu+1}$  for every  $\nu \in \mathbb{N}$ , the sequence  $(S_\nu)_{\nu \in \mathbb{N}}$  is an increasing sequence of strictly pseudoconvex domains with boundary of class  $\mathcal{C}^\infty$ . Since  $\lim_{\nu \rightarrow \infty} s^D(z^{(\nu)}) = 1$  we have  $\bigcup_{\nu \in \mathbb{N}} S_\nu = \mathbb{B}$ , hence  $(S_\nu)_{\nu \in \mathbb{N}}$  is an exhaustion of the ball. Therefore according to Lemma 2.7 we deduce the following:

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} \left| \left( \frac{|\langle v, w \rangle_0^{S_\nu}|}{|v|_0^{S_\nu} |w|_0^{S_\nu}} \right)^2 - \left( \frac{|\langle v, w \rangle_0^{\mathbb{B}}|}{|v|_0^{\mathbb{B}} |w|_0^{\mathbb{B}}} \right)^2 \right| \xrightarrow{\nu \rightarrow \infty} 0,$$

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |Bis^{D_\nu}(0; v, w) - Bis^{\mathbb{B}}(0; v, w)| \xrightarrow{\nu \rightarrow \infty} 0.$$

Moreover, since each  $g^{(\nu)}$  is holomorphic and injective, the linear map  $\partial_{z^{(\nu)}} g^{(\nu)}$  is invertible,

hence:

$$\sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left| \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{S_\nu}|}{|v^{(\nu)}|_0^{S_\nu} |w^{(\nu)}|_0^{S_\nu}} \right)^2 - \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{\mathbb{B}}|}{|v^{(\nu)}|_0^{\mathbb{B}} |w^{(\nu)}|_0^{\mathbb{B}}} \right)^2 \right| \xrightarrow{\nu \rightarrow \infty} 0, \quad (2.23)$$

$$\sup_{v,w \in \mathbb{C}^n \setminus \{0\}} |Bis^{S_\nu}(0; v^{(\nu)}, w^{(\nu)}) - Bis^{\mathbb{B}}(0; v^{(\nu)}, w^{(\nu)})| \xrightarrow{\nu \rightarrow \infty} 0. \quad (2.24)$$

Using the transformation formula (1.13) and the triangle inequality we obtain for every integer  $\nu \in \mathbb{N}$ :

$$\begin{aligned} & \sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left| Bis^D(z^{(\nu)}; v, w) + 1 + \left( \frac{|\langle v, w \rangle_{z^{(\nu)}}^D|}{|v|_{z^{(\nu)}}^D |w|_{z^{(\nu)}}^D} \right)^2 \right| \\ &= \sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left| Bis^{D_\nu}(0; v^{(\nu)}, w^{(\nu)}) + 1 + \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{D_\nu}|}{|v^{(\nu)}|_0^{D_\nu} |w^{(\nu)}|_0^{D_\nu}} \right)^2 \right| \\ &\leq \sup_{v,w \in \mathbb{C}^n \setminus \{0\}} |Bis^{D_\nu}(0; v^{(\nu)}, w^{(\nu)}) - Bis^{S_\nu}(0; v^{(\nu)}, w^{(\nu)})| \\ &+ \sup_{v,w \in \mathbb{C}^n \setminus \{0\}} |Bis^{S_\nu}(0; v^{(\nu)}, w^{(\nu)}) - Bis^{\mathbb{B}}(0; v^{(\nu)}, w^{(\nu)})| \\ &+ \sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left| Bis^{\mathbb{B}}(0; v^{(\nu)}, w^{(\nu)}) + 1 + \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{S_\nu}|}{|v^{(\nu)}|_0^{S_\nu} |w^{(\nu)}|_0^{S_\nu}} \right)^2 \right| \\ &+ \sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left| \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{S_\nu}|}{|v^{(\nu)}|_0^{S_\nu} |w^{(\nu)}|_0^{S_\nu}} \right)^2 - \left( \frac{|\langle v^{(\nu)}, w^{(\nu)} \rangle_0^{D_\nu}|}{|v^{(\nu)}|_0^{D_\nu} |w^{(\nu)}|_0^{D_\nu}} \right)^2 \right|. \end{aligned}$$

From condition (2.22), respectively condition (2.24), condition (2.21), the first term of the right hand side, respectively the second, the fourth, tends to 0 as  $\nu$  tends to  $+\infty$ . Moreover the Kähler-Einstein metric we work with satisfies  $Bis^{\mathbb{B}}(0; v, w) = -1 - \left( \frac{|\langle v, w \rangle_0^{\mathbb{B}}|}{|v|_0^{\mathbb{B}} |w|_0^{\mathbb{B}}} \right)^2$  for every  $v, w \in \mathbb{C}^2 \setminus \{0\}$ . We combine this remark with condition (2.23) to deduce that the third term of the right hand side tends to 0 as  $\nu$  tends to  $+\infty$ . Therefore we have proved that there exists a subsequence  $(z^{(\nu_k)})_{k \in \mathbb{N}} \in D^{\mathbb{N}}$  such that

$$\sup_{v,w \in \mathbb{C}^n \setminus \{0\}} \left| Bis^D(z^{(\nu_k)}; v, w) + 1 + \left( \frac{|\langle v, w \rangle_{z^{(\nu_k)}}^D|}{|v|_{z^{(\nu_k)}}^D |w|_{z^{(\nu_k)}}^D} \right)^2 \right| \xrightarrow{k \rightarrow +\infty} 0.$$

We obtain the conclusion of Theorem 2.6 by applying the above reasoning to any subsequence of  $(z^{(\nu'_k)})_{\nu \in \mathbb{N}} \in D^{\mathbb{N}}$ .  $\square$

**Remark 2.22.** *The previous approach also works when  $[g_{i\bar{j}}]$  is the Bergman metric. In particular Theorem 2.6 also holds for  $\frac{1}{n+1}$  times the Bergman metric. Precise estimates of the holomorphic sectional curvatures, Ricci curvatures and scalar curvature can be found in [62] and may be adapted to obtain precise estimates on the holomorphic bisectonal curvatures of the Bergman metric. However the approach developed there cannot be applied to obtain estimates on the curvatures of the Kähler-Einstein metric.*

# Chapter 3

## Study of the Kähler-Einstein metric in pseudoconvex domains in $\mathbb{C}^2$

### Abstract

In this chapter we study the Kähler-Einstein metric and its holomorphic bisectional curvatures in pseudoconvex domains in  $\mathbb{C}^2$ . First we briefly review the known results in the case of Thullen domains and study the boundary regularity of the Kähler-Einstein potential at weakly pseudoconvex points. Then we prove analogous results in tube domains. Using the estimates about the holomorphic bisectional curvatures obtained in these two types of domains we prove that the holomorphic bisectional curvatures of the Kähler-Einstein metric at certain boundary points of bounded convex domains in  $\mathbb{C}^2$  are pinched between two negative constants in the case of a non-tangential approach. In Section 3.4 we provide with a partial study of the Kähler-Einstein metric in homogeneous polynomial domains  $D'_H$  where  $H$  is non negative.

**Convention** In this chapter we work only with the Kähler-Einstein metric with Ricci curvature  $-3$ , that we denote by  $g^D$  or by  $g$  when there is no confusion.

### Introduction

In Chapter 2 we saw that every bounded pseudoconvex domain  $D$  admits a unique complete Kähler-Einstein metric induced by a potential  $g \in \mathcal{C}^\omega(D)$  which satisfies Equation (2.2) with boundary condition (2.4) by exhausting the domain  $D$  by strictly pseudoconvex domains with smooth boundary, and that the curvature behaviour of the metric induced by  $g$  is well known at strictly pseudoconvex boundary points. However our approach



leaves open the question of the behaviour of the Kähler-Einstein metric and its curvatures at weakly pseudoconvex boundary points. J.S. Bland studied the Kähler-Einstein metric in Thullen domains in  $\mathbb{C}^{n+1}$  for  $n \in \mathbb{N}^*$  and proved that the Riemannian sectional curvatures of the Kähler-Einstein metric are negatively pinched (see [5]). He also obtained estimates for the Kähler-Einstein metric. Using his work we obtain an asymptotic expansion of the Kähler-Einstein potential in Thullen domains:

**Theorem 3.1.** *Let  $n \in \mathbb{N}^*$ , let  $p \geq 1$ , and set  $E_p^n := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}, |z|^2 + |w|^{2p} < 1\}$ . Let  $g \in \mathcal{C}^\omega(E_p^n)$  be the Kähler-Einstein potential solution of Equation (2.2) with boundary condition (2.4) on  $E_p^n$ . Then  $(1 - |z|^2)^{\frac{p-1}{(n+1)p}} e^{-g} \in \mathcal{C}^\infty(\overline{E_p^n} \setminus \{|z| = 1\})$ .*

The set  $\overline{E_p^n} \setminus \{|z| = 1\}$  is exactly the reunion of the Thullen domain and its strictly pseudoconvex boundary points. The asymptotic expansion obtained in Theorem 3.1 is very similar to the asymptotic expansion obtained by J. Lee and R. Melrose in the case of strictly pseudoconvex domains with boundary of class  $\mathcal{C}^\infty$ . We may use the regularity of  $e^{-g}$  to compare the Kähler-Einstein metric and the Bergman metric in the Thullen domains in  $\mathbb{C}^2$ , which give another proof that the Bergman metric of Thullen domains in  $\mathbb{C}^2$  is not an Einstein metric (see Corollary 3.10).

Drawing inspiration from the case of Thullen domains in  $\mathbb{C}^n$ , we study the Kähler-Einstein metric and curvatures in tube domains  $T'_p = \{z \in \mathbb{C}^2, \operatorname{Re}(z_1) + \operatorname{Re}(z_2)^{2p} < 0\}$  for  $p \in \mathbb{N}^*$ . We prove the following regarding the behaviour of the holomorphic bisectional curvatures at weakly pseudoconvex boundary points:

**Theorem 3.2.** *There exist positive constants  $0 < c \leq C$  and  $0 < \alpha < 1$  such that the following holds for every  $z \in T'_p \cap \left( \left\{ \frac{\operatorname{Re}(z_2)^{2p}}{-\operatorname{Re}(z_1)} \leq \alpha \right\} \cup \left\{ 1 - \alpha \leq \frac{\operatorname{Re}(z_2)^{2p}}{-\operatorname{Re}(z_1)} < 1 \right\} \right)$ :*

$$\forall v, w \in \mathbb{C}^2 \setminus \{0\}, \quad -C \leq \operatorname{Bis}(g)(z; v, w) \leq -c.$$

The region  $T'_p \cap \left\{ \frac{\operatorname{Re}(z_2)^{2p}}{-\operatorname{Re}(z_1)} \leq \alpha \right\}$  contains the axis  $\{z_2 = 0\}$ . Consequently the holomorphic bisectional curvatures of the Kähler-Einstein metric in tube domains and in Thullen domains in  $\mathbb{C}^2$  are negatively pinched in a neighbourhood of  $\{z_2 = 0\}$ , as it is the case for the Bergman metric (see Chapter 4). Using a rescaling we derive:

**Theorem 3.3.** *Let  $D \subset \mathbb{C}^2$  be a bounded convex domain with boundary of class  $\mathcal{C}^\infty$ . Let  $q \in \partial D$  be a point of finite type such that a local model at  $q$  is either a Thullen domain or a tube domain. There exists positive constants  $0 < c \leq C$  such that for every non*

tangential cone  $\Lambda$  with vertex at  $q$  and every sequence  $(z^{(\nu)})_{\nu \in \mathbb{N}} \in (D \cap \Lambda)^{\mathbb{N}}$  we have:

$$-C \leq \liminf_{\nu \rightarrow \infty} \inf_{v, w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}(g)(z^{(\nu)}; v, w), \limsup_{\nu \rightarrow \infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}(g)(z^{(\nu)}; v, w) \leq -c.$$

To prove Theorem 3.3 we use a scaling method. This technique applies at every boundary point of bounded convex domains in  $\mathbb{C}^2$ , but in the general case the local model of  $\partial D$  at  $q$  is of the form  $D'_H = \{z \in \mathbb{C}^2 / \text{Re}(z_1) + H(z_2) < 0\}$  where  $H$  is a real-valued homogeneous convex polynomial. We do not know whether the holomorphic bisectonal curvatures of the Kähler-Einstein metric of these domains are negatively pinched along the axis  $\{z_2 = 0\}$  (except for  $H(z) = |z|^{2p}$  and  $H(z) = \text{Re}(z)^{2p}$  where  $p \in \mathbb{N}^*$ ). Nonetheless we prove the following partial result:

**Theorem 3.4.** *Let  $p \in \mathbb{N}^*$ . Let  $H$  be a real-valued homogeneous polynomial function of degree  $2p$  which is subharmonic but not harmonic. Assume that there exists a complete Kähler-Einstein potential  $g \in \mathcal{C}^\omega(D'_H)$  solving Equation (2.2) and satisfying condition (2.4) on  $D'_H$ . Set  $K := \frac{2p+1}{3}$ . Then:*

$$\max_{v \in \mathbb{C}^n \setminus \{0\}} H(g)((-1, 0); v) = \frac{-3 - \frac{1}{K} + \frac{p-1}{pK} \frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0)}{2},$$

$$\max_{v, w \in \mathbb{C}^n \setminus \{0\}} \text{Bis}(g)((-1, 0); v, w) = \max \left\{ -\frac{1}{K}, \frac{-3 + \frac{1}{K} + \frac{p-1}{pK} \frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0)}{2} \right\},$$

$$\min_{v, w \in \mathbb{C}^n \setminus \{0\}} \text{Bis}(g)((-1, 0); v, w) = \min \left\{ -3 + \frac{1}{K}, \frac{-3 - \frac{1}{K} - \frac{p-1}{pK} \frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0)}{2} \right\},$$

$$\min_{v \in \mathbb{C}^n \setminus \{0\}} H(g)((-1, 0); v) = \min_{v, w \in \mathbb{C}^n \setminus \{0\}} \text{Bis}(g)((-1, 0); v, w).$$

This chapter is organised as follows. In Section 3.1 we recall some results obtained by J.S. Bland regarding the Kähler-Einstein metric in Thullen domains  $E_p^n$  and prove Theorem 3.1. In Section 3.2 we study the Kähler-Einstein metric in tube domains and prove Theorems 3.2 and 3.3. In Section 3.4 we prove Theorem 3.4.

## 3.1 Study of the Kähler-Einstein in Thullen domains

This section is mostly a summary of the results obtained by J.S. Bland about the Kähler-Einstein metric and its curvatures in Thullen domains in  $\mathbb{C}^{n+1}$  (see [5]). We recall them and use them to compare the Kähler-Einstein metric with the Bergman metric in Thullen domains. We also give more precise estimates about the holomorphic bisectional curvatures of the metric in certain regions.

### 3.1.1 Preliminary notations and remarks

For every integer  $n \in \mathbb{N} \setminus \{0, 1\}$ , and every real number  $p \geq 1$ , let

$$E_p^n := \{|z_1|^2 + \dots + |z_{n-1}|^2 + |z_n|^{2p} < 1\}.$$

The domain  $E_p^n$  is bounded and pseudoconvex, and has boundary of class  $\mathcal{C}^2$ . In particular there exists a unique Kähler-Einstein potential  $g \in \mathcal{C}^\omega(E_p^n)$  that solves Equation (2.2) on  $E_p^n$  and satisfies the boundary condition (2.4).

Define

$$\begin{aligned} \pi_1 : \mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C} &\longrightarrow \mathbb{C}^{n-1} \\ (z_1, \dots, z_n) &\longmapsto (z_1, \dots, z_{n-1}), \\ \\ \pi_2 : \mathbb{C}^n &\longrightarrow \mathbb{C} \\ (z_1, \dots, z_n) &\longmapsto z_n, \end{aligned}$$

and define  $X := \frac{|\pi_2|^2}{(1 - |\pi_1|^2)^{\frac{1}{p}}}$  on  $\mathbb{B} \times \mathbb{C} \subset \mathbb{C}^n$  so that  $E_p^n$  is exactly the set  $\{X < 1\}$ . It

is important to note that the function  $X$  is invariant under the action of  $Aut(E_p^n)$ .

The results of this Section give an expression of the potential  $g$  and its curvature coefficients in terms of functions of one real parameter applied to the "orbits parametrization function"  $X$ .

### 3.1.2 Overview of the already known results

In this subsection we recall two results stated in [5] regarding the expression of the Kähler-Einstein potential  $g$  and its curvature coefficients.

**Theorem 3.5** (Theorem 1 in [5]). *There exists a function  $Y \in \mathcal{C}^\infty([0, 1])$  such that the metric the following relations hold on  $E_p^n$ :*

$$[g_{i\bar{j}}] = Y \circ X \left[ (\text{Log} \circ X)_{i\bar{j}} \right] + Y^{(1)} \circ X \left[ \frac{X_i X_{\bar{j}}}{X} \right] \quad (3.1)$$

$$\text{Det}(g_{i\bar{j}}) = \frac{\left( \frac{Y \circ X}{p} \right)^{n-1} Y^{(1)} \circ X}{(1 - |\pi_1|^2)^{n + \frac{1}{p}}}. \quad (3.2)$$

Moreover,  $Y$  satisfies  $Y(0) = \frac{np+1}{n+1}$  and the following differential equation for every number  $x \in [0, 1[$ :

$$xY^{(1)}(x)Y(x)^{n-1} = Y(x)^{n+1} - pY(x)^n + \alpha, \quad (3.3)$$

where  $\alpha = \frac{p-1}{n+1}Y(0)^n$ . The functions  $x \mapsto Y(x)(1-x)$  and  $x \mapsto Y^{(1)}(x)(1-x)^2$  are bounded on  $[0, 1[$ .

**Remark 3.6.** • Define  $F := g_{|\{0\} \times ]-1, 1[}$ . Then  $F \in \mathcal{C}^\omega(]-1, 1[)$  and a careful analysis of the proof of Theorem 3.5 gives  $Y(x) = Y(0) + xF^{(1)}(x)$  for all  $x \in [0, 1[$ . This is why we added that  $Y \in \mathcal{C}^\infty([0, 1])$  compared to the original statement of Theorem 1 in [5].

- Observe that  $\text{Log} \circ X$  is not well defined on  $\{X = 0\}$  (or equivalently on  $\{z_2 = 0\}$ ). However by pluriharmonicity of  $\text{Log} \circ |\pi_2|^2$  we can naturally extend  $\left[ (\text{Log} \circ X)_{i\bar{j}} \right]$  on  $\{X = 0\}$  by setting  $\left[ (\text{Log} \circ X)_{i\bar{j}} \right] = \left[ \text{Log} \left( \frac{1}{(1 - |\pi_1|^2)^{\frac{1}{p}}} \right) \right]_{i\bar{j}}$  on  $\{X = 0\}$ .
- Likewise, the matrix  $\left[ \frac{X_i X_{\bar{j}}}{X} \right]$  is not well defined on  $\{X = 0\}$ , but we can naturally extend it on  $\{X = 0\}$  by using the expression of  $\left[ \frac{X_i X_{\bar{j}}}{X} \right]$  on  $\{X \neq 0\}$ .
- From relation (3.1) one sees that  $g_{2\bar{2}} = Y^{(1)} \circ X \frac{|X_2|^2}{X} > 0$  on  $E_p^n$  so that  $Y$  is an increasing function on  $[0, 1[$ . Since  $Y(0) > 0$  one has  $Y > 0$  on  $[0, 1[$ . Moreover one has  $\lim_{x \rightarrow 1^-} Y(x) = +\infty$ .

In [5] the author uses Theorem 3.5 to estimate the length of vectors for the Kähler-Einstein metric induced by  $g$  in regions of the form  $\{X \leq c\} \subset E_p^n$  for  $c \in [0, 1[$ , and also to estimate the blow-up rate of the volume of the metric in the same regions (see Theorem 3 in [5]).

By differentiating the metric and using the Kähler-Einstein condition one obtains the following expression of the curvature coefficients:

**Theorem 3.7** (Theorem 2 in [5]). *The coefficients of the curvature tensor of the Kähler-Einstein metric induced by the potential  $g$  satisfy the following on  $E_p^n$  for every integer  $1 \leq i, j, k, l \leq n$ :*

$$R_{i\bar{j}k\bar{l}} = - \left( 1 - \frac{n\alpha}{(Y \circ X)^{n+1}} \right) (g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) - n(n+1)\alpha \frac{(Y^{(1)} \circ X)^2}{(Y \circ X)^{n+1}} \frac{X_i X_{\bar{j}} X_k X_{\bar{l}}}{X^2} \\ - \frac{(n+1)\alpha}{(Y \circ X)^{n-1}} \left( (Log \circ X)_{i\bar{j}} (Log \circ X)_{k\bar{l}} + (Log \circ X)_{i\bar{l}} (Log \circ X)_{k\bar{j}} \right). \quad (3.4)$$

Using Theorem 3.7 J.S. Bland proved that the Riemannian sectional curvatures of the Kähler-Einstein metric induced by  $g$  are pinched between negative constants on  $E_p^n$  (see Theorem 4 in [5]).

**Remark 3.8.** *We can also define  $E_p^n$  (and the function  $X$ ) when  $p \in ]0, 1[$  in an obvious way. In that case, the boundary of  $E_p^n$  is not of class  $\mathcal{C}^2$  at boundary points  $z \in \partial E_p^n$  with  $z_n = 0$ , but  $E_p^n$  is still pseudoconvex in the sense that there exists an exhaustion of  $E_p^n$  by bounded strictly pseudoconvex domains with boundary of class  $\mathcal{C}^\infty$ . In particular there exists a Kähler-Einstein potential  $g \in \mathcal{C}^\omega(E_p^n)$  that solves Equation (2.2) with boundary condition (2.4). In those cases Theorems 3.5 and 3.7 still hold, but the holomorphic sectional curvatures of the Kähler-Einstein metric of  $E_p^n$  is not pinched between negative constants on  $E_p^n$  (see Theorem 4 in [5]).*

### 3.1.3 Asymptotic expansion of the Kähler-Einstein potential

We use Theorem 3.5 to obtain the regularity of  $e^{-g}$ . This is very similar to the analogue result in the case of bounded strictly pseudoconvex domains obtained by J. Lee and R. Melrose in [45]:

**Theorem 3.9.** *Let  $g$  be the Kähler-Einstein potential solution of Equation (2.2) with condition (2.4). There exists a positive function  $\eta \in \mathcal{C}^\infty([0, 1])$  with  $\eta(1) = 1$  and such that one has the following on  $E_p^n$*

$$e^{-g} = p^{\frac{n-1}{n+1}} (1 - X) \eta \circ X (1 - |\pi_1|^2)^{\frac{np+1}{p(n+1)}}.$$

*Proof of Theorem 3.9.* We use the notations introduced in Theorem 3.5. The Monge-Ampère Equation (2.2) combined with Equation (3.3) directly gives the relation  $e^{(n+1)g} = \frac{\left(\frac{Y \circ X}{p}\right)^{n-1} Y^{(1)} \circ X}{(1 - |\pi_1|^2)^{n+\frac{1}{p}}}$  on  $E_p^n$  hence  $e^{-g} = \left(\left(\frac{Y \circ X}{p}\right)^{n-1} Y^{(1)} \circ X\right)^{-\frac{1}{n+1}} (1 - |\pi_1|^2)^{\frac{np+1}{p(n+1)}}$ . Set  $\eta(x) =: (1-x)^{n+1} Y(x)^{n-1} Y^{(1)}(x)$  for  $x \in [0, 1[$ . Since  $Y$  and  $Y^{(1)}$  are positive functions on  $[0, 1[$ , the conclusion of Theorem 3.9 easily follows if we prove that  $\eta \in \mathcal{C}^\infty([0, 1])$  and is a positive function at  $x = 1$ .

We first prove that  $\frac{1}{Y} \in \mathcal{C}^\infty([0, 1])$ , then that  $x \mapsto (1-x)Y(x) \in \mathcal{C}^\infty([0, 1])$  and finally obtain the conclusion.

• Set  $h := \frac{1}{Y}$  on  $[0, 1[$  and  $h(1) := 0$ . Since  $Y > 0$  on  $[0, 1[$  and  $\lim_{x \rightarrow 1^-} Y(x) = +\infty$  we have  $h \in \mathcal{C}([0, 1]) \cap \mathcal{C}^\infty([0, 1])$ . We divide Equation (3.3) by  $-Y^{n+1}(x)$  to obtain the following for every  $x \in [0, 1[$ :

$$xh^{(1)}(x) = -1 + ph(x) - \alpha h(x)^{n+1}. \quad (3.5)$$

Therefore by classical ODE theory we deduce that  $h \in \mathcal{C}^\infty([0, 1])$ , hence  $\frac{1}{Y} \in \mathcal{C}^\infty([0, 1])$ .

• Let  $H$  be the primitive of  $h$  that vanishes at 1, and let  $G$  be the primitive of  $h^{n+1}$  that vanishes at 1, so that  $x \mapsto \frac{H(x)}{1-x}, x \mapsto \frac{G(x)}{1-x} \in \mathcal{C}^\infty([0, 1])$ . We take the primitives that vanish at 1 on both sides of Equation (3.5) and subtract  $H$  to obtain the following for every  $x \in [0, 1[$ :

$$xh(x) = 1 - x - (p+1)H(x) + \alpha G(x),$$

hence

$$x \frac{h(x)}{1-x} = 1 - (p+1) \frac{H(x)}{1-x} + \alpha \frac{G(x)}{1-x}, \quad (3.6)$$

so that  $x \mapsto \frac{h(x)}{1-x} \in \mathcal{C}^\infty([0, 1])$  or equivalently  $x \mapsto (1-x)Y(x) \in \mathcal{C}^\infty([0, 1])$  by definition of  $h$ . Letting  $x$  tend to  $1^-$  on both sides of Equation (3.6) directly gives  $\lim_{x \rightarrow 1^-} (1-x)Y(x) = 1 > 0$ . Moreover by the product rule one has:

$$\forall x \in [0, 1], \quad (1-x)(x \mapsto (1-x)Y(x))^{(1)}(x) = -(1-x)Y(x) + (1-x)^2 Y^{(1)}(x),$$

hence  $(x \mapsto (1-x)^2 Y^{(1)}(x)) \in \mathcal{C}^\infty([0, 1])$  and  $\lim_{x \rightarrow 1^-} (1-x)^2 Y^{(1)}(x) = 1 > 0$ . This gives the desired conclusion.  $\square$

As a corollary of Theorem 3.9 we compare the Bergman metric and the Kähler-Einstein metric on  $E_p^2$ . The result roughly indicates that the Bergman metric and the Kähler-Einstein metric blow up at the same rate at any boundary point:

**Corollary 3.10.** *Let  $K$  be the Bergman kernel of  $E_p^2$ , and set  $g' := \frac{\text{Log}(K)}{3}$ . Then  $e^{g'-g} \in C^\infty(\overline{E_p^2} \setminus \{|z|=1\})$ .*

*Proof of Corollary 3.10.* From relation (4.1) in [2] we have on  $E_p^2$ :

$$K = \frac{p+1}{p\pi^2} \frac{1-rX}{(1-X)^3(1-|\pi_1|^2)^{2+\frac{1}{p}}},$$

where  $r = \frac{p-1}{p+1}$ . Therefore we have:

$$e^{g'-g} = \left(\frac{p+1}{\pi^2}\right)^{\frac{1}{3}} \eta \circ X (1-rX)^{\frac{1}{3}},$$

so that the result directly follows from Theorem 3.9. □

**Remark 3.11.** *Let us explain why it is natural to compare the metrics  $g$  and  $g'$  in Theorem 3.10. It is well known (see [37]) that given a pseudoconvex domain  $D$ , the holomorphic sectional curvatures of the Bergman metric tend to  $-\frac{2}{3}$  at any smooth strictly pseudoconvex boundary point (provided that the Bergman kernel induces a Kähler metric). Hence according to formula (1.11) the Ricci form is asymptotically close to the opposite of the metric tensor. On the other hand, the Kähler-Einstein metric we work with satisfies  $\text{Ric}(g) = -3g$ . Thus to compare these metrics it is relevant to rescale one of them.*

## 3.2 Study of the Kähler-Einstein metric in pseudoconvex tube domains

In this Section we study the Kähler-Einstein metric in tube domains. In order to simplify the forthcoming computations, we work in the domains

$$T_p := \{z \in \mathbb{C}^2, \text{Re}(4pz_1) + \text{Re}(z_2)^{2p} < 1\}$$

where  $p \in \mathbb{N}^*$ . The biholomorphic affine map of  $\mathbb{C}^2$

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (z_1, z_2) &\longmapsto (4pz_1 - 1, z_2) \end{aligned}$$

maps  $T_p$  to  $T'_p$ . In particular the automorphism group of  $T_p$  and the geometric properties  $\partial T_p$  are well known (see Section 1.2 and Section 1.3). We use the invariance property of the Kähler-Einstein metric and the structure of the automorphism group of  $T_p$  to reduce the study of the Kähler-Einstein potential on  $T_p$  to the study of an auxiliary function of one real parameter satisfying an ordinary differential equation.

### 3.2.1 Geometry of $T_p$ and parametrisation of the orbits of $T_p$

We recall the description of the automorphism group of the tube, denoted by  $Aut(T_p)$ :

**Proposition 3.12.** *The automorphism group  $Aut(T_p)$  of  $T_p$  is generated by the following affine maps:*

- *Translations:*  $\tau_u(z_1, z_2) = (z_1 + iu_1, z_2 + iu_2)$ , where  $u \in \mathbb{R}^2$ ,
- *Dilations:*  $d_\lambda(z_1, z_2) = \left( \frac{\lambda(4pz_1-1)+1}{4p}, \lambda^{\frac{1}{2p}} z_2 \right)$ , where  $\lambda > 0$ ,
- *The symmetry of complex axis  $\{z_2 = 0\}$ :*  $s(z_1, z_2) = (z_1, -z_2)$ .

The translations have a Jacobian equal to the identity matrix. Also,  $Jac_{\mathbb{C}}(d_\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{\frac{1}{2p}} \end{bmatrix}$  and  $Jac_{\mathbb{C}}(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

Let us denote by  $\pi_1^{\mathbb{R}}$  and  $\pi_2^{\mathbb{R}}$  the following maps:

$$\begin{aligned} \pi_1^{\mathbb{R}} : \quad \mathbb{C}^2 &\longrightarrow \mathbb{R} \\ (z_1, z_2) &\longmapsto Re(z_1), \end{aligned}$$

$$\begin{aligned} \pi_2^{\mathbb{R}} : \quad \mathbb{C}^2 &\longrightarrow \mathbb{R} \\ (z_1, z_2) &\longmapsto Re(z_2). \end{aligned}$$

Let  $X := \frac{\pi_2^{\mathbb{R}}}{(1 - 4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}}}$ . This function is well defined on the set  $\{z \in \mathbb{C}^2 / Re(4pz_1) < 1\}$  which contains  $T_p$ .

Moreover, observe that it satisfies the following properties:

- $X \in C^\infty(\{z \in \mathbb{C}^2 / Re(4pz_1) < 1\})$ ,
- $X$  is a parametrization of the orbits of  $T_p$  under the action of  $Aut(T_p)$ , in the sense that

$$\forall F \in Aut(T_p), \quad \forall z \in T_p, \quad X(F(z)) = X(z) \text{ and } X_{|\{0\} \times ]-1, 1[} \text{ is injective,}$$

- $X(T_p) = ]-1, 1[$ ,
- $q \in \{|X| = 1\}$  if and only if  $q$  is a strictly pseudoconvex boundary point of  $\partial T_p$ .



Let us relate the regions introduced in Theorem 3.2 to the notions of tangential and non-tangential convergences. Let  $\theta \in ]0, \frac{\pi}{2}[$ . We denote by

$$\Lambda(\theta) := \left\{ z \in T_p / \frac{\sqrt{\operatorname{Im}(z_1)^2 + |z_2|^2}}{\left(\frac{1}{4p} - \operatorname{Re}(z_1)\right)} \leq \tan(\theta) \right\}$$

the half cone of vertex  $\left(\frac{1}{4p}, 0\right)$ , of axis  $\mathbb{R} \times \{0\}$  and of angle  $\theta$ .

Let  $(z^{(n)})_{n \in \mathbb{N}} \in T_p^{\mathbb{N}}$  such that  $z^{(\nu)} \xrightarrow{\nu \rightarrow +\infty} \left(\frac{1}{4p}, 0\right)$ . Recall that  $(z^{(\nu)})_{\nu \in \mathbb{N}}$  converges non tangentially to  $\left(\frac{1}{4p}, 0\right)$  if there exists a constant  $\theta \in ]0, \frac{\pi}{2}[$  and an integer  $\nu_0 \in \mathbb{N}$  such that for every  $\nu \geq \nu_0$  we have  $z_\nu \in \Lambda(\theta)$ , and that  $(z^{(\nu)})_{\nu \in \mathbb{N}}$  converges tangentially to  $\left(\frac{1}{4p}, 0\right)$  if for every constant  $\theta \in ]0, \frac{\pi}{2}[$  and there exists an integer  $\nu_0 \in \mathbb{N}$  such that for every  $\nu \geq \nu_0$  we have  $z_\nu \notin \Lambda(\theta)$ .

Now observe that we have:

$$\forall z \in T_p, \quad 4p |X(z)| (1 - \operatorname{Re}(4pz_1))^{\frac{1}{2p}-1} \leq \frac{\sqrt{\operatorname{Im}(z_1)^2 + |z_2|^2}}{\left(\frac{1}{4p} - \operatorname{Re}(z_1)\right)},$$

hence we deduce that for every sequence  $(z^{(\nu)})_{n \in \mathbb{N}} \in T_p^{\mathbb{N}}$  that converges to  $\left(\frac{1}{4p}, 0\right)$  and for every  $0 < \alpha < 1$  we have:

$$\begin{aligned} \forall 0 < \theta < \frac{\pi}{2}, (z^{(\nu)}) \in \Lambda(\theta)^{\mathbb{N}} &\Rightarrow \exists \nu_0 \in \mathbb{N}, \quad \forall \nu \geq \nu_0, \quad z^{(\nu)} \in \{|X| \leq \alpha\}, \\ (z^{(\nu)}) \in \{1 - \alpha \leq |X| < 1\}^{\mathbb{N}} &\Rightarrow \forall 0 < \theta < \frac{\pi}{2}, \quad \exists \nu_0 \in \mathbb{N}, \quad \forall \nu \geq \nu_0, \quad z^{(\nu)} \notin \Lambda(\theta). \end{aligned}$$

In particular, Theorem 1 gives the non-tangential behaviour of the bisectonal curvatures at weakly pseudoconvex boundary points of  $T_p$ , and also gives a "hyper-tangential" behaviour of the bisectonal curvatures at weakly pseudoconvex boundary points of  $T_p$ .

We conclude this subsection with the following Proposition, which directly follows from Proposition 3.12 and the definition of  $X$ :

**Proposition 3.13.** *Let  $z \in T_p$  and define  $\psi^{(z)} := d_{\frac{1}{1 - \operatorname{Re}(4pz_1)}} \circ \tau_{-(\operatorname{Im}(z_1), \operatorname{Im}(z_2))} \in \operatorname{Aut}(T_p)$ . Then  $\psi^{(z)}$  satisfies  $\psi^{(z)}(z) = (0, X(z))$ , and  $\operatorname{Det}(\operatorname{Jac}_{\mathbb{C}}(\psi^{(z)})) = \frac{1}{(1 - \operatorname{Re}(4pz_1))^{\frac{2p+1}{2p}}}$ .*

Proposition 3.13 enables to reduce the study of the metric and its curvatures on  $T_p$  to the study of the same quantities on the set  $\{0\} \times ]-1, 1[ + i\mathbb{R}^2$  (see for instance Propositions 3.14 and 3.16).

### 3.2.2 The invariance property and an expression of the Kähler-Einstein potential in terms of a special auxiliary function

The invariance property of the Kähler-Einstein metric under the action of  $Aut(T_p)$  gives a first reduction of the potential  $g$ :

**Proposition 3.14.** *Let*

$$\begin{aligned} F : ]-1, 1[ &\longrightarrow \mathbb{R} \\ x &\longmapsto g(0, x), \end{aligned}$$

and set  $K := \frac{2p+1}{3}$ . Then the following holds on  $T_p$ :

$$g = F \circ X + \frac{K}{p} \text{Log} \left( \frac{1}{1 - 4p\pi_1^{\mathbb{R}}} \right). \quad (3.7)$$

*Proof of Proposition 3.14.* The Kähler-Einstein metric is invariant under the action of  $Aut(T_p)$ , which means that:

$$\forall \psi \in Aut(T_p), \quad [g_{i\bar{j}}] = {}^t \text{Jac}_{\mathbb{C}}(\psi) [g_{i\bar{j}} \circ \psi] \overline{\text{Jac}_{\mathbb{C}}(\psi)}. \quad (3.8)$$

We apply the function  $\text{Log} \circ \text{Det}$  to both sides of Equation (3.8) and use the Monge-Ampère Equation (2.2) to deduce the following transformation formula:

$$\forall \psi = (\psi_1, \psi_2) \in Aut(T_p), \quad g = g \circ \psi + \frac{2}{3} \text{Log} |\text{Det}(\text{Jac}_{\mathbb{C}}(\psi))|. \quad (3.9)$$

Let  $z \in T_p$ . We use Equation (3.9) with the function  $\psi = \psi^{(z)}$  given in Proposition 3.13 and obtain the result.  $\square$

The function  $F$  inherits from the Kähler-Einstein potential  $g$  some regularity properties.

**Proposition 3.15.** *The function  $F$  is real analytic on  $]-1, 1[$ , strictly convex, and even. Moreover,  $e^{-F} \in \mathcal{C}^{3+\delta}([-1, 1])$  for every number  $\delta \in [0, \frac{1}{2}[$ .*

*Proof of Proposition 3.15.* The relation  $F(x) = g(0, x)$  for every number  $x \in ]-1, 1[$  directly implies that  $F \in \mathcal{C}^\omega(]-1, 1[)$  and  $e^{-F} \in \mathcal{C}^{3+\delta}([-1, 1])$  for every number  $\delta \in [0, \frac{1}{2}[$ . In particular, by differentiating Equation (3.7) twice at the point  $(0, x) \in T_p$ , we obtain  $F^{(2)}(x) = 4g_{2\bar{2}}(0, x) > 0$  because  $g$  is strictly plurisubharmonic on  $T_p$ . Hence  $F$  is strictly convex on  $]-1, 1[$ . To prove that  $F$  is even on  $]-1, 1[$ , we use the automorphism  $s$  introduced in Proposition 3.12 to deduce that for every  $-1 < x < 1$ , we have  $F(x) = g(0, x) = F(X(0, -x)) + \frac{K}{p} \text{Log}(1) = F(-x)$ , hence the result.  $\square$

### 3.2.3 The Kähler-Einstein condition and two differential equations satisfied by $F$

We use relation (3.7) and the Monge-Ampère Equation (2.2) to obtain a first differential equation satisfied by the function  $F$ :

**Proposition 3.16.** *Denote  $f := F^{(1)}$ . Then the metric  $[g_{i\bar{j}}]$  satisfies the following on  $T_p$ :*

$$[g_{i\bar{j}}] = \begin{bmatrix} \frac{X^2 f^{(1)} \circ X + (2p+1)Xf \circ X + 4pK}{(1-4p\pi_1^{\mathbb{R}})^2} & \frac{Xf^{(1)} \circ X + f \circ X}{2(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} \\ \frac{Xf^{(1)} \circ X + f \circ X}{2(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} & \frac{f^{(1)} \circ X}{4(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{p}}} \end{bmatrix}, \quad (3.10)$$

$$\text{Det}(g_{i\bar{j}}) = \frac{Z(X)}{(1-4p\pi_1^{\mathbb{R}})^{\frac{3K}{p}}}, \quad (3.11)$$

where the function  $Z$  is defined by  $Z(x) := \frac{f^{(1)}(x)((2p-1)xf(x)+4pK)-f(x)^2}{4}$  for every number  $x \in ]-1, 1[$ . Moreover,  $Z$  satisfies the following:

$$Z = e^{3F} \text{ on } ]-1, 1[. \quad (3.12)$$

*Proof of Proposition 3.16.* On  $T_p$  we have:

$$[X_i] = [X_{\bar{i}}] = \begin{bmatrix} \frac{X}{1-4p\pi_1^{\mathbb{R}}} \\ \frac{1}{2(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}}} \end{bmatrix},$$

$$[X_i X_{\bar{j}}] = \begin{bmatrix} \frac{X^2}{(1-4p\pi_1^{\mathbb{R}})^2} & \frac{X}{2(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}+1}} \\ \frac{X}{2(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}+1}} & \frac{1}{4(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{p}}} \end{bmatrix},$$

$$[X_{i\bar{j}}] = \begin{bmatrix} \frac{(2p+1)X}{(1-4p\pi_1^{\mathbb{R}})^2} & \frac{1}{2(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}+1}} \\ \frac{1}{2(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}+1}} & 0 \end{bmatrix}.$$

Differentiating Equation (3.7), we directly deduce:

$$[g_{i\bar{j}}] = f \circ X [X_{i\bar{j}}] + f^{(1)} \circ X [X_i X_{\bar{j}}] + \frac{4Kp}{(1-4p\pi_1^{\mathbb{R}})^2} E_{11},$$

$$= \begin{bmatrix} \frac{X^2 f^{(1)} \circ X + (2p+1)f \circ X + 4pK}{(1-4p\pi_1^{\mathbb{R}})^2} & \frac{Xf^{(1)} \circ X + f \circ X}{2(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} \\ \frac{Xf^{(1)} \circ X + f \circ X}{2(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} & \frac{f^{(1)} \circ X}{4(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{p}}} \end{bmatrix}.$$

Then we apply the function Det to Equation (3.10) and directly obtain Equation (3.11). Finally, recall that according to Equations (2.2) and (3.7) one has on  $T_p$ :

$$\begin{aligned} \text{Det}(g_{i\bar{j}}) &= e^{3g}, \\ &= e^{3F \circ X - \frac{3K}{p} \text{Log}(1 - 4p\pi_1^{\mathbb{R}})}, \\ &= \frac{e^{3F \circ X}}{(1 - 4p\pi_1^{\mathbb{R}})^{2 + \frac{1}{p}}}, \end{aligned}$$

hence Equation (3.12). □

We use Equation (3.12) to obtain a differential equation satisfied by  $f$  and  $f^{(1)}$ :

**Proposition 3.17.** *The function  $f$  satisfies the following equation for every  $x \in ]-1, 1[$ :*

$$\begin{aligned} ((2p - 1)xf(x) + 4pK) f^{(1)}(x) &= (2p - 1)xf(x)^3 + (6pK + 1) f(x)^2 \\ &\quad - 2(p + 1) \int_0^x f(t)^3 dt + 4e^{3F(0)}. \end{aligned} \quad (3.13)$$

*Proof of Proposition 3.17.* Let  $x \in ]-1, 1[$ . We put the definition of the function  $Z$  into Equation (3.12), multiply both sides by  $12f$  and integrate from 0 to  $x$  to obtain:

$$\begin{aligned} 4(3f(x)e^{3F(x)}) &= 3(2p - 1)xf(x)^2 f^{(1)}(x) + 12pKf(x)f^{(1)}(x) - 3f(x)^3, \\ 4Z(x) = 4e^{3F(x)} &= 3(2p - 1) \int_0^x tf(t)^2 f^{(1)}(t) dt + 6pKf(x)^2 - 3 \int_0^x f(t)^3 dt + 4e^{3F(0)}. \end{aligned}$$

We integrate by part the first term of the right hand side:

$$\int_0^x tf(t)^2 f^{(1)}(t) dt = \left[ \frac{tf(t)^3}{3} \right]_0^x - \frac{1}{3} \int_0^x f(t)^3 dt = \frac{xf(x)^3}{3} - \frac{1}{3} \int_0^x f(t)^3 dt.$$

Using the definition of  $Z$  again, we obtain:

$$\begin{aligned} ((2p - 1)xf(x) + 4pK) f^{(1)}(x) - f(x)^2 &= (2p - 1)xf(x)^3 + 6pKf(x)^2 \\ &\quad - 2(p + 1) \int_0^x f(t)^3 dt + 4e^{3F(0)}, \\ ((2p - 1)xf(x) + 4pK) f^{(1)}(x) &= (2p - 1)xf(x)^3 + (6pK + 1) f(x)^2 \\ &\quad - 2(p + 1) \int_0^x f(t)^3 dt + 4e^{3F(0)}, \end{aligned}$$

hence relation (3.13). □

### 3.2.4 Asymptotic analysis of the auxiliary function

In this subsection we use condition (2.4), Proposition 3.15 and Equation (3.13) to study the function  $F$  and its derivatives. Since  $F$  is an even function, we only study it on the set  $[0, 1[$ .

We want to point out that Propositions 3.18, 3.20, Corollary 3.21 and part of Proposition 3.22 can also be deduced from Theorem 2.4 because the function  $F$  is the restriction of the Kähler-Einstein potential  $g$  to the set  $\{0\} \times ]-1, 1[$ , and  $\partial T_p$  is smooth and strictly pseudoconvex at  $(0, 1)$ . Here, we only use Equation (3.13) and the interior regularity of  $F$  to derive these results.

From the strict convexity of  $F$  and condition (2.4) we have the following:

**Proposition 3.18.** *Every derivative of  $F$  is unbounded in a neighbourhood of  $1^-$ . Moreover,  $f(x) \xrightarrow{x \rightarrow 1^-} +\infty$ .*

*Proof of Proposition 3.18.* If there existed some integer  $k \in \mathbb{N}$  such that  $F^{(k)}$  is bounded in a neighbourhood of  $1^-$ , then  $g(0, \cdot)$  would be bounded in a neighbourhood of  $1^-$ , which would be in contradiction with the hypothesis (2.4), hence every derivative of  $F$  is unbounded in a neighbourhood of  $1^-$ .

Since  $F$  is a strictly convex, even function in  $] -1, 1[$ ,  $f = F^{(1)}$  is increasing on  $[0, 1[$  and positive on  $]0, 1[$ . Since it is unbounded, we directly deduce that  $f(x) \xrightarrow{x \rightarrow 1^-} +\infty$ . Hence the result.  $\square$

We use the following lemma to deduce the asymptotic behaviour of  $f^{(1)}$  at  $x = 1^-$ :

**Lemma 3.19.** *Let  $F \in \mathcal{C}^1(]0; 1[)$  be a convex function satisfying  $\lim_{y \rightarrow 1^-} F^{(1)}(y) = +\infty$ .*

*Then:  $\lim_{y \rightarrow 1^-} \frac{F(y)}{F^{(1)}(y)} = 0$ .*

*Proof of Lemma 3.19.* Since  $F$  satisfies  $\lim_{y \rightarrow 1^-} F^{(1)}(y) = +\infty$ , there exists a constant  $a \in ]0, 1[$  such that  $F^{(1)} > 0$  on  $]a, 1[$ . Let  $a < x < y < 1$ . Then  $F^{(1)}(y) > 0$  and  $F(x) \leq F(y)$ . We apply the fundamental theorem of calculus to the function  $F$  to deduce the following:

$$0 \leq F(y) - F(x) = \int_x^y F^{(1)}(t) dt \leq (y - x)F^{(1)}(y),$$

so that  $\frac{F(x)}{F^{(1)}(y)} \leq \frac{F(y)}{F^{(1)}(y)} \leq (y - x)$ . Hence we deduce:

$$\forall x \in ]0; 1[, \quad 0 \leq \liminf_{y \rightarrow 1^-} \frac{F(y)}{F^{(1)}(y)} \leq \limsup_{y \rightarrow 1^-} \frac{F(y)}{F^{(1)}(y)} \leq 1 - x,$$

so that we obtain  $\lim_{y \rightarrow 1^-} \frac{F(y)}{F^{(1)}(y)} = 0$  by letting  $x$  tend to  $1^-$ , hence the result.  $\square$

We use Equation (3.13), Proposition 3.18 and Lemma 3.19 to obtain the asymptotic of  $f$  to the first order at  $x = 1^-$ :

**Proposition 3.20.** *We have:  $\lim_{x \rightarrow 1^-} \frac{f^{(1)}}{f^2}(x) = \lim_{x \rightarrow 1^-} f(x)(1-x) = 1$ .*

*Proof of Proposition 3.20.* Let  $x > 0$ . Since  $f(0) = 0$  and  $f$  is increasing, we have  $f(x) > 0$ . We divide Equation (3.13) both sides by  $f(x)$  to obtain the following:

$$\begin{aligned} \left( (2p-1)x + \frac{4pK}{f(x)} \right) \frac{f^{(1)}(x)}{f(x)^2} &= (2p-1)x \\ &+ \frac{6pK+1}{f(x)} - 2(p+1) \frac{\int_0^x f(t)^3 dt}{f(x)^3} + \frac{4e^{3F(0)}}{f(x)^3}. \end{aligned}$$

Let us prove that  $\lim_{x \rightarrow 1^-} \frac{\int_0^x f(t)^3 dt}{f(x)^3} = 0$ . Define  $\tilde{f}(x) := \int_0^x f(t)^3 dt$  for  $x \in [0, 1[$ . Then  $\tilde{f} \in \mathcal{C}^1([0, 1[)$ , is convex and satisfies  $\lim_{x \rightarrow 1^-} \tilde{f}^{(1)}(x) = +\infty$ . We apply Lemma 3.19 to  $\tilde{f}$  to

deduce that  $\lim_{x \rightarrow 1^-} \frac{\tilde{f}}{\tilde{f}^{(1)}}(x) = 0$ .

Define  $b(x) := \frac{6pK+1}{f(x)} - 2(p+1) \frac{\int_0^x f(t)^3 dt}{f(x)^3} + \frac{4e^{3F(0)}}{f(x)^3}$  for  $x \in [0, 1[$ . Then  $b \in \mathcal{C}([0, 1[)$

and  $\lim_{x \rightarrow 1^-} b(x) = 0$ . Hence  $B := \int_x^1 b(t) dt$  is well defined and  $B \in \mathcal{C}^1([0, 1[)$ . Let  $x \in ]0, 1[$ .

We integrate between  $x$  and 1 to obtain:

$$\begin{aligned} (2p-1) \frac{x}{f(x)} + (2p-1) \int_x^1 \frac{dt}{f(t)} + \frac{2pK}{f(x)^2} &= \int_x^1 \left( (2p-1)t + \frac{4pK}{f(t)} \right) \frac{f^{(1)}(t)}{f^2(t)} dt, \\ &= \frac{2p-1}{2} (1-x^2) + B(x), \end{aligned}$$

$$(2p-1) \left( 1 + \frac{2pK}{f(x)} \right) \frac{x}{f(x)(1-x)} + (2p-1) \frac{\int_x^1 \frac{dt}{f(t)}}{1-x} = \frac{2p-1}{2} (1+x) + \frac{B(x)}{1-x}. \quad (3.14)$$

Note that  $\int_x^1 \frac{dt}{f(t)}$  is the primitive of the function  $\frac{1}{f} \in \mathcal{C}([0, 1])$ , so that  $\lim_{x \rightarrow 1^-} \frac{\int_x^1 \frac{dt}{f(t)}}{1-x} = 0$ .

Likewise by construction of  $B$  we have  $\lim_{x \rightarrow 1^-} \frac{B(x)}{1-x} = 0$ , therefore we can let  $x$  tend to  $1^-$

in Equation (3.14) to deduce  $\lim_{x \rightarrow 1^-} \frac{2p-1}{f(x)(1-x)} = \lim_{x \rightarrow 1^-} (2p-1) \left( 1 + \frac{2pK}{f(x)} \right) \frac{x}{f(x)(1-x)} = 2p-1$ , hence  $\lim_{x \rightarrow 1^-} f(x)(1-x) = 1$ .  $\square$

Proposition 3.20 directly gives the asymptotic behaviour of  $Z$ , and also an asymptotic expansion of  $F$ :

**Corollary 3.21.** *We have:  $\lim_{x \rightarrow 1^-} (1-x)^3 Z(x) = \frac{2p-1}{4}$ , and  $F(x) = \text{Log} \left( \frac{1}{1-x} \right) + \frac{\text{Log} \left( \frac{2p-1}{4} \right)}{3} + o(1)$ .*

*Proof of Corollary 3.21.* Proposition 3.20 and the definition of the function  $Z$  directly gives the first result. We apply formula (3.11) to deduce that  $\lim_{x \rightarrow 1^-} ((1-x)e^{F(x)})^3 = \frac{2p-1}{4}$ , hence  $\lim_{x \rightarrow 1^-} F(x) - \text{Log} \left( \frac{1}{1-x} \right) = \frac{\text{Log} \left( \frac{2p-1}{4} \right)}{3}$ , hence the second result.  $\square$

Corollary 3.21 is enough to deduce the asymptotic behaviour of the potential  $g$ , or equivalently its volume form (see Proposition 3.16). In order to estimate the curvatures of the Kähler-Einstein metric, we also need the asymptotic behaviour of higher derivatives of  $F$  at  $x = 1^-$ . We have the following:

**Proposition 3.22.** *For every integer  $k \in \mathbb{N}$ , one has the following:*

$$\lim_{x \rightarrow 1^-} (1-x)^{k+3} Z^{(k)}(x) = \frac{2p-1}{8} (k+2)! \text{ and } \lim_{x \rightarrow 1^-} f^{(k)}(x) (1-x)^{k+1} = k!.$$

*Proof.* Proof of Proposition 3.22 We argue by induction. Proposition 3.20 and Corollary 3.21 ensure that the formulas are true for  $k = 0$ . Let  $k \geq 0$  be an integer and assume that the formulas are true for any integer  $0 \leq l \leq k$ . We differentiate Equation (3.12)  $k+1$  times to obtain  $Z^{(k+1)} = (Z^{(1)})^{(k)} = 3 \sum_{l=0}^k \binom{k}{l} f^{(l)} Z^{(k-l)}$ , hence the following:

$$\begin{aligned} \frac{(1-x)^{k+4} Z^{(k+1)}(x)}{3} &\xrightarrow{x \rightarrow 1^-} \sum_{l=0}^k \binom{k}{l} \lim_{x \rightarrow 1^-} ((1-x)^{l+1} f^{(l)}(x)) \lim_{x \rightarrow 1^-} ((1-x)^{k-l+3} Z^{(k-l)}(x)), \\ &= \frac{2p-1}{8} \sum_{l=0}^k \binom{k}{l} l!(k+2-l)!, \\ &= \frac{2p-1}{8} k! \sum_{l=0}^k (k+2-l)(k+1-l), \\ &= \frac{2p-1}{8} k! \sum_{l=1}^{k+1} l(l+1) = \frac{2p-1}{8} (k+3)!. \end{aligned}$$

We differentiate Equation (3.12)  $k$  times to obtain:

$$\begin{aligned}
\frac{(k+2)!}{2} &= \lim_{x \rightarrow 1^-} 4(1-x)^{k+3} \frac{Z^{(k)}}{2p-1}(x) \\
&= \lim_{x \rightarrow 1^-} \sum_{l=0}^k \binom{k}{l} ((1-x)^{l+2} f^{(l+1)}(x)) ((1-x)^{k+1-l} f^{(k-l)}(x)) \\
&= \sum_{l=0}^{k-1} \binom{k}{l} (l+1)!(k-l)! + \lim_{x \rightarrow 1^-} ((1-x)^{k+2} f^{(k+1)}(x)) \\
&= k! \sum_{l=0}^{k-1} (l+1) + \lim_{x \rightarrow 1^-} ((1-x)^{k+2} f^{(k+1)}(x)) \\
&= \frac{k(k+1)!}{2} + \lim_{x \rightarrow 1^-} ((1-x)^{k+2} f^{(k+1)}(x)),
\end{aligned}$$

hence  $\lim_{x \rightarrow 1^-} ((1-x)^{k+2} f^{(k+1)}(x)) = \frac{(k+2)!}{2} - \frac{k(k+1)!}{2} = (k+1)!$  as stated.  $\square$

**Remark 3.23.** • We do not have an asymptotic expansion of  $F$  or  $e^{-F}$  to higher order.

We conjecture that:

$$\exists (\eta_k)_{k \in \mathbb{N}} \in \mathcal{C}^\infty([0, 1])^{\mathbb{N}}, \quad e^{-F(x)} \underset{x \rightarrow 1^-}{\sim} (1-x) \sum_{k=0}^{+\infty} \eta_k ((1-x)^3 \text{Log}(1-x))^k, \quad (3.15)$$

with  $\lim_{x \rightarrow 1^-} \eta_1(x) \neq 0$  except for  $p = 1$ . Especially, apart from the case of the ball ( $p = 1$ ), one would have  $e^{-F} \notin \mathcal{C}^4([0, 1])$  so that the regularity given in Proposition 3.15 would be almost optimal.

Conjecture (3.15) is motivated by results of J. Lee and R. Melrose and of R. Graham in the case of smooth strictly pseudoconvex domains, and by J. Kamimoto in the case of the Bergman metric in tube domains (see [28, 34, 45]).

### 3.2.5 Holomorphic bisectonal curvatures when $X \rightarrow 0$

We estimate the holomorphic bisectonal curvatures of the Kähler-Einstein metric  $g$  that we denote by  $Bis(z; v, w)$  (or by  $Bis(v, w)$  if we omit the point at which we compute it) for the rest of this section. The following Proposition simplifies the expression of the holomorphic bisectonal curvatures in tube domains:

**Proposition 3.24.** Let  $v = (v_1, v_2) \in \mathbb{C}^2$ ,  $w = (w_1, w_2) \in \mathbb{C}^2$  such that  $|v|_g = |w|_g = 1$ . Let  $\alpha \in \mathbb{R}$ , respectively  $\beta \in \mathbb{R}$  satisfying  $v_1 \bar{v}_2 = |v_1 v_2| e^{i\alpha}$ , respectively  $w_1 \bar{w}_2 = |w_1 w_2| e^{i\beta}$ .



Then the following holds on  $T_p$ :

$$\begin{aligned}
Bis(v, w) &= R_{1\bar{1}1\bar{1}}|v_1|^2|w_1|^2 \\
&+ 2R_{1\bar{1}1\bar{2}}|v_1||w_1|(|v_1||w_2|\cos(\beta) + |v_2||w_1|\cos(\alpha)) \\
&+ R_{1\bar{1}2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2 + 2|v_1||v_2||w_1||w_2|\cos(\alpha - \beta)) \\
&+ 2R_{1\bar{2}1\bar{2}}|v_1||v_2||w_1||w_2|\cos(\alpha + \beta) \\
&+ 2R_{1\bar{2}2\bar{2}}|v_2||w_2|(|v_1||w_2|\cos(\alpha) + |v_2||w_1|\cos(\beta)) \\
&+ R_{2\bar{2}2\bar{2}}|v_2|^2|w_2|^2.
\end{aligned} \tag{3.16}$$

*Proof of Proposition 3.24.* From the expression of the curvature coefficients (1.7) and the fact that  $g$  and all its complex derivatives are real numbers we derive that for  $1 \leq i, j, k, l \leq 2$ , one has  $R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{j\bar{i}l\bar{k}}$ . Hence we can simplify formula (1.9) by gathering the terms depending on the number of 2 occurring in the 4-uple  $(i, j, k, l)$ :

$$\begin{aligned}
Bis(v, w) &= R_{1\bar{1}1\bar{1}}|v_1|^2|w_1|^2 \\
&+ R_{1\bar{1}1\bar{2}}(|v_1|^2(w_1\bar{w}_2 + \bar{w}_1w_2) + (v_1\bar{v}_2 + \bar{v}_1v_2)|w_1|^2) \\
&+ R_{1\bar{1}2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2 + v_1\bar{v}_2\bar{w}_1w_2 + \bar{v}_1v_2w_1\bar{w}_2) \\
&+ R_{1\bar{2}1\bar{2}}(v_1\bar{v}_2w_1\bar{w}_2 + \bar{v}_1v_2\bar{w}_1w_2) \\
&+ R_{1\bar{2}2\bar{2}}((v_1\bar{v}_2 + \bar{v}_1v_2)|w_2|^2 + |v_2|^2(w_1\bar{w}_2 + \bar{w}_1w_2)) \\
&+ R_{2\bar{2}2\bar{2}}|v_2|^2|w_2|^2, \\
&= R_{1\bar{1}1\bar{1}}|v_1|^2|w_1|^2 \\
&+ 2R_{1\bar{1}1\bar{2}}|v_1||w_1|(|v_1||w_2|\cos(\beta) + |v_2||w_1|\cos(\alpha)) \\
&+ R_{1\bar{1}2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2 + 2|v_1||v_2||w_1||w_2|\cos(\alpha - \beta)) \\
&+ 2R_{1\bar{2}1\bar{2}}|v_1||v_2||w_1||w_2|\cos(\alpha + \beta) \\
&+ 2R_{1\bar{2}2\bar{2}}|v_2||w_2|(|v_1||w_2|\cos(\alpha) + |v_2||w_1|\cos(\beta)) \\
&+ R_{2\bar{2}2\bar{2}}|v_2|^2|w_2|^2.
\end{aligned}$$

□

First we compute the curvature coefficients at the origin in the following:

**Proposition 3.25.** *The curvature coefficients satisfy the following at the origin:*

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= -32p^3K, \\ R_{1\bar{2}1\bar{2}} &= (p-1)f^{(1)}(0), \\ R_{1\bar{1}2\bar{2}} &= -pf^{(1)}(0), \\ R_{2\bar{2}2\bar{2}} &= \left(-3 + \frac{1}{K}\right) \frac{f^{(1)}(0)^2}{16}, \end{aligned}$$

all the other coefficients being equal to 0.

*Proof of Proposition 3.25.* Recall that  $F$  is an even function, hence  $f = F^{(1)} = g_2(0, \cdot)$  is an odd function. From this we directly deduce that the following quantities vanish at the origin:

$$g_{12}, \quad g_{112}, \quad g_{222}, \quad g_{1112}, \quad g_{1222},$$

and since  $g$  depends only on the real parts of its arguments, the same quantities with conjugate on some of the indices also vanish at the origin. From the relation (1.7) we deduce that the curvature coefficients satisfy the following at  $z = 0$ :

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= -g_{1\bar{1}1\bar{1}} + g_{11\bar{1}\bar{1}}g^{1\bar{1}}g_{1\bar{1}\bar{1}}, \\ R_{1\bar{2}1\bar{2}} &= -g_{1\bar{2}1\bar{2}} + g_{11\bar{1}\bar{2}}g^{1\bar{1}}g_{1\bar{2}\bar{2}}, \\ R_{1\bar{1}2\bar{2}} &= -g_{1\bar{1}2\bar{2}} + g_{12\bar{2}\bar{2}}g^{2\bar{2}}g_{2\bar{1}\bar{2}}, \\ R_{2\bar{2}2\bar{2}} &= -g_{2\bar{2}2\bar{2}} + g_{22\bar{1}\bar{2}}g^{1\bar{1}}g_{1\bar{2}\bar{2}}, \\ R_{1\bar{1}1\bar{2}} &= R_{1\bar{2}2\bar{2}} = 0. \end{aligned}$$

We use formula (3.10) to compute the derivatives of  $g$  at the origin. We have, at  $z = 0$ :

$$[g_{i\bar{j}}] = \begin{bmatrix} 4pK & 0 \\ 0 & \frac{f^{(1)}(0)}{4} \end{bmatrix}, \quad [g^{i\bar{j}}] = \begin{bmatrix} \frac{1}{4pK} & 0 \\ 0 & \frac{4}{f^{(1)}(0)} \end{bmatrix},$$

$$g_{1\bar{1}1\bar{1}} = 16p^2K, \quad g_{1\bar{2}2\bar{2}} = \frac{f^{(1)}(0)}{2}, \quad g_{1\bar{1}1\bar{1}} = 96p^3K, \quad g_{1\bar{1}2\bar{2}} = (p+1)f^{(1)}(0), \quad g_{2\bar{2}2\bar{2}} = \frac{f^{(3)}(0)}{16}.$$

Thus we obtain:

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= -32p^3K, \\ R_{1\bar{1}2\bar{1}2} &= (p-1)f^{(1)}(0), \\ R_{1\bar{1}2\bar{2}2} &= -pf^{(1)}(0), \\ R_{2\bar{2}2\bar{2}2} &= -\frac{f^{(3)}(0)}{16} + \frac{f^{(1)}(0)^2}{16pK}. \end{aligned}$$

According to Equation (3.12), we have  $Z^{(2)}(0) = 3f^{(1)}(0)Z(0)$ , that is  $4pKf^{(3)}(0) + 4(p-1)f^{(1)}(0)^2 = 12pKf^{(1)}(0)^2$ , hence  $R_{2\bar{2}2\bar{2}2} = \left(-3 + \frac{1}{K}\right) \frac{f^{(1)}(0)^2}{16}$ .  $\square$

From the computations of Proposition 3.25 we deduce the precise upper and lower bounds for the holomorphic bisectional curvatures and holomorphic sectional curvatures at the origin. Before proving them, we need the following lemma:

**Lemma 3.26.** *Let  $-3 < A < -\frac{3}{2}$  and  $B \geq 0$  and define*

$$\begin{aligned} C : [-1, 1]^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \begin{aligned} &A(x^2y^2 + (1-x^2)(1-y^2)) \\ &-(3+A)(x^2(1-y^2) + y^2(1-x^2)) \\ &+ 2(3+A+B)x\sqrt{1-x^2}y\sqrt{1-y^2}. \end{aligned} \end{aligned}$$

Then:

$$\begin{aligned} \max_{0 \leq x, y \leq 1} C(x, y) &= \max \left\{ -(3+A), \frac{A+B}{2} \right\}, \\ \min_{0 \leq x, y \leq 1} C(x, -y) &= \min_{0 \leq x \leq 1} C(x, -x) = \min \left\{ A, -3 - \frac{A+B}{2} \right\}, \end{aligned}$$

*Proof of Lemma 3.26.* Observe that  $C \in \mathcal{C}([-1, 1]^2) \cap \mathcal{C}^\infty(]-1, 1[^2)$ . Trivial computations give  $\min_{\partial([-1, 1]^2)} C = C(0, 0) = A$  and  $\max_{\partial([-1, 1]^2)} C = C(1, 0) = -(3+A)$ . We study the critical values of  $C$  on  $] -1, 1[^2$ . Let  $(x, y) \in ] -1, 1[^2$  be such that  $d_{(x, y)}C = 0$ . This is equivalent to  $\frac{\partial C}{\partial x}(x, y) = \frac{\partial C}{\partial y}(x, y) = 0$ , that is:

$$\begin{cases} (3+2A)x(2y^2-1) + (3+A+B)y\sqrt{1-y^2}\frac{1-2x^2}{\sqrt{1-x^2}} = 0 \\ (3+2A)y(2x^2-1) + (3+A+B)x\sqrt{1-x^2}\frac{1-2y^2}{\sqrt{1-y^2}} = 0. \end{cases}$$

setting  $\lambda := \frac{3+2A}{3+A+B} < 0$ , this amounts to

$$\begin{cases} \lambda x \sqrt{1-x^2}(2y^2-1) = -y \sqrt{1-y^2}(1-2x^2) \\ \frac{1}{\lambda} x \sqrt{1-x^2}(2y^2-1) = -y \sqrt{1-y^2}(1-2x^2). \end{cases} \quad (3.17)$$

In particular since  $-1 < x, y < 1$  this implies that either  $\lambda^2 = 1$  or  $x(2y^2 - 1) = 0$ . We first deal with the case  $\lambda^2 \neq 1$ .

If  $x = 0$  then from (3.17) we deduce that  $y = 0$ , and  $C(0, 0) = \min_{\partial([-1, 1]^2)} C$ . If  $2y^2 = 1$ , then from (3.17) we deduce that  $2x^2 = 1$ . Computations yields:

$$C\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = C\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{A+B}{2}, \quad C\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = C\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = -3 - \frac{A+B}{2},$$

so that we obtain the conclusion for points 1. and 2. by comparing with the values of  $C$  on  $\partial([-1, 1]^2)$ .

Now assume that  $\lambda^2 = 1$ , meaning  $3+2A = -(3+A+B)$ . Then  $C$  takes the following simpler expression

$$C(x, y) = A \left( xy - \sqrt{1-x^2}\sqrt{1-y^2} \right)^2 - (3+A) \left( x\sqrt{1-y^2} + y\sqrt{1-x^2} \right)^2.$$

For  $-1 \leq x, y \leq 1$ , let  $-\pi < \theta, \varphi \leq \pi$  such that  $x = \cos(\theta)$  and  $y = \cos(\varphi)$ . Then we see that

$$C(x, y) = A \cos(\theta - \varphi)^2 - (3+A) \sin(\theta - \varphi)^2 = A - (3+2A) \sin(\theta - \varphi)^2,$$

thus we obtain the same conclusion as in the case  $\lambda^2 \neq 1$ . This concludes the proof.  $\square$

**Proposition 3.27.** *Let  $v, w \in \mathbb{C}^2 \setminus \{0\}$  be two vectors. Then we have:*

$$-3 + \frac{3}{2p+1} \leq \text{Bis}(0; v, w) \leq -\frac{3}{2p+1} \quad \text{and} \quad H(0; v) \leq -\frac{3}{2} - \frac{1}{2pK}.$$

Moreover,

$$\text{Bis}(0; (1, 0), (1, 0)) = -3 + \frac{3}{2p+1}, \quad \text{Bis}(0; (1, 0), (0, 1)) = -\frac{3}{2p+1},$$

$$H\left(0; \left(\frac{1}{\sqrt{4pK}}, \frac{\sqrt{f^{(1)}(0)}}{2}\right)\right) = -\frac{3}{2} - \frac{1}{2pK}.$$

*Proof of Proposition 3.27.* In this proof, all the computations are implicitly done at  $z = 0$ . In order to apply Lemma 3.26, we set  $A := -3 + \frac{3}{2p+1}$  and  $B := \frac{p-1}{pK}$ . Using

Proposition 3.25 we have:

$$\begin{aligned}\frac{R_{1\bar{1}1\bar{1}}}{g_{1\bar{1}}^2} &= -\frac{2p}{K} = A, \\ \frac{R_{2\bar{2}2\bar{2}}}{g_{2\bar{2}}^2} &= -\frac{2p}{K} = A, \\ \frac{R_{1\bar{1}2\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} &= -(3+A), \\ \frac{R_{1\bar{2}1\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} &= \frac{p-1}{pK} = B.\end{aligned}$$

Let  $v, w \in \mathbb{C}^2$  be two vectors satisfying  $|v|_g = |w|_g = 1$ , and let  $\alpha, \beta \in \mathbb{R}$  be such that  $v_1\bar{v}_2 = |v_1||v_2|e^{i\alpha}$  and  $w_1\bar{w}_2 = |w_1||w_2|e^{i\beta}$ . Then we have:

$$\begin{aligned}Bis(v, w) &= R_{1\bar{1}1\bar{1}}|v_1|^2|w_1|^2 + R_{1\bar{1}2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2 + 2|v_1||v_2||w_1||w_2|\cos(\alpha - \beta)) \\ &\quad + 2|v_1||v_2||w_1||w_2|R_{1\bar{2}1\bar{2}}\cos(\alpha + \beta) + R_{2\bar{2}2\bar{2}}|v_2|^2|w_2|^2 \\ &= A(g_{1\bar{1}}^2|v_1|^2|w_1|^2 + g_{2\bar{2}}^2|v_2|^2|w_2|^2) - (3+A)g_{1\bar{1}}g_{2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2) \\ &\quad + 2g_{1\bar{1}}g_{2\bar{2}}|v_1||v_2||w_1||w_2|(B\cos(\alpha + \beta) - (3+A)\cos(\alpha - \beta)).\end{aligned}$$

Noting that the particular case  $\alpha = \beta = \frac{\pi}{2}$  respectively  $\alpha = \frac{\pi}{2} = -\beta$  gives the minimum, respectively the maximum, of  $Bis(v, w)$  with respect to  $\alpha, \beta$  we deduce

$$\begin{aligned}&A(g_{1\bar{1}}^2|v_1|^2|w_1|^2 + g_{2\bar{2}}^2|v_2|^2|w_2|^2) - (3+A)g_{1\bar{1}}g_{2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2) \\ &\quad - 2(B + (3+A))g_{1\bar{1}}g_{2\bar{2}}|v_1||v_2||w_1||w_2| \\ &\leq Bis(v, w) \\ &\leq A(g_{1\bar{1}}^2|v_1|^2|w_1|^2 + g_{2\bar{2}}^2|v_2|^2|w_2|^2) - (3+A)g_{1\bar{1}}g_{2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2) \\ &\quad + 2(B + (3+A))g_{1\bar{1}}g_{2\bar{2}}|v_1||v_2||w_1||w_2|.\end{aligned}$$

We set  $x := \sqrt{g_{1\bar{1}}}|v_1|$ ,  $y := \sqrt{g_{1\bar{1}}}|w_1|$  so that the above inequalities rephrase into

$$C(x, -y) \leq Bis(v, w) \leq C(x, y),$$

where  $C$  is the function defined in Lemma 3.26. We apply Lemma 3.26 and obtain the extremas for the bisectonal curvatures at the origin and also the minimum for the holomorphic sectional curvatures. For the maximum of the holomorphic sectional curvatures, we set  $x^2 := t$  to obtain

$$\begin{aligned}H(v) = Bis(v, v) &\leq A(t^2 + (1-t)^2) - 2(3+A)t(1-t) + 2(B - (3+A))t(1-t) \\ &= 2(B - 3(A+2))t(1-t) + A.\end{aligned}$$

Since  $B - 3(A + 2) \geq -3(A + 2) > 0$ , the maximum of the polynomial function  $P : t \mapsto 2(B - 3(A + 2))t(1 - t) + A$  on  $[0, 1]$  is achieved at  $t = \frac{1}{2}$ . Hence we deduce

$$\max_{|v|_g=1} H(v) = \frac{-3 - \frac{1}{K} + \frac{p-1}{pK}}{2} = -\frac{3}{2} - \frac{1}{2pK}.$$

□

We can deduce from Proposition 3.27 part of Theorem 3.2:

**Theorem 3.28.** *There exist positive constants  $0 < c \leq C$  and  $\alpha > 0$  such that*

$$\forall v, w \in \mathbb{C}^2 \setminus \{0\}, \quad \forall z \in \{|X| \leq \alpha\}, \quad -C \leq \text{Bis}(z; v, w) \leq -c.$$

*Proof of Theorem 3.28.* Since the map

$$\begin{aligned} ]-1, 1[ \times S(0, 1)^2 &\longrightarrow \mathbb{R} \\ (x, v, w) &\longmapsto \text{Bis}((0, x); v, w) \end{aligned}$$

is continuous, it is uniformly continuous on every subset of the form  $J \times S(0, 1)^2$  where  $J \subset ]-1, 1[$  is a compact set. Especially, we deduce that for every positive number  $\epsilon > 0$  there exists a positive constant  $\alpha > 0$  such we have the following:

$$\forall (x, v, w) \in [-\alpha, \alpha] \times S(0, 1)^2, \quad |\text{Bis}((0, x); v, w) - \text{Bis}(0; v, w)| \leq \epsilon,$$

and consequently, according to (1.10)

$$\forall x \in [-\alpha, \alpha], \quad \sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |\text{Bis}((0, x); v, w) - \text{Bis}(0; v, w)| \leq \epsilon. \quad (3.18)$$

Take  $\epsilon := \frac{3}{2(2p+1)}$  and let  $\alpha > 0$  be such that (3.18) holds and let  $z \in \{|X| \leq \alpha\}$ . Let  $v, w \in \mathbb{C}^n \setminus \{0\}$  and set  $v^{(z)} := \partial_z \psi^{(z)}(v)$ ,  $w^{(z)} := \partial_z \psi^{(z)}(w)$ . Using the transformation formula (1.13) and Proposition 3.27 we have:

$$\begin{aligned} \text{Bis}(z; v, w) &= \text{Bis}((0, X(z)); v^{(z)}, w^{(z)}) \\ &= \text{Bis}((0, X(z)); v^{(z)}, w^{(z)}) - \text{Bis}(0; v^{(z)}, w^{(z)}) + \text{Bis}(0; v^{(z)}, w^{(z)}) \\ &\leq \sup_{v, w \in \mathbb{C}^n \setminus \{0\}} |\text{Bis}((0, X(z)); v, w) - \text{Bis}(0; v, w)| - \frac{3}{2p+1} \\ &\leq -\frac{3}{2(2p+1)}, \end{aligned}$$

and likewise we prove that  $-3 + \frac{1}{2p+1} \leq \text{Bis}(z; v, w)$ . We obtain the conclusion of Theorem 3.28 by taking  $c = \frac{3}{2(2p+1)}$  and  $C = 3 + \frac{1}{2p+1}$ . □

**Remark 3.29.** *A careful examination of the proof of Theorem 3.28 shows that*

$$\forall v, w \in \mathbb{C}^n \setminus \{0\}, \quad -3 + \frac{1}{K} \leq \liminf_{X(z) \rightarrow 0} \text{Bis}(z; v, w) \leq \limsup_{X(z) \rightarrow 0} \text{Bis}(z; v, w) \leq -\frac{1}{K}.$$

### 3.2.6 Holomorphic bisectonal curvatures when $|X| \rightarrow 1$

In this subsection, we use the asymptotic behaviour of  $F$  obtained in Proposition 3.22 up to order 4 to prove the other part of Theorem 3.2. It will follow from the computation of  $\lim_{x \rightarrow 1^-} \text{Bis}((0, x); v, w)$ :

**Theorem 3.30.** *There exist positive constants  $0 < c \leq C$  and  $\alpha > 0$  such that*

$$\forall v, w \in \mathbb{C}^2 \setminus \{0\}, \quad \forall z \in \{|1 - |X|| \leq \alpha\}, \quad -C \leq \text{Bis}(z; v, w) \leq -c.$$

*Proof of Theorem 3.30.* Because of the invariance of  $T_p$  under the symmetry  $s$  introduced in Proposition 3.12, it is enough to prove that there exist positive constants  $0 < c \leq C$  and  $0 < \alpha < 1$  such that:

$$\forall v, w \in \mathbb{C}^2 \setminus \{0\}, \quad \forall z \in \{1 - \alpha \leq X < 1\}, \quad -C \leq \text{Bis}(z; v, w) \leq -c.$$

First we prove the following:

$$\lim_{x \rightarrow 1^-} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \left( \text{Bis}((0, x); v, w) + 1 + \frac{|\langle v, w \rangle_{(0, x)}|^2}{|v|_{(0, x)}^2 |w|_{(0, x)}^2} \right) = 0. \quad (3.19)$$

The conclusion then follows from the invariance property 1.13. In the sequel, functions on  $T_p$  are computed at the point  $(0, x)$ , and functions on  $] - 1, 1[$  are computed at the point  $x$  with  $0 < x < 1$ .

First we prove that  $\frac{R_{i\bar{j}k\bar{l}}}{f^4} \underset{x \rightarrow 1^-}{\sim} -2X_i X_{\bar{j}} X_k X_{\bar{l}}$  and  $\frac{g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}}{f^4} \underset{x \rightarrow 1^-}{\sim} -2X_i X_{\bar{j}} X_k X_{\bar{l}}$ .

Let  $1 \leq i, j, k, l, \alpha, \beta \leq 2$ . We differentiate relation (3.10) to obtain:

$$\begin{aligned}
g_{i\bar{j}} &= f^{(1)} X_i X_{\bar{j}} + f X_{i\bar{j}} + \frac{K}{p} \text{Log} \left( \frac{1}{1 - \text{Re}(4pz_1)} \right)_{i\bar{j}}, \\
Zg^{\alpha\bar{\beta}} &= (-1)^{\alpha+\beta} \left( f^{(1)} X_{\bar{3-\alpha}} X_{3-\beta} + f X_{\bar{3-\alpha}3-\beta} + \frac{K}{p} \text{Log} \left( \frac{1}{1 - \text{Re}(4pz_1)} \right)_{\bar{3-\alpha}3-\beta} \right), \\
g_{i\bar{j}k} &= f^{(2)} X_i X_{\bar{j}} X_k + f^{(1)} (X_{i\bar{j}} X_k + X_{ik} X_{\bar{j}} + X_{k\bar{j}} X_i) \\
&\quad + f X_{i\bar{j}k} + \frac{K}{p} \text{Log} \left( \frac{1}{1 - \text{Re}(4pz_1)} \right)_{i\bar{j}k}, \\
g_{i\bar{j}k\bar{l}} &= f^{(3)} X_i X_{\bar{j}} X_k X_{\bar{l}} \\
&\quad + f^{(2)} (X_{i\bar{j}} X_k X_{\bar{l}} + X_{ik} X_{\bar{j}} X_{\bar{l}} + X_{i\bar{l}} X_{\bar{j}} X_k + X_{k\bar{j}} X_i X_{\bar{l}} + X_{k\bar{l}} X_i X_{\bar{j}} + X_{\bar{j}\bar{l}} X_i X_k) \\
&\quad + f^{(1)} (X_{i\bar{j}k} X_{\bar{l}} + X_{i\bar{j}\bar{l}} X_k + X_{ik\bar{l}} X_{\bar{j}} + X_{\bar{j}k\bar{l}} X_i + X_{i\bar{j}} X_{k\bar{l}} + X_{ik} X_{\bar{j}\bar{l}} + X_{i\bar{l}} X_{k\bar{j}}) \\
&\quad + f X_{i\bar{j}k\bar{l}} + \frac{K}{p} \text{Log} \left( \frac{1}{1 - \text{Re}(4pz_1)} \right)_{i\bar{j}k\bar{l}} \\
&\quad \underset{x \rightarrow 1^-}{\sim} f^{(3)} X_i X_{\bar{j}} X_k X_{\bar{l}}, \\
&\quad \underset{x \rightarrow 1^-}{\sim} 6f^4 X_i X_{\bar{j}} X_k X_{\bar{l}}.
\end{aligned}$$

In the expression  $g_{ik\bar{\alpha}} g^{\alpha\bar{\beta}} g_{\beta\bar{j}\bar{l}}$ , the contribution from a term of the form  $(-1)^\alpha X_{\bar{\alpha}} X_{3-\alpha}$  or  $(-1)^\beta X_\beta X_{3-\beta}$  is 0. Thus, the leading term in  $\sum_{1 \leq \alpha, \beta \leq 2} g_{ik\bar{\alpha}} g^{\alpha\bar{\beta}} g_{\beta\bar{j}\bar{l}}$  as  $x$  tends to  $1^-$  is:

$$\begin{aligned}
\frac{(f^{(2)})^2 f}{Z} X_i X_{\bar{j}} X_k X_{\bar{l}} \sum_{1 \leq \alpha, \beta \leq 2} (-1)^{\alpha+\beta} X_{\bar{3-\alpha}3-\beta} X_{\bar{\alpha}} X_\beta &= \frac{(f^{(2)})^2 f}{Z} X_i X_{\bar{j}} X_k X_{\bar{l}} \frac{2p-1}{4}, \\
&\underset{x \rightarrow 1^-}{\sim} 4f^4 X_i X_{\bar{j}} X_k X_{\bar{l}}.
\end{aligned}$$

Therefore we deduce that  $R_{i\bar{j}k\bar{l}} \underset{x \rightarrow 1^-}{\sim} -2f^4 X_i X_{\bar{j}} X_k X_{\bar{l}}$ . Moreover we have  $g_{i\bar{j}} \underset{x \rightarrow 1^-}{\sim} f^2 X_i X_{\bar{j}}$ , which leads to  $g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} \underset{x \rightarrow 1^-}{\sim} 2f^4 X_i X_{\bar{j}} X_k X_{\bar{l}}$ . Hence:

$$\begin{aligned}
&\sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \left( \text{Bis}((0, x); v, w) + 1 + \frac{|\langle v, w \rangle_{(0, x)}|^2}{|v|_{(0, x)}^2 |w|_{(0, x)}^2} \right) \\
&= \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \frac{\sum_{1 \leq i, j, k, l \leq 2} \left( \frac{R_{i\bar{j}k\bar{l}}}{f(x)^4} + \frac{g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}}{f(x)^4} \right) v_i \bar{v}_j w_k \bar{w}_l}{\left( \sum_{1 \leq i, j \leq 2} \frac{g_{i\bar{j}}(0, x)}{f(x)^2} v_i \bar{v}_j \right) \left( \sum_{1 \leq i, j \leq 2} \frac{g_{i\bar{j}}(0, x)}{f(x)^2} w_i \bar{w}_j \right)}, \\
&\xrightarrow{x \rightarrow 1^-} 0,
\end{aligned}$$



hence formula (3.19). Since for every  $v, w \in \mathbb{C}^2 \setminus \{0\}$  and every point  $z \in T_p$  we have

$$-2 \leq -1 - \frac{|\langle v, w \rangle_z|^2}{|v|_z^2 |w|_z^2} \leq -1,$$

we may conclude as in the end of the proof of Theorem 3.28.  $\square$

### 3.3 Behaviour of the Kähler-Einstein metric in some convex domains

In this section we use the estimates of the holomorphic bisectional curvatures of the Kähler-Einstein metric in tube domains and in Thullen domains to prove Theorem 3.3. In the proof of Theorem 3.3 we use an other invariant metric, namely the Kobayashi metric. We briefly recall some results about the Kobayashi metric which are needed in our proof of Theorem 3.3.

For a domain  $D \subset \mathbb{C}^n$ , a point  $z \in D$  and a vector  $v \in \mathbb{C}^n \setminus \{0\}$  the Kobayashi pseudo-metric at point  $z$  and vector  $v$  is defined by :

$$Kob^D(z, v) := \inf \{ |\xi|, \exists f \in \mathcal{H}(\Delta, D) \text{ satisfying } f(0) = z \text{ and } \partial_0 f(\xi) = v \}.$$

If  $D$  is a convex domain not containing a complex line, then  $Kob^D(z, v) > 0$  for every  $(z, v) \in D \times (\mathbb{C}^n \setminus \{0\})$ . We refer to [64] for more precise estimates of the Kobayashi metric.

We will use the following result in the proof of Theorem 3.3 (see [26]) let  $(D_\nu)_{\nu \in \mathbb{N}}$  be a sequence of bounded convex sets with boundary of class  $\mathcal{C}^\infty$ . Assume that  $(D_\nu)_{\nu \in \mathbb{N}}$  converges in the local Hausdorff topology to a  $\mathbb{C}$ -proper convex domain  $D_\infty$ , and let  $K \subset D_\infty$  be a compact set. Then there exists an integer  $\nu_K \in \mathbb{N}$  such that for every integer  $\nu \geq \nu_K$  one has  $K \subset D_\nu$ , and the sequence  $(Kob^{D_\nu})_{\nu \geq \nu_K}$  converges uniformly to  $Kob^{D_\infty}$  on  $K \times S(0, 1)$ . In that case we say that  $(Kob^{D_\nu})_{\nu \in \mathbb{N}}$  converges uniformly on compact sets of  $D_\infty$  to  $Kob^{D_\infty}$ .

In the following, for a domain  $D \subset \mathbb{C}^2$ , we use the notation  $g^D$  to denote the Kähler-Einstein potential of  $D$  solution of Equation (2.2) with boundary condition (2.4), and  $Bis^D$  to denote its holomorphic bisectional curvatures.

We have all the tools to prove Theorem 3.3:

*Proof of Theorem 3.3.* We use a rescaling method to change the study of the boundary behaviour of the holomorphic bisectional curvatures into the study of the interior convergence for the sequence of the rescaled Kähler-Einstein metrics.

Our hypothesis on the local expression of  $\partial D$  at  $q$  and the convexity of  $D$  imply that there exist an affine map  $\psi \in \text{Aut}(\mathbb{C}^2)$  and a neighbourhood  $U$  of  $q$  such that  $\psi(q) = 0$  and

$$\psi(D \cap U) = \{ \text{Re}(z_1) + H(z_2) + O(|z_2|^{2p+1} + |z_1||z|) < 0 \} \cap \psi(U),$$

with either  $H(z) = |z|^{2p}$  or  $H(z) = \text{Re}(z)^{2p}$  for every  $z \in \mathbb{C}$  (see also point 2. of Remark 1.14). Since  $\psi$  maps  $D$  to  $\psi(D)$  biholomorphically, we have the following by the invariance property of the Kähler-Einstein metric:

$$\forall z \in D, \quad \forall v, w \in \mathbb{C}^2 \setminus \{0\}, \quad \text{Bis}^D(z; v, w) = \text{Bis}^{\psi(D)}(\psi(z); \partial_z \psi(v), \partial_z \psi(w)).$$

Moreover the sequence  $(\psi(z^{(\nu)}))_{\nu \in \mathbb{N}}$  converges non tangentially to  $\psi(q) = 0$  because  $\psi$  is an affine invertible map. Thus up to replacing  $D$  with  $\psi(D)$  and  $U$  with  $\psi(U)$  we may assume that  $q = 0$  and  $D \cap U = \{ \text{Re}(z_1) + H(z_2) + O(|z_2|^{2p+1} + |z_1||z|) < 0 \} \cap U$ .

In this setting the condition of non-tangential convergence of  $(z^{(\nu)})_{\nu \in \mathbb{N}} \in D^{\mathbb{N}}$  means that

$\left( \frac{-\text{Re}(z_1^{(\nu)})}{|z^{(\nu)}|} \right)_{\nu \in \mathbb{N}}$  is bounded from below by a positive constant, thus up to taking a

subsequence we may assume that  $\left( \frac{z^{(\nu)}}{|z^{(\nu)}|} \right)_{\nu \in \mathbb{N}}$  converges to a point  $z^{(\infty)}$  with  $\text{Re}(z_1^{(\infty)}) < 0$ . Let

$$\begin{aligned} \Lambda^{(\nu)} : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto \left( \frac{z_1 - z_1^{(\nu)}}{\text{Re}(-z_1^{(\infty)})|z^{(\nu)}|}, \frac{z_2 - z_2^{(\nu)}}{\left(\text{Re}(-z_1^{(\infty)})|z^{(\nu)}|\right)^{\frac{1}{2p}}} \right), \end{aligned}$$

and set  $D_\nu := \Lambda^{(\nu)}(D)$ . From results in [25],  $(D_\nu)_{\nu \in \mathbb{N}}$  converges to the  $\mathbb{C}$ -proper convex domain  $D_\infty := \{ \text{Re}(z_1) + H(z_2) < 1 \}$  in the local Hausdorff topology. From results in [26] we deduce that the sequence  $(\text{Kob}^{D_\nu})_{\nu \in \mathbb{N}}$  converges uniformly on compact sets of  $D_\infty$  to  $\text{Kob}^{D_\infty}$ .

Also, observe that  $\Lambda^{(\nu)}(z^{(\nu)}) = 0$ . Using this and the transformation formula (1.13), we obtain for every  $z \in D$  and  $v, w \in \mathbb{C}^2 \setminus \{0\}$ :

$$\text{Bis}^D(z^{(\nu)}; (\partial_{z^{(\nu)}} \Lambda^{(\nu)})^{-1}(v), (\partial_{z^{(\nu)}} \Lambda^{(\nu)})^{-1}(w)) = \text{Bis}^{D_\nu}(0; v, w). \quad (3.20)$$

Assume momentarily that for every compact  $K \subset D_\infty$  the sequence  $(g^{(\nu)})_{\nu \geq \nu_K}$  converges to the Kähler-Einstein potential  $g^{D_\infty}$  of  $D_\infty$  in  $\mathcal{C}^4(K)$ . Then according to formula (1.7) and relation (3.20) we deduce:

$$\sup_{v,w \in \mathbb{C}^2 \setminus \{0\}} \left| \text{Bis}^D \left( z^{(\nu)}; (\partial_{z^{(\nu)}} \Lambda^{(\nu)})^{-1}(v), (\partial_{z^{(\nu)}} \Lambda^{(\nu)})^{-1}(w) \right) - \text{Bis}^{D_\infty}(0; v, w) \right| \xrightarrow{\nu \rightarrow \infty} 0,$$

so that we obtain Theorem 3.3 using relation (1.10), Theorem 3.2 and results in [5]. Thus it remains to prove that for every compact  $K \subset D_\infty$  the sequence  $(g^{(\nu)})_{\nu \geq \nu_K}$  converges to the Kähler-Einstein potential  $g^{D_\infty}$  of  $D_\infty$  in  $\mathcal{C}^4(K)$ . In fact we prove that for every integer  $k \in \mathbb{N}$ , the sequence  $(g^{(\nu)})_{\nu \geq \nu_K}$  converges to  $g^{D_\infty}$  in  $\mathcal{C}^k(K)$ . Observe that by uniqueness of the Kähler-Einstein potential  $g^{E, D_\infty}$  and by the theorem of Arzelà-Ascoli it is enough to prove that for every integer  $k \in \mathbb{N}$  the sequence  $(g^{(\nu)})_{\nu \geq \nu_K}$  is bounded in  $\mathcal{C}^k(K)$  (see the discussion preceding Lemma 2.7 for details). Thus we are interested in obtaining  $\mathcal{C}^k(K)$  estimates of the family  $(g^{(\nu)})_{\nu \geq \nu_K}$  of solutions to equation (2.2). It relies on obtaining estimates of Sobolev norms of the sequences  $\left( \text{Log} \left| g_{i\bar{j}}^{D_\nu} \right| \right)_{\nu \in \mathbb{N}}$  and  $(\Delta g^{D_\nu})_{\nu \in \mathbb{N}}$  on bounded subdomains of  $D_\infty$ , where  $\Delta$  denotes the Laplacian operator for the Euclidean metric on  $\mathbb{C}^2$  (see the proof in [57, Lemma 3] for more details).

Since  $D$  is a bounded convex domain with boundary of class  $\mathcal{C}^\infty$ , it follows from [38] that there exists a number  $0 < a$  such that  $D$  satisfies the  $a$ -squeezing property. Since  $D_\nu$  is biholomorphic to  $D$ ,  $D_\nu$  also satisfies the  $a$ -squeezing property. From Proposition 3 in [57] we deduce that there exist constants  $0 < c \leq C$  such that for every integer  $\nu \in \mathbb{N}$  we have  $c \text{Kob}^{D_\nu} \leq \left[ g_{i\bar{j}}^{D_\nu} \right] \leq C \text{Kob}^{D_\nu}$  on  $D_\nu$  in the sense that for every  $z \in D_\nu$  and every  $v \in \mathbb{C}^2$  the inequalities  $c \text{Kob}^{D_\nu}(z, v) \leq |v|_z^{D_\nu} \leq C \text{Kob}^{D_\nu}(z, v)$  hold. Moreover since the sequence of bounded convex sets  $(D_\nu)_{\nu \in \mathbb{N}}$  converges to the convex set  $D_\infty$  in the local Hausdorff topology, the sequence  $(\text{Kob}^{D_\nu})_{\nu \in \mathbb{N}}$  converges uniformly on compact sets of  $D_\infty$  to  $\text{Kob}^{D_\infty}$ . Therefore we obtain the uniform estimates by following line by line the proof of Lemma 3 in [57] (by replacing the balls  $B_{\frac{a}{2}}(x), B_a(x)$  with bounded domains included in  $D_\infty$ ).  $\square$

**Remark 3.31.** *If  $D \subset \mathbb{C}^2$  is a smoothly bounded convex domain with boundary point of infinite type  $q$ , there exists a sequence of points in  $D$  converging to  $q$  non tangentially such that the limit domain is biholomorphic to the bidisc. In particular the holomorphic bisectional curvatures of the Kähler-Einstein metric are not uniformly bounded from above by a negative constant along that sequence. The same phenomenon holds for the Bergman metric.*

### 3.4 Partial results for the study of the Kähler-Einstein metric in postive model domains

Let  $p \in \mathbb{N}^*$  be an integer, and let  $H$  be a real valued homogenous polynomial of degree  $2p$ , subharmonic and not harmonic in  $\mathbb{C}$ . Let  $D_H := \{Re(4pz_1) + H(z_2) < 1\}$ . We assume that  $D_H$  admits a complete Kähler-Einstein metric induced by a potential  $g$  satisfying Equation (2.2) with boundary condition (2.4) on  $D_H$ . For simplicity we also assume that  $H$  is positive on  $\mathbb{C}^*$ . This is the case if  $D_H$  is convex and not biholomorphic to a tube domain (see Lemma 1.16). Nonetheless the forthcoming study can easily be adapted to any pseudoconvex domain of the form  $D_H$  without restriction on the sign of  $H$ . In this Section we study the behaviour of  $g$  on  $D_H$ . To do so we adapt the ideas developed in Section 3.2 and use some notations introduced therein. We also refer to Section 1.3 for information regarding the geometry of  $D_H$ .

Denote by  $\pi_2^{\mathbb{C}}$  the map

$$\begin{aligned} \pi_2^{\mathbb{C}} : \quad \mathbb{C}^2 &\longrightarrow \mathbb{R} \\ (z_1, z_2) &\longmapsto z_2. \end{aligned}$$

and set  $X := \frac{\pi_2^{\mathbb{C}}}{(1 - 4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}}}$ . This function is well defined on the set  $\{z \in \mathbb{C}^2 / Re(4pz_1) < 1\}$  which contains  $D_H$ . It satisfies the following properties:

- $X \in C^\infty(\{z \in \mathbb{C}^2 / Re(4pz_1) < 1\})$ ,
- $\forall F \in Aut(D_H), \quad \forall z \in D_H, \quad X(F(z)) = X(z)$ ,
- $D_H = (H \circ X)^{-1}([0, 1[)$ .

**Remark 3.32.** *In comparison to the cases of Thullen domains and tube domains, we do not know if  $(H \circ X)|_{\{0\} \times H^{-1}([0, 1[)}$  is injective (for instance take  $H(z) = Re(z)^p Im(z)^p$  with  $p \in 2\mathbb{N}^*$ ). Thus we cannot think of  $H \circ X$  as a parametrization of the orbits of  $D_H$  under the action of its automorphism group.*

#### 3.4.1 The Kähler-Einstein condition and two differential equations satisfied by $F$

Let

$$\begin{aligned} F : \quad H^{-1}([0, 1[) &\longrightarrow \mathbb{R} \\ z &\longmapsto g(0, z), \end{aligned}$$

and define  $K := \frac{2p+1}{3}$ . Then the following holds on  $D_H$ :

$$g = F \circ X + \frac{K}{p} \text{Log} \left( \frac{1}{1 - 4p\pi_1^{\mathbb{R}}} \right). \quad (3.21)$$

We use Equation (3.21) and the Monge-Ampère Equation (2.2) to obtain a first differential equation satisfied by the function  $F$ :

**Proposition 3.33.** *Denote  $f := \frac{\partial F}{\partial z}$ . Then the metric  $[g_{i\bar{j}}]$  satisfies the following on  $D_H$ :*

$$[g_{i\bar{j}}] = \begin{bmatrix} 2 \frac{|X|^2 \frac{\partial f}{\partial \bar{z}} \circ X + \text{Re}(X^2 \frac{\partial f}{\partial z} \circ X) + (2p+1) \text{Re}(Xf \circ X) + 2pK}{(1-4p\pi_1^{\mathbb{R}})^2} & \frac{X \frac{\partial f}{\partial \bar{z}} \circ X + X \overline{\frac{\partial f}{\partial z} \circ X} + f \circ X}{(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} \\ \frac{\overline{X} \frac{\partial f}{\partial \bar{z}} \circ X + X \frac{\partial f}{\partial z} \circ X + f \circ X}{(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} & \frac{\frac{\partial f}{\partial \bar{z}} \circ X}{(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{p}}} \end{bmatrix}, \quad (3.22)$$

$$\text{Det}(g_{i\bar{j}}) = \frac{Z(X)}{(1 - 4p\pi_1^{\mathbb{R}})^{\frac{3K}{p}}}, \quad (3.23)$$

where the function  $Z$  is defined by

$$Z(z) := \left( |z| \frac{\partial f}{\partial \bar{z}}(z) \right)^2 + 4p \frac{\partial f}{\partial \bar{z}}(z) (\text{Re}(zf(z)) + K) - \left| z \frac{\partial f}{\partial z}(z) + f(z) \right|^2$$

for every  $z \in H^{-1}([0, 1])$ . Moreover,  $Z$  satisfies the following:

$$Z = e^{3F} \text{ on } H^{-1}([0, 1]). \quad (3.24)$$

*Proof of Proposition 3.33.* On  $D_H$  we have:

$$[X_i] = \begin{bmatrix} \frac{X}{1-4p\pi_1^{\mathbb{R}}} \\ \frac{1}{(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}}} \end{bmatrix}, \quad [X_{\bar{j}}] = \begin{bmatrix} \frac{X}{1-4p\pi_1^{\mathbb{R}}} \\ 0 \end{bmatrix}, \quad [X_{i\bar{j}}] = \frac{(2p+1)X}{(1-4p\pi_1^{\mathbb{R}})^2} E_{11},$$

$$[X_{i\bar{j}}] = \begin{bmatrix} \frac{(2p+1)X}{(1-4p\pi_1^{\mathbb{R}})^2} & 0 \\ \frac{1}{(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}+1}} & 0 \end{bmatrix}, \quad [X_{ij}] = \begin{bmatrix} \frac{(2p+1)X}{(1-4p\pi_1^{\mathbb{R}})^2} & \frac{1}{(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}+1}} \\ \frac{1}{(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{2p}+1}} & 0 \end{bmatrix}.$$

Differentiating Equation (3.21), using the chain rule and the fact that  $F$  and  $\frac{\partial f}{\partial \bar{z}}$  are real

valued functions, we successively obtain:

$$[(F \circ X)_i] = f \circ X [X_i] + \bar{f} \circ X [\bar{X}_i],$$

$$[(f \circ X)_{\bar{j}}] = \frac{\partial f}{\partial \bar{z}} \circ X [\bar{X}_{\bar{j}}] + \frac{\partial f}{\partial z} \circ X [X_{\bar{j}}], \quad [(\bar{f} \circ X)_{\bar{j}}] = \frac{\partial \bar{f}}{\partial z} \circ X [\bar{X}_{\bar{j}}] + \frac{\partial \bar{f}}{\partial \bar{z}} \circ X [X_{\bar{j}}],$$

$$[g_{i\bar{j}}] = \frac{\partial f}{\partial \bar{z}} \circ X [X_i \bar{X}_{\bar{j}}] + \frac{\partial f}{\partial z} \circ X [X_i X_{\bar{j}}] + \frac{\partial \bar{f}}{\partial z} \circ X [\bar{X}_i \bar{X}_{\bar{j}}] + \frac{\partial \bar{f}}{\partial \bar{z}} \circ X [\bar{X}_i X_{\bar{j}}]$$

$$+ f \circ X [X_{i\bar{j}}] + \bar{f} \circ X [\bar{X}_{i\bar{j}}] + \frac{4Kp}{(1-4p\pi_1^{\mathbb{R}})^2} E_{11},$$

$$= \begin{bmatrix} 2 \frac{|X|^2 \frac{\partial f}{\partial \bar{z}} \circ X + \operatorname{Re}(X^2 \frac{\partial f}{\partial z} \circ X) + (2p+1) \operatorname{Re}(X f \circ X) + 2pK}{(1-4p\pi_1^{\mathbb{R}})^2} & \frac{X \frac{\partial f}{\partial \bar{z}} \circ X + X \frac{\partial \bar{f}}{\partial z} \circ X + f \circ X}{(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} \\ \frac{\bar{X} \frac{\partial f}{\partial \bar{z}} \circ X + X \frac{\partial f}{\partial z} \circ X + f \circ X}{(1-4p\pi_1^{\mathbb{R}})^{1+\frac{1}{2p}}} & \frac{\frac{\partial f}{\partial \bar{z}} \circ X}{(1-4p\pi_1^{\mathbb{R}})^{\frac{1}{p}}} \end{bmatrix}.$$

Then we apply the function  $\operatorname{Det}$  to Equation (3.22) and directly obtain Equation (3.23). Finally, recall that according to Equations (2.2) and (3.21) one has on  $D_H$ :

$$\begin{aligned} \operatorname{Det}(g_{i\bar{j}}) &= e^{3g} \\ &= e^{3F \circ X - \frac{3K}{p} \operatorname{Log}(1-4p\pi_1^{\mathbb{R}})} \\ &= \frac{e^{3F \circ X}}{(1-4p\pi_1^{\mathbb{R}})^{2+\frac{1}{p}}}, \end{aligned}$$

hence Equation (3.24). □

We do not have a generalisation of Proposition 3.17, nor an asymptotic expansion of  $F$  as in Thullen domains and tube domains.

### 3.4.2 Curvatures estimates at the origin

We prove Theorem 3.4, which generalises the results obtained in the case of Thullen domains and tube domains and reduces the study of the sign of these curvatures to the study of the quantity  $\frac{|g_{2\bar{2}}|}{g_{2\bar{2}}}(0)$ .

*Proof of Theorem 3.4.* In this proof, all the functions are implicitly computed at  $z = 0$  unless stated. First we simplify the expression of the curvature coefficients. We use the invariance of the Kähler-Einstein metric under the symmetry  $s$  to obtain  $g_{1\bar{2}} = g_{1\bar{1}\bar{2}} = g_{2\bar{2}} = g_{1\bar{1}\bar{2}} = g_{1\bar{2}\bar{2}} = 0$ . Then we use relation (1.7) to deduce that  $R_{1\bar{1}\bar{1}\bar{1}} = R_{1\bar{2}\bar{2}\bar{2}} = 0$ .

Putting the Kähler-Einstein condition  $[Ric(g)_{i\bar{j}}] = -3 [g_{i\bar{j}}] = -3 \begin{bmatrix} g_{1\bar{1}} & 0 \\ 0 & g_{2\bar{2}} \end{bmatrix}$  into relation (1.11) we also obtain:

$$-3 = \frac{R_{1\bar{1}\bar{1}\bar{1}}}{g_{1\bar{1}}^2} + \frac{R_{1\bar{1}\bar{2}\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} = \frac{R_{2\bar{2}\bar{1}\bar{1}}}{g_{1\bar{1}}g_{2\bar{2}}} + \frac{R_{2\bar{2}\bar{2}\bar{2}}}{g_{2\bar{2}}^2} = \frac{R_{1\bar{1}\bar{2}\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} + \frac{R_{2\bar{2}\bar{2}\bar{2}}}{g_{2\bar{2}}^2}. \quad (3.25)$$

We use the invariance of the Kähler-Einstein metric under the dilations  $d_\lambda$  ( $\lambda > 0$ ) to compute  $\frac{R_{1\bar{1}\bar{1}\bar{1}}}{g_{1\bar{1}}^2}$ . From the expression of the curvature coefficients (1.7) we have  $R_{1\bar{1}\bar{1}\bar{1}} = -g_{1\bar{1}\bar{1}\bar{1}} + \frac{|g_{11\bar{1}\bar{1}}|^2}{g_{1\bar{1}}}$ . Moreover from relation (3.21) we obtain at the origin  $g = F$ ,  $g_1 = 2K$ ,  $g_{1\bar{1}} = 4pK$ ,  $g_{1\bar{1}\bar{1}} = 16p^2K$  and  $g_{1\bar{1}\bar{1}\bar{1}} = 96p^3K$  hence  $\frac{R_{1\bar{1}\bar{1}\bar{1}}}{g_{1\bar{1}}^2} = -3 + \frac{1}{K}$ . We set  $A := \frac{R_{1\bar{1}\bar{1}\bar{1}}}{g_{1\bar{1}}^2}$ . Observe that  $-3 < A < -\frac{3}{2}$ .

Now we deal with  $R_{1\bar{2}\bar{1}\bar{2}}$ . Observe that at the origin relation (1.7) gives  $R_{1\bar{2}\bar{1}\bar{2}} = -g_{1\bar{2}\bar{1}\bar{2}} + \frac{g_{11\bar{1}}g_{1\bar{2}\bar{2}}}{g_{1\bar{1}}}$ . We use relation (3.22) to compute  $g_{1\bar{2}\bar{1}\bar{2}}$  and  $g_{1\bar{2}\bar{2}}$ . Since we want to compute the coefficients at  $z = 0$  we may look at  $g_{i\bar{j}}(0, z_2)$  for suitable indices  $i, j$ , then differentiate with respect to  $z_2$  and specify at  $z_2 = 0$ . For instance we have  $g_{1\bar{2}}(0, z_2) = z_2g_{2\bar{2}}(0, z_2) + \bar{z}_2g_{\bar{2}\bar{2}}(0, z_2) + g_{\bar{2}}(0, z_2)$  for  $z_2 \in \mathbb{C}$  small, hence by applying  $\frac{\partial}{\partial \bar{z}_2}$  and putting  $z_2 = 0$  we obtain  $g_{1\bar{2}\bar{2}} = 2g_{\bar{2}\bar{2}}$ . Using the "trick"  $g_{11}(z) = g_{1\bar{1}}(z)$  for all  $z \in D_H$  we obtain  $g_{1\bar{2}\bar{1}\bar{2}} = 4(p+1)g_{\bar{2}\bar{2}}$  in a similar fashion. Thus  $\frac{R_{1\bar{2}\bar{1}\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} = \frac{4(p-1)}{g_{1\bar{1}}g_{2\bar{2}}} = \frac{(p-1)}{pK} \frac{g_{\bar{2}\bar{2}}}{g_{2\bar{2}}} = \frac{(p-1)}{pK} \frac{\bar{g}_{2\bar{2}}}{g_{2\bar{2}}}$ . We set  $B := \frac{(p-1)}{pK} \left| \frac{g_{2\bar{2}}}{g_{2\bar{2}}} \right| \geq 0$ , and let  $\gamma \in \mathbb{R}$  such that  $\frac{R_{1\bar{2}\bar{1}\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} = Be^{i\gamma}$ .

Let  $v = (v_1, v_2) \in \mathbb{C}^2$ ,  $w = (w_1, w_2) \in \mathbb{C}^2$  such that  $|v|_g = |w|_g = 1$ . Let  $\alpha \in \mathbb{R}$  be an argument of  $v_1\bar{v}_2$  (respectively  $\beta \in \mathbb{R}$  an argument of  $w_1\bar{w}_2$ ). Adapting the proof of Proposition 3.24 we easily obtain:

$$\begin{aligned} Bis(v, w) &= R_{1\bar{1}\bar{1}\bar{1}}|v_1|^2|w_1|^2 + R_{1\bar{1}\bar{2}\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2 + 2|v_1||v_2||w_1||w_2|\cos(\alpha - \beta)) \\ &\quad + 2|v_1||v_2||w_1||w_2|Re(R_{1\bar{2}\bar{1}\bar{2}}e^{i(\alpha+\beta)}) + R_{2\bar{2}\bar{2}\bar{2}}|v_2|^2|w_2|^2 \\ &= A(g_{1\bar{1}}^2|v_1|^2|w_1|^2 + g_{2\bar{2}}^2|v_2|^2|w_2|^2) - (3 + A)g_{1\bar{1}}g_{2\bar{2}}(|v_1|^2|w_2|^2 + |v_2|^2|w_1|^2) \\ &\quad + 2g_{1\bar{1}}g_{2\bar{2}}|v_1||v_2||w_1||w_2|(B\cos(\alpha + \beta + \gamma) - (3 + A)\cos(\alpha - \beta)). \end{aligned}$$

Noting that the particular case  $\alpha = \beta = \frac{\pi - \gamma}{2}$  (respectively  $\alpha = -\frac{\pi + \gamma}{2} = \beta - \pi$ ) gives

the minimum (respectively the maximum) of  $Bis(v, w)$  with respect to  $\alpha, \beta$  we deduce

$$\begin{aligned}
& A (g_{1\bar{1}}^2 |v_1|^2 |w_1|^2 + g_{2\bar{2}}^2 |v_2|^2 |w_2|^2) - (3 + A) g_{1\bar{1}} g_{2\bar{2}} (|v_1|^2 |w_2|^2 + |v_2|^2 |w_1|^2) \\
& - 2 (B + (3 + A)) g_{1\bar{1}} g_{2\bar{2}} |v_1| |v_2| |w_1| |w_2| \\
& \leq Bis(v, w) \\
& \leq A (g_{1\bar{1}}^2 |v_1|^2 |w_1|^2 + g_{2\bar{2}}^2 |v_2|^2 |w_2|^2) - (3 + A) g_{1\bar{1}} g_{2\bar{2}} (|v_1|^2 |w_2|^2 + |v_2|^2 |w_1|^2) \\
& + 2 (B + (3 + A)) g_{1\bar{1}} g_{2\bar{2}} |v_1| |v_2| |w_1| |w_2|.
\end{aligned}$$

We set  $x := \sqrt{g_{1\bar{1}}} |v_1|$ ,  $y := \sqrt{g_{1\bar{1}}} |w_1|$  so that the above inequalities rephrase into

$$C(x, -y) \leq Bis(v, w) \leq C(x, y),$$

where  $C$  is the function defined in Lemma 3.26. We apply Lemma 3.26 and obtain the extrema for the bisectonal curvatures at the origin and also the minimum for the holomorphic sectional curvatures. For the maximum of the holomorphic sectional curvatures, we set  $x^2 := t$  to obtain

$$\begin{aligned}
H(v) = Bis(v, v) & \leq A (t^2 + (1 - t)^2) - 2(3 + A)t(1 - t) + 2(B - (3 + A))t(1 - t) \\
& = 2(B - 3(A + 2))t(1 - t) + A.
\end{aligned}$$

Since  $B - 3(A + 2) \geq -3(A + 2) > 0$ , the maximum of the polynomial function  $P : t \mapsto 2(B - 3(A + 2))t(1 - t) + A$  on  $[0, 1]$  is achieved at  $t = \frac{1}{2}$ . Hence we deduce

$$\max_{v \in \mathbb{C}^2 \setminus \{0\}} H(0; v) = \frac{-3 - \frac{1}{K} + \frac{p-1}{pK} \frac{|g_{2\bar{2}}|}{g_{2\bar{2}}}}{2}. \text{ The proof is ended. } \quad \square$$

We notice that  $\max_{v, w \in \mathbb{C}^2 \setminus \{0\}} Bis(0; v, w) = \max \left( -\frac{1}{K}, \max_{v \in \mathbb{C}^2 \setminus \{0\}} H(0; v) + \frac{1}{K} \right)$ . Consequently we have the following criterion

$$\max_{v, w \in \mathbb{C}^2 \setminus \{0\}} Bis(0; v, w) < 0 \iff \max_{v \in \mathbb{C}^2 \setminus \{0\}} H(0; v) < -\frac{1}{K} \iff (p-1) \frac{|g_{2\bar{2}}|}{g_{2\bar{2}}}(0) < 2p^2.$$

Moreover we also have  $\max_{v \in \mathbb{C}^2 \setminus \{0\}} H(0; v) < 0 \iff (p-1) \frac{|g_{2\bar{2}}|}{g_{2\bar{2}}}(0) < 2p(p+1)$ .



# Chapter 4

## Study of the Bergman metric in pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$

### Abstract

We prove that the holomorphic bisectional curvatures of the Bergman metric of Thullen domains and tube domains in  $\mathbb{C}^2$  are negatively pinched on the axis  $\{z_2 = 0\}$ . We use these results to prove that for every bounded pseudoconvex complete Reinhardt domain of finite type in  $\mathbb{C}^2$  there exists a neighbourhood of the boundary on which the holomorphic bisectional curvatures of the Bergman metric of the domain are negatively pinched.

**Change of notations** In this chapter we work only with the Bergman metric. Given an open set  $U \subset \mathbb{C}^n$ , we denote by  $K$  the Bergman kernel of  $U$ , and (whenever they are well defined) by  $g$ , respectively  $\langle \cdot, \cdot \rangle$ ,  $H$ ,  $Bis$  the potential, respectively the hermitian scalar product, the holomorphic sectional curvature, the holomorphic bisectional curvature of the Bergman metric of  $U$ . When multiple open sets are considered at the same time, we use the notations  $K^U$ , respectively  $g^U$ ,  $\langle \cdot, \cdot \rangle^U$ ,  $H^U$ ,  $Bis^U$  to avoid confusions.

### Introduction

Years after the work of P. Klembeck regarding the curvatures of the Bergman metric in strictly pseudoconvex domains with boundary of class  $\mathcal{C}^\infty$ , K.-T. Kim and J. Yu. proved

the following:

**Theorem 4.1** (Theorem 1 in [37]). *Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain, let  $q \in \partial D$  such that there exists a neighbourhood  $U$  of  $q$  such that  $\partial D \cap U$  is of class  $\mathcal{C}^2$ . Moreover assume that  $q$  is a strictly pseudoconvex boundary point for  $\partial D$ . Then*

$$\lim_{z \rightarrow q} \sup_{v \in \mathbb{C}^n \setminus \{0\}} \left| H(z; v) + \frac{2}{n+1} \right| = 0.$$

It was observed by many authors in the litterature (see for instance [35]) that a similar result holds for the holomorphic bisectional curvatures. In particular every strictly pseudoconvex boundary point  $q$  of a bounded pseudoconvex domain  $D$  with boundary of class  $\mathcal{C}^\infty$  there exists a neighbourhood  $U$  of  $q$  such that the holomorphic bisectional curvatures of the Bergman metric of  $D$  are negatively pinched in  $D \cap U$ .

In comparison, the behaviour of the Bergman metric and its holomorphic bisectional curvatures at weakly pseudoconvex boundary points is not clearly understood. In 1975 S. Kobayshi proved that the holomorphic sectional curvatures of any bounded pseudoconvex domain in  $\mathbb{C}^n$  are bounded from above by 2, in 1989 J. McNeal proved that the holomorphic sectional curvatures of any bounded pseudoconvex domain of finite type in  $\mathbb{C}^2$  are bounded, and recently S. Yoo proved that the holomorphic bisectional curvatures of bounded pseudoconvex domains of finite type in  $\mathbb{C}^2$  and of bounded convex domains of finite type in  $\mathbb{C}^n$  are bounded from below (see [40, 46, 58]). On the other hand G. Herbort exhibited a bounded pseudoconvex domain of finite type in  $\mathbb{C}^3$  for which the holomorphic sectional curvatures are not bounded from below (see [30]). However the question of the existence of a negative upper bound for the holomorphic bisectional curvatures of the Bergman metric of bounded pseudoconvex domains of finite type in  $\mathbb{C}^n$  is still relevant and unanswered even in the simpler case of domains in  $\mathbb{C}^2$  that we stick to from now on. K. Azukawa and N. Suzuki studied the Bergman metric in Thullen domains  $D_p := \{z \in \mathbb{C}^2, |z_1| < 1, |z_2|^2 < (1 - |z_1|^2)^p\}$  (where  $p \in ]0, 1[$ ) and proved that the maximum of the holomorphic sectional curvatures of the Bergman metric is negative (see [2]). S. Fu essentially proved in [22] that the holomorphic sectional curvatures of the Bergman metric of tube domains  $T'_p = \{z \in \mathbb{C}^2, \operatorname{Re}(z_1) + \operatorname{Re}(z_2)^{2p} < 0\}$  (where  $p \in \mathbb{N}^*$ ) are bounded from above by a negative constant along the axis  $\{z_2 = 0\}$ .

Thullen and tube domains (in their unbounded polynomial representation described in Section 1.3) serve as local models for the boundary of complete Reinhardt domains. Recall that a domain  $D \subset \mathbb{C}^n$  is complete Reinhardt if it satisfies  $(a_1 z_1, \dots, a_n z_n) \in D$

for every  $z \in D$  and  $a \in \Delta^n$ . Using this observation and the estimates of the Bergman curvatures in the model domains, he proved the following:

**Theorem 4.2** (Theorem 6.4 in [22]). *Let  $D \subset \mathbb{C}^2$  be a bounded pseudoconvex complete Reinhardt domain with boundary of class  $C^\infty$  and of finite type. Then there exists a neighbourhood  $U$  of  $\partial D$  and two constants  $0 < c < C$  such that  $-C \leq H \leq -c$  on  $D \cap U \times \mathbb{C}^2 \setminus \{0\}$ .*

In this chapter we prove a version of the above theorem regarding the holomorphic bisectonal curvatures of the Bergman metric. To do so we first study the behaviour of the holomorphic bisectonal curvatures in the model domains and prove:

**Theorem 4.3.** *Let  $p \in \mathbb{N}^*$ . Then*

$$-\infty < \min_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}^{E_p}(0; v, w), \quad \max_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}^{E_p}(0; v, w) < 0,$$

$$-\infty < \min_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}^{T'_p}((-1, 0); v, w), \quad \max_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}^{T'_p}((-1, 0); v, w) < 0.$$

We prove Theorem 4.3 in Section 4.1. In Section 4.2 we extend the result obtained by S. Fu to the holomorphic bisectonal curvatures, namely we prove:

**Theorem 4.4.** *Let  $D \subset \mathbb{C}^2$  be a bounded pseudoconvex complete Reinhardt domain with boundary of class  $C^\infty$  and of finite type. Then there exists a neighbourhood  $U$  of  $\partial D$  and two constants  $0 < c < C$  such that  $-C \leq \text{Bis} \leq -c$  on  $D \cap U \times (\mathbb{C}^2 \setminus \{0\})^2$ .*

## 4.1 Estimates of the Bergman curvatures in model domains

Here we prove Theorem 4.3. We use the following result:

**Lemma 4.5.** *Let  $D \subset \mathbb{C}^2$  be a domain and let  $[g_{i\bar{j}}]$  be a Kähler metric of class  $C^2$  on  $D$ . Let  $z \in D$ . Assume that at point  $z$  the matrix  $[g_{i\bar{j}}]$  is diagonal, the curvature coefficients  $R_{i\bar{j}k\bar{l}}$  are real numbers and that  $R_{1\bar{1}1\bar{2}} = R_{1\bar{2}2\bar{2}} = 0$ . Let  $v, w \in \mathbb{C}^2$  to vector with unit length with respect to the metric at point  $z$ . Then the holomorphic bisectonal curvature*

$Bis(v, w)$  of  $D$  with respect to  $[g_{i\bar{j}}]$  at point  $z$  and between vectors  $v$  and  $w$  satisfies:

$$\begin{aligned} Bis(v, w) &= \frac{R_{1\bar{1}1\bar{1}}}{g_{1\bar{1}}^2} x^2 y^2 \\ &+ \frac{R_{1\bar{1}2\bar{2}}}{g_{1\bar{1}} g_{2\bar{2}}} \left( x^2 (1 - y^2) + (1 - x^2) y^2 + 2x\sqrt{1 - x^2}y\sqrt{1 - y^2} \cos(\alpha - \beta) \right) \\ &+ 2 \frac{R_{1\bar{2}1\bar{2}}}{g_{1\bar{1}} g_{2\bar{2}}} x\sqrt{1 - x^2}y\sqrt{1 - y^2} \cos(\alpha + \beta) \\ &+ \frac{R_{2\bar{2}2\bar{2}}}{g_{2\bar{2}}^2} (1 - x^2) (1 - y^2), \end{aligned}$$

where  $x = \sqrt{g_{1\bar{1}}} |v_1|$ ,  $y = \sqrt{g_{1\bar{1}}} |w_1|$ ,  $\alpha \in \mathbb{R}$  satisfies  $v_1 \bar{v}_2 = |v_1 v_2| e^{i\alpha}$  and  $\beta \in \mathbb{R}$  satisfies  $w_1 \bar{w}_2 = |w_1 w_2| e^{i\beta}$ .

*Proof of Lemma 4.5.* From the expression of the curvature coefficients (1.7) and the hypothesis, we have for  $1 \leq i, j, k, l \leq 2$ :  $R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{j\bar{i}l\bar{k}}$ . Hence we may simplify formula (1.9) by gathering the terms depending on the number of 2 occurring in the 4-uple  $(i, j, k, l)$ :

$$\begin{aligned} Bis(v, w) &= R_{1\bar{1}1\bar{1}} |v_1|^2 |w_1|^2 \\ &+ R_{1\bar{1}1\bar{2}} (|v_1|^2 (w_1 \bar{w}_2 + \bar{w}_1 w_2) + (v_1 \bar{v}_2 + \bar{v}_1 v_2) |w_1|^2) \\ &+ R_{1\bar{1}2\bar{2}} (|v_1|^2 |w_2|^2 + |v_2|^2 |w_1|^2 + v_1 \bar{v}_2 \bar{w}_1 w_2 + \bar{v}_1 v_2 w_1 \bar{w}_2) \\ &+ R_{1\bar{2}1\bar{2}} (v_1 \bar{v}_2 w_1 \bar{w}_2 + \bar{v}_1 v_2 \bar{w}_1 w_2) \\ &+ R_{1\bar{2}2\bar{2}} ((v_1 \bar{v}_2 + \bar{v}_1 v_2) |w_2|^2 + |v_2|^2 (w_1 \bar{w}_2 + \bar{w}_1 w_2)) \\ &+ R_{2\bar{2}2\bar{2}} |v_2|^2 |w_2|^2 \\ &= R_{1\bar{1}1\bar{1}} |v_1|^2 |w_1|^2 \\ &+ R_{1\bar{1}2\bar{2}} (|v_1|^2 |w_2|^2 + |v_2|^2 |w_1|^2 + 2|v_1| |v_2| |w_1| |w_2| \cos(\alpha - \beta)) \\ &+ 2R_{1\bar{2}1\bar{2}} |v_1| |v_2| |w_1| |w_2| \cos(\alpha + \beta) \\ &+ R_{2\bar{2}2\bar{2}} |v_2|^2 |w_2|^2. \end{aligned}$$

Moreover, from the fact that  $[g_{i\bar{j}}]$  is diagonal at  $z$  and that  $v$  and  $w$  have unit length we deduce  $g_{1\bar{1}} |v_1|^2 + g_{2\bar{2}} |v_2|^2 = 1$ , respectively  $g_{1\bar{1}} |w_1|^2 + g_{2\bar{2}} |w_2|^2 = 1$ , hence  $\sqrt{g_{2\bar{2}}} |v_2| = \sqrt{1 - g_{1\bar{1}} |v_1|^2}$ , respectively  $\sqrt{g_{2\bar{2}}} |w_2| = \sqrt{1 - g_{1\bar{1}} |w_1|^2}$ . Setting  $x = \sqrt{g_{1\bar{1}}} |v_1|$  and  $y = \sqrt{g_{1\bar{1}}} |w_1|$  we directly obtain the conclusion.  $\square$

We now prove Theorem 4.3. To do so we use computations done in [22] and [2] respectively. It relies on the fact that the curvature coefficients of the Bergman metric satisfy the relations  $R_{i\bar{j}k\bar{l}} = (g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) - \hat{R}_{i\bar{j}k\bar{l}}$  for every integers  $1 \leq i, j, k, l \leq n$ , where

$$\hat{R}_{i\bar{j}k\bar{l}} = \frac{KK_{i\bar{j}k\bar{l}} - K_{ik}K_{j\bar{l}}}{K^2} - \frac{\sum_{1 \leq \alpha, \beta \leq n} g^{\alpha\bar{\beta}} (KK_{ik\bar{\alpha}} - K_{ik}K_{\bar{\alpha}}) (KK_{j\bar{l}\beta} - K_{j\bar{l}}K_{\beta})}{K^4}.$$

We prove the existence of negatives upper and lower bounds for the holomorphic bisectional curvatures of the Bergman metric in the tube domain  $T'_p$ .

**Proposition 4.6.** *Let  $p \in \mathbb{N}^*$ . In the tube domain  $T'_p$ , one has*

$$-\infty < \min_{v, w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}((-1, 0); v, w), \quad \max_{v, w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}((-1, 0); v, w) < 0.$$

*Proof.* Notice that the biholomorphism  $z \mapsto iz$  maps  $T'_p$  to

$$\{z \in \mathbb{C}^2, \text{Im}(z_1) + \text{Im}(z_2)^{2p} < 0\}$$

and sends  $(-1, 0)$  to  $(-i, 0)$  we may use the computations done in [22] by simply replacing the point  $(-i, 0)$  with  $(-1, 0)$ . To simplify notations we write  $\text{Bis}(v, w)$  instead of  $\text{Bis}((-1, 0); v, w)$ . Because of relation (1.10), it is enough to prove that  $-\infty < \text{Bis}(v, w) < 0$  for every  $v, w \in \mathbb{C}^2$  with unit length with respect to the Bergman metric at point  $(-1, 0)$ . At point  $z = (-1, 0)$  we have:

$$\begin{aligned} [g_{i\bar{j}}] &= \begin{bmatrix} \frac{2p+1}{4p} & 0 \\ 0 & \frac{B_p}{A_p} \end{bmatrix}, \\ \hat{R}_{1\bar{1}1\bar{1}} &= \frac{(2p+1)(3p+1)}{8p^2}, \quad \hat{R}_{1\bar{1}1\bar{2}} = 0, \quad \hat{R}_{1\bar{1}2\bar{2}} = \frac{B_p p + 1}{A_p 2p}, \\ \hat{R}_{1\bar{2}1\bar{2}} &= -\frac{B_p p - 1}{A_p 4p^2}, \quad \hat{R}_{1\bar{2}2\bar{2}} = 0, \quad \hat{R}_{2\bar{2}2\bar{2}} = \frac{B_p^2}{A_p^2} \left( \lambda_p - 1 - \frac{1}{p(2p+1)} \right), \end{aligned}$$

where  $A_p, B_p$  and  $\lambda_p$  are positive numbers (see the bottom of pages 412 and 414 in [22] for their explicit definition). Hence the following relations:

$$\begin{aligned} \frac{R_{1\bar{1}1\bar{1}}}{g_{1\bar{1}}^2} &= -\frac{2p}{2p+1}, \quad \frac{R_{1\bar{1}2\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} = -\frac{1}{2p+1}, \\ \frac{R_{1\bar{2}1\bar{2}}}{g_{1\bar{1}}g_{2\bar{2}}} &= \frac{p-1}{p(2p+1)}, \quad \frac{R_{2\bar{2}2\bar{2}}}{g_{2\bar{2}}^2} = -\left( \lambda_p - 3 - \frac{1}{p(2p+1)} \right), \end{aligned}$$

Applying Lemma 4.5 we obtain:

$$\begin{aligned}
Bis(v, w) &= -\frac{2p}{2p+1}x^2y^2 \\
&\quad -\frac{1}{2p+1}\left(x^2(1-y^2) + (1-x^2)y^2 + 2x\sqrt{1-x^2}y\sqrt{1-y^2}\cos(\alpha-\beta)\right) \\
&\quad +2\frac{p-1}{p(2p+1)}x\sqrt{1-x^2}y\sqrt{1-y^2}\cos(\alpha+\beta) \\
&\quad -\left(\lambda_p - 3 - \frac{1}{p(2p+1)}\right)(1-x^2)(1-y^2).
\end{aligned}$$

Especially we directly obtain  $-\infty < \min_{v,w \in \mathbb{C}^2 \setminus \{0\}} Bis(v, w)$ . Moreover since  $R_{1\bar{1}2\bar{2}} \leq 0 \leq R_{1\bar{2}1\bar{2}}$  we deduce:

$$\begin{aligned}
Bis(v, w) &\leq -\frac{2p}{2p+1}x^2y^2 \\
&\quad -\frac{1}{2p+1}\left(x^2(1-y^2) + (1-x^2)y^2 - 2x\sqrt{1-x^2}y\sqrt{1-y^2}\right) \\
&\quad +2\frac{p-1}{p(2p+1)}x\sqrt{1-x^2}y\sqrt{1-y^2} \\
&\quad -\left(\lambda_p - 3 - \frac{1}{p(2p+1)}\right)(1-x^2)(1-y^2) \\
&= -\frac{2p}{2p+1}x^2y^2 \\
&\quad -\frac{1}{2p+1}\left(x\sqrt{1-y^2} - \sqrt{1-x^2}y\right)^2 \\
&\quad +2\frac{p-1}{p(2p+1)}x\sqrt{1-x^2}y\sqrt{1-y^2} \\
&\quad -\left(\lambda_p - 3 - \frac{1}{p(2p+1)}\right)(1-x^2)(1-y^2)
\end{aligned}$$

Using the Hölder inequality  $2x\sqrt{1-x^2}y\sqrt{1-y^2} \leq x^2y^2 + (1-x^2)(1-y^2)$  we obtain

$$\begin{aligned}
Bis(v, w) &\leq -\left(\frac{2p}{2p+1} - \frac{p-1}{p(2p+1)}\right)x^2y^2 \\
&\quad -\frac{1}{2p+1}\left(x\sqrt{1-y^2} - \sqrt{1-x^2}y\right)^2 \\
&\quad -\left(\lambda_p - 3 - \frac{1}{p(2p+1)} - \frac{p-1}{p(2p+1)}\right)(1-x^2)(1-y^2) \\
&= -\frac{p(2p-1)+1}{p(2p+1)}x^2y^2 \\
&\quad -\frac{1}{2p+1}\left(x\sqrt{1-y^2} - \sqrt{1-x^2}y\right)^2 \\
&\quad -\left(\lambda_p - 3 - \frac{1}{2p+1}\right)(1-x^2)(1-y^2).
\end{aligned}$$

According to [22] we have  $\lambda_p \geq \frac{7}{2}$ , hence  $\lambda_p - 3 - \frac{1}{2p+1} \geq \frac{2p-1}{2(2p+1)} > 0$  so that  $Bis(v, w) < 0$  as it is less or equal to a sum of three non-positive quantities that does not vanish for the same values of  $(x, y)$ .  $\square$

Now we turn our attention to the case of Thullen domains. In [2] the authors work in the domain  $D_p$  (defined in the introduction of the current chapter) with  $p \in ]0, 1]$ . If  $\frac{1}{p} \in \mathbb{N}$  then  $D_p = E_{\frac{1}{p}}$  so that Proposition 4.7 gives a more general result that we need.

**Proposition 4.7.** *Let  $p \in ]0, 1]$ . In the domain  $D_p$ , one has:*

$$-\infty < \min_{v, w \in \mathbb{C}^2 \setminus \{0\}} Bis(0; v, w), \quad \max_{v, w \in \mathbb{C}^2 \setminus \{0\}} Bis(0; v, w) < 0.$$

**Remark 4.8.** *In [2] the definition of the curvature coefficients is the opposite of the one we use in this thesis (compare relation (1.9) with the definition of the curvature coefficients given at bottom of page 1 in [2]). This explains the difference of sign between the quantities obtained in [2] and the same quantities appearing in the following proof.*

*Proof of Proposition 4.7.* To simplify notations we write  $Bis(v, w)$  instead of  $Bis(0; v, w)$ . Because of relation (1.10), it is enough to prove that  $Bis(v, w) < 0$  for every  $v, w \in \mathbb{C}^2$  with unit length with respect to the Bergman metric at 0. We use the computations done in [2]. We have  $r = \frac{1-p}{1+p}$ , and the following relations:

$$\begin{aligned} [g_{i\bar{j}}] &= \begin{bmatrix} \frac{3+r}{1+r} & 0 \\ 0 & 3-r \end{bmatrix}, \\ \hat{R}_{1\bar{1}1\bar{1}} &= -4 \frac{(3+r)(2+r)}{(1+r)^2}, \quad \hat{R}_{1\bar{1}1\bar{2}} = 0, \quad \hat{R}_{1\bar{1}2\bar{2}} = -4 \frac{3-r}{1+r}, \\ \hat{R}_{1\bar{2}1\bar{2}} &= 0, \quad \hat{R}_{1\bar{2}2\bar{2}} = 0, \quad \hat{R}_{2\bar{2}2\bar{2}} = -12(2-r), \end{aligned}$$

thus we obtain the following expressions for the curvature coefficients:

$$\frac{R_{1\bar{1}1\bar{1}}}{g_{1\bar{1}}^2} = -2 \frac{1+r}{3+r}, \quad \frac{R_{1\bar{1}2\bar{2}}}{g_{1\bar{1}} g_{2\bar{2}}} = -\frac{1-r}{3+r}, \quad \frac{R_{2\bar{2}2\bar{2}}}{g_{2\bar{2}}^2} = -2 \frac{3-r^2}{(3-r)^2}.$$

We apply Lemma 4.5 to obtain the following:

$$\begin{aligned} Bis(v, w) &= -2 \frac{1+r}{3+r} x^2 y^2 \\ &\quad - \frac{1-r}{3+r} \left( x^2 (1-y^2) + (1-x^2) y^2 + 2x\sqrt{1-x^2} y\sqrt{1-y^2} \cos(\alpha - \beta) \right) \\ &\quad - 2 \frac{3-r^2}{(3-r)^2} (1-x^2) (1-y^2). \end{aligned}$$

Consequently we directly obtain  $-\infty < \min_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}(v,w)$ . Moreover since  $p \in ]0, 1]$  we have  $3 - r^2, 1 - r > 0$ . Doing as in the proof of Proposition 4.6 we obtain

$$\text{Bis}(v,w) \leq -2 \frac{1+r}{3+r} x^2 y^2 - \frac{1-r}{3+r} \left( x \sqrt{1-y^2} - \sqrt{1-x^2} y \right)^2 - 2 \frac{3-r^2}{(3-r)^2} (1-x^2)(1-y^2)$$

hence  $\text{Bis}(v,w) < 0$  as it is less or equal to a sum of three non-positive quantities that does not vanish for the same values of  $(x,y)$ . This concludes the proof.  $\square$

**Remark 4.9.** *The proof of Proposition 4.6 does not give the maximum of the bisectonal curvature at the origin in the tube domains, whereas we easily derive from the proof of Proposition 4.7 that  $\max_{v,w \in \mathbb{C}^2 \setminus \{0\}} \text{Bis}^{D_p}(0;v,w) = \max \left\{ -2 \frac{1+r}{3+r}, -\frac{1-r}{3+r}, -2 \frac{3-r^2}{(3-r)^2} \right\}$  where  $r$  is as in the proof of Proposition 4.7.*

## 4.2 Estimate of Bergman curvatures in complete Reinhardt domains of finite type in $\mathbb{C}^2$

### 4.2.1 Localisation of the Bergman metric

Let  $D \subset \mathbb{C}^n$  be an open set and let  $\mathcal{A}^2(D) := \mathcal{H}(D, \mathbb{C}) \cap \mathcal{L}^2(D)$  be the Bergman space of  $D$  with weight 2. Given a point  $z \in D$  and two vectors  $v, w \in \mathbb{C}^n \setminus \{0\}$  we define the following:

$$\begin{aligned} I_0^D(z) &:= \inf \left\{ \|f\|_{L^2(D)}^2 / f \in \mathcal{A}^2(D), f(z) = 1 \right\}, \\ I_1^D(z, v) &:= \inf \left\{ \|f\|_{L^2(D)}^2 / f \in \mathcal{A}^2(D), f(z) = 0, \partial_z f(v) = 1 \right\}, \\ I_2^D(z, v, w) &:= \inf \left\{ \|f\|_{L^2(D)}^2 / f \in \mathcal{A}^2(D), f(z) = 0, \partial_z f = 0, \sum_{1 \leq i, j \leq n} f_{ij}(z) v_i w_j = 1 \right\}. \end{aligned}$$

The Bergman kernel, the metric it induces and its holomorphic bisectonal curvatures may be expressed in terms of the above integrals. The following formulas are attributed to S. Bergman, B.A. Fuks and A. Pagano (see Theorem 2.1. in [58]):

$$K^D(z, z) = \frac{1}{I_0^D(z)} \tag{4.1}$$

$$|v|_z^D = \frac{I_1^D(z)}{I_1^D(z, v)} \tag{4.2}$$

$$\text{Bis}^D(z; v, w) = 2 - \frac{I_1^D(z, v) I_1^D(z, w)}{I_0^D(z) I_2^D(z, v, w)}. \tag{4.3}$$



The above formulas are used to "localise" the Bergman metric and its curvatures at peak boundary points of  $D$ . Recall that a point  $q \in \partial D$  is a local peak point if there exists an open neighbourhood  $U \subset \mathbb{C}^n$  of  $q$  and a map  $f \in \mathcal{C}(\overline{D} \cap U, \mathbb{C}) \cap \mathcal{H}(D \cap U, \mathbb{C})$  such that  $f(q) = 1$  and  $|f(z)| < 1$  for every  $z \in \overline{D} \cap U \setminus \{q\}$  (the function  $f$  is called a peaking function for  $q$ ).

**Theorem 4.10** (Theorem 4 in [37]). *Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain, let  $q \in \partial D$  and let  $U \subset \mathbb{C}^n$  be an open neighbourhood of  $q$ . If  $q$  is a local peak point then for every  $v, w \in \mathbb{C}^n \setminus \{0\}$  the following holds:*

$$\lim_{z \rightarrow q} \frac{I_0^D(z)}{I_0^{D \cap U}(z)} - 1 = \lim_{z \rightarrow q} \sup_{v \in \mathbb{C}^2 \setminus \{0\}} \left| \frac{I_1^D(z, v)}{I_1^{D \cap U}(z, v)} - 1 \right| = \lim_{z \rightarrow q} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \left| \frac{I_2^D(z, v, w)}{I_2^{D \cap U}(z, v, w)} - 1 \right| = 0.$$

Especially, under the assumptions of Theorem 4.10 (and provided that the Bergman kernel on  $D \cap U$  induces a Kähler metric on  $D \cap U$ ), one has:

$$\lim_{z \rightarrow q} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \left| \frac{Bis^D(z; v, w)}{Bis^{D \cap U}(z; v, w)} - 1 \right| = 0. \quad (4.4)$$

E. Bedford and J.E. Forneaess proved that every boundary point of a bounded pseudoconvex domain of finite type in  $\mathbb{C}^2$  is a peak point, and that the peaking function depends continuously on  $q$  (see Theorem 3.1. and Remark 3.4. in [3]). It follows that if the neighbourhood  $U$  has a uniform size with respect to the point  $q$  (for instance if  $U = B(q, R)$  for some fixed  $R > 0$ ) then the limit (4.4) is uniform in  $q$ .

## 4.2.2 The scaling of bounded pseudoconvex Reinhardt domain of finite type in $\mathbb{C}^2$

If  $\partial D$  is pseudoconvex and the type at  $q$  is finite there exists an integer  $p \in \mathbb{N}$ , an homogeneous polynomial of degree  $2p$   $H$  which is subharmonic but not harmonic, an open set  $U \subset \mathbb{C}^2$  containing  $q$  and a biholomorphic map  $\Phi : U \rightarrow \Phi(U)$  such that  $\Phi(q) = 0$  and  $\Phi(D \cap U) = M'_H \cap \Phi(U)$  where

$$M'_H := \left\{ z \in \mathbb{C}^2, \operatorname{Re}(z_1) + H(z_2) + \underset{z \rightarrow 0}{O}(|z_2|^{2p+1} + |z_1||z|) < 0 \right\}.$$

If  $D$  is a Reinhardt domain, we can assume that  $M'_H$  has a simpler form (see the expressions (6.10) and (6.25) in [22] and the proof of Theorem 4.11).

Let  $q \in \partial D$  be a point of finite type  $2p$ . Every complex vector  $v \in \mathbb{C}^2$  admits a unique decomposition  $v = v_T + v_N$  where  $v_T \in T_q^{\mathbb{C}} \partial D$  and there exists a complex number  $\lambda$  such

that  $v_N = \lambda(d_1(q), d_2(q))$ . Using that decomposition we define for every positive number  $\lambda > 0$  the following anisotropic dilation  $\Pi_\lambda$ :

$$\begin{aligned} \Pi_\lambda : \quad \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ v = v_T + v_N &\longmapsto \lambda^{\frac{1}{2p}} v_T + \lambda v_N. \end{aligned}$$

The key result to prove Theorem 4.4 is the following result which generalises Theorem 6.1 in [22]:

**Theorem 4.11.** *Let  $D \subset \mathbb{C}^2$  be a bounded pseudoconvex complete Reinhardt domain with boundary of class  $\mathcal{C}^\infty$ . Let  $z^{(\infty)} \in \partial D$  be a point of finite type  $2p$ . Let  $\Lambda$  be a nontangential cone with vertex at  $z^{(\infty)}$ . Then for every sequence  $(z^{(\nu)})_{\nu \in \mathbb{N}} \in (D \cap \Lambda)^\mathbb{N}$  converging to  $z^{(\infty)}$  the following holds:*

$$\lim_{\nu \rightarrow +\infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} |Bis^D(z^{(\nu)}; \Pi_{d^{(\nu)}}(v), \Pi_{d^{(\nu)}}(w)) - Bis^{M_p}(q; v', w')| = 0, \quad (4.5)$$

where:

1. the domain  $M_p$  is the Thullen domain  $E_p$  if one of the coordinates of  $z^{(\infty)}$  is 0 and the tube domain  $T'_p$  otherwise,
2. if  $M_p = E_p$  then  $q = (0, 0)$ , and if  $M_p = T'_p$  then  $q = (-1, 0)$ ,
3. for every vector  $v \in \mathbb{C}^2 \setminus \{0\}$  (respectively  $w \in \mathbb{C}^2 \setminus \{0\}$ ) the vector  $v' \in \mathbb{C}^2 \setminus \{0\}$  (respectively  $w' \in \mathbb{C}^2 \setminus \{0\}$ ) depends only on  $v$  (respectively  $w$ ) and the geometry of  $\partial D$  at  $z^{(\infty)}$ .

We refer to relations (6.4) , (6.6) and (6.8) in [22] for the definition of  $v'$  and  $w'$ .

*Proof of Theorem 4.11.* The proof is an adaptation of the proof of [22, Theorem 6.1.]. It follows the same steps and uses the same technical results, namely localising, scaling, and proving the interior convergence of a sequence of Bergman kernels. The only change is that we work with the holomorphic bisectional curvatures instead of the holomorphic sectional curvatures. We sketch the main steps of the proof of [22, Theorem 6.1.] and point out the new ideas we use.

- Case  $z_1^{(\infty)} = 0$  or  $z_2^{(\infty)} = 0$ : since  $D$  is a pseudoconvex complete Reinhardt domain, the type at  $z^{(\infty)}$  is equal to  $2p$ , and one of the coordinates of  $z^{(\infty)}$  is equal to 0, there exists

of a neighbourhood  $U$  of  $z^{(\infty)}$  and a biholomorphic map  $\Phi : U \rightarrow \Phi(U)$  such that  $\Phi(z^{(\infty)}) = 0$  and  $\Phi(D \cap U) = E'_p \cap \Phi(U)$  with

$$E'_p := \left\{ z \in \mathbb{C}^2, \operatorname{Re}(z_1) + |z_2|^{2p} + O_{z \rightarrow 0}(|z_2|^{2p+1} + |z_1||z|) < 0 \right\}$$

(see definitions (6.9), (6.10) in [22]). For every integer  $\nu \in \mathbb{N}$ , we set  $z^{(\nu)} := \Phi(z^{(\infty)})$ ,  $d^{(\nu)} := d(z^{(\nu)}, \Phi(\partial D \cap U))$  and consider the rescaling map

$$F^{(\nu)} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \\ z \mapsto \left( \frac{z_1 - z_1^{(\nu)}}{d^{(\nu)}}, \frac{z_2 - z_2^{(\nu)}}{d^{(\nu)\frac{1}{2p}}} \right).$$

Let  $M$  be the Mobius map  $M(z) := \left( \frac{z_1 + 1}{z_1 - 1}, 2^{\frac{1}{p}} \frac{z_2}{(z_1 - 1)^{\frac{1}{p}}} \right)$  defined for every  $z \in \mathbb{C}^2$  such that  $z_1 \neq 1$ , and set  $G^{(\nu)} := M \circ F^{(\nu)} \circ \Phi$ ,  $D_\nu := G^{(\nu)}(D \cap U)$ ,  $v^{(\nu)} := \partial_{z^{(\nu)}} G^{(\nu)}(\Pi_{d^{(\nu)}}(v))$  and  $w^{(\nu)} := \partial_{z^{(\nu)}} G^{(\nu)}(\Pi_{d^{(\nu)}}(w))$ . Applying the invariance formula (1.13) we have the following for every  $v, w \in \mathbb{C}^2 \setminus \{0\}$ :

$$\operatorname{Bis}^{D \cap U}(z^{(\nu)}; \Pi_{d^{(\nu)}}(v), \Pi_{d^{(\nu)}}(w)) = \operatorname{Bis}^{D_\nu}(0; v^{(\nu)}, w^{(\nu)}). \quad (4.6)$$

According to relations (6.17) to (6.20) in [22], the sequence  $(v^{(\nu)})_{\nu \in \mathbb{N}}$ , respectively  $(w^{(\nu)})_{\nu \in \mathbb{N}}$  converges to a vector  $v' \in \mathbb{C}^2 \setminus \{0\}$ , respectively to a vector  $w' \in \mathbb{C}^2 \setminus \{0\}$ .

Moreover according to relation (6.14) in [22] the sequence of domains  $(D_\nu)_{\nu \in \mathbb{N}}$  converges to the Thullen domain  $E_p$  in the local Hausdorff topology. This implies that the sequence of Bergman kernels  $(K^{D_\nu})_{\nu \in \mathbb{N}}$  converges uniformly on compact sets of  $E_p$  to  $K^{E_p}$  (see the relation at the top of page 422 in [22]). From this convergence we derive the pointwise convergence of the derivatives of  $K^{D_\nu}$  up to the order 4 at point  $0 \in E_p$  (see Theorems 5.3., 5.4. and Remark 5.7. in [22]). Since for every integer  $1 \leq i, j, k, l \leq 2$  the coefficients  $g_{i\bar{j}}$  and  $R_{i\bar{j}k\bar{l}}$  are rational fractions of these quantities (see the definition of the Bergman metric and also relation (1.9)) we obtain the following

$$\lim_{\nu \rightarrow +\infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} |\operatorname{Bis}^{D_\nu}(0; v^{(\nu)}, w^{(\nu)}) - \operatorname{Bis}^{E_p}(0; v', w')|. \quad (4.7)$$

Combining relations (4.6) and (4.7) we deduce

$$\lim_{\nu \rightarrow +\infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} |\operatorname{Bis}^{D \cap U}(z^{(\nu)}; \Pi_{d^{(\nu)}}(v), \Pi_{d^{(\nu)}}(w)) - \operatorname{Bis}^{E_p}(0; v', w')| = 0.$$

Finally, we apply Theorem 4.10 to obtain

$$\lim_{\nu \rightarrow +\infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} |Bis^D(z^{(\nu)}; \Pi_{d^{(\nu)}}(v), \Pi_{d^{(\nu)}}(w)) - Bis^{E_p}(0; v', w')| = 0.$$

This proves the result (4.5) in the case that one of the coordinates of  $z^{(\infty)}$  is equal to 0.

- Case  $z_1^{(\infty)} z_2^{(\infty)} \neq 0$ : since  $D$  is a pseudoconvex complete Reinhardt domain, the type at  $z^{(\infty)}$  is equal to  $2p$ , and none of the coordinates of  $z^{(\infty)}$  are 0, there exists a neighbourhood  $U$  of  $z^{(\infty)}$  and a biholomorphic map  $\Phi : U \rightarrow \Phi(U)$  such that  $\Phi(z^{(\infty)}) = 0$  and  $\Phi(D \cap U) = T_p'' \cap \Phi(U)$  with

$$T_p'' := \left\{ Re(z_1) + Re(z_2)^{2p} < \underset{z \rightarrow 0}{O} (|Re(z_2)|^{2p+1} + |Re(z_1)|(|Re(z_1)| + |Re(z_2)|)) \right\}$$

(see definitions (6.23), (6.24) and (6.25) in [22]). For every integer  $\eta \in \mathbb{N}$ , we set  $z^{(\nu)} := \Phi(z^{(\nu)})$ ,  $d^{(\eta)} := d(z^{(\nu)}, \Phi(\partial D \cap U))$  and consider the rescaling map

$$F^{(\nu)} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \\ z \longmapsto \left( \frac{z_1 - z_1^{(\nu)}}{d^{(\nu)}} - 1, \frac{z_2 - z_2^{(\nu)}}{d^{(\nu) \frac{1}{2p}}} \right).$$

Set  $G^{(\nu)} := F^{(\nu)} \circ \Phi$ ,  $D_\nu := G^{(\nu)}(D \cap U)$ ,  $v^{(\nu)} := \partial_{z^{(\nu)}} G^{(\nu)}(\Pi_{d^{(\nu)}}(v))$  and  $w^{(\nu)} := \partial_{z^{(\nu)}} G^{(\nu)}(\Pi_{d^{(\nu)}}(w))$ . Applying the transformation formula (1.13) we obtain the following for every  $v, w \in \mathbb{C}^2 \setminus \{0\}$ :

$$Bis^{D \cap U}(z^{(\nu)}; \Pi_{d^{(\nu)}}(v), \Pi_{d^{(\nu)}}(w)) = Bis^{D_\nu}((-1, 0); v^{(\nu)}, w^{(\nu)}). \quad (4.8)$$

According to relations (6.41) and (6.42) in [22], the sequence  $(v^{(\nu)})_{\nu \in \mathbb{N}}$ , respectively  $(w^{(\nu)})_{\nu \in \mathbb{N}}$  converges to a vector  $v' \in \mathbb{C}^2 \setminus \{0\}$ , respectively to a vector  $w' \in \mathbb{C}^2 \setminus \{0\}$ .

Moreover according to (6.24) and (6.29) in [22] the sequence of domains  $(G^{(\nu)}(T_p''))_{\nu \in \mathbb{N}}$  converges to the tube domain  $T_p'$  in the local Hausdorff topology. This implies that (up to a possible shrinking of  $U$ ) the sequence of Bergman kernels  $(K^{D_\nu})_{\nu \in \mathbb{N}}$  converges uniformly on compact sets of  $T_p'$  to  $K^{T_p'}$  (see claim 6.2. in [22]). From this convergence we derive the pointwise convergence of the derivatives of  $K^{D_\nu}$  up to the order 4 at point  $(-1, 0) \in T_p'$  (see Theorem 5.4. and Remark 5.7. in [22]), hence the following

$$\lim_{\nu \rightarrow +\infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \left| Bis^{D_\nu}((-1, 0); v, w) - Bis^{T_p'}((-1, 0); v', w') \right| = 0. \quad (4.9)$$

Combining relations (4.8) and (4.9) we easily deduce that

$$\lim_{\nu \rightarrow +\infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \left| \text{Bis}^{D \cap U} (z^{(\nu)}; \Pi_{d^{(\nu)}}(v), \Pi_{d^{(\nu)}}(w)) - \text{Bis}^{T'_p}((-1, 0); v', w') \right| = 0.$$

Finally, we apply Theorem 4.10 to obtain

$$\lim_{\nu \rightarrow +\infty} \sup_{v, w \in \mathbb{C}^2 \setminus \{0\}} \left| \text{Bis}^D (z^{(\nu)}; \Pi_{d^{(\nu)}}(v), \Pi_{d^{(\nu)}}(w)) - \text{Bis}^{T'_p}((-1, 0); v', w') \right| = 0.$$

This proves the result (4.5) in the case that the coordinates of  $z^{(\infty)}$  are non zero.

This concludes the proof of Theorem 4.11. □

The proof of Theorem 4.4 follows from Theorem 4.11 and Theorem 4.3:

*Proof of Theorem 4.4.* The proof works exactly as in [22, Theorem 6.4.], we just replace the estimates on the holomorphic sectional curvatures of the Bergman metric in Thullen and tube domains obtained in [22, Theorem 4.6.] and [2, Corollary 2] with the estimates on the holomorphic bisectional curvatures obtained in Theorem 4.7 and Theorem 4.6. □

# Prospects

If  $D \subset \mathbb{C}^n$  is a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$  and  $q$  is either a strictly pseudoconvex boundary point or a point such that the squeezing function of  $D$  tends to 1 at  $q$ , then the curvature behaviour

$$\sup_{v, w \in \mathbb{C}^n \setminus \{0\}} \left( \text{Bis}(g^D)(z; v, w) + \left( 1 + \left( \frac{|\langle v, w \rangle_{g^D, z}|}{\langle v, v \rangle_{g^D, z} \langle w, w \rangle_{g^D, z}} \right)^2 \right) \right) \xrightarrow{z \rightarrow q} 0 \quad (4.10)$$

holds, for either the Kähler-Einstein metric with Ricci curvature  $-(n+1)$  or  $\frac{1}{n+1}$  times the Bergman metric of  $D$  (see Theorems 2.5, 2.6, Remark 2.22 and [15, 37]). One may ask whether condition (4.10) implies that  $q$  is a ball-like boundary point. More precisely:

### Question 3. Sufficient curvature condition for "ball-like" points

Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$ . Let  $q \in \partial D$ . Let  $[g_{i\bar{j}}]$  be a complete Kähler metric of class  $\mathcal{C}^2$  on  $D$  such that condition (4.10) holds at  $q$ . Are the following satisfied

- (1)  $s^D(z) \xrightarrow{z \rightarrow q} 1$ , or
- (2)  $\partial D$  is strictly pseudoconvex at  $q$ ?

The answer to Question 3 seems to be known and affirmative only if we assume that  $D$  is convex and that the behaviour (4.10) holds uniformly on  $\partial D$ , see [8, Theorem 1.4.] and [63, Theorem 1.7.]. Moreover in this case condition (4.10) for the holomorphic sectional curvatures is sufficient for (1) or (2) to hold. For non convex domains one may study a similar question replacing (4.10) with a condition on the holomorphic sectional curvatures.

In some classes of pseudoconvex domains conditions (1) and (2) are related. For a bounded pseudoconvex domain  $D$  with boundary of class  $\mathcal{C}^2$  the strict pseudoconvexity of  $\partial D$  implies condition (1) at every boundary point (see [16]). Additionally if  $D$  is

convex with boundary of class  $\mathcal{C}^{2+\alpha}$ ,  $\alpha > 0$ , and condition (1) holds at every boundary point, then  $\partial D$  is strictly pseudoconvex (see [63]). Yet there is no local version of these results. Additionally one notices that the Fornaess-Wold domain is bounded, convex, and has a boundary of class  $\mathcal{C}^2$  with weakly pseudoconvex point at which condition (1) holds. In light of these results one may ask the following:

**Question 4. Equivalence of "ball-like" conditions**

Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain. Let  $\alpha \in ]0, 1[$  and  $q \in \partial D$  be such that  $\partial D$  is of class  $\mathcal{C}^{2+\alpha}$  in a neighbourhood of  $q$ . Are conditions (1) and (2) equivalent?

In contrast with the strictly pseudoconvex case, very few results are known regarding the boundary behaviour of the holomorphic bisectional curvatures of the Kähler-Einstein metric and of the Bergman metric at weakly pseudoconvex boundary points of smoothly bounded pseudoconvex domains of finite type. We can ask:

**Question 5. Negativity of bisectional curvatures for finite type domains**

Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$  and let  $q \in \partial D$  be a point of finite type. Does there exist a neighbourhood  $U$  of  $q$  such that

$$\sup_{\substack{z \in D \cap U \\ v, w \in \mathbb{C}^n \setminus \{0\}}} \text{Bis}(g)(z; v, w) < 0,$$

where  $g$  is either the Bergman metric of  $D$  or the Kähler-Einstein metric of  $D$ ?

One expects that a positive answer to Question 5 suffices to obtain the existence of a neighbourhood  $U$  of  $\partial D$  in which the holomorphic bisectional curvatures of the metric are bounded from above by a negative constant. Notice that the hypotheses in Question 5 do not imply  $\inf_{\substack{z \in D \cap U \\ v, w \in \mathbb{C}^n \setminus \{0\}}} \text{Bis}(g)(z; v, w) > -\infty$  as the following example shows. Let  $a, m \in \mathbb{N}^*$  such that  $2a < m$  and consider the smooth bounded pseudoconvex domain of finite type

$$D^{\mathcal{H}} = \{z \in \mathbb{C}^3 / \text{Re}(z_1) + |z_1|^2 + |z_1|^{2m} + |z_2 z_3|^{2a} + |z_3|^{2m} < 0\}.$$

G. Herbort proved in [30] that there exists a vector  $v \in \mathbb{C}^3 \setminus \{0\}$  and a sequence of points  $(z^{(\nu)})_{\nu \in \mathbb{N}} \in (D^{\mathcal{H}})^{\mathbb{N}}$  such that  $H(g_B^{D^{\mathcal{H}}})(z^{(\nu)}; v) \xrightarrow{\nu \rightarrow +\infty} -\infty$ . However he noted that  $D^{\mathcal{H}}$  is not  $h$ -extendible (see [61] for a definition, a geometric characterisation and various examples of  $h$ -extendible domains), and observed that the holomorphic sectional curvatures of the Bergman metric of any  $h$ -extendible domains are bounded from below

(see also [7, 43, 58]). This motivates the study of the Bergman metric and its curvatures in such domains to answer affirmatively the following question, which generalises Question 1, for  $h$ -extendible domains:

**Question 6. Negative pinching of curvatures in  $h$ -extendible domains**

Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain with boundary of class  $C^\infty$  and let  $q \in \partial D$ . Assume that  $D$  is  $h$ -extendible at  $q$ . Does there exist a neighbourhood  $U$  of  $q$  such that

$$-\infty < \inf_{\substack{z \in D \cap U \\ v, w \in \mathbb{C}^n \setminus \{0\}}} \text{Bis}(g)(z; v, w) \leq \sup_{\substack{z \in D \cap U \\ v, w \in \mathbb{C}^n \setminus \{0\}}} \text{Bis}(g)(z; v, w) < 0,$$

where  $g$  is either the Bergman metric of  $D$  or the Kähler-Einstein metric of  $D$ ?

Once the answer to Question 5 or 6 is affirmative, we may focus on the local behaviour of the holomorphic bisectional curvatures and compare them to their analogue in model domains which is the content of Question 2. In  $h$ -extendible domains the curvatures of the Bergman metric behave like their analogue in the local model, which answers affirmatively Question 2 in this case. More precisely, if  $D \subset \mathbb{C}^n$  is  $h$ -extendible at a boundary point  $q$  and if  $D'_H$  is a local model at  $q$  (see point 1. of Remark 1.14 and also [60]), then for every nontangential cone  $\Lambda$  with vertex at  $q$  the following holds:

$$\lim_{\Lambda \ni z \rightarrow q} \sup_{v \in \mathbb{C}^n \setminus \{0\}} \left| \text{Bis}(g_B^D)(z; \Pi_{d(z, \partial D)}(v)) - \text{Bis}(g_B^{D'_H})((-1, 0, \dots, 0); v') \right| = 0, \quad (4.11)$$

where for every vector  $v \in \mathbb{C}^n \setminus \{0\}$  the vector  $v' \in \mathbb{C}^n \setminus \{0\}$  depends only on  $v$  and the geometry of  $\partial D$  at  $q$ , and for every  $\lambda > 0$  the map  $\Pi_\lambda$  is an anisotropic dilation of factor  $\lambda$  with weight depending only on the geometry (more precisely on the Catlin multitype) of  $\partial D$  at  $q$ . We refer to [7] for a more precise statement. Consequently, the only remaining question for  $h$ -extendible domains concerns the negative pinching of the holomorphic bisectional curvatures of the Bergman metric of  $D'_H$  (see Section 1.3). This is part of the following more general question:

**Question 7. Bisectional curvature estimates in model domains**

Let  $H$  be a weighted homogeneous polynomial function, plurisubharmonic in  $\mathbb{C}^{n-1}$ . Assume that the Bergman metric (respectively the Kähler-Einstein metric) of  $D'_H$  is well-defined and complete. What conditions ensure that its holomorphic bisectional curvature are negatively pinched on  $D'_H \cap (\mathbb{C} \times \{0\})^{n-1}$ ?



The study of the curvatures of invariant metrics in polynomial domains used as model domains seems to be a key point in the study of the curvatures in pseudoconvex domains. The answer to Question 7 is known only when the local model is a Thullen domain or a tube domain in  $\mathbb{C}^2$ . We expect the situation to be simpler for the Kähler-Einstein metric than for the Bergman metric. This is justified by the fact that the Kähler-Einstein metric satisfies by definition a certain curvature condition (see relation (1.11)). This is also supported by the comparison of the curvature bounds obtained for Thullen domains and tube domains (see Theorems 3.4, 4.3 and also Remark 4.9). However there is no localisation result as (4.11) for the holomorphic (bi)sectional curvatures of the Kähler-Einstein metric at  $h$ -extendible points. Nonetheless if  $D \subset \mathbb{C}^2$  and is convex, no localisation is needed and we have a partial result, see Theorem 3.3. Its proof may be adapted to every boundary point of a smoothly bounded convex domain of finite type in  $\mathbb{C}^n$ , see [25]. In fact the sequence of scaled domains converges globally to the model domain  $D'_H = \{z = (z, z') \in \mathbb{C} \times \mathbb{C}^{n-1} / Re(z) + H(z') < 0\}$  where  $H$  is a weighted homogeneous convex polynomial function. Moreover the uniform squeezing property of convex domains and the stability of the Kobayashi metric in the class of  $\mathbb{C}$ -proper convex domains in  $\mathbb{C}^n$  imply the stability of the Kähler-Einstein metric of  $D'_H$  (see [26, 57]). Therefore, the study of the non tangential behaviour of the holomorphic bisectional curvatures of the Kähler-Einstein metric at boundary points of smoothly bounded convex domains of finite type reduces to the study of the same quantities in  $D'_H$ . For a homogeneous non negative polynomial function  $H$  of degree  $2p$  ( $p \in \mathbb{N}^*$ ) which is subharmonic but not harmonic in  $\mathbb{C}$  we expressed the pinching of the curvatures of the Kähler-Einstein metric  $[g_{i\bar{j}}]$  of  $D'_H$  in terms of  $\frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0)$  (see Theorem 3.4). Proving the inequality  $\frac{|g_{22}|}{g_{2\bar{2}}}(-1, 0) < 2p$  when  $H$  is convex would suffice to answer affirmatively Question 5 (and equivalently Question 1) for the Kähler-Einstein metric in the case of smoothly bounded convex domains of finite type in  $\mathbb{C}^2$ . Incidentally, this would give a characterisation of the finite type for bounded convex domains in  $\mathbb{C}^2$  with smooth boundary (see [8, Theorem 1.2.]).

Going back to the case of a smoothly bounded pseudoconvex domain, one may ask whether the holomorphic bisectional curvatures of the Bergman metric or of the Kähler-Einstein metric satisfy a localisation property as obtained in [37] (see also 4.10). For the Bergman metric the localisation follows from the Bergman-Fuks integral formulae which

have no analogue in the case of the Kähler-Einstein metric. In general we may study:

**Question 8. Localisation of Kähler potentials**

Let  $D \subset \mathbb{C}^n$  be a bounded pseudoconvex domain with boundary of class  $\mathcal{C}^\infty$ . Let  $g^D$  (respectively  $g^{D \cap U}$ ) be the potential either of the Bergman metric or of the Kähler-Einstein metric of  $D$  (respectively of  $D \cap U$ ). Let  $q \in \partial D$  be a point of finite type. What conditions on  $q$  imply the existence of a neighbourhood  $U$  of  $q$  and of an integer  $k \geq 5$  such that

$$e^{-g^D} - e^{-g^{D \cap U}} = O_{z \rightarrow q} \left( d(z, \partial D)^k \right)?$$

Notice that a positive answer with  $k = 3$  is sufficient to localise the metrics. If one can localise the holomorphic bisectional curvatures at a boundary point  $q$ , then one may use a scaling method, and under suitable conditions prove the sequence of rescaled domains converges in the local Hausdorff topology to some model domain. Consequently, the study of the boundary behavior of the curvatures reduces to the study of the curvatures in some interior point of a model domain provided that the sequence of associated metric converges in a certain sense to the metric of the model domain. This brings to the question of the stability of the Bergman metric and of the Kähler-Einstein metric:

**Question 9. Convergence of metrics under deformation of domains**

Let  $(D_\nu)_{\nu \in \mathbb{N} \cup \{\infty\}}$  be a family of bounded complete Kobayashi hyperbolic domains. Let  $\left( \left[ g_{i\bar{j}}^{(\nu)} \right] \right)_{\nu \in \mathbb{N} \cup \{\infty\}}$  be either the family of complete Bergman metrics or the family of complete Kähler-Einstein metrics associated to the family  $(D_\nu)_{\nu \in \mathbb{N} \cup \{\infty\}}$ .

- Assume that  $(D_\nu)_{\nu \in \mathbb{N}}$  converges to  $D_\infty$  in the sense of the Caratheodory kernel convergence (see [29, Subsection 9.2.2]). Does  $\left( \left[ g_{i\bar{j}}^{(\nu)} \right] \right)_{\nu \in \mathbb{N}}$  converge to  $\left[ g_{i\bar{j}}^{(\infty)} \right]$  uniformly on compact sets of  $D_\infty$ ?
- More generally, what notion of convergence on the sequence  $(D_\nu)_{\nu \in \mathbb{N}}$  to  $D_\infty$  ensures the local uniform convergence of the metrics  $\left( \left[ g_{i\bar{j}}^{(\nu)} \right] \right)_{\nu \in \mathbb{N}}$ ?

For the Bergman metric, if the sequence  $(D_\nu)_{\nu \in \mathbb{N}}$  satisfies the hypothesis of Question 9 and is increasing then the sequence of associated Bergman metrics converges to the Bergman metric of  $D_\infty$ , uniformly on compact sets of  $D_\infty$ , and the same result holds for the Kähler-Einstein metrics (see [52, 12, 48]). Moreover if all the domains are convex and the sequence converges to  $D_\infty$  in the sense of the local Hausdorff topology, then the sequence  $\left( \left[ g_{i\bar{j}}^{(\nu)} \right] \right)_{\nu \in \mathbb{N}}$  converge to  $\left[ g_{i\bar{j}}^{(\infty)} \right]$  uniformly on compact sets of  $D_\infty$  (see [26]). Using the convergence results obtained in [52], H. Boas, E. Straube and J. Yu studied the

convergence of Bergman kernels and curvatures for families of bumpings of local models in  $\mathbb{C}^n$  (see [7] for a precise statement and associated notions). These families of domains naturally appear when studying the behaviour of invariant metrics at  $h$ -extendible points of pseudoconvex domains.

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