On the notion of geodesic curvature in sub-Riemannian geometry

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"Il faut rendre à César ce qui est à César et à Dieu ce qui est à Dieu."

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Chapter 1

Introduction

1.1 English

One of the major concepts of geometry that was developed in its modern form in the 19th century is that of curvature. One can distinguish two types of curvatures. On the one hand, intrinsic curvatures that measure properties of geometric spaces and on the other hand, extrinsic curvatures that contain information on how geometric objects are embedded in another space. The famous Egregium Theorem, proved by Carl Friedrich Gauss, states that the Gaussian curvature of a surface embedded in a three dimensional Euclidean space only depends on the metric properties of the surface itself. This means that the Gaussian curvature of a surface is in fact an intrinsic notion of curvature. More generally, to extend the notion of intrinsic curvature that Gauss studied for surfaces to Riemannian spaces, the relevant notion is that of Riemannian curvature tensors. The first example of extrinsic curvature that we introduce is the geodesic curvature of a curve in a Riemannian space.

Let us consider I an open interval in $\mathbb{R}$ containing zero, $\mathcal{N}$ a Riemannian manifold and $\zeta : I \to \mathcal{N}$ a curve that is parametrized by arc length. ζ is said to be a geodesic when the distance between any point $\zeta(s)$ and any other point $\zeta(t)$ where t is close enough to s is equal to $|t - s|$. Before we define the notion of geodesic curvature that measures how far a curve is from being a geodesic, we explain how geodesics are characterized.

In a Riemannian space, there is a unique linear connection that is torsion-free and that is compatible with the metric. This connection is called the Levi-Civita connection and we denote it as ∇^{LC}.

Proposition 1.1.1. *Let $\mathcal{N}$ be a Riemannian manifold and let $\zeta : I \to \mathcal{N}$ be a smooth curve parametrized by arc length. We have that ζ is a geodesic if, and only if the equation*

$$\nabla^{LC}_{\zeta'} \zeta' = 0$$

is satisfied along the trajectory of ζ.

For $\zeta : I \to \mathcal{N}$ a smooth curve parametrized at speed one in a Riemannian space, the geodesic curvature of ζ is therefore naturally defined as the function

$$\begin{array}{ccc} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \left\| \nabla^{LC}_{\zeta'(t)} \zeta' \right\|. \end{array}$$

We now want to understand the geodesic curvature as a measure of how far a curve is from being a geodesic in a metric sense. Notice that by definition if $\zeta : I \to \mathcal{N}$ is a geodesic then for t close enough to zero, we have

$$\mathrm{dist}^2_M \left(\zeta(0), \zeta(t)\right) = t^2.$$

For an arbitrary curve parametrized by arc length, the previous formula is modified with a correction where the geodesic curvature appears.

Proposition 1.1.2. *Let $\mathcal{N}$ be a Riemannian manifold and let $\zeta : I \to \mathcal{N}$ be a smooth curve parametrized by arc length. We have*

$$\mathrm{dist}^2_M \left(\zeta(0), \zeta(t)\right) = t^2 - \frac{\left\| \nabla^{LC}_{\zeta'(0)} \zeta' \right\|^2}{12} t^4 + o\left(t^4\right).$$

One can ask if it is possible to extend these notions of curvature we briefly presented in Riemannian geometry to more general settings. In this thesis we focus more particularly on sub-Riemannian spaces.

Let us start by explaining what a sub-Riemannian manifold is. To endow a smooth manifold M with a sub-Riemannian structure, we choose a sub-bundle of its tangent bundle that we denote as Δ and that we call the *distribution*. A vector that is tangent to the distribution is said to be *horizontal* and a vector field is said to be horizontal when its evaluation at each point is horizontal. In order for M to be a sub-Riemannian manifold, we endow the distribtution with a metric tensor g which is a bilinear form on TM that is symmetric and positive-definite. We also require the Hörmander condition that means that the Lie algebra generated by the horizontal vector fields reaches every vector tangent to M.

A sub-Riemannian manifold M is a length space whose distance we now define. We say that a Lipschitz curve $\zeta : I \to M$ is horizontal when its velocity is almost everywhere horizontal and when the decomposition of its velocity on an orthonormal frame that smoothly depends on the point of M is measurable and essentially bounded. It is possible to define the length of a horizontal curve $\zeta : I \to M$ between times t and s with $t > s$ in I as

$$\mathrm{length}\left(\zeta_{|_{[s,t]}}\right) = \int_s^t \sqrt{g_{\zeta(u)}\left(\zeta'(u), \zeta'(u)\right)} \mathrm{d}u.$$

That way we can define the distance between two points in a sub-Riemannian manifold as the infimum of all lengths of curves that join them. This sub-Riemannian distance satisfies several convenient properties. First of all,

thanks to the Hörmander condition that we imposed on the distribution, the topology induced by the distance on a sub-Riemannian manifold is equal to the initial topology of the manifold . Another important property of the sub-Riemannian distance is that any pair of points in a complete and connected sub-Riemannian manifold are connected by a length minimizing curve. Proofs of these fundamental properties can be found in the textbook [ABB19, Chapter 3] for example.

The dichotomy between intrinsic and extrinsic curvature that we already mentioned also holds in sub-Riemannian geometry. As regards intrinsic curvature of sub-Riemannian spaces, it has been intensively studied from different perspectives, some of them involving a metric interpretation of the curvatre, and others using optimal transport or the heat equation. A brief summary of some existing works concerning intrinsic curvature in sub-Riemannian geometry can be found in the introduction of [ABR18]. As regards extrinsic curvature, it can be defined for several types of subspaces. For instance, the paper [RR08] focuses on curvature of surfaces in the Heisenberg group. This thesis is dedicated to studying geodesic curvature of curves in sub-Riemannian geometry, on which there is not yet an abundant litterature. To the best of our knowledge, previous works that study the geodesic curvature of curves focus on curves inside the simplest sub-Riemannian structures, namely the Heisenberg groups. In [DV16] and [BTV17] a notion of geodesic curvature has been introduced in the context of a Gauss-Bonnet-like formula. In [CFH18, CHL17] the authors find complete invariants for regular curves in the Heisenberg groups. The goal of this thesis is to introduce a notion of geodesic curvature in the contact sub-Riemannian setting and to understand this curvature as an object that contains metric information by proving properties similar to Proposition 1.1.2.

Now that we introduced the main definitions of sub-Riemannian geometry, we are ready to present the results we obtain in this thesis. But before we do this, let us first summarize the main goals that we aim at, in order to keep in mind where we are heading. The central idea in this thesis is to measure through a notion of geodesic curvature how far a smooth curve is from beeing a geodesic. Since a smooth curve has to be of finite length and therefore horizontal in order to be a geodesic, we choose to restrict the study only to smooth horizontal curves. Moreover, since it is the trajectory of a curve and not its parametriztion that makes it a geodesic, we focus only on smooth horizontal curves that are parametrized by arc length. The question now is, what do we expect from a notion of geodesic curvature ? First of all, that a smooth horizontal curve parametrized by arc length is a geodesic if and only if its geodesic curvature is identically zero. Once we have defined a notion of geodesic curvature such that the previous characterization of geodesics is satisfied, we enquire about its metric interpretation. More precisely, we link the sub-Riemannian geodesic curvature and the asymptotics of the distance between two close points along a curve to obtain results similar

to Proposition 1.1.2 in the sub-Riemannian setting.

Statement of the results We begin this thesis with a chapter devoted to the simplest example of sub-Riemannian manifold, namely the Heisenberg group. The results we present concerning the Heisenberg group can also be found in the paper [Koh19]. The Heisenberg group $\mathbb{H}$ can be represented as $\mathbb{R}^3$ with coordinates (x, y, z) that we endow with a distribution generated by the the orthonormal frame

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}.$$

Let us consider a smooth horizontal curve parametrized by arc length $\zeta : I \to \mathbb{H}$. If we denote by π the projection from the Heisenberg group to the (x, y)-plane parallely to the z-axis, then the curve $\pi \circ \zeta$ is also parametrized by arc length in the Euclidean plane. We define k_ζ the sub-Riemannian geodesic curvature of the curve ζ as the derivative of the Euclidean curvature of $\pi \circ \zeta$. If we denote as $(x_\zeta(t), y_\zeta(t), z_\zeta(t))$ the coordinates of $\zeta(t)$, we can write

$$k_\zeta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x'_\zeta(t)y''_\zeta(t) - y'_\zeta(t)x''_\zeta(t)\right).$$

Curvature k_ζ is identically zero if and only if ζ is a geodesic in $\mathbb{H}$. We moreover obtain a metric interpretation of the geodesic curvature.

Proposition 1.1.3. *Let $\zeta :]-T, T[\to \mathbb{H}$ be a smooth horizontal curve that is parametrized by arc length. We have*

$$\mathrm{dist}^2_{\mathbb{H}}\left(\zeta(0), \zeta(t)\right) = t^2 - \frac{k^2_\zeta(0)}{720}t^6 + \mathcal{O}\left(t^7\right).$$

In order to explore other sub-Riemannian spaces than $\mathbb{H}$, we must use tools we introduce in the third chapter. More specifically, we start the third chapter by explaining the contact framework, in which we set up to generalize the result we obtained in the Heisenberg group. A contact manifold is a space in which the distribution is the kernel of a differential one form ω such that $\mathrm{d}\omega$ restricted to $\ker\omega$ is non degenerate. Contact sub-Riemannian manifolds are practical to work in since they are endowed with several canonical object.

First of all in any contact sub-Riemannian space there exists a linear connection ∇ called the Tanno connection that respects the metric but is *not* torsion-free.

In a contact sub-Riemannian manifold, we also can naturally define a fiberwise endomorphism $J : \Delta \to \Delta$ such that for any horizontal vector fields X and Y,

$$g\left(X, JY\right) = \mathrm{d}\omega\left(X, Y\right).$$

In particular J sends each horizontal vector V onto a vector orthogonal to V.

A last important object that is associated to a contact structure is the Reeb vector field X_0 such that

$$i_{X_0} \mathrm{d}\omega = 0 \text{ and } \omega(X_0) = 1.$$

Let us consider $\zeta : I \to M$ a smooth horizontal curve parametrized by arc length in a contact sub-Riemannian manifold. The first natural idea is to study $\nabla_{\zeta'(t)}\zeta'$ since in the Riemannian case it is the relevant quantity to consider to distinguish geodesics from other curves. We start by noticing that since ζ is parametrized by arc length and as the Tanno connection is compatible with g, we have for all $t \in I$,

$$g_{\zeta(t)}\left(\nabla_{\zeta'(t)}\zeta', \zeta'(t)\right) = 0.$$

For a more precise description of $\nabla_{\zeta'(t)}\zeta'$, we decompose it on the direction of $\langle J\zeta'(t)\rangle$ and on $\langle J\zeta'(t)\rangle^{\perp}$ where $\langle\cdot\rangle$ denotes the span of a family of vectors. Let us introduce a few notations. For $V \in \Delta_q$, we denote by $\Pi_V^{\perp} : \Delta_q \to \Delta_q$ the orthogonal projection on $\langle V\rangle^{\perp}$. We define the characteristic deviation function of ζ as

$$\begin{array}{rccc} h_\zeta : & I & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & g_{\zeta(t)}\left(\nabla_{\zeta'(t)}\zeta', J\zeta'(t)\right), \end{array}$$

as well as the function

$$\begin{array}{rccc} k_{\zeta,1} : & I & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \left\|\Pi^{\perp}_{J\zeta'(t)}\left(\nabla_{\zeta'(t)}\zeta'\right)\right\|. \end{array}$$

However, the knowledge of both of the previous functions along a horizontal curve is not sufficient to say whether or not this curve is a geodesic. In the example of the Heisenberg group, it is possible to show that along every smooth horizontal curve parametrized by arc length $\zeta : I \to \mathbb{H}$, $k_{\zeta,1}$ is always identically zero and that this same curve ζ is a geodesic if, and only if h_ζ is a constant function. This gives us an indication that if we want to introduce a third function that vanishes along geodesics, it may contain the derivative of the characteristic deviation. In fact, for $\zeta : I \to M$ a smooth horizontal curve parametrized by arc length we define

$$\begin{array}{rccc} k_{\zeta,2} : & I & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \frac{\mathrm{d}}{\mathrm{d}t}\frac{h_\zeta(t)}{\|J\zeta'(t)\|^2} + g\left(\mathrm{Tor}\left(\zeta'(t), X_0\right), \zeta'(t)\right), \end{array}$$

where Tor stands for the torsion of the Tanno connection.

By reformulating the interpretation of geodesics in terms of projections of integral lines of a Hamiltonian vector field we prove the following.

Proposition 1.1.4. *Let M be a contact sub-Riemannian space and $\zeta : I \to M$ be a smooth horizontal curve parametrized by arc length. We have ζ is a geodesic if, and only if both $k_{\zeta,1}$ and $k_{\zeta,2}$ are identically zero.*

This last proposition allows us to call $k_{\zeta,1}$ and $k_{\zeta,2}$ the first and the second geodesic curvature.

The fourth chapter is devoted to the study of the specific case of three dimensional contact sub-Riemannian manifolds. When M is a three dimensional contact sub-Riemannian manifold, the key simplication that happens is that for any curve $\zeta : I \to M$ and at any time $t \in I$,

$$\nabla_{\zeta'(t)}\zeta' = \frac{h_\zeta(t)}{\|J\zeta'(t)\|^2} J\zeta'(t).$$

We can notice two consequences of the previous identity.

Proposition 1.1.5. *Let M be a three-dimensional contact sub-Riemannian structure and $\zeta : I \to M$ be a smooth horizontal curve parametrized by arc length. We have*

$$k_{\zeta,1} = 0.$$

Proposition 1.1.6. *Let M be a three-dimensional contact sub-Riemannian structure. If $\zeta_1, \zeta_2 : I \to M$ are two smooth horizontal curves parametrized by arc length such that*

(i) $\zeta_1(0) = \zeta_2(0)$ *and* $\zeta_1'(0) = \zeta_2'(0)$,

(ii) $h_{\zeta_1}(t) = h_{\zeta_2}(t)$ *for every* $t \in I$.

Then $\zeta_1(t) = \zeta_2(t)$ *for every* $t \in I$.

We then generalize Proposition 1.1.3 to 3D contact sub-Riemannian manifolds.

Theorem 1.1.7. *Let M be a three-dimensional contact sub-Riemannian manifold and let $\zeta : I \to M$ be a smooth horizontal curve parametrized by arc length. We have*

$$\mathrm{dist}^2_M(\zeta(0), \zeta(t)) = t^2 - \frac{k^2_{\zeta,2}(0)}{720} t^6 + o\left(t^6\right).$$

We emphasize the fact that an important part of the proof of the previous theorem actually consists in proving the regularity *at time* $t = 0$ of

$$\mathrm{dist}^2_M(\zeta(0), \zeta(t)).$$

Actually, in higher dimensions, we are not able to prove that this function has the same level of regularity than in three dimensions.

Theorem 1.1.7 can be compared to Proposition 1.1.2. We notice that the main order correction appears at a higher order in three dimensional contact sub-Riemannian spaces than in a Riemannian space. We can roughly interpret this phenomenon by saying that in 3D contact sub-Riemannian spaces there is a one-parameter family of geodesics parametrized by arc length that leave from a point in a fixed horizontal direction whereas in the Riemannian case, there is only one geodesic parametrized by arc length that leaves from a point in a chosen direction. Since there are "more" geodesics that can approximate a curve at a point in a 3D contact sub-Riemannian space than in a Riemannian space, an arbitrary curve can be "more closely" approximated by geodesics in three dimensional contact sub-Riemannian spaces than in the Riemannian case. That is why it is not surprising to observe that the term that indicates how far a curve is to being a geodesics appears further in the expansion in Theorem 1.1.7 than in Proposition 1.1.2. However, this phenomenon disappears in contact sub-Riemannian spaces in higher dimension (than three) since curves are authorized to be "further from" geodesics as they can have a non-vanishing geodesic curvature k_1.

In the last chapter of this thesis, having in mind the same questions as in three dimensional contact sub-Riemannian spaces, we explore what happens in higher dimensions. The first result we obtain looks quite like Theorem 1.1.7 but only contains a Taylor expansion at order 3.

Theorem 1.1.8. *Let M be a contact sub-Riemannian manifold. If $\zeta : I \to M$ is a smooth horizontal curve parametrized by arc length then for $t > 0$,*

$$\mathrm{dist}_M\left(\zeta(0), \zeta(t)\right) = t - \frac{k_{\zeta,1}(0)^2}{12} t^3 + o\left(t^3\right).$$

Under the hypothesis of the previous theorem, it is natural to ask if we can go deeper in the expansion of $\mathrm{dist}\left(\zeta(s), \zeta(t)\right)$ in the case where $k_{\zeta,1}$ vanishes along curve ζ ? The answer is yes in the case where $J^2 = -\mathrm{Id}$, provided we assume sufficient regularity of $\mathrm{dist}\left(\zeta(s), \zeta(t)\right)$ along the diagonal.

Theorem 1.1.9. *Let M be a contact sub-Riemannian manifold such that $J^2 = -\mathrm{Id}$. If $\zeta : I \to M$ is a smooth horizontal curve parametrized by arc length such that for every $s \in I$ and $t > s$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{dist}_M\left(\zeta(s), \zeta(t)\right) = 1 - K(s)\,(t-s)^4 + o\left((t-s)^4\right), \tag{1.1}$$

then

$$K(s) = \frac{k_{\zeta,2}^2(s)}{288}.$$

Notice that when the hypothesis of Theorem 1.1.9 are satisfied we can integrate (1.1) and we actually obtain that

$$\mathrm{dist}_M\left(\zeta(s), \zeta(t)\right) = (t-s) - \frac{k_{\zeta,2}^2(s)}{1440}\,(t-s)^5 + o\left((t-s)^5\right).$$

A first corollary of Theorem 1.1.9 consists in a new characterization of geodesics.

Corollary 1.1.10. *Let M be a contact sub-Riemannian manifold such that $J^2 = -\mathrm{Id}$ and let $\zeta : I \to M$ be a smooth horizontal curve parametrized by arc length.*

Curve ζ is a geodesic if, and only if for every $s \in I$ and $t > s$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{dist}_M\left(\zeta(s), \zeta(t)\right) = 1 + o\left(t^4\right).$$

Although the previous result is a quite satisfactory consequence of Theorem 1.1.9 from a theoretical point of view, we would also like to apply Theorem 1.1.9 to curves that are not geodesics. We show in section 5.5 that in the $(2n+1)$-dimensional Heisenberg groups, the explicit expression of the distance from a fixed point allows us check that the hypotheses of Theorem 1.1.9 are satisfied along any smooth horizontal curve parametrized by arc length.

Theorem 1.1.11. *Let $\zeta : I \to \mathbb{H}_{2n+1}$ be a smooth horizontal curve that is parametrized by arc length such that $k_{\zeta,1}$ is identically zero. We have*

$$\mathrm{dist}^2_{\mathbb{H}_{2n+1}}\left(\zeta(0), \zeta(t)\right) = t^2 - \frac{k^2_{\zeta,2}(0)}{720}t^6 + \mathcal{O}\left(t^7\right).$$

Moreover, the geodesic curvature $k_{\zeta,2}(0)$ is simply equal to $h'_\zeta(0)$.

From the results we obtain in this thesis, new questions quite naturally arise.

Since one of the major difficulties in this work is to prove the regularity of the squared distance between two close points on a smooth horizontal curve, it is natural to ask : are there examples of smooth horizontal curves in dimension strictly greater than three for which the squared distance between two close points on this curve is not regular enough to have a Taylor expansion at order 6 ?

It would also be interesting to try to generalize the results of this work. If we want to apply the same techniques as those we use in this thesis in more general sub-Riemannian spaces than contact spaces, we must possess a connection with which we can carry on computations efficiently and understand precisely the theory of Jacobi fields.

Another possible generalization of this work would be to define not only geodesic curvature that characterizes geodesics, but a complete set of Frenet-Serret invariants that characterize a curve and to study the properties of these invariants.

1.2 Français

Une notion géométrique fondamentale développée sous sa forme moderne au dix-neuvième siècle est la courbure. On distingue deux types de courbures. Tout d'abord, les courbures intrisèques qui mesurent les propriétés géométriques d'un espace ; mais on définit aussi des courbures extrinsèques qui apportent une information géométrique sur le plongement de notre objet dans l'espace ambiant. Le Théorème Egreguim de Gauss montre par exemple que la courbure gaussienne d'une surface plongée dans un espace euclidien de dimension trois dépend seulement des propriétés métriques de ladite surface. Ainsi, la courbure gaussienne d'une surface se révèle-t-elle être une notion intrinsèque de courbure. Une généralisation naturelle de la notion de courbure intrinsèque étudiée par Gauss dans le cas des surface est, dans les espaces riemanniens, celle de tenseur de courbure riemannienne. Quant aux courbures extrinsèques, la courbure géodésique d'une courbe d'un espace riemannien en est un premier exemple.

Soient I un intervalle ouvert de $\mathbb{R}$ contenant zéro, $\mathcal{N}$ une variété riemannienne et $\zeta : I \to \mathcal{N}$ une courbe paramétrée par longueur d'arc. ζ est qualifée de géodésique quand la distance entre chacun de ses points $\zeta(s)$ et des points $\zeta(t)$ suffisament proches est égale à $|t-s|$. Commençons par caractériser les géodésiques avant de définir la notion de courbure géodésique qui quantifie à quel point une courbe est loin d'être une géodésique.

Dans une variété riemannienne, il existe une unique connexion linéaire sans torsion qui est compatible avec la métrique. Il s'agit de la connexion de la notion de Levi-Civita, noté ∇^{LC}.

Proposition 1.2.1. *Soient $\mathcal{N}$ une variété riemannienne et $\zeta : I \to \mathcal{N}$ paramétrée par longueur d'arc. Alors ζ est une géodésique si et seulement si l'équation*

$$\nabla^{LC}_{\zeta'} \zeta' = 0$$

est vérifiée le long de ζ.

Ainsi, pour une courbe lisse $\zeta : I \to \mathcal{N}$ paramétrée à vitesse 1 dans une variété riemannienne, la courbure géodésique de ζ est naturellement définie comme suit :

$$\begin{array}{ccc} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \left\| \nabla^{LC}_{\zeta'(t)} \zeta' \right\|. \end{array}$$

On cherche désormais à interpréter la courbure géodésique comme une mesure, au sens métrique du terme, de l'éloignement de cette courbe par rapport à la notion de géodésique. Notons que par définition, si $\zeta : I \to \mathcal{N}$ est une géodésique, alors pour t suffisament proche de zéro, on a :

$$\mathrm{dist}^2_M (\zeta(0), \zeta(t)) = t^2.$$

Pour une courbe quelconque paramétrée par longueur d'arc la formule précédente admet un terme de correction où apparait la courbure géodésique.

Proposition 1.2.2. *Soient $\mathcal{N}$ une variété riemannienne et $\zeta : I \to \mathcal{N}$ une courbe lisse paramétrée par longueur d'arc. On a*

$$\mathrm{dist}_M^2\left(\zeta(0), \zeta(t)\right) = t^2 - \frac{\left\|\nabla^{LC}_{\zeta'(0)}\zeta'\right\|^2}{12} t^4 + o\left(t^4\right).$$

On peut se demander si les notions de courbure que nous avons brièvement introduites dans le cadre riemannien s'étendent à un contexte plus général. Cette thèse s'intéresse au cas des espaces sous-riemanniens.

Commençons par définir ce qu'est une variété sous-riemannienne. Pour munir une variété lisse d'une structure sous-riemannienne, il convient de choisir un sous-fibré vectoriel de son fibré tangent que l'on notera Δ appelé la *distribution*. Un vecteur tangent à la distribution est dit *horizontal* et un champ de vecteurs est dit horizontal lorsque son évaluation en chaque point est horizontale. Pour compléter la définition d'une variété sous-Riemannienne M, on munit la distribution d'un tenseur métrique g qui est une forme bilinéaire, symétrique, définie-positive sur TM. Enfin, on demande que la condition de Hörmander soit vérifiée, c'est à dire que l'algèbre de Lie générée par les champs de vecteurs horizontaux atteigne chaque vecteur tangent à M.

Toute variété sous-riemannienne M est un espace de longueur dont la distance est définie comme suit. On dit qu'une courbe lipschitzienne $\zeta \to M$ est horizontale lorsque sa vitesse est presque partout horizontale et lorsque la décomposition de sa vitesse dans une base orthonormée qui dépend de manière lisse du point de M est mesurable et essentiellement bornée. Il est alors possible de définir la longueur d'une courbe horizontale $\zeta : I \to M$ entre les instants t et s $(t > s)$ de I en posant

$$\text{longueur}\left(\zeta_{|_{[s,t]}}\right) = \int_s^t \sqrt{g_{\zeta(u)}\left(\zeta'(u), \zeta'(u)\right)}\mathrm{d}u.$$

Ainsi peut-on définir la distance entre deux points d'une variété sous riemannienne comme l'infimum des longueur de courbes les joignant. Cette distance vérifie plusieurs propriétés utiles. Tout d'abord la condition de Hörmander imposée à la distribution assure que la topologie induite par la distance sur une variété sous-riemannienne est égale à la topologie de la variété initiale. Une autre propriété importante de la distance sous-riemannienne est que tout couple de points d'une variété sous-riemannienne connexe et complète est joint par une courbe minimsant la longueur. Des preuves de ces résutats fondamentaux peuvent être par exemple trouvées dans [ABB19, Chapitre 3].

La dichotomie déjà soulignée entre courbures intrinsèque et extrinsèque existe également en géométrie sous-riemannienne. Les courbures intrinsèques

en géométrie sous-riemannienne ont été étudiées sous différents angles : intérprétation métrique, transport optimal, équation de la chaleur par exemple. L'introduction de [ABR18] dresse un panorama de certains de ces travaux. En ce qui concerne les courbures extrinsèques, elles peuvent être définies pour plusieurs types d'espaces. Par exemple, l'article [RR08] traite de courbure de surfaces dans le groupe de Heisenberg. Cette présente thèse est quant à elle dédiée á l'étude de la courbure géodésique de courbes en géométrie sous-Riemannienne, au sujet de laquelle la littérature n'est pour l'instant pas très abondante. A notre connaissance, les travaux existant sur la courbure géodésique dans ce contexte se cantonnent aux cas les plus élémentaires de structures sous-riemanniennes, à savoir les groupes de Heisenberg. Dans [DV16] et [BTV17] une notion de courbure géodésique est introduite pour étudier l'equivalent de Gauss-Bonnet. Dans [CFH18, CHL17] les auteurs définissent des invariants complets pour les courbes régulières dans les groupes de Heisenberg. Le but de cette thèse est de définir une courbure géodésique dans le contexte de la géométrie sous-riemannienne de contact et de comprendre l'information métrique contenue dans cette courbure en démontrant des propriétés similaires à la Proposition 1.2.2.

Les définitions de base de la géométrie sous-riemannienne ayant été explicitées, on peut desormais présenter les résultats obtenus dans ce manuscrit. L'idée centrale est de chercher à mesurer à travers le spectre d'une courbure géodésique, l'éloignement d'une courbe lisse par rapport au fait d'être une géodésique. Mais puisqu'une courbe lisee doit être de longueur finie, et donc horizontale pour être une géodésique, on restreint notre étude au cas des courbes lisses horizontales. De plus, c'est la trajectoire d'une courbe et non son paramétrage qui en fait une géodésique. On peut donc considérer que la courbe lisse horizontale est paramétrée par longueur d'arc. On doit maintenant préciser ce qu'on attend d'une courbure géodésique. On souhaite en premier lieu qu'une courbe horizontale lisse paramétrée par longueur d'arc soit une géodésique si et seulement sa courbure est identiquement nulle. On peut ensuite rechercher son interprétation métrique en liant la courbure géodésique à l'asymptotique de la distance entre deux points proches de la courbe.

Résultats Le premier chapitre de cette thèse est consacré à l'exemple le plus élémentaire de variété sous-riemannienne, le groupe de Heisenberg. Ce chapitre reprend les résultat de l'article [Koh19]. Le groupe de Heisenberg $\mathbb{H}$ peut être représenté comme $\mathbb{R}^3$ de coordonnées (x, y, z) que l'on munit de la distribution générée par la base orthonormée

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}.$$

Si on considère une courbe lisse horizontale paramétrée par longueur d'arc $\zeta \to \mathbb{H}$ et qu'on note π la projection du groupe de Heisenberg sur le plan

(x, y) parallèlement à l'axe des z, alors la courbe $\pi \circ \zeta$ est aussi paramétrée par longueur d'arc dans le plan euclidien. On définit k_ζ la courbure géodésique sous-riemannienne de ζ comme étant la dérivée de la courbure euclidienne de $\pi \circ \zeta$. En écrivant $(x_\zeta(t), y_\zeta(t), z_\zeta(t))$ les coordonnées de $\zeta(t)$, on a

$$k_\zeta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x'_\zeta(t)y''_\zeta(t) - y'_\zeta(t)x''_\zeta(t)\right).$$

La courbure k_ζ est identiquement nulle si et seulement si ζ est une géodésique dans $\mathbb{H}$. De plus, on obtient une interprétation métrique de la courbure géodésique.

Proposition 1.2.3. *Considérons $\zeta :]-T, T[\to \mathbb{H}$ une courbe horizontale lisse paramétrée par longueur d'arc. On a*

$$\mathrm{dist}^2_{\mathbb{H}}\left(\zeta(0), \zeta(t)\right) = t^2 - \frac{k_\zeta^2(0)}{720}t^6 + \mathcal{O}\left(t^7\right).$$

Afin de pouvoir étudier d'autres espaces sous-riemanniens que $\mathbb{H}$, on introduit des outils adaptés dans le troisième chapitre. Ce chapitre commence par une description du cadre de la géométrie sous-riemannienne de contact propice à la généralisation du développement obtenu dans le groupe de Heisenberg. Une variété de contact est un espace dans lequel la distribution est le noyau d'une 1-forme differentielle ω telle que $\mathrm{d}\omega$ restreinte à $\ker\omega$ est non dégénérée. Les variétés sous-riemanniennes de contact sont un cadre pratique puisqu'elles sont munies de plusieurs objets canoniques.

Tout d'abord, tout espace sous-riemannien de contact admet une connexion linéaire ∇ appelée connexion de Tanno qui préserve la métrique *mais* admet de la torsion.

Une variété sous-riemannienne de contact possède également un endomorphisme linéaire sur chaque fibre $J : \Delta \to \Delta$ qui vérifie pour tous champs de vecteurs horizontaux X et Y,

$$g\left(X, JY\right) = \mathrm{d}\omega\left(X, Y\right).$$

En particulier J envoie tout vecteur horizontal V sur un vecteur qui lui est orthogonal.

Un dernier objet notable associé à une structure de contact est le champ de vecteur de Reeb X_0 qui satisfait

$$i_{X_0}\mathrm{d}\omega = 0 \text{ et } \omega\left(X_0\right) = 1.$$

Considérons $\zeta : I \to M$ une courbe lisse horizontale paramétrée par longueur d'arc dans une variété sous-riemannienne de contact. La première

idée naturelle est d'étudier $\nabla_{\zeta'(t)}\zeta'$ car dans le cas riemannien, c'est la quantité qui permet de distinguer les géodésiques parmi toutes les courbes. Notons tout d'abord que puisque ζ est paramétrée par la longueur d'arc et que la connexion de Tanno est compatible avec g, on a pour tout $t \in I$,

$$g_{\zeta(t)}\left(\nabla_{\zeta'(t)}\zeta', \zeta'(t)\right) = 0.$$

Afin de décrire $\nabla_{\zeta'(t)}\zeta'$ plus précisément, on décompose ce vecteur suivant les directions of $\langle J\zeta'(t)\rangle$ et $\langle J\zeta'(t)\rangle^{\perp}$ où $\langle\cdot\rangle$ désigne l'espace vectoriel engendré par une famille de vecteurs. Introduisons quelques notations supplémentaires. Pour $V \in T_qM$, on note $\Pi_V^{\perp} : \Delta_q \to \Delta_q$ la projection orthogonale sur $\langle V\rangle^{\perp}$. On définit de plus la déviation caractéristique de ζ comme étant la fonction

$$\begin{array}{rccc} h_\zeta : & I & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & g_{\zeta(t)}\left(\nabla_{\zeta'(t)}\zeta', J\zeta'(t)\right), \end{array}$$

de même que la fonction

$$\begin{array}{rccc} k_{\zeta,1} : & I & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \left\|\Pi^{\perp}_{J\zeta'(t)}\left(\nabla_{\zeta'(t)}\zeta'\right)\right\|. \end{array}$$

Malgré tout, la connaissance de ces deux fonctions le long d'une courbe horizontale ne suffit pas à déterminer si il s'agit ou non d'une géodésique. Dans le cas du groupe de Heisenberg, on peut montrer que le long de toute courbe lisse horizontale paramétrée par longueur d'arc $\zeta : I \to \mathbb{H}$, $k_{\zeta,1}$ est identiquement nulle. De plus, toujours dans cet espace ζ est une géodésique si et seulement si h_ζ est une fonction constante. Cela suggère que si on souhaite trouver une fonction s'annulant le long des géodésiques, on peut s'attendre à ce qu'elle contienne la dérivée de la déviation caractérique. De fait, pour $\zeta : I \to M$ une courbe lisse horizontale paramétrée par longueur d'arc, on définit

$$\begin{array}{rccc} k_{\zeta,2} : & I & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \frac{\mathrm{d}}{\mathrm{d}t}\frac{h_\zeta(t)}{\|J\zeta'(t)\|^2} + g\left(\mathrm{Tor}\left(\zeta'(t), X_0\right), \zeta'(t)\right), \end{array}$$

où Tor désigne la torsion de la connexion de Tanno.

En reformulant l'interprétation des géodésiques comme des projections des lignes intégrales du champ hamiltonien, on démontre le résultat suivant.

Proposition 1.2.4. *Soient M un espace sous-riemannien de contact et $\zeta : I \to M$ une courbe lisse paramétrée par la longueur d'arc. ζ est une géodésique si et seulement les fonctions $k_{\zeta,1}$ et $k_{\zeta,2}$ sont toutes deux identiquement nulles.*

Ce dernier résultat justifie la terminologie première et seconde courbure géodésique pour $k_{\zeta,1}$ et $k_{\zeta,2}$.

Le quatrième chapitre traite le cas particulier de la dimension trois. Si M est une variété de contact sous-riemannienne de dimension trois, une simplification remarquable se produit. Pour toute courbe $\zeta : I \to M$ et tout instant $t \in I$,

$$\nabla_{\zeta'(t)}\zeta' = \frac{h_\zeta(t)}{\|J\zeta'(t)\|^2} J\zeta'(t).$$

Notons deux consequences de l'identité précédente.

Proposition 1.2.5. *Soient M une variété sous-riemannienne de contact tridimensionelle et $\zeta : I \to M$ une courbe lisse horizontale pramétrée par longueur d'arc. Alors*

$$k_{\zeta,1} = 0.$$

Proposition 1.2.6. *Soit M une variété sous-riemannienne de contact de dimension trois. Si $\zeta_1, \zeta_2 : I \to M$ sont deux courbes lisses horizontales paramétrées par longueur d'arc vérifiant*

(i) $\zeta_1(0) = \zeta_2(0)$ *et* $\zeta_1'(0) = \zeta_2'(0)$,

(ii) $h_{\zeta_1}(t) = h_{\zeta_2}(t)$ *pour tout* $t \in I$.

On a $\zeta_1(t) = \zeta_2(t)$ *pour tout* $t \in I$.

On généralise ensuite la Proposition 1.2.3 aux variétés sous-riemanniennes de contact tridimensionnelles.

Théorème 1.2.1. *Soient M une variété de contact sous-riemannienne tridimensionelle et $\zeta : I \to M$ une courbe horizontale lisse paramétrée par le longueur d'arc. On a*

$$\mathrm{dist}^2_M(\zeta(0), \zeta(t)) = t^2 - \frac{k^2_{\zeta,2}(0)}{720} t^6 + o\left(t^6\right).$$

Insistons sur le fait qu'une part importante de la preuve du précédent théorème consiste en l'étude de la régularité *à l'instant* $t = 0$ de

$$\mathrm{dist}^2_M(\zeta(0), \zeta(t)).$$

En dimension supérieure, nous n'avons pas pu étblir que cette fonction avait le même niveau de régularité qu'en dimension trois.

Le Théorème 1.2.1 doit être comparé avec la Proposition 1.2.2. Remarquons que le terme de correction prépondérant est d'ordre supérieur dans le cas sous-Riemannien de contact tridimensionel que dans le cas des variétés riemanniennes. Ceci peut être interprété en voyant que dans le cas de contact, il existe une famille à un paramétre de géodésiques paramétrées par la

longueur d'arc qui partent d'un point dans une direction horizontale fixée, quand dans le cas riemannien il n'y a qu'une seule telle géodésique. Il y a donc "plus" de géodésiques approximant une courbe en un point dans le cas de contact 3D que dans le cas riemannien. Dès lors, cette courbe sera "mieux approximée" par l'une de ces géodésiques. C'est pourquoi il n'est pas surprenant d'observer que le terme indiquant l'éloignement de la courbe à être une géodésique apparait plus loin dans le développement du Théorème 1.2.1 que dans la Proposition 1.2.2. Cependant, ce phénomène disparait dans les espaces sous-riemanniens de contact en dimension supérieure à trois puisque les courbes sont autorisées à être plus éloignées des géodésiques car elles peuvent avoir une courbure géodésique k_1 non identiquement nulle.

Dans le dernier chapitre du manuscrit, on s'intéresse à l'extension de ces résultats en dimension supérieure à trois. La première propriété obtenue est similaire au Théorème 1.2.1 sauf que c'est un développement de Taylor d'ordre 3.

Théorème 1.2.2. *Soit M une variété de contact sous-riemannienne. Si $\zeta : I \to M$ est une courbe lisse horizontale paramétrée par longueur d'arc alors pour tout $t > 0$,*

$$\mathrm{dist}_M\left(\zeta(0), \zeta(t)\right) = t - \frac{k_{\zeta,1}(0)^2}{12} t^3 + o\left(t^3\right).$$

Sous les hypothèses du théorème précédent il est naturel de se demander si il est possible de pousser plus loin le développement de $\mathrm{dist}\left(\zeta(s), \zeta(t)\right)$ au cas où $k_{\zeta,1}$ s'annule le long de ζ. On peut apporter une réponse positive si $J^2 = -\mathrm{Id}$ et si on suppose $\mathrm{dist}\left(\zeta(s), \zeta(t)\right)$ suffisamment régulière le long de la diagonale.

Théorème 1.2.3. *Soit M une variété de contact sous-riemannienne vérifiant $J^2 = -Id$. Si $\zeta : I \to M$ est une courbe lisse horizontale paramétrée par longueur d'arc vérifiant, pour tout $s \in I$ et tout $t > s$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{dist}_M\left(\zeta(s), \zeta(t)\right) = 1 - K(s)\left(t-s\right)^4 + o\left(\left(t-s\right)^4\right), \tag{1.2}$$

alors

$$K(s) = \frac{k_{\zeta,2}^2(s)}{288}.$$

Notons que sous les hypothèses du Théorème 1.2.3, on peut intégrer (1.2) et on obtient alors

$$\mathrm{dist}_M\left(\zeta(s), \zeta(t)\right) = (t-s) - \frac{k_{\zeta,2}^2(s)}{1440}\left(t-s\right)^5 + o\left(\left(t-s\right)^5\right).$$

Un corollaire du Théorème 1.2.3 est une nouvelle caractérisation des géodésiques.

Corollaire 1.2.7. *Soient M une variété de contact sous-riemannienne vérifiant $J^2 = -Id$ et $\zeta : I \to M$ une courbe lisse horizontale paramétrée par longueur d'arc.*

La courbe ζ est une géodésique si et seulement si pour tout $s \in I$ et tout $t > s$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{dist}_M\left(\zeta(s), \zeta(t)\right) = 1 + o\left(t^4\right).$$

Si le résultat précédent est une conséquence théorique remarquable au Théorème 1.2.3, on souhaiterais aussi l'appliquer aux courbes qui ne sont pas des géodésiques. On montre dans la section 5.5 que dans le groupe de Heisenberg de dimension $2n+1$, l'expression explicite de la distance à un point fixé nous permet de vérifier que les hypothèses du Théorème 1.2.3 sont satisfaites le long de toute courbe lisse horizontale paramétrée par longueur d'arc.

Théorème 1.2.4. *Soit $\zeta : I \to \mathbb{H}_{2n+1}$ une courbe lisse horizontale paramétrée par la longueur d'arc et vérifiant que $k_{\zeta,1}$ est identiquement nulle. On a*

$$\mathrm{dist}^2_{\mathbb{H}_{2n+1}}\left(\zeta(0), \zeta(t)\right) = t^2 - \frac{k^2_{\zeta,2}(0)}{720}t^6 + \mathcal{O}\left(t^7\right).$$

De plus, la courbure géodésique $k_{\zeta,2}(0)$ est égale à $h'_\zeta(0)$.

Des résultats obtenus dans cette thèse émergent de nouvelles questions.

Une des difficultés majeures étant d'établir la régularité de la distance au carré entre deux points proches le long d'une courbe, on s'interroge naturellement sur l'éventuelle existence de courbes en dimension strictement supérieure à trois telles que cette distance n'est pas suffisamment régulière pour en obtenir un développement à l'ordre 6.

Il serait également intéressant de généraliser certains résultats. Si l'on souhaite appliquer des techniques similaire à celles utilisées en contact dans un cadre plus large, on doit trouver une connexion adaptée au calcul et comprendre précisément la théorie des champs de Jacobi dans l'espace considéré.

Une autre possibilité d'extension de ce travail serait de définir, au delà de la courbure géodésique, un ensemble d'invariants de Frenet-Serret caractérisant entièrement une courbe et d'étudier les propriétés de ces invariants.

Le reste de ce manuscrit sera rédigé dans la langue de Shakespeare.

Chapter 2

A first example : the Heisenberg group

In this chapter, we follow the ideas developped in [Koh19] concerning the notion of geodesic curvature in the simplest example of sub-Riemannian structure, namely the Heisenberg group. The reason why we start by focusing on that group is that the properties of its distance function as well as of its geodesics are explicitly known, and lead to a quite straightforward solution to the problem of the distance between two points on a horizontal curve, which we study throughout this text.

2.1 The Heisenberg group

We briefly present the properties of the Heisenberg group that are of some use to our proof. For a deeper insight into this sub-Riemannian space we refer to [Mon02], [Bel96], [ABB19] and [Rif14]. There are several ways to present the Heisenberg group $\mathbb{H}$. The one we choose is to see it as $\mathbb{R}^3$ with coordinates x, y and z endowed with a sub-Riemannian structure whose distribution is spanned by the orthonormal frame

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}. \tag{2.1}$$

We also define

$$X_0 = \frac{\partial}{\partial z}. \tag{2.2}$$

We call g the metric on the distribution whose orthonormal frame is (X_1, X_2).

We say that a curve $\zeta :]-T, T[\to \mathbb{H}$ is horizontal if ζ is a Lipshitz curve that is almost everywhere tangent to the distribution, whose speed defined with respect to the orthonormal frame (X_1, X_2) is measurable and essentially bounded.

We can compute the length of a horizontal curve by integrating its norm along the curve. The distance between two points is defined as the infimum of length of curves that link those two points. This infimum happens to be a minimum.

We also emphasize the fact that the Heisenberg group is in fact a Lie group on which the sub-Riemannian structure is left-invariant, where the group law $*$ is given by :

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - y_1 x_2) \right). \tag{2.3}$$

Remark 2.1.1. In order to study properties of curves that only depend on the sub-Riemannian distance, it is sufficient to consider curves that leave from the origin at time zero, since every other curve can be sent to such a curve by the isometry that corresponds to the left-multiplication by the inverse of the initial point.

Another interesting piece of information about the Heisenberg group is the expression of the geodesics in this space that we can find in [ABB19, chapter 4, section 4.4.3]. We recall that a geodesic is a horizontal curve $\zeta : \mathbb{R} \to \mathbb{H}$ parametrized at constant speed such that for any t in $\mathbb{R}$ and for s in $\mathbb{R}$ close enough to t, the length of curve ζ between times t and s is equal to the distance between $\zeta(t)$ and $\zeta(s)$.

It is sufficient to give the expression of geodesics parametrized by arc length leaving from the origin since the Heisenberg group is a Lie group, it follows that all the other geodesics will be left translations and reparametrizations of these geodesics.

Proposition 2.1.2. *A curve $\zeta : \mathbb{R} \to \mathbb{H}$ is a geodesic parametrized by arc length leaving from the origin at time zero if, and only if, there exist two real numbers ω and θ_0 such that the coordinates $(x(t), y(t), z(t))$ of $\zeta(t)$ are*

$$\begin{cases} x(t) &= \frac{\sin(\omega t + \theta_0) - \sin(\theta_0)}{\omega}, \\ y(t) &= \frac{\cos(\theta_0) - \cos(\omega t + \theta_0)}{\omega}, \\ z(t) &= \frac{\omega t - \sin(\omega t)}{2\omega^2}, \end{cases}$$

for $\omega \neq 0$. When $\omega = 0$ these formulas become :

$$\begin{cases} x(t) &= t\cos(\theta_0), \\ y(t) &= t\sin(\theta_0), \\ z(t) &= 0. \end{cases}$$

The previous proposition has several other formulations

Corollary 2.1.3. *A horizontal curve $\zeta : \mathbb{R} \to \mathbb{H}$ is a geodesic parametrized by arc length if, and only if its projection on the plane (x, y) parallely to the z-axis is a circle or a straight line parametrized at speed one for the canonical metric on $\mathbb{R}^2$.*

Proof. We start by noticing that for any smooth curve $\zeta : \mathbb{R} \to \mathbb{R}^2$ that leaves from the origin at time zero, there exists a unique horizontal lift $\overline{\zeta} : \mathbb{R} \to \mathbb{H}$ of ζ through the projection parallely to the z-axis which leaves from the origin at time zero. It is therefore possible to characterize if a curve in the Heisenberg group that leaves from the origin at time zero is a geodesic parametrized by arc length by checking if its projection coincides with the projection of a geodesic leaving from the origin and parametrized by arc length. By using Proposition 2.1.2, we deduce that a smooth curve that leaves from the origin at time zero is a geodesic if, and only if its projection along the $z-$axis coincides with circles leaving from the origin and that are parametrized at speed one.

Now what about a curve ζ in the Heisenberg group that does not leave from the origin ? Since the Heisenberg group is a Lie group, we can say that ζ is a geodesic parametrized by arc length if and only if its left translation by $\zeta(0)^{-1}$ -which leaves from the origin at time zero- is a geodesic parametrized by arc length. This is equivalent to saying that the projection along the $z-$axis of the left translation by $\zeta(0)^{-1}$ of ζ is a circle. But according to the expression of the group law in the Heisenberg group (2.3), the projection along the z-axis of a left translation is nothing but a translation in $\mathbb{R}^2$ of the projection along the z-axis. We therefore deduce the result we are looking for. □

In order to state a last characterization of geodesics, we introduce a new notation.

Definition 2.1.4. To any horizontal curve parametrized by arc length $\zeta :]-T, T[\to \mathbb{H}$ we associate $\theta_\zeta :]-T, T[\to \mathbb{R}/2\pi\mathbb{Z}$ such that for any $t \in]-T, T[$,

$$\zeta'(t) = \cos\left(\theta_\zeta(t)\right) X_1 + \sin\left(\theta_\zeta(t)\right) X_2.$$

We now can rewrite Corollary 2.1.3 as

Corollary 2.1.5. *A horizontal curve parametrized by arc length $\zeta : \mathbb{R} \to \mathbb{H}$ is a geodesic if, and only if θ_ζ is an affine function.*

Proof. We directly show the equivalence.

$\theta_\zeta : \mathbb{R} \to \mathbb{R}$ is an affine function of time, if, and only if there exists $(a, b) \in \mathbb{R}^2$ such that for every $t \in \mathbb{R}$, $\zeta'(t) = \cos(at + b) X_1 + \sin(at + b) X_2$.

Now thanks to the expression of X_1 and X_2 given in (2.1), we can say that $\theta_\zeta : \mathbb{R} \to \mathbb{R}$ is an affine function of time, if, and only if there exists $(a, b) \in \mathbb{R}^2$ such that for every $t \in \mathbb{R}$ the projection of $\zeta'(t)$ on $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ parallely to $\frac{\partial}{\partial z}$ is

$$\cos(at + b)\frac{\partial}{\partial x} + \sin(at + b)\frac{\partial}{\partial y}.$$

We then notice that another formulation of the previous assertion consists in saying that $\theta_\zeta : \mathbb{R} \to \mathbb{R}$ is an affine function of time, if, and only if the projection of ζ on the $(x,y)-$plane parallely to the $z-$axis is a circle parametrized by arc length.

We conclude by applying Corollary 2.1.3. □

Since we expect the geodesic curvature of a curve to vanish all along a curve if and only if the curve is a geodesic, we are lead by the previous corollary to a natural definition of geodesic curvature in the Heisenberg group.

Definition 2.1.6. Let $\zeta :]-T,T[\to \mathbb{H}$ be a horizontal curve parametrized by arc length. We define its geodesic curvature as

$$\begin{array}{rccc} k_\zeta : &]-T,T[& \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \ddot{\theta}_\zeta(t). \end{array}$$

2.2 Expansion of the distance

Before we actually study the influence of the geodesic curvature on the distance between two close points on a curve, we must prove a technical result.

Proposition 2.2.1. *Let $\zeta :]-T,T[\to \mathbb{H}$ be a smooth horizontal curve parametrized by arc length. We denote by $x(t)$, $y(t)$ and $z(t)$ the coordinates of $\zeta(t)$. They are $\mathcal{C}^\infty$ smooth functions of t and*

$$\dot{z}(0) = \ddot{z}(0) = 0.$$

Moreover,

- Either *for every integer $i \geqslant 1$, $\theta^{(i)}(0) = 0$* and in this case *for all integers j, $z^{(j)}(0) = 0$ and for every i integer greater or equal to two, $x^{(i)}(0) = 0$ and $y^{(i)}(0) = 0$.*
- Or *there exists an integer $i \geqslant 1$ such that $\theta^{(i)}(0) \neq 0$* which entails that *for $t > 0$ close enough to zero, $\dot{\theta}(t)$ is non-vanishing and the two following identities hold true :*

$$x^2(t) + y^2(t) = 4 \int_0^t \int_0^u \left(-\dot{\theta}(s)\dot{z}(s) + \frac{1}{2} \right) \mathrm{d}s\mathrm{d}u, \tag{2.4}$$

$$\dddot{z}(t) = \ddot{\theta}(t) \int_0^t \left(-\dot{\theta}(s)\dot{z}(s) + \frac{1}{2} \right) \mathrm{d}s - \dot{\theta}^2(t)\dot{z}(t) + \frac{\dot{\theta}(t)}{2}. \tag{2.5}$$

Proof. First let us notice that $x(0) = y(0) = z(0) = 0$ since ζ leaves from $(0,0,0)$ by definition. The smoothness of x and y with respect to time comes from the fact that θ is $\mathcal{C}^\infty$ smooth and that

$$x(t) = \int_0^t \cos(\theta(s))\mathrm{d}s \qquad \text{and} \qquad y(t) = \int_0^t \sin(\theta(s))\mathrm{d}s.$$

The coordinate $z(t)$ is also a smooth function of t according to its expression

$$z(t) = \int_0^t -\frac{y(s)}{2}\cos(\theta(s)) + \frac{x(s)}{2}\sin(\theta(s))\mathrm{d}s.$$

Then we show that z satisfies a differential equation.

We start by writing :

$$\dot{z} = \frac{x\dot{y} - y\dot{x}}{2}, \qquad \ddot{x} = -\dot{\theta}\dot{y}, \qquad \ddot{y} = \dot{\theta}\dot{x}. \tag{2.6}$$

In particular, $\dot{z}(0) = 0$ and if we differentiate z once more

$$\ddot{z} = \frac{\dot{\theta}\,(x\dot{x} + y\dot{y})}{2}. \tag{2.7}$$

This implies that $\ddot{z}(0) = 0$ and if we go further in the differentiation

$$\dddot{z} = \frac{\ddot{\theta}\,(x\dot{x} + y\dot{y}) + \dot{\theta}(\overbrace{\dot{x}^2 + \dot{y}^2}^{=1}) + \dot{\theta}^2(-x\dot{y} + y\dot{x})}{2}.$$

If we multiply this last identity by $\dot{\theta}$ and combine it with (2.6) and (2.7), we obtain

$$\dot{\theta}\dddot{z} = \ddot{\theta}\ddot{z} - \dot{\theta}^3\dot{z} + \frac{\dot{\theta}^2}{2}. \tag{2.8}$$

Moreover, (2.6) and (2.7) allow us to assert that if for every integer i greater or equal to one $\theta^{(i)}(0) = 0$, then for every i integer greater or equal to two, $x^{(i)}(0) = 0$, $y^{(i)}(0) = 0$ and $z^{(i)}(0) = 0$.

On the other hand if we consider θ such that there exists an integer $i \geqslant 1$ that satisfies $\theta^{(i)}(0) \neq 0$ then for $t > 0$ close enough to zero, $\dot{\theta}(t)$ is non-vanishing and for such t we can divide the differential equation (2.8) by $\dot{\theta}^2(t)$ and find out that :

$$\frac{\partial}{\partial t}\left(\frac{\ddot{z}(t)}{\dot{\theta}(t)}\right) = -\dot{\theta}(t)\dot{z}(t) + \frac{1}{2}.$$

Therefore the difference between $\frac{\ddot{z}(t)}{\dot{\theta}(t)}$ and $\int_0^t \left(-\dot{\theta}(s)\dot{z}(s) + \frac{1}{2}\right)\mathrm{d}s$ is a constant. But since (2.7) holds we know that

$$\frac{\ddot{z}(t)}{\dot{\theta}(t)} = \frac{x(t)\dot{x}(t) + y(t)\dot{y}(t)}{2} \xrightarrow{t\to 0} 0.$$

So that for $t > 0$ small enough such that $\dot{\theta}(t) \neq 0$:

$$\frac{\ddot{z}(t)}{\dot{\theta}(t)} = \int_0^t \left(-\dot{\theta}(s)\dot{z}(s) + \frac{1}{2}\right)\mathrm{d}s. \tag{2.9}$$

But through (2.7), we are able to find a second expression for $\frac{\ddot{z}(t)}{\dot{\theta}(t)}$:

$$\frac{\ddot{z}(t)}{\dot{\theta}(t)} = \frac{1}{4}\frac{\partial}{\partial t}\left(x^2(t)+y^2(t)\right).$$

As a consequence of the two previous formula, for $t>0$ small enough

$$\frac{\partial}{\partial t}\left(x^2(t)+y^2(t)\right) = 4\int_0^t \left(-\dot{\theta}(s)\dot{z}(s)+\frac{1}{2}\right)\mathrm{d}s.$$

The fact that $\frac{\partial}{\partial t}\left(x^2(t)+y^2(t)\right)$ is continuous and that $x^2(0)+y^2(0)=0$ is sufficient to be sure that for t small enough

$$x^2(t)+y^2(t) = 4\int_0^t\int_0^u \left(-\dot{\theta}(s)\dot{z}(s)+\frac{1}{2}\right)\mathrm{d}s\mathrm{d}u.$$

Finally, still in the case where there exists an integer $i \geqslant 1$ such that $\theta^{(i)}(0)\neq 0$, we consider $t>0$ small enough to have $\dot{\theta}(t)\neq 0$ and we divide the differential equation (2.8) we have already established by $\dot{\theta}(t)$:

$$\dddot{z}(t) = \ddot{\theta}(t)\frac{\ddot{z}(t)}{\dot{\theta}(t)} - \dot{\theta}^2(t)\dot{z}(t) + \frac{\dot{\theta}(t)}{2}.$$

Then we replace $\frac{\ddot{z}(t)}{\dot{\theta}(t)}$ using (2.9) and we find out that :

$$\dddot{z}(t) = \ddot{\theta}(t)\int_0^t \left(-\dot{\theta}(s)\dot{z}(s)+\frac{1}{2}\right)\mathrm{d}s - \dot{\theta}^2(t)\dot{z}(t) + \frac{\dot{\theta}(t)}{2}.$$

□

We are now ready to explain what role the geodesic curvature of a curve plays in the distance separating two of its points.

Proposition (1.1.3). *If $\zeta :]-T,T[\to\mathbb{H}$ is a smooth horizontal curve that is parametrized by arc length then*

$$\mathrm{dist}^2_{\mathbb{H}}(\zeta(0),\zeta(t)) = t^2 - \frac{\left(k_\zeta(0)\right)^2}{720}t^6 + \mathcal{O}(t^7).$$

Proof. We know from [ABR18, Chapter 5, section 5.7, identity (5.24)] that the squared distance between $\zeta(t)$ and the origin, which is also $\zeta(0)$ can be expressed as

$$d^2_{\mathbb{H}}(\zeta(0),\zeta(t)) = \frac{x^2(t)+y^2(t)}{\mathrm{sinc}^2\circ\phi\left(\frac{z(t)}{x^2(t)+y^2(t)}\right)}, \tag{2.10}$$

where

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x},$$

with $\mathrm{sinc}(0) = 1$, and where ϕ is the inverse function of

$$\begin{array}{rccc} \psi: & [-\pi, \pi] & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \frac{1}{4}\left(\frac{u}{\sin^2(u)} - \cot(u)\right). \end{array}$$

We notice that we can rewrite

$$\psi(u) = \frac{2u - \sin(2u)}{4(1 - \cos(2u))}.$$

Then we check that

$$\psi(u) = \frac{u}{6} + \frac{u^3}{45} + \mathcal{O}\left(u^5\right).$$

And since ψ is odd and analytic, so is $\phi = \psi^{-1}$ and

$$\phi(u) = 6u + \alpha u^3 + \mathcal{O}\left(u^5\right).$$

Now

$$u = \psi \circ \phi(u) = u + \left(\frac{24}{5} + \frac{\alpha}{6}\right) u^3 + \mathcal{O}\left(u^5\right),$$

so $\alpha = -\frac{144}{5}$ and

$$\phi(u) = 6u - \frac{144}{5} u^3 + \mathcal{O}\left(u^5\right).$$

We recall that sinc, the cardinal sine function is defined as the entire function such that $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$ for all x different from 0, which implies that

$$\mathrm{sinc}(u) = 1 - \frac{u^2}{6} + \frac{u^4}{120} + \mathcal{O}\left(u^6\right).$$

We are then able to compute

$$\frac{1}{\mathrm{sinc}^2 \circ \phi(u)} = 1 + 12u^2 - \frac{144}{5} u^4 + \mathcal{O}\left(u^5\right). \tag{2.11}$$

Now we will need to know the Taylor expansion of z at time zero. We are interested only in the case where there exists an integer $i \geqslant 1$ such that $\theta^{(i)}(0) \neq 0$. Indeed, in the other case, we have already noticed in Proposition 2.2.1 that for all integers i, $z^{(i)}(0) = 0$. First, by Proposition 2.2.1, we have that $z(0) = \dot{z}(0) = \ddot{z}(0) = 0$. Then we write (2.5)

$$\dddot{z} = \ddot{\theta} \int_0^t \left(-\dot{\theta}(s)\dot{z}(s) + \frac{1}{2}\right) \mathrm{d}s - \dot{\theta}^2 \dot{z} + \frac{\dot{\theta}}{2}.$$

We evaluate this identity at zero and find out that :

$$\dddot{z}(0) = \frac{\dot{\theta}(0)}{2}. \tag{2.12}$$

Then we differentiate (2.5) and evaluate the identity at zero to obtain

$$z^{(4)}(0) = \ddot{\theta}(0). \tag{2.13}$$

Similarly when we differentiate (2.5) twice and look at what we find at $t = 0$ we get

$$z^{(5)}(0) = \frac{3\theta^{(3)}(0)}{2} - \frac{\dot{\theta}^3(0)}{2}. \tag{2.14}$$

These formulae for the first differentials of z at zero entail that

$$z(t) = \frac{\dot{\theta}(0)}{12}t^3 + \frac{\ddot{\theta}(0)}{24}t^4 + \left(\frac{\theta^{(3)}(0)}{80} - \frac{\dot{\theta}^3(0)}{240}\right)t^5 + \mathcal{O}(t^6). \tag{2.15}$$

A last ingredient we will need in order to complete the proof is the expression of the first differentials of $x^2 + y^2$ at zero. In order to find these differentials, we use Proposition 2.2.1 :

$$x^2(t) + y^2(t) = 4\int_0^t \int_0^u \left(-\dot{\theta}(s)\dot{z}(s) + \frac{1}{2}\right) \mathrm{d}s\mathrm{d}u.$$

This identity enables us to compute the derivatives of $x^2(t) + y^2(t)$.

$$x^2(0) + y^2(0) = \frac{\partial}{\partial t}\Big|_{t=0} \left(x^2(t) + y^2(t)\right) = 0,$$

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0} \left(x^2(t) + y^2(t)\right) = 2 \text{ (since } \dot{z}(0) = 0 \text{ by Proposition 2.2.1),}$$

and for $n \geqslant 3$:

$$\frac{\partial^n}{\partial t^n}\left(x^2(t) + y^2(t)\right) = -4\sum_{i=0}^{n-2} \binom{n-2}{i} \theta^{(i+1)} z^{(n-i-1)}.$$

Now we remember that $\dot{z}(0) = \ddot{z}(0) = 0$ by Proposition (2.2.1) that $z^{(3)}(0)$, $z^{(4)}(0)$ and $z^{(5)}(0)$ are given by (2.12), (2.13) and (2.14) so we get

$$\frac{\partial^3}{\partial t^3}\Big|_{t=0} \left(x^2(t) + y^2(t)\right) = 0,$$

and

$$\frac{\partial^4}{\partial t^4}\Big|_{t=0} \left(x^2(t) + y^2(t)\right) = -4\dot{\theta}(0)z^{(3)}(0) = -2\dot{\theta}^2(0),$$

and also

$$\frac{\partial^5}{\partial t^5}\Big|_{t=0}\left(x^2(t)+y^2(t)\right) = -12\ddot{\theta}(0)z^{(3)}(0) - 4\dot{\theta}(0)z^{(4)}(0) = -10\dot{\theta}(0)\ddot{\theta}(0),$$

and finally

$$\begin{aligned}\frac{\partial^6}{\partial t^6}\Big|_{t=0}\left(x^2(t)+y^2(t)\right) &= -24\dddot{\theta}(0)z^{(3)}(0) - 16\ddot{\theta}(0)z^{(4)}(0) - 4\dot{\theta}(0)z^{(5)}(0)\\ &= -18\dot{\theta}(0)\dddot{\theta}(0) - 16\ddot{\theta}^2(0) + 2\dot{\theta}^4(0).\end{aligned}$$

From the knowledge of the six first derivatives of $x^2(t)+y^2(t)$, we deduce the following Taylor expansion

$$\begin{aligned}x^2(t)+y^2(t) =& t^2 - \frac{\dot{\theta}^2(0)}{12}t^4 - \frac{\dot{\theta}(0)\ddot{\theta}(0)}{12}t^5\\ &+ \left(-\frac{\dot{\theta}(0)\ddddot{\theta}(0)}{40} - \frac{\ddot{\theta}^2(0)}{45} + \frac{\dot{\theta}^4(0)}{360}\right)t^6 + \mathcal{O}(t^7). \qquad (2.16)\end{aligned}$$

Remark 2.2.2. The expansions that are given by (2.15) and (2.16) are still valid in the case where for all integers $i \geqslant 1$, $\theta^{(i)}(0) = 0$, according to the first point in Proposition 2.2.1.

We make use of (2.15) and (2.16) and we find out that

$$\frac{z(t)}{x^2(t)+y^2(t)} = \frac{\dot{\theta}(0)}{12}t + \frac{\ddot{\theta}(0)}{24}t^2 + \left(\frac{\theta^{(3)}(0)}{80} + \frac{\dot{\theta}^3(0)}{360}\right)t^3 + \mathcal{O}\left(t^4\right).$$

By composing the previous Taylor expansion with (2.11), we can write :

$$\begin{aligned}\frac{1}{\mathrm{sinc}^2 \circ \phi\left(\frac{z(t)}{x^2(t)+y^2(t)}\right)} =& 1 + \frac{\dot{\theta}^2(0)}{12}t^2 + \frac{\dot{\theta}(0)\ddot{\theta}(0)}{12}t^3\\ &+ \left(\frac{\ddot{\theta}^2(0)}{48} + \frac{\dot{\theta}(0)\theta^{(3)}(0)}{40} + \frac{\dot{\theta}^4(0)}{240}\right)t^4 + \mathcal{O}\left(t^5\right).\end{aligned}$$

We consider the product of the expansion we just found and of the expansion given by (2.16), and thanks to (2.10) we are able to conclude that

$$d^2_{\mathbb{H}}(\zeta(0),\zeta(t)) = t^2 - \frac{\left(\ddot{\theta}^2(0)\right)^2}{720}t^6 + \mathcal{O}(t^7).$$

□

Chapter 3

Some tools and properties in sub-Riemannian geometry

In this chapter, we present the contact setting in which our study takes place. We then recall some properties of the distance function in sub-Riemannian geometry as well as the Hamiltonian interpretation of sub-Riemannian geodesics. We also introduce, along any smooth horizontal curve parametrized by arc length, a real valued function, which we call the characteristic deviation of the curve. We show that this characteristic deviation can be used to reformulate the Hamiltonian geodesic equation.

3.1 Contact geometry

Adressing the problem of the asymptotics of the distance between two close points along a curve in a general sub-Riemannian manifold is somewhat too complicated. The object of this section is to present the specific spaces in which we work in this paper, namely contact sub-Riemannian manifolds.

We say that (M, ω, g) is sub-Riemannian contact space when M is a manifold endowed with a sub-Riemannian structure whose distribution is the kernel of the differential 1-form ω, with the condition that $\mathrm{d}\omega$ restricted to $\ker \omega$ is non degenerate.

In the following we always consider sub-Riemannian contact manifolds that are complete.

In a contact sub-Riemannian space there exist several object that we now present. We start with the Reeb vector field.

Definition 3.1.1. Let us consider (M, ω) a contact space, its Reeb vector field is the unique field $X_0 \in \Gamma(TM)$ that satisfies

$$i_{X_0} \mathrm{d}\omega = 0 \text{ and } \omega(X_0) = 1.$$

Definition 3.1.2. Let (M, ω, g) be a sub-Riemannian contact space, we define the map

$$J : \Delta \to \Delta$$

which acts on each fiber of Δ as an antisymmetric linear operator such that for any horizontal vector fields X and Y

$$g(X, JY) = \mathrm{d}\omega(X, Y).$$

In contact sub-Riemannian spaces, there also exists a canonical connection [Tan89].

Theorem 3.1.3. *Let us consider (M, ω, g) a sub-Riemannian contact space. There exists a unique linear connection*

$$\nabla : \Gamma(TM) \times TM \longrightarrow TM$$

that satisfies

(i) $\nabla\omega = 0$,
(ii) $\nabla X_0 = 0$,
(iii) $\nabla g = 0$,
(iv) $\mathrm{Tor}(X, Y) = \mathrm{d}\omega(X, Y) X_0$ *for X and Y any horizontal fields,*
(v) $\mathrm{Tor}(X_0, JX) = -J\mathrm{Tor}(X_0, X)$ *for X any horizontal field,*

where Tor *is the torsion of ∇.*

To get familiar with the Tanno connection in the context of intrinsic curvature of contact sub-Riemannian manifolds, one can read [ABR17]. We here stress than in more general sub-Riemannian structures than contact spaces, the question of finding an adapted connection is complex [BR17, BGMR18, MG17].

Finally we introduce a tensor.

Definition 3.1.4. Let (M, ω, g) be a contact sub-Riemannian manifold. In this space the Tanno tensor is defined as

$$\begin{array}{rccc} Q: & \Gamma(\Delta) \times \Gamma(TM) & \longrightarrow & \Gamma(\Delta) \\ & (X, Y) & \longmapsto & (\nabla_Y J) X, \end{array}$$

where it is to be understood that

$$(\nabla_Y J) X = (\nabla_Y JX) - J(\nabla_Y X).$$

3.2 Smooth points

Definition 3.2.1. For M a sub-Riemannian manifold and $\zeta :]-T, T[\to M$ a smooth horizontal curve parametrized by arc length, we define the set of smooth points around $\zeta(s)$, that we denote by Σ_s as the set of points q in M such that

(i) there exists a unique minimizing geodesic $\gamma_{s,q}$ parametrized by arc length that leaves from $\zeta(s)$ at time 0 and that reaches q at time $\mathrm{dist}(\zeta(s), q)$,

(ii) $\gamma_{s,q}$ is stricly normal,

(iii) points $\zeta(s)$ and q are non-conjugate along $\gamma_{s,q}$.

On a set of smooth points, one can define a radial field.

Definition 3.2.2. For M a sub-Riemannian manifold and $\zeta :]-T, T[\to M$ a smooth horizontal curve parametrized by arc length, the radial field around $\zeta(s)$ is the vector field whose evaluation at each point has norm one that is defined as

$$\begin{array}{rccc} \Gamma_s : & \Sigma_s & \longrightarrow & TM \\ & q & \longmapsto & \gamma'_{s,q}\left(\mathrm{dist}(\zeta(s), q)\right). \end{array}$$

The set Σ_s as well as the field Γ_s are closely linked to the distance on M.

Theorem 3.2.3. *Let M be a sub-Riemannian manifold and $\zeta :]-T, T[\to M$ be a smooth horizontal curve parametrized by arc length. For $s \in]-T, T[$, the set Σ_s of smooth points around $\zeta(s)$*

(i) is an open set,

(ii) is equal to the set of points on the neighbourhood of which the function

$$\begin{array}{rccc} \tilde{\delta}_s : & M & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \mathrm{dist}\left(\zeta(s), q\right) \end{array}$$

is smooth.
In addition, over the set Σ_s, the field Γ_s is equal to the horizontal gradient of $\tilde{\delta}_s$, by which we mean that for any horizontal vector V tangent to Σ_s,

$$g\left(\Gamma_s, V\right) = \mathrm{d}\tilde{\delta}_s\left(V\right).$$

Proof. For a proof of the properties of the set Σ_s, namely (i) and (ii), we recall that we only work on contact sub-Riemannian spaces that are complete and we refer to [ABB19, Proposition 11.4 and Theorem 11.8 combined with Proposition 3.47].

To show that Γ_s is the gradient of $\tilde{\delta}_s$, we assume that it is not the case at some point q of Σ_s.

Let us consider $\gamma_{s,q}$ the minimizing geodesic parametrized by arc length such that $\gamma_{s,q}(0) = \zeta(s)$ and $\gamma_{s,q}\left(\tilde{\delta}_s(q)\right) = q$. For times t smaller than $\tilde{\delta}_s(q)$, $\gamma_{s,q}$ is also minimizing between $\zeta(s)$ and $\gamma_{s,q}(t)$, which means that

$$\tilde{\delta}_s\left(\gamma_{s,q}(t)\right) = t. \tag{3.1}$$

Now for times $t < \tilde{\delta}_s(q)$ close enough to $\tilde{\delta}_s(q)$, since the set Σ_s is open, we have that $\gamma_{s,q}(t)$ belongs to Σ_s on which $\tilde{\delta}_s$ is smooth. Therefore, for such times t, by differentiating (3.1) with respect to time,

$$1 = \mathrm{d}\tilde{\delta}_s\left(\Gamma_s\left(\gamma_{s,q}(t)\right)\right) = g_{\gamma_{s,q}(t)}\left(\mathrm{grad}\tilde{\delta}_s, \Gamma_s\right). \tag{3.2}$$

We recall that by definition of Σ_s, $\gamma_{s,q}$ is stricly normal, therefore $\gamma_{s,q}$ is smooth and field Γ_s which corresponds to the speed of $\gamma_{s,q}$ is smooth along $\gamma_{s,q}$. As $\tilde{\delta}_s$ is smooth in Σ_s, therefore $\mathrm{grad}\tilde{\delta}_s$ is smooth along $\gamma_{s,q}(t)$ for times $t \leqslant \tilde{\delta}_s(q)$ close enough to $\tilde{\delta}_s(q)$.

Since we assumed that $\Gamma_s(q) \neq \mathrm{grad}\tilde{\delta}_s(q)$ and according to the smoothness properties we just explained, we deduce that for $t \leqslant \tilde{\delta}_s(q)$ close enough to $\tilde{\delta}_s(q)$,

$$\Gamma_s\left(\gamma_{s,q}(t)\right) \neq \mathrm{grad}\tilde{\delta}_s\left(\gamma_{s,q}(t)\right). \tag{3.3}$$

If we combine (3.2), (3.3) and the fact that $\|\Gamma_s\| = 1$, we find out that for $t < \tilde{\delta}_s(q)$ close enough to $\tilde{\delta}_s(q)$,

$$\left\|\mathrm{grad}\tilde{\delta}_s\left(\gamma_{s,q}(t)\right)\right\| > 1. \tag{3.4}$$

Let us fix $t_* < \tilde{\delta}_s(q)$ that satisfies (3.4). We build a continuous curve $\tilde{\gamma}$ that is composed of two pieces that are geodesics, in such a way that

$$\tilde{\gamma}_{|[0,t_*]} = \gamma_{s,q}{}_{|[0,t_*]}, \tag{3.5}$$

and such that the curve $\tilde{\gamma}$ restricted to times t greater or equal than t_* is a geodesic parametrized by arc length that satisfies

$$\tilde{\gamma}'(t_*^+) = \frac{\mathrm{grad}\tilde{\delta}_s\left(\tilde{\gamma}(t_*)\right)}{\left\|\mathrm{grad}\tilde{\delta}_s\left(\tilde{\gamma}(t_*)\right)\right\|}. \tag{3.6}$$

Thanks to (3.5) and since $\gamma_{s,q}$ is a geodesic parametrized by arc length that is minimizing between $\zeta(s)$ and $\tilde{\gamma}(t_*)$ we find out that

$$\tilde{\delta}_s\left(\tilde{\gamma}(t_*)\right) = t_*. \tag{3.7}$$

By (3.4), (3.5) and (3.6), for times $t > t_*$ close enough to t_*,

$$\operatorname{dist}\left(\tilde{\gamma}(0), \tilde{\gamma}(t)\right) = \tilde{\delta}_s\left(\tilde{\gamma}(t)\right) > t,$$

which is impossible since $\tilde{\gamma}$ is parametrized by arc length. Our initial assumption that Γ_s does not coincide with the gradient of $\tilde{\delta}_s$ on Σ_s is thereby disproved. □

In order to use the property we just proved concerning the gradient of the distance on a set of smooth points, we must understand the geometry of this set of smooth points more precisely. As we show in the following, this geometry of smooth points is closely linked to the notion of cut locus.

Definition 3.2.4. For $\gamma : \mathbb{R} \to M$ a geodesic parametrized by arc length and t_0 any real number, we define the cut time associated to t_0 that we denote by $CT(\gamma, t_0)$ as the quantity :

$$\sup\left\{t > t_0 : \text{ the length of } \gamma_{|_{[t_0,t]}} \text{ equals the distance from } \gamma(t_0) \text{ to } \gamma(t)\right\}.$$

The cut point along γ associated to $\gamma(t_0)$ is the point

$$CP(\gamma, \gamma(t_0)) = \gamma\left(CT(\gamma, t_0)\right).$$

Around a point q, we denote by $CL(q)$ the cut locus around q, which is the set of all cut points $CP(\gamma, p)$ where γ varies in the set of geodesic parametrized by arc length that go through q.

Another interpretation of the notion of cut time is given by the following theorem whose proof can be found in [ABB19, chapter 8, Theorem 8.72].

Theorem 3.2.5. *Let us assume that M is a sub-Riemannian space that contains no abnormal minimizers.*

If $\gamma : \mathbb{R} \to M$ is a geodesic parametrized by arc length and if t is the cut time associated to t_0 along γ then

(i) either t is the infimum of $\tilde{t} > t_0$ -or equivalently "minimum of $\tilde{t} > t_0$"- such that $\gamma\left(\tilde{t}\right)$ is conjugate to $\gamma(t_0)$ along γ,

(ii) or there exists $\tilde{\gamma} : [t_0, t] \to M$ a geodesic different from $\gamma_{|_{[t_0,t]}}$ that satisfies

$$\tilde{\gamma}(t_0) = \gamma(t_0) \text{ and } \tilde{\gamma}(t) = \gamma(t)$$

such that both the lengths of $\tilde{\gamma}$ and $\gamma_{|_{[t_0,t]}}$ between their extremities are equal to the distance separating $\gamma(t_0)$ and $\gamma(t)$,

Conversly if $t > t_0$ are two real numbers such that either (i) or (ii) is satisfied then there exists t_* in $]t_0, t]$ such that t_* is the cut time associated to t_0 along γ.

Corollary 3.2.6. *Let us assume that M is a sub-Riemannian space in which there are no abnormal minimizers. For $\gamma : \mathbb{R} \to M$ a geodesic parametrized by arc length, $\mathrm{CT}(\gamma, t_0)$ is equal to the minimal time t for which $\gamma(t)$ is outside the set of smooth points with respect to $\gamma(t_0)$.*

Proof. On the one hand, by using Theorem 3.2.5, $\gamma(CT(\gamma, t_0))$ does not satisfy the conditions required by Definition 3.2.1 to be in the set of smooth points with respect to $\gamma(t_0)$.

On the other hand, for every $t < CT(\gamma, t_0)$, neither point (i) nor point (ii) in Theorem 3.2.5 is satisfied. The fact that for every $t < CT(\gamma, t_0)$, (i) does not hold simply means that there are no times $t < CT(\gamma, t_0)$ such that $\gamma(t)$ is conjugate to $\gamma(t_0)$. Moreover, by definition of the cut time, for $t < CT(\gamma, t_0)$, γ is minimizing between $\gamma(t_0)$ and $\gamma(t)$. It follows that we can state the fact that (ii) is false by saying that there is a unique minimizing geodesic linking $\gamma(t_0)$ with $\gamma(t)$. As a consequence, for every $t < CT(\gamma, t_0)$, by Definition 3.2.1, $\gamma(t)$ belongs to the set of smooth points with respect to $\gamma(t_0)$. □

Corollary 3.2.7. *If M is a sub-Riemannian manifold in which there are no abnormal geodesics, and $\zeta :]-T, T[\to M$ is a smooth horizontal curve parametrized by arc length, then for $s \in]-T, T[$, the two following disjoint sets form a partition of M,*

$$M = \Sigma_s \sqcup CL(\zeta(s)).$$

Proof. For $q \in M$, let us consider $\hat{\gamma}_{\zeta(s),q}$ a minimizing geodesic parametrized by arc length that links $\zeta(s)$ and q -which is not necessarily unique. Since $\hat{\gamma}_{\zeta(s),q}$ is minimizing, its cut time is greater than or equal to $\tilde{\delta}_s(q)$, the distance between $\zeta(s)$ and q. There are two cases

(i) The cut time of $\hat{\gamma}_{\zeta(s),q}$ is equal to $\tilde{\delta}_s(q)$, which implies that $q \in CL(\zeta(s))$ and, by applying Corollary 3.2.6, that $q \notin \Sigma_s$.

(ii) The cut time of $\hat{\gamma}_{\zeta(s),q}$ is strictly greater than $\tilde{\delta}_s(q)$. According to Corollary 3.2.6, this means that $q \in \Sigma_s$.

Let us assume that $q \in CL(\zeta(s))$ to show that this is impossible. If $q \in CL(\zeta(s))$, there must be $\tilde{\gamma}$ a geodesic that reaches q at its cut time. Since by definition of the cut time, $\tilde{\gamma}$ is minimizing between $\zeta(s)$ and $\tilde{\gamma}(t)$ for times strictly smaller than its cut time, we deduce by continuity of the distance that it is also minimizing between $\zeta(s)$ and q. However, $\tilde{\gamma}$ is different from $\hat{\gamma}_{\zeta(s),q}$ since both curves do not share the same cut time. By applying the converse in Theorem 3.2.5, we find out that the cut time associated to $\hat{\gamma}_{\zeta(s),q}$ is smaller than $\tilde{\delta}_s(q)$, which is a contradiction.

□

3.3 Characterisitic deviation

Before we define the characteristic deviation of a curve, let us introduce a usefull notation.

Definition 3.3.1. For $\zeta :]-T,T[\to M$ a horizontal curve parametrized by arc length in a sub-Riemannian manifold and for $s \in]-T,T[$, we say that a vector field T is a normalized extension of the velocity of ζ around $\zeta(s)$ if it is horizontal, has norm one at each point and if there exists $\varepsilon > 0$ such that for every $t \in]s-\varepsilon, s+\varepsilon[$,

$$\zeta'(t) = \mathsf{T}(\zeta(t)).$$

We now present a central object in our study, which is the characteristic deviation of a horizontal curve parametrized by arc length.

Definition 3.3.2. Let M be a contact sub-Riemannian manifold. The characteristic deviation of $\zeta :]-T,T[\to M$ a smooth horizontal curve parametrized by arc length is the function

$$\begin{array}{rccc} h_\zeta : &]-T,T[& \longrightarrow & \mathbb{R} \\ & t & \longmapsto & g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T}, J\mathsf{T}\right), \end{array}$$

where T is a normalized extension of the velocity of ζ.

The characteristic deviation can be computed using other formulae.

Lemma 3.3.3. *If M is a sub-Riemannian contact space and if $\zeta :]-T,T[\to M$ is a smooth horizontal curve parametrized by arc length then for any t in $]-T,T[$,*

$$h_\zeta(t) = -g_{|_{\zeta(t)}}\left([\mathsf{T}, J\mathsf{T}] + \mathrm{Tor}\left(\mathsf{T}, J\mathsf{T}\right), \mathsf{T}\right).$$

Proof. First of all, let us show that

$$g\left(Q\left(\mathsf{T},\mathsf{T}\right),\mathsf{T}\right) = 0. \tag{3.8}$$

This is the consequence of

$$\begin{aligned} 0 = \mathsf{T}\mathrm{d}\omega\left(\mathsf{T},\mathsf{T}\right) &= \mathsf{T}g\left(\mathsf{T}, J\mathsf{T}\right) = g\left(\nabla_{\mathsf{T}}\mathsf{T}, J\mathsf{T}\right) + g\left(\mathsf{T}, \nabla_{\mathsf{T}}\left(J\mathsf{T}\right)\right) \\ &= g\left(\nabla_{\mathsf{T}}\mathsf{T}, J\mathsf{T}\right) + g\left(\mathsf{T}, J\nabla_{\mathsf{T}}\mathsf{T}\right) + g\left(\mathsf{T}, Q\left(\mathsf{T},\mathsf{T}\right)\right) \\ &= g\left(\mathsf{T}, Q\left(\mathsf{T},\mathsf{T}\right)\right) \text{ by antisymmetry of } J. \end{aligned}$$

Now we can compute

$$\begin{aligned} -g_{|_{\zeta(t)}}\left([\mathsf{T}, J\mathsf{T}] + \mathrm{Tor}\left(\mathsf{T}, J\mathsf{T}\right), \mathsf{T}\right) &= -g_{|_{\zeta(t)}}\left(\nabla_{\mathsf{T}}\left(J\mathsf{T}\right) - \nabla_{J\mathsf{T}}\mathsf{T}, \mathsf{T}\right) \\ &= -g_{|_{\zeta(t)}}\left(\nabla_{\mathsf{T}}\left(J\mathsf{T}\right), \mathsf{T}\right) \text{ since } \|\mathsf{T}\| = 1. \\ &= -g_{|_{\zeta(t)}}\left(J\nabla_{\mathsf{T}}\mathsf{T} + Q(\mathsf{T},\mathsf{T}), \mathsf{T}\right) \\ &= g_{|_{\zeta(t)}}\left(\nabla_{\mathsf{T}}\mathsf{T}, J\mathsf{T}\right), \end{aligned}$$

where the last line corresponds to the result we were looking for and comes from (3.8) and from the antisymmetry of J. □

A particular example of characteristic deviation is that that is defined along geodesics, for which we introduce a specific notation.

Definition 3.3.4. Let us consider M a contact sub-Riemannian space and $\zeta :]-T, T[\to M$ a smooth horizontal curve parametrized by arc length. For s in $]-T, T[$, we define the geodesic deviation around $\zeta(s)$ as

$$\begin{array}{rccc} h_{0,s}: & \Sigma_s & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & h_{\gamma_{s,q}}\left(\tilde{\delta}_s\left(q\right)\right). \end{array}$$

Lemma 3.3.5. *Let M be a contact sub-Riemannian manifold and let $\zeta :]-T, T[\to M$ be a smooth horizontal curve parametrized by arc length. If we consider $s \in]-T, T[$ and $q \in \Sigma_s$ then*

$$h_{0,s}(q) = \left\|J\Gamma_s(q)\right\|^2 X_0(q)\tilde{\delta}_s.$$

Proof. By definition of J,

$$g\left(\Gamma_s, J\Gamma_s\right) = \mathrm{d}\omega\left(\Gamma_s, \Gamma_s\right) = 0.$$

As a consequence and since according to Theorem 3.2.3, Γ_s is the gradient of $\tilde{\delta}_s$ over Σ_s,

$$\begin{aligned} 0 &= \Gamma_s\left(J\Gamma_s\tilde{\delta}_s\right) - J\Gamma_s\left(\Gamma_s\tilde{\delta}_s\right) = \left[\Gamma_s, J\Gamma_s\right]\tilde{\delta}_s \\ &= -\mathrm{Tor}\left(\Gamma_s, J\Gamma_s\right)\tilde{\delta}_s + g\left(\left[\Gamma_s, J\Gamma_s\right] + \mathrm{Tor}\left(\Gamma_s, J\Gamma_s\right), \Gamma_s\right), \\ &= -\mathrm{Tor}\left(\Gamma_s, J\Gamma_s\right)\tilde{\delta}_s - h_{0,s} \end{aligned} \tag{3.9}$$

where the second line of the previous computation comes from the fact that Γ_s is the gradient of $\tilde{\delta}_s$, and the last line follows from Lemma 3.3.3. Moreover according to Theorem 3.1.3,

$$\mathrm{Tor}\left(\Gamma_s, J\Gamma_s\right) = -\mathrm{d}\omega\left(J\Gamma_s, \Gamma_s\right)X_0 = -g\left(J\Gamma, J\Gamma\right)X_0. \tag{3.10}$$

We combine (3.9) and (3.10) to obtain the result. □

3.4 Hamiltonian interpretation of the geometric invariants

Here we briefly recall the main results of the Hamiltonian framework applied to M a contact sub-Riemannian manifold of dimension $2n+1$. For more details see [ABB19, Chapter 4].

In what follows, we denote by $\pi : T^*M \to M$ the canonical projection from T^*M to M. Given $X \in \Gamma(TM)$ a smooth vector field on M, we introduce

$$\begin{array}{rccc} h_X : & T^*M & \to & \mathbb{R} \\ & \lambda & \longmapsto & \langle \lambda, X(\pi(\lambda)) \rangle . \end{array} \tag{3.11}$$

For $\left(\tilde{X}_i\right)_{i=1}^{2n}$ an orthonormal frame of the distribution, we define the *sub-Riemannian Hamiltonian* $H : T^*M \to \mathbb{R}$ by the following formula

$$H = \frac{1}{2} \sum_{i=1}^{2n} h_{X_i}^2. \tag{3.12}$$

One can show that H actually does not depend on the choice of the frame.

Now we introduce the tautological 1-form τ on T^*M such that for every $\lambda \in T^*M$,

$$\begin{array}{rccc} \tau_\lambda : & T_\lambda (T^*M) & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \langle \lambda, \pi_* w \rangle . \end{array} \tag{3.13}$$

The differential of the tautological 1-form $\sigma = \mathrm{d}\tau$ is a canonical symplectic form. This canonical symplectic form allows us to build a Hamiltonian vector field $\vec{h}$ from a smooth function $h : TM \to \mathbb{R}$ through the following identity.

$$\sigma(\cdot, \vec{h}) = \mathrm{d}h. \tag{3.14}$$

Now if we choose any coordinate system $(x_i)_{i=0}^{2n}$ on M, we extend it to coordinates (p, x) on T^*M such that (p_i, x_i) represents a covector λ whose expression is

$$\lambda = \sum_{i=0}^{2n} p_i \mathrm{d}x_i \tag{3.15}$$

and whose projection on M has coordinates (x_i).

It is interesting to notice that the tautological 1-form evaluated at a point λ of T^*M whose coordinates are (p_i, x_i) has exactly the same expression as λ itself, namely (3.15). Hence the term "tautological". From this remark we deduce both the expression of the canonical symplectic structure and of the Hamiltonian vector field associated to a Hamiltonian function in the (p, x) system of coordinates.

$$\sigma = \sum_{i=0}^{2n} \mathrm{d}p_i \wedge \mathrm{d}x_i. \tag{3.16}$$

We thereby deduce an expression for $\vec{h}$.

$$\vec{h} = \sum_{i=0}^{2n} \frac{\partial h}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial p_i}. \tag{3.17}$$

It is also convenient to introduce a frame on the cotangent bundle T^*M that is adapted to the choice of an orthonormal frame of the distribution $(X_1, ...X_{2n})$ - to which we add the Reeb vector field X_0 to obtain a frame of TM.

First, to every vector field X over M we associate a vector field over T^*M that we denote as $\overline{X}$ and that satisfies for any λ in T^*M and for any j in $[\![0, 2n]\!]$,

$$\pi_*(\overline{X}(\lambda)) = X(\pi(\lambda)) \text{ and } \overline{X}h_{X_j} = 0. \tag{3.18}$$

Then we introduce the frame of $T(T^*M)$ defined by

$$(\overline{X}_0, \overline{X}_1, ..., \overline{X}_{2n}, \frac{\partial}{\partial h_{X_0}}, \frac{\partial}{\partial h_{X_1}}, ..., \frac{\partial}{\partial h_{X_{2n}}}).$$

Notice that $\frac{\partial}{\partial h_{X_i}}$ denotes the vector field tangent to T^*M such that

$$\pi_*\left(\frac{\partial}{\partial h_{X_i}}\right) = 0, \quad \text{and} \quad \frac{\partial}{\partial h_{X_i}} h_{X_j} = \delta_{i,j}. \tag{3.19}$$

Similarly, we can lift a function $f : M \to \mathbb{R}$ to a function $T^*M \to \mathbb{R}$ that is defined as $\overline{f} = f \circ \pi$. In particular, one can define the coordinates of the Lie brackets of the elements of a frame of TM as well as their lift to T^*M.

Definition 3.4.1. We consider, on a contact sub-Riemannian space M an orthonormal frame $(X_1, X_2, ..., X_{2n})$ of the distribution that we complete with the Reeb vector field X_0 to form a frame of TM.

For $(i, j, k) \in [\![0, 2n]\!]$, we denote either by $c_{X_i,X_j}^{X_k}$ or simply by $c_{i,j}^k$ the k-th coordinate of the Lie bracket

$$[X_i, X_j].$$

In other words,

$$c_{X_i,X_j}^{X_0} = \omega([X_i, X_j])$$

and for $k \neq 0$,

$$c_{X_i,X_j}^{X_k} = g([X_i, X_j] - \omega([X_i, X_j])X_0, X_k).$$

Moreover $\overline{c}_{i,j}^k : T^*M \to M$ stands for $c_{i,j}^k \circ \pi$ where we recall that $\pi : T^*M \to M$ is the canonical projection.

We now can give another expression of $\vec{H}$.

Proposition 3.4.2. *Let us consider M a contact sub-Riemannian space on which we choose $(X_1, X_2, ..., X_{2n})$ an orthonormal frame of the distribution, to which we add the Reeb vector field X_0 to form a frame of the whole tangent space TM.*

The sub-Riemannian Hamiltonian vector field $\vec{H}$ can be written as

$$\vec{H} = \sum_{i=1}^{2n} h_{X_i} \overline{X}_i + \sum_{i=1}^{2n} \sum_{j,k=0}^{2n} \overline{c}_{i,j}^{k} h_{X_i} h_{X_k} \frac{\partial}{\partial h_{X_j}}.$$

Proof. Let us fix coordinates $(x_i)_{i=0}^{2n}$ on M. As we explained in our brief summary of Hamiltonian geometry, these coordinates induce coordinates (p, x) on T^*M. We now link coordinates $(x_i)_{i=0}^{2n}$ with the frame $(X_i)_{i=0}^{2n}$ by writing

$$X_i = \sum_{k=0}^{2n} \beta_{i,k} \frac{\partial}{\partial x_k}. \tag{3.20}$$

By combining the previous identity with (3.11) and (3.15) we deduce that

$$h_{X_i} = \sum_{k=0}^{2n} \overline{\beta}_{i,k} p_k. \tag{3.21}$$

We then can apply (3.17) to obtain

$$\vec{h}_{X_i} = \sum_{k=0}^{2n} \overline{\beta}_{i,k} \frac{\partial}{\partial x_k} - \sum_{j,k=0}^{2n} p_k \frac{\partial \overline{\beta}_{i,k}}{\partial x_j} \frac{\partial}{\partial p_j}. \tag{3.22}$$

We are now able to compute

$$\begin{aligned} \vec{H} &= \overbrace{\longrightarrow}^{\text{by applying (3.14).}} \frac{1}{2} \sum_{i=1}^{2n} h_{X_i}^2 = \sum_{i=1}^{2n} h_{X_i} \vec{h}_{X_i} \\ &= \sum_{i=1}^{2n} h_{X_i} \left(\sum_{k=0}^{2n} \overline{\beta}_{i,k} \frac{\partial}{\partial x_k} - \sum_{j,k=0}^{2n} p_k \frac{\partial \overline{\beta}_{i,k}}{\partial x_j} \frac{\partial}{\partial p_j} \right). \end{aligned} \tag{3.23}$$

We first notice that

$$\begin{aligned}\pi_*\left(\vec{H}(p,x)\right) &= \sum_{i=1}^{2n} h_{X_i}(p,x) \sum_{k=0}^{2n} \beta_{i,k}(x) \frac{\partial}{\partial x_k} \\ &= \sum_{i=1}^{2n} h_{X_i}(p,x) X_i(x) \text{ by (3.20).} \\ &= \pi_*\left(\left(\sum_{i=1}^{2n} h_{X_i}\overline{X}_i + \sum_{i=1}^{2n}\sum_{j,k=0}^{2n} \overline{c}_{i,j}^k h_{X_i} h_{X_k} \frac{\partial}{\partial h_{X_j}}\right)(p,x)\right).\end{aligned} \tag{3.24}$$

Moreover for any $l \in [\![0, 2n]\!]$, thanks to (3.21) and (3.23),

$$\begin{aligned}\vec{H}(p,x)\, h_{X_l} &= \sum_{i=1}^{2n} h_{X_i}\left(\sum_{j,k=0}^{2n} p_j \overline{\beta}_{i,k} \frac{\partial \overline{\beta}_{l,j}}{\partial x_k} - \sum_{j,k=0}^{2n} p_k \overline{\beta}_{l,j} \frac{\partial \overline{\beta}_{i,k}}{\partial x_j}\right) \\ &= \sum_{i=1}^{2n} h_{X_i}\left(\sum_{j=0}^{2n} p_j \overline{X}_i \overline{\beta}_{l,j} - \sum_{k=0}^{2n} p_k \overline{X}_l \overline{\beta}_{i,k}\right) \text{ by (3.20),} \\ &= \sum_{i=1}^{2n} h_{X_i} \sum_{j=0}^{2n} p_j \left(\overline{X}_i \overline{\beta}_{l,j} - \overline{X}_l \overline{\beta}_{i,j}\right) \\ &= \sum_{i=1}^{2n} h_{X_i} \sum_{j=0}^{2n} p_j \mathrm{d}x_j \left(\overline{[X_i, X_l]}\right) \text{ as a consequence of (3.20),} \\ &= \sum_{i=1}^{2n} h_{X_i} h_{[X_i,X_l]}, \text{ by applying (3.11) and (3.15),} \\ &= \sum_{i=1}^{2n}\sum_{k=0}^{2n} \overline{c}_{i,l}^k h_{X_i} h_{X_k}, \text{ by linearity of } h.\end{aligned} \tag{3.25}$$

Recalling (3.18) and (3.19), we see that the proof of the proposition we are focusing on is contained in (3.24) and (3.25). □

Theorem 3.4.3. *Let us consider M a contact sub-Riemannian manifold. A smooth curve*

$$\zeta :]-T, T[\to M$$

is a geodesic parametrized at constant speed if and only if there exists

$$\overline{\zeta} :]-T, T[\to T^*M$$

a lift of ζ through the canonical projection $\pi : T^*M \to M$ such that $\overline{\zeta}$ is an integral line of the Hamiltonian vector field $\vec{H}$.

We now reformulate the previous theorem, by stating a result which can also be found in [ABR17, Lemma 6.7].

Proposition 3.4.4. *We consider M a contact sub-Riemannian manifold and $\zeta :]-T,T[\to M$ a smooth horizontal curve parametrized by arc length. We have :*

(i) ζ is a geodesic if, and only if, for every t in $]-T,T[$,

$$\begin{cases} \nabla_{\mathsf{T}(\zeta(t))}\mathsf{T} &= \frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|^2} J\mathsf{T}\left(\zeta(t)\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|^2} &= -g_{\zeta(t)}\left(\mathrm{Tor}\left(\mathsf{T}, X_0\right), \mathsf{T}\right) \end{cases} \tag{3.26}$$

where T is a horizontal and normalized extention of the velocity of ζ.

(ii) If ζ is a geodesic whose lift defined in Theorem 3.4.3 we denote by $\overline{\zeta}$, then we have

$$h_\zeta(t) = h_{X_0}\left(\overline{\zeta}(t)\right). \tag{3.27}$$

Proof. First step. Let us start by assuming that ζ is indeed a geodesic to show that it implies (3.26) and (3.27). In order to do this, we choose an orthonormal frame of the distribution $(X_1, X_2, ...X_{2n})$ such that

$$X_1 = \mathsf{T} \text{ and } X_2 = \frac{J\mathsf{T}}{\|J\mathsf{T}\|}. \tag{3.28}$$

We now recall that according to Theorem 3.4.3, ζ possesses a lift $\overline{\zeta}$ which is an integral line of the Hamiltonian vector field $\vec{H}$. In particular for $t \in]-T,T[$,

$$\pi_*\left(\vec{H}\left(\overline{\zeta}(t)\right)\right) = \mathsf{T}\left(\zeta(t)\right), \tag{3.29}$$

where the expression of $\vec{H}$ is given by Proposition 3.4.2.

$$\vec{H} = \sum_{i=1}^{2n} h_{X_i}\overline{X}_i + \sum_{i=1}^{2n}\sum_{j,k=0}^{2n} \overline{c}_{i,j}^{k} h_{X_i} h_{X_k} \frac{\partial}{\partial h_{X_j}}. \tag{3.30}$$

We now can reformulate (3.29) thanks to (3.30), (3.28) and (3.18) and we obtain

$$\begin{cases} h_{X_1}\left(\overline{\zeta}(t)\right) &= 1, \\ h_{X_i}\left(\overline{\zeta}(t)\right) &= 0 \text{ for every } i \geqslant 2. \end{cases} \tag{3.31}$$

We can therefore simplify (3.30) along $\overline{\zeta}$ by writing

$$\vec{H}\left(\overline{\zeta}(t)\right) = \overline{\mathsf{T}}\left(\overline{\zeta}(t)\right) + \sum_{j=0}^{2n}\left(\overline{c}_{1,j}^{1} + \overline{c}_{1,j}^{0} h_{X_0}\right)\left(\overline{\zeta}(t)\right)\frac{\partial}{\partial h_{X_j}}\left(\overline{\zeta}(t)\right). \tag{3.32}$$

To go further in the simplification of $\vec{H}$ we write, for $j \in [\![0, 2n]\!]$,

$$c^0_{1,j} = \omega([X_1, X_j]) = \overbrace{X_1\omega(X_j) - X_j\omega(X_1)}^{=0} - \mathrm{d}\omega(X_1, X_j) \tag{3.33}$$
$$= \begin{cases} 0 \text{ if } j = 0 \text{ since by defintion, } X_0 \in \ker \mathrm{d}\omega, \\ g(X_j, JX_1) = g(X_j, J\mathsf{T}) = \delta_{j,2} \|J\mathsf{T}\| \text{ otherwise.} \end{cases}$$

As a consequence, we can rewrite (3.32) as

$$\vec{H}\left(\overline{\zeta}(t)\right) = \overline{\mathsf{T}}\left(\overline{\zeta}(t)\right) + \sum_{j=0}^{2n} \left(\overline{c}^1_{1,j} + \delta_{j,2}\overline{\|J\mathsf{T}\|} h_{X_0}\right)\left(\overline{\zeta}(t)\right) \frac{\partial}{\partial h_{X_j}}\left(\overline{\zeta}(t)\right).$$

Since $\overline{\zeta}$ follows the flow of $\vec{H}$, we deduce from the previous identity that for any $j \in [\![0, 2n]\!]$,

$$\frac{\partial}{\partial t} h_{X_j}\left(\overline{\zeta}(t)\right) = c^1_{1,j}(\zeta(t)) + \delta_{j,2}\|J\mathsf{T}(\zeta(t))\| h_{X_0}\left(\overline{\zeta}(t)\right). \tag{3.34}$$

But thanks to (3.31), for $j \geqslant 1$, $\frac{\partial}{\partial t} h_{X_j}\left(\overline{\zeta}(t)\right) = 0$ so (3.34) implies

$$\begin{cases} \frac{\partial}{\partial t} h_{X_0}\left(\overline{\zeta}(t)\right) &= c^1_{1,0}(\zeta(t)), \\ 0 &= c^1_{1,2}(\zeta(t)) + \|J\mathsf{T}(\zeta(t))\| h_{X_0}\left(\overline{\zeta}(t)\right), \\ 0 &= c^1_{1,j}(\zeta(t)) \text{ for every } j \in [\![3, 2n]\!]. \end{cases} \tag{3.35}$$

Let us now compute the Lie brackets that appear in the previous system. We notice that for $j \in [\![2, 2n]\!]$,

$$[X_1, X_j] + \mathrm{Tor}(X_1, X_j) = \nabla_{X_1} X_j - \nabla_{X_j} X_1$$

is the projection of $[X_1, X_j]$ on the distribution parallely to the Reeb vector field. Indeed, according to Theorem 3.1.3 i. the right hand side of the previous expression is horizontal and Theorem 3.1.3 iv. implies that $\mathrm{Tor}(X_1, X_j)$ is parallel to the Reeb vector field.

This last remark means that for $j \in [\![2, 2n]\!]$,

$$c^1_{1,j} = g([X_1, X_j] + \mathrm{Tor}(X_1, X_j), X_1) = g\left(\nabla_{X_1} X_j - \nabla_{X_j} X_1, X_1\right) \tag{3.36}$$

If we combine (3.36) with Lemma 3.3.3 we obtain

$$c^1_{1,2}(\zeta(t)) = -\frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|}. \tag{3.37}$$

Still using (3.36), we find out that for $j \in [\![3, 2n]\!]$,

$$c^1_{1,j} = g\left(\nabla_{\mathsf{T}} X_j - \nabla_{X_j}\mathsf{T}, \mathsf{T}\right) \tag{3.38}$$
$$= g(\nabla_{\mathsf{T}} X_j, \mathsf{T}) - \frac{1}{2} X_j g(\mathsf{T}, \mathsf{T}) \text{ according to Theorem 3.1.3 iii.}$$
$$= g(\nabla_{\mathsf{T}} X_j, \mathsf{T}) \text{ since } \mathsf{T} \text{ has norm one.}$$
$$= \mathsf{T} g(X_j, \mathsf{T}) - g(X_j, \nabla_{\mathsf{T}}\mathsf{T}) \text{ by Theorem 3.1.3 iii.}$$
$$= -g(X_j, \nabla_{\mathsf{T}}\mathsf{T}) \text{ as } X_j \perp \mathsf{T}.$$

Moreover

$$g\left(\nabla_{\mathsf{T}}\mathsf{T},\mathsf{T}\right)=\frac{1}{2}\mathsf{T}\overbrace{g\left(\mathsf{T},\mathsf{T}\right)}^{=1}\text{ as a consequence of Theorem 3.1.3 iii.}\qquad(3.39)$$
$$=0.$$

We must also compute $c^1_{1,0}$ in order to understand the first equation in (3.35). First of all, thanks to (3.33), we know that $[\mathsf{T},X_0]$ is horizontal, so we can write

$$\begin{aligned}c^1_{1,0}&=g\left(\left[\mathsf{T},X_0\right],\mathsf{T}\right)=g\left(\nabla_{\mathsf{T}}X_0-\nabla_{X_0}\mathsf{T}-\mathrm{Tor}\left(\mathsf{T},X_0\right),\mathsf{T}\right)\qquad(3.40)\\&=-g\left(\nabla_{X_0}\mathsf{T},\mathsf{T}\right)-g\left(\mathrm{Tor}\left(\mathsf{T},X_0\right),\mathsf{T}\right)\text{ by Theorem 3.1.3 ii.}\\&=-\frac{1}{2}X_0g\left(\mathsf{T},\mathsf{T}\right)-g\left(\mathrm{Tor}\left(\mathsf{T},X_0\right),\mathsf{T}\right)\text{ by Theorem 3.1.3 iii.}\\&=-g\left(\mathrm{Tor}\left(\mathsf{T},X_0\right),\mathsf{T}\right)\text{ since }\mathsf{T}\text{ has norm one.}\end{aligned}$$

We now come back to (3.35), that we reformulate by applying (3.37), (3.38) and (3.40) and to which we add two equations, namely (3.39), and also the formula defining h_ζ in Definition 3.3.2. We obtain

$$\left\{\begin{array}{rcl}\frac{\partial}{\partial t}h_{X_0}\left(\overline{\zeta}(t)\right)&=&-g_{\zeta(t)}\left(\mathrm{Tor}\left(\mathsf{T},X_0\right),\mathsf{T}\right),\\0&=&-\frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|^2}+h_{X_0}\left(\overline{\zeta}(t)\right),\\0&=&g_{\zeta(t)}\left(X_j,\nabla_{\mathsf{T}}\mathsf{T}\right)\text{ for every }j\in[\![3,2n]\!].\\0&=&g_{\zeta(t)}\left(\mathsf{T},\nabla_{\mathsf{T}}\mathsf{T}\right)\\h_\zeta(t)&=&g_{\zeta(t)}\left(J\mathsf{T},\nabla_{\mathsf{T}}\mathsf{T}\right)\end{array}\right.$$

This last system exactly corresponds to (3.26) and (3.27), which we wanted to prove are true if we assume that ζ is a geodesic.

Second step. We now must prove the converse of the first part of the proposition. We assume that $\zeta:]-T,T[\to M$ is a smooth horizontal curve parametrized by arc length such that (3.26) holds for every $t\in]-T,T[$. We must show that ζ is a geodesic.

Let us consider $(X_1,X_2,...,X_{2n})$ an orthonormal frame of the distribution. We define the covector p_0 in T^*M that satisfies

$$\pi\left(p_0\right)=\zeta(0),\,h_{X_0}\left(p_0\right)=\frac{h_\zeta(0)}{\|J\mathsf{T}\left(\zeta(0)\right)\|^2}\qquad(3.41)$$

and such that

$$h_{X_i}\left(p_0\right)=g_{\zeta(0)}\left(\mathsf{T},X_i\right),\text{ for every }i\in[\![1,2n]\!].\qquad(3.42)$$

We build

$$\overline{\zeta}:\quad]-T,T[\quad\to\quad T^*M$$

as the integral line of the Hamiltonian vector field $\vec{H}$ that leaves at time 0 from point p_0.

We deduce from Theorem 3.4.3 that curve

$$\tilde{\zeta} = \pi \circ \overline{\zeta}$$

is a geodesic parametrized at constant speed. Moreover, the expression of $\vec{H}$ given by Proposition 3.4.2 combined with the defintion of p_0 enables us to write that

$$\tilde{\zeta}'(0) = \pi_* \left(\vec{H}\left(p_0\right) \right) = \sum_{i=1}^{2n} g_{\zeta(0)}\left(\mathsf{T}, X_i\right) X_i = \mathsf{T}\left(\zeta(0)\right). \tag{3.43}$$

Furthermore, what we have proven in the "first step" of the proof of this proposition allows us to say that

$$\begin{aligned} h_{\tilde{\zeta}}(0) &= \left\| J\tilde{\zeta}'(0) \right\|^2 h_{X_0}\left(\overline{\zeta}(0)\right) \\ &= \left\| J\mathsf{T}\left(\zeta(0)\right) \right\|^2 h_{X_0}\left(p_0\right) \text{according to (3.43)}, \\ &= h_{\zeta}(0) \text{ by definition of } p_0, \end{aligned} \tag{3.44}$$

The "first step" of the proof also shows that $\tilde{\zeta}$ satisfies (3.26) at each time $t \in]-T, T[$. But we assumed that ζ also satisfies the differential equation (3.26) for the same initial conditions according to (3.43) and (3.44). Therefore we deduce that

$$\zeta = \tilde{\zeta}.$$

This means that ζ is a geodesic. □

The characterization of geodesics we find in Proposition 3.4.4 can be reformulated in terms of the concept of geodesic curvatures, that we now introduce.

Definition 3.4.5. For $\zeta :]-T, T[\to M$ a smooth horizontal curve in a contact sub-Riemannian manifold whose speed is extended by the normalized field T, we define

(i) the first geodesic curvature that is

$$\begin{array}{rccc} k_{\zeta,1} : &]-T,T[& \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \left\| \Pi^{\perp}_{J\mathsf{T}(\zeta(s))} \left(\nabla_{\mathsf{T}(\zeta(s))} \mathsf{T} \right) \right\| \end{array}$$

where, for any V in $\Delta_{\zeta(s)}$,

$$\Pi^{\perp}_{V} : \Delta_{\zeta(s)} \longrightarrow \Delta_{\zeta(s)}$$

stands for the orthogonal projection on $\langle V \rangle^{\perp}$.

(ii) the second geodesic curvature which is

$$\begin{array}{rccc} k_{\zeta,2}: &]-T,T[& \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \frac{\mathrm{d}}{\mathrm{d}t}\frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|^2} + g_{\zeta(t)}\left(\mathrm{Tor}\left(\mathsf{T},X_0\right),\mathsf{T}\right). \end{array}$$

To simplify notations, for $v \in TM$ such that $\|v\| = 1$, we denote by $\eta(v)$ the term $g\left(\mathrm{Tor}\left(v,X_0\right),v\right)$ that appears in the defintion of $k_{\zeta,2}$.

Remark 3.4.6. Thanks to identity (3.40) we find in the proof of Proposition 3.4.4, we can give another expression for the term $\eta(v)$ we introduced in the previous definition. For $v \in T_xM$ such that $\|v\|$ has norm one, if we consider V a horizontal vector field which has norm one at each point and such that $V(x) = v$ then

$$\eta(v) = -g_x\left(\left[V,X_0\right],V\right).$$

We now write Proposition 3.4.4 differently, using geodesic curvatures.

Proposition (1.1.4)**.** *Let M be a contact sub-Riemannian manifold. A horizontal curve parametrized by a arc length $\zeta:]-T,T[\to \mathbb{R}$ is a geodesic if, and only if both function $k_{\zeta,1}$ and $k_{\zeta,2}$ are identically zero.*

Proof. First of all, notice that the vanishing of $k_{\zeta,2}$ is equivalent to the second equation in (3.26).

Now let us consider T a normalized horizontal field extending the velocity of ζ. According to Definition 3.3.2 we know that

$$\frac{h_\zeta(t)}{\|J\mathsf{T}\left(\zeta(t)\right)\|^2} J\mathsf{T}\left(\zeta(t)\right)$$

is the orthogonal projection of $\nabla_{\mathsf{T}(\zeta(t))}\mathsf{T}$ on the direction of $J\mathsf{T}\left(\zeta(t)\right)$. This means that the first equation in (3.26) corresponds to saying that $\nabla_{\mathsf{T}(\zeta(t))}$ is aligned with $J\mathsf{T}\left(\zeta(t)\right)$, which is also equivalent to the fact that k_ζ^1 vanishes. □

Apart from the characterization of geodesics through geodesic curvatures, Proposition 3.4.4 allows to study geodesics leaving from a central point. First of all, if we combine 3.4.4 with Definition 3.2.2 and Defintion 3.3.4 we obtain the following.

Corollary 3.4.7. *let M be a contact sub-Riemannian space and $\zeta:]-T,T[\to M$ be a smooth horizontal curve parametrizd by arc length. For $s \in]-T,T[$ and $q \in \Sigma_s$,*

$$\nabla_{\Gamma_s(q)}\Gamma_s = \frac{h_{0,s}(q)}{\|J\Gamma_s(q)\|^2} J\Gamma_s \text{ and } \Gamma_s(q)\frac{h_{0,s}(q)}{\|J\Gamma_s(q)\|^2} = -\eta\left(\Gamma_s(q)\right),$$

Proposition 3.4.4 also tells us that the exponential map we are about to define is well-defined.

Definition 3.4.8. Let M be a three dimensional sub-Riemannian manifold. For $p \in M$ by denoting the set of horizontal vectors at p of norm one as S_pM we define the exponential map at p as

$$\exp_p : S_pM \times \mathbb{R} \times \mathbb{R} \longrightarrow M$$

where for $V \in S_p$ and $h \in \mathbb{R}$ fixed, $t \mapsto \exp_p(V, h, t)$ is the geodesic parametrized by arc length that leaves at time zero from q with a velocity equal to V and an initial Hamiltonian lift whose coordinate h_{X_0} is equal to h.

Proposition 3.4.9. *The exponential map around any point in a contact sub-Riemannian manifold is smooth.*

Proof. For $V \in S_pM$ and $h \in \mathbb{R}$ fixed, we know from Theorem 3.4.3 that the geodesic $t \mapsto \exp_p(V, h, t)$ is the projection of some integral line $\overline{\gamma}_{V,h} : \mathbb{R} \to T^*M$ of the Hamiltonian vector field $\vec{H}$. The projection on TM of $\vec{H}\left(\overline{\gamma}_{V,h}(0)\right)$ must coincide with V so by observing the expression of $\vec{H}$ that is given in Proposition 3.4.2, we find out that the for $i \in [|1, 2n|]$,

$$h_{X_i}\left(\overline{\gamma}_{V,h}(0)\right) = g(V, X_i).$$

Moreover, by definition $h_{X_0}\left(\overline{\gamma}_{V,h}(0)\right) = h$ so $\overline{\gamma}_{V,h}(0)$ smoothly depends on (V, h). Since $t \mapsto \exp_p(V, h, t)$ is the projection of the integral line of the smooth Hamiltonian field $\vec{H}$ which starts at $\overline{\gamma}_{V,h}(0)$, the proof is complete. □

Chapter 4

Curvature of horizontal curves in the three dimensional contact setting

In this chapter we restrict our study to three dimensional contact spaces. The higher dimensional case is postponed to the next chapter. In the 3D contact setting, our goal is to understand how the geodesic curvature appears in the Taylor expansion of the distance between a fixed point on a curve and an other point of the curve tending to this fixed point.

4.1 Isoperimetric problems

This first section is devoted to presenting examples of three dimensional contact sub-Riemannian manifolds. Let (N, g_N) be a two-dimensional Riemannian manifold and A be a 1-form on N. For $x, y \in N$ set

$$\Omega^N_{x,y} = \{\alpha : [0,T] \to N \mid \alpha \in C^\infty, \alpha(0) = x, \alpha(t) = y\}.$$

The *isoperimetric problem* on M associated to A, is the following

$$\inf\{\ell(\alpha) \mid \alpha \in \Omega^N_{x,y}, \int_\alpha A = c\}, \tag{4.1}$$

where c is a real constant and x, y are points on N. If one chooses A in such a way that $dA = \mathrm{vol}_N$ then one recovers the classical problem of minimizing the length of a curve spanning a fixed area.

One can introduce the sub-Riemannian structure on $M = N \times \mathbb{R}$ by lifting a curve α on N to a curve $\zeta(t) = (\alpha(t), z(t))$ where

$$z(t) = \int_0^t A(\dot\alpha(s))ds,$$

The lifted curves ζ are tangent to the distribution defined as $\Delta = \ker \omega$ where $\omega = dz - A$. Notice that ω is contact if and only if dA is never vanishing on N. If $\pi : M \to N$ denotes the canonical projection, then π_* restricts to an isomorphism between Δ and TN. Denoting $g = \pi^* g_N$ the pull-back of the metric of N on the distribution Δ, problem (4.1) rewrites as

$$\inf\{\ell_{SR}(\zeta) \mid \zeta \text{ horizontal}, \zeta(0) = (x, 0), \zeta(T) = (y, c)\}, \tag{4.2}$$

Proposition 4.1.1. *Let $\zeta : [0, T] \to M$ be the smooth horizontal lift of a smooth curve $\alpha : [0, T] \to N$. Then $h_\zeta(t) = \kappa_\alpha^N(t)$, where κ_α^N is the Riemannian geodesic curvature of α on N.*

Proof. Fix an orthonormal basis (X_1, X_2) for the distribution and write

$$\dot{\zeta} = \cos(\theta) X_1 + \sin(\theta) X_2$$

Then it is easy to see that

$$\dot{\alpha} = \cos(\theta) Y_1 + \sin(\theta) Y_2$$

where $Y_i := \pi_* X_i$ is an orthonormal basis for the Riemannian metric on N. Then one can observe that the formula

$$h_\zeta = \dot{\theta} + c_{12}^1 \cos\theta + c_{12}^2 \sin\theta.$$

actually coincides with the geodesic curvature of α on N. □

4.2 Specific properties in the three dimensional setting

4.2.1 Characteristic deviation and geodesic curvature

In the three dimensional case, the characteristic deviation and the geodesic curvature have specific properties they do not possess in higher dimension.

Proposition 4.2.1. *Let M be a three dimensional contact sub-Riemannian manifold and let $\zeta :]-T, T[\to M$ be a smooth horizontal curve parametrized by arc length whose velocity we extend by a normalized vector field T. For $t \in]-T, T[$ we have*

$$\nabla_{\mathsf{T}(t)} \mathsf{T} = \frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|^2} J\mathsf{T}(\zeta(t)).$$

Proof. We start by pointing out that T has norm one everywhere. Moreover, the Tanno connection differentiates horizontal vector fields into horizontal

vector fields and is compatible with the metric according to Theorem 3.1.3. So we can write

$$g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T},\mathsf{T}\right)=\frac{1}{2}\mathsf{T}\left(\zeta(t)\right)g\left(\mathsf{T},\mathsf{T}\right)=0.$$

We also notice that by definition of J we have

$$g_{\zeta(t)}\left(\mathsf{T},J\mathsf{T}\right)=\mathrm{d}\omega_{\zeta(t)}\left(\mathsf{T},\mathsf{T}\right)=0.$$

The distribution is two dimensional and both $J\mathsf{T}\left(\zeta(t)\right)$ and $\nabla_{\mathsf{T}}\mathsf{T}\left(\zeta(t)\right)$ are orthogonal to a same vector so they are coolinear. We now use Definition 3.3.2 to conclude. □

From the previous proposition, we deduce results concerning the geodesic curvature and characteristic deviation.

Proposition (1.1.5)**.** *Let M be a three dimensional contact sub-Riemannian manifold. Any smooth horizontal curve parametrized by arc length $\zeta:]-T,T[\to M$ has a first geodesic deviation that is identically zero.*

The previous proposition means that the only relevant geodesic deviation in 3D is the second geodesic deviation so we simply call it *the geodesic deviation* and we denote it as k_ζ instead of $k_{\zeta,2}$.

Proposition (1.1.6)**.** *Let M be a complete 3D sub-Riemannian contact structure. Given $x\in M$, a unit vector $v\in\Delta_x$ and a smooth function $\varphi:I\to\mathbb{R}$, there exists a unique smooth horizontal curve $\zeta:I\to M$ parametrized by arc length such that $\zeta(0)=x$, $\dot\zeta(0)=v$, and $h_\zeta(t)=\varphi(t)$ for all $t\in I$.*

Proof. (i). Let $\zeta_1,\zeta_2:I\to M$ be two smooth horizontal curves parametrized by arc length such that $\zeta(0)=x$, $\dot\zeta(0)=v$ and with the same characteristic deviation $\varphi=h_{\zeta_1}=h_{\zeta_2}$. It follows that $\zeta_1:I\to TM$ and $\zeta_2:I\to TM$ both satisfy the same Cauchy problem

$$\nabla_{\dot\zeta}\dot\zeta=\varphi(t)J\dot\zeta.$$

with the same initial conditions. Hence $\zeta_1=\zeta_2$.

(ii). Fix $x\in M$, a unit vector $v\in T_xM$, and a smooth function $\varphi:I\to\mathbb{R}$. Since M is complete, there exists $\zeta:I\to TM$ a smooth solution to the second order Cauchy problem:

$$\nabla_{\dot\zeta}\dot\zeta=\varphi(t)J\dot\zeta,\quad \zeta(0)=x,\dot\zeta(0)=v.$$

We are left to show that ζ is horizontal and has unit speed. Since by definition of the Tanno connection, $\nabla\omega=0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(\dot\zeta(t))=\omega(\nabla_{\dot\zeta(t)}\dot\zeta)=\omega(\varphi(t)J\dot\zeta(t))=0,$$

which implies that $\dot{\zeta}(t)$ is horizontal for any t in I. Moreover since by definition of the Tanno connection, $\nabla g = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\dot{\zeta}(t), \dot{\zeta}(t)) = 2g(\nabla_{\dot{\zeta}(t)}\dot{\zeta}, \dot{\zeta}(t)) = 2g(\varphi(t)J\dot{\zeta}(t), \dot{\zeta}(t)) = 0,$$

which means that $\dot{\zeta}(t)$ is indeed a unit vector for every $t \in I$. □

4.2.2 Endomorphism J

Endomorphism J also has properties that are specifically linked to dimension three. Indeed, J can be interpreted in a very natural way in the 3D contact sub-Riemannian case. By definition of J, for all $X \in \Delta$,

$$g(X, JX) = \mathrm{d}\omega(X, X) = 0.$$

But when we are considering M a three dimensional contact sub-Riemannian manifold the distribution is two-dimensional so the previous identity implies that for any $q \in M$, $J_q = \alpha(q) R_{\frac{\pi}{2},q}$ where $R_{\frac{\pi}{2},q}$ is the rotation of angle $\frac{\pi}{2}$ in Δ_q, and $\alpha(q)$ is a real number. But if instead of considering ω, we choose the contact form to be equal to $\tilde{\omega} = \alpha\omega$, we simply have

$$J = R_{\frac{\pi}{2}}. \tag{4.3}$$

From now on, in the 3D contact sub-Riemannian case, we assume that (4.3) holds.

4.2.3 Direction-dependant functions

In the general contact case, we have already introduced such a direction-dependant function, namely η whose definition can be found in 3.4.5 and which one can also compute using the formula presented in Remark 3.4.6. We now introduce another such function that is closely linked to η, and that proves usefull in the three-dimensional setting.

Definition 4.2.2. Let M be a contact sub-Riemannian manifold. For $v \in \Delta$ such that $\|v\| = 1$, we define

$$\iota(v) = g(\mathrm{Tor}(v, X_0), Jv),$$

where we recall that Tor stands for the torsion of the Tanno connection.

Notice that since $\mathrm{Tor}(v, X_0)$ is horizontal the previous definition makes sense. This was already neccesary to introduce η. Like in Remark 3.4.6, we can obtain an expression of ι in terms of Lie brackets.

Proposition 4.2.3. *Let M be a contact sub-Riemannian manifold. If $v \in T_xM$ is such that $\|v\| = 1$ and if V is a horizontal vector field that has norm one everywhere with $V(x) = v$ then*

$$\iota(v) = \frac{1}{2}\left(g_x\left([X_0, JV], V\right) + g_x\left([X_0, V], JV\right)\right).$$

Proof. First of all, by applying the definition of ι, the antisymmetry of J and point (v) in Theorem 3.1.3,

$$\iota(v) = \frac{1}{2}\left(g_x\left(\mathrm{Tor}\left(JV, X_0\right), V\right) + g_x\left(\mathrm{Tor}\left(V, X_0\right), JV\right)\right).$$

Thanks to [ABR17, Lemma 6.8 c)] the previous identity is transformed into

$$\iota(v) = \frac{1}{2}\left(g_x\left(\mathrm{Tor}\left(JV, X_0\right), V\right) + g_x\left(\mathrm{Tor}\left(V, X_0\right), JV\right) + g_x\left(Q\left(V, X_0\right), V\right)\right),$$

where we recall that Q is the Tanno tensor introduced in Defintion 3.1.4. We now express what the Tanno tensor represents to write

$$\begin{aligned}
\iota(v) &= \frac{1}{2}\left(g_x\left(\mathrm{Tor}\left(JV, X_0\right), V\right) + g_x\left(\mathrm{Tor}\left(V, X_0\right), JV\right)\right. \\
&\qquad \left. + g_x\left(\nabla_{X_0} JV - J\nabla_{X_0} V, V\right)\right), \\
&= \frac{1}{2}\left(g_x\left(\mathrm{Tor}\left(JV, X_0\right), V\right) + g_x\left(\mathrm{Tor}\left(V, X_0\right), JV\right)\right. \\
&\qquad \left. + g_x\left(\nabla_{X_0} JV, V\right) + g_x\left(\nabla_{X_0} V, JV\right)\right) \text{ as } J \text{ is antisymmetric by def.} \\
&= \frac{1}{2}\left(g_x\left(\nabla_{X_0} JV - \nabla_{JV} X_0 + \mathrm{Tor}\left(JV, X_0\right), V\right)\right. \\
&\qquad \left. + g_x\left(\nabla_{X_0} V - \nabla_V X_0 + \mathrm{Tor}\left(V, X_0\right), JV\right)\right),
\end{aligned}$$

where the last line follows from the fact that $\nabla . X_0 = 0$ according to Theorem 3.1.3, and allows us to conclude. □

Now the link between η and ι is given by the following proposition.

Proposition 4.2.4. *Let M be a three-dimensional contact sub-Riemannian manifold in which J is a rotation of angle $\frac{\pi}{2}$. For $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, let us denote by*

$$R_\alpha^\Delta : \Delta \to \Delta \text{ and } R_\alpha^{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$$

the fiberwise rotations of angle α in Δ and $\mathbb{R}^2$ respectively.

If $v \in TM$ has norm one then

$$\begin{pmatrix} \eta\left(R_\alpha^\Delta(v)\right) \\ \iota\left(R_\alpha^\Delta(v)\right) \end{pmatrix} = R_{-2\alpha}^{\mathbb{R}^2} \begin{pmatrix} \eta\left(v\right) \\ \iota\left(v\right) \end{pmatrix}.$$

Proof. Thanks to 4.3 we can write

$$R_\alpha^\Delta(v) = \cos(\alpha)\, v + \sin(\alpha)\, Jv. \tag{4.4}$$

We then apply the definition of η (Definition 3.4.5) and the defintion of ι (Definition 4.2.2) to (4.4) in order to compute $\begin{pmatrix} \eta\left(R_\alpha^\Delta(v)\right) \\ \iota\left(R_\alpha^\Delta(v)\right) \end{pmatrix}$. To conclude from the formula we find, we notice that $J^2 = -\mathrm{Id}$ according to 4.3 and we use point (v) in Theorem 3.1.3. □

Another property of ι is its link to the radial vector field and to the geodesic deviation with respect to a point. We recall that those two notions are introduced in Definitions 3.2.2 and 3.3.4.

Lemma 4.2.5. *Let M be a contact sub-Riemannian manifold such that J is an isometry and let us consider $\zeta :]-T, T[\to M$ a smooth horizontal curve parametrized by arc length. Over Σ_s we have*

$$J\Gamma_s h_{0,s} = c^{\Gamma_s}_{J\Gamma_s, X_0} = -2\iota(\Gamma_s) - c^{J\Gamma_s}_{\Gamma_s, X_0},$$

where we use the notations introduced in Definition 3.4.1.

Proof. According to Lemma (3.3.5) and since by assumption J is an isometry, over Σ_s we have

$$\begin{aligned}
J\Gamma_s h_{0,s} &= J\Gamma_s\left(X_0\tilde{\delta}_s\right), \\
&= J\Gamma_s\left(X_0\tilde{\delta}_s\right) - X_0\left(J\Gamma_s\tilde{\delta}_s\right) \text{ thanks to Theorem 3.2.3}, \\
&= [J\Gamma_s, X_0]\,\delta_s, \\
&= c^{\Gamma_s}_{J\Gamma_s, X_0}\Gamma_s\tilde{\delta}_s + c^{J\Gamma_s}_{J\Gamma_s, X_0}J\Gamma_s\tilde{\delta}_s + c^{X_0}_{J\Gamma_s, X_0}X_0\tilde{\delta}_s, \\
&= c^{\Gamma_s}_{J\Gamma_s, X_0} + c^{X_0}_{J\Gamma_s, X_0}X_0\tilde{\delta}_s \text{ still from Theorem 3.2.3}, \\
&= c^{\Gamma_s}_{J\Gamma_s, X_0} + \omega\left([J\Gamma_s, X_0]\right)X_0\tilde{\delta}_s \\
&= c^{\Gamma_s}_{J\Gamma_s, X_0},
\end{aligned} \tag{4.5}$$

where the last line comes from applying the Cartan formula to the term $\omega([J\Gamma_s, X_0])$. Now that we know (4.5), we deduce

$$J\Gamma_s h_{0,s} = -2\iota(\Gamma_s) - c^{J\Gamma_s}_{\Gamma_s, X_0},$$

by applying 4.2.3. □

4.3 General strategy

We work in M a three-dimensional contact manifold in which J is the rotation of angle $\frac{\pi}{2}$. Let us consider $\zeta : I \to \Sigma_0 \cup \{p\}$ an injective horizontal curve parametrized by arc length such that $\zeta(0) = p$. Without loss of generality, we study the expansion of the distance between $\zeta(0)$ and $\zeta(t)$ only at time zero. That is why we can write Γ instead of $\Gamma_0$for the field. Similarly, Σ stands for Σ_0, h_0 for $h_{0,0}$ and $\tilde{\delta}$ represents $\tilde{\delta}_0$. We recall that the objects we cited are defined in Definitions 3.2.2, 3.2.1, 3.3.4 and Theorem 3.2.3.

We also define $\theta_\zeta : I \setminus \{0\} \to \mathbb{R}/2\pi\mathbb{Z}$ as the angle such that :

$$\zeta'(t) = \cos\left(\theta_\zeta(t)\right)\Gamma\left(\zeta(t)\right) + \sin\left(\theta_\zeta(t)\right)J\Gamma\left(\zeta(t)\right) \tag{4.6}$$

for every t in I.

Assuming regularity, it it is possible to obtain a Taylor expansion of the distance along curve γ in terms of function θ_γ.

Proposition 4.3.1. *Let us consider $\zeta : I \to \Sigma_0 \cup \{p\}$, a smooth horizontal curve parametrized by arc length such that $\zeta(0) = p$ and that is injective. Define $\theta_\zeta : I \setminus \{0\}$ such that*

$$\zeta'(t) = \cos\left(\theta_\zeta(t)\right)\Gamma\left(\zeta(t)\right) + \sin\left(\theta_\zeta(t)\right)J\Gamma\left(\zeta(t)\right), \tag{4.7}$$

for every t in I. Assume that θ_ζ can be extended at zero to a $\mathcal{C}^2$ function such that $\theta_\zeta(0) = \theta'_\zeta(0) = 0$. Then

$$\operatorname{dist}(\zeta(t), \zeta(0)) = t - \frac{\theta''_\zeta(0)^2}{40}t^5 + o(t^5). \tag{4.8}$$

Proof. We write

$$\begin{aligned}
\tilde{\delta}\left(\zeta(t)\right) &= \int_0^t \mathrm{d}\tilde{\delta}(\zeta'(s))\mathrm{d}s, \\
&= \int_0^t g(\Gamma(\zeta(s)), \zeta'(s))\mathrm{d}s \text{ by Theorem 3.2.3}, \\
&= \int_0^t \cos\left(\theta_\zeta(s)\right)\mathrm{d}s, \\
&= \int_0^t \cos\left(\frac{\theta''_\zeta(0)}{2}s^2 + o(s^2)\right)\mathrm{d}s, \\
&= t - \frac{\left(\theta''_\zeta(0)\right)^2}{40}t^5 + o(t^5).
\end{aligned}$$

□

The goal of the following section is to show that these assumptions are in fact valid for ζ any smooth horizontal curve parametrized by arc length. Moreover, we recover the geometric meaning of the coefficient appearing in (4.8).

We start by linking the characteristic deviation and the geodesic curvature with θ_ζ.

Proposition 4.3.2. *Let us consider M a contact sub-Riemannian manifold in which (4.3) is satisfied. If $\zeta : I \to \Sigma_0 \cup \{p\}$ is a smooth horizontal curve parametrized by arc length in M such that $\zeta(0) = p$ and that is injective, then for any t in $I \setminus \{0\}$,*

$$h_\zeta(t) = \theta'_\zeta(t) + \cos(\theta_\zeta(t))h_0(\zeta(t)) - \sin(\theta_\zeta(t))c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)) \tag{4.9}$$

and

$$\begin{aligned} k_\zeta(t) = \eta(\zeta'(t)) + \theta''_\zeta(t) - \theta'_\zeta(t)(\sin(\theta_\zeta(t))h_0(\zeta(t)) + \cos(\theta_\zeta(t))c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t))) \\ + \cos^2(\theta_\zeta(t))(\sin(2\theta_\zeta(t))\iota(\zeta'(t)) - \cos(2\theta_\zeta(t))\eta(\zeta'(t))) \\ - \sin(2\theta_\zeta(t))\left(\sin(2\theta_\zeta(t))\eta(\zeta'(t)) + \cos(2\theta_\zeta(t))\iota(\zeta'(t)) + \frac{1}{2}c^{J\Gamma}_{\Gamma,X_0}\right) \\ - \frac{\sin(2\theta_\zeta(t))}{2}(\Gamma c^{J\Gamma}_{\Gamma,J\Gamma})(\zeta(t)) - \sin^2(\theta_\zeta(t))(J\Gamma c^{J\Gamma}_{\Gamma,J\Gamma})(\zeta(t)). \end{aligned} \tag{4.10}$$

Proof. Let us consider T a smooth horizontal vector field over Σ_0 extending the speed of ζ. We denote by $\psi : \Sigma_0 \to \mathbb{R}/2\pi\mathbb{Z}$ the angle such that

$$\mathsf{T} = \cos(\psi)\Gamma + \sin(\psi)J\Gamma. \tag{4.11}$$

In particular, thanks to Definition 4.6, along ζ, $\psi(\zeta(t)) = \theta_\zeta(t)$. We compute for any t in $I \setminus \{0\}$,

$$\begin{aligned} h_\zeta(t) &= -c^{\mathsf{T}}_{\mathsf{T},J\mathsf{T}} = -g(\zeta'(t), [\mathsf{T}, J\mathsf{T}](\zeta(t))) \\ &= -g(\zeta'(t), -\operatorname{grad}\psi(\zeta(t)) + [\Gamma, J\Gamma](\zeta(t))) \text{ by applying 4.11,} \\ &= \theta'_\zeta(t) - g\left(\cos(\theta_\zeta(t))\Gamma + \sin(\theta_\zeta(t))J\Gamma, [\Gamma, J\Gamma](\zeta(t))\right) \\ &= \theta'_\zeta(t) - \cos(\theta_\zeta(t))c^{\Gamma}_{\Gamma,J\Gamma}(\zeta(t)) - \sin(\theta_\zeta(t))c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)) \\ &= \theta'_\zeta(t) + \cos(\theta_\zeta(t))h_0(\zeta(t)) - \sin(\theta_\zeta(t))c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)). \end{aligned}$$

This proves (4.9). To obtain the formula for the geodesic curvature, we use Definition 3.4.5, with the understanding that hypothesis (4.3) holds. For t in $I \setminus \{0\}$,

$$\begin{aligned} k_\zeta &= \eta(\zeta'(t)) + h'_\zeta(t) \\ &= \eta(\zeta'(t)) + \theta''_\zeta(t) - \theta'_\zeta(t)(\sin(\theta_\zeta(t))h_0(\zeta(t)) + \cos(\theta_\zeta(t))c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t))) \\ &\quad + \cos(\theta_\zeta(t))(\mathrm{d}h_0)(\zeta'(t)) - \sin(\theta_\zeta(t))(\mathrm{d}c^{J\Gamma}_{\Gamma,J\Gamma})(\zeta'(t)). \end{aligned} \tag{4.12}$$

We now focus on the two terms $(\mathrm{d}c^{J\Gamma}_{\Gamma,J\Gamma})(\zeta'(t))$ and $(\mathrm{d}h_0)(\zeta'(t))$. We replace vector $\zeta'(t)$ by its expression in the frame $(\Gamma, J\Gamma)$ in terms of θ_ζ. We obtain

$$(\mathrm{d}c^{J\Gamma}_{\Gamma,J\Gamma})(\zeta'(t)) = \cos(\theta_\zeta(t))(\Gamma c^{J\Gamma}_{\Gamma,J\Gamma})(\zeta(t)) + \sin(\theta_\zeta(t))(J\Gamma c^{J\Gamma}_{\Gamma,J\Gamma})(\zeta(t)) \tag{4.13}$$

and

$$\begin{aligned}(\mathrm{d}h_0)(\zeta'(t)) &= \cos(\theta_\zeta(t))(\Gamma h_0)(\zeta(t)) + \sin(\theta_\zeta(t))(J\Gamma h_0)(\zeta(t)) \\ &= -\sin(\theta_\zeta(t))(2\iota(\Gamma(\zeta(t))) + c^{J\Gamma}_{\Gamma,X_0}(\zeta(t))) - \cos(\theta_\zeta(t))\eta(\Gamma(\zeta(t))),\end{aligned} \tag{4.14}$$

where the previous line comes from Lemma 4.2.3, and Corollary 3.4.7 where it is understood that J is an isometry since (4.3) is satisfied.

Now we carry on the computation of (4.14) by combining Proposition 4.2.4 with the definition of θ_ζ as the angle between Γ and the speed of ζ and we deduce that

$$\begin{aligned}\cos(\theta_\zeta(t))(\mathrm{d}h_0)(\zeta'(t)) = \cos^2(\theta_\zeta(t))&(\sin(2\theta_\zeta(t))\iota(\zeta'(t)) - \cos(2\theta_\zeta(t))\eta(\zeta'(t))) \\ &- \sin(2\theta_\zeta(t))\Big(\sin(2\theta_\zeta(t))\eta(\zeta'(t)) + \cos(2\theta_\zeta(t))\iota(\zeta'(t)) \\ &+\frac{1}{2}c^{J\Gamma}_{\Gamma,X_0}\Big).\end{aligned} \tag{4.15}$$

The proof is completed by combining (4.12), (4.13) and (4.15). □

4.4 Continuity of the geodesic deviation and asymptotics for the Lie brackets

We express the characteristic deviation in a particular adapted set of coordinates called normal coordinates, as introduced in [EAGK96].

Proposition 4.4.1. *If p is a point in M, there exist a neighbourhood U of p and coordinates (x, y, z) on U as well as smooth functions $u, v : U \to \mathbb{R}$ that satisfy*

$$u(0,0,z) = v(0,0,z) = \frac{\partial v}{\partial x}(0,0,z) = \frac{\partial v}{\partial y}(0,0,z) = 0,$$

such that the two vector fields

$$\begin{aligned}X_1 &= \left(\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}\right) + uy\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) - v\frac{y}{2}\frac{\partial}{\partial z}, \\ X_2 &= \left(\frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}\right) - ux\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + v\frac{x}{2}\frac{\partial}{\partial z},\end{aligned}$$

form an orthonormal frame of the distribution.

A first property of this set of coordinates is the following fact that can be easily checked from the definition of the Reeb vector field.

Lemma 4.4.2. *In normal coordinates, we have* $[X_1, X_2](0) = X_0(0) = \frac{\partial}{\partial z}$.

In normal coordinates the characteristic deviation of a horizontal curve parametrized by arc length leaving from the origin is computed very easily.

Proposition 4.4.3. *Let us consider* (x, y, z) *a system of normal coordinates on* M. *If* $\zeta : I \to M$ *is a smooth horizontal curve parametrized by arc length with* $\zeta(t) = (x(t), y(t), z(t))$ *such that* $\zeta(0) = (0, 0, 0)$ *then*

$$\dot{z}(0) = \ddot{z}(0) = 0, \quad z^{(3)}(0) = \frac{h_\zeta(0)}{2}.$$

In particular we have

$$h_\zeta(0) = \lim_{t \to 0} \frac{12 z(t)}{(x^2(t) + y^2(t))^{3/2}}.$$

Proof. Let us consider T a smooth horizontal unitary vector field extending the speed of ζ and let $\psi : U \to \mathbb{R}/2\pi\mathbb{Z}$ such that $\mathsf{T} = \cos(\psi) X_1 + \sin(\psi) X_2$. We have, for every t in I :

$$\begin{aligned} h_\zeta(0) &= -g([\mathsf{T}, J\mathsf{T}](\zeta(0)) - X_0(\zeta(0)), \mathsf{T}(\zeta(0))) \\ &= g(\operatorname{grad}\psi(\zeta(0)) - [X_1, X_2](\zeta(0)) + X_0(\zeta(0)), \mathsf{T}(\zeta(0))) \\ &= g(\operatorname{grad}\psi(\zeta(0)), \mathsf{T}(\zeta(0))) \text{ thanks to Proposition 4.4.2.} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \psi(\zeta(t)). \end{aligned}$$

Moreover,

$$\begin{aligned} \dot{x} &= \cos(\psi \circ \zeta)(1 + (u \circ \zeta) y^2) - \sin(\psi \circ \zeta)(u \circ \zeta) xy, \\ \dot{y} &= \sin(\psi \circ \zeta)(1 + (u \circ \zeta) x^2) - \cos(\psi \circ \zeta)(u \circ \zeta) xy, \\ \dot{z} &= (-\cos(\psi \circ \zeta)\frac{y}{2} + \sin(\psi \circ \zeta)\frac{x}{2})(1 + v \circ \zeta). \end{aligned}$$

By differentiating these expressions we find out that :

$$\dot{z}_\zeta(0) = \ddot{z}_\zeta(0) = 0, \qquad z_\zeta^{(3)}(0) = \frac{(\psi \circ \zeta)'(0)}{2} = \frac{h_\zeta(0)}{2}.$$

□

We now intend to use the expression of h_0 in terms of normal coordinates to prove that $h_0(t)$ is continuous at $t = 0$. The reason why we are interested in the regularity of h_0 is that the function h_0 appears in the equations in Proposition 4.3.2. We start with some lemmas and definitions.

Lemma 4.4.4. *Let $\Xi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ smooth function such that for any x in $\mathbb{R}^d$,*

$$\Xi(x,0) = \frac{\partial}{\partial y}\bigg|_{y=0} \Xi(x,y) = ... = \frac{\partial^{n-1}}{\partial y^{n-1}}\bigg|_{y=0} \Xi(x,y) = 0,$$

then the function $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ defined by $(x,y) \mapsto \frac{\Xi(x,y)}{y^n}$ is smooth.

Proof. We write the Taylor-Lagrange formula :

$$\begin{aligned}\frac{\Xi(x,y)}{y^n} &= \frac{1}{y^n}\int_0^y \frac{\frac{\partial^n}{\partial z^n}_{|z=s}\Xi(x,z)}{(n-1)!}(y-s)^{n-1}\,\mathrm{d}s \\ &= \int_0^y \frac{\frac{\partial^n}{\partial z^n}_{|z=s}\Xi(x,z)}{(n-1)!}\left(1-\frac{s}{y}\right)^{n-1}\frac{\mathrm{d}s}{y} \\ &= \int_0^1 \frac{\frac{\partial^n}{\partial z^n}_{|z=uy}\Xi(x,z)}{(n-1)!}(1-u)^{n-1}\,\mathrm{d}u,\end{aligned}$$

which is smooth. □

Lemma 4.4.5. *Let M be a three dimensional sub-Riemannian contact manifold. If we consider normal coordinates (x,y,z) around $p \in M$, such as defined in Proposition 4.4.1, the map :*

$$\begin{array}{rccc}\mathfrak{q}: & S_pM \times \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (Z,h,t) & \longmapsto & \frac{12z(\exp(Z,h,t))}{(x^2(\exp(Z,h,t))+y^2(\exp(Z,h,t)))^{\frac{3}{2}}}\end{array}$$

is smooth and its evaluation at a point of the form $(Z,h,0)$ is h.

Proof. Let us first recall that the exponential map we defined in Definition 3.4.8 is smooth according to Proposition 3.4.9. Thanks to Proposition 4.4.3 combined with Lemma 4.4.4 we learn that

$$\begin{array}{ccc}S_pM \times \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (Z,h,t) & \longmapsto & \frac{z(\exp(Z,h,t))}{t^3}\end{array}$$

is smooth and is equal to $\frac{h}{12}$ at any point $(Z,h,0)$.

Moreover, for any (Z,h) in $S_pM \times \mathbb{R}$, we have that $t \mapsto \exp(Z,h,t)$ is parametrized by arc length, and leaves from p at time 0. By the expression of the orthonormal frame of the distribution in normal coordinates around p given in Proposition 4.4.1, we deduce that

$$x^2(\exp(Z,h,t)) + y^2(\exp(Z,h,t)) \overset{\substack{(Z,h) \text{ fixed} \\ t \to 0}}{\sim} t^2.$$

Therefore, by using Lemma 4.4.4 once more, we obtain that

$$\begin{array}{ccc}S_pM \times \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (Z,h,t) & \longmapsto & \frac{x^2(\exp(Z,h,t))+y^2(\exp(Z,h,t))}{t^2}\end{array}$$

is smooth and is equal to 1 at any point $(Z,h,0)$. □

By applying the same proof technique as the one we just used, we also prove the following.

Lemma 4.4.6. *Let M be a three dimensional sub-Riemannian contact manifold. Let us consider $\zeta : I \to M$ a smooth horizontal curve parametrized by arc length. If (x, y, z) are normal coordinates around $p \in M$, such as defined in Proposition 4.4.1 then*

$$\frac{12z\left(\zeta(t)\right)}{\left(x^2\left(\zeta(t)\right) + y^2\left(\zeta(t)\right)\right)^{\frac{3}{2}}}$$

is smooth at $t = 0$, where it is equal to $h_\zeta(0)$

We can now express a key proposition. Before reading the following proposition, notice that the geodesic deviation that we present in Definition 3.3.4 and that is denoted by h_0 (as an abreviation of $h_{0,0}$) must be carefully distingushed from the characteristic deviation that we introduce in Definition 3.3.2 and that depends on a curve (that can be a a minimizing geodesic such as introduced in Definition 3.2.1).

Proposition 4.4.7. *Let M be a three-dimensional contact sub-Riemannian manifold in which J is the rotation of angle $\frac{\pi}{2}$ of the distribution. If $\zeta : I \to M$ is a smooth horizontal curve parametrized by arc length such that $\zeta(0) = p$ then for $t \neq 0$ close enough to zero, $\zeta(t)$ belongs to Σ_0 and we have*

$$h_{\gamma_{\zeta(t)}}(0) \overset{t\to 0}{\longrightarrow} h_\zeta(0) \text{ and } h_0\left(\zeta(t)\right) \overset{t\to 0}{\longrightarrow} h_\zeta(0).$$

Proof. For $t \in I$, we consider $\gamma_{\zeta(t)}$ a minimizing geodesic such that $\gamma_{\zeta(t)}(0) = p$ and $\gamma_{\zeta(t)}\left(\delta\left(\zeta(t)\right)\right) = \zeta(t)$. This notation extends that given in Defintion 3.2.1 outside Σ_0, but we have to pay a price : $\gamma_{\zeta(t)}$ is not necessarily unique. Let us now assume that

$$h_{\gamma_{\zeta(t)}}(0) \overset{t\to 0}{\not\to} h_\zeta(0).$$

There exist $\varepsilon > 0$ and $(t_n) \in I^{\mathbb{N}}$ a sequence that tends to zero such that

$$|h_{\gamma_{\zeta(t_n)}}(0) - h_\zeta(0)| \geqslant \varepsilon.$$

Without loss of generality, we can assume that

$$h_{\gamma_{\zeta(t_n)}}(0) \geqslant \varepsilon + h_\zeta(0).$$

For λ_1 and λ_2 two real numbers, we set

$$\mathfrak{R}_{\lambda_1,\lambda_2} = \left\{ (x, y, z) \in M \mid \lambda_1 \geqslant \frac{12z}{(x^2 + y^2)^{\frac{3}{2}}} \geqslant \lambda_2 \right\}.$$

If we apply Lemma 4.4.6 to $\gamma_{\zeta(t_n)}$ we obtain that for s small enough, $\gamma_{\zeta(t_n)}(s)$ is above $\mathfrak{R}_{h_\zeta(0)+\frac{3}{4}\varepsilon,h_\zeta(0)+\frac{1}{4}\varepsilon}$. Similarly, if we apply Lemma 4.4.6 to ζ, we find out that for n large enough $\zeta(t_n)$ is below that same set $\mathfrak{R}_{h_\zeta(0)+\frac{3}{4}\varepsilon,h_\zeta(0)+\frac{1}{4}\varepsilon}$. Since $\gamma_{\zeta(t_n)}$ goes through the point $\zeta(t_n)$ at time $\delta(\zeta(t_n))$, this means that for n large enough, $\gamma_{\zeta(t_n)}$ crosses the region $\mathfrak{R}_{h_\zeta(0)+\frac{3}{4}\varepsilon,h_\zeta(0)+\frac{1}{4}\varepsilon}$ from its superior boundary to its inferior boundary.

Another property of the trajectory of $\gamma_{\zeta(t_n)}$ is a consequence of the fact that it is a minimizing geodesic between p and $\zeta(t_n)$: between time 0 and time $\delta(\zeta(t_n))$, $\gamma_{\zeta(t_n)}$ stays inside the ball $\mathcal{B}(\delta(\zeta(t_n)))$ centered at p of radius $\delta(\zeta(t_n))$.

We are now going to prove that the properties of $\gamma_{\zeta(t_n)}$ that we just explained are in fact incompatible.

We set

$$\rho_1 = \frac{1}{2}\min\left\{\text{cut time of } t \mapsto \exp_p(V,h,t) \ : \ V \in S_pM, |h - h_\zeta(0)| \leqslant \varepsilon\right\},$$

Thanks to [ABB19, Proposition 8.76] $\rho_1 > 0$.

We also define

$$\rho_2 = \frac{1}{2}\min\left\{t : \ \exp_p\left(S_p \times \left\{h_\zeta(0) + \frac{\varepsilon}{2}\right\} \times \{t\}\right) \not\subset \mathfrak{R}_{h_\zeta(0)+\frac{3}{4}\varepsilon,h_\zeta(0)+\frac{1}{4}\varepsilon}\right\}$$

Thanks to Lemma 4.4.5, $\rho_2 > 0$. We now set $\rho = \min\{\rho_1, \rho_2\}$.

By definition of ρ, the surface

$$\mathcal{S} := \exp\left(S_pM \times \left\{h_\zeta(0) + \frac{\varepsilon}{2}\right\} \times [0,\rho]\right)$$

is made of geodesics leaving from p that have not yet reached the cut locus. Therefore, these geodesics are length minimizing at times smaller than ρ, which implies that at time ρ they reach the sphere centered at p of radius ρ. Moreover since the geodesics that form the surface $\mathcal{S}$ have not yet reached the cut locus, they can not cross any other length minimizing geodesics leaving from p. But since $\mathcal{S}$ is included in $\mathfrak{R}_{h_\zeta(0)+\frac{3}{4}\varepsilon,h_\zeta(0)+\frac{1}{4}\varepsilon}$, also by definition of ρ, the surface $\mathcal{S}$ is an obstruction that prevents minimizing geodesics contained in $\mathcal{B}(\rho)$ from crossing $\mathfrak{R}_{h_\zeta(0)+\frac{3}{4}\varepsilon,h_\zeta(0)+\frac{1}{4}\varepsilon}$ from its superior side to its inferior side.

We have contradicted the properties of $\gamma_{\zeta(t_n)}$ for n great enough to have $\delta(\zeta(t_n)) \leqslant \rho$. As a consequence, our initial assumption was false, and we obtain that

$$h_{\gamma_{\zeta(t)}}(0) \xrightarrow{t\to 0} h_\zeta(0). \tag{4.16}$$

In particular for t small enough, $h_{\gamma_{\zeta(t)}}(0)$ belongs to $[h_\zeta(0) - 1, h_\zeta(0) + 1]$. Now according to [ABB19, Proposition 8.76], the cut time is continuous with respect to geodesics leaving from a point so there exists $T > 0$ such

that for times t small enough, the cut time of $\gamma_{\zeta(t)}$ is greater than T. As a consequence, for times t small enough, a minimizing geodesic that comes from $\zeta(0)$ and reaches $\zeta(t)$ has not yet reached its cut time. According to Proposition 3.2.7, this means that for t small enough, $\zeta(t) \in \Sigma_0$.

To finish the proof, we apply Proposition 1.1.4 -whose proof is in section 3.4- to the geodesic $\gamma_{\zeta(t)}$ in order to write

$$k_{\gamma_{\zeta(t)},2} = 0.$$

By Definition 3.4.5 combined with the fact that J is an isometry since (4.3) is satisfied, the vanishing of $k_{\gamma_{\zeta(t)},2}$ can be rewritten as

$$h'_{\gamma_{\zeta(t)}} = -\eta \circ \gamma'_{\zeta(t)}.$$

Since the absolute value of η of a vector is locally bounded we find out that :

$$|\overbrace{h_{\gamma_{\zeta(t)}}(\delta_0(t))}^{h_0(\zeta(t))} - h_{\gamma_{\zeta(t)}}(0)| \leqslant \delta_0(\zeta(t)) \sup_{\substack{x\in\mathcal{B}(t),\\ v\in T_xM}} \eta(v),$$

where $\delta_0(t)$ stands for the distance between $\zeta(t)$ and the origin and where $\mathcal{B}(t)$ is the closed sub-Riemannian ball centered at the origin of radius t. To conclude, we combine the last inequality with (4.16). □

Now that we studied the continuity of the geodesic deviation along a curve, we must also study the Lie bracket terms that appear in the equations of Proposition 4.3.2. By using the notation $\tilde{\delta}$ introduced in Theorem 3.2.3 (where as we already said, it is understood that $\tilde{\delta}$ stands for $\tilde{\delta}_0$, the distance from $\zeta(0)$) we write the following asymptotics.

Proposition 4.4.8. *Let $\zeta : I \to \Sigma_0 \cup \{p\}$ be a smooth horizontal curve parametrized by arc length such that $\zeta(0) = p$ and such that for $t \neq 0$, $\zeta(t) \neq p$. Then for $t \to 0$*

(a) $\tilde{\delta}(\zeta(t))\, c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)) \longrightarrow -4,$

(b) $\tilde{\delta}^2(\zeta(t))\, c^{J\Gamma}_{\Gamma,X_0}(\zeta(t)) \longrightarrow -6,$

(c) $\tilde{\delta}^2(\zeta(t))\, \Gamma c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)) \longrightarrow 4,$

(d) $\tilde{\delta}^2(\zeta(t))\, J\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)) = O(1).$

The proof of Proposition 4.4.8 is contained in Appendix A.

4.5 Studying θ

4.5.1 Regularity of the angle θ along a smooth curve

We now go back to the regularity properties of the function θ_ζ, which we recall is introduced in Definition 4.6. We first prove two technical lemmas.

Lemma 4.5.1. *It is impossible that for some $t_1 > 0$, $\cos(\theta_\zeta(t)) < 0$ for every t in $(0, t_1)$.*

Proof. We proceed to prove this by contradiction. We assume there exists $t_1 > 0$ such that $\cos(\theta_\zeta(t)) < 0$ for every t in $(0, t_1)$. By combining Theorem 3.2.3 and Definition 4.6 we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\delta}(\zeta(t)) = \cos(\theta_\zeta(t)).$$

And $\tilde{\delta}(\zeta(0)) = 0$ by definition, so for t positive small enough,

$$\tilde{\delta}(\zeta(t)) < 0,$$

which is impossible. □

Lemma 4.5.2. *If $\zeta : I \to \Sigma_0 \cup \{p\}$ is a smooth horizontal curve parametrized by arc length such that $\zeta(0) = p$ and such that for $t \neq 0$, $\zeta(t) \neq p$, then there exists $B > 0$ such that for any $(t_1, t_2) \in]0, B[^2$ with $t_1 \leqslant t_2$:*

$$|\sin(\theta_\zeta(t_1))| \leqslant t_2 \frac{\max_{s\in]0,t_2]} |h_\zeta(s) - \cos(\theta_\zeta(s))\, h_0(\zeta(s))|}{3}.$$

Proof. By applying Proposition 4.4.8, there exists $B > 0$ small enough to have for every t in $]0, B[$:

$$\tilde{\delta}(\zeta(t))\, c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)) < -3. \tag{4.17}$$

Let us consider t_2 in $]0, B[$. We decompose the circle $\mathbb{R}/2\pi\mathbb{Z}$ into three zones.

$$\begin{aligned}
\mathcal{Z}_1(t_2) &= |\sin|^{-1}\left(\left(t_2 \frac{\max_{s\in]0,t_2]} |h_\zeta(s) - \cos(\theta_\zeta(s))\, h_0(\zeta(s))|}{3}, +\infty\right)\right),\\
\mathcal{Z}_2(t_2) &= ((\mathbb{R}/2\pi\mathbb{Z}) \setminus \mathcal{Z}_1(t_2)) \cap \cos^{-1}([0, +\infty)),\\
\mathcal{Z}_3(t_2) &= ((\mathbb{R}/2\pi\mathbb{Z}) \setminus \mathcal{Z}_1(t_2)) \cap \cos^{-1}((-\infty, 0)).
\end{aligned}$$

Let us recall that, according to Proposition 4.3.2, the angle θ_ζ satisfies the differential equation

$$\theta'_\zeta(t) = h_\zeta(t) - \cos(\theta_\zeta(t))\, h_0(\zeta(t)) + \sin(\theta_\zeta(t))\, c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)). \tag{4.18}$$

Let us now compare the different terms at the right hand side of the previous differential equation, in the case where $t \leqslant t_2$ and $\theta_\zeta(t)$ belongs to $\mathcal{Z}_1$. We have

$$\begin{aligned}\left|\sin\left(\theta_\zeta(t)\right) c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right)\right| &= \left|\frac{\sin\left(\theta_\zeta(t)\right)}{\tilde{\delta}\left(\zeta(t)\right)}\tilde{\delta}\left(\zeta(t)\right) c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right)\right| \\ &\geqslant \left|3\frac{\sin\left(\theta_\zeta(t)\right)}{\tilde{\delta}\left(\zeta(t)\right)}\right| \text{ by (4.17)}, \\ &\geqslant \frac{t_2}{\tilde{\delta}\left(\zeta(t)\right)} \max_{s\in]0,t_2]} \left|h_\zeta(s) - \cos\left(\theta_\zeta(s)\right) h_0\left(\zeta(s)\right)\right|.\end{aligned}$$

But $\|\zeta'\| = 1$ so $0 \leqslant \tilde{\delta}\left(\zeta(t)\right) \leqslant t \leqslant t_2$ and

$$\left|\sin\left(\theta_\zeta(t)\right) c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right)\right| \geqslant \max_{s\in]0,t_2]} \left|h_\zeta(s) - \cos\left(\theta_\zeta(s)\right) h_0\left(\zeta(s)\right)\right|.$$

This implies, by (4.18) that in the case where $t \leqslant t_2$ and $\theta_\zeta(t)$ belongs to $\mathcal{Z}_1$, $\theta'_\zeta(t)$ has the same sign as $\sin\left(\theta_\zeta(t)\right) c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right)$. But by (4.17), this means that $\theta'_\zeta(t)$ and $\sin\left(\theta_\zeta(t)\right)$ have opposite signs, which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos\left(\theta_\zeta(t)\right)\right) \geqslant 0.$$

To summarize, for $t \leqslant t_2$, the time depending vector field associated to the differential equation (4.18) is oriented in the direction of increasing $\cos \circ\, \theta_\zeta$ on the zone $\mathcal{Z}_1$.

Let us now prove the lemma by contradiction. Let us assume there exists $t_1 \leqslant t_2$ such that

$$\left|\sin\left(\theta_\zeta(t_1)\right)\right| > t_2 \frac{\max_{s\in]0,t_2]} \left|h_\zeta(s) - \cos\left(\theta_\zeta(s)\right) h_0\left(\zeta(s)\right)\right|}{3}.$$

This means that $\theta_\zeta(t_1)$ belongs to $Z_1\left(t_2\right)$ with $t_1 \leqslant t_2$. By considering the orientation of the time depending vector field associated to the differential equation (4.18) on the set $\mathcal{Z}_1\left(t_2\right)$ that we proved a few lines ago, as well as the relative poistion of the sets $\mathcal{Z}_i\left(t_2\right)$, we deduce there are two cases for what happens at times smaller that t_1.

• *In the first case* there exists a time t_3 smaller than t_1 such that $\theta_\zeta\left(t_3\right)$ belongs to $\mathcal{Z}_3$, which implies that,

$$\theta_\zeta\left(t\right) \in \mathcal{Z}_3 \text{ for all times } t \in (0, t_3] \tag{4.19}$$

since if this was not the case, then there should exist a positive time smaller than t_3 when the curve belongs to $\mathcal{Z}_1$ which is incompatible with the orientation of the vector field associated to the differential equation (4.18) on the zone $\mathcal{Z}_1$. Now by definition of $\mathcal{Z}_3$, (4.19) implies that for all positive times t smaller than t_3,

$$\cos\left(\theta_\zeta(t)\right) < 0$$

This is impossible, according to Lemma 4.5.1.

• *In the second case,* for all times t smaller or equal than t_1, $\theta_\zeta(t)$ belongs to $\mathcal{Z}_1(t_2)$, and in this case, $\cos\circ\theta_\zeta$ is increasing on $(0,t_1]$ and $\cos(\theta_\zeta(t))$ converges when t tends to zero.

Now the limit of $\cos(\theta_\zeta(t))$ when t tends to zero is different from 1, or else since $\cos\circ\theta_\zeta$ is increasing on $(0,t_1]$, $\cos\circ\theta_\zeta$ would be constant equal to 1, but this is incompatible with the fact that θ_ζ belongs to $\mathcal{Z}_1(t_2)$.

If the limit of $\cos(\theta_\zeta(t))$ when t tends to zero is also different from -1, then we find a contradiction by integrating (4.18).

$$\theta_\zeta(t)-\theta_\zeta(t_1)=\int_t^{t_1} h_\zeta(s)-\cos(\theta_\zeta(s))\,h_0(\zeta(s))+\sin(\theta_\zeta(s))\,c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(s))\,\mathrm{d}s$$

More precisely, the contradiction comes from the fact that the left hand side of the previous identity converges, but that the right hand side of this same identity diverges. Indeed, according to Proposition 4.4.7, the term $h_\zeta(s)-\cos(\theta_\zeta(s))\,h_0(\zeta(s))$ is bounded for s small, but the third term inside the integral explodes for small times :

$$\begin{aligned}\left|\sin(\theta_\zeta(s))\,c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(s))\right| &\geqslant \left|3\frac{\sin(\theta_\zeta(s))}{\tilde{\delta}(\zeta(s))}\right| \text{ by (4.17)}\\ &\geqslant \left|3\frac{\sin(\theta_\zeta(s))}{s}\right| \text{ as } \zeta \text{ is parametrized by arc length.}\end{aligned}$$

Since $\sin(\theta_\zeta(s))$ converges to a non zero limit when s goes to zero, we deduce the divergence of the integral we are focusing on.

Now that we have obtained a contradiction in the case where the limit of $\cos(\theta_\zeta(t))$ is different from -1 when t tends to zero, we treat the case where this limit is equal to -1. In particular, for t small enough, $\cos(\theta_\zeta(t))<0$. This cannot happen according to Lemma 4.5.1.

In both cases, we obtain a contradiction and our assumption was false. We are done with the proof of the lemma. □

4.5.2 First order

Proposition 4.5.3. *If $\zeta : I \to \Sigma_0\cup\{p\}$ is a smooth horizontal curve parametrized by arc length such that $\zeta(0)=p$ and such that for $t\neq 0$, $\zeta(t)\neq p$, then the function $\theta_\zeta : I\setminus\{0\}\to\mathbb{R}/2\pi\mathbb{Z}$ can be extended to a $\mathcal{C}^1$ function $I\to\mathbb{R}/2\pi\mathbb{Z}$ that vanishes at time zero and whose derivative also vanishes at time zero. Moreover,*

$$\tilde{\delta}(\zeta(t)) \overset{t\to 0}{\sim} t.$$

Proof. We study what happens for positive times. The result for negative times is a consequence by reversing time. We apply Lemma 4.5.2 and we

find out that there exists B positive such that for any t in $(0, B)$,

$$|\sin(\theta_\zeta(t))| \leqslant t\frac{\max_{s\in]0,t]}|h_\zeta(s) - \cos(\theta_\zeta(s))\, h_0(\zeta(s))|}{3}. \tag{4.20}$$

We deduce from Proposition 4.4.7 that the quantity

$$\frac{\max_{s\in]0,t]}|h_\zeta(s) - \cos(\theta_\zeta(s))\, h_0(\zeta(s))|}{3}$$

is bounded for t small enough. This implies by using (4.20) that

$$\sin(\theta_\zeta(t)) \overset{t\to 0}{\longrightarrow} 0.$$

This means that $\cos(\theta_\zeta(t))$ tends to 1 or -1 as t goes to zero. But by applying Lemma 4.5.1, we discover that $\cos(\theta_\zeta(t))$ cannot tend to -1. As a consequence,

$$\cos(\theta_\zeta(t)) \overset{t\to 0}{\longrightarrow} 1 \text{ and } \theta_\zeta(t) \overset{t\to 0}{\longrightarrow} 0. \tag{4.21}$$

From now on we choose to extend θ_ζ by defining

$$\theta_\zeta(0) = 0.$$

A first interesting consequence of the fact that $\theta_\zeta(t)$ converges to zero when t tends to zero is the asymptotics of $\tilde{\delta}(\zeta(t))$. Indeed, by combining Theorem 3.2.3 and Definition 4.6 we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\delta}(\zeta(t)) = \cos(\theta_\zeta(t)),$$

which implies, since $\tilde{\delta}(\zeta(0)) = 0$, that

$$\tilde{\delta}(\zeta(t)) = \int_0^t \cos(\theta_\zeta(s))\,\mathrm{d}s \overset{t\to 0}{\sim} t. \tag{4.22}$$

To go futher, we once again make use of (4.20) but we analyse the terms of this inequality more precisely. We still apply 4.4.7 to obtain the convergence of the term h_0, but we now know that the cosine term tends to one. We deduce that

$$\left|\frac{\sin(\theta_\zeta(t))}{t}\right| \leqslant \frac{\max_{s\in]0,t]}|h_\zeta(s) - \cos(\theta_\zeta(s))\, h_0(\zeta(s))|}{3} \overset{t\to 0}{\longrightarrow} 0. \tag{4.23}$$

This means that the function θ_ζ is differentiable at time zero and that

$$\theta'_\zeta(0) = 0.$$

To show that θ_ζ is $\mathcal{C}^1$, we recall the first differential that is proved in Proposition 4.3.2 and that is valid for $t > 0$, namely

$$\theta'_\zeta(t) = h_\zeta(t) - \cos(\theta_\zeta(t))\, h_0(\zeta(t)) + \sin(\theta_\zeta(t))\, c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)),$$

which it is more convenient to rewrite as :

$$\theta'_\zeta(t) = h_\zeta(t) - \cos(\theta_\zeta(t))\, h_0(\zeta(t)) + \frac{\sin(\theta_\zeta(t))}{t}\frac{t}{\tilde{\delta}(\zeta(t))}\tilde{\delta}(\zeta(t))\, c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)).$$

Now we can analyse each term in the right hand side of the previous identity, by applying Propositions 4.4.7, 4.4.8 and by using (4.21), (4.22) and (4.23), we deduce that

$$\theta'_\zeta(t) \overset{t>0, t\to 0}{\longrightarrow} 0 = \theta'_\zeta(0).$$

We have studied the regularity of θ_ζ at time zero.

The smoothness of this function at any other time where the curve ζ stays inside Σ is a consequence of the fact that the decomposition given in Definition 4.6 is that of the smooth velocity of the curve ζ on the frame $(\Gamma, J\Gamma)$ that is itself smooth thanks to Theorem 3.2.3. □

4.5.3 Second order

Lemma 4.5.4. *We have for every $t > 0$*

$$k_\zeta(t) = \theta''_\zeta(t) + \frac{4\theta'_\zeta(t)}{t} + \frac{2\theta_\zeta(t)}{t^2} + r(t) \tag{4.24}$$

where $r(t) \to 0$ for $t \to 0$.

Proof. For the sake of simplicity, we focus on $t > 0$. We recall that from Proposition 4.3.2 the curvature $k_\zeta(t)$ can be expressed as:

$$\begin{aligned}
k_\zeta(t) = {} & \eta\left(\zeta'(t)\right) + \theta''_\zeta(t) \\
& - \theta'_\zeta(t)\left(\sin(\theta_\zeta(t))\, h_0(\zeta(t)) + \cos(\theta_\zeta(t))\, c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t))\right) \\
& + \cos^2(\theta_\zeta(t))\left(\sin(2\theta_\zeta(t))\, \iota\left(\zeta'(t)\right) - \cos(2\theta_\zeta(t))\, \eta\left(\zeta'(t)\right)\right) \\
& - \sin(2\theta_\zeta(t))\left(\sin(2\theta_\zeta(t))\, \eta\left(\zeta'(t)\right) + \cos(2\theta_\zeta(t))\, \iota\left(\zeta'(t)\right) + \frac{1}{2}c^{J\Gamma}_{\Gamma,X_0}\right) \\
& - \frac{\sin(2\theta_\zeta(t))}{2}\left(\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\right)(\zeta(t)) - \sin^2(\theta_\zeta(t))\left(J\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\right)(\zeta(t)).
\end{aligned} \tag{4.25}$$

Let us rewrite the three quantities

$$-\theta'_\zeta(t)\cos(\theta_\zeta(t))\, c^{J\Gamma}_{\Gamma,J\Gamma}(\zeta(t)), \tag{4.26}$$

$$-\frac{\sin(2\theta_\zeta(t))}{2}\left(c^{J\Gamma}_{\Gamma,X_0}(\theta_\zeta(t)) + \left(\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\right)(\zeta(t))\right), \tag{4.27}$$

$$-\sin^2(\theta_\zeta(t))\left(J\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\right)(\zeta(t)) \tag{4.28}$$

as follows, respectively

$$\frac{4\theta'_\zeta(t)}{t} + \frac{\theta'_\zeta(t)}{t}\left(\frac{-t}{\tilde{\delta}\left(\zeta(t)\right)}\cos\left(\theta_\zeta(t)\right)\tilde{\delta}\left(\zeta(t)\right)c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right) - 4\right), \tag{4.29}$$

$$\frac{2\theta_\zeta(t)}{t^2} + \frac{\theta_\zeta(t)}{t^2}\left(\frac{-t^2}{\tilde{\delta}^2\left(\zeta(t)\right)}\frac{\sin\left(2\theta_\zeta(t)\right)}{2\theta_\zeta(t)}\tilde{\delta}^2\left(\zeta(t)\right)\left(c^{J\Gamma}_{\Gamma,X_0} + \left(\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\right)\right)\left(\zeta(t)\right) - 2\right), \tag{4.30}$$

$$-\left(\frac{\sin\left(\theta_\zeta(t)\right)}{t}\right)^2\frac{t^2}{\tilde{\delta}^2\left(\zeta(t)\right)}\tilde{\delta}^2\left(\zeta(t)\right)\left(J\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\right)\left(\zeta(t)\right). \tag{4.31}$$

Using a Taylor expansion, together with the asymptotics of Propositions 4.4.8 and 4.5.3, we obtain (4.24) with $r(t)$ which tends to zero when $t \to 0$. □

Proposition 4.5.5. *If $\zeta : I \to \Sigma_0 \cup \{p\}$ is a smooth horizontal curve parametrized by arc length such that $\zeta(0) = p$ and such that for $t \neq 0$, $\zeta(t) \neq p$, then the function θ_ζ (extended at zero in such a way that $\theta_\zeta(0) = 0$) is $\mathcal{C}^2$ on I and*

$$\theta''_\zeta(0) = \frac{k_\zeta(0)}{6}.$$

Proof. Let us define, for t belonging to $(-T, T) \setminus \{0\}$

$$\mathfrak{F}(t) = \theta'_\zeta(t) + 2\frac{\theta_{\zeta(t)}}{t} - \frac{k_\zeta(0)t}{3}. \tag{4.32}$$

According to Proposition 4.5.3, $\theta'_\zeta(t) \to 0$ hence $\mathfrak{F}(t)$ tends to zero as $t \to 0$ (as the three terms tends to zero). We can reformulate (4.24) by writing

$$\mathfrak{F}'(t) + 2\frac{\mathfrak{F}(t)}{t} \xrightarrow{t\to 0} 0 \tag{4.33}$$

Now we consider $\varepsilon > 0$, and $t_\varepsilon > 0$ small enough to have :

$$\left|\mathfrak{F}'(t) + 2\frac{\mathfrak{F}(t)}{t}\right| \leqslant \varepsilon, \text{ for every } t \in (0, t_\varepsilon]. \tag{4.34}$$

At this point, we proceed by contradiction and *we assume* there exists t_* in $(0, t_\varepsilon]$ such that

$$\left|2\frac{\mathfrak{F}(t_*)}{t_*}\right| > \varepsilon. \tag{4.35}$$

By combining (4.34) and (4.35), we obtain that $\frac{\mathfrak{F}(t_*)}{t_*}$ and $\mathfrak{F}'(t_*)$ have opposite signs which implies, since $t_* > 0$, that $\left(\mathfrak{F}^2\right)'(t_*) \leqslant 0$. As a consequence

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=t_*}\left|2\frac{\mathfrak{F}(t)}{t}\right| \leqslant 0$$

Therefore (4.35) holds for every $t \in (0, t_*]$, and $\left(\mathfrak{F}^2\right)'(t) \leqslant 0$ for every $t \in (0, t_*]$. This is incompatible with the fact that $\mathfrak{F}(t) \xrightarrow{t\to 0} 0$. Therefore the assumption was false and for all t in $(0, t_\varepsilon]$,

$$\left|2\frac{\mathfrak{F}(t)}{t}\right| \leqslant \varepsilon.$$

As a consequence $\mathfrak{F}$ is differentiable at zero and $\mathfrak{F}'(0) = 0$. Moreover thanks to the previous inequality, we can use (4.33) to obtain that

$$\mathfrak{F}'(t) \xrightarrow{t\to 0} 0.$$

But thanks to (4.32), this means that

$$\theta_\zeta''(t) + 2\frac{\theta_\zeta'(t)}{t} - 2\frac{\theta_\zeta(t)}{t^2} - \frac{k_\zeta(0)}{3} \xrightarrow{t\to 0} 0.$$

And if we combine this limit with (4.24), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\theta_\zeta'(t) - t\frac{k_\zeta(0)}{6}\right) + 3\frac{\theta_\zeta'(t) - t\frac{k_\zeta(0)}{6}}{t} \xrightarrow{t\to 0} 0,$$

where $\mathfrak{G}(t) := \theta_\zeta'(t) - t\frac{k_\zeta(0)}{6} \xrightarrow{t\to 0} 0$ in accordance with Proposition 4.5.3. By reasoning with $\mathfrak{G}$ exactly as we did with $\mathfrak{F}$, we deduce that

$$\mathfrak{G}'(t) \xrightarrow{t\to 0} 0,$$

which proves that θ_ζ' is $\mathcal{C}^1$ and that $\theta_\zeta''(0) = \frac{k_\zeta(0)}{6}$, as required. □

4.5.4 Asymptotics of the distance between two close points along a horizontal curve

Theorem (1.1.7). *Let M be a three-dimensional contact sub-Riemannian manifold and let $\zeta : I \to M$ be a smooth horizontal curve parametrized by arc length. We have*

$$\mathrm{dist}_M^2(\zeta(0), \zeta(t)) = t^2 - \frac{k_{\zeta,2}^2(0)}{720}t^6 + o\left(t^6\right).$$

Proof. According to Proposition 4.4.7, for times $t \neq 0$ close enough to zero, $\zeta(t)$ belongs to Σ_0. Moreover, since ζ is parametrized by arc length, $\zeta'(0) \neq 0$ and for $t \neq 0$ close enough to zero $\zeta(t) \neq p$. This means that the hypotheses of Proposition 4.5.3 and Proposition 4.5.5 are satisfied by curve ζ locally around $t = 0$. If we apply both those propositions, we deduce that the hypotheses of Proposition 4.3.1 are satisfied with the additional property that

$$\theta_\zeta''(0) = \frac{k_\zeta(0)}{6}.$$

We apply Proposition 4.3.1 and we find out that

$$\begin{aligned}\tilde{\delta}\left(\zeta(t)\right) &= t - \frac{\left(\theta''_{\zeta}(0)\right)^2}{40}t^5 + o(t^5)\\ &= t - \frac{\left(k_{\zeta}(0)\right)^2}{1440}t^5 + o(t^5),\end{aligned}$$

which completes the proof. □

Chapter 5

Curvature of horizontal curves in the general contact setting

In this chapter, as we already did in three dimensions, we study the Taylor expansion of the distance between two close points on a horizontal curve in an arbitrary contact sub-Riemannian manifold. The general idea of the proof is the same than in 3D, although it is more technical. The main difference with the 3D case is that in the general contact setting, we are not able to prove the same regularity for the distance between points along the curve as we are able to do in three dimensional spaces, so we have to require it in the hypotheses for some theorems. Nevertheless, in special cases such as $(2n+1)-$Heisenberg groups, we are able to obtain as good results as in 3D.

5.1 A differential equation

In this section, we introduce a differential equation in order to compute, for s fixed and t close enough to s, the asymptotics of the distance between $\zeta(s)$ and $\zeta(t)$. First of all, we separate the radial and the orthoradial components of ζ'.

Definition 5.1.1. For $\zeta :]-T, T[\to M$ a horizontal curve parametrized by arc length and for $(s,t) \in]-T,T[^2$ such that $\zeta(t) \in \Sigma_s$, we define the adapted decomposition of the velocity of ζ around the point $\zeta(s)$ and at time t as

$$\zeta'(t) = f(t)\Gamma_s\left(\zeta(t)\right) + \mathscr{S}_s(t),$$

where $\mathscr{S}_s(t)$ is orthogonal to $\Gamma_s\left(\zeta(t)\right)$.

We can deduce a more precise form for the adapted decomposition of the velocity of ζ around the point $\zeta(s)$ from the properties of the field Γ_s. Before, we do this, we quickly recall the notations we use for the distance from a point $\zeta(s)$ on a curve $\zeta :]-T,T[\to M$. $\tilde{\delta}_s$ is the function introduced in

the statement of Theorem 3.2.3 that associates to a point in M its distance to $\zeta(s)$ while δ_s maps $t \in]-T,T[$ to $\tilde{\delta}_s(\zeta(t))$.

Proposition 5.1.2. *Let us consider $\zeta :]-T,T[\to M$ a horizontal curve parametrized by arc length and $s \in]-T,T[$ such that for some ε, at time $t \in]s-\varepsilon, s+\varepsilon[\in \Sigma_s$, then the adapted decomposition reads*

$$\zeta'(t) = \delta_s'(t)\Gamma_s\left(\zeta(t)\right) + \mathscr{S}_s(t),$$

where $\mathscr{S}_s(t)$ is orthogonal to $\Gamma_s\left(\zeta(t)\right)$.

Proof. We combine Definition 5.1.1 with Theorem 3.2.3 to deduce

$$\delta_s'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\delta}_s\left(\zeta(t)\right) = g_{\zeta(t)}\left(\mathrm{grad}\tilde{\delta}_s, \zeta'(t)\right) = g_{\zeta(t)}\left(\Gamma_s, f(t)\Gamma_s + \mathscr{S}_s(t)\right) = f(t).$$

□

Notice that as ζ is parametrized by arc length, we have $\|\zeta'(t)\| = 1$ for every time t, and since we also know that Γ_s has norm one wherever it is defined, by using Pythagoras' Theorem we deduce from Proposition 5.1.2 that

$$1 = \left(\delta_s'(t)\right)^2 + \|\mathscr{S}_s(t)\|^2, \tag{5.1}$$

for t close enough to s.

This last identity allows us to study the expansion as t goes to s of $\|\mathscr{S}_s(t)\|$ instead of the expansion of $\delta_s(t)$, which is what we initially wanted to study.

Now the question is : how do we study the asymptotics of $\|\mathscr{S}_s(t)\|$ as t goes to s ? The following proposition consists in presenting a differential equation on $\|\mathscr{S}_s(t)\|$ whose terms we must understand to solve our problem.

Proposition 5.1.3. *Let $\zeta :]-T,T[\to M$ be a horizontal curve parametrized by arc length and s be an element of $]-T,T[$ such that for every t close enough to s to have $\zeta(t) \in \Sigma_s$. At such a time t, we choose*

$$\left(\Gamma_s\left(\zeta(t)\right), X_2\left(\zeta(t)\right), X_3\left(\zeta(t)\right), ..., X_m\left(\zeta(t)\right)\right)$$

to be an orthonormal frame of the distribution that smoothly depends on t for $t \neq s$. Let us moreover consider $\tilde{\nabla}$ a connection that differentiates horizontal vector fields into horizontal vector fields and that respects the metric g.

For $i \geqslant 2$ and t such that $\zeta(t) \in \Sigma_s$, we have

$$\begin{aligned} g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathscr{S}_s, X_i\right) = &-\dot{\delta}_s(t)\sum_{j=2}^{m} g_{\zeta(t)}\left(\tilde{\nabla}_{X_j}\Gamma_s, X_i\right) g_{\zeta(t)}\left(\mathscr{S}_s, X_j\right) \\ &+ g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(t)\tilde{\nabla}_{\Gamma_s}\Gamma_s, X_i\right). \end{aligned}$$

Proof. For $i \geqslant 2$,

$$\begin{aligned}
g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathscr{S}_s, X_i\right) &= g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\left(\mathsf{T} - \dot{\delta}_s(t)\Gamma_s\right), X_i\right) \text{ by Proposition 5.1.2,} \\
&= g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s(t)\tilde{\nabla}_{\mathsf{T}}\Gamma_s - \ddot{\delta}_s(t)\Gamma_s, X_i\right) \\
&= g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s(t)\tilde{\nabla}_{\mathsf{T}}\Gamma_s, X_i\right) \text{ as } \Gamma_s \perp X_i, \\
&= g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s(t)\tilde{\nabla}_{\dot{\delta}_s(t)\Gamma_s + \mathscr{S}_s}\Gamma_s, X_i\right) \text{ by Prop. 5.1.2,} \\
&= g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(t)\tilde{\nabla}_{\Gamma_s}\Gamma_s, X_i\right) - \dot{\delta}_s(t) g_{\zeta(t)}\left(\tilde{\nabla}_{\mathscr{S}_s}\Gamma_s, X_i\right) \\
&= -\dot{\delta}_s(t) g_{\zeta(t)}\left(\tilde{\nabla}_{\sum_{j=2}^m g_{\zeta(t)}(\mathscr{S}, X_j)X_j}\Gamma_s, X_i\right) \\
&\quad + g_{\zeta(t)}\left(\tilde{\nabla}_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(t)\tilde{\nabla}_{\Gamma_s}\Gamma_s, X_i\right),
\end{aligned}$$

which corresponds to the result we are looking for. □

5.2 Normal coordinates

5.2.1 Characteristic deviation in normal coordinates

We present an adapted set of coordinates in which we compute the reduced deviation that is similar to the one we used in three-dimensional spaces whose existence is proved in [AG01, Theorem 6.6 and Proposition 6.3].

Theorem 5.2.1. *Let us consider M a sub-Riemannian contact space and $q_0 \in M$. There exists a neighbourhood $\mathscr{N}$ of q_0 on which one can define*

i. smooth coordinates

$$(x_1, y_1, x_2, y_2, ..., x_n, y_n, z),$$

whose origin is q_0,

ii. a smooth orthonormal frame of the distribution on $\mathscr{N}$

$$(X_1, Y_1, X_2, Y_2, ..., X_n, Y_n),$$

iii. smooth real valued functions

$$(u_{i,X}, u_{i,Y}, v_{i,X}, v_{i,Y}, w_{i,X}, w_{i,Y}, \alpha_i)_{i=1}^n$$

such that

$$\begin{aligned}
X_i &= (1 + u_{i,X})\frac{\partial}{\partial x_i} + w_{i,X}\frac{\partial}{\partial y_i} + \alpha_i\left(\frac{y_i}{2} + v_{i,X}\right)\frac{\partial}{\partial z}, \\
Y_i &= w_{i,Y}\frac{\partial}{\partial x_i} + (1 + u_{i,Y})\frac{\partial}{\partial y_i} - \alpha_i\left(\frac{x_i}{2} + v_{i,Y}\right)\frac{\partial}{\partial z},
\end{aligned}$$

where

$$u_{i,j}(q_0) = v_{i,j}(q_0) = w_{i,j}(q_0) = 0,$$

$$\mathrm{d}u_{i,j}(q_0) = \mathrm{d}v_{i,j}(q_0) = \mathrm{d}w_{i,j}(q_0) = 0,$$

and where the α_i functions do not vanish near q_0.

A first remark concerning the frame we just introduced is its link to J.

Definition 5.2.2. If we consider coordinates such as they are introduced in Theorem 5.2.1 on a neighbourhood $\mathcal{N}$ of a point q_0 then at each point q in $\mathcal{N}$ we define $\mathscr{J}(q)$ the matrix of J_q in the frame

$$(X_1(q), Y_2(q), X_2(q), Y_2(q), ..., X_n(q), Y_n(q)).$$

Proposition 5.2.3. *Let us consider a contact sub-Riemannian manifold (M, ω, g) on which we define around a point q_0 coordinates such as explained in Theorem 5.2.1. We have that $\mathscr{J}(q_0)$ has the following block structure*

$$\mathscr{J}(q_0) = A \begin{pmatrix} \begin{array}{cc} 0 & \alpha_1(q_0) \\ -\alpha_1(q_0) & 0 \end{array} & & (0) \\ & \ddots & \\ (0) & & \begin{array}{cc} 0 & \alpha_n(q_0) \\ -\alpha_n(q_0) & 0 \end{array} \end{pmatrix}$$

where A is the scalar coefficient such that

$$X_0(q_0) = \frac{1}{A}\frac{\partial}{\partial z}.$$

Proof. We start by checking that the following differential 1-form cancels the distribution.

$$\begin{aligned} \sigma = dz + \sum_i \alpha_i & \frac{\left((1+u_{i,X})\left(\frac{x}{2}+v_{i,Y}\right) + \left(\frac{y}{2}+v_{i,X}\right)w_{i,Y}\right)}{(1+u_{i,X})(1+u_{i,Y}) - w_{i,X}w_{i,Y}}\,dy \\ & -\alpha_i \frac{\left((1+u_{i,Y})\left(\frac{y}{2}+v_{i,X}\right) + \left(\frac{x}{2}+v_{i,Y}\right)w_{i,X}\right)}{(1+u_{i,X})(1+u_{i,Y}) - w_{i,X}w_{i,Y}}\,dx. \end{aligned}$$

As a consequence, there exists a smooth function $h : \mathcal{N} \to \mathbb{R}$ such that

$$\omega = h\sigma.$$

Therefore

$$\mathrm{d}\omega_{|\Delta} = h\mathrm{d}\sigma_{|\Delta},$$

which implies that

$$\mathrm{d}\omega\left(q_0\right)_{|\Delta} = h\left(q_0\right) \sum_i \alpha_i \mathrm{d}x_i\left(q_0\right) \wedge \mathrm{d}y_i\left(q_0\right)_{|\Delta},$$

which yields the result by choosing $A = h\left(q_0\right)$. Indeed, we directly check that the expression for X_0 given in the proposition satisfies the definition of the Reeb vector field. We also combine the expression of $\mathrm{d}\omega$ we just found to the fact that

$$\left(X_i\left(q_0\right), Y_i\left(q_0\right)\right) = \left(\frac{\partial}{\partial x_i}\left(q_0\right), \frac{\partial}{\partial y_i}\left(q_0\right)\right),$$

to check that the expression of J that we give satisfies the definition of J. □

We are now ready to express the reduced deviation in the coordinates we just studied.

Proposition 5.2.4. *Let us consider M a contact sub-Riemannian manifold and $\zeta :]-T, T[\to M$ a smooth horizontal curve parametrized by arc length. If we choose $s \in]-T, T[$ and define local coordinates around $\zeta(s)$ such as described in Theorem 5.2.1, then*

$$h_\zeta(s) = 2A\ddot{z}(s) + \mathfrak{h}\left(\mathsf{T}\left(\zeta(s)\right)\right),$$

where $\mathfrak{h}$ is a smooth function whose explicit expression is given in Appendix B and where $A \in \mathbb{R}$ is defined in Proposition 5.2.3.

We postpone the proof of this proposition to Appendix B.

5.2.2 Characteristic deviation and geodesic deviation

Theorem 5.2.5. *We consider (M, ω, g) a contact sub-Riemannian manifold and $\zeta :]-T, T[\to M$ a smooth horizontal curve parametrized by arc length. For every s in $]-T, T[$, there exists I an open interval that contains s such that for $t \in I$, $\zeta(t)$ belongs to Σ_s and we have*

$$h_{0,s}\left(\zeta(t)\right) \overset{t \to s}{\longrightarrow} h_\zeta\left(s\right).$$

Proof. The idea is to adapt the proof of Proposition 4.4.7 that is valid in dimension three to make it work in an arbitrary contact space.

To do this, we must identify what in the previous proof cannot be directly generalized to an arbitrary dimension. The main difference is the proposition we apply to compute the characteristic deviation of a smooth curve ζ parametrized by arc length in terms of normal coordinates. Indeed Proposition 4.4.3 tells us that the characteristic deviation of such a curve ζ in dimension three only depends on the evolution of its z coordinate while in higher dimension, the formula given in Proposition 5.2.4 to compute the

characteristic deviation of a curve also contains a term that depends on the initial direction of that curve.

Moreover, we notice that the only object in the proof of 4.4.7 that depends on Proposition 4.4.3 is $\mathfrak{R}_{\lambda_1,\lambda_2}$ so that is the object that we must redefine here.

In fact, in the general contact case, we can do the exact same proof as the proof in 3D of Proposition 4.4.7 by simply replacing $\mathfrak{R}_{\lambda_1,\lambda_2}$ by the set

$$\mathfrak{Q}_{\lambda_1,\lambda_2} = \left\{ (x_i, y_i, z) \in M \mid \lambda_1 \geqslant \frac{12Az}{\left(\sum_i x_i^2 + y_i^2\right)^{\frac{3}{2}}} - \mathfrak{h}\left(V\left(x_i, y_i\right)\right) \geqslant \lambda_2 \right\},$$

where $\mathfrak{h}$ is the function defined in Proposition 5.2.4 and where $V\left(x_i, y_i\right)$ represents the horizontal vector of norm one that is tangent to the origin of the normal coordinates and that points in the direction of (x_i, y_i).

□

5.3 Second differential of the squared distance from a point

We have set to study the differential equation in Proposition 5.1.3 that is satisfied along $\zeta :]-T, T[\to M$ a smooth horizontal curve parametrized by arc length. In order to do this, we need to understand the asymptotics of fields of the form

$$\nabla_X \Gamma_s\left(\zeta(t)\right)$$

when t goes to s.

We start with a lemma that links the quantity we are focusing on to the second differential of the squared distance from $\zeta(s)$.

Lemma 5.3.1. *If X and Y are two smooth vector fields on Σ_s, then*

$$\tilde{\delta}_s g\left(\nabla_X \Gamma_s, Y\right) = \frac{1}{2} XY\tilde{\delta}_s^2 - \left(X\tilde{\delta}_s\right)\left(Y\tilde{\delta}_s\right) - \tilde{\delta}_s g\left(\Gamma_s, \nabla_X Y\right).$$

Proof. We write

$$\begin{aligned}
\tilde{\delta}_s g\left(\nabla_X \Gamma_s, Y\right) &= \tilde{\delta}_s X g\left(\Gamma, Y\right) - \tilde{\delta}_s g\left(\Gamma_s, \nabla_X Y\right) \\
&= \tilde{\delta}_s XY\tilde{\delta}_s - \tilde{\delta}_s g\left(\Gamma_s, \nabla_X Y\right) \\
&= \frac{1}{2} XY\tilde{\delta}_s^2 - \left(X\tilde{\delta}_s\right)\left(Y\tilde{\delta}_s\right) - \tilde{\delta}_s g\left(\Gamma_s, \nabla_X Y\right).
\end{aligned}$$

□

In order to use the previous lemma to compute the asymptotics of $\nabla_X \Gamma_s$, we must understand the term $XY\tilde{\delta}_s^2$. The second differential of the distance from a point has already been studied in [ABR17] and [ABR18] but in a somewhat different setting than ours. We adapt the proof of the precited papers to our own framework. In order to do this, we must introduce the normal moving frame along the Hamiltonian lift of a geodesic that was introduced by Zelenko and Li [ZL09].

Theorem 5.3.2. *[ZL09] Let us consider M a $(2n+1)-$dimensional contact sub-Riemannian manifold and $\overline{\gamma} : \mathbb{R} \to T^*M$ an integral line of the Hamiltonian vector field $\vec{H}$. Along the trajectory of $\overline{\gamma}$ there exist $(E_i, F_i)_{i=0}^{2n}$ a smooth frame of T^*M and a one-parameter family of matrices $R(t)$ that smoothly depend on t such that*

$$\pi_* (E_i) = 0, \ \sigma (E_i, F_j) - \delta_{i,j} = \sigma (E_i, E_j) = \sigma (F_i, F_j) = 0,$$

and

$$\begin{cases} \mathcal{L}_{\vec{H}} E_0 (\overline{\gamma}(t)) &= E_1 (\overline{\gamma}(t)), \\ \mathcal{L}_{\vec{H}} E_i (\overline{\gamma}(t)) &= -F_i (\overline{\gamma}(t)) & \text{for } i \geqslant 1, \\ \mathcal{L}_{\vec{H}} F_1 (\overline{\gamma}(t)) &= \sum_{j=0}^{2n} R_{1,j}(t) E_j (\overline{\gamma}(t)) - F_0 (\overline{\gamma}(t)), \\ \mathcal{L}_{\vec{H}} F_i (\overline{\gamma}(t)) &= \sum_{j=0}^{2n} R_{i,j}(t) E_j (\overline{\gamma}(t)) & \text{for } i \neq 1. \end{cases}$$

Such a frame $(E_i, F_i)_{i=0}^{2n}$ is called a normal moving frame.

Moreover, if $(E_i, F_i)_{i=0}^{2n}$ is a normal moving frame along $\overline{\gamma}$, any other frame $\left(\tilde{E}_i, \tilde{F}_i\right)_{i=0}^{2n}$ is a normal moving frame if and only if there exists O a constant orthogonal matrix such that

$$\left(\tilde{E}_i, \tilde{F}_i\right)_{i=2}^{2n} = (OE_i, OF_i)_{i=2}^{2n}.$$

The geometric interpretation of a normal moving frame is made easier by the following lemma.

Lemma 5.3.3. *Inside a contact sub-Riemannian space M, let us consider $\gamma : \mathbb{R} \to M$ a geodesic parametrized by arc length that corresponds to the projection of $\overline{\gamma} : \mathbb{R} \to T^*M$. For $(E_i, F_j)_{i=0}^{2n}$ a normal moving frame, we have that*

$$(\pi_* (F_i (\overline{\gamma}(t))))_{i=1}^{2n}$$

is an orthonormal frame of the distribution evaluated at $\gamma(t)$.

Moreover, $\pi_* \left(F_1 \left(\overline{\zeta}(t)\right)\right)$ is colinear to $J\gamma'(t)$.

Furthermore, it is possible to choose a normal moving frame such that F_2 coincides with $\vec{H}$. In this case, for every $t \in \mathbb{R}$, $\pi_* \left(F_2 \left(\overline{\zeta}(t)\right)\right)$ is equal to $\gamma'(t)$.

Proof. The fact that

$$\left(\pi_*\left(F_i\left(\overline{\gamma}(t)\right)\right)\right)_{i=1}^{2n}$$

is an orthonormal frame of the distribution is proved in [ABR18, Lemma 7.9].

We still need to prove that $\pi_*\left(F_1\left(\overline{\gamma}(t)\right)\right)$ is colinear to $J\gamma'(t)$. Let us choose s close enough to t so that $\gamma(s)$ is a smooth point around $\gamma(t)$, which is possible thanks to Corollary 3.2.6. By symmetry with respect to q_1 and q_2 two points of the definition of "q_1 is a smooth point around q_2", we can say that $\gamma(t)$ is a smooth point around $\gamma(s)$. As a consequence next to $\gamma(t)$ we can consider $\mathbf{G}$, the field representing the velocity of the minimizing geodesic parametrized by arc length that leaves from $\gamma(s)$ and reaches the point we are considering.

We now can write, for $i \in [\![1, 2n]\!]$,

$$\begin{aligned} g_{\gamma(t)}\left(\pi_* F_i, J\gamma'\right) &= \mathrm{d}\omega_{\gamma(t)}\left(\pi_* F_i, \gamma'\right) \\ &= \omega\left(\mathcal{L}_{\mathbf{G}} \pi_* F_i\right) \text{ by Cartan and since } \mathbf{G} \text{ and } \pi_* F_i \in \ker\omega. \\ &= \omega\left(\pi_* \mathcal{L}_{\vec{H}} F_i\right) \text{ as a consequence of Theorem 3.4.3.} \\ &= \begin{cases} 0 & \text{if } i \in [\![2, 2n]\!], \\ -\omega\left(F_0\left(\gamma(t)\right)\right) & \text{if } i = 1. \end{cases} \end{aligned} \tag{5.2}$$

Since we already know that $\left(\pi_*\left(F_i\left(\overline{\gamma}(t)\right)\right)\right)_{i=1}^{2n}$ is orthonormal, the only possibility for (5.2) to be satisfied is that $\pi_*\left(F_1\left(\overline{\gamma}(t)\right)\right)$ is colinear to $J\gamma'(t)$.

The fact that there exists a normal moving frame along $\overline{\gamma}$ such that F_2 coincides with the Hamiltonian vector field $\vec{H}$ can be derived from the computations in [ABR18, Section 7.5.4], not only in the three dimensional case, but more generally in the contact case. □

The notion of normal moving frame along the Hamiltonian lift of a geodesic is used to prove the following. Since the proof of this next result is quite technical and is not one of the main goals of this thesis, we postpone it to Appendix C.

Proposition 5.3.4. *Let M be a contact sub-Riemannian space and $\zeta :]-T, T[\to M$ be a smooth horizontal curve parametrized by arc length. In an orthonormal frame of the distribution whose two first elements are the fields $\left(\Gamma_s, \frac{J\Gamma_s}{\|J\Gamma_s\|}\right)$, the matrix of the bilinear form*

$$\begin{array}{ccc} T_{\zeta(t)}M \times T_{\zeta(t)}M & \longrightarrow & \mathbb{R} \\ (X, Y) & \longmapsto & g_{\zeta(t)}\left(\nabla_X \Gamma_s, Y\right) \end{array}$$

is equal to

$$\left(\begin{array}{cc|ccc} 0 & 0 & & & \\ 0 & -\frac{4}{\delta_s(t)} & & (0) & \\ \hline & & -\frac{1}{\delta_s(t)} & & (0) \\ & (0) & & \ddots & \\ & & (0) & & -\frac{1}{\delta_s(t)} \end{array}\right) + \mathcal{O}(1).$$

5.4 Asymptotics of the distance between two close points along a horizontal curve

We start this section by writing the simplest form of asymptotics of the distance between two points along a horizontal curve parametrized by arc length. That corresponds to the result we can find in [Mon01, Theorem 1.3.5].

Proposition 5.4.1. *[Mon01, Theorem 1.3.5]. Let M be a contact sub-Riemannian manifold and $\zeta :]-T,T[\to M$ be a smooth horizontal curve parametrized by arc length. For any $s \in]-T,T[$ we have*

$$\frac{\delta_s(t)}{t-s} \overset{t\to s^+}{\longrightarrow} 1.$$

We now have all the tools to understand the equation in Proposition 5.1.3.

Lemma 5.4.2. *Let M be a contact sub-Riemannian space and $\zeta :]-T,T[\to M$ be a smooth horizontal curve parametrized by arc length. For any $s \in]-T,T[$ we have*

$$\dot{\delta}_s(t) \overset{t\to s}{\longrightarrow} 1, \quad \Gamma_s(\zeta(t)) \overset{t\to s}{\longrightarrow} \mathsf{T}(\zeta(s)) \text{ and } \|\mathscr{S}_s(t)\| = \mathcal{O}(t-s).$$

Proof. In this proof, we only consider times greater than s. The proof for times smaller than s is similar.

In terms of notations, let us write T to stand for a normalized horizontal vector field extending the velocity of ζ, and let us consider $(\Gamma_s, X_2, ..., X_{2n})$ an orthonormal frame of the distribution along ζ. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\mathscr{S}_s(t)\| =& \frac{1}{2\|\mathscr{S}_s(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathscr{S}_s(t)\|^2, \\ =& g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathscr{S}_s, \frac{\mathscr{S}_s(t)}{\|\mathscr{S}_s(t)\|}\right) \text{ since } \nabla \text{ is compatible with } g, \\ =& -\dot{\delta}_s(t)\sum_{i,j=2}^{2n} g_{\zeta(t)}\left(\nabla_{X_j}\Gamma_s, X_i\right) g_{\zeta(t)}\left(\mathscr{S}_s, X_j\right) g_{\zeta(t)}\left(\frac{\mathscr{S}_s}{\|\mathscr{S}_s\|}, X_i\right) \\ &+ g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(t)h_{0,s}\frac{J\Gamma_s}{\|J\Gamma_s\|^2}, \frac{\mathscr{S}_s}{\|\mathscr{S}_s\|}\right), \end{aligned} \tag{5.3}$$

where the last line of the previous identity comes from Proposition 5.1.3 and Corollary 3.4.7.

Now the term

$$\nabla_{\mathsf{T}(\zeta(t))}\mathsf{T} - \dot{\delta}_s^2(t) h_{0,s}\left(\zeta(t)\right) \frac{J\Gamma_s\left(\zeta(t)\right)}{\left\|J\Gamma_s\left(\zeta(t)\right)\right\|^2} \tag{5.4}$$

that appears in (5.3) is bounded for t near s since T is smooth along ζ, Γ_s has norm one and since $h_{0,s}\left(\zeta(t)\right) \xrightarrow{t\to s} h_\zeta(s)$, according to Theorem 5.2.5.

Equation (5.3) combined with the boundedness of (5.4) and Proposition 5.3.4 allows us to state that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathscr{S}_s(t)\right\| = -\frac{\dot{\delta}_s(t)}{\delta_s(t)}\left(\left\|\mathscr{S}_s(t)\right\| + \frac{3}{\left\|\mathscr{S}_s\right\|} g_{\zeta(s)}\left(\frac{J\Gamma}{\|J\Gamma\|}, \mathscr{S}_s\right)^2\right) + \mathcal{O}(1). \tag{5.5}$$

Now we notice that according to (5.1), for $t \in]-T, s[\cup]s, T[$,

$$1 = \dot{\delta}_s^2(t) + \left\|\mathscr{S}_s(t)\right\|^2. \tag{5.6}$$

As a consequence, another formulation of (5.5) is

$$\frac{\ddot{\delta}_s(t)\dot{\delta}_s(t)}{\sqrt{1-\dot{\delta}_s^2(t)}} = \frac{\dot{\delta}_s(t)}{\delta_s(t)}\left(\sqrt{1-\dot{\delta}_s^2(t)} + \frac{3}{\left\|\mathscr{S}_s\right\|} g_{\zeta(s)}\left(\frac{J\Gamma}{\|J\Gamma\|}, \mathscr{S}_s\right)^2\right) + \mathcal{O}(1). \tag{5.7}$$

Let us assume that $\dot{\delta}_s(t)$ does not tend to one as t tends to s^+ in order to obtain a contradiction.

Let us first notice that $\dot{\delta}_s(t)$ cannot tend to -1 as t tends to s^+ or by integration we would obtain that δ_s would assume negative values, which is impossible for a distance. The assumption that $\dot{\delta}_s(t)$ does not tend to one as t tends to s^+ is therefore equivalent to saying that $\left|\dot{\delta}_s(t)\right|$ does not tend to one as t tends to s^+. This means that there exists $\varepsilon > 0$ such that for any $\eta > 0$ there exists $t \in]s, s+\eta[$ such that

$$\left|\dot{\delta}_s(t)\right| \leqslant 1 - \varepsilon. \tag{5.8}$$

Now for times $t > s$ that satisfy condition (5.8), since $\delta_s(t) = \operatorname{dist}\left(\zeta(s), \zeta(t)\right) \leqslant t - s$ we have

$$\begin{aligned}\frac{1}{\delta_s(t)}\left(\sqrt{1-\dot{\delta}_s^2(t)} + \frac{3}{\left\|\mathscr{S}_s\right\|} g_{\zeta(s)}\left(\frac{J\Gamma}{\|J\Gamma\|}, \mathscr{S}_s\right)^2\right) &\geqslant \frac{\sqrt{1-\dot{\delta}_s^2(t)}}{\delta_s(t)} \\ &\geqslant \frac{\sqrt{2\varepsilon - \varepsilon^2}}{t-s}.\end{aligned}$$

As a consequence, for $t_* > s$ that satisfies condition (5.8) close enough to s, the term $\mathcal{O}(1)$ in (5.7) can be neglected in comparison with the other term at the right hand side of the identity (5.7) and for such a t_*,

$$\ddot{\delta}_s(t_*) > 0.$$

This implies that for such a t_*, $\dot{\delta}_s(t)$ is "trapped" in the interval $[-1, 1-\varepsilon]$ for $s < t \leqslant t_*$. By integrating the inequality $\dot{\delta}_s(t) \leqslant 1-\varepsilon$ for $t \leqslant t_*$ we obtain

$$\delta_s(t) \leqslant (1-\varepsilon)(t-s). \tag{5.9}$$

But (5.9) cannot occur, since according to Proposition 5.4.1.

$$\frac{\delta_s(t)}{t-s} \overset{t\to s^+}{\longrightarrow} 1.$$

We obtain a contradiction and our initial assumption that $\dot{\delta}_s(t)$ does not tend to one as t tends to s^+ is therefore disproved. By using (5.6) we obtain

$$\dot{\delta}_s(t) \overset{t\to s}{\longrightarrow} 1 \text{ and } \|\mathscr{S}_s(t)\| \overset{t\to s}{\longrightarrow} 0. \tag{5.10}$$

The convergence of $\Gamma_s(\zeta(t))$ to $\mathsf{T}(\zeta(s))$ as t goes to s is then a consequence of Proposition 5.1.2 and of (5.10).

We still need to prove that for $s \in]-T, T[$, $\|\mathscr{S}_s(t)\| = \mathcal{O}(t-s)$. To do so, we come back to (5.3), from which we deduce by using Proposition 5.3.4, that for t close enough to s to have $\dot{\delta}_s(t) \geqslant 0$, the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathscr{S}_s(t)\| \leqslant \frac{\dot{\delta}_s(t)}{\delta_s(t)}\|\mathscr{S}_s(t)\| + g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(t)h_{0,s}\frac{J\Gamma_s}{\|J\Gamma_s\|^2}, \frac{\mathscr{S}_s}{\|\mathscr{S}_s\|}\right) + \mathcal{O}(\|\mathscr{S}_s(t)\|).$$

The previous identity can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta_s(t)\|\mathscr{S}_s(t)\|) \leqslant \delta_s(t) g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(t)h_{0,s}\frac{J\Gamma_s}{\|J\Gamma_s\|^2}, \frac{\mathscr{S}_s}{\|\mathscr{S}_s\|}\right) + \mathcal{O}(\delta_s(t)\|\mathscr{S}_s(t)\|). \tag{5.11}$$

Now since $\|\mathscr{S}_s\| \leqslant 1$, we have $\delta_s(t)\|\mathscr{S}_s(t)\| \overset{t\to s}{\longrightarrow} 0$. As a consequence, by integrating (5.11) between time s and time t we find out that

$$\delta_s(t)\|\mathscr{S}_s(t)\| \leqslant \int_s^t \delta_s(u) g_{\zeta(u)}\left(\nabla_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(u)h_{0,s}\frac{J\Gamma_s}{\|J\Gamma_s\|^2}, \frac{\mathscr{S}_s}{\|\mathscr{S}_s\|}\right)\mathrm{d}u + \int_s^t \mathcal{O}(\delta_s(u)\|\mathscr{S}_s(u)\|)\,\mathrm{d}u. \tag{5.12}$$

We already noticed during this proof that the term $\nabla_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(u)h_{0,s}J\Gamma_s$ is bounded, so by applying Grönwall's Lemma to (5.12) we obtain

$$\delta_s(t)\left\|\mathscr{S}_s(t)\right\| \leqslant \int_s^t \delta_s(u)g_{\zeta(u)}\left(\nabla_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(u)h_{0,s}\frac{J\Gamma_s}{\|J\Gamma_s\|^2}, \frac{\mathscr{S}_s}{\|\mathscr{S}_s\|}\right)\mathrm{d}u + \int_s^t u^2\mathrm{d}u.$$

By once again using the boundedness of $\nabla_{\mathsf{T}}\mathsf{T} - \dot{\delta}_s^2(u)h_{0,s}\frac{J\Gamma_s}{\|J\Gamma_s\|^2}$ and the fact that $\delta_s(t) \sim t - s$ we deduce that

$$\left\|\mathscr{S}_s(t)\right\| = \mathcal{O}\left(t - s\right).$$

□

We carry on our study of the differential equation that appears in Proposition 5.1.3 by studying the frame in which the identity is written.

Lemma 5.4.3. *Let M be a contact sub-Riemannian manifold, and $\zeta :]-T, T[\to M$ be a smooth horizontal curve parametrized by arc length. Along ζ, we consider a smooth orthonormal frame of the distribution*

$$\left(\mathsf{T}, \frac{J\mathsf{T}}{\|J\mathsf{T}\|}, \tilde{X}_3, \tilde{X}_4, ..., \tilde{X}_{2n}\right).$$

By applying the Gram-Schmidt process from left to right to

$$\left(\Gamma_s, \frac{J\Gamma_s}{\|J\Gamma_s\|}, \tilde{X}_3, \tilde{X}_4, ..., \tilde{X}_{2n}\right),$$

we build a new frame

$$(X_i)_{1=1}^{2n}.$$

This new frame, which satisfies $X_1 = \Gamma_s$, $X_2 = \frac{J\Gamma_s}{\|J\Gamma_s\|}$ is continuous at $\zeta(s)$. Moreover, for $i \in [\![1, 2n]\!]$ we have,

$$\nabla_{\mathsf{T}(\zeta(t))}X_i$$

is bounded as t tends to s.

Proof. The first fact we point out is that we know from Lemma 5.4.2 that

$$\Gamma_s\left(\zeta(t)\right) \overset{t\to s}{\longrightarrow} \mathsf{T}\left(\zeta(s)\right) \tag{5.13}$$

As a consequence, since

$$\left(\mathsf{T}, \frac{J\mathsf{T}}{\|J\mathsf{T}\|}, \tilde{X}_3, \tilde{X}_4, ..., \tilde{X}_{2n}\right).$$

is chosen to be a frame of the distribution along ζ, then for times t close enough to s,

$$\left(\Gamma_s\left(\zeta(t)\right), \frac{J\Gamma_s}{\|J\Gamma_s\|}\left(\zeta(t)\right), \tilde{X}_3\left(\zeta(t)\right), ..., \tilde{X}_{2n}\left(\zeta(t)\right)\right),$$

is also a frame of the distribution. It therefore makes sense to apply the Gram-Schmidt algorithm to the previous family. We add that (5.13) also means that the family to which we apply the Gram-Schmidt algorithm is continuous, which entails that the family $(X_i)_{i=1}^{2n}$ is also continuous.

Now along ζ

$$\begin{aligned}\nabla_{\mathsf{T}(\zeta(t))}\Gamma_s &= \dot{\delta}_s(t)\nabla_{\Gamma_s(\zeta(t))}\Gamma_s + \nabla_{\mathscr{S}_s(t)}\Gamma_s \text{ according to Proposition 5.1.2,}\\ &= \dot{\delta}_s(t)h_{0,s}\left(\zeta(t)\right)\frac{J\Gamma_s\left(\zeta(t)\right)}{\|J\Gamma_s\left(\zeta(t)\right)\|^2} + \nabla_{\mathscr{S}_s(t)}\Gamma_s \text{ by Corollary 3.4.7.}\end{aligned}$$

The expression of $\nabla_{\mathsf{T}(\zeta(t))}\Gamma_s$ that we just obtained, combined with Theorem 5.2.5, Proposition 5.3.4 and Lemma 5.4.2, shows that $\nabla_{\mathsf{T}(\zeta(t))}\Gamma_s$ is bounded as t tends to s.

By definition of the Gram-Schmidt process, all the vector fields X_i have an explicit expression in terms of Γ_s, $J\Gamma_s$ and the vector fields $\tilde{X}_j$. By using these expressions and the Tanno tensor to compute $\nabla_{\mathsf{T}(\zeta(t))}X_i$ and since we already proved that $\nabla_{\mathsf{T}(\zeta(t))}\Gamma_s$ is bounded as t tends to s, we can conclude that all the fields $\nabla_{\mathsf{T}(\zeta(t))}X_i$ are bounded as t tends to s. □

The reason why the previous Lemma is relevant is that it provides us with a frame where we can rewrite Proposition 5.1.3.

Lemma 5.4.4. *Let M be a contact sub-Riemannian manifold and $\zeta :]-T, T[\to M$ be a smooth horizontal curve parametrized by arc length. We denote by T a normalized horizontal extension of the velocity of ζ. We also consider along ζ an orthonormal frame of the distribution*

$$\left(\Gamma_s, \frac{J\Gamma_s}{\|J\Gamma_s\|}, X_3, ..., X_{2n}\right)$$

such as described in Lemma 5.4.3.

We have

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_s^4(t)g_{\zeta(t)}\left(\mathscr{S}_s, \frac{J\Gamma_s}{\|J\Gamma_s\|}\right)\right) &= \delta_s^4(t)\left(\frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|} - \frac{h_{0,s}(\zeta(t))}{\|J\Gamma_s(\zeta(t))\|}\right)\\ &\quad + \mathcal{O}\left(\delta_s^4(t)\left\|\mathscr{S}_s(t)\right\|\right),\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_s(t)g_{\zeta(t)}\left(\mathscr{S}_s, X_i\right)\right) &= \delta_s(t)g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T}, X_i\right) + \mathcal{O}\left(\delta_s(t)\left\|\mathscr{S}_s(t)\right\|\right), \end{cases}$$

where i belongs to $[\![3, 2n]\!]$.

Proof. For $i \in [\![2, 2n]\!]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} g_{\zeta(t)}\left(\mathscr{S}_s, X_i\right) &= g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathscr{S}_s, X_i\right) - g_{\zeta(t)}\left(\mathscr{S}_s, \nabla_{\mathsf{T}} X_i\right) \\
&= g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathscr{S}_s, X_i\right) + \mathcal{O}\left(\|\mathscr{S}_s(t)\|\right) \text{ by Lemma 5.4.3.} \\
&= -\dot{\delta}_s(t) \sum_{j=2}^{m} g_{\zeta(t)}\left(\nabla_{X_j} \Gamma_s, X_i\right) g_{\zeta(t)}\left(\mathscr{S}_s, X_j\right) + \mathcal{O}\left(\|\mathscr{S}_s(t)\|\right) \\
&+ g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathsf{T} - \dot{\delta}_s^2(t) \nabla_{\Gamma_s} \Gamma_s, X_i\right) \text{ by Proposition 5.1.3.}
\end{aligned}$$

We now use Proposition 5.3.4 to carry on our computation

$$\left\{\begin{array}{rcl}
\frac{\mathrm{d}}{\mathrm{d}t} g_{\zeta(t)}\left(\mathscr{S}_s, \frac{J\Gamma_s}{\|J\Gamma_s\|}\right) &=& -4\frac{\dot{\delta}_s(t)}{\delta_s(t)} g_{\zeta(t)}\left(\mathscr{S}_s, \frac{J\Gamma_s}{\|J\Gamma_s\|}\right) + \mathcal{O}\left(\|\mathscr{S}_s(t)\|\right) \\
& & + g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathsf{T} - \dot{\delta}_s^2(t) \nabla_{\Gamma_s} \Gamma_s, \frac{J\Gamma_s}{\|J\Gamma_s\|}\right), \\
\frac{\mathrm{d}}{\mathrm{d}t} g_{\zeta(t)}\left(\mathscr{S}_s, X_i\right) &=& -\frac{\dot{\delta}_s(t)}{\delta_s(t)} g_{\zeta(t)}\left(\mathscr{S}_s, X_i\right) + \mathcal{O}\left(\|\mathscr{S}_s(t)\|\right) \\
& & + g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathsf{T} - \dot{\delta}_s^2(t) \nabla_{\Gamma_s} \Gamma_s, X_i\right) \text{ for } i \in [\![3, 2n]\!].
\end{array}\right. \tag{5.14}$$

To transform this system let us point out several facts. First of all, according to (5.1) we have

$$\dot{\delta}_s(t) = 1 + \mathcal{O}\left(\|\mathscr{S}_s(t)\|^2\right). \tag{5.15}$$

Then we write

$$\begin{aligned}
g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathsf{T}, \frac{J\Gamma_s}{\|J\Gamma_s\|}\right) &= g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathsf{T}, \frac{J\left(\dot{\delta}_s(t)\Gamma_s\right)}{\left\|J\left(\dot{\delta}_s(t)\Gamma_s\right)\right\|}\right) \\
&= g_{\zeta(t)}\left(\nabla_{\mathsf{T}} \mathsf{T}, \frac{J\mathsf{T}}{\|J\mathsf{T}\|}\right) + \mathcal{O}\left(\|\mathscr{S}_s(t)\|\right) \text{ by Prop. 5.1.2,} \\
&= \frac{h_\zeta(t)}{\|J\mathsf{T}\left(\zeta(t)\right)\|} + \mathcal{O}\left(\|\mathscr{S}_s(t)\|\right) \text{ by Definition 3.3.2.}
\end{aligned} \tag{5.16}$$

To obtain the result that we stated in the proposition we are focusing on, we start by considering system (5.14). We multiply its first equation by $\delta_s^4(t)$ and we multiply the family of equations indexed on $i \in [\![3, 2n]\!]$ by $\delta_s(t)$. To the equations we just found, we apply (5.15), (5.16), Corollary 3.4.7 and the boundedness of $h_{0,s}\left(\zeta(t)\right)$ that comes from Theorem 5.2.5 and we reach the conclusion we were looking for. □

We are ready to state a theorem.

Theorem (1.1.8). *Let M be a contact sub-Riemannian manifold. If $\zeta :]-T,T[\to M$ is a smooth horizontal curve parametrized by arc length then for every $s\in]-T,T[$ and $t\in]s,T[$,*

$$\delta_s(t)=(t-s)-\frac{k_{\zeta,1}^2(s)}{12}(t-s)^3+o\left((t-s)^3\right),$$

where we recall that $k_{\zeta,1}$ is the first geodesic curvature introduced in Definition 3.4.5.

Proof. We start by pointing out that according to Theorem 5.2.5 and Lemma 5.4.2,

$$\frac{h_\zeta(t)}{\|J\mathsf{T}(\zeta(t))\|}-\frac{h_{0,s}(\zeta(t))}{\|J\Gamma_s(\zeta(t))\|}\xrightarrow{s\to t}0 \tag{5.17}$$

We deduce from Lemma 5.4.4 combined with (5.17) and Lemma 5.4.2 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_s^4(t)g_{\zeta(t)}\left(\mathscr{S}_s,\frac{J\Gamma_s}{\|J\Gamma_s\|}\right)\right)=o\left(\delta_s^4(t)\right) \tag{5.18}$$

We recall that according to Proposition 5.4.1, $\delta_s(t)\overset{t\to s}{\sim}t-s$ so by integrating (5.18) between s and t, we obtain

$$g_{\zeta(t)}\left(\mathscr{S}_s,\frac{J\Gamma_s}{\|J\Gamma_s\|}\right)=o\left(t-s\right). \tag{5.19}$$

Let us now consider $(X_i)_{i=1}^{2n}$ an orthonormal frame of the distribution such as described in Lemma 5.4.3. We insist on the fact that according to the lemma where the frame $(X_i)_{i=1}^{2n}$ is introduced, this frame is continuous at $\zeta(s)$. We use Lemma 5.4.2 to deduce from Lemma 5.4.4 that for $i\in[\![3,2n]\!]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_s(t)g_{\zeta(t)}\left(\mathscr{S}_s,X_i\right)\right)=(t-s)\,g\left(\nabla_{\mathsf{T}(\zeta(s))}\mathsf{T},X_i(\zeta(s))\right)+o\left((t-s)\right).$$

By integrating the previous Taylor expansion bewteen s and t we deduce that

$$g_{\zeta(t)}\left(\mathscr{S}_s,X_i\right)=\frac{t-s}{2}g\left(\nabla_{\mathsf{T}(\zeta(s))}\mathsf{T},X_i(\zeta(s))\right)+o\left((t-s)\right). \tag{5.20}$$

We summarize (5.19) and (5.20) by saying that

$$\mathscr{S}_s\left(t\right)=\frac{t-s}{2}\Pi^\perp_{J\mathsf{T}(\zeta(s))}\left(\nabla_{\mathsf{T}(\zeta(s))}\mathsf{T}\right)+o\left((t-s)\right),$$

where $\Pi^\perp_{J\mathsf{T}(\zeta(s))}$ stands for the orthogonal projection on $\langle J\mathsf{T}\left(\zeta(s)\right)\rangle^\perp$. This last identity allows us to conclude by applying (5.1) and Definition 3.4.5. □

We now go deeper in the expansion of the distance between two close points of a horizontal curve ζ parametrized by arc length. More precisely we focus on the case where the first corrective term in the expansion of the distance between $\zeta(s)$ and $\zeta(t)$, that we computed in the previous theorem, vanishes all along ζ. To do this without getting lost in computations, we assume that $J^2 = -\mathrm{Id}$.

Theorem (1.1.9)**.** *Let M be a contact sub-Riemannian manifold such that $J^2 = -\mathrm{Id}$. If $\zeta :]-T, T[\to M$ is a smooth horizontal curve parametrized by arc length such that for every $s \in]-T, T[$ and $t > s$,*

$$\dot{\delta}_s(t) = 1 - K(s)\,(t-s)^4 + o\left((t-s)^4\right),$$

then

$$K(s) = \frac{k_{\zeta,2}^2(s)}{288},$$

where we recall that $k_{\zeta,2}$ is the second geodesic curvature introduced in Definition 3.4.5.

Proof. Under the hypotheses we made, for every $s \in]-T, T[$,

$$\delta_s(t) = (t-s) + o\left((t-s)^3\right),$$

so using Theorem 1.1.8 (previously proved in section 5.4) we deduce that $\nabla_{\mathsf{T}(\zeta(s))}\mathsf{T}$ is colinear to $J\mathsf{T}(\zeta(s))$. Since we assumed that $J^2 = -\mathrm{Id}$, J is an isometry. So by Definition 3.3.2 we have for every $s \in]-T, T[$,

$$\nabla_{\mathsf{T}(\zeta(s))}\mathsf{T} = h_\zeta(s) J\mathsf{T}(\zeta(s)). \tag{5.21}$$

Now if we consider $(X_i)_{i=1}^{2n}$ an orthonormal frame of the distribution such as defined in Lemma 5.4.3, we recall that by defintion, along ζ, $X_2 = J\Gamma_s$. So for $i \in [\![3, 2n]\!]$ we can write

$$\begin{aligned} g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T}, X_i\right) &= g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T} - h_\zeta(t)\dot{\delta}_s(t) J\Gamma_s, X_i\right) \\ &= g_{\zeta(t)}\left(h_\zeta(t) J\left(\mathsf{T} - \dot{\delta}_s(t)\Gamma_s\right), X_i\right) \text{ by (5.21)}, \\ &= g_{\zeta(t)}\left(h_\zeta(t) J\mathscr{S}_s), X_i\right) \text{ by applying Proposition 5.1.2.} \end{aligned}$$

It follows that for $i \in [\![3, 2n]\!]$,

$$g_{\zeta(t)}\left(\nabla_{\mathsf{T}}\mathsf{T}, X_i\right) = \mathcal{O}\left(\mathscr{S}_s(t)\right).$$

As a consequence we can apply Lemma 5.4.4:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_s(t) g_{\zeta(t)}\left(\mathscr{S}_s, X_i\right)\right) &= \mathcal{O}\left(\delta_s(t)\left\|\mathscr{S}_s(t)\right\|\right) \\ &= \mathcal{O}\left(\delta_s(t)\left\|\sqrt{1-\dot{\delta}_s^2(t)}\right\|\right) \text{ according to (5.1)}, \\ &= \mathcal{O}\left((t-s)^3\right),\end{aligned} \tag{5.22}$$

where the last line comes from the fact that $\delta_s(t) \sim t-s$ and from the hypothesis concerning the Taylor expansion of $\dot{\delta}_s$. We integrate (5.22) between times s and t and we find out that for $i \in [\![3, 2n]\!]$,

$$g_{\zeta(t)}\left(\mathscr{S}_s, X_i\right) = \mathcal{O}\left((t-s)^3\right). \tag{5.23}$$

But according to (5.1) and the hypothesis on the expansion of $\dot{\delta}_s$,

$$\left\|\mathscr{S}_s(t)\right\| = \sqrt{1-\dot{\delta}_s^2(t)} = \sqrt{2K(s)}\,(t-s)^2 + o\left((t-s)^2\right). \tag{5.24}$$

If we decompose $\mathscr{S}_s(t) = \sum_{i=2}^{2n} g_{\zeta(t)}\left(\mathscr{S}_s(t), X_i\right) X_i$ and take into account both (5.23) and (5.24), we are able to quantify the asymptotics of $\mathscr{S}_s(t)$ in the direction of X_2. Now X_2 is equal to $J\Gamma_s$ because $J\Gamma_s$ has norm one since $J^2 = -\mathrm{Id}$. We end up with

$$g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right) = \pm\sqrt{2K(s)}\,(t-s)^2 + o\left((t-s)^2\right). \tag{5.25}$$

Now we rewrite the first equation of the system in Lemma (5.4.4) by taking into account that J is an isometry since $J^2 = -\mathrm{Id}$. we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_s^4(t) g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right)\right) = \delta_s^4(t)\left(h_\zeta(t) - h_{0,s}\left(\zeta(t)\right)\right) + \mathcal{O}\left(\delta_s^4(t)\left\|\mathscr{S}_s(t)\right\|\right)$$

Similarly to (5.22) and (5.23), we deduce from the previous identity that

$$g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right) = \frac{1}{\delta_s^4(t)}\int_s^t \delta_s^4(u)\left(h_\zeta(u) - h_{0,s}\left(\zeta(u)\right)\right)\mathrm{d}u + \mathcal{O}\left((t-s)^3\right). \tag{5.26}$$

We now focus more particularly on the term $h_\zeta(t) - h_{0,s}\left(\zeta(t)\right)$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(h_\zeta(t) - h_{0,s}\left(\zeta(t)\right)\right) =& \dot{h}_\zeta(t) - \left(\dot{\delta}_s(t)\Gamma_s\left(\zeta(t)\right) + \mathscr{S}_s\left(\zeta(t)\right)\right) h_{0,s} \\ =& \dot{h}_\zeta(t) + \dot{\delta}_s(t) g_{\zeta(t)}\left(\mathrm{Tor}\left(\Gamma_s, X_0\right), \Gamma_s\right) \\ & - \mathscr{S}_s\left(\zeta(t)\right) h_{0,s} \text{ by applying Corollary 3.4.7.}\end{aligned} \tag{5.27}$$

We need to compute $\mathscr{S}_s(\zeta(t))\, h_{0,s}$. Once again, since $J^2 = -\mathrm{Id}$, J is an isometry so we can apply Lemma 3.3.5 to write, for any vector $V \in T\Sigma_s$,

$$\begin{aligned} Vh_{0,s} &= VX_0\tilde{\delta}_s, \\ &= VX_0\tilde{\delta}_s - X_0(\zeta(t))\, V\tilde{\delta}_s, \end{aligned} \tag{5.28}$$

because $V \perp \Gamma_s = \mathrm{grad}\tilde{\delta}_s$, according to Proposition 5.1.2 and Theorem 3.2.3. We carry on computation (5.28),

$$\begin{aligned} Vh_{0,s} &= [V, X_0]\,\tilde{\delta}_s, \\ &= g([V, X_0], \Gamma_s) \text{ as } \Gamma_s = \mathrm{grad}\tilde{\delta}_s \text{ by Theorem 3.2.3}, \\ &= g(\nabla_V X_0 - \nabla_{X_0} V - \mathrm{Tor}(V, X_0), \Gamma_s) \\ &= -g(\nabla_{X_0} V + \mathrm{Tor}(V, X_0), \Gamma_s) \text{ as } X_0 = 0 \text{ by Theo. 3.1.3}, \\ &= -g(\nabla_{X_0} V, \Gamma_s) - g_{\zeta(t)}(\mathrm{Tor}(V, X_0), \Gamma_s) \\ &= -X_0 g(V, \Gamma_s) + g(V, \nabla_{X_0}\Gamma_s) \\ &\quad - g(\mathrm{Tor}(V, X_0), \Gamma_s) \text{ as } \nabla \text{ is compatible with } g, \\ &= g(V, \nabla_{X_0}\Gamma_s) - g(\mathrm{Tor}(V, X_0), \Gamma_s) \text{ as } V \perp \Gamma_s. \end{aligned} \tag{5.29}$$

In order to understand the term $g(V, \nabla_{X_0}\Gamma_s)$ in the previous computation, we start by writting

$$\begin{aligned} [\Gamma_s, J\Gamma_s] &= \nabla_{\Gamma_s} J\Gamma_s - \nabla_{J\Gamma_s}\Gamma_s - \mathrm{Tor}(\Gamma_s, J\Gamma_s), \\ &= \nabla_{\Gamma_s} J\Gamma_s - \nabla_{J\Gamma_s}\Gamma_s - \mathrm{d}\omega(\Gamma_s, J\Gamma_s) X_0 \text{ by Theorem 3.1.3}, \\ &= \nabla_{\Gamma_s} J\Gamma_s - \nabla_{J\Gamma_s}\Gamma_s - g(\Gamma_s, J^2\Gamma_s) X_0 \text{ by definition of } J, \\ &= \nabla_{\Gamma_s} J\Gamma_s - \nabla_{J\Gamma_s}\Gamma_s + X_0 \text{ since } J^2 = -\mathrm{Id} \text{ and } \|\Gamma_s\| = 1, \\ &= Q(\Gamma_s, J\Gamma_s) + J\nabla_{\Gamma_s}\Gamma_s - \nabla_{J\Gamma_s}\Gamma_s + X_0 \text{ by Definition 3.1.4}, \\ &= Q(\Gamma_s, J\Gamma_s) - h_{0,s}\Gamma_s - \nabla_{J\Gamma_s}\Gamma_s + X_0 \text{ by Cor. 3.4.7 as } J^2 = -\mathrm{Id}. \end{aligned} \tag{5.30}$$

From (5.30) we extract an expression of X_0 which we can use to compute

$$\begin{aligned} \nabla_{X_0}\Gamma_s &= \nabla_{[\Gamma_s, J\Gamma_s]}\Gamma_s + \nabla_{\nabla_{J\Gamma_s}\Gamma_s}\Gamma_s - \nabla_{Q(\Gamma_s, J\Gamma_s)}\Gamma_s + h_{0,s}\nabla_{\Gamma_s}\Gamma_s \\ &= \nabla_{[\Gamma_s, J\Gamma_s]}\Gamma_s + \nabla_{\nabla_{J\Gamma_s}\Gamma_s}\Gamma_s - \nabla_{Q(\Gamma_s, J\Gamma_s)}\Gamma_s + h_{0,s}^2 J\Gamma_s \text{ by Cor. 3.4.7}, \\ &= \nabla_{\Gamma_s}(\nabla_{J\Gamma_s}\Gamma_s) - \nabla_{J\Gamma_s}(\nabla_{\Gamma_s}\Gamma_s) - \mathrm{Curv}(\Gamma_s, J\Gamma_s, \Gamma_s) \\ &\quad + \nabla_{\nabla_{J\Gamma_s}\Gamma_s}\Gamma_s - \nabla_{Q(\Gamma_s, J\Gamma_s)}\Gamma_s + h_{0,s}^2 J\Gamma_s, \end{aligned}$$

where Curv represents the curvature tensor of connection ∇, that is for example used in Riemannian geometry with the Levi-Civita connection instead of the Tanno connection. We continue the previous computation by writing

that

$$\begin{aligned}\nabla_{X_0}\Gamma_s &= \nabla_{\Gamma_s}\left(\nabla_{J\Gamma_s}\Gamma_s\right) - \nabla_{J\Gamma_s}\left(h_{0,s}J\Gamma\right) - \mathrm{Curv}\left(\Gamma_s, J\Gamma_s, \Gamma_s\right)\\ &\quad + \nabla_{\nabla_{J\Gamma_s}\Gamma_s}\Gamma_s - \nabla_{Q(\Gamma_s,J\Gamma_s)}\Gamma_s + h_{0,s}^2 J\Gamma_s \text{ by Cor. 3.4.7 as } \|J\| = 1,\\ &= \nabla_{\Gamma_s}\left(\nabla_{J\Gamma_s}\Gamma_s\right) - \left(J\Gamma_s h_{0,s}\right)J\Gamma - h_{0,s}\nabla_{J\Gamma_s}J\Gamma - \mathrm{Curv}\left(\Gamma_s, J\Gamma_s, \Gamma_s\right)\\ &\quad + \nabla_{\nabla_{J\Gamma_s}\Gamma_s}\Gamma_s - \nabla_{Q(\Gamma_s,J\Gamma_s)}\Gamma_s + h_{0,s}^2 J\Gamma_s\\ &= \nabla_{\Gamma_s}\left(\nabla_{J\Gamma_s}\Gamma_s\right) - \left(J\Gamma_s h_{0,s}\right)J\Gamma - h_{0,s}J\nabla_{J\Gamma_s}\Gamma + Q\left(\Gamma_s, J\Gamma_s\right)\\ &\quad - \mathrm{Curv}\left(\Gamma_s, J\Gamma_s, \Gamma_s\right) + \nabla_{\nabla_{J\Gamma_s}\Gamma_s}\Gamma_s - \nabla_{Q(\Gamma_s,J\Gamma_s)}\Gamma_s + h_{0,s}^2 J\Gamma_s.\end{aligned}$$

We used Definition 3.1.4 to write the last line of the previous computation. For the next step of this computation, we use $(X_i)_{i=1}^{2n}$ an orthonormal frame of the distribution that comes from the process described in Lemma 5.4.3. We write

$$\begin{aligned}\nabla_{X_0}\Gamma_s &= \nabla_{\Gamma_s}\left(\nabla_{J\Gamma_s}\Gamma_s\right) - \left(J\Gamma_s h_{0,s}\right)J\Gamma - h_{0,s}J\nabla_{J\Gamma_s}\Gamma + Q\left(\Gamma_s, J\Gamma_s\right)\\ &\quad - \mathrm{Curv}\left(\Gamma_s, J\Gamma_s, \Gamma_s\right) + \sum_{i=2}^{2n} g\left(\nabla_{J\Gamma_s}\Gamma_s, X_i\right)\nabla_{X_i}\Gamma_s - \nabla_{Q(\Gamma_s,J\Gamma_s)}\Gamma_s\\ &\quad + h_{0,s}^2 J\Gamma_s.\end{aligned} \tag{5.31}$$

Now we need to understand the asymptotics along curve ζ of each of the eight terms at the right hand side of (5.31) in order to discover the asymptotics of $\nabla_{X_0(\zeta(t))}\Gamma_s$ as t goes to s. We simply explain which are the key properties that enable us to evaluate the asymptotics of each term. The major tool we use is Proposition 5.3.4. Moreover, to show that five of the eight terms at the right hand side of (5.31) are in fact negligible when $\zeta(t)$ goes to $\zeta(s)$, we notice that

i. $h_{0,s}\left(\zeta(t)\right)$ is bounded as t goes to s thanks to Theorem 5.2.5.

ii. terms $\mathrm{Curv}\left(\Gamma_s, J\Gamma_s, \Gamma_s\right)\left(\zeta(t)\right)$ and $Q\left(\Gamma_s, J\Gamma_s\right)$ are bounded for t going to s.

iii. term $Q\left(\Gamma_s, J\Gamma_s\right)$ is horizontal by Definition 3.1.4 since the differential of a horizontal vector fields with respect to the Tanno connection is horizontal by Theorem 3.1.3.

If we combine all the element we just presented with equation (5.31), we find out that the higest order terms at the right hand side of (5.31) are $\nabla_{\Gamma_s}\left(\nabla_{J\Gamma_s}\Gamma_s\right)$, $\left(J\Gamma_s h_{0,s}\right)J\Gamma$ and the term $g\left(\nabla_{J\Gamma_s}\Gamma_s, J\Gamma_s\right)\nabla_{J\Gamma_s}\Gamma_s$ in the sum. More precisely, we find out that

$$\nabla_{X_0(\zeta(t))}\Gamma_s = -\left(J\Gamma_s h_{0,s}\right)J\Gamma_s + \frac{12}{\left(t-s\right)^2}J\Gamma_s + \mathcal{O}\left(\frac{1}{t-s}\right).$$

We can replace $J\Gamma_s h_{0,s}$ by its expression given by (5.29) in the previous formula and we obtain

$$\nabla_{X_0(\zeta(t))}\Gamma_s + g_{\zeta(t)}\left(\nabla_{X_0}\Gamma_s, J\Gamma_s\right) J\Gamma_s = \frac{12}{(t-s)^2} J\Gamma_s + \mathcal{O}\left(\frac{1}{t-s}\right).$$

As a consequence,

$$\nabla_{X_0(\zeta(t))}\Gamma_s = \frac{6}{(t-s)^2} J\Gamma_s + \mathcal{O}\left(\frac{1}{t-s}\right).$$

We now apply (5.29) and we find out that

$$\mathscr{S}_s\left(\zeta(t)\right) h_{0,s} = \frac{6}{(t-s)^2} g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right) + \mathcal{O}\left(\frac{\left\|\mathscr{S}_s\left(\zeta(t)\right)\right\|}{t-s}\right). \tag{5.32}$$

If we use (5.1) to express $\left\|\mathscr{S}_s\left(\zeta(t)\right)\right\|$ in terms of $\dot{\delta}_s\left(\zeta(t)\right)$ - whose Taylor expansion is given as a hypothesis of the Theorem we are proving - we deduce from (5.32) that

$$\mathscr{S}_s\left(\zeta(t)\right) h_{0,s} = \frac{6}{(t-s)^2} g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right) + \mathcal{O}\left(t-s\right). \tag{5.33}$$

At this point, combining (5.33) and Lemma 5.4.2, it is possible to deduce from (5.27) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(h_\zeta(t) - h_{0,s}\left(\zeta(t)\right)\right) =& \dot{h}_\zeta(s) + g_{\zeta(s)}\left(\mathrm{Tor}\left(\mathsf{T}, X_0\right), \mathsf{T}\right) \\ &- \frac{6}{(t-s)^2} g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right) + o\left(1\right).\end{aligned}$$

In the previous formula, we can replace the term $\dot{h}_\zeta(s) + g_{\zeta(s)}\left(\mathrm{Tor}\left(\mathsf{T}, X_0\right), \mathsf{T}\right)$ by the second geodesic curvature $k_{\zeta,2}(s)$ according to Definition 3.4.5 and the fact that J is an isometry since by assumption, $J^2 = -\mathrm{Id}$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(h_\zeta(t) - h_{0,s}\left(\zeta(t)\right)\right) = k_{\zeta,2}(s) - \frac{6}{(t-s)^2} g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right) + o\left(1\right). \tag{5.34}$$

We can now integrate (5.34) between t and s to find an expression of $h_\zeta(t) - h_{0,s}\left(\zeta(t)\right)$ which we substitute to $h_\zeta(t) - h_{0,s}\left(\zeta(t)\right)$ in identity (5.26). Then we replace all the occurences of $g_{\zeta(t)}\left(\mathscr{S}_s, J\Gamma_s\right)$ by its expression given by (5.25) in the equality we obtain by this process. We are left with an equation linking two Taylor expansion from which we deduce the result we wanted to prove. □

Corollary (1.1.10). *Let M be a contact sub-Riemannian space such that $J^2 = -\mathrm{Id}$ and $\zeta :]-T, T[\to M$ be a smooth horizontal curve parametrized by arc length.*

Curve ζ is a geodesic if, and only if for every time $s \in]-T,T[$ and for $t > s$,

$$\dot{\delta}_s(t) = 1 + o\left((t-s)^4\right).$$

Proof. On the one hand, if ζ is a geodesic, then for any $s \in]-T,T[$ and for $t > s$ close enough to s,

$$\delta_s(t) = t - s.$$

As a consequence, for any $s \in]-T,T[$ and for $t > s$ close enough to s,

$$\dot{\delta}_s(t) = 1.$$

On the other hand, let us assume that for every $s \in]-T,T[$ and for $t > s$,

$$\dot{\delta}_s(t) = 1 + o\left((t-s)^4\right). \tag{5.35}$$

By integrating the previous Taylor expansion between s and t we find out that for every $s \in]-T,T[$ and for $t > s$,

$$\delta_s(t) = (t-s) + o\left((t-s)^5\right).$$

Thanks to Theorem 1.1.8 (previously proved in section 5.4), this means that for every $s \in]-T,T[$, $\nabla_{\mathsf{T}(\zeta(s))}\mathsf{T}$ is parallel to $J\mathsf{T}(\zeta(s))$, where T is a normalized horizontal extension of the speed of ζ. By Definition 3.3.2, this means that for any $s \in]-T,T[$

$$\nabla_{\mathsf{T}(\zeta(s))}\mathsf{T} = \frac{h_\zeta(s)}{\|J\mathsf{T}(\zeta(s))\|^2} J\mathsf{T}(\zeta(s)). \tag{5.36}$$

Moreover, thanks to Theorem 1.1.9 (previously proved in section 5.4), we deduce from (5.35) that for every $s \in]-T,T[$,

$$\dot{h}_\zeta(s) - g_{\zeta(s)}\left(\mathrm{Tor}\left(\mathsf{T}, X_0\right), \mathsf{T}\right) = 0. \tag{5.37}$$

Notice that since $J^2 = -\mathrm{Id}$, J is an isometry and $\|J\Gamma_s\| = 1$, (5.36) and (5.37) correspond to the necessary and sufficient conditions for ζ to be a geodesic that we find in Proposition 3.4.4. □

Corollary 5.4.5. *Let M be a contact sub-Riemannian manifold such that $J^2 = -\mathrm{Id}$ and $\zeta :]-T,T[\to M$ be a smooth horizontal curve parametrized by arc length whose velocity we extend by the normalized horizontal field T. We assume that for every $s \in]-T,T[$, vector $\Gamma_s(\zeta(t))$ smoothly depends on $t \geqslant s$, even at $t = s$.*

If for every $s \in]-T,T[$ and for $t > s$,

$$\delta_s(t) = (t-s) + o\left((t-s)^3\right),$$

then for any $s \in]-T,T[$ and for $t > s$,

$$\delta_s(t) = (t-s) - \frac{k_{\zeta,2}^2(s)}{1440}(t-s)^5 + \mathcal{O}\left(t^6\right).$$

Proof. Let us start by noticing that for $s \in]-T, T[$, the smoothness of $\Gamma_s(t)$ for times $t \geqslant s$ entails the same regularity for $\delta_s(t)$ and $\mathscr{S}_s(t)$. Indeed, by Theorem 3.2.3,

$$\dot{\delta}_s(t) = g_{\zeta(t)}(\mathsf{T}, \Gamma_s),$$

and by Proposition 5.1.2,

$$\mathscr{S}_s(t) = \mathsf{T}(\zeta(t)) - \dot{\delta}_s(t)\Gamma_s(\zeta(t)).$$

Now, according to the regularity of δ_s we just proved, it is possible for every $s \in]-T, T[$, to differentiate the Taylor expansion of $\delta_s(t)$ that we assumed as a hypothesis. For $t \geqslant s$, we find

$$\dot{\delta}_s(t) = 1 + o\left((t-s)^2\right). \tag{5.38}$$

Now we use (5.1) and we deduce from (5.38) that

$$\|\mathscr{S}_s(t)\| = o(t-s). \tag{5.39}$$

But since we started by proving that $\mathscr{S}_s$ is smooth, even at time s, we deduce from (5.39) that

$$\|\mathscr{S}_s(t)\| = \mathcal{O}\left((t-s)^2\right). \tag{5.40}$$

We combine (5.1) with (5.40) and we find out that

$$\dot{\delta}_s(t) = 1 + \mathcal{O}\left((t-s)^4\right)$$

for $t \geqslant s$. We know that $\dot{\delta}_s(t)$ is smooth for $t \geqslant s$ even at $t = s$. As a consequence, we can write a more precise Taylor expansion than the previous one.

$$\dot{\delta}_s(t) = 1 - K(s)(t-s)^4 + \mathcal{O}\left((t-s)^5\right), \tag{5.41}$$

for $t \geqslant s$. The value of $K(s)$ for any $s \in]-T, T[$ is given by Theorem 1.1.9 (previously proved in section 5.4), so we only need to integrate (5.41) to obtain the result we were looking for. □

5.5 The $(2n+1)-$Heisenberg structures

In this section we consider, for n a positive integer, the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_{2n+1}$. Our goal is to prove that in this space, the hypotheses of Corollary 5.4.5 are satisfied along any smooth horizontal curve parametrized by arc length, which allows us to deduce Theorem 1.1.11.

$\mathbb{H}_{2n+1}$ is $\mathbb{R}^{2n+1}$ whose coordinates we denote as $(x_1, y_1, ..., x_n, y_n, z)$ and which is endowed with a group law given by

$$(x_i, y_i, z) * (\tilde{x}_i, \tilde{y}_i, \tilde{z}) = \left(x_i + \tilde{x}_i, y_i + \tilde{y}_i, z + \tilde{z} + \frac{1}{2} \sum_i x_i \tilde{y}_i - y_i \tilde{x}_i \right).$$

The $(2n+1)$-dimensional Heisenberg group is endowed with a left-invariant contact sub-Riemannian structure whose distribution is spanned by the orthonormal frame

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial z}, \qquad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial z}.$$

A smooth horizontal curve $\zeta : \mathbb{R} \to \mathbb{H}_{2n+1}$ whose expression in coordinates we denote as $(x_{\zeta,i}, y_{\zeta,i}, z_\zeta)$ is a geodesic parametrized by arc length and leaving from the origin of coordinates at time zero if, and only if there exists $(\omega, \theta_i, r_i) \in \mathbb{R}^{2n+1}$ that satisfies $\sum_i r_i^2 = 1$ such that

$$\left\{ \begin{array}{rcl} x_{\zeta,i}(t) & = & r_i \frac{\cos(\omega t + \theta_i) - \cos(\theta_i)}{\omega}, \\ y_{\zeta,i}(t) & = & r_i \frac{\sin(\omega t + \theta_i) - \sin(\theta_i)}{\omega}, \\ z_\zeta(t) & = & \frac{\omega t - \sin(\omega t)}{\omega^2}. \end{array} \right. \tag{5.42}$$

For more details concerning $\mathbb{H}_{2n+1}$, see [ABB19, Section 13.2].

Corollary 5.5.1. *If $\zeta :]-T, T[\to \mathbb{H}_{2n+1}$ is a smooth horizontal curve parametrized by arc length then for every $s \in]-T, T[$, $\Gamma_s(\zeta(t))$ smoothly depends on $t \geqslant s$ even at $t = s$.*

Proof. Without loss of generality, we choose $s = 0$ and we assume that in the set of coordinates (x_i, y_i, z) that we introduced on $\mathbb{H}_{2n+1}$, $\zeta(0)$ has all its coordinates equal to zero. According to the expression of the geodesics leaving from the origin at time zero, for any point q in Σ_0,

$$\left\{ \begin{array}{rcl} x_i(q) & = & r_i(q) \frac{\cos(\omega(q)\delta_0(q) + \theta_i(q)) - \cos(\theta_i(q))}{\omega(q)}, \\ y_i(q) & = & r_i(q) \frac{\sin(\omega(q)\delta_0(q) + \theta_i(q)) - \sin(\theta_i(q))}{\omega(q)}, \\ z(q) & = & \frac{\omega(q)\delta(q) - \sin(\omega(q)\delta_0(q))}{\omega^2(q)}, \end{array} \right. \tag{5.43}$$

where $\omega(q)$, $\theta_i(q)$ and $r_i(q)$ are the parameters that define the minimizing geodesic that leaves from the origin at arrives at q, with the condition that $\sum_i r_i^2(q) = 1$. Now we can write

$$x_i^2(q) + y_i^2(q) = r_i^2(q) \frac{2 - 2\cos(\omega(q)\delta_0(q))}{\omega^2(q)} = 4 r_i^2(q) \frac{\sin^2\left(\frac{\omega(q)\delta_0(q)}{2}\right)}{\omega^2(q)}.$$

It follows that along ζ,

$$\operatorname{sinc}^2\left(\frac{\omega(\zeta(t))\,\delta_0(\zeta(t))}{2}\right) = \frac{\sum_i x_i^2(\zeta(t)) + y_i^2(\zeta(t))}{\delta_0^2(\zeta(t))} \xrightarrow{t \to 0} 1, \tag{5.44}$$

where sinc stands for the cardinal sine function and where the previous limit comes from Proposition 5.4.1. The limit given in (5.44) implies that

$$\omega\left(\zeta(t)\right)\delta_0\left(\zeta(t)\right) \overset{t\to 0}{\longrightarrow} 0. \tag{5.45}$$

Moreover we have

$$\frac{\omega\left(\zeta(t)\right)\delta\left(\zeta(t)\right) - \sin\left(\omega\left(\zeta(t)\right)\delta_0\left(\zeta(t)\right)\right)}{8\sin^2\left(\frac{\omega(\zeta(t))\delta_0(\zeta(t))}{2}\right)} = \frac{z\left(\zeta(t)\right)}{\sum_i x_i^2\left(\zeta(t)\right) + y_i^2\left(\zeta(t)\right)},$$

If we consider

$$\begin{array}{rccc} \phi: &]-\pi,\pi[& \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \frac{x-\sin(x)}{8\sin^2\left(\frac{x}{2}\right)}, \end{array}$$

ϕ is locally invertible around zero. We can therefore write

$$\omega\left(\zeta(t)\right)\delta_0\left(\zeta(t)\right) = \phi^{-1}\left(\frac{z\left(\zeta(t)\right)}{\sum_i x_i^2\left(\zeta(t)\right) + y_i^2\left(\zeta(t)\right)}\right), \tag{5.46}$$

Let us justify that the previous expression of $\omega\left(\zeta(t)\right)\delta_0\left(\zeta(t)\right)$ is smooth at $t=0$. First of all, both $z\left(\zeta(t)\right)$ and $\sum_i x_i^2\left(\zeta(t)\right) + y_i^2\left(\zeta(t)\right)$ are smooth since ζ is smooth. Since ζ is parametrized by arc length, we have

$$\sum_i x_i^2\left(\zeta(0)\right) + y_i^2\left(\zeta(0)\right) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \sum_i x_i^2\left(\zeta(t)\right) + y_i^2\left(\zeta(t)\right) = 0 \text{ and}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} \sum_i x_i^2\left(\zeta(t)\right) + y_i^2\left(\zeta(t)\right) = 2.$$

Moreover since ζ is horizontal and leaves from the origin at time zero we have that

$$z(\zeta(0)) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} z(\zeta(t)) = 0.$$

We have all the elements to apply Lemma 4.4.4 to prove the smoothness of $\frac{z(\zeta(t))}{\sum_i x_i^2(\zeta(t)) + y_i^2(\zeta(t))}$, which entails that $\omega\left(\zeta(t)\right)\delta_0\left(\zeta(t)\right)$ is smooth by using (5.46). Now

$$\delta_0^2\left(\zeta(t)\right) = \frac{2z\left(\zeta(t)\right)\omega^2\left(\zeta(t)\right)\delta_0^2\left(\zeta(t)\right)}{\omega\left(\zeta(t)\right)\delta_0\left(\zeta(t)\right) - \sin\left(\omega\left(\zeta(t)\right)\delta_0\left(\zeta(t)\right)\right)}, \tag{5.47}$$

But the numerator of the right hand side of (5.47) vanishes at a higher order than is denominator since the left hand side of (5.47) tends to zero. We can

therefore apply Lemma 4.4.4 and we find out that δ_0^2 is also smooth along ζ. But $\delta_0^2(\zeta(0)) = 0$ and by Proposition 5.4.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|_{t=0^+}} \delta_0^2(\zeta(t)) = 0 \text{ and } \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|_{t=0^+}} \delta_0^2(\zeta(t)) = 2.$$

Now for $t \geqslant 0$,

$$\delta_0(\zeta(t)) = t\sqrt{\frac{\delta_0^2(\zeta(t))}{t^2}},$$

so by applying Lemma 4.4.4, $\delta_0(\zeta(t))$ is smooth for $t \geqslant 0$.

We have just proven that $\omega(\zeta(t))\,\delta_0(\zeta(t))$ and $\delta_0(\zeta(t))$ are smooth for $t \geqslant 0$. In addition $\omega(\zeta(0))\,\delta_0(\zeta(0)) = 0$ and according to Proposition 5.4.1 $\frac{\mathrm{d}}{\mathrm{d}t}_{|_{t=0^+}} \delta_0(\zeta(t)) = 1$. As a consequence, we can apply 4.4.4 to deduce that

$$\omega(\zeta(t)) = \frac{\omega(\zeta(t))\,\delta_0(\zeta(t))}{\delta_0(\zeta(t))}$$

is smooth for $t \geqslant 0$.

At this point, for $q \in \Sigma_0$ we rewrite the equations in (5.43) concerning x_i and y_i as

$$\begin{pmatrix} x_i(q) \\ y_i(q) \end{pmatrix} = \frac{r_i(q)}{\omega(q)} \begin{pmatrix} \cos(\omega(q)\delta_0(q)) - 1 & -\sin(\omega(q)\delta_0(q)) \\ \sin(\omega(q)\delta_0(q)) & \cos(\omega(q)\delta_0(q)) - 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_i(q)) \\ \sin(\theta_i(q)) \end{pmatrix}.$$

The previous identity implies that if $\omega(q)\delta(q)$ is not a multiple of 2π,

$$\begin{pmatrix} \cos(\theta_i(q)) \\ \sin(\theta_i(q)) \end{pmatrix} = \frac{\omega(q)}{r_i(q)} \begin{pmatrix} \cos(\omega(q)\delta_0(q)) - 1 & -\sin(\omega(q)\delta_0(q)) \\ \sin(\omega(q)\delta_0(q)) & \cos(\omega(q)\delta_0(q)) - 1 \end{pmatrix}^{-1} \begin{pmatrix} x_i(q) \\ y_i(q) \end{pmatrix}. \tag{5.48}$$

Moreover for $q \in \Sigma_0$, $\Gamma_0(q)$ is by definition the velocity of the minimizing geodesic parametrized by arc length that comes from the origin, so according to the expression on geodesics leaving from the origin in $\mathbb{H}_{2n+1}$ given in (5.42), the projection on the coordinates (x_i, y_i) of Γ_0 is

$$\begin{aligned} (\Gamma_0(q))_i &= \begin{pmatrix} -r_i(q)\sin(\omega(q)\delta_0(q) + \theta_i(q)) \\ r_i(q)\cos(\omega(q)\delta_0(q) + \theta_i(q)) \end{pmatrix} \\ &= r_i(q) \begin{pmatrix} -\sin(\omega(q)\delta_0(q)) & -\cos(\omega(q)\delta_0(q)) \\ \cos(\omega(q)\delta_0(q)) & -\sin(\omega(q)\delta_0(q)) \end{pmatrix} \begin{pmatrix} \cos(\theta_i(q)) \\ \sin(\theta_i(q)) \end{pmatrix}. \end{aligned} \tag{5.49}$$

If in (5.49) we replace $\begin{pmatrix} \cos(\theta_i(q)) \\ \sin(\theta_i(q)) \end{pmatrix}$ by its expression given by (5.48), we find a new formula for $(\Gamma_0(q))_i$. Form this new expression of $(\Gamma_0(q))_i$, it follows that $\Gamma_0(\zeta(t))$ is smooth for $t \geqslant 0$ since $\omega(\zeta(t))\,\delta_0(\zeta(t))$ and $\omega(\zeta(t))$ are smooth for $t \geqslant 0$. □

Appendix A

Jacobi fields and asymptotics of Lie brackets in the 3D case

In this section, M is a three dimensional contact sub-Riemannian manifold. We fix $p \in M$ a privileged point. We slightly adapt the notation $\tilde{\delta}$ to this section by saying that it represents the function that maps a point to its distance to p.

We consider $\overrightarrow{J}$ a vector field defined along $\overline{\gamma} : I \to T^*M$ an integral line of the Hamiltonian field. We say that $\overrightarrow{J}$ is a *Jacobi field* when the Lie derivative $\mathcal{L}_{\overrightarrow{H}} \overrightarrow{J}$ in the direction of $\overrightarrow{H}$ vanishes along $\overline{\gamma}$.

For $\gamma : I \to M$ a geodesic and $\overrightarrow{\mathcal{J}}$ a vector field along γ, we say that $\overrightarrow{\mathcal{J}}$ is a *projected Jacobi field* when there exists $\overline{\gamma} : I \to T^*M$ a lift of γ that is an integral line for the Hamiltonian field and $\overrightarrow{J}$ a Jacobi field along $\overline{\gamma}$ such that the natural projection of $\overrightarrow{J}$ on TM is equal to $\overrightarrow{\mathcal{J}}$.

Notice that we distinguish Jacobi fields tangent to M from those tangent to T^*M by using a calligraphic typography instead of a roman one.

The notion of conjugate points is closely linked to the concept of Jacobi fields.

We consider $\overline{\gamma} : I \to T^*M$ an integral line of the Hamiltonian vector field. We say that two points $\overline{\gamma}(t_1)$ and $\overline{\gamma}(t_2)$ are conjugate along $\overline{\gamma}$ when there exists a Jacobi field $\overrightarrow{J}$ that is not identically zero along $\overline{\gamma}$ such that

$$\mathrm{d}\pi\left(\overrightarrow{J}\left(\overline{\gamma}(t_1)\right)\right) = 0 \text{ and } \mathrm{d}\pi\left(\overrightarrow{J}\left(\overline{\gamma}(t_2)\right) = 0\right).$$

Similarly, for $\gamma : I \to M$ a geodesic, we say that points $\gamma(t_1)$ and $\gamma(t_2)$ are conjugate along γ when there exists a projected Jacobi field $\overrightarrow{\mathcal{J}}$ along γ that is not identically zero and such that

$$\overrightarrow{\mathcal{J}}\left(\gamma(t_1)\right) = 0 \text{ and } \overrightarrow{\mathcal{J}}\left(\gamma(t_2)\right) = 0.$$

Since Jacobi fields are defined along lifts of geodesics, it is important to understand these lifts of geodesics. The following theorem summarizes

several results concerning these lifts. [ABB19, 11.6, 11.9 and 11.10] [ABR17, Theorem 4.1, Rem 4.2]

Theorem A.0.1. *Let $p \in \mathcal{M}$ be a reference point, and let us consider $\Sigma_p \subset M$ the set of smooth points around p, which we recall is the set of points at which the distance function from p is smooth, or equivalently the complement of the cut locus.*

For $q \in \Sigma_p$ let us consider γ_q the minimizing geodesic parametrized by arc length that leaves from p at time zero and that reaches q, and $\overline{\gamma}_q : \mathbb{R} \to T^*\mathcal{M}$ the lift of γ_q that is an integral line of the Hamiltonian vector field $\vec{H}$ that is provided by Theorem 3.4.3.

$\overline{\gamma}_q$ is contained in the set $H^{-1}\left(\frac{1}{2}\right)$, since the geodesic of which it is a lift is parametrized by arc length. Moreover $\overline{\gamma}_q$ can be computed as follows :

$$\overline{\gamma}_q\left(\tilde{\delta}(q)\right) = \mathrm{d}_q\tilde{\delta},$$

where $\tilde{\delta}$ stands for the distance function from p.

We start by establishing a general property of Jacobi fields. Before we do this, we introduce technical notations.

Definition A.0.2. We set

$$\mathfrak{S} = \left(H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M\right) \bigcup \left\{ d_q\tilde{\delta} / q \in \Sigma_p \right\}.$$

We can interpret what this set represents by applying Theorem A.0.1. Indeed $\mathfrak{S}$ can be decomposed as the union of the integral lines of that Hamiltonian flow that are the lifts to T^*M of geodesics leaving from p that are parametrized by arc length and that have not yet reached their cut time with respect to p. $\mathfrak{S}$ can be sent through a smooth diffeomorphism to a subset of $\left(H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M\right) \times \mathbb{R}$.

Proposition A.0.3. *For X a vector field tangent to T^*M, V a covector in T^*M and $t \in \mathbb{R}$, let us denote by*

$$\Phi_X^t(V)$$

the covector in T^*M that is reached at time t by the integral line of X that leaves from V at time 0. The map

$$\begin{array}{rccc} F: & \mathfrak{S} & \longrightarrow & F(\mathfrak{S}) \subset \left(H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M\right) \times \mathbb{R} \\ & V & \longmapsto & \left(\Phi_{-\vec{H}}^{\tilde{\delta}(\pi(V))}(V), \tilde{\delta}(\pi(V))\right) \end{array}$$

is a diffeomorphism whose inverse is :

$$\begin{array}{rccc} F^{-1}: & F(\mathfrak{S}) & \longrightarrow & \mathfrak{S} \\ & \left(V, \overline{\delta}\right) & \longmapsto & \Phi_{\vec{H}}^{\overline{\delta}}(V). \end{array}$$

We can see $\left(V, \overline{\delta}\right)$ as coordinates on the set $\mathfrak{S}$. Function $\tilde{\delta}$ is thereby transported from its initial domain Σ_p to a new domain $\mathfrak{S}$, in a most natural fashion since $\overline{\delta} = \tilde{\delta} \circ \pi$.

At this point, it is possible to link several notations that we have introduced in this paper.

Lemma A.0.4. *If we consider $\left(V, \overline{\delta}\right)$ as coordinates on $\mathfrak{S}$ then*

$$\frac{\partial}{\partial \overline{\delta}} = \overrightarrow{\overline{H}} \tag{A.1}$$

and over $\mathfrak{S}$,

$$\mathrm{d}\pi \circ \overrightarrow{\overline{H}} = \Gamma \circ \pi. \tag{A.2}$$

Proof. Identity (A.1) follows from the expression of map F^{-1} (see Proposition A.0.3) while identity (A.2) holds over $\mathfrak{S}$ as a consequence of the definition of our set $\mathfrak{S}$ (Definition A.0.2) combined with Theorem A.0.1. □

Before we prove a result concerning Jacobi fields, let us state two usefull lemmas concerning Lie brackets.

Lemma A.0.5. *Let X, Y be two horizontal normalized vector fields. Assume*

$$Y = \cos(\psi)X + \sin(\psi)JX,$$

for some smooth function $\psi : M \to \mathbb{R}/2\pi\mathbb{Z}$. Then

$$[Y, JY] = [X, JX] - \operatorname{grad}\psi.$$

Here, if $f : M \to \mathbb{R}$ is a smooth function, we denote by $\operatorname{grad} f$ its *horizontal gradient* which is the horizontal vector field such that $\mathrm{d}f(X) = g(\operatorname{grad} f, X)$ for any smooth horizontal X. It is easy to check that

$$\operatorname{grad} f = (X_1 f)X_1 + (X_2 f)X_2. \tag{A.3}$$

where (X_1, X_2) is any orthonormal frame of the distribution. The proof of Lemma A.0.5 follows by direct computation.

The next lemma follows from (3.18) and (3.19).

Lemma A.0.6. *If $\{X_1, X_2\}$ is an orthonormal frame of the distribution then for every $i, j, k = 0, 1, 2$ we have*

$$\left[\overline{X}_i, \overline{X}_j\right] = \overline{[X_i, X_j]}, \tag{A.4}$$

which implies

$$c^{\overline{X}_k}_{\overline{X}_i, \overline{X}_j} = \overline{c}^{X_k}_{X_i, X_j}. \tag{A.5}$$

Moreover

$$\left[\frac{\partial}{\partial h_{X_i}}, \overline{X}_j\right] = \left[\frac{\partial}{\partial h_{X_i}}, \frac{\partial}{\partial h_{X_j}}\right] = 0. \tag{A.6}$$

Lemma A.0.7. *Let us consider $\{X_1, X_2\}$ an orthonormal frame of the distribution. There exist two smooth vector fields $\overrightarrow{J}^{\perp} : \mathfrak{S} \to T\mathfrak{S}$ and $\overrightarrow{J}^{0} : \mathfrak{S} \to T\mathfrak{S}$ which we decompose as*

$$\overrightarrow{J}^{i} = \alpha^i \overline{X}_1 + \beta^i \overline{X}_2 + \sigma^i \overline{X}_0 + j_1^i \frac{\partial}{\partial h_{X_1}} + j_2^i \frac{\partial}{\partial h_{X_2}} + j_0^i \frac{\partial}{\partial h_{X_0}},$$

for $i \in \{\perp, 0\}$, and that satisfy :

(i) Fields $\overrightarrow{J}^{\perp}$ and $\overrightarrow{J}^{0}$ are Jacobi fields in the sense that

$$[\overrightarrow{J}^{i}, \overrightarrow{H}] = 0.$$

(ii) For every $i \in \{\perp, 0\}$ and for every fixed V in $H^{-1}\left(\frac{1}{2}\right) \cap T *_p M$,

$$\sigma^{\perp} \circ F^{-1}\left(V, \overline{\delta}\right) \sim \frac{\overline{\delta}^2}{2} \text{ and } \sigma^{0} \circ F^{-1}\left(V, \overline{\delta}\right) \sim -\frac{\overline{\delta}^3}{6}.$$

(iii) For every $i \in \{\perp, 0\}$ and for every covector V in $H^{-1}\left(\frac{1}{2}\right) \cap T_p^* M$,

$$\mathrm{d}\pi\left(\overrightarrow{J}^{i}(V)\right) = 0.$$

Moreover, functions σ^i are smooth and do not depend on the choice of $\{X_1, X_2\}$.

Proof. By combining the expression of $\overrightarrow{H}$ given by Proposition 3.4.2 and that of $\overrightarrow{J}^{i}$, to which we add Lemma A.0.5, we can reformulate the condition $[\overrightarrow{J}^{i}, \overrightarrow{H}] = 0$ by writing it on the frame $\left(\overline{X}_0, \overline{X}_1, \overline{X}_2, \frac{\partial}{\partial h_{X_0}}, \frac{\partial}{\partial h_{X_1}}, \frac{\partial}{\partial h_{X_2}}\right)$. If we project the equation on each of the six directions, the system we obtain is :

$$\left\{\begin{array}{rcl}
\overrightarrow{H}\sigma^i & = & h_{X_2}\alpha^i - h_{X_1}\beta^i \\
\overrightarrow{H}\alpha^i & = & \left(h_{X_2}\alpha^i - h_{X_1}\beta^i\right)\overline{c}^1_{1,2} + h_{X_1}\sigma^i\eta\left(X_1\right) + h_{X_2}\sigma^i\overline{c}^1_{0,2} + j_1^i \\
\overrightarrow{H}\beta^i & = & \left(h_{X_2}\alpha^i - h_{X_1}\beta^i\right)\overline{c}^2_{1,2} + h_{X_2}\sigma^i\eta\left(X_2\right) + h_{X_1}\sigma^i\overline{c}^2_{0,1} + j_2^i \\
\overrightarrow{H}j_1^i & = & -\sum_k \left(h_{X_2}h_{X_k}\left(\alpha^i X_1 + \beta^i X_2 + \sigma^i X_0\right)\overline{c}^k_{1,2}\right. \\
 & & \left. \qquad + \overline{c}^k_{1,2}\left(j_2^i h_{X_k} + h_{X_2} j_k^i\right)\right) \\
\overrightarrow{H}j_2^i & = & -\sum_k \left(h_{X_1}h_{X_k}\left(\alpha^i X_1 + \beta^i X_2 + \sigma^i X_0\right)\overline{c}^k_{2,1}\right. \\
 & & \left. \qquad + \overline{c}^k_{2,1}\left(j_1^i h_{X_k} + h_{X_1} j_k^i\right)\right) \\
\overrightarrow{H}j_0^i & = & \sum_{j,k,j\neq 0}\left(h_{X_j}h_{X_k}\left(\alpha^i X_1 + \beta^i X_2 + \sigma^i X_0\right)\overline{c}^k_{0,j}\right. \\
 & & \left. \qquad + \overline{c}^k_{0,j}\left(j_j^i h_{X_k} + h_{X_j} j_k^i\right)\right).
\end{array}\right. \tag{A.7}$$

In order to define the vector fields $\overrightarrow{J}^{\perp}$ and $\overrightarrow{J}^{0}$, it is sufficient to define their values on $H^{-1}\left(\frac{1}{2}\right) \cap T_p^* M$ and the values of $\overrightarrow{J}^{\perp}$ and $\overrightarrow{J}^{0}$ on the whole

space F then follow from differential equation (A.7). We define $\overrightarrow{J}^{\perp}$ as equal to

$$h_{X_2}\frac{\partial}{\partial h_{X_1}} - h_{X_1}\frac{\partial}{\partial h_{X_2}}$$

on $H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M$ while $\overrightarrow{J}^0$ coincides with $\frac{\partial}{\partial h_{X_0}}$ on that same space.

We now use (A.7) to establish the asymptotics of σ^0 and $\sigma^{\perp}$ by computing the successive differentials of σ and evaluating them at zero. We find out that for any V in $H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M$,

$$\begin{aligned}
\sigma^i(V) &= 0 \\
\overrightarrow{H}\left(\sigma^i\right)(V) &= h_{X_2}(V)\underbrace{\alpha^i(V)}_{=0} - h_{X_1}(V)\underbrace{\beta^i(V)}_{=0} = 0 \\
\overrightarrow{H}^2\left(\sigma^i\right)(V) &= h_{X_2}(V)\overrightarrow{H}\left(\alpha^i\right)(V) - h_{X_1}(V)\overrightarrow{H}\left(\beta^i\right)(V) \\
&= h_{X_2}(V)j_1^i(V) - h_{X_1}(V)j_2^i(V).
\end{aligned}$$

In particular,

$$\overrightarrow{H}^2\left(\sigma^{\perp}\right)(V) = h_{X_2}^2(V) + h_{X_1}^2(V) = 1 \text{ since } V \in H^{-1}\left(\frac{1}{2}\right),$$

and $\overrightarrow{H}^2\left(\sigma^0\right)(V) = 0$.

Furthermore,

$$\begin{aligned}
\overrightarrow{H}^3\left(\sigma^0\right)(V) &= h_{X_2}(V)\overrightarrow{H}\left(j_1^i\right)(V) - h_{X_1}(V)\overrightarrow{H}\left(j_2^i\right)(V) \\
&= -h_{X_2}^2(V) - h_{X_1}^2(V) = -1 \text{ since } V \in H^{-1}\left(\frac{1}{2}\right).
\end{aligned}$$

Now, as we noticed in Lemma A.0.4,

$$\overrightarrow{H}^n\left(\sigma^i\right)(V) = \frac{\partial^n}{\partial\overline{\delta}^n}_{|_{\overline{\delta}=0}} \sigma^i \circ F^{-1}\left(V, \overline{\delta}\right),$$

which is sufficient to conclude concerning the asymptotics we had set to establish.

The fact that functions σ^i are smooth and independant of the choice of (X_1, X_2) simply comes from the formula:

$$\sigma^i = \omega \circ \mathrm{d}\pi\left(\overrightarrow{J}^i\right).$$

□

Before we can apply the previous results concerning Jacobi fields to study the asymptotics of the Lie brackets, we need to prove a property concerning the set $\mathfrak{S}$.

Lemma A.0.8. *The following inclusion holds true:*

$$\mathfrak{S} \setminus \left(H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M \right) \subset h_\Gamma^{-1}(1) \cap h_{J\Gamma}^{-1}(0).$$

Proof. Let us consider any covector in $\mathfrak{S} \setminus T_p^*M$. It can be written as $\mathrm{d}_q\tilde{\delta}$ for a certain q in Σ_p by Definition A.0.2. Now if we choose $(\Gamma, J\Gamma)$ as a frame of the distribution, by using (3.11), we can write thanks to Theorem 3.2.3,

$$h_\Gamma\left(\mathrm{d}_q\tilde{\delta}\right) = \mathrm{d}_q\tilde{\delta}(\Gamma) = 1, \qquad h_{J\Gamma}\left(\mathrm{d}_q\tilde{\delta}\right) = \mathrm{d}_q\tilde{\delta}(J\Gamma) = 0.$$

□

We are now able to compute the asymptotics of the Lie brackets of the elements of the frame $(\Gamma, J\Gamma, X_0)$.

Proposition A.0.9. *Quantities $\overline{\delta} c_{\Gamma,J\Gamma}^{J\Gamma}$ and $\overline{\delta}^2 c_{\Gamma,X_0}^{J\Gamma}$ (a priori defined on $\mathfrak{S} \setminus T_p^*M$) can be smoothly extended to $\mathfrak{S}$ and are respectively equal to -4 and -6 over $H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M$.*

Proof. Let us focus on the fields $\overrightarrow{J}^0$ and $\overrightarrow{J}^\perp$ that we introduced in Lemma A.0.7. Over $\mathfrak{S} \setminus T_p^*M$, we write them in an adapted frame :

$$\overrightarrow{J}^i = \alpha^i \overline{\Gamma} + \beta^i \overline{J\Gamma} + \sigma^i \overline{X}_0 + j_1^i \frac{\partial}{\partial h_\Gamma} + j_2^i \frac{\partial}{\partial h_{J\Gamma}} + j_0^i \frac{\partial}{\partial h_{X_0}},$$

First of all, $\overrightarrow{J}^0$ and $\overrightarrow{J}^\perp$ are tangent to $\mathfrak{S} \setminus T_p^*M$ which is contained in $h_\Gamma^{-1}(1) \cap h_{J\Gamma}^{-1}(0)$ according to Lemma A.0.8. Therefore, their components in the directions of $\frac{\partial}{\partial h_\Gamma}$ and $\frac{\partial}{\partial h_{J\Gamma}}$ vanish, which means that

$$j_1^i = j_2^i = 0. \tag{A.8}$$

Now if we use identity (A.8) and the fact that at each point of $\mathfrak{S} \setminus T_p^*M$, $h_\Gamma = 1$ and $h_{J\Gamma} = 0$, we can simplify system (A.7) that rules the fields $\overrightarrow{J}^0$ and $\overrightarrow{J}^\perp$, in the case where system (A.7) is written with $(X_1, X_2) = (\Gamma, J\Gamma)$. If we combine the first and the third equation of the simplified sytem (A.7), we obtain the following equation

$$0 = \overrightarrow{H}^2 \sigma^i + \overline{c}_{\Gamma,J\Gamma}^{J\Gamma} \overrightarrow{H} \sigma^i + \overline{c}_{X_0,\Gamma}^{J\Gamma} \sigma^i.$$

Since this last equation is satisfied by σ^0 and $\sigma^\perp$ we find out that

$$\begin{pmatrix} -\overrightarrow{H}^2 \sigma^\perp \\ -\frac{\overrightarrow{H}^2 \sigma^0}{\overline{\delta}} \end{pmatrix} = \begin{pmatrix} \frac{\overrightarrow{H}\sigma^\perp}{\overline{\delta}} & \frac{\sigma^\perp}{\overline{\delta}^2} \\ \frac{\overrightarrow{H}\sigma^0}{\overline{\delta}^2} & \frac{\sigma^0}{\overline{\delta}^3} \end{pmatrix} \begin{pmatrix} \overline{\delta}\overline{c}_{\Gamma,J\Gamma}^{J\Gamma} \\ \overline{\delta}^2 \overline{c}_{X_0,\Gamma}^{J\Gamma} \end{pmatrix} \tag{A.9}$$

The matrix of this system as well as its right hand side are smooth over $\mathfrak{S}$ by applying Lemma 4.4.4 to the asymptotics given in Lemma A.0.7 when $\overline{\delta}$ goes to zero (where we recall the remark we made in Lemma A.0.4 that $\frac{\partial}{\partial\overline{\delta}} = \overrightarrow{H}$). Moreover, thanks to the asymptotics in Lemma A.0.7, we deduce that the left hand side of system (A.9) and its matrix respectively tend to

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{6} \end{pmatrix}$$

when $\overline{\delta}$ tends to zero. As a consequence, by inversing system (A.9) we obtain that functions $\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ and $\overline{\delta}^2\overline{c}^{J\Gamma}_{X_0,\Gamma}$ that were a priori defined on $\mathfrak{S} \setminus T^*_pM$, can in fact be extended to the whole domain $\mathfrak{S}$ in a smooth manner. By inversing system (A.9) and taking its limit as $\overline{\delta}$ goes to zero, we even discover the values of $\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ and $\overline{\delta}^2\overline{c}^{J\Gamma}_{X_0,\Gamma}$ on the set $\overline{\delta}^{-1}(0) = H^{-1}\left(\frac{1}{2}\right) \cap T^*_pM$. □

We can also write several other results that are similar to the previous proposition.

Proposition A.0.10. *Function $\overline{\delta}^2\overrightarrow{H}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ that is a priori defined on $\mathfrak{S}\setminus T^*_pM$ can be extended to a smooth function on the domain $\mathfrak{S}$ and its evaluation is equal to 4 on $H^{-1}\left(\frac{1}{2}\right) \cap T^*_pM$.*

Proof. We know from Proposition A.0.9 that $\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ can be extended to a smooth function on $\mathfrak{S}$ that is equal to -4 on $\overline{\delta}^{-1}(0)$. Since $\overrightarrow{H}$ is also smooth, we can write

$$\overline{\delta}\overrightarrow{H}\left(\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right) = \delta\overbrace{\left(\overrightarrow{H}\overline{\delta}\right)}^{=1}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma} + \overline{\delta}^2\left(\overrightarrow{H}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right)$$

So

$$\overline{\delta}^2\left(\overrightarrow{H}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right) = \overline{\delta}\overrightarrow{H}\left(\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right) - \overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$$

has a smooth extension on $\mathfrak{S}$ that is equal to 4 on $\overline{\delta}^{-1}(0) = H^{-1}\left(\frac{1}{2}\right) \cap T^*_pM$. □

Proposition A.0.11. *Function $\overline{\delta}^2\overline{J\Gamma}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ that is a priori defined on $\mathfrak{S} \setminus T^*_pM$ can be extended to a smooth function on the domain $\mathfrak{S}$.*

Proof. Let us consider the fields $\overrightarrow{J}^0$ and $\overrightarrow{J}^\perp$ over $\mathfrak{S}$ that we introduced in Lemma A.0.7. We have for i equal to 0 or $\perp$, that $\left[\overrightarrow{H}, \overrightarrow{J}^i\right] = 0$ which

implies that

$$\begin{aligned} 0 &= \left[\overrightarrow{H}, \overrightarrow{J}^i\right] \overline{\delta} \\ &= \overrightarrow{H}\overrightarrow{J}^i\overline{\delta} - \overrightarrow{J}^i \overbrace{\overrightarrow{H}\overline{\delta}}^{=1}. \end{aligned}$$

So $\overrightarrow{J}^i\overline{\delta}$ is constant on the integral lines of $\overrightarrow{H}$. But on $H^{-1}\left(\frac{1}{2}\right) \cap T_p^*M$, the function $\overrightarrow{J}^i\overline{\delta}$ is equal to zero. Therefore

$$\overrightarrow{J}^i\overline{\delta} = 0 \tag{A.10}$$

on $\mathfrak{S}$. As a consequence, for any V in $\mathfrak{S}$, $\mathrm{d}\pi\left(\overrightarrow{J}^i(V)\right)$ belongs to the kernel of $\mathrm{d}\tilde{\delta}$. Moreover, for any V in $\mathfrak{S}$ the vector

$$\xi(V) := \mathrm{d}\pi\left(\sigma^\perp(V)\overrightarrow{J}^0(V) - \sigma^0(V)\overrightarrow{J}^\perp(V)\right) \in TM \tag{A.11}$$

belongs to the distribution, since its component in the direction of the Reeb vector field X_0 is canceled. Now since vector $\xi(V)$ is a linear combination of vectors in the kernel of $\mathrm{d}\tilde{\delta}$, it is also in the kernel of $\mathrm{d}\tilde{\delta}$ and we deduce that vector $\xi(V)$ is orthogonal to the sub-Riemannian gradient of $\tilde{\delta}$. By Theorem 3.2.3, this means that $\xi(V)$ is colinear to $J\Gamma$. We define

$$\Upsilon : \mathfrak{S} \longrightarrow \mathbb{R}$$

such that for any V in $\mathfrak{S}$,

$$\xi(V) = \Upsilon(V) J\Gamma\left(\pi(V)\right). \tag{A.12}$$

Now we decide to write the fields $\overrightarrow{J}^i$ over $\mathfrak{S} \setminus T_p^*M$ as :

$$\overrightarrow{J}^i = \alpha^i\overline{\Gamma} + \beta^i\overline{J\Gamma} + \sigma^i\overline{X}_0 + j_1^i\frac{\partial}{\partial h_\Gamma} + j_2^i\frac{\partial}{\partial h_{J\Gamma}} + j_0^i\frac{\partial}{\partial h_{X_0}}.$$

If we combine the latter with (A.11) and (A.12), we obtain that for any V in $\mathfrak{S} \setminus T_p^*M$,

$$\Upsilon(V) = \sigma^\perp(V)\beta^0(V) - \sigma^0(V)\beta^\perp(V).$$

Now thanks to Lemma A.0.8, h_Γ is identically one and $h_{J\Gamma}$ is identically zero on $\mathfrak{S} \setminus T_p^*M$, so the first equation of (A.7) becomes

$$\overrightarrow{H}\sigma^i = -\beta^i.$$

As a consequence, for every V in $\mathfrak{S} \setminus T_p^*M$,

$$\Upsilon(V) = \sigma^0(V)\overrightarrow{H}\sigma^\perp(V) - \sigma^\perp(V)\overrightarrow{H}\sigma^0(V). \tag{A.13}$$

At this point, we must remember what our goal consists in. We are computing the asymptotics of $\overline{\delta}^2\overline{J\Gamma}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ over $\mathfrak{S}\setminus T^*_pM$. Since the function $\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ is constant on the fiber of T^*M, we can replace $\overline{J\Gamma}$ in the expression $\overline{\delta}^2\overline{J\Gamma}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ by any vector field $\mathfrak{S}\setminus T^*_pM \to T^*M$ that has the same projection as $\overline{J\Gamma}$ through $\mathrm{d}\pi$, namely that is projected onto $J\Gamma$ through $\mathrm{d}\pi$.

But by combining (A.11) and (A.12), we obtain that for all covectors V in $\mathfrak{S}\setminus T^*_pM$,

$$\mathrm{d}\pi\left(\frac{\sigma^\perp(V)\overrightarrow{J}^0(V)-\sigma^0(V)\overrightarrow{J}^\perp(V)}{\Upsilon(V)}\right)=J\Gamma\left(\pi(V)\right).$$

As a consequence,

$$\begin{aligned}\overline{\delta}^2\overline{J\Gamma}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}&=\overline{\delta}^2\frac{\sigma^\perp\overrightarrow{J}^0\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}-\sigma^0\overrightarrow{J}^\perp\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}}{\Upsilon}\\&=\overline{\delta}\frac{\sigma^\perp\overrightarrow{J}^0\left(\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right)-\sigma^0\overrightarrow{J}^\perp\left(\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right)}{\Upsilon}\text{ by (A.10),}\\&=\overline{\delta}\frac{\sigma^\perp\overrightarrow{J}^0\left(\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right)-\sigma^0\overrightarrow{J}^\perp\left(\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right)}{\sigma^0\overrightarrow{H}\sigma^\perp-\sigma^\perp\overrightarrow{H}\sigma^0}\text{ according to (A.13).}\end{aligned}\tag{A.14}$$

We have all the elements we need to conclude. By applying Proposition A.0.9, $\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ can be extended to a smooth function defined on $\mathfrak{S}$ and its value on $H^{-1}\left(\frac{1}{2}\right)\cap T^*_pM$ is constant. Therefore, all functions $\overrightarrow{J}^i\left(\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\right)$ can be extended to smooth functions on $\mathfrak{S}$ that vanish at every point of $H^{-1}\left(\frac{1}{2}\right)\cap T^*_pM$. We combine this with the smoothness and the asymptotics of functions σ^i that come from Lemma A.0.7, and we now understand the asymptotics of all the terms in (A.14). We simply apply Lemma 4.4.4 and we obtain the fact that $\overline{\delta}^2\overline{J\Gamma}c^{J\Gamma}_{\Gamma,J\Gamma}$ can be extended to a smooth function on the domain $\mathfrak{S}$. □

A.1 Proof of Proposition 4.4.8

Let us start by proving the first identity. We consider

$$\begin{array}{rccc}\overline{\zeta}:&I\setminus\{0\}&\longrightarrow&\mathfrak{S}\\&t&\longmapsto&\mathrm{d}_{\zeta(t)}\delta,\end{array}$$

which is a lift of ζ. We recall that $\overline{\delta}=\tilde{\delta}\circ\pi$ and that $\overline{c}^k_{i,j}=c^k_{i,j}\circ\pi$. Therefore,

$$\tilde{\delta}\left(\zeta(t)\right)c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right)=\overline{\delta}\left(\overline{\zeta}(t)\right)\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\left(\overline{\zeta}(t)\right),\tag{A.15}$$

and we may study the right hand side of the previous identity instead of the left hand side.

Now by Theorem A.0.1 for every t in $I\backslash\{0\}$, $\overline{\zeta}(t) = \mathrm{d}_{\zeta(t)}\tilde{\delta}$ is the evaluation at time $\tilde{\delta}(\zeta(t))$ of the integral line of the Hamiltonian flow $\overline{\gamma}_{\zeta(t)} : I \setminus \{0\} \to T^*M$, that is a lift of $\gamma_{\zeta(t)}$, the minimizing geodesic parametried by arc length linking p to $\zeta(t)$.

In particular by using diffeomorphism F introduced in Proposition A.0.3,

$$\overline{\zeta}(t) = F^{-1}\left(\overline{\gamma}_{\zeta(t)}(0), \tilde{\delta}\left(\zeta(t)\right)\right). \tag{A.16}$$

But using Proposition 4.4.7 combined with point (ii) in Proposition 3.4.4,

$$h_{X_0}\left(\overline{\gamma}_{\zeta(t)}(0)\right) \overset{t\to 0}{\longrightarrow} h_\zeta(0). \tag{A.17}$$

Moreover, since for every t, $\gamma_{\zeta(t)}$ is parametrized by arc length, then by Theorem A.0.1, its lift $\overline{\gamma}_{\zeta(t)}$ must be contained in $H^{-1}\left(\frac{1}{2}\right)$, which implies that for (X_1, X_2) any choice of frame of the distribution,

$$h_{X_1}\left(\overline{\gamma}_{\zeta(t)}(0)\right) + h_{X_2}\left(\overline{\gamma}_{\zeta(t)}(0)\right) = 1. \tag{A.18}$$

Furthermore, since ζ is parametrized by arc length and leaves from p at time zero, $\tilde{\delta}\left(\zeta(t)\right) \in [0, |t|]$ for every time t, so

$$\tilde{\delta}\left(\zeta(t)\right) \overset{t\to 0}{\longrightarrow} 0. \tag{A.19}$$

By putting together (A.16), (A.17), (A.18) and (A.19), we obtain that for t small enough, $\overline{\zeta}(t)$ belongs to a compact subset of $\mathfrak{S}$. On this compact set, thanks to Proposition A.0.9, function $\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ is uniformly continuous. Now since $\overline{\delta}\left(\overline{\zeta}(t)\right) = \tilde{\delta}\left(\zeta(t)\right) \overset{t\to 0}{\longrightarrow} 0$, and as $\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ is equal to -4 on $\overline{\delta}^{-1}(0) = H^{-1}\left(\frac{1}{2}\right) \cap T^*_pM$, we deduce from the uniform continuity of $\overline{\delta}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}$ on the compact set containing the trajectory of $\overline{\zeta}$ for times small enough, that

$$\overline{\delta}\left(\overline{\zeta}(t)\right)\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\left(\overline{\zeta}(t)\right) \overset{t\to 0}{\longrightarrow} -4.$$

Now we use (A.15) to obtain the first limit in the proposition we are proving.

To prove the other three asymptotics we essentially do the same proof where we may use Propositions A.0.10 and A.0.11 instead of Proposition A.0.9 and where we replace (A.15) by

$$\begin{aligned}
\tilde{\delta}^2\left(\zeta(t)\right) c^{J\Gamma}_{\Gamma,X_0}\left(\zeta(t)\right) &= \overline{\delta}^2\left(\overline{\zeta}(t)\right)\overline{c}^{J\Gamma}_{\Gamma,X_0}\left(\overline{\zeta}(t)\right),\\
\tilde{\delta}^2\left(\zeta(t)\right)\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right) &= \overline{\delta}^2\left(\overline{\zeta}(t)\right)\overrightarrow{H}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\left(\overline{\zeta}(t)\right),\\
\text{and } \tilde{\delta}^2\left(\zeta(t)\right) J\Gamma c^{J\Gamma}_{\Gamma,J\Gamma}\left(\zeta(t)\right) &= \overline{\delta}^2\left(\overline{\zeta}(t)\right)\overline{J\Gamma}\overline{c}^{J\Gamma}_{\Gamma,J\Gamma}\left(\overline{\zeta}(t)\right).
\end{aligned}$$

The proof of these three identity is the same as that of (A.15), to which we add the facts that $\Gamma \circ \pi = \mathrm{d}\pi \circ \overrightarrow{H}$ (see Lemma A.0.4) and $J\Gamma \circ \pi = \mathrm{d}\pi \circ \overline{J\Gamma}$ (by 3.18).

Appendix B

Proof of Proposition 5.2.4

In fact, we prove a more precise result than the one stated in Proposition 5.2.4, since we compute an explicit expression for $\mathfrak{h}$, namely

$$\begin{aligned} h_\zeta(s) =& 2A\left(\ddot{z}(s) - \sum_i \alpha_i(s)\left(\dot{x}_i(s)\mathrm{d}^2_{\zeta(s)}v_{1,X}\left(\mathsf{T}\right) - \dot{y}_i(s)\mathrm{d}^2_{\zeta(s)}v_{1,Y}\left(\mathsf{T}\right)\right)\right) \\ & - \begin{pmatrix} \dot{x}_1(s) \\ \dot{y}_1(s) \\ \vdots \\ \dot{x}_n(s) \\ \dot{y}_n(s) \end{pmatrix}^T \mathrm{d}_{\zeta(s)}\mathscr{J}\left(\mathsf{T}\right) \begin{pmatrix} \dot{x}_1(s) \\ \dot{y}_1(s) \\ \vdots \\ \dot{x}_n(s) \\ \dot{y}_n(s) \end{pmatrix}. \end{aligned}$$

We start by defining matrix $\mathfrak{I}$ whose block made of the intersection of columns $2j-1$ and $2j$ and of lines $2i-1$ and $2i$ is

$$\begin{pmatrix} X_j x_i & Y_j x_i \\ X_j y_j & Y_j y_j \end{pmatrix}.$$

According to Theorem 5.2.1, $\mathfrak{I}$ smoothly depends on its evaluation point and is such that

$$\mathfrak{I}\left(\zeta(s)\right) = \mathrm{Id} \text{ and } \mathrm{d}\mathfrak{I}\left(\zeta(s)\right) = 0. \tag{B.1}$$

In this proof instead of writing $x_i\left(\zeta(t)\right)$, we write $x_i(t)$. Let us also write $\dot{x}_i(t)$ to denote $\mathsf{T}\left(\zeta(t)\right)x_i$. We proceed similarly for all other functions.

The decomposition of vector $\mathsf{T}\left(\zeta(t)\right)$ in the frame (X_i, Y_i) then reads

$$\mathfrak{I}^{-1}\left(t\right)\begin{pmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \\ \dot{x}_2(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{x}_n(t) \\ \dot{y}_n(t) \end{pmatrix}. \tag{B.2}$$

From which we deduce by using the expressions of X_i and Y_i given by Theorem 5.2.1 that

$$\dot{z}(t) = \mathsf{T}(t)\, z = \begin{pmatrix} \alpha_1(t)\left(\frac{y_1(t)}{2} + v_{1,X}(t)\right) \\ -\alpha_1(t)\left(\frac{x_1(t)}{2} + v_{1,Y}(t)\right) \\ \alpha_2(t)\left(\frac{y_2(t)}{2} + v_{2,X}(t)\right) \\ -\alpha_2(t)\left(\frac{x_2(t)}{2} + v_{2,Y}(t)\right) \\ \vdots \\ \alpha_n(t)\left(\frac{y_n(t)}{2} + v_{n,X}(t)\right) \\ -\alpha_n(t)\left(\frac{x_n(t)}{2} + v_{n,Y}(t)\right) \end{pmatrix}^T \mathfrak{I}^{-1} \begin{pmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \\ \dot{x}_2(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{x}_n(t) \\ \dot{y}_n(t) \end{pmatrix}.$$

As a consequence, and by applying (B.1) and the fact that $\zeta(s)$ is the origin of coordinates,

$$\dot{z}(s) = \ddot{z}(s) = 0$$

and

$$\dddot{z}(s) = \sum_i \alpha_i(s) \frac{\ddot{x}_i(s)\dot{y}_i(s) - \ddot{y}_i(s)\dot{x}_i(s)}{2} + \sum_i \alpha_i(s) \left(\dot{x}_i(s) \mathrm{d}^2_{\zeta(s)} v_{1,X}(\mathsf{T}) - \dot{y}_i(s) \mathrm{d}^2_{\zeta(s)} v_{1,Y}(\mathsf{T})\right). \tag{B.3}$$

We now link this value of $\dddot{z}(s)$ with $h_\zeta(s)$. Thanks to Lemma 3.3.3, we know that we can study the projection of $[\mathsf{T}, J\mathsf{T}]$ in the direction of T. Along ζ, we already know the decomposition of T on the frame (X_i, Y_i) from (B.2). Moreover, using (B.2) and Definition 5.2.2, the decomposition of $J\mathsf{T}(t)$ reads

$$\mathscr{J}(t)\, \mathfrak{I}^{-1}(t) \begin{pmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \\ \dot{x}_2(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{x}_n(t) \\ \dot{y}_n(t) \end{pmatrix}.$$

However, in order to compute $[\mathsf{T}, J\mathsf{T}]$, it is not sufficent to define the field T along the curve ζ but we must extend it outside the curve ζ. In order to extend T we start by choosing V any vector field that coincides with $J\mathsf{T}$ along ζ. Then we define T as a smooth vector that extends the velocity of ζ whose expression in the frame (X_i, Y_i) is constant along each integral line

of V. Now by decomposing T and $J\mathsf{T}$ in the frame (X_i, Y_i) we can compute

$$\begin{aligned}
[\mathsf{T}, J\mathsf{T}](\zeta(s)) = &\begin{pmatrix} X_1(\zeta(s)) \\ Y_1(\zeta(s)) \\ \vdots \\ X_n(\zeta(s)) \\ Y_n(\zeta(s)) \end{pmatrix}^T \frac{\partial}{\partial t}_{|t=s} \left(\mathscr{J}(t)\, \mathfrak{I}^{-1}(t) \begin{pmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \\ \vdots \\ \dot{x}_n(t) \\ \dot{y}_n(t) \end{pmatrix} \right) \\
&\underbrace{+ f \frac{\partial}{\partial z}(\zeta(s))}_{\text{term coming from the } [X_i, Y_i]\text{'s}} \qquad \text{(B.4)} \\
= &\begin{pmatrix} X_1(\zeta(s)) \\ Y_1(\zeta(s)) \\ \vdots \\ X_n(\zeta(s)) \\ Y_n(\zeta(s)) \end{pmatrix}^T \left(\mathrm{d}_{\zeta(s)} \mathscr{J}(\mathsf{T}) \begin{pmatrix} \dot{x}_1(s) \\ \dot{y}_1(s) \\ \vdots \\ \dot{x}_n(s) \\ \dot{y}_n(s) \end{pmatrix} + \mathscr{J}(s) \begin{pmatrix} \ddot{x}_1(s) \\ \ddot{y}_1(s) \\ \vdots \\ \ddot{x}_n(s) \\ \ddot{y}_n(s) \end{pmatrix} \right) \\
&+ f A X_0(\zeta(s)),
\end{aligned}$$

where we replaced $\frac{\partial}{\partial z}$ by AX_0 by using Proposition 5.2.3.

Now let us recall that according to Lemma 3.3.3,

$$h_\zeta(s) = -g_{|_{\zeta(s)}}([\mathsf{T}, J\mathsf{T}] + \mathrm{Tor}(\mathsf{T}, J\mathsf{T}), \mathsf{T}). \qquad \text{(B.5)}$$

In order to deduce the expression of $h_\zeta(s)$ from that of $[\mathsf{T}, J\mathsf{T}](\zeta(s))$ we therefore must understand the $\mathrm{Tor}(\mathsf{T}, J\mathsf{T})$ term.

First of all, we know from Theorem 3.1.3 iv. that $\mathrm{Tor}(\mathsf{T}, J\mathsf{T})$ is parallel to the Reeb vector field. Moreover we notice that

$$[\mathsf{T}, J\mathsf{T}] + \mathrm{Tor}(\mathsf{T}, J\mathsf{T}) = \nabla_{\mathsf{T}}(J\mathsf{T}) - \nabla_{J\mathsf{T}}\mathsf{T},$$

where the right hand side is horizontal since by Theorem 3.1.3 i. the differential of a horizontal vector field with respect to the Tanno connection is horizontal. As a consequence, $[\mathsf{T}, J\mathsf{T}] + \mathrm{Tor}(\mathsf{T}, J\mathsf{T})$ is the projection of $[\mathsf{T}, J\mathsf{T}]$ on the distribution parallely to the Reeb vector field. Therefore, by (B.4), the expression of $[\mathsf{T}, J\mathsf{T}] + \mathrm{Tor}(\mathsf{T}, J\mathsf{T})$ in the frame (X_i, Y_i) is

$$\mathrm{d}_{\zeta(s)} \mathscr{J}(\mathsf{T}) \begin{pmatrix} \dot{x}_1(s) \\ \dot{y}_1(s) \\ \vdots \\ \dot{x}_n(s) \\ \dot{y}_n(s) \end{pmatrix} + \mathscr{J}(s) \begin{pmatrix} \ddot{x}_1(s) \\ \ddot{y}_1(s) \\ \vdots \\ \ddot{x}_n(s) \\ \ddot{y}_n(s) \end{pmatrix}.$$

In this last expression we can replace $\mathscr{J}(s)$ by its expression given by Proposition 5.2.3. If we combine the decomposition of $[\mathsf{T}, J\mathsf{T}] + \mathrm{Tor}(\mathsf{T}, J\mathsf{T})$ in the

frame (X_i, Y_i) that we now obtain, with that of T that is given by (B.2) and (B.1) and with (B.5), we can conclude that

$$\begin{aligned} h_\zeta(s) =& A \sum_i \alpha_i(s) \left(\ddot{x}_i(s)\dot{y}_i(s) - \ddot{y}_i(s)\dot{x}_i(s)\right) \\ & - \begin{pmatrix} \dot{x}_1(s) \\ \dot{y}_1(s) \\ \vdots \\ \dot{x}_n(s) \\ \dot{y}_n(s) \end{pmatrix}^T \mathrm{d}_{\zeta(s)} \mathscr{J}(\mathsf{T}) \begin{pmatrix} \dot{x}_1(s) \\ \dot{y}_1(s) \\ \vdots \\ \dot{x}_n(s) \\ \dot{y}_n(s) \end{pmatrix}. \end{aligned} \tag{B.6}$$

Identities (B.3) and (B.6) entail the result.

Appendix C

Computing the second differential of the squared distance from a point

We start by explaining how normal moving frames such as described along one single integral line of the the Hamiltonian vector field in Theorem 5.3.2 are linked along different integral lines of this Hamiltonian vector field.

Proposition C.0.1. *Let M be a contact sub-Riemannian manifold. We consider $q_0 \in M$. Let us lift Σ_{q_0} the set of smooth points around q_0 to the cotangent bundle by considering $\overline{\Sigma}_{q_0} \subset T^*M$ that stands for the union of all Hamiltonian lifts of geodesics that leave from q_0, are parametrized by arc length and are restricted to the times stricly smaller than the cut time with respect to q_0.*

There exists a smooth frame $(E_i, F_i)_{i=0}^{2n}$ of $T(T^*M)$ that is defined over $\overline{\Sigma}_{q_0}$ such that the restrictions of this frame to the integral lines of $\vec{H}$ are normal moving frames whose vector F_2 is equal to $\vec{H}$ and whose associated R-matrix is also smooth over $\overline{\Sigma}_{q_0}$.

To such a frame $(E_i, F_i)_{i=0}^{2n}$ of $T(T^*M)$ that is defined over $\overline{\Sigma}_{q_0}$ we can associate a frame of TM

$$(f_i) = \pi_* (F_i)$$

over Σ_{q_0} since the canonical projection $\pi : T^*M \to M$ is a diffeomorphism between $\overline{\Sigma}_{q_0} \setminus T^*_{q_0}M \subset T^*M$ and Σ_{q_0}.

Proof. The fact that the vector spaces $\langle E_0 \rangle$, $\langle F_0 \rangle$, $\langle E_1 \rangle$, $\langle F_1 \rangle$, $\langle E_i \rangle_{i=2}^{2n}$ and $\langle F_i \rangle_{i=2}^{2n}$ as well as the endomorphism that is represented by the matrix R can be chosen smoothly on $\overline{\Sigma}_{q_0}$ is a consequence of explicit computation that we can find in [ABR17, Section 6] and in [LZ11]. In addition, this construction provides us with an actual smooth frame $(E_i, F_i)_{i=0}^{2n}$. However

we must check that it is possible to choose F_2 to be equal to $\vec{H}$. First of all, the smooth vector field $\vec{H}$ belongs to $\langle F_i \rangle_{i=2}^{2n}$ thanks to the fact that along *one* geodesic $\vec{H}$ can be chosen equal to F_2 according to lemma 5.3.3 and since the space $\langle F_i \rangle_{i=2}^{2n}$ is independant of the chosen normal moving frame. In particular, for $p \in \left\{\lambda \in TM : \left\|\vec{H}(\lambda)\right\| = 1\right\}$ it is possible to chose an orthogonal matrix $O(p)$ that smoothly depends on p and that sends $(F_i(p))_{i=2}^{2n}$ onto a family of vectors whose first element is $\vec{H}(p)$. Now from Theorem 5.3.2 this allows us to build over $\overline{\Sigma}_{q_0}$ a smooth frame $(E_i, F_i)_{i=0}^{2n}$ that is a normal moving frame along the integral lines of $\vec{H}$ and that coincides with $\overline{H}$ over $\left\{\lambda \in TM : \left\|\vec{H}(\lambda)\right\| = 1\right\}$. Thanks to [ABR18, Section 7.5.4] adapted to the contact case, field F_2 actually conicides with $\vec{H}$ on $\overline{\Sigma}_{q_0}$.

The proof is complete but let us just point out that papers [ABR17, Section 6] and [LZ11] we use actually do not treat the complete contact sub-Riemannian case, but [ABR17, Section 6] assumes that $J^2 = -\mathrm{Id}$ and [LZ11] assumes some symmetry. Nevertheless, both paper refer to a general algorithm given in [ZL09] that may be used to prove the regularity in the general contact case. □

Lemma C.0.2. *Let M be a contact sub-Riemannian manifold. We consider $q_0 \in M$.*

For any geodesic γ_0 that is parametrized by arc length and that leaves from q_0 at time zero, there exist

(i) an interval I containing zero,

(ii) an open set U that contains $\gamma_0(I)$,

(iii) an open set $\mathscr{G}$ of geodesics parametrized by arc length that leave from q_0 at time zero ("open" in the sense that the evaluation at time zero of the Hamiltonian lifts of each of these geodesics forms a open set),

(iv) and a family of orthonormal smooth frames (f_i^γ) that are defined over U and indexed on $\gamma \in \mathscr{G}$

such that

(i) for every $\gamma \in \mathscr{G}$, frame (f_i^γ) coincides with frame (f_i) that is described in Proposition C.0.1 along $\gamma(I)$,

(ii) frame (f_i^γ) smoothly depends on the geodesic γ, by which we mean that it smoothly depends on the evaluation at time zero of the Hamiltonian lifts of γ.

Proof. We prove this lemma by explicitly building the vector fields f_i^γ by only using smooth operations. First of all, there exist $I \subset \mathbb{R}$ a time interval that contains zero such that γ_0 is injective on I and a diffeomorphism $\mathfrak{M}$:

$U \to I \times V \subset \mathbb{R}^{2n+1}$ that sends $\gamma_0(t)$ onto $(t,0)$. Moreover, there exists an open set $\mathscr{G}$ of geodesics parametrized by arc length that leave from q_0 ("open" in the sense we mentionned in the statement of the result) that contains γ_0 such that for every $\gamma \in \mathscr{G}$, the trajectory $\mathfrak{M}(\gamma(I))$ is transverse to the sets $\{t\} \times V$. Then we define the vector field $\tilde{f}_i^\gamma$ on U that coincides with f_i along $\gamma(I)$ and such that the image of $\tilde{f}_i^\gamma$ through $\mathfrak{M}$ is constant on the sets $\{t\} \times V$. Finally we project the field $\tilde{f}_i^\gamma$ on the distribution parallely to the Reeb vector field and normalize this projection to obtain the field f_i^γ.

The smoothness of f_i^γ as a function of γ is a consequence of the smoothness of the vector field F_i described in Proposition C.0.1 from which we build f_i and therefore indirectly f_i^γ. □

We must now explain how a Hamiltonian perspective involving a normal moving frame can help us to prove Proposition 5.3.4.

C.1 Proof of Proposition 5.3.4

Let us start by introducing the notion of second differential of a real-valued function which is defined on a manifold. For

$$\mathfrak{f} : M \longrightarrow \mathbb{R},$$

we have

$$\mathrm{d}\mathfrak{f} : M \longrightarrow T^*M,$$

so for every q in M we can define

$$\begin{array}{rccc} \mathrm{d}_q^2\mathfrak{f} : & T_qM & \longrightarrow & T_{\mathrm{d}_q f}T^*M \\ & v & \longmapsto & \mathrm{d}_q\mathrm{d}\mathfrak{f}(v). \end{array}$$

Now that we have set the proper definition for the second differential of a real-valued function, let us point out an important property of the squared distance. If q belongs to Σ_s then the integral line of the Hamiltonian flow that reaches the covector

$$\frac{1}{2t}\mathrm{d}_q\delta_s^2$$

at time t is a lift of the minimizing geodesic that leaves from $\zeta(s)$ at time zero and whose speed is compatible with the fact of reaching q at time t. We can for example find a proof of this property for $t = 1$ in [ABB19, Proposition 11.4] to which one can add the time parameter t by applying [ABB19, Remark 4.26].

As a consequence for every q in Σ_s and t in $\mathbb{R} \setminus \{0\}$,

$$\pi\left(\mathrm{e}^{-t\vec{H}}\,\frac{1}{2t}\mathrm{d}_q\delta_s^2\right) = \zeta(s),$$

and

$$\pi_* \, \mathrm{e}_*^{-t\vec{H}} \, \frac{1}{2t} \mathrm{d}_q^2 \delta_s^2 = 0. \tag{C.1}$$

At this point in the proof, let us focus on one single geodesic γ that is parametrized by arc length and leaves from $\zeta(s)$ at time zero. Later on in the proof, when we have enough information about the asymptotics we are studying along one geodesic then we can link what happens along several geodesics to come back to the curve ζ we are actually interested in. Until then, let us "forget about" the notations that come from Proposition C.0.1 and Lemma C.0.2, since we simply use the notations in Theorem 5.3.2.

Theorem 3.4.3 provides us with $\overline{\gamma}$, a lift of γ that is an integral line of $\vec{H}$. Along $\overline{\gamma}$, we consider a normal moving frame $(E_i, F_i)_{i=0}^{2n}$ such as described in Theorem 5.3.2. Along γ, we also can define a frame

$$(f_i)_{i=0}^{2n} := \pi_* \, (F_i)_{i=0}^{2n} \, .$$

Now notice that according to Theorem 5.3.2, at every point $\overline{\gamma}(t)$, the family $(E_i(t))_{i=0}^{2n}$ is a frame of the vertical subspace of $T_{\overline{\gamma}(t)} T^*M$, in other words of the set of vectors in $T_{\overline{\gamma}(t)} T^*M$ that vanish through π_*. This last remark combined with (C.1) allows us to write for every time $t \in \mathbb{R}$,

$$\mathrm{e}_*^{-t\vec{H}} \, \mathrm{d}^2_{\gamma(t)} \frac{1}{2t} \delta_s^2 \begin{pmatrix} f_0 \left(\gamma(t)\right) \\ f_1 \left(\gamma(t)\right) \\ \vdots \\ f_{2n} \left(\gamma(t)\right) \end{pmatrix} = \theta \left(t\right) \begin{pmatrix} E_0 \left(\overline{\gamma}(t)\right) \\ E_1 \left(\overline{\gamma}(t)\right) \\ \vdots \\ E_{2n} \left(\overline{\gamma}(t)\right) \end{pmatrix}, \tag{C.2}$$

where $\theta(t)$ is an element of $M_{2n+1}\left(\mathbb{R}\right)$. For the sake of simplicity, in this proof we simply write $f(t)$ to stand for the vector whose components are the $f_i \left(\gamma(t)\right)$, and we do the same for E.

At this point, we introduce, for every time $t \in \mathbb{R}$ two matrices $A(t)$ and $B(t)$ both in $M_{2n+1}\left(\mathbb{R}\right)$ such that

$$\mathrm{e}^{t\vec{H}} \, E(0) = A(t)E(t) + B(t)F(t). \tag{C.3}$$

So we can write, for $t \in \mathbb{R}$,

$$\begin{aligned} f(t) &= \pi_* \mathrm{d}^2_{\gamma(t)} \frac{1}{2t} \delta_s^2 \left(f(t)\right), \\ &= \pi_* \, \mathrm{e}^{t\vec{H}} \, \theta(t) E(0) \text{ according to (C.2)}, \\ &= \theta(t) \pi_* \, \mathrm{e}^{t\vec{H}} \, E(0), \\ &= \theta(t) \pi_* \left(A(t)E(t) + B(t)F(t)\right) \text{ thanks to (C.3)}, \\ &= \theta(t) B(t) f(t) \text{ as } E \text{ is vertical and } \pi(F) = f \text{ by definition.} \end{aligned}$$

Since $f(t)$ is a frame of $T_{\gamma(t)}M$, we deduce from the previous identity that

$$\theta(t) = B^{-1}(t). \tag{C.4}$$

We carry on our computations.

$$\begin{aligned}
\mathcal{L}_{\vec{H}} \mathrm{d}^2_{\gamma(t)} \frac{1}{2t} \delta_s^2 \left(f(t)\right) &= \mathrm{e}^{t\vec{H}} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-t\vec{H}} \frac{1}{2t} \mathrm{d}^2_{\gamma(t)} \delta_s^2 \left(f(t)\right), \\
&= \mathrm{e}^{t\vec{H}} \frac{\mathrm{d}}{\mathrm{d}t} B^{-1}(t) E(0) \text{ by combining (C.2) and (C.4)}, \\
&= \left(B^{-1}\right)'(t) \, \mathrm{e}^{t\vec{H}} \, E(0), \\
&= \left(B^{-1}\right)'(t) \left(A(t)E(t) + B(t)F(t)\right) \text{ by using (C.3)}.
\end{aligned} \tag{C.5}$$

We continue by searching for another expression of $\mathcal{L}_{\vec{H}} \frac{1}{2t} \mathrm{d}^2_{\gamma(t)} \delta_s^2 f(t)$ than the one given by the previous identity. In order to do this, along $\overline{\gamma}$ instead of considering the frame $(E_i, F_i)_{i=0}^{2n}$, we use the frame $\left(\overline{f}_i, \frac{\partial}{\partial h_{f_i}}\right)_{i=0}^{2n}$. We recall that this frame is defined by identities (3.18) and (3.19).

Now an important identity that we find in [ABR18, Lemma 8.2] reads

$$\mathrm{d}^2_{\gamma(t)} \frac{1}{2t} \delta_s^2 \left(f_i(t)\right) = \overline{f}_i(t) + \sum_{j=0}^{2n} f_i(t) \left(f_j \frac{1}{2t} \delta_s^2\right) \frac{\partial}{\partial h_{f_j}}, \tag{C.6}$$

for $t \in \mathbb{R}$ and $i \in [\![0, 2n]\!]$.

Let us link the frames $(E_i, F_i)_{i=0}^{2n}$ and $\left(\overline{f}_i, \frac{\partial}{\partial h_{f_i}}\right)_{i=0}^{2n}$ along the curve $\overline{\gamma}$ in order to state (C.6) in the frame $(E_i, F_i)_{i=0}^{2n}$.

We start by noticing that along $\overline{\gamma}$ for $i \in [\![0, 2n]\!]$ by Theorem 5.3.2 and (3.19),

$$\pi_* \left(E_i\right) = 0 = \pi_* \left(\frac{\partial}{\partial h_{f_i}}\right), \tag{C.7}$$

and that for $j \in [\![0, 2n]\!]$,

$$\begin{aligned}
\sigma\left(E_i, F_j\right) &= \delta_{i,j} \text{ as explained in Theorem 5.3.2}, \\
&= \frac{\partial}{\partial h_{f_i}} h_{f_j} \\
&= \frac{\partial}{\partial h_{f_i}} \tau\left(F_j\right) \text{ by (3.11) and (3.13)}, \\
&= \tau\left(\left[\frac{\partial}{\partial h_{f_i}}, F_j\right]\right) + F_j \tau\left(\frac{\partial}{\partial h_{f_i}}\right) + \mathrm{d}\tau\left(\frac{\partial}{\partial h_{f_i}}, F_j\right) \\
&= \mathrm{d}\tau\left(\frac{\partial}{\partial h_{f_i}}, F_j\right) \text{ by (3.13) as } \pi_*\left[\frac{\partial}{\partial h_{f_i}}, F_j\right] = \pi_*\left(\frac{\partial}{\partial h_{f_i}}\right) = 0, \\
&= \sigma\left(\frac{\partial}{\partial h_{f_i}}, F_j\right) \text{ by definition of } \sigma.
\end{aligned} \tag{C.8}$$

We combine (C.7) and (C.8) and we deduce that along $\overline{\gamma}$,

$$E_i = \frac{\partial}{\partial h_{f_i}}. \tag{C.9}$$

Then we write that along $\overline{\gamma}$, for $i \in [\![1, 2n]\!]$,

$$\begin{aligned} F_i &= -\mathcal{L}_{\vec{H}} E_i \text{ according to Theorem 5.3.2,} \\ &= -\mathcal{L}_{\vec{H}} \frac{\partial}{\partial h_{f_i}} \text{ by applying (C.9),} \\ &= \overline{f}_i + \sum_{j=0}^{2n} \left(\overline{c}^0_{i,j} h_{f_0} + \sum_{k=1}^{2n} \left(\overline{c}^k_{i,j} + \overline{c}^i_{k,j} \right) h_{f_k} \right) \frac{\partial}{\partial h_{f_j}}, \end{aligned} \tag{C.10}$$

where the last line is provided by Proposition 3.4.2.

By assembling (C.6), (C.9) and (C.10) we find out that along $\overline{\gamma}$ and for $i \in [\![1, 2n]\!]$,

$$\begin{aligned} \mathrm{d}^2_{\gamma(t)} \frac{1}{2t} \delta_s^2 \left(f_i(t) \right) =& F_i(t) + \sum_{j=0}^{2n} f_i(t) \left(f_j \frac{1}{2t} \delta_s^2 \right) E_j(t) \\ &- \sum_{j=0}^{2n} \left(\overline{c}^0_{i,j} h_{f_0} + \sum_{k=1}^{2n} \left(\overline{c}^k_{i,j} + \overline{c}^i_{k,j} \right) h_{f_k} \right) E_j(t). \end{aligned} \tag{C.11}$$

For $i \in [\![1, 2n]\!]$, if we take the Lie differential of the previous identity in the direction of $\vec{H}$, we obtain an expression of $\mathcal{L}_{\vec{H}} \mathrm{d}^2_{\gamma(t)} \frac{1}{2t} \delta_s^2 \left(f_i(t) \right)$ along $\overline{\gamma}$. But (C.5) gives us another expression of that same quantity. Now, for $i \in [\![1, 2n]\!]$ we consider the equation that identifies both the expressions of $\mathcal{L}_{\vec{H}} \mathrm{d}^2_{\gamma(t)} \frac{1}{2t} \delta_s^2 \left(f_i(t) \right)$ given by (C.5) and (C.11). Then we project the identity we obtain on the vector space generated by the F_j's and we have

$$\begin{aligned} \left(\left(B^{-1} \right)' (t) B(t) F(t) \right)_i =& - \delta_{i,1} F_0(t) - \sum_{j=1}^{2n} f_i(t) \left(f_j \frac{1}{2t} \delta_s^2 \right) F_j(t) \\ &+ \sum_{j=1}^{2n} \left(\overline{c}^0_{i,j} h_{f_0} + \sum_{k=1}^{2n} \left(\overline{c}^k_{i,j} + \overline{c}^i_{k,j} \right) h_{f_k} \right) F_j(t), \end{aligned} \tag{C.12}$$

along $\overline{\gamma}$, for any $i \in [\![1, 2n]\!]$.

By observing the previous identity, we see that in order to understand the asymptotics of the second differential of the squared distance from $\zeta(s)$, we can study the behaviour of matrix $B(t)$ for small times t. To do this, along $\overline{\gamma}$ we take the Lie differential of (C.3) in the direction of $\vec{H}$ and we obtain

$$0 = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} A(t) \,\middle|\, \tfrac{\mathrm{d}}{\mathrm{d}t} B(t) \right) \begin{pmatrix} E(t) \\ F(t) \end{pmatrix} + \left(A(t) \,\middle|\, B(t) \right) \begin{pmatrix} \mathcal{L}_{\vec{H}} E(t) \\ \mathcal{L}_{\vec{H}} F(t) \end{pmatrix}.$$

In the previous identity, we replace $\mathcal{L}_{\vec{H}}E(t)$ and $\mathcal{L}_{\vec{H}}F(t)$ by their expressions in terms of $E(t)$ and $F(t)$ given in Theorem 5.3.2, which leads us to writing

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{array}{c|c} A & B \end{array}\right)(t) = -\left(\begin{array}{c|c} A & B \end{array}\right)(t) \left(\begin{array}{cc||c|c} 0 & 1 \quad 0 \quad \cdots \quad 0 & 0 & 0 \\ \hline & & & \\ & (0) & 0 & -I \\ & & & \\ \hline & & 0 & \\ & & -1 & \\ & R(t) & 0 & (0) \\ & & \vdots & \\ & & 0 & \end{array}\right). \tag{C.13}$$

The previous differential equation on $A(t)$ and $B(t)$ combined with the initial conditions that are implied by (C.3) namely $A(0) = \mathrm{Id}$ and $B(0) = 0$, allow us to understand the asymptotics of $B(t)$ and of $\left(B^{-1}\right)'(t)B(t)$. More precisely, by studying the differential equation on $A(t)$ and $B(t)$ we obtain that

$$\begin{aligned} B(t) = {} & t\left(\begin{array}{cc|c} 0 & 0 & (0) \\ 0 & 1 & \\ \hline (0) & & \mathrm{I} \end{array}\right) + \frac{t^2}{2}\left(\begin{array}{cc|c} 0 & -1 & (0) \\ 1 & 0 & \\ \hline (0) & & (0) \end{array}\right) \\ & + \frac{t^3}{6}\left[\left(\begin{array}{cc|c} -1 & 0 & (0) \\ 0 & 0 & \\ \hline (0) & & (0) \end{array}\right) - \left(\begin{array}{cc|c} 0 & 0 & (0) \\ 0 & 1 & \\ \hline (0) & & \mathrm{I} \end{array}\right) R(t) \left(\begin{array}{cc|c} 0 & 0 & (0) \\ 0 & 1 & \\ \hline (0) & & \mathrm{I} \end{array}\right)\right] \\ & + \mathcal{O}\left(t^4\right). \end{aligned}$$

It follows that in a $(2n+1)-$dimensional space,

$$\det\left(B(t)\right) = \frac{t^{2n+3}}{12} + \mathcal{O}\left(t^{2n+4}\right).$$

We then use the adjugate matrix of B to compute

$$B^{-1}(t) = \left(\begin{array}{cc|c} \frac{12}{t^3} + \mathcal{O}\left(\frac{1}{t}\right) & \frac{6}{t^2} + \mathcal{O}(1) & \mathcal{O}(1) \\ -\frac{6}{t^2} + \mathcal{O}(1) & \frac{-2}{t} + \mathcal{O}(1) & \\ \hline \mathcal{O}(1) & & \frac{1}{t}\mathrm{I} + \mathcal{O}(1) \end{array}\right).$$

Now we can compute

$$B^{-1}(t)B'(t) = \left(\begin{array}{c|ccc} \cdot & \multicolumn{3}{c}{\cdot} \\ \hline & \frac{4}{t} & \multicolumn{2}{c}{(0)} \\ \cline{2-4} & & \frac{1}{t} & (0) \\ \cdot & (0) & & \ddots \\ & & (0) & & \frac{1}{t} \end{array}\right) + \mathcal{O}(1).$$

Then we notice that $\left(B^{-1}\right)'(t)B(t) = -B^{-1}(t)B'(t)$ and we are able to write

$$\left(B^{-1}\right)'(t)B(t) = \left(\begin{array}{c|ccc} \cdot & \multicolumn{3}{c}{\cdot} \\ \hline & -\frac{4}{t} & \multicolumn{2}{c}{(0)} \\ \cline{2-4} & & -\frac{1}{t} & (0) \\ \cdot & (0) & & \ddots \\ & & (0) & & -\frac{1}{t} \end{array}\right) + \mathcal{O}(1). \tag{C.14}$$

Let us now come back to the study of ζ. To do this, we apply the process we just explained along each minimizing geodesic parametrized by arc length leaving from $\zeta(s)$ at time zero, by choosing the frame $(E_i, F_i)_{i=0}^{2n}$ to be equal to the frame that is described in Proposition C.0.1. Now according to Proposition C.0.1, the matrix R is smooth for this choice of frame, so if we do the same reasoning as we did along one single geodesic γ again, but this time along any compact set of geodesics parametrized by arc length leaving from the $\zeta(s)$ at time zero, result (C.14) still holds uniformely with respect to the geodesic we are considering.

But according to Theorem 5.2.5 the minimizing geodesics parametrized by arc length that reach $\zeta(t)$ for $t \in [s, s+\varepsilon]$ are limited to a compact set ("compact" in terms of initial Hamiltonian lift). As a consequence, if we define $\tilde{B}_q$ along γ_q exactly as B was defined along γ, then along ζ we can rewrite (C.14) as

$$\left(\tilde{B}^{-1}_{\zeta(t)}\right)'\left(\delta_s(t)\right)\tilde{B}_{\zeta(t)}\left(\delta_s(t)\right) = \frac{-1}{\delta_s(t)}\left(\begin{array}{c|ccc} \cdot & \multicolumn{3}{c}{\cdot} \\ \hline & 4 & \multicolumn{2}{c}{(0)} \\ \cline{2-4} & & 1 & (0) \\ \cdot & (0) & & \ddots \\ & & (0) & & 1 \end{array}\right) + \mathcal{O}(1). \tag{C.15}$$

Now we recall Lemma C.0.2 and we replace the fields f_i by the fields $f_i^{\gamma_{s,\zeta(t)}}$ in identity (C.12), that we evaluate at $\zeta(t)$. If we combine the identity we obtain with (C.15) and with Lemma 5.3.1, we are almost done with the proof. We just need to understand all the terms that are involved.

The terms in (C.12) that involve Lie brackets $c_{i,j}^k$ and Hamiltonian coefficients h_i are bounded since the fields $f_i^{\gamma_{s,\zeta(t)}}$ whose Lie brackets we consider are smooth, smoothly depend on the geodesic $\gamma_{s,\zeta(t)}$ along which they are evaluated, which are restricted to a compact set of geodesics by Theorem 5.2.5. For the same reason, the term

$$g_{\zeta(t)}\left(\Gamma_s, \nabla_{f_i^{\gamma_{s,\zeta(t)}}} f_j^{\gamma_{s,\zeta(t)}}\right)$$

that comes from Lemma C.0.2 is bounded. Finally, by construction vector $f_2^{\gamma_{s,\zeta(t)}}\left(\zeta(t)\right)$ coincides with $\Gamma_s\left(\zeta(t)\right)$ and Lemma 5.3.3 tells us that $\left(f_i^{\gamma_{s,\zeta(t)}}\left(\zeta(t)\right)\right)_{i=1}^{2n}$

is an orthonormal frame of $\Delta_{\zeta(t)}$ such that $f_1^{\gamma_s,\zeta(t)}(\zeta(t))$ is equal to $J\Gamma_s(\zeta(t))$. In particular, thanks to Theorem 3.2.3, we can compute another term that appears in Lemma C.0.2 :

$$f_2^{\gamma_s,\zeta(t)}(\zeta(t))\,\tilde{\delta}_s = 1 \text{ and } f_i^{\gamma_s,\zeta(t)}(\zeta(t))\,\tilde{\delta}_s = 0, \text{ for } i \neq 2.$$

Now that we know the asymptotics of every term that is involved, we can combine (C.12) with (C.15) and with Lemma 5.3.1 to prove Proposition 5.3.4.

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Titre : De la notion de courbure géodésique en géométrie sous-Riemannienne

Mots clés : géométrie sous-Riemannienne, courbure géodésique, distance

Résumé : Dans cette thèse, on présente une notion de courbure géodésique pour les courbes lisses horizontales dans une variété sous-Riemannienne de contact, qui indique dans quelle mesure une courbe est différente d'une géodésique. Cette courbure géodésique se présente sous la forme de deux fonctions qui sont toutes deux identiquement nulles le long d'une courbe lisse horizontale si et seulement si cette dernière courbe est une géodésique. Le résultat principal de cette thèse réside dans l'interprétation métrique que l'on donne de ces fonctions de courbure. Cette interprétation consiste à extraire la courbure géodésique des premiers termes de correction dans le développement limité de la distance sous-Riemannienne entre deux points proches le long de la courbe.

Title : On the notion of geodesic curvature in sub-Riemannian geometry

Keywords : sub-Riemannian geometry, geodesic curvature, distance

Abstract : We present a notion of geodesic curvature for smooth horizontal curves in a contact sub-Riemannian manifold, measuring how far a horizontal curve is from being a geodesic. This geodesic curvature consists in two functions that both vanish along a smooth horizontal curve if and only if this curve is a geodesic. The main result of this thesis is the metric interpretation of these geodesic curvature functions. This interpretation consists in seeing the geodesic curvature functions as the first corrective coefficients in the Taylor expansion of the sub-Riemannian distance between two close points on the curve.

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