Study of the large time behaviour of partial differential equations using probabilistic methods

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THESE DE DOCTORAT DE

L'UNIVERSITE DE RENNES 1
COMUE UNIVERSITE BRETAGNE LOIRE

ECOLE DOCTORALE N° 601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : *Mathématiques et leurs interactions*

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Etude du comportement en temps long d'équations aux dérivées partielles par des méthodes probabilistes

Thèse présentée et soutenue à Rennes, le 28 mai 2019
Unité de recherche : IRMAR

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Introduction

1.1 Quelques rappels sur les EDSR

1.1.1 Explications sur la forme générale des EDSR

Considérons un mouvement brownien W, k-dimensionnel, défini sur un espace probabilisé complet $(\Omega, \mathcal{F}, \mathbb{P})$, dont on note $(\mathcal{F}_t)_{t\in[0,T]}$ la filtration naturelle augmentée. Il serait alors naturel d'appeler équation différentielle stochastique rétrograde (abrégée en EDSR) le problème suivant :

$$\left\{ \begin{array}{l} \mathrm{d}Y_t = -f(t, Y_t)\,\mathrm{d}t, \quad t \in [0, T], \\ Y_T = \xi, \end{array} \right. \tag{1.1}$$

où ξ est une variable aléatoire réelle $\mathcal{F}_T$-mesurable, c'est-à-dire, une variable aléatoire réelle dont la valeur est connue lorsqu'on dispose de l'information disponible à l'instant T. Cet instant T est appelé horizon de l'EDSR (1.1). Parmi les EDSR de cette forme, on peut, par exemple, souhaiter résoudre :

$$\left\{ \begin{array}{l} \mathrm{d}Y_t = 0, \quad t \in [0, T], \\ Y_T = \xi. \end{array} \right. \tag{1.2}$$

Le processus constant $Y_t = \xi$, $t \in [0, T]$ semble être un bon candidat pour être solution de l'équation (1.2). Pourtant, il présente l'inconvénient majeur de ne pas être adapté, dès lors que la variable aléatoire terminale ξ n'est pas $\mathcal{F}_0$-mesurable. En effet, cela signifierait alors que la valeur prise par le processus Y au temps 0 dépendrait d'évènements postérieurs à cet instant. Il s'agit d'une situation qu'on souhaite éviter, et, par conséquent, on impose au processus solution d'une EDSR d'être adapté à la filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Pour considérer un processus adapté à une filtration sans modifier sa valeur terminale (au temps T), une méthode possible est de considérer son espérance conditionnellement à l'information disponible à chaque instant. Définissons donc un nouveau processus Y' par la formule :

$$Y'_t = \mathbb{E}\left[Y_t | \mathcal{F}_t\right] = \mathbb{E}\left[\xi | \mathcal{F}_t\right].$$

On souhaiterait alors pouvoir différentier temporellement Y'_t ; mais ceci n'est pas immédiatement possible, au vu de sa définition. Néanmoins, le théorème de représentation des martingales browniennes de carré intégrable nous fournit l'existence d'un processus $(Z_t)_{t\in[0,T]}$, à valeurs dans $\mathbb{R}^{m\times k}$, prévisible par rapport à la filtration $(\mathcal{F}_t)_{t\in[0,T]}$, et tel que :

$$\mathbb{E}\left[\xi | \mathcal{F}_t\right] = \mathbb{E}[\xi] + \int_0^t Z_s\,\mathrm{d}W_s,$$

si on suppose préalablement que $\mathbb{E}\left[\xi^2\right] < \infty$. Ainsi, le couple $(\mathbb{E}\left[\xi | \mathcal{F}_t\right], Z_t)$ résout l'EDSR :

$$\left\{ \begin{array}{l} \mathrm{d}Y_t = Z_t\,\mathrm{d}W_t, \quad t \in [0, T], \\ Y_T = \xi. \end{array} \right. \tag{1.3}$$

Cette difficulté liée à l'adaptabilité de la solution rend donc naturel le fait d'appeler EDSR toute équation du type :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, \mathrm{d}s - \int_t^T Z_s \,\mathrm{d}W_s, \quad t \in [0,T], \tag{1.4}$$

où la fonction $f : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ est mesurable pour les tribus $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^{m\times k})$ et $\mathcal{B}(\mathbb{R}^m)$, $\mathcal{P}$ désignant la tribu des évènements prévisibles. On dit que la fonction f est le générateur de l'EDSR (1.4).

Définition 1.1. On appelle solution de l'EDSR (1.4) tout couple de processus (Y,Z) à valeurs dans $\mathbb{R}^m \times \mathbb{R}^{m\times k}$, où Y est continu et adapté, où Z est prévisible, et tel que $\mathbb{P}$-p.s., pour tout $t \in [0,T]$, on a :

$$\int_0^T \left[|f(s, Y_s, Z_s)| + |Z_s|^2 \right] \mathrm{d}s < \infty \quad \text{et} \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, \mathrm{d}s - \int_t^T Z_s \,\mathrm{d}W_s.$$

1.1.2 Théorème fondamental

La première apparition des EDSR dans la littérature remonte à un article de Bismut [7], dans lequel était traité le cas des EDSR à générateur linéaire. Toutefois, le résultat fondamental dans la théorie des EDSR a été prouvé par Pardoux et Peng dans leur article [49], et permet de traiter le cas d'équations dont le générateur n'est pas linéaire, mais seulement lipschitzien en y et en z.

Théorème 1.2 (Pardoux, Peng - 1990)**.** *Supposons le générateur f lipschitzien par rapport à (y,z), uniformément en (t,ω), et*

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(s,0,0)|^2 \,\mathrm{d}s \right] < \infty. \tag{1.5}$$

Alors l'EDSR (1.4) admet une unique solution telle que le processus Z soit de carré intégrable.

On constate une grande proximité entre les hypothèses du théorème 1.2 et celles du théorème de Cauchy-Lipschitz pour la résolution globale d'équations différentielles ordinaires. La preuve qui en est faite dans [49] est principalement basée sur l'utilisation du lemme de Gronwall. Néanmoins, il est possible d'utiliser un argument de point fixe, en considérant un espace complet adéquat (voir à ce propos l'article [25] d'El Karoui, Peng et Quenez en 1997).

Par la suite, ce résultat a pu être amélioré par divers affaiblissements d'hypothèses. Par exemple, dans [52], en utilisant notamment des arguments de point fixe et le lemme de Gronwall, Pardoux remplace dans les hypothèses le fait que le générateur soit lipschitzien en y par une hypothèse de continuité en y et la condition de monotonie suivante :

$$\langle y - y', f(t,y,z) - f(t,y',z) \rangle \leqslant -\mu |y - y'|^2, \quad \mathbb{P}\text{-p.s.}, \tag{1.6}$$

où μ est un nombre réel indépendant de t, y, y' et z. L'approche de Pardoux a ensuite été généralisée au cas L^p, avec $p > 1$, dans [8], c'est-à-dire, lorsqu'on remplace la condition (1.5) par :

$$\mathbb{E}\left[|\xi|^p + \int_0^T |f(s,0,0)|^p \,\mathrm{d}s \right] < \infty.$$

Il est à noter que le cas $p = 1$ est également étudié dans [8].

D'autres améliorations ont pu être menées dans le cas unidimensionnel ($k = 1$). En effet, quand Y est un processus à valeurs réelles, on dispose du théorème de comparaison, dont la première apparition remonte à [55].

Théorème 1.3 (Théorème de comparaison - Peng - 1992)**.** *Soient (ξ^1, f^1) et (ξ^2, f^2) deux couples vérifiant les hypothèses du théorème 1.2 ; on désigne par (Y^1, Z^1) et (Y^2, Z^2) les solutions associées à ces jeux de données. Si on suppose que :*

- $\xi^1 \geqslant \xi^2 \quad \mathbb{P}$*-p.s. ;*
- $f^1\left(t, Y_t^2, Z_t^2\right) \geqslant f^2\left(t, Y_t^2, Z_t^2\right) \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}t$*-p.p. ;*

alors $\mathbb{P}$-p.s., pour tout $t \in [0,T]$, $Y_t^1 \geqslant Y_t^2$. De plus, la comparaison est stricte : si $Y_0^1 = Y_0^2$, alors $Y^1 = Y^2$ $\mathbb{P}$-p.s.

À l'aide du théorème de comparaison, Lepeltier et San Martín construisent dans [39] des solutions maximales et minimales dans le cas où le générateur est supposé uniquement continu et à croissance linéaire selon y et z. Aussi, c'est ce résultat qui a permis l'émergence de la théorie des EDSR quadratiques. On appelle EDSR quadratiques les EDSR dont le générateur est supposé non plus lipschitzien en z mais à croissance quadratique en z. Les premiers travaux dans ce domaine sont dus à Kobylanski [38]. N'ayant pas travaillé sur ce sujet, le lecteur est renvoyé vers la thèse [31] pour une introduction détaillée à ce domaine.

1.1.3 Passage à l'horizon infini

Une autre source d'enrichissements du théorème 1.2 de Pardoux et Peng a consisté à s'intéresser à l'horizon T de l'EDSR.

En utilisant l'hypothèse de monotonie (1.6), Peng s'intéresse dans [54] à une EDSR d'horizon aléatoire, c'est-à-dire une EDSR dont la condition terminale est $Y_\tau = \xi$, où τ est un temps d'arrêt ; autrement dit :

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\,\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s\,\mathrm{d}W_s, \quad t \in [0,T]. \tag{1.7}$$

Les travaux de Peng sur les EDSR d'horizon aléatoire ont été ensuite poursuivis par ceux de Darling et Pardoux [21]. En supposant que le générateur satisfait l'hypothèse de monotonie (1.6), qu'il est lipschitzien en z, continu en y et qu'il vérifie une majoration du type :

$$\forall t \in \mathbb{R}_+, \forall y \in \mathbb{R}^m, \forall z \in \mathbb{R}^{m\times k}, \; |f(t,y,z)| \leqslant f_t + C(|y| + |z|),$$

où la constante C et le processus positif $(f_t)_{t\geqslant 0}$ vérifient les conditions suivantes :

$$\mathbb{E}\left[|\xi|^2 + \int_0^\tau f_t^2\,\mathrm{d}t\right] < \infty \quad \text{et} \quad \mu > \frac{C^2}{2},$$

ils montrent que l'EDSR :

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\,\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s\,\mathrm{d}W_s, \quad t \geqslant 0, \tag{1.8}$$

admet une unique solution telle que Z est un processus de carré intégrable sur $[0,\tau]$.

Mentionnons également le résultat de Briand et Hu [9], obtenu dans le cas unidimensionnel. En supposant que $\xi = 0$, que f est uniformément lipschitzienne en y et z, et monotone en y et que l'application $f(\bullet, 0, 0)$ est bornée, ils montrent que l'EDSR (1.8) admet une unique solution telle que Y est continu borné et Z est de carré localement intégrable sur $[0,\tau]$. Il est à souligner que, dans leurs travaux, Briand et Hu supposent seulement $\mu > 0$, hypothèse de monotonie plus faible que dans [21]. Par la suite, ceci a été généralisé par Royer [57].

En prenant une condition terminale nulle et un horizon infini, on obtient l'EDSR suivante :

$$Y_t = \int_t^{+\infty} f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^{+\infty} Z_s\,\mathrm{d}W_s, \quad t \geqslant 0. \tag{1.9}$$

Par soustraction, la formulation (1.9) équivaut à :

$$Y_t = Y_T + \int_t^{T} f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^{T} Z_s\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < \infty, \tag{1.10}$$

et cette dernière écriture permet notamment de n'avoir à évoquer qu'une intégrabilité locale pour le processus $|Z|^2$, à la manière du résultat de Briand et Hu [9].

1.1.4 EDSR markoviennes

Un autre champ largement étudié est celui des EDSR markoviennes. Ici, le générateur et la condition terminale ne dépendent de l'aléa ω que par le biais d'un processus solution d'une EDS. Considérons l'EDS issue de x à l'instant t :

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\,\mathrm{d}W_r, \quad s \geqslant t. \tag{1.11}$$

Pourvu que b et σ soient globalement lipschitziens, on sait que cette EDS admet une unique solution (à ce propos, voir par exemple [19]). En se donnant deux fonctions déterministes f et g, on souhaite considérer l'EDSR :

$$Y_s^{t,x} = g\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\,\mathrm{d}r - \int_s^T Z_r^{t,x}\,\mathrm{d}W_r, \quad s \in [t,T]. \tag{1.12}$$

Pardoux et Peng se sont intéressés à ces EDSR dans [50]. La qualification d'EDSR markovienne provient du fait qu'en utilisant l'unicité de la solution à l'EDSR, et la propriété de Markov vérifiée par le processus X, la solution de (1.12) vérifie les relations :

$$Y_s^{t,X_t^{r,x}} = Y_s^{r,x} \quad \text{et} \quad Z_s^{t,X_t^{r,x}} = Z_s^{r,x}.$$

Par ailleurs, le système formé par les équations (1.11) et (1.12) est dit découplé, car les processus Y et Z n'apparaissent pas dans l'EDS ; le cas de systèmes couplés a également été étudié (voir l'article de Ma, Protter et Yong [42]). Les EDSR markoviennes sont également étudiées en horizon infini (voir par exemple [36]).

1.1.5 Des motivations diverses pour l'étude des EDSR

Les différentes applications des EDSR (résolution d'EDP, de problèmes de contrôle stochastique ou de mathématiques financières) ont contribué à apporter davantage d'intérêt à l'étude des EDSR depuis le théorème fondamental de Pardoux et Peng [49].

Résolution d'EDP de Hamilton-Jacobi-Bellman *via* la formule de Feynman-Kac

Considérons l'EDP linéaire :

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + K(t,x)u(t,x) = 0, & t \in [0,T],\ x \in \mathbb{R}^d, \\ u(T,x) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.13}$$

où $\mathcal{L}$ est un opérateur différentiel du second ordre, agissant sur les fonctions suffisamment régulières par la formule :

$$\mathcal{L}h(t,x) := \frac{1}{2}\mathrm{tr}\left(\sigma(t,x)\sigma(t,x)^*\nabla^2 h(t,x)\right) + \langle b(t,x), \nabla h(t,x)\rangle, \tag{1.14}$$

et où $b : \mathbb{R}^d \to \mathbb{R}^d$ et $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$. Par la formule d'Itô, l'opérateur différentiel $\mathcal{L}$ est relié à l'EDS :

$$X_s^{t,x} = x + \int_t^s b\left(r, X_r^{t,x}\right)\,\mathrm{d}r + \int_t^s \sigma\left(r, X_r^{t,x}\right)\,\mathrm{d}W_r. \tag{1.15}$$

Alors, si les entrées b, σ, K et g vérifient certaines hypothèses, on peut montrer que la fonction :

$$u : (t,x) \mapsto \mathbb{E}\left[\mathrm{e}^{\int_t^T K\left(r, X_r^{t,x}\right)\mathrm{d}r} g\left(X_T^{t,x}\right)\right] \tag{1.16}$$

est solution classique de l'EDP (1.13). Ce résultat a été démontré bien avant le développement de la théorie des EDSR (voir [37]).

Pardoux et Peng [50] font le lien entre les EDSR markoviennes et les EDP semi-linéaires paraboliques du type :

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),(\nabla u.\sigma)(t,x)) = 0, & t \in [0,T],\ x \in \mathbb{R}^d, \\ u(T,x) = g(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.17}$$

Théorème 1.4 (Pardoux, Peng - 1992). *Si $u \in \mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$ est solution de l'EDP (1.17), alors, pour tout $(t,x) \in [0,T] \times \mathbb{R}^d$, le couple $\left(Y_s^{t,x}, Z_s^{t,x}\right) := \left(u\left(s, X_s^{t,x}\right), (\nabla u.\sigma)\left(s, X_s^{t,x}\right)\right)$ est solution de l'EDSR :*

$$Y_s = g\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r, Z_r\right) \mathrm{d}r - \int_s^T Z_r \,\mathrm{d}W_r, \quad s \in [t,T], \tag{1.18}$$

où X est la diffusion de générateur infinitésimal $\mathcal{L}$ (voir (1.15)). Réciproquement, sous de bonnes hypothèses de régularité sur l'ensemble des paramètres, la fonction $u : (t,x) \mapsto Y_t^{t,x}$ est une fonction déterministe et est solution classique de l'EDP (1.17).

Remarque 1.5. On retrouve effectivement la formule de Feynman-Kac (1.16) dans le cas où on prend pour générateur la fonction $f : (t,x,y,z) \mapsto K(t,x)y$.

Le sens direct du théorème 1.4 se prouve à l'aide de la formule d'Itô. Les hypothèses de régularité du théorème précédent peuvent être assouplies ; cela impose en contrepartie de travailler avec des solutions de viscosité, et cela nécessite d'avoir démontré au préalable un théorème de comparaison (comme le théorème 1.3).

Définition 1.6. On appelle solution de viscosité de l'EDP (1.17) toute fonction u satisfaisant :

- u est continue et vérifie $u(T, \bullet) = g$;
- pour toute fonction $\phi \in \mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$, et pour tout point (t_0, x_0) de maximum local de $u - \phi$, on a

$$\partial_t \phi\,(t_0,x_0) + \mathcal{L}\phi\,(t_0,x_0) + f\left(t_0, x_0, u\,(t_0,x_0), (\nabla\phi.\sigma)\,(t_0,x_0)\right) \geqslant 0;$$

- pour toute fonction $\phi \in \mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$, et pour tout point (t_0, x_0) de minimum local de $u - \phi$, on a

$$\partial_t \phi\,(t_0,x_0) + \mathcal{L}\phi\,(t_0,x_0) + f\left(t_0, x_0, u\,(t_0,x_0), (\nabla\phi.\sigma)\,(t_0,x_0)\right) \leqslant 0.$$

La justification de cette définition repose sur le principe du maximum ; on renvoie à [2] pour plus de détails. L'obtention d'une formule de Feynman-Kac est une problématique présente dans la plupart des articles traitant d'EDSR déjà mentionnés.

Application à un problème de contrôle stochastique

Les problèmes de contrôle stochastique (voir par exemple [27]) consistent généralement en la minimisation du coût d'un processus solution d'une EDS sur laquelle on a la possibilité d'exercer un contrôle. Plus précisément, considérons l'EDS suivante :

$$X_t^{x,a} = x + \int_0^t \left[b\left(s, X_s^{x,a}\right) + R\left(s, X_s^{x,a}, a_s\right)\right] \mathrm{d}s + \int_0^t \sigma\left(s, X_s^{x,a}\right) \mathrm{d}W_s, \quad t \in [0,T], \tag{1.19}$$

où le contrôle qu'on peut exercer se traduit par le processus $(a_t)_{t \in [0,T]}$ prenant ses valeurs dans un espace métrique séparable U, et où R est la fonction traduisant l'impact de notre contrôle au sein de l'EDS. On définit une fonction appelée coût (et qu'on souhaite donc minimiser) :

$$J(x,a) = \mathbb{E}\left[g\left(X_T^{x,a}\right) + \int_0^T L\left(s, X_s^{x,a}, a_s\right) \mathrm{d}s\right].$$

Dans le cas où σ ne prend que des valeurs inversibles, on définit l'hamiltonien de ce problème de contrôle stochastique par :

$$f(t,x,z) = \inf_{a \in U} \left\{L(t,x,a) + z\sigma(t,x)^{-1}R(s,x,a)\right\}.$$

On considère alors le processus X^x, solution de l'EDS (1.19) avec $R \equiv 0$, et l'EDSR markovienne générée par l'hamiltonien f et de condition terminale définie par g :

$$Y_t^x = g\left(X_T^x\right) + \int_t^T f\left(s, X_s^x, Z_s^x\right) \mathrm{d}s - \int_t^T Z_s^x \,\mathrm{d}W_s, \quad t \in [0, T]. \tag{1.20}$$

En remarquant que Y_0^x est égal à son espérance (car non-aléatoire), on montre facilement que $Y_0^x \leqslant J(x, a)$ pour tout contrôle a. De plus, à l'aide de quelques hypothèses supplémentaires, le lemme de Filippov (voir [45]) permet de montrer l'existence d'un contrôle optimal $\bar{a}$ vérifiant $J(x, \bar{a}) = Y_0^x$, et exprimer $\bar{a}_t$ comme fonction déterministe de X_t^x et Z_t^x.

Avant l'émergence des EDSR, et dans un cadre plus restreint, de tels problèmes avaient déjà été étudiés à l'aide d'EDS (voir à ce titre [1]).

Application en mathématiques financières

Considérons ici un investisseur souhaitant répartir sa richesse parmi différents actifs. Le marché auquel il a accès est constitué :

- d'une obligation sans risque, de rendement r_t et de valeur unitaire P_t^0, satisfaisant l'EDS :

$$\mathrm{d}P_t^0 = r_t P_t^0 \,\mathrm{d}t;$$

- de k actions, cotées en continu, et de valeurs unitaires notées $P_t^1, \ldots, P_t^k$ satisfaisant, pour tout $1 \leqslant i \leqslant k$:

$$\mathrm{d}P_t^i = P_t^i \left(b_t^i \,\mathrm{d}t + \sum_{j=1}^k \sigma_t^{i,j} \,\mathrm{d}W_t^j \right).$$

Les processus r, b et σ sont supposés bornés et prévisibles, et, par ailleurs, la fonction $t \mapsto \sigma_t^{-1}$ est bornée. On suppose également que l'investisseur est suffisamment petit pour que ses transactions n'affectent pas le marché. On note V_t la valeur totale de ses investissements à la date t et, pour tout $1 \leqslant i \leqslant k$, π_t^i désigne la somme investie sur l'action i. Enfin, la somme investie sur l'obligation est notée ρ_t. On suppose que l'investisseur n'effectue ni retrait ni versement (stratégie auto-financée) et sa richesse vérifie donc l'équation :

$$\mathrm{d}V_t = \frac{\rho_t}{P_t^0} \,\mathrm{d}P_t^0 + \sum_{i=1}^n \frac{\pi_t^i}{P_t^i} \,\mathrm{d}P_t^i.$$

Dans [25], les auteurs montrent que le processus V vérifie alors l'EDS :

$$\mathrm{d}V_t = r_t V_t \,\mathrm{d}t + \pi_t^* \sigma_t (\theta_t \,\mathrm{d}t + \mathrm{d}W_t), \tag{1.21}$$

où le processus θ est appelé prime de risque et vérifie $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$ p.p. l'égalité $\theta_t = \sigma_t^{-1}(b_t - r_t \mathbb{1})$, $\mathbb{1}$ étant le vecteur de $\mathbb{R}^k$ dont toutes les composantes valent 1.

Définition 1.7. On appelle stratégie auto-financée tout couple (V, π) satisfaisant $\mathbb{P}$-p.s. :

$$\mathrm{d}V_t = r_t V_t \,\mathrm{d}t + \pi_t^* \sigma_t (\theta_t \,\mathrm{d}t + \mathrm{d}W_t) \quad \text{et} \quad \int_0^T |\pi_t^* \sigma_t|^2 \,\mathrm{d}t < \infty.$$

Cette stratégie est dite réplicable lorsque la richesse V_t est $\mathbb{P}$-p.s. positive ou nulle à chaque instant.

Une option européenne est un contrat qui paie un montant ξ à une date de maturité fixée T (où ξ est une variable aléatoire $\mathcal{F}_T$-mesurable). La valeur d'une telle option est fixée suivant le principe suivant : si on investit la valeur initiale de l'option parmi les actifs proposés, le montant du portefeuille à l'instant T doit être juste suffisant pour garantir le paiement de ξ.

Définition 1.8. On appelle stratégie de couverture de ξ toute stratégie auto-financée réplicable (V, π), et telle que $V_T = \xi$. On note $\mathcal{H}(\xi)$ l'ensemble des stratégies de couverture de ξ ; si cet ensemble est non-vide, on dit alors que ξ est couvrable. Le prix équitable de l'option européenne payant ξ à maturité T est défini, lorsque ξ est couvrable, par :

$$X_t = \inf \{x \geqslant 0 | \exists (V, \pi) \in \mathcal{H}(\xi), V_t = x\} .$$

Avant même le théorème fondamental de Pardoux et Peng en 1990, il était possible de résoudre des EDSR linéaires d'horizon fini, ce qui menait au résultat qui suit (démontré dans [25]).

Proposition 1.9. *Soit ξ une variable aléatoire positive, $\mathcal{F}_T$-mesurable et de carré intégrable. Alors, il existe une unique stratégie de couverture de ξ, notée (X, π), c'est-à-dire vérifiant :*

$$\mathrm{d}X_t = (r_t X_t + \pi_t^* \sigma_t \theta_t)\,\mathrm{d}t + \pi_t^* \theta_t\,\mathrm{d}W_t \quad \textit{et} \quad X_T = \xi.$$

Le processus X désigne à chaque instant le prix équitable de l'option européenne payant ξ à maturité T.

Par la suite, ce modèle a été enrichi. Par exemple, Cvitanic and Karatzas [18] ont considéré le cas d'un investisseur ayant le droit d'emprunter de l'argent à chaque instant à un taux R_t. Ce processus R est supposé prévisible et borné ; par ailleurs, $R_t > r_t$, ce qui traduit le fait qu'il n'est pas raisonnable d'emprunter de l'argent pour l'investir sur l'obligation. En conséquence, la stratégie (V, π) satisfait alors l'équation :

$$\mathrm{d}V_t = r_t V_t\,\mathrm{d}t + \pi_t^* \sigma_t (\theta_t\,\mathrm{d}t + \mathrm{d}W_t) - (R_t - r_t) \left(V_t - \sum_{i=1}^{n} \pi_t^i \right)^{-} \mathrm{d}t,$$

le montant emprunté à l'instant t étant $\left(V_t - \sum_{i=1}^{n} \pi_t^i\right)^{-}$.

Le résultat suivant est énoncé dans le cadre présenté dans l'article de Cvitanic et Karatzas ; et il se prouve à l'aide du théorème de résolution des EDSR non-linéaires de Pardoux et Peng.

Proposition 1.10. *Soit ξ une variable aléatoire positive, $\mathcal{F}_T$-mesurable et de carré intégrable. Alors, il existe une unique stratégie de couverture de ξ, notée (X, π), c'est-à-dire vérifiant :*

$$\mathrm{d}X_t = \left(r_t X_t + \pi_t^* \sigma_t \theta_t + \left(X_t - \sum_{i=1}^{n} \pi_t^i \right)^{-} \right) \mathrm{d}t + \pi_t^* \theta_t\,\mathrm{d}W_t \quad \textit{et} \quad X_T = \xi.$$

Le processus X désigne à chaque instant le prix équitable de l'option européenne payant ξ à maturité T, l'émetteur de l'option étant autorisé à emprunter de l'argent au taux R_t.

1.2 EDSR ergodiques

On a déjà évoqué en section 1.1.3 les EDSR d'horizon infini. Une hypothèse cruciale dans l'étude de ces EDSR est la condition de monotonie (1.6). Évidemment, lorsque le générateur f ne dépend plus de y (comme c'est le cas, par exemple, pour l'EDSR (1.20) visant à traiter le problème de contrôle stochastique d'horizon fini), cette condition de monotonie n'est plus vérifiée. Les EDSR ergodiques, introduites par Fuhrman, Hu et Tessitore [26], font apparaître une nouvelle inconnue λ, appelée constante d'ergodicité, et sont de la forme :

$$Y_t = Y_T + \int_t^T [f(X_s, Z_s) - \lambda] \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < \infty. \tag{1.22}$$

La solution d'une EDSR ergodique est donc un triplet (Y, Z, λ) ; l'apparition de l'inconnue λ peut-être conçue comme un relâchement compensatoire à la perte de monotonie en y de f.

Sous certaines hypothèses, Fuhrman, Hu et Tessitore font le lien entre ces EDSR ergodiques, et des EDP ergodiques :

$$\mathcal{L}v(x) + f(x, \nabla v(x)\sigma) = \lambda, \quad x \in E, \tag{1.23}$$

d'inconnue (v, λ), et où l'opérateur différentiel $\mathcal{L}$ a été défini en (1.14). Notamment, ils construisent une solution *mild* à cette EDPE à partir d'une solution de l'EDSRE (1.22), et réciproquement, obtiennent une solution de l'EDSRE à l'aide d'une solution de l'EDPE. La notion de solution *mild* (faisant intervenir une fonction seulement Gateaux-différentiable comme solution à un opérateur différentiel du second ordre) en dimension infinie peut être remplacée en dimension finie par celle de solution de viscosité, déjà abordée dans la définition 1.6.

Dans [26] sont également évoquées des questions d'unicité. On peut déjà remarquer que le processus Y peut varier librement d'une constante et rester solution de (1.22) ; tout comme la fonction v dans (1.23), cette équation ne faisant apparaître que les dérivées de v, et jamais la fonction v en elle-même. Ainsi, sous de bonnes hypothèses, il est possible de montrer l'unicité de la constante λ et l'unicité à une constante près du processus Y (et en parallèle, l'unicité à une constante près de la fonction v).

Les EDSR ergodiques permettent également de traiter des problèmes de contrôle stochastique ergodique, consistant à minimiser une fonction coût de la forme :

$$J(x, a) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T L(X_s^a, a_s) \, \mathrm{d}s \right].$$

Plus tard, Hu, Madec et Richou [35], ont montré l'utilité des EDSR ergodiques dans la compréhension du comportement en temps long d'EDSR d'horizon fini.

Les EDSR ergodiques sont également utilisées pour leurs applications en mathématiques financières (voir à ce propos [13], [33] ou encore [41]).

1.2.1 Résultats connus

Le premier article portant sur les EDSR ergodiques est celui de Fuhrman, Hu et Tessitore [26]. L'EDS sous-jacente est la suivante :

$$\begin{cases} \mathrm{d}X_t = AX_t \, \mathrm{d}t + F(X_t) \, \mathrm{d}t + G \, \mathrm{d}W_t, & t \geqslant 0, \\ X_0 = x, \end{cases} \tag{1.24}$$

où le processus solution prend ses valeurs dans un espace de Banach E, où A est le générateur d'un semi-groupe de contractions fortement continu $\left(\mathrm{e}^{tA}\right)_{t \geqslant 0}$, F est une fonction mesurable bornée, G désigne un opérateur linéaire borné et W est un processus de Wiener.

La dérive doit vérifier une hypothèse assez forte : pour un certain $\eta > 0$, $A + F + \eta \mathrm{Id}$ est un opérateur dissipatif ; sans rentrer dans le détail de la définition de la dissipativité d'opérateurs sur les espaces de Banach généraux, ceci signifie, dans le cas particulier $E = \mathbb{R}^d$ que

$$\forall x, x' \in E, \ \langle x - x', A(x - x') + F(x) - F(x') \rangle \leqslant -\eta |x - x'|^2.$$

Ceci permet alors de montrer que les trajectoires prises par X se resserrent à vitesse exponentielle :

$$\forall t \geqslant 0, \forall x, x' \in E, \left| X_t^x - X_t^{x'} \right| \leqslant \mathrm{e}^{-\eta t} |x - x'|. \tag{1.25}$$

Puis, ils introduisent l'EDSR auxiliaire d'horizon infini :

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [f(X_s^x, Z_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}] \,\mathrm{d}s - \int_t^T Z_s^{x,\alpha} \,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < \infty, \tag{1.26}$$

où la fonction f est supposée lipschitzienne par rapport à x et z et où $f(\bullet, 0)$ est supposée bornée.

L'EDSR (1.26) est donc une EDSR d'horizon infini à générateur monotone en y. Ce type d'EDSR a déjà été étudié par Briand et Hu [9], puis par Royer [57]. On sait donc qu'elle admet une unique solution telle que $Y^{x,\alpha}$ est un processus continu borné et $Z^{x,\alpha}$ est localement de carré intégrable. De plus, on dispose d'une borne explicite sur $Y^{x,\alpha}$: $\frac{\|f(\bullet,0)\|}{\alpha}$. Ils définissent la fonction $v^\alpha : x \mapsto Y_0^{x,\alpha}$, puis, en utilisant la continuité de v^α et la structure markovienne de l'EDSR, ils obtiennent la relation $Y_t^{x,\alpha} = v^\alpha(X_t^x)$.

L'étape suivante de leur raisonnement consiste à montrer que les fonctions v^α sont uniformément lipschitziennes (et forment donc, par conséquent, une famille équicontinue). Pour ce faire, après un changement de mesure de probabilité adéquat, et après avoir calculé $\mathrm{d}(\mathrm{e}^{-\alpha t}|Y_t^x - Y_t^{x'}|)$, ils montrent que, sous une certaine mesure de probabilité $\widetilde{\mathbb{P}}$, pour tout $T > 0$,

$$\left| Y_0^x - Y_0^{x'} \right| \leqslant \mathrm{e}^{-\alpha T} \widetilde{\mathbb{E}} \left[\left| Y_T^x - Y_T^{x'} \right| \right] + \widetilde{\mathbb{E}} \left[\int_0^T \mathrm{e}^{-\alpha s} \left| f(X_s^x, Z_s^{x,\alpha}) - f\left(X_s^{x'}, Z_s^{x,\alpha}\right) \right| \mathrm{d}s \right],$$

ce qui permet de conclure en faisant tendre T vers l'infini, en utilisant le fait que f est lipschitzien selon x et le resserrement exponentiel des trajectoires de X.

Par une extraction diagonale, il est alors possible de construire une suite $(\alpha_n)_n$ de limite nulle, telle que :

$$\overline{v}^{\alpha_n}(x) := v^{\alpha_n}(x) - v^{\alpha_n}(0) \underset{n \to \infty}{\longrightarrow} \overline{v}(x) \quad \text{et} \quad \alpha_n v^{\alpha_n}(0) \underset{n \to \infty}{\longrightarrow} \overline{\lambda},$$

pour un réel $\overline{\lambda}$ convenable et une certaine fonction $\overline{v} : E \to \mathbb{R}$. Réécrivons alors l'équation (1.26) sous la forme :

$$\overline{v}^{\alpha_n}(X_t^x) = \overline{v}^{\alpha_n}(X_T^x) + \int_t^T [f(X_s^x, Z_s^{x,\alpha_n}) - \alpha_n \overline{v}^{\alpha_n}(X_s^x) - \alpha_n v^{\alpha_n}(0)] \,\mathrm{d}s - \int_t^T Z_s^{x,\alpha_n} \,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < \infty. \tag{1.27}$$

En utilisant la formule d'Itô, les auteurs montrent alors que le processus Z^{x,α_n} admet lui aussi une limite et, par convergence dominée, ils obtiennent :

$$\overline{v}(X_t^x) = \overline{v}(X_T^x) + \int_t^T \left[f(X_s^x, Z_s^x) - \overline{\lambda} \right] \mathrm{d}s - \int_t^T Z_s^x \,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < \infty, \tag{1.28}$$

et ont donc construit une solution à l'EDSR ergodique (1.22).

Par la suite, Debussche, Hu et Tessitore [22] ont cherché à affaiblir l'hypothèse de dissipativité. Considérant toujours l'EDS sous-jacente (1.11) (à valeurs cette fois dans un espace de Hilbert H), ils ont supposé que seul $A + \eta\mathrm{Id}$ était dissipatif (pour un certain $\eta > 0$), et que F était lipschitzien et borné. L'opérateur $A + F$ est alors dit faiblement dissipatif, et on perd le resserrement exponentiel des trajectoires (1.25). Pour y remédier, les auteurs établissent un *basic coupling estimate*, par des arguments de couplage :

$$\left| \mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](x') \right| \leqslant C \left(1 + |x|^2 + |x'|^2 \right) \mathrm{e}^{-\nu t}, \tag{1.29}$$

où ϕ est une fonction mesurable bornée quelconque et où $\mathcal{P}$ désigne le semigroupe de Kolmogorov de X, c'est-à-dire, $\mathcal{P}_t[\phi](x) = \mathbb{E}[\phi(X_t^x)]$. Par ailleurs, ils expliquent qu'il n'est pas nécessaire de supposer que F soit lipschitzien : il suffit que F soit borné et limite simple d'une suite de fonctions lipschitziennes bornées par une constante commune pour conserver un tel résultat. Dans ce cas, l'EDS (1.11) admet seulement une solution faible (si l'espace de Hilbert H est de dimension finie) ou une solution de martingale (si H est de dimension infinie). Le *basic coupling estimate* devient : si (X^x, W^x) et $(X^{x'}, W^{x'})$ sont deux solutions de martingales sous des probabilités respectives $\mathbb{P}^x$ et $\mathbb{P}^{x'}$, alors :

$$\left|\mathbb{E}^x\left[\phi\left(X_t^x\right)\right] - \mathbb{E}^{x'}\left[\phi\left(X_t^{x'}\right)\right]\right| \leqslant C\left(1+|x|^2+|x'|^2\right)\mathrm{e}^{-\nu t}.$$

Le *basic coupling estimate* permet de montrer (sans avoir à supposer que le générateur f soit lipschitzien en x) que les fonctions $v^\alpha : x \mapsto Y_0^{x,\alpha}$ vérifient :

$$\exists C > 0,\ \forall \alpha > 0,\ \forall x, x' \in H,\ \left|v^\alpha(x) - v^\alpha(x')\right| \leqslant C\left(1+|x|^2+|x'|^2\right). \tag{1.30}$$

Il est alors possible (moyennant quelques hypothèses supplémentaires : inversiblité de G et Gateaux-différentiabilité de F), par une formule de Bismut-Elworthy (voir [28]), d'en déduire une borne sur la différentielle de Gateaux de v^α, puis, après utilisation du théorème des accroissements finis, d'obtenir :

$$\exists C > 0,\ \forall \alpha > 0,\ \forall x, x' \in H,\ \left|v^\alpha(x) - v^\alpha(x')\right| \leqslant C\left(1+|x|^2+|x'|^2\right)|x-x'|. \tag{1.31}$$

La construction d'une solution à l'EDSR ergodique (1.22) est alors similaire à celle de l'article [26].

Sous des hypothèses proches (le seul changement réside dans le fait que la fonction $f(\bullet, 0)$ n'est plus supposée bornée, mais à croissance polynomiale de degré μ), Hu, Madec et Richou poursuivent l'étude des EDSR ergodiques dans [35]. Ils relient le comportement en temps long de l'EDSR d'horizon fini :

$$Y_t^{T,x} = g\left(X_T^x\right) + \int_t^T f\left(X_s^x, Z_s^{T,x}\right) \mathrm{d}s - \int_t^T Z_s^{T,x}\, \mathrm{d}W_s, \quad t \in [0,T], \tag{1.32}$$

où g est une fonction mesurable à croissance polynomiale de degré μ, à celui de l'EDSR ergodique :

$$Y_t^x = Y_T^x + \int_t^T \left[f\left(X_s^x, Z_s^x\right) - \lambda\right] \mathrm{d}s - \int_t^T Z_s^x\, \mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < \infty. \tag{1.33}$$

En effet, la différence entre les équations (1.32) et (1.33) s'écrit :

$$Y_0^{T,x} - Y_0^x - \lambda T = g\left(X_T^x\right) - v\left(X_T^x\right) + \int_0^T \left[f\left(X_s^x, Z_s^{T,x}\right) - f\left(X_s^x, Z_s^x\right)\right] \mathrm{d}s - \int_0^T \left[Z_s^{T,x} - Z_s^x\right] \mathrm{d}W_s. \tag{1.34}$$

Par un changement de probabilité (il est possible d'appliquer le théorème de Girsanov, car f est supposée lipschitzienne selon z), et en prenant l'espérance, ils montrent alors que :

$$\left|Y_0^{T,x} - Y_0^x - \lambda T\right| \leqslant C\left(1+|x|^\mu\right), \tag{1.35}$$

avec une constante C indépendante de T, ce qui permet de conclure sur le comportement au premier ordre de $Y_0^{T,x}$:

$$\left|\frac{Y_0^{T,x}}{T} - \lambda\right| \leqslant \frac{C\left(1+|x|^\mu\right)}{T}.$$

Pour obtenir des résultats plus fins, ils définissent une nouvelle fonction par la relation :

$$w_T(0,x) = Y_0^{T,x} - \lambda T - Y_0^x.$$

L'équation (1.35) rend licite une extraction diagonale, et donc permet d'exhiber une suite $(T_i)_i$ croissante et de limite infinie, telle que, sur un sous-ensemble dénombrable dense D de H, les fonctions w_{T_i} convergent simplement sur D vers une certaine fonction w. L'utilisation de la formule de Bismut-Elworthy fournit :

$$\left|w_T(0,x) - w_T(0,x')\right| \leqslant C\left(1+|x|^{1+\mu}+|x'|^{1+\mu}\right)|x-x'|,$$

et permet dans un premier temps d'étendre cette convergence simple à l'espace H tout entier. Ensuite, le *basic coupling estimate* :

$$\left|w_T(0,x) - w_T(0,x')\right| \leqslant C\left(1+|x|^2+|x'|^2\right)\mathrm{e}^{-\nu T},$$

permet de montrer que la fonction limite w est forcément une constante (notée L dans la suite). Le théorème d'Ascoli, le *basic coupling estimate* et un changement de probabilité permettent de passer de la convergence de la sous-suite (w_{T_i}) à celle de toute la famille (w_T), et d'obtenir le comportement au second ordre :

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to\infty}{\longrightarrow} L. \tag{1.36}$$

Enfin, les mêmes outils permettent d'obtenir le comportement au troisième ordre :

$$\left|Y_0^{T,x} - \lambda T - Y_0^x - L\right| \leqslant C\left(1+|x|^\mu\right)\mathrm{e}^{-\nu T}. \tag{1.37}$$

Ce comportement en temps long peut ensuite s'appliquer au comportement en temps long de la solution de l'EDP (1.17), qu'on peut réécrire, après changement de temps $t \leftrightarrow T-t$:

$$\begin{cases} \partial_t u_T(t,x) = \mathcal{L}u_T(t,x) + f(t,x,u_T(t,x),(\nabla u_T.G)(t,x)), & t\in[0,T],\ x\in\mathbb{R}^d, \\ u_T(0,x) = g(x), & x\in\mathbb{R}^d, \end{cases} \tag{1.38}$$

ainsi qu'à la solution du problème de contrôle stochastique présenté en section 1.1.5.

1.2.2 Résultats nouveaux

Dans le chapitre 2, nous étendons les résultats de [35] au cas où l'EDS sous-jacente (à valeurs dans un espace de dimension finie) possède un bruit multiplicatif non-borné. La démarche est similaire.

On considère l'EDS à valeurs dans $\mathbb{R}^d$:

$$X_t^x = x + \int_0^t \Xi\left(X_s^x\right)\,\mathrm{d}s + \int_0^t \sigma\left(X_s^x\right)\,\mathrm{d}W_s, \quad t\geqslant 0, \tag{1.39}$$

où la dérive Ξ est supposée lipschitzienne et faiblement dissipative ; en particulier :

$$\langle \Xi(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2;$$

et où $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ est lipschitzienne et telle que la fonction $x \mapsto \sigma(x)^{-1}$ est bornée. Notamment, σ est à croissance au plus linéaire et vérifie une inégalité du type :

$$|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2,$$

où $|\bullet|_F$ désigne la norme de Frobenius sur l'ensemble des matrices de format $d\times d$. Remarquons que la dernière majoration peut être obtenue avec un coefficient r_2 aussi proche de zéro que l'on souhaite dans le cas particulier où σ est à croissance sous-linéaire :

$$|\sigma(x)|_F \leqslant C\left(1+|x|^\alpha\right), \quad \alpha\in]0,1[,$$

en vertu de l'inégalité de Young.

Le principal changement dans l'étude de l'EDS demeure dans l'obtention du *basic coupling estimate*. En effet, la méthode de couplage telle qu'utilisée dans [22] ou [35] échoue dans le cas d'un bruit multiplicatif. Notre preuve est basée sur l'utilisation d'un théorème de mélange exponentiel démontré dans [24]. Quand r_2 est suffisamment petit par rapport à η_2, c'est-à-dire, quand la faible dissipativité de Ξ l'emporte sur la croissance linéaire de σ, nous montrons que les fonctions $x\mapsto|x|^\mu$ sont des fonctions de Lyapunov pour l'opérateur $\mathcal{L}$ défini comme le générateur infinitésimal du semigroupe de Kolmogorov associé à l'EDS (1.39). On rappelle que $\mathcal{L}$ agit sur n'importe quelle fonction ϕ de classe $\mathcal{C}^2$ par :

$$\mathcal{L}\phi(x) := \frac{1}{2}\mathrm{tr}\left(\sigma(x)\sigma(x)^*\nabla^2\phi(x)\right) + \langle\Xi(x),\nabla\phi(x)\rangle.$$

L'autre propriété nécessaire à l'application de [24] est l'irréductibilité de X ; ce résultat bien connu est par exemple démontré dans [20].

Nous résolvons ensuite l'EDSR ergodique :

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] \, ds - \int_t^T Z_s^x \, dW_s, \quad 0 \leqslant t \leqslant T < \infty, \tag{1.40}$$

où la fonction ψ est globalement lipschitzienne par rapport à $\left(x, z\sigma(x)^{-1}\right)$:

$$\left|\psi(x,z) - \psi(x',z')\right| \leqslant K_x|x - x'| + K_z|z\sigma(x)^{-1} - z'\sigma(x')^{-1}|.$$

La raison pour laquelle on ne suppose pas ψ lipschitzienne par rapport à z trouve sa justification dans l'étude du comportement en temps long, voir ci-après. À cause de ce changement par rapport à [22] ou [35], l'utilisation d'une formule de Bismut-Elworthy devient plus technique et nécessite de supposer que ψ est aussi lipschitzienne par rapport à x (hypothèse absente de [22] et [35]). Remarquons que dans le cas où σ est borné, cela revient exactement à supposer que ψ est lipschitzienne par rapport à (x,z). Par ailleurs, on suppose que $\sqrt{r_2}K_z\|\sigma^{-1}\|_\infty + \frac{1}{2}r_2 < \eta_2$ (hypothèse quantitative permettant de s'assurer que la fonction $x \mapsto |x|^2$ est une fonction de Lyapunov pour $\mathcal{L}$).

Notre méthode emploie des arguments similaires à [35] : l'étude se fait via l'EDSR auxiliaire :

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T [\psi(X_s^x, Z_s^{\alpha,x}) - \alpha Y_s^{\alpha,x}] \, ds - \int_t^T Z_s^{\alpha,x} \, dW_s, \quad 0 \leqslant t \leqslant T < \infty, \tag{1.41}$$

dont la solution vérifie les mêmes estimées et converge de la même manière. Certaines preuves nécessitent comme ingrédient supplémentaire une approximation des coefficients par des fonctions plus régulières (ou par une suite de fonctions bornées en ce qui concerne σ).

Le comportement en temps long des EDSR de la forme :

$$Y_t^{T,x} = g(X_T^x) + \int_t^T \psi\left(X_s^x, Z_s^{T,x}\right) ds - \int_t^T Z_s^{T,x} \, dW_s, \quad t \in [0,T], \tag{1.42}$$

demande, pour les second et troisième ordres, d'approximer les coefficients de l'EDS (1.39) et de l'EDSR ergodique (1.40), ce qui rend la preuve plus technique. Par ailleurs, la preuve du second ordre nécessite de pouvoir majorer une quantité du type :

$$\sup_{x \in \mathbb{R}^d} |\psi(x, \nabla u.\sigma(x)) - \psi(x, \nabla \widetilde{u}.\sigma(x))|,$$

ce qui justifie, la fonction σ n'étant pas supposée bornée, notre hypothèse sur le caractère Lipschitz de ψ. Sous de bonnes hypothèses sur la fonction g, on montre la vitesse de convergence :

$$\left|Y_0^{T,x} - \lambda T - Y_0^x - L\right| \leqslant C\left(1 + |x|^p\right) e^{-\nu T}.$$

Une fois obtenu ce résultat, il nous est possible de l'appliquer à la résolution d'un problème de contrôle stochastique, ou au comportement en temps long de la solution d'une EDP.

Ce chapitre est organisé de la manière suivante. En section 2.3, nous étudions l'EDS (1.39), dont la principale nouveauté réside dans l'obtention du *basic coupling estimate*. Dans la section 2.4, nous montrons l'existence et l'unicité de la solution à l'EDSRE (1.40). Puis, la partie 2.5 est dédiée au résultat de comportement en temps long des EDSR de la forme (1.42). En 2.6, nous évoquons deux applications : un problème de contrôle stochastique ergodique, puis le comportement en temps long de la solution de viscosité d'une EDP. Enfin, la section 2.7 est un appendice consacré au caractère lipschitzien de la fonction v^α définie à l'aide de l'EDSR auxiliaire (1.41).

1.3 EDSR à sauts

Les EDSR évoquées depuis le début de cette introduction ont toutes un point commun : leur solution fait apparaître un processus Y continu. La deuxième partie de cette thèse s'intéresse à des EDSR ergodiques à sauts ; nous mentionnons donc dans cette section ce qu'est une EDSR à sauts et quels sont les résultats principaux s'y rapportant.

Considérons un espace de Wiener-Poisson, c'est-à-dire un espace probabilisé complet, muni d'un mouvement brownien W et d'une mesure aléatoire de Poisson homogène $\mathcal{N}$ (sur $[0,T] \times \mathbb{R}^l \backslash \{0\}$), indépendante de W. De plus, on définit $(\mathcal{F}_t)_{t\in[0,T]}$ comme étant la filtration engendrée par W et $\mathcal{N}$ et augmentée de façon à satisfaire les hypothèses habituelles. Il est alors possible de définir un processus $N_t(A)$ (parfois aussi noté $N(t,A)$), pour tout $A \subset \mathbb{R}^l \backslash \{0\}$ mesurable, comme étant le nombre aléatoire de "sauts de taille A" jusqu'au temps t, c'est-à-dire, $N_t(A) = N(t,A) = \mathcal{N}([0,t] \times A)$. De plus, on définit l'intensité du processus N par la relation :

$$\nu(A) = \mathbb{E}\left[N(1,A)\right], \tag{1.43}$$

et, dès que la quantité $\nu(A)$ est finie, l'homogénéité de $\mathcal{N}$ assure que le processus :

$$(N(t,A) - t\nu(A))_{t\in[0,T]}$$

est une martingale. On appelle ν la mesure intensité de N et on définit le processus de Poisson compensé par :

$$\widetilde{N}(t,A) := N(t,A) - t\nu(A).$$

Adapté à ce cadre, le théorème de représentation des martingales nous invite à résoudre des EDSR de la forme :

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s, U_s\right) \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^l\backslash\{0\}} U_s(e)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}e), \quad t \in [0,T], \tag{1.44}$$

et dont on cherche des solutions (Y, Z, U) vérifiant :

- Y est un processus adapté, càdlàg et tel que $\mathbb{E}\left[\sup_{s\in[0,T]} |Y_s|^2\right] < \infty$;
- Z est un processus prévisible et tel que $\mathbb{E}\left[\int_0^T |Z_s|^2 \, \mathrm{d}s\right] < \infty$;
- U est un processus $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^l\backslash\{0\})$-mesurable et tel que $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}^l\backslash\{0\}} |U_s(e)|^2 \, \nu(\mathrm{d}e)\mathrm{d}s\right] < \infty$, où $\mathcal{P}$ désigne l'ensemble des évènements prévisibles sur $\Omega \times [0,T]$.

1.3.1 Résultats connus

Similairement au théorème 1.2 de Pardoux et Peng, Tang et Li [58] démontrent l'existence et l'unicité pour des équations de la forme (1.44).

Théorème 1.11 (Tang, Li - 1994). *Supposons le générateur f lipschitzien par rapport à (y,z,u), uniformément en (t,ω), et*

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(s,0,0,0)|^2 \, \mathrm{d}s\right] < \infty. \tag{1.45}$$

Alors l'EDSR (1.44) admet une unique solution.

Par la suite, d'autres résultats connus dans le cadre d'EDSR d'horizon fini sur un espace de Wiener ont été étendu aux EDSR à sauts.

Buckdahn et Pardoux [11] se sont intéressés à l'étude d'EDSR markoviennes à sauts. En se donnant des fonctions suffisamment régulières et la solution d'une EDS :

$$X_t^x = x + \int_0^t b\left(X_s^x\right)\, \mathrm{d}s + \int_0^t \sigma\left(X_s^x\right)\, \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^l\setminus\{0\}} \beta\left(X_{s^-}^x, e\right)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}e), \quad t \geqslant 0, \tag{1.46}$$

la formule d'Itô fournit une passerelle entre l'EDSR :

$$Y_t^x = g\left(X_T^x\right) + \int_t^T f\left(X_s^x, Y_s^x, Z_s^x, U_s^x\right)\, \mathrm{d}s - \int_t^T Z_s^x\, \mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^l\setminus\{0\}} U_s^x(e)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}e), \quad t \in [0,T], \tag{1.47}$$

et l'équation intégro-différentielle partielle (EIDP) :

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + f\left(x, u(t,x), (\nabla u.\sigma)(t,x), u(t, x+\beta(x,\bullet)) - u(t,x)\right) = 0, & t \in [0,T], x \in \mathbb{R}^d, \\ u(T,x) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.48}$$

et où $\mathcal{L}$ désigne le semigroupe de Kolmogorov de l'EDS (1.46), c'est-à-dire, pour tout ϕ de classe $\mathcal{C}^2$:

$$\begin{aligned} \mathcal{L}\phi(x) = \frac{1}{2}\mathrm{tr}\left(\sigma(x)\sigma(x)^*\nabla^2\phi(x)\right) + \langle b(x), \nabla\phi(x)\rangle \\ + \int_{\mathbb{R}^l\setminus\{0\}} \left[\phi(x+\beta(x,e)) - \phi(x) - \langle \beta(x,e), \nabla\phi(x)\rangle\right] \nu(\mathrm{d}e). \end{aligned}$$

Comme dans le cas des EDSR définies sur un espace de Wiener, l'étape suivante consiste à affaiblir les hypothèses de régularité et à considérer des solutions de viscosité. C'est l'objet de l'article [3] de Barles, Buckdahn et Pardoux. La preuve du théorème de comparaison qui suit est basée sur la formule d'Itô et s'obtient de la même manière que la preuve du théorème 1.3.

Théorème 1.12 (Barles, Buckdahn, Pardoux - 1997)**.** *Soit $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ une fonction $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R})$-mesurable, satisfaisant les hypothèses suivantes :*

1. $\mathbb{E}\left[\int_0^T |h(t,0,0,0)|^2\, \mathrm{d}t\right] < \infty$ *;*
2. *$q \mapsto h(t,y,z,q)$ est croissante, quel que soit (ω, t, y, z) ;*
3. *h est lipschitzienne par rapport à (y,z,q) uniformément en (t,ω).*

De plus, soit $\gamma : \Omega \times [0,T] \times \mathbb{R}^l\setminus\{0\} \to \mathbb{R}$ une fonction $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^l\setminus\{0\})$-mesurable vérifiant :

$$\forall e \in \mathbb{R}^l\setminus\{0\},\ 0 \leqslant \gamma_t(e) \leqslant C(1 \wedge |e|).$$

On pose

$$f(t,y,z,u) = h\left(t, y, z, \int_{\mathbb{R}^l\setminus\{0\}} u(e)\gamma_t(e)\, \nu(\mathrm{d}e)\right),$$

et on considère deux conditions terminales de carré intégrable ξ^1 et ξ^2. On désigne par $\left(Y^i, Z^i, U^i\right)$ la solution de l'EDSR associée au jeu de données $\left(\xi^i, f\right)$, pour $i = 1$ ou 2. Si $\xi^1 \geqslant \xi^2$ $\mathbb{P}$-p.s., alors $\mathbb{P}$-p.s., pour tout $t \in [0,T]$, $Y_t^1 \geqslant Y_t^2$.

Ce théorème de comparaison impose le fait que le générateur f doive dépendre du processus U d'une façon bien spécifique ; les auteurs de [3] fournissent un contre-exemple qui montre que ce théorème de comparaison n'est pas vrai si on autorise f à dépendre de U de façon quelconque. Dès lors, sans supposer que les entrées de l'EDSR markovienne (1.47) soient de classe $\mathcal{C}^3$, ils obtiennent l'existence et l'unicité d'une solution de viscosité, en se plaçant dans un cadre permettant l'utilisation du théorème de comparaison 1.12.

Dans leur article [53], Pardoux, Pradeilles et Rao traitent d'un cas particulier d'EDSR à sauts où la mesure intensité de N ne charge que les entiers compris entre 1 et k. Ils définissent un processus de Markov par la relation :

$$N_s^{t,n} = n + \sum_{l=1}^{k-1} l\mathcal{N}(]t,s] \times \{l\}) \quad \text{modulo } k, \quad s \geqslant t, \tag{1.49}$$

et considèrent l'EDSR markovienne à sauts :

$$\begin{aligned} Y_s^{t,x,n} = g_{N_T^{t,n}}\left(X_T^{t,x,n}\right) + \int_s^T f_{N_r^{t,n}}\left(r, X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n}, H_r^{t,x,n}\right) \mathrm{d}r - \int_s^T Z_r^{t,x,n} \,\mathrm{d}W_r \\ - \sum_{l=1}^{k-1} \int_s^T H_r^{t,x,n}(l)\, \widetilde{N}(\mathrm{d}s, \{l\}), \quad s \in [t,T], \end{aligned} \tag{1.50}$$

où le processus $X^{t,x,n}$ est solution de l'EDS :

$$X_s^{t,x,n} = x + \int_t^s b_{N_r^{t,n}}\left(r, X_r^{t,x,n}\right) \mathrm{d}r + \int_t^s \sigma_{N_r^{t,n}}\left(r, X_r^{t,x,n}\right) \mathrm{d}W_r, \quad s \geqslant t. \tag{1.51}$$

Comme à l'accoutumée, l'unicité de la solution de l'EDSR (1.50), permet de montrer que :

$$Y_s^{t,x,n} = u_{N_s^{t,n}}\left(s, X_s^{t,x,n}\right),$$

où la fonction u_n est définie par $u_n(t,x) = Y_t^{t,x,n}$; puis d'en déduire l'identification :

$$H_s^{t,x,n}(l) = u_{N_{s^-}^{t,n}+l}\left(s, X_s^{t,x,n}\right) - u_{N_{s^-}^{t,n}}\left(s, X_s^{t,x,n}\right).$$

Ensuite, à la manière du théorème 1.4, ils montrent que la famille de fonctions $(u_1, \ldots, u_k)$ est la solution de viscosité du système d'EDP couplées :

$$\begin{cases} \partial_t u_i(t,x) + \mathcal{L}^i u_i(t,x) + f_i\left(t, x, u_i(t,x), (\partial_x u_i.\sigma_i)(t,x), (u_{i+l}(t,x) - u_i(t,x))_{1\leqslant l \leqslant k-1}\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T], x \in \mathbb{R}^d, 1 \leqslant i \leqslant k, \\ u_i(T,x) = g_i(x), \qquad\qquad\qquad\qquad\qquad\quad x \in \mathbb{R}^d, 1 \leqslant i \leqslant k, \end{cases} \tag{1.52}$$

dont l'écriture peut être simplifiée moyennant le changement de variable :

$$f_i\left(t, x, y_i, z, (y_{i+l} - y_i)_{1\leqslant l\leqslant k-1}\right) = \widetilde{f}_i\left(t, x, (y_l)_{1\leqslant l \leqslant k}, z\right), \tag{1.53}$$

et où $\mathcal{L}^i$ est le générateur du semigroupe de Kolmogorov de l'EDS :

$$X_s = x + \int_t^s b_i(r, X_r)\, \mathrm{d}r + \int_t^s \sigma_i(r, X_r)\, \mathrm{d}W_r, \quad s \geqslant t. \tag{1.54}$$

1.3.2 Résultats nouveaux sur les systèmes d'EDP

Dans le chapitre 3, nous nous inspirons du cadre de Pardoux, Pradeilles et Rao [53], et nous faisons une étude similaire à celle du chapitre 2.

On considère l'EDS suivante (où le processus N^n est défini en (1.49)) :

$$X_t^{x,n} = x + \int_0^t b_{N_s^n}\left(X_s^{x,n}\right) \mathrm{d}s + \int_0^t \sigma_{N_s^n}\left(X_s^{x,n}\right) \mathrm{d}W_s, \quad t \geqslant 0, \tag{1.55}$$

où, pour tout $1 \leqslant i \leqslant k$, les fonctions b_i sont lipschitziennes et faiblement dissipatives et où les fonctions σ_i sont lipschitziennes, à valeurs inversibles et telles que les $x \mapsto \sigma_i(x)^{-1}$ sont bornées. Comme dans le chapitre 2, on dispose donc des inégalités suivantes :

$$\langle b_i(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2 \quad \text{et} \quad |\sigma_i(x)|_F^2 \leqslant r_1 + r_2|x|^2.$$

Le point important dans l'étude de l'EDS (1.55) est l'obtention d'un *basic coupling estimate*. Cloez et Hairer [14] se sont déjà intéressés à des résultats de mélange exponentiel pour des processus de Markov avec *random switching*. Ici, ce *random switching* provient du processus de Poisson N^n, indépendant de W, qui provoque ponctuellement une modification des fonctions intervenant dans l'équation suivie par le processus $X^{x,n}$. De façon résumée, leur théorème indique que si chacun des états $1 \leqslant i \leqslant k$ est récurrent pour N^n (c'est le cas : N^n est un processus de Poisson) et que s'il existe un certain état $1 \leqslant l \leqslant k$ pour lequel l'EDS non-*switchée* :

$$\widetilde{X}_t^{x,l} = x + \int_0^t b_l \left(\widetilde{X}_s^{x,l}\right) \mathrm{d}s + \int_0^t \sigma_l \left(\widetilde{X}_s^{x,l}\right) \mathrm{d}W_s, \quad t \geqslant 0, \tag{1.56}$$

admet un *basic coupling estimate* (résultat déjà démontré dans le chapitre 2), alors on dispose d'un résultat similaire pour l'EDS (1.55), à savoir :

$$|\mathcal{P}_t[\phi](x,n) - \mathcal{P}_t[\phi](y,m)| \leqslant C\left(1 + |x|^2 + |y|^2\right) \mathrm{e}^{-\nu t}, \tag{1.57}$$

où $\mathcal{P}_t[\phi](x,n) = \mathbb{E}[\phi(X_t^{x,n})]$.

Ce résultat s'avère crucial dans la résolution de l'EDSR ergodique : pour tous $0 \leqslant t \leqslant T < \infty$,

$$Y_t^{x,n} = Y_T^{x,n} + \int_t^T \left\{ f_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right) - \lambda \right\} \mathrm{d}s - \int_t^T Z_s^{x,n} \,\mathrm{d}W_s - \sum_{l=1}^{k-1} \int_t^T H_s^{x,n}(l) \,\mathrm{d}M_s(l), \tag{1.58}$$

où, de façon semblable au chapitre 2, les fonctions f_i sont supposées lipschitziennes par rapport à $\left(x, z\sigma_i(x)^{-1}, h\right)$. Dans ce chapitre, on a noté M la mesure de Poisson compensée. Là encore, le fait de ne pas supposer f_i lipschitzienne par rapport à (x,z,h) est une conséquence du fait que σ_i n'est pas bornée. On maintient l'hypothèse quantitative $\sqrt{r_2}\|f\|_{\mathrm{lip},z}\|\sigma^{-1}\|_\infty + \frac{1}{2}r_2 < \eta_2$. Néanmoins, pour pouvoir utiliser le théorème de Girsanov, on a besoin d'une hypothèse supplémentaire pour traiter l'intégrale selon la mesure de Poisson compensée : pour tous l, x, z, h et h', les coordonnées du vecteur $(k-1)$-dimensionnel :

$$\frac{f_l(x,z,h) - f_l(x,z,h')}{|h-h'|^2}(h-h')$$

sont toutes strictement supérieures à -1.

Encore une fois, l'étude de l'EDSR ergodique (1.58) se fait à l'aide d'une EDSR auxiliaire monotone : pour tous $0 \leqslant t \leqslant T < \infty$,

$$Y_t^{\alpha,x,n} = Y_T^{\alpha,x,n} + \int_t^T \left\{ f_{N_s^n}\left(X_s^{x,n}, Z_s^{\alpha,x,n}, H_s^{\alpha,x,n}\right) - \alpha Y_s^{\alpha,x,n} \right\} \mathrm{d}s - \int_t^T Z_s^{\alpha,x,n} \,\mathrm{d}W_s - \sum_{l=1}^{k-1} \int_t^T H_s^{\alpha,x,n}(l) \,\mathrm{d}M_s(l), \tag{1.59}$$

et les démarches sont similaires en tout point au chapitre 2. La seule réelle innovation dans la démonstration de l'existence d'une solution à l'EDSR ergodique est la suivante. Comme dans [26], par une extraction diagonale, on exhibe une suite $\left(\alpha_p\right)$ qui tend vers 0 et telle que :

$$\text{pour tout } 1 \leqslant l \leqslant k,\ \alpha_p Y_0^{\alpha_p,0,l} \text{ converge.}$$

Cependant, la limite obtenue dépend, *a priori*, du paramètre l. Par passage à la limite dans l'équation (1.59), le long d'une certaine sous-suite, on construit trois processus $Y^{x,n}$, $Z^{x,n}$ et $H^{x,n}$ tels que :

$$Y_t^{x,n} = Y_T^{x,n} + \int_t^T \left\{ f_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right) - \lambda_{N_s^n} \right\} \mathrm{d}s - \int_t^T Z_s^{x,n} \,\mathrm{d}W_s - \sum_{l=1}^{k-1} \int_t^T H_s^{x,n}(l) \,\mathrm{d}M_s(l), \tag{1.60}$$

pour tous $0 \leqslant t \leqslant T < \infty$. En multipliant l'équation (1.59) de part et d'autre par α, puis en prenant la limite le long de la bonne sous-suite, on obtient alors que, pour tous $0 \leqslant t \leqslant T < \infty$, $\lambda_{N_t^n} = \lambda_{N_T^n}$, c'est-à-dire : $\lambda_1 = \cdots = \lambda_k$; d'où l'existence d'une solution à l'équation (1.58). Comme précédemment, $Y^{x,n}$ s'exprime comme fonction de $X^{x,n}$:

$$\forall t \geqslant 0,\ Y_t^{x,n} = v_{N_t^n}\left(X_t^{x,n}\right),$$

et cette même famille de fonctions $(v_1, \dots, v_k)$ permet d'exprimer le processus $H^{x,n}$ en fonction $X^{x,n}$:

$$\forall t \geqslant 0,\ H_t^{x,n}(l) = v_{N_{t^-}^n + l}\left(X_t^{x,n}\right) - v_{N_{t^-}^n}\left(X_t^{x,n}\right).$$

Par la suite, l'unicité de la solution se démontre comme au chapitre 2. Puis, on démontre un résultat de comportement en temps long qui relie l'EDSR ergodique (1.58) à l'EDSR d'horizon fini suivante :

$$Y_t^{T,x,n} = \xi^T + \int_t^T f_{N_s^n}\left(X_s^{x,n}, Z_s^{T,x,n}, H_s^{T,x,n}\right) \mathrm{d}s - \int_t^T Z_s^{x,n}\, \mathrm{d}W_s - \sum_{l=1}^{k-1} \int_t^T H_s^{T,x,n}(l)\, \mathrm{d}M_s(l), \quad t \in [0,T]. \tag{1.61}$$

Enfin, nous montrons l'existence et l'unicité de la solution de viscosité du système d'EDP ergodiques suivant :

$$\mathcal{L}^i v_i(x) + f_i\left(x, (\nabla v_i.\sigma_i)(x), (v_{i+l}(x) - v_i(x))_{1 \leqslant l \leqslant k-1}\right) = \lambda, \quad t \in [0,T], x \in \mathbb{R}^d, 1 \leqslant i \leqslant k. \tag{1.62}$$

Cette solution (v, λ) s'obtient en résolvant l'EDSR ergodique. Par ailleurs, le théorème de comportement en temps long permet de relier ce système d'EDP ergodique au système :

$$\begin{cases} \partial_t u_i(t,x) + \mathcal{L}^i u_i(t,x) + f_i\left(x, (\nabla u_i.\sigma_i)(t,x), (u_{i+l}(t,x) - u_i(t,x))_{1 \leqslant l \leqslant k-1}\right) = 0, \\ \hfill t \in [0,T], x \in \mathbb{R}^d, 1 \leqslant i \leqslant k, \\ u_i(T,x) = g_i(x), \hfill x \in \mathbb{R}^d, 1 \leqslant i \leqslant k. \end{cases} \tag{1.63}$$

On remarque que l'absence du processus Y dans l'EDSR ergodique (1.58) se traduit par l'absence de la fonction v_i dans l'EDP ergodique (1.62). En conséquence, il n'est pas possible ici de réaliser un changement de variable esthétique dans les équations (1.62) et (1.63), alors que c'était le cas dans les travaux de Pardoux, Pradeilles et Rao [53] (voir (1.53)).

Ce chapitre est organisé comme suit. En section 3.2, nous étudions l'EDS (1.55) et obtenons un *basic coupling estimate*. La section 3.3 est dédiée à l'étude de l'EDSR auxiliaire (1.59), qui nous permet d'établir l'exitence et l'unicité de la solution à l'EDSR ergodique (1.58) en section 3.4. On démontre dans la partie 3.5 un théorème de comportement en temps long. La section 3.6 s'intéresse à deux applications de ces résultats : un problème de contrôle stochastique et la résolution d'un système d'EDP de Hamilton-Jacobi-Bellman couplées entre elles. Enfin, la section 3.7 est un appendice présentant divers résultats techniques.

1.3.3 Résultats nouveaux sur les EIDP

Le chapitre 4 s'intéresse à un autre type d'EDSR ergodiques à sauts. Comme dans le chapitre 3, l'EDS sous-jacente fait intervenir une mesure aléatoire de Poisson ; cependant, ici, elle est à l'origine de sauts dans l'EDS elle-même. Plus précisément, on considère l'EDS à bruit additif :

$$X_t^x = x + \int_0^t \Xi\left(X_s^x\right) \mathrm{d}s + \int_0^t \sigma\, \mathrm{d}L_s, \quad t \geqslant 0, \tag{1.64}$$

où L désigne une martingale de Lévy, c'est-à-dire un processus de la forme :

$$L_t = QW_t + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} y\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y).$$

Le jeu d'hypothèse reposant sur cette EDS est le suivant : la dérive Ξ est une fonction lipschitzienne et faiblement dissipative, les matrices σ et Q sont inversibles et la mesure ν (définie en (1.43)) est finie et admet un moment fini d'ordre 2. Ces hypothèses de travail nous permettent à la fois d'obtenir des propriétés essentielles (déjà utilisées dans le chapitre 2) sur la norme S^2 de X par exemple ; mais aussi d'utiliser directement le *basic coupling estimate* déjà démontré par Cohen et Fedyashov [17], dans leur pré-publication consacrée à l'étude d'EDSR ergodiques à sauts.

Le but est ensuite de démontrer l'existence et l'unicité de la solution de l'EDSR ergodique :

$$Y_t^x = Y_T^x + \int_t^T \{f(X_s^x, Z_s^x, U_s^x) - \lambda\} \,\mathrm{d}s - \int_t^T Z_s^x \,\mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^d \setminus \{0\}} U_s^x(y) \,\widetilde{N}(\mathrm{d}s, \mathrm{d}y), \quad 0 \leqslant t \leqslant T < \infty, \tag{1.65}$$

où le générateur f est une fonction lipschitzienne en ses trois paramètres x, z et u. Aussi, on suppose que f vérifie la propriété :

$$f(x,z,u) - f(x,z,u') \leqslant (u-u') \cdot \gamma_t^{x,z,u,u'}, \tag{1.66}$$

où le point central $\cdot$ désigne le produit scalaire dans $\mathrm{L}^2(\nu)$ et où γ est une fonction mesurable et telle qu'il existe deux constantes $C_1 \in]-1,0]$ et $C_2 \geqslant 0$, vérifiant :

$$C_1(1 \wedge |v|) \leqslant \gamma_t^{x,z,u,u'}(v) \leqslant C_2(1 \wedge |v|).$$

Cette nouvelle hypothèse nous permet de construire un processus γ qui, étant donnés des processus X, Z, U et U', satisfait :

$$f(X_t, Z_t, U_t) - f(X_t, Z_t, U_t') = (U_t - U_t') \cdot \gamma_t,$$

égalité qui s'avère utile lors de l'utilisation du théorème de Girsanov pour changer de mesure de probabilité. Encore une fois, l'étude de l'EDSR ergodique (1.65) est réalisée par le biais d'une EDSR auxiliaire monotone, vérifiée pour tous $0 \leqslant t \leqslant T < \infty$:

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T \{f(X_s^x, Z_s^{\alpha,x}, U_s^{\alpha,x}) - \alpha Y_s^{\alpha,x}\} \,\mathrm{d}s - \int_t^T Z_s^{\alpha,x} \,\mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^d \setminus \{0\}} U_s^{\alpha,x}(y) \,\widetilde{N}(\mathrm{d}s, \mathrm{d}y). \tag{1.67}$$

Similairement à ce qui est démontré au cours du chapitre 3, la fonction v intervenant dans l'expression de Y^x en fonction de X^x :

$$\forall t \geqslant 0,\ Y_t^x = v(X_t^x),$$

intervient également dans l'expression de U^x en fonction de X^x :

$$\forall t \geqslant 0,\ U_t^x(y) = v\left(X_{t^-}^x + \sigma y\right) - v\left(X_{t^-}^x\right).$$

Après avoir établi existence et unicité de la solution de l'EDSR ergodique (1.65), on s'intéresse aux équations intégro-différentielles partielles :

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + f(x, (\nabla u.\sigma)(t,x), (u(t, x+\sigma\bullet) - u(x)) \cdot \delta) = 0, & t \in [0,T], x \in \mathbb{R}^d, \\ u(T,x) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.68}$$

$$\text{et} \quad \mathcal{L}v(x) + f(x, \nabla v(x)\sigma Q, (v(x+\sigma\bullet) - v(x)) \cdot \delta) = \lambda, \quad x \in \mathbb{R}^d, \tag{1.69}$$

où l'opérateur $\mathcal{L}$ est le générateur du semigroupe de Kolmogorov de l'EDS (1.64), agissant sur les fonctions ϕ de classe $\mathcal{C}^2$ par la relation :

$$\mathcal{L}\phi(x) = \langle \Xi(x), \nabla\phi(x) \rangle + \frac{1}{2}\mathrm{tr}\left(\sigma Q Q^* \sigma^* \nabla^2 \phi(x)\right) + \int_{\mathbb{R}^d \setminus \{0\}} \{\phi(x+\sigma y) - \phi(x) - \langle \nabla\phi(x), \sigma y \rangle\} \,\mathrm{d}y.$$

On remarque l'apparition d'une nouvelle donnée δ dans l'équation (1.69) ; ceci est dû au fait qu'il est nécessaire d'ajouter une hypothèse supplémentaire sur la façon dont f dépend de u, pour pouvoir utiliser un théorème de comparaison et démontrer l'existence d'une solution de viscosité.

Le chapitre 4 est organisé de la façon suivante. La section 4.2 s'intéresse à l'étude de l'EDS (1.64). En section 4.3, nous montrons l'existence et l'unicité de la solution de l'EDSR ergodique (1.65). La partie 4.4 est consacrée à l'existence et l'unicité de la solution pour les EIDP (1.68) et (1.69). Le théorème de comportement en temps long est démontré dans la partie 4.5 et directement appliqué aux deux dernières équations. La section 4.6 aborde un problème de contrôle stochastique, et enfin, la partie 4.7 est un appendice où sont développées quelques preuves techniques.

Chapitre 2

EDSR ergodiques dont l'EDS sous-jacente est à bruit multiplicatif et non-borné

Résumé

Nous étudions dans ce chapitre des EDSRE pour lesquelles le bruit de l'EDS sous-jacente est supposé multiplicatif à croissance linéaire. Le fait que ce bruit ne soit pas borné est contrebalancé par une hypothèse de dissipativité faible portant sur la dérive de cette EDS. De plus, cette EDS est non-dégénérée. Nous étudions l'existence et l'unicité de la solution de l'EDSRE et nous appliquons nos résultats à un problème de contrôle ergodique optimal. En particulier, nous obtenons le comportement en temps long de la solution de viscosité d'une équation de Hamilton-Jacobi-Bellman, avec une vitesse de convergence exponentielle.

Mots-clés : équation différentielle stochastique rétrograde ergodique ; bruit multiplicatif et non-borné ; équation HJB ; comportement en temps long ; vitesse de convergence.

Abstract

We study in this chapter some EBSDEs such that the noise of its underlying SDE is assumed multiplicative and of linear growth. The fact that this noise is unbounded is balanced with an assumption of weak dissipativity for the drift of the SDE. Moreover, this SDE is non-degenerate. We study existence and uniqueness of the solution of the EBSDE and we apply our results to an ergodic optimal control problem. In particular, we obtain the large time behaviour of the viscosity solution of an Hamilton-Jacobi-Bellman equation, with an exponential rate of convergence.

Key words: ergodic backward stochastic differential equation; multiplicative and unbounded noise; HJB equation; large time behaviour; rate of convergence.

This chapter will be published in *Stochastic Processes and their Applications*
(article currently in press and available online),
under the title: Ergodic BSDE with unbounded and multiplicative underlying diffusion and application to large time behaviour of viscosity solution of HJB equation.

2.1 Introduction

We study the following EBSDE in finite dimension and infinite horizon: for all $t, T \in \mathbb{R}_+$ such that $0 \leqslant t \leqslant T < \infty$,

$$Y_t^x = Y_T^x + \int_t^T \{\psi(X_s^x, Z_s^x) - \lambda\} \, ds - \int_t^T Z_s^x \, dW_s, \tag{2.1}$$

where the unknown is the triplet (Y^x, Z^x, λ), with:

- Y^x a real-valued and progressively measurable process;
- Z^x an $(\mathbb{R}^d)^*$-valued and progressively measurable process;
- λ a real number.

The given data of our equation consists in:

- W, an $\mathbb{R}^d$-valued standard Brownian motion;
- $x \in \mathbb{R}^d$;
- X^x, an $\mathbb{R}^d$-valued process, starting from x, and solution of the SDE: for all $t \in \mathbb{R}_+$,

$$X_t^x = x + \int_0^t \Xi(X_s^x) \, ds + \int_0^t \sigma(X_s^x) \, dW_s; \tag{2.2}$$

- $\psi : \mathbb{R}^d \times (\mathbb{R}^d)^* \to \mathbb{R}$ a measurable function.

This class of ergodic BSDEs was first introduced by Fuhrman, Hu and Tessitore in [26], in order to study optimal ergodic control problem. In that paper, the main assumption is the strong dissipativity of Ξ, that is to say:

$$\exists \eta > 0, \, \forall x, x' \in \mathbb{R}^d, \, \langle \Xi(x) - \Xi(x'), x - x' \rangle \leqslant -\eta |x - x'|^2.$$

The strong dissipativity assumption was then dropped off in [22] and replaced by a weak dissipativity assumption: in other words, Ξ can be written as the sum of a dissipative function and of a bounded function.

A few years later, in [35], under similar assumptions, the large time behaviour of BSDEs in finite horizon T: for all $t \in [0, T]$,

$$Y_t^{T,x} = g(X_T^x) + \int_t^T \psi\left(X_s^x, Z_s^{T,x}\right) ds - \int_t^T Z_s^{T,x} \, dW_s, \tag{2.3}$$

was studied and linked with ergodic BSDEs. The authors prove the existence of a constant $L \in \mathbb{R}$, such that for all $x \in \mathbb{R}^d$,

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T \to \infty}{\longrightarrow} L;$$

moreover, they obtain an exponential rate on convergence.

Those BSDEs are linked with Hamilton-Jacobi-Bellman equations. Indeed, the solution of equation (2.3) can be written as $Y_t^{T,x} = u(T - t, X_t^x)$, where u is the solution of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) &= \mathcal{L}u(t,x) + \psi(x, \nabla u(t,x)\sigma(x)) & \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) &= g(x) & \forall x \in \mathbb{R}^d, \end{cases} \tag{2.4}$$

and where $\mathcal{L}$ is the generator of the Kolmogorov semigroup of X^x, solution of (2.2). Also, the ergodic BSDE (2.1) admits a solution such that $Y_t^x = v(X_t^x)$ satisfies

$$\mathcal{L}v(x) + \psi(x, \nabla v(x)\sigma(x)) - \lambda = 0 \qquad \forall x \in \mathbb{R}^d. \tag{2.5}$$

Because the functions u and v built by solving BSDEs or EBSDEs are not, in general, of class $\mathcal{C}^2$, we only solve these equations in a weak sense. In [35], the authors prove that, under their assumptions, the function v is of class $\mathcal{C}^1$ and they are able to work with mild solutions. But in this article, we only prove the continuity of v, so we link the solution of the EBSDE (2.1) with the viscosity solution of the ergodic PDE (2.5). Viscosity solutions of ergodic PDEs have already been widely studied (see [29]).

The large time behaviour of such PDEs has already been widely studied, for example in [12] and the references therein: but we do not make here the same assumptions. The authors work on the torus $\mathbb{R}^n/\mathbb{Z}^n$, and they need the Hamiltonian ψ to be uniformly convex; on the other hand, they are less restrictive about the matrix $\sigma(x)\sigma(x)^*$ which only needs to be nonnegative definite. They prove the convergence of $u(t,\bullet) - v - \lambda t$ as $t \to \infty$, but they do not have any rate of convergence. The same method has been used in [48] to study the large time behaviour of the solution to the obstacle problem for degenerate viscous Hamilton-Jacobi equations.

But in [26], [22] or [35], the diffusion coefficient σ of the forward process X^x is always supposed to be constant. The main contribution in this article is that the function σ is here assumed to be Lipschitz continuous, invertible, and such that σ^{-1} is bounded. Moreover, we will need the linear growth of σ to be small enough, with respect to the weak dissipativity of the drift Ξ: consequently, in average, the forward process X^x is attracted to the origin. The key point is a coupling estimate for a multiplicative noise driven diffusion X^x obtained by the irreducibility of X^x (see [20], [16] or [24]): the proof (see Theorem 2.7) is different from [44], where the author used a bridge in equation (B.6) and assumed σ to fulfil $\langle y, \sigma(x+y) - \sigma(x)\rangle \leqslant \Lambda|y|$ with $\Lambda \geqslant 0$ not too large. Then, we apply this result to auxiliary monotone BSDEs in infinite horizon:

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T \left\{\psi\left(X_s^x, Z_s^{\alpha,x}\right) - \alpha Y_s^{\alpha,x}\right\} \mathrm{d}s - \int_t^T Z_s^{\alpha,x}\,\mathrm{d}W_s,$$

where $\alpha \in \mathbb{R}_+^*$ (see [9] or [57]); we get that $Y_0^{\alpha,x}$ is locally Lipschitz with respect to x, and it allows us to prove the convergence of $(Y^{\alpha,x} - Y_0^{\alpha,0}, Z^{\alpha,x}, \alpha Y_0^{\alpha,0})$ to a solution (Y^x, Z^x, λ) of the EBSDE, for every $x \in \mathbb{R}^d$. We can prove uniqueness for λ, but we can only expect uniqueness of (Y^x, Z^x) as measurable functions of X^x.

In [35], the large time behaviour is obtained when ψ is assumed to be Lipschitz continuous with respect to z only. But we could not extend this result when σ is unbounded: we need here ψ to be Lipschitz continuous with respect to $z\sigma(x)^{-1}$ (it is not equivalent to being Lipschitz continuous with respect to z when σ is unbounded). The price to pay is that we also require the existence of a constant K_x such that:

$$|\psi(x,z) - \psi(x',z')| \leqslant K_x|x-x'| + K_z\left|z\sigma(x)^{-1} - z'\sigma(x')^{-1}\right|,$$

in order to keep the implication "$Y_0^{\alpha,x}$ has quadratic growth, so its increments have also quadratic growth". This part of our work is somewhat technical and is presented in the appendix.

The paper is organised as follows. Some notations are introduced in section 2.2. In section 2.3, we study a SDE slightly more general than the one satisfied by X^x: indeed, it will be the SDE satisfied by X^x after a change of probability space, due to Girsanov's theorem. In section 2.4, we prove that, in a way, our EBSDE admits a unique solution, which allows us also to solve an ergodic PDE. The link between this solution and the solutions of some finite horizon BSDEs is presented in section 2.5. Section 2.6 is devoted to an application of our results to an optimal ergodic control problem. Finally, the appendix presents how we keep the estimates of the increments of $Y_0^{\alpha,x}$ with respect to x, despite our twisted Lipschitz assumption on ψ.

2.2 Notations

Throughout this paper, $(W_t)_{t\geqslant 0}$ will denote a d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geqslant 0$, let $\mathcal{F}_t$ the σ-algebra generated by W_s, $0 \leqslant s \leqslant t$, and augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$. We write elements of $\mathbb{R}^d$ as vectors, and the star * stands for transposition. The Euclidean norm on $\mathbb{R}^d$ and $(\mathbb{R}^d)^*$ will be denoted by $|\bullet|$. For a matrix of $\mathcal{M}_d(\mathbb{R})$, we denote by $|\bullet|_F$ its Frobenius norm, that is to say the square root of the sum of the square of its coefficients. The Lipschitz constant of a function is written $\|\bullet\|_{\text{lip}}$ and we also write $\|f\|_\infty := \sup_x |f(x)|$ for a bounded function f. Finally, for any process, $|\bullet|_{t,T}^{*,p}$ stands for $\sup\limits_{s\in[t,T]} |\bullet_s|^p$, and the exponent p is omitted when equal to 1.

2.3 The SDE

We consider the SDE:
$$\begin{cases} \mathrm{d}X_t^x &= \Xi\left(t, X_t^x\right) \mathrm{d}t + \sigma\left(X_t^x\right) \mathrm{d}W_t, \\ X_0^x &= x. \end{cases} \tag{2.6}$$

Assumption 2.1. We suppose that

- $\Xi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous w.r.t. x uniformly in t: precisely, there exist some constants ξ_1 and ξ_2 such that for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, $|\Xi(t,x)| \leqslant \xi_1 + \xi_2|x|$;
- Ξ is weakly dissipative, *i.e.* $\Xi(t,x) = \xi(x) + b(t,x)$, where ξ is dissipative (that is to say $\langle \xi(x), x\rangle \leqslant -\eta|x|^2$, with $\eta > 0$) and locally Lipschitz and b is bounded; this way, there exists $\eta_1, \eta_2 > 0$, such that $\forall t \in \mathbb{R}_+$, $\forall x \in \mathbb{R}^d$, $\langle \Xi(t,x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$;
- $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ is Lipschitz continuous, the function $x \mapsto \sigma(x)^{-1}$ is bounded and for all $x \in \mathbb{R}^d$, $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$.

Remark 2.2.

1. Later on (from Proposition 2.5), we will need to assume that r_2 is sufficiently small.
2. If σ has a sublinear growth, *i.e.* $|\sigma(x)|_F \leqslant C\left(1 + |x|^\alpha\right)$ with $\alpha \in (0,1)$, then, by Young's inequality, we have $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$, and r_2 can be chosen as close to 0 as possible. For example, we can choose $\sigma(x) = ((1+|x|)\mathbb{1}_{|x|\leqslant 1} + (1+|x|^\alpha)\mathbb{1}_{|x|>1})\mathrm{I}_d$, where $\alpha \in (0,1)$.
3. $\sigma(x) = \sqrt{\dfrac{r_2}{2d}}(1+|x|)\mathrm{I}_d$ is another example of function satisfying Assumption 2.1. Indeed,
   - for any $x \in \mathbb{R}^d$, $\sigma(x) \in \mathrm{GL}_d(\mathbb{R})$ and $\sigma(x)^{-1} = \dfrac{\sqrt{2d}}{\sqrt{r_2}(1+|x|)}\mathrm{I}_d$ is bounded;
   - σ is Lipschitz;
   - moreover, σ is unbounded and $|\sigma(x)|_F^2 = \dfrac{r_2}{2}(1+|x|)^2 \leqslant r_2 + r_2|x|^2$.

Theorem 2.3. *Under Assumption 2.1, for all $p \in [2, +\infty)$ and for all $T \in (0, +\infty)$, there exists a unique process $X^x \in \mathrm{L}^p_{\mathcal{P}}(\Omega, \mathcal{C}([0,T], \mathbb{R}^d))$ strong solution to (2.6).*

Proof. See Theorem 7.4 of [19]. □

Proposition 2.4. *Under Assumption 2.1, for all $p \in (0, +\infty)$, and $T > 0$, we have:*

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\left[|X_t^x|^p\right] \leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |X_t^x|^p\right] \leqslant C\left(1 + |x|^p\right),$$

where C only depends on p, T, r_1, r_2, ξ_1 and ξ_2.

Proof. This is a straightforward consequence of Burkholder-Davis-Gundy's inequality and Gronwall's lemma. ☐

Proposition 2.5. *Suppose that Assumption 2.1 holds. Let $p \in (0, +\infty)$, $T > 0$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ a bounded function. The process $\widetilde{W}_t^x = W_t - \int_0^t \gamma(s, X_s^x)\, \mathrm{d}s$ is a Brownian motion under the corresponding Girsanov probability $\widetilde{\mathbb{P}}^{x,T}$, on the interval $[0,T]$. When $\sqrt{r_2}\|\gamma\|_\infty + \frac{(p \vee 2) - 1}{2} r_2 < \eta_2$, we get:*

$$\sup_{T \geqslant 0} \widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right] \leqslant C\left(1 + |x|^p\right),$$

where C only depends on p, η_1, η_2, r_1, r_2 and $\|\gamma\|_\infty$.

Proof. We start the proof with the case when $p > 2$.
We will suppose that we have $\sqrt{r_2}\|\gamma\|_\infty + (p-1)\frac{r_2}{2} < \eta_2$. By Itô's formula,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] \\
&\quad = p\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left\langle X_t^x, \Xi(t, X_t^x) + \sigma(X_t^x)\gamma(t, X_t^x)\right\rangle\right] + \frac{p}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left|\sigma(X_t^x)\right|_F^2\right] \\
&\qquad\qquad + \frac{p(p-2)}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-4}\sum_{i=1}^d\left(\sum_{j=1}^d (X_t^x)_j\, \sigma(X_t^x)_{i,j}\right)^2\right] \\
&\quad \leqslant p\widetilde{\mathbb{E}}^{x,T}\left[\eta_1|X_t^x|^{p-2} - \eta_2|X_t^x|^p + |X_t^x|^{p-1}\left|\sigma(X_t^x)\right|_F\left|\gamma(t, X_t^x)\right|\right] + \frac{p}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left|\sigma(X_t^x)\right|_F^2\right] \\
&\qquad\qquad + \frac{p(p-2)}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left|\sigma(X_t^x)\right|_F^2\right] \\
&\quad \leqslant \left(-p\eta_2 + p\sqrt{r_2}\|\gamma\|_\infty + \frac{p(p-1)}{2}r_2\right)\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] + p\sqrt{r_1}\|\gamma\|_\infty\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-1}\right] \\
&\qquad\qquad + \left(p\eta_1 + \frac{p(p-1)}{2}r_1\right)\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\right].
\end{aligned}$$

But, using Young's inequality, we can show, for every $\varepsilon > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] \leqslant \lambda_\varepsilon\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] + \sqrt{r_1}\|\gamma\|_\infty\varepsilon^{-p} + \left(2\eta_1 + (p-1)r_1\right)\varepsilon^{-\frac{p}{2}}.$$

We set $\lambda_\varepsilon := -p\eta_2 + p\sqrt{r_2}\|\gamma\|_\infty + \frac{p(p-1)}{2}r_2 + (p-1)\sqrt{r_1}\|\gamma\|_\infty\varepsilon^{\frac{p}{p-1}} + \left((p-2)\eta_1 + \frac{(p-2)(p-1)}{2}r_1\right)\varepsilon^{\frac{p}{p-2}}$ (for ε small enough) and this quantity is negative. Hence, for ε small enough,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda_\varepsilon t}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right]\right) &\leqslant \mathrm{e}^{-\lambda_\varepsilon t}\left(\sqrt{r_1}\|\gamma\|_\infty\varepsilon^{-p} + \left(2\eta_1 + (p-1)r_1\right)\varepsilon^{-\frac{p}{2}}\right) \\
\widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right] &\leqslant |x|^p + \frac{1}{|\lambda_\varepsilon|}\left(\sqrt{r_1}\|\gamma\|_\infty\varepsilon^{-p} + \left(2\eta_1 + (p-1)r_1\right)\varepsilon^{-\frac{p}{2}}\right).
\end{aligned}$$

We have been able to conclude the case $p > 2$ because $\lambda_\varepsilon < 0$. For the case when $p = 2$, Itô's formula gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^2\right] = 2\widetilde{\mathbb{E}}^{x,T}\left[\left\langle X_t^x, \Xi(t, X_t^x) + \sigma(X_t^x)\gamma(t, X_t^x)\right\rangle\right] + \widetilde{\mathbb{E}}^{x,T}\left[\left|\sigma(X_t^x)\right|_F^2\right].$$

And we conclude the proof in the same manner as that in the case when $p > 2$. When $p \in (0,2)$, and under the Assumption $\sqrt{r_2}\|\gamma\|_\infty + \frac{r_2}{2} < \eta_2$, we have

$$\sup_{T \geqslant 0} \widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right] \leqslant C\left(1 + |x|^p\right),$$

by use of Jensen's inequality (with the concavity of $x \mapsto |x|^{\frac{p}{2}}$) and the inequality $(1+z)^\alpha \leqslant 1 + z^\alpha$ for $z \geqslant 0$ and $\alpha \in (0,1)$. $\square$

Proposition 2.6. *Under Assumption 2.1, the process X^x is irreducible, that is to say:*

$$\forall t > 0,\, \forall x, z \in \mathbb{R}^d,\, \forall r > 0,\, \mathbb{P}\left(|X_t^x - z| < r\right) > 0.$$

Proof. The proof is very similar to Theorem 7.3.1 of [20]. $\square$

Theorem 2.7. *Suppose Assumption 2.1 holds. Let $\mu \geqslant 2$ be such that $\dfrac{\mu-1}{2} r_2 < \eta_2$ and $\phi : \mathbb{R}^d \to \mathbb{R}^d$ a measurable function, satisfying $|\phi(x)| \leqslant c_\phi\left(1 + |x|^\mu\right)$. Then we have,*

$$\forall x, y \in \mathbb{R}^d,\, \forall t \geqslant 0,\, |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leqslant \widehat{c} c_\phi \left(1 + |x|^\mu + |y|^\mu\right) \mathrm{e}^{-\widehat{\nu} t},$$

where $\mathcal{P}_t[\phi](x) = \mathbb{E}\left[\phi\left(X_t^x\right)\right]$ (Kolmogorov semigroup of X) and $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2 and μ.

Proof. For reader's convenience, we recall here Theorem A.2 of [24].

Lemma. *Let X be an hypoelliptic diffusion on $\mathbb{R}^d$. Suppose the following additional conditions hold.*

1. *There exists a function $V : \mathbb{R}^d \to \mathbb{R}_+$ satisfying:*
   (a) *$V(x) \to \infty$ as $|x| \to \infty$;*
   (b) *the level sets $\left\{x \in \mathbb{R}^d \middle| V(x) \leqslant c\right\}$ are precompact for all positive c;*
   (c) *there exist positive constants a and b and a compact set C so that*

   $$\forall x \in \mathbb{R}^d,\, LV(x) < -aV(x) + b\mathbb{1}_C(x),$$

   where L is the generator of the Kolmogorov semigroup of X.
2. *There exists a point x^* such that for any open neighbourhood U of x^*, there exists a sampling rate $h > 0$, so that for any $x \in \mathbb{R}^d$ there exists an $n > 0$ with $\mathbb{P}(X_{nh}^x \in U) > 0$.*

Then the diffusion X has an invariant measure π and this measure is unique. Furthermore, there exist positive constants B and r so that for all x,

$$\|\mathcal{P}_t(x, \bullet) - \pi\|_V := \sup_{f \in \mathcal{V}} \left| \int_{\mathbb{R}^d} f(y)\, \mathcal{P}_t(x, \mathrm{d}y) - \int_{\mathbb{R}^d} f(y)\, \pi(\mathrm{d}y) \right| \leqslant B(1 + V(x))\mathrm{e}^{-rt},$$

where $\mathcal{V} = \{$measurable f with $|f(x)| < 1 + V(x)\}$.

First, hypoellipticity is a consequence of the boundedness of σ^{-1}. Secondly, we choose the function $V : x \mapsto |x|^\mu$, and we have:

- $|x|^\mu \to \infty$ as $|x| \to \infty$;
- $\left\{x \in \mathbb{R}^d \middle| |x|^\mu \leqslant c\right\}$ is the closed ball, centred in the origin and of radius $c^{1/\mu}$, so it is a compact set;

- $LV(x) = \mu|x|^{\mu-2}\langle x, \Xi(0,x)\rangle + \dfrac{\mu}{2}|x|^{\mu-2}|\sigma(x)|_F^2 + \dfrac{\mu(\mu-2)}{2}|x|^{\mu-4}\sum_{i=1}^d\left(\sum_{j=1}^d x_j\sigma(x)_{i,j}\right)^2$

$$\begin{aligned}
&\leqslant \left(-\mu\eta_2 + \frac{\mu(\mu-1)}{2}r_2\right)|x|^\mu + \left(\mu\eta_1 + \frac{\mu(\mu-1)}{2}r_1\right)|x|^{\mu-2}\\
&\leqslant \left(-\mu\eta_2 + \frac{\mu(\mu-1)}{2}r_2\right)|x|^\mu + \left(\mu\eta_1 + \frac{\mu(\mu-1)}{2}r_1\right)\left(|x|^{\mu-2}\frac{|x|^2}{R^2}\mathbb{1}_{|x|>R} + R^{\mu-2}\mathbb{1}_{|x|\leqslant R}\right)\\
&\leqslant \left(-\mu\eta_2 + \frac{\mu(\mu-1)}{2}r_2 + \frac{\mu}{R^2}\eta_1 + \frac{\mu(\mu-1)}{2R^2}r_1\right)|x|^\mu + \left(\mu\eta_1 + \frac{\mu(\mu-1)}{2}r_1\right)R^{\mu-2}\mathbb{1}_{|x|\leqslant R}
\end{aligned}$$

  and $-\mu\eta_2 + \dfrac{\mu(\mu-1)}{2}r_2 + \dfrac{\mu}{R^2}\eta_1 + \dfrac{\mu(\mu-1)}{2R^2}r_1 < 0$ for any R large enough.

The last condition of Theorem A.2 of [24] is satisfied because of the irreducibility of X^x (we can take $x^* = 0$, see Proposition 2.6). Then, we get:

$$|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leqslant \widehat{c}c_\phi\,(1+|x|^\mu+|y|^\mu)\,\mathrm{e}^{-\widehat{\nu}t}.$$

Finally, $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2 and μ (see [46] and [47]). □

Assumption 2.8. We write $\Xi(t,x) = \xi(x) + \rho(t,x)$ where ξ and ρ satisfy the following:

- ρ is a bounded function;
- ξ is Lipschitz continuous and satisfies $\langle\xi(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$, with η_1 and η_2 two positive constants;
- ρ is the pointwise limit of a sequence (ρ_n) of $\mathcal{C}^1$ functions, with bounded derivatives w.r.t. x and uniformly bounded by $\|\rho\|_\infty$.

As before, we still suppose the following:

- $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ is Lipschitz continuous;
- the function $x \mapsto \sigma(x)^{-1}$ is bounded;
- $\forall x \in \mathbb{R}^d$, $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$.

Corollary 2.9. *Suppose Assumption 2.8 holds. Let $\mu \geqslant 2$ be such that $\sqrt{r_2}\|\rho\|_\infty + \dfrac{\mu-1}{2}r_2 < \eta_2$ and $\phi : \mathbb{R}^d \to \mathbb{R}^d$ measurable with $|\phi(x)| \leqslant c_\phi\,(1+|x|^\mu)$. Then we have,*

$$\forall x,y\in\mathbb{R}^d,\ \forall t\geqslant 0,\ |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leqslant \widehat{c}c_\phi\,(1+|x|^\mu+|y|^\mu)\,\mathrm{e}^{-\widehat{\nu}t},$$

where $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2, μ and $\|\rho\|_\infty$.

Proof. We set σ_n a function close to σ on the centered ball of $\mathbb{R}^d$ of radius n, equal to I_d outside the centred ball of radius $n+1$ and of class $\mathcal{C}^1$ with bounded derivatives on $\mathbb{R}^d$; on the ring between the radius n and $n+1$, σ_n is chosen in such a way that $\sigma^{-1}\sigma_n$ is bounded, independently from n. This way, the function $\Xi_n : (t,x) \mapsto \xi(x) + \sigma_n(x)\rho_n(t,x)$ is Lipschitz continuous w.r.t. x. We denote $X^{n,x}$ the solution of the SDE:

$$\begin{cases} \mathrm{d}X_t^{n,x} = \Xi_n\left(t, X_t^{n,x}\right)\,\mathrm{d}t + \sigma\left(X_t^{n,x}\right)\,\mathrm{d}W_t,\\ X_0^{n,x} = x.\end{cases}$$

We can write, for every $\varepsilon > 0$:

$$\langle\Xi_n(t,x), x\rangle \leqslant \eta_1 - \eta_2|x|^2 + \left(\sqrt{r_1} + \sqrt{r_2}|x|\right)\|\rho\|_\infty|x| \leqslant \left[\eta_1 + \frac{\|\rho\|_\infty^2 r_1}{2\varepsilon^2}\right] - \left[\eta_2 - \sqrt{r_2}\|\rho\|_\infty - \frac{\varepsilon^2}{2}\right]|x|^2.$$

Then, when ε is small enough, for any ϕ with polynomial growth of degree μ, Theorem 2.7 tells us that $\widehat{c}_n$ and $\widehat{\nu}_n$ are independent of n, and we have:

$$\exists \widehat{c}, \widehat{\nu} > 0,\ \forall n \in \mathbb{N}^*,\ \forall x, y \in \mathbb{R}^d,\ \forall t \geqslant 0,\ \left|\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] - \mathbb{E}\left[\phi\left(X_t^{n,y}\right)\right]\right| \leqslant \widehat{c} c_\phi \left(1 + |x|^\mu + |y|^\mu\right) \mathrm{e}^{-\widehat{\nu} t}.$$

Our goal is to take the limit; let us show that $\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\phi\left(X_t^x\right)\right]$. Let U be the solution of the SDE

$$U_t^x = x + \int_0^t \xi\left(U_s^x\right) \mathrm{d}s + \int_0^t \sigma\left(U_s^x\right) \mathrm{d}W_s.$$

We can write: $X_t^{n,x} = x + \int_0^t \xi\left(X_s^{n,x}\right) \mathrm{d}s + \int_0^t \sigma\left(X_s^{n,x}\right) \mathrm{d}W_s^{(n)}$, where

$$W_t^{(n)} = W_t + \int_0^t \sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right) \mathrm{d}s$$

is a Brownian motion under the probability $\mathbb{P}^{(n)} = p_T^n\left(X^{n,x}\right)\mathbb{P}$ on $[0,T]$ and where

$$p_t^n\left(X^{n,x}\right) = \exp\left(-\int_0^t \left\langle \sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right), \mathrm{d}W_s\right\rangle \right.$$
$$\left. -\frac{1}{2}\int_0^t \left|\sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right)\right|^2 \mathrm{d}s\right).$$

Similarly, $X_t^x = x + \int_0^t \xi\left(X_s^x\right) \mathrm{d}s + \int_0^t \sigma\left(X_s^x\right) \mathrm{d}W_s^{(\infty)}$, where $W_t^{(\infty)} = W_t + \int_0^t \rho\left(s, X_s^x\right) \mathrm{d}s$ is a Brownian motion under the probability $\mathbb{P}^{(\infty)} = p_T^\infty\left(X^x\right)\mathbb{P}$ on $[0,T]$ and where

$$p_t^\infty\left(X^x\right) = \exp\left(-\int_0^t \left\langle \rho\left(s, X_s^x\right), \mathrm{d}W_s\right\rangle - \frac{1}{2}\int_0^t \left|\rho\left(s, X_s^x\right)\right|^2 \mathrm{d}s\right).$$

By uniqueness in law of the solutions of the SDEs, we get the equalities:

$$\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] = \mathbb{E}^{(n)}\left[p_t^n\left(X^{n,x}\right)^{-1} \phi\left(X_t^{n,x}\right)\right] = \mathbb{E}\left[p_t^n\left(U^x\right)^{-1} \phi\left(U_t^x\right)\right];$$

$$\mathbb{E}\left[\phi\left(X_t^x\right)\right] = \mathbb{E}^{(\infty)}\left[p_t^\infty\left(X^x\right)^{-1} \phi\left(X_t^x\right)\right] = \mathbb{E}\left[p_t^\infty\left(U^x\right)^{-1} \phi\left(U_t^x\right)\right].$$

But $\rho_n(t,x) \underset{n\to\infty}{\longrightarrow} \rho(t,x)$, so we have $p_t^n\left(U^x\right)^{-1} \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} p_t^\infty\left(U^x\right)^{-1}$. We just have to show that the sequence $\left(p_t^n\left(U^x\right)^{-1}\right)_{n\in\mathbb{N}^*}$ is uniformly integrable, in order to show that $\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\phi\left(X_t^x\right)\right]$. We have:

$$\mathbb{E}\left[\left(p_t^n\left(U^x\right)^{-1}\right)^2\right]$$
$$= \mathbb{E}\left[\exp\left(2\int_0^t \left\langle \sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right), \mathrm{d}W_s\right\rangle + \int_0^t \left|\sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right)\right|^2 \mathrm{d}s\right)\right]$$
$$\leqslant \mathbb{E}\left[\exp\left(\int_0^t \left\langle 4\sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right), \mathrm{d}W_s\right\rangle - \frac{1}{2}\int_0^t \left|4\sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right)\right|^2 \mathrm{d}s\right)\right]^{\frac{1}{2}}$$
$$\mathbb{E}\left[\exp\left(10\int_0^t \left|\sigma\left(X_s^{n,x}\right)^{-1} \sigma_n\left(X_s^{n,x}\right) \rho_n\left(s, X_s^{n,x}\right)\right|^2 \mathrm{d}s\right)\right]^{\frac{1}{2}}$$
$$\leqslant \exp\left(5t\left(d^2 + \left\|\sigma^{-1}\right\|_\infty^2\right)\|\rho\|_\infty\right) < \infty.$$

□

2.4 The EBSDE

We consider the following EBSDE:

$$\forall 0 \leqslant t \leqslant T < \infty,\ Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\, \mathrm{d}s - \int_t^T Z_s^x\, \mathrm{d}W_s \tag{2.7}$$

and we make the following assumptions.

Assumption 2.10.

- $\forall x \in \mathbb{R}^d$, $|\psi(x,0)| \leqslant M_\psi(1+|x|)$;
- $\forall x, x' \in \mathbb{R}^d$, $\forall z, z' \in \mathbb{R}^d$, $|\psi(x,z) - \psi(x',z')| \leqslant K_x |x - x'| + K_z \left|z\sigma(x)^{-1} - z'\sigma(x')^{-1}\right|$;
- $\Xi : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous, $|\Xi(x)| \leqslant \xi_1 + \xi_2|x|$;
- $\Xi(x) = \xi(x) + b(x)$, with ξ dissipative and locally Lipschitz and b bounded;
- $\langle \Xi(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$ for two positive constants η_1, η_2;
- $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ is Lipschitz continuous;
- $x \mapsto \sigma(x)^{-1}$ is bounded and $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$;
- $\sqrt{r_2} K_z \left\|\sigma^{-1}\right\|_\infty + \dfrac{r_2}{2} < \eta_2$.

Remark 2.11. In most papers, the function ψ is assumed to be Lipschitz continuous. We make a slight modification of this assumption in order to have some information about the second and third behaviour (we refer to Theorem 4.4 of [35] for the meaning of these behaviours). But ψ is still Lipschitz continuous w.r.t. z, with a constant equal to $K_z \left\|\sigma^{-1}\right\|_\infty$. Moreover, ψ is still continuous w.r.t. x.

2.4.1 Existence of a solution

Theorem 2.3 ensures that the process X^x is well defined. We introduce a new parameter $\alpha > 0$ and we consider a new BSDE of infinite horizon:

$$\forall 0 \leqslant t \leqslant T < \infty,\ Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T [\psi(X_s^x, Z_s^{\alpha,x}) - \alpha Y_s^{\alpha,x}]\, \mathrm{d}s - \int_t^T Z_s^{\alpha,x}\, \mathrm{d}W_s. \tag{2.8}$$

Lemma 2.12. *Under Assumption 2.10, for every $x \in \mathbb{R}^d$ and $\alpha > 0$, there exists a unique solution $(Y^{\alpha,x}, Z^{\alpha,x})$ to BSDE (2.8), such that $Y^{\alpha,x}$ is a continuous process bounded in L^1 and $Z^{\alpha,x} \in \mathrm{L}^2_{\mathcal{P},loc}(\Omega, \mathrm{L}^2(0,\infty;(\mathbb{R}^d)^*))$. Also, for every $t \geqslant 0$, $\left|Y_t^{\alpha,x}\right| \leqslant \dfrac{C}{\alpha}(1+|X_t^x|)$, $\mathbb{P}$-a.s., where C only depends on M_ψ, η_1, η_2, r_1, r_2, K_z and $\left\|\sigma^{-1}\right\|_\infty$. The function $v^\alpha : x \mapsto Y_0^{\alpha,x}$ is continuous, and for every $t \geqslant 0$, $Y_t^{\alpha,x} = v^\alpha(X_t^x)$ $\mathbb{P}$-a.s.*

Proof. For the upper bound for $Y^{\alpha,x}$, see Theorem 2.1 of [57]. The main difference is that we do not require $\psi(\bullet, 0)$ to be bounded. Uniqueness only needs boundedness in L^1 and not almost surely boundedness. The continuity of v^α is a consequence of a straightforward adaptation of the Theorem 2.1 of [57] when $\psi(\bullet, 0)$ is no more assumed to be bounded and proposition 2.1 of [25]. The representation of $Y^{\alpha,x}$ by X^x comes from Lemma 4.4 of [57]. □

Lemma 2.13. *Under Assumption 2.10, for every $\alpha \in (0,1]$, we have:*

$$\forall x, x' \in \mathbb{R}^d,\ \left|v^\alpha(x) - v^\alpha(x')\right| \leqslant C\left(1 + \frac{1}{\alpha}\right)\left(1 + |x| + |x'|\right)|x - x'|, \tag{2.9}$$

where C only depends on K_x, K_z, M_ψ, η_1, η_2, r_1, r_2, $\left\|\sigma^{-1}\right\|_\infty$, ξ_1, $\|\Xi\|_{\mathrm{lip}}$ and $\|\sigma\|_{\mathrm{lip}}$.

Proof. See Theorem 2.35, given in the appendix, with $f : (x,y,z) \mapsto \psi(x,z) - \alpha y$ and $g = v^\alpha$. It uses the estimate of Lemma 2.12. □

Remark 2.14.

1. If σ and $\psi(\bullet, 0)$ are bounded and Ξ and σ are of class $\mathcal{C}^1$, then, Theorem 3.2 of [36] tells us that v^α is of class $\mathcal{C}^1$. Then, we can apply Theorem 3.1 of [43] and write $Z_t^{\alpha,x} = \partial_x v^\alpha\left(X_t^x\right)\sigma\left(X_t^x\right)$.
2. The Lipschitz constant given by the last lemma depends on α, and this constant goes to infinity as α goes to zero.

Lemma 2.15. *Let $\psi : \mathbb{R}^d \times (\mathbb{R}^d)^* \to \mathbb{R}$ continuous w.r.t. the first variable and Lipschitz continuous w.r.t. the second one. Let $\zeta, \zeta' : \mathbb{R}_+ \times \mathbb{R}^d \to (\mathbb{R}^d)^*$ be such that for every $t \geqslant 0$, $\zeta(t, \bullet)$ and $\zeta'(t, \bullet)$ are continuous, and set:*

$$\widetilde{\Gamma}(t,x) = \begin{cases} \dfrac{\psi(x,\zeta(t,x)) - \psi(x,\zeta'(t,x))}{|\zeta(t,x) - \zeta'(t,x)|^2}(\zeta(t,x) - \zeta'(t,x)), & \text{if } \zeta(t,x) \neq \zeta'(t,x), \\ 0, & \text{if } \zeta(t,x) = \zeta'(t,x). \end{cases}$$

*Then, there exists a uniformly bounded sequence of $\mathcal{C}^1$ functions w.r.t. x with bounded derivatives $(\widetilde{\Gamma}_n)_{n\geqslant 1}$ (*i.e. *for all n, $\widetilde{\Gamma}_n$ has bounded derivatives w.r.t. x – the bound of derivatives can depend on n – and* $\sup_{n\geqslant 1}\|\widetilde{\Gamma}_n(t,\bullet)\|_\infty < \infty$ *for every $t \geqslant 0$), such that $\widetilde{\Gamma}_n \underset{n\to\infty}{\longrightarrow} \widetilde{\Gamma}$ pointwise.*

Proof. See Lemma 3.7 of [35]. □

Proposition 2.16. *Under Assumption 2.10, there exists a constant C, such that for every $\alpha \in (0,1]$, we have:*

$$\forall x, x' \in \mathbb{R}^d,\ \left|v^\alpha(x) - v^\alpha(x')\right| \leqslant C\left(1 + |x|^2 + |x'|^2\right).$$

The constant C only depends on η_1, η_2, r_1, r_2, K_z, $\left\|\sigma^{-1}\right\|_\infty$ and M_ψ.

Proof.

1. We approximate σ by a sequence $(\sigma^\varepsilon)_{\varepsilon>0}$ of functions satisfying:
   - σ^ε converges pointwise towards σ over $\mathbb{R}^d$;
   - σ^ε is bounded (the bound can depend on ε) and $|\sigma^\varepsilon(x)|_F^2 \leqslant r_1 + r_2|x|^2$;
   - σ^ε is of class $\mathcal{C}^1$ and $\|\sigma^\varepsilon\|_{\mathrm{lip}} \leqslant \|\sigma\|_{\mathrm{lip}}$;
   - $x \mapsto \sigma^\varepsilon(x)^{-1}$ is bounded, the bound is independent of ε.

   We also approximate Ξ by a sequence $(\Xi^\varepsilon)_{\varepsilon>0}$ of $\mathcal{C}^1$ functions which converges uniformly. One can check that the functions Ξ^ε are "uniformly weakly dissipative", because the functions $\Xi^\varepsilon - \xi$ are uniformly bounded. Moreover, ψ is approximated by a sequence (ψ^ε) of functions satisfying:

   - $|\psi^\varepsilon(x,z) - \psi^\varepsilon(x',z')| \leqslant K_x|x - x'| + K_z\left|z\sigma^\varepsilon(x)^{-1} - z'\sigma^\varepsilon(x')^{-1}\right|$;

- $\psi^\varepsilon(\bullet, 0)$ is bounded;
- $|\psi^\varepsilon(x,0)| \leqslant M_\psi(1+|x|)$;
- (ψ^ε) converges pointwise towards ψ.

We consider the BSDE:

$$Y_t^{\varepsilon,\alpha,x} = Y_T^{\varepsilon,\alpha,x} + \int_t^T \left\{\psi^\varepsilon\left(X_s^{\varepsilon,x}, Z_s^{\varepsilon,\alpha,x}\right) - \alpha Y_s^{\varepsilon,\alpha,x}\right\} \mathrm{d}s - \int_t^T Z_s^{\varepsilon,\alpha,x}\, \mathrm{d}W_s,$$

where the process $X^{\varepsilon,x}$ satisfies the following equation:

$$X_t^{\varepsilon,x} = x + \int_0^t \Xi^\varepsilon\left(X_s^{\varepsilon,x}\right) \mathrm{d}s + \int_0^t \sigma^\varepsilon\left(X_s^{\varepsilon,x}\right) \mathrm{d}W_s.$$

This BSDE has a unique solution (see Lemma 2.12), and $\left|Y_t^{\varepsilon,\alpha,x}\right| \leqslant \frac{C}{\alpha}\left(1+|X_t^x|\right)$ $\mathbb{P}$-a.s. and for every $t \geqslant 0$. Thanks to Theorem 3.2 of [36], we have $Z_t^{\varepsilon,\alpha,x} = \nabla v^{\varepsilon,\alpha}\left(X_t^{\varepsilon,x}\right)\sigma^\varepsilon\left(X_t^{\varepsilon,x}\right)$ $\mathbb{P}$-a.s. and for a.e. $t \geqslant 0$, and $v^{\varepsilon,\alpha}$ is of class $\mathcal{C}^1$. We define:

$$\Gamma^{\varepsilon,\alpha}(x) = \begin{cases} \dfrac{\psi^\varepsilon\left(x, \nabla v^{\varepsilon,\alpha}(x)\sigma^\varepsilon(x)\right) - \psi^\varepsilon(x,0)}{\left|\nabla v^{\varepsilon,\alpha}(x)\sigma^\varepsilon(x)\right|^2}\nabla v^{\varepsilon,\alpha}(x)\sigma^\varepsilon(x), & \text{if } \nabla v^{\varepsilon,\alpha}(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, thanks to Lemma 2.15, we can approximate $\Gamma^{\varepsilon,\alpha}$ in such a way that we can use Corollary 2.9.

We can rewrite:

$$-\mathrm{d}Y_t^{\varepsilon,\alpha,x} = \left\{\psi^\varepsilon\left(X_t^{\varepsilon,x},0\right) + Z_t^{\varepsilon,\alpha,x}\Gamma^{\varepsilon,\alpha}\left(X_t^{\varepsilon,x}\right)^* - \alpha Y_t^{\varepsilon,\alpha,x}\right\} \mathrm{d}t - Z_t^{\varepsilon,\alpha,x}\, \mathrm{d}W_t.$$

But $\Gamma^{\varepsilon,\alpha}$ is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$, and there exists a probability $\widehat{\mathbb{P}}^{\varepsilon,\alpha,x,T}$ under which the process $\widehat{W}_t^{\varepsilon,\alpha,x} = W_t - \int_0^t \Gamma^{\varepsilon,\alpha}\left(X_s^{\varepsilon,x}\right)^* \mathrm{d}s$ is a Brownian motion on $[0,T]$. Finally, we get the equality:

$$v^{\varepsilon,\alpha}(x) = \widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\mathrm{e}^{-\alpha T}v^{\varepsilon,\alpha}\left(X_T^{\varepsilon,x}\right) + \int_0^T \mathrm{e}^{-\alpha s}\psi^\varepsilon\left(X_s^{\varepsilon,x},0\right) \mathrm{d}s\right].$$

On the one hand, using proposition 2.5:

$$\left|\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\mathrm{e}^{-\alpha T}v^{\varepsilon,\alpha}\left(X_T^{\varepsilon,x}\right)\right]\right| \leqslant \mathrm{e}^{-\alpha T}\frac{C}{\alpha}\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\left|X_T^{\varepsilon,x}\right|\right] \underset{T\to\infty}{\longrightarrow} 0.$$

On the other hand, X^ε satisfies the following SDE under $\widehat{\mathbb{P}}^{\varepsilon,\alpha,x,T}$:

$$\mathrm{d}X_t^{\varepsilon,x} = \left[\Xi^\varepsilon\left(X_t^{\varepsilon,x}\right) + \sigma^\varepsilon\left(X_t^{\varepsilon,x}\right)\Gamma^{\varepsilon,\alpha}\left(X_t^{\varepsilon,x}\right)^*\right] \mathrm{d}t + \sigma^\varepsilon\left(X_t^{\varepsilon,x}\right) \mathrm{d}\widehat{W}_t^{\varepsilon,\alpha,x}.$$

Thanks to Corollary 2.9, we get:

$$\left|\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x},0\right)\right] - \widehat{\mathbb{E}}^{\varepsilon,\alpha,x',T}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x'},0\right)\right]\right| \leqslant 2M_\psi\widehat{c}\mathrm{e}^{-\widehat{\nu}t}\left(1+|x|^2+|x'|^2\right),$$

where $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2, K_z and $\left\|\sigma^{-1}\right\|_\infty$. As a consequence, we get:

$$\begin{aligned} \left|v^{\varepsilon,\alpha}(x) - v^{\varepsilon,\alpha}(x')\right| &= \lim_{T\to\infty}\left|\int_0^T \mathrm{e}^{-\alpha t}\left(\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x},0\right)\right] - \widehat{\mathbb{E}}^{\varepsilon,\alpha,x',t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x'},0\right)\right]\right) \mathrm{d}t\right| \\ &\leqslant \int_0^\infty \mathrm{e}^{-\alpha t}\left|\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x},0\right)\right] - \widehat{\mathbb{E}}^{\varepsilon,\alpha,x',t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x'},0\right)\right]\right| \mathrm{d}t \\ &\leqslant 2\int_0^\infty \mathrm{e}^{-\alpha t}\widehat{c}M_\psi\mathrm{e}^{-\widehat{\nu}t}\left(1+|x|^2+|x'|^2\right) \mathrm{d}t \leqslant 2\frac{\widehat{c}M_\psi}{\widehat{\nu}}\left(1+|x|^2+|x'|^2\right). \end{aligned} \tag{2.10}$$

2. Now, our goal is to take the limit when $\varepsilon \to 0$. Let D be a dense and countable subset of $\mathbb{R}^d$. By a diagonal argument, there exists a positive sequence $(\varepsilon_n)_n$ such that $(v^{\varepsilon_n,\alpha})_n$ converges pointwise over D to a function $\overline{v}^\alpha$. Because the constant C in equation (2.9) does not depend on ε, $\overline{v}^\alpha$ satisfies the same inequality. Let (K_n) be an increasing sequence of compact sets whose diameter goes to infinity. The function $\overline{v}^\alpha$ is uniformly continuous on $K_n \cap D$, so it has an extension, still called $\overline{v}^\alpha$ for simplicity, which is continuous on K_n. Passing to the limit as n goes to infinity, we get a continuous function on $\mathbb{R}^d$, and it is the pointwise limit of the sequence $(v^{\varepsilon_n,\alpha})_n$. We denote $\overline{Y}_t^{\alpha,x} = \overline{v}^\alpha(X_t^x)$; we have $\left|\overline{Y}_t^{\alpha,x}\right| \leqslant \dfrac{C}{\alpha}(1+|X_t^x|)$. Using the Lemma 2.1 of [43], we have, for every $T>0$:

$$\mathbb{E}\left[|X^{\varepsilon_n,x} - X^x|_{0,T}^{*,2}\right] \leqslant C\left\{\|\Xi - \Xi^{\varepsilon_n}\|_\infty^2 T + \mathbb{E}\left[\int_0^T |\sigma^{\varepsilon_n}(X_t^x) - \sigma(X_t^x)|^2 \, \mathrm{d}t\right]\right\}.$$

By dominated convergence, this quantity goes to 0 as n goes to infinity. Also by dominated convergence, we get:

$$\mathbb{E}\left[\int_0^T \left|Y_t^{\varepsilon_n,\alpha,x} - \overline{Y}_t^{\alpha,x}\right|^2 \mathrm{d}t\right] \underset{n\to\infty}{\longrightarrow} 0 \text{ and } \mathbb{E}\left[\left|Y_T^{\varepsilon_n,\alpha,x} - \overline{Y}_T^{\alpha,x}\right|^2\right] \underset{n\to\infty}{\longrightarrow} 0.$$

3. We will show that there exists a process $\overline{Z}^{\alpha,x}$ belonging to $L^2_{\mathcal{P},\mathrm{loc}}(\Omega, L^2(0,\infty;(\mathbb{R}^d)^*))$ which satisfies $\mathbb{E}\left[\int_0^T \left|Z_t^{\varepsilon_n,\alpha,x} - \overline{Z}_t^{\alpha,x}\right|^2 \mathrm{d}t\right] \underset{n\to\infty}{\longrightarrow} 0$, for every $T>0$. Indeed, $\left(\overline{Y}^{\alpha,x}, \overline{Z}^{\alpha,x}\right)$ is solution of BSDE (2.8); by uniqueness of the solution, $v^\alpha \equiv \overline{v}^\alpha$ and taking the limit in the equation (2.10) gives the result.

Let $n \leqslant m \in \mathbb{N}$, we define $\widetilde{Y} = Y^{\varepsilon_n,\alpha,x} - Y^{\varepsilon_m,\alpha,x}$ and $\widetilde{Z} = Z^{\varepsilon_n,\alpha,x} - Z^{\varepsilon_m,\alpha,x}$. We have:

$$\mathrm{d}\widetilde{Y}_t = \alpha \widetilde{Y}_t \, \mathrm{d}t + \underbrace{\left(\psi^{\varepsilon_m}\left(X_t^{\varepsilon_m,x}, Z_t^{\varepsilon_m,\alpha,x}\right) - \psi^{\varepsilon_n}\left(X_t^{\varepsilon_n,x}, Z_t^{\varepsilon_n,\alpha,x}\right)\right)}_{=:\psi_t} \mathrm{d}t + \widetilde{Z}_t \, \mathrm{d}W_t.$$

Thanks to Itô's formula, we obtain:

$$\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] \leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] - 2\mathbb{E}\left[\int_0^T \widetilde{Y}_t \psi_t \, \mathrm{d}t\right].$$

But, we have:

$$|\psi_t| \leqslant M_\psi\left(2 + |X_t^{\varepsilon_m,x}| + |X_t^{\varepsilon_n,x}|\right) + K_z\left(\left|Z_t^{\varepsilon_m,x}\sigma^{\varepsilon_m}\left(X_t^{\varepsilon_m,x}\right)^{-1}\right| + \left|Z_t^{\varepsilon_n,x}\sigma^{\varepsilon_n}\left(X_t^{\varepsilon_n,x}\right)^{-1}\right|\right).$$

So:

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] \leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] &+ 2TM_\psi \mathbb{E}\left[\left|\widetilde{Y}\right|_{0,T}^*\left(2 + |X^{\varepsilon_m,x}|_{0,T}^* + |X^{\varepsilon_n,x}|_{0,T}^*\right)\right] \\ &+ 2K_z\left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\left|\widetilde{Y}\right|_{0,T}^* \int_0^T \left(|Z_t^{\varepsilon_m,x}| + |Z_t^{\varepsilon_n,x}|\right) \mathrm{d}t\right].\end{aligned}$$

Now, we need to estimate the bounds for $\mathbb{E}\left[|X^{\varepsilon_n,x}|_{0,T}^{*,2}\right]$ and for $\mathbb{E}\left[\int_0^T |Z_t^{\varepsilon_n,\alpha,x}|^2 \, \mathrm{d}t\right]$ independently of n. Thanks to Proposition 2.4, we have $\mathbb{E}\left[|X^{\varepsilon_n,x}|_{0,T}^{*,2}\right] \leqslant C\left(1+|x|^2\right)$, where C only depends on T, r_1, r_2, ξ_1 and ξ_2. Also, we have:

$$\mathbb{E}\left[\int_0^T |Z_t^{\varepsilon_n,\alpha,x}|^2 \, \mathrm{d}t\right] \leqslant \mathbb{E}\left[|Y_T^{\varepsilon_n,\alpha,x}|^2\right] + 2\mathbb{E}\left[\int_0^T \left|Y_t^{\varepsilon_n,\alpha,x}\psi^{\varepsilon_n}\left(X_t^{\varepsilon_n,x}, Z_t^{\varepsilon_n,\alpha,x}\right)\right| \mathrm{d}t\right].$$

But, using the estimate of Lemma 2.12,

$$\begin{aligned}\left|Y_t^{\varepsilon_n,\alpha,x}\psi^{\varepsilon_n}\left(X_t^{\varepsilon_n,x},Z_t^{\varepsilon_n,\alpha,x}\right)\right| &\leqslant \frac{C}{\alpha}\left(1+\left|X_t^{\varepsilon_n,x}\right|\right)\left(M_\psi\left(1+\left|X_t^{\varepsilon_n,x}\right|\right)+K_z\left\|\sigma^{-1}\right\|_\infty\left|Z_t^{\varepsilon_n,\alpha,x}\right|\right)\\ &\leqslant 2\left(\frac{CM_\psi}{\alpha}+\frac{C^2K_z^2\|\sigma^{-1}\|_\infty^2}{\alpha^2}\right)\left(1+|X_t^{\varepsilon_n,x}|^2\right)+\frac{1}{4}\left|Z_t^{\varepsilon_n,\alpha,x}\right|^2.\end{aligned}$$

Finally,

$$\mathbb{E}\left[\int_0^T\left|Z_t^{\varepsilon_n,\alpha,x}\right|^2\mathrm{d}t\right]\leqslant\left\{4\frac{C^2}{\alpha^2}+8T\left(\frac{CM_\psi}{\alpha}+\frac{C^2K_z^2\|\sigma^{-1}\|_\infty^2}{\alpha^2}\right)\right\}\left(1+\mathbb{E}\left[|X^{\varepsilon_n,x}|_{0,T}^{*,2}\right]\right);$$

it proves, after using Proposition 2.4, that the sequence $(Z^{\varepsilon_n,\alpha,x})_n$ is a Cauchy sequence in $\mathrm{L}^2_{\mathcal{P}}(\Omega,\mathrm{L}^2([0,T],(\mathbb{R}^d)^*))$; we can define its limit process $\overline{Z}^{\alpha,x}\in\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega,\mathrm{L}^2(0,\infty;(\mathbb{R}^d)^*))$, and it satisfies the convergence we claimed.

□

Proposition 2.17. *Under Assumption 2.10, there exists a constant C, such that for every $\alpha\in(0,1]$, we have:*

$$\forall x,x'\in\mathbb{R}^d,\ \left|v^\alpha(x)-v^\alpha(x')\right|\leqslant C\left(1+|x|^2+|x'|^2\right)|x-x'|.$$

The constant C only depends on η_1, η_2, r_1, r_2, K_z, $\left\|\sigma^{-1}\right\|_\infty$, M_ψ, K_x, $\|\Xi\|_{\mathrm{lip}}$ and $\|\sigma\|_{\mathrm{lip}}$.

Proof. See Theorem 2.35 given in the appendix, with $f:(x,y,z)\mapsto\psi(x,z)-\alpha y$ and $g=v^\alpha$. □

Theorem 2.18 (Existence of solutions to the EBSDE)**.** *Under Assumption 2.10, there exist a real number $\overline{\lambda}$, a locally Lipschitz function $\overline{v}$, which satisfies $\overline{v}(0)=0$, and a process $\overline{Z}^x\in\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega,\mathrm{L}^2(0,\infty;(\mathbb{R}^d)^*))$ such that if we define $\overline{Y}_t^x=\overline{v}(X_t^x)$, then the EBSDE (2.7) is satisfied by $(\overline{Y}^x,\overline{Z}^x,\overline{\lambda})$ $\mathbb{P}$-a.s. and for all $0\leqslant t\leqslant T<\infty$. Moreover, there exists $C>0$ such that for all $x\in\mathbb{R}^d$, $|\overline{v}(x)|\leqslant C\left(1+|x|^2\right)$, and there exists $\overline{\zeta}$ measurable such that $\overline{Z}_t^x=\overline{\zeta}(X_t^x)$ $\mathbb{P}$-a.s. and for a.e. $t\geqslant 0$.*

Proof. The strategy is the same as in Theorem 4.4 of [26]. We give a sketch of the proof here for completeness.

Step 1: Construction of $\overline{v}$ by a diagonal procedure.
For every $\alpha>0$, we define $\overline{v}^\alpha(x)=v^\alpha(x)-v^\alpha(0)$; we recall that $|\overline{v}^\alpha(x)|\leqslant C\left(1+|x|^2\right)$ and $|\alpha v^\alpha(0)|\leqslant C$, with C independent of α. Let D be a countable dense set in $\mathbb{R}^d$; by a diagonal argument, we can construct a sequence (α_n), such that $(\overline{v}^{\alpha_n})_n$ converges pointwise over D to a function $\overline{v}$ and $\alpha_n v^{\alpha_n}(0)\underset{n\to\infty}{\longrightarrow}\overline{\lambda}$, for a convenient real number $\overline{\lambda}$. Moreover, thanks to the previous proposition:

$$\exists C>0,\ \forall\alpha\in(0,1],\ \forall x,x'\in\mathbb{R}^d,\ \left|\overline{v}^\alpha(x)-\overline{v}^\alpha(x')\right|\leqslant C\left(1+|x|^2+|x'|^2\right)|x-x'|.$$

Because it is uniformly continuous on every compact subset of D, $\overline{v}$ has an extension which is continuous on $\mathbb{R}^d$. Then, we can show that $\overline{v}$ is the pointwise limit of the functions $\overline{v}^{\alpha_n}$ on $\mathbb{R}^d$, and then $\overline{v}$ is locally Lipschitz continuous and has quadratic growth.

Step 2: Construction of the process $\overline{Z}^x$.
We will show that $(Z^{\alpha_n,x})_n$ is Cauchy in $\mathrm{L}^2_{\mathcal{P}}(\Omega;\mathrm{L}^2([0,T];(\mathbb{R}^d)^*))$ for every $T>0$. Then, we will be able to define $\overline{Z}^x\in\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega;\mathrm{L}^2(0,\infty;(\mathbb{R}^d)^*))$. When $n\leqslant m\in\mathbb{N}$, we set $\widetilde{Y}=\overline{Y}^{\alpha_n,x}-\overline{Y}^{\alpha_m,x}$ and $\widetilde{Z}=Z^{\alpha_n,x}-Z^{\alpha_m,x}$; we have:

$$\mathrm{d}\widetilde{Y}_t=-\widetilde{\psi}_t\,\mathrm{d}t+\left(\alpha_nY_t^{\alpha_n,x}-\alpha_mY_t^{\alpha_m,x}\right)\mathrm{d}t+\widetilde{Z}_t\,\mathrm{d}W_t,$$

where $\widetilde{\psi}_t = \psi\left(X_t^x, Z_t^{\alpha_n,x}\right) - \psi\left(X_t^x, Z_t^{\alpha_m,x}\right)$. Thanks to Itô's formula:

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] &= \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] - |Y_0|^2 + 2\mathbb{E}\left[\int_0^T \widetilde{\psi}_t \widetilde{Y}_t \,\mathrm{d}t\right] - 2\mathbb{E}\left[\int_0^T \left(\alpha_n Y_t^{\alpha_n,x} - \alpha_m Y_t^{\alpha_m,x}\right) \widetilde{Y}_t \,\mathrm{d}t\right] \\ &\leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] + 2K_z \left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right| \left|\widetilde{Z}_t\right| \,\mathrm{d}t\right] + 4M_\psi \mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right| \,\mathrm{d}t\right].\end{aligned}$$

By the Cauchy-Schwarz inequality, and noting that

$$\left|\widetilde{Y}_t\right| \left|\widetilde{Z}_t\right| \leqslant K_z \left\|\sigma^{-1}\right\|_\infty \left|\widetilde{Y}_t\right|^2 + \frac{1}{4K_z \left\|\sigma^{-1}\right\|_\infty} \left|\widetilde{Z}_t\right|^2,$$

we obtain:

$$\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] \leqslant 2\mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] + 4K_z^2 \left\|\sigma^{-1}\right\|_\infty^2 \mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right|^2 \mathrm{d}t\right] + 8M_\psi \sqrt{T} \mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right|^2 \mathrm{d}t\right]^{\frac{1}{2}}.$$

Dominated convergences (using Propositions 2.16 and 2.4) give us the result claimed.

Step 3: $\left(\overline{Y}^x, \overline{Z}^x, \overline{\lambda}\right)$ is a solution to the EBSDE (2.7).
Taking the limit in the BSDE satisfied by $(Y^{\alpha_n,x}, Z^{\alpha_n,x})$ gives us:

$$\overline{Y}_t^x = \overline{Y}_T^x + \int_t^T \left[\psi\left(X_s^x, \overline{Z}_s^x\right) - \overline{\lambda}\right] \mathrm{d}s - \int_t^T \overline{Z}_s^x \,\mathrm{d}W_s.$$

Step 4: $\overline{Z}^x$ can be represented as a measurable function of X^x. We fix $T > 0$. Here, we denote $\Delta\bullet := \bullet^{\alpha,x} - \bullet^{\alpha,x'}$. By standard calculations, for every $\alpha \in (0,1]$, $x, x' \in \mathbb{R}^d$:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T |\Delta Z_t|^2 \,\mathrm{d}t\right] \\ &\leqslant \mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right] + 2K_x T \mathbb{E}\left[|\Delta Y|_{0,T}^* |\Delta X|_{0,T}^*\right] + 2K_z \left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^* |\Delta Z_t| \,\mathrm{d}t\right] \\ &\qquad + 4K_z \left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^* \left|Z_t^{\alpha,x'}\right| \mathrm{d}t\right] \\ &\leqslant 2\mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right] + 4K_x T \mathbb{E}\left[|\Delta Y|_{0,T}^* |\Delta X|_{0,T}^*\right] + 2K_z^2 \left\|\sigma^{-1}\right\|_\infty^2 T \mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right] \\ &\qquad + 8K_z \left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^* \left|Z_t^{\alpha,x'}\right| \mathrm{d}t\right]\end{aligned}$$

But, using the Cauchy-Schwarz inequality and the estimate at the end of the proof of Proposition 2.16, we get:

$$\mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^* \left|Z_t^{\alpha,x'}\right| \mathrm{d}t\right] \leqslant C\mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right]^{\frac{1}{2}} (1 + |x|).$$

Using Lemma 2.1 of [43] and Propositions 2.4 and 2.17, we finally get:

$$\mathbb{E}\left[\int_0^T \left|Z_t^{\alpha,x} - Z_t^{\alpha,x'}\right|^2 \mathrm{d}t\right] \leqslant \mathcal{C}\left(1 + |x|^4 + |x'|^4\right) |x - x'|^2, \tag{2.11}$$

where $\mathcal{C}$ is independent of x and x', but depends on α and T. For every $x \in \mathbb{R}^d$, the sequence $\left(\mathbb{E}\left[\int_0^T \left|Z_t^{\alpha_n,x} - Z_t^{\alpha_m,x}\right|^2 \mathrm{d}t\right]\right)_{n \leqslant m \in \mathbb{N}}$ is bounded (it converges). By a diagonal procedure, there exists a subsequence $(\alpha_n') \subset (\alpha_n)$ such that:

$$\forall x \in D,\ \forall n \leqslant m \in \mathbb{N},\ \mathbb{E}\left[\int_0^T \left|Z_t^{\alpha_n',x} - Z_t^{\alpha_m',x}\right|^2 \mathrm{d}t\right] \leqslant 2^{-n}.$$

Equation (2.11) extends this inequality to $\mathbb{R}^d$. The Borel-Cantelli theorem gives that for a.e. $t \in [0,T]$, $Z_t^{\alpha'_n,x} \underset{n\to\infty}{\longrightarrow} \overline{Z}_t^x$ $\mathbb{P}$-a.s. Set

$$\overline{\zeta}(x) = \begin{cases} \lim_n \zeta^{\alpha'_n}(x), & \text{if the limit exists,} \\ 0, & \text{elsewhere.} \end{cases}$$

For a.e. $t \in [0,T]$, X_t^x belongs $\mathbb{P}$-a.s. to the set where $\lim_n \zeta^{\alpha'_n}(x)$ exists; $\overline{Z}_t^x = \overline{\zeta}\left(X_t^x\right)$ $\mathbb{P}$-a.s. and for a.e. $t \in [0,T]$.

□

2.4.2 Uniqueness of the solution

Theorem 2.19 (Uniqueness of the parameter λ). *Let $p > 0$; we suppose that*

$$\sqrt{r_2}K_z \left\|\sigma^{-1}\right\|_\infty + [(p \vee 2) - 1]\frac{r_2}{2} < \eta_2$$

and that Assumption 2.10 holds. We suppose that, for some $x \in \mathbb{R}^d$, (Y', Z', λ') verifies the EBSDE (2.7) $\mathbb{P}$-a.s. and for all $0 \leqslant t \leqslant T < \infty$, where Y' is a progressively measurable continuous process, $Z' \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega, L^2(0,\infty;(\mathbb{R}^d)^*))$ and $\lambda' \in \mathbb{R}$. Finally, we assume that there exists $c_x > 0$ (that may depend on x) such that

$$\forall t \geqslant 0, |Y'_t| \leqslant c_x \left(1 + |X_t^x|^p\right).$$

Then $\lambda' = \overline{\lambda}$.

Proof. We define $\widetilde{\lambda} = \lambda' - \overline{\lambda}$, $\widetilde{Y} = Y' - \overline{Y}$ and $\widetilde{Z} = Z' - \overline{Z}$. We have:

$$\widetilde{\lambda} = \frac{\widetilde{Y}_T - \widetilde{Y}_0}{T} + \frac{1}{T}\int_0^T \left[\psi\left(X_t^x, Z'_t\right) - \psi\left(X_t^x, \overline{Z}_t^x\right)\right] \mathrm{d}t - \frac{1}{T}\int_0^T \widetilde{Z}_t \,\mathrm{d}W_t.$$

We denote:

$$\gamma_t^x = \begin{cases} \dfrac{\psi\left(X_t^x, Z'_t\right) - \psi\left(X_t^x, \overline{Z}_t^x\right)}{\left|\widetilde{Z}_t\right|^2}\widetilde{Z}_t, & \text{if } \widetilde{Z}_t \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

There exists a probability $\widetilde{\mathbb{P}}^{x,T}$ under which $\widetilde{W}_t^x = W_t - \displaystyle\int_0^t \gamma_s^{x*}\,\mathrm{d}s$ is a Brownian motion on $[0,T]$. Then,

$$\widetilde{\lambda} = \frac{1}{T}\widetilde{\mathbb{E}}^{x,T}\left[\widetilde{Y}_T - \widetilde{Y}_0\right] \leqslant \frac{1}{T}\left\{c_x\left(2 + |x|^p + \widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right]\right) + C\left(2 + |x|^2 + \widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^2\right]\right)\right\}.$$

We conclude by taking the limit $T \to \infty$ and using Proposition 2.5. □

Theorem 2.20 (Uniqueness of the functions v and ζ). *Let $p > 0$; we suppose that*

$$\sqrt{r_2}K_z \left\|\sigma^{-1}\right\|_\infty + [(p \vee 2) - 1]\frac{r_2}{2} < \eta_2$$

and that Assumption 2.10 holds.
Let (v, ζ) and $(\widetilde{v}, \widetilde{\zeta})$ be two couples of functions with:

- *$v, \widetilde{v} : \mathbb{R}^d \to \mathbb{R}$ are continuous, $|v(x)| \leqslant C\left(1 + |x|^p\right)$, $|\widetilde{v}(x)| \leqslant C\left(1 + |x|^p\right)$ and $v(0) = \widetilde{v}(0) = 0$;*
- *$\zeta, \widetilde{\zeta} : \mathbb{R}^d \to (\mathbb{R}^d)^*$ are measurable.*

We also assume that for some constants λ, $\widetilde{\lambda}$, and for all $x \in \mathbb{R}^d$, the triplets $(v(X_t^x), \zeta(X_t^x), \lambda)$ and $(\widetilde{v}(X_t^x), \widetilde{\zeta}(X_t^x), \widetilde{\lambda})$ verify the EBSDE (2.7).
Then $\lambda = \widetilde{\lambda}$, $v = \widetilde{v}$ and $\zeta(X_t^x) = \widetilde{\zeta}(X_t^x)$ $\mathbb{P}$-a.s. and for a.e. $t \geqslant 0$ and for all $x \in \mathbb{R}^d$.

Proof. By Theorem 2.19, we already know that $\lambda = \widetilde{\lambda}$. We denote $Y_t^x = v(X_t^x)$, $Z_t^x = \zeta(X_t^x)$, $\widetilde{Y}_t^x = \widetilde{v}(X_t^x)$ and $\widetilde{Z}_t^x = \widetilde{\zeta}(X_t^x)$. We approximate the functions Ξ, σ, ψ, v and $\widetilde{v}$ by sequences (Ξ^ε), (σ^ε), (ψ^ε), (v^ε) and $(\widetilde{v}^\varepsilon)$ of $\mathcal{C}^1$ functions with bounded derivatives that converge uniformly (only on every compact set for the approximations of v and $\widetilde{v}$). We require that for all $\varepsilon > 0$, the bounds on the derivatives of Ξ^ε, σ^ε and ψ^ε are independent of ε and v^ε, $\widetilde{v}^\varepsilon$ have polynomial growth (with constant C and exponent p, all independent of ε). In the sequel, $x \in \mathbb{R}^d$, $T > 0$ and $\varepsilon > 0$ are fixed. Let us consider the following BSDEs in finite time horizon:

$$\begin{cases} \mathrm{d}Y_t^{\varepsilon,x} = -[\psi^\varepsilon(X_t^{\varepsilon,x}, Z_t^{\varepsilon,x}) - \lambda]\,\mathrm{d}t + Z_t^{\varepsilon,x}\,\mathrm{d}W_t, \\ Y_T^{\varepsilon,x} = v^\varepsilon(X_T^{\varepsilon,x}), \end{cases}$$

$$\begin{cases} \mathrm{d}\widetilde{Y}_t^{\varepsilon,x} = -[\psi^\varepsilon(X_t^{\varepsilon,x}, \widetilde{Z}_t^{\varepsilon,x}) - \lambda]\,\mathrm{d}t + \widetilde{Z}_t^{\varepsilon,x}\,\mathrm{d}W_t, \\ \widetilde{Y}_T^{\varepsilon,x} = \widetilde{v}^\varepsilon(X_T^{\varepsilon,x}), \end{cases}$$

where $X_t^{\varepsilon,x} = x + \int_0^t \Xi^\varepsilon(X_s^{\varepsilon,x})\,\mathrm{d}s + \int_0^t \sigma^\varepsilon(X_s^{\varepsilon,x})\,\mathrm{d}W_s$. We denote

$$\Delta Y_t^x = Y_t^x - \widetilde{Y}_t^x,\ \Delta Z_t^x = Z_t^x - \widetilde{Z}_t^x,\ \Delta Y_t^{\varepsilon,x} = Y_t^{\varepsilon,x} - \widetilde{Y}_t^{\varepsilon,x} \text{ and } \Delta Z_t^{\varepsilon,x} = Z_t^{\varepsilon,x} - \widetilde{Z}_t^{\varepsilon,x}.$$

This way, we get:

$$\Delta Y_0^x - \mathbb{E}[\Delta Y_T^x] = \mathbb{E}\left[\int_0^T \left[\psi(X_t^x, Z_t^x) - \psi\left(X_t^x, \widetilde{Z}_t^x\right)\right]\mathrm{d}t\right],$$

$$\Delta Y_0^{\varepsilon,x} - \mathbb{E}\left[\Delta Y_T^{\varepsilon,x}\right] = \mathbb{E}\left[\int_0^T \left[\psi^\varepsilon\left(X_t^{\varepsilon,x}, Z_t^{\varepsilon,x}\right) - \psi^\varepsilon\left(X_t^{\varepsilon,x}, \widetilde{Z}_t^{\varepsilon,x}\right)\right]\mathrm{d}t\right].$$

By subtraction, it leads us to:

$$\begin{aligned}\left|\Delta Y_0^x - \Delta Y_0^{\varepsilon,x}\right| \leqslant \mathbb{E}\left[\int_0^T \left|\psi(X_t^x, Z_t^x) - \psi^\varepsilon\left(X_t^{\varepsilon,x}, Z_t^{\varepsilon,x}\right)\right|\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T \left|\psi\left(X_t^x, \widetilde{Z}_t^x\right) - \psi^\varepsilon\left(X_t^{\varepsilon,x}, \widetilde{Z}_t^{\varepsilon,x}\right)\right|\mathrm{d}t\right] \\ + \mathbb{E}\left[\left|v(X_T^x) - v^\varepsilon(X_T^{\varepsilon,x})\right|\right] + \mathbb{E}\left[\left|\widetilde{v}(X_T^x) - \widetilde{v}^\varepsilon(X_T^{\varepsilon,x})\right|\right].\end{aligned}$$

Set $\delta X_t^{\varepsilon,x} = X_t^x - X_t^{\varepsilon,x}$, $\delta Y_t^{\varepsilon,x} = Y_t^x - Y_t^{\varepsilon,x}$ and $\delta Z_t^{\varepsilon,x} = Z_t^x - Z_t^{\varepsilon,x}$. We have:

$$\left|\psi(X_t^x, Z_t^x) - \psi^\varepsilon\left(X_t^{\varepsilon,x}, Z_t^{\varepsilon,x}\right)\right| \leqslant \left(K_x + K_z |Z_t^x| \|\sigma\|_{\mathrm{lip}} \|\sigma^{-1}\|_\infty^2\right) |\delta X_t^{\varepsilon,x}| + K_z \|\sigma^{-1}\|_\infty \left|\delta Z_t^{\varepsilon,x}\right| + \|\psi - \psi^\varepsilon\|_\infty.$$

Using Lemma 2.1 of [43], we get:

$$\mathbb{E}\left[|\delta X^{\varepsilon,x}|_{0,T}^{*,2}\right] \leqslant C_T\left(\|\Xi - \Xi^\varepsilon\|_\infty^2 + \|\sigma - \sigma^\varepsilon\|_\infty^2\right).$$

Our next goal is to estimate $\mathbb{E}\left[\int_0^T \left|\delta Z_t^{\varepsilon,x}\right|^2 \mathrm{d}t\right]$. We see that $\left(\delta Y_t^{\varepsilon,x}, \delta Z_t^{\varepsilon,x}\right)$ is solution of the BSDE:

$$\begin{cases} \mathrm{d}\delta Y_t^{\varepsilon,x} = -\left[\psi(X_t^x, Z_t^x) - \psi^\varepsilon\left(X_t^{\varepsilon,x}, Z_t^{\varepsilon,x}\right)\right]\mathrm{d}t + \delta Z_t^{\varepsilon,x}\,\mathrm{d}W_t, \\ \delta Y_T^{\varepsilon,x} = v\left(X_T^x\right) - v^\varepsilon\left(X_T^{\varepsilon,x}\right). \end{cases}$$

Using Lemma 2.2 of [43], we get:

$$\mathbb{E}\left[\int_0^T \left|\delta Z_t^{\varepsilon,x}\right|^2 \mathrm{d}t\right] \leqslant C_T \mathbb{E}\left[\left|v\left(X_T^x\right) - v^\varepsilon\left(X_T^{\varepsilon,x}\right)\right|^2 + \int_0^T \left|\psi(X_t^x, Z_t^x) - \psi^\varepsilon\left(X_t^{\varepsilon,x}, Z_t^x\right)\right|^2 \mathrm{d}t\right].$$

We need to prove that $\mathbb{E}\left[\left|v\left(X_T^x\right) - v^\varepsilon\left(X_T^{\varepsilon,x}\right)\right|^2\right] \underset{\varepsilon \to 0}{\longrightarrow} 0$:

- on the one hand, using uniform local Lipschitz property,

$$\mathbb{E}\left[\left|v^\varepsilon(X_T^x)-v^\varepsilon(X_T^{\varepsilon,x})\right|^2\right]\leqslant C\left(1+\mathbb{E}\left[|X^x|_{0,T}^{*,8}\right]^{1/2}+\mathbb{E}\left[|X^{\varepsilon,x}|_{0,T}^{*,8}\right]^{1/2}\right)\mathbb{E}\left[|\delta X^{\varepsilon,x}|_{0,T}^{*,4}\right]^{1/2}\underset{\varepsilon\to 0}{\longrightarrow}0,$$

- on the other hand, for any $R>0$, we have

$$\begin{aligned}
\mathbb{E}\left[|(v-v^\varepsilon)(X_T^x)|^2\,\mathbb{1}_{|X_T^x|\leqslant R}\right]&\leqslant \|v-v^\varepsilon\|_{\infty,\mathcal{B}_R}^2,\\
\mathbb{E}\left[|(v-v^\varepsilon)(X_T^x)|^2\,\mathbb{1}_{|X_T^x|> R}\right]&\leqslant \mathbb{E}\left[|(v-v^\varepsilon)(X_T^x)|^4\right]^{1/2}\mathbb{P}\left(|X_T^x|>R\right)^{1/2}\\
&\leqslant C\left(1+\mathbb{E}\left[|X_T^x|^8\right]^{1/2}\right)\frac{\mathbb{E}\left[|X_T^x|^2\right]^{1/2}}{R}\leqslant\frac{\tilde{C}(1+|x|^5)}{R}.
\end{aligned}$$

For any $R>0$, $\limsup_{\varepsilon\to 0}\mathbb{E}\left[|(v-v^\varepsilon)(X_T^x)|^2\right]\leqslant\dfrac{\tilde{C}(1+|x|^5)}{R}$, i.e. $\mathbb{E}\left[\left|(v-v^\varepsilon)(X_T^x)\right|^2\right]\underset{\varepsilon\to 0}{\longrightarrow}0.$

Finally, we obtain: $\left|\Delta Y_0^x-\Delta Y_0^{\varepsilon,x}\right|\underset{\varepsilon\to 0}{\longrightarrow}0$. By Theorem 3.1 of [43], because ψ^ε, v^ε and $\widetilde{v}^\varepsilon$ have bounded derivatives, there exist continuous functions ζ_T^ε and $\widetilde{\zeta}_T^\varepsilon$ such that: $Z_t^{\varepsilon,x}=\zeta_T^\varepsilon\left(t,X_t^{\varepsilon,x}\right)$ and $\widetilde{Z}_t^{\varepsilon,x}=\widetilde{\zeta}_T^\varepsilon\left(t,X_t^{\varepsilon,x}\right)$ $\mathbb{P}$-a.s. and for a.e. $t\in[0,T]$. As usual, we linearise our BSDE; so we set:

$$\Gamma_T^\varepsilon(t,x)=\begin{cases}\dfrac{\psi^\varepsilon\left(x,\zeta_T^\varepsilon(t,x)\right)-\psi^\varepsilon\left(x,\widetilde{\zeta}_T^\varepsilon(t,x)\right)}{\left|\zeta_T^\varepsilon(t,x)-\widetilde{\zeta}_T^\varepsilon(t,x)\right|^2}\left(\zeta_T^\varepsilon(t,x)-\widetilde{\zeta}_T^\varepsilon(t,x)\right), & \text{if } \zeta_T^\varepsilon(t,x)\neq\widetilde{\zeta}_T^\varepsilon(t,x),\\ 0, & \text{otherwise.}\end{cases}$$

The process $\Gamma_T^\varepsilon\left(t,X_t^{\varepsilon,x}\right)^*$ is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$; by Girsanov's theorem,

$$W_t^{\varepsilon,x,T}=W_t-\int_0^t\Gamma_T^\varepsilon\left(s,X_s^{\varepsilon,x}\right)^*\,\mathrm{d}s$$

is a Brownian motion on $[0,T]$ under the corresponding Girsanov probability $\mathbb{Q}_T^{\varepsilon,x}$. $\Delta Y^{\varepsilon,x}$ is a $\mathbb{Q}_T^{\varepsilon,x}$-martingale and we get:

$$\Delta Y_0^{\varepsilon,x}=\mathbb{E}^{\mathbb{Q}_T^{\varepsilon,x}}\left[\Delta Y_T^{\varepsilon,x}\right]=\mathbb{E}^{\mathbb{Q}_T^{\varepsilon,x}}\left[\left(v^\varepsilon-\widetilde{v}^\varepsilon\right)\left(X_T^{\varepsilon,x}\right)\right]=\mathcal{P}_T^\varepsilon\left[v^\varepsilon-\widetilde{v}^\varepsilon\right](x),$$

where $\mathcal{P}_T^\varepsilon$ is the Kolmogorov semigroup of the SDE:

$$\mathrm{d}U_t^x=\left[\Xi^\varepsilon\left(U_t^x\right)+\sigma^\varepsilon\left(U_t^x\right)\Gamma_T^\varepsilon\left(t,U_t^x\right)^*\right]\,\mathrm{d}t+\sigma^\varepsilon\left(U_t^x\right)\,\mathrm{d}W_t.$$

Using Lemma 2.15 and Corollary 2.9, we get:

$$\left|\mathcal{P}_T^\varepsilon\left[v^\varepsilon-\widetilde{v}^\varepsilon\right](x)-\mathcal{P}_T^\varepsilon\left[v^\varepsilon-\widetilde{v}^\varepsilon\right](0)\right|\leqslant C\left(1+|x|^{p\vee 2}\right)\mathrm{e}^{-\nu T},$$

where ν and C are independent of ε (because the polynomial growths of v^ε and $\widetilde{v}^\varepsilon$ do not depend on ε). Finally,

$$\forall\varepsilon>0,\ \forall T>0,\ \forall x\in\mathbb{R}^d,\ \left|\Delta Y_0^{\varepsilon,x}-\Delta Y_0^{\varepsilon,0}\right|\leqslant C\left(1+|x|^{p\vee 2}\right)\mathrm{e}^{-\nu T}.$$

By taking the limit as ε goes to 0:

$$\forall T>0,\ \forall x\in\mathbb{R}^d,\ \left|\left(v-\widetilde{v}\right)(x)\right|=\left|\Delta Y_0^x-\Delta Y_0^0\right|\leqslant C\left(1+|x|^{p\vee 2}\right)\mathrm{e}^{-\nu T}.$$

Taking the limit as T goes to infinity leads us to $v=\widetilde{v}$. Then, uniqueness of ζ is the consequence of Itô's formula. □

2.5 Large time behaviour

In this section, we always suppose that

$$\sqrt{r_2}K_z \left\|\sigma^{-1}\right\|_\infty + \frac{p-1}{2}r_2 < \eta_2,$$

with $p \geqslant 2$, and we keep working under Assumption 2.10. Indeed, we have seen in the previous section that, under this assumption, there exists a unique triplet (v, ζ, λ) such that $(v(X_t^x), \zeta(X_t^x), \lambda)$ is a solution to EBSDE (2.7), v is continuous with quadratic growth, $v(0) = 0$ and ζ is measurable. Let ξ^T be a real random variable $\mathcal{F}_T$-measurable and such that $\left|\xi^T\right| \leqslant C(1 + \left|X_T^x\right|^p)$. It will allow us to use Proposition 2.5. We denote by $(Y_t^{T,x}, Z_t^{T,x})$ the solution of the BSDE in finite horizon:

$$Y_t^{T,x} = \xi^T + \int_t^T \psi\left(X_s^x, Z_s^{T,x}\right) \mathrm{d}s - \int_t^T Z_s^{T,x}\,\mathrm{d}W_s.$$

Theorem 2.21. *We have the following inequality:*

$$\left|\frac{Y_0^{T,x}}{T} - \lambda\right| \leqslant \frac{C\left(1+|x|^p\right)}{T},$$

where the constant C is independent of x and T; and in particular: $\frac{Y_0^{T,x}}{T} \underset{T\to\infty}{\longrightarrow} \lambda$, uniformly in any bounded subset of $\mathbb{R}^d$.

Proof. For all $x \in \mathbb{R}^d$ and $T > 0$, we write:

$$\left|\frac{Y_0^{T,x}}{T} - \lambda\right| \leqslant \left|\frac{Y_0^{T,x} - Y_0^x - \lambda T}{T}\right| + \left|\frac{Y_0^x}{T}\right|.$$

First of all, $\left|Y_0^x\right| = |v(x)| \leqslant C\left(1+|x|^2\right)$. Also, by the usual linearisation technique, we have:

$$Y_0^{T,x} - Y_0^x - \lambda T = \xi^T - v\left(X_T^x\right) + \int_0^T \left(Z_s^{T,x} - Z_s^x\right)\beta_s^{T*}\,\mathrm{d}s - \int_0^T \left(Z_s^{T,x} - Z_s^x\right)\mathrm{d}W_s,$$

where

$$\beta_t^T = \begin{cases} \dfrac{\psi\left(X_t^x, Z_t^{T,x}\right) - \psi\left(X_t^x, Z_t^x\right)}{\left|Z_t^{T,x} - Z_t^x\right|^2}\left(Z_t^{T,x} - Z_t^x\right), & \text{if } Z_t^{T,x} \neq Z_t^x, \\ 0, & \text{otherwise.} \end{cases}$$

The process β^T is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$ and by Girsanov's theorem, there exists a probability measure $\mathbb{Q}^T$ under which $\widetilde{W}_t^T = W_t - \int_0^t \beta_s^{T*}\,\mathrm{d}s$ is a Brownian motion on $[0,T]$. This way, we can see that:

$$\left|Y_0^{T,x} - Y_0^x - \lambda T\right| = \left|\mathbb{E}^{\mathbb{Q}^T}\left[\xi^T - v\left(X_T^x\right)\right]\right| \leqslant \mathbb{E}^{\mathbb{Q}^T}\left[\left|\xi^T\right|\right] + \mathbb{E}^{\mathbb{Q}^T}\left[\left|v\left(X_T^x\right)\right|\right] \leqslant C\left(1 + \mathbb{E}^{\mathbb{Q}^T}\left[|X_T^x|^p\right]\right).$$

Thanks to Proposition 2.5, we get: $\sup_{T\geqslant 0} \mathbb{E}^{\mathbb{Q}^T}\left[|X_T^x|^p\right] \leqslant \kappa\left(1+|x|^p\right)$, where κ is independent of x. □

Theorem 2.22. *We suppose that $\xi^T = g\left(X_T^x\right)$, where $g : \mathbb{R}^d \to \mathbb{R}$ has polynomial growth:*

$$\forall x \in \mathbb{R}^d,\ |g(x)| \leqslant C\left(1+|x|^p\right)$$

and satisfies:

$$\forall x, x' \in \mathbb{R}^d,\ |g(x)-g(x')| \leqslant C\left(1+|x|^p+|x'|^p\right)|x-x'|.$$

Then, there exists $L \in \mathbb{R}$, such that: $\forall x \in \mathbb{R}^d$, $Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to\infty}{\longrightarrow} L$. Furthermore,

$$\forall x \in \mathbb{R}^d,\ \forall T>0,\ \left|Y_0^{T,x} - \lambda T - Y_0^x - L\right| \leqslant C\left(1+|x|^p\right)\mathrm{e}^{-\nu T}.$$

Proof. We will consider the following equations:

$$\left\{\begin{array}{rcl} \mathrm{d}Y_s^{T,t,x} &=& -\psi(X_s^{t,x}, Z_s^{T,t,x})\,\mathrm{d}s + Z_s^{T,t,x}\mathrm{d}W_s \\ Y_T^{T,t,x} &=& g(X_T^{t,x}) \end{array}\right. \longrightarrow \begin{array}{l} Y_s^{T,t,x} = u_T(s, X_s^{t,x}), \\ \text{see Theorem 4.1 of [25];} \end{array}$$

$$\left\{\begin{array}{rcl} \mathrm{d}Y_s^{t,x} &=& -\{\psi(X_s^{t,x}, Z_s^{t,x})-\lambda\}\,\mathrm{d}s + Z_s^{t,x}\mathrm{d}W_s \\ Y_T^{t,x} &=& v(X_T^{t,x}) \end{array}\right. \longrightarrow \begin{array}{l} Y_s^{t,x} = v(X_s^{t,x}), \\ \text{solution of the EBSDE;} \end{array}$$

$$\left\{\begin{array}{rcl} \mathrm{d}Y_s^{n,t,x} &=& -\psi^n(X_s^{n,t,x}, Z_s^{n,t,x})\,\mathrm{d}s + Z_s^{n,t,x}\mathrm{d}W_s \\ Y_T^{n,t,x} &=& g^n(X_T^{n,t,x}) \end{array}\right. \longrightarrow \begin{array}{l} Y_s^{n,t,x} = u_T^n(s, X_s^{n,t,x}), \\ \text{where } u_T^n \text{ is } \mathcal{C}^1 \text{ (see appendix);} \end{array}$$

$$\left\{\begin{array}{rcl} \mathrm{d}\widetilde{Y}_s^{n,t,x} &=& -\psi^n(X_s^{n,t,x}, \widetilde{Z}_s^{n,t,x})\,\mathrm{d}s + \widetilde{Z}_s^{n,t,x}\mathrm{d}W_s \\ \widetilde{Y}_T^{n,t,x} &=& v^n(X_T^{n,t,x}) \end{array}\right. \longrightarrow \begin{array}{l} \widetilde{Y}_s^{n,t,x} = \widetilde{u}_T^n(s, X_s^{n,t,x}), \\ \text{where } \widetilde{u}_T^n \text{ is } \mathcal{C}^1 \text{ (see appendix);} \end{array}$$

where $g^n, v^n \in \mathcal{C}_b^1$ converge uniformly on every compact towards g and v; also, we take $(\Xi^n)_n$ and $(\sigma^n)_n$ some sequences of $\mathcal{C}_b^1$ functions that converge uniformly towards Ξ and σ and define $X^{n,t,x}$ as the solution of the SDE

$$\left\{\begin{array}{rcl} \mathrm{d}X_s^{n,t,x} &=& \Xi^n(X_s^{n,t,x})\,\mathrm{d}s + \sigma^n(X_s^{n,t,x})\,\mathrm{d}W_s, \\ X_t^{n,t,x} &=& x. \end{array}\right.$$

Step 1: Approximation of the function ψ.

We set $\rho(x,q) = \psi(x, q\sigma(x))$; ρ is Lipschitz, and we approximate it by a sequence of $\mathcal{C}_b^1$ functions (ρ^n) which converges uniformly on $\mathbb{R}^d \times (\mathbb{R}^d)^*$. Then, we define a function β^n satisfying:

$$\beta^n(x,q) = \left\{\begin{array}{ll} \rho^n(x,q), & \text{if } |q\sigma(x)| \leqslant n, \\ 0, & \text{if } |q\sigma(x)| \geqslant f(n), \end{array}\right.$$

where $f(n)$ is chosen such that β^n and ρ^n have the same Lipschitz constant. This way, $\beta^n \underset{n\to\infty}{\longrightarrow} \rho$, uniformly on every compact, and $\beta^n \in \mathcal{C}_b^1$. We set $\psi^n(x,z) = \beta^n(x, z\sigma^n(x)^{-1})$ and $\psi^n \in \mathcal{C}_b^1$. Moreover, we have the following:

$$\begin{aligned} |(\psi-\psi^n)(x,z)| &\leqslant \left|(\rho-\rho^n)(x, z\sigma(x)^{-1})\right|\mathbb{1}_{|z|\leqslant n} + 2|\rho(x, z\sigma(x)^{-1})|\mathbb{1}_{|z|>n} \\ &\qquad + \left|\beta^n(x, z\sigma(x)^{-1}) - \beta^n(x, z\sigma^n(x)^{-1})\right| \\ &\leqslant \|\rho-\rho^n\|_\infty + 2(M_\psi(1+|x|) + K_z\|\sigma^{-1}\|_\infty|z|)\mathbb{1}_{|z|>n} + K_z\|\sigma^{-1}\|_\infty^2\|\sigma-\sigma^n\|_\infty|z| \end{aligned}$$

Step 2: Growth of w_T^n.

We set $w_T^n(t,x) = u_T^n(t,x) - \widetilde{u}_T^n(t,x)$ and $w_T(t,x) = u_T(t,x) - \lambda(T-t) - v(x)$. By uniqueness of viscosity solutions (see [25]), we have $u_T^n(0,x) = u_{T+S}^n(S,x)$ and $\widetilde{u}_T^n(0,x) = \widetilde{u}_{T+S}^n(S,x)$.

Lemma 2.23. *We have:* $\forall \varepsilon > 0$, $\forall T > 0$, $\exists \widetilde{C}_T > 0$, $\forall x \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$,

$$|v(x) - \widetilde{u}^n_T(0,x) + \lambda T|^2$$
$$\leqslant \widetilde{C}_T \left\{ \mathbb{E}\left[\left| v\left(X^x_T\right) - v^n\left(X^{n,x}_T\right)\right|^2 \right] + \|\rho - \rho^n\|^2_\infty + \left(\|\Xi - \Xi^n\|^2_\infty + \|\sigma - \sigma^n\|^2_\infty + \frac{1}{n^\varepsilon} \right) \left(1 + |x|^{2(2+\varepsilon)}\right) \right\}.$$

Proof. We set $\delta \widetilde{Y}^n_s = Y^x_s - \widetilde{Y}^{n,x}_s + \lambda(T-s)$ and $\delta \widetilde{Z}^n_s = Z^x_s - \widetilde{Z}^{n,x}_s$; then,

$$\mathrm{d}\delta \widetilde{Y}^n_s = \mathrm{d}Y^x_s - \mathrm{d}\widetilde{Y}^{n,x}_s - \lambda \mathrm{d}s = \left\{ \psi^n\left(X^{n,x}_s, \widetilde{Z}^{n,x}_s\right) - \psi\left(X^x_s, Z^x_s\right) \right\} \mathrm{d}s + \delta \widetilde{Z}^n_s \, \mathrm{d}W_s.$$

We will use [8]; to use the same notations, we set $f(s,y,z) = \psi\left(X^x_s, Z^x_s\right) - \psi^n\left(X^{n,x}_s, Z^x_s - z\right)$. We have:

$$\begin{aligned} |f(s,y,z)| \leqslant{}& \|\rho - \rho^n\|_\infty + \left(2M_\psi\left(1 + |X^x_s|\right) + 2K_z \left\|\sigma^{-1}\right\|_\infty |Z^x_s|\right) \mathbb{1}_{|Z^x_s| > n} \\ &+ K_z |Z^x_s| \left\|\sigma^{-1}\right\|^2_\infty \|\sigma - \sigma^n\|_\infty + K_x \left|X^x_s - X^{n,x}_s\right| \\ &+ K_z \left|Z^x_s\right| \|\sigma\|_{\mathrm{lip}} \left\|\sigma^{-1}\right\|^2_\infty \left|X^x_s - X^{n,x}_s\right| + K_z \left\|\sigma^{-1}\right\|_\infty |z| \qquad =: f_s + \lambda |z|. \end{aligned}$$

We define $F = \int_0^T f_s \, \mathrm{d}s$; because $F \in \mathrm{L}^2$ and $\delta \widetilde{Y}^n \in \mathcal{S}^2$, we get, $\mathbb{E}\left[\left|\delta \widetilde{Y}^n\right|^{*,2}_{0,T} \right] \leqslant C e^{2\lambda^2 T} \mathbb{E}\left[\left|\delta \widetilde{Y}^n_T\right|^2 + F^2 \right]$. We just need an upper bound for $\mathbb{E}\left[F^2\right]$:

$$\begin{aligned} \mathbb{E}\left[F^2\right] \leqslant{}& 6T^2 K_x^2 \mathbb{E}\left[|X^x - X^{n,x}|^{*,2}_{0,T} \right] + 6K_z^2 \|\sigma\|^2_{\mathrm{lip}} \left\|\sigma^{-1}\right\|^4_\infty \mathbb{E}\left[|X^x - X^{n,x}|^{*,2}_{0,T} \left(\int_0^T |Z^x_s| \, \mathrm{d}s \right)^2 \right] \\ &+ 6T^2 \|\rho - \rho^n\|^2_\infty + 48 M_\psi^2 \mathbb{E}\left[\left(1 + |X^x|^{*,2}_{0,T}\right) \left(\int_0^T \mathbb{1}_{|Z^x_s| > n} \, \mathrm{d}s \right)^2 \right] \\ &+ 24 K_z^2 \left\|\sigma^{-1}\right\|^2_\infty \mathbb{E}\left[\left(\int_0^T |Z^x_s| \mathbb{1}_{|Z^x_s| > n} \, \mathrm{d}s \right)^2 \right] + 6K_z^2 \left\|\sigma^{-1}\right\|^4_\infty \|\sigma^n - \sigma\|^2_\infty \mathbb{E}\left[\left(\int_0^T |Z^x_s| \, \mathrm{d}s \right)^2 \right]. \end{aligned}$$

Thanks to Lemma 2.1 of [43], we have:

$$\forall \mu \geqslant 2, \ \exists C_{T,\mu} > 0, \ \mathbb{E}\left[|X^x - X^{n,x}|^{*,\mu}_{0,T} \right] \leqslant C_{T,\mu} \left(\|\Xi - \Xi^n\|^\mu_\infty + \|\sigma - \sigma^n\|^\mu_\infty \right).$$

Moreover, we have, for every $\alpha \in (0,2]$:

$$\mathbb{E}\left[\left(\int_0^T \mathbb{1}_{|Z^x_s| > n} \, \mathrm{d}s \right)^2 \right] \leqslant T \int_0^T \mathbb{P}\left(|Z^x_s| > n\right) \, \mathrm{d}s \leqslant \frac{T}{n^\alpha} \int_0^T \mathbb{E}\left[|Z^x_s|^\alpha\right] \, \mathrm{d}s \leqslant \frac{T^{2-\frac{\alpha}{2}}}{n^\alpha} \mathbb{E}\left[\int_0^T |Z^x_s|^2 \, \mathrm{d}s \right]^{\frac{\alpha}{2}}.$$

We also recall – see [8] – for $\mu > 1$: $\mathbb{E}\left[\left(\int_0^T |Z^x_s|^2 \, \mathrm{d}s \right)^{\frac{\mu}{2}} \right] \leqslant C_{\mu,T}\left(1 + |x|^{2\mu}\right)$, because v has quadratic growth. Using the Cauchy-Schwarz inequality and the above remarks, we get:

$$\mathbb{E}\left[F^2\right] \leqslant C_T \left(\|\rho - \rho^n\|^2_\infty + \left(\|\Xi - \Xi^n\|^2_\infty + \|\sigma - \sigma^n\|^2_\infty + \frac{1}{n^\varepsilon} \right) \left(1 + |x|^{2(2+\varepsilon)}\right) \right),$$

for every $\varepsilon > 0$. □

Lemma 2.24. *We have:* $\forall \varepsilon > 0$, $\forall T > 0$, $\exists C_T > 0$, $\forall x \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$,

$$|u_T(0,x) - u^n_T(0,x)|^2$$
$$\leqslant C_T \left\{ \mathbb{E}\left[\left| g\left(X^x_T\right) - g^n\left(X^{n,x}_T\right)\right|^2 \right] + \|\rho - \rho^n\|^2_\infty + \left(\|\Xi - \Xi^n\|^2_\infty + \frac{1}{n^\varepsilon} \right) \left(1 + |x|^{(2+\varepsilon)p}\right) \right\}.$$

Proof. The proof is essentially the same; the main difference is that the exponent p appears in the following inequality (see [8]): $\mathbb{E}\left[\left(\int_0^T \left|Z_s^{T,0,x}\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \leqslant C_{p,T}\left(1+\mathbb{E}\left[\left|Y_T^{T,0,x}\right|^p\right]\right)$. $\square$

We keep in mind the following results:

$$\begin{aligned}
|w_T^n(0,x) - w_T(0,x)|^2 \leqslant C_T \Big\{ &\mathbb{E}\left[\left|g\left(X_T^x\right) - g^n\left(X_T^{n,x}\right)\right|^2 + \left|v\left(X_T^x\right) - v^n\left(X_T^{n,x}\right)\right|^2\right] \\
&+ \|\rho - \rho^n\|_\infty^2 + \left(\|\Xi - \Xi^n\|_\infty^2 + \|\sigma - \sigma^n\|_\infty^2 + \frac{1}{n^\varepsilon}\right)\left(1 + |x|^{(2+\varepsilon)p}\right)\Big\}, \\
|w_T(0,x)| \leqslant C\left(1 + |x|^p\right).&
\end{aligned}$$

Step 3: Variation of $w_T^n(0,\bullet)$.
We write:

$$Y_0^{n,x} - \widetilde{Y}_0^{n,x} = Y_T^{n,x} - \widetilde{Y}_T^{n,x} + \int_0^T \left\{\psi^n\left(X_s^{n,x}, Z_s^{n,x}\right) - \psi^n\left(X_s^{n,x}, \widetilde{Z}_s^{n,x}\right)\right\} \mathrm{d}s - \int_0^T \left\{Z_s^{n,x} - \widetilde{Z}_s^{n,x}\right\} \mathrm{d}W_s.$$

We have $Z_s^{n,x} = \partial_x u_T^n\left(s, X_s^{n,x}\right)\sigma\left(X_s^{n,x}\right)$ and $\widetilde{Z}_s^{n,x} = \partial_x \widetilde{u}_T^n\left(s, X_s^{n,x}\right)\sigma\left(X_s^{n,x}\right)$ (see Theorem 3.1 of [43]). Consequently, $Z_s^{n,x} - \widetilde{Z}_s^{n,x} = \partial_x w_T^n\left(s, X_s^{n,x}\right)\sigma^n\left(X_s^{n,x}\right)$. We define the following function:

$$\beta_T^n(t,x) = \begin{cases} \dfrac{\psi^n\left(x, \partial_x u_T^n(t,x)\sigma^n(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}_T^n(t,x)\sigma^n(x)\right)}{\left|\partial_x w_T^n(t,x)\sigma^n(x)\right|^2} & \left(\partial_x w_T^n(t,x)\sigma^n(x)\right)^*, \\ & \text{if } t < T \text{ and } \partial_x w_T^n(t,x) \neq 0, \\ \dfrac{\psi^n\left(x, \partial_x u_T^n(T,x)\sigma^n(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}_T^n(T,x)\sigma^n(x)\right)}{\left|\partial_x w_T^n(T,x)\sigma^n(x)\right|^2} & \left(\partial_x w_T^n(T,x)\sigma^n(x)\right)^*, \\ & \text{if } t \geqslant T \text{ and } \partial_x w_T^n(T,x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The function β_T^n is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$; $\widetilde{W}_t = W_t - \int_0^t \beta_T^n\left(s, X_s^{n,x}\right)\mathrm{d}s$ is a Brownian motion on $[0,T]$ under the probability $\mathbb{Q}_T^n$. Because

$$\psi^n\left(X_s^{n,x}, Z_s^{n,x}\right) - \psi^n\left(X_s^{n,x}, \widetilde{Z}_s^{n,x}\right) = \left(Z_s^{n,x} - \widetilde{Z}_s^{n,x}\right)\beta_T^n\left(s, X_s^{n,x}\right),$$

we have:

$$w_T^n(0,x) = Y_0^{n,x} - \widetilde{Y}_0^{n,x} = \mathbb{E}^{\mathbb{Q}_T^n}\left[Y_T^{n,x} - \widetilde{Y}_T^{n,x}\right] = \mathbb{E}^{\mathbb{Q}_T^n}\left[g^n\left(X_T^{n,x}\right) - v^n\left(X_T^{n,x}\right)\right] = \mathcal{P}_T^n\left[g^n - v^n\right](x),$$

where $\mathcal{P}^n$ is the Kolmogorov semigroup of the SDE:

$$\mathrm{d}U_t = \Xi^n\left(U_t\right)\mathrm{d}t + \sigma^n\left(U_t\right)\left(\mathrm{d}W_t + \beta_T^n\left(t, U_t\right)\mathrm{d}t\right).$$

By Lemma 2.15 and corollary 2.9, we get:

$$|w_T^n(0,x) - w_T^n(0,y)| = \left|\mathcal{P}_T^n\left[g^n - v^n\right](x) - \mathcal{P}_T^n\left[g^n - v^n\right](y)\right| \leqslant \widehat{c}_n c_{g^n - v^n}\left(1 + |x|^p + |y|^p\right)\mathrm{e}^{-\widehat{\nu}_n T},$$

where $\widehat{c}_n$ and $\widehat{\nu}_n$ depend only on η_1, η_2, r_1, r_2, K_z and $\|\sigma^{-1}\|_\infty$; also $c_{g^n - v^n}$ is the constant appearing in the p-polynomial growth of $g^n - v^n$: it only depends on $g - v$ and is independent of n. So:

$$\exists \widehat{c}, \widehat{\nu} > 0,\ \forall T > 0,\ \forall x, y \in \mathbb{R}^d,\ \forall n \in \mathbb{N}^*,\ |w_T^n(0,x) - w_T^n(0,y)| \leqslant \widehat{c}\left(1 + |x|^p + |y|^p\right)\mathrm{e}^{-\widehat{\nu} T}. \tag{2.12}$$

Step 4: Upper bound for $\partial_x w^n_T(0,\bullet)$.

We use Theorem 4.2 of [43]; for every $T' \in (t,T]$, we have:

$$\partial_x u^n_T(t,x) = \mathbb{E}\left[u^n_T\left(T', X^{n,t,x}_{T'}\right) N^{n,t,x}_{T'} + \int_t^{T'} \psi^n\left(X^{n,t,x}_s, Z^{n,t,x}_s\right) N^{n,t,x}_s \,\mathrm{d}s\right],$$

$$\partial_x \widetilde{u}^n_T(t,x) = \mathbb{E}\left[\widetilde{u}^n_T\left(T', X^{n,t,x}_{T'}\right) N^{n,t,x}_{T'} + \int_t^{T'} \psi^n\left(X^{n,t,x}_s, \widetilde{Z}^{n,t,x}_s\right) N^{n,t,x}_s \,\mathrm{d}s\right],$$

where $N^{n,t,x}_s = \dfrac{1}{s-t}\left(\displaystyle\int_t^s \left(\sigma^n\left(X^{n,t,x}_r\right)^{-1} \nabla X^{n,t,x}_r\right)^* \mathrm{d}W_r\right)^*$. By subtraction:

$$\partial_x w^n_T(t,x) = \mathbb{E}\left[w^n_{T-T'}\left(0, X^{n,t,x}_{T'}\right) N^{n,t,x}_{T'} + \int_t^{T'} \left\{\psi^n\left(X^{n,t,x}_s, Z^{n,t,x}_s\right) - \psi^n\left(X^{n,t,x}_s, \widetilde{Z}^{n,t,x}_s\right)\right\} N^{n,t,x}_s \,\mathrm{d}s\right].$$

By Itô's formula, we prove that:

$$\mathrm{d}\left|\nabla X^{n,t,x}_s\right|^2_F \leqslant \left(2\|\Xi\|_{\mathrm{lip}} + \|\sigma\|^2_{\mathrm{lip}}\right)\left|\nabla X^{n,t,x}_s\right|^2_F \,\mathrm{d}s + \mathrm{d}M_s,$$

with M a martingale starting at 0. We set $\lambda = 2\|\Xi\|_{\mathrm{lip}} + \|\sigma\|^2_{\mathrm{lip}}$ and we get:

$$\mathbb{E}\left[\left|\nabla X^{n,t,x}_s\right|^2_F\right] \leqslant d^2 \mathrm{e}^{\lambda(s-t)}.$$

Then, for any $s \in [t,T']$,

$$\mathbb{E}\left[\left|N^{n,t,x}_s\right|^2\right] \leqslant \left(\frac{1}{s-t}\right)^2 \mathbb{E}\left[\int_t^s \left|\sigma^n\left(X^{n,t,x}_r\right)^{-1}\nabla X^{n,t,x}_r\right|^2_F \mathrm{d}r\right] \leqslant \frac{d^2\left\|\sigma^{-1}\right\|^2_\infty}{s-t}\mathrm{e}^{\lambda(T'-t)}.$$

Recalling that $(x,q) \mapsto \psi^n(x, q\sigma^n(x))$ is Lipschitz, we obtain:

$$\begin{aligned}
&|\partial_x w^n_T(t,x)| \\
&\leqslant \mathbb{E}\left[\left|w^n_{T-T'}\left(0, X^{n,t,x}_{T'}\right)\right|\left|N^{n,t,x}_{T'}\right|\right] + \int_t^{T'} \mathbb{E}\left[\left|\psi^n\left(X^{n,t,x}_s, Z^{n,t,x}_s\right) - \psi^n\left(X^{n,t,x}_s, \widetilde{Z}^{n,t,x}_s\right)\right|\left|N^{n,t,x}_s\right|\right] \mathrm{d}s \\
&\leqslant \mathbb{E}\left[\left|w^n_{T-T'}\left(0, X^{n,t,x}_{T'}\right)\right|^2\right]^{\frac12} \mathbb{E}\left[\left|N^{n,t,x}_{T'}\right|^2\right]^{\frac12} + K_z \int_t^{T'} \mathbb{E}\left[\left|\partial_x w^n_T\left(s, X^{n,t,x}_s\right)\right|^2\right]^{\frac12} \mathbb{E}\left[\left|N^{n,t,x}_s\right|^2\right]^{\frac12} \mathrm{d}s \\
&\leqslant \mathbb{E}\left[\left|w^n_{T-T'}\left(0, X^{n,t,x}_{T'}\right)\right|^2\right]^{\frac12} \frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}}\mathrm{e}^{\frac{\lambda}{2}(T'-t)} \\
&\qquad + K_z d\left\|\sigma^{-1}\right\|_\infty \mathrm{e}^{\frac{\lambda T'}{2}} \int_t^{T'} \mathbb{E}\left[\left|\partial_x w^n_T\left(s, X^{n,t,x}_s\right)\right|^2\right]^{\frac12} \frac{\mathrm{d}s}{\sqrt{s-t}}.
\end{aligned}$$

Thanks to Step 2 and Proposition 2.5, we have the following upper bound (for any $\varepsilon > 0$):

$$\begin{aligned}
&\mathbb{E}\left[\left|w^n_{T-T'}\left(0, X^{n,t,x}_{T'}\right)\right|^2\right] \\
&\leqslant C\left(1+|x|^{2p}\right) + C_{T-T'}\left(\Delta^{n,x}_t + \|\rho-\rho^n\|^2_\infty + \left(\|\Xi-\Xi^n\|^2_\infty + \|\sigma-\sigma^n\|^2_\infty + \frac{1}{n^\varepsilon}\right)\left(1+|x|^{(2+\varepsilon)p}\right)\right),
\end{aligned}$$

where $\Delta^{n,x}_t = \mathbb{E}\left[\left|v\left(X^{0,X^{n,t,x}_{T'}}_{T-T'}\right) - v^n\left(X^{n,0,X^{n,t,x}_{T'}}_{T-T'}\right)\right|^2 + \left|g\left(X^{0,X^{n,t,x}_{T'}}_{T-T'}\right) - g^n\left(X^{n,0,X^{n,t,x}_{T'}}_{T-T'}\right)\right|^2\right]$. So, for any $\varepsilon > 0$,

$$\begin{aligned}
&|\partial_x w^n_T(t,x)| \\
&\leqslant \left\{C\left(1+|x|^p\right) + C_{T-T'}\left(\sqrt{\Delta^{n,x}_t} + \|\rho-\rho^n\|_\infty + \left(\|\Xi-\Xi^n\|_\infty + \|\sigma-\sigma^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)\left(1+|x|^{(1+\varepsilon)p}\right)\right)\right\} \\
&\qquad \times \frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}}\mathrm{e}^{\frac{\lambda}{2}(T'-t)} + K_z d\left\|\sigma^{-1}\right\|_\infty \mathrm{e}^{\frac{\lambda T'}{2}} \int_t^{T'} \mathbb{E}\left[\left|\partial_x w^n_T\left(s, X^{t,x}_s\right)\right|^2\right]^{\frac12} \frac{\mathrm{d}s}{\sqrt{s-t}}.
\end{aligned}$$

Let us take $\zeta > p$; we set $\varphi_T^n(t) = \sup_{x\in\mathbb{R}^d} \frac{\left|\partial_x w_T^n(t,x)\right|}{1+|x|^\zeta}$. Using the appendix, we see that:

$$|\partial_x w_T^n(t,x)| \leqslant |\partial_x u_T^n(t,x)| + |\partial_x \widetilde{u}_T^n(t,x)| \leqslant C\left(1+|x|^p\right).$$

This proves that φ_T^n is well defined on $[0,T)$; moreover, it is bounded on every set $[0,T']$, for every $T' < T$. Also, for every $s \in [t,T']$, $\mathbb{E}\left[\left|\partial_x w_T^n\left(s, X_s^{t,x}\right)\right|^2\right]^{\frac{1}{2}} \leqslant \varphi_T^n(s) C\left(1+|x|^\zeta\right)$, where C is independent of T and x. We get, for ε small enough,

$$\begin{aligned}
&\frac{\left|\partial_x w_T^n(t,x)\right|}{1+|x|^\zeta} \\
&\leqslant \left\{ C + C_{T-T'}\left(\frac{\sqrt{\Delta_t^{n,x}}}{1+|x|^\zeta} + \|\rho-\rho^n\|_\infty + \left(\|\Xi-\Xi^n\|_\infty + \|\sigma-\sigma^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)\right)\right\} \frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}} \mathrm{e}^{\frac{\lambda}{2}(T'-t)} \\
&\qquad + C_{T'}\int_t^{T'} \frac{\varphi_T^n(s)}{\sqrt{s-t}}\,\mathrm{d}s.
\end{aligned}$$

We can take the supremum when $x \in \mathbb{R}^d$, and by a change of variable, it can be rewritten as:

$$\begin{aligned}
&\varphi_T^n(T'-t) \\
&\leqslant \underbrace{\left\{ C + C_{T-T'}\left(\sup_{x\in\mathbb{R}^d}\frac{\sqrt{\Delta_{T'-t}^{n,x}}}{1+|x|^\zeta} + \|\rho-\rho^n\|_\infty + \left(\|\Xi-\Xi^n\|_\infty + \|\sigma-\sigma^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)\right)\right\} \frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{t}} \mathrm{e}^{\frac{\lambda}{2}t}}_{=a(t)} \\
&\qquad + C_{T'}\int_0^{t} \frac{\varphi_T^n(T'-u)}{\sqrt{t-u}}\,\mathrm{d}u,
\end{aligned}$$

where $0 < t \leqslant T' < T$. We use the Lemma 7.1.1 of [30]; indeed, the function a is locally integrable on $(0,T')$, since $\sup_{x\in\mathbb{R}^d}\frac{\sqrt{\Delta_{T'-t}^{n,x}}}{1+|x|^\zeta}$ is bounded independently of t. We get

$$\varphi_T^n(T'-t) \leqslant a(t) + C_{T'}\int_0^t E'\left(C_{T'}(t-u)\right)a(u)\,\mathrm{d}u,$$

and the integral is well defined, since $E(z) = \sum_{n=0}^{\infty}\frac{z^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ is $\mathcal{C}^1$ and $E'(z) \underset{z\to 0^+}{\sim} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{z}}$. Choose $t = T'$ and get:

$$\begin{aligned}
&|\partial_x w_T^n(0,x)| \\
&\quad\leqslant \left[C + C_{T-T',T'}\gamma_{T'}^n(0) + C_{T'}\int_0^{T'} E'\left(C_{T'}(T'-u)\right)\left\{C + C_{T-T'}\gamma_{T'}^n(T'-u)\right\}\frac{\mathrm{d}u}{\sqrt{u}}\right]\left(1+|x|^\zeta\right),
\end{aligned}$$

where we denoted $\gamma_{T'}^n(t) = \sup_{x\in\mathbb{R}^d}\frac{\sqrt{\Delta_t^{n,x}}}{1+|x|^\zeta} + \|\rho-\rho^n\|_\infty + C\left(\|\Xi-\Xi^n\|_\infty + \|\sigma-\sigma^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)$.

Step 5: Taking the limit when $n\to\infty$. From Theorem 2.21 and equation (2.12), we get easily:

$$\exists C > 0,\ \forall T > 0,\ \forall x \in \mathbb{R}^d,\ |w_T(0,x)| \leqslant C\left(1+|x|^p\right), \tag{2.13}$$

$$\exists \widehat{c}, \widehat{\nu} > 0,\ \forall T > 0,\ \forall x,y \in \mathbb{R}^d,\ |w_T(0,x) - w_T(0,y)| \leqslant \widehat{c}\left(1+|x|^p+|y|^p\right)\mathrm{e}^{-\widehat{\nu}T}. \tag{2.14}$$

The function w_T has no reason to be $\mathcal{C}^1$, so we use the mean value theorem:

$$\frac{\left|w_T^n(0,x) - w_T^n(0,y)\right|}{\left(1+|x|^\zeta+|y|^\zeta\right)|x-y|} \leqslant \left[C_{T'}\int_0^{T'} E'\left(C_{T'}(T'-u)\right)\left\{C + C_{T-T'}\gamma_{T'}^n(T'-u)\right\}\frac{\mathrm{d}u}{\sqrt{u}} + C + C_{T-T',T'}\gamma_{T'}^n(0)\right].$$

Our next goal will be to get an upper bound for $\gamma_{T'}^n$. Let $R>0$; we denote $\alpha = \zeta - p$;

$$\begin{aligned}
\mathbb{E}\left[\left|(g-g^n)\left(X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right)\right|^2 \mathbb{1}_{\left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right|>R}\right] &\leqslant \mathbb{E}\left[\left|(g-g^n)\left(X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right)\right|^4\right]^{\frac12}\mathbb{P}\left(\left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right|>R\right)^{\frac12} \\
&\leqslant C\mathbb{E}\left[\left(1+\left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right|^p\right)^4\right]^{\frac12}\frac{\mathbb{E}\left[\left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right|^{2\alpha}\right]^{\frac12}}{R^\alpha} \\
&\leqslant C_{T-T',T'}\frac{1+|x|^{2p+\alpha}}{R^\alpha}.
\end{aligned}$$

By Proposition 2.4 and Lemma 2.1 of [43], we get:

$$\mathbb{E}\left[\left|g\left(X_{T-T'}^{0,X_{T'}^{n,t,x}}\right) - g\left(X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right)\right|^2\right] \leqslant C_{T-T',T'}\left(1+|x|^{2p}\right)\left(\|\Xi-\Xi^n\|_\infty^2 + \|\sigma-\sigma^n\|_\infty^2\right).$$

We can do exactly the same with g and g^n replaced by v and v^n. This way, for any $R>0$:

$$\sup_{x\in\mathbb{R}^d}\frac{\sqrt{\Delta_{T'-t}^{n,x}}}{1+|x|^\zeta} \leqslant C_{T-T',T'}\left(R^{-\frac{\alpha}{2}} + \|\Xi-\Xi^n\|_\infty + \|\sigma-\sigma^n\|_\infty\right) + \|g-g^n\|_{\infty,\mathcal{B}(0,R)} + \|v-v^n\|_{\infty,\mathcal{B}(0,R)}.$$

To sum up, we have:

$$\forall\alpha>0,\ \forall T'>0,\ \exists C_{T'}>0,\ \forall T>T',\ \exists C_{T-T',T'}>0,\ \forall x,y\in\mathbb{R}^d,\ \forall R>0,\ \forall n\in\mathbb{N}^*,$$

$$\begin{aligned}
&\left|w_T^n(0,x)-w_T^n(0,y)\right|\left(1+|x|^{p+\alpha}+|y|^{p+\alpha}\right)^{-1}|x-y|^{-1} \\
&\leqslant C + C_{T-T',T'}\left(R^{-\frac{\alpha}{2}} + \|\Xi-\Xi^n\|_\infty + \|\sigma-\sigma^n\|_\infty + \|g-g^n\|_{\infty,\mathcal{B}(0,R)} + \|v-v^n\|_{\infty,\mathcal{B}(0,R)}\right) \\
&\quad + \int_0^{T'}\left\{C_{T'} + C_{T-T',T'}\left(R^{-\frac{\alpha}{2}} + \|\Xi-\Xi^n\|_\infty + \|\sigma-\sigma^n\|_\infty + \|g-g^n\|_{\infty,\mathcal{B}(0,R)} + \|v-v^n\|_{\infty,\mathcal{B}(0,R)}\right)\right\} \\
&\qquad\qquad \times E'\left(C_{T'}(T'-u)\right)\frac{\mathrm{d}u}{\sqrt{u}}.
\end{aligned}$$

We take the limit as n and then R go to the infinity: for every $\alpha>0$,

$$\forall T'>0,\ \exists\mathcal{C}_{T'}>0,\ \forall T>T',\ \forall x,y\in\mathbb{R}^d,\ |w_T(0,x)-w_T(0,y)| \leqslant \mathcal{C}_{T'}\left(1+|x|^{p+\alpha}+|y|^{p+\alpha}\right)|x-y|.$$

Step 6: Convergence of $w_T(0,\bullet)$ when $T\to\infty$.
Thanks to Equation (2.13), by a diagonal argument, there exist an increasing sequence $(T_i)_{i\in\mathbb{N}}$, with limit $+\infty$; a function $w : D\to\mathbb{R}$, with $D\subset\mathbb{R}^d$ countable and dense, such that $\left(w_{T_i}(0,\bullet)\right)_i$ converges pointwise over D to w. Equation (2.14), with $x,y\in D$, tells us that w is equal to a constant L on D. The same equation, with $x\in\mathbb{R}^d$ and $y\in D$, gives us that $w=L$ on the whole $\mathbb{R}^d$. Let K be a compact in $\mathbb{R}^d$; we define $\mathcal{A}=\{w_T(0,\bullet)|_K | T>1\}\subset\mathcal{C}(K,\mathbb{R})$; it is equicontinuous, because $w_T(0,\bullet)$ is uniformly Lipschitz. For all $x\in K$, the set $\mathcal{A}(x)=$

$\{w_T(0,x)|T>1\}$ is bounded (see equation (2.13). By Ascoli's theorem, $\mathcal{A}$ is relatively compact in $\mathcal{C}(K,\mathbb{R})$. Let $x \in K$,

$$\left|w_{T_i}(0,x) - L\right| \leqslant \left|w_{T_i}(0,x) - w_{T_i}(0,0)\right| + \left|w_{T_i}(0,0) - L\right| \leqslant \widehat{c}\,(1+|x|^p)\,\mathrm{e}^{-\widehat{\nu} T_i} + \left|w_{T_i}(0,0) - L\right|,$$

which proves that the functions $w_{T_i}(0,\bullet)$ converge uniformly to L on K. So L is an accumulation point of $\mathcal{A}$. Since $\mathcal{A}$ is relatively compact, if we prove that L is the unique accumulation point of $\mathcal{A}$, then, we get the uniform convergence of $w_T(0,\bullet)$ to L on K. It works for any compact $K \subset \mathbb{R}^d$, and L does not depend on K. We conclude:

$$\forall x \in \mathbb{R}^d,\ w_T(0,x) \underset{T\to\infty}{\longrightarrow} L.$$

Step 7: L is the unique accumulation point of $\mathcal{A}$.

Let $w_{\infty,K}$ be an accumulation point of $\mathcal{A}$, *i.e.* $\left\| w_{T_i'}(0,\bullet)|_K - w_{\infty,K} \right\|_\infty \underset{i\to\infty}{\longrightarrow} 0$, for a sequence $\left(T_i'\right)$. Again, the equation (2.14) tells us that $w_{\infty,K}$ is equal to a constant L_K on K. Our goal is now to show that $L = L_K$. Now, Y^n and $\widetilde{Y}^n$ will follow the same equations as before, but their terminal conditions will be given at time $T+S$. So,

$$w^n_{T+S}(0,x) = Y^n_0 - \widetilde{Y}^n_0 = Y^n_S - \widetilde{Y}^n_S + \int_0^S \left\{ \psi^n\left(X^{n,x}_r, Z^n_r\right) - \psi^n\left(X^{n,x}_r, \widetilde{Z}^n_r\right) \right\} \mathrm{d}r - \int_0^S \left\{ Z^n_r - \widetilde{Z}^n_r \right\} \mathrm{d}W_r.$$

We define the function:

$$\beta^n_{T,S}(t,x) = \begin{cases} \dfrac{\psi^n\left(x, \partial_x u^n_{T+S}(t,x)\sigma^n(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}^n_{T+S}(t,x)\sigma^n(x)\right)}{\left|\partial_x w^n_{T+S}(t,x)\sigma^n(x)\right|^2} \left(\partial_x w^n_{T+S}(t,x)\sigma^n(x)\right)^*, & \\ \qquad\qquad \text{if } t < S \text{ and } \partial_x w^n_{T+S}(t,x) \neq 0, & \\ \dfrac{\psi^n\left(x, \partial_x u^n_{T+S}(S,x)\sigma^n(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}^n_{T+S}(S,x)\sigma^n(x)\right)}{\left|\partial_x w^n_{T+S}(S,x)\sigma^n(x)\right|^2} \left(\partial_x w^n_{T+S}(S,x)\sigma^n(x)\right)^*, & \\ \qquad\qquad \text{if } t \geqslant S \text{ and } \partial_x w^n_{T+S}(S,x) \neq 0, & \\ 0, \qquad\qquad \text{otherwise.} & \end{cases}$$

This function is bounded by $K_z \left\|\sigma^{-1}\right\|_\infty$; $\widetilde{W}_t = W_t - \displaystyle\int_0^t \beta^n_{T,s}\left(r, X^x_r\right) \mathrm{d}r$ is a Brownian motion on $[0,S]$ under the probability $\mathbb{Q}^{T,S,n,x}$. This way, we can write:

$$\begin{aligned} w^n_{T+S}(0,x) &= \mathbb{E}^{\mathbb{Q}^{T,S,n,x}}\left[Y^n_S - \widetilde{Y}^n_S\right] = \mathbb{E}^{\mathbb{Q}^{T,S,n,x}}\left[w^n_{T+S}\left(S, X^{n,x}_S\right)\right] = \mathbb{E}^{\mathbb{Q}^{T,S,n,x}}\left[w^n_T\left(0, X^{n,x}_S\right)\right] \\ &= \mathcal{P}^n_S\left[w^n_T(0,\bullet)\right](x), \end{aligned}$$

where $\mathcal{P}^n$ is the Kolmogorov semigroup of the SDE:

$$\mathrm{d}U_t = \Xi^n\left(U_t\right)\mathrm{d}t + \sigma^n\left(U_t\right)\left(\mathrm{d}W_t + \beta^n_{T,S}\left(t,U_t\right)\mathrm{d}t\right).$$

Without loss of generality, we can suppose that $T_i > T_i'$, for every $i \in \mathbb{N}$. We take $T = T_i'$ and $S = T_i - T_i'$. This way:

$$\forall x \in \mathbb{R}^d,\ \mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) = w^n_{T_i}(0,x) \underset{n\to\infty}{\longrightarrow} w_{T_i}(0,x) \underset{i\to\infty}{\longrightarrow} L.$$

Now, to prove that $L = L_K$, it suffices to show that $\lim\limits_{i\to\infty} \limsup\limits_{n\to\infty} \mathcal{P}^n_{T_i-T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) = L_K$. We have:

$$\begin{aligned} &\left|\mathcal{P}^n_{T_i-T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - L_K\right| \\ &\qquad \leqslant \left|\mathcal{P}^n_{T_i-T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - w^n_{T_i'}(0,x)\right| + \left|w^n_{T_i'}(0,x) - w_{T_i'}(0,x)\right| + \left|w_{T_i'}(0,x) - L_K\right|, \end{aligned}$$

$$\limsup_{n\to\infty}\left|\mathcal{P}^n_{T_i-T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x)-L_K\right|\leqslant\limsup_{n\to\infty}\left|\mathcal{P}^n_{T_i-T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x)-w^n_{T_i'}(0,x)\right|+\left|w_{T_i'}(0,x)-L_K\right|.$$

Thanks to Lemma 2.15, $\beta^n_{T,S}$ is the limit of $\mathcal{C}^1$-functions, uniformly bounded, with bounded derivatives $\left(\beta^{n,m}_{T,S}\right)_{m\geqslant 0}$. We define $U^{m,x}$ as the solution of the SDE:

$$\mathrm{d}U^{m,x}_t=\Xi^n\left(U^{m,x}_t\right)\,\mathrm{d}t+\sigma^n\left(U^{m,x}_t\right)\left(\mathrm{d}W_t+\beta^{n,m}_{T,S}\left(t,U^{m,x}_t\right)\,\mathrm{d}t\right).$$

Thus,

$$\begin{aligned}\left|\mathcal{P}^n_{T_i-T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x)-w^n_{T_i'}(0,x)\right|&=\left|\lim_{m\to\infty}\mathbb{E}\left[w^n_{T_i'}\left(0,U^{m,x}_{T_i-T_i'}\right)\right]-w^n_{T_i'}(0,x)\right|\\&\leqslant\limsup_{m\to\infty}\mathbb{E}\left[\left|w^n_{T_i'}\left(0,U^{m,x}_{T_i-T_i'}\right)-w^n_{T_i'}(0,x)\right|\right]\\&\leqslant\widehat{c}\left(1+\limsup_{m\to\infty}\mathbb{E}\left[\left|U^{m,x}_{T_i-T_i'}\right|^p\right]+|x|^p\right)\mathrm{e}^{-\widehat{\nu}T_i'},\end{aligned}$$

and the constants $\widehat{c}$ and $\widehat{\nu}$ are independent of n, m and T_i'. Thanks to Proposition 2.5, we have $\mathbb{E}\left[\left|U^{m,x}_t\right|^p\right]\leqslant C\left(1+|x|^p\right)$, with C independent of n, m and t. So,

$$\left|\mathcal{P}^n_{T_i-T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x)-w^n_{T_i'}(0,x)\right|\leqslant\widetilde{C}\left(1+|x|^p\right)\mathrm{e}^{-\nu T_i'},$$

with $\widetilde{C}$ and ν independent of n and T_i'. And we can conclude that $L=L_K$.

Step 8: Speed of convergence. Let $x\in\mathbb{R}^d$ and $T>0$. We have:

$$\begin{aligned}|w_T(0,x)-L|&=\lim_{V\to\infty}|w_T(0,x)-w_V(0,x)|=\lim_{V\to\infty}\lim_{n\to\infty}|w^n_T(0,x)-w^n_V(0,x)|\\&=\lim_{V\to\infty}\lim_{n\to\infty}\left|w^n_T(0,x)-\mathcal{P}^n_{V-T}\left[w^n_T(0,\bullet)\right](x)\right|,\end{aligned}$$

where $\mathcal{P}^n$ is the Kolmogorov semigroup of the SDE:

$$\mathrm{d}U_t=\Xi^n\left(U_t\right)\,\mathrm{d}t+\sigma^n\left(U_t\right)\left(\mathrm{d}W_t+\beta^n_{T,V-T}\left(t,U_t\right)\,\mathrm{d}t\right).$$

Like before, we use Lemma 2.15 and Proposition 2.5, and we get:

$$\begin{aligned}|w_T(0,x)-L|&=\lim_{V\to\infty}\lim_{n\to\infty}\left|w^n_T(0,x)-\lim_{m\to\infty}\mathbb{E}\left[w^n_T\left(0,U^{m,x}_{V-T}\right)\right]\right|\\&\leqslant\limsup_{V\to\infty}\limsup_{n\to\infty}\limsup_{m\to\infty}\mathbb{E}\left[\left|w^n_T(0,x)-w^n_T\left(0,U^{m,x}_{V-T}\right)\right|\right]\leqslant\widetilde{C}\left(1+|x|^p\right)\mathrm{e}^{-\nu T}.\end{aligned}$$

□

2.6 Application to Optimal ergodic problem and HJB equation

2.6.1 Optimal ergodic problem

In this section, we apply our results to an ergodic control problem. The proofs of the following results are so similar to the ones of [35] that we omit them. As before, we consider a process X^x satisfying:

$$X^x_t=x+\int_0^t\Xi\left(X^x_s\right)\,\mathrm{d}s+\int_0^t\sigma\left(X^x_s\right)\,\mathrm{d}W_s,$$

where Ξ and σ are Lipschitz continuous, Ξ is weakly dissipative with $\langle\Xi(x),x\rangle\leqslant\eta_1-\eta_2|x|^2$, the function $x\mapsto\sigma(x)^{-1}$ is bounded and $|\sigma(x)|_F^2\leqslant r_1+r_2|x|^2$. Moreover, we pick $\mu\geqslant 2$ such that $\frac{\mu-1}{2}r_2<\eta_2$. Let U be a separable metric space; we call "control" any progressively measurable U-valued process. We consider some measurable functions satisfying the following assumptions:

- $R : U \to \mathbb{R}^d$ is bounded;
- $L : \mathbb{R}^d \times U \to \mathbb{R}$ satisfies : $\exists C > 0$, $\forall a \in U$, $\forall x, x' \in \mathbb{R}^d$, $|L(x,a) - L(x',a)| \leqslant C|x - x'|$ and $|L(x,a)| \leqslant C(1 + |x|)$;
- $g : \mathbb{R}^d \to \mathbb{R}$ is continuous, has polynomial growth and is locally Lipschitz continuous: for every x and x' in $\mathbb{R}^d$, $|g(x)| \leqslant C(1 + |x|^\mu)$ and $|g(x) - g(x')| \leqslant C(1 + |x|^\mu + |x'|^\mu)|x - x'|$.

Let X^x be the solution of the SDE (2.6); for any control a and horizon $T > 0$, we set:

$$\rho_T^{x,a} = \exp\left(\int_0^T \sigma(X_t^x)^{-1} R(a_t)\, \mathrm{d}W_t - \frac{1}{2}\int_0^T \left|\sigma(X_t^x)^{-1} R(a_t)\right|^2 \mathrm{d}t\right) \text{ and } \mathbb{P}_T^{x,a} = \rho_T^{x,a}\mathbb{P} \text{ over } \mathcal{F}_T.$$

We define the finite horizon cost as:

$$J^T(x,a) = \mathbb{E}_T^{x,a}\left[\int_0^T L(X_t^x, a_t)\, \mathrm{d}t\right] + \mathbb{E}_T^{x,a}\left[g(X_T^x)\right],$$

and the optimal control problem is to minimise $J^T(x,a)$ over the controls $a^T : \Omega \times [0,T] \to U$. We also define another cost, called "ergodic cost":

$$J(x,a) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_T^{x,a}\left[\int_0^T L(X_t^x, a_t)\, \mathrm{d}t\right],$$

and the associated optimal control problem is to minimise $J(x,a)$ over the controls $a : \Omega \times \mathbb{R}_+ \to U$. Due to Girsanov's theorem, $W_t^{a,x} = W_t - \int_0^t \sigma(X_s^x)^{-1} R(a_s)\, \mathrm{d}s$ is a Brownian motion on $[0,T]$ under $\mathbb{P}_T^{x,a}$ and

$$\mathrm{d}X_t^x = \left[\Xi(X_t^x) + R(a_t)\right] \mathrm{d}t + \sigma(X_t^x)\, \mathrm{d}W_t^{x,a}.$$

We define the Hamiltonian in the following way:

$$\psi(x,z) = \inf_{a\in U}\left\{L(x,a) + z\sigma(x)^{-1}R(a)\right\}. \tag{2.15}$$

We recall that, if this infimum is attained for every x and z, by Filippov's theorem (see [45]), there exists a measurable function $\gamma : \mathbb{R}^d \times (\mathbb{R}^d)^* \to U$ such that: $\psi(x,z) = L(x,\gamma(x,z)) + z\sigma(x)^{-1}R(\gamma(x,z))$.

Lemma 2.25. *The Hamiltonian ψ satisfies:*

- $\forall x \in \mathbb{R}^d$, $|\psi(x,0)| \leqslant C(1 + |x|)$;
- $\forall x, x' \in \mathbb{R}^d$, $\forall z, z' \in (\mathbb{R}^d)^*$, $|\psi(x,z) - \psi(x',z')| \leqslant \|L\|_{\mathrm{lip}}|x - x'| + \|R\|_\infty \left|z\sigma(x)^{-1} - z'\sigma(x')^{-1}\right|$.

Lemma 2.26. *For every control a, we have: $J^T(x,a) \geqslant Y_0^{T,x}$, where $Y^{T,x}$ is part of the solution of the finite horizon BSDE:*

$$Y_t^{T,x} = g(X_T^x) + \int_t^T \psi\left(X_s^x, Z_s^{T,x}\right) \mathrm{d}s - \int_t^T Z_s^{T,x}\, \mathrm{d}W_s, \quad \forall t \in [0,T]. \tag{2.16}$$

Moreover, if the infimum is attained for every x and z in equation (2.15), we have $J\left(x, \overline{a}^T\right) = Y_0^{T,x}$, where we set $\overline{a}_t^T = \gamma(X_t^x, \nabla u(t, X_t^x)\sigma(X_t^x))$.

Lemma 2.27. *For every control a, we have: $J(x,a) \geqslant \lambda$, where λ is part of the solution of the EBSDE:*

$$Y_t^x = Y_T^x + \int_t^T \left\{\psi(X_s^x, Z_s^x) - \lambda\right\} \mathrm{d}s - \int_t^T Z_s^x\, \mathrm{d}W_s, \quad \forall 0 \leqslant t \leqslant T < \infty.$$

Moreover, if the infimum is attained for every x and z in equation (2.15), we have $J(x,\overline{a}) = \lambda$, where we set $\overline{a}_t = \gamma(X_t^x, \nabla v(X_t^x)\sigma(X_t^x))$.

Theorem 2.28. *For every control a, we have:*

$$\liminf_{T\to\infty} \frac{J^T(x,a)}{T} \geqslant \lambda.$$

Moreover, if the infimum is attained for every x and z in equation (2.15), we have

$$\left| J^T\left(x, \bar{a}^T\right) - J\left(x, \bar{a}\right) T - Y_0^x - L \right| \leqslant C\left(1 + |x|^2\right) \mathrm{e}^{-\nu T}.$$

Proof. This is a straightforward consequence of the previous lemmas and Theorem 2.22. □

Remark 2.29. All the results of this subsection can be rephrased in terms of viscosity solution of PDEs (2.17) and (2.19).

2.6.2 Large time behaviour of viscosity solution of HJB equation

We consider the ergodic PDE:

$$\mathcal{L}v(x) + \psi\left(x, \nabla v(x)\sigma(x)\right) - \lambda = 0, \tag{2.17}$$

where $\mathcal{L}$ is the generator of the Kolmogorov semigroup of X^x, solution of (2.2). We recall that the couple (v, λ) is a viscosity subsolution (resp. supersolution) if:

- $v : \mathbb{R}^d \to \mathbb{R}$ is a continuous function with polynomial growth;
- for any function $\phi \in \mathcal{C}^2\left(\mathbb{R}^d, \mathbb{R}\right)$, for every $x \in \mathbb{R}^d$ of local maximum (resp. minimum) of $v - \phi$:

$$\mathcal{L}\phi(x) + \psi(x, \nabla\phi(x)\sigma(x)) - \lambda \geqslant 0 \quad \text{(resp. } \leqslant 0\text{)}.$$

Proposition 2.30 (Existence of ergodic viscosity solution)**.** *Under Assumption 2.10, the couple $\left(\overline{v}, \overline{\lambda}\right)$ obtained with the solution given in Theorem 2.18 is a viscosity solution of Equation (2.17).*

Proof. Note that we already know that $\overline{v}$ is continuous and has quadratic growth. The proof of this result is classical and can easily be adapted from Theorem 4.3 of [51]. □

Proposition 2.31 (Uniqueness of ergodic viscosity solution)**.** *Let $p > 0$; we suppose that*

$$\sqrt{r_2} K_z \|\sigma^{-1}\|_\infty + [(p \vee 2) - 1]\frac{r_2}{2} < \eta_2$$

and that Assumption 2.10 holds.
Then uniqueness holds for viscosity solutions (v, λ) of Equation (2.17) in the class of viscosity solutions such that $\exists a \in \mathbb{R}^d, v(a) = \widetilde{v}(a)$ and v and $\widetilde{v}$ have polynomial growth of at most degree p.

Proof. The proof is the same as in Lemma 3.18 of [34]. Let (v, λ) and $(\widetilde{v}, \widetilde{\lambda})$ be two viscosity solutions of (2.17). We fix $T > 0$, and we consider the solution $\left(Y^{T,x}, Z^{T,x}\right)$ of the BSDE

$$Y_t^{T,x} = v\left(X_T^x\right) + \int_t^T \left\{\psi\left(X_s^x, Z_s^{T,x}\right) - \lambda\right\} \mathrm{d}s - \int_t^T Z_s^{T,x}\, \mathrm{d}W_s, \quad t \in [0,T],$$

linked to the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + \psi\left(x, \nabla u(t,x)\sigma(x)\right) = \lambda & \forall (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(T,x) = v(x) & \forall x \in \mathbb{R}^d. \end{cases} \tag{2.18}$$

Because this PDE has a unique viscosity solution (see [51]), we can claim that $v(x) = Y_0^{T,x}$. We define a couple $\left(\widetilde{Y}^{T,x}, \widetilde{Z}^{T,x}\right)$ by replacing in the previous equation (v, λ) by $(\widetilde{v}, \widetilde{\lambda})$. Then, for any $T > 0$ and $x \in \mathbb{R}^d$:

$$(v - \widetilde{v})(x) = (v - \widetilde{v})(X_T^x) + \int_0^T \left\{\psi\left(X_s^x, Z_s^{T,x}\right) - \psi\left(X_s^x, \widetilde{Z}_s^{T,x}\right)\right\} \mathrm{d}s - \left(\lambda - \widetilde{\lambda}\right) T - \int_0^T \left\{Z_s^{T,x} - \widetilde{Z}_s^{T,x}\right\} \mathrm{d}W_s.$$

We set

$$\beta_s = \begin{cases} \dfrac{\psi\left(X_s^x, Z_s^{T,x}\right) - \psi\left(X_s^x, \widetilde{Z}_s^{T,x}\right)}{\left|Z_s^{T,x} - \widetilde{Z}_s^{T,x}\right|^2} \left(Z_s^{T,x} - \widetilde{Z}_s^{T,x}\right)^*, & \text{if } Z_s^{T,x} \neq \widetilde{Z}_s^{T,x}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the process β is bounded by $K_z \left\|\sigma^{-1}\right\|_\infty$, by Girsanov's theorem, there exists a new probability measure $\mathbb{Q}^T$ equivalent to $\mathbb{P}$ and under which $W - \int_0^\bullet \beta_s \,\mathrm{d}s$ is a Brownian motion. Then:

$$\frac{(v - \widetilde{v})(x)}{T} = \frac{\mathbb{E}^{\mathbb{Q}^T}\left[(v - \widetilde{v})(X_T^x)\right]}{T} - \left(\lambda - \widetilde{\lambda}\right).$$

Thanks to Proposition 2.5 and the polynomial growth of v and $\widetilde{v}$, letting $T \to \infty$ gives us $\lambda = \widetilde{\lambda}$. Applying the same argument as that in Theorem 2.20, we deduce the uniqueness claimed. □

We recall the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + \psi\left(x, \nabla u(t,x)\sigma(x)\right) = 0 & \forall (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(T,x) = g(x) & \forall x \in \mathbb{R}^d, \end{cases} \tag{2.19}$$

whose viscosity solution is linked to BSDE (2.16) via $Y_t^{T,x} = u\left(T - t, X_t^x\right)$. We can rephrase Theorem 2.22 as:

Theorem 2.32. *We consider Equations (2.17) and (2.19) and we suppose that Assumption 2.10 holds; moreover, we assume that $g : \mathbb{R}^d \to \mathbb{R}^d$ has polynomial growth and is locally Lipschitz continuous: for every x and x' in $\mathbb{R}^d$, $|g(x)| \leqslant C\left(1 + |x|^\mu\right)$ and $|g(x) - g(x')| \leqslant C\left(1 + |x|^\mu + |x'|^\mu\right)|x - x'|$ with $\mu \geqslant 2$ and $\sqrt{r_2} K_z \|\sigma^{-1}\|_\infty + \dfrac{\mu - 1}{2} r_2 < \eta_2$.*
Then, there exists $L \in \mathbb{R}$, such that: $\forall x \in \mathbb{R}^d$, $u(T,x) - \lambda T - v(x) \underset{T \to \infty}{\longrightarrow} L$. Furthermore,

$$\forall x \in \mathbb{R}^d,\ \forall T > 0,\ |u(T,x) - \lambda T - v(x) - L| \leqslant C\left(1 + |x|^\mu\right) \mathrm{e}^{-\nu T}.$$

Acknowledgment. The authors would like to thank both referees for their careful reading and constructive suggestions, which helped them to considerably improve the presentation of the paper.

2.7 Appendix about the Lipschitz continuity of v^α

We consider the forward-backward system, where X is a vector in $\mathbb{R}^d$, Y a real, Z a row with d real coefficients, and W a Brownian motion in $\mathbb{R}^d$:

$$X_s^{t,x} = x + \int_t^s b\left(X_r^{t,x}\right) \mathrm{d}r + \int_t^s \sigma\left(X_r^{t,x}\right) \mathrm{d}W_r, \tag{2.20}$$

$$Y_s^{t,x} = g\left(X_T^{t,x}\right) + \int_s^T f\left(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) \mathrm{d}r - \int_s^T Z_r^{t,x} \,\mathrm{d}W_r. \tag{2.21}$$

The first result is an adaptation to our case of the Theorem 3.1 of [43].

Proposition 2.33. *We make the following assumptions:*

- $\sigma \in \mathcal{C}^1(\mathbb{R}^d, \mathrm{GL}_d(\mathbb{R}))$ *and* $b \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ *have bounded derivatives;*
- *the function* $\sigma(\bullet)^{-1}$ *is bounded;*
- $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ *has bounded derivatives;*
- *if we define* $h : (x, y, p) \mapsto f(x, y, p\sigma(x))$*, then* $h \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^d)^*, \mathbb{R})$ *has bounded derivatives (the bounds of the derivatives of h will be denoted* H_x*,* H_y *and* H_p *w.r.t. x, y and p);*
- $\forall x \in \mathbb{R}^d$, $|f(x, 0, 0)| \leqslant M_f\,(1 + |x|^\mu)$.

Then, the function $u : (t, x) \mapsto Y_t^{t,x}$ *is of class* $\mathcal{C}^1$ *w.r.t. x and*

$$\forall s \in [t, T],\ Z_s^{t,x} = \partial_x u\left(s, X_s^{t,x}\right)\sigma\left(X_s^{t,x}\right)\ \mathbb{P}\text{-a.s.}$$

Proof. We will prove the proposition in the case $d = 1$, to simplify presentation.

Step 1: We have, for all $p \geqslant 1$:

$$\mathbb{E}\left[\left(\int_t^T \left|Z_r^{t,x}\right|^2 \mathrm{d}r\right)^p\right] \leqslant C e^{2ap(T-t)} \left\{\mathbb{E}\left[\left|Y^{t,x}\right|_{t,T}^{*,2p}\right] + C_T\left(1 + |x|^{2p\mu}\right)\right\}, \tag{2.22}$$

where C comes from Lemma 3.1 of [8] and C_T depends only on p, μ, T, r_1, $\|\sigma\|_{\mathrm{lip}}$, $|b(0)|$, $\|b\|_{\mathrm{lip}}$ and M_f.

Step 2: We will show: $\forall p \geqslant 2$, $\forall x \in \mathbb{R}^d$, $\mathbb{E}\left[\left|Y^{t,x+\varepsilon} - Y^{t,x}\right|_{t,T}^{*,p} + \left(\int_t^T \left|Z_s^{t,x+\varepsilon} - Z_s^{t,x}\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \underset{\varepsilon \to 0}{\longrightarrow} 0.$

Let $(t, x) \in [0, T] \times \mathbb{R}$ be fixed. We define, for any $\varepsilon \neq 0$,

$$\nabla X^\varepsilon = \frac{X^{t,x+\varepsilon} - X^{t,x}}{\varepsilon},\ \nabla Y^\varepsilon = \frac{Y^{t,x+\varepsilon} - Y^{t,x}}{\varepsilon},\ \nabla Z^\varepsilon = \frac{Z^{t,x+\varepsilon} - Z^{t,x}}{\varepsilon}.$$

We also define the solution of the variational system:

$$\nabla X_s = 1 + \int_t^s \partial_x b\left(X_r^{t,x}\right)\nabla X_r\,\mathrm{d}r + \int_t^s \partial_x \sigma\left(X_r^{t,x}\right)\nabla X_r\,\mathrm{d}W_r; \tag{2.23}$$

$$\nabla Y_s = \partial_x g\left(X_T^{t,x}\right)\nabla X_T + \int_s^T \left[\partial_x f\left(\Theta_r^{t,x}\right)\nabla X_r + \partial_y f\left(\Theta_r^{t,x}\right)\nabla Y_r + \partial_z f\left(\Theta_r^{t,x}\right)\nabla Z_r\right]\mathrm{d}r - \int_s^T \nabla Z_r\,\mathrm{d}W_r, \tag{2.24}$$

where Θ stands for (X, Y, Z). For every $s \in [t, T]$, we have:

$$\nabla Y_s^\varepsilon = g_x^\varepsilon \nabla X_T^\varepsilon + \int_s^T \left[f_x^\varepsilon(r)\nabla X_r^\varepsilon + f_y^\varepsilon(r)\nabla Y_r^\varepsilon + f_z^\varepsilon(r)\nabla Z_r^\varepsilon\right]\mathrm{d}r - \int_s^T \nabla Z_r^\varepsilon\,\mathrm{d}W_r, \tag{2.25}$$

where $g_x^\varepsilon = \int_0^1 \partial_x g\left(X_T^{t,x} + w\varepsilon\nabla X_T^\varepsilon\right)\mathrm{d}w$, $f_x^\varepsilon(r) = \int_0^1 \partial_x f\left(\Theta_r^{t,x} + w\varepsilon(\nabla X_r^\varepsilon, 0, 0)\right)\mathrm{d}w$,

$f_y^\varepsilon(r) = \int_0^1 \partial_y f\left(\Theta_r^{t,x} + w\varepsilon(\nabla X_r^\varepsilon, \nabla Y_r^\varepsilon, 0)\right)\mathrm{d}w$ and $f_z^\varepsilon(r) = \int_0^1 \partial_z f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w$.

In order to apply Proposition 3.2 of [8] later; we set

$$i(r, y, z) := \int_0^1 \partial_x f\left(\Theta_r^{t,x} + w\varepsilon\nabla(\nabla X_r^\varepsilon, 0, 0)\right)\mathrm{d}w\,\nabla X_r^\varepsilon + \int_0^1 \partial_y f\left(\Theta_r^{t,x} + w\varepsilon(\nabla X_r^\varepsilon, \nabla Y_r^\varepsilon, 0)\right)\mathrm{d}w\,y + \int_0^1 \partial_z f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w\,z.$$

This way, we can check that:

$$|i(r,y,z)| \leqslant \underbrace{\left(H_x + H_p\,|Z_r^{t,x}|\,\left\|\sigma^{-1}\right\|_{\text{lip}}\right)|\nabla X_r^\varepsilon|}_{=i_r} + H_y|y| + H_p\left\|\sigma^{-1}\right\|_\infty |z|,$$

$$I := \int_t^T i_r\,\mathrm{d}r \leqslant C\left(1 + \int_t^T |Z_r^{t,x}|\,\mathrm{d}r\right)|\nabla X^\varepsilon|_{t,T}^*,$$

$$\mathbb{E}\left[|I|^p\right] \leqslant C\left(1 + \mathbb{E}\left[\left(\int_t^T |Z_r^{t,x}|^2\,\mathrm{d}r\right)^p\right]^{\frac{1}{2}}\right)\mathbb{E}\left[|\nabla X^\varepsilon|_{t,T}^{*,2p}\right]^{\frac{1}{2}}.$$

By Step 1 and the convergence of ∇X^ε in $\mathcal{S}^p$, $\mathbb{E}\left[|I|^p\right]$ is clearly bounded independently of ε. Using Proposition 3.2 of [8], we are now able to say that, for a great enough,

$$\mathbb{E}\left[|\nabla Y^\varepsilon|_{t,T}^{*,p} + \left(\int_t^T |\nabla Z_r^\varepsilon|^2\,\mathrm{d}r\right)^{\frac{p}{2}}\right] \leqslant Ce^{ap(T-t)}\left(\|\partial_x g\|_\infty\,\mathbb{E}\left[|\nabla X_T^\varepsilon|^p\right] + \mathbb{E}\left[I^p\right]\right).$$

Step 3: Some dominated convergences. Thanks to the previous step, we have, for all $p \geqslant 2$,

$$\mathbb{E}\left[\left|g_x^\varepsilon - g_x^0\right|^p\right] \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad \mathbb{E}\left[\left(\int_t^T \left|f_x^\varepsilon(r) - f_x^0(r)\right|^2\mathrm{d}r\right)^{\frac{p}{2}}\right] \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

$$\mathbb{E}\left[\int_t^T \left|f_y^\varepsilon(r) - f_y^0(r)\right|^p\mathrm{d}r\right] \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad \mathbb{E}\left[\int_t^T \left|f_z^\varepsilon(r) - f_z^0(r)\right|^p\mathrm{d}r\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

In fact, the first, third and fourth convergences are easy to prove since f have bounded derivatives. Also, the inequality

$$\left|\partial_x f\left(\Theta_r^{t,x} + w\varepsilon(\nabla X_r^\varepsilon,0,0)\right) - \partial_x f\left(\Theta_r^{t,x}\right)\right| \leqslant 2H_x + 2H_p\left\|\sigma^{-1}\right\|_{\text{lip}}|Z_r^{t,x}|$$

allows us to use dominated convergence to prove the second convergence.

Step 4: We will show that: $\forall x \in \mathbb{R}^d$, $\mathbb{E}\left[|\nabla Y^\varepsilon - \nabla Y|_{t,T}^{*,2} + \left(\int_t^T |\nabla Z_s^\varepsilon - \nabla Z_s|^2\,\mathrm{d}s\right)\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$
Combining equations (2.24) and (2.25), we get:

$$\begin{aligned}\Delta Y_s^\varepsilon &= g_x^\varepsilon \Delta X_T^\varepsilon + \left(g_x^\varepsilon - g_x^0\right)\nabla X_T \\ &\quad + \int_s^T \left[f_x^\varepsilon(r)\nabla X_r^\varepsilon - f_x^0(r)\nabla X_r + f_y^\varepsilon(r)\Delta Y_r^\varepsilon + f_z^\varepsilon(r)\Delta Z_r^\varepsilon + \beta^\varepsilon(r)\right]\mathrm{d}r - \int_s^T \Delta Z_r^\varepsilon\,\mathrm{d}W_r,\end{aligned}$$

where we set

$$\beta^\varepsilon(s) = \left[f_y^\varepsilon(r) - f_y^0(r)\right]\nabla Y_r + \left[f_z^\varepsilon(r) - f_z^0(r)\right]\nabla Z_r,$$

$\Delta X_s^\varepsilon = \nabla X_s^\varepsilon - \nabla X_s$, $\Delta Y_s^\varepsilon = \nabla Y_s^\varepsilon - \nabla Y_s$ and $\Delta Z_s^\varepsilon = \nabla Z_s^\varepsilon - \nabla Z_s$. Lemma 2.2 of [43] gives:

$$\begin{aligned}\mathbb{E}\left[|\Delta Y^\varepsilon|_{t,T}^{*,2} + \int_t^T |\Delta Z_r^\varepsilon|^2\,\mathrm{d}r\right] \leqslant C\mathbb{E}\Bigg[&\left|g_x^\varepsilon \Delta X_T^\varepsilon + \left(g_x^\varepsilon - g_x^0\right)\nabla X_T\right|^2 \\ &+ \int_t^T \left|f_x^\varepsilon(r)\nabla X_r^\varepsilon - f_x^0(r)\nabla X_r + \beta^\varepsilon(r)\right|^2\mathrm{d}r\Bigg].\end{aligned}$$

First,

$$\mathbb{E}\left[\left|g_x^\varepsilon\Delta X_T^\varepsilon + \left(g_x^\varepsilon - g_x^0\right)\nabla X_T\right|^2\right] \leqslant 2\|\partial_x g\|_\infty\,\mathbb{E}\left[|\Delta X_T^\varepsilon|^2\right] + 2\mathbb{E}\left[\left|g_x^\varepsilon - g_x^0\right|^4\right]^{\frac{1}{2}}\mathbb{E}\left[|\nabla X_T|^4\right]^{\frac{1}{2}} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

by Lemma 2.1 of [43]. By dominated convergence, we get $\mathbb{E}\left[\int_t^T |\beta^\varepsilon(r)|^2\, \mathrm{d}r\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$, and then

$$\mathbb{E}\left[\int_t^T \left|f_x^\varepsilon(r)\nabla X_r^\varepsilon - f_x^0(r)\nabla X_r\right|^2 \mathrm{d}r\right] \leqslant 2\mathbb{E}\left[|\Delta X^\varepsilon|_{t,T}^{*,4}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_t^T \left|f_x^0(r)\right|^2 \mathrm{d}r\right)^2\right]^{\frac{1}{2}}$$
$$+ 2\mathbb{E}\left[|\nabla X^\varepsilon|_{t,T}^{*,4}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_t^T \left|f_x^\varepsilon(r) - f_x^0(r)\right|^2 \mathrm{d}r\right)^2\right]^{\frac{1}{2}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Step 5: We conclude that $\partial_x u$ exists and that $\partial_x u(t,x) = \nabla Y_t^{t,x}$, for all (t,x). See Theorem 3.1 of [43].

Step 6: We will show that $\partial_x u$ is continuous. Let $(t_i, x_i) \in [0,T]\times\mathbb{R}^d$, $i=1,2$, with $t_1 < t_2$. To simplify, we write:

$$\Theta^i = \left(X^i, Y^i, Z^i\right) = \left(X^{t_i,x_i}, Y^{t_i,x_i}, Z^{t_i,x_i}\right),\ f_x^i(r) = \partial_x f\left(\Theta_r^i\right),\ f_y^i(r) = \partial_y f\left(\Theta_r^i\right),\ f_z^i(r) = \partial_z f\left(\Theta_r^i\right),$$
$$g_x^i = \partial_x g\left(X_T^i\right),\ b_x^i(r) = \partial_x b\left(X_r^i\right) \text{ and } \sigma_x^i(r) = \partial_x \sigma\left(X_r^i\right).$$

We set $\widetilde{\Delta} X_r = \nabla X_r^1 - \nabla X_r^2$, $\widetilde{\Delta} Y_r = \nabla Y_r^1 - \nabla Y_r^2$, $\widetilde{\Delta} Z_r = \nabla Z_r^1 - \nabla Z_r^2$ and $\widetilde{\Delta}_{12}[\varphi] = \varphi^1 - \varphi^2$, for any φ.

$$\begin{aligned}
&\left|\partial_x u\left(t_1,x_1\right) - \partial_x u\left(t_2,x_2\right)\right| \\
&= \left|\mathbb{E}\left[g_x^1\nabla X_T^1 + \int_{t_1}^T \left\{f_x^1(r)\nabla X_r^1 + f_y^1(r)\nabla Y_r^1 + f_z^1(r)\nabla Z_r^1\right\} \mathrm{d}r\right]\right. \\
&\qquad \left. - \mathbb{E}\left[g_x^2\nabla X_T^2 + \int_{t_2}^T \left\{f_x^2(r)\nabla X_r^2 + f_y^2(r)\nabla Y_r^2 + f_z^2(r)\nabla Z_r^2\right\} \mathrm{d}r\right]\right| \\
&\leqslant \mathbb{E}\left[\left|g_x^1\nabla X_T^1 - g_x^2\nabla X_T^2\right|\right] + \mathbb{E}\left[\int_{t_1}^{t_2} \left\{\left|f_x^1(r)\right|\left|\nabla X_r^1\right| + \left|f_y^1(r)\right|\left|\nabla Y_r^1\right| + \left|f_z^1(r)\right|\left|\nabla Z_r^1\right|\right\} \mathrm{d}r\right] \\
&\quad + \mathbb{E}\left[\int_{t_2}^T \left\{\left|f_x^1(r)\widetilde{\Delta} X_r\right| + \left|\widetilde{\Delta}_{12}\left[f_x\right](r)\right|\left|\nabla X_r^2\right| + \|\partial_y f\|_\infty \left|\widetilde{\Delta} Y_r\right| + \left|\widetilde{\Delta}_{12}\left[f_y\right](r)\right|\left|\nabla Y_r^2\right|\right.\right. \\
&\qquad \left.\left. + \|\partial_z f\|_\infty \left|\widetilde{\Delta} Z_r\right| + \left|\widetilde{\Delta}_{12}\left[f_z\right](r)\right|\left|\nabla Z_r^2\right|\right\} \mathrm{d}r\right]
\end{aligned}$$

First of all, $\mathbb{E}\left[\int_{t_1}^{t_2} \left\{\left|f_x^1(r)\right|\left|\nabla X_r^1\right| + \|\partial_y f\|_\infty\left|\nabla Y_r^1\right| + \|\partial_z f\|_\infty\left|\nabla Z_r^1\right|\right\} \mathrm{d}r\right] \underset{t_1\to t_2}{\longrightarrow} 0$, by dominated convergence, because $\mathbb{E}\left[\left|\nabla X^1\right|_{t,T}^{*,2} + \left|\nabla Y^1\right|_{t,T}^{*,2} + \int_t^T \left|\nabla Z_s^1\right|^2 \mathrm{d}s + \int_t^T \left|Z_s^1\right|^2 \mathrm{d}s\right] < \infty$. Using Lemma 2.2 of [43],

$$\begin{aligned}
\mathbb{E}\left[\left|\widetilde{\Delta} Y\right|_{t_2,T}^{*,2} + \int_{t_2}^T \left|\widetilde{\Delta} Z_s\right|^2 \mathrm{d}s\right] \leqslant C\Bigg\{&\|\partial_x g\|_\infty^2\, \mathbb{E}\left[\left|\widetilde{\Delta} X_T\right|^2\right] + \mathbb{E}\left[\left|\widetilde{\Delta}_{12}\left[g_x\right]\right|^2\left|\nabla X_T^2\right|^2\right] \\
&+ \mathbb{E}\left[\int_{t_2}^T \left(\left|f_x^1(r)\right|^2\left|\widetilde{\Delta} X_r\right|^2 + \left|\widetilde{\Delta}_{12}\left[f_x\right](r)\right|^2\left|\nabla X_r^2\right|^2\right.\right. \\
&\left.\left. + \left|\widetilde{\Delta}_{12}\left[f_y\right](r)\right|^2\left|\nabla Y_r^2\right|^2 + \left|\widetilde{\Delta}_{12}\left[f_z\right](r)\right|^2\left|\nabla Z_r^2\right|^2\right) \mathrm{d}r\right]\Bigg\}.
\end{aligned}$$

We can adapt Step 2, we replace (t,x) by (t_2,x_2) and $x+h$ by $X_{t_2}^{t_1,x_1}$ and get:

$$\forall p \geqslant 2,\ \mathbb{E}\left[\left|X^1 - X^2\right|_{t_2,T}^{*,p} + \left|Y^1 - Y^2\right|_{t_2,T}^{*,p} + \left(\int_{t_2}^T \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \longrightarrow 0.$$

Using dominated convergence, for any function φ bounded, $\mathbb{E}\left[\left|\widetilde{\Delta}_{12}[\varphi]\right|_{t_2,T}^{*,p}\right] \longrightarrow 0$. Since $\mathbb{E}\left[\left(\int_{t_2}^{T}\left|Z_r^i\right|^2 \mathrm{d}r\right)^2\right] < \infty$, we have $\mathbb{E}\left[\int_{t_2}^{T}\left\{\left|f_x^1(r)\widetilde{\Delta}X_r\right|^2 + \left|\widetilde{\Delta}_{12}\left[f_x\right](r)\nabla X_r^2\right|^2\right\}\mathrm{d}r\right] \longrightarrow 0$. To sum up, we have shown that $\left|\partial_x u\left(t_1, x_1\right) - \partial_x u\left(t_2, x_2\right)\right| \longrightarrow 0$ when $t_1 \to t_2$ and $x_1 \to x_2$. We can prove it when $t_2 \to t_1$ and $x_2 \to x_1$: $\partial_x u$ is continuous on $[0,T] \times \mathbb{R}^d$.

Step 7: To show the relation $\forall s \in [t,T]$, $Z_s^{t,x} = \partial_x u\left(s, X_s^{t,x}\right)\sigma\left(X_s^{t,x}\right)$ $\mathbb{P}$-a.s., we can do the same as in Theorem 3.1 of [43]. We approximate b, σ, h and g by functions b^ε, σ^ε, h^ε and g^ε which are of class $\mathcal{C}^\infty$ with bounded derivatives (and the bound is independent of ε), and converge uniformly. Then, $(\sigma^\varepsilon)^{-1}$ converges uniformly to σ^{-1}. We define a function f^ε (we want it to approximate f and to be smooth) by $f^\varepsilon(x,y,z) = h^\varepsilon\left(x, y, z\sigma^\varepsilon(x)^{-1}\right)$ and this approximation satisfies the inequality $|f^\varepsilon(x,y,z) - f(x,y,z)| \leqslant \|h^\varepsilon - h\|_\infty + H_p|z|\left\|(\sigma^\varepsilon)^{-1} - \sigma^{-1}\right\|_\infty$. Like before, results from [8] are useful to adapt the end of the proof in [43].

□

Lemma 2.34. *Under the same assumptions, for every $T > t$, we have the following inequality:*

$$|\partial_x u(t,x)| \leqslant \mathcal{C}\left(1 + |x|^\mu + \mathbb{E}\left[\left|g\left(X_T^{t,x}\right)\right|^4\right]^{\frac{1}{4}}\right),$$

where $\mathcal{C}$ depends on T, H_x, H_y, H_p, M_f, $\left\|\sigma^{-1}\right\|_\infty$, $|b(0)|$, $|\sigma(0)|$, $\|b\|_{\mathrm{lip}}$, $\|\sigma\|_{\mathrm{lip}}$ and μ.

Proof. Our proof is based on the ideas developed in Theorem 3.3 of [56]. In the following, we set $\Theta_r^{t,x} = (X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$. We can rewrite the equation verified by $Y^{t,x}$ and $Z^{t,x}$, under differential form:

$$\mathrm{d}Y_s^{t,x} = -f\left(\Theta_s^{t,x}\right)\mathrm{d}s + \sum_{j=1}^{d}\left[Z_s^{t,x}\right]^{(j)}\mathrm{d}W_s^j, \tag{2.26}$$

where $\left[Z_s^{t,x}\right]^{(j)}$ stands for the j^{th} coefficient of the row $Z_s^{t,x}$. By differentiation of (2.26) w.r.t. x and discounting method,

$$\begin{aligned}
&\mathrm{d}\left(\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_s^{t,x}\right)\\
&\quad= \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\left(-\partial_x f\left(\Theta_s^{t,x}\right)\nabla X_s^{t,x}\,\mathrm{d}s - \sum_{j=1}^{d}\partial_{z_j} f\left(\Theta_s^{t,x}\right)\nabla\left[Z_s^{t,x}\right]^{(j)}\mathrm{d}s + \sum_{j=1}^{d}\nabla\left[Z_s^{t,x}\right]^{(j)}\mathrm{d}W_s^j\right).
\end{aligned}$$

We set, for every $j \in [\![1,d]\!]$, $\mathrm{d}\widetilde{W}_s^j = \mathrm{d}W_s^j - \partial_{z_j} f\left(\Theta_s^{t,x}\right)\mathrm{d}s$; $\widetilde{W}$ is a Brownian motion under a new probability denoted $\mathbb{Q}$. Between s and T (for $s \in [t,T]$), the integral form of this equation is:

$$\begin{aligned}
&\mathrm{e}^{\int_t^T \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_T^{t,x} - \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_s^{t,x}\\
&\quad= -\int_s^T \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\partial_x f\left(\Theta_r^{t,x}\right)\nabla X_r^{t,x}\,\mathrm{d}r + \sum_{j=1}^{d}\int_s^T \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\nabla\left[Z_r^{t,x}\right]^{(j)}\mathrm{d}\widetilde{W}_r^j.
\end{aligned} \tag{2.27}$$

We define $F_s^{t,x} = \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_s^{t,x} + \int_t^s \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\partial_x f\left(\Theta_r^{t,x}\right)\nabla X_r^{t,x}\,\mathrm{d}r$, and then Equation (2.27) becomes:

$$F_s^{t,x} = F_T^{t,x} - \sum_{j=1}^{d}\int_s^T \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\nabla\left[Z_r^{t,x}\right]^{(j)}\mathrm{d}\widetilde{W}_r^j, \tag{2.28}$$

and it tells us that $F^{t,x}$ is a $\mathbb{Q}$-martingale. We recall, for every $s \in [t,T]$, that $Y_s^{t,x} = u(s, X_s^{t,x})$, $\nabla Y_s^{t,x} = \partial_x u(s, X_s^{t,x}) \nabla X_s^{t,x}$ and $Z_s^{t,x} = \partial_x u(s, X_s^{t,x}) \sigma(X_s^{t,x}) = \nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(X_s^{t,x})$. Indeed, $\nabla X_s^{t,x}$ is invertible. We can show that $\nabla X^{t,x}$ is the solution of the linear SDE (because b and σ are $\mathcal{C}^1$ with bounded derivatives):

$$\mathrm{d}\nabla X_s^{t,x} = \partial_x b\left(X_s^{t,x}\right) \nabla X_s^{t,x}\,\mathrm{d}s + \sum_{j=1}^{d} \partial_x^j \sigma\left(X_s^{t,x}\right) \nabla X_s^{t,x}\,\mathrm{d}W_s^j,$$

whose solution is given by

$$\nabla X_s^{t,x} = \exp\left(\int_s^t \left\{\partial_x b\left(X_r^{t,x}\right) - \frac{1}{2}\sum_{j=1}^{d} \partial_x^j \sigma\left(X_r^{t,x}\right)\left[\partial_x^j \sigma\left(X_r^{t,x}\right)\right]^*\right\} \mathrm{d}r + \sum_{j=1}^{d}\int_t^s \partial_x^j \sigma\left(X_r^{t,x}\right)\,\mathrm{d}W_r^j\right),$$

where the notation $\partial_x^j \sigma(X_s^{t,x})$ stands for the $(d \times d)$-matrix $\begin{pmatrix} \partial_x[\sigma(X_s^{t,x})]^{(1,j)} \\ \hline \vdots \\ \hline \partial_x[\sigma(X_s^{t,x})]^{(d,j)} \end{pmatrix}$, and where $[\sigma(X_s^{t,x})]^{(i,j)}$ is the coefficient at line i and column j in the matrix $\sigma(X_s^{t,x})$. This way, we see that $(\nabla X_s^{t,x})^{-1}$ is solution of the following linear SDE:

$$\mathrm{d}(\nabla X_s^{t,x})^{-1} = (\nabla X_s^{t,x})^{-1}\left\{-\partial_x b(X_s^{t,x}) + \frac{1}{2}\sum_{j=1}^{d} \partial_x^j \sigma(X_s^{t,x})[\partial_x^j \sigma(X_s^{t,x})]^*\right\}\mathrm{d}s - (\nabla X_s^{t,x})^{-1}\sum_{j=1}^{d} \partial_x^j \sigma(X_s^{t,x})\,\mathrm{d}W_s^j. \tag{2.29}$$

In the following, for every $s \in [t,T]$, we set $R_s^{t,x} = Z_s^{t,x}\sigma(X_s^{t,x})^{-1} = \nabla Y_s^{t,x}(\nabla X_s^{t,x})^{-1} = \partial_x u(s, X_s^{t,x})$, $\beta_s^{t,x} = \displaystyle\int_t^s \mathrm{e}^{\int_t^r \partial_y f(\Theta_u^{t,x})\,\mathrm{d}u}\partial_x f(\Theta_r^{t,x})\nabla X_r^{t,x}\,\mathrm{d}r(\nabla X_s^{t,x})^{-1}$, $\widetilde{R}_s^{t,x} = F_s^{t,x}(\nabla X_s^{t,x})^{-1} = \mathrm{e}^{\int_t^s \partial_y f(\Theta_r^{t,x})\,\mathrm{d}r}R_s^{t,x} + \beta_s^{t,x}$. Using the integration by parts formula, combining (2.28) and (2.29), one gets:

$$\begin{aligned}
\mathrm{d}\widetilde{R}_s^{t,x} = \widetilde{R}_s^{t,x}&\left\{-\partial_x b\left(X_s^{t,x}\right) + \frac{1}{2}\sum_{j=1}^{d} \partial_x^j \sigma\left(X_s^{t,x}\right)\left[\partial_x^j \sigma\left(X_s^{t,x}\right)\right]^* - \sum_{j=1}^{d} \partial_{z_j} f\left(\Theta_s^{t,x}\right)\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\mathrm{d}s \\
&- \sum_{j=1}^{d} \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1}\partial_x^j \sigma\left(X_s^{t,x}\right)\,\mathrm{d}s \\
&+ \sum_{j=1}^{d}\left\{\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\mathrm{d}\widetilde{W}_s^j.
\end{aligned} \tag{2.30}$$

Then, let us take a new parameter $\lambda \in \mathbb{R}$; we have:

$$\mathrm{d}\left|\mathrm{e}^{\lambda s}\widetilde{R}_s^{t,x}\right|^2 \tag{2.31}$$

$$\begin{aligned}
= \mathrm{e}^{2\lambda s}\widetilde{R}_s^{t,x}&\left\{2\lambda \mathrm{I}_d - 2\partial_x b\left(X_s^{t,x}\right) + \sum_{j=1}^{d} \partial_x^j \sigma\left(X_s^{t,x}\right)\left[\partial_x^j \sigma\left(X_s^{t,x}\right)\right]^* - 2\sum_{j=1}^{d} \partial_{z_j} f\left(\Theta_s^{t,x}\right)\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^*\mathrm{d}s \\
&-2\mathrm{e}^{2\lambda s}\sum_{j=1}^{d} \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1}\partial_x^j \sigma\left(X_s^{t,x}\right)\left(\widetilde{R}_s^{t,x}\right)^*\mathrm{d}s \\
&+2\mathrm{e}^{2\lambda s}\sum_{j=1}^{d}\left\{\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^*\mathrm{d}\widetilde{W}_s^j \\
&+\mathrm{e}^{2\lambda s}\sum_{j=1}^{d}\left\{\left[\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right]\right. \\
&\left.\times\left[\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right]^*\right\}\mathrm{d}s.
\end{aligned} \tag{2.32}$$

If we denote $\gamma = e^{\int_t^s \partial_y f(\Theta_r^{t,x})\,dr} \nabla \left[Z_s^{t,x}\right]^{(j)} (\nabla X_s^{t,x})^{-1}$ and $\delta = \widetilde{R}_s^{t,x} \partial_x^j \sigma(X_s^{t,x})$, we remark that we have the equality $-2\gamma\delta^* + |\gamma - \delta|^2 = |\gamma - 2\delta|^2 - 3|\delta|^2$. Thus,

$$
\begin{aligned}
& d\left|e^{\lambda s}\widetilde{R}_s^{t,x}\right|^2 \qquad (2.33)\\
&= 2e^{2\lambda s}\widetilde{R}_s^{t,x}\left\{\lambda I_d - \partial_x b\left(X_s^{t,x}\right) - \sum_{j=1}^d \partial_x^j \sigma\left(X_s^{t,x}\right)\left[\partial_x^j\sigma\left(X_s^{t,x}\right)\right]^* - \sum_{j=1}^d \partial_{z_j} f\left(\Theta_s^{t,x}\right)\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^* ds\\
&\quad + 2e^{2\lambda s}\sum_{j=1}^d \left\{e^{\int_t^s \partial_y f(\Theta_r^{t,x})\,dr}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j\sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^* d\widetilde{W}_s^j\\
&\quad + e^{2\lambda s}\sum_{j=1}^d \left|e^{\int_t^s \partial_y f(\Theta_r^{t,x})\,dr}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - 2\widetilde{R}_s^{t,x}\partial_x^j\sigma\left(X_s^{t,x}\right)\right|^2 ds. \qquad (2.34)
\end{aligned}
$$

This way, we can see that, for λ great enough (that is to say bigger than something depending only on H_p, $\left\|\sigma^{-1}\right\|_\infty$ and on bounds over the derivatives of b and σ), the process $\left(\left|e^{\lambda s}\widetilde{R}_s^{t,x}\right|^2\right)_{s\in[t,T]}$ is a $\mathbb{Q}$-submartingale. So, we get:

$$
\left|R_t^{t,x}\right|^2 (T-t) \leqslant e^{-2\lambda t}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{2\lambda s}\left|\widetilde{R}_s^{t,x}\right|^2 ds\right] \leqslant e^{2\lambda(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|\widetilde{R}_s^{t,x}\right|^2 ds\right].
$$

But $\widetilde{R}_s^{t,x} = e^{\int_t^s \partial_y f(\Theta_r^{t,x})\,dr} R_s^{t,x} + \beta_s^{t,x}$ and $\partial_y f$ is bounded by a constant H_y, so

$$
\left|R_t^{t,x}\right|^2 (T-t) \leqslant 2e^{2(\lambda+H_y)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|R_s^{t,x}\right|^2 ds\right] + 2e^{2\lambda(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|\beta_s^{t,x}\right|^2 ds\right].
$$

On the one hand, we have:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|R_s^{t,x}\right|^2 ds\right] &\leqslant \left\|\sigma^{-1}\right\|_\infty^2 \mathbb{E}\left[e^{\int_t^T \partial_z f(\Theta_s^{t,x})\,dW_s - \frac{1}{2}\int_t^T \left|\partial_z f(\Theta_s^{t,x})\right|^2 ds}\int_t^T \left|Z_s^{t,x}\right|^2 ds\right]\\
&\leqslant \left\|\sigma^{-1}\right\|_\infty^2 e^{\frac{1}{2}H_p^2\left\|\sigma^{-1}\right\|_\infty^2 (T-t)}\mathbb{E}\left[\left(\int_t^T \left|Z_s^{t,x}\right|^2 ds\right)^2\right]^{\frac{1}{2}} \qquad (2.35)
\end{aligned}
$$

We recall equation (2.22): $\mathbb{E}\left[\left(\int_t^T \left|Z_r^{t,x}\right|^2 dr\right)^2\right] \leqslant Ce^{4a(T-t)}\left\{\mathbb{E}\left[\left|Y^{t,x}\right|_{t,T}^{*,4}\right] + C_T\left(1+|x|^{4\mu}\right)\right\}$. By using the Lemma 2.2 of [43], we have, where C is a constant:

$$
\mathbb{E}\left[\left|Y^{t,x}\right|_{t,T}^{*,4}\right] \leqslant C\mathbb{E}\left[\left|g\left(X_T^{t,x}\right)\right|^4 + \int_t^T \left|f\left(X_r^{t,x},0,0\right)\right|^4 dr\right].
$$

But, using Proposition 2.4 and the assumption made on $f(\bullet,0,0)$, we get:

$$
\mathbb{E}\left[\int_t^T \left|f\left(X_r^{t,x},0,0\right)\right|^4 dr\right] \leqslant C\left(1+|x|^{4\mu}\right),
$$

where C only depends on μ, M_f, T, $|\sigma(0)|$, $\|\sigma\|_{\text{lip}}$, $|b(0)|$ and $\|b\|_{\text{lip}}$. On the other hand, using the Lemma 2.1 of [43], we can show that $\nabla X^{t,x}$ and $(\nabla X^{t,x})^{-1}$ are in $\mathcal{S}^p$, for all $p<\infty$. Also, we have $|\partial_x f(\Theta_r^{t,x})| \leqslant H_x + H_p|Z_r^{t,x}|\|\partial_x\sigma^{-1}\|_\infty$. Then, with Cauchy-Schwarz inequality and a priori estimates of [8], we get that $\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|\beta_s^{t,x}\right|^2 ds\right]$ is bounded. We recall that $R_t^{t,x} = \partial_x u(t,x)$. We have the conclusion:

$$
|\partial_x u(t,x)|^2 \leqslant C\left(1+|x|^{2\mu} + \mathbb{E}\left[\left|g\left(X_T^{t,x}\right)\right|^4\right]^{\frac{1}{2}}\right).
$$

□

Theorem 2.35. *We make the following assumptions:*

- *σ and b are Lipschitz continuous;*
- *the function $\sigma(\bullet)^{-1}$ is bounded;*
- *$f \in \mathcal{C}^0(\mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^d)^*, \mathbb{R})$;*
- *$\forall x \in \mathbb{R}^d$, $|f(x,0,0)| \leqslant M_f\,(1+|x|^\mu)$ and $|g(x)| \leqslant M_g\,(1+|x|^\nu)$;*
- *we can write $f(x,y,z) = \psi(x,z) - \alpha y$, with $\alpha \in (0,1]$ and if we define $h : (x,p) \mapsto \psi(x, p\sigma(x))$, then h is Lipschitz continuous (with bounds denoted H_x and H_p w.r.t. x and p).*

Under those assumptions, the function $u : (t,x) \mapsto Y_t^{t,x}$ satisfies the following inequality, for every $T > t$:

$$\left|u(t,x) - u(t,x')\right| \leqslant \mathcal{C}\left(1 + |x|^{\mu\vee\nu} + |x'|^{\mu\vee\nu}\right)|x - x'|,$$

where $\mathcal{C}$ depends on T, H_x, H_p, M_f, M_g, $\left\|\sigma^{-1}\right\|_\infty$, $|b(0)|$, $|\sigma(0)|$, $\|b\|_{\text{lip}}$, $\|\sigma\|_{\text{lip}}$ and μ.

Proof. We approximate the functions h, g, b and σ by some uniformly converging sequences $(h^\varepsilon)_{\varepsilon>0}$, $(g^\varepsilon)_{\varepsilon>0}$, $(b^\varepsilon)_{\varepsilon>0}$ and $(\sigma^\varepsilon)_{\varepsilon>0}$ of functions of class $\mathcal{C}^1$ with bounded derivatives. We can assume that there exists a bound, independent of ε, of all the derivatives of h^ε, b^ε and σ^ε, for every $\varepsilon > 0$. We define an approximation f^ε of f by $f^\varepsilon(x,y,z) = h^\varepsilon\left(x, z\sigma(x)^{-1}\right) - \alpha y$ and we can assume that $f^\varepsilon(\bullet,0,0) = h^\varepsilon(\bullet,0,0)$ and g^ε have polynomial growth independently of ε, with constants M_f and M_g and exponents μ and ν. For every $(t,x) \in [0,T) \times \mathbb{R}^d$, $|\partial_x u^\varepsilon(t,x)| \leqslant \mathcal{C}\,(1+|x|^{\mu\vee\nu})$, where $\mathcal{C}$ does not depend on ε or x. Using the mean-value theorem, it comes that :

$$\forall t \in [0,T),\ \forall x, x' \in \mathbb{R}^d,\ \left|u^\varepsilon(t,x) - u^\varepsilon(t,x')\right| \leqslant \mathcal{C}\left(1 + |x|^{\mu\vee\nu} + |x'|^{\mu\vee\nu}\right)|x - x'|. \tag{2.36}$$

Then, for every $T > 1$, the set $\{u^\varepsilon(t,\bullet) | \varepsilon > 0, t \in [0, T-1]\}$ is equicontinuous; also it is pointwise bounded: we can show it by writing the equation verified by $Y^{\varepsilon,t,x}$ and using Lemma 2.2 of [43]. Let K be a compact subset of $\mathbb{R}^d$. By the Arzelà-Ascoli theorem, we get, for a sequence $(\varepsilon_n^{(K)})_{n\geqslant 0}$ which has 0 as limit:

$$\forall t \in [0, T-1],\ u^{\varepsilon_n^{(K)}}(t,\bullet)\big|_K \xrightarrow[n\to\infty]{\|\cdot\|_\infty} u_K(t,\bullet).$$

Let $x \in K$; we have: $u^{\varepsilon_n^{(K)}}(t,x) \underset{n\to\infty}{\longrightarrow} u_K(t,x)$ and $u^{\varepsilon_n^{(K)}}(t,x) = Y_t^{\varepsilon_n^{(K)},t,x} \underset{n\to\infty}{\longrightarrow} Y_t^{t,x}$. This way, we see that the limit $u_K(t,x) = Y_t^{t,x}$ does not depend on K; and the convergence of $u^{\varepsilon_n^{(K)}}(t,\bullet)$ is uniform on K. The following triangle inequality gives the result claimed after taking the limit as n goes to infinity (because $\mathcal{C}$ is independent of K): for every $x, x' \in K$, $t \in [0, T-1]$ and $n \in \mathbb{N}$,

$$|u(t,x) - u(t,x')| \leqslant \left|u(t,x) - u^{\varepsilon_n^{(K)}}(t,x)\right| + \mathcal{C}\left(1 + |x|^{\mu\vee\nu} + |x'|^{\mu\vee\nu}\right)|x - x'| + \left|u^{\varepsilon_n^{(K)}}(t,x') - u(t,x')\right|.$$

$\square$

Chapitre 3

EDSR ergodiques randomisées et application au comportement en temps long de la solution de viscosité d'un système d'équations HJB

Résumé

Nous étudions dans ce chapitre des EDSRE, dont les coefficients peuvent varier au cours du temps suivant une mesure de Poisson. L'EDS sous-jacente a un bruit multiplicatif non-dégénéré et ne comporte aucun saut ; la dissipativité faible de la dérive contrebalance la croissance linéaire du terme diffusif. Nous étudions l'existence et l'unicité de la solution de l'EDSRE et nous appliquons nos résultats à un problème de contrôle ergodique optimal. Finalement, nous obtenons le comportement en temps long de la solution de viscosité d'un système d'équations de Hamilton-Jacobi-Bellman couplées, avec une vitesse de convergence exponentielle.

Mots-clés : équation différentielle stochastique rétrograde ergodique ; bruit multiplicatif et non-borné ; mesure de Poisson ; système d'équations HJB ; comportement en temps long ; vitesse de convergence.

Abstract

We study in this chapter some EBSDEs, whose coefficients may vary from time to time due to a Poisson measure. The underlying SDE has a multiplicative and non-degenerate noise and does not have any jump; the weak dissipativity of the drift balances the linear growth of the diffusion term. We study existence and uniqueness of the solution of the EBSDE and we apply our results to an ergodic optimal control problem. Finally, we obtain the large time behaviour of the viscosity solution of a system of coupled Hamilton-Jacobi-Bellman equations, with an exponential rate of convergence.

Key words: ergodic backward stochastic differential equation; multiplicative and unbounded noise; Poisson measure; system of HJB equations; large time behaviour; rate of convergence.

3.1 Introduction

We study the following EBSDE in infinite horizon: for all $t, T \in \mathbb{R}_+$, such that $0 \leqslant t \leqslant T < \infty$,

$$Y_t^{x,n} = Y_T^{x,n} + \int_t^T \left\{ f_{N_s^n}(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}) - \lambda \right\} \mathrm{d}s - \int_t^T Z_s^{x,n} \, \mathrm{d}W_s - \int_t^T H_s^{x,n} \, \mathrm{d}M_s, \tag{3.1}$$

where the unknown is the quadruplet $(Y^{x,n}, Z^{x,n}, H^{x,n}, \lambda)$, with:

- $Y^{x,n}$ a real-valued and progressively measurable process;
- $Z^{x,n}$ an $(\mathbb{R}^d)^*$-valued and progressively measurable process;
- $H^{x,n}$ an $(\mathbb{R}^{k-1})^*$-valued and progressively measurable process;
- λ a real number.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a right continuous and complete stochastic basis. The given data of our equation consists in:

- W, a d-dimensional standard Brownian motion;
- $\mathbb{K} = [\![1, k]\!]$, the set of integers between 1 and k, and $\mathbb{L} = \mathbb{K} \backslash \{k\}$;
- $\mathcal{N}$, a Poisson random measure on $\mathbb{R}_+ \times \mathbb{L}$, independent of W, where $\mathbb{L}$ is equipped with the field $\mathcal{L}$ of all subsets of $\mathbb{L}$, such that, if we set $\mathcal{M}([0,t] \times A) = \mathcal{N}([0,t] \times A) - t\mu\#A$, $\mathcal{M}([0,\bullet] \times A)$ is a $(\mathcal{F}_t)_t$-martingale for all $A \subset \mathcal{L}$ and some fixed $\mu > 0$;
- $x \in \mathbb{R}^d$ and $n \in \mathbb{K}$;
- for every $s \geqslant t \geqslant 0$ and $n \in \mathbb{K}$, we set $N_s^{t,n} = n + \sum_{l \in \mathbb{L}} l \, \mathcal{N}(]t,s] \times \{l\})$ modulo k;
- for every $s \in \mathbb{R}_+$ and $l \in \mathbb{L}$, we set $M_s(l) = \mathcal{N}(]0,s] \times \{l\}) - \mu s$;
- $X^{x,n}$, an $\mathbb{R}^d$-valued process, starting from x and equation n, which is the solution of the SDE: for all $t \in \mathbb{R}_+$,
$$X_t^{x,n} = x + \int_0^t b_{N_s^n}(X_s^{x,n}) \, \mathrm{d}s + \int_0^t \sigma_{N_s^n}(X_s^{x,n}) \, \mathrm{d}W_s; \tag{3.2}$$
- $f_i : \mathbb{R}^d \times (\mathbb{R}^d)^* \times (\mathbb{R}^{k-1})^* \to \mathbb{R}$ some measurable functions.

Ergodic BSDEs with an additive Brownian noise, and in the case $k = 1$, have already been widely studied. This class of BSDEs was first studied by Fuhrman, Hu and Tessitore in [26] in order to study optimal ergodic control problem; their main assumption was the strong dissipativity of b, that is to say:

$$\exists \eta > 0, \, \forall x, x' \in \mathbb{R}^d, \, \langle b(x) - b(x'), x - x' \rangle \leqslant -\eta |x - x'|^2;$$

this assumption was then weakened in [22], where b can be written as the sum of a disspative function and a bounded function. Under similar assumptions, the authors of [35] studied the large time behaviour of finite horizon BSDEs:

$$Y_t^{T,x} = g(X_T^x) + \int_t^T f(X_s^x, Z_s^{T,x}) \, \mathrm{d}s - \int_t^T Z_s^{T,x} \, \mathrm{d}W_s, \quad t \in [0, T], \tag{3.3}$$

and they proved that it was linked with ergodic BSDEs. More precisely, they proved the existence of a constant $L \in \mathbb{R}$, such that for all $x \in \mathbb{R}^d$,

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T \to \infty}{\longrightarrow} L;$$

and they even obtained an exponential rate of convergence. More recently, the two authors of this paper proved that this convergence and this rate still hold when the noise is multiplicative (see [32]).

Itô's formula gives a bridge between those BSDEs and Hamilton-Jacobi-Bellman equations. Indeed, the solution of equation (3.3) can be written as $Y_t^{T,x} = u(T-t, X_t^x)$, where u is the viscosity solution of the Cauchy problem:

$$\begin{cases} \partial_t u(t,x) = \mathcal{L}u(t,x) + f(x, \partial_x u(t,x)\sigma(x)), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.4}$$

and where $\mathcal{L}$ is the generator of the Kolmogorov semigroup of X^x, solution of a given SDE.

Also, the ergodic BSDE has a solution such that we can write $Y_t^x = v(X_t^x)$ where (v, λ) is the viscosity solution of:

$$\mathcal{L}v(x) + f(x, \nabla v(x)\sigma(x)) = \lambda, \quad \forall x \in \mathbb{R}^d. \tag{3.5}$$

Our goal in this article is to study coupled Hamilton-Jacobi-Bellman equations:

$$\begin{cases} \partial_t u_i(t,x) = \mathcal{L}^i u_i(t,x) + f_i(x, \partial_x u_i(t,x)\sigma_i(x), (u_{i+l}(t,x) - u_i(t,x))_{l \in \mathbb{L}}), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,\ i \in \mathbb{K}, \\ u_i(0,x) = g_i(x), & x \in \mathbb{R}^d,\ i \in \mathbb{K}, \end{cases} \tag{3.6}$$

and to see that its viscosity solution is related asymptoticly to the viscosity solution (v, λ) of the ergodic problem

$$\mathcal{L}^i v_i(x) + f_i(x, \nabla v_i(x)\sigma_i(x), (v_{i+l}(x) - v_i(x))_{l \in \mathbb{L}}) = \lambda, \quad x \in \mathbb{R}^d,\ i \in \mathbb{K}. \tag{3.7}$$

Such systems of equations have already been linked to finite horizon BSDEs (see [53]). A Poisson random measure decides which dynamic to follow at any time: this fact implies a new dependance of every coefficient of the BSDE and its underlying SDE, see equations (3.1) and (3.2).

In the case $k = 1$ (i.e. when there is only one equation), viscosity solutions of such PDEs and their large time behaviour have already been investigated (see e.g. [29]). The case $k > 1$ has been considered in [40]. Ley and Nguyen in [40] look for the large time behaviour of nonlinear parabolic systems, such as (3.6), but we do not make here the same assumptions. The authors in [40] work on the torus $\mathbb{R}^n / \mathbb{Z}^n$ and they require the coupling between the equations to be linear. On the other hand, they allow the parameters to depend on a new parameter θ, and they present some assumptions on the Hamiltonians that permit to work with degenerate equations (only one σ_i needs to be uniformly elliptic, instead of every σ_i in our article). They prove the convergence of $u_i(t, \bullet) - v_i - \lambda t$ as $t \to \infty$, but they do not have any rate of convergence. Similar results are obtained in [12], where the Hamiltonians are convex, the equations defined on a torus and can be degenerate.

In [33], the authors get the large time behaviour of a system of PDEs, with quadratic growth Hamiltonians; the coupling between the different equations has also a form more specific than in (3.6). They make a link between such a system and a system of quadratic BSDEs, driven by a forward process satisfying an SDE with additive noise and dissipative drift.

Compared to [32], the new contribution of this article is that we consider not only one Hamilton-Jacobi-Bellman equation, but a system of coupled equations. In order to keep, in average, the process X near the origin, we require the drifts b_i to be sufficiently weakly dissipative regarding the linear growth of σ_i. The method is close to the one adopted in [32]: first, exponential mixing for X is obtained by irreducibility; then, it gives us the convergence of the solution of the discounted BSDE of infinite horizon

$$Y_t^{\alpha,x,n} = Y_T^{\alpha,x,n} + \int_t^T \left\{ f_{N_s^n}(X_s^{x,n}, Z_s^{\alpha,x,n}, H_s^{\alpha,x,n}) - \alpha Y_s^{\alpha,x,n} \right\} \mathrm{d}s - \int_t^T Z_s^{\alpha,x,n}\, \mathrm{d}W_s - \int_t^T H_s^{\alpha,x,n}\, \mathrm{d}M_s, \tag{3.8}$$

i.e. the convergence of $(Y^{\alpha,x,n} - Y_0^{\alpha,0,n}, Z^{\alpha,x,n}, H^{\alpha,x,n}, \alpha Y_0^{\alpha,0,n})$ to a solution $(Y^{x,n}, Z^{x,n}, H^{x,n}, \lambda)$ of the EBSDE, for every $x \in \mathbb{R}^d$ and $n \in \mathbb{K}$. We can prove uniqueness for λ, and also for $Y^{x,n}$, $Z^{x,n}$ and $H^{x,n}$, but under certain conditions. Like in [32], the unboundedness of σ_i implies that we require f_i to be Lipschitz continuous with respect to $z\sigma_i(x)^{-1}$: the price to pay is some technical difficulty which is treated in the appendix.

The paper is organised as follows. In section 3.2, we study an SDE slightly more general than (3.2): indeed, it is the SDE satisfied by X after a change of probability space, due to Girsanov's theorem. Section 3.3 is devoted to the auxiliary discounted BSDE of infinite horizon (3.8). In section 3.4, we prove existence and uniqueness for the solution of the ergodic BSDE (3.1). The large time behaviour of the solution of

$$Y_t^{T,x,n} = g_{N_T^n}(X_T^{x,n}) + \int_t^T f_{N_s^n}(X_s^{x,n}, Z_s^{T,x,n}, H_s^{T,x,n})\,\mathrm{d}s - \int_t^T Z_s^{T,x,n}\,\mathrm{d}W_s - \int_t^T H_s^{T,x,n}\,\mathrm{d}M_s$$

will be presented in section 3.5. Finally, section 3.6 applies the results of this paper to an optimal ergodic control problem and to the solution of the system (3.6).

3.2 The SDE

We consider the SDE:

$$\begin{cases} \mathrm{d}X_t^{x,n} = b_{N_t^n}(t, X_t^{x,n})\,\mathrm{d}t + \sigma_{N_t^n}(X_t^{x,n})\,\mathrm{d}W_t, \\ X_0^{x,n} = x. \end{cases} \tag{3.9}$$

Assumption 3.1.

- The functions $b_l : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_l : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ are Lipschitz continuous w.r.t. x (uniformly in t for b_l), for all $l \in \mathbb{K}$. We write $\|b\|_{\mathrm{lip}}$ and $\|\sigma\|_{\mathrm{lip}}$ their Lipschitz constants.
- The function b has linear growth: $\exists \xi_1, \xi_2 > 0$, $\forall l \in \mathbb{K}$, $\forall t \in \mathbb{R}_+$, $\forall x \in \mathbb{R}^d$, $|b_l(t,x)| \leqslant \xi_1 + \xi_2|x|$.
- For any $l \in \mathbb{K}$, the function b_l is weakly dissipative; that is to say $b_l(t,x) = b_l^\sharp(x) + b_l^\flat(t,x)$, where $b_l^\sharp$ is η-dissipative and locally Lipschitz continuous and $b_l^\flat$ is bounded. This way,
$$\exists \eta_1, \eta_2 > 0,\ \forall l \in \mathbb{K},\ \forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{R}^d,\ \langle b_l(t,x), x\rangle \leqslant \eta_1 - \eta_2|x|^2.$$
- The functions $x \mapsto \sigma_l(x)^{-1}$, $l \in \mathbb{K}$, are bounded by a common constant $\|\sigma^{-1}\|_\infty$.
- The function σ has linear growth: $\exists r_1, r_2 > 0$, $\forall l \in \mathbb{K}$, $|\sigma_l(x)|_F^2 \leqslant r_1 + r_2|x|^2$, with $r_2 < 2\eta_2$.

Remark 3.2. We recall that, for any matrix A, $|A|_F$ denotes its Frobenius norm.

Proposition 3.3. *Under Assumption 3.1, for all $p \in (0, +\infty)$, and $T > 0$, we have:*

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\left[|X_t^x|^p\right] \leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |X_t^x|^p\right] \leqslant C\left(1+|x|^p\right),$$

where C only depends on p, T, r_1, r_2, ξ_1 and ξ_2.

Proof. This is a straightforward consequence of Burkholder-Davis-Gundy's inequality and Gronwall's lemma. □

Proposition 3.4. *Suppose that Assumption 3.1 holds true. Let $T > 0$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ a bounded function. The process $\widetilde{W}_t^{x,n} = W_t - \int_0^t \gamma(s, X_s^{x,n})\, \mathrm{d}s$ is a Brownian motion under a probability measure $\widetilde{\mathbb{P}}^{x,n,T}$, on the interval $[0,T]$. When $\sqrt{r_2}\|\gamma\|_\infty + \frac{r_2}{2} < \eta_2$, we get:*

$$\sup_{T \geqslant 0} \widetilde{\mathbb{E}}^{x,n,T}\left[|X_T^{x,n}|^2\right] \leqslant C\left(1 + |x|^2\right),$$

where C only depends on η_1, η_2, r_1, r_2 and $\|\gamma\|_\infty$.

Proof. See the proposition 5 of [32]. □

Theorem 3.5. *Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function, satisfying $|\phi(x)| \leqslant c_\phi(1 + |x|^2)$. Then, we have*

$$\forall x, y \in \mathbb{R}^d,\ \forall n, m \in \mathbb{K},\ \forall t \geqslant 0,\ |\mathcal{P}_t[\phi](x,n) - \mathcal{P}_t[\phi](y,m)| \leqslant \widehat{c} c_\phi (1 + |x|^2 + |y|^2)\mathrm{e}^{-\widehat{\nu} t},$$

where $\mathcal{P}_t[\phi](x,n) = \mathbb{E}[\phi(X_t^{x,n})]$ (Kolmogorov semigroup of X) and $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1 and r_2.

Proof. For reader's convenience, we recall here Theorem 1.7 of [14].

Lemma. *Let (E, d) be a Polish space and F a finite set. Let $(Z^{(n)})_{n \in F}$ be a family of E-valued Markov processes represented by their semigroups $(\mathcal{P}^{(n)})_{n \in F}$. Finally, let $(a(i,j))_{i,j \in F}$ be a family of non-negative numbers. Given a starting point $(x, i) \in E \times F$ and exponential r.v. T_j of mean $a(i,j)^{-1}$, we set $T = \min_{j \in F} T_j$. For every $t < T$, we set $X_t^{x,i} = Z_t^{(i),x}$ and $I_t^i = i$. Also, $X_T^{x,i} = X_{T-}^{x,i}$; there exists a unique $j \in F$ such that $T = T_j$ and we set $I_T^i = j$. We take $(X_T^{x,i}, I_T^i)$ as a new starting point at time T. We make the following assumptions:*

1. *the family $(a(i,j))_{i,j \in F}$ yields the transition rates of an irreducible and positive recurrent Markov chain;*
2. *there exists $K > 0$, and a function $V : E \to \mathbb{R}_+$ such that:*

$$\forall i \in F,\ \exists \lambda(i) \in \mathbb{R},\ \forall t \in \mathbb{R}_+,\ \forall x \in E,\ \mathcal{P}_t^{(i)} V(x) \leqslant \mathrm{e}^{-\lambda(i)t} V(x) + K;$$

3. *I has an invariant probability measure ν verifying $\sum_{i \in F} \nu(i)\lambda(i) > 0$;*
4. *there exists $i_0 \in F$ such that the sublevel sets $\{x \in E | V(x) \leqslant K\}$ of V are small for $\mathcal{P}^{(i_0)}$,* i.e.,

$$\exists i_0 \in F,\ \forall K > 0,\ \exists t > 0,\ \exists \varepsilon > 0,\ \forall x, y \in E,\ [V(x) \vee V(y) \leqslant K] \Rightarrow \mathrm{d}_{TV}(\delta_x \mathcal{P}_t^{(i_0)}, \delta_y \mathcal{P}_t^{(i_0)}) \leqslant 1 - \varepsilon.$$

Then there exists a probability measure π and positive constants C and λ such that for all x and i,

$$\|\mathcal{P}_t((x,i), \bullet) - \pi\|_V := \sup_{f \in \mathcal{V}} \left| \int_{E \times F} f(y)\, \mathcal{P}_t((x,i), \mathrm{d}y) - \int_{E \times F} f(y)\, \pi(\mathrm{d}y) \right| \leqslant C(1 + V(x))\mathrm{e}^{-\lambda t},$$

where $\mathcal{V} = \{$measurable f with $|f(x)| < 1 + V(x)\}$.

Here, $E = \mathbb{R}^d$ and $F = \mathbb{K}$. The process I is the Poisson process N and $a(i,j) = \mu$ for any i and j (a gives the transition rates of an irreducible and positive recurrent Markov chain). For any $n \in \mathbb{K}$, $Z^{(n)}$ is the solution of the SDE:

$$\begin{cases} \mathrm{d}Z_t^{(n),x} = b_n(t, Z_t^{(n),x})\, \mathrm{d}t + \sigma_n(Z_t^{(n),x})\, \mathrm{d}W_t, \\ Z_0^{(n),x} = x, \end{cases}$$

and we need to prove that $x \mapsto |x|^2$ is a Lyapunov function for any of those underlying SDEs. Using Itô's formula, we easily obtain:

$$\mathrm{d}|Z_t^{(n),x}|^2 \leqslant \left(2\eta_1 - 2\eta_2|Z_t^{(n),x}|^2 + r_1 + r_2|Z_t^{(n),x}|^2\right) \mathrm{d}t + 2\langle Z_t^{(n),x}, \sigma_n(Z_t^{(n),x})\,\mathrm{d}W_t\rangle.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{(2\eta_2-r_2)t}\mathbb{E}\left[|Z_t^{(n),x}|^2\right]\right) \leqslant (2\eta_1 + r_1)\mathrm{e}^{(2\eta_2-r_2)t}$$

gives us

$$\mathbb{E}\left[|Z_t^{(n),x}|^2\right] \leqslant \mathrm{e}^{-(2\eta_2-r_2)t}|x|^2 + \frac{2\eta_1 + r_1}{2\eta_2 - r_2}.$$

For any i, we get $\lambda(i) = 2\eta_2 - r_2 > 0$. To conclude, we recall that, for any n, we have exponential mixing for the SDE satisfied by $Z^{(n),x}$ (see the theorem 7 of [32]), which ensures that the balls are small for the Kolmogorov semigroup of $Z^{(n),x}$. The dependence of the constants can be checked using the articles [46] and [47]. □

Corollary 3.6. *Let $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded and progressively measurable function. Moreover, we assume that γ is the pointwise limit of a sequence (γ_ε) of functions of class $\mathcal{C}_b^1$ w.r.t. x, and uniformly bounded by $\|\gamma\|_\infty$. Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function with $|\phi(x)| \leqslant c_\phi\left(1+|x|^2\right)$. There exists a probability measure $\widetilde{\mathbb{P}}^{x,n,T}$ under which the process*

$$\widetilde{W}^{x,n} = W_t - \int_0^t \gamma(s, X_s^{x,n})\,\mathrm{d}s$$

is a Brownian motion on $[0,T]$. Then we have,

$$\forall x, y \in \mathbb{R}^d,\ \forall n, m \in \mathbb{K},\ \forall t \geqslant 0,\ \left|\widetilde{\mathbb{E}}^{x,n,t}\left[\phi(X_t^{x,n})\right] - \widetilde{\mathbb{E}}^{y,m,t}\left[\phi(X_t^{y,m})\right]\right| \leqslant \widehat{c}c_\phi\left(1+|x|^2+|y|^2\right)\mathrm{e}^{-\widehat{\nu}t},$$

where $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2 and $\|\gamma\|_\infty$.

Proof. For every $l \in \mathbb{K}$, we set $\sigma_{l,\varepsilon}$ a function ε-close (in uniform norm) to σ_l on the centered ball of $\mathbb{R}^d$ of radius ε^{-1}, equal to I_d outside the centered ball of radius $\varepsilon^{-1}+1$ and of class $\mathcal{C}^1$ with bounded derivatives on $\mathbb{R}^d$; on the ring between the radius ε^{-1} and $\varepsilon^{-1}+1$, $\sigma_{l,\varepsilon}$ is chosen such that $\sigma_l^{-1}\sigma_{l,\varepsilon}$ is bounded, independently of ε. This way, the function $\Xi_{l,\varepsilon} : (t,x) \mapsto b_l(x) + \sigma_{l,\varepsilon}(x)\gamma_{l,\varepsilon}(t,x)$ is Lipschitz continuous w.r.t. x. We denote $X^{x,n,\varepsilon}$ the solution of the SDE:

$$\begin{cases} \mathrm{d}X_t^{x,n,\varepsilon} = \Xi_{N_t^n,\varepsilon}\left(t, X_t^{x,n,\varepsilon}\right)\,\mathrm{d}t + \sigma_{N_t^n}\left(X_t^{x,n,\varepsilon}\right)\,\mathrm{d}\widetilde{W}_t^{x,n}, \\ X_0^{x,n,\varepsilon} = x. \end{cases}$$

We can write, for every $\delta > 0$:

$$\langle \Xi_{l,\varepsilon}(t,x), x\rangle \leqslant \eta_1 - \eta_2|x|^2 + (\sqrt{r_1} + \sqrt{r_2}|x|)\,\|\gamma\|_\infty|x| \leqslant \left[\eta_1 + \frac{\|\gamma\|_\infty^2 r_1}{2\delta^2}\right] - \left[\eta_2 - \sqrt{r_2}\|\gamma\|_\infty - \frac{\delta^2}{2}\right]|x|^2.$$

Then, when δ is small enough, for any ϕ with quadratic growth, Theorem 3.5 tells us that $\widehat{c}_\varepsilon$ and $\widehat{\nu}_\varepsilon$ are independent of ε, and we have positive constants $\widehat{c}$ and $\widehat{\nu}$, such that for every (x,n) and (y,m) in $\mathbb{R}^d \times \mathbb{K}$,

$$\forall \varepsilon > 0,\ \forall t \geqslant 0,\ \left|\widetilde{\mathbb{E}}^{x,n,t}\left[\phi\left(X_t^{x,n,\varepsilon}\right)\right] - \widetilde{\mathbb{E}}^{y,m,t}\left[\phi\left(X_t^{y,m,\varepsilon}\right)\right]\right| \leqslant \widehat{c}c_\phi\left(1+|x|^2+|y|^2\right)\mathrm{e}^{-\widehat{\nu}t}.$$

Our goal is to take the limit; let us show that $\widetilde{\mathbb{E}}^{x,n,t}\left[\phi\left(X_t^{x,n,\varepsilon}\right)\right] \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{\mathbb{E}}^{x,n,t}\left[\phi\left(X_t^{x,n}\right)\right]$.

We can write: $X_t^{x,n,\varepsilon} = x + \int_0^t b_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)\mathrm{d}s + \int_0^t \sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)\mathrm{d}W_s^{(\varepsilon)}$, where

$$W_t^{(\varepsilon)} = \widetilde{W}_t^{x,n} + \int_0^t \sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1}\sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right)\gamma_\varepsilon\left(s, X_s^{x,n,\varepsilon}\right)\mathrm{d}s$$

is a Brownian motion under the probability $\mathbb{P}^{(\varepsilon)} = p_t^\varepsilon \left(X^{x,n,\varepsilon}\right) \widetilde{\mathbb{P}}^{x,n,t}$ on $[0,t]$ and where

$$p_t^\varepsilon \left(X^{x,n,\varepsilon}\right) = \exp\left(-\int_0^t \left\langle \sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1} \sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right) \gamma_\varepsilon\left(s, X_s^{x,n,\varepsilon}\right), \mathrm{d}\widetilde{W}_s^{x,n}\right\rangle \right.$$
$$\left. -\frac{1}{2}\int_0^t \left|\sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1} \sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right) \gamma_\varepsilon\left(s, X_s^{x,n,\varepsilon}\right)\right|^2 \mathrm{d}s\right).$$

By uniqueness in law of the solutions of the SDEs, we get the equalities:

$$\widetilde{\mathbb{E}}^{x,n,t}\left[\phi\left(X_t^{x,n,\varepsilon}\right)\right] = \mathbb{E}^{(\varepsilon)}\left[p_t^\varepsilon\left(X^{x,n,\varepsilon}\right)^{-1}\phi\left(X_t^{x,n,\varepsilon}\right)\right] = \mathbb{E}\left[p_t^\varepsilon\left(X^{x,n}\right)^{-1}\phi\left(X_t^{x,n}\right)\right].$$

But $\gamma_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \gamma$, so we have $p_t^\varepsilon\left(X^{x,n}\right)^{-1} \underset{\varepsilon\to 0}{\overset{\mathbb{P}}{\longrightarrow}} \exp\left(-\int_0^t \langle\gamma(s,X_s^{x,n}), \mathrm{d}\widetilde{W}_s^{x,n}\rangle - \frac{1}{2}\int_0^t |\gamma(s,X_s^{x,n})|^2\,\mathrm{d}s\right)$. In order to show that $\widetilde{\mathbb{E}}^{x,n,t}\left[\phi\left(X_t^{x,n,\varepsilon}\right)\right] \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{\mathbb{E}}^{x,n,t}\left[\phi\left(X_t^{x,n}\right)\right]$, the last thing to prove is the uniform integrability of the family $\left(p_t^\varepsilon\left(X^{x,n}\right)^{-1}\right)_{\varepsilon>0}$. We have:

$$\begin{aligned}
\widetilde{\mathbb{E}}^{x,n,t}\left[\left(p_t^\varepsilon\left(X^{x,n}\right)^{-1}\right)^2\right] &= \widetilde{\mathbb{E}}^{x,n,t}\left[\exp\left(2\int_0^t \left\langle \sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1}\sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right)\gamma_\varepsilon\left(s,X_s^{x,n,\varepsilon}\right), \mathrm{d}\widetilde{W}_s^{x,n}\right\rangle\right.\right.\\
&\qquad\left.\left. + \int_0^t \left|\sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1}\sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right)\gamma_\varepsilon\left(s,X_s^{x,n,\varepsilon}\right)\right|^2\mathrm{d}s\right)\right]\\
&\leqslant \widetilde{\mathbb{E}}^{x,n,t}\left[\exp\left(\int_0^t \left\langle 4\sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1}\sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right)\gamma_\varepsilon\left(s,X_s^{x,n,\varepsilon}\right), \mathrm{d}\widetilde{W}_s^{x,n}\right\rangle\right.\right.\\
&\qquad\left.\left. -\frac{1}{2}\int_0^t \left|4\sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1}\sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right)\gamma_\varepsilon\left(s,X_s^{x,n,\varepsilon}\right)\right|^2\mathrm{d}s\right)\right]^{\frac{1}{2}}\\
&\qquad\times \widetilde{\mathbb{E}}^{x,n,t}\left[\exp\left(10\int_0^t \left|\sigma_{N_s^n}\left(X_s^{x,n,\varepsilon}\right)^{-1}\sigma_{N_s^n,\varepsilon}\left(X_s^{x,n,\varepsilon}\right)\gamma_\varepsilon\left(s,X_s^{x,n,\varepsilon}\right)\right|^2\mathrm{d}s\right)\right]^{\frac{1}{2}}\\
&\leqslant \exp\left(5t\left(d^2 + \left\|\sigma^{-1}\right\|_\infty^2\right)\|\gamma\|_\infty\right) < \infty.
\end{aligned}$$

The proof is complete. □

3.3 The infinite horizon BSDE

We consider in this section the infinite horizon BSDE: for every $0 \leqslant t \leqslant T < \infty$,

$$Y_t^{\alpha,x,n} = Y_T^{\alpha,x,n} + \int_t^T \{f_{N_s^n}(X_s^{x,n}, Z_s^{\alpha,x,n}, H_s^{\alpha,x,n}) - \alpha Y_s^{\alpha,x,n}\}\,\mathrm{d}s - \int_t^T Z_s^{\alpha,x,n}\,\mathrm{d}W_s - \int_t^T H_s^{\alpha,x,n}\,\mathrm{d}M_s. \quad (3.10)$$

Assumption 3.7.

- The process X satisfies Assumption 3.1, with a drift b independent of time t.
- If we define $\rho_l : (x,p,h) \mapsto f_l(x, p\sigma_l(x), h)$, then ρ_l is Lipschitz continuous for every $l \in \mathbb{K}$.
- As a consequence, the functions f_l are Lipschitz continuous with constants denoted $\|f\|_{\mathrm{lip},z}$ and $\|f\|_{\mathrm{lip},h}$ w.r.t. z and h, for every $l \in \mathbb{K}$ (note that $\|f\|_{\mathrm{lip},z} \leqslant \|\rho\|_{\mathrm{lip},p}\|\sigma^{-1}\|_\infty$).
- $\forall l \in \mathbb{L}$, $\forall z \in (\mathbb{R}^d)^*$, $\forall h, h' \in (\mathbb{R}^{k-1})^*$, the vector
$$\frac{f_l(x,z,h) - f_l(x,z,h')}{|h-h'|^2}(h-h')$$
has all its coefficients in $(-1, +\infty)$.

- The functions $f_l(\bullet, 0, 0)$ have at most linear growth: $|f_l(x, 0, 0)| \leqslant M_f(1 + |x|)$.
- $\sqrt{r_2}\|f\|_{\mathrm{lip},z} + \frac{r_2}{2} < \eta_2$.

Theorem 3.8. *For every $x \in \mathbb{R}^d$, $n \in \mathbb{K}$ and $\alpha > 0$, Equation (3.10) admits a unique solution $(Y^{\alpha,x,n}, Z^{\alpha,x,n}, H^{\alpha,x,n})$ such that $Y^{\alpha,x,n}$ is continuous and bounded in L^1, $Z^{\alpha,x,n} \in \mathrm{L}^2_{\mathrm{loc}}(\Omega, \mathrm{L}^2(0, \infty; (\mathbb{R}^d)^*))$ and $H^{\alpha,x,n} \in \mathrm{L}^2_{\mathrm{loc}}(\Omega, \mathrm{L}^2(0, \infty; (\mathbb{R}^{k-1})^*))$. Moreover, we have*

$$|Y_t^{\alpha,x,n}| \leqslant \frac{C}{\alpha}(1 + |X_t^{x,n}|),$$

where the constant C only depends on η_1, η_2, r_1, r_2, $\|f\|_{\mathrm{lip},z}$ and M_f.

Proof.

Step 1: We consider the finite horizon BSDE

$$Y_t^T = \int_t^T \{f_{N_s^n}(X_s^{x,n}, Z_s^T, H_s^T) - \alpha Y_s^T\}\,\mathrm{d}s - \int_t^T Z_s^T\,\mathrm{d}W_s - \int_t^T H_s^T\,\mathrm{d}M_s, \quad t \in [0, T]. \tag{3.11}$$

It has a unique solution (see [53]), we want a bound independent of T for Y^T. We set:

$$\beta_t = \begin{cases} \frac{f_{N_t^n}(X_t^{x,n}, Z_t^T, H_t^T) - f_{N_t^n}(X_t^{x,n}, 0, H_t^T)}{|Z_t^T|^2}(Z_t^T)^* & \text{if } Z_t^T \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$\gamma_t = \begin{cases} \frac{f_{N_t^n}(X_t^{x,n}, 0, H_t^T) - f_{N_t^n}(X_t^{x,n}, 0, 0)}{|H_t^T|^2}(H_t^T)^* & \text{if } H_t^T \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The processes β and γ are bounded; moreover, $1 + \gamma$ is positive. So, we are able to define a probability measure $\mathbb{Q}_T$ by the Doléans-Dade exponential:

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\int_0^\bullet (\beta_s)^*\mathrm{d}W_s + \int_0^\bullet (\gamma_s)^*\mathrm{d}M_s\right)_{\bullet \in [0,T]}.$$

Under $\mathbb{Q}_T$, we can write:

$$\mathrm{d}Y_t^T = \{\alpha Y_t^T - f_{N_t^n}(X_t^{x,n}, 0, 0)\}\,\mathrm{d}t + Z_t^T\,\mathrm{d}W_t^{\mathbb{Q}_T} + H_t^T\,\mathrm{d}M_t^{\mathbb{Q}_T},$$

where $W^{\mathbb{Q}_T}$ and $M^{\mathbb{Q}_T}$ are a Brownian motion and a compensated Poisson measure under $\mathbb{Q}_T$. After using Itô and Tanaka formula (see [15]), we get

$$\mathrm{e}^{-\alpha t}|Y_t^T| \leqslant \mathbb{E}^{\mathbb{Q}_T}\left[\int_t^T \mathrm{sgn}(Y_s^T)\mathrm{e}^{-\alpha s} f_{N_s^n}(X_s^{x,n}, 0, 0)\,\mathrm{d}s \middle| \mathcal{F}_t\right],$$

and we finally obtain $|Y_t^T| \leqslant \frac{C}{\alpha}(1 + |X_t^{x,n}|)$, where the constant C only depends on η_1, η_2, r_1, r_2, $\|f\|_{\mathrm{lip},z}$ and M_f (see the proposition 5 of [32]).

Step 2: $(Y^T)_T$ is a Cauchy sequence. For $t \leqslant S \leqslant T$, we have:

$$Y_t^T - Y_t^S = Y_S^T + \int_t^S \{f_{N_s^n}(X_s^{x,n}, Z_s^T, H_s^T) - \alpha Y_s^T - f_{N_s^n}(X_s^{x,n}, Z_s^S, H_s^S) - \alpha Y_s^S\}\,\mathrm{d}s$$
$$- \int_t^S (Z_s^T - Z_s^S)\,\mathrm{d}W_s - \int_t^S (H_s^T - H_s^S)\,\mathrm{d}M_s.$$

We set:

$$\beta_t^{T,S} = \begin{cases} \frac{f_{N_t^n}(X_t^{x,n},Z_t^T,H_t^T) - f_{N_t^n}(X_t^{x,n},Z_t^S,H_t^T)}{|Z_t^T - Z_t^S|^2}(Z_t^T - Z_t^S)^* & \text{if } Z_t^T \neq Z_t^S, \\ 0 & \text{otherwise;} \end{cases}$$

$$\gamma_t^{T,S} = \begin{cases} \frac{f_{N_t^n}(X_t^{x,n},Z_t^S,H_t^T) - f_{N_t^n}(X_t^{x,n},Z_t^S,H_t^S)}{|H_t^T - H_t^S|^2}(H_t^T - H_t^S)^* & \text{if } H_t^T \neq H_t^S, \\ 0 & \text{otherwise.} \end{cases}$$

After a change of probability measure, one finds:

$$\mathrm{d}(Y_t^T - Y_t^S) = -\alpha(Y_t^T - Y_t^S)\,\mathrm{d}t - (Z_t^T - Z_t^S)\,\mathrm{d}W_t^{\mathbb{Q}_{T,S}} - (H_t^T - H_t^S)\,\mathrm{d}M_t^{\mathbb{Q}_{T,S}}.$$

Taking the integral and the expected value, we obtain:

$$|Y_t^T - Y_t^S| \leqslant \mathrm{e}^{\alpha(t-S)}\mathbb{E}^{\mathbb{Q}_{T,S}}\left[|Y_S^T|\middle|\mathcal{F}_t\right] \leqslant \mathrm{e}^{\alpha(t-S)}\frac{C}{\alpha}(1 + |X_t^{x,n}|).$$

So, $(Y^T)_T$ is a Cauchy sequence in $\mathrm{L}^2(\Omega, \mathcal{C}(\mathbb{R}_+, \mathbb{R}))$, admits a limit Y and satisfies

$$\forall S \geqslant 0,\, \forall t \leqslant S,\, |Y_t - Y_t^S| \leqslant \mathrm{e}^{\alpha(t-S)}\frac{C}{\alpha}(1 + |X_t^{x,n}|).$$

The convergence of Y^T is uniform on every compact set of $\mathbb{R}_+$, so Y is continuous.

Step 3: $(Z^T)_T$ and $(H^T)_T$ are also Cauchy sequences. Using Itô's formula, for every $t \in [0, S]$:

$$\begin{aligned} &\mathrm{e}^{-2\alpha t}\mathbb{E}[|Y_t^T - Y_t^S|^2] - |Y_0^T - Y_0^S|^2 \\ &\qquad = -2\mathbb{E}\left[\int_0^t \mathrm{e}^{-2\alpha s}(Y_s^T - Y_s^S)\{(Z_s^T - Z_s^S)\beta_s^{T,S} + (H_s^T - H_s^S)\gamma_s^{T,S}\}\,\mathrm{d}s\right] \\ &\qquad\quad + 2\mathbb{E}\left[\int_0^t \mathrm{e}^{-2\alpha s}|Z_s^T - Z_s^S|^2\,\mathrm{d}s\right] + \mu\mathbb{E}\left[\int_0^t \mathrm{e}^{-2\alpha s}|H_s^T - H_s^S|^2\,\mathrm{d}s\right]. \end{aligned}$$

We recall the inequalities

$$2(Y_s^T - Y_s^S)(Z_s^T - Z_s^S)\beta_s^{T,S} \leqslant |Z_s^T - Z_s^S|^2 + \|f\|_{\mathrm{lip},z}^2|Y_s^T - Y_s^S|^2,$$

$$2(Y_s^T - Y_s^S)(H_s^T - H_s^S)\gamma_s^{T,S} \leqslant \frac{\mu}{2}|H_s^T - H_s^S|^2 + \frac{2}{\mu}\|f\|_{\mathrm{lip},h}^2|Y_s^T - Y_s^S|^2,$$

that give us:

$$\begin{aligned} &\mathbb{E}\left[\int_0^t \mathrm{e}^{-2\alpha s}|Z_s^T - Z_s^S|^2\,\mathrm{d}s\right] + \frac{\mu}{2}\mathbb{E}\left[\int_0^t \mathrm{e}^{-2\alpha s}|H_s^T - H_s^S|^2\,\mathrm{d}s\right] \\ &\qquad \leqslant \mathrm{e}^{-2\alpha t}\mathbb{E}[|Y_t^T - Y_t^S|^2] + \left(\|f\|_{\mathrm{lip},z}^2 + \frac{2}{\mu}\|f\|_{\mathrm{lip},h}^2\right)\mathbb{E}\left[\int_0^t \mathrm{e}^{-2\alpha s}|Y_s^T - Y_s^S|^2\,\mathrm{d}s\right]. \end{aligned}$$

We have proved that the sequences (Z^T) and (H^T) are Cauchy, and their limits satisfy equation (3.10).

Step 4: Uniqueness. Let (Y^1, Z^1, H^1) and (Y^2, Z^2, H^2) be two solutions, such that Y^1 and Y^2 are bounded in L^1. We write $\Delta Y = Y^1 - Y^2$, etc. and we have the equality

$$\Delta Y_t = \Delta Y_T + \int_t^T \{f_{N_s^n}(X_s^{x,n}, Z_s^1, H_s^1) - f_{N_s^n}(X_s^{x,n}, Z_s^2, H_s^2) - \alpha\Delta Y_s\}\,\mathrm{d}s - \int_t^T \Delta Z_s\,\mathrm{d}W_s - \int_t^T \Delta H_s\,\mathrm{d}M_s.$$

We set:

$$\beta_t = \begin{cases} \frac{f_{N_t^n}(X_t^{x,n},Z_t^1,H_t^1) - f_{N_t^n}(X_t^{x,n},Z_t^2,H_t^1)}{|\Delta Z_t|^2}\Delta Z_t^* & \text{if } \Delta Z_t \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$\gamma_t = \begin{cases} \frac{f_{N_t^n}(X_t^{x,n}, Z_t^2, H_t^1) - f_{N_t^n}(X_t^{x,n}, Z_t^2, H_t^2)}{|\Delta H_t|^2} \Delta H_t^* & \text{if } \Delta H_t \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Again, under a certain probability measure $\mathbb{Q}_T$, one has:

$$|\Delta Y_t| = \mathrm{e}^{\alpha(t-T)} \left| \mathbb{E}^{\mathbb{Q}_T} \left[\Delta Y_T | \mathcal{F}_t \right] \right| \leqslant \mathrm{e}^{\alpha(t-T)} M,$$

for every $T > t$. So $\Delta Y = 0$ and by Itô's formula $\Delta Z = 0$ and $\Delta H = 0$.

□

For any $\alpha > 0$, we define the functions $v_n^\alpha : x \mapsto Y_0^{\alpha,x,n}$.

Proposition 3.9. *The functions v_n^α are continuous; moreover, we have the following representations*

$$\forall t \geqslant 0,\ Y_t^{\alpha,x,n} = v_{N_t^n}^\alpha(X_t^{x,n}) \text{ and } H_t^{\alpha,x,n}(l) = v_{N_{t-}^n + l}^\alpha(X_t^{x,n}) - v_{N_{t-}^n}^\alpha(X_t^{x,n}).$$

Proof. Let $(x_m)_m$ be a sequence of limit $x \in \mathbb{R}^d$. We use the same type of approximations as in Theorem 3.8:

$$Y_t^{T,y} = \int_t^T \{ f_{N_s^n}(X_s^{y,n}, Z_s^{T,y}, H_s^{T,y}) - \alpha Y_s^{T,y} \} \, \mathrm{d}s - \int_t^T Z_s^{T,y} \, \mathrm{d}W_s - \int_t^T H_s^{T,y} \, \mathrm{d}M_s, \quad \forall t \in [0,T].$$

Thanks to Step 2 of Theorem 3.8, we already know that:

$$\forall m \in \mathbb{N},\ \forall T > 0,\ |v_n^\alpha(x_m) - v_n^\alpha(x)| \leqslant |Y_0^{T,x_m} - Y_0^{T,x}| + \mathrm{e}^{-\alpha T} \frac{C}{\alpha} (2 + |x| + |x_m|).$$

Then, due to Lemma 3.29, we get:

$$|Y_0^{T,x_m} - Y_0^{T,x}|^2 \leqslant C_T \mathbb{E} \left[\int_0^T |f_{N_s^n}(X_s^{x_m,n}, Z_s^{T,x}, H_s^{T,x}) - f_{N_s^n}(X_s^{x,n}, Z_s^{T,x}, H_s^{T,x})|^2 \, \mathrm{d}s \right].$$

Finally, Lemma 3.28 and continuity of f with respect to x ensure

$$\forall T > 0,\ \limsup_{m \to \infty} |v_n^\alpha(x_m) - v_n^\alpha(x)| \leqslant \mathrm{e}^{-\alpha T} \frac{C}{\alpha} (2 + |x| + |x_m|),$$

which concludes the continuity of v_n^α. The representation of $Y^{\alpha,x,n}$ is somewhat classical and left to the reader, who should have a look at [57] for further explanations. Finally, the representation of $H^{\alpha,x,n}$ is a straightforward consequence of the lemma 2.2 of [53]. □

Proposition 3.10. *The functions v_n^α are locally Lipschitz continuous:*

$$\forall x, x' \in \mathbb{R}^d,\ |v_n^\alpha(x) - v_n^\alpha(x')| \leqslant C \left(1 + \frac{1}{\alpha} \right) (1 + |x| + |x'|) |x - x'|,$$

where C only depends on μ, k, $\left\| \sigma^{-1} \right\|_\infty$ and the Lipschitz constants of ρ, b and σ.

Proof. We approach the functions b, σ and f by uniformly converging sequences (b^ε), (σ^ε) and (f^ε) of $\mathcal{C}_b^1$ functions. We suppose that those approximations have the same Lipschitz constants as b, σ and f respectively, and that b^ε and σ^ε satisfy the same linear growth assumptions as b and σ. Moreover, let (g^ε) be a sequence of $\mathcal{C}_b^1$ functions converging uniformly to v^α on every compact set. The functions g^ε

are supposed to have the same linear growth as v^α: $|g^\varepsilon_l(x)| \leqslant \frac{C}{\alpha}(1+|x|)$. We consider the following BSDE, where $0 \leqslant t \leqslant s \leqslant 1$:

$$Y^{\varepsilon,t,x,n}_s = g^\varepsilon_{N^{t,n}_1}(X^{\varepsilon,t,x,n}_1) + \int_s^1 \{f^\varepsilon_{N^{t,n}_r}(X^{\varepsilon,t,x,n}_r, Z^{\varepsilon,t,x,n}_r, H^{\varepsilon,t,x,n}_r) - \alpha Y^{\varepsilon,t,x,n}_r\}\,\mathrm{d}r - \int_s^1 Z^{\varepsilon,t,x,n}_r\,\mathrm{d}W_r - \int_s^1 H^{\varepsilon,t,x,n}_r\,\mathrm{d}M_r,$$

and where

$$X^{\varepsilon,t,x,n}_s = x + \int_t^s b^\varepsilon_{N^{t,n}_r}(X^{\varepsilon,t,x,n}_r)\,\mathrm{d}r + \int_t^s \sigma^\varepsilon_{N^{t,n}_r}(X^{\varepsilon,t,x,n}_r)\,\mathrm{d}W_r.$$

Due to [53], it has a unique solution, and, if we do the same calculations as in the first step of Theorem 3.8, we get:

$$|Y^{\varepsilon,t,x,n}_s| \leqslant \frac{C}{\alpha}(1+|X^{\varepsilon,t,x,n}_s|).$$

Also, if we define $u^\varepsilon_n(t,x) = Y^{\varepsilon,t,x,n}_t$, we have the following representations:

- $Y^{\varepsilon,t,x,n}_s = u^\varepsilon_{N^{t,n}_s}(s, X^{\varepsilon,t,x,n}_s)$;
- $Z^{\varepsilon,t,x,n}_s = \partial_x u^\varepsilon_{N^{t,n}_s}(s, X^{\varepsilon,t,x,n}_s)\sigma^\varepsilon_{N^{t,n}_s}(X^{\varepsilon,t,x,n}_s)$;
- $H^{\varepsilon,t,x,n}_s(l) = u^\varepsilon_{N^{t,n}_{s^-}+l}(s, X^{\varepsilon,t,x,n}_s) - u^\varepsilon_{N^{t,n}_{s^-}}(s, X^{\varepsilon,t,x,n}_s)$.

In particular, the coefficients of $H^{\varepsilon,t,x,n}$ are bounded by $2\dfrac{C}{\alpha}(1+|X^{\varepsilon,t,x,n}|)$. Using Lemma 3.32, we get:

$$|\partial_x u^\varepsilon_n(t,x)| \leqslant C\left(\sqrt[4]{\mathbb{E}} + \sqrt[4]{\mathbb{E}^{\mathbb{Q}^\varepsilon}}\right)\left[1 + |X^{\varepsilon,t,x,n}|^{*,4}_{t,1}\right],$$

where the constant C is independent of x, and where $\mathbb{Q}^\varepsilon$ is given by:

$$\frac{\mathrm{d}\mathbb{Q}^\varepsilon}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\sum_{j=1}^d \int_t^\bullet \partial_{z_j} f^\varepsilon_{N^{t,n}_s}(X^{\varepsilon,t,x,n}_s, Z^{\varepsilon,t,x,n}_s, H^{\varepsilon,t,x,n}_s)\mathrm{d}W^j_s + \sum_{l=1}^{k-1}\int_t^\bullet \partial_{h_l} f^\varepsilon_{N^{t,n}_s}(X^{\varepsilon,t,x,n}_s, Z^{\varepsilon,t,x,n}_s, H^{\varepsilon,t,x,n}_s)\mathrm{d}M_s(l)\right)_{\bullet\in[t,1]}.$$

But, using Lemma 3.28, we obtain, for all $p \geqslant 2$:

$$\mathbb{E}\left[|X^{\varepsilon,t,x,n}|^{*,p}_{t,1}\right] \leqslant \mathcal{C}(p, \|b\|_{\mathrm{lip}}, \|\sigma\|_{\mathrm{lip}})\left\{|x|^p + (1-t)(\xi^p_1 + \sqrt{r_1}^p)\right\}.$$

Moreover, doing the same as in equation (3.33), we find:

$$\mathbb{E}^{\mathbb{Q}^\varepsilon}\left[|X^{\varepsilon,t,x,n}|^{*,4}_{t,1}\right] \leqslant \exp\left(\frac{\|f\|^2_{\mathrm{lip},z} + \mu(k-1)\left(\mathrm{e}^{2\|f\|_{\mathrm{lip},h}+1}\right)}{2}(1-t)\right)\mathbb{E}\left[|X^{\varepsilon,t,x,n}|^{*,8}_{t,1}\right]^{1/2}.$$

We put everything together, then use the mean value theorem and find that

$$|u^\varepsilon_n(t,x) - u^\varepsilon_n(t,x')| \leqslant C(1+|x|+|x'|)|x-x'|,$$

where C is independent of x and ε. The last thing to prove is the pointwise convergence of $u^\varepsilon_n(0,\bullet)$ to v^α_n. Using Lemma 3.29, we have:

$$\mathbb{E}\left[|Y^{\alpha,x,n} - Y^{\varepsilon,0,x,n}|^{*,2}_{0,1}\right] \leqslant C\mathbb{E}\left[\left|v^\alpha_{N^n_1}(X^{x,n}_1) - g^\varepsilon_{N^n_1}(X^{\varepsilon,x,n}_1)\right|^2\right] + \mathbb{E}\left[\int_0^1 \left|f_{N^n_s}(X^{x,n}_s, Z^{\alpha,x,n}_s, H^{\alpha,x,n}_s) - f^\varepsilon_{N^n_s}(X^{\varepsilon,x,n}_s, Z^{\alpha,x,n}_s, H^{\alpha,x,n}_s)\right|^2\mathrm{d}s\right].$$

It is easy to see that the second term converges to 0, due to the convergence of f^ε and Lemma 3.28. For the first term, we can see that:

$$\mathbb{E}\left[\left|v^\alpha_{N^n_1}(X_1^{x,n}) - g^\varepsilon_{N^n_1}(X_1^{\varepsilon,x,n})\right|^2\right] \leqslant 2\mathbb{E}\left[\left|v^\alpha_{N^n_1}(X_1^{x,n}) - v^\alpha_{N^n_1}(X_1^{\varepsilon,x,n})\right|^2\right] + 2\mathbb{E}\left[\left|v^\alpha_{N^n_1}(X_1^{\varepsilon,x,n}) - g^\varepsilon_{N^n_1}(X_1^{\varepsilon,x,n})\right|^2\right],$$

and again, the first term goes to 0 using Lemma 3.28, and dominated convergence along a subsequence. For the second one, let us take $R > 0$, we have:

$$\mathbb{E}\left[\left|v^\alpha_{N^n_1}(X_1^{\varepsilon,x,n}) - g^\varepsilon_{N^n_1}(X_1^{\varepsilon,x,n})\right|^2 \mathbb{1}_{|X_1^{\varepsilon,x,n}|\leqslant R}\right] \leqslant \|v^\alpha - g^\varepsilon\|_{\infty,\mathcal{B}_R},$$

which will go to 0, due to the uniform convergence of g^ε on compact sets, and

$$\mathbb{E}\left[\left|v^\alpha_{N^n_1}(X_1^{\varepsilon,x,n}) - g^\varepsilon_{N^n_1}(X_1^{\varepsilon,x,n})\right|^2 \mathbb{1}_{|X_1^{\varepsilon,x,n}|> R}\right] \leqslant \frac{C(1+|x|^3)}{R},$$

using Cauchy-Schwarz and Markov inequalities, plus Lemma 3.28; the constant C is independent of ε and R. We are able to conclude this proof. □

Lemma 3.11. *Let $\psi : \mathbb{R}^d \times E^* \to \mathbb{R}$ continuous w.r.t. the first variable and Lipschitz continuous w.r.t. the second one, where E can be $\mathbb{R}^d$ or $\mathbb{R}^{k-1}$. Let $\zeta, \zeta' : \mathbb{R}_+ \times \mathbb{R}^d \to E^*$ be such that for every $t \geqslant 0$, $\zeta(t,\bullet)$ and $\zeta'(t,\bullet)$ are continuous, and set:*

$$\widetilde{\Gamma}(t,x) = \begin{cases} \dfrac{\psi(x,\zeta(t,x)) - \psi(x,\zeta'(t,x))}{|\zeta(t,x)-\zeta'(t,x)|^2}(\zeta(t,x)-\zeta'(t,x)), & \text{if } \zeta(t,x) \neq \zeta'(t,x), \\ 0, & \text{if } \zeta(t,x) = \zeta'(t,x). \end{cases}$$

Then, there exists a uniformly bounded sequence of $\mathcal{C}^1$ functions w.r.t. x with bounded derivatives $(\widetilde{\Gamma}_n)_{n\geqslant 1}$ (i.e. for all n, $\widetilde{\Gamma}_n$ has bounded derivatives w.r.t. x – the bound of derivatives can depend on n – and $\sup_{n\geqslant 1}\|\widetilde{\Gamma}_n(t,\bullet)\|_\infty < \infty$ for every $t \geqslant 0$), such that $\widetilde{\Gamma}_n \underset{n\to\infty}{\longrightarrow} \widetilde{\Gamma}$ pointwise.

Proof. See the Lemma 3.7 of [35]: we can approximate the Lipschitz functions by $\mathcal{C}^1$ functions with bounded derivatives and construct a new sequence having the required regularity. □

Proposition 3.12. *Under Assumption 3.7, there exists a constant C, such that for every $\alpha \in (0,1]$, we have:*

$$\forall l \in \mathbb{K}, \forall x, x' \in \mathbb{R}^d, \left|v^\alpha_l(x) - v^\alpha_l(x')\right| \leqslant C\left(1+|x|^2+|x'|^2\right).$$

The constant C only depends on η_1, η_2, r_1, r_2, $\|f\|_{\text{lip},z}$ and M_f.

Proof.

1. We approximate σ_l by a sequence $\left(\sigma^\varepsilon_l\right)_{\varepsilon>0}$ of functions satisfying:
   - σ^ε_l converges pointwise towards σ_l over $\mathbb{R}^d$;
   - $\left|\sigma^\varepsilon_l(x)\right|_F^2 \leqslant r_1 + r_2|x|^2$;
   - σ^ε_l is of class $\mathcal{C}^1$ and $\left\|\sigma^\varepsilon_l\right\|_{\text{lip}} \leqslant \|\sigma_l\|_{\text{lip}}$;
   - $x \mapsto \sigma^\varepsilon_l(x)^{-1}$ is bounded, the bound is independent of ε.

We also approximate b_l by a sequence $\left(b_l^\varepsilon\right)_{\varepsilon>0}$ of $\mathcal{C}^1$ functions which converges uniformly. One can check that the functions b_l^ε can be chosen "uniformly weakly dissipative". Moreover, $\rho_l : (x,z,h) \mapsto f_l(x, z\sigma_l(x)^{-1}, h)$ is approximated by a sequence $\left(\rho_l^\varepsilon\right)$ of $\mathcal{C}_b^1$ functions converging uniformly. We consider the BSDE of infinite horizon:

$$Y_t^{\varepsilon,\alpha,x,n} = Y_T^{\varepsilon,\alpha,x,n} + \int_t^T \left\{ f^\varepsilon_{N_s^n}\left(X_s^{\varepsilon,x,n}, Z_s^{\varepsilon,\alpha,x,n}, H_s^{\varepsilon,\alpha,x,n}\right) - \alpha Y_s^{\varepsilon,\alpha,x,n} \right\} \mathrm{d}s - \int_t^T Z_s^{\varepsilon,\alpha,x,n}\, \mathrm{d}W_s - \int_t^T H_s^{\varepsilon,\alpha,x,n}\, \mathrm{d}M_s,$$

where the process $X^{\varepsilon,x,n}$ satisfies the following equation:

$$X_t^{\varepsilon,x,n} = x + \int_0^t b^\varepsilon_{N_s^n}\left(X_s^{\varepsilon,x,n}\right) \mathrm{d}s + \int_0^t \sigma^\varepsilon_{N_s^n}\left(X_s^{\varepsilon,x,n}\right) \mathrm{d}W_s.$$

This BSDE has a unique solution, and we can write $Y_t^{\varepsilon,\alpha,x,n} = v^{\varepsilon,\alpha}_{N_t^n}(X_t^{\varepsilon,x,n})$. Due to Proposition 3.10, we can find, for every $\varepsilon > 0$ a sequence $(v^{\delta,\varepsilon,\alpha})_\delta$ of $\mathcal{C}^1$ functions converging uniformly to $v^{\varepsilon,\alpha}$ and satisfying

$$\forall l \in \mathbb{K},\, \forall x \in \mathbb{R}^d,\, |v_l^{\delta,\varepsilon,\alpha}(x)| \leqslant \frac{C}{\alpha}(1+|x|).$$

We consider the finite horizon BSDE:

$$Y_t^{\delta,T,\varepsilon,\alpha,x,n} = v^{\delta,\varepsilon,\alpha}_{N_T^n}(X_T^{\varepsilon,x,n}) + \int_t^T \{ f^\varepsilon_{N_s^n}(X_s^{\varepsilon,x,n}, Z_s^{\delta,T,\varepsilon,\alpha,x,n}, H_s^{\delta,T,\varepsilon,\alpha,x,n}) - \alpha Y_s^{\delta,T,\varepsilon,\alpha,x,n} \} \mathrm{d}s - \int_t^T Z_s^{\delta,T,\varepsilon,\alpha,x,n}\, \mathrm{d}W_s - \int_t^T H_s^{\delta,T,\varepsilon,\alpha,x,n}\, \mathrm{d}M_s.$$

This BSDE has a unique solution and satisfies the inequality $|Y_t^{\delta,T,\varepsilon,\alpha,x,n}| \leqslant \frac{C}{\alpha}\left(1+\left|X_t^{\varepsilon,x,n}\right|\right)$, for every $t \geqslant 0$ and $\mathbb{P}$-a.s. Due to Lemma 3.31, we can write $Y_t^{\delta,T,\varepsilon,\alpha,x,n} = u^{\delta,T,\varepsilon,\alpha}_{N_t^n}(t, X_t^{\varepsilon,x,n})$, $Z_t^{\delta,T,\varepsilon,\alpha,x,n} = \partial_x u^{\delta,T,\varepsilon,\alpha}_{N_t^n}(t, X_t^{\varepsilon,x,n}) \sigma^\varepsilon_{N_t^n}(X_t^{\varepsilon,x,n})$ and $H_t^{\delta,T,\varepsilon,\alpha,x,n}(l) = u^{\delta,T,\varepsilon,\alpha}_{N_{t^-}^n+l}(t, X_t^{\varepsilon,x,n}) - u^{\delta,T,\varepsilon,\alpha}_{N_{t^-}^n}(t, X_t^{\varepsilon,x,n})$ $\mathbb{P}$-a.s. and for a.e. $t \geqslant 0$. We define:

$$z_n^{\delta,T,\varepsilon,\alpha}(t,x) = \partial_x u_n^{\delta,T,\varepsilon,\alpha}(t,x)\sigma_n^\varepsilon(x), \qquad h_m^{\delta,T,\varepsilon,\alpha}(t,x) = \left(u_{m+l}^{\delta,T,\varepsilon,\alpha}(t,x) - u_m^{\delta,T,\varepsilon,\alpha}(t,x)\right)_{l\in\mathbb{K}},$$

$$\Gamma_n^{\delta,T,\varepsilon,\alpha}(t,x) = \begin{cases} \dfrac{f_n^\varepsilon\left(x, z_n^{\delta,T,\varepsilon,\alpha}(t,x), 0\right) - f_n^\varepsilon(x,0,0)}{\left|z_n^{\delta,T,\varepsilon,\alpha}(t,x)\right|^2} z_n^{\delta,T,\varepsilon,\alpha}(t,x), & \text{if } z_n^{\delta,T,\varepsilon,\alpha}(t,x) \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\widetilde{\Gamma}_{n,m}^{\delta,T,\varepsilon,\alpha}(t,x) = \begin{cases} \dfrac{f_n^\varepsilon\left(x, z_n^{\delta,T,\varepsilon,\alpha}(t,x), h_m^{\delta,T,\varepsilon,\alpha}(t,x)\right) - f_n^\varepsilon\left(x, z_n^{\delta,T,\varepsilon,\alpha}(t,x), 0\right)}{\left|h_m^{\delta,T,\varepsilon,\alpha}(t,x)\right|^2} h_m^{\delta,T,\varepsilon,\alpha}(t,x), \\ \qquad\qquad\qquad \text{if } h_m^{\delta,T,\varepsilon,\alpha}(t,x) \neq 0, \\ 0, \qquad\qquad\qquad \text{otherwise}. \end{cases}$$

We can rewrite:

$$\mathrm{d}Y_t^{\delta,T,\varepsilon,\alpha,x,n} = -\Big\{ f^\varepsilon_{N_t^n}\left(X_t^{\varepsilon,x,n}, 0, 0\right) + Z_t^{\delta,T,\varepsilon,\alpha,x,n} \Gamma^{\delta,T,\varepsilon,\alpha}_{N_t^n}\left(t, X_t^{\varepsilon,x,n}\right)^* + H_t^{\delta,T,\varepsilon,\alpha,x,n} \widetilde{\Gamma}^{\delta,T,\varepsilon,\alpha}_{N_t^n, N_{t^-}^n}\left(t, X_t^{\varepsilon,x,n}\right)^* - \alpha Y_t^{\delta,T,\varepsilon,\alpha,x,n} \Big\} \mathrm{d}t + Z_t^{\delta,T,\varepsilon,\alpha,x,n}\, \mathrm{d}W_t + H_t^{\delta,T,\varepsilon,\alpha,x,n}\, \mathrm{d}M_t.$$

Due to Girsanov's theorem, there exists a probability measure $\widetilde{\mathbb{P}}^{\delta,T,\varepsilon,\alpha,x,n}$ under which:

$$u_n^{\delta,T,\varepsilon,\alpha}(0,x) = \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x,n}\left[\mathrm{e}^{-\alpha T} v^{\delta,\varepsilon,\alpha}_{N^n_T}\left(X^{\varepsilon,x,n}_T\right) + \int_0^T \mathrm{e}^{-\alpha s} f^{\varepsilon}_{N^n_s}\left(X^{\varepsilon,x,n}_s, 0, 0\right)\,\mathrm{d}s\right].$$

On the one hand, using Proposition 3.4: $\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x,n}\left[\mathrm{e}^{-\alpha T} v^{\delta,\varepsilon,\alpha}_{N^n_T}\left(X^{\varepsilon,x,n}_T\right)\right]\right| \leqslant \mathrm{e}^{-\alpha T} C(1+|x|)$.
On the other hand, thanks to Corollary 3.6, we get:

$$\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x,n}\left[f^{\varepsilon}_{N^n_t}\left(X^{\varepsilon,x,n}_t,0,0\right)\right] - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y,n}\left[f^{\varepsilon}_{N^n_t}\left(X^{\varepsilon,y,n}_t,0,0\right)\right]\right| \leqslant \widehat{c}\mathrm{e}^{-\widehat{\nu} t}\left(1+|x|^2+|y|^2\right),$$

where $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2, $\|f\|_{\mathrm{lip},z}$ and M_f. As a consequence, we get:

$$\begin{aligned}
&|v_n^{\varepsilon,\alpha}(x) - v_n^{\varepsilon,\alpha}(y)| \qquad (3.12)\\
&= \lim_{\delta\to 0}\left|u_n^{\delta,T,\varepsilon,\alpha}(0,x) - u_n^{\delta,T,\varepsilon,\alpha}(0,y)\right| \qquad \text{(see Lemma 3.29)}\\
&= \lim_{T\to\infty}\lim_{\delta\to 0}\Bigg|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x,n}\left[\mathrm{e}^{-\alpha T} v^{\delta,\varepsilon,\alpha}_{N^n_T}(X^{\varepsilon,x,n}_T) + \int_0^T \mathrm{e}^{-\alpha s} f^{\varepsilon}_{N^n_s}(X^{\varepsilon,x,n}_s,0,0)\,\mathrm{d}s\right]\\
&\qquad\qquad - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y,n}\left[\mathrm{e}^{-\alpha T} v^{\delta,\varepsilon,\alpha}_{N^n_T}(X^{\varepsilon,y,n}_T) + \int_0^T \mathrm{e}^{-\alpha s} f^{\varepsilon}_{N^n_s}(X^{\varepsilon,y,n}_s,0,0)\,\mathrm{d}s\right]\Bigg|\\
&\leqslant \limsup_{T\to\infty}\limsup_{\delta\to 0}\mathrm{e}^{-\alpha T}\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x,n}\left[v^{\delta,\varepsilon,\alpha}_{N^n_T}(X^{\varepsilon,x,n}_T)\right] - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y,n}\left[v^{\delta,\varepsilon,\alpha}_{N^n_T}(X^{\varepsilon,y,n}_T)\right]\right|\\
&\quad + \limsup_{T\to\infty}\limsup_{\delta\to 0}\int_0^T \mathrm{e}^{-\alpha s}\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x,n}\left[f^{\varepsilon}_{N^n_s}(X^{\varepsilon,x,n}_s,0,0)\right] - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y,n}\left[f^{\varepsilon}_{N^n_s}(X^{\varepsilon,y,n}_s,0,0)\right]\right|\mathrm{d}s\\
&\leqslant 0 + \int_0^\infty \mathrm{e}^{-\alpha s}\widehat{c}(1+|x|^2+|y|^2)\mathrm{e}^{-\widehat{\nu} s}\,\mathrm{d}s\\
&\leqslant \frac{\widehat{c}}{\widehat{\nu}}(1+|x|^2+|y|^2). \qquad (3.13)
\end{aligned}$$

2. Now, our goal is to take the limit when $\varepsilon \to 0$. Let D be a dense and countable subset of $\mathbb{R}^d$. By a diagonal argument, there exists a positive sequence $(\varepsilon_p)_p$ such that $(v_l^{\varepsilon_p,\alpha})_p$ converges pointwise over D to a function $\overline{v}_l^{\alpha}$, for every $l \in \mathbb{K}$. Because the constant C in Proposition 3.10 does not depend on ε, $\overline{v}^{\alpha}$ satisfies the same inequality. Let (K_r) be a sequence of compact sets whose diameter goes to infinity. The function $\overline{v}_l^{\alpha}$ is uniformly continuous on $K_r \cap D$, so it has an extension $\overline{v}_l^{\alpha}$ which is continuous on K_r. Passing to the limit as r goes to infinity, we get a continuous function on $\mathbb{R}^d$, and it is the pointwise limit of the sequence $(v_l^{\varepsilon_p,\alpha})_p$. We denote $\overline{Y}_t^{\alpha,x,n} = \overline{v}^{\alpha}_{N^n_t}(X^{x,n}_t)$; we have $|\overline{Y}_t^{\alpha,x,n}| \leqslant \frac{C}{\alpha}(1+|X^{x,n}_t|)$. Using Lemma 3.28, we have, for every $T>0$:

$$\mathbb{E}\left[|X^{\varepsilon_p,x,n} - X^{x,n}|^{*,2}_{0,T}\right] \underset{p\to\infty}{\longrightarrow} 0.$$

Then, by dominated convergence, we get:

$$\mathbb{E}\left[\int_0^T \left|Y_t^{\varepsilon_p,\alpha,x,n} - \overline{Y}_t^{\alpha,x,n}\right|^2\mathrm{d}t\right] \underset{p\to\infty}{\longrightarrow} 0 \text{ and } \mathbb{E}\left[\left|Y_T^{\varepsilon_p,\alpha,x} - \overline{Y}_T^{\alpha,x,n}\right|^2\right] \underset{p\to\infty}{\longrightarrow} 0.$$

3. We will show that there exist some processes $\overline{Z}^{\alpha,x,n}$ and $\overline{H}^{\alpha,x,n}$ belonging to $\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega,\mathrm{L}^2\left(0,\infty;E^*\right)\right)$ (where $E=\mathbb{R}^d$ or $\mathbb{R}^{k-1}$) which satisfy

$$\mathbb{E}\left[\int_0^T \left|Z_t^{\varepsilon_p,\alpha,x,n} - \overline{Z}_t^{\alpha,x,n}\right|^2\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T \left|H_t^{\varepsilon_p,\alpha,x,n} - \overline{H}_t^{\alpha,x,n}\right|^2\mathrm{d}t\right] \underset{p\to\infty}{\longrightarrow} 0,$$

for every $T > 0$. Indeed, $\left(\overline{Y}^{\alpha,x,n}, \overline{Z}^{\alpha,x,n}, \overline{H}^{\alpha,x,n}\right)$ is solution of the BSDE (3.10) ; by uniqueness of the solution, $v^\alpha \equiv \overline{v}^\alpha$ and taking the limit in the equation (3.13) gives the result. Let us choose $p \leqslant m \in \mathbb{N}$, we define $\widetilde{Y} = Y^{\varepsilon_p,\alpha,x,n} - Y^{\varepsilon_m,\alpha,x,n}$, $\widetilde{Z} = Z^{\varepsilon_p,\alpha,x,n} - Z^{\varepsilon_m,\alpha,x,n}$ and $\widetilde{H} = H^{\varepsilon_p,\alpha,x,n} - H^{\varepsilon_m,\alpha,x,n}$. We have:

$$d\widetilde{Y}_t = \alpha \widetilde{Y}_t\, dt + \underbrace{\left(f^{\varepsilon_m}_{N^n_t}\left(X^{\varepsilon_m,x,n}_t, Z^{\varepsilon_m,\alpha,x,n}_t, H^{\varepsilon_m,\alpha,x,n}_t\right) - f^{\varepsilon_p}_{N^n_t}\left(X^{\varepsilon_p,x,n}_t, Z^{\varepsilon_p,\alpha,x,n}_t, H^{\varepsilon_p,\alpha,x,n}_t\right)\right)}_{=:f_t} dt + \widetilde{Z}_t\, dW_t + \widetilde{H}_t\, dM_t.$$

Thanks to Itô's formula, we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\{|\widetilde{Z}_t|^2 + \mu|\widetilde{H}_t|^2\right\} dt\right] \\
&\quad \leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] - 2\mathbb{E}\left[\int_0^T \widetilde{Y}_t f_t\, dt\right] \\
&\quad \leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] + 2TM_f\mathbb{E}\left[\left|\widetilde{Y}\right|^*_{0,T}\left(2 + |X^{\varepsilon_m,x,n}|^*_{0,T} + |X^{\varepsilon_p,x,n}|^*_{0,T}\right)\right] \\
&\qquad\qquad + 2\|f\|_{\mathrm{lip},z}\mathbb{E}\left[\left|\widetilde{Y}\right|^*_{0,T}\int_0^T \left(\left|Z^{\varepsilon_m,\alpha,x,n}_t\right| + \left|Z^{\varepsilon_p,\alpha,x,n}_t\right|\right) dt\right] \\
&\qquad\qquad + 2\|f\|_{\mathrm{lip},h}\mathbb{E}\left[\left|\widetilde{Y}\right|^*_{0,T}\int_0^T \left(\left|H^{\varepsilon_m,\alpha,x,n}_t\right| + \left|H^{\varepsilon_p,\alpha,x,n}_t\right|\right) dt\right].
\end{aligned}$$

Now, we want to bound $\mathbb{E}\left[|X^{\varepsilon_p,x,n}|^{*,2}_{0,T}\right]$ and $\mathbb{E}\left[\int_0^T \left\{\left|Z^{\varepsilon_p,\alpha,x,n}_t\right|^2 + \mu\left|H^{\varepsilon_p,\alpha,x,n}_t\right|^2\right\} dt\right]$ independently of p. Thanks to proposition 3.3, we have $\mathbb{E}\left[|X^{\varepsilon_p,x,n}|^{*,2}_{0,T}\right] \leqslant C\left(1 + |x|^2\right)$, where C only depends on T, r_1, r_2, ξ_1 and ξ_2. Also, we have:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\{\left|Z^{\varepsilon_p,\alpha,x,n}_t\right|^2 + \mu\left|Z^{\varepsilon_p,\alpha,x,n}_t\right|^2\right\} dt\right] \\
&\qquad\qquad \leqslant \mathbb{E}\left[\left|Y^{\varepsilon_p,\alpha,x,n}_T\right|^2\right] + 2\mathbb{E}\left[\int_0^T \left|Y^{\varepsilon_p,\alpha,x,n}_t f^{\varepsilon_p}_{N^n_t}\left(X^{\varepsilon_p,x,n}_t, Z^{\varepsilon_p,\alpha,x,n}_t, H^{\varepsilon_p,\alpha,x,n}_t\right)\right| dt\right].
\end{aligned}$$

But we have the following upperbound:

$$\begin{aligned}
&\left|Y^{\varepsilon_p,\alpha,x,n}_t f^{\varepsilon_p}_{N^n_t}\left(X^{\varepsilon_p,x,n}_t, Z^{\varepsilon_p,\alpha,x,n}_t, H^{\varepsilon_p,\alpha,x,n}_t\right)\right| \\
&\qquad \leqslant \frac{C}{\alpha}\left(1 + \left|X^{\varepsilon_p,x,n}_t\right|\right)\left(M_f\left(1 + \left|X^{\varepsilon_p,x,n}_t\right|\right) + \|f\|_{\mathrm{lip},z}\left|Z^{\varepsilon_p,\alpha,x,n}_t\right| + \|f\|_{\mathrm{lip},h}\left|H^{\varepsilon_p,\alpha,x,n}_t\right|\right) \\
&\qquad \leqslant \mathcal{C}\left(1 + \left|X^{\varepsilon_p,x,n}_t\right|^2\right) + \frac{1}{4}\left|Z^{\varepsilon_p,\alpha,x,n}_t\right|^2 + \frac{\mu}{4}\left|H^{\varepsilon_p,\alpha,x,n}_t\right|^2.
\end{aligned}$$

Finally, $\mathbb{E}\left[\int_0^T \left\{\left|Z^{\varepsilon_p,\alpha,x,n}_t\right|^2 + \mu\left|H^{\varepsilon_p,\alpha,x,n}_t\right|^2\right\} dt\right] \leqslant \mathcal{C}'\left(1 + |x|^2\right)$, with $\mathcal{C}'$ independent of p. The sequences $\left(Z^{\varepsilon_p,\alpha,x,n}\right)_p$ and $\left(H^{\varepsilon_p,\alpha,x,n}\right)_p$ are Cauchy in $\mathrm{L}^2_{\mathcal{P}}\left(\Omega, \mathrm{L}^2\left([0,T], E^*\right)\right)$; they admit limit processes $\overline{Z}^{\alpha,x,n} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty; \left(\mathbb{R}^d\right)^*\right)\right)$ and $\overline{H}^{\alpha,x,n} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty; \left(\mathbb{R}^{k-1}\right)^*\right)\right)$; the convergence we claimed holds.

□

Proposition 3.13. *Under Assumption 3.7, there exists a constant C, such that for every $\alpha \in (0,1]$, we have:*

$$\forall l \in \mathbb{K},\, \forall x, x' \in \mathbb{R}^d,\, \left|v_l^\alpha(x) - v_l^\alpha(x')\right| \leqslant C\left(1 + |x|^2 + |x'|^2\right)|x - x'|.$$

The constant C only depends on η_1, η_2, r_1, r_2, M_f, μ, k, $\|\sigma^{-1}\|_\infty$ and the Lipschitz constants of b, σ and ρ.

Proof. We set $\overline{v}^\alpha = v^\alpha - v^\alpha(0)$. We approach $\overline{v}^\alpha$ by a sequence $(g^{\varepsilon,\alpha})_\varepsilon$ of $\mathcal{C}_b^1$ functions, converging uniformly on every compact set and satisfying the same quadratic growth:

$$|g_l^{\varepsilon,\alpha}(x)| \leqslant C(1 + |x|^2), \quad \text{where } C \text{ only depends on } \eta_1,\, \eta_2,\, r_1,\, r_2,\, \|f\|_{\mathrm{lip},z} \text{ and } M_f.$$

We keep the same approximations of b, σ and f as in Proposition 3.10. We consider the following BSDE:

$$Y_s^{\varepsilon,\alpha,x,n,t} = g^{\varepsilon,\alpha}_{N_1^{n,t}}(X_T^{\varepsilon,x,n,t}) + \int_s^1 \left\{f^\varepsilon_{N_r^{n,t}}(X_r^{\varepsilon,x,n,t}, Z_r^{\varepsilon,\alpha,x,n,t}, H_r^{\varepsilon,\alpha,x,n,t}) - \alpha(Y_r^{\varepsilon,\alpha,x,n,t} + v^\alpha_{N_r^{n,t}}(0))\right\}\mathrm{d}r$$
$$- \int_s^1 Z_r^{\varepsilon,\alpha,x,n,t}\,\mathrm{d}W_r - \int_s^1 H_r^{\varepsilon,\alpha,x,n,t}\,\mathrm{d}M_r,$$

and define $u_n^{\varepsilon,\alpha}(t,x) = Y_t^{\varepsilon,\alpha,x,n,t}$, for $t \in [0,1]$. Using Lemma 3.32, we obtain,

$$|\partial_x u_n^{\varepsilon,\alpha}(t,x)|$$
$$\leqslant C\left(\sqrt[4]{\mathbb{E}} + \sqrt[4]{\mathbb{E}^{\mathbb{Q}^{\varepsilon,\alpha}}}\right)\left[1 + |Y^{\varepsilon,\alpha,x,n}|_{0,1}^{*,4} + |H^{\varepsilon,\alpha,x,n}|_{0,1}^{*,4} + \left(\int_0^1 |f^\varepsilon_{N_r^n}(X_r^{\varepsilon,x,n}, 0, 0) - \alpha v^\alpha_{N_r^n}(0)|\,\mathrm{d}r\right)^4\right],$$

where the constant C is independent of α. But, since $\alpha \leqslant 1$, by Itô's formula,

$$\mathbb{E}\left[|Y_s^{\varepsilon,\alpha,x,n,t}|^2 + \int_s^1 \left\{|Z_r^{\varepsilon,\alpha,x,n,t}|^2 + \mu|H_r^{\varepsilon,\alpha,x,n,t}|^2\right\}\mathrm{d}r\right]$$
$$\leqslant \mathbb{E}\left[\left|g^{\varepsilon,\alpha}_{N_1^{n,t}}(X_1^{\varepsilon,\alpha,x,n,t})\right|^2\right] + 2\mathbb{E}\left[\int_s^1 \left\{|Y_r^{\varepsilon,\alpha,x,n,t}|\left|f^\varepsilon_{N_r^{n,t}}(X_r^{\varepsilon,x,n,t}, Z_r^{\varepsilon,\alpha,x,n,t}, H_r^{\varepsilon,\alpha,x,n,t})\right| + \alpha\left|v^\alpha_{N_r^{n,t}}(0)\right|\right\}\mathrm{d}r\right]$$
$$\leqslant C(1+|x|^2) + C\mathbb{E}\left[\int_s^1 |Y_r^{\varepsilon,\alpha,x,n,t}|^2\,\mathrm{d}r\right] + \mathbb{E}\left[\int_s^1 \left\{|Z_r^{\varepsilon,\alpha,x,n,t}|^2 + \mu\left|H_r^{\varepsilon,\alpha,x,n,t}\right|^2\right\}\mathrm{d}r\right].$$

It leads us to $\mathbb{E}\left[|Y_s^{\varepsilon,\alpha,x,n,t}|^2\right] \leqslant C(1+|x|^2)$, with a constant C independent of α (note that $\alpha v_l^\alpha(0)$ is bounded independently of α). This way, we obtain $|u_n^{\varepsilon,\alpha}(t,x)| \leqslant C(1+|x|^2)$, and, due to the representation of $H^{\varepsilon,\alpha,x,n}$ in terms of $u^{\varepsilon,\alpha}$, we find:

$$|\partial_x u_n^{\varepsilon,\alpha}(t,x)| \leqslant C\left(\sqrt[4]{\mathbb{E}} + \sqrt[4]{\mathbb{E}^{\mathbb{Q}^{\varepsilon,\alpha}}}\right)\left[1 + |X^{\varepsilon,x,n}|_{0,1}^{*,8}\right],$$

where the constant C is still independent of α. Mean value theorem tells us

$$|u_n^{\varepsilon,\alpha}(t,x) - u_n^{\varepsilon,\alpha}(t,x')| \leqslant C(1 + |x|^2 + |x'|^2)|x - x'|.$$

To conclude, we just need to take the limit as ε goes to 0. The key is the same as in Proposition 3.10; denoting $\overline{Y}_t^{\alpha,x,n} = \overline{v}^\alpha_{N_t^n}(X_t^{x,n})$, we see that $(\overline{Y}^{\alpha,x,n}, Z^{\alpha,x,n}, H^{\alpha,x,n})$ is the solution of the BSDE

$$\overline{Y}_t^{\alpha,x,n} = \overline{v}^\alpha_{N_1^n}(X_1^{x,n}) + \int_t^1 \left\{f_{N_r^n}(X_r^{x,n}, Z_r^{\alpha,x,n}, H_r^{\alpha,x,n}) - \alpha\left(\overline{Y}_r^{\alpha,x,n} + v^\alpha_{N_r^n}(0)\right)\right\}\mathrm{d}r - \int_t^1 Z_r^{\alpha,x,n}\,\mathrm{d}W_r$$
$$- \int_t^1 H_r^{\alpha,x,n}\,\mathrm{d}M_r,$$

so $u_n^{\varepsilon,\alpha} \underset{\varepsilon\to 0}{\longrightarrow} \overline{v}_n^\alpha$. ∎

3.4 The ergodic BSDE

We consider the ergodic BSDE:

$$Y_t^{x,n} = Y_T^{x,n} + \int_t^T \{f_{N_s^n}(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}) - \lambda\} \,\mathrm{d}s - \int_t^T Z_s^{x,n} \,\mathrm{d}W_s - \int_t^T H_s^{x,n} \,\mathrm{d}M_s, \quad 0 \leqslant t \leqslant T < \infty. \tag{3.14}$$

Theorem 3.14 (Existence). *Under Assumption 3.7, there exist:*

- *a real number λ;*
- *a family of locally Lipschitz functions $(v_l)_{l\in\mathbb{K}}$ satisfying $v_l(0) = 0$;*
- *a process $Z^{x,n} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty;(\mathbb{R}^d)^*\right)\right)$;*

such that, if we define $Y_t^{x,n} = v_{N_t^n}(X_t^{x,n})$ and $H_t^{x,n}(l) = v_{N_{t-}^n+l}(X_t^{x,n}) - v_{N_{t-}^n}(X_t^{x,n})$, EBSDE (3.14) is satisfied by $(Y^{x,n}, Z^{x,n}, H^{x,n}, \lambda)$ $\mathbb{P}$-a.s. and for all $0 \leqslant t \leqslant T < \infty$.
Moreover, there exists $C > 0$, such that for all $x \in \mathbb{R}^d$ and $l \in \mathbb{K}$, $|v_l(x)| \leqslant C(1+|x|^2)$.

Proof.

Step 1: Construction of v_l, $l \in \mathbb{K}$, by a diagonal procedure.
For every $\alpha > 0$ and $l \in \mathbb{K}$, we define $\overline{v}_l^\alpha(x) = v_l^\alpha(x) - v_l^\alpha(0)$. We know that there exists a constant C, independent of α, such that $|\overline{v}_l^\alpha(x)| \leqslant C(1+|x|^2)$ and $|\alpha v_l^\alpha(0)| \leqslant C$. Let D be a countable subset of $\mathbb{R}^d$; by a diagonal argument, we can construct a sequence (α_p) converging to 0, such that for every $l \in \mathbb{K}$, $(\overline{v}_l^{\alpha_p})_p$ converges pointwise to a function v_l over D and $\alpha_p v_l^{\alpha_p}(0) \underset{p\to\infty}{\longrightarrow} \lambda_l$, for a convenient real number λ_l. Moreover, the functions $\overline{v}_l^\alpha$ are uniformly locally Lipschitz:

$$\exists C > 0,\ \forall \alpha \in (0,1],\ \forall x, x' \in \mathbb{R}^d,\ \forall l \in \mathbb{K},\ |\overline{v}_l^\alpha(x) - \overline{v}_l^\alpha(x')| \leqslant C(1+|x|^2+|x'|^2)|x-x'|. \tag{3.15}$$

Then, v_l is uniformly continuous on every compact subset of D, so it has a continuous extension over $\mathbb{R}^d$ (we still call it v_l). Then, we can show that v_l is the pointwise limit of the functions $\overline{v}_l^{\alpha_p}$ over $\mathbb{R}^d$, v_l is locally Lipschitz continuous and has quadratic growth.

Step 2: Construction of some processes $Z^{x,n}$ and $H^{x,n}$.
We will show that $(Z^{\alpha_p,x,n})_p$ and $(H^{\alpha_p,x,n})_p$ are Cauchy in $\mathrm{L}^2_{\mathcal{P}}\left(\Omega, \mathrm{L}^2\left([0,T], E^*\right)\right)$, where $E = \mathbb{R}^d$ or $\mathbb{R}^{k-1}$, for every $T > 0$. When $p \leqslant m \in \mathbb{N}$, we set $\widetilde{Y} = \overline{Y}^{\alpha_p,x,n} - \overline{Y}^{\alpha_m,x,n}$, $\widetilde{Z} = Z^{\alpha_p,x,n} - Z^{\alpha_m,x,n}$ and $\widetilde{H} = H^{\alpha_p,x,n} - H^{\alpha_m,x,n}$. By Itô's formula, writing $R_s = \alpha_m Y_s^{\alpha_m,x,n} - \alpha_p Y_s^{\alpha_p,x,n}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \{|\widetilde{Z}_s|^2 + \mu|\widetilde{H}_s|^2\}\,\mathrm{d}s\right] \\
&= \mathbb{E}\left[|\widetilde{Y}_T|^2\right] - |\widetilde{Y}_0|^2 \\
&\qquad + 2\mathbb{E}\left[\int_0^T \widetilde{Y}_s \left\{f_{N_s^n}(X_s^{x,n}, Z_s^{\alpha_p,x,n}, H_s^{\alpha_p,x,n}) - f_{N_s^n}(X_s^{x,n}, Z_s^{\alpha_m,x,n}, H_s^{\alpha_m,x,n}) + R_s\right\}\,\mathrm{d}s\right] \\
&\leqslant \mathbb{E}\left[|\widetilde{Y}_T|^2\right] + \left(2\|f\|^2_{\mathrm{lip},z} + \frac{2\|f\|^2_{\mathrm{lip},h}}{\mu}\right)\mathbb{E}\left[\int_0^T |\widetilde{Y}_s|^2\,\mathrm{d}s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T \{|\widetilde{Z}_s|^2 + \mu|\widetilde{H}_s|^2\}\,\mathrm{d}s\right] \\
&\qquad + 2\mathbb{E}\left[\int_0^T |\widetilde{Y}_s|^2\,\mathrm{d}s\right]^{1/2} \mathbb{E}\left[\int_0^T |R_s|^2\,\mathrm{d}s\right]^{1/2}.
\end{aligned}$$

Using the fact that $|R_s| \leqslant C(1+|X_s^{x,n}|)$, we are able to prove the existence of the limits of $(Z^{\alpha_p,x,n})_p$ and $(H^{\alpha_p,x,n})_p$ in $\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty;E^*\right)\right)$.

Step 3: Taking the limit in the BSDE satisfied by $\left(\overline{Y}^{\alpha_p,x,n}, Z^{\alpha_p,x,n}, H^{\alpha_p,x,n}\right)$ leads us to:

$$Y_t^{x,n} = Y_T^{x,n} + \int_t^T \{f_{N_s^n}(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}) - \lambda_{N_s^n}\}\,\mathrm{d}s - \int_t^T Z_s^{x,n}\,\mathrm{d}W_s - \int_t^T H_s^{x,n}\,\mathrm{d}M_s.$$

Step 4: $\lambda_1 = \cdots = \lambda_k$.

For every $\alpha > 0$, and for all $0 \leqslant t \leqslant T < \infty$, we can write:

$$\alpha\overline{v}^{\alpha}_{N_t^n}(X_t^{x,n}) + \alpha v^{\alpha}_{N_t^n}(0) = \alpha\overline{v}^{\alpha}_{N_T^n}(X_T^{x,n}) + \alpha v^{\alpha}_{N_T^n}(0) + \alpha\int_t^T \{f_{N_s^n}(X_s^{x,n}, Z_s^{\alpha,x,n}, H_s^{\alpha,x,n}) - \alpha Y_s^{\alpha,x,n}\}\,\mathrm{d}s \\ - \alpha\int_t^T Z_s^{\alpha,x,n}\,\mathrm{d}W_s - \alpha\int_t^T H_s^{\alpha,x,n}\,\mathrm{d}M_s.$$

Taking the limit along the subsequence (α_p), we get $\lambda_{N_t^n} = \lambda_{N_T^n}$ for every $0 \leqslant t \leqslant T < \infty$, so $\lambda_1 = \cdots = \lambda_k =: \lambda$.

Step 5: Almost surely convergences.

For every $x \in \mathbb{R}^d$, the sequence $\left(\mathbb{E}\left[\int_0^T \left\{\left|Z_t^{\alpha_p,x,n} - Z_t^{\alpha_m,x,n}\right|^2 + \mu\left|H_t^{\alpha_p,x,n} - H_t^{\alpha_m,x,n}\right|^2\right\}\mathrm{d}t\right]\right)_{p\leqslant m\in\mathbb{N}}$ is bounded (because it converges). By a diagonal procedure, there exists a subsequence $(\alpha'_p) \subset (\alpha_p)$ such that:

$$\forall x \in D,\ \forall p \leqslant m \in \mathbb{N},\ \mathbb{E}\left[\int_0^T \left\{\left|Z_t^{\alpha'_p,x,n} - Z_t^{\alpha'_m,x,n}\right|^2 + \mu\left|H_t^{\alpha'_p,x,n} - H_t^{\alpha'_m,x,n}\right|^2\right\}\mathrm{d}t\right] < 2^{-p}.$$

We claim that this result can be extended to every $x \in \mathbb{R}^d$; let us fix $T > 0$ and denote $\Delta\bullet = \bullet^{\alpha,x,n} - \bullet^{\alpha,x',n}$, for any $\alpha > 0$ and $x, x' \in \mathbb{R}^d$. Then, by usual techniques:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \{|\Delta Z_t|^2 + \mu|\Delta H_t|^2\}\,\mathrm{d}t\right] \\ &\leqslant 2\mathbb{E}\left[|\Delta Y_T|^2\right] + \left(4\|f\|^2_{\mathrm{lip},z} + \frac{4\|f\|^2_{\mathrm{lip},h}}{\mu}\right)\mathbb{E}\left[\int_0^T |\Delta Y_t|^2\,\mathrm{d}t\right] \\ &\qquad + 4\mathbb{E}\left[\int_0^T |\Delta Y_t||\Delta X_t|\left\{\|\rho\|_{\mathrm{lip},x} + \|\rho\|_{\mathrm{lip},p}\|\sigma^{-1}\|^2_\infty\|\sigma\|_{\mathrm{lip}}|Z_t^{\alpha,x,n}|\right\}\,\mathrm{d}t\right].\end{aligned}$$

We can use Lemma 3.28, Proposition 3.13 and the estimate of the end of the proof of Proposition 3.12, to prove that:

$$\mathbb{E}\left[\int_0^T \{|\Delta Z_t|^2 + \mu|\Delta H_t|^2\}\,\mathrm{d}t\right] \leqslant C(1 + |x|^4 + |x'|^4)|x - x'|^2,$$

with a constant C independent of x and x'. So, we are now able to write:

$$\forall x \in \mathbb{R}^d,\ \forall p \leqslant m \in \mathbb{N},\ \mathbb{E}\left[\int_0^T \left\{\left|Z_t^{\alpha'_p,x,n} - Z_t^{\alpha'_m,x,n}\right|^2 + \mu\left|H_t^{\alpha'_p,x,n} - H_t^{\alpha'_m,x,n}\right|^2\right\}\mathrm{d}t\right] < 2^{-p}.$$

Finally, Borel-Cantelli theorem gives us, for a.e. $t \in [0, T]$:

$$Z_t^{\alpha'_p,x,n} \underset{p\to\infty}{\longrightarrow} Z_t^{x,n} \quad \text{and} \quad H_t^{\alpha'_p,x,n} \underset{p\to\infty}{\longrightarrow} H_t^{x,n}, \quad \mathbb{P}\text{-a.s.}$$

Step 6: Representation of H.

For every $T > 0$, $(Y^{x,n}, Z^{x,n}, H^{x,n})$ is the solution the finite horizon BSDE: for every $t \in [0, T]$,

$$Y_t^{x,n} = v_{N_T^n}(X_T^{x,n}) + \int_t^T \left\{f_{N_s^n}(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}) - \lambda\right\}\,\mathrm{d}s - \int_t^T Z_s^{x,n}\,\mathrm{d}W_s - \int_t^T H_s^{x,n}\,\mathrm{d}M_s.$$

The formula $H_t^{x,n}(l) = v_{N^n_{t-}+l}(X_t^{x,n}) - v_{N^n_{t-}}(X_t^{x,n})$ comes from Lemma 2.2 in [53]. □

Theorem 3.15 (Uniqueness for λ). *We suppose that:*

- *Assumption 3.7 holds;*
- *$(Y^{x,n}, Z^{x,n}, H^{x,n}, \lambda)$ is the solution built in Theorem 3.14;*
- *for some $x \in \mathbb{R}^d$ and $n \in \mathbb{K}$, (Y', Z', H', λ') satifies the EBSDE (3.14) $\mathbb{P}$-a.s. for every $0 \leqslant t \leqslant T < \infty$, where Y' is a progressively measurable process, $Z', H' \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega, \mathrm{L}^2(0,\infty; E^*))$ (with $E = \mathbb{R}^d$ or $\mathbb{R}^{k-1}$) and $\lambda' \in \mathbb{R}$;*
- *$\exists c_{x,n} > 0$, $\forall t \geqslant 0$, $|Y'_t| \leqslant c_{x,n}(1 + |X^{x,n}_t|^2)$ $\mathbb{P}$-a.s.*

Then, $\lambda = \lambda'$.

Proof. We define $\overline{\bullet} = \bullet^{x,n} - \bullet'$, for $\bullet = Y$, Z, H or λ. We have:

$$\overline{\lambda} = \frac{\overline{Y}_T - \overline{Y}_0}{T} + \frac{1}{T}\int_0^T \left\{ f_{N^n_s}(X^{x,n}_s, Z^{x,n}_s, H^{x,n}_s) - f_{N^n_s}(X^{x,n}_s, Z'_s, H'_s) \right\} \mathrm{d}s - \frac{1}{T}\int_0^T \overline{Z}_s \,\mathrm{d}W_s - \frac{1}{T}\int_0^T \overline{H}_s \,\mathrm{d}M_s.$$

Thanks to Girsanov's lemma, there exists a probability measure $\widehat{\mathbb{P}}$ under which

$$\overline{\lambda} = \frac{1}{T}\widehat{\mathbb{E}}\left[\overline{Y}_T - \overline{Y}_0\right] \leqslant \frac{C}{T}(1 + |x|^2),$$

with a constant C independent of T, due to Proposition 3.4. We conclude by taking the limit when $T \to \infty$. □

Theorem 3.16 (Uniqueness for Y, Z and H). *We suppose that:*

- *Assumption 3.7 holds;*
- *for every $l \in \mathbb{K}$, $v_l, \widehat{v}_l : \mathbb{R}^d \to \mathbb{R}$ are continuous, have quadratic growth, and $v_l(0) = \widehat{v}_l(0) = 0$;*
- *for every $x \in \mathbb{R}^d$ and $n \in \mathbb{K}$, $Z^{x,n}, H^{x,n} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega, \mathrm{L}^2(0,\infty; E^*))$ (with $E = \mathbb{R}^d$ or $\mathbb{R}^{k-1}$);*
- *there exist λ, $\widehat{\lambda} \in \mathbb{R}$, such that, for every $(x,n) \in \mathbb{R}^d \times \mathbb{K}$, the tuples $(v_{N^n}(X^{x,n}), Z^{x,n}, H^{x,n}, \lambda)$ and $(\widehat{v}_{N^n}(X^{x,n}), \widehat{Z}^{x,n}, \widehat{H}^{x,n}, \widehat{\lambda})$ satisfy the EBSDE (3.14).*

Then, $\lambda = \widehat{\lambda}$, $v = \widehat{v}$, and for all $(x,n) \in \mathbb{R}^d \times \mathbb{K}$, for a.e. $t \geqslant 0$, $Z^{x,n}_t = \widehat{Z}^{x,n}_t$ and $H^{x,n}_t = \widehat{H}^{x,n}_t$ $\mathbb{P}$-a.s.

Proof. We already know that $\lambda = \widehat{\lambda}$. We write $Y^{x,n}_t = v_{N^n_t}(X^{x,n}_t)$ and $\widehat{Y}^{x,n}_t = \widehat{v}_{N^n_t}(X^{x,n}_t)$. We approach b, σ, f, v and $\widehat{v}$ by $\mathcal{C}^1_b$-functions which converge uniformly (on every compact set for v and $\widehat{v}$). Moreover, we can assume that b^ε, σ^ε and f^ε are uniformly Lipschitz continuous and that v^ε and $\widehat{v}^\varepsilon$ are uniformly locally Lipschitz continuous, in the sense of equation (3.15). We consider the following equations, for every $t \in [0,T]$:

$$X^{x,n,\varepsilon}_t = x + \int_0^t b^\varepsilon_{N^n_s}(X^{x,n,\varepsilon}_s)\,\mathrm{d}s + \int_0^t \sigma^\varepsilon_{N^n_s}(X^{x,n,\varepsilon}_s)\,\mathrm{d}W_s,$$

$$Y^{x,n,\varepsilon}_t = v^\varepsilon_{N^n_T}(X^{x,n,\varepsilon}_T) + \int_t^T \left\{ f^\varepsilon_{N^n_s}(X^{x,n,\varepsilon}_s, Z^{x,n,\varepsilon}_s, H^{x,n,\varepsilon}_s) - \lambda \right\} \mathrm{d}s - \int_t^T Z^{x,n,\varepsilon}_s \,\mathrm{d}W_s - \int_t^T H^{x,n,\varepsilon}_s \,\mathrm{d}M_s,$$

$$\widehat{Y}^{x,n,\varepsilon}_t = \widehat{v}^\varepsilon_{N^n_T}(X^{x,n,\varepsilon}_T) + \int_t^T \left\{ f^\varepsilon_{N^n_s}(X^{x,n,\varepsilon}_s, \widehat{Z}^{x,n,\varepsilon}_s, \widehat{H}^{x,n,\varepsilon}_s) - \lambda \right\} \mathrm{d}s - \int_t^T \widehat{Z}^{x,n,\varepsilon}_s \,\mathrm{d}W_s - \int_t^T \widehat{H}^{x,n,\varepsilon}_s \,\mathrm{d}M_s.$$

Step 1: Convergence of the approximations.
We set $\Delta\bullet = \bullet - \widehat{\bullet}$, and we get:

$$\begin{aligned}
\left|\Delta Y_0^{x,n} - \Delta Y_0^{x,n,\varepsilon}\right| \leqslant \mathbb{E}\left[\left|v_{N_T^n}(X_T^{x,n}) - v^\varepsilon_{N_T^n}(X_T^{x,n,\varepsilon})\right|\right] + \mathbb{E}\left[\left|\widehat{v}_{N_T^n}(X_T^{x,n}) - \widehat{v}^\varepsilon_{N_T^n}(X_T^{x,n,\varepsilon})\right|\right] \\
+ \mathbb{E}\left[\int_0^T \left|f_{N_t^n}(X_t^{x,n}, Z_t^{x,n}, H_t^{x,n}) - f^\varepsilon_{N_t^n}(X_t^{x,n,\varepsilon}, Z_t^{x,n,\varepsilon}, H_t^{x,n,\varepsilon})\right| \mathrm{d}t\right] \\
+ \mathbb{E}\left[\int_0^T \left|f_{N_t^n}(X_t^{x,n}, \widehat{Z}_t^{x,n}, \widehat{H}_t^{x,n}) - f^\varepsilon_{N_t^n}(X_t^{x,n,\varepsilon}, \widehat{Z}_t^{x,n,\varepsilon}, \widehat{H}_t^{x,n,\varepsilon})\right| \mathrm{d}t\right].
\end{aligned}$$

Setting $\delta\bullet = \bullet^{x,n} - \bullet^{x,n,\varepsilon}$, we have the upperbound:

$$\begin{aligned}
&\left|f_{N_t^n}(X_t^{x,n}, Z_t^{x,n}, H_t^{x,n}) - f^\varepsilon_{N_t^n}(X_t^{x,n,\varepsilon}, Z_t^{x,n,\varepsilon}, H_t^{x,n,\varepsilon})\right| \\
&\quad \leqslant \|f - f_\varepsilon\|_\infty + \|\rho\|_{\mathrm{lip},x}|\delta X_t| + \|\rho\|_{\mathrm{lip},p}|Z_t^{x,n}|\|\sigma^{-1}\|_\infty^2\|\sigma\|_{\mathrm{lip}}|\delta X_t| + \|f\|_{\mathrm{lip},z}|\delta Z_t| + \|f\|_{\mathrm{lip},h}|\delta H_t|.
\end{aligned}$$

But, thanks to the lemmas 3.28 and 3.29, we have:

$$\mathbb{E}\left[|\delta X|_{0,T}^{*,p}\right] \leqslant C_p\left(\|b - b^\varepsilon\|_\infty^p + \|\sigma - \sigma^\varepsilon\|_\infty^p\right) \quad \text{and}$$

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\{|\delta Z_t|^2 + \mu|\delta H_t|^2\right\} \mathrm{d}t\right] \\
&\quad \leqslant C\mathbb{E}\left[\left|v_{N_T^n}(X_T^{x,n}) - v^\varepsilon_{N_T^n}(X_T^{x,n,\varepsilon})\right|^2 + \int_0^T \left|f_{N_t^n}(X_t^{x,n}, Z_t^{x,n}, H_t^{x,n}) - f^\varepsilon_{N_t^n}(X_t^{x,n,\varepsilon}, Z_t^{x,n}, H_t^{x,n})\right|^2 \mathrm{d}t\right].
\end{aligned}$$

Obviously, the same estimates hold with $\widehat{}$ on v, Z and H. To prove that $\left|\Delta Y_0^{x,n} - \Delta Y_0^{x,n,\varepsilon}\right| \underset{\varepsilon\to 0}{\longrightarrow} 0$, we only need to show that $\mathbb{E}\left[\left|v_{N_T^n}(X_T^{x,n}) - v^\varepsilon_{N_T^n}(X_T^{x,n,\varepsilon})\right|^2\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$:

- on the one hand, using uniform local Lipschitz property,

$$\begin{aligned}
&\mathbb{E}\left[\left|v^\varepsilon_{N_T^n}(X_T^{x,n}) - v^\varepsilon_{N_T^n}(X_T^{x,n,\varepsilon})\right|^2\right] \\
&\qquad \leqslant C\left(1 + \mathbb{E}\left[|X^{x,n}|_{0,T}^{*,8}\right]^{1/2} + \mathbb{E}\left[|X^{x,n,\varepsilon}|_{0,T}^{*,8}\right]^{1/2}\right)\mathbb{E}\left[|\delta X|_{0,T}^{*,4}\right]^{1/2} \underset{\varepsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

- on the other hand, for any $R > 0$, we have

$$\mathbb{E}\left[\left|(v - v^\varepsilon)_{N_T^n}(X_T^{x,n})\right|^2 \mathbb{1}_{|X_T^{x,n}|\leqslant R}\right] \leqslant \|v - v^\varepsilon\|_{\infty,\mathcal{B}_R}^2 \quad \text{and}$$

$$\begin{aligned}
\mathbb{E}\left[\left|(v - v^\varepsilon)_{N_T^n}(X_T^{x,n})\right|^2 \mathbb{1}_{|X_T^{x,n}|> R}\right] &\leqslant \mathbb{E}\left[\left|(v - v^\varepsilon)_{N_T^n}(X_T^{x,n})\right|^4\right]^{1/2} \mathbb{P}\left(|X_T^{x,n}| > R\right)^{1/2} \\
&\leqslant C\left(1 + \mathbb{E}\left[|X_T^{x,n}|^8\right]^{1/2}\right)\frac{\mathbb{E}\left[|X_T^{x,n}|^2\right]^{1/2}}{R} \leqslant \frac{\tilde{C}(1 + |x|^5)}{R}.
\end{aligned}$$

For any $R > 0$, $\limsup_{\varepsilon\to 0} \mathbb{E}\left[\left|(v - v^\varepsilon)_{N_T^n}(X_T^{x,n})\right|^2\right] \leqslant \frac{\tilde{C}(1 + |x|^5)}{R}$, i.e.

$$\mathbb{E}\left[\left|(v - v^\varepsilon)_{N_T^n}(X_T^{x,n})\right|^2\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Step 2: Convergence of the approximations to the same limit.

Approximations have been made in such a way that $Z^{x,n,\varepsilon}$ and $\widehat{Z}^{x,n,\varepsilon}$ can be written as continuous functions of $X^{x,n,\varepsilon}$; we write:

$$Z_t^{x,n,\varepsilon} = \zeta^\varepsilon_{N_t^n}(t, X_t^{x,n,\varepsilon}), \quad \widehat{Z}_t^{x,n,\varepsilon} = \widehat{\zeta}^\varepsilon_{N_t^n}(t, X_t^{x,n,\varepsilon}),$$

$$H_t^{x,n,\varepsilon} = h^\varepsilon_{N^n_{t-}}(t, X_t^{x,n,\varepsilon}) \quad \text{and} \quad \widehat{H}_t^{x,n,\varepsilon} = \widehat{h}^\varepsilon_{N^n_{t-}}(t, X_t^{x,n,\varepsilon}).$$

We set:

$$\Gamma^\varepsilon_{n,m}(t,x) = \begin{cases} \frac{f_n^\varepsilon(x,\zeta_n^\varepsilon(t,x),\widehat{h}^\varepsilon_m(t,x)) - f_n^\varepsilon(x,\widehat{\zeta}_n^\varepsilon(t,x),\widehat{h}^\varepsilon_m(t,x))}{|(\zeta_n^\varepsilon - \widehat{\zeta}_n^\varepsilon)(t,x)|^2}(\zeta_n^\varepsilon - \widehat{\zeta}_n^\varepsilon)(t,x)^* & \text{if } \zeta_n^\varepsilon(t,x) \neq \widehat{\zeta}_n^\varepsilon(t,x), \\ 0 & \text{otherwise;} \end{cases}$$

$$\overline{\Gamma}^\varepsilon_{n,m}(t,x) = \begin{cases} \frac{f_n^\varepsilon(x,\zeta_n^\varepsilon(t,x),h^\varepsilon_m(t,x)) - f_n^\varepsilon(x,\zeta_n^\varepsilon(t,x),\widehat{h}^\varepsilon_m(t,x))}{|(h_m^\varepsilon - \widehat{h}_m^\varepsilon)(t,x)|^2}(h_m^\varepsilon - \widehat{h}_m^\varepsilon)(t,x)^* & \text{if } h_m^\varepsilon(t,x) \neq \widehat{h}_m^\varepsilon(t,x), \\ 0 & \text{otherwise.} \end{cases}$$

There exists a probability measure $\mathbb{P}^{x,n,\varepsilon,T}$ under which we can write:

$$\Delta Y_0^{x,n,\varepsilon} = \mathbb{E}^{x,n,\varepsilon,T}\left[\Delta Y_T^{x,n,\varepsilon}\right] = \mathbb{E}^{x,n,\varepsilon,T}\left[(v^\varepsilon - \widehat{v}^\varepsilon)_{N_T^n}(X_T^{x,n,\varepsilon})\right].$$

Using Corollary 3.6, we get

$$\left|\mathbb{E}^{x,n,\varepsilon,T}\left[(v^\varepsilon - \widehat{v}^\varepsilon)_{N_T^n}(X_T^{x,n,\varepsilon})\right] - \mathbb{E}^{0,n,\varepsilon,T}\left[(v^\varepsilon - \widehat{v}^\varepsilon)_{N_T^n}(X_T^{0,n,\varepsilon})\right]\right| \leqslant c(1+|x|^2)\mathrm{e}^{-\nu T}.$$

Because c and ν do not depend on ε, passing to the limit gives us:

$$|(v - \widehat{v})_n(x)| = |\Delta Y_0^{x,n} - \Delta Y_0^{0,n}| \leqslant c(1+|x|^2)\mathrm{e}^{-\nu T}.$$

We finally obtain $v = \widehat{v}$ when $T \to \infty$. Then, uniqueness for Z and H is the consequence of Itô's formula.

□

3.5 Large time behaviour

In this section, we keep working under Assumption 3.7. We already know that there exists a unique solution $(Y^{x,n}, Z^{x,n}, H^{x,n}, \lambda)$ to equation (3.14). Let ξ^T be a real random variable $\mathcal{F}_T$-measurable and C a positive constant such that $|\xi^T| \leqslant C(1+|X_T^{x,n}|^2)$ $\mathbb{P}$-a.s. We consider the solution $(Y^{T,x,n}, Z^{T,x,n}, H^{T,x,n})$ of the finite horizon BSDE:

$$Y_t^{T,x,n} = \xi^T + \int_t^T f_{N_s^n}(X_s^{x,n}, Z_s^{T,x,n}, H_s^{T,x,n})\,\mathrm{d}s - \int_t^T Z_s^{T,x,n}\,\mathrm{d}W_s - \int_t^T H_s^{T,x,n}\,\mathrm{d}M_s, \qquad t \in [0,T].$$

Theorem 3.17. *We have the following inequality:*

$$\left|\frac{Y_0^{T,x,n}}{T} - \lambda\right| \leqslant \frac{C\left(1+|x|^2\right)}{T},$$

where the constant C is independent of x, n and T; and in particular: $\dfrac{Y_0^{T,x,n}}{T} \underset{T\to\infty}{\longrightarrow} \lambda$, uniformly in any bounded subset of $\mathbb{R}^d$.

Proof. For all $x \in \mathbb{R}^d$, $n \in \mathbb{K}$ and $T > 0$, we write:

$$\left| \frac{Y_0^{T,x,n}}{T} - \lambda \right| \leqslant \left| \frac{Y_0^{T,x,n} - Y_0^{x,n} - \lambda T}{T} \right| + \left| \frac{Y_0^{x,n}}{T} \right|.$$

First of all, $\left|Y_0^{x,n}\right| = |v_n(x)| \leqslant C\left(1+|x|^2\right)$. Also, by the usual linearisation technique, we have:

$$\begin{aligned} Y_0^{T,x,n} - Y_0^{x,n} - \lambda T = \xi^T - v_{N_T^n}\left(X_T^{x,n}\right) + \int_0^T \left\{ \left(Z_s^{T,x,n} - Z_s^{x,n}\right)\beta_s + \left(H_s^{T,x,n} - H_s^{x,n}\right)\gamma_s \right\} \mathrm{d}s \\ - \int_0^T \left(Z_s^{T,x,n} - Z_s^{x,n}\right) \mathrm{d}W_s - \int_0^T \left(H_s^{T,x,n} - H_s^{x,n}\right) \mathrm{d}M_s, \end{aligned}$$

where

$$\beta_s = \begin{cases} \dfrac{f_{N_s^n}\left(X_s^{x,n}, Z_s^{T,x,n}, H_s^{x,n}\right) - f_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right)}{\left|Z_s^{T,x,n} - Z_s^{x,n}\right|^2} \left(Z_s^{T,x,n} - Z_s^{x,n}\right)^*, & \text{if } Z_s^{T,x,n} \neq Z_s^{x,n}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\gamma_s = \begin{cases} \dfrac{f_{N_s^n}\left(X_s^{x,n}, Z_s^{T,x,n}, H_s^{T,x,n}\right) - f_{N_s^n}\left(X_s^{x,n}, Z_s^{T,x,n}, H_s^{x,n}\right)}{\left|H_s^{T,x,n} - H_s^{x,n}\right|^2} \left(H_s^{T,x,n} - H_s^{x,n}\right)^*, & \text{if } H_s^{T,x,n} \neq H_s^{x,n}, \\ 0, & \text{otherwise.} \end{cases}$$

By Girsanov's theorem, there exists a probability measure $\mathbb{Q}^T$ under which $\widetilde{W}_t = W_t - \int_0^t \beta_s \, \mathrm{d}s$ is a Brownian motion on $[0, T]$, and such that we can write:

$$\begin{aligned} \left|Y_0^{T,x,n} - Y_0^{x,n} - \lambda T\right| = \left|\mathbb{E}^{\mathbb{Q}^T}\left[\xi^T - v_{N_T^n}\left(X_T^{x,n}\right)\right]\right| &\leqslant \mathbb{E}^{\mathbb{Q}^T}\left[\left|\xi^T\right|\right] + \mathbb{E}^{\mathbb{Q}^T}\left[\left|v_{N_T^n}\left(X_T^{x,n}\right)\right|\right] \\ &\leqslant C\left(1 + \mathbb{E}^{\mathbb{Q}^T}\left[\left|X_T^{x,n}\right|^2\right]\right). \end{aligned}$$

Due to Proposition 3.4, we get: $\sup_{T \geqslant 0} \mathbb{E}^{\mathbb{Q}^T}\left[\left|X_T^{x,n}\right|^2\right] \leqslant \kappa\left(1+|x|^2\right)$, where κ is independent of x and n. □

Theorem 3.18. *We suppose that $\xi^T = g_{N_T^n}\left(X_T^{x,n}\right)$, where all the functions $g_n : \mathbb{R}^d \to \mathbb{R}$ have quadratic growth: $\forall (x,n) \in \mathbb{R}^d \times \mathbb{K}$, $|g_n(x)| \leqslant C\left(1+|x|^2\right)$ and satisfy the following local Lipschitz condition: $\forall x, x' \in \mathbb{R}^d$, $\forall n \in \mathbb{K}$, $|g_n(x) - g_n(x')| \leqslant C\left(1+|x|^2+|x'|^2\right)|x-x'|$.*
Then, there exists $L \in \mathbb{R}$, such that: $\forall (x,n) \in \mathbb{R}^d \times \mathbb{K}$, $Y_0^{T,x,n} - \lambda T - Y_0^{x,n} \underset{T\to\infty}{\longrightarrow} L$. Furthermore,

$$\forall (x,n) \in \mathbb{R}^d \times \mathbb{K}, \forall T > 0, \left|Y_0^{T,x,n} - \lambda T - Y_0^{x,n} - L\right| \leqslant C\left(1+|x|^2\right)\mathrm{e}^{-\nu T}.$$

Proof. We will consider the following equations:

$$\begin{cases} \mathrm{d}Y_s^{T,t,x,n} &= -f_{N_s^{t,n}}\left(X_s^{t,x,n}, Z_s^{T,t,x,n}, H_s^{T,t,x,n}\right)\mathrm{d}s + Z_s^{T,t,x,n}\,\mathrm{d}W_s + H_s^{T,t,x,n}\,\mathrm{d}M_s \\ Y_T^{T,t,x,n} &= g_{N_T^{t,n}}\left(X_T^{t,x,n}\right) \end{cases}$$

$$\longrightarrow Y_s^{T,t,x,n} = u^T_{N_s^{t,n}}\left(s, X_s^{t,x,n}\right), \text{ see Lemma 2.2 of [53];}$$

$$\begin{cases} \mathrm{d}Y_s^{t,x,n} &= -\left\{ f_{N_s^{t,n}}\left(X_s^{t,x,n}, Z_s^{t,x,n}, H_s^{t,x,n}\right) - \lambda \right\} \mathrm{d}s + Z_s^{t,x,n}\,\mathrm{d}W_s + H_s^{t,x,n}\,\mathrm{d}M_s \\ Y_T^{t,x,n} &= v_{N_T^{t,n}}\left(X_T^{t,x,n}\right) \end{cases}$$

$$\longrightarrow Y_s^{t,x,n} = v_{N_s^{t,n}}\left(X_s^{t,x,n}\right), \text{ solution of the EBSDE;}$$

$$\begin{cases} \mathrm{d}Y_s^{\varepsilon,T,t,x,n} &= -f^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,t,x,n}, Z_s^{\varepsilon,T,t,x,n}, H_s^{\varepsilon,T,t,x,n}\right) \mathrm{d}s + Z_s^{\varepsilon,T,t,x,n}\,\mathrm{d}W_s + H_s^{\varepsilon,T,t,x,n}\,\mathrm{d}M_s \\ Y_T^{\varepsilon,T,t,x,n} &= g^{\varepsilon}_{N_T^{t,n}}\left(X_T^{\varepsilon,t,x,n}\right) \end{cases}$$

$$\longrightarrow Y_s^{\varepsilon,T,t,x,n} = u^{\varepsilon,T}_{N_s^{t,n}}\left(s, X_s^{\varepsilon,t,x,n}\right), \text{ where } u^{\varepsilon,T} \text{ is } \mathcal{C}^1 \text{ w.r.t. } x \text{ (see appendix);}$$

$$\begin{cases} \mathrm{d}\widetilde{Y}_s^{\varepsilon,T,t,x,n} &= -f^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,T,t,x,n}, \widetilde{Z}_s^{\varepsilon,T,t,x,n}, \widetilde{H}_s^{\varepsilon,T,t,x,n}\right) \mathrm{d}s + \widetilde{Z}_s^{\varepsilon,T,t,x,n}\,\mathrm{d}W_s + \widetilde{H}_s^{\varepsilon,T,t,x,n}\,\mathrm{d}M_s \\ \widetilde{Y}_T^{\varepsilon,T,t,x,n} &= v^{\varepsilon}_{N_T^{t,n}}\left(X_T^{\varepsilon,t,x,n}\right) \end{cases}$$

$$\longrightarrow \widetilde{Y}_s^{\varepsilon,T,t,x,n} = \widetilde{u}^{\varepsilon,T}_{N_s^{t,n}}\left(s, X_s^{\varepsilon,t,x,n}\right), \text{ where } \widetilde{u}^{\varepsilon,T} \text{ is } \mathcal{C}^1 \text{ w.r.t. } x \text{ (see appendix);}$$

where $g^{\varepsilon}, v^{\varepsilon} \in \mathcal{C}_b^1$ converge uniformly on every compact towards g and v; also, we take (b^{ε}) and (σ^{ε}) two sequences of $\mathcal{C}_b^1$ functions that converge uniformly towards b and σ and define $X^{\varepsilon,t,x,n}$ as the solution of the SDE:

$$\begin{cases} \mathrm{d}X_s^{\varepsilon,t,x,n} &= b^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,t,x,n}\right) \mathrm{d}s + \sigma^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,t,x,n}\right) \mathrm{d}W_s, \\ X_t^{\varepsilon,t,x,n} &= x. \end{cases}$$

Step 1: Approximation of the function f.

We set $\rho_i(x,p,h) = f_i(x, p\sigma_i(x), h)$; ρ is Lipschitz continuous, and we approximate it by a sequence of $\mathcal{C}_b^1$ functions (ρ^{ε}) which converges uniformly. Then, we define a function $\widetilde{\rho}^{\varepsilon}$ satisfying:

$$\widetilde{\rho}^{\varepsilon}_i(x,p,h) = \begin{cases} \rho_i^{\varepsilon}(x,p,h), & \text{if } |p\sigma_i(x)| \leqslant \frac{1}{\varepsilon}, \\ 0, & \text{if } |p\sigma_i(x)| \geqslant r(\varepsilon), \end{cases}$$

where $r(\varepsilon)$ is chosen in such a way that $\widetilde{\rho}^{\varepsilon}$ and ρ^{ε} have the same Lipschitz constant. Also, where $\frac{1}{\varepsilon} < |p\sigma_i(x)| < r(\varepsilon)$, $\widetilde{\rho}^{\varepsilon}(x,p,h)$ is chosen such that $\widetilde{\rho}^{\varepsilon}$ is of class $\mathcal{C}_b^1$ and $|\widetilde{\rho}^{\varepsilon}(x,p,h)| \leqslant |\rho^{\varepsilon}(x,p,h)|$. This way, $\widetilde{\rho}^{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \rho$, uniformly on every compact set. We set $f_i^{\varepsilon}(x,z,h) = \widetilde{\rho}_i^{\varepsilon}\left(x, z\sigma_i^{\varepsilon}(x)^{-1}, h\right)$ and $f^{\varepsilon} \in \mathcal{C}_b^1$. Moreover, we have the following upperbound:

$$\begin{aligned} |(f - f^{\varepsilon})_i(x,z,h)| &\leqslant |(\rho - \widetilde{\rho}^{\varepsilon})_i(x, z\sigma_i(x)^{-1}, h)| + |\widetilde{\rho}_i^{\varepsilon}(x, z\sigma_i(x)^{-1}, h) - \widetilde{\rho}_i^{\varepsilon}(x, z\sigma_i^{\varepsilon}(x)^{-1}, h)| \\ &\leqslant |(\rho - \rho^{\varepsilon})_i(x, z\sigma_i(x)^{-1}, h)|\mathbb{1}_{|z|\leqslant\varepsilon^{-1}} + 2|\rho(x, z\sigma_i(x)^{-1}, h)|\mathbb{1}_{|z|>\varepsilon^{-1}} \\ &\qquad + \|\rho\|_{\mathrm{lip},p}|z||\sigma_i(x)^{-1} - \sigma_i^{\varepsilon}(x)^{-1}| \\ &\leqslant \|\rho - \rho^{\varepsilon}\|_{\infty} + 2\left\{ M_f(1+|x|) + \|\rho\|_{\mathrm{lip},p}|z\sigma_i(x)^{-1}| + \|\rho\|_{\mathrm{lip},h}|h| \right\}\mathbb{1}_{|z|>\varepsilon^{-1}} \\ &\qquad + \|\rho\|_{\mathrm{lip},p}|z|\|\sigma^{-1}\|_{\infty}^2\|\sigma^{\varepsilon} - \sigma\|_{\infty}. \end{aligned}$$

Step 2: Growth of $w^{\varepsilon,T}$.

We set $w^{\varepsilon,T}(t,x) = u^{\varepsilon,T}(t,x) - \widetilde{u}^{\varepsilon,T}(t,x)$ and $w^T(t,x) = u^T(t,x) - \lambda(T-t) - v(x)$. By uniqueness of viscosity solutions (see Theorem 5.1 in [53]), we have $u^{\varepsilon,T}(0,x) = u^{\varepsilon,T+S}(S,x)$ and $\widetilde{u}^{\varepsilon,T}(0,x) = \widetilde{u}^{\varepsilon,T+S}(S,x)$.

Lemma 3.19. *We have:* $\forall \alpha \in (0,2]$, $\forall T>0$, $\exists C_T>0$, $\forall (x,n) \in \mathbb{R}^d \times \mathbb{K}$, $\forall \varepsilon > 0$,

$$\left|v_n(x) - \widetilde{u}_n^{\varepsilon,T}(0,x) + \lambda T\right|^2 \leqslant \widetilde{C}_T \left\{ \mathbb{E}\left[\left|v_{N_T^n}\left(X_T^{x,n}\right) - v^\varepsilon_{N_T^n}\left(X_T^{\varepsilon,x,n}\right)\right|^2\right] + \|\rho-\rho^\varepsilon\|_\infty^2 \right.$$
$$\left. + \left(\|b-b^\varepsilon\|_\infty^2 + \|\sigma-\sigma^\varepsilon\|^2_\infty + \varepsilon^{\alpha/2}\right)\left(1+|x|^{4+\alpha}\right)\right\}.$$

Proof. We set $\delta Y_s = Y_s^{x,n} - \widetilde{Y}_s^{\varepsilon,T,x,n} + \lambda(T-s)$, $\delta Z_s = Z_s^{x,n} - \widetilde{Z}_s^{\varepsilon,T,x,n}$ and $\delta H_s = H_s^{x,n} - \widetilde{H}_s^{\varepsilon,T,x,n}$; then,

$$\mathrm{d}\delta Y_s = \left\{ f^\varepsilon_{N_s^n}\left(X_s^{\varepsilon,x,n}, \widetilde{Z}_s^{\varepsilon,T,x,n}, \widetilde{H}_s^{\varepsilon,T,x,n}\right) - f_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right)\right\}\mathrm{d}s + \delta Z_s\,\mathrm{d}W_s + \delta H_s\,\mathrm{d}M_s.$$

We will use Lemma 3.30; we set

$$F(s,z,h) = f_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right) - f^\varepsilon_{N_s^n}\left(X_s^{\varepsilon,x,n}, Z_s^{x,n}-z, H_s^{x,n}-h\right),$$

and we can write:

$$\mathrm{d}\delta Y_s = -F(s,\delta Z_s,\delta H_s)\,\mathrm{d}s + \delta Z_s\,\mathrm{d}W_s + \delta H_s\,\mathrm{d}M_s.$$

We have:

$$\begin{aligned}
&|F(s,z,h)| \\
&\quad\leqslant \left|f_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right) - f^\varepsilon_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right)\right| \\
&\qquad\qquad + \left|f^\varepsilon_{N_s^n}\left(X_s^{x,n}, Z_s^{x,n}, H_s^{x,n}\right) - f^\varepsilon_{N_s^n}\left(X_s^{\varepsilon,x,n}, Z_s^{x,n}-z, H_s^{x,n}-h\right)\right| \\
&\quad\leqslant F_s + \|\rho\|_{\mathrm{lip},p}\|\sigma^{-1}\|_\infty|z| + \|\rho\|_{\mathrm{lip},h}|h|,
\end{aligned}$$

where we set

$$\begin{aligned}
F_s = \|\rho-\rho^\varepsilon\|_\infty &+ 2\left\{M_f(1+|X_s^{x,n}|) + \|\rho\|_{\mathrm{lip},p}\|\sigma^{-1}\|_\infty|Z_s^{x,n}| + \|\rho\|_{\mathrm{lip},h}|H_s^{x,n}|\right\}\mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}} \\
&+ \|\rho\|_{\mathrm{lip},p}|Z_s^{x,n}|\|\sigma^{-1}\|^2_\infty\|\sigma^\varepsilon-\sigma\|_\infty + \|\rho\|_{\mathrm{lip},x}|X_s^{x,n}-X_s^{\varepsilon,x,n}| \\
&+ \|\rho\|_{\mathrm{lip},p}|Z_s^{x,n}|\|\sigma\|_{\mathrm{lip}}\|\sigma^{-1}\|^2_\infty|X_s^{x,n}-X_s^{\varepsilon,x,n}|.
\end{aligned}$$

We get $\mathbb{E}\left[|\delta Y|^{*,2}_{0,T}\right] \leqslant C_T\mathbb{E}\left[|\delta Y_T|^2 + \left(\int_0^T F_s\,\mathrm{d}s\right)^2\right]$. But,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T F_s\,\mathrm{d}s\right)^2\right] \leqslant 7T\|\rho-\rho^\varepsilon\|^2_\infty &+ 28M_f^2\mathbb{E}\left[\left(\int_0^T (1+|X_s^{x,n}|)\mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}}\,\mathrm{d}s\right)^2\right] \\
&+ 28\|\rho\|^2_{\mathrm{lip},p}\|\sigma^{-1}\|^2_\infty\mathbb{E}\left[\left(\int_0^T |Z_s^{x,n}|\mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}}\,\mathrm{d}s\right)^2\right] \\
&+ 28\|\rho\|^2_{\mathrm{lip},h}\mathbb{E}\left[\left(\int_0^T |H_s^{x,n}|\mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}}\,\mathrm{d}s\right)^2\right] \\
&+ 7\|\rho\|^2_{\mathrm{lip},p}\|\sigma^{-1}\|^4_\infty\|\sigma^\varepsilon-\sigma\|^2_\infty\mathbb{E}\left[\left(\int_0^T |Z_s^{x,n}|\,\mathrm{d}s\right)^2\right] \\
&+ 7\|\rho\|^2_{\mathrm{lip},x}\mathbb{E}\left[\left(\int_0^T |X_s^{x,n}-X_s^{\varepsilon,x,n}|\,\mathrm{d}s\right)^2\right] \\
&+ 7\|\rho\|^2_{\mathrm{lip},p}\|\sigma\|^2_{\mathrm{lip}}\|\sigma^{-1}\|^4_\infty\mathbb{E}\left[\left(\int_0^T |X_s^{x,n}-X_s^{\varepsilon,x,n}|\,\mathrm{d}s\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^T F_s\,\mathrm{d}s\right)^2\right] \leqslant C_T \Bigg\{ & \|\rho-\rho^\varepsilon\|_\infty^2 + \left(1+\mathbb{E}\left[|X^{x,n}|_{0,T}^{*,4}\right]^{1/2}\right)\mathbb{E}\left[\left(\int_0^T \mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}}\,\mathrm{d}s\right)^4\right]^{1/2} \\
& + \mathbb{E}\left[\left(\int_0^T |Z_s^{x,n}|^2\,\mathrm{d}s\right)^2\right]^{1/2} \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}}\,\mathrm{d}s\right)^2\right]^{1/2} \\
& + \mathbb{E}\left[|H^{x,n}|_{0,T}^{*,4}\right]^{1/2} \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}}\,\mathrm{d}s\right)^4\right]^{1/2} \\
& + \|\sigma-\sigma^\varepsilon\|_\infty^2 \mathbb{E}\left[\int_0^T |Z_s^{x,n}|^2\,\mathrm{d}s\right] + \mathbb{E}\left[|X^{x,n}-X^{\varepsilon,x,n}|_{0,T}^{*,2}\right] \\
& + \mathbb{E}\left[|X^{x,n}-X^{\varepsilon,x,n}|_{0,T}^{*,4}\right]^{1/2} \mathbb{E}\left[\left(\int_0^T |Z_s^{x,n}|\,\mathrm{d}s\right)^4\right]^{1/2} \Bigg\}.
\end{aligned}
$$

We recall the following inequalities:

- from Proposition 3.4: $\mathbb{E}\left[|X^{x,n}|_{0,T}^{*,4}\right] \leqslant C_T(1+|x|^4)$;
- from Lemma 3.28: $\mathbb{E}\left[|X^{x,n}-X^{\varepsilon,x,n}|_{0,T}^{*,p}\right] \leqslant C_{p,T}\left(\|b-b^\varepsilon\|_\infty^p + \|\sigma-\sigma^\varepsilon\|_\infty^p\right)$;
- from quadratic growth of v: $\mathbb{E}\left[|H^{x,n}|_{0,T}^{*,4}\right] \leqslant C(1+|x|^8)$;
- from Lemma 3.30: $\mathbb{E}\left[\left(\int_0^T |Z_s^{x,n}|^2\,\mathrm{d}s\right)^2\right] \leqslant C_T(1+|x|^8)$.

Moreover, we have, for every $\alpha \in (0,2]$ and for $p \in \{2,4\}$:

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^T \mathbb{1}_{|Z_s^{x,n}|>\varepsilon^{-1}}\,\mathrm{d}s\right)^p\right] &\leqslant T^{p-1}\int_0^T \mathbb{P}\left(|Z_s^{x,n}|>\varepsilon^{-1}\right)\,\mathrm{d}s \leqslant T^{p-1}\varepsilon^\alpha \int_0^T \mathbb{E}\left[|Z_s^{x,n}|^\alpha\right]\,\mathrm{d}s \\
&\leqslant T^{p-\frac{\alpha}{2}}\varepsilon^\alpha \mathbb{E}\left[\int_0^T |Z_s^{x,n}|^2\,\mathrm{d}s\right]^{\alpha/2}.
\end{aligned}
$$

□

The proof of the following lemma is exactly the same.

Lemma 3.20. *We have:* $\forall \alpha \in (0,2]$, $\forall T>0$, $\exists C_T>0$, $\forall (x,n)\in \mathbb{R}^d\times\mathbb{K}$, $\forall \varepsilon>0$,

$$
\begin{aligned}
\left|u_n^T(0,x)-u_n^{\varepsilon,T}(0,x)\right|^2 \leqslant \widetilde{C}_T \Bigg\{ & \mathbb{E}\left[\left|g_{N_T^n}\left(X_T^{x,n}\right)-g^\varepsilon_{N_T^n}\left(X_T^{\varepsilon,x,n}\right)\right|^2\right] + \|\rho-\rho^\varepsilon\|_\infty^2 \\
& + \left(\|b-b^\varepsilon\|_\infty^2 + \|\sigma-\sigma^\varepsilon\|_\infty^2 + \varepsilon^{\alpha/2}\right)\left(1+|x|^{4+\alpha}\right)\Bigg\}.
\end{aligned}
$$

We keep in mind the following results:

$$
\begin{aligned}
\left|w_n^{\varepsilon,T}(0,x)-w_n^T(0,x)\right|^2 \leqslant C_T \Bigg\{ & \mathbb{E}\left[\left|g_{N_T^n}\left(X_T^{x,n}\right)-g^\varepsilon_{N_T^n}\left(X_T^{\varepsilon,x,n}\right)\right|^2 + \left|v_{N_T^n}\left(X_T^{x,n}\right)-v^\varepsilon_{N_T^n}\left(X_T^{\varepsilon,x,n}\right)\right|^2\right] \\
& + \|\rho-\rho^\varepsilon\|_\infty^2 + \left(\|b-b^\varepsilon\|_\infty^2 + \|\sigma-\sigma^\varepsilon\|_\infty^2 + \varepsilon^{\alpha/2}\right)\left(1+|x|^{4+\alpha}\right)\Bigg\}, \\
\left|w_n^T(0,x)\right| \leqslant C\left(1+|x|^2\right). &
\end{aligned}
$$

Step 3: Variation of $w^{\varepsilon,T}(0,\bullet)$.

We write:

$$\begin{aligned}Y_0^{\varepsilon,T,x,n} - \widetilde{Y}_0^{\varepsilon,T,x,n} = Y_T^{\varepsilon,T,x,n} - \widetilde{Y}_T^{\varepsilon,T,x,n} \\ + \int_0^T \left\{ f^{\varepsilon}_{N_s^n}\left(X_s^{\varepsilon,x,n}, Z_s^{\varepsilon,T,x,n}, H_s^{\varepsilon,T,x,n}\right) - f^{\varepsilon}_{N_s^n}\left(X_s^{\varepsilon,x,n}, \widetilde{Z}_s^{\varepsilon,T,x,n}, \widetilde{H}_s^{\varepsilon,T,x,n}\right)\right\} \mathrm{d}s \\ - \int_0^T \left\{ Z_s^{\varepsilon,T,x,n} - \widetilde{Z}_s^{\varepsilon,T,x,n}\right\} \mathrm{d}W_s - \int_0^T \left\{ H_s^{\varepsilon,T,x,n} - \widetilde{H}_s^{\varepsilon,T,x,n}\right\} \mathrm{d}M_s.\end{aligned}$$

We use Girsanov's theorem exactly the same way as in the step 2 of Theorem 3.16, and we get:

$$w_n^{\varepsilon,T}(0,x) = Y_0^{\varepsilon,T,x,n} - \widetilde{Y}_0^{\varepsilon,T,x,n} = \mathbb{E}^{\mathbb{Q}^x}\left[Y_T^{\varepsilon,T,x,n} - \widetilde{Y}_T^{\varepsilon,T,x,n}\right] = \mathbb{E}^{\mathbb{Q}^x}\left[g^{\varepsilon}_{N_T^n}\left(X_T^{\varepsilon,x,n}\right) - v^{\varepsilon}_{N_T^n}\left(X_T^{\varepsilon,x,n}\right)\right].$$

By Lemma 3.11 and corollary 3.6, we get:

$$\begin{aligned}\left|w_n^{\varepsilon,T}(0,x) - w_n^{\varepsilon,T}(0,y)\right| &= \left|\mathbb{E}^{\mathbb{Q}^x}\left[(g^{\varepsilon} - v^{\varepsilon})_{N_T^n}\left(X_T^{\varepsilon,x,n}\right)\right] - \mathbb{E}^{\mathbb{Q}^y}\left[(g^{\varepsilon} - v^{\varepsilon})_{N_T^n}\left(X_T^{\varepsilon,y,n}\right)\right]\right| \\ &\leqslant \widehat{c}\left(1 + |x|^2 + |y|^2\right) \mathrm{e}^{-\widehat{\nu} T},\end{aligned}$$

where $\widehat{c}$ and $\widehat{\nu}$ are independent of ε. So:

$$\exists \widehat{c}, \widehat{\nu} > 0,\ \forall T > 0,\ \forall x, y \in \mathbb{R}^d,\ \forall n \in \mathbb{K},\ \forall \varepsilon > 0,\ \left|w_n^{\varepsilon,T}(0,x) - w_n^{\varepsilon,T}(0,y)\right| \leqslant \widehat{c}\left(1 + |x|^2 + |y|^2\right) \mathrm{e}^{-\widehat{\nu} T}. \tag{3.16}$$

Step 4: Upperbound for $\partial_x w^{\varepsilon,T}(0,\bullet)$.

We use the Bismut-Elworthy formula (see [59]); for every $T' \in (t,T]$, we have:

$$\partial_x u_n^{\varepsilon,T}(t,x) = \mathbb{E}\left[u^{\varepsilon,T}_{N^{t,n}_{T'}}\left(T', X_{T'}^{\varepsilon,t,x,n}\right) R_{T'}^{\varepsilon,t,x,n} + \int_t^{T'} f^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,t,x,n}, Z_s^{\varepsilon,T,t,x,n}, H_s^{\varepsilon,T,t,x,n}\right) R_s^{\varepsilon,t,x,n}\, \mathrm{d}s\right],$$

$$\partial_x \widetilde{u}_n^{\varepsilon,T}(t,x) = \mathbb{E}\left[\widetilde{u}^{\varepsilon,T}_{N^{t,n}_{T'}}\left(T', X_{T'}^{\varepsilon,t,x,n}\right) R_{T'}^{\varepsilon,t,x,n} + \int_t^{T'} f^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,t,x,n}, \widetilde{Z}_s^{\varepsilon,T,t,x,n}, \widetilde{H}_s^{\varepsilon,T,t,x,n}\right) R_s^{\varepsilon,t,x,n}\, \mathrm{d}s\right],$$

where $R_s^{\varepsilon,t,x,n} = \dfrac{1}{s-t}\left(\displaystyle\int_t^s \left(\sigma^{\varepsilon}_{N_r^{t,n}}\left(X_r^{\varepsilon,t,x,n}\right)^{-1} \nabla X_r^{\varepsilon,t,x,n}\right)^* \mathrm{d}W_r\right)^*$. By subtraction:

$$\begin{aligned}&\partial_x w_n^{\varepsilon,T}(t,x) \\ &\quad = \mathbb{E}\Bigg[w^{\varepsilon,T-T'}_{N^{t,n}_{T'}}\left(0, X_{T'}^{\varepsilon,t,x,n}\right) R_{T'}^{\varepsilon,t,x,n} \\ &\qquad + \int_t^{T'}\left\{f^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,t,x,n}, Z_s^{\varepsilon,T,t,x,n}, H_s^{\varepsilon,T,t,x,n}\right) - f^{\varepsilon}_{N_s^{t,n}}\left(X_s^{\varepsilon,t,x,n}, \widetilde{Z}_s^{\varepsilon,T,t,x,n}, \widetilde{H}_s^{\varepsilon,T,t,x,n}\right)\right\} R_s^{\varepsilon,t,x,n}\, \mathrm{d}s\Bigg].\end{aligned}$$

Using Itô's formula, we can prove that $\mathbb{E}\left[\left|\nabla X_s^{\varepsilon,t,x,n}\right|_F^2\right] \leqslant d^2 \mathrm{e}^{\kappa(s-t)}$, where $\kappa = 2\|b\|_{\mathrm{lip}} + \|\sigma\|_{\mathrm{lip}}^2$. Then, for any $s \in [t,T']$,

$$\mathbb{E}\left[\left|R_s^{\varepsilon,t,x,n}\right|^2\right] \leqslant \left(\frac{1}{s-t}\right)^2 \mathbb{E}\left[\int_t^s \left|\sigma^{\varepsilon}_{N_r^{t,n}}\left(X_r^{\varepsilon,t,x,n}\right)^{-1} \nabla X_r^{\varepsilon,t,x,n}\right|_F^2 \mathrm{d}r\right] \leqslant \frac{d^2 \left\|\sigma^{-1}\right\|_\infty^2}{s-t} \mathrm{e}^{\kappa(T'-t)}.$$

Recalling the Lipschitz continuity of f, we have:

$$\begin{aligned}&\left|f^{\varepsilon}_{N_s^{t,n}}(X_s^{\varepsilon,t,x,n}, Z_s^{\varepsilon,T,t,x,n}, H_s^{\varepsilon,T,t,x,n}) - f^{\varepsilon}_{N_s^{t,n}}(X_s^{\varepsilon,t,x,n}, \widetilde{Z}_s^{\varepsilon,T,t,x,n}, \widetilde{H}_s^{\varepsilon,T,t,x,n})\right| \\ &\quad \leqslant \|\rho\|_{\mathrm{lip},p}\left|Z_s^{\varepsilon,T,t,x,n}\sigma^{\varepsilon}_{N_s^{t,n}}(X_s^{\varepsilon,t,x,n})^{-1} - \widetilde{Z}_s^{\varepsilon,T,t,x,n}\sigma^{\varepsilon}_{N_s^{t,n}}(X_s^{\varepsilon,t,x,n})^{-1}\right| + \|f\|_{\mathrm{lip},h}\left|H_s^{\varepsilon,T,t,x,n} - \widetilde{H}_s^{\varepsilon,T,t,x,n}\right| \\ &\quad \leqslant \|\rho\|_{\mathrm{lip},p}\left|\partial_x w^{\varepsilon,T}_{N_s^{t,n}}(s, X_s^{\varepsilon,t,x,n})\right| + \|f\|_{\mathrm{lip},h}\left(\sum_{l\in\mathbb{L}}\left|w^{\varepsilon,T}_{N^{t,n}_{s^-}+l}(s, X_s^{\varepsilon,t,x,n}) - w^{\varepsilon,T}_{N^{t,n}_{s^-}}(s, X_s^{\varepsilon,t,x,n})\right|^2\right)^{1/2}.\end{aligned}$$

Using Cauchy-Schwarz inequality, we find:

$$
\begin{aligned}
&\left|\partial_x w_n^{\varepsilon,T}(t,x)\right| \\
&\quad\leqslant \mathbb{E}\left[\left|w^{\varepsilon,T-T'}_{N^{t,n}_{T'}}\left(0,X^{\varepsilon,t,x,n}_{T'}\right)\right|^2\right]^{1/2}\frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}}\mathrm{e}^{\kappa T'/2} \\
&\qquad+\|\rho\|_{\mathrm{lip},p}d\left\|\sigma^{-1}\right\|_\infty\mathrm{e}^{\kappa T'/2}\int_t^{T'}\mathbb{E}\left[\left|\partial_x w^{\varepsilon,T}_{N^{t,n}_s}\left(s,X^{\varepsilon,t,x,n}_s\right)\right|^2\right]^{1/2}\frac{\mathrm{d}s}{\sqrt{s-t}} \\
&\qquad+\|f\|_{\mathrm{lip},h}d\left\|\sigma^{-1}\right\|_\infty\mathrm{e}^{\kappa T'/2}\int_t^{T'}\mathbb{E}\left[\sum_{l\in\mathbb{L}}\left|w^{\varepsilon,T}_{N^{t,n}_{s^-}+l}(s,X^{\varepsilon,t,x,n}_s)-w^{\varepsilon,T}_{N^{t,n}_{s^-}}(s,X^{\varepsilon,t,x,n}_s)\right|^2\right]^{1/2}\frac{\mathrm{d}s}{\sqrt{s-t}}.
\end{aligned}
$$

Thanks to step 2, we have the following upperbound (for any $\alpha\in(0,2]$):

$$
\begin{aligned}
&\mathbb{E}\left[\left|w_l^{\varepsilon,T-s}\left(0,X^{\varepsilon,t,x,n}_s\right)\right|^2\right] \\
&\quad\leqslant C\left(1+|x|^4\right)+C_{T-s}\left(\Delta_t^{\varepsilon,x,s,n,l}+\|\rho-\rho^\varepsilon\|_\infty^2+\left(\|b-b^\varepsilon\|_\infty^2+\|\sigma-\sigma^\varepsilon\|_\infty^2+\varepsilon^{\alpha/2}\right)\left(1+|x|^{4+\alpha}\right)\right),
\end{aligned}
$$

where

$$
\Delta_t^{\varepsilon,x,s,n,l}=\mathbb{E}\left[\left|v_{N^l_{T-s}}\left(X^{X^{\varepsilon,t,x,n}_s,l}_{T-s}\right)-v^\varepsilon_{N^l_{T-s}}\left(X^{\varepsilon,X^{\varepsilon,t,x,n}_s,l}_{T-s}\right)\right|^2+\left|g_{N^l_{T-s}}\left(X^{X^{\varepsilon,t,x,n}_s,l}_{T-s}\right)-g^\varepsilon_{N^l_{T-s}}\left(X^{\varepsilon,X^{\varepsilon,t,x,n}_s,l}_{T-s}\right)\right|^2\right].
$$

So, we obtain,

$$
\begin{aligned}
&\left|\partial_x w_n^{\varepsilon,T}(t,x)\right| \\
&\quad\leqslant\Bigg\{C\left(1+|x|^2\right)+C_{T-T'}\Bigg(\sqrt{\Delta_t^{\varepsilon,T',x,n,N^{t,n}_{T'}}}+\|\rho-\rho^\varepsilon\|_\infty \\
&\qquad\qquad+\left(\|b-b^\varepsilon\|_\infty+\|\sigma-\sigma^\varepsilon\|_\infty+\varepsilon^{\alpha/4}\right)\left(1+|x|^{2+\frac{\alpha}{2}}\right)\Bigg)\Bigg\}\frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}}\mathrm{e}^{\kappa T'/2} \\
&\qquad+\|\rho\|_{\mathrm{lip},p}d\|\sigma^{-1}\|_\infty\mathrm{e}^{\kappa T'/2}\int_t^{T'}\mathbb{E}\left[\left|\partial_x w^{\varepsilon,T}_{N^{t,n}_s}\left(s,X^{\varepsilon,t,x,n}_s\right)\right|^2\right]^{1/2}\frac{\mathrm{d}s}{\sqrt{s-t}} \\
&\qquad+Ck^{3/2}\|f\|_{\mathrm{lip},h}d\|\sigma^{-1}\|_\infty\mathrm{e}^{\kappa T'/2}(1+|x|^2)\sqrt{T'} \\
&\qquad+C_{T-t}\sqrt{k}\|f\|_{\mathrm{lip},h}d\|\sigma^{-1}\|_\infty\mathrm{e}^{\kappa T'/2}\sqrt{T'} \\
&\qquad\quad\times\left\{\sum_{l\in\mathbb{K}}\sqrt{\Delta_t^{\varepsilon,T',x,n,l}}+k\left(\|\rho-\rho^\varepsilon\|_\infty+\left(\|b-b^\varepsilon\|_\infty+\|\sigma-\sigma^\varepsilon\|_\infty+\varepsilon^{\alpha/4}\right)\left(1+|x|^{2+\frac{\alpha}{2}}\right)\right)\right\}.
\end{aligned}
$$

Let us take $\zeta>2$; we set $\varphi^{\varepsilon,T}(t)=\sup_{(x,n)\in\mathbb{R}^d\times\mathbb{K}}\dfrac{\left|\partial_x w_n^{\varepsilon,T}(t,x)\right|}{1+|x|^\zeta}$. Using the appendix 3.7.4, we see that:

$$
\left|\partial_x w_n^{\varepsilon,T}(t,x)\right|\leqslant\left|\partial_x u_n^{\varepsilon,T}(t,x)\right|+\left|\partial_x\widetilde{u}_n^{\varepsilon,T}(t,x)\right|\leqslant C\left(1+|x|^2\right).
$$

This proves that $\varphi^{\varepsilon,T}$ is well defined on $[0,T)$; also, for every $s\in[t,T']$,

$$
\mathbb{E}\left[\left|\partial_x w^{\varepsilon,T}_{N^{t,n}_s}\left(s,X^{\varepsilon,t,x,n}_s\right)\right|^2\right]^{1/2}\leqslant C\left(1+|x|^\zeta\right)\varphi^{\varepsilon,T}(s),
$$

where C is independent of T and x. We get:

$$\begin{aligned}
&\frac{\left|\partial_x w_n^{\varepsilon,T}(t,x)\right|}{1+|x|^\zeta} \\
&\qquad \leqslant \frac{C_{T'}}{\sqrt{T'-t}} + \frac{C_{T,T'}}{\sqrt{T'-t}} \left(\frac{\sqrt{\Delta_t^{\varepsilon,T',x,n,N_{T'}^{t,n}}}}{1+|x|^\zeta} + \|\rho-\rho^\varepsilon\|_\infty + \|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty + \varepsilon^{\alpha/4} \right) \\
&\qquad\quad + C_{T'} \int_t^{T'} \frac{\varphi^{\varepsilon,T}(s)}{\sqrt{s-t}}\, \mathrm{d}s + C_{T'} \\
&\qquad\quad + C_{T-t,T'} \left\{ \sum_{l\in\mathbb{K}} \frac{\sqrt{\Delta_t^{\varepsilon,T',x,n,l}}}{1+|x|^\zeta} + \|\rho-\rho^\varepsilon\|_\infty + \|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty + \varepsilon^{\alpha/4} \right\}.
\end{aligned}$$

When $2+\frac{\alpha}{2}<\zeta$, we can take the supremum for $x\in\mathbb{R}^d$ and $n\in\mathbb{K}$, and by a change of variable, it can be rewritten as:

$$\varphi^{\varepsilon,T}(T'-t) \leqslant a(t) + C_{T'} \int_0^t \frac{\varphi^{\varepsilon,T}(T'-u)}{\sqrt{t-u}}\, \mathrm{d}u,$$

where $0<t\leqslant T'<T$ and

$$\begin{aligned}
a(t) &:= C_{T'}\left(1+\frac{1}{\sqrt{t}}\right) + \left(\frac{C_{T,T'}}{\sqrt{t}} + C_{T-T'+t,T'}\right)\left(\|\rho-\rho^\varepsilon\|_\infty + \|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty + \varepsilon^{\alpha/4}\right) \\
&\quad + \frac{C_{T,T'}}{\sqrt{t}} \sup_{(x,n)\in\mathbb{R}^d\times\mathbb{K}} \frac{\sqrt{\Delta_{T'-t}^{\varepsilon,T',x,n,N_{T'}^{T'-t,n}}}}{1+|x|^\zeta} + C_{T-T'+t,T'} \sum_{l\in\mathbb{K}} \sup_{(x,n)\in\mathbb{R}^d\times\mathbb{K}} \frac{\sqrt{\Delta_{T'-t}^{\varepsilon,T',x,n,l}}}{1+|x|^\zeta}.
\end{aligned}$$

Lemma 7.1.1 of [30] gives us $\varphi^{\varepsilon,T}(T'-t) \leqslant a(t) + C_{T'} \int_0^t E'\left(C_{T'}(t-u)\right) a(u)\, \mathrm{d}u$, and the integral is well defined, since $E(z) = \sum_{n=0}^{\infty} \frac{z^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ is $\mathcal{C}^1$ and $E'(z) \underset{z\to 0^+}{\sim} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{z}}$. Choose $t=T'$ and find:

$$\frac{\left|\partial_x w_n^{\varepsilon,T}(0,x)\right|}{1+|x|^\zeta} \leqslant C_{T'} + C_{T,T'}\gamma_{T'}^{\varepsilon,T}(T') + \int_0^{T'} E'\left(C_{T'}(T'-u)\right)\left\{C_{T'} + C_{T,T'}\gamma_{T'}^{\varepsilon,T}(u)\right\}\left(1+\frac{1}{\sqrt{u}}\right) \mathrm{d}u,$$

where we denoted

$$\begin{aligned}
\gamma_{T'}^{\varepsilon,T}(u) &= \|\rho-\rho^\varepsilon\|_\infty + \|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty + \varepsilon^{\alpha/4} \\
&\quad + \sup_{(x,n)\in\mathbb{R}^d\times\mathbb{K}} \frac{\sqrt{\Delta_{T'-u}^{\varepsilon,T',x,n,N_{T'}^{T'-u,n}}}}{1+|x|^\zeta} + \sum_{l\in\mathbb{K}} \sup_{(x,n)\in\mathbb{R}^d\times\mathbb{K}} \frac{\sqrt{\Delta_{T'-u}^{\varepsilon,T',x,n,l}}}{1+|x|^\zeta}.
\end{aligned}$$

Step 5: Taking the limit when $\varepsilon\to 0$. From Theorem 3.17 and equation (3.16), we get easily:

$$\exists C>0,\ \forall T>0,\ \forall x\in\mathbb{R}^d,\ \forall n\in\mathbb{K},\ \left|w_n^T(0,x)\right| \leqslant C\left(1+|x|^2\right), \tag{3.17}$$

$$\exists \widehat{c},\widehat{\nu}>0,\ \forall T>0,\ \forall x,y\in\mathbb{R}^d,\ \forall n\in\mathbb{K},\ \left|w_n^T(0,x)-w_n^T(0,y)\right| \leqslant \widehat{c}\left(1+|x|^2+|y|^2\right)\mathrm{e}^{-\widehat{\nu}T}. \tag{3.18}$$

The function w_n^T has no reason to be $\mathcal{C}^1$, so we use the mean value theorem:

$$\frac{\left|w_n^{\varepsilon,T}(0,x) - w_n^{\varepsilon,T}(0,y)\right|}{\left(1+|x|^\zeta+|y|^\zeta\right)|x-y|} \leqslant C_{T'} + C_{T,T'}\gamma_{T'}^{\varepsilon,T}(T') + \int_0^{T'} E'\left(C_{T'}(T'-u)\right)\left\{C_{T'} + C_{T,T'}\gamma_{T'}^{\varepsilon,T}(u)\right\}\left(1+\frac{1}{\sqrt{u}}\right)\mathrm{d}u.$$

Our next goal will be to get an upperbound for $\gamma_{T'}^{\varepsilon,T}$. Let $R>0$;

$$\begin{aligned}
&\mathbb{E}\left[\left|(g-g^\varepsilon)_{N^l_{T-T'}}\left(X_{T-T'}^{\varepsilon,X_{T'}^{\varepsilon,t,x,n},l}\right)\right|^2 \mathbb{1}_{\left|X_{T-T'}^{\varepsilon,X_{T'}^{\varepsilon,t,x,n},l}\right|>R}\right] \\
&\qquad\leqslant \mathbb{E}\left[\left|(g-g^\varepsilon)_{N^l_{T-T'}}\left(X_{T-T'}^{\varepsilon,X_{T'}^{\varepsilon,t,x,n},l}\right)\right|^4\right]^{1/2}\mathbb{P}\left(\left|X_{T-T'}^{\varepsilon,X_{T'}^{\varepsilon,t,x,n},l}\right|>R\right)^{1/2} \\
&\qquad\leqslant C\mathbb{E}\left[\left(1+\left|X_{T-T'}^{\varepsilon,X_{T'}^{\varepsilon,t,x,n},l}\right|^2\right)^4\right]^{1/2}\frac{\mathbb{E}\left[\left|X_{T-T'}^{\varepsilon,X_{T'}^{\varepsilon,t,x,n},l}\right|^{2(\zeta-2)}\right]^{1/2}}{R^{\zeta-2}} \\
&\qquad\leqslant C\frac{1+|x|^{2+\zeta}}{R^{\zeta-2}}.
\end{aligned}$$

Using that g is locally Lipschitz continuous and Lemma 3.28, we get:

$$\begin{aligned}
&\mathbb{E}\left[\left|g_{N^l_{T-T'}}\left(X_{T-T'}^{X_{T'}^{\varepsilon,t,x,n},l}\right) - g_{N^l_{T-T'}}\left(X_{T-T'}^{\varepsilon,X_{T'}^{\varepsilon,t,x,n},l}\right)\right|^2\right] \\
&\qquad\leqslant C_{T-T',T'}\left(1+|x|^4\right)\left(\|b-b^\varepsilon\|_\infty^2 + \|\sigma-\sigma^\varepsilon\|_\infty^2\right).
\end{aligned}$$

We can do exactly the same with g and g^ε replaced by v and v^ε. This way, for any $R>0$:

$$\frac{\sqrt{\Delta_{T'-u}^{\varepsilon,T',x,n,l}}}{1+|x|^\zeta} \leqslant CR^{1-\frac{\zeta}{2}} + C_{T-T',T'}\left(\|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty\right) + \|g-g^\varepsilon\|_{\infty,\mathcal{B}(0,R)} + \|v-v^\varepsilon\|_{\infty,\mathcal{B}(0,R)}.$$

It leads us to:

$$\begin{aligned}
\gamma_{T'}^{\varepsilon,T}(u) \leqslant C_{T,T'}\Big(&\|\rho-\rho^\varepsilon\|_\infty + \|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty + \varepsilon^{\alpha/4} + R^{1-\frac{\zeta}{2}} + \|g-g^\varepsilon\|_{\infty,\mathcal{B}(0,R)} \\
&+ \|v-v^\varepsilon\|_{\infty,\mathcal{B}(0,R)}\Big).
\end{aligned}$$

To sum up:

$$\forall\alpha>0,\ \forall T'>0,\ \exists C_{T'}>0,\ \forall T>T',\ \exists C_{T,T'}>0,\ \forall x,y\in\mathbb{R}^d,\ \forall n\in\mathbb{K},\ \forall R>0,\ \forall\varepsilon>0,$$

$$\begin{aligned}
&\left|w_n^{\varepsilon,T}(0,x) - w_n^{\varepsilon,T}(0,y)\right|\left(1+|x|^{2+\alpha}+|y|^{2+\alpha}\right)^{-1}|x-y|^{-1} \\
&\quad\leqslant C_{T'} + C_{T,T'}\Big(\|\rho-\rho^\varepsilon\|_\infty + \|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty + \varepsilon^{\alpha/4} + R^{1-\frac{\zeta}{2}} + \|g-g^\varepsilon\|_{\infty,\mathcal{B}(0,R)} \\
&\qquad\qquad + \|v-v^\varepsilon\|_{\infty,\mathcal{B}(0,R)}\Big) \\
&\quad + \int_0^{T'}\Big\{C_{T'} + C_{T,T'}\Big(\|\rho-\rho^\varepsilon\|_\infty + \|b-b^\varepsilon\|_\infty + \|\sigma-\sigma^\varepsilon\|_\infty + \varepsilon^{\alpha/4} + R^{1-\frac{\zeta}{2}} \\
&\qquad\qquad + \|g-g^\varepsilon\|_{\infty,\mathcal{B}(0,R)} + \|v-v^\varepsilon\|_{\infty,\mathcal{B}(0,R)}\Big)\Big\}E'\left(C_{T'}(T'-u)\right)\left(1+\frac{1}{\sqrt{u}}\right)\mathrm{d}u.
\end{aligned}$$

We take the limit as $\varepsilon \to 0$ and then $R \to \infty$: for every $\alpha > 0$ and $T' > 0$,

$$\exists \mathcal{C}_{T'} > 0,\ \forall T > T',\ \forall x, y \in \mathbb{R}^d,\ \forall n \in \mathbb{K},\ \left| w_n^T(0,x) - w_n^T(0,y) \right| \leqslant \mathcal{C}_{T'} \left(1 + |x|^{2+\alpha} + |y|^{2+\alpha} \right) |x-y|.$$

Step 6: Convergence of $w^T(0,\bullet)$ when $T \to \infty$.

Thanks to equation (3.17), by a diagonal argument, there exist an increasing sequence $(T_i)_{i\in\mathbb{N}}$, with limit $+\infty$; some functions $w_n : D \to \mathbb{R}$, with $D \subset \mathbb{R}^d$ countable and dense, such that the sequences $(w_n^{T_i}(0,\bullet))_i$ converge pointwise over D to the functions w_n. Equation (3.18), with $x, y \in D$, tells us that the functions w_n are equal to the same constant L on D. The same equation, with $x \in \mathbb{R}^d$ and $y \in D$, gives us that $w_n = L$ on $\mathbb{R}^d$, for every $n \in \mathbb{K}$. Let K be a compact subset of $\mathbb{R}^d$; we define $\mathcal{A} = \left\{ \left(w_1^T(0,\bullet), \ldots, w_k^T(0,\bullet) \right) |_K \middle| T > 1 \right\} \subset \mathcal{C}(K, \mathbb{R}^{\mathbb{K}})$; it is equicontinuous. For all $x \in K$, the set $\mathcal{A}(x) = \left\{ \left(w_1^T(0,x), \ldots, w_k^T(0,x) \right) \middle| T > 1 \right\}$ is bounded (see equation (3.17). By Ascoli's theorem, $\mathcal{A}$ is relatively compact in $\mathcal{C}(K, \mathbb{R}^{\mathbb{K}})$. Let $x \in K$ and $n \in \mathbb{K}$,

$$\left| w_n^{T_i}(0,x) - L \right| \leqslant \left| w_n^{T_i}(0,x) - w_n^{T_i}(0,0) \right| + \left| w_n^{T_i}(0,0) - L \right| \leqslant \widehat{c} \left(1 + |x|^2 \right) \mathrm{e}^{-\widehat{\nu} T_i} + \left| w_{T_i}(0,0) - L \right|,$$

which proves that the functions $w_n^{T_i}(0,\bullet)$ converge uniformly to L on K. So $(L, \ldots, L)$ is an accumulation point of $\mathcal{A}$. Since $\mathcal{A}$ is relatively compact, if we prove that $(L, \ldots, L)$ is the unique accumulation point of $\mathcal{A}$, then, we get the uniform convergence of $w_n^T(0,\bullet)$ to L on K. It works for any compact set $K \subset \mathbb{R}^d$, and L does not depend on K. We conclude:

$$\forall x \in \mathbb{R}^d,\ \forall n \in \mathbb{K},\ w_n^T(0,x) \underset{T \to \infty}{\longrightarrow} L.$$

Step 7: $(L, \ldots, L)$ is the only accumulation point of $\mathcal{A}$.

Let $w^{\infty,K}$ be an accumulation point of $\mathcal{A}$, *i.e.* there exists an increasing sequence (T_i') with limit $+\infty$, such that

$$\forall n \in \mathbb{K},,\ \left\| w_n^{T_i'}(0,\bullet)|_K - w_n^{\infty,K} \right\|_\infty \underset{i \to \infty}{\longrightarrow} 0.$$

Again, Equation (3.18) tells us that $w^{\infty,K}$ is equal to a constant $(L_K, \ldots, L_K)$. Our goal is now to show that $L = L_K$. We consider:

$$\begin{aligned}
w_n^{\varepsilon,T+S}(0,x) &= Y_0^{\varepsilon,T+S,x,n} - \widetilde{Y}_0^{\varepsilon,T+S,x,n} \\
&= Y_S^{\varepsilon,T+S,x,n} - \widetilde{Y}_S^{\varepsilon,T+S,x,n} \\
&\quad + \int_0^S \left\{ f^\varepsilon_{N_r^n} \left(X_r^{\varepsilon,x,n}, Z_r^{\varepsilon,T+S,x,n}, H_r^{\varepsilon,T+S,x,n} \right) - f^\varepsilon_{N_r^n} \left(X_r^{\varepsilon,x,n}, \widetilde{Z}_r^{\varepsilon,T+S,x,n}, \widetilde{H}_r^{\varepsilon,T+S,x,n} \right) \right\} \mathrm{d}r \\
&\quad - \int_0^S \left\{ Z_r^{\varepsilon,T+S,x,n} - \widetilde{Z}_r^{\varepsilon,T+S,x,n} \right\} \mathrm{d}W_r - \int_0^S \left\{ H_r^{\varepsilon,T+S,x,n} - \widetilde{H}_r^{\varepsilon,T+S,x,n} \right\} \mathrm{d}M_r.
\end{aligned}$$

We define the function:

$$\Gamma_{n,m}^{\varepsilon,T,S}(t,x) = \begin{cases} \dfrac{f_n^\varepsilon \left(x, \partial_x u_n^{\varepsilon,T+S}(t,x) \sigma_n^\varepsilon(x), \underline{u}_m^{\varepsilon,T+S}(t,x) \right) - f_n^\varepsilon \left(x, \partial_x \widetilde{u}_n^{\varepsilon,T+S}(t,x) \sigma_n^\varepsilon(x), \underline{u}_m^{\varepsilon,T+S}(t,x) \right)}{\left| \partial_x w_n^{\varepsilon,T+S}(t,x) \sigma_n^\varepsilon(x) \right|^2} & \\
\qquad \times \left(\partial_x w_n^{\varepsilon,T+S}(t,x) \sigma_n^\varepsilon(x) \right)^*, & \text{if } t < S \text{ and } \partial_x w_n^{\varepsilon,T+S}(t,x) \neq 0, \\
0, & \text{if } t \leqslant S \text{ and } \partial_x w_n^{\varepsilon,T+S}(t,x) = 0, \\
\Gamma_{n,m}^{\varepsilon,T,S}(S,x), & \text{if } t > S,
\end{cases}$$

where $\underline{u}_m^{\varepsilon,T+S}(t,x) = \left(u_{m+l}^{\varepsilon,T+S}(t,x) - u_m^{\varepsilon,T+S}(t,x) \right)_{l \in \mathbb{L}}$. Using Girsanov's theorem, there exists a probability measure $\mathbb{Q}$ under which we can write:

- $\widetilde{W}_t = W_t - \int_0^t \Gamma^{\varepsilon,T,S}_{N^n_r,N^n_{r-}}\left(r, X^{\varepsilon,x,n}_r\right)\,\mathrm{d}r$ is a $\mathbb{Q}$-Brownian motion over $[0,S]$;
- $w^{\varepsilon,T+S}_n(0,x) = \mathbb{E}^{\mathbb{Q}}\left[Y^{\varepsilon,T+S,x,n}_S - \widetilde{Y}^{\varepsilon,T+S,x,n}_S\right]$.

This way, we can write:

$$w^{\varepsilon,T+S}_n(0,x) = \mathbb{E}^{\mathbb{Q}}\left[w^{\varepsilon,T+S}_{N^n_S}\left(S, X^{\varepsilon,x,n}_S\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[w^{\varepsilon,T}_{N^n_S}\left(0, X^{\varepsilon,x,n}_S\right)\right] = \mathcal{P}^{\varepsilon}_S\left[w^{\varepsilon,T}_{\bullet}(0,\bullet)\right](x,n),$$

where $\mathcal{P}^{\varepsilon}$ is the Kolmogorov semigroup of the SDE:

$$U^{x,n}_t = b^{\varepsilon}_{N^n_t}\left(U^{x,n}_t\right)\mathrm{d}t + \sigma^{\varepsilon}_{N^n_t}\left(U^{x,n}_t\right)\left(\mathrm{d}W_t + \Gamma^{\varepsilon,T,S}_{N^n_t,N^n_{t-}}\left(t, U^{x,n}_t\right)\mathrm{d}t\right).$$

Without any loss of generality, we can suppose that $T_i > T'_i$, for every $i \in \mathbb{N}$. We take $T = T'_i$ and $S = T_i - T'_i$. This way:

$$\forall x \in \mathbb{R}^d,\ \forall n \in \mathbb{K},\ \mathcal{P}^{\varepsilon}_{T_i-T'_i}\left[w^{\varepsilon,T'_i}_{\bullet}(0,\bullet)\right](x,n) = w^{\varepsilon,T_i}_n(0,x) \xrightarrow[\varepsilon\to 0]{} w^{T_i}_n(0,x) \xrightarrow[i\to\infty]{} L.$$

Now, to prove that $L = L_K$, we only need to show that

$$\lim_{i\to\infty}\limsup_{\varepsilon\to 0}\mathcal{P}^{\varepsilon}_{T_i-T'_i}\left[w^{\varepsilon,T'_i}_{\bullet}(0,\bullet)\right](x,n) = L_K.$$

We have:

$$\begin{aligned}
&\left|\mathcal{P}^{\varepsilon}_{T_i-T'_i}\left[w^{\varepsilon,T'_i}_{\bullet}(0,\bullet)\right](x,n) - L_K\right| \\
&\quad\leqslant \left|\mathcal{P}^{\varepsilon}_{T_i-T'_i}\left[w^{\varepsilon,T'_i}_{\bullet}(0,\bullet)\right](x,n) - \mathbb{E}\left[w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}(0,x)\right]\right| + \left|\mathbb{E}\left[w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}(0,x)\right] - \mathbb{E}\left[w^{T'_i}_{N^n_{T_i-T'_i}}(0,x)\right]\right| \\
&\qquad + \left|\mathbb{E}\left[w^{T'_i}_{N^n_{T_i-T'_i}}(0,x)\right] - L_K\right|, \\
&\limsup_{\varepsilon\to 0}\left|\mathcal{P}^{\varepsilon}_{T_i-T'_i}\left[w^{\varepsilon,T'_i}_{\bullet}(0,\bullet)\right](x,n) - L_K\right| \\
&\qquad\leqslant \limsup_{\varepsilon\to 0}\left|\mathcal{P}^{\varepsilon}_{T_i-T'_i}\left[w^{\varepsilon,T'_i}_{\bullet}(0,\bullet)\right](x,n) - \mathbb{E}\left[w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}(0,x)\right]\right| + \left|\mathbb{E}\left[w^{T'_i}_{N^n_{T_i-T'_i}}(0,x)\right] - L_K\right|.
\end{aligned}$$

Thanks to Lemma 3.11, $\Gamma^{\varepsilon,T,S}_{n,m}$ is the limit of $\mathcal{C}^1$-functions, uniformly bounded, with bounded derivatives $\left(\Gamma^{\varepsilon,T,S,p}_{n,m}\right)_{p\geqslant 0}$. We consider the SDE:

$$\mathrm{d}U^{x,n,p}_t = b^{\varepsilon}_{N^n_t}\left(U^{x,n,p}_t\right)\mathrm{d}t + \sigma^{\varepsilon}_{N^n_t}\left(U^{x,n,p}_t\right)\left(\mathrm{d}W_t + \Gamma^{\varepsilon,T,S,p}_{N^n_t,N^n_{t-}}\left(t, U^{x,n,p}_t\right)\mathrm{d}t\right).$$

Thus,

$$\begin{aligned}
&\left|\mathcal{P}^{\varepsilon}_{T_i-T'_i}\left[w^{\varepsilon,T'_i}_{\bullet}(0,\bullet)\right](x) - \mathbb{E}\left[w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}(0,x)\right]\right| \\
&\qquad = \left|\lim_{p\to\infty}\mathbb{E}\left[w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}\left(0, U^{x,n,p}_{T_i-T'_i}\right)\right] - \mathbb{E}\left[w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}(0,x)\right]\right| \\
&\qquad \leqslant \limsup_{p\to\infty}\mathbb{E}\left[\left|w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}\left(0, U^{x,n,p}_{T_i-T'_i}\right) - w^{\varepsilon,T'_i}_{N^n_{T_i-T'_i}}(0,x)\right|\right] \\
&\qquad \leqslant \widehat{c}\left(1 + \limsup_{p\to\infty}\mathbb{E}\left[\left|U^{x,n,p}_{T_i-T'_i}\right|^2\right] + |x|^2\right)\mathrm{e}^{-\widehat{\nu}T'_i},
\end{aligned}$$

and the constants $\widehat{c}$ and $\widehat{\nu}$ are independent of ε, p and T_i'. Thanks to Proposition 3.4, we have $\mathbb{E}\left[\left|U_t^{x,n,p}\right|^\mu\right] \leqslant C\left(1+|x|^2\right)$, with C independent of ε, p and t. So,

$$\left|\mathcal{P}^\varepsilon_{T_i-T_i'}\left[w_\bullet^{\varepsilon,T_i'}(0,\bullet)\right](x) - \mathbb{E}\left[w^{\varepsilon,T_i'}_{N^n_{T_i-T_i'}}(0,x)\right]\right| \leqslant C\left(1+|x|^2\right)\mathrm{e}^{-\nu T_i'},$$

with C and ν independent of ε and T_i'. And we can conclude that $L = L_K$.

Step 8: Speed of convergence.
Let $x\in\mathbb{R}^d$, $n\in\mathbb{K}$ and $T>0$. For any $V\geqslant T$, we set $m(V) = n - N^0_{V-T}$, and we get the equality $N^{m(V)}_{V-T} = m(V) + N^0_{V-T} = n$. We have:

$$\begin{aligned}\left|w_n^T(0,x) - L\right| &= \lim_{V\to\infty}\left|w_n^T(0,x) - \mathbb{E}\left[w^V_{m(V)}(0,x)\right]\right| = \lim_{V\to\infty}\lim_{\varepsilon\to 0}\left|w_n^{\varepsilon,T}(0,x) - \mathbb{E}\left[w^{\varepsilon,V}_{m(V)}(0,x)\right]\right| \\ &= \lim_{V\to\infty}\lim_{\varepsilon\to 0}\left|w_n^{\varepsilon,T}(0,x) - \mathbb{E}\left[w^{\varepsilon,T}_{N^{m(V)}_{V-T}}(0,U^{x,m(V)}_{V-T})\right]\right|,\end{aligned}$$

where $\mathrm{d}U_t^{x,n} = b^\varepsilon_{N^n_t}\left(U_t^{x,n}\right)\mathrm{d}t + \sigma^\varepsilon_{N^n_t}\left(U_t^{x,n}\right)\left(\mathrm{d}W_t + \Gamma^{\varepsilon,T,V-T}_{N^n_t,N^n_{t-}}\left(t,U_t^{x,n}\right)\mathrm{d}t\right)$. Like before, we use Lemma 3.11 and proposition 3.4, and we get:

$$\begin{aligned}|w_T(0,x) - L| &= \lim_{V\to\infty}\lim_{\varepsilon\to 0}\left|w_n^{\varepsilon,T}(0,x) - \lim_{p\to\infty}\mathbb{E}\left[w_n^{\varepsilon,T}\left(0,U^{x,m(V),p}_{V-T}\right)\right]\right| \\ &\leqslant \limsup_{V\to\infty}\limsup_{\varepsilon\to 0}\limsup_{p\to\infty}\mathbb{E}\left[\left|w_n^{\varepsilon,T}(0,x) - w_n^{\varepsilon,T}\left(0,U^{x,m(V),p}_{V-T}\right)\right|\right] \\ &\leqslant C\left(1+|x|^2\right)\mathrm{e}^{-\nu T}.\end{aligned}$$

□

3.6 Some applications

3.6.1 Optimal ergodic control problem

We consider in this subsection a process $X^{x,n}$ satisfying the SDE

$$X_t^{x,n} = x + \int_0^t b_{N^n_s}(X_s^{x,n})\,\mathrm{d}s + \int_0^t \sigma_{N^n_s}(X_s^{x,n})\,\mathrm{d}W_s, \quad t\geqslant 0,$$

under the following assumptions, for all $i\in\mathbb{K}$:

- b_i and σ_i are Lipschitz continuous, with $|\sigma_i(x)|_F^2 \leqslant r_1 + r_2|x|^2$;
- b_i is weakly dissipative, with $\langle b_l(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$;
- $x\mapsto \sigma_l(x)^{-1}$ is bounded.

Let U be a separable metric space; we call "control" any progressively measurable U-valued process. For every $i\in\mathbb{K}$, let us take:

- $R_i : U \to (\mathbb{R}^d)^*$ a bounded function, with $\sqrt{r_2}\|R\|_\infty\|\sigma^{-1}\|_\infty + \dfrac{r_2}{2} < \eta_2$;
- $L_i : \mathbb{R}^d\times U\to\mathbb{R}$ such that: $\exists C>0$, $\forall a\in U$, $\forall x,x'\in\mathbb{R}^d$, $|L_i(x,a) - L_i(x',a)| \leqslant C|x-x'|$ and $|L_i(x,a)| \leqslant C(1+|x|)$;

- $g_i : \mathbb{R}^d \to \mathbb{R}$ satisfying: $\exists C > 0$, $\forall x, x' \in \mathbb{R}^d$, $|g_i(x) - g_i(x')| \leqslant C(1 + |x|^2 + |x'|^2)|x - x'|$ and $|g_i(x)| \leqslant C(1 + |x|^2)$.

For any control a and horizon $T > 0$, we set $\mathbb{P}_T^{x,n,a} = \rho_T^{x,n,a}\mathbb{P}$ over $\mathcal{F}_T$, where

$$\rho_T^{x,n,a} = \exp\left(\int_0^T \sigma_{N_t^n}(X_t^{x,n})^{-1} R_{N_t^n}(a_t)\,\mathrm{d}W_t - \frac{1}{2}\int_0^T \left|\sigma_{N_t^n}(X_t^{x,n})^{-1} R_{N_t^n}(a_t)\right|^2 \mathrm{d}t\right).$$

We define the finite horizon cost as

$$J^T(x,n,a) = \mathbb{E}_T^{x,n,a}\left[\int_0^T L_{N_t^n}(X_t^{x,n}, a_t)\,\mathrm{d}t\right] + \mathbb{E}_T^{x,n,a}\left[g_{N_T^n}(X_T^{x,n})\right],$$

and the associated problem is to minimise $J^T(x,n,a)$ over the set of controls $a : \Omega \times [0,T] \to U$. We also define the ergodic cost as

$$J(x,n,a) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_T^{x,n,a}\left[\int_0^T L_{N_t^n}(X_t^{x,n}, a_t)\,\mathrm{d}t\right]$$

and the goal is to minimise $J(x,n,a)$ over the set of controls $a : \Omega \times \mathbb{R}_+ \to U$. We define

$$W_t^{x,n,a} = W_t - \int_0^t \sigma_{N_s^n}(X_s^{x,n})^{-1} R_{N_s^n}(a_s)\,\mathrm{d}s;$$

due to Girsanov's theorem, $W^{x,n,a}$ is a Brownian motion under $\mathbb{P}_T^{x,n,a}$ on $[0,T]$, for every $T > 0$. We define the Hamiltonian, for any $i \in \mathbb{K}$, $x \in \mathbb{R}^d$ and $z \in (\mathbb{R}^d)^*$:

$$f_i(x,z) = \inf_{a\in U}\left\{L_i(x,a) + z\sigma_i(x)^{-1}R_i(a)\right\}.$$

We recall that, if this infimum is attained for every x and z, by Filippov's theorem (see [45]), there exists a measurable function γ_i such that $f_i(x,z) = L_i(x, \gamma_i(x,z)) + z\sigma_i(x)^{-1}R_i(\gamma_i(x,z))$.

Lemma 3.21. *The Hamiltonian f_i satisfies, for every $i \in \mathbb{K}$:*

- $\forall x \in \mathbb{R}^d$, $|f_i(x,0)| \leqslant C(1 + |x|)$;
- $\forall x, x' \in \mathbb{R}^d$, $\forall z, z' \in (\mathbb{R}^d)^*$, $|f_i(x,z) - f_i(x',z')| \leqslant C|x - x'| + \|R\|_\infty |z\sigma_i(x)^{-1} - z'\sigma_i(x')^{-1}|$.

Proposition 3.22 (Optimal control in finite horizon)**.** *For every control a, we have $J^T(x,n,a) \geqslant Y_0^{T,x,n}$ where $Y^{T,x,n}$ comes from the finite horizon BSDE:*

$$Y_t^{T,x,n} = g_{N_T^n}(X_T^{x,n}) + \int_t^T f_{N_s^n}(X_s^{x,n}, Z_s^{T,x,n})\,\mathrm{d}s - \int_t^T Z_s^{T,x,n}\,\mathrm{d}W_s - \int_t^T H_s^{T,x,n}\,\mathrm{d}M_s, \quad t \in [0,T].$$

Moreover, under Filippov's condition, we have $J^T(x,n,\bar{a}^T) = Y_0^{T,x,n}$, for $\bar{a}_t^T = \gamma_{N_t^n}(X_t^{x,n}, Z_t^{T,x,n})$.

Proof. Under the probability measure $\mathbb{P}_T^{x,n,a}$, we can write:

$$Y_0^{T,x,n} = \mathbb{E}_T^{x,n,a}\left[g_{N_T^n}(X_T^{x,n})\right] + \mathbb{E}_T^{x,n,a}\left[\int_0^T \left\{f_{N_s^n}(X_s^{x,n}, Z_s^{T,x,n}) - Z_s^{T,x,n}\sigma_{N_s^n}(X_s^{x,n})^{-1}R_{N_s^n}(a_s)\right\}\mathrm{d}s\right].$$

And then:

$$\begin{aligned} J^T(x,n,a) &= Y_0^{T,x,n} + \mathbb{E}_T^{x,n,a}\left[\int_0^T \left\{L_{N_s^n}(X_s^{x,n}, a_s) + Z_s^{T,x,n}\sigma_{N_s^n}(X_s^{x,n})^{-1}R_{N_s^n}(a_s) - f_{N_s^n}(X_s^{x,n}, Z_s^{T,x,n})\right\}\mathrm{d}s\right] \\ &\geqslant Y_0^{T,x,n}, \end{aligned}$$

thanks to the definition of f. We see that we even have an equality if we take $a = \bar{a}^T$. □

Proposition 3.23 (Optimal ergodic control)**.** *For every control a, we have $J(x,n,a) \geqslant \lambda$ where λ comes from the ergodic BSDE:*

$$Y_t^{x,n} = Y_T^{x,n} + \int_t^T \left\{ f_{N_s^n}(X_s^{x,n}, Z_s^{x,n}) - \lambda \right\} \mathrm{d}s - \int_t^T Z_s^{x,n}\,\mathrm{d}W_s - \int_t^T H_s^{x,n}\,\mathrm{d}M_s, \quad 0 \leqslant t \leqslant T < \infty.$$

Moreover, under Filippov's condition, we have $J(x,n,\overline{a}) = \lambda$, for $\overline{a}_t = \gamma_{N_t^n}(X_t^{x,n}, Z_t^{x,n})$.

Proof. Under the probability measure $\mathbb{P}_T^{x,n,a}$, we can write:

$$Y_0^{x,n} = \mathbb{E}_T^{x,n,a}\left[Y_T^{x,n}\right] + \mathbb{E}_T^{x,n,a}\left[\int_0^T \left\{ f_{N_s^n}(X_s^{x,n}, Z_s^{x,n}) - Z_s^{x,n}\sigma_{N_s^n}(X_s^{x,n})^{-1}R_{N_s^n}(a_s)\right\} \mathrm{d}s\right] - \lambda T.$$

And then:

$$\begin{aligned} J(x,n,a) &= \lambda + \limsup_{T\to\infty} \frac{Y_0^{x,n} - \mathbb{E}_T^{x,n,a}\left[Y_T^{x,n}\right]}{T} \\ &\quad + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_T^{x,n,a}\left[\int_0^T \left\{ L_{N_s^n}(X_s^{x,n}, a_s) + Z_s^{x,n}\sigma_{N_s^n}(X_s^{x,n})^{-1}R_{N_s^n}(a_s) - f_{N_s^n}(X_s^{x,n}, Z_s^{x,n})\right\} \mathrm{d}s\right] \\ &\geqslant \lambda, \end{aligned}$$

because we can apply Proposition 3.4, and thanks to the definition of f. We see that we even have an equality if we take $a = \overline{a}$. □

Theorem 3.24 (Large time behaviour for optimal control problem)**.** *For every control a, we have:*

$$\liminf_{T\to\infty} \frac{J^T(x,n,a)}{T} \geqslant \lambda.$$

Moreover, under Filippov's condition, we have:

$$\left| J^T(x,n,\overline{a}^T) - J(x,n,\overline{a})T - Y_0^{x,n} - L \right| \leqslant C(1+|x|^2)\mathrm{e}^{-\nu T}.$$

Proof. This is a straightforward consequence of the previours results of this subsection and of Theorem 3.18. □

3.6.2 Large time behaviour of viscosity solution of a system of coupled HJB equations

In this subsection, we keep working under Assumption 3.7. If we consider the finite horizon BSDE: for every $s \in [t,T]$,

$$Y_s^{T,t,x,n} = g_{N_T^{t,n}}(X_T^{t,x,n}) + \int_s^T f_{N_r^{t,n}}(X_r^{t,x,n}, Z_r^{T,t,x,n}, H_r^{T,t,x,n})\,\mathrm{d}r - \int_s^T Z_r^{T,t,x,n}\,\mathrm{d}W_r - \int_s^T H_r^{T,t,x,n}\,\mathrm{d}M_r,$$

and if we set $u_n^T(t,x) = Y_t^{T,t,x,n}$, then u^T is the unique viscosity solution of the system of the following PDEs (see [53]): for every $i \in \mathbb{K}$,

$$\begin{cases} \partial_t u_i(t,x) + \mathcal{L}^i u_i(t,x) + f_i\left(x, \partial_x u_i(t,x)\sigma_i(x), (u_{i+l}(t,x) - u_i(t,x))_{l\in\mathbb{L}}\right) = 0, & (t,x) \in [0,T]\times\mathbb{R}^d, \\ u_i(T,x) = g_i(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.19}$$

satisfying

$$\exists A > 0,\ \forall t \in [0,T],\ \lim_{|x|\to\infty} |u(t,x)|\mathrm{e}^{-A(\ln|x|)^2} = 0.$$

We recall that $\mathcal{L}^i$, for any $i \in \mathbb{K}$, stands for the generator of the Kolmogorov semigroup of the SDE: $\mathrm{d}U_t^{x,i} = b_i(U_t^{x,i})\,\mathrm{d}t + \sigma_i(U_t^{x,i})\,\mathrm{d}W_t$. We also consider the system of ergodic PDEs:

$$\mathcal{L}^i v_i(x) + f_i\left(x, \nabla v_i(x)\sigma_i(x), (v_{i+l}(x) - v_i(x))_{l\in\mathbb{L}}\right) = \lambda, \qquad x \in \mathbb{R}^d,\ i \in \mathbb{K}. \tag{3.20}$$

We recall that the couple (v, λ) is a viscosity subsolution (resp. supersolution) of the system (3.20) if, for all $i \in \mathbb{K}$:

- $v_i : \mathbb{R}^d \to \mathbb{R}$ is a continuous function with polynomial growth;
- for any function $\phi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, for every $x \in \mathbb{R}^d$ of local maximum (resp. minimum) of $v_i - \phi$:

$$\mathcal{L}^i\phi(x) + f_i\left(x, \nabla\phi(x)\sigma_i(x), (v_{i+l}(x) - v_i(x))_{l\in\mathbb{L}}\right) \geqslant \lambda \quad (\text{resp. } \leqslant \lambda).$$

Proposition 3.25 (Existence of ergodic viscosity solution)**.** *The couple (v, λ) obtained with the solution given in Theorem 3.14 is a viscosity solution of the system (3.20).*

Proof. We fix $i \in \mathbb{K}$ and $\phi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$. Let $x_0 \in \mathbb{R}^d$ be a local maximum of $v_i - \phi$. By contradiction, we suppose:

$$\mathcal{L}^i\phi(x_0) + f_i\left(x_0, \nabla\phi(x_0)\sigma_i(x_0), (v_{i+l}(x_0) - v_i(x_0))_{l\in\mathbb{L}}\right) - \lambda =: -\delta < 0.$$

Without any loss of generality, we assume that $v_i(x_0) = \phi(x_0)$. By continuity, there exists $\alpha > 0$, such that

$$\forall x \in \overline{\mathcal{B}_\alpha(x_0)},\ v_i(x) \leqslant \phi(x) \text{ and } \mathcal{L}^i\phi(x) + f_i\left(x, \nabla\phi(x)\sigma_i(x), (v_{i+l}(x) - v_i(x))_{l\in\mathbb{L}}\right) - \lambda =: \psi_i(x) \leqslant -\frac{\delta}{2}.$$

We define a stopping time τ by:

$$\tau = \inf\left\{s \in \mathbb{R}_+ \middle| |X_s^{x_0,i} - x_0| > \alpha\right\} \wedge \inf\left\{s \in \mathbb{R}_+ \middle| N_s^i \neq i\right\} \wedge \alpha.$$

Before the stopping time τ, we can write the EBSDE as:

$$\begin{aligned} Y_{s\wedge\tau}^{x_0,i} &= v_i(X_\tau^{x_0,i}) + \int_{s\wedge\tau}^{\tau} \left\{f_{N_r^i}(X_r^{x_0,i}, Z_r^{x_0,i}, H_r^{x_0,i}) - \lambda\right\} \mathrm{d}r - \int_{s\wedge\tau}^{\tau} Z_r^{x_0,i}\,\mathrm{d}W_r \\ &= v_i(X_\tau^{x_0,i}) + \int_{s\wedge\tau}^{\tau} \left\{f_i\left(X_r^{x_0,i}, Z_r^{x_0,i}, \left(v_{i+l}(X_r^{x_0,i}) - v_i(X_r^{x_0,i})\right)_{l\in\mathbb{L}}\right) - \lambda\right\} \mathrm{d}r - \int_{s\wedge\tau}^{\tau} Z_r^{x_0,i}\,\mathrm{d}W_r. \end{aligned}$$

We set $\widetilde{Y}_s = Y_{s\wedge\tau}^{x_0,i}$ and $\widetilde{Z}_s = Z_s^{x_0,i}\mathbb{1}_{s\leqslant\tau}$. We get, for every $s \in [0, \alpha]$,

$$\widetilde{Y}_s = v_i(X_\tau^{x_0,i}) + \int_s^\alpha \mathbb{1}_{r\leqslant\tau}\left\{f_i\left(X_r^{x_0,i}, \widetilde{Z}_r, \left(v_{i+l}(X_r^{x_0,i}) - v_i(X_r^{x_0,i})\right)_{l\in\mathbb{L}}\right) - \lambda\right\} \mathrm{d}r - \int_s^\alpha \widetilde{Z}_r\,\mathrm{d}W_r.$$

Applying Itô's formula before τ, we get, for every $s \in [0, \alpha]$:

$$\phi(X_{s\wedge\tau}^{x_0,i}) = \phi(X_\tau^{x_0,i}) - \int_{s\wedge\tau}^{\tau} \mathcal{L}^i\phi(X_r^{x_0,i})\,\mathrm{d}r - \int_{s\wedge\tau}^{\tau} \nabla\phi(X_r^{x_0,i})\sigma_i(X_r^{x_0,i})\,\mathrm{d}W_r.$$

We set $\widehat{Y}_s = \phi(X_{s\wedge\tau}^{x_0,i})$ and $\widehat{Z}_s = \nabla\phi(X_s^{x_0,i})\sigma_i(X_s^{x_0,i})\mathbb{1}_{s\leqslant\tau}$. Thus, for every $s \in [0, \alpha]$,

$$\begin{aligned} \widehat{Y}_s &= \phi(X_\tau^{x_0,i}) - \int_s^\alpha \mathbb{1}_{r\leqslant\tau}\mathcal{L}^i\phi(X_r^{x_0,i})\,\mathrm{d}r - \int_s^\alpha \widehat{Z}_r\,\mathrm{d}W_r \\ &= \phi(X_\tau^{x_0,i}) - \int_s^\alpha \widehat{Z}_r\,\mathrm{d}W_r \\ &\quad + \int_s^\alpha \mathbb{1}_{r\leqslant\tau}\left\{f_i\left(X_r^{x_0,i}, \nabla\phi(X_r^{x_0,i})\sigma_i(X_r^{x_0,i}), \left(v_{i+l}(X_r^{x_0,i}) - v_i(X_r^{x_0,i})\right)_{l\in\mathbb{L}}\right) - \lambda - \psi_i(X_r^{x_0,i})\right\} \mathrm{d}r \\ &= \phi(X_\tau^{x_0,i}) + \int_s^\alpha \mathbb{1}_{r\leqslant\tau}\left\{f_i\left(X_r^{x_0,i}, \widehat{Z}_r, \left(v_{i+l}(X_r^{x_0,i}) - v_i(X_r^{x_0,i})\right)_{l\in\mathbb{L}}\right) - \lambda - \psi_i(X_r^{x_0,i})\right\} \mathrm{d}r - \int_s^\alpha \widehat{Z}_r\,\mathrm{d}W_r \end{aligned}$$

Using the comparison theorem (see [51]), we get $\widehat{Y}_0 > \widetilde{Y}_0$, which yields a contradiction. □

Proposition 3.26 (Uniqueness of ergodic viscosity solution)**.** *Uniqueness holds for viscosity solutions (v,λ) of the system (3.20) in the class of viscosity solutions such that $v(0)=0$ and v has quadratic growth.*

Proof. Let (v,λ) and $(\widetilde{v},\widetilde{\lambda})$ be two viscosity solutions of (3.20). We fix $T>0$, and we consider the solution $\left(Y^{T,x,n},Z^{T,x,n},H^{T,x,n}\right)$ of the BSDE: for every $t\in[0,T]$,

$$Y_t^{T,x,n}=v_{N_T^n}\left(X_T^{x,n}\right)+\int_t^T\left\{f_{N_s^n}\left(X_s^{x,n},Z_s^{T,x,n},H_s^{T,x,n}\right)-\lambda\right\}\mathrm{d}s-\int_t^T Z_s^{T,x,n}\,\mathrm{d}W_s-\int_t^T H_s^{T,x,n}\,\mathrm{d}M_s.$$

Using [53], $Y_t^{T,x,n}=u^T_{N_t^n}(t,X_t^{x,n})$, where u^T is the unique viscosity solution of the system of PDEs: for any $i\in\mathbb{K}$,

$$\begin{cases}\partial_t u_i(t,x)+\mathcal{L}^i u_i(t,x)+f_i\left(x,\partial_x u_i(t,x)\sigma_i(x),(u_{i+l}(t,x)-u_i(t,x))_{l\in\mathbb{L}}\right)=\lambda, & (t,x)\in[0,T]\times\mathbb{R}^d,\\ u_i(T,x)=v_i(x), & x\in\mathbb{R}^d.\end{cases}$$

But v is solution of the same system of PDEs, so we can claim that $Y_0^{T,x,n}=v_n(x)$. We define a triple $(\widetilde{Y}^{T,x,n},\widetilde{Z}^{T,x,n},\widetilde{H}^{T,x,n})$ by replacing in the previous equation (v,λ) by $(\widetilde{v},\widetilde{\lambda})$. Then, for any $T>0$ and $(x,n)\in\mathbb{R}^d\times\mathbb{K}$:

$$\begin{aligned}(v-\widetilde{v})_n(x)=(v-\widetilde{v})_{N_T^n}\left(X_T^{x,n}\right)&+\int_0^T\left\{f_{N_s^n}\left(X_s^{x,n},Z_s^{T,x,n},H_s^{T,x,n}\right)-f_{N_s^n}\left(X_s^{x,n},\widetilde{Z}_s^{T,x,n},\widetilde{H}_s^{T,x,n}\right)\right\}\mathrm{d}s\\&-\left(\lambda-\widetilde{\lambda}\right)T-\int_0^T\left\{Z_s^{T,x,n}-\widetilde{Z}_s^{T,x,n}\right\}\mathrm{d}W_s-\int_0^T\left\{H_s^{T,x,n}-\widetilde{H}_s^{T,x,n}\right\}\mathrm{d}M_s.\end{aligned}$$

By Girsanov's theorem, there exists a new probability measure $\mathbb{Q}^T$ under which we can write:

$$\frac{(v-\widetilde{v})_n(x)}{T}=\frac{\mathbb{E}^{\mathbb{Q}^T}\left[(v-\widetilde{v})_{N_T^n}\left(X_T^{x,n}\right)\right]}{T}-\left(\lambda-\widetilde{\lambda}\right).$$

Thanks to proposition 3.4 and the quadratic growth of v and $\widetilde{v}$, letting $T\to\infty$ gives us $\lambda=\widetilde{\lambda}$. Applying the same argument as that in Theorem 3.16, we deduce that $v=\widetilde{v}$. □

We can formulate Theorem 3.18 as:

Theorem 3.27. *We consider the systems (3.20) and (3.19).*
Then, there exists $L\in\mathbb{R}$, such that: $\forall x\in\mathbb{R}^d$, $u_n^T(0,x)-\lambda T-v_n(x)\underset{T\to\infty}{\longrightarrow}L$. Furthermore,

$$\forall x\in\mathbb{R}^d,\ \forall n\in\mathbb{K},\ \forall T>0,\ \left|u_n^T(0,x)-\lambda T-v_n(x)-L\right|\leqslant C\left(1+|x|^2\right)\mathrm{e}^{-\nu T}.$$

3.7 Appendix

3.7.1 Some standard estimates for SDEs and BSDEs

The following results are well-known in the literature, and can be found for example in [43].

Lemma 3.28. *We consider the SDE:*

$$X_t=x+\int_0^t\{\widetilde{b}(s,X_s)+h_s^0\}\,\mathrm{d}s+\int_0^t\{\widetilde{\sigma}(s,X_s)+h_s^1\}\,\mathrm{d}W_s,$$

with adapted coefficients. We suppose that $\widetilde{b}$ and $\widetilde{\sigma}$ are Lipschitz w.r.t. x and that $\widetilde{b}(t,0)=0$ and $\widetilde{\sigma}(t,0)=0$ a.s. Then, for every $p\geqslant 2$, there exists a constant C_p depending only on T, $\|\widetilde{b}\|_{\mathrm{lip}}$ and $\|\widetilde{\sigma}\|_{\mathrm{lip}}$, such that:

$$\mathbb{E}[|X|_{0,T}^{*,p}]\leqslant C_p\left(|x|^p+\mathbb{E}\left[\int_0^T\{|h_s^0|^p+|h_s^1|^p\}\,\mathrm{d}s\right]\right).$$

Lemma 3.29. *We consider the BSDE:*

$$Y_t = \xi + \int_t^T \{\widetilde{f}(s, Y_s, Z_s, H_s) + h_s\} \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s - \int_t^T H_s \, \mathrm{d}M_s,$$

with an adapted driver. We suppose that $\widetilde{f}$ is Lipschitz w.r.t. y, z and h and that $\widetilde{f}(t, 0, 0, 0) = 0$ a.s. Then, there exists a constant C depending only on T and $\|\widetilde{f}\|_{\mathrm{lip}}$, such that:

$$\mathbb{E}\left[|Y|_{0,T}^{*,2} + \int_0^T |Z_t|^2 \, \mathrm{d}t + \mu \int_0^T |H_t|^2 \, \mathrm{d}t\right] \leqslant C\mathbb{E}\left[|\xi|^2 + \int_0^T |h_s|^2 \, \mathrm{d}s\right].$$

3.7.2 L^4 estimate for finite horizon BSDEs

The following lemma is inspired from the lemma 3.1 of [8].

Lemma 3.30. *We consider the BSDE*

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r, H_r) \, \mathrm{d}r - \int_t^T Z_r \, \mathrm{d}W_r - \int_t^T H_r \, \mathrm{d}M_r,$$

with the following conditions:

- *f and ξ are such that this BSDE admits a unique solution;*
- *this solution satisfies $\mathbb{E}\left[|Y|_{0,T}^{*,4}\right] < \infty$ and $\mathbb{E}\left[|H|_{0,T}^{*,4}\right] < \infty$;*
- *$f(t, y, z, h) \leqslant f_t + \alpha|y| + \beta|z| + \gamma|h|$ with $\alpha, \beta, \gamma \in \mathbb{R}$ and f a non-negative progressively measurable process and $\mathbb{E}\left[\left(\int_0^T f_r \, \mathrm{d}r\right)^4\right] < \infty$.*

Then, there exists a numerical constant C, such that for any $a \geqslant \alpha + \beta^2 + \frac{\gamma^2}{2\mu}$,

$$\mathbb{E}\left[\left(\int_0^T \mathrm{e}^{2ar}|Z_r|^2 \, \mathrm{d}r\right)^2\right] \leqslant C(1 + \mu T)\mathbb{E}\left[\sup_{t\in[0,T]} \mathrm{e}^{4at}|Y_t|^4 + \left(\int_0^T \mathrm{e}^{ar} f_r \, \mathrm{d}r\right)^4 + \sup_{t\in[0,T]} \mathrm{e}^{4at}|H_t|^4\right].$$

Proof. Let us choose $a \geqslant \alpha + \beta^2 + \frac{\gamma^2}{2\mu}$ and set $\widetilde{Y}_t = \mathrm{e}^{at} Y_t$, $\widetilde{Z}_t = \mathrm{e}^{at} Z_t$ and $\widetilde{H}_t = \mathrm{e}^{at} H_t$. We have

$$\widetilde{Y}_t = \widetilde{\xi} + \int_t^T \widetilde{f}(r, \widetilde{Y}_r, \widetilde{Z}_r, \widetilde{H}_r) \, \mathrm{d}r - \int_t^T \widetilde{Z}_r \, \mathrm{d}W_r - \int_t^T \widetilde{H}_r \, \mathrm{d}M_r,$$

where $\widetilde{\xi} = \mathrm{e}^{aT}\xi$ and $\widetilde{f}(t, y, z, h) = \mathrm{e}^{at} f(t, \mathrm{e}^{-at}y, \mathrm{e}^{-at}z, \mathrm{e}^{-at}h) - ay$. The function $\widetilde{f}$ satisfies the hypotheses with $\widetilde{f}_t = \mathrm{e}^{at} f_t$, $\widetilde{\alpha} = \alpha - a$, $\widetilde{\beta} = \beta$ and $\widetilde{\gamma} = \gamma$. So, by a change of variable, we can reduce our problem to the case $a = 0$ and $\alpha + \beta^2 + \frac{\gamma^2}{2\mu} \leqslant 0$. For every $n \in \mathbb{N}^*$, we define the stopping time:

$$\tau_n = \inf\left\{t \in [0, T] \middle| \int_0^t |Z_r|^2 \, \mathrm{d}r \geqslant n\right\} \wedge T.$$

Using Itô's formula, we get:

$$|Y_0|^2 + \int_0^{\tau_n} |Z_r|^2 \, \mathrm{d}r + \mu \int_0^{\tau_n} |H_r|^2 \, \mathrm{d}r = |Y_{\tau_n}|^2 + 2\int_0^{\tau_n} Y_r f(r, Y_r, Z_r, H_r) \, \mathrm{d}r - 2\int_0^{\tau_n} Y_r Z_r \, \mathrm{d}W_r$$
$$+ \sum_{l=1}^{k-1} \int_0^{\tau_n^+} \left(|Y_{r^-} + H_{r^-}(l)|^2 - |Y_{r^-}|^2\right) \mathrm{d}M_r(l).$$

But we have $2yf(t,y,z,h) \leqslant 2|y|f_t + \frac{|z|^2}{2} + \mu|h|^2$, and we get:

$$\frac{1}{2}\int_0^{\tau_n} |Z_r|^2 \,\mathrm{d}r \leqslant |Y_{\tau_n}|^2 + 2\int_0^{\tau_n} |Y_r| f_r \,\mathrm{d}r - 2\int_0^{\tau_n} Y_r Z_r \,\mathrm{d}W_r + \sum_{l=1}^{k-1} \int_0^{\tau_n^+} \left(2Y_{r^-}H_{r^-}(l) + H_{r^-}(l)^2\right) \,\mathrm{d}M_r(l).$$

After standard calculations, it can be reduced to:

$$\int_0^{\tau_n} |Z_r|^2 \,\mathrm{d}r \leqslant 4|Y|_{0,T}^{*,2} + 2\left(\int_0^{\tau_n} f_r \,\mathrm{d}r\right)^2 + 4\left|\int_0^{\tau_n} Y_r Z_r \,\mathrm{d}W_r\right| \\ + 4\left|\sum_{l=1}^{k-1}\int_0^{\tau_n^+} Y_{r^-}H_{r^-}(l)\,\mathrm{d}M_r(l)\right| + 2\left|\sum_{l=1}^{k-1}\int_0^{\tau_n^+} H_{r^-}(l)^2\,\mathrm{d}M_r(l)\right|.$$

And then

$$\left(\int_0^{\tau_n} |Z_r|^2 \,\mathrm{d}r\right)^2 \leqslant 20|Y|_{0,T}^{*,4} + 10\left(\int_0^{\tau_n} f_r \,\mathrm{d}r\right)^4 + 20\left|\int_0^{\tau_n} Y_r Z_r \,\mathrm{d}W_r\right|^2 \\ + 20\left|\sum_{l=1}^{k-1}\int_0^{\tau_n^+} Y_{r^-}H_{r^-}(l)\,\mathrm{d}M_r(l)\right|^2 + 10\left|\sum_{l=1}^{k-1}\int_0^{\tau_n^+} H_{r^-}(l)^2\,\mathrm{d}M_r(l)\right|^2.$$

Using the BDG inequalities,

$$20\mathbb{E}\left[\left|\int_0^{\tau_n} Y_r Z_r \,\mathrm{d}W_r\right|^2\right] \leqslant C\mathbb{E}\left[\int_0^{\tau_n} |Y_r|^2|Z_r|^2\,\mathrm{d}r\right] \leqslant \frac{C^2}{2}\mathbb{E}\left[|Y|_{0,T}^{*,4}\right] + \frac{1}{2}\mathbb{E}\left[\left(\int_0^{\tau_n} |Z_r|^2\,\mathrm{d}r\right)^2\right],$$

$$20\mathbb{E}\left[\left|\sum_{l=1}^{k-1}\int_0^{\tau_n^+} Y_{r^-}H_{r^-}(l)\,\mathrm{d}M_r(l)\right|^2\right] \leqslant C\mathbb{E}\left[\sum_{l=1}^{k-1}\int_0^{\tau_n^+} |Y_{r^-}|^2 H_{r^-}(l)^2\,\mathrm{d}N_r(l)\right] \leqslant C\mu T\mathbb{E}\left[|Y|_{0,T}^{*,2}|H|_{0,T}^{*,2}\right],$$

$$10\mathbb{E}\left[\left|\sum_{l=1}^{k-1}\int_0^{\tau_n^+} H_{r^-}(l)^2\,\mathrm{d}M_r(l)\right|^2\right] \leqslant C\mathbb{E}\left[\sum_{l=1}^{k-1}\int_0^{\tau_n^+} H_{r^-}(l)^4\,\mathrm{d}N_r(l)\right] \leqslant C\mu T\mathbb{E}\left[|H|_{0,T}^{*,4}\right].$$

Finally, we obtain:

$$\frac{1}{2}\mathbb{E}\left[\left(\int_0^{\tau_n} |Z_r|^2\,\mathrm{d}r\right)^2\right] \leqslant C(1+\mu T)\mathbb{E}\left[|Y|_{0,T}^{*,4} + \left(\int_0^T f_r\,\mathrm{d}r\right)^4 + |H|_{0,T}^{*,4}\right].$$

And Fatou's lemma implies

$$\mathbb{E}\left[\left(\int_0^{T} |Z_r|^2\,\mathrm{d}r\right)^2\right] \leqslant C(1+\mu T)\mathbb{E}\left[|Y|_{0,T}^{*,4} + \left(\int_0^T f_r\,\mathrm{d}r\right)^4 + |H|_{0,T}^{*,4}\right].$$

□

3.7.3 About the regularity of the function u for finite horizon BSDEs

We consider the following forward-backward system:

$$X_s^{t,x,n} = x + \int_t^s b_{N_r^{t,n}}\left(X_r^{t,x,n}\right)\,\mathrm{d}r + \int_t^s \sigma_{N_r^{t,n}}\left(X_r^{t,x,n}\right)\,\mathrm{d}W_r, \tag{3.21}$$

$$Y_s^{t,x,n} = g_{N_T^{t,n}}\left(X_T^{t,x,n}\right) + \int_s^T f_{N_r^{t,n}}\left(X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n}, H_r^{t,x,n}\right)\,\mathrm{d}r - \int_s^T Z_r^{t,x,n}\,\mathrm{d}W_r - \int_s^T H_r^{t,x,n}\,\mathrm{d}M_r. \tag{3.22}$$

Lemma 3.31. *We make the following assumptions:*

- *b, σ and g are of class $\mathcal{C}^1$ with bounded derivatives;*
- *if we define $\rho_l : (x, y, p, h) \mapsto f_l(x, y, p\sigma_l(x), h)$, then ρ is also of class $\mathcal{C}^1_b$;*
- *for all $l \in \mathbb{L}$, $\sigma_l(\mathbb{R}^d) \subset \mathrm{GL}_d(\mathbb{R})$ and $x \mapsto \sigma_l(x)^{-1}$ is bounded;*
- *the solution of (3.21, 3.22) verifies $\mathbb{E}\left[|Y^{t,x,n}|^{*,4}_{t,T} + |H^{t,x,n}|^{*,4}_{t,T} + \left(\int_t^T \left|f_{N_r^{t,n}}(X_r^{t,x,n}, 0, 0, 0)\right| \mathrm{d}r\right)^4\right] < \infty.$*

Then, the functions $u_n : (t, x) \mapsto Y_t^{t,x,n}$ are of class $\mathcal{C}^1$ w.r.t. x, for every $n \in \mathbb{K}$, and

$$\forall s \in [t, T],\ Z_s^{t,x,n} = \partial_x u_{N_s^{t,n}}(s, X_s^{t,x,n})\sigma_{N_s^{t,n}}(X_s^{t,x,n}).$$

Proof. We will prove this result in the case $d = 1$.

Step 1: Let $(t, x, n) \in [0, T] \times \mathbb{R} \times \mathbb{K}$ and $\varepsilon \neq 0$. We set: $\nabla \bullet^\varepsilon = \frac{\bullet^{t,x+\varepsilon,n} - \bullet^{t,x,n}}{\varepsilon}$, for $\bullet = X$, Y, Z or H. Also, we define $\Theta = (X, Y, Z, H)$. We consider the variational system:

$$\nabla X_s = 1 + \int_t^s \partial_x b_{N_r^{t,n}}\left(X_r^{t,x,n}\right) \nabla X_r \, \mathrm{d}r + \int_t^s \partial_x \sigma_{N_r^{t,n}}\left(X_r^{t,x,n}\right) \nabla X_r \, \mathrm{d}W_r; \tag{3.23}$$

$$\begin{aligned}
\nabla Y_s = {} & \partial_x g_{N_T^{t,n}}\left(X_T^{t,x,n}\right) \nabla X_T \\
& + \int_s^T \Big[\partial_x f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right) \nabla X_r + \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right) \nabla Y_r + \partial_z f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right) \nabla Z_r \\
& \qquad\qquad + \partial_h f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right) \nabla H_r\Big] \, \mathrm{d}r \\
& - \int_s^T \nabla Z_r \, \mathrm{d}W_r - \int_s^T \nabla H_r \, \mathrm{d}M_r.
\end{aligned} \tag{3.24}$$

Moreover, for every $s \in [t, T]$, we have,

$$\begin{aligned}
\nabla Y_s^\varepsilon = g_x^\varepsilon \nabla X_T^\varepsilon + \int_s^T \left[f_x^\varepsilon(r)\nabla X_r^\varepsilon + f_y^\varepsilon(r)\nabla Y_r^\varepsilon + f_z^\varepsilon(r)\nabla Z_r^\varepsilon + f_h^\varepsilon(r)\nabla H_r^\varepsilon\right] \mathrm{d}r \\
- \int_s^T \nabla Z_r^\varepsilon \, \mathrm{d}W_r - \int_s^T \nabla H_r^\varepsilon \, \mathrm{d}M_r,
\end{aligned} \tag{3.25}$$

where we use the following notations

$$g_x^\varepsilon = \int_0^1 \partial_x g_{N_T^{t,n}}(X_T^{t,x,n} + w\varepsilon\nabla X_T^\varepsilon) \, \mathrm{d}w,$$

$$f_h^\varepsilon(r) = \int_0^1 \partial_h f_{N_r^{t,n}}(X_r^{t,x+\varepsilon,n}, Y_r^{t,x+\varepsilon,n}, Z_r^{t,x+\varepsilon,n}, H_r^{t,x,n} + w\varepsilon\nabla H_r^\varepsilon) \, \mathrm{d}w,$$

$$f_z^\varepsilon(r) = \int_0^1 \partial_z f_{N_r^{t,n}}(X_r^{t,x+\varepsilon,n}, Y_r^{t,x+\varepsilon,n}, Z_r^{t,x,n} + w\varepsilon\nabla Z_r^\varepsilon, H_r^{t,x,n}) \, \mathrm{d}w,$$

$$f_y^\varepsilon(r) = \int_0^1 \partial_y f_{N_r^{t,n}}(X_r^{t,x+\varepsilon,n}, Y_r^{t,x,n} + w\varepsilon\nabla Y_r^\varepsilon, Z_r^{t,x,n}, H_r^{t,x,n}) \, \mathrm{d}w \text{ and}$$

$$f_x^\varepsilon(r) = \int_0^1 \partial_x f_{N_r^{t,n}}(X_r^{t,x,n} + w\varepsilon\nabla X_r^\varepsilon, Y_r^{t,x,n}, Z_r^{t,x,n}, H_r^{t,x,n}) \, \mathrm{d}w.$$

Then, we set $\Delta\bullet^\varepsilon = \nabla\bullet^\varepsilon - \nabla\bullet$, for $\bullet = X, Y, Z$ or H. We have:

$$\Delta Y^\varepsilon_s = g^\varepsilon_x \nabla X^\varepsilon_T - g^0_x \nabla X_T - \int_s^T \Delta Z^\varepsilon_r \,\mathrm{d}W_r - \int_s^T \Delta H^\varepsilon_r \,\mathrm{d}M_r$$
$$+ \int_s^T \left[f^\varepsilon_x(r)\nabla X^\varepsilon_r - f^0_x(r)\nabla X_r + f^\varepsilon_y(r)\Delta Y^\varepsilon_r + f^\varepsilon_z(r)\Delta Z^\varepsilon_r + f^\varepsilon_h(r)\Delta H^\varepsilon_r + \beta^\varepsilon(r) \right] \mathrm{d}r,$$

where we set $\beta^\varepsilon(s) = \left[f^\varepsilon_y(r) - f^0_y(r) \right] \nabla Y_r + \left[f^\varepsilon_z(r) - f^0_z(r) \right] \nabla Z_r + \left[f^\varepsilon_h(r) - f^0_h(r) \right] \nabla H_r$. Using Lemma 3.29, we get the upperbound:

$$\mathbb{E}\left[|\Delta Y^\varepsilon|^{*,2}_{t,T} + \int_t^T \{ |\Delta Z^\varepsilon_s|^2 + \mu |\Delta H^\varepsilon_s|^2 \} \,\mathrm{d}s \right]$$
$$\leqslant C\mathbb{E}\left[|g^\varepsilon_x \nabla X^\varepsilon_T - g^0_x \nabla X_T|^2 + \int_t^T |f^\varepsilon_x(r)\nabla X^\varepsilon_r - f^0_x(r)\nabla X_r + \beta^\varepsilon(r)|^2 \,\mathrm{d}r \right].$$

Step 2: Behaviour of ∇X and ΔX^ε.

Due to Lemma 3.28, $\mathbb{E}\left[|\nabla X|^{*,p}_{t,T} \right]$ is bounded by a constant depending only on p, T and the Lipschitz constants of b and σ. Moreover, we can write:

$$\nabla X^\varepsilon_s = 1 + \int_t^s b^\varepsilon_x(r)\nabla X^\varepsilon_r \,\mathrm{d}r + \int_t^s \sigma^\varepsilon_x(r)\nabla X^\varepsilon_r \,\mathrm{d}r,$$

where $b^\varepsilon_x(r) = \int_0^1 \partial_x b_{N^{t,n}_r}(X^{t,x,n}_r + w\varepsilon\nabla X^\varepsilon_r)\,\mathrm{d}w$ and $\sigma^\varepsilon_x(r) = \int_0^1 \partial_x \sigma_{N^{t,n}_r}(X^{t,x,n}_r + w\varepsilon\nabla X^\varepsilon_r)\,\mathrm{d}w$. So we have $\Delta X^\varepsilon_s = \int_t^s \{ b^\varepsilon_x(r)\nabla X^\varepsilon_r - b^0_x(r)\nabla X_r \} \,\mathrm{d}r + \int_t^s \{ \sigma^\varepsilon_x(r)\nabla X^\varepsilon_r - \sigma^0_x(r)\nabla X_r \} \,\mathrm{d}W_r$, and after Lemma 3.28:

$$\mathbb{E}\left[|\Delta X^\varepsilon|^{*,p}_{t,T} \right] \leqslant C\mathbb{E}\left[\int_t^T \{ |(b^\varepsilon_x(r) - b^0_x(r))\nabla X_r|^p + |(\sigma^\varepsilon_x(r) - \sigma^0_x(r))\nabla X_r|^p \} \,\mathrm{d}r \right] \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

by dominated convergence.

Step 3: Behaviour of ΔY^ε, ΔZ^ε and ΔH^ε.

Using Step 2, the fact that $g \in \mathcal{C}^1_b$ and dominated convergence, one finds:

$$\mathbb{E}\left[|g^\varepsilon_x \Delta X^\varepsilon_T + (g^\varepsilon_x - g^0_x)\nabla X_T|^2 \right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Also, by dominated convergence,

$$\mathbb{E}\left[\left(\int_t^T |f^\varepsilon_x(r) - f^0_x(r)|^2 \,\mathrm{d}r \right)^2 \right] \underset{\varepsilon\to 0}{\longrightarrow} 0$$

(the domination is $16 \left(\int_t^T (\|\rho\|_{\mathrm{lip},x} + \|\rho\|_{\mathrm{lip},p} |Z^{t,x,n}_r| \|\sigma^{-1}\|^2_\infty \|\sigma\|_{\mathrm{lip}})^2 \,\mathrm{d}r \right)^2$, and Lemma 3.30 allows us to claim that it is integrable). Again Lemma 3.30 and Cauchy-Schwarz inequality are useful to prove that

$$\mathbb{E}\left[\int_t^T |f^\varepsilon_x(r)\nabla X^\varepsilon_r - f^0_x(r)\nabla X_r|^2 \,\mathrm{d}r \right] \underset{\to 0}{\longrightarrow} 0.$$

Also, using the dominated convergence theorem, and the Lipschitz continuity of f w.r.t. y, z and h, $\mathbb{E}\left[\int_t^T |\beta^\varepsilon(r)|^2 \,\mathrm{d}r \right] \underset{\varepsilon\to 0}{\longrightarrow} 0$. Finally,

$$\mathbb{E}\left[|\Delta Y^\varepsilon|^{*,2}_{t,T} + \int_t^T |\Delta Z^\varepsilon_s|^2 \,\mathrm{d}s + \mu \int_t^T |\Delta H^\varepsilon_s|^2 \,\mathrm{d}s \right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Step 4: we conclude that $\partial_x u_n$ exists and that $\partial_x u_n(t,x) = \nabla Y_t$, using the Blumenthal 0-1 law (see e.g. [43]).

Step 5: we will show that $\partial_x u_n$ is continuous. Let $(t_i, x_i) \in [0,T] \times \mathbb{R}^d$, $i = 1,2$, with $t_1 < t_2$. To simplify, we write:

$$\Theta^i = \left(X^i, Y^i, Z^i, H^i\right) = \left(X^{t_i,x_i,n}, Y^{t_i,x_i,n}, Z^{t_i,x_i,n}, H^{t_i,x_i,n}\right),$$

$$f_x^i(r) = \partial_x f_{N_r^{t_i,n}}\left(\Theta_r^i\right),\ f_y^i(r) = \partial_y f_{N_r^{t_i,n}}\left(\Theta_r^i\right),\ f_z^i(r) = \partial_z f_{N_r^{t_i,n}}\left(\Theta_r^i\right),\ f_h^i(r) = \partial_h f_{N_r^{t_i,n}}\left(\Theta_r^i\right),$$

$$g_x^i = \partial_x g_{N_T^{t_i,n}}\left(X_T^i\right),\ b_x^i(r) = \partial_x b_{N_r^{t_i,n}}\left(X_r^i\right) \text{ and } \sigma_x^i(r) = \partial_x \sigma_{N_r^{t_i,n}}\left(X_r^i\right).$$

We set $\widetilde{\Delta}\bullet_r = \nabla \bullet_r^1 - \nabla \bullet_r^2$ for $\bullet = X$, Y, Z or H, and for any function φ, $\widetilde{\Delta}_{12}[\varphi] = \varphi^1 - \varphi^2$.

$$\begin{aligned}
&\left|\partial_x u_n(t_1,x_1) - \partial_x u_n(t_2,x_2)\right| \\
&\quad = \Bigg|\mathbb{E}\left[g_x^1 \nabla X_T^1 + \int_{t_1}^T \left\{f_x^1(r)\nabla X_r^1 + f_y^1(r)\nabla Y_r^1 + f_z^1(r)\nabla Z_r^1 + f_h^1(r)\nabla H_r^1\right\} \mathrm{d}r\right] \\
&\qquad - \mathbb{E}\left[g_x^2 \nabla X_T^2 + \int_{t_2}^T \left\{f_x^2(r)\nabla X_r^2 + f_y^2(r)\nabla Y_r^2 + f_z^2(r)\nabla Z_r^2 + f_h^2(r)\nabla H_r^2\right\} \mathrm{d}r\right]\Bigg| \\
&\quad \leqslant \mathbb{E}\left[\left|g_x^1 \nabla X_T^1 - g_x^2 \nabla X_T^2\right|\right] \\
&\qquad + \mathbb{E}\left[\int_{t_1}^{t_2} \left\{\left|f_x^1(r)\right|\left|\nabla X_r^1\right| + \left|f_y^1(r)\right|\left|\nabla Y_r^1\right| + \left|f_z^1(r)\right|\left|\nabla Z_r^1\right| + \left|f_h^1(r)\right|\left|\nabla H_r^1\right|\right\} \mathrm{d}r\right] \\
&\qquad + \mathbb{E}\Bigg[\int_{t_2}^T \Big\{\left|f_x^1(r)\widetilde{\Delta}X_r\right| + \left|\widetilde{\Delta}_{12}[f_x](r)\nabla X_r^2\right| + \|f\|_{\mathrm{lip},y}\left|\widetilde{\Delta}Y_r\right| + \left|\widetilde{\Delta}_{12}[f_y](r)\right|\left|\nabla Y_r^2\right| \\
&\qquad\quad + \|f\|_{\mathrm{lip},z}\left|\widetilde{\Delta}Z_r\right| + \left|\widetilde{\Delta}_{12}[f_z](r)\right|\left|\nabla Z_r^2\right| + \|f\|_{\mathrm{lip},h}\left|\widetilde{\Delta}H_r\right| + \left|\widetilde{\Delta}_{12}[f_h](r)\right|\left|\nabla H_r^2\right|\Big\} \mathrm{d}r\Bigg].
\end{aligned}$$

First,

$$\mathbb{E}\left[\int_{t_1}^{t_2} \left\{\left|f_x^1(r)\right|\left|\nabla X_r^1\right| + \|f\|_{\mathrm{lip},y}\left|\nabla Y_r^1\right| + \|f\|_{\mathrm{lip},z}\left|\nabla Z_r^1\right| + \|f\|_{\mathrm{lip},h}\left|\nabla H_r^1\right|\right\} \mathrm{d}r\right] \underset{t_1 \to t_2}{\longrightarrow} 0,$$

since

$$\mathbb{E}\left[\left|\nabla X^1\right|_{t,T}^{*,2} + \left|\nabla Y^1\right|_{t,T}^{*,2} + \int_t^T \left|\nabla Z_s^1\right|^2 \mathrm{d}s + \int_t^T \left|\nabla H_s^1\right|^2 \mathrm{d}s + \int_t^T \left|Z_s^1\right|^2 \mathrm{d}s\right] < \infty$$

(by dominated convergence). Using Lemma 3.29,

$$\begin{aligned}
&\mathbb{E}\left[\left|\widetilde{\Delta}Y\right|_{t_2,T}^{*,2} + \int_{t_2}^T \left\{\left|\widetilde{\Delta}Z_s\right|^2 + \mu\left|\widetilde{\Delta}H_s\right|^2\right\} \mathrm{d}s\right] \\
&\leqslant C\Bigg\{\|\partial_x g\|_\infty^2 \,\mathbb{E}\left[\left|\widetilde{\Delta}X_T\right|^2\right] + \mathbb{E}\left[\left|\widetilde{\Delta}_{12}[g_x]\right|^2\left|\nabla X_T^2\right|^2\right] + \mathbb{E}\Bigg[\int_{t_2}^T \Bigg(\left|f_x^1(r)\widetilde{\Delta}X_r\right|^2 + \left|\widetilde{\Delta}_{12}[f_x](r)\nabla X_r^2\right|^2 \\
&\qquad + \left|\widetilde{\Delta}_{12}[f_y](r)\right|^2\left|\nabla Y_r^2\right|^2 + \left|\widetilde{\Delta}_{12}[f_z](r)\right|^2\left|\nabla Z_r^2\right|^2 + \left|\widetilde{\Delta}_{12}[f_h](r)\right|^2\left|\nabla H_r^2\right|^2\Bigg) \mathrm{d}r\Bigg]\Bigg\}.
\end{aligned}$$

We can adapt Step 2, we replace (t,x) by (t_2,x_2) and $x+\varepsilon$ by $X_{t_2}^{t_1,x_1,n}$ and get, for every $p \geqslant 2$,

$$\mathbb{E}\left[\left(\left|X^1 - X^2\right|_{t_2,T}^{*,p} + \left|Y^1 - Y^2\right|_{t_2,T}^{*,2} + \int_{t_2}^T \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s + \mu\int_{t_2}^T \left|H_s^1 - H_s^2\right|^2 \mathrm{d}s\right)\mathbb{1}_{N_{t_2}^{t_1,n}=n}\right] \longrightarrow 0.$$

Also, for every $p \geqslant 2$,

$$\mathbb{E}\left[\left(\left|X^1-X^2\right|_{t_2,T}^{*,p}+\left|Y^1-Y^2\right|_{t_2,T}^{*,2}+\int_{t_2}^T\left|Z_s^1-Z_s^2\right|^2\mathrm{d}s+\mu\int_{t_2}^T\left|H_s^1-H_s^2\right|^2\mathrm{d}s\right)\mathbb{1}_{N_{t_2}^{t_1,n}\neq n}\right]$$
$$\leqslant 2\underbrace{\mathbb{E}\left[\left|X^1-X^2\right|_{t_2,T}^{*,2p}+\left|Y^1-Y^2\right|_{t_2,T}^{*,4}+\left(\int_{t_2}^T\left|Z_s^1-Z_s^2\right|^2\mathrm{d}s\right)^2+\mu^2\left(\int_{t_2}^T\left|H_s^1-H_s^2\right|^2\mathrm{d}s\right)^2\right]^{1/2}}_{<\infty}$$
$$\times\underbrace{\mathbb{P}(N_{t_2}^{t_1,n}\neq n)^{1/2}}_{\to 0}.$$

By dominated convergence, for $\varphi = f_y$, f_z or f_h, $\mathbb{E}\left[\left|\widetilde{\Delta}_{12}[\varphi]\right|_{t_2,T}^{*,p}\right] \longrightarrow 0$. Since

$$\mathbb{E}\left[\left(\int_{t_2}^T\left|Z_r^i\right|^2\mathrm{d}r\right)^2\right]<\infty,$$

we have

$$\mathbb{E}\left[\int_{t_2}^T\left\{\left|f_x^1(r)\widetilde{\Delta}X_r\right|^2+\left|\widetilde{\Delta}_{12}\left[f_x\right](r)\nabla X_r^2\right|^2\right\}\mathrm{d}r\right]\longrightarrow 0$$

(using Lemma 3.28 for $\widetilde{\Delta}X$, and a convergence dominated thanks to Lemma 3.30). To sum up, we have shown that $|\partial_x u_n(t_1,x_1)-\partial_x u_n(t_2,x_2)| \longrightarrow 0$ when $t_1 \to t_2$ and $x_1 \to x_2$. We can prove it when $t_2 \to t_1$ and $x_2 \to x_1$: $\partial_x u_n$ is continuous on $[0,T]\times\mathbb{R}^d$.

Step 7: to show the relation $\forall s \in [t,T]$, $Z_s^{t,x,n} = \partial_x u_{N_s^{t,n}}\left(s, X_s^{t,x,n}\right)\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)$ $\mathbb{P}$-a.s., we can do the same as in the Theorem 3.1 of [43]. We approximate b, σ, ρ and g by functions b^ε, σ^ε, ρ^ε and g^ε which are of class $\mathcal{C}^\infty$ with bounded derivatives (and the bound is independent of ε), and converge uniformly. Then, $(\sigma^\varepsilon)^{-1}$ converges uniformly to σ^{-1}. We define a function f^ε (we want it to approximate f and to be smooth) by $f_l^\varepsilon(x,y,z,h) = \rho_l^\varepsilon\left(x,y,z\sigma_l^\varepsilon(x)^{-1},h\right)$ and it satisfies $|f^\varepsilon(x,y,z,h)-f(x,y,z,h)| \leqslant \|\rho^\varepsilon-\rho\|_\infty + H_p|z|\left\|(\sigma^\varepsilon)^{-1}-\sigma^{-1}\right\|_\infty$. Like before, Lemma 3.30 is useful to adapt the end of the proof in [43].

□

3.7.4 About the estimate of ∇Y

Lemma 3.32. *We make the following assumptions:*

- *b and σ are of class $\mathcal{C}^1$ with bounded derivatives;*
- *if we define $\rho_l : (x,y,p,h) \mapsto f_l(x,y,p\sigma_l(x),h)$, then ρ is of class $\mathcal{C}^1$ with bounded derivatives;*
- *g is of class $\mathcal{C}^1$ with derivatives which grow at most polynomially;*
- *$f(\bullet,0,0,0)$ and g have at most polynomial growth;*
- *for all $l \in \mathbb{L}$, $\sigma_l(\mathbb{R}^d) \subset \mathrm{GL}_d(\mathbb{R})$ and $x \mapsto \sigma_l(x)^{-1}$ is bounded;*
- *for all $l \in \mathbb{L}$, $\partial_{h_l} f$ takes its values in $(-1,+\infty)$.*

Then the functions u_n are differentiable w.r.t. x and when the right hand side is finite, we have :

$$|\partial_x u_n(t,x)| \leqslant C\left(\sqrt[4]{\mathbb{E}}+\sqrt[4]{\mathbb{E}^{\mathbb{Q}}}\right)\left[1+|Y^{t,x,n}|_{t,T}^{*,4}+|H^{t,x,n}|_{t,T}^{*,4}+\left(\int_t^T\left|f_{N_s^{t,n}}\left(X_s^{t,x,n},0,0,0\right)\right|\mathrm{d}s\right)^4\right],$$

where the constant C depends on $\|\rho\|_{\mathrm{lip},x}$, $\|\rho\|_{\mathrm{lip},p}$, $\|f\|_{\mathrm{lip},y}$, $\|f\|_{\mathrm{lip},h}$, μ, T, k, $\left\|\sigma^{-1}\right\|_{\infty}$, $\|b\|_{\mathrm{lip}}$ and $\|\sigma\|_{\mathrm{lip}}$. The probability measure $\mathbb{Q}$ is defined by the Doléans-Dade exponential (where $\Theta = (X, Y, Z, H)$):

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\sum_{j=1}^{d}\int_t^{\bullet}\partial_{z_j} f_{N_s^{t,n}}(\Theta_s^{t,x,n})\mathrm{d}W_s^j + \sum_{l=1}^{k-1}\int_t^{\bullet}\partial_{h_l} f_{N_s^{t,n}}(\Theta_s^{t,x,n})\mathrm{d}M_s(l)\right)_{\bullet\in[t,T]}.$$

Proof. In what follows, we write $\Theta_r^{t,x,n} = \left(X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n}, H_r^{t,x,n}\right)$. We can write equation (3.22), under differential form:

$$\mathrm{d}Y_s^{t,x,n} = -f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\,\mathrm{d}s + \sum_{j=1}^{d} Z_s^{t,x,n}(j)\,\mathrm{d}W_s^j + \sum_{l=1}^{k-1} H_s^{t,x,n}(l)\,\mathrm{d}M_s(l), \tag{3.26}$$

where $Z_s^{t,x,n}(j)$ stands for the j^{th} coefficient of the row $Z_s^{t,x,n}$. Due to the regularity of the coefficients, we are allowed to differentiate (3.26) w.r.t. x and after a use of discounting method,

$$\begin{aligned}
&\mathrm{d}\left(\mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Y_s^{t,x,n}\right)\\
&\quad= \mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\Bigg(-\partial_x f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\nabla X_s^{t,x,n}\,\mathrm{d}s - \sum_{j=1}^{d}\partial_{z_j} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\nabla Z_s^{t,x,n}(j)\,\mathrm{d}s\\
&\qquad - \sum_{l=1}^{k-1}\partial_{h_l} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\nabla H_s^{t,x,n}(l)\,\mathrm{d}s + \sum_{j=1}^{d}\nabla Z_s^{t,x,n}(j)\,\mathrm{d}W_s^j + \sum_{l=1}^{k-1}\nabla H_s^{t,x,n}(l)\,\mathrm{d}M_s(l)\Bigg).
\end{aligned}$$

After a change of probability measure, between s and T (for $s \in [t, T]$), the integral form of this equation can be written as:

$$\begin{aligned}
&\mathrm{e}^{\int_t^T \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Y_T^{t,x,n} - \mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Y_s^{t,x,n}\\
&\quad= -\int_s^T \mathrm{e}^{\int_t^r \partial_y f_{N_u^{t,n}}\left(\Theta_u^{t,x,n}\right)\mathrm{d}u}\partial_x f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\nabla X_r^{t,x,n}\,\mathrm{d}r + \sum_{j=1}^{d}\int_s^T \mathrm{e}^{\int_t^r \partial_y f_{N_u^{t,n}}\left(\Theta_u^{t,x,n}\right)\mathrm{d}u}\nabla Z_r^{t,x,n}(j)\,\mathrm{d}\widetilde{W}_r^j\\
&\qquad + \sum_{l=1}^{k-1}\int_s^T \mathrm{e}^{\int_t^r \partial_y f_{N_u^{t,n}}\left(\Theta_u^{t,x,n}\right)\mathrm{d}u}\nabla H_r^{t,x,n}(l)\,\mathrm{d}\widetilde{M}_r(l).
\end{aligned}\tag{3.27}$$

We define $F_s^{t,x,n} = \mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Y_s^{t,x,n} + \int_t^s \mathrm{e}^{\int_t^r \partial_y f_{N_u^{t,n}}\left(\Theta_u^{t,x,n}\right)\mathrm{d}u}\partial_x f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\nabla X_r^{t,x,n}\,\mathrm{d}r$, and then equation (3.27) becomes:

$$F_s^{t,x,n} = F_T^{t,x,n} - \sum_{j=1}^{d}\int_s^T \mathrm{e}^{\int_t^r \partial_y f_{N_u^{t,n}}\left(\Theta_u^{t,x,n}\right)\mathrm{d}u}\nabla Z_r^{t,x,n}(j)\,\mathrm{d}\widetilde{W}_r^j - \sum_{l=1}^{k-1}\int_s^T \mathrm{e}^{\int_t^r \partial_y f_{N_u^{t,n}}\left(\Theta_u^{t,x,n}\right)\mathrm{d}u}\nabla H_r^{t,x,n}(l)\,\mathrm{d}\widetilde{M}_r(l), \tag{3.28}$$

and it tells us that $F^{t,x,n}$ is a $\mathbb{Q}$-martingale. We recall that for every $s \in [t, T]$, we have the following expressions:

$$\begin{gathered}
Y_s^{t,x,n} = u_{N_s^{t,n}}\left(s, X_s^{t,x,n}\right), \quad \nabla Y_s^{t,x,n} = \partial_x u_{N_s^{t,n}}\left(s, X_s^{t,x,n}\right)\nabla X_s^{t,x,n},\\
Z_s^{t,x,n} = \partial_x u_{N_s^{t,n}}\left(s, X_s^{t,x,n}\right)\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right) = \nabla Y_s^{t,x,n}\left(\nabla X_s^{t,x,n}\right)^{-1}\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\\
\text{and}\quad H_s^{t,x,n}(l) = u_{N_{s^-}^{t,n}+l}(s, X_s^{t,x,n}) - u_{N_{s^-}^{t,n}}(s, X_s^{t,x,n}).
\end{gathered}$$

Indeed, $\nabla X_s^{t,x,n}$ is invertible. We can show that $\nabla X^{t,x,n}$ is the solution of the linear SDE (because b and σ are $\mathcal{C}^1$ with bounded derivatives):

$$\mathrm{d}\nabla X_s^{t,x,n} = \partial_x b_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\nabla X_s^{t,x,n}\,\mathrm{d}s + \sum_{j=1}^{d}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\nabla X_s^{t,x,n}\,\mathrm{d}W_s^j,$$

whose solution is

$$\nabla X_s^{t,x,n} = \exp\left(\int_s^t \left\{\partial_x b_{N_r^{t,n}}\left(X_r^{t,x,n}\right) - \frac{1}{2}\sum_{j=1}^d \partial_x^j \sigma_{N_r^{t,n}}\left(X_r^{t,x,n}\right)\left[\partial_x^j \sigma\left(X_r^{t,x,n}\right)\right]^*\right\} \mathrm{d}r \right.$$
$$\left. + \sum_{j=1}^d \int_t^s \partial_x^j \sigma_{N_r^{t,n}}\left(X_r^{t,x,n}\right) \mathrm{d}W_r^j\right),$$

where the notation $\partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)$ stands for the $(d \times d)$-matrix

$$\begin{pmatrix} \partial_x \left[\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right]^{(1,j)} \\ \hline \vdots \\ \hline \partial_x \left[\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right]^{(d,j)} \end{pmatrix},$$

and where $\left[\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right]^{(i,j)}$ is the coefficient at line i and column j in the matrix $\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)$. This way, we see that $\left(\nabla X_s^{t,x,n}\right)^{-1}$ is solution of the following linear SDE:

$$\begin{aligned} \mathrm{d}\left(\nabla X_s^{t,x,n}\right)^{-1} = \left(\nabla X_s^{t,x,n}\right)^{-1} &\left\{-\partial_x b_{N_s^{t,n}}\left(X_s^{t,x,n}\right) + \frac{1}{2}\sum_{j=1}^d \partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\left[\partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right]^*\right\} \mathrm{d}s \\ &- \left(\nabla X_s^{t,x,n}\right)^{-1} \sum_{j=1}^d \partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right) \mathrm{d}W_s^j. \end{aligned} \tag{3.29}$$

For every $s \in [t,T]$, we set

$$R_s^{t,x,n} = Z_s^{t,x,n}\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)^{-1} = \nabla Y_s^{t,x,n}\left(\nabla X_s^{t,x,n}\right)^{-1} = \partial_x u_{N_s^{t,n}}\left(s, X_s^{t,x,n}\right),$$

$$\beta_s^{t,x,n} = \int_t^s \mathrm{e}^{\int_t^r \partial_y f_{N_u^{t,n}}\left(\Theta_u^{t,x,n}\right)\mathrm{d}u}\partial_x f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\nabla X_r^{t,x,n}\,\mathrm{d}r\left(\nabla X_s^{t,x,n}\right)^{-1},$$

and finally we define the process

$$\widetilde{R}_s^{t,x,n} = F_s^{t,x,n}\left(\nabla X_s^{t,x,n}\right)^{-1} = \mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r} R_s^{t,x,n} + \beta_s^{t,x,n}.$$

Using the integration by parts formula, combining (3.28) and (3.29), one gets:

$$\begin{aligned} \mathrm{d}\widetilde{R}_s^{t,x,n} = \widetilde{R}_s^{t,x,n} &\left\{-\partial_x b_{N_s^{t,n}}\left(X_s^{t,x,n}\right) + \frac{1}{2}\sum_{j=1}^d \partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\left[\partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right]^* \right. \\ &\left. - \sum_{j=1}^d \partial_{z_j} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right\} \mathrm{d}s \\ &- \sum_{j=1}^d \mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1}\partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\mathrm{d}s \\ &+ \sum_{j=1}^d \left\{\mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1} - \widetilde{R}_s^{t,x,n}\partial_x^j \sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right\}\mathrm{d}\widetilde{W}_s^j \\ &+ \sum_{l=1}^{k-1} \mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla H_s^{t,x,n}(l)\left(\nabla X_s^{t,x,n}\right)^{-1}\mathrm{d}\widetilde{M}_s(l). \end{aligned} \tag{3.30}$$

Then, let us take a new parameter $\lambda \in \mathbb{R}$; we have:

$$
\begin{aligned}
&\mathrm{d}\left|\mathrm{e}^{\lambda s}\widetilde{R}_s^{t,x,n}\right|^2 \\
&\quad = \mathrm{e}^{2\lambda s}\widetilde{R}_s^{t,x,n}\Bigg\{2\lambda \mathrm{I}_d - 2\partial_x b_{N_s^{t,n}}\left(X_s^{t,x,n}\right) + \sum_{j=1}^{d}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\left[\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right]^* \\
&\qquad\qquad -2\sum_{j=1}^{d}\partial_{z_j}f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\Bigg\}\left(\widetilde{R}_s^{t,x,n}\right)^*\,\mathrm{d}s \\
&\quad\quad - 2\mathrm{e}^{2\lambda s}\sum_{j=1}^{d}\mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\left(\widetilde{R}_s^{t,x,n}\right)^*\,\mathrm{d}s \\
&\quad\quad + 2\mathrm{e}^{2\lambda s}\sum_{j=1}^{d}\left\{\mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1} - \widetilde{R}_s^{t,x,n}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right\}\left(\widetilde{R}_s^{t,x,n}\right)^*\,\mathrm{d}\widetilde{W}_s^j \\
&\quad\quad + \mathrm{e}^{2\lambda s}\sum_{j=1}^{d}\left|\mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1} - \widetilde{R}_s^{t,x,n}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right|^2\,\mathrm{d}s \\
&\quad\quad + \mathrm{e}^{2\lambda s}\sum_{l=1}^{k-1}\left\{\left|\widetilde{R}_s^{t,x,n} + \mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla H_s^{t,x,n}(l)\left(\nabla X_s^{t,x,n}\right)^{-1}\right|^2 - \left|\widetilde{R}_s^{t,x,n}\right|^2\right\}\mathrm{d}\widetilde{M}_s(l) \\
&\quad\quad + \mu\mathrm{e}^{2\lambda s}\sum_{l=1}^{k-1}\left|\mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla H_s^{t,x,n}(l)\left(\nabla X_s^{t,x,n}\right)^{-1}\right|^2\,\mathrm{d}s.
\end{aligned}
\tag{3.31}
$$

If we denote $\gamma_j = \mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1}$ and $\delta_j = \widetilde{R}_s^{t,x,n}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)$, we remark that we have the inequality $-2\gamma_j\delta_j^* + |\gamma_j - \delta_j|^2 = |\gamma_j - 2\delta_j|^2 - 3|\delta_j|^2$. Thus,

$$
\begin{aligned}
&\mathrm{d}\left|\mathrm{e}^{\lambda s}\widetilde{R}_s^{t,x,n}\right|^2 \\
&\quad = 2\mathrm{e}^{2\lambda s}\widetilde{R}_s^{t,x,n}\Bigg\{\lambda \mathrm{I}_d - \partial_x b_{N_s^{t,n}}\left(X_s^{t,x,n}\right) - \sum_{j=1}^{d}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\left[\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right]^* \\
&\qquad\qquad -\sum_{j=1}^{d}\partial_{z_j}f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\partial_x^j\sigma\left(X_s^{t,x,n}\right)\Bigg\}\left(\widetilde{R}_s^{t,x,n}\right)^*\,\mathrm{d}s \\
&\quad\quad + \mathrm{e}^{2\lambda s}\sum_{j=1}^{d}\left|\mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1} - 2\widetilde{R}_s^{t,x,n}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right|^2\,\mathrm{d}s \\
&\quad\quad + \mu\mathrm{e}^{2\lambda s}\sum_{l=1}^{k-1}\left|\mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla H_s^{t,x,n}(l)\left(\nabla X_s^{t,x,n}\right)^{-1}\right|^2\,\mathrm{d}s \\
&\quad\quad + 2\mathrm{e}^{2\lambda s}\sum_{j=1}^{d}\left\{\mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla Z_s^{t,x,n}(j)\left(\nabla X_s^{t,x,n}\right)^{-1} - \widetilde{R}_s^{t,x,n}\partial_x^j\sigma_{N_s^{t,n}}\left(X_s^{t,x,n}\right)\right\}\left(\widetilde{R}_s^{t,x,n}\right)^*\,\mathrm{d}\widetilde{W}_s^j \\
&\quad\quad + \mathrm{e}^{2\lambda s}\sum_{l=1}^{k-1}\left\{\left|\widetilde{R}_s^{t,x,n} + \mathrm{e}^{\int_t^s\partial_y f_{N_r^{t,n}}\left(\Theta_r^{t,x,n}\right)\mathrm{d}r}\nabla H_s^{t,x,n}(l)\left(\nabla X_s^{t,x,n}\right)^{-1}\right|^2 - \left|\widetilde{R}_s^{t,x,n}\right|^2\right\}\mathrm{d}\widetilde{M}_s(l).
\end{aligned}
\tag{3.32}
$$

This way, we can see that, for λ great enough (that is to say bigger than something depending only on $\|b\|_{\mathrm{lip}}$, $\|\sigma\|_{\mathrm{lip}}$ and $\|f\|_{\mathrm{lip},z}$ — note that f is Lipschitz continuous w.r.t. z and $\|f\|_{\mathrm{lip},z} \leqslant \|\rho\|_{\mathrm{lip},p}\|\sigma^{-1}\|_\infty$), the process $\left(\left|\mathrm{e}^{\lambda s}\widetilde{R}_s^{t,x,n}\right|^2\right)_{s\in[t,T]}$ is a $\mathbb{Q}$-submartingale. So, we get:

$$
\left|R_t^{t,x,n}\right|^2(T-t) \leqslant \mathrm{e}^{-2\lambda t}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{2\lambda s}\left|\widetilde{R}_s^{t,x,n}\right|^2\,\mathrm{d}s\right] \leqslant \mathrm{e}^{2\lambda(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T\left|\widetilde{R}_s^{t,x,n}\right|^2\,\mathrm{d}s\right].
$$

But $\widetilde{R}_s^{t,x,n} = \mathrm{e}^{\int_t^s \partial_y f_{N_r^{t,n}}(\Theta_r^{t,x,n})\,\mathrm{d}r} R_s^{t,x,n} + \beta_s^{t,x,n}$ and $\partial_y f$ is bounded by $\|f\|_{\mathrm{lip},y}$, so

$$\left|R_t^{t,x,n}\right|^2 (T-t) \leqslant 2\mathrm{e}^{2(\lambda+\|f\|_{\mathrm{lip},y})(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |R_s^{t,x,n}|^2\,\mathrm{d}s\right] + 2\mathrm{e}^{2\lambda(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |\beta_s^{t,x,n}|^2\,\mathrm{d}s\right].$$

On the one hand, we have:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |R_s^{t,x,n}|^2\,\mathrm{d}s\right]\\
&\leqslant \left\|\sigma^{-1}\right\|_\infty^2 \mathbb{E}\left[\left(\int_t^T |Z_s^{t,x,n}|^2\,\mathrm{d}s\right)\exp\left(\int_t^T \partial_z f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\mathrm{d}W_s - \frac{1}{2}\int_t^T \left|\partial_z f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\right|^2\mathrm{d}s\right)\right.\\
&\qquad\left.\exp\left(\sum_{l=1}^{k-1}\int_t^T \partial_{h_l} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\mathrm{d}M_s(l) - \mu\int_t^T\left(\mathrm{e}^{\partial_{h_l} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)}-1\right)\mathrm{d}s\right)\right]\\
&\leqslant \left\|\sigma^{-1}\right\|_\infty^2 \mathbb{E}\left[\left(\int_t^T |Z_s^{t,x,n}|^2\,\mathrm{d}s\right)^2\right]^{1/2}\\
&\quad \mathbb{E}\left[\exp\left(\int_t^T 2\partial_z f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\mathrm{d}W_s - \frac{1}{2}\int_t^T \left|2\partial_z f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\right|^2\mathrm{d}s\right)\exp\left(\int_t^T \left|\partial_z f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\right|^2\mathrm{d}s\right)\right.\\
&\quad \exp\left(\sum_{l=1}^{k-1}\int_t^T 2\partial_{h_l} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)\mathrm{d}M_s(l) - \mu\int_t^T\left(\mathrm{e}^{2\partial_{h_l} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)}-1\right)\mathrm{d}s\right)\\
&\quad \left.\exp\left(\mu\sum_{l=1}^{k-1}\int_t^T\left(\mathrm{e}^{2\partial_{h_l} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)} - 2\mathrm{e}^{\partial_{h_l} f_{N_s^{t,n}}\left(\Theta_s^{t,x,n}\right)}+1\right)\mathrm{d}s\right)\right]^{1/2}\\
&\leqslant \left\|\sigma^{-1}\right\|_\infty^2 \exp\left(\frac{\|f\|_{\mathrm{lip},z}^2 + \mu(k-1)\left(\mathrm{e}^{2\|f\|_{\mathrm{lip},h}+1}\right)}{2}(T-t)\right)\mathbb{E}\left[\left(\int_t^T |Z_s^{t,x}|^2\,\mathrm{d}s\right)^2\right]^{1/2}
\end{aligned}\tag{3.33}$$

Using Lemma 3.30, we have (where C depends on the Lipschitz constants of f w.r.t. y, z and h, μ and T):

$$\mathbb{E}\left[\left(\int_t^T |Z_s^{t,x,n}|^2\,\mathrm{d}s\right)^2\right] \leqslant C\mathbb{E}\left[|Y^{t,x,n}|_{t,T}^{*,4} + |H^{t,x,n}|_{t,T}^{*,4} + \left(\int_t^T \left|f_{N_s^{t,n}}\left(X_s^{t,x,n},0,0,0\right)\right|\mathrm{d}s\right)^4\right].$$

On the other hand, using Lemma 2.1 of [43], we can bound $\mathbb{E}\left[|\nabla X^{t,x,n}|_{t,T}^{*,4}\right]$ and $\mathbb{E}\left[\left|\left(\nabla X^{t,x,n}\right)^{-1}\right|_{t,T}^{*,4}\right]$, independently of x. Also, we have $|\partial_x f(\Theta_r^{t,x,n})| \leqslant \|\rho\|_{\mathrm{lip},x} + \|\rho\|_{\mathrm{lip},p}|Z_r^{t,x,n}|\|\sigma\|_{\mathrm{lip}}\|\sigma^{-1}\|_\infty^2$, and we can use Cauchy-Schwarz inequality to prove:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |\beta_s^{t,x,n}|^2\,\mathrm{d}s\right]\\
&\quad\leqslant \mathrm{e}^{2\|f\|_{\mathrm{lip},y}(T-t)}\mathbb{E}^{\mathbb{Q}}\left[|\nabla X^{t,x,n}|_{t,T}^{*,8}\right]^{1/4}\mathbb{E}^{\mathbb{Q}}\left[\left|\left(\nabla X^{t,x,n}\right)^{-1}\right|_{t,T}^{*,8}\right]^{1/4}(T-t)\\
&\qquad\qquad \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^s \{2\|\rho\|_{\mathrm{lip},x}^2 + 2\|\rho\|_{\mathrm{lip},p}^2|Z_r^{t,x,n}|^2\|\sigma\|_{\mathrm{lip}}^2\|\sigma^{-1}\|_\infty^4\}\,\mathrm{d}r\right)^2\right]^{1/2}\mathrm{d}s\\
&\quad\leqslant \mathcal{C}(\|f\|_{\mathrm{lip},y}, T-t, \|b\|_{\mathrm{lip}}, \|\sigma\|_{\mathrm{lip}}, \|f\|_{\mathrm{lip},z})\\
&\qquad\qquad \int_t^T\left\{8(T-t)^2\|\rho\|_{\mathrm{lip},x}^4 + 8\|\rho\|_{\mathrm{lip},p}^4\|\sigma\|_{\mathrm{lip}}^4\|\sigma^{-1}\|_\infty^8\mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^s |Z_r^{t,x,n}|^2\,\mathrm{d}r\right)^2\right]\right\}^{1/2}\mathrm{d}s.
\end{aligned}$$

Due to Lemma 3.30, we have (where C depends on the Lipschitz constants of f w.r.t. y, z and h, μ and T):

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^T |Z_r^{t,x,n}|^2\,\mathrm{d}r\right)^2\right]$$
$$\leqslant C\mathbb{E}^{\mathbb{Q}}\left[|Y^{t,x,n}|_{t,T}^{*,4} + |H^{t,x,n}|_{t,T}^{*,4} + \left(\int_t^T \left\{\left|f_{N_r^{t,n}}(X_r^{t,x,n},0,0,0)\right| + \|f\|_{\mathrm{lip},z} + \|f\|_{\mathrm{lip},h}\right\}\,\mathrm{d}r\right)^4\right].$$

We recall that $R_t^{t,x,n} = \partial_x u_n(t,x)$. We have the conclusion:

$$|\partial_x u_n(t,x)|^2 \leqslant C'\left(\sqrt{\mathbb{E}} + \sqrt{\mathbb{E}^{\mathbb{Q}}}\right)\left[1 + |Y^{t,x,n}|_{t,T}^{*,4} + |H^{t,x,n}|_{t,T}^{*,4} + \left(\int_t^T \left|f_{N_r^{t,n}}\left(X_r^{t,x,n},0,0,0\right)\right|\,\mathrm{d}r\right)^4\right],$$

where C' depends on $\|f\|_{\mathrm{lip},y}$, $\|f\|_{\mathrm{lip},h}$, μ, k, T, $\|b\|_{\mathrm{lip}}$, $\|\sigma\|_{\mathrm{lip}}$, $\|\rho\|_{\mathrm{lip},x}$, $\|\rho\|_{\mathrm{lip},p}$ and $\|\sigma^{-1}\|_\infty$. $\square$

Chapitre 4

EDSR ergodique à bruit Lévy et application au comportement en temps long de la solution de viscosité d'une EIDP

Résumé

Nous étudions dans ce chapitre des EDSRE pour lesquelles le bruit de l'EDS sous-jacente a un bruit donné par un processus de Lévy et une dérive faiblement dissipative. Nous étudions l'existence et l'unicité de la solution de telles EDSRE et nous appliquons nos résultats à un problème de contrôle ergodique optimal. En particulier, nous obtenons le comportement en temps long de la solution de viscosité d'une équation intégro-différentielle partielle, avec une vitesse de convergence exponentielle.

Mots-clés : équation différentielle stochastique rétrograde ergodique ; bruit Lévy ; équation intégro-différentielle partielle ; comportement en temps long ; vitesse de convergence.

Abstract

We study in this chapter an EBSDE whose underlying SDE has a Lévy noise and a weakly dissipative drift. We study existence and uniqueness of solution of such EBSDEs and we apply our results to an ergodic optimal control problem. Finally, we obtain the large time behaviour of the viscosity solution of a partial integro-differential equation, with an exponential rate of convergence.

Key words: ergodic backward stochastic differential equation; Lévy noise; partial integro-differential equation; large time behaviour; rate of convergence.

4.1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a right continuous and complete stochastic basis. The given data of our equation consists in:

- L, a Lévy d-dimensional martingale; it means that we can write $L_t = QW_t + \int_0^t \int_B y\widetilde{N}(\mathrm{d}s, \mathrm{d}y)$, where $Q \in \mathcal{M}_d(\mathbb{R})$, W is a standard d-dimensional Brownian motion, and $\widetilde{N}$ is the compensated Poisson random measure of N (whose intensity measure can be written as $\nu(\mathrm{d}y)\mathrm{d}t$), well defined on $\mathbb{R}_+ \times B$, with $B = \mathbb{R}^d \backslash \{0\}$ the Blackwell space;
- $x \in \mathbb{R}^d$;
- X^x, an $\mathbb{R}^d$-valued process, starting from x, which is the solution of the SDE: for all $t \in \mathbb{R}_+$,

$$X_t^x = x + \int_0^t \Xi(X_s^x)\,\mathrm{d}s + \int_0^t \sigma\,\mathrm{d}L_s; \tag{4.1}$$

- $f : \mathbb{R}^d \times (\mathbb{R}^d)^* \times \mathrm{L}^2(\nu) \to \mathbb{R}$ a measurable function.

We study the following EBSDE in infinite horizon: for all $t, T \in \mathbb{R}_+$, such that $0 \leqslant t \leqslant T < \infty$,

$$Y_t^x = Y_T^x + \int_t^T \{f(X_s^x, Z_s^x, U_s^x) - \lambda\}\,\mathrm{d}s - \int_t^T Z_s^x\,\mathrm{d}W_s - \int_t^T \int_B U_s^x(y)\,\widetilde{N}(\mathrm{d}s, \mathrm{d}y), \tag{4.2}$$

where the unknown is the quadruplet (Y^x, Z^x, U^x, λ), with:

- Y^x a real-valued and progressively measurable process;
- Z^x an $(\mathbb{R}^d)^*$-valued and progressively measurable process;
- U^x an $\mathrm{L}^2(\nu)$-valued and progressively measurable process;
- λ a real number.

Ergodic BSDEs, with an additive Brownian noise, have already been widely studied. This class of BSDEs was first studied by Fuhrman, Hu and Tessitore in [26] in order to study optimal ergodic control problem; their main assumption was the strong dissipativity of Ξ, that is to say:

$$\exists \eta > 0,\ \forall x, x' \in \mathbb{R}^d,\ \langle \Xi(x) - \Xi(x'), x - x' \rangle \leqslant -\eta |x - x'|^2.$$

This assumption was then weakened in [22], where Ξ can be written as the sum of a disspative function and a bounded function. Under similar assumptions, the authors of [35] studied the large time behaviour of finite horizon BSDEs:

$$Y_t^{T,x} = g(X_T^x) + \int_t^T f(X_s^x, Z_s^{T,x})\,\mathrm{d}s - \int_t^T Z_s^{T,x}\,\mathrm{d}W_s, \quad t \in [0, T], \tag{4.3}$$

and they proved that it was linked with ergodic BSDEs. More precisely, they proved the existence of a constant $L \in \mathbb{R}$, such that for all $x \in \mathbb{R}^d$,

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to\infty}{\longrightarrow} L;$$

and they even obtained an exponential rate of convergence. More recently, the two authors of this paper proved that this convergence and this rate still hold when the noise is multiplicative (see [32]).

Ergodic BSDEs with jumps have already been investigated by Cohen and Fedyashov in [17]. They obtain existence and uniqueness of the solution of ergodic BSDEs using an auxiliary discounted equation; for the reader convenience, we detailed here the proofs of some of their results. Moreover, due to Itô's formula and a comparison theorem (see e.g. [23]), under some additional assumptions, the ergodic BSDE

$$Y_t^x = Y_T^x + \int_t^T \{f(X_s^x, Z_s^x, U_s^x \cdot \delta) - \lambda\} \,\mathrm{d}s - \int_t^T Z_s^x \,\mathrm{d}W_s - \int_t^T \int_B U_s^x(y) \,\widetilde{N}(\mathrm{d}s, \mathrm{d}y), \quad 0 \leqslant t \leqslant T < \infty \tag{4.4}$$

has a solution such that we can write $Y_t^x = v(X_t^x)$, where (v, λ) is the viscosity solution of:

$$\mathcal{L}v(x) + f(x, \nabla v(x)\sigma Q, \mathcal{J}v(x)) = \lambda, \quad \forall x \in \mathbb{R}^d, \tag{4.5}$$

where $\mathcal{L}$ is the generator of the Kolmogorov semigroup of X^x, and $\mathcal{J}$ a jump operator.

Exactly the same way (see [11]), the solution of the finite horizon BSDE

$$Y_t^{T,x} = g(X_T^x) + \int_t^T f(X_s^x, Z_s^{T,x}, U_s^{T,x} \cdot \delta) \,\mathrm{d}s - \int_t^T Z_s^{T,x} \,\mathrm{d}W_s - \int_t^T \int_B U_s^{T,x}(y) \,\widetilde{N}(\mathrm{d}s, \mathrm{d}y), \quad t \in [0, T], \tag{4.6}$$

can be written as $Y_t^{T,x} = u(T-t, X_t^x)$, where u is the viscosity solution of the Cauchy problem:

$$\left\{ \begin{array}{lcll} \partial_t u(t,x) & = & \mathcal{L}u(t,x) + f(x, \partial_x u(t,x)\sigma Q, \mathcal{J}u(t,x)) & \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) & = & g(x) & \forall x \in \mathbb{R}^d. \end{array} \right. \tag{4.7}$$

Uniqueness of the viscosity solution of such PIDEs can be used in order to get the large time behaviour of the solution of BSDE (4.6), with an exponential rate of convergence. These results can be directly applied to the corresponding PIDEs, to get:

$$u(T,x) - \lambda T - v(x) \underset{T \to \infty}{\longrightarrow} L. \tag{4.8}$$

In [6], Barles, Ley and Topp study the large time behaviour of PIDEs in a periodic setting; under their assumptions, they prove a convergence similar to (4.8), but without any rate of convergence. This topic is also investigated in [4] and [5].

Moreover, our results can also be applied to an optimal control problem.

The paper is organised as follows. In section 2, we study an SDE slightly more general than (4.1): indeed, it is the SDE satisfied by X after a change of probability space, due to Girsanov's theorem. Section 3 is devoted to an auxiliary discounted BSDE of infinite horizon; then, we prove existence and uniqueness for the solution of the ergodic BSDE (4.2). Section 4 consists of existence and uniqueness of the viscosity solution for the problems (4.7) and (4.5). The large time behaviour of the solution of (4.6) will be presented in section 5, and is directly applied to the large time behaviour for the viscosity solution of (4.7). Finally, section 6 applies the results of this paper to an optimal ergodic control problem.

4.2 The forward SDE

We consider the following SDE:

$$\mathrm{d}X_t = \Xi(t, X_t) \,\mathrm{d}t + \sigma \,\mathrm{d}L_t, \qquad X_{t_0} = x \tag{4.9}$$

and we make the following assumptions:

- $\Xi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous w.r.t. x, uniformly in t; $|\Xi(t,x)| \leqslant \xi_1 + \xi_2 |x|$;
- Ξ is weakly dissipative, *i.e.* $\Xi(t,x) = \xi(x) + b(t,x)$, where ξ is η-dissipative, locally Lipschitz and b is bounded;
- $\sigma Q \in \mathrm{GL}_d(\mathbb{R})$;
- ν is a finite measure and $m_{p_0}(\nu) := \displaystyle\int_B |y|^{p_0} \,\nu(\mathrm{d}y) < \infty$, for some $p_0 \geqslant 2$.

Theorem 4.1. *For all $p \in [2, p_0]$ and $T \in (t_0, \infty)$, there exists a unique process $X^{t_0,x} \in \mathrm{L}^p_{\mathcal{P}}(\Omega, \mathcal{C}([t_0, T], \mathbb{R}^d))$ strong solution to (4.9).*

Proof. We fix $T > t_0$, and we define $\mathcal{X}_T$ the set of predictable processes $Y : [t_0, T] \times \Omega \to \mathbb{R}^d$ such that

$$\|Y\|_T := \sup_{t \in [t_0,T]} \mathbb{E}[|Y_t|^p]^{1/p} < \infty.$$

For every $\beta > 0$ and $Y \in \mathcal{X}_T$, we set

$$\|Y\|_{T,\beta} := \sup_{t \in [t_0,T]} \mathrm{e}^{-\beta t} \mathbb{E}[|Y_t|^p]^{1/p}.$$

$(\mathcal{X}_T, \| \bullet \|_T)$ is a Banach space, and for every $\beta > 0$, the norms $\| \bullet \|_{T,\beta}$ and $\| \bullet \|_T$ are equivalent. We define the function $I : Y \mapsto x + \int_{t_0}^t \Xi(s, Y_s)\, \mathrm{d}s + \int_{t_0}^t \sigma Q\, \mathrm{d}W_s + \int_{t_0}^t \int_B \sigma y\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y)$. When $Y \in \mathcal{X}_T$, we have:

$$\begin{aligned}
\|I(Y)\|^p_{T,\beta} &\leqslant \sup_{t \in [t_0,T]} \mathrm{e}^{-p\beta t} \mathbb{E}\left[\left|\int_{t_0}^t \Xi(s, Y_s)\, \mathrm{d}s\right|^p\right] + \sup_{t \in [t_0,T]} \mathrm{e}^{-p\beta t} \mathbb{E}\left[\left|\int_{t_0}^t \sigma Q\, \mathrm{d}W_s\right|^p\right] \\
&\qquad + \sup_{t \in [t_0,T]} \mathrm{e}^{-p\beta t} \mathbb{E}\left[\left|\int_{t_0}^t \int_B \sigma y\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y)\right|^p\right] \\
&\leqslant \sup_{t \in [t_0,T]} \mathrm{e}^{-p\beta t} t^{p-1} \mathbb{E}\left[\int_{t_0}^t |\Xi(s, Y_s)|^p\, \mathrm{d}s\right] + \mathbb{E}\left[\sup_{t \in [t_0,T]} \left|\int_{t_0}^t \sigma Q\, \mathrm{d}W_s\right|^p\right] \\
&\qquad + \mathbb{E}\left[\sup_{t \in [t_0,T]} \left|\int_{t_0}^t \int_B \sigma y\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y)\right|^p\right] \\
&\leqslant \sup_{t \in [t_0,T]} \mathrm{e}^{-p\beta t} (2t)^{p-1} \mathbb{E}\left[\int_{t_0}^t (\xi_1^p + \xi_2^p |Y_s|^p)\, \mathrm{d}s\right] + C_p \left(\int_{t_0}^T |\sigma Q|_F^2\, \mathrm{d}s\right)^{p/2} \\
&\qquad + C_p \mathbb{E}\left[\left(\int_{t_0}^T \int_B |\sigma y|^2\, N(\mathrm{d}s, \mathrm{d}y)\right)^{p/2}\right] \\
&\leqslant (2T)^{p-1} \mathbb{E}\left[\int_{t_0}^T (\xi_1^p + \xi_2^p |Y_s|^p)\, \mathrm{d}s\right] + C_p |\sigma|_F^p |Q|_F^p T^{p/2} \\
&\qquad + C_p |\sigma|_F^p \mathbb{E}\left[\left(\int_B |y|^2\, N([t_0, T], \mathrm{d}y)\right)^{p/2}\right],
\end{aligned}$$

where the constant C_p is given by Burkholder-Davis-Gundy inequality. But, for $J \sim \mathcal{P}((T - t_0)\nu(B))$ and X an independent i.i.d.r.v. sequence of common law $\dfrac{\nu}{\nu(B)}$ over B, we get, using Minkowski inequality:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_B |y|^2\, N([t_0, T], \mathrm{d}y)\right)^{p/2}\right] &= \mathbb{E}\left[\left(\sum_{j=1}^J |X_j|^2\right)^{p/2}\right] = \sum_{j=1}^\infty \mathbb{P}(J = j) \mathbb{E}\left[\left(\sum_{k=1}^j |X_k|^2\right)^{p/2}\right] \\
&\leqslant \sum_{j=1}^\infty \mathbb{P}(J = j) \left(\sum_{k=1}^j \mathbb{E}[|X_k|^p]^{2/p}\right)^{p/2} \leqslant \sum_{j=1}^\infty \mathbb{P}(J = j) j^{p/2} \mathbb{E}[|X_1|^p] \\
&= \mathbb{E}[J^{p/2}] \int_B |y|^p\, \frac{\nu(\mathrm{d}y)}{\nu(B)}.
\end{aligned}$$

This way, we have, for a constant C depending only on p, T, $|Q|_F$, $\nu(B)$, ξ_1, ξ_2 and $|\sigma|_F$:

$$\|I(Y)\|^p_{T,\beta} \leqslant C(1 + \|Y\|^p_{T,\beta}).$$

This proves that $I(Y) \in \mathcal{X}_T$. In order to prove that I is a contraction, we see that, for $Y, Z \in \mathcal{X}_T$, we have:

$$\begin{aligned}
\|I(Y) - I(Z)\|_{T,\beta}^p &= \sup_{t\in[t_0,T]} \mathrm{e}^{-p\beta t}\mathbb{E}\left[\left|\int_{t_0}^t \{\Xi(s,Y_s) - \Xi(s,Z_s)\}\,\mathrm{d}s\right|^p\right] \\
&\leqslant \sup_{t\in[t_0,T]} \mathrm{e}^{-p\beta t} t^{p-1}\mathbb{E}\left[\int_{t_0}^t |\Xi(s,Y_s) - \Xi(s,Z_s)|^p\,\mathrm{d}s\right] \\
&\leqslant T^{p-1}\|\Xi\|_{\mathrm{lip}}^p \sup_{t\in[t_0,T]} \mathrm{e}^{-p\beta t}\int_{t_0}^t \mathbb{E}[|Y_s - Z_s|^p]\,\mathrm{d}s \\
&\leqslant T^{p-1}\|\Xi\|_{\mathrm{lip}}^p \|Y - Z\|_{T,\beta}^p \sup_{t\in[t_0,T]} \mathrm{e}^{-p\beta t}\int_{t_0}^t \mathrm{e}^{p\beta s}\,\mathrm{d}s \\
&\leqslant \frac{T^{p-1}\|\Xi\|_{\mathrm{lip}}^p}{p\beta}\|Y - Z\|_{T,\beta}^p.
\end{aligned}$$

Taking β large enough ends the proof. □

Proposition 4.2. *For all $p \in (0, p_0]$ and $T > t_0$, we have:*

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} |X_t^{t_0,x}|^p\right] \leqslant C(1 + |x|^p),$$

where C only depends on p, T, ξ_1, ξ_2, $|\sigma|_F$, $|Q|_F$, $m_p(\nu)$ and $\nu(B)$.

Proof. This is a straightforward consequence of Burkholder-Davis-Gundy inequality and Gronwall's lemma. □

Proposition 4.3. *Let $T > t_0$, $\beta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathrm{L}^2(\nu)$ be two bounded functions, continuous in the second variable, and we suppose that $1 + \gamma$ is non-negative. Using Girsanov theorem, we define a new probability measure $\mathbb{Q}_T$ by:*

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\int_{t_0}^{\bullet} \langle \beta(s, X_s^{t_0,x}), \mathrm{d}W_s\rangle + \int_{t_0}^{\bullet}\int_B \gamma(s, X_{s^-}^{t_0,x})(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y)\right)_{\bullet\in[t_0,T]},$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential. Then, for all $p \in [2, p_0]$, we have:

$$\sup_{T\geqslant t_0} \mathbb{E}^{\mathbb{Q}_T}\left[|X_T^{t_0,x}|^p\right] \leqslant C(1 + |x|^p),$$

and the constant C only depends on p, η, $\|b\|_\infty$, $|\sigma|_F$, $|Q|_F$, $m_1(\nu)$, $m_2(\nu)$, $m_p(\nu)$, $\|\beta\|_\infty$ and $\|\gamma\|_\infty$.

Proof. The case $p = 2$ is easier, so we will focus on the case $p > 2$.
By Itô's formula, we get, for $0 \leqslant t \leqslant T$:

$$\begin{aligned}
|X_t^{t_0,x}|^p - |x|^p &= p\int_{t_0}^t |X_s^{t_0,x}|^{p-2}\langle X_s^{t_0,x}, \Xi(s, X_s^{t_0,x})\rangle\,\mathrm{d}s + p\int_{t_0}^t |X_s^{t_0,x}|^{p-2}\langle X_s^{t_0,x}, \sigma Q\,\mathrm{d}W_s\rangle \\
&+ \frac{p(p-2)}{2}\int_{t_0}^t |X_s^{t_0,x}|^{p-4}|(X_s^{t_0,x})^*\sigma Q|^2\,\mathrm{d}s + \frac{p}{2}\int_{t_0}^t |X_s^{t_0,x}|^{p-2}|\sigma Q|_F^2\,\mathrm{d}s \\
&+ \int_{t_0}^{t^+}\int_B \left\{|X_{s^-}^{t_0,x} + \sigma y|^p - |X_{s^-}^{t_0,x}|^p\right\}(N(\mathrm{d}s,\mathrm{d}y) - \nu(\mathrm{d}y)\mathrm{d}s) \\
&+ \int_{t_0}^t \int_B \left\{|X_s^{t_0,x} + \sigma y|^p - |X_s^{t_0,x}|^p - p|X_s^{t_0,x}|^{p-2}\langle X_s^{t_0,x}, \sigma y\rangle\right\}\nu(\mathrm{d}y)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
|X_t^{t_0,x}|^p - |x|^p &= p\int_{t_0}^t |X_s^{t_0,x}|^{p-2}\langle X_s^{t_0,x}, \Xi(s,X_s^{t_0,x})\rangle\,\mathrm{d}s + p\int_{t_0}^t |X_s^{t_0,x}|^{p-2}\langle X_s^{t_0,x}, \sigma Q\,(\mathrm{d}W_s - \beta(s,X_s^{t_0,x})\mathrm{d}s)\rangle \\
&+ p\int_{t_0}^t |X_s^{t_0,x}|^{p-2}\langle X_s^{t_0,x}, \sigma Q\beta(s,X_s^{t_0,x})\rangle\,\mathrm{d}s + \frac{p(p-2)}{2}\int_{t_0}^t |X_s^{t_0,x}|^{p-4}|(X_s^{t_0,x})^*\sigma Q|^2\,\mathrm{d}s \\
&+ \frac{p}{2}\int_{t_0}^t |X_s^{t_0,x}|^{p-2}|\sigma Q|_F^2\,\mathrm{d}s + \int_{t_0}^{t^+}\int_B \left\{|X_{s^-}^{t_0,x}+\sigma y|^p - |X_{s^-}^{t_0,x}|^p\right\}\gamma(s,X_{s^-}^{t_0,x})(y)\nu(\mathrm{d}y)\mathrm{d}s \\
&+ \int_{t_0}^{t^+}\int_B \left\{|X_{s^-}^{t_0,x}+\sigma y|^p - |X_{s^-}^{t_0,x}|^p\right\}(N(\mathrm{d}s,\mathrm{d}y) - (1+\gamma(s,X_{s^-}^{t_0,x})(y))\nu(\mathrm{d}y)\mathrm{d}s) \\
&+ \int_{t_0}^t\int_B \left\{|X_s^{t_0,x}+\sigma y|^p - |X_s^{t_0,x}|^p - p|X_s^{t_0,x}|^{p-2}\langle X_s^{t_0,x},\sigma y\rangle\right\}\nu(\mathrm{d}y)\mathrm{d}s
\end{aligned}
$$

We can take the expectation, and then the derivative w.r.t. t:

$$
\begin{aligned}
\partial_t\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^{t_0,x}|^p\right] &= p\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^{t_0,x}|^{p-2}\langle X_t^{t_0,x},\Xi(t,X_t^{t_0,x})\rangle\right] + p\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^{t_0,x}|^{p-2}\langle X_t^{t_0,x},\sigma Q\beta(t,X_t^{t_0,x})\rangle\right] \\
&+ \frac{p(p-2)}{2}\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^{t_0,x}|^{p-4}|(X_t^{t_0,x})^*\sigma Q|^2\right] + \frac{p}{2}\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^{t_0,x}|^{p-2}|\sigma Q|_F^2\right] \\
&+ \mathbb{E}^{\mathbb{Q}_T}\left[\int_B \left\{|X_t^{t_0,x}+\sigma y|^p - |X_t^{t_0,x}|^p\right\}\gamma(t,X_{t^-}^{t_0,x})(y)\nu(\mathrm{d}y)\right] \\
&+ \mathbb{E}^{\mathbb{Q}_T}\left[\int_B \left\{|X_t^{t_0,x}+\sigma y|^p - |X_t^{t_0,x}|^p - p|X_t^{t_0,x}|^{p-2}\langle X_t^{t_0,x},\sigma y\rangle\right\}\nu(\mathrm{d}y)\right]
\end{aligned}
$$

Using Taylor-Lagrange and Cauchy-Schwarz inequalities, we obtain:

$$
\left||X_t^{t_0,x}+\sigma y|^p - |X_t^{t_0,x}|^p\right| \leqslant \sup_{w\in[0,1]} p|X_t^{t_0,x}+w\sigma y|^{p-1}|\sigma y| \leqslant p2^{p-2}\left(|X_t^{t_0,x}|^{p-1} + |\sigma y|^{p-1}\right)|\sigma y|
$$

$$
\begin{aligned}
\left||X_t^{t_0,x}+\sigma y|^p - |X_t^{t_0,x}|^p - p|X_t^{t_0,x}|^{p-2}\langle X_t^{t_0,x},\sigma y\rangle\right| &\leqslant \sup_{w\in[0,1]}\frac{p(p-1)}{2}|X_t^{t_0,x}+w\sigma y|^{p-2}|\sigma y|^2 \\
&\leqslant \frac{p(p-1)}{2}(1\vee 2^{p-3})\left(|X_t^{t_0,x}|^{p-2} + |\sigma y|^{p-2}\right)|\sigma y|^2
\end{aligned}
$$

This way:

$$
\begin{aligned}
\partial_t\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^{t_0,x}|^p\right] &= -p\eta\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^x|^p\right] + p(\|b\|_\infty + |\sigma|_F|Q|_F\|\beta\|_\infty + 2^{p-2}|\sigma|_F m_1(\nu)\|\gamma\|_\infty)\mathbb{E}^{\mathbb{Q}_T}\left[|X_t^x|^{p-1}\right] \\
&+ \frac{p(p-1)}{2}|\sigma|_F^2\left(|Q|_F^2 + (1\vee 2^{p-3})m_2(\nu)\right)\mathbb{E}\left[|X_t^{t_0,x}|^{p-2}\right] \\
&+ \left(p2^{p-2}\|\gamma\|_\infty + \frac{p(p-1)}{2}(1\vee 2^{p-3})\right)|\sigma|_F^p m_p(\nu)
\end{aligned}
$$

Using Young's inequality, we can show that, for some $\varepsilon > 0$, we have:

$$
\partial_t\mathbb{E}\left[|X_t^{t_0,x}|^p\right] \leqslant -\varepsilon\mathbb{E}\left[|X_t^{t_0,x}|^p\right] + C_\varepsilon,
$$

where C_ε depends only on p, η, $\|b\|_\infty$, $|\sigma|_F$, $|Q|_F$, $m_1(\nu)$, $m_2(\nu)$, $m_p(\nu)$, $\|\beta\|_\infty$ and $\|\gamma\|_\infty$. So:

$$
\partial_t\left(\mathrm{e}^{\varepsilon t}\mathbb{E}\left[|X_t^{t_0,x}|^p\right]\right) \leqslant C_\varepsilon\mathrm{e}^{\varepsilon t}, \text{ and then } \mathbb{E}\left[|X_t^{t_0,x}|^p\right] \leqslant C_\varepsilon(1+|x|^p).
$$

□

Remark 4.4. This result is still true when $p \in (0,2)$, using Jensen inequality and $(1+z)^\alpha \leqslant 1+z^\alpha$ for $z \geqslant 0$ and $\alpha \in (0,1)$. The only difference is that the dependence on $m_p(\nu)$ disappears.

We recall the definition of the semigroup $\mathcal{P}$ of the SDE (4.9):

$$\mathcal{P}_{t,t_0}[\phi](x) = \mathbb{E}[\phi(X_t^{t_0,x})].$$

As usual, when only one time parameter is mentioned, it means that the initial time $t_0 = 0$.

Theorem 4.5. *Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function satisfying $|\phi(x)| \leqslant c_\phi(1 + |x|^2)$. Then, we have:*

$$\forall x, y \in \mathbb{R}^d,\ \forall t \geqslant 0,\ |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leqslant \widehat{c} c_\phi (1 + |x|^2 + |y|^2) \mathrm{e}^{-\theta t}, \tag{4.10}$$

where $\widehat{c}$ and θ only depend on η, $\|b\|_\infty$, $\|\sigma\|_\infty$, $|Q|_F$ and $m_2(\nu)$.

Proof. See Theorem 5 of [17], with a weighted total variation. □

Corollary 4.6. *Let $\beta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathrm{L}^2(\nu)$ be two bounded and progressively measurable functions. We also assume that $1 + \gamma$ is non-negative. Using Girsanov theorem, we define a new probability measure by:*

$$\frac{\mathrm{d}\mathbb{P}^{x,t}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\int_0^\bullet \langle \beta(s, X_s^x), \mathrm{d}W_s \rangle + \int_0^\bullet \int_B \gamma(s, X_{s-}^x)(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y) \right)_{\bullet \in [0,t]},$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential. Moreover, we assume that β and γ are the pointwise limits of some sequences (β_ε) and (γ_ε) of functions of class $\mathcal{C}_b^1$ w.r.t. x, and uniformly bounded by $\|\beta\|_\infty$ and $\|\gamma\|_\infty$. Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function with $|\phi(x)| \leqslant c_\phi\left(1 + |x|^2\right)$. Then we have,

$$\forall x, y \in \mathbb{R}^d,\ \forall t \geqslant 0,\ \left| \widetilde{\mathbb{E}}^{x,t}\left[\phi(X_t^x)\right] - \widetilde{\mathbb{E}}^{y,t}\left[\phi(X_t^y)\right] \right| \leqslant \widehat{c} c_\phi \left(1 + |x|^2 + |y|^2\right) \mathrm{e}^{-\theta t},$$

where $\widehat{c}$ and θ only depend on η, $\|b\|_\infty$, $\|\sigma\|_\infty$, $|Q|_F$, $m_2(\nu)$, $\|\beta\|_\infty$ and $\|\gamma\|_\infty$.

Proof. See Lemma 4 of [17]. □

Lemma 4.7. *Let $\psi : \mathbb{R}^d \times \left(\mathbb{R}^d\right)^* \to \mathbb{R}$ continuous w.r.t. the first variable and Lipschitz continuous w.r.t. the second one. Let $\zeta, \zeta' : \mathbb{R}_+ \times \mathbb{R}^d \to \left(\mathbb{R}^d\right)^*$ be such that for every $t \geqslant 0$, $\zeta(t, \bullet)$ and $\zeta'(t, \bullet)$ are continuous, and set:*

$$\widetilde{\Gamma}(t, x) = \begin{cases} \dfrac{\psi(x, \zeta(t,x)) - \psi(x, \zeta'(t,x))}{|\zeta(t,x) - \zeta'(t,x)|^2}(\zeta(t,x) - \zeta'(t,x)), & \text{if } \zeta(t,x) \neq \zeta'(t,x), \\ 0, & \text{if } \zeta(t,x) = \zeta'(t,x). \end{cases}$$

Then, there exists a uniformly bounded sequence of $\mathcal{C}^1$ functions w.r.t. x with bounded derivatives $\left(\widetilde{\Gamma}_n\right)_{n \geqslant 1}$ (i.e. for all n, $\widetilde{\Gamma}_n$ has bounded derivatives w.r.t. x – the bound of derivatives can depend on n – and $\sup_{n \geqslant 1} \left\| \widetilde{\Gamma}_n(t, \bullet) \right\|_\infty < \infty$ for every $t \geqslant 0$), such that $\widetilde{\Gamma}_n \underset{n \to \infty}{\longrightarrow} \widetilde{\Gamma}$ pointwise.

Proof. See the Lemma 3.7 of [35]: we can approximate the Lipschitz functions by $\mathcal{C}^1$ functions with bounded derivatives and construct a new sequence having the required regularity. □

4.3 The backward SDEs

4.3.1 The infinite horizon BSDE

We consider the infinite horizon BSDE: for every $0 \leqslant t \leqslant T < \infty$,

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T \{f(X_s^x, Z_s^{\alpha,x}, U_s^{\alpha,x}) - \alpha Y_s^{\alpha,x}\}\, \mathrm{d}s - \int_t^T Z_s^{\alpha,x}\, \mathrm{d}W_s - \int_t^T \int_B U_s^{\alpha,x}(y) \widetilde{N}(\mathrm{d}s, \mathrm{d}y), \quad (4.11)$$

and we make the following assumptions, for all $T > 0$:

- the process X satisfies the assumptions of the previous section, with a drift Ξ independent of t;
- f is Lipschitz continuous;
- $f(x,z,u) - f(x,z,u') \leqslant (u-u') \cdot \gamma_t^{x,z,u,u'}$, ($\cdot$ is the scalar product in $\mathrm{L}^2(\nu)$) with γ measurable and there exist $-1 < C_1 \leqslant 0$ and $C_2 \geqslant 0$ such that $C_1(1 \wedge |v|) \leqslant \gamma_t^{x,z,u,u'}(v) \leqslant C_2(1 \wedge |v|)$.

Lemma 4.8. *Under those assumptions, there exists a process γ such that:*

$$f(X_t, Z_t, U_t) - f(X_t, Z_t, U_t') = (U_t - U_t') \cdot \gamma_t.$$

Proof. See Lemma 6 of [17]. □

Theorem 4.9. *For every $x \in \mathbb{R}^d$ and $\alpha > 0$, equation (4.11) admits a unique solution $(Y^{\alpha,x}, Z^{\alpha,x}, U^{\alpha,x})$, such that $Y^{\alpha,x}$ is a continuous and bounded process, and with $Z^{\alpha,x}$ and $U^{\alpha,x}$ progressively measurable and square-integrable w.r.t. W and $\widetilde{N}$. Moreover, we have:*

$$\exists C > 0,\ \forall \alpha > 0,\ \forall x \in \mathbb{R}^d,\ |Y_t^{\alpha,x}| \leqslant \frac{C}{\alpha}(1 + |X_t^x|).$$

Proof. See Theorem 8 of [17]. The upperbound is somewhat classical (see e.g. [32]). □

For any $\alpha > 0$, we define a function $v^\alpha : x \to Y_0^{\alpha,x}$.

Proposition 4.10. *The function v^α is continuous and we have the following representations:*

$$\forall t \geqslant 0,\ Y_t^{\alpha,x} = v^\alpha(X_t^x) \text{ and } U_t^{\alpha,x}(y) = v^\alpha(X_{t^-}^x + \sigma y) - v^\alpha(X_{t^-}^x)\ \nu\text{-a.e.}$$

Proof. For continuity and representation of $Y^{\alpha,x}$, see [57]. Let us check the identification of $U^{\alpha,x}$. We set

$$\widetilde{U}_t^{\alpha,x}(y) = v^\alpha(X_{t^-}^x + \sigma y) - v^\alpha(X_{t^-}^x).$$

We get:

$$\int_B U_t^{\alpha,x}(y)\, \Delta N(t, \mathrm{d}y) = \Delta Y_t^{\alpha,x} = v^\alpha(X_t^x) - v^\alpha(X_{t^-}^x) = \int_B \widetilde{U}_t^{\alpha,x}(y)\, \Delta N(t, \mathrm{d}y).$$

Hence, for any $T > 0$, $\mathbb{E}\left[\int_0^T \int_B (U_t^{\alpha,x}(y) - \widetilde{U}_t^{\alpha,x}(y))^2\, N(\mathrm{d}t, \mathrm{d}y)\right] = 0$ which implies:

$$\mathbb{E}\left[\int_0^T \int_B (U_t^{\alpha,x}(y) - \widetilde{U}_t^{\alpha,x}(y))^2\, \nu(\mathrm{d}y)\, \mathrm{d}t\right] = 0.$$

□

Proposition 4.11. *For every $\alpha > 0$, we have:*

$$\exists C > 0, \forall x, x' \in \mathbb{R}^d, \ |v^\alpha(x) - v^\alpha(x')| \leqslant C(1 + |x| + |x'|)|x - x'|.$$

Proof. We approximate v^α by a sequence $(v^{\varepsilon,\alpha})$ of $\mathcal{C}_b^1$-functions converging uniformly on every compact set. The functions f and Ξ are also approached by some sequences of $\mathcal{C}_b^1$-functions converging uniformly. We consider the BSDE:

$$Y_t^x = v^{\alpha,\varepsilon}(X_1^{\varepsilon,x}) + \int_t^1 \{f^\varepsilon(X_s^{\varepsilon,x}, Z_s^x, U_s^x) - \alpha Y_s^x\} \, \mathrm{d}s - \int_t^1 Z_s^x \, \mathrm{d}W_s - \int_t^1 \int_B U_s^x(y) \, \widetilde{N}(\mathrm{d}s, \mathrm{d}y), \quad t \in [0,1],$$

and there exists a function u^ε such that $Y_0^x = u^\varepsilon(x)$. Using Girsanov's theorem, there exists a probability measure $\mathbb{Q}$ under which we can write:

$$u^\varepsilon(x) = \mathbb{E}^{\mathbb{Q}} \left[\mathrm{e}^{-\alpha} v^{\alpha,\varepsilon}(X_1^{\varepsilon,x}) + \int_0^1 \mathrm{e}^{-\alpha s} f^\varepsilon(X_s^{\varepsilon,x}, 0, 0) \, \mathrm{d}s \right].$$

Then, using a Bismut-Elworthy formula (see [59]):

$$\partial_x u^\varepsilon(x) = \mathbb{E}^{\mathbb{Q}} \left[\mathrm{e}^{-\alpha} v^{\alpha,\varepsilon}(X_1^{\varepsilon,x}) R_1^{\varepsilon,x} + \int_t^T \mathrm{e}^{-\alpha s} f^\varepsilon(X_s^{\varepsilon,x}, 0, 0) R_s^{\varepsilon,x} \, \mathrm{d}s \right],$$

where $R_s^{\varepsilon,x} = \dfrac{1}{s} \displaystyle\int_0^s \sigma^{-1} \nabla X_r^{\varepsilon,x} \, \mathrm{d}W_r^{\mathbb{Q}}$. After standard calculations, $\mathbb{E}\left[|R_s^{\varepsilon,x}|^2 \right] \leqslant \dfrac{d^2 |\sigma^{-1}|_F^2}{s} \mathrm{e}^{2\|\Xi\|_{\mathrm{lip}}}$. Using Hölder's inequality, we find:

$$\begin{aligned} |\partial_x u^\varepsilon(x)| &\leqslant \mathrm{e}^{-\alpha} \mathbb{E}^{\mathbb{Q}} \left[\left| v^{\alpha,\varepsilon}(X_1^{\varepsilon,x}) \right|^2 \right]^{1/2} d|\sigma^{-1}|_F \mathrm{e}^{\|\Xi\|_{\mathrm{lip}}} + \int_0^1 \mathrm{e}^{-\alpha s} \mathbb{E}^{\mathbb{Q}} \left[|f^\varepsilon(X_s^{\varepsilon,x}, 0, 0)|^2 \right]^{1/2} \frac{d|\sigma^{-1}|_F}{\sqrt{s}} \mathrm{e}^{\|\Xi\|_{\mathrm{lip}}} \, \mathrm{d}s \\ &\leqslant C(1 + |x|). \end{aligned}$$

Due to the mean value theorem, we have:

$$|u^\varepsilon(x) - u^\varepsilon(x')| \leqslant C(1 + |x| + |x'|)|x - x'|.$$

Finally, Lemma 4.33 gives us what we claimed, because, for any $R > 0$:

$$\begin{aligned} \mathbb{E}\left[|u^\varepsilon(x) - v^\alpha(x)|^2 \right] &\leqslant \mathbb{E}\left[|v^{\varepsilon,\alpha}(X_1^{\varepsilon,x}) - v^\alpha(X_1^{\varepsilon,x})|^2 + \|f - f^\varepsilon\|_\infty \right] \\ &\leqslant \|v^{\varepsilon,\alpha} - v^\alpha\|_{\infty,\mathcal{B}_R}^2 + C(1 + |x|^2) \mathbb{P}(|X_1^{\varepsilon,x}| > R)^{1/2} + \|f - f^\varepsilon\|_\infty, \end{aligned}$$

and by Markov's inequality, $\mathbb{P}(|X_1^{\varepsilon,x}| > R) \leqslant \dfrac{C(1 + |x|^2)}{R}$. □

Proposition 4.12. *There exists a constant $C > 0$, such that for every $\alpha > 0$, we have:*

$$\forall x, x' \in \mathbb{R}^d, \ |v^\alpha(x) - v^\alpha(x')| \leqslant C(1 + |x|^2 + |x'|^2).$$

Proof.

1. We approximate Ξ by a sequence $(\Xi^\varepsilon)_{\varepsilon > 0}$ of $\mathcal{C}^1$ functions which converges uniformly. One can check that the functions Ξ^ε can be chosen "uniformly weakly dissipative". Moreover, f is

approximated by a sequence (f^ε) of $\mathcal{C}_b^1$ functions converging uniformly. We consider the BSDE of infinite horizon:

$$Y_t^{\varepsilon,\alpha,x} = Y_T^{\varepsilon,\alpha,x} + \int_t^T \{f^\varepsilon(X_s^{\varepsilon,x}, Z_s^{\varepsilon,\alpha,x}, U_s^{\varepsilon,\alpha,x}) - \alpha Y_s^{\varepsilon,\alpha,x}\}\, \mathrm{d}s - \int_t^T Z_s^{\varepsilon,\alpha,x}\, \mathrm{d}W_s - \int_t^T \int_B U_s^{\varepsilon,\alpha,x}(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y),$$

where the process $X^{\varepsilon,x}$ satisfies the following equation:

$$X_t^{\varepsilon,x} = x + \int_0^t \Xi^\varepsilon(X_s^{\varepsilon,x})\, \mathrm{d}s + \int_0^t \sigma\, \mathrm{d}L_s.$$

This BSDE has a unique solution, and we can write $Y_t^{\varepsilon,\alpha,x} = v^{\varepsilon,\alpha}(X_t^{\varepsilon,x})$. Due to Proposition 4.11, we can find, for every $\varepsilon > 0$ a sequence $(v^{\delta,\varepsilon,\alpha})_\delta$ of $\mathcal{C}^1$ functions converging uniformly to $v^{\varepsilon,\alpha}$ and satisfying

$$\forall x \in \mathbb{R}^d,\ |v^{\delta,\varepsilon,\alpha}(x)| \leqslant \frac{C}{\alpha}(1+|x|).$$

We consider the finite horizon BSDE:

$$Y_t^{\delta,T,\varepsilon,\alpha,x} = v^{\delta,\varepsilon,\alpha}(X_T^{\varepsilon,x}) + \int_t^T \{f^\varepsilon(X_s^{\varepsilon,x}, Z_s^{\delta,T,\varepsilon,\alpha,x}, U_s^{\delta,T,\varepsilon,\alpha,x}) - \alpha Y_s^{\delta,T,\varepsilon,\alpha,x}\}\, \mathrm{d}s - \int_t^T Z_s^{\delta,T,\varepsilon,\alpha,x}\, \mathrm{d}W_s - \int_t^T \int_B H_s^{\delta,T,\varepsilon,\alpha,x}(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y).$$

This BSDE has a unique solution and satisfies $\left|Y_t^{\delta,T,\varepsilon,\alpha,x}\right| \leqslant \frac{C}{\alpha}\left(1+\left|X_t^{\varepsilon,x}\right|\right)$ $\mathbb{P}$-a.s., for every $t \geqslant 0$. Also, we can write

$$Y_t^{\delta,T,\varepsilon,\alpha,x} = u^{\delta,T,\varepsilon,\alpha}(t, X_t^{\varepsilon,x}), \quad Z_t^{\delta,T,\varepsilon,\alpha,x} = \partial_x u^{\delta,T,\varepsilon,\alpha}(t, X_t^{\varepsilon,x})\sigma \quad \text{and}$$

$$U_t^{\delta,T,\varepsilon,\alpha,x}(y) = u^{\delta,T,\varepsilon,\alpha}(t, X_{t^-}^{\varepsilon,x} + \sigma y) - u^{\delta,T,\varepsilon,\alpha}(t, X_{t^-}^{\varepsilon,x})$$

$\mathbb{P}$-a.s. and for a.e. $t \geqslant 0$. We define:

$$z^{\delta,T,\varepsilon,\alpha}(t,x) = \partial_x u^{\delta,T,\varepsilon,\alpha}(t,x)\sigma, \quad h^{\delta,T,\varepsilon,\alpha}(t,x) : y \mapsto u^{\delta,T,\varepsilon,\alpha}(t, x+\sigma y) - u^{\delta,T,\varepsilon,\alpha}(t,x),$$

$$\Gamma^{\delta,T,\varepsilon,\alpha}(t,x) = \begin{cases} \frac{f^\varepsilon\left(x, z^{\delta,T,\varepsilon,\alpha}(t,x), 0\right) - f^\varepsilon(x,0,0)}{\left|z^{\delta,T,\varepsilon,\alpha}(t,x)\right|^2} z^{\delta,T,\varepsilon,\alpha}(t,x), & \text{if } z^{\delta,T,\varepsilon,\alpha}(t,x) \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\widetilde{\Gamma}^{\delta,T,\varepsilon,\alpha}(t,x,\overline{x}) = \begin{cases} \frac{f^\varepsilon\left(x, z^{\delta,T,\varepsilon,\alpha}(t,\overline{x}), h^{\delta,T,\varepsilon,\alpha}(t,x)\right) - f^\varepsilon\left(x, z^{\delta,T,\varepsilon,\alpha}(t,\overline{x}), 0\right)}{\left\|h^{\delta,T,\varepsilon,\alpha}(t,x)\right\|_\nu^2} h^{\delta,T,\varepsilon,\alpha}(t,x), & \text{if } h^{\delta,T,\varepsilon,\alpha}(t,x) \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

We can rewrite:

$$\mathrm{d}Y_t^{\delta,T,\varepsilon,\alpha,x} = -\Big\{f^\varepsilon\left(X_t^{\varepsilon,x}, 0, 0\right) + Z_t^{\delta,T,\varepsilon,\alpha,x}\Gamma^{\delta,T,\varepsilon,\alpha}\left(t, X_t^{\varepsilon,x}\right)^* + U_t^{\delta,T,\varepsilon,\alpha,x}\widetilde{\Gamma}^{\delta,T,\varepsilon,\alpha}\left(t, X_{t^-}^{\varepsilon,x}, X_t^{\varepsilon,x}\right)^* - \alpha Y_t^{\delta,T,\varepsilon,\alpha,x}\Big\}\, \mathrm{d}t + Z_t^{\delta,T,\varepsilon,\alpha,x}\, \mathrm{d}W_t + U_t^{\delta,T,\varepsilon,\alpha,x}(y)\, \widetilde{N}(\mathrm{d}t, \mathrm{d}y).$$

Due to Girsanov's theorem, there exists a probability measure $\widetilde{\mathbb{P}}^{\delta,T,\varepsilon,\alpha,x}$ under which:

$$u^{\delta,T,\varepsilon,\alpha}(0,x) = \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x}\left[\mathrm{e}^{-\alpha T} v^{\delta,\varepsilon,\alpha}\left(X_T^{\varepsilon,x}\right) + \int_0^T \mathrm{e}^{-\alpha s} f^\varepsilon\left(X_s^{\varepsilon,x}, 0, 0\right)\, \mathrm{d}s\right].$$

On the one hand, using Proposition 4.3: $\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x}\left[\mathrm{e}^{-\alpha T}v^{\delta,\varepsilon,\alpha}\left(X_T^{\varepsilon,x}\right)\right]\right| \leqslant \mathrm{e}^{-\alpha T}C(1+|x|)$.

On the other hand, thanks to Corollary 4.6, we get:

$$\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x}\left[f^{\varepsilon}\left(X_t^{\varepsilon,x},0,0\right)\right] - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y}\left[f^{\varepsilon}\left(X_t^{\varepsilon,y},0,0\right)\right]\right| \leqslant \widehat{c}\mathrm{e}^{-\theta t}\left(1+|x|^2+|y|^2\right),$$

where $\widehat{c}$ and θ only depend on η, $\|b\|_{\infty}$, $|\sigma|_F$, $\|f\|_{\mathrm{lip},z}$ and M_f. As a consequence, we get:

$$\begin{aligned}
&|v^{\varepsilon,\alpha}(x) - v^{\varepsilon,\alpha}(y)| \\
&\quad = \lim_{\delta\to 0}\left|u^{\delta,T,\varepsilon,\alpha}(0,x) - u^{\delta,T,\varepsilon,\alpha}(0,y)\right| \qquad \text{(see Lemma 4.33)} \\
&\quad = \lim_{T\to\infty}\lim_{\delta\to 0}\Bigg|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x}\left[\mathrm{e}^{-\alpha T}v^{\delta,\varepsilon,\alpha}(X_T^{\varepsilon,x}) + \int_0^T \mathrm{e}^{-\alpha s}f^{\varepsilon}(X_s^{\varepsilon,x},0,0)\,\mathrm{d}s\right] \\
&\qquad\qquad - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y}\left[\mathrm{e}^{-\alpha T}v^{\delta,\varepsilon,\alpha}(X_T^{\varepsilon,y}) + \int_0^T \mathrm{e}^{-\alpha s}f^{\varepsilon}(X_s^{\varepsilon,y},0,0)\,\mathrm{d}s\right]\Bigg| \\
&\quad \leqslant \limsup_{T\to\infty}\limsup_{\delta\to 0}\mathrm{e}^{-\alpha T}\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x}\left[v^{\delta,\varepsilon,\alpha}(X_T^{\varepsilon,x})\right] - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y}\left[v^{\delta,\varepsilon,\alpha}(X_T^{\varepsilon,y})\right]\right| \\
&\qquad + \limsup_{T\to\infty}\limsup_{\delta\to 0}\int_0^T \mathrm{e}^{-\alpha s}\left|\widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,x}\left[f^{\varepsilon}(X_s^{\varepsilon,x},0,0)\right] - \widetilde{\mathbb{E}}^{\delta,T,\varepsilon,\alpha,y}\left[f^{\varepsilon}(X_s^{\varepsilon,y},0,0)\right]\right|\mathrm{d}s \\
&\quad \leqslant 0 + \int_0^{\infty}\mathrm{e}^{-\alpha s}\widehat{c}(1+|x|^2+|y|^2)\mathrm{e}^{-\theta s}\,\mathrm{d}s \\
&\quad \leqslant \frac{\widehat{c}}{\theta}(1+|x|^2+|y|^2).
\end{aligned} \tag{4.12}$$

2. Now, our goal is to take the limit when $\varepsilon \to 0$. Let D be a dense and countable subset of $\mathbb{R}^d$. By a diagonal argument, there exists a positive sequence $(\varepsilon_p)_p$ such that $(v^{\varepsilon_p,\alpha})_p$ converges pointwise over D to a function $\overline{v}^{\alpha}$. Because the constant C in Proposition 4.11 does not depend on ε, $\overline{v}^{\alpha}$ satisfies the same inequality. Let (K_r) be a sequence of compact sets whose diameter goes to infinity. The function $\overline{v}^{\alpha}$ is uniformly continuous on $K_r \cap D$, so it has an extension $\overline{v}^{\alpha}$ which is continuous on K_r. Passing to the limit as r goes to infinity, we get a continuous function on $\mathbb{R}^d$, and it is the pointwise limit of the sequence $(v^{\varepsilon_p,\alpha})_p$. We denote $\overline{Y}_t^{\alpha,x} = \overline{v}^{\alpha}\left(X_t^x\right)$; we have $|\overline{Y}_t^{\alpha,x}| \leqslant \frac{C}{\alpha}\left(1+|X_t^x|\right)$. Using Lemma 4.32, we have, for every $T>0$:

$$\mathbb{E}\left[|X^{\varepsilon_p,x} - X^x|_{0,T}^{*,2}\right] \underset{p\to\infty}{\longrightarrow} 0.$$

Then, by dominated convergence, we get:

$$\mathbb{E}\left[\int_0^T\left|Y_t^{\varepsilon_p,\alpha,x,n} - \overline{Y}_t^{\alpha,x,n}\right|^2\mathrm{d}t\right] \underset{p\to\infty}{\longrightarrow} 0 \text{ and } \mathbb{E}\left[\left|Y_T^{\varepsilon_p,\alpha,x} - \overline{Y}_T^{\alpha,x,n}\right|^2\right] \underset{p\to\infty}{\longrightarrow} 0.$$

3. We will show that there exist some processes $\overline{Z}^{\alpha,x}$ and $\overline{U}^{\alpha,x}$ belonging to $\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega,\mathrm{L}^2\left(0,\infty;E\right)\right)$ (where $E=(\mathbb{R}^d)^*$ or $\mathrm{L}^2(\nu)$) which satisfy

$$\mathbb{E}\left[\int_0^T\left|Z_t^{\varepsilon_p,\alpha,x} - \overline{Z}_t^{\alpha,x}\right|^2\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T\left\|U_t^{\varepsilon_p,\alpha,x} - \overline{U}_t^{\alpha,x}\right\|_{\nu}^2\mathrm{d}t\right] \underset{p\to\infty}{\longrightarrow} 0,$$

for every $T>0$. Indeed, $\left(\overline{Y}^{\alpha,x},\overline{Z}^{\alpha,x},\overline{U}^{\alpha,x}\right)$ is solution of the BSDE (4.11) ; by uniqueness of the solution, $v^{\alpha} \equiv \overline{v}^{\alpha}$ and taking the limit in the equation (4.12) gives the result.

Let us choose $p \leqslant m \in \mathbb{N}$, we define $\widetilde{Y} = Y^{\varepsilon_p,\alpha,x} - Y^{\varepsilon_m,\alpha,x}$, $\widetilde{Z} = Z^{\varepsilon_p,\alpha,x} - Z^{\varepsilon_m,\alpha,x}$ and $\widetilde{U} = U^{\varepsilon_p,\alpha,x} - U^{\varepsilon_m,\alpha,x}$. We have:

$$\mathrm{d}\widetilde{Y}_t = \alpha \widetilde{Y}_t \,\mathrm{d}t + \underbrace{\left(f^{\varepsilon_m}\left(X_t^{\varepsilon_m,x}, Z_t^{\varepsilon_m,\alpha,x}, U_t^{\varepsilon_m,\alpha,x}\right) - f^{\varepsilon_p}\left(X_t^{\varepsilon_p,x}, Z_t^{\varepsilon_p,\alpha,x}, U_t^{\varepsilon_p,\alpha,x}\right)\right)}_{=:f_t} \mathrm{d}t + \widetilde{Z}_t \,\mathrm{d}W_t + \widetilde{U}_t(y)\,\widetilde{N}(\mathrm{d}t,\mathrm{d}y).$$

Thanks to Itô's formula, we obtain:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\{|\widetilde{Z}_t|^2 + \|\widetilde{U}_t\|_\nu^2\right\}\mathrm{d}t\right]
&\leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] - 2\mathbb{E}\left[\int_0^T \widetilde{Y}_t f_t \,\mathrm{d}t\right] \\
&\leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] + 2TM_f\mathbb{E}\left[\left|\widetilde{Y}\right|_{0,T}^* \left(2 + |X^{\varepsilon_m,x}|_{0,T}^* + |X^{\varepsilon_p,x}|_{0,T}^*\right)\right] \\
&\quad + 2\|f\|_{\mathrm{lip},z}\mathbb{E}\left[\left|\widetilde{Y}\right|_{0,T}^*\int_0^T \left(|Z_t^{\varepsilon_m,\alpha,x}| + \left|Z_t^{\varepsilon_p,\alpha,x}\right|\right)\mathrm{d}t\right] \\
&\quad + 2\|f\|_{\mathrm{lip},u}\mathbb{E}\left[\left|\widetilde{Y}\right|_{0,T}^*\int_0^T \left(\|U_t^{\varepsilon_m,\alpha,x}\|_\nu + \left\|U_t^{\varepsilon_p,\alpha,x}\right\|_\nu\right)\mathrm{d}t\right].
\end{aligned}$$

Now, we want to bound $\mathbb{E}\left[|X^{\varepsilon_p,x}|_{0,T}^{*,2}\right]$ and $\mathbb{E}\left[\int_0^T\left\{\left|Z_t^{\varepsilon_p,\alpha,x}\right|^2 + \left\|U_t^{\varepsilon_p,\alpha,x}\right\|_\nu^2\right\}\mathrm{d}t\right]$ independently of p. Thanks to proposition 4.2, we have $\mathbb{E}\left[|X^{\varepsilon_p,x}|_{0,T}^{*,2}\right] \leqslant C\left(1+|x|^2\right)$, where C only depends on T, $|\sigma|_F$ and the linear growth of Ξ. Also, we have:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T\left\{\left|Z_t^{\varepsilon_p,\alpha,x}\right|^2 + \left\|U_t^{\varepsilon_p,\alpha,x}\right\|_\nu^2\right\}\mathrm{d}t\right]
&\leqslant \mathbb{E}\left[\left|Y_T^{\varepsilon_p,\alpha,x}\right|^2\right] + 2\mathbb{E}\left[\int_0^T \left|Y_t^{\varepsilon_p,\alpha,x} f^{\varepsilon_p}\left(X_t^{\varepsilon_p,x}, Z_t^{\varepsilon_p,\alpha,x}, U_t^{\varepsilon_p,\alpha,x}\right)\right|\mathrm{d}t\right].
\end{aligned}$$

But we have the following upperbound,

$$\begin{aligned}
&\left|Y_t^{\varepsilon_p,\alpha,x} f^{\varepsilon_p}\left(X_t^{\varepsilon_p,x}, Z_t^{\varepsilon_p,\alpha,x}, U_t^{\varepsilon_p,\alpha,x}\right)\right| \\
&\qquad\leqslant \frac{C}{\alpha}\left(1+\left|X_t^{\varepsilon_p,x}\right|\right)\left(M_f\left(1+\left|X_t^{\varepsilon_p,x}\right|\right) + \|f\|_{\mathrm{lip},z}\left|Z_t^{\varepsilon_p,\alpha,x}\right| + \|f\|_{\mathrm{lip},u}\left|U_t^{\varepsilon_p,\alpha,x}\right|\right) \\
&\qquad\leqslant \mathcal{C}\left(1+\left|X_t^{\varepsilon_p,x}\right|^2\right) + \frac{1}{4}\left|Z_t^{\varepsilon_p,\alpha,x}\right|^2 + \frac{1}{4}\left\|U_t^{\varepsilon_p,\alpha,x}\right\|_\nu^2.
\end{aligned}$$

Finally, $\mathbb{E}\left[\int_0^T\left\{\left|Z_t^{\varepsilon_p,\alpha,x}\right|^2 + \left\|U_t^{\varepsilon_p,\alpha,x}\right\|_\nu^2\right\}\mathrm{d}t\right] \leqslant \mathcal{C}'\left(1+|x|^2\right)$, with $\mathcal{C}'$ independent of p. The sequences $\left(Z^{\varepsilon_p,\alpha,x}\right)_p$ and $\left(U^{\varepsilon_p,\alpha,x}\right)_p$ are Cauchy in $\mathrm{L}^2_{\mathcal{P}}(\Omega, \mathrm{L}^2([0,T],E))$; they admit limit processes $\overline{Z}^{\alpha,x} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega, \mathrm{L}^2(0,\infty;(\mathbb{R}^d)^*))$ and $\overline{U}^{\alpha,x,n} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega,\mathrm{L}^2(0,\infty;\mathrm{L}^2(\nu)))$; the convergence we claimed holds.

□

Proposition 4.13. *There exists a constant $C > 0$, such that for every $\alpha > 0$, we have:*

$$\forall x, x' \in \mathbb{R}^d,\ |v^\alpha(x) - v^\alpha(x')| \leqslant C(1+|x|^2+|x'|^2)|x-x'|.$$

Proof. The proof is very similar to the one of Proposition 4.11. We make the same approximations, and because $\alpha > 0$, we have:

$$\begin{aligned}|\partial_x u^\varepsilon(x)| &\leqslant \mathbb{E}^{\mathbb{Q}}\left[\left|v^{\alpha,\varepsilon}(X_1^{\varepsilon,x})\right|^2\right]^{1/2} d|\sigma^{-1}|_F e^{\|\Xi\|_{\text{lip}}} + \int_0^1 \mathbb{E}^{\mathbb{Q}}\left[|f^\varepsilon(X_s^{\varepsilon,x},0,0)|^2\right]^{1/2} \frac{d|\sigma^{-1}|_F}{\sqrt{s}} e^{\|\Xi\|_{\text{lip}}} \,\mathrm{d}s \\ &\leqslant C(1+|x|^2),\end{aligned}$$

with C independent of α and x. Due to the mean value theorem, we have:

$$|u^\varepsilon(x) - u^\varepsilon(x')| \leqslant C(1+|x|^2+|x'|^2)|x-x'|.$$

And we conclude again using Lemma 4.33 and Markov's inequality. □

4.3.2 The ergodic BSDE

We consider the following ergodic BSDE:

$$Y_t^x = Y_T^x + \int_t^T \{f(X_s^x, Z_s^x, U_s^x) - \lambda\}\,\mathrm{d}s - \int_t^T Z_s^x\,\mathrm{d}W_s - \int_t^T \int_B U_s^x(y)\widetilde{N}(\mathrm{d}s,\mathrm{d}y), \qquad \forall 0 \leqslant t \leqslant T < \infty, \tag{4.13}$$

and we require f to satisfy the assumptions of the previous subsection.

Theorem 4.14 (Existence)**.** *There exist:*

- *a real number λ;*
- *a locally Lipschitz function v satisfying $v(0) = 0$;*
- *a process $Z^x \in L^2_{\mathcal{P},\text{loc}}(\Omega, L^2(0,\infty;(\mathbb{R}^d)^*))$;*

such that, if we define $Y_t^x = v(X_t^x)$ and $U_t^x(y) = v(X_{t^-}^x + \sigma y) - v(X_{t^-}^x)$, the EBSDE (4.13) is satisfied by (Y^x, Z^x, U^x, λ) $\mathbb{P}$-a.s. and for all $0 \leqslant t \leqslant T < \infty$.
Moreover, there exists $C > 0$, such that for all $x \in \mathbb{R}^d$, $|v(x)| \leqslant C(1+|x|^2)$.

Proof.

Step 1: Construction of v by a diagonal procedure.
For every $\alpha > 0$, we define $\overline{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0)$. We know that there exists a constant C, independent of α, such that $|\overline{v}^\alpha(x)| \leqslant C(1+|x|^2)$ and $|\alpha v^\alpha(0)| \leqslant C$. Let D be a countable subset of $\mathbb{R}^d$; by a diagonal argument, we can construct a sequence (α_p) converging to 0, such that $(\overline{v}^{\alpha_p})_p$ converges pointwise to a function v_l over D and $\alpha_p v^{\alpha_p}(0) \underset{p\to\infty}{\longrightarrow} \lambda$, for a convenient real number λ. Moreover, the functions $\overline{v}^\alpha$ are uniformly locally Lipschitz:

$$\exists C > 0,\ \forall \alpha > 0,\ \forall x, x' \in \mathbb{R}^d,\ |\overline{v}^\alpha(x) - \overline{v}^\alpha(x')| \leqslant C(1+|x|^2+|x'|^2)|x-x'|. \tag{4.14}$$

Then, v is uniformly continuous on every compact subset of D, so it has a continuous extension over $\mathbb{R}^d$ (we still call it v). Then, we can show that v is the pointwise limit of the functions $\overline{v}^{\alpha_p}$ over $\mathbb{R}^d$, v is locally Lipschitz continuous and has quadratic growth.

Step 2: Construction of some processes Z^x and U^x.
We will show that $(Z^{\alpha_p,x})_p$ and $(U^{\alpha_p,x})_p$ are Cauchy in $L^2_{\mathcal{P}}\left(\Omega, L^2([0,T],E)\right)$, where $E = (\mathbb{R}^d)^*$ or $L^2(\nu)$, for every $T > 0$. When $p \leqslant m \in \mathbb{N}$, we set $\widetilde{Y} = \overline{Y}^{\alpha_p,x} - \overline{Y}^{\alpha_m,x}$, $\widetilde{Z} = Z^{\alpha_p,x} - Z^{\alpha_m,x}$ and

$\widetilde{U} = U^{\alpha_p,x} - U^{\alpha_m,x}$. By Itô's formula, writing $R_s = \alpha_m Y_s^{\alpha_m,x} - \alpha_p Y_s^{\alpha_p,x}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \{|\widetilde{Z}_s|^2 + \|\widetilde{U}_s\|_\nu^2\}\,\mathrm{d}s\right] \\
&\quad = \mathbb{E}\left[|\widetilde{Y}_T|^2\right] - |\widetilde{Y}_0|^2 + 2\mathbb{E}\left[\int_0^T \widetilde{Y}_s\left\{f(X_s^x, Z_s^{\alpha_p,x}, U_s^{\alpha_p,x}) - f(X_s^x, Z_s^{\alpha_m,x}, U_s^{\alpha_m,x}) + R_s\right\}\,\mathrm{d}s\right] \\
&\quad \leqslant \mathbb{E}\left[|\widetilde{Y}_T|^2\right] + \left(2\|f\|_{\mathrm{lip},z}^2 + 2\|f\|_{\mathrm{lip},u}^2\right)\mathbb{E}\left[\int_0^T |\widetilde{Y}_s|^2\,\mathrm{d}s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T \{|\widetilde{Z}_s|^2 + \|\widetilde{H}_s\|_\nu^2\}\,\mathrm{d}s\right] \\
&\qquad\qquad + 2\mathbb{E}\left[\int_0^T |\widetilde{Y}_s|^2\,\mathrm{d}s\right]^{1/2}\mathbb{E}\left[\int_0^T |R_s|^2\,\mathrm{d}s\right]^{1/2}.
\end{aligned}$$

Using the fact that $|R_s| \leqslant C(1+|X_s^x|)$, we are able to prove the existence of the limits of $(Z^{\alpha_p,x})_p$ and $(H^{\alpha_p,x})_p$ in $\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty; E\right)\right)$.

Step 3: Taking the limit in the BSDE satisfied by $\left(\overline{Y}^{\alpha_p,x}, Z^{\alpha_p,x}, H^{\alpha_p,x}\right)$ leads us to:

$$Y_t^x = Y_T^x + \int_t^T \{f(X_s^x, Z_s^x, U_s^x) - \lambda\}\,\mathrm{d}s - \int_t^T Z_s^x\,\mathrm{d}W_s - \int_t^T\int_B U_s^x(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y).$$

Step 4: Almost surely convergences.

For every $x \in \mathbb{R}^d$, the sequence $\left(\mathbb{E}\left[\int_0^T \left\{\left|Z_t^{\alpha_p,x} - Z_t^{\alpha_m,x}\right|^2 + \left\|U_t^{\alpha_p,x} - U_t^{\alpha_m,x}\right\|_\nu^2\right\}\,\mathrm{d}t\right]\right)_{p\leqslant m\in\mathbb{N}}$ is bounded (because this sequence converges). By a diagonal procedure, there exists a subsequence $(\alpha'_p) \subset (\alpha_p)$ such that:

$$\forall x \in D,\ \forall p \leqslant m \in \mathbb{N},\ \mathbb{E}\left[\int_0^T \left\{\left|Z_t^{\alpha'_p,x} - Z_t^{\alpha'_m,x}\right|^2 + \left\|U_t^{\alpha'_p,x} - U_t^{\alpha'_m,x}\right\|_\nu^2\right\}\,\mathrm{d}t\right] < 2^{-p}.$$

We claim that this result can be extended to every $x \in \mathbb{R}^d$; let us fix $T > 0$ and denote $\Delta\bullet = \bullet^{\alpha,x} - \bullet^{\alpha,x'}$, for any $\alpha > 0$ and $x, x' \in \mathbb{R}^d$. Then, by usual techniques:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \{|\Delta Z_t|^2 + \|\Delta U_t\|_\nu^2\}\,\mathrm{d}t\right] &\leqslant 2\mathbb{E}\left[|\Delta Y_T|^2\right] + \left(4\|f\|_{\mathrm{lip},z}^2 + 4\|f\|_{\mathrm{lip},u}^2\right)\mathbb{E}\left[\int_0^T |\Delta Y_t|^2\,\mathrm{d}t\right] \\
&\qquad + 4\|f\|_{\mathrm{lip},x}\mathbb{E}\left[\int_0^T |\Delta Y_t||\Delta X_t|\,\mathrm{d}t\right].
\end{aligned}$$

We can use Lemma 4.32, Proposition 4.13 and the estimate of the end of the proof of Proposition 4.12, to prove that:

$$\mathbb{E}\left[\int_0^T \{|\Delta Z_t|^2 + \|\Delta U_t\|_\nu^2\}\,\mathrm{d}t\right] \leqslant C(1+|x|^4+|x'|^4)|x-x'|^2,$$

with a constant C independent of x and x'. So, we are now able to write:

$$\forall x \in \mathbb{R}^d,\ \forall p \leqslant m \in \mathbb{N},\ \mathbb{E}\left[\int_0^T \left\{\left|Z_t^{\alpha'_p,x} - Z_t^{\alpha'_m,x}\right|^2 + \left\|U_t^{\alpha'_p,x} - U_t^{\alpha'_m,x}\right\|^2\right\}\,\mathrm{d}t\right] < 2^{-p}.$$

Finally, Borel-Cantelli theorem gives us, for a.e. $t \in [0,T]$:

$$Z_t^{\alpha'_p,x,n} \underset{p\to\infty}{\longrightarrow} Z_t^{x,n} \quad \text{and} \quad U_t^{\alpha'_p,x} \underset{p\to\infty}{\longrightarrow} U_t^x, \quad \mathbb{P}\text{-a.s.}$$

Representation of U comes from the same trick as Proposition 4.10.

□

Theorem 4.15 (Uniqueness for λ)**.** *We suppose that:*

- (Y^x, Z^x, U^x, λ) *is the solution built in Theorem 4.14;*
- *for some* $x \in \mathbb{R}^d$, (Y', Z', U', λ') *satifies the EBSDE (4.13)* $\mathbb{P}$*-a.s. and for every* $0 \leqslant t \leqslant T < \infty$, *where* Y' *is a progressively measurable process,* $Z', U' \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega, \mathrm{L}^2(0,\infty; E))$ *(with* $E = (\mathbb{R}^d)^*$ *or* $\mathrm{L}^2(\nu)$*) and* $\lambda' \in \mathbb{R}$*;*
- $\exists c_x > 0$, $\forall t \geqslant 0$, $|Y'_t| \leqslant c_x(1 + |X^x_t|^2)$ $\mathbb{P}$*-a.s.*

Then, $\lambda = \lambda'$.

Proof. We define $\overline{\bullet} = \bullet^x - \bullet'$, for $\bullet = Y$, Z, U or λ. We have:

$$\overline{\lambda} = \frac{\overline{Y}_T - \overline{Y}_0}{T} + \frac{1}{T}\int_0^T \left\{ f(X^x_s, Z^x_s, U^x_s) - f(X^x_s, Z'_s, U'_s) \right\} \mathrm{d}s - \frac{1}{T}\int_0^T \overline{Z}_s \,\mathrm{d}W_s - \frac{1}{T}\int_0^T \int_B \overline{U}_s(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y).$$

Thanks to Girsanov's lemma, there exists a probability measure $\widehat{\mathbb{P}}$ under which

$$\overline{\lambda} = \frac{1}{T}\widehat{\mathbb{E}}\left[\overline{Y}_T - \overline{Y}_0\right] \leqslant \frac{C}{T}(1 + |x|^2),$$

with a constant C independent of T, due to Proposition 4.3. We conclude by taking the limit $T \to \infty$. □

Theorem 4.16 (Uniqueness for Y, Z and U)**.** *We suppose that:*

- $v, \widehat{v} : \mathbb{R}^d \to \mathbb{R}$ *are continuous, have quadratic growth, and* $v(0) = \widehat{v}(0) = 0$*;*
- *for every* $x \in \mathbb{R}^d$, $Z^x, U^x \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}(\Omega, \mathrm{L}^2(0,\infty; E))$ *(with* $E = (\mathbb{R}^d)^*$ *or* $\mathrm{L}^2(\nu)$*);*
- *there exist* $\lambda, \widehat{\lambda} \in \mathbb{R}$, *such that, for all* $x \in \mathbb{R}^d$, $(v(X^x), Z^x, U^x, \lambda)$ *and* $(\widehat{v}(X^x), \widehat{Z}^x, \widehat{U}^x, \widehat{\lambda})$ *satisfy the EBSDE (4.13).*

Then, $\lambda = \widehat{\lambda}$, $v = \widehat{v}$, *and for all* $x \in \mathbb{R}^d$, *for a.e.* $t \geqslant 0$, $Z^x_t = \widehat{Z}^x_t$ *and* $U^x_t = \widehat{U}^x_t$ $\mathbb{P}$*-a.s.*

Proof. We already know that $\lambda = \widehat{\lambda}$. We write $Y^x_t = v(X^x_t)$ and $\widehat{Y}^x_t = \widehat{v}(X^x_t)$. We approach Ξ, f, v and $\widehat{v}$ by $\mathcal{C}^1_b$-functions which converge uniformly (on every compact set for v and $\widehat{v}$). Moreover, we can assume that Ξ^ε and f^ε are uniformly Lipschitz continuous and that v^ε and $\widehat{v}^\varepsilon$ are uniformly locally Lipschitz continuous, in the sense of equation (4.14). We consider the following equations, for every $t \in [0, T]$:

$$X^{x,\varepsilon}_t = x + \int_0^t \Xi^\varepsilon(X^{x,\varepsilon}_s)\,\mathrm{d}s + \int_0^t \sigma\,\mathrm{d}W_s + \int_0^t \int_B \sigma y\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y),$$

$$Y^{x,\varepsilon}_t = v^\varepsilon(X^{x,\varepsilon}_T) + \int_t^T \left\{ f^\varepsilon(X^{x,\varepsilon}_s, Z^{x,\varepsilon}_s, U^{x,\varepsilon}_s) - \lambda \right\} \mathrm{d}s - \int_t^T Z^{x,\varepsilon}_s \,\mathrm{d}W_s - \int_t^T \int_B U^{x,\varepsilon}_s(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y),$$

$$\widehat{Y}^{x,\varepsilon}_t = \widehat{v}^\varepsilon(X^{x,\varepsilon}_T) + \int_t^T \left\{ f^\varepsilon(X^{x,\varepsilon}_s, \widehat{Z}^{x,\varepsilon}_s, \widehat{U}^{x,\varepsilon}_s) - \lambda \right\} \mathrm{d}s - \int_t^T \widehat{Z}^{x,\varepsilon}_s \,\mathrm{d}W_s - \int_t^T \int_B \widehat{U}^{x,\varepsilon}_s(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y).$$

Step 1: Convergence of the approximations.

We set $\Delta\bullet = \bullet - \widehat{\bullet}$, and we get:

$$\begin{aligned} \left|\Delta Y^x_0 - \Delta Y^{x,\varepsilon}_0\right| \leqslant{}& \mathbb{E}\left[\left|v(X^x_T) - v^\varepsilon(X^{x,\varepsilon}_T)\right|\right] + \mathbb{E}\left[\int_0^T \left|f(X^x_t, Z^x_t, U^x_t) - f^\varepsilon(X^{x,\varepsilon}_t, Z^{x,\varepsilon}_t, U^{x,\varepsilon}_t)\right| \mathrm{d}t\right] \\ &+ \mathbb{E}\left[\left|\widehat{v}(X^x_T) - \widehat{v}^\varepsilon(X^{x,\varepsilon}_T)\right|\right] + \mathbb{E}\left[\int_0^T \left|f(X^x_t, \widehat{Z}^x_t, \widehat{U}^x_t) - f^\varepsilon(X^{x,\varepsilon}_t, \widehat{Z}^{x,\varepsilon}_t, \widehat{U}^{x,\varepsilon}_t)\right| \mathrm{d}t\right]. \end{aligned}$$

Setting $\delta\bullet = \bullet^x - \bullet^{x,\varepsilon}$, we have the upperbound:

$\left|f(X_t^x, Z_t^x, U_t^x) - f^\varepsilon(X_t^{x,\varepsilon}, Z_t^{x,\varepsilon}, U_t^{x,\varepsilon})\right| \leqslant \|f - f_\varepsilon\|_\infty + \|f\|_{\mathrm{lip},x}|\delta X_t| + \|f\|_{\mathrm{lip},z}|\delta Z_t| + \|f\|_{\mathrm{lip},u}\|\delta U_t\|_\nu$.

But, thanks to the lemmas 4.32 and 4.33, we have:

$$\mathbb{E}\left[|\delta X|_{0,T}^{*,p}\right] \leqslant C_p\|\Xi - \Xi^\varepsilon\|_\infty^p,$$

$$\mathbb{E}\left[\int_0^T \left\{|\delta Z_t|^2 + \|\delta U_t\|_\nu^2\right\} \mathrm{d}t\right]$$
$$\leqslant C\mathbb{E}\left[\left|v(X_T^x) - v^\varepsilon(X_T^{x,\varepsilon})\right|^2 + \int_0^T \left|f(X_t^x, Z_t^x, U_t^x) - f^\varepsilon(X_t^{x,\varepsilon}, Z_t^x, U_t^x)\right|^2 \mathrm{d}t\right].$$

Obviously, the same estimates hold with $\widehat{\ }$ on v, Z and U. To prove that $\left|\Delta Y_0^x - \Delta Y_0^{x,\varepsilon}\right| \underset{\varepsilon\to 0}{\longrightarrow} 0$, we only need to show that $\mathbb{E}\left[\left|v(X_T^x) - v^\varepsilon(X_T^{x,\varepsilon})\right|^2\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$:

- on the one hand, using uniform local Lipschitz property,

$$\mathbb{E}\left[\left|v^\varepsilon(X_T^x) - v^\varepsilon(X_T^{x,\varepsilon})\right|^2\right] \leqslant C\left(1 + \mathbb{E}\left[|X^x|_{0,T}^{*,8}\right]^{1/2} + \mathbb{E}\left[|X^{x,\varepsilon}|_{0,T}^{*,8}\right]^{1/2}\right)\mathbb{E}\left[|\delta X|_{0,T}^{*,4}\right]^{1/2} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

- on the other hand, for any $R > 0$, we have

$$\mathbb{E}\left[|(v - v^\varepsilon)(X_T^x)|^2 \mathbb{1}_{|X_T^x| \leqslant R}\right] \leqslant \|v - v^\varepsilon\|_{\infty,\mathcal{B}_R}^2,$$

$$\mathbb{E}\left[|(v - v^\varepsilon)(X_T^x)|^2 \mathbb{1}_{|X_T^x| > R}\right] \leqslant \mathbb{E}\left[|(v - v^\varepsilon)(X_T^x)|^4\right]^{1/2} \mathbb{P}\left(|X_T^x| > R\right)^{1/2}$$
$$\leqslant C\left(1 + \mathbb{E}\left[|X_T^x|^8\right]^{1/2}\right)\frac{\mathbb{E}\left[|X_T^x|^2\right]^{1/2}}{R} \leqslant \frac{\tilde{C}(1 + |x|^5)}{R}.$$

For any $R > 0$, $\limsup\limits_{\varepsilon\to 0} \mathbb{E}\left[|(v - v^\varepsilon)(X_T^x)|^2\right] \leqslant \dfrac{\tilde{C}(1 + |x|^5)}{R}$, i.e. $\mathbb{E}\left[\left|(v - v^\varepsilon)(X_T^x)\right|^2\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Step 2: Convergence of the approximations to the same limit.
Approximations have been made in such a way that $Z^{x,\varepsilon}$ and $\widehat{Z}^{x,\varepsilon}$ can be written as continuous functions of $X^{x,\varepsilon}$; we write:

$$Z_t^{x,\varepsilon} = \zeta^\varepsilon(t, X_t^{x,\varepsilon}), \quad \widehat{Z}_t^{x,\varepsilon} = \widehat{\zeta}^\varepsilon(t, X_t^{x,\varepsilon}), \quad U_t^{x,\varepsilon} = h^\varepsilon(t, X_{t^-}^{x,\varepsilon}) \quad \text{and} \quad \widehat{U}_t^{x,\varepsilon} = \widehat{h}^\varepsilon(t, X_{t^-}^{x,\varepsilon}).$$

We set:

$$\Gamma^\varepsilon(t,x,\overline{x}) = \begin{cases} \frac{f^\varepsilon(x,\zeta^\varepsilon(t,x),\widehat{h}^\varepsilon(t,\overline{x})) - f^\varepsilon(x,\widehat{\zeta}^\varepsilon(t,x),\widehat{h}^\varepsilon(t,\overline{x}))}{|(\zeta^\varepsilon - \widehat{\zeta}^\varepsilon)(t,x)|^2}(\zeta^\varepsilon - \widehat{\zeta}^\varepsilon)(t,x)^* & \text{if } \zeta^\varepsilon(t,x) \neq \widehat{\zeta}^\varepsilon(t,x), \\ 0 & \text{otherwise;} \end{cases}$$

$$\overline{\Gamma}^\varepsilon(t,x,\overline{x}) = \begin{cases} \frac{f^\varepsilon(x,\zeta^\varepsilon(t,x),h^\varepsilon(t,\overline{x})) - f^\varepsilon(x,\zeta^\varepsilon(t,x),\widehat{h}^\varepsilon(t,\overline{x}))}{|(h^\varepsilon - \widehat{h}^\varepsilon)(t,\overline{x})|^2}(h^\varepsilon - \widehat{h}^\varepsilon)(t,\overline{x})^* & \text{if } h_m^\varepsilon(t,\overline{x}) \neq \widehat{h}_m^\varepsilon(t,\overline{x}), \\ 0 & \text{otherwise.} \end{cases}$$

There exists a probability measure $\mathbb{P}^{x,\varepsilon,T}$ under which we can write:

$$\Delta Y_0^{x,\varepsilon} = \mathbb{E}^{x,\varepsilon,T}\left[\Delta Y_T^{x,\varepsilon}\right] = \mathbb{E}^{x,\varepsilon,T}\left[(v^\varepsilon - \widehat{v}^\varepsilon)(X_T^{x,\varepsilon})\right].$$

Using Corollary 4.6, we get

$$\left|\mathbb{E}^{x,\varepsilon,T}\left[(v^\varepsilon - \widehat{v}^\varepsilon)(X_T^{x,\varepsilon})\right] - \mathbb{E}^{0,\varepsilon,T}\left[(v^\varepsilon - \widehat{v}^\varepsilon)(X_T^{0,\varepsilon})\right]\right| \leqslant c(1 + |x|^2)\mathrm{e}^{-\theta T}.$$

Because c and θ do not depend on ε, passing to the limit gives us:

$$|(v-\widehat{v})(x)| = |\Delta Y_0^x - \Delta Y_0^0| \leqslant c(1+|x|^2)\mathrm{e}^{-\theta T}.$$

We finally obtain $v = \widehat{v}$ when $T \to \infty$. Then, uniqueness for Z and U is the consequnce of Itô's formula.

□

4.4 Link with viscosity solutions of PIDEs

4.4.1 Finite horizon case

We consider in this section the forward-backward system:

$$\begin{cases} \mathrm{d}X_s^{t,x} = \Xi(X_s^{t,x})\,\mathrm{d}s + \sigma Q\,\mathrm{d}W_s + \int_B \sigma y\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y), & X_t^{t,x} = x, \\ \mathrm{d}Y_s^{T,t,x} = -f(X_s^{t,x}, Z_s^{T,t,x}, U_s^{T,t,x}\cdot\delta)\,\mathrm{d}s + Z_s^{T,t,x}\mathrm{d}W_s + \int_B U_s^{T,t,x}(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y), & Y_T^{T,t,x} = g(X_T^{t,x}), \end{cases} \tag{4.15}$$

where:

- Ξ, σ, Q and ν satisfy the assumptions of section 1;
- f and g are Lipschitz continuous;
- the mapping $p \mapsto f(x,z,p)$ is non-decreasing for each (x,z);
- there exists a constant $C > 0$, such that for every $y \in B$, $0 \leqslant \delta(y) \leqslant C(1\wedge|y|)$.

We introduce the following operators:

$$\mathcal{L}\varphi(x) = \langle \Xi(x), \nabla\varphi(x)\rangle + \frac{1}{2}\mathrm{tr}(\sigma QQ^*\sigma^*\nabla^2\varphi(x)) + \int_B \{\varphi(x+\sigma y) - \varphi(x) - \langle\nabla\varphi(x), \sigma y\rangle\}\,\nu(\mathrm{d}y),$$

$$\mathcal{J}\varphi(x) = \int_B \{\varphi(x+\sigma y) - \varphi(x)\}\delta(y)\,\nu(\mathrm{d}y),$$

which are well defined for functions φ of class $\mathcal{C}^2$. We also consider the PIDE:

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + f(x, \partial_x u(t,x)\sigma Q, \mathcal{J}u(t,x)) = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(T,x) = g(x), & x \in \mathbb{R}^d. \end{cases} \tag{4.16}$$

Definition 4.17 (Viscosity solution)**.** We say that $u \in \mathcal{C}([0,T]\times\mathbb{R}^d)$ is

1. a viscosity subsolution to (4.16) if

$$u(T,x) \leqslant g(x), \qquad x \in \mathbb{R}^d,$$

and for any $\phi \in \mathcal{C}^3_{l,b}([0,T]\times\mathbb{R}^d)$, whenever (t,x) is a local maximum point of $u-\phi$, we have,

$$\partial_t\phi(t,x) + \mathcal{L}\phi(t,x) + f(x, \partial_x\phi(t,x)\sigma Q, \mathcal{J}\phi(t,x)) \geqslant 0.$$

2. a viscosity supersolution to (4.16) if

$$u(T,x) \geqslant g(x), \qquad x \in \mathbb{R}^d,$$

and for any $\phi \in \mathcal{C}^3_{l,b}([0,T]\times\mathbb{R}^d)$, whenever (t,x) is a local minimum point of $u-\phi$, we have,

$$\partial_t\phi(t,x) + \mathcal{L}\phi(t,x) + f(x, \partial_x\phi(t,x)\sigma Q, \mathcal{J}\phi(t,x)) \leqslant 0.$$

3. a viscosity solution to (4.16) if it is both viscosity sub and supersolution to (4.16).

Remark 4.18. $\mathcal{C}^3_{l,b}([0,T]\times\mathbb{R}^d)$ is the set of real-valued functions continuously differentiable up to the third order and whose derivatives of order 1 to 3 are bounded.

Proposition 4.19 (Existence of viscosity solution)**.** *The function $u:(t,x)\mapsto Y^{T,t,x}_t$ defined by (4.15) is a viscosity solution to the PIDE (4.16).*

Proof. Let $\phi\in\mathcal{C}^3_{l,b}([0,T]\times\mathbb{R}^d)$ and (t_0,x_0) a point of local maximum of $u-\phi$. We suppose that

$$\partial_t\phi(t_0,x_0)+\mathcal{L}\phi(t_0,x_0)+f(x_0,\partial_x\phi(t_0,x_0)\sigma Q,\mathcal{J}\phi(t_0,x_0))=-\varepsilon<0$$

and we will find a contradiction. Without any loss of generality, we suppose that $u(t_0,x_0)=\phi(t_0,x_0)$. By continuity, we have: $\exists\alpha\in(0,T-t_0)$, $\forall t\in[t_0,t_0+\alpha]$, $\forall x\in\overline{\mathcal{B}_\alpha(x_0)}$,

$$u(t,x)\leqslant\phi(t,x)\text{ and }\partial_t\phi(t,x)+\mathcal{L}\phi(t,x)+f(x,\partial_x\phi(t,x)\sigma Q,\mathcal{J}\phi(t,x))=:\psi(t,x)\leqslant-\frac{\varepsilon}{2}.$$

We set $\tau=\inf\left\{u\geqslant t_0\middle||X^{t_0,x_0}_u-x_0|>\alpha\right\}\wedge\alpha$. The solution of (4.15) satisfies, for all $s\in[t_0,t_0+\alpha]$,

$$\begin{aligned}Y^{T,t_0,x_0}_{s\wedge\tau}=u(\tau,X^{t_0,x_0}_\tau)+\int_{s\wedge\tau}^{\tau}f(X^{t_0,x_0}_r,Z^{T,t_0,x_0}_r,U^{T,t_0,x_0}_r\cdot\delta)\,\mathrm{d}r-\int_{s\wedge\tau}^{\tau}Z^{T,t_0,x_0}_r\,\mathrm{d}W_r\\-\int_{s\wedge\tau}^{\tau}\int_B U^{T,t_0,x_0}_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y).\end{aligned}$$

Setting $\widetilde{Y}_r=Y^{T,t_0,x_0}_{r\wedge\tau}$, $\widetilde{Z}_r=Z^{T,t_0,x_0}_r\mathbb{1}_{r\leqslant\tau}$ and $\widetilde{U}_r=U^{T,t_0,x_0}_r\mathbb{1}_{r\leqslant\tau}$ makes the last equation become:

$$\begin{aligned}\forall s\in[t_0,t_0+\alpha],\ \widetilde{Y}_s=u(\tau,X^{t_0,x_0}_\tau)-\int_s^{t_0+\alpha}f(X^{t_0,x_0}_r,\widetilde{Z}_r,\widetilde{U}_r\cdot\delta)\mathbb{1}_{r\leqslant\tau}\,\mathrm{d}r-\int_s^{t_0+\alpha}\widetilde{Z}_r\,\mathrm{d}W_r\\-\int_s^{t_0+\alpha}\int_B\widetilde{U}_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y).\end{aligned}$$

Also, by Itô's formula, we find, for any $s\in[t_0,t_0+\alpha]$,

$$\begin{aligned}\phi(s\wedge\tau,X^{t_0,x_0}_{s\wedge\tau})=\phi(\tau,X^{t_0,x_0}_\tau)&-\int_{s\wedge\tau}^{\tau}\left\{\partial_t\phi(r,X^{t_0,x_0}_r)+\mathcal{L}\phi(r,X^{t_0,x_0}_r)\right\}\mathrm{d}r\\&-\int_{s\wedge\tau}^{\tau}\partial_x\phi(r,X^{t_0,x_0}_r)\sigma Q\mathrm{d}W_r\\&-\int_{s\wedge\tau^+}^{\tau^+}\int_B\{\phi(r,X^{t_0,x_0}_{r^-}+\sigma y)-\phi(r,X^{t_0,x_0}_{r^-})\}\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y).\end{aligned}$$

Setting $Y'_r=\phi(r,X^{t_0,x_0}_{r\wedge\tau})$, $Z'_r=\partial_x\phi(r,X^{t_0,x_0}_r)\sigma Q\mathbb{1}_{r\leqslant\tau}$ and $U'_r:y\mapsto\{\phi(r,X^{t_0,x_0}_{r^-}+\sigma y)-\phi(r,X^{t_0,x_0}_{r^-})\}\mathbb{1}_{r\leqslant\tau^+}$ makes the last equation become: $\forall s\in[t_0,t_0+\alpha]$,

$$\begin{aligned}Y'_s&=\phi(\tau,X^{t_0,x_0}_\tau)-\int_s^{t_0+\alpha}\left\{\partial_t\phi(r,X^{t_0,x_0}_r)+\mathcal{L}\phi(r,X^{t_0,x_0}_r)\right\}\mathbb{1}_{r\leqslant\tau}\,\mathrm{d}r\\&\qquad-\int_s^{t_0+\alpha}Z'_r\,\mathrm{d}W_r-\int_s^{t_0+\alpha}\int_B U'_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y)\\&=\phi(\tau,X^{t_0,x_0}_\tau)+\int_s^{t_0+\alpha}\left\{f(X^{t_0,x_0}_r,\partial_x\phi(r,X^{t_0,x_0}_r)\sigma Q,\mathcal{J}\phi(r,X^{t_0,x_0}_r))-\psi(r,X^{t_0,x_0}_r)\right\}\mathbb{1}_{r\leqslant\tau}\,\mathrm{d}r\\&\qquad-\int_s^{t_0+\alpha}Z'_r\,\mathrm{d}W_r-\int_s^{t_0+\alpha}\int_B U'_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y)\\&=\phi(\tau,X^{t_0,x_0}_\tau)+\int_s^{t_0+\alpha}\left\{f(X^{t_0,x_0}_r,Z'_r,U'_r\cdot\delta)-\psi(r,X^{t_0,x_0}_r)\right\}\mathbb{1}_{r\leqslant\tau}\,\mathrm{d}r\\&\qquad-\int_s^{t_0+\alpha}Z'_r\,\mathrm{d}W_r-\int_s^{t_0+\alpha}\int_B U'_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y)\end{aligned}$$

Finally, we use the comparison theorem (see e.g. [23]), and we find that $\widetilde{Y}_0 < Y'_0$, that is to say $v(x_0) < \phi(x_0)$, which is a contradiction. So v is a viscosity subsolution, and we show exactly the same way that it is also a supersolution. □

Proposition 4.20 (Uniqueness of viscosity solution)**.** *The PIDE (4.16) admits a unique viscosity solution in the class of solutions that satisfy the growth condition*

$$\lim_{|x|\to\infty} |u(t,x)| \mathrm{e}^{-c\ln(\sqrt{1+|x|^2})^2} = 0, \quad 0 \leqslant t \leqslant T,\, c > 0.$$

Proof. See theorem 5.1 of [10]. □

4.4.2 Ergodic case

We consider the ergodic PIDE:

$$\mathcal{L}v(x) + f(x, \nabla v(x)\sigma Q, \mathcal{J}v(x)) = \lambda. \tag{4.17}$$

We recall that the couple (v,λ) is a viscosity subsolution (resp. supersolution) if:

- $v : \mathbb{R}^d \to \mathbb{R}$ is continuous;
- for any function $\phi \in \mathcal{C}^3_{l,b}(\mathbb{R}^d)$, and for every $x \in \mathbb{R}^d$ of local maximum (resp. minimum) of $v - \phi$:
$$\mathcal{L}\phi(x) + f(x, \nabla\phi(x)\sigma Q, \mathcal{J}\phi(x)) \geqslant \lambda \quad (\text{resp. } \leqslant \lambda).$$

Proposition 4.21 (Existence of ergodic viscosity solution)**.** *The couple (v,λ) obtained in Theorem 4.14 is a viscosity solution of equation (4.17).*

Proof. This proof is very close from the one of Proposition 4.19, so we omit it here. □

Proposition 4.22 (Uniqueness of ergodic viscosity solution)**.** *Uniqueness holds for viscosity solutions (v,λ) of equation (4.17) in the class of viscosity solutions such that $\exists a \in \mathbb{R}^d$, $v(a) = \overline{v}(a)$ and v and $\overline{v}$ have quadratic growth.*

Proof. Let (v,λ) and $(\overline{v},\overline{\lambda})$ be two viscosity solutions of (4.17). We fix $T > 0$, and we consider the BSDEs:

$$Y^{T,x}_t = v(X^x_T) + \int_t^T \{f(X^x_r, Z^{T,x}_r, U^{T,x}_r \cdot \delta) - \lambda\}\,\mathrm{d}r - \int_t^T Z^{T,x}_r\,\mathrm{d}W_r - \int_t^T\int_B U^{T,x}_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y),$$

$$\overline{Y}^{T,x}_t = \overline{v}(X^x_T) + \int_t^T \{f(X^x_r, \overline{Z}^{T,x}_r, \overline{U}^{T,x}_r \cdot \delta) - \overline{\lambda}\}\,\mathrm{d}u - \int_t^T \overline{Z}^{T,x}_r\,\mathrm{d}W_r - \int_t^T\int_B \overline{U}^{T,x}_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y).$$

We approach f, v and $\overline{v}$ by $\mathcal{C}^1_b$ functions, and we suppose that f^ε converges uniformly to f on $\mathbb{R}^d \times (\mathbb{R}^d)^* \times \mathbb{R}$ and that v^ε and $\overline{v}^\varepsilon$ converge uniformly to v and $\overline{v}$ on every compact set. We also consider the following BSDEs:

$$Y^{\varepsilon,T,x}_t = v^\varepsilon(X^x_T) + \int_t^T \{f^\varepsilon(X^x_r, Z^{\varepsilon,T,x}_r, U^{\varepsilon,T,x}_r \cdot \delta) - \lambda\}\,\mathrm{d}r - \int_t^T Z^{\varepsilon,T,x}_r\,\mathrm{d}W_r - \int_t^T\int_B U^{\varepsilon,T,x}_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y),$$

$$\overline{Y}^{\varepsilon,T,x}_t = \overline{v}^\varepsilon(X^x_T) + \int_t^T \{f^\varepsilon(X^x_r, \overline{Z}^{\varepsilon,T,x}_r, \overline{U}^{\varepsilon,T,x}_r \cdot \delta) - \lambda\}\,\mathrm{d}r - \int_t^T \overline{Z}^{\varepsilon,T,x}_r\,\mathrm{d}W_r - \int_t^T\int_B \overline{U}^{\varepsilon,T,x}_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y).$$

We define the functions u^ε_T and $\overline{u}^\varepsilon_T$ as in Lemma 4.34 and we set:

$$\beta^\varepsilon_T(t,x) = \begin{cases} \frac{f^\varepsilon(x,\partial_x u^\varepsilon_T(t,x)\sigma Q,\underline{u^\varepsilon_T}(t,x)\cdot\delta) - f^\varepsilon(x,\partial_x \overline{u}^\varepsilon_T(t,x)\sigma Q,\underline{u^\varepsilon_T}(t,x)\cdot\delta)}{|\partial_x u^\varepsilon_T(t,x)\sigma Q - \partial_x \overline{u}^\varepsilon_T(t,x)\sigma Q|^2}((\partial_x u^\varepsilon_T(t,x) - \partial_x \overline{u}^\varepsilon_T(t,x))\sigma Q)^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \partial_x u^\varepsilon_T(t,x) \neq \partial_x \overline{u}^\varepsilon_T(t,x), \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

$$\gamma^\varepsilon_T(t,x) = \begin{cases} \frac{f^\varepsilon(x,\partial_x \overline{u}^\varepsilon_T(t,x)\sigma Q,\underline{u^\varepsilon_T}(t,x)\cdot\delta) - f^\varepsilon(x,\partial_x \overline{u}^\varepsilon_T(t,x)\sigma Q,\underline{\overline{u}^\varepsilon_T}(t,x)\cdot\delta)}{|\underline{u^\varepsilon_T}(x)\cdot\delta - \underline{\overline{u}^\varepsilon_T}(x)\cdot\delta|^2}(\underline{u^\varepsilon_T}(t,x) - \underline{\overline{u}^\varepsilon_T}(t,x))\cdot\delta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \underline{u^\varepsilon_T}(t,x) \neq \underline{\overline{u}^\varepsilon_T}(t,x), \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

There exists a probability measure $\mathbb{Q}^\varepsilon_T$ equivalent to $\mathbb{P}$, under which we can write:

$$Y^{\varepsilon,T,x}_0 - \overline{Y}^{\varepsilon,T,x}_0 = \mathbb{E}^{\mathbb{Q}^\varepsilon_T}[(v^\varepsilon - \overline{v}^\varepsilon)(X^x_T)] - (\lambda - \overline{\lambda})T,$$

and then, dividing by T and doing $T \to \infty$, Proposition 4.3 leads us to $\lambda = \overline{\lambda}$. So, we have

$$Y^{\varepsilon,T,x}_0 - \overline{Y}^{\varepsilon,T,x}_0 = \mathcal{P}^{\varepsilon,T}_T[v^\varepsilon - \overline{v}^\varepsilon](x),$$

where $\mathcal{P}^{\varepsilon,T}$ stands for the Kolmogorov semigroup of the SDE:

$$\mathrm{d}V_t = \left\{\Xi(V_t) + \sigma Q \beta^\varepsilon_T(t,V_t) + \sigma \int_B y\gamma^\varepsilon_T(t,V_t)(y)\,\nu(\mathrm{d}y)\right\}\mathrm{d}t + \sigma Q\,\mathrm{d}W_t + \sigma\int_B y\,\widetilde{N}(\mathrm{d}t,\mathrm{d}y).$$

Using Lemma 4.7 and Corollary 4.6, we obtain:

$$\left|Y^{\varepsilon,T,x}_0 - \overline{Y}^{\varepsilon,T,x}_0 - Y^{\varepsilon,T,a}_0 - \overline{Y}^{\varepsilon,T,a}_0\right| \leqslant C(1+|x|^2+|a|^2)\mathrm{e}^{-\theta T}, \tag{4.18}$$

where the constant C is independant of T, x and ε. We would like to take the limit as ε goes to zero;

$$\begin{aligned} Y^{T,x}_t - Y^{\varepsilon,T,x}_t = (v - v^\varepsilon)(X^x_T) &+ \int_t^T \{f(X^x_r, Z^{T,x}_r, U^{T,x}_r\cdot\delta) - f^\varepsilon(X^x_r, Z^{\varepsilon,T,x}_r, U^{\varepsilon,T,x}_r\cdot\delta)\}\,\mathrm{d}r \\ &- \int_t^T \{Z^{T,x}_r - Z^{\varepsilon,T,x}_r\}\,\mathrm{d}W_r - \int_t^T\int_B \{U^{T,x}_r - U^{\varepsilon,T,x}_r\}(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y). \end{aligned}$$

Using Lemma 4.33, we have:

$$\left|Y^{T,x}_0 - Y^{\varepsilon,T,x}_0\right|^2 \leqslant \mathbb{E}\left[\left|Y^{T,x} - Y^{\varepsilon,T,x}\right|^{*,2}_{0,T}\right] \leqslant C_T(\mathbb{E}[|(v-v^\varepsilon)(X^x_T)|^2] + \|f - f^\varepsilon\|^2_\infty),$$

but, for any $R > 0$, using Markov's inequality, the polynomial growth of $v - v^\varepsilon$ and Proposition 4.3, we find that

$$\mathbb{E}[|(v-v^\varepsilon)(X^x_T)|^2] \leqslant \|v - v^\varepsilon\|^2_{\infty,\mathcal{B}_R} + C\frac{1+|x|^3}{R},$$

so, $\limsup\limits_{\varepsilon\to 0}\left|Y^{T,x}_0 - Y^{\varepsilon,T,x}_0\right|^2 \leqslant C_T\frac{1+|x|^3}{R}$, for every $R > 0$. Finally, $Y^{\varepsilon,T,x}_0 \underset{\varepsilon\to 0}{\longrightarrow} Y^{T,x}_0$. Taking the limit as $\varepsilon \to 0$ in equation (4.18) leads us to

$$|v(x) - \overline{v}(x)| \leqslant C(1+|x|^2+|a|^2)\mathrm{e}^{-\theta T},$$

because we assume that $v(a) = \overline{v}(a)$. Taking the limit as $T \to \infty$ ends the proof that $v = \overline{v}$. $\square$

4.5 Large time behaviour

In this section, we work under the assumptions made in the beginning of subsection 4.4.1.

Theorem 4.23 (First behaviour). *We consider the unique solution (Y^x, Z^x, U^x, λ) of the EBSDE (4.13). Let ξ^T be a real random variable $\mathcal{F}_T$-measurable and such that $|\xi^T| \leqslant C(1+|X_T^x|^2)$. We consider the unique solution $(Y^{T,x}, Z^{T,x}, U^{T,x})$ of the finite horizon BSDE:*

$$Y_t^{T,x} = \xi^T + \int_t^T f(X_s^x, Z_x^{T,x}, U_s^{T,x} \cdot \delta)\, \mathrm{d}s - \int_t^T Z_s^{T,x}\, \mathrm{d}W_s - \int_t^T \int_B U_s^{T,x}(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y).$$

Then, we have the following inequality:

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leqslant \frac{C(1+|x|^2)}{T},$$

where the constant C is independent of T and x ; in particular, we get the convergence $\frac{Y_0^{T,x}}{T} \underset{T\to\infty}{\longrightarrow} \lambda$, uniformly in any bounded subset of $\mathbb{R}^d$.

Proof. For all $x \in \mathbb{R}^d$ and $T > 0$, we write:

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leqslant \left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| + \left| \frac{Y_0^x}{T} \right|.$$

First of all, $|Y_0^x| = |v(x)| \leqslant C(1+|x|^2)$; we also have:

$$\begin{aligned} Y_0^{T,x} - Y_0^x - \lambda T = \xi^T - Y_T^x &+ \int_0^T \{ f(X_s^x, Z_s^{T,x}, U_s^{T,x} \cdot \delta) - f(X_s^x, Z_s^x, U_s^x \cdot \delta) \}\, \mathrm{d}s \\ &- \int_0^T \{ Z_s^{T,x} - Z_s^x \}\, \mathrm{d}W_s - \int_0^T \int_B \{ U_s^{T,x}(y) - U_s^x(y) \}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y). \end{aligned}$$

We set:

$$\beta_t = \begin{cases} \frac{f(X_t^x, Z_t^{T,x}, U_t^{T,x} \cdot \delta) - f(X_t^x, Z_t^x, U_t^{T,x} \cdot \delta)}{|Z_t^{T,x} - Z_t^x|^2} (Z_t^{T,x} - Z_t^x)^* & \text{if } Z_t^{T,x} \neq Z_t^x, \\ 0 & \text{otherwise.} \end{cases}$$

The function β is bounded by $\|f\|_{\mathrm{lip},z}$. Moreover, thanks to the Lemma 4.8, there exists a process γ such that:

$$f(X_t^x, Z_t^x, U_t^{T,x}) - f(X_t^x, Z_t^x, U_t^x) = (U_t^{T,x} - U_t^x) \cdot \gamma_t.$$

This way, the last equation can be rewritten as:

$$\begin{aligned} Y_0^{T,x} - Y_0^x - \lambda T = \xi^T - Y_T^x &- \int_0^T \{ Z_s^{T,x} - Z_s^x \} \{ \mathrm{d}W_s - \beta_s \mathrm{d}s \} \\ &- \int_0^T \int_B \{ U_s^{T,x}(y) - U_s^x(y) \} \{ N(\mathrm{d}s, \mathrm{d}y) - (1 + \gamma_s(y)) \nu(\mathrm{d}y) \mathrm{d}s \}, \end{aligned}$$

and we obtain, under a new probability measure $\mathbb{Q}_T$:

$$Y_0^{T,x} - Y_0^x - \lambda T = \mathbb{E}^{\mathbb{Q}^T}[\xi^T - Y_T^x].$$

Proposition 4.3 ends the proof. □

Theorem 4.24 (Second and third behaviours)**.** *In addition, we suppose that $\xi^T = g(X_T^x)$, where the function $g : \mathbb{R}^d \to \mathbb{R}$ has quadratic growth: $\forall x \in \mathbb{R}^d$, $|g(x)| \leqslant C(1+|x|^2)$ and satisfies:*

$$\forall x, x' \in \mathbb{R}^d,\ |g(x) - g(x')| \leqslant C(1+|x|^2+|x'|^2)|x-x'|.$$

Moreover, we require the measure ν to have a finite moment of order strictly greater than 6.
Then, there exists $L \in \mathbb{R}$, such that: $\forall x \in \mathbb{R}^d$, $Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to\infty}{\longrightarrow} L$. Furthermore,

$$\forall x \in \mathbb{R}^d,\ \forall T > 0,\ \left|Y_0^{T,x} - \lambda T - Y_0^x - L\right| \leqslant C(1+|x|^2+|x'|^2)\mathrm{e}^{-\theta T}.$$

Proof. We recall the ergodic BSDE:

$$\mathrm{d}Y_s^{t,x} = -\{f(X_s^{t,x}, Z_s^{t,x}, U_s^{t,x}\cdot\delta) - \lambda\}\mathrm{d}s + Z_s^{t,x}\,\mathrm{d}W_s + \int_B U_s^{t,x}(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y),$$

where $\mathrm{d}X_s^{t,x} = \Xi(X_s^{t,x})\,\mathrm{d}s + \sigma Q\,\mathrm{d}W_s + \sigma\int_B y\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y)$ and $X_t^{t,x} = x$, and whose solution satisfies $Y_s^{t,x} = v(X_s^{t,x})$. We want to approach the solution of this EBSDE by the following finite horizon BSDE:

$$\mathrm{d}Y_s^{T,t,x} = -f(X_s^{t,x}, Z_s^{T,t,x}, U_s^{T,t,x}\cdot\delta)\,\mathrm{d}s + Z_s^{T,t,x}\,\mathrm{d}W_s + \int_B U_s^{T,t,x}(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y), \quad Y_T^{T,t,x} = g(X_T^{t,x}),$$

and whose solution satisfies $Y_s^{T,t,x} = u_T(s, X_s^{t,x})$. We will need some regularity that the functions v and u_T may not have; we approach Ξ, f, v and g by sequences of $\mathcal{C}_b^1$ functions. We suppose that f^n and Ξ^n converge uniformly to f and Ξ on their whole domain of definition and that g^n and v^n converge uniformly on every compact set of $\mathbb{R}^d$. We consider the following finite horizon BSDEs:

$$\mathrm{d}Y_s^{n,T,t,x} = -f^n(X_s^{n,t,x}, Z_s^{n,T,t,x}, U_s^{n,T,t,x}\cdot\delta)\,\mathrm{d}s + Z_s^{n,T,t,x}\,\mathrm{d}W_s + \int_B U_s^{n,T,t,x}(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y),$$

$$\mathrm{d}\overline{Y}_s^{n,T,t,x} = -f^n(X_s^{n,t,x}, \overline{Z}_s^{n,T,t,x}, \overline{U}_s^{n,T,t,x}\cdot\delta)\,\mathrm{d}s + \overline{Z}_s^{n,T,t,x}\,\mathrm{d}W_s + \int_B \overline{U}_s^{n,T,t,x}(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y),$$

with terminal conditions $Y_T^{n,T,t,x} = g^n(X_T^{n,t,x})$ and $\overline{Y}_T^{n,T,t,x} = v^n(X_T^{n,t,x})$, and where

$$\mathrm{d}X_s^{n,t,x} = \Xi^n(X_s^{n,t,x})\,\mathrm{d}s + \sigma Q\,\mathrm{d}W_s + \sigma\int_B y\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y), \quad X_t^{n,t,x} = x.$$

Lemma 4.34 gives us $\mathcal{C}_b^{0,1}$ functions u_T^n and $\overline{u}_T^n$ such that $Y_s^{n,T,t,x} = u_T^n(s, X_s^{n,t,x})$ and $\overline{Y}_s^{n,T,t,x} = \overline{u}_T^n(s, X_s^{n,t,x})$. We set $w_T(t,x) = u_T(t,x) - v(x) - \lambda(T-t)$ and $w_T^n(t,x) = u_T^n(t,x) - \overline{u}_T^n(t,x)$; uniqueness for viscosity solutions of PIDEs tells us that we have the equalities $u_T^n(0,x) = u_{T+S}^n(S,x)$ and $\overline{u}_T^n(0,x) = \overline{u}_{T+S}^n(S,x)$.

Step 1: Growth of w_T^n.

Lemma 4.25. *We have: $\forall T > 0$, $\exists \overline{C}_T > 0$, $\forall x \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$,*

$$|v(x) - \overline{u}_T^n(0,x) + \lambda T|^2 \leqslant \overline{C}_T\left(\mathbb{E}\left[|v(X_T^x) - v^n(X_T^{n,x})|^2\right] + \|f - f^n\|_\infty^2 + \|\Xi - \Xi^n\|_\infty^2\right).$$

Proof.
We set $\Delta^n\overline{Y}_s = Y_s^x - \overline{Y}_s^{n,T,x} + \lambda(T-s)$, $\Delta^n\overline{Z}_s = Z_s^x - \overline{Z}_s^{n,T,x}$, $\Delta^n\overline{U}_s = U_s^x - \overline{U}_s^{n,T,x}$ and $\Delta^n X_s = X_s^x - X_s^{n,x}$. We have:

$$\mathrm{d}\Delta^n\overline{Y}_s = -\{f(X_s^x, Z_s^x, U_s^x\cdot\delta) - f^n(X_s^{n,x}, Z_s^{n,T,x}, U_s^{n,T,x}\cdot\delta)\}\,\mathrm{d}s + \Delta^n\overline{Z}_s\,\mathrm{d}W_s + \int_B \Delta^n\overline{U}_s(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y).$$

Using Lemma 4.33, we have

$$\mathbb{E}\left[|\Delta^n \overline{Y}|_{0,T}^{*,2} + \int_0^T \{|\Delta^n \overline{Z}_s|^2 + \|\Delta^n \overline{U}_s\|_\nu^2\} \, \mathrm{d}s\right] \leqslant \mathcal{C}_T \left(\mathbb{E}\left[|v(X_T^x) - v^n(X_T^{n,x})|^2\right] + \|f - f^n\|_\infty^2 + \|f\|_{\mathrm{lip},x}^2 |\Delta^n X|_{0,T}^{*,2}\right),$$

and Lemma 4.32 tells us

$$\mathbb{E}\left[|\Delta^n X|_{0,T}^{*,2}\right] \leqslant \mathcal{C}'_T \|\Xi - \Xi^n\|_\infty^2,$$

which concludes. □

The following lemma can be proved exactly the same way as the previous one.

Lemma 4.26. *We have:* $\forall T > 0$, $\exists C_T > 0$, $\forall x \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$,

$$|u_T(0,x) - u_T^n(0,x)|^2 \leqslant C_T \left(\mathbb{E}\left[|g(X_T^x) - g^n(X_T^{n,x})|^2\right] + \|f - f^n\|_\infty^2 + \|\Xi - \Xi^n\|_\infty^2\right).$$

We keep in mind the following results:

$$\begin{aligned} &|w_T^n(0,x) - w_T(0,x)|^2 \\ &\qquad \leqslant C_T \left(\mathbb{E}\left[|v(X_T^x) - v^n(X_T^{n,x})|^2 + |g(X_T^x) - g^n(X_T^{n,x})|^2\right] + \|f - f^n\|_\infty^2 + \|\Xi - \Xi^n\|_\infty^2\right), \\ &|w_T(0,x)| \leqslant C(1 + |x|^{2\vee p}). \end{aligned}$$

Step 2: Variation of $w_T^n(0,\bullet)$.
We write:

$$Y_0^{n,T,x} - \overline{Y}_0^{n,T,x} + \int_0^T \{Z_s^{n,T,x} - \overline{Z}_s^{n,T,x}\} \, \mathrm{d}W_s + \int_0^T \int_B \{U_s^{n,T,x} - \overline{U}_s^{n,T,x}\}(y) \, \widetilde{N}(\mathrm{d}s, \mathrm{d}y)$$
$$= (g^n - v^n)(X_T^{n,x}) + \int_0^T \{f^n(X_s^{n,x}, Z_s^{n,T,x}, U_s^{n,T,x} \cdot \delta) - f^n(X_s^{n,x}, \overline{Z}_s^{n,T,x}, \overline{U}_s^{n,T,x} \cdot \delta)\} \, \mathrm{d}s.$$

We set:

$$\beta_{T,S}^n(t,x) = \begin{cases} \dfrac{f^n(x, \partial_x u_{T+S}^n(t,x)\sigma Q, \underline{u_{T+S}^n}(t,x)\cdot\delta) - f^n(x, \partial_x \overline{u}_{T+S}^n(t,x)\sigma Q, \underline{u_{T+S}^n}(t,x)\cdot\delta)}{|\partial_x u_{T+S}^n(t,x)\sigma Q - \partial_x \overline{u}_{T+S}^n(t,x)\sigma Q|^2} \\ \qquad \times((\partial_x u_{T+S}^n(t,x) - \partial_x \overline{u}_{T+S}^n(t,x))\sigma Q)^* & \text{if } t \leqslant S \text{ and } \partial_x u_{T+S}^n(t,x) \neq \partial_x \overline{u}_{T+S}^n(t,x), \\ 0 & \text{if } t \leqslant S \text{ and } \partial_x u_{T+S}^n(t,x) = \partial_x \overline{u}_{T+S}^n(t,x), \\ \beta_{T,S}^n(S,x) & \text{if } t > S. \end{cases}$$

$$\gamma_{T,S}^n(t,x) = \begin{cases} \dfrac{f^n(x, \partial_x \overline{u}_{T+S}^n(t,x)\sigma Q, \underline{u_{T+S}^n}(t,x)\cdot\delta) - f^n(x, \partial_x \overline{u}_{T+S}^n(t,x)\sigma Q, \underline{\overline{u}_{T+S}^n}(t,x)\cdot\delta)}{|\underline{u_{T+S}^n}(x)\cdot\delta - \underline{\overline{u}_{T+S}^n}(x)\cdot\delta|^2} (\underline{u_{T+S}^n}(t,x) - \underline{\overline{u}_{T+S}^n}(t,x))\cdot\delta \\ & \text{if } t \leqslant S \text{ and } \underline{u_{T+S}^n}(t,x) \neq \underline{\overline{u}_{T+S}^n}(t,x), \\ 0 & \text{if } t \leqslant S \text{ and } \underline{u_{T+S}^n}(t,x) = \underline{\overline{u}_{T+S}^n}(t,x), \\ \gamma_{T,S}^n(S,x) & \text{if } t > S. \end{cases}$$

Because $\beta_{0,T}^n$ and $\gamma_{0,T}^n$ are bounded, we have, under a probability measure $\mathbb{Q}_T^n$ equivalent to $\mathbb{P}$:

$$w_T^n(0,x) = Y_0^{n,T,x} - \overline{Y}_0^{n,T,x} = \mathbb{E}^{\mathbb{Q}_T^n}[(g^n - v^n)(X_T^{n,x})] = \mathcal{P}_T^{n,T}[g^n - v^n](x),$$

where $\mathcal{P}^{n,T}$ is the Kolmogorov semigroup of the SDE:

$$\mathrm{d}V_t = \left\{\Xi^n(V_t) + \sigma Q \beta_T^n(t, V_t) + \sigma \int_B y \gamma_T^n(t, V_t)(y) \, \nu(\mathrm{d}y)\right\} \mathrm{d}t + \sigma Q \, \mathrm{d}W_t + \sigma \int_B y \, \widetilde{N}(\mathrm{d}t, \mathrm{d}y).$$

Corollary 4.6 and Lemma 4.7 give us:

$$\exists \widehat{c}, \varepsilon > 0,\ \forall T > 0,\ \forall x, y \in \mathbb{R}^d,\ \forall n \in \mathbb{N}^*,\ |w_T^n(0,x) - w_T^n(0,y)| \leqslant \widehat{c}(1 + |x|^2 + |y|^2)\mathrm{e}^{-\theta T}; \quad (4.19)$$

indeed, the constants of polynomial growths of g^n and v^n can be assumed independent from n (because their limits g and v have polynomial growth). Note that the use of Lemma 4.7 requires that the function $x \mapsto \underline{u_T^n}(t,x)$ is continuous; one can easily prove it when $m_4(\nu) < \infty$.

Step 3: Upperbound for $\partial_x w_T^n(0,\bullet)$.
We recall the Bismut-Elworthy formula (see Theorem 3.3 of [59]):

$$\begin{aligned}\partial_x w_T^n(t,x) = \mathbb{E}\Big[& w_{T-T'}^n(0, X_{T'}^{n,t,x}) N_{T'}^{n,t,x} \\ &+ \int_t^{T'} \{f^n(X_s^{n,t,x}, Z_s^{n,T,t,x}, U_s^{n,T,t,x}\cdot\delta) - f^n(X_s^{n,t,x}, \overline{Z}_s^{n,T,t,x}, \overline{U}_s^{n,T,t,x}\cdot\delta)\} N_s^{n,t,x}\,\mathrm{d}s\Big],\end{aligned}$$

where $N_s^{n,t,x} = \dfrac{1}{s-t}\displaystyle\int_s^t \langle (\sigma Q)^{-1}\nabla X_r^{n,t,x}, \mathrm{d}W_r\rangle$, and for every $T' \in (t,T]$. For all $s \in [t,T']$, we have:

$$\mathbb{E}\left[|N_s^{n,t,x}|^2\right] \leqslant \frac{|(\sigma Q)^{-1}|_F^2}{s-t}\mathbb{E}\left[|\nabla X^{n,t,x}|_{t,T'}^{*,2}\right].$$

But $\nabla X_s^{n,t,x} = \mathrm{I}_d + \displaystyle\int_t^s \partial_x \Xi^n(X_r^{n,t,x})\nabla X_r^{n,t,x}\,\mathrm{d}r$, so $\mathbb{E}\left[|\nabla X^{n,t,x}|_{t,T'}^{*,2}\right] \leqslant C_{T',p}$, independently of x and n. This way, we have shown that:

$$\mathbb{E}\left[|N_s^{n,t,x}|^2\right] \leqslant \frac{C_{T'}|(\sigma Q)^{-1}|_F^2}{s-t}.$$

Putting it in the equation given by Bismut-Elworthy formula gives us:

$$\begin{aligned}
&|\partial_x w_T^n(t,x)| \\
&\quad\leqslant \mathbb{E}\left[|w_{T-T'}(0, X_{T'}^{n,t,x})|^2\right]^{1/2}\sqrt{\frac{C_{T'}}{T'-t}}|(\sigma Q)^{-1}|_F \\
&\qquad + \int_t^{T'}\left\{\|f\|_{\mathrm{lip},z}\mathbb{E}\left[|Z_s^{n,T,t,x} - \overline{Z}_s^{n,T,t,x}|^2\right]^{1/2} + \|f\|_{\mathrm{lip},u}\|\delta\|_\nu \mathbb{E}\left[\|U_s^{n,T,t,x} - \overline{U}_s^{n,T,t,x}\|_\nu^2\right]^{1/2}\right\} \\
&\qquad\qquad \times\sqrt{\frac{C_{T'}}{s-t}}|(\sigma Q)^{-1}|_F\,\mathrm{d}s \\
&\quad\leqslant \mathbb{E}\left[|w_{T-T'}(0, X_{T'}^{n,t,x})|^2\right]^{1/2}\sqrt{\frac{C_{T'}}{T'-t}}|(\sigma Q)^{-1}|_F \\
&\qquad + |\sigma|_F\|f\|_{\mathrm{lip},z}\sqrt{C_{T'}}|(\sigma Q)^{-1}|_F\int_t^{T'}\mathbb{E}\left[|\partial_x w_T^n(s, X_s^{n,t,x})|^2\right]^{1/2}\frac{\mathrm{d}s}{\sqrt{s-t}} \\
&\qquad + |\sigma|_F\|f\|_{\mathrm{lip},u}\|\delta\|_\nu\sqrt{C_{T'}}|(\sigma Q)^{-1}|_F \\
&\qquad\qquad \times\int_t^{T'}\mathbb{E}\left[\int_B |y|^2\int_0^1 |\partial_x w_T^n(s, X_s^{n,t,x} + r\sigma y)|^2\,\mathrm{d}r\,\nu(\mathrm{d}y)\right]^{1/2}\frac{\mathrm{d}s}{\sqrt{s-t}}.
\end{aligned}$$

Because the derivatives of g and v have polynomial, we can assume that the derivatives of g^n

and v^n have the same polynomial growth. Using Lemma 4.33, we find that:

$$\begin{aligned}\mathbb{E}\left[|\nabla Y^{n,T,t,x}|^{*,2}_{t,T'}\right] &\leqslant C_{T'}\mathbb{E}\left[|\partial_x g^n(X^{n,t,x}_{T'})\nabla X^{n,t,x}_{T'}|^2\right.\\ &\qquad\qquad\left.+\int_t^{T'}|\partial_x f^n(X^{n,t,x}_s, Z^{n,T,t,x}_s, U^{n,T,t,x}_s\cdot\delta)\nabla X^{n,t,x}_s|^2\,\mathrm{d}s\right]\\ &\leqslant C_{T'}\mathbb{E}\left[\left\{C^2\left(1+|X^{n,t,x}_{T'}|^2\right)^2+\|f\|^2_{\mathrm{lip},x}(T'-t)\right\}|\nabla X^{n,t,x}|^{*,2}_{t,T'}\right]\\ &\leqslant \mathcal{C}_{T'}\left(1+|x|^4\right)\end{aligned}$$

So, we get the inequalities $|\partial_x u^n_T(t,x)| \leqslant \mathcal{C}_{T'}\left(1+|x|^2\right)$ and $|\partial_x \overline{u}^n_T(t,x)| \leqslant \mathcal{C}_{T'}\left(1+|x|^2\right)$, so we deduce $|\partial_x w^n_T(t,x)| \leqslant \mathcal{C}_{T'}\left(1+|x|^2\right)$. Let $\zeta > 2$, we are now able to define

$$\varphi^n_T(t) = \sup_{x\in\mathbb{R}^d}\frac{|\partial_x w^n_T(t,x)|}{1+|x|^\zeta},$$

and we have:

$$\mathbb{E}\left[|\partial_x w^n_T(s, X^{n,t,x}_s)|^2\right]^{1/2} \leqslant \mathbb{E}\left[\varphi^n_T(s)^2\left(1+|X^{n,t,x}_s|^\zeta\right)^2\right] \leqslant C\varphi^n_T(s)\left(1+|x|^\zeta\right),$$

where C is independent of s, n and x. Also:

$$\begin{aligned}&\mathbb{E}\left[\int_B |y|^2\int_0^1 |\partial_x w^n_T(s, X^{n,t,x}_s + r\sigma y)|^2\,\mathrm{d}r\,\nu(\mathrm{d}y)\right]^{1/2}\\ &\qquad\leqslant \mathbb{E}\left[\int_B |y|^2\int_0^1 \varphi^n_T(s)^2\left(1+|X^{n,t,x}_s + r\sigma y|^\zeta\right)^2\,\mathrm{d}r\,\nu(\mathrm{d}y)\right]^{1/2}\\ &\qquad\leqslant \varphi^n_T(s)\mathbb{E}\left[\int_B |y|^2\left(2+4^\zeta\left(|X^{n,t,x}_s|^{2\zeta}+\frac{|\sigma|_F^{2\zeta}|y|^{2\zeta}}{2\zeta+1}\right)\right)\nu(\mathrm{d}y)\right]^{1/2}\\ &\qquad\leqslant C\varphi^n_T(s)\left(1+|x|^\zeta\right),\end{aligned}$$

when $m_{2+2\zeta}(\nu) < \infty$. Consequently,

$$|\partial_x w^n_T(t,x)| \leqslant \frac{\mathcal{C}_{T'}}{\sqrt{T'-t}}\mathbb{E}\left[|w^n_{T-T'}(0, X^{n,t,x}_{T'})|^2\right]^{1/2} + \mathcal{C}_{T'}\int_t^{T'}\varphi^n_T(s)\frac{\mathrm{d}s}{\sqrt{s-t}}\left(1+|x|^\zeta\right).$$

But $\mathbb{E}\left[|w^n_{T-T'}(0, X^{n,t,x}_{T'})|^2\right] \leqslant C\left(1+|x|^4\right) + C_{T-T'}\left(\Delta^{n,x}_t + \|\Xi-\Xi^n\|^2_\infty + \|f-f^n\|^2_\infty\right)$, where

$$\Delta^{n,x}_t := \mathbb{E}\left[\left|v\left(X^{0,X^{n,t,x}_{T'}}_{T-T'}\right) - v^n\left(X^{n,0,X^{n,t,x}_{T'}}_{T-T'}\right)\right|^2 + \left|g\left(X^{0,X^{n,t,x}_{T'}}_{T-T'}\right) - g^n\left(X^{n,0,X^{n,t,x}_{T'}}_{T-T'}\right)\right|^2\right].$$

We finally obtain successively:

$$\begin{aligned}|\partial_x w^n_T(t,x)| &\leqslant \frac{\mathcal{C}_{T'}}{\sqrt{T'-t}}\left(1+|x|^2\right) + \frac{\mathcal{C}_{T',T-T'}}{\sqrt{T'-t}}\left(\sqrt{\Delta^{n,x}_t} + \|\Xi-\Xi^n\|_\infty + \|f-f^n\|_\infty\right)\\ &\qquad + \mathcal{C}_{T'}\int_t^{T'}\varphi^n_T(s)\frac{\mathrm{d}s}{\sqrt{s-t}}\left(1+|x|^\zeta\right)\end{aligned}$$

$$\begin{aligned}\frac{|\partial_x w^n_T(t,x)|}{1+|x|^\zeta} &\leqslant \frac{\mathcal{C}_{T'}}{\sqrt{T'-t}} + \frac{\mathcal{C}_{T',T-T'}}{\sqrt{T'-t}}\left(\frac{\sqrt{\Delta^{n,x}_t}}{1+|x|^\zeta} + \|\Xi-\Xi^n\|_\infty + \|f-f^n\|_\infty\right)\\ &\qquad + \mathcal{C}_{T'}\int_t^{T'}\varphi^n_T(s)\frac{\mathrm{d}s}{\sqrt{s-t}}\end{aligned}$$

$$\varphi_T^n(t) \leqslant \frac{\mathcal{C}_{T'}}{\sqrt{T'-t}} + \frac{\mathcal{C}_{T',T-T'}}{\sqrt{T'-t}} \left(\sup_{x\in\mathbb{R}^d} \frac{\sqrt{\Delta_t^{n,x}}}{1+|x|^\zeta} + \|\Xi - \Xi^n\|_\infty + \|f-f^n\|_\infty \right) + \mathcal{C}_{T'} \int_t^{T'} \varphi_T^n(s) \frac{\mathrm{d}s}{\sqrt{s-t}}$$

$$\varphi_T^n(T'-t) \leqslant \frac{\mathcal{C}_{T'}}{\sqrt{t}} + \frac{\mathcal{C}_{T',T-T'}}{\sqrt{t}} \left(\sup_{x\in\mathbb{R}^d} \frac{\sqrt{\Delta_{T'-t}^{n,x}}}{1+|x|^\zeta} + \|\Xi - \Xi^n\|_\infty + \|f-f^n\|_\infty \right) + \mathcal{C}_{T'} \int_0^{t} \varphi_T^n(T'-u) \frac{\mathrm{d}u}{\sqrt{t-u}}$$

Using Lemma 7.1.1 of [30], and chosing $t = T'$:

$$\begin{aligned}
&\frac{|\partial_x w_T^n(0,x)|}{1+|x|^\zeta} \\
&\leqslant \mathcal{C}_{T'} + \mathcal{C}_{T-T',T'} \left(\sup_{v\in\mathbb{R}^d} \frac{\sqrt{\Delta_0^{n,v}}}{1+|v|^\zeta} + \|\Xi - \Xi^n\|_\infty + \|f-f^n\|_\infty \right) \\
&\quad + \mathcal{C}_{T'} \int_0^{T'} E'(\mathcal{C}_{T'}(T'-u)) \left(\mathcal{C}_{T'} + \mathcal{C}_{T-T',T'} \left(\sup_{v\in\mathbb{R}^d} \frac{\sqrt{\Delta_{T'-u}^{n,v}}}{1+|v|^\zeta} + \|\Xi - \Xi^n\|_\infty + \|f-f^n\|_\infty \right) \right) \frac{\mathrm{d}u}{\sqrt{u}},
\end{aligned}$$

where E is the function defined in [30], *i.e.* $E(z) = \sum_{n=0}^{\infty} \frac{z^{n/2}}{\Gamma(\frac{n}{2}+1)}$.

Step 4: Taking the limit when $n \to \infty$.
From Theorem 4.23 and equation (4.19), we get easily:

$$\exists C > 0,\ \forall T > 0,\ \forall x \in \mathbb{R}^d,\ |w_T(0,x)| \leqslant C \left(1 + |x|^2\right),$$

$$\exists \widehat{c}, \varepsilon > 0,\ \forall T > 0,\ \forall x, y \in \mathbb{R}^d,\ |w_T(0,x) - w_T(0,y)| \leqslant \widehat{c} \left(1 + |x|^2 + |y|^2\right) \mathrm{e}^{-\theta T}.$$

The function w_T has no reason to be of class $\mathcal{C}^1$; so we use the mean value theorem before taking the limit in the inequality obtained in Step 3. The interested reader should have a look at Step 5 in the proof of Theorem 22 of [32] to know how we obtain, for every $\alpha > 0$ small enough, and for every $T' > 0$,

$$\exists C_{T'} > 0,\ \forall T > T',\ \forall x, y \in \mathbb{R}^d,\ |w_T(0,x) - w_T(0,y)| \leqslant C_{T'} \left(1 + |x|^{2+\alpha} + |y|^{2+\alpha}\right) |x-y|.$$

Step 5: Convergence of $w_T(0,\bullet)$ when $T \to \infty$.
Let K be a compact subset of $\mathbb{R}^d$. We need to show that the set $\mathcal{A} = \{w_T(0,\bullet)|_K | T > 1\}$ has a unique accumulation point to prove

$$\forall x \in \mathbb{R}^d,\ w_T(0,x) \underset{T\to\infty}{\longrightarrow} L,$$

where L is the pointwise limit of w_{T_i} on $\mathbb{R}^d$, with (T_i) an increasing sequence (see Step 6 of Theorem 22 of [32] for further explanations). Note that L is a constant.

Step 6: L is the unique accumulation point of $\mathcal{A}$.
Let $w_{\infty,K}$ be another accumulation point of $\mathcal{A}$, *i.e.* $\|w_{T_i'}(0,\bullet)|_K - w_{\infty,K}\|_\infty \underset{i\to\infty}{\longrightarrow} 0$, for an increasing sequence (T_i'). Again, we can easily prove that $w_{\infty,K}$ is equal to a constant L_K on K. Our

goal is now to show that $L = L_K$. We have:

$$\begin{aligned} w^n_{T+S}(0,x) &= Y^{n,T+S,x}_S - \overline{Y}^{n,T+S,x}_S \\ &\quad + \int_0^S \left\{ f^n(X^{n,x}_r, Z^{n,T+S,x}_r, U^{n,T+S,x}_r \cdot \delta) - f^n(X^{n,x}_r \overline{Z}^{n,T+S,x}_r, \overline{U}^{n,T+S,x}_r \cdot \delta) \right\} \mathrm{d}r \\ &\quad - \int_0^S \left\{ Z^{n,T+S,x}_r - \overline{Z}^{n,T+S,x}_r \right\} \mathrm{d}W_r - \int_0^S \int_B \left\{ U^{n,T+S,x}_r - \overline{U}^{n,T+S,x}_r \right\}(y)\, \widetilde{N}(\mathrm{d}r, \mathrm{d}y). \end{aligned}$$

Under a probability $\mathbb{Q}^{T,S,n,x}$ equivalent to $\mathbb{P}$, one can write:

$$\begin{aligned} w_{T+S}(0,x) &= \mathbb{E}^{\mathbb{Q}^{T,S,n}} \left[Y^{n,T+S,x}_S - \overline{Y}^{n,T+S,x}_S \right] = \mathbb{E}^{\mathbb{Q}^{T,S,n}} \left[w^n_{T+S}(S, X^{n,x}_S) \right] = \mathbb{E}^{\mathbb{Q}^{T,S,n}} \left[w^n_T(0, X^{n,x}_S) \right] \\ &= \mathcal{P}^{T,S,n}_S [w^n_T(0,\bullet)](x), \end{aligned}$$

where $\mathcal{P}^{T,S,n}$ is the Kolmogorov semigroup of the SDE:

$$\mathrm{d}V_t = \left\{ \Xi^n(V_t) + \sigma Q \beta^n_{T,S}(t,V_t) + \sigma \int_B y \gamma^n_{T,S}(t,V_t)(y)\, \nu(\mathrm{d}y) \right\} \mathrm{d}t + \sigma Q\, \mathrm{d}W_t + \sigma \int_B y\, \widetilde{N}(\mathrm{d}t, \mathrm{d}y).$$

Without any loss of generality, we can suppose that $T_i > T'_i$, for every $i \in \mathbb{N}$. We take $T = T'_i$ and $S = T_i - T'_i$. This way:

$$\forall x \in \mathbb{R}^d,\ \mathcal{P}^{T'_i, T_i - T'_i, n}_{T_i - T'_i} \left[w^n_{T'_i}(0,\bullet) \right](x) = w^n_{T_i}(0,x) \underset{n\to\infty}{\longrightarrow} w_{T_i}(0,x) \underset{i\to\infty}{\longrightarrow} L.$$

Now, to prove that $L = L_K$, it suffices to show that $\lim\limits_{i\to\infty} \limsup\limits_{n\to\infty} \mathcal{P}^{T'_i, T_i - T'_i, n}_{T_i - T'_i} \left[w^n_{T'_i}(0,\bullet) \right](x) = L_K$. We have:

$$\begin{aligned} \left| \mathcal{P}^{T'_i, T_i - T'_i, n}_{T_i - T'_i} \left[w^n_{T'_i}(0,\bullet) \right](x) - L_K \right| &\leqslant \left| \mathcal{P}^{T'_i, T_i - T'_i, n}_{T_i - T'_i} \left[w^n_{T'_i}(0,\bullet) \right](x) - w^n_{T'_i}(0,x) \right| \\ &\quad + \left| w^n_{T'_i}(0,x) - w_{T'_i}(0,x) \right| + \left| w_{T'_i}(0,x) - L_K \right|, \end{aligned}$$

$$\begin{aligned} \limsup_{n\to\infty} \left| \mathcal{P}^{T'_i, T_i - T'_i, n}_{T_i - T'_i} \left[w^n_{T'_i}(0,\bullet) \right](x) - L_K \right| &\leqslant \limsup_{n\to\infty} \left| \mathcal{P}^{T'_i, T_i - T'_i, n}_{T_i - T'_i} \left[w^n_{T'_i}(0,\bullet) \right](x) - w^n_{T'_i}(0,x) \right| \\ &\quad + \left| w_{T'_i}(0,x) - L_K \right|. \end{aligned}$$

Thanks to Lemma 4.7, $\beta^n_{T,S}$ and $\gamma^n_{T,S}$ are the limits of $\mathcal{C}^1$-functions, uniformly bounded, with bounded derivatives $\left(\beta^{n,m}_{T,S}\right)_{m\geqslant 0}$ and $\left(\gamma^{n,m}_{T,S}\right)_{m\geqslant 0}$. We define V^m as the solution of the SDE:

$$\begin{aligned} \mathrm{d}V^m_t = \left\{ \Xi^n(V^m_t) + \sigma Q \beta^{n,m}_{T,S}(t, V^m_t) + \sigma \int_B y \gamma^{n,m}_{T,S}(t, V^m_t)(y)\, \nu(\mathrm{d}y) \right\} \mathrm{d}t + \sigma Q\, \mathrm{d}W_t \\ + \sigma \int_B y\, \widetilde{N}(\mathrm{d}t, \mathrm{d}y), \end{aligned}$$

with $V^m_0 = x$. Thus,

$$\begin{aligned} \left| \mathcal{P}^{T'_i, T_i - T'_i, n}_{T_i - T'_i} \left[w^n_{T'_i}(0,\bullet) \right](x) - w^n_{T'_i}(0,x) \right| &= \left| \lim_{m\to\infty} \mathbb{E} \left[w^n_{T'_i}\left(0, V^m_{T_i - T'_i}\right) \right] - w^n_{T'_i}(0,x) \right| \\ &\leqslant \limsup_{m\to\infty} \mathbb{E} \left[\left| w^n_{T'_i}\left(0, V^m_{T_i - T'_i}\right) - w^n_{T'_i}(0,x) \right| \right] \\ &\leqslant \widehat{c} \left(1 + \limsup_{m\to\infty} \mathbb{E} \left[\left| V^m_{T_i - T'_i} \right|^2 \right] + |x|^2 \right) \mathrm{e}^{-\theta T'_i}, \end{aligned}$$

and the constants $\widehat{c}$ and ε are independent of n, m and T'_i. Thanks to proposition 4.3, we are able to conclude that $L = L_K$.

Step 8: Speed of convergence. Let $x \in \mathbb{R}^d$ and $T > 0$. We have:

$$\begin{aligned}|w_T(0,x) - L| &= \lim_{S\to\infty} |w_T(0,x) - w_S(0,x)| = \lim_{S\to\infty}\lim_{n\to\infty} |w_T^n(0,x) - w_S^n(0,x)| \\ &= \lim_{S\to\infty}\lim_{n\to\infty} \left| w_T^n(0,x) - \mathcal{P}_{S-T}^{T,S-T,n}\left[w_T^n(0,\bullet)\right](x)\right|.\end{aligned}$$

Like before, we use Lemma 4.7 and proposition 4.3, and we get:

$$\begin{aligned}|w_T(0,x) - L| &= \lim_{S\to\infty}\lim_{n\to\infty}\left| w_T^n(0,x) - \lim_{m\to\infty}\mathbb{E}\left[w_T^n\left(0, V_{S-T}^m\right)\right]\right| \\ &\leqslant \limsup_{S\to\infty}\limsup_{n\to\infty}\limsup_{m\to\infty}\mathbb{E}\left[\left|w_T^n(0,x) - w_T^n\left(0,V_{S-T}^m\right)\right|\right] \leqslant \widetilde{C}\left(1+|x|^2\right)\mathrm{e}^{-\theta T}.\end{aligned}$$

□

We can rephrase the previous theorem as:

Theorem 4.27. *We consider the systems (4.16) and (4.17).*
Then, there exists $L \in \mathbb{R}$, such that: $\forall x \in \mathbb{R}^d$, $u_T(0,x) - \lambda T - v(x) \underset{T\to\infty}{\longrightarrow} L$. Furthermore,

$$\forall x \in \mathbb{R}^d,\ \forall T > 0,\ |u_T(0,x) - \lambda T - v(x) - L| \leqslant C\left(1+|x|^2\right)\mathrm{e}^{-\theta T}.$$

4.6 Application to an Optimal control problem

We consider in this subsection a process X^x satisfying the SDE

$$X_t^x = x + \int_0^t \Xi(X_s^x)\,\mathrm{d}s + \int_0^t \sigma Q\,\mathrm{d}W_s + \int_0^t\int_B \sigma y\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y), \quad t \geqslant 0,$$

under the following assumptions:

- Ξ is Lipschitz continuous and weakly dissipative;
- $\sigma Q \in \mathrm{GL}_d(\mathbb{R})$.

Let U be a separable metric space; we call "control" any progressively measurable U-valued process. Let us take:

- $R_1 : U \to (\mathbb{R}^d)^*$ and $R_2 : U \to \mathbb{R}_+$ two bounded functions;
- $L : \mathbb{R}^d \times U \to \mathbb{R}$ such that: $\exists C > 0$, $\forall a \in U$, $\forall x, x' \in \mathbb{R}^d$, $|L(x,a) - L(x',a)| \leqslant C|x - x'|$ and $|L(x,a)| \leqslant C(1+|x|)$;
- $\delta \in \mathrm{L}^2(\nu)$, such that $\forall y \in B$, $0 \leqslant \delta(y) \leqslant C(1 \wedge |y|)$;
- $g : \mathbb{R}^d \to \mathbb{R}$ satisfying: $\exists C > 0$, $\forall x, x' \in \mathbb{R}^d$, $|g(x) - g(x')| \leqslant C(1+|x|^2+|x'|^2)|x-x'|$ and $|g(x)| \leqslant C(1+|x|^2)$.

For any control a and horizon $T > 0$, we set:

$$\rho_T^a = \mathcal{E}\left(\int_0^\bullet R_1(a_t)\,\mathrm{d}W_t + \int_0^\bullet\int_B R_2(a_t)\delta(y)\,\widetilde{N}(\mathrm{d}t,\mathrm{d}y)\right)_{\bullet\in[0,T]} \text{ and } \mathbb{P}_T^a = \rho_T^a\mathbb{P} \text{ over } \mathcal{F}_T.$$

We define the finite horizon cost as

$$J^T(x,a) = \mathbb{E}_T^a\left[\int_0^T L(X_t^x, a_t)\,\mathrm{d}t\right] + \mathbb{E}_T^a\left[g(X_T^x)\right],$$

and the associated problem is to minimise $J^T(x,a)$ over the set of controls $a : \Omega \times [0,T] \to U$. We also define the ergodic cost as

$$J(x,a) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_T^a \left[\int_0^T L(X_t^x, a_t)\,\mathrm{d}t \right]$$

and the goal is to minimise $J(x,a)$ over the controls $a : \Omega \times \mathbb{R}_+ \to U$. We define

$$W_t^a = W_t - \int_0^t R_1(a_s)\,\mathrm{d}s;$$

due to Girsanov's theorem, W^a is a Brownian motion under $\mathbb{P}_T^a$ on $[0,T]$, for every $T > 0$. Also, the compensator of the Poisson measure N under $\mathbb{P}_T^a$ is $(1 - R_2(a_t)\delta(y))\nu(\mathrm{d}y)\mathrm{d}t$. We define the Hamiltonian, for any $x \in \mathbb{R}^d$, $z \in (\mathbb{R}^d)^*$ and $u \in \mathrm{L}^2(\nu)$:

$$f(x,z,p) = \inf_{a\in U} \left\{ L(x,a) + zR_1(a) + R_2(a)p \right\}.$$

We recall that, is this infimum is attained for every x, z and u, by Filippov's theorem (see [45]), there exists a measurable function γ such that $f(x,z,p) = L(x,\gamma(x,z,p)) + zR_1(\gamma(x,z,p)) + R_2(\gamma(x,z,p))p$.

Lemma 4.28. *The Hamiltonian f satisfies:*

- $\forall x \in \mathbb{R}^d$, $|f(x,0,0)| \leqslant C(1+|x|)$;
- $\forall x,x' \in \mathbb{R}^d$, $\forall z,z' \in (\mathbb{R}^d)^*$, $\forall p,p' \in \mathbb{R}$,
$$|f(x,z,p) - f(x',z',p')| \leqslant C|x-x'| + \|R_1\|_\infty |z-z'| + \|R_2\|_\infty |p-p'|;$$
- $\forall x \in \mathbb{R}^d$, $\forall z \in (\mathbb{R}^d)^*$, $p \mapsto f(x,z,p)$ *is non-decreasing.*

Proposition 4.29 (Optimal control in finite horizon). *For every control a, we have $J^T(x,a) \geqslant Y_0^{T,x}$ where $Y^{T,x}$ comes from the finite horizon BSDE:*

$$Y_t^{T,x} = g(X_T^x) + \int_t^T f(X_s^x, Z_s^{T,x}, U_s^{T,x}\cdot\delta)\,\mathrm{d}s - \int_t^T Z_s^{T,x}\,\mathrm{d}W_s - \int_t^T \int_B U_s^{T,x}(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y), \quad t \in [0,T].$$

Moreover, under Filippov's condition, we have $J^T(x,\bar{a}^T) = Y_0^{T,x}$, for $\bar{a}_t^T = \gamma(X_t^x, Z_t^{T,x}, U_t^{T,x}\cdot\delta)$.

Proof. Under the probability measure $\mathbb{P}_T^a$, we can write:

$$Y_0^{T,x} = \mathbb{E}_T^a\left[g(X_T^x)\right] + \mathbb{E}_T^a\left[\int_0^T \left\{ f(X_s^x, Z_s^{T,x}, U_s^{T,x}\cdot\delta) - Z_s^{T,x}R_1(a_s) - R_2(a_s)U_s^{T,x}\cdot\delta \right\} \mathrm{d}s\right].$$

And then:

$$\begin{aligned} J^T(x,a) &= Y_0^{T,x} + \mathbb{E}_T^a\left[\int_0^T \left\{ L(X_s^x, a_s) + Z_s^{T,x}R_1(a_s) + R_2(a_s)U_s^{T,x}\cdot\delta - f(X_s^x, Z_s^{T,x}, U_s^{T,x}\cdot\delta) \right\} \mathrm{d}s\right] \\ &\geqslant Y_0^{T,x}, \end{aligned}$$

thanks to the definition of f. We see that we even have an equality if we take $a = \bar{a}^T$. □

Proposition 4.30 (Optimal ergodic control). *For every control a, we have $J(x,a) \geqslant \lambda$ where λ comes from the ergodic BSDE:*

$$Y_t^x = Y_T^x + \int_t^T \left\{ f(X_s^x, Z_s^x, U_s^x\cdot\delta) - \lambda \right\} \mathrm{d}s - \int_t^T Z_s^x\,\mathrm{d}W_s - \int_t^T \int_B U_s^x(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y), \quad 0 \leqslant t \leqslant T < \infty.$$

Moreover, under Filippov's condition, we have $J(x,\bar{a}) = \lambda$, for $\bar{a}_t = \gamma(X_t^x, Z_t^x, U_t^x\cdot\delta)$.

Proof. Under the probability measure $\mathbb{P}^a_T$, we can write:

$$Y^x_0 = \mathbb{E}^a_T\left[Y^x_T\right] + \mathbb{E}^a_T\left[\int_0^T \{f(X^x_s, Z^x_s, U^x_s\cdot\delta) - Z^x_s R_1(a_s) - R_2(a_s)U^x_s\cdot\delta\}\,\mathrm{d}s\right] - \lambda T.$$

And then:

$$\begin{aligned}J(x,a) &= \lambda + \limsup_{T\to\infty}\frac{Y^{x,n}_0 - \mathbb{E}^a_T\left[Y^x_T\right]}{T}\\ &\quad + \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}^a_T\left[\int_0^T \{L(X^x_s,a_s) + Z^x_s R_1(a_s) + R_2(a_s)U^x_s\cdot\delta - f(X^x_s, Z^x_s, U^x_s\cdot\delta)\}\,\mathrm{d}s\right]\\ &\geqslant \lambda,\end{aligned}$$

because we can apply Proposition 4.2, and thanks to the definition of f. We see that we even have an equality if we take $a = \bar{a}$. □

Theorem 4.31 (Large time behaviour for optimal control problem)**.** *For every control a, we have:*

$$\liminf_{T\to\infty}\frac{J^T(x,a)}{T} \geqslant \lambda.$$

Moreover, under Filippov's condition, and when ν has a finite moment of order strictly greater than 6, we have:

$$\left|J^T(x,\bar{a}^T) - J(x,\bar{a})T - Y^x_0 - L\right| \leqslant C(1+|x|^2)\mathrm{e}^{-\theta T}.$$

Proof. This is a straightforward consequence of the previous results of this subsection and of Theorem 4.24. □

4.7 Appendix

4.7.1 Some standard estimates

Lemma 4.32. *We consider the SDE:*

$$X_t = x + \int_0^t\{\widetilde{b}(s,X_s) + h^0_s\}\,\mathrm{d}s + \int_0^t\{\widetilde{\sigma}(s,X_s) + h^1_s\}\,\mathrm{d}W_s + \int_0^t\int_B\{\widetilde{\rho}(s,X_s) + h^2_s\}(y)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}y),$$

with adapted coefficients.
We suppose that $\widetilde{b}$, $\widetilde{\sigma}$ and $\widetilde{\rho}$ are Lipschitz w.r.t. the second variable and that $\widetilde{b}(t,0) = 0$, $\widetilde{\sigma}(t,0) = 0$ and $\widetilde{\rho}(t,0) = 0$ a.s.
Then, there exists a constant C depending only on T, $\|\widetilde{b}\|_{\mathrm{lip}}$, $\|\widetilde{\sigma}\|_{\mathrm{lip}}$ and $\|\widetilde{\rho}\|_{\mathrm{lip}}$, such that:

$$\mathbb{E}[|X|^{*,2}_{0,T}] \leqslant C\left(|x|^2 + \mathbb{E}\left[\int_0^T\{|h^0_s|^2 + |h^1_s|^2 + \|h^2_s\|^2_\nu\}\,\mathrm{d}s\right]\right).$$

Proof. The use of Burkholder-Davis-Gundy inequality gives us:

$$\begin{aligned}\mathbb{E}[|X|^{*,2}_{0,t}] &\leqslant 4|x|^2 + \mathbb{E}\left[\int_0^t\{8t|h^0_s|^2 + 32|h^1_s|^2 + 32\|h^2_s\|^2_\nu\}\,\mathrm{d}s\right]\\ &\qquad + (8t\|\widetilde{b}\|^2_{\mathrm{lip}} + 32\|\widetilde{\sigma}\|^2_{\mathrm{lip}} + 32\|\widetilde{\rho}\|^2_{\mathrm{lip}})\mathbb{E}\left[\int_0^t |X_s|^2\,\mathrm{d}s\right]\\ &\leqslant 4|x|^2 + \mathbb{E}\left[\int_0^T\{8T|h^0_s|^2 + 32|h^1_s|^2 + 32\|h^2_s\|^2_\nu\}\,\mathrm{d}s\right]\\ &\qquad + (8T\|\widetilde{b}\|^2_{\mathrm{lip}} + 32\|\widetilde{\sigma}\|^2_{\mathrm{lip}} + 32\|\widetilde{\rho}\|^2_{\mathrm{lip}})\int_0^t \mathbb{E}[|X|^{*,2}_{0,s}]\,\mathrm{d}s,\end{aligned}$$

and we conclude by Gronwall lemma. □

Lemma 4.33. *We consider the BSDE:*

$$Y_t = \xi + \int_t^T \{\widetilde{f}(s, Y_s, Z_s, U_s) + h_s\}\, \mathrm{d}s - \int_t^T Z_s\, \mathrm{d}W_s - \int_t^T \int_B U_s(y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}y),$$

with an adapted driver.
We suppose that $\widetilde{f}$ is Lipschitz w.r.t. y, z and u and that $\widetilde{f}(t,0,0,0) = 0$ a.s.
Then, there exists a constant C depending only on T and $\|\widetilde{f}\|_{\mathrm{lip}}$, such that:

$$\mathbb{E}\left[|Y|^{*,2}_{0,T} + \int_0^T |Z_t|^2\, \mathrm{d}t + \int_0^T \|U_t\|^2_\nu\, \mathrm{d}t\right] \leqslant C\mathbb{E}\left[|\xi|^2 + \int_0^T |h_s|^2\, \mathrm{d}s\right].$$

Proof. Using Lemma 19.1.4 and Theorem 19.1.6 of [15], we get:

$$\begin{aligned}
\mathbb{E}\left[|Y|^{*,2}_{0,T} + \int_0^T \{|Z_t|^2 + \|U_t\|^2_\nu\}\, \mathrm{d}t\right] &\leqslant 4\mathbb{E}\left[|\xi|^2 + \int_0^T |h_s|^2\, \mathrm{d}s\right] \\
&\qquad + 4(\|\widetilde{f}\|_{\mathrm{lip}} + 4) \int_0^T \mathbb{E}[|Y_s|^2 + |Z_s|^2 + \|U_s\|^2_\nu]\, \mathrm{d}s \\
&\leqslant 4(1 + (\|\widetilde{f}\|_{\mathrm{lip}} + 4)(T+2))\mathbb{E}[|\xi|^2] \\
&\qquad + 4\left(1 + \frac{\|\widetilde{f}\|_{\mathrm{lip}} + 4}{2\|\widetilde{f}\|_{\mathrm{lip}}} \mathrm{e}^{\beta T}(T+2)\right) \mathbb{E}\left[\int_0^T |h_s|^2\, \mathrm{d}s\right],
\end{aligned}$$

where $\beta = 4\|\widetilde{f}\|_{\mathrm{lip}} + \frac{1}{2}$. □

4.7.2 Regularity of the solution of a finite horizon BSDE

Lemma 4.34. *We consider the forward/backward system:*

$$\begin{cases}
X^{t,x}_s = x + \int_t^s \Xi(r, X^{t,x}_r)\, \mathrm{d}r + \int_t^s \sigma(r, X^{t,x}_r) Q\, \mathrm{d}W_r + \int_t^s \int_B \sigma(r, X^{t,x}_r) y\, \widetilde{N}(\mathrm{d}r, \mathrm{d}y), \\
Y^{t,x}_s = g(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r, U^{t,x}_r)\, \mathrm{d}r - \int_s^T Z^{t,x}_r\, \mathrm{d}W_r - \int_s^T \int_B U^{t,x}_r(y)\, \widetilde{N}(\mathrm{d}r, \mathrm{d}y).
\end{cases}$$

We make the following assumptions:

- $b \in \mathcal{C}^{0,1}_b([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ *and* $\sigma \in \mathcal{C}^{0,1}_b([0,T] \times \mathbb{R}^d, \mathrm{GL}_d(\mathbb{R}))$;
- $g \in \mathcal{C}^1_b(\mathbb{R}^d, \mathbb{R})$ *and* $f \in \mathcal{G}^{0,1}_b([0,T] \times \mathbb{R}^d \times \mathbb{R} \times (\mathbb{R}^d)^* \times \mathrm{L}^2(\nu), \mathbb{R})$.

Then we can show that the function $u : (t,x) \mapsto Y^{t,x}_t$ is of class $\mathcal{C}^{0,1}_b([0,T] \times \mathbb{R}^d, \mathbb{R})$. Moreover, we have the following representations for a.e. t, ω and y:

- $Z^{t,x}_s = \partial_x u(s, X^{t,x}_s)\sigma(s, X^{t,x}_s)$;
- $U^{t,x}_s(y) = u(s, X^{t,x}_s + \sigma(s, X^{t,x}_s)y) - u(s, X^{t,x}_s)$.

Proof. To simplify the presentation of the proof, we assume $d = 1$.
Let $(t,x) \in [0,T] \times \mathbb{R}$ and $h \neq 0$. We define $\nabla \bullet^h = \frac{\bullet^{t,x+h} - \bullet^{t,x}}{h}$, for $\bullet = X$, Y, Z or U. Using Lemma 4.33 and then Lemma 4.32, we get:

$$\mathbb{E}\left[|\nabla X^h|^{*,2}_{t,T} + |\nabla Y^h|^{*,2}_{t,T} + \int_t^T \{|\nabla Z^h_s|^2 + \|\nabla U^h_s\|^2_\nu\}\, \mathrm{d}s\right] \leqslant C,$$

where the constant C is independent of h and x (we use here the fact that f and g are Lipschitz continuous). So, we have the following convergence, uniformly in x:

$$\mathbb{E}\left[|X^{t,x+h}-X^{t,x}|_{t,T}^{*,2}+|Y^{t,x+h}-Y^{t,x}|_{t,T}^{*,2}+\int_t^T\{|Z_s^{t,x+h}-Z_s^{t,x}|^2+\|U_s^{t,x+h}-U_s^{t,x}\|_\nu^2\}\,\mathrm{d}s\right]\underset{h\to 0}{\longrightarrow}0.$$

Moreover, we have the following BSDE:

$$\begin{aligned}\nabla Y_s^h&+\int_s^T\nabla Z_r^h\,\mathrm{d}W_r+\int_s^T\int_B\nabla U_r^h(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y)\\&=\widetilde{g}_x^h\nabla X_T^h+\int_s^T\left\{\widetilde{f}_x^h(r)\nabla X_r^h+\widetilde{f}_y^h(r)\nabla Y_r^h+\widetilde{f}_z^h(r)\nabla Z_r^h\right.\\&\qquad\left.+\frac{f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},U_r^{t,x+h})-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},U_r^{t,x})}{h}\right\}\mathrm{d}r,\end{aligned}$$

where

$$\widetilde{g}_x^h=\int_0^1\partial_x g(X_T^{t,x}+wh\nabla X_T^h)\,\mathrm{d}w,\quad \widetilde{f}_x^h(r)=\int_0^1\partial_x f(r,X_r^{t,x}+wh\nabla X_r^h,Y_r^{t,x+h},Z_r^{t,x+h},U_r^{t,x+h})\,\mathrm{d}w,$$

$$\widetilde{f}_y^h(r)=\int_0^1\partial_y f(r,X_r^{t,x},Y_r^{t,x}+wh\nabla Y_r^h,Z_r^{t,x+h},U_r^{t,x+h})\,\mathrm{d}w\quad\text{and}$$
$$\widetilde{f}_z^h(r)=\int_0^1\partial_z f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}+wh\nabla Z_r^h,U_r^{t,x+h})\,\mathrm{d}w.$$

Similarly, we denote $\widetilde{g}_x^0=\partial_x g(X_T^{t,x})$, $\widetilde{f}_x^0(r)=\partial_x f(r,\Theta_r^{t,x})$, $\widetilde{f}_y^0(r)=\partial_y f(r,\Theta_r^{t,x})$ and $\widetilde{f}_z^0(r)=\partial_z f(r,\Theta_r^{t,x})$, where Θ stands for (X,Y,Z,U). We define the variational system:

$$\begin{cases}\nabla X_s=1+\int_t^s\partial_x b(r,X_r^{t,x})\nabla X_r\,\mathrm{d}r+\int_t^s\partial_x\sigma(r,X_r^{t,x})Q\nabla X_r\,\mathrm{d}W_r+\int_t^s\int_B\partial_x\sigma(r,X_r^{t,x})y\nabla X_r\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y),\\ \nabla Y_s=\widetilde{g}_x^0\nabla X_T+\int_s^T\{\widetilde{f}_x^0(r)\nabla X_r+\widetilde{f}_y^0(r)\nabla Y_r+\widetilde{f}_z^0(r)\nabla Z_r+\nabla_u f(r,\Theta_r^{t,x})\cdot\nabla U_r\}\,\mathrm{d}r-\int_s^T\nabla Z_r\,\mathrm{d}W_r\\ \qquad\qquad-\int_s^T\int_B\nabla U_r(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y),\end{cases}$$

where the dot $\cdot$ stands for the scalar product in the space $\mathrm{L}^2(\nu)$. In the following, we set $\Delta\bullet^h=\nabla\bullet^h-\nabla\bullet$, for $\bullet=X$, Y, Z or U. Using Lemma 4.33, we find that

$$\begin{aligned}\mathbb{E}&\left[|\Delta Y^h|_{t,T}^{*,2}+\int_t^T\{|\Delta Z_s^h|^2+\|\Delta U_s^h\|_\nu^2\}\,\mathrm{d}s\right]\\&\leqslant C_T\mathbb{E}\left[|\widetilde{g}_x^h\Delta X_T^h+(\widetilde{g}_x^h-\widetilde{g}_x^0)\nabla X_T|^2+\int_t^T|\widetilde{f}_x^h(r)\Delta X_r^h+\varepsilon^h(r)|^2\,\mathrm{d}r\right],\end{aligned}$$

where

$$\begin{aligned}\varepsilon^h(r)&=(\widetilde{f}_x^h(r)-\widetilde{f}_x^0(r))\nabla X_r+(\widetilde{f}_y^h(r)-\widetilde{f}_y^0(r))\nabla Y_r+(\widetilde{f}_z^h(r)-\widetilde{f}_z^0(r))\nabla Z_r\\&\quad+\frac{f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},U_r^{t,x}+h\nabla U_r)-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},U_r^{t,x})}{h}-\nabla_u f(r,\Theta_r^{t,x})\cdot\nabla U_r.\end{aligned}$$

But $\mathbb{E}[|\Delta X^h|_{t,T}^{*,2}]\underset{h\to 0}{\longrightarrow}0$, by dominated convergence, using the continuity of $\partial_x b$ and $\partial_x\sigma$, and the fact that $m_2(\nu)<\infty$. Then, using the Lipschitz continuity of f and g, we deduce that:

$$\mathbb{E}\left[|\Delta X^h|_{t,T}^{*,2}+|\Delta Y^h|_{t,T}^{*,2}+\int_t^T\{|\Delta Z_s^h|^2+\|\Delta U_s^h\|_\nu^2\}\,\mathrm{d}s\right]\underset{h\to 0}{\longrightarrow}0.$$

Doing the same as in Theorem 3.1 of [43], we can write:

$$\partial_x u(t,x) = \mathbb{E}\left[\partial_x g(X_T^{t,x})\nabla X_T \right.$$
$$\left. + \int_t^T \{\partial_x f(r,\Theta_r^{t,x})\nabla X_r + \partial_y f(r,\Theta_r^{t,x})\nabla Y_r + \partial_z f(r,\Theta_r^{t,x})\nabla Z_r + \nabla_u f(r,\Theta_r^{t,x})\cdot\nabla U_r\}\,\mathrm{d}r\right].$$

The proof of the continuity of $\partial_x u$ is very similar to the Step 6 of Proposition 33 of [32], so we omit here its proof. To conclude, we want now to prove the expressions of Z and U as functions of X. First of all, we define $\rho(t,x,y) = \sigma(t,x)y$.

- We approach b, σ, ρ, f and g by sequences of $\mathcal{C}_0^\infty$ functions which converge uniformly. Moreover, we can suppose that the derivatives of b^ε, σ^ε, ρ^ε, f^ε and g^ε are bounded independently of ε, because b, σ, ρ, f and g are Lipschitz continuous. We consider the forward-backward system:

$$\begin{cases} X_s^\varepsilon = x + \int_t^s b^\varepsilon(r,X_r^\varepsilon)\,\mathrm{d}r + \int_t^s \sigma^\varepsilon(r,X_r^\varepsilon)Q\,\mathrm{d}W_r + \int_t^s\int_B \rho^\varepsilon(r,X_r^\varepsilon,y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y), \\ Y_s^\varepsilon = g^\varepsilon(X_t^\varepsilon) + \int_s^T f^\varepsilon(r,\Theta_r^\varepsilon)\,\mathrm{d}r - \int_s^T Z_r^\varepsilon\,\mathrm{d}W_r - \int_s^T\int_B U_r^\varepsilon(y)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}y). \end{cases}$$

- We set $u^\varepsilon(t,x) = Y_t^\varepsilon$. Theorem 4.1 of [11] tells us that u^ε is the solution of the PIDE

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \mathcal{L}_t^\varepsilon u^\varepsilon(t,x) + f^\varepsilon(t,x,u^\varepsilon(t,x),\partial_x u^\varepsilon(t,x)\sigma^\varepsilon(t,x)Q, u^\varepsilon(t,x+\rho^\varepsilon(t,x,\bullet)) - u^\varepsilon(t,x)) = 0, \\ u^\varepsilon(T,x) = g^\varepsilon(x). \end{cases}$$

  Then, using Theorem 19.5.1 of [15], we find that

$$Z_s^\varepsilon = \partial_x u^\varepsilon(s,X_s^\varepsilon)\sigma^\varepsilon(s,X_s^\varepsilon)Q \quad \text{and} \quad U_s^\varepsilon : y \mapsto u^\varepsilon(s,X_s^\varepsilon + \rho^\varepsilon(s,X_s^\varepsilon,y)) - u^\varepsilon(s,X_s^\varepsilon).$$

  The conclusion consists on taking the limit as $\varepsilon \to 0$, which is similar to the end of the proof of Theorem 3.1 of [43].

□

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UNIVERSITE BRETAGNE LOIRE / MATHSTIC

UNIVERSITÉ DE RENNES 1

Titre : Etude du comportement en temps long d'équations aux dérivées partielles par des méthodes probabilistes

Mots clés : EDSR, EDP, comportement en temps long, contrôle optimal.

Résumé : Cette thèse s'intéresse à une étude des EDSR ergodiques, avec pour principal objectif leur application à l'étude du comportement en temps long de certaines EDP. Dans un premier temps, nous démontrons des résultats (qui sont déjà connus dans le cadre où l'EDS sous-jacente est à bruit additif) dans un cadre de bruit sous-jacent multiplicatif. Par la suite, l'introduction d'un nouvel aléa via un processus de Poisson nous permet de nous intéresser non plus au comportement en temps long d'une seule EDP, mais au comportement en temps long d'un système d'EDP couplées. Enfin, lorsque l'EDS sous-jacente est bruitée par un processus de Lévy, le lien est fait avec des équations intégro-différentielles partielles. L'application de ces équations à la résolution de problèmes de contrôle optimal est également présentée.

Title : Study of the large time behaviour of partial differential equations using probabilistic methods

Keywords: BSDE, PDE, large time behaviour, optimal control.

Abstract: In this thesis, we are interested in studying ergodic BSDEs, and our main goal is to apply our results to the large time behaviour of some PDEs. First, we prove some results (already known in the case where the underlying SDE has an additive noise) in the case of an underlying multiplicative noise. Then, we introduce a Poisson process and it leads us to the large time behaviour of a system of coupled PDEs. Finally, when the underlying SDE has a Lévy noise, we make a link with partial integro-differential equations. We also apply these equations to solve some optimal control problems.