p-adic and mod p local-global compatibility for GLn(p)

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Résumé

Titre: la compatibilité local-global p-adique et modulo p pour $\mathrm{GL}_n(\mathbb{Q}_p)$

Cette thèse est consacrée à deux aspects du programme de Langlands local p-adique et de la compatibilité local-global p-adique.

Dans la première partie, j'étudie la question de savoir comment extraire, d'un certain sous-espace Hecke-isotypique de formes automorphes modulo p, suffisament d'invariants d'une représentation galoisienne. Soient p un nombre premier, $n > 2$ un entier, et F un corps à multiplication complexe dans lequel p est complètement décomposé. Supposons qu'une représentation galoisienne automorphe continue $\overline{r} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ est triangulaire supérieure, Fontaine–Laffaille et suffisament générique (dans un certain sens) en une place w au-dessus de p. On montre, en admettant un résultat d'élimination de poids de Serre prouvé dans [LLMPQ], que la classe d'isomorphisme de $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ est déterminée par l'action de $\mathrm{GL}_n(F_w)$ sur un espace de formes automorphes modulo p découpé par l'idéal maximal associée à $\overline{r}$ dans une algèbre de Hecke. En particulier, on montre que la partie sauvagement ramifiée de $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ est déterminée par l'action de sommes de Jacobi (vus comme éléments de $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$) sur cet espace.

La deuxième partie de ma thèse vise à établir une relation entre les résultats précédents de [Schr11], [Bre17] et [BD18]. Soient E une extension finie de $\mathbb{Q}_p$ suffisamment grande et $\rho_p : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_3(E)$ une représentation p-adique semi-stable telle que la représentation de Weil-Deligne $\mathrm{WD}(\rho_p)$ associée a un opérateur de monodromie N de rang 2 et que la filtration de Hodge associée est non-critique. On sait que la filtration de Hodge de ρ_p dépend de trois invariants dans E. On construit une famille de représentations localement analytiques $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ qui dépend de trois invariants $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ et telle que chaque représentation contient la représentation localement algébrique $\mathrm{Alg} \otimes \mathrm{Steinberg}$ déterminée par ρ_p. Quand ρ_p provient, pour un groupe unitaire convenable $G_{/\mathbb{Q}}$, d'une représentation automorphe π de $G(\mathbb{A}_{\mathbb{Q}_p})$ avec un niveau fixé U^p premier avec p, on montre (sous quelques hypothèses techniques) qu'il existe une unique représentation localement analytique dans la famille ci-dessus qui est une sous-représentation du sous-espace Hecke-isotypique associé dans la cohomologie complétée de niveau U^p. On rappelle que [Bre17] a construit une famille de représentations localement analytiques qui dépend de quatre invariants (voir (4) dans [Bre17]) avec une propriété similaire. On donne un critère purement de théorie de représentation: si une représentation Π dans la famille de Breuil se plonge dans un certain sous-espace Hecke-isotypique de la cohomologie complétée, alors elle se plonge nécessairement dans une $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ pour certains choix de $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ qui sont déterminés explicitement par Π. De plus, certains sous-quotients naturels de $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ permettent de construite un complexe de représentations localement analytiques qui ≪réalise≫ l'objet dérivé abstrait $\Sigma(\lambda, \underline{\mathscr{L}})$ defini dans [Schr11]. Par conséquent, la famille de représentations localement analytiques $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ fournit une relation entre les $\mathscr{L}$- invariants supérieurs étudiés dans [Bre17] et [BD18], et la fonction dilogarithme p-adique qui est apparue dans la construction de $\Sigma(\lambda, \underline{\mathscr{L}})$ dans [Schr11].

Mots clé : programme de Langlands local p-adique et modulo p, compatibilité local-global, Fontaine–Laffaille, sommes de Jacobi, représentations de Steinberg, famille de représentations localement analytiques, fonction dilogarithme p-adique

Abstract

Title: p-adic and mod p local-global compatibility for $\mathrm{GL}_n(\mathbb{Q}_p)$

This thesis is devoted to two aspects of the p-adic local Langlands program and p-adic local-global compatibility.

In the first part, I study the problem of how to capture enough invariants of a local Galois representation from a certain Hecke-isotypic subspace of mod p automorphic forms. Let p be a prime number, $n > 2$ an integer, and F a CM field in which p splits completely. Assume that a continuous automorphic Galois representation $\overline{r} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is upper-triangular and satisfies certain genericity conditions at a place w above p, and that every subquotient of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ of dimension > 2 is Fontaine–Laffaille generic. We show that the isomorphism class of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is determined by $\mathrm{GL}_n(F_w)$-action on a space of mod p algebraic automorphic forms cut out by the maximal ideal of a Hecke algebra associated to $\overline{r}$, assuming a weight elimination result which is now a theorem of Bao V. Le Hung to be proven in [LLMPQ]. In particular, we show that the wildly ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is determined by the action of Jacobi sum operators (seen as elements of $\mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$) on this space.

The second part of my thesis aims at clarifying the relation between previous results in [Schr11], [Bre17] and [BD18]. Let E be a sufficiently large finite extension of $\mathbb{Q}_p$ and ρ_p be a p-adic semi-stable representation $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_3(E)$ such that the Weil–Deligne representation $\mathrm{WD}(\rho_p)$ associated with it has rank two monodromy operator N and the Hodge filtration associated with it is non-critical. We know that the Hodge filtration of ρ_p depends on three invariants in E. We construct a family of locally analytic representations $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ depending on three invariants $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ with each of the representation containing the locally algebraic representation $\mathrm{Alg} \otimes \mathrm{Steinberg}$ determined by ρ_p. When ρ_p comes from an automorphic representation π of $G(\mathbb{A}_{\mathbb{Q}_p})$ with a fixed level U^p prime to p for a suitable unitary group $G_{/\mathbb{Q}}$, we show (under some technical assumption) that there is a unique locally analytic representation in the above family that occurs as a subrepresentation of the associated Hecke-isotypic subspace in the completed cohomology with level U^p. We recall that [Bre17] constructed a family of locally analytic representations depending on four invariants (c.f. (4) in [Bre17]) with a similar property. We give a purely representation theoretic criterion: if a representation Π in Breuil's family embeds into a certain Hecke-isotypic subspace of completed cohomology, then it must equally embed into $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ for certain choices of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ determined explicitly by Π. Moreover, certain natural subquotients of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ allow us to construct a complex of locally analytic representations that realizes the abstract derived object $\Sigma(\lambda, \underline{\mathscr{L}})$ in [Schr11]. Consequently, the family of locally analytic representations $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ give a relation between the higher $\mathscr{L}$-invariants studied in [Bre17] as well as [BD18] and the p-adic dilogarithm function which appears in the construction of $\Sigma(\lambda, \underline{\mathscr{L}})$ in [Schr11].

Key words: p-adic and mod p local Langlands program, local-global compatibility, Fontaine–Laffaille, Jacobi sums, Steinberg representation, family of locally analytic representations, p-adique dilogarithm function

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Chapter 1

Introduction générale

1.1 Conjecture de compatibilité locale-globale p-adique

Soit p un nombre premier, K une extension finie de $\mathbf{Q}_p$ et E une autre extension finie de $\mathbf{Q}_p$ qui est suffisamment grande. On note $\mathcal{O}_K$ (resp. ϖ_K, resp. k) l'anneau des entiers (resp. une uniformisante, resp. le corps résiduel) de K; On note $\mathcal{O}_E$ (resp. ϖ_E, resp. k_E) l'anneau des entiers (resp. une uniformisante, resp. le corps résiduel) de E. Le programme de Langlands local p-adique initié par Breuil dans [Bre03a], [Bre03b] et [Bre04] vise à associer une représentation linéaire de $\mathrm{GL}_n(K)$ $p\mathrm{LL}(\rho)$ sur un espace de Banach p-adique à une représentation galoisienne p-adique $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(E)$. Cette correspondance de Langlands locale p-adique est connue pour $\mathrm{GL}_2(\mathbf{Q}_p)$ et est dû à Colmez dans [Col10]; sa compatibilité avec la cohomologie étale complétée de la tour de courbes modulaires est montrée par Emerton dans [Eme06]. L'application pLL reste encore très mystérieuse quand $K \neq \mathbf{Q}_p$ ou quand $n \geq 3$. On s'attend (c.f. [Bre10a]) à ce que pLL soit compatible avec la reduction modulo ϖ_E, avec $\ll$les familles p-adiques$\gg$, etc. Nous utilisons dans la suite la notation $\overline{p\mathrm{LL}}$ pour désigner l'application (qui est seulement conjecturale pour $K \neq \mathbf{Q}_p$ ou $n \geq 3$) qui associe à une représentation galoisienne $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(k_E)$ une representation lisse admissible de $\mathrm{GL}_n(K)$ à coefficients dans k_E.

On commence par donner une formulation plus précise de cette compatibilité local-global conjecturale pour pLL, qui est une généralisation naturelle des idées dans [Eme06] (sauf que nous ne connaissons plus l'existence de pLL si $K \neq \mathbf{Q}_p$ ou si $n \geq 3$). À partir de maintenant on fixe un corps F^+ totalement réel et une extension F quadratique totalement imaginaire de F^+. On fixe une place finie v_0 de F^+ qui divise p, qui est scindée dans F et qui vérifie $K \cong F^+_{v_0} \cong F_{w_0}$ où w_0 est une place finie de F au-dessus de v_0. On fixe une telle place w_0 de F à partir de maintenant. On fixe un groupe unitaire $\mathcal{G}$ sur F^+ tel que

(i) $\mathcal{G} \otimes_{F^+} F \cong \mathrm{GL}_{n/F}$;

(ii) $\mathcal{G}(F^+ \otimes_{\mathbf{Q}} \mathbf{R})$ est compact.

On fixe aussi un sous-groupe compact ouvert $U^{v_0} \subset \mathcal{G}(\mathbb{A}_{F^+}^{\infty,v_0})$ et un $\mathcal{O}_E$-module A de type fini muni de sa topologie p-adique. Puis on considère l'espace des formes automorphes p-adiques continues (resp. localement constantes) à valeurs dans A sur $\mathcal{G}(\mathbb{A}_{F^+}^{\infty})$ de un niveau fixé U^{v_0} premier avec v_0:

$$\widehat{S}(U^{v_0}, A) := \{f : \mathcal{G}(F^+)\backslash\mathcal{G}(\mathbb{A}_{F^+}^{\infty})/U^{v_0} \to A, \text{ continue } \}$$

$$(\text{ resp. } S(U^{v_0}, A) := \{f : \mathcal{G}(F^+)\backslash\mathcal{G}(\mathbb{A}_{F^+}^{\infty})/U^{v_0} \to A, \text{ localement constante } \}\ ).$$

En particulier si A est un module de p-torsion muni de la topologie discrète, alors les deux espaces définis ci-dessus coïncident. Chaque espace ci-dessus admet une action d'une algèbre de Hecke universelle $\mathbb{T}$ sur $\mathcal{O}_E$ (engendrée librement par des opérateurs de Hecke indexés par les places finies v

de F au-dessus d'un ensemble $D(U^{v_0})$ de places finies de F^+ qui sont totalement décomposées dans F, premières avec p, telles que $U_v := U^{v_0} \cap \mathcal{G}(F_v^+)$ est un sous-groupe compact ouvert maximal de $\mathcal{G}(F_v^+)$); et il y a aussi une action continue (resp. lisse) de $\mathrm{GL}_n(K)$ sur $\widehat{S}(U^{v_0}, A)$ (resp. sur $S(U^{v_0}, A)$) qui provient de la translation à droite sur $\mathcal{G}(F^+)\backslash\mathcal{G}(\mathbb{A}_{F^+}^{\infty})/U^{v_0}$. Les actions de $\mathrm{GL}_n(K)$ sur les deux espaces commutent avec les actions de $\mathbb{T}$. Pour une représentation galoisienne p-adique continue $r_A : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(A)$ qui est conjuguée auto-duale et non-ramifiée à toute place de F au-dessus d'un $v \in D(U^{v_0})$, on peut lui associer un idéal $\mathfrak{p}_{r_A} \subset \mathbb{T} \otimes_{\mathcal{O}_E} A$; Le sous-espace propre $\widehat{S}(U^{v_0}, A)[\mathfrak{p}_{r_A}]$ (resp. $S(U^{v_0}, A)[\mathfrak{p}_{r_A}]$) défini par cet idéal admet naturellement une action continue (resp. lisse) de $\mathrm{GL}_n(K)$. On note LL l'application donnée par la correspondance de Langlands locale classique qui envoie une représentation de Weil-Deligne, Frobenius semi-simple de dimension n, vers la représentation lisse irréductible de $\mathrm{GL}_n(K)$ (c.f. [HT01], [He00] et [Sch13]). On pose

$$r := r_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} E,\ \overline{r} := r_{k_E} = r_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} k_E,\ \rho := r|_{\mathrm{Gal}(\overline{K}/K)},\ \overline{\rho} := \overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$$

et on utilise la notation abrégée

$$\begin{aligned}
\widehat{\Pi}(r) &:= \left(\widehat{S}(U^{v_0}, \mathcal{O}_E)[\mathfrak{p}_{r_{\mathcal{O}_E}}]\right) \otimes_{\mathcal{O}_E} E \\
\Pi(\overline{r}) &:= S(U^{v_0}, k_E)[\mathfrak{p}_{r_{k_E}}] \\
\Pi^{\mathrm{sm}}(r) &:= \left(S(U^{v_0}, \mathcal{O}_E)[\mathfrak{p}_{r_{\mathcal{O}_E}}]\right) \otimes_{\mathcal{O}_E} E \quad .
\end{aligned}$$

On omet le niveau U^{v_0} dans la notation pour simplicité. On définit $\Pi^{\mathrm{alg}}(r)$ comme le sous-espace de $\widehat{\Pi}(r)$ des vecteurs localement algébriques. On remarque que $\Pi^{\mathrm{alg}}(r)$ est naturellement une représentation localement algébrique de $\mathrm{GL}_n(K)$. On note $\mathrm{WD}(\rho)$ (resp. $\mathrm{Alg}(\rho)$) l'application qui associe une représentation de Weil–Deligne (resp. une représentation algébrique de $\mathrm{GL}_n(K)$ de dimension finie) à une représentation galoisienne p-adique ρ potentiellement semi-stable (resp. l'ensemble des poids de Hodge–Tate de ρ) via la théorie de Fontaine de [Fon94] (resp. via la section 5 de [Bre16]). A ce stade, la compatibilité local-global classique en p (qui a été demontrée dans [BLGGT12] et [BLGGT14] sous plus d'hypothèses techniques) peut se formuler grossièrement comme suit :

Théorème 1.1.1. *Supposons que*

$$\Pi^{\mathrm{alg}}(r) \neq 0.$$

Alors ρ est potentiellement semi-stable et il existe un entier $d > 0$ qui ne dépend que de r et de U^{v_0} tel que

$$\Pi^{\mathrm{alg}}(r) \cong (\mathrm{LL} \circ \mathrm{WD}(\rho) \otimes_E \mathrm{Alg}(\rho))^{\oplus d}. \tag{1.1.2}$$

Il est naturel d'imaginer que la compatibilité local-global p-adique conjecturale devrait avoir la forme suivante:

Espoir 1.1.3. *Supposons que*

$$\Pi(\overline{r}) \neq 0.$$

Alors il existe un entier $d > 0$ qui ne dépend que de r et de U^{v_0} tel que

$$\widehat{\Pi}(r) \cong p\mathrm{LL}(\rho)^{\oplus d} \tag{1.1.4}$$

et

$$\Pi(\overline{r}) \cong \overline{p\mathrm{LL}}(\overline{\rho})^{\oplus d}. \tag{1.1.5}$$

En particulier, l'Espoir 1.1.3 implique :

Conjecture 1.1.6. *La structure de $\widehat{\Pi}(r)$ (resp. de $\Pi(\overline{r})$) comme représentation de Banach p-adique unitaire admissible (resp. représentation lisse admissible à coefficients dans k_E) de $\mathrm{GL}_n(K)$ détermine la classe d'isomorphisme de ρ (resp. de $\overline{\rho}$) et ne dépend que de cette classe.*

On insiste sur le fait que ρ n'est pas nécessairement potentiellement semi-stable dans l'Espoir 1.1.3 et dans la Conjecture 1.1.6. En revanche, comme les applications pLL et $\overline{p\mathrm{LL}}$ sont très mystérieuses quand $K \neq \mathbf{Q}_p$ ou quand $n \geq 3$, on ne considère que les cas où ρ est potentiellement semi-stable dans le reste de cette introduction de sorte que le Théorème 1.1.1 soit disponible. Il est assez difficile d'étudier directement les représentations de Banach p-adiques unitaires de $\mathrm{GL}_n(K)$. On a essentiellement deux types d'objets par lesquels il est peut-être plus facile de commencer. Ma thèse est donc divisée en deux parties et dans chaque partie on étudie l'un de ces deux types d'objets. La première partie de ma thèse traite les représentations $\Pi(\overline{r})$ lisses et admissibles à coefficients dans k_E (travail en commun avec Chol Park) et la seconde partie traite les représentations $\Pi^{\mathrm{an}}(r)$ localement analytiques définies comme l'ensemble des vecteurs localement analytiques (l'ensemble des vecteurs sur lesquels $\mathrm{GL}_n(K)$ agit par des fonctions localement analytiques sur le groupe $\mathrm{GL}_n(K)$ à valeurs vectorielles) dans $\widehat{\Pi}(r)$.

1.2 Compatibilité local-global modulo p pour $\mathrm{GL}_n(\mathbf{Q}_p)$

On commence par $\Pi(\overline{r})$. Plusieures questions naturelles se posent sur $\Pi(\overline{r})$:

(i) Est-ce qu'on a $\Pi(\overline{r}) \neq 0$?

(ii) Quelle est la structure de $\Pi(\overline{r})$?

(iii) Quel est le lien explicite entre $\Pi(\overline{r})$ et $\overline{\rho} = \overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$ (on rappelle que $K = F_{w_0}$)?

L'assertion $\Pi(\overline{r}) \neq 0$ revient essentiellement à dire, en terminologie plus classique, que $\overline{r}$ est modulaire (ou automorphe). Le cas où $F = \mathbf{Q}$ et $\mathcal{G} = \mathrm{GL}_2$ est connu et découle de la conjecture de modularité de Serre prouvée par Khare–Wintenberger dans [KW09]. Malheureusement (i) reste toujours non résolu en général et dans cette thèse il sera parfois nécessaire de le mettre dans l'hypothèse. On suppose que (i) est vraie et on considère la question (ii). La réponse complète à (ii) est connue dans le cas où $F = \mathbf{Q}$, $\mathcal{G} = \mathrm{GL}_2$, elle est dû à Emerton (qui repose sur l'existence de pLL pour $\mathrm{GL}_2(\mathbf{Q}_p)$) mais est toujours ouverte quand $K \neq \mathbf{Q}_p$ ou quand $n \geq 3$. Un des obstacles principaux est l'absence d'une classification complète des représentations lisses irréductibles de $\mathrm{GL}_n(K)$ à coefficients dans k_E (sauf le cas $\mathrm{GL}_2(\mathbf{Q}_p)$ qui est connu d'après [BL94] et [Bre03a]). Plus précisément, les résultats de [BP12], [Hu10] et [Schr15] montrent que la classification des représentations qui n'apparaissent pas comme sous-quotients d'inductions paraboliques (elles sont appelées supercuspidales) est très difficile même pour $\mathrm{GL}_2(\mathbf{Q}_{p^2})$. En revanche, puisque l'on sait comment classifier toutes les représentations paraboliquement induites par les caractères d'un sous-groupe de Borel (voir [Her11] pour les cas $\mathrm{GL}_n(K)$), on peut déjà construire en utilisant [BH15] une représentation lisse $\Pi^{\mathrm{ord}}(\overline{\rho})$ sur k_E d'une longueur finie (qui ne dépend que de $\overline{\rho}$) qui se plonge toujours dans $\Pi(\overline{r})$ si $K = \mathbf{Q}_p$ et si $\overline{\rho}$ est *ordinaire* (i.e. $\overline{\rho}$ a son image dans un sous-groupe de Borel de $\mathrm{GL}_n(k_E)$). Mais il est clair, compte tenue de la construction dans [BH15], que $\Pi^{\mathrm{ord}}(\overline{\rho})$ n'est pas suffisant pour déterminer $\overline{\rho}$ en général. Notons que Hu montre dans [Hu17] qu'une représentations supercuspidale apparaît nécessairement dans $\Pi(\overline{r})$ même si $K = \mathbf{Q}_{p^2}$ et $n = 2$.

Par conséquent, on a besoin d'une méthode qui nous permet de trouver suffisament d'informations dans $\Pi(\overline{r})$ pour déterminer $\overline{\rho}$. Une approche naturelle est de restreindre l'action de $\mathrm{GL}_n(K)$ à $\mathrm{GL}_n(\mathcal{O}_K)$. Il y a au moins deux raisons pour faire ceci : les représentations irréductibles de $\mathrm{GL}_n(\mathcal{O}_K)$ à coefficients dans k_E (donc de $\mathrm{GL}_n(k)$) sont faciles à classifier grace à la section 2 de la partie II de [Jan03]; au moins la classification des représentations irréductibles de $\mathrm{GL}_n(k)$ à coefficients dans E est bien connue d'après la théorie de Deligne–Lusztig de [DL76]. En plus on a

Théorème 1.2.1 (voir [Jan81] et la Proposition 4.1.3 de [LLL16]). *Pour toute représentation de Deligne-Lusztig τ suffisament générique et pour toute $\mathcal{O}_E$-réseau $\tau^\circ \subsetneq \tau$, L'ensemble des facteurs de Jordan–Hölder*

$$\mathrm{JH}_{\mathrm{GL}_n(k)}\left(\tau^\circ \otimes_{\mathcal{O}_E} k_E\right)$$

est indépendant du choix de τ° et admet une description purement combinatoire via le groupe de Weyl affine de $\mathrm{GL}_n/_k$.

Ici par *suffisamment générique* on veut dire que, quand on paramétrise une représentation de Deligne–Lusztig par un poids et un élément du groupe de Weyl comme dans [Jan81] (voir aussi la Section 4 de [Her09]), ce poids doit être suffisamment éloigné du bord de l'alcôve qui le contient.

La première étape pour comprendre $\Pi(\overline{r})|_{\mathrm{GL}_n(\mathcal{O}_K)}$ consiste à caractériser le socle

$$\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$$

ce qui est déjà un problème profond et qui est historiquement une reformulation de la partie ≪poids≫ de la conjecture de Serre lorsque $F = \mathbf{Q}$ et $\mathcal{G} = \mathrm{GL}_2$. Les conjectures sur la structure de $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ sont formulée par Buzzard–Diamond–Jarvis dans [BDJ10] si $K = \mathbf{Q}_{p^f}$ et $n = 2$, par Herzig [Her09] si $K = \mathbf{Q}_p$ et n arbitraire, et par Gee–Herzig–Savitt [GHS] dans certains cas plus généraux. On s'attend à ce qu'il y ait une relation profonde entre la structure de $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ et les fibre spéciales de différents anneaux de déformations potentiellement semi-stables de $\overline{\rho}$ (conjecture de Breuil–Mézard–Emerton–Gee, c.f. [BM02], [BM14] et [EG14]).

Quand $K = \mathbf{Q}_{p^f}$ la structure de $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ a été déterminée pour $n = 2$ dans [GK14] et pour $n = 3$ dans [EGH13], [HLM17], [MP17], [LMP] et [LLHLMa]. On pose

$$\mathrm{K}(1) := \mathrm{Ker}\left(\mathrm{GL}_n(\mathcal{O}_K) \twoheadrightarrow \mathrm{GL}_n(k)\right)$$

et on remarque qu'il est naturel, du point de vue de la théorie des représentations, d'étudier $\Pi(\overline{r})^{\mathrm{K}(1)}$ qui contient évidemment $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ comme une sous-représentation. La représentation $\Pi(\overline{r})^{\mathrm{K}(1)}$ est un point de départ (déjà non-trivial) pour reconstruire $\Pi(\overline{r})$ comme une représentation de $\mathrm{GL}_n(K)$. Mais malheureusement les résultats de [HW18], [LMS16] et [Le17] si $K = \mathbf{Q}_{p^f}$ et $n = 2$, ainsi que des travaux en préparation des auteurs de [LLHLMa] si $K = \mathbf{Q}_{p^f}$ et $n = 3$ suggèrent que $\Pi(\overline{r})^{\mathrm{K}(1)}$, comme représentation de $\mathrm{GL}_n(k)$, est toujours insuffisant pour déterminer $\overline{\rho}$, surtout dans le cas où $\overline{\rho}$ est ordinaire et indécomposable. On a l'exemple suivant

Exemple 1.2.2. *On prend $K = \mathbf{Q}_{p^f}$, $n = 2$, et $\overline{\rho}$ une représentation galoisienne ordinaire de la forme suivante*

$$\begin{pmatrix} \overline{\chi}_2 & * \\ 0 & \overline{\chi}_1 \end{pmatrix}.$$

Supposons en plus que $\overline{\chi}_1\overline{\chi}_2^{-1}$ satisfait une hypothèse générique. Il s'ensuit d'un calcul standard de caractéristique d'Euler–Poincaré en cohomologie galoisienne que

$$\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbf{Q}_{p^f}}/\mathbf{Q}_{p^f})}(\overline{\chi}_2,\ \overline{\chi}_1) = f$$

et par conséquent $\overline{\rho}$ est déterminé par $\overline{\rho}^{\mathrm{ss}}$ et $f-1$ invariants à valeurs dans $k_E \cup \{\infty\}$ à isomorphisme près. On sait que $\overline{\rho}$ est Fontaine–Laffaille dans ce cas, alors on peut utiliser les modules de Fontaine–Laffaille (voir [FL82]) pour définir un ensemble d'invariants dans $k_E \cup \{\infty\}$ (voir Lemme 2.1.1 de [BD14]) qui détermine la classe d'isomorphisme de $\overline{\rho}$. Il se trouve que la structure de $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ ne peut pas déterminer ces invariants (s'ils ne sont ni 0 ni ∞). Plus généralement, supposons que toutes les representations à coefficients dans k_E de $\mathrm{Gal}(\overline{K}/K)$ de dimension n peuvent être paramétrisées par un certain espace de module et qu'un tel espace admet une stratification par des sous-schémas localement fermés donnés par certaines conditions explicites, alors on s'attend à ce que $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ nous permette seulement de dire dans quelle strate $\overline{\rho}$ se trouve, au lieu de dire quel point de la strate $\overline{\rho}$ correspond précisement.

On définit le sous-groupe d'Iwahori I, resp. le pro-p sous-groupe d'Iwahori $I(1)$, de $\mathrm{GL}_n(K)$ comme la préimage de l'ensemble des matrices triangulaires supérieures (resp. unipotentes) de $\mathrm{GL}_n(k)$ via

la surjection $\mathrm{GL}_n(\mathcal{O}_K) \twoheadrightarrow \mathrm{GL}_n(k)$. Il est bien connu que le normalisateur de I dans $\mathrm{GL}_n(K)$ est engendré par I et l'élément suivant:

$$\Xi_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \varpi_K & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_n(K).$$

Puisque Ξ_n et $\mathrm{GL}_n(\mathcal{O}_K)$ engendrent $\mathrm{GL}_n(K)$, on s'attend naturellement à ce que Ξ_n joue un rôle crucial, quelle que soit la méthode utilisée, dans la reconstruction d'une représentation lisse irréductible de $\mathrm{GL}_n(K)$ à partir da sa restriction à $\mathrm{GL}_n(\mathcal{O}_K)$.

Dans [BD14], les auteurs ont montré que (si $K = \mathbf{Q}_{p^f}$ et $n = 2$) $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ et l'action de Ξ_2 sur $\Pi(\overline{r})^{I(1)}$ détermine $\overline{\rho}$ à isomorphisme près; c'est le premier résultat sur la détermination des valeurs des *invariants de Fontaine–Laffaille* (ceci est résumé dans l'Exemple 1.2.2). Le résultat de [BD14] a été récemment généralisé au cas où $K = \mathbf{Q}_p$ et $n = 3$ dans [HLM17] quand $\overline{\rho}$ est ordinaire et Fontaine–Laffaille, et dans [LMP] quand $\overline{\rho}$ a une sous représentation ou un quotient irréductible de dimension deux. Ces deux approches pour $n = 3$ considèrent l'action de Ξ_3 sur $\Pi(\overline{r})^{I(1)}$ et requièrent des hypothèses génériques techniques supplémentaires sur $\overline{\rho}$ qui consistent essentiellement à garantir que

$$\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r}) \text{ a une longueur minimale}$$

si on fixe $\overline{\rho}^{\mathrm{ss}}$ et si on fait varier les paramètres d'extension de $\overline{\rho}$. La première partie de ma thèse consiste à généraliser les résultats de [HLM17] au cas où $K = \mathbf{Q}_p$ et n est arbitraire. Ceci est fait dans le Chapitre 3; il s'agit d'un travail en commun avec Chol Park.

Théorème 1.2.3. *Supposons que $\Pi(\overline{r}) \neq 0$. Alors la structure de répresentation lisse admissible de $\mathrm{GL}_n(\mathbf{Q}_p)$ à coefficients dans k_E de $\Pi(\overline{r})$ détermine $\overline{\rho}$ à isomorphisme près si $\overline{\rho}$ est Fontaine-Laffaille, ordinaire et suffisamment générique.*

Remarquons que le cas où $K = \mathbf{Q}_{p^f}$ et $n = 3$ a récemment aussi été obtenu par Enns dans [En]. Un autre résultat important dans cette direction est le Théorème 7.8 de [Sch15] où Scholze a utilisé une méthode géométrique complètement différente de la nôtre pour montrer que $\Pi(\overline{r})$ détermine toujours $\overline{\rho}$ pour $n = 2$ et K arbitraire.

Maintenant on donne un énoncé plus précis du Théorème 1.2.3 et on renvoie les lecteurs au Chapitre 3 pour plus de détails. On pose $K = \mathbf{Q}_p$ à partir de maintenant. On fixe $\overline{r}$ tel que

$$\Pi(\overline{r}) \neq 0$$

et suppose que $\overline{\rho}$ est ordinaire. Alors il existe une suite de sous-représentations

$$0 \subsetneq \overline{\rho}_{1,1} \subsetneq \overline{\rho}_{1,2} \subsetneq \cdots \subsetneq \overline{\rho}_{1,n-1} \subsetneq \overline{\rho}_{1,n} = \overline{\rho}$$

telle que

$$\overline{\chi}_1 := \overline{\rho}_{1,1} \text{ and } \overline{\chi}_i := \overline{\rho}_{1,i}/\overline{\rho}_{1,i-1}$$

sont de dimension 1 pour tout $2 \leq i \leq n$. Si on suppose que $\overline{\chi}_{i-1}\overline{\chi}_i^{-1}$ satisfait certaines hypothèses (pas très fortes) qui ressemblent à celle qui a été mentionnée dans l'Exemple 1.2.2, alors on a

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\chi}_{i-1}) = \mathrm{Ext}^2_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\chi}_{i-1}) = 0$$

et

$$\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\chi}_{i-1}) = 1$$

pour tout $2 \leq i \leq n$. Par un dévissage on en déduit que

$$\dim_{k_E} \mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\rho}_{1,i-1}) = i-1$$

qui dit grosso modo que $\overline{\rho}_i$ est déterminé par $\overline{\chi}_i$, $\overline{\rho}_{1,i-1}$ et $i-2$ invariants supplémentaires pour tout $2 \leq i \leq n$. Par une récurrence rapide on sait que $\overline{\rho}$ est déterminé par

$$\overline{\rho}^{\mathrm{ss}} = \oplus_{i=1}^n \overline{\chi}_i$$

et $\frac{(n-1)(n-2)}{2}$ invariants. Selon une classification complète des caractères de $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ à valeurs dans k_E via la théorie de corps de classes, on sait qu'il existe $\overline{a}_i \in \mathbf{Z}/(p-1)\mathbf{Z}$ tel que

$$\overline{\chi}_i \omega^{-\overline{a}_i - i + 1}$$

est non ramifié pour tout $1 \leq i \leq n$ où ω est la réduction modulo p du caractère cyclotomique

$$\varepsilon : \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p) \twoheadrightarrow \mathbf{Z}_p^\times.$$

Il se trouve que la condition que

$$\overline{\rho} \text{ est Fontaine–Laffaille}$$

est essentiellement équivalente à l'existence d'entiers $a_i \in \mathbf{Z}$ dont les images dans $\mathbf{Z}/(p-1)\mathbf{Z}$ sont les $\overline{a}_i$ et vérifiant

$$a_1 + p - 1 > a_n > a_{n-1} > \cdots > a_1.$$

La théorie de [FL82] associe à $\overline{\rho}$ un *module de Fontaine–Laffaille* $\mathrm{FL}(\overline{\rho})$. On peut définir un certain ensemble d'invariants $\mathrm{FL}_{i,j}(\overline{\rho})$, pour toute paire d'entiers (i,j) tels que $1 \leq i < i+1 < j \leq n$, comme certaines fonctions rationnelles explicites en entrées de la matrice de Frobenius du module $\mathrm{FL}(\overline{\rho})$ et prouver que les $\mathrm{FL}_{i,j}(\overline{\rho})$ ne dépendent que de la classe d'isomorphisme de $\mathrm{FL}(\overline{\rho})$. On observe que $\overline{\rho}$ est déterminé par $\overline{\rho}^{\mathrm{ss}}$ et ces invariants $\mathrm{FL}_{i,j}(\overline{\rho})$, et que la démonstration du Théorème 1.2.3 se réduit au problème de détecter tous les invariants $\mathrm{FL}_{i,j}(\overline{\rho})$ dans $\Pi(\overline{r})$.

Une idée générale consiste à étudier $\Pi(\overline{r})$ via $\Pi^{\mathrm{sm}}(r)$ pour de différents relèvements r d'un $\overline{r}$ fixé tels que les représentations galoisiennes locales $\rho = r|_{\mathrm{Gal}(\overline{F_{w_0}}/F_{w_0})}$ satisfassent des conditions supplémentaires. Plus précisément, pour une représentation de Deligne-Lusztig τ de $\mathrm{GL}_n(\mathbf{F}_p)$ donnée, il existe un type inertiel modérément ramifié $\mathrm{LL}^{-1}(\tau)$ (qui est une représentation du sous-groupe d'inertie $I_{\mathbf{Q}_p} \subsetneq \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ qui se factorise par le quotient modérément ramifié) qui lui correspond via la correspondance de Langlands locale inertielle (voir Section 3 de [CEGGPS]). La notation $\mathrm{LL}^{-1}(\tau)$ provient de la compatibilité entre la correspondance de Langlands locale classique et la correspondance de Langlands locale inertielle. Alors on considère tous les relèvements ρ de $\overline{\rho}$ avec les poids de Hodge-Tate $\{0, 1, \cdots, n-1\}$ tels que

$$\mathrm{WD}(\rho)|_{I_{\mathbf{Q}_p}} \cong \mathrm{LL}^{-1}(\tau). \tag{1.2.4}$$

On remarque que tout ρ correspond à un E-point du schéma formel donné par un anneau local noethérien complet $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau),\{0,\cdots,n-1\}}$. C'est un cas spécial de la construction des anneaux de déformation potentiellement semi-stables de [Kis08] pour les types inertiels et les poids de Hodge–Tate plus généraux.

Pour toute paire d'entiers (i,j) tels que $1 \leq i < i+1 < j \leq n$, on choisit une représentation de Deligne–Lusztig $\tau_{i,j}$ puis on prend une représentation galoisienne $r_{i,j} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E)$ telle que

(i) $r_{i,j}$ est automorphe et non-ramifié en toute place finie de F au-dessus de $v \in D(U^{v_0})$;

(ii) $r_{i,j}$ contient un $\mathcal{O}_E$-réseau $r_{i,j}^\circ$ invariant dont la réduction modulo ϖ_E est $\overline{r}$;

(iii) $\rho_{i,j} := r_{i,j}|_{\mathrm{Gal}(\overline{F_{w_0}}/F_{w_0})}$ correspond à un morphisme $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau_{i,j}),\{0,\cdots,n-1\}} \to E$.

En particulier on a le diagramme commutatif suivant:

$$\begin{array}{ccccc}
\widehat{\Pi}(r_{i,j}) & \hookleftarrow & \widehat{S}(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}] & \twoheadrightarrow & \Pi(\overline{r_{i,j}}) \\
\uparrow & & \uparrow & & \uparrow \\
\Pi^{\mathrm{sm}}(r_{i,j}) & \hookleftarrow & S(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}] & & \\
\uparrow & & \uparrow & & \\
\tau_{i,j} & \hookleftarrow & \tau^\circ_{i,j} & \twoheadrightarrow & \overline{\tau_{i,j}}
\end{array} \tag{1.2.5}$$

où

$$\tau^\circ_{i,j} := \tau_{i,j} \cap S(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}] \subsetneq \Pi^{\mathrm{sm}}(r_{i,j})$$

et

$$\overline{\tau_{i,j}} := \tau^\circ_{i,j} \otimes_{\mathcal{O}_E} k_E.$$

En prenant les sous-espaces invariants par $I(1)$, le diagramme (1.2.5) induit un autre diagramme:

$$\begin{array}{ccccc}
\Pi^{\mathrm{sm}}(r_{i,j})^{I(1)} & \hookleftarrow & S(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}]^{I(1)} & \twoheadrightarrow & \Pi(\overline{r_{i,j}})^{I(1)} \\
\uparrow & & \uparrow & & \uparrow \\
\tau_{i,j}^{I(1)} & \hookleftarrow & (\tau^\circ_{i,j})^{I(1)} & \twoheadrightarrow & \overline{\tau_{i,j}}^{I(1)}
\end{array} \quad . \tag{1.2.6}$$

Le fait que

$$\tau_{i,j}^{I(1)} \neq 0$$

implique que $\tau_{i,j}$ est une représentation de la série principale de $\mathrm{GL}_n(\mathbf{F}_p)$. Cela revient à dire que $\tau_{i,j}$ provient d'une induction parabolique d'un caractère de $T(\mathbf{F}_p)$. Chaque terme de (1.2.6) admet une action de l'algèbre de Hecke–Iwahori qui contient Ξ_n et $n-1$ opérateurs $U_n^m \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Q}_p)]$ pour $1 \leq m \leq n-1$. On note P_m le parabolique standard de GL_n qui contient le sous-groupe de Borel triangulaire supérieur et qui a des blocs de Levi de la forme $\mathrm{GL}_m \times \mathrm{GL}_{n-m}$. On note N_m le radical unipotent de P_m. Le lemme suivant résume les deux propriétés principales de U_n^m.

Lemme 1.2.7. *On a*

$$(\Xi_n)^m \circ U_n^m \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)].$$

Etant donnée une représentation irréductible lisse Π_m *(resp.* Π_{n-m}*) de* $\mathrm{GL}_m(\mathbf{Q}_p)$ *(resp. de* $\mathrm{GL}_{n-m}(\mathbf{Q}_p)$*) dont le caractère central est* ω_{Π_m} *(resp.* $\omega_{\Pi_{n-m}}$*), on a*

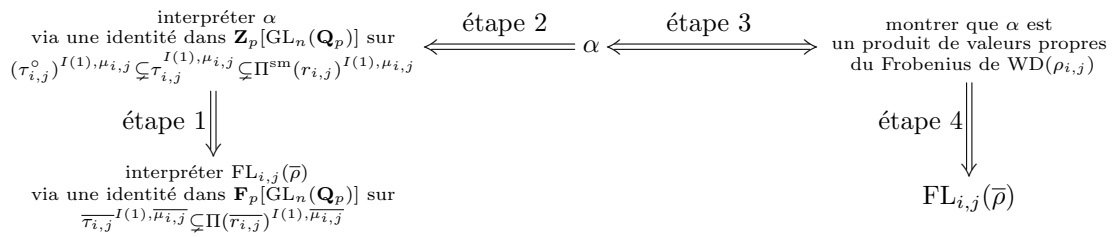
$$U_n^m = \omega_{\Pi_m}(p)^{-1}$$

restreint à l'image de

$$\Pi_m \otimes_E \Pi_{n-m} \hookrightarrow \left(\mathrm{Ind}_{P_m(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)} (\Pi_m \otimes_E \Pi_{n-m})\right)^{N_m(\mathbf{Z}_p)}.$$

On note $\mu_{i,j}$ un caractère $T(\mathbf{F}_p) \to \mathbf{Z}_p^\times$ (à déterminer plus tard) et $\overline{\mu_{i,j}} : T(\mathbf{F}_p) \to \mathbf{F}_p^\times$ sa réduction modulo p. Le groupe $T(\mathbf{F}_p)$ agit naturellement sur l'espace $\Pi^{\mathrm{sm}}(r_{i,j})^{I(1)}$. On note $\Pi^{\mathrm{sm}}(r_{i,j})^{I(1),\mu_{i,j}}$ le sous espace propre associé au caractère $\mu_{i,j}$. Étant donnée une valeur propre $\alpha \in E^\times$ de U_n^m sur

$\Pi^{\mathrm{sm}}(r_{i,j})^{I(1),\mu_{i,j}}$, on cherche un diagramme:



(1.2.8)

qui a déjà apparu dans [BD14]. On insiste sur le fait que k est un entier qui vérifie $1 \leq k \leq n-1$ et qu'il est déterminé par la paire (i,j). La partie la plus standard du diagramme 1.2.8 est l'étape 3 qui est essentiellement une égalité entre valeurs propres de U_n^m et produits de certaines valeurs propres du Frobenius de $\mathrm{WD}(\rho_{i,j})$ et qui découle directement de la correspondance de Langlands locale classique et de la compatibilité local-global classique (voir Théorème 1.1.1). L'étape 4 qui réalise l'invariant $\mathrm{FL}_{i,j}(\overline{\rho})$ comme la reduction modulo ϖ_E d'un produit de valeurs propres du Frobenius (multiplié par une puissance convenable de p), est prouvée par des calculs techniques de la théorie de Hodge p-adique entière, notamment via des modules de Breuil et de Kisin. La plupart de l'étape 4 est faite dans Section 3.3. Il est nécessaire d'insister sur une différence importante entre l'étape 3 et l'étape 4. L'étape 3 situe complètement en caractéristique 0 à coefficients dans E, ne dépend que de $\mathrm{WD}(\rho)$ et n'exige pas de conditions supplémentaires sur $\overline{\rho}$. En revanche l'étape step 4 repose sur une condition technique supplémentaire sur $\overline{\rho}$ qui sera rappelée dans la Définition 1.2.9.

On peut associer à une paire $(\overline{\rho}, \tau_{i,j})$ un élément $\widetilde{w}(\overline{\rho}, \tau_{i,j})$ du groupe de Weyl affine $\widetilde{W}$ de GL_n en utilisant la théorie des ≪shapes≫ qui est essentiellement développée dans [LLHLMa]. Pour toute représentation galoisienne semi-simple

$$\overline{\rho}_0 : \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p) \to \mathrm{GL}_n(k_E)$$

on définit l'ensemble suivant

$$\Omega(\overline{\rho}_0, \tau_{i,j}) := \{\widetilde{w}(\overline{\rho}, \tau_{i,j}) \mid \overline{\rho}^{\mathrm{ss}} \cong \overline{\rho}_0\}.$$

On considère la longueur

$$\ell\left(\widetilde{w}(\overline{\rho}, \tau_{i,j})\right)$$

par rapport au système de Coxeter standard de $\widetilde{W}$.

Définition 1.2.9. *On dit que $\overline{\rho}$ est $\tau_{i,j}$-générique si la longueur de $\widetilde{w}(\overline{\rho}, \tau_{i,j})$ est maximale parmi celles des éléments de $\Omega(\overline{\rho}^{\mathrm{ss}}, \tau_{i,j})$.*

On peut construire (via les modules de Fontaine–Laffaille) un k_E-schéma $M_{\overline{\rho}_0}$ dont les k_E-points paramétrisent toute la représentation de Fontaine–Laffaille $\overline{\rho}$ avec une semi-simplification fixée $\overline{\rho}^{\mathrm{ss}} \cong \overline{\rho}_0$. On a alors une stratification naturelle $\mathscr{S}$ de $M_{\overline{\rho}_0}$ indexée par l'ensemble $\Omega(\overline{\rho}_0, \tau_{i,j})$ (ceci découle des travaux en cours des auteurs de [LLHLMa] sur l'espace de module des modules de Kisin avec données de descente modérément ramifiées). Il s'avère que $M_{\overline{\rho}_0}$ est irréductible et il existe un unique élément de longueur maximale dans $\Omega(\overline{\rho}_0, \tau_{i,j})$ qui correspond à l'unique strate ouverte (non vide) dans $\mathscr{S}$, ce qui implique que $\tau_{i,j}$-*générique* est en effet une hypothèse générique.

Il découle du Lemme 1.2.7 que

$$U_n^m = \alpha_{i,j} \tag{1.2.10}$$

sur $\Pi^{\mathrm{sm}}(r)^{I(1),\mu_{i,j}}$ pour une certaine constante $\alpha_{i,j} \in E^{\times}$. On pose

$$h_{i,j} := \mathrm{val}_p(\alpha_{i,j}),\ \widetilde{\alpha_{i,j}} := p^{-h_{i,j}}\alpha_{i,j} \in \mathcal{O}_E^{\times}$$

et on note $\overline{\alpha_{i,j}}$ la réduction modulo ϖ_E de $\widetilde{\alpha_{i,j}}$. Le résultat principal issu des étapes 3 et 4 en (1.2.8) est:

Proposition 1.2.11. *On a l'égalité*

$$\overline{\alpha_{i,j}} = \mathrm{FL}_{i,j}(\overline{\rho})^{-1} \in k_E^\times \tag{1.2.12}$$

si $\overline{\rho}$ est $\tau_{i,j}$-générique.

Il reste à clarifier les étapes 1 et 2 en (1.2.8). Il nous faut deux opérateurs

$$\widehat{S}^{i,j},\ \widehat{S}^{i,j,\prime} \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{F}_p)]$$

et leurs réductions modulo p

$$S^{i,j},\ S^{i,j,\prime} \in \mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$$

qui ne dépendent que du choix de $\tau_{i,j}$ et de $\mu_{i,j}$. Alors l'égalité dans $\mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Q}_p)]$ dont on a besoin est de la forme

$$\widehat{S}^{i,j,\prime} \circ (\Xi_n)^m \circ U_n^m = \widehat{c}_{i,j}\widehat{S}^{i,j} \tag{1.2.13}$$

(sur $\Pi^{\mathrm{sm}}(r)^{I(1),\mu_{i,j}}$) où $\widehat{c}_{i,j} \in E^\times$ est une constante qui ne dépend que du choix de $\tau_{i,j}$, $\mu_{i,j}$ et k. On insiste que l'égalité (1.2.13) peut être calculé dans $\mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$ parce que

$$(\Xi_n)^m \circ U_n^m \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$$

d'après le Lemme 1.2.7. On suppose que

$$p^{-h_{i,j}}\widehat{c}_{i,j} \in \mathcal{O}_E^\times$$

et on note $c_{i,j}$ les réductions modulo ϖ_E de $p^{-h_{i,j}}\widehat{c}_{i,j}$. Alors en combinant (1.2.13) et (1.2.10), on obtient:

$$\widehat{S}^{i,j,\prime} \circ (\Xi_n)^m = \alpha_{i,j}^{-1}\widehat{c}_{i,j}\widehat{S}^{i,j} \tag{1.2.14}$$

sur $\Pi^{\mathrm{sm}}(r)^{I(1),\mu_{i,j}}$ dont les réductions modulo ϖ_E sont

$$S^{i,j,\prime} \circ (\Xi_n)^m = c_{i,j}\mathrm{FL}_{i,j}(\overline{\rho})S^{i,j} \tag{1.2.15}$$

sur $\Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}}$ par la Proposition 1.2.11, quitte à faire l'hypothèse supplémentaire suivante:

Hypothèse 1.2.16. *On a*

$$\dim_{k_E}\Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}} = 1$$

et

$$\overline{\tau_{i,j}}^{I(1),\overline{\mu_{i,j}}} \xrightarrow{\sim} \Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}}.$$

On observe que l'Hypothèse 1.2.16 exige un choix optimal du niveau U^{v_0}. Sinon on aurait besoin de la remplacer par une autre hypothèse qui assure qu'un certain module sur l'algèbre de Hecke $\mathbb{T}$ est libre sur son support. Pour supprimer l'Hypothèse 1.2.16, on a besoin d'une version forte de la condition *$\tau_{i,j}$-générique*. En fait, il découle d'un calcul des modules de Kisin avec données de descente modérément ramifiées (qui généralise les calculs de [LLHLMa] et sera fait dans [LLMPQ]) qu'il existe un sous-schéma ouvert non vide $M_{\overline{\rho}_0}^{\mathrm{sm}}$ de $M_{\overline{\rho}_0}$ contenu dans une strate ouverte $\tau_{i,j}$-générique tel que $\overline{\rho}$ est un k_E-point de $M_{\overline{\rho}_0}^{\mathrm{sm}}$ si et seulement si $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau_{i,j}),\{0,\cdots,n-1\}}$ est formellement lisse. L'existence même de $M_{\overline{\rho}_0}^{\mathrm{sm}}$ donne une restriction très forte sur le choix des $\tau_{i,j}$.

Définition 1.2.17. *On dit que $\overline{\rho}$ est fortement $\tau_{i,j}$-générique s'il est donné par un k_E-point de $M_{\overline{\rho}_0}^{\mathrm{sm}}$.*

La proposition suivante (qui est très liée à la conjecture de Breuil–Mézard–Emerton–Gee) sera prouvée dans [LLMPQ] et découle essentiellement d'un foncteur construit par la méthode du *patching* de Taylor–Wilies–Kisin (voir la Section 7.3 de [LLHLMa] pour le cas de $\mathrm{GL}_3(\mathbf{Q}_{p^f})$).

Proposition 1.2.18. *L'Hypothèse 1.2.16 est vraie pour un choix convenable de U^{v_0} si $\overline{\rho}$ est fortement $\tau_{i,j}$-générique.*

Il faut comprendre pourquoi les opérateurs $\widehat{S}^{i,j}, \widehat{S}^{i,j,\prime} \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{F}_p)]$ (satisfaisant (1.2.13)) existent et s'il est possible de les construire *explicitement* et de calculer la constante $\widehat{c}_{i,j}$. L'existence de certains $\widehat{S}^{i,j}$ et $\widehat{S}^{i,j,\prime}$ qui satisfont (1.2.13) est relativement formelle et découle essentiellement de l'Hypothèse 1.2.16 mais les opérateurs peuvent être très méchants et par conséquent on ne peut rien dire sur la constante $\widehat{c}_{i,j}$. En revanche la construction explicite d'une seule paire de tels opérateurs est très technique et la partie la plus difficile est de montrer que $S^{i,j}$ et $S^{i,j,\prime}$ ne s'annulent pas sur $\Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}}$. Dans le Chapitre 3, on donne en effet (avec des notations différentes) une construction explicite de ces opérateurs, on prouve la non-annulation mentionnée ci-dessus dans la Section 3.4.8 et on calcule la constante $\widehat{c}_{i,j}$ dans la Section 3.4.5.

On conclut que l'identité (1.2.15) montre que l'invariant $\mathrm{FL}_{i,j}(\overline{\rho})$ apparaît dans $\Pi(\overline{r})$ si $\overline{\rho}$ est *fortement $\tau_{i,j}$-générique*, et que $\Pi(\overline{r})$ détermine $\overline{\rho}$ à isomorphisme près si $\overline{\rho}$ est *fortement $\tau_{i,j}$-générique* pour toute paire (i,j) telle que $1 \leq i < i+1 < j \leq n$.

Il y a plusieurs généralisations naturelles du Théorème 1.2.3 qui ne sont pas incluses dans cette thèse. On en donne un résumé rapide. On suppose dans la suite que $\overline{\rho}^{\mathrm{ss}}$ est *suffisamment générique*. Les généralisations suivantes sont pour la plupart des travaux en cours avec Viet Bao Le Hung, Daniel Le, Stefano Morra et Chol Park.

La première étape est de supprimer la condition *ordinaire*. La nouvelle difficulté se divise en deux parties : (i) on a besoin d'une famille d'invariants de Fontaine–Laffaille et d'un convenable τ pour tout invariant afin que la stratégie en (1.2.8) s'applique; (ii) il faut construire explicitement certaines paires d'opérateurs $\widehat{S}, \widehat{S}' \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{F}_p)]$ pour tout invariant afin qu'un analogue de (1.2.13) soit vrai pour une constante explicite $\widehat{c}$. On remarque que il y a beaucoup plus de cas de $\overline{\rho}$. La seconde étape est de éliminer certaines conditions τ-*générique* qui sont analogues à la Définition 1.2.9. La difficulté de cette étape du fait que $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau),\{0,\cdots,n-1\}}$ ait peu de chance d'être formellement lisse sans une hypothèse générique forte sur $\overline{\rho}$. Ainsi la Proposition 1.2.11, l'Hypothèse 1.2.16 et (1.2.15) s'écroulent complètement en général. L'observation cruciale est que l'on peut déterminer $\overline{\rho}$ en utilisant un ensemble d'invariants et un ensemble de poids de Serre modulaires. La dernière étape consiste à éliminer la condition *Fontaine–Laffaille*. Le problème est que $\overline{\rho}$ admet d'habitude des relèvements pour beaucoup moins de choix possibles de τ si on fixe les poids de Hodge–Tate. La solution naturelle est de considérer les relèvements (de $\overline{\rho}$) dont les poids de Hodge–Tate sont plus généraux.

Remarque 1.2.19. *Il serait possible de généraliser toutes les discussions ci-dessus au cas où $K = \mathbf{Q}_{p^f}$ en utilisant les travaux récents de Enns dans [En]. Si K est ramifié au-dessus de $\mathbf{Q}_p$, la situation semble beaucoup plus compliquée parce que $\Pi(\overline{r})^{\mathrm{K}(1)}$ est en général trop petit et que l'on aurait besoin de types inertiels sauvagement ramifiés pour trouver suffisamment d'invariants de $\overline{\rho}$ dans $\Pi(\overline{r})$. Le seul résultat connu sans restriction sur la ramification de K est dû à Scholze [Sch15] pour $n = 2$ et K arbitraire où une méthode complètement différente et plus géométrique est employée. L'auteur ne sait pas comment généraliser la méthode de [Sch15] au cas n arbitraire pour déterminer $\overline{\rho}$.*

1.3 Dilogarithme et $\mathscr{L}$-invariants supérieurs pour $\mathrm{GL}_3(\mathbf{Q}_p)$

Maintenant on passe à la représentation localement analytique $\Pi^{\mathrm{an}}(r)$ qui est l'objet central de la seconde partie de ma thèse. On peut utiliser la théorie des modules de Verma (autrement dit la théorie de catégorie de Bernstein–Gelfand–Gelfand $\mathcal{O}$ expliquée par exemple dans [Hum08]) pour étudier $\Pi^{\mathrm{an}}(r)$. Très peu est connu sur la classification complète d'objets irréductibles dans la catégorie $\mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_n(K),E}$ des représentations localement analytiques admissibles de $\mathrm{GL}_n(K)$. On considère la sous-catégorie pleine $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$ (qui est beaucoup mieux comprise) de $\mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_n(K),E}$ engendré par les sous-quotients des séries principales localement analytiques $\left(\mathrm{Ind}_{B(K)}^{\mathrm{GL}_n(K)}\chi\right)^{\mathrm{an}}$ pour χ un caractère

localement analytique arbitraire de $T(K)$. Quand on dit qu'une sous-catégorie pleine est engendrée par une famille d'objets, on veut dire qu'elle est la sous-catégorie pleine minimale (tout objet est de longueur finie) qui contient cette famille d'objets et qui est stable par extensions et sous-quotients. On suppose que r est non ramifié en toute place finie de F au-dessus de $D(U^{v_0})$ et que

$$\Pi^{\mathrm{alg}}(r) \neq 0. \tag{1.3.1}$$

On remarque que l'équation (1.3.1) revient à dire que r *est automorphe* et qu'en particulier ρ est potentiellement semi-stable. On définit

$$\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$$

comme l'ensemble des classes d'isomorphismes d'objets irréductibles de $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$ et

$$\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r) \subseteq \mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$$

comme le sous-ensemble des objets irréductibles Π^{irr} tels qu'il existe un plongement

$$\Pi^{\mathrm{irr}} \hookrightarrow \Pi^{\mathrm{an}}(r).$$

On définit la multiplicité:

$$d_{\Pi^{\mathrm{irr}}} := \dim_E \mathrm{Hom}_{\mathrm{GL}_n(K)}\left(\Pi^{\mathrm{irr}},\ \Pi^{\mathrm{an}}(r)\right)$$

qui est finie pour tout $\Pi^{\mathrm{irr}} \in \mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r)$ puisque $\Pi^{\mathrm{an}}(r)$ est admissible. Si ρ est crystalline et satisfait une certaine hypothèse de généricité (faible), alors l'ensemble $\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r)$ est explicitement déterminé par certaines données combinatoires données par la filtration de Hodge de ρ par [BHS17]. Sous les mêmes conditions, [BH18] construit une certaine sous-représentation de $\Pi^{\mathrm{an}}(r)$ qui est maximale parmi toutes les représentations qui satisfont une certaine condition de multiplicité une et qui appartiennent à $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$. Si ρ est crystalline, les résultats de la Section 3.3 de [Bre17] et de [BH18] suggèrent que la question de comment construire une sous-représentation suffisamment large de $\Pi^{\mathrm{an}}(r)$ pour déterminer complètement ρ est encore mystérieux (surtout quand la filtration de Hodge de ρ se trouve dans une position suffisamment générique). La situation est assez différente (et meilleure) si ρ est semi-stable et non-crystalline, au moins quand $n = 3$. On suppose à partir de maintenant que $F^+ = \mathbf{Q}$ (ainsi $K = \mathbf{Q}_p$ et $v_0 = p$) et que le (φ, N)-module $\underline{D}$ associé à $\mathrm{WD}(\rho)$ a la forme suivante:

$$\underline{D} = Ee_0 \oplus Ee_1 \oplus Ee_2 \tag{1.3.2}$$

où

$$N(e_0) = 0,\ N(e_1) = e_0,\ N(e_2) = e_1$$

et

$$\varphi(e_i) = \mathrm{ur}(\alpha)p^{i-2}e_i$$

pour $0 \leq i \leq 2$ et un certain $\alpha \in E^\times$. Il découle de la correspondance de Langlands locale classique que

$$\mathrm{LL} \circ \mathrm{WD}(\rho) = \mathrm{St}_3^\infty \otimes_E (\mathrm{ur}(\alpha) \circ \det_3)$$

où St_3^∞ est la représentation de Steinberg lisse de $\mathrm{GL}_3(\mathbf{Q}_p)$ définie comme l'unique quotient irréductibles de la série principale lisse:

$$\left(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)} 1\right)^\infty,$$

$\det_3$ est le caractère déterminant $\mathrm{GL}_3(\mathbf{Q}_p) \twoheadrightarrow \mathbf{Q}_p^\times$ et $\mathrm{ur}(\alpha)$ est le caratère non ramifée qui envoie p à α. On note $\underline{k} = (k_1, k_2, k_3) \in \mathbf{Z}^3$ avec

$$k_1 > k_2 > k_3$$

l'ensemble des poids de Hodge–Tate de ρ. On définit

$$\Pi^{\mathrm{alg}}(\underline{k},\underline{D}) := \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\otimes_E (\mathrm{ur}(\alpha)\circ\det_3)$$

où $\overline{L}(\lambda)$ est la représentation algébrique de dimension finie de $\mathrm{GL}_3(\mathbf{Q}_p)$ dont le poids le plus haut est

$$\lambda = (\lambda_1,\lambda_2,\lambda_3) = (k_1-2, k_2-1, k_3) \tag{1.3.3}$$

par rapport au sous-groupe de Borel triangulaire supérieur $\overline{B}(\mathbf{Q}_p)$. Le Théorème 1.1.1 implique l'égalité suivante:

$$\Pi^{\mathrm{alg}}(r) = \Pi^{\mathrm{alg}}(\underline{k},\underline{D})^{\oplus d}$$

pour un entier $d = d_{\Pi^{\mathrm{alg}}(\underline{k},\underline{D})}$ qui dépend de U^p et de r. On dit que ρ est *non-critique* (voir le (ii) de la Remarque 6.1.4 de [Bre17]) si la filtration de Hodge de ρ satisfait une hypothèse générique explicite qui implique l'égalité suivante:

$$\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r) = \{\Pi^{\mathrm{alg}}(\underline{k},\underline{D})\}. \tag{1.3.4}$$

Si ρ est non-critique, alors il existe un triplet d'invariants $(\mathscr{L}_{i,j})_{0\le i<j\le 2}\in E^3$ tel que la filtration de Hodge (faiblement admissible) $\mathrm{Fil}^\bullet$ sur $\underline{D}$ a la forme suivante:

$$\mathrm{Fil}^i\underline{D} = \begin{cases} \underline{D} & \text{si } i\le k_3 \\ E(e_2+\mathscr{L}_{1,2}e_1+\mathscr{L}_{0,2}e_0)\oplus E(e_1+\mathscr{L}_{0,1}e_0) & \text{si } k_3+1\le i\le k_2 \\ E(e_2+\mathscr{L}_{1,2}e_1+\mathscr{L}_{0,2}e_0) & \text{si } k_2+1\le i\le k_1 \\ 0 & \text{si } i\ge k_1+1 \end{cases} . \tag{1.3.5}$$

Dans l'étude de pLL l'un des objectifs principaux est de trouver dans le cas (1.3.5) tous les invariants $\mathscr{L}_{i,j}$ dans $\Pi^{\mathrm{an}}(r)$ pour $0\le i<j\le 2$. On appelle $\mathscr{L}$-invariants ces $\mathscr{L}_{i,j}$ et $\mathscr{L}$-invariants simples les $\mathscr{L}_{0,1}$ et les $\mathscr{L}_{1,2}$. On appelle $\mathscr{L}_{0,2}$ *le $\mathscr{L}$-invariant supérieur* de ρ. Ce qui fait ce cas (1.3.5) si spécial est le fait que l'on espère trouver tous les invariants $\mathscr{L}_{i,j}$ dans une sous-représentation de $\Pi^{\mathrm{an}}(r)$ qui appartient à $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$. Le cas de $\mathrm{GL}_2(\mathbf{Q}_p)$ a été étudié en premier par Breuil dans [Bre04] et dans [Bre10b], et c'était l'un des points de départ de pLL. Afin de retrouver explicitement l'unique *$\mathscr{L}$-invariant* de $\Pi^{\mathrm{an}}(r)$ quand ρ est semi-stable non-crystalline de dimension deux, on doit considérer la fonction logarithme p-adique pour construire une base d'un certain groupe de Ext^1. Un résultat récent de Ding dans [Ding18] généralise [Bre04] et [Bre10b] à tous les $\mathscr{L}$-invariants simples (qui sont les généralisations de $\mathscr{L}_{0,1}$ et $\mathscr{L}_{1,2}$) de $\mathrm{GL}_n(K)$ pour K arbitraire. Il reste donc à traiter $\mathscr{L}_{0,2}$. Dans [Schr11], Schraen a étudié quelques représentations localement analytiques de $\mathrm{GL}_3(\mathbf{Q}_p)$ pour essayer de réaliser le module filtré $\mathrm{Fil}^\bullet\underline{D}$ dans le complexe de de Rham de l'espace de Drinfeld de dimension deux. L'un des ingrédients principaux de [Schr11] est un calcul des groupes de cohomologie localement analytiques. Schraen a construit un objet $\Sigma(\lambda,\underline{\mathscr{L}})'$ (voir la Définition 5.19 de [Schr11]) dans la catégorie dérivée $\mathcal{D}\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$ de la catégorie abélienne $\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}$ des modules abstraits sur l'algèbre de distribution localement analytique $D(\mathrm{GL}_3(\mathbf{Q}_p),E)$. La motivation de cette construction est de chercher une famille raisonnable de représentations localement analytiques de $\mathrm{GL}_3(\mathbf{Q}_p)$ qui dépend de trois invariants, puisque ρ dépend de trois invariants. Schraen construit d'abord une famille de représentations localement analytiques $\Sigma(\lambda,\mathscr{L},\mathscr{L}')$ (voir (1.11) de [Schr11]) qui dépendent de deux invariants. Puis il observe qu'un troisième invariant se trouve dans le groupe

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p)}(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L},\mathscr{L}')) \tag{1.3.6}$$

mais pas dans

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L},\mathscr{L}')).$$

Il semble donc naturel dans un premier temps de remplacer une famille de représentations localement analytiques par une famille de complexes $\Sigma(\lambda,\underline{\mathscr{L}})'$. Grosso modo, on espère rajouter $\overline{L}(\lambda)$ pour obtenir le troisième invariant mais on n'arrive à le faire qu'au sens d'un complexe abstrait dans

[Schr11]. Une observation cruciale de Breuil est que l'on peut peut-être étendre chaque $\Sigma(\lambda, \mathscr{L}, \mathscr{L}')$ en une représentation localement analytique plus grande qui contient plus d'invariants, en ajoutant d'autres représentations que $\overline{L}(\lambda)$. Breuil a construit dans [Bre17] deux représentations localement analytiques $\Pi^1(\underline{k}, \underline{D})$ et $\Pi^2(\underline{k}, \underline{D})$ dans $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$ qui ne dépendent que de $\underline{k}$ et $\underline{D}$ telles que

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi^i(\underline{k}, \underline{D}),\ \Pi^{\mathrm{alg}}(\underline{k}, \underline{D})\right) = 3 \tag{1.3.7}$$

pour $i = 1, 2$. On a besoin des conditions suivantes:

Condition 1.3.8. *On suppose que U^p et r satisfont:*

(i) $\Pi^{\mathrm{alg}}(r) \neq 0$;

(ii) $\mathrm{WD}(\rho)$ *est donné par le cas $n = 3$ de (1.3.2);*

(iii) *ρ est non-critique;*

(iv) *la représentation automorphe correspondante à r n'apparaît qu'une fois dans l'espace des formes automorphes sur $\mathcal{G}(\mathbb{A}^{\infty}_{F^+})$.*

Alors il est montré dans le Théorème 6.2.1 dans [Bre17] que pour toute paire fixée (U^p, r) qui satisfait la Condition 1.3.8, il existe une unique représentation localement analytique Π de $\mathrm{GL}_3(\mathbf{Q}_p)$ de la forme

$$\Pi^{\mathrm{alg}}(\underline{k}, \underline{D}) \begin{array}{c} \diagup\ \Pi^1(\underline{k}, \underline{D}) \\ \diagdown\ \Pi^2(\underline{k}, \underline{D}) \end{array} \tag{1.3.9}$$

telle que

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi \otimes_E \varepsilon^2 \circ \det_3,\ \Pi^{\mathrm{an}}(r)\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi^{\mathrm{alg}}(\underline{k}, \underline{D}) \otimes_E \varepsilon^2 \circ \det_3,\ \Pi^{\mathrm{an}}(r)\right). \tag{1.3.10}$$

Ici on utilise la notation suivante: $V \text{ — } W$ pour une extension non-scindée de W par V. Une observation cruciale est que Π dépend de quatre invariants selon (1.3.7) mais que ρ ne dépend que de trois invariants selon l'égalité (1.3.5). L'isomorphisme (1.3.10) implique l'existence d'une unique représentation Π de la forme (1.3.9) qui apparaît comme une sous-représentation de $\Pi^{\mathrm{an}}(r)$. Puisque ρ ne dépend que trois $\mathscr{L}$-invariants quand $n = 3$, on ne s'attend pas à ce que toutes les représentations localement analytiques de la forme (1.3.9) se plongent dans $\Pi^{\mathrm{an}}(r)$ pour certains U^p et r. Une question naturelle que l'on se pose est la suivante:

Question 1.3.11. *Y-a-t-il un critère purement de théorie de représentation qui détermine si une représentation de la forme (1.3.9) se plonge dans un certain $\Pi^{\mathrm{an}}(r)$?*

Une observation cruciale est que $\overline{L}(\lambda)$ est exactement l'critère que l'on voulait dans la Question 1.3.11 pour réduire le nombre d'invariants de Π. Plus précisément, le fait qu'une telle représentation localement analytique Π soit incluse dans une représentation plus grande avec $\overline{L}(\lambda)$ dans le cosocle (avec multiplicité deux) donne un critère pour que Π se plonge dans la cohomologie complétée. Une remarque importante de [Schr11] est que la fonction dilogarithme p-adique apparaît dans l'espace de dimension deux (1.3.6) (voir la Corollaire 5.17 de [Schr11]). Par conséquent, Breuil a suggéré qu'il devait y avoir une relation explicite entre [Schr11], [Bre17] et [BD18], plus précisément la fonction dilogarithme p-adique devrait être reliée à l'unique $\mathscr{L}$-invariant supérieur $\mathscr{L}_{0,2}$ de ρ, et cette relation devrait être analogue à celle entre la fonction logarithme p-adique et les $\mathscr{L}$-invariants simples. On arrive donc naturellement à la question suivante:

Question 1.3.12. *Quelle est la relation précise entre la fonction dilogarithme p-adique et l'unique $\mathscr{L}$-invariant supérieur $\mathscr{L}_{0,2}$ de ρ?*

Le résultat principal de la seconde partie de ma thèse est une tentative de répondre à ces deux questions à la fois, en utilisant la construction d'une famille de représentations localement analytiques $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ qui dépend de trois invariants $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ (voir le Théorème 4.1.1 pour une description intuitive de la représentation). L'exposant 'min, +' signifie que $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ est *légèrement plus grande que la représentation localement analytique minimale de* $\mathrm{GL}_3(\mathbf{Q}_p)$ *qui "contient" la fonction dilogarithme p-adique.* On remarque que λ et $\underline{k}$ se déterminent l'un l'autre par (1.3.3). Les propriétés principales de $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ sont résumées dans les trois énoncés ci-dessous.

Théorème 1.3.13. *Etant donnés U^p et r qui satisfont la Condition 1.3.8, il existe un unique choix de paramètres $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ tel que*

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E(\mathrm{ur}(\alpha)\otimes_E\varepsilon^2)\circ\det_3,\ \Pi^{\mathrm{an}}(r)\right)\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi^{\mathrm{alg}}(\underline{k},\underline{D})\otimes_E\varepsilon^2\circ\det_3,\ \Pi^{\mathrm{an}}(r)\right). \quad (1.3.14)$$

Remarque 1.3.15. *On sait que*

$$\mathscr{L}_1=\mathscr{L}_{1,2},\ \mathscr{L}_2=\mathscr{L}_{0,1} \quad (1.3.16)$$

d'après la définition de $\mathscr{L}_1$ et $\mathscr{L}_2$, d'après le Théorème 1.2 de [Ding18] et d'après certains arguments de la théorie de Hodge p-adique qui apparaissent dans le cas de $\mathrm{GL}_2(\mathbf{Q}_p)$ *(voir la Remarque 3.1 de [Ding18]). L'énoncé du Théorème 1.3.13 implique que $\mathscr{L}_3$ est uniquement déterminé par U^p et r, mais a priori non par ρ. On croit fortement à l'existence d'un plongement*

$$\Pi(D)\hookrightarrow\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3) \quad (1.3.17)$$

pour certains $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ uniquement déterminés par ρ où $\Pi(D)$ est la représentation localement analytique construite à partir de ρ dans [BD18]. Mais on ne sait pas comment le prouver en utilisant une méthode purement locale. Si ρ est ordinaire (autrement dit si elle a des poids de Hodge–Tate consécutifs), alors le Théorème 1.1 de [BD18] prouve que $\Pi(D)$ se plonge en fait dans $\Pi^{\mathrm{an}}(r)$, ce qui avec notre Théorème 1.3.13 impliquent que (1.3.17) existe et que $\mathscr{L}_3$ ne dépend que de ρ.

On donne ci-dessous une réponse positive à la Question 1.3.11 (au moins dans une direction). Elle découle facilement du Théorème 6.2.1 de [Bre17] et de notre Théorème 1.3.13.

Corollaire 1.3.18. *On fixe une représentation Π de la forme (1.3.9). Si l'on a un plongement*

$$\Pi\otimes_E\varepsilon^2\circ\det_3\hookrightarrow\Pi^{\mathrm{an}}(r)$$

pour un choix convenable de U^p et de r, alors on a un plongement

$$\Pi\hookrightarrow\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\otimes_E\mathrm{ur}(\alpha)\circ\det_3$$

pour certains $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ uniquement déterminés par Π.

Le résultat suivant est une conséquence naturelle de la construction des $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$. On trouvera plus de détails dans le Théorème 4.1.7.

Théorème 1.3.19. *Il existe un complexe explicite de représentations localement analytiques de* $\mathrm{GL}_3(\mathbf{Q}_p)$ *dont le dual fort réalise l'objet $\Sigma(\lambda,\underline{\mathscr{L}})'$.*

Chapter 2

General Introduction

2.1 p-adic local-global compatibility conjecture

We fix a prime number p, a finite extension $[K : \mathbf{Q}_p] < +\infty$ and another sufficiently large finite extension $[E : \mathbf{Q}_p] < +\infty$. We use the notation $\mathcal{O}_K$ (resp. ϖ_K, resp. k) for the ring of integer (resp. a uniformizer, resp. the residual field) of K. Similarly, We use the notation $\mathcal{O}_E$ (resp. ϖ_E, resp. k_E) for the ring of integer (resp. a uniformizer, resp. the residual field) of E. The so-called p-adic local Langlands correspondence initiated by Breuil in [Bre03a], [Bre03b] and [Bre04] is conjecturally a bijection between a certain set of p-adic Banach linear representations of $\mathrm{GL}_n(K)$ and the set of p-adic continuous Galois representations $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(E)$. We use the notation $p\mathrm{LL}$ for the map from ρ to the conjectural p-adic Banach representation $p\mathrm{LL}(\rho)$ corresponding to it. The full p-adic local Langlands correspondence is known for $\mathrm{GL}_2(\mathbf{Q}_p)$ essentially due to Colmez in [Col10], and its compatibility with certain p-adic interpolation of étale cohomology (the so-called completed cohomology) of tower of modular curves is known by Emerton in [Eme06]. The map $p\mathrm{LL}$ is largely mysterious for $K \neq \mathbf{Q}_p$ or $n \geq 3$ and only a few partial results are known. It is widely expected (c.f. [Bre10a]) that $p\mathrm{LL}$ should be compatible with reduction modulo ϖ_E, 'p-adic families' and so on, hence we use the notation $\overline{p\mathrm{LL}}$ for the map (which is highly conjectural if $K \neq \mathbf{Q}_p$ or $n \geq 3$) that associate a admissible smooth k_E-representation of $\mathrm{GL}_n(K)$ with a Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(k_E)$.

We start to give more precise formulation of the conjectural local-global compatibility for $p\mathrm{LL}$, which is a natural generalization of the ideas in [Eme06] (except that we no longer know $p\mathrm{LL}$ if $K \neq \mathbf{Q}_p$ or $n \geq 3$). From now on we fix a totally real field F^+ together with a totally imaginary quadratic extension F of F^+. We fix a finite place v_0 of F^+ that divides p, splits in F and satisfies $K \cong F^+_{v_0} \cong F_{w_0}$ where w_0 is a fixed finite place of F above v_0. We fix a unitary group $\mathcal{G}$ over F^+ such that

(i) $\mathcal{G} \otimes_{F^+} F \cong \mathrm{GL}_n/_F$;

(ii) $\mathcal{G}(F^+ \otimes_{\mathbf{Q}} \mathbf{R})$ is compact.

We also fix an open compact subgroup $U^{v_0} \subset \mathcal{G}(\mathbb{A}_{F^+}^{\infty,v_0})$ and a finitely generated $\mathcal{O}_E$-module A endowed with its p-adic topology. Then we consider the space of p-adic (resp. locally constant) A-valued automorphic forms on $\mathcal{G}(\mathbb{A}_{F^+}^{\infty})$ with a fixed level U^{v_0} prime to v_0:

$$\widehat{S}(U^{v_0}, A) := \{f : \mathcal{G}(F^+)\backslash\mathcal{G}(\mathbb{A}_{F^+}^{\infty})/U^{v_0} \to A, \text{ continuous }\}$$

$$(\text{resp. } S(U^{v_0}, A) := \{f : \mathcal{G}(F^+)\backslash\mathcal{G}(\mathbb{A}_{F^+}^{\infty})/U^{v_0} \to A, \text{ locally constant }\}).$$

Note in particular that if A is a p-torsion module and has discrete topology, then the two spaces defined above coincide. Each space above carries an action of a universal Hecke algebra $\mathbb{T}$ over $\mathcal{O}_E$ (freely

generated by Hecke operators indexed by finite places of F lying above a set $D(U^{v_0})$ of finite places of F^+ which splits in F, are prime to p and where $U_v := U^{v_0} \cap \mathcal{G}(F_v^+)$ is a maximal open compact subgroup of $\mathcal{G}(F_v^+)$), as well as an action of $\mathrm{GL}_n(K)$ which is continuous (resp. smooth) on $\widehat{S}(U^{v_0}, A)$ (resp. on $S(U^{v_0}, A)$) and comes from right translation on $\mathcal{G}(F^+)\backslash\mathcal{G}(\mathbb{A}_{F^+}^{\infty})/U^{v_0})$. The action of $\mathrm{GL}_n(K)$ on both spaces commute with the action of $\mathbb{T}$. Given a p-adic continuous Galois representation $r_A : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(A)$ which is absolutely irreducible, conjugate self-dual and unramified at each place of F lying above a certain $v \in D(U^{v_0})$, we can associate an ideal $\mathfrak{p}_{r_A} \subset \mathbb{T} \otimes_{\mathcal{O}_E} A$ with r_A, then the torsion subspace $\widehat{S}(U^{v_0}, A)[\mathfrak{p}_{r_A}]$ (resp. $S(U^{v_0}, A)[\mathfrak{p}_{r_A}]$) cut out by this ideal naturally carries a continuous (resp. smooth) action of $\mathrm{GL}_n(K)$. We use the notation LL for the map given by classical local Langlands correspondence that sends an n-dimensional Frobenius semi-simple Weil–Deligne representation to its corresponding irreducible smooth representation of $\mathrm{GL}_n(K)$ (c.f. [HT01], [He00] and [Sch13]). We set

$$r := r_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} E,\ \overline{r} := r_{k_E} = r_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} k_E,\ \rho := r|_{\mathrm{Gal}(\overline{K}/K)},\ \overline{\rho} := \overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$$

and use the shorten notation

$$\begin{aligned}
\widehat{\Pi}(r) &:= \left(\widehat{S}(U^{v_0}, \mathcal{O}_E)[\mathfrak{p}_{r_{\mathcal{O}_E}}]\right) \otimes_{\mathcal{O}_E} E \\
\Pi(\overline{r}) &:= S(U^{v_0}, k_E)[\mathfrak{p}_{r_{k_E}}] \\
\Pi^{\mathrm{sm}}(r) &:= \left(S(U^{v_0}, \mathcal{O}_E)[\mathfrak{p}_{r_{\mathcal{O}_E}}]\right) \otimes_{\mathcal{O}_E} E \quad .
\end{aligned}$$

Note that we omit the level U^{v_0} from the notation for simplicity. We define $\Pi^{\mathrm{alg}}(r)$ as the subspace of $\widehat{\Pi}(r)$ consisting of locally algebraic vectors and note that $\Pi^{\mathrm{alg}}(r)$ is naturally a locally algebraic representation of $\mathrm{GL}_n(K)$. We use the notation $\mathrm{WD}(\rho)$ (resp. $\mathrm{Alg}(\rho)$) for the map that associates a Weil–Deligne representation (resp. a finite dimensional algebraic representation of $\mathrm{GL}_n(K)$) to a potentially semi-stable p-adic Galois representation ρ (resp. the set of Hodge–Tate weights of ρ) via Fontaine's theory in [Fon94] (resp. via Section 5 of [Bre16]). At this stage, the classical local-global compatibility at p (which is proven in [BLGGT12] and [BLGGT14] with further technical assumptions) can be roughly formulated as the following

Theorem 2.1.1. *Assume that*

$$\Pi^{\mathrm{alg}}(r) \neq 0.$$

Then ρ is potentially semi-stable and there exists an integer $d > 0$ depending only on r and U^{v_0} such that

$$\Pi^{\mathrm{alg}}(r) \cong \left(\mathrm{LL} \circ \mathrm{WD}(\rho) \otimes_E \mathrm{Alg}(\rho)\right)^{\oplus d}. \tag{2.1.2}$$

It is not too difficult to imagine that the conjectural p-adic local-global compatibility should very roughly have the following form

Hope 2.1.3. *Assume that*

$$\Pi(\overline{r}) \neq 0.$$

Then there exists an integer $d > 0$ depending only on r and U^{v_0} such that

$$\widehat{\Pi}(r) \cong p\mathrm{LL}(\rho)^{\oplus d} \tag{2.1.4}$$

and

$$\Pi(\overline{r}) \cong \overline{p\mathrm{LL}}(\overline{\rho})^{\oplus d}. \tag{2.1.5}$$

In particular, Hope 2.1.3 predicts that

Conjecture 2.1.6. *The structure of $\widehat{\Pi}(r)$ (resp. of $\Pi(\overline{r})$) as a p-adic admissible Banach E-representation (resp. admissible smooth k_E-representation) of $\mathrm{GL}_n(K)$ determines and depends only on the isomorphism class of ρ (resp. of $\overline{\rho}$).*

We emphasize that ρ needs not be potentially semi-stable in Hope 2.1.3 and Conjecture 2.1.6. On the other hand, as the maps pLL and $\overline{p\mathrm{LL}}$ are highly mysterious if $K \neq \mathbf{Q}_p$ or $n \geq 3$, we need to start with the cases when ρ is indeed potentially semi-stable and thus Theorem 2.1.1 is available. As a result, we will essentially consider only the cases when ρ is potentially semi-stable in the rest of this introduction. It is quite difficult to study the p-adic Banach representations for $\mathrm{GL}_n(K)$ directly, and we have essentially two kinds of possibly easier objects to start with. My thesis is thus divided into two parts and each part studies one kind of objects. The first part of my thesis studies the admissible smooth k_E-representation $\Pi(\overline{r})$ (which is joint work with Chol Park) while the second part of my thesis studies the locally analytic representation $\Pi^{\mathrm{an}}(r)$ defined as the set of locally analytic vectors (vectors upon which elements of $\mathrm{GL}_n(K)$ act via vector-valued locally analytic functions on the group $\mathrm{GL}_n(K)$) in $\widehat{\Pi}(r)$.

2.2 Mod p local-global compatibility for $\mathrm{GL}_n(\mathbf{Q}_p)$

We firstly start with $\Pi(\overline{r})$. There are several natural questions to ask about $\Pi(\overline{r})$:

(i) do we have $\Pi(\overline{r}) \neq 0$?

(ii) what is the structure of $\Pi(\overline{r})$?

(iii) what is the explicit relation between $\Pi(\overline{r})$ and $\overline{\rho} = \overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$(remember that $K = F_{w_0}$)?

The statement $\Pi(\overline{r}) \neq 0$ is essentially equivalent to the more classical terminology '$\overline{r}$ is modular (or automorphic)'. This is known for the case $F = \mathbf{Q}$ and $\mathcal{G} = \mathrm{GL}_2$ as a result of Serre's conjecture which was proven by Khare–Wintenberger in [KW09]. Unfortunately, (i) is still widely open in general and therefore will be put as an assumption in my thesis whenever necessary. We assume that (i) holds and consider the question (ii). The complete answer to (ii) is known for the case $F = \mathbf{Q}$, $\mathcal{G} = \mathrm{GL}_2$ due to Emerton (which relies crucially on pLL for $\mathrm{GL}_2(\mathbf{Q}_p)$) and is typically still widely open if $K \neq \mathbf{Q}_p$ or $n \geq 3$. One of the main obstacles is a lack of complete classification of irreducible smooth k_E-representations of $\mathrm{GL}_n(K)$ except the case $\mathrm{GL}_2(\mathbf{Q}_p)$ which is known due to [BL94] and [Bre03a]. More precisely, evidences from [BP12], [Hu10] and [Schr15] show that classification of representations that do not occur as subquotients of parabolic inductions (which are called supercuspidal) is very difficult even for $\mathrm{GL}_2(\mathbf{Q}_{p^2})$. On the other hand, as one knows how to classify all the representations parabolically induced from characters of a Borel subgroup (c.f. [Her11] for general parabolic induction for $\mathrm{GL}_n(K)$), one can already construct by [BH15] a quite big finite length smooth k_E-representation $\Pi^{\mathrm{ord}}(\overline{\rho})$ (depending only on $\overline{\rho}$) that always embeds into $\Pi(\overline{r})$ if $K = \mathbf{Q}_p$ and $\overline{\rho}$ is *ordinary* (i.e. $\overline{\rho}$ has its image in a Borel subgroup of $\mathrm{GL}_n(k_E)$). However, it is clear from the construction of [BH15] that $\Pi^{\mathrm{ord}}(\overline{\rho})$ is not sufficient to determine $\overline{\rho}$ in general. Note that Hu shows in [Hu17] that a supercuspidal representation necessarily appears in $\Pi(\overline{r})$ even in the case $K = \mathbf{Q}_{p^2}$ and $n = 2$ and it is widely expected that the extra information of $\overline{\rho}$ involves the extensions between $\Pi^{\mathrm{ord}}(\overline{\rho})$ and certain supercuspidal representations.

As a result, one needs a method to 'walk around' supercuspidal representations but nevertheless to be able to find enough information in $\Pi(\overline{r})$ to determine $\overline{\rho}$. One natural approach is to restrict the action to $\mathrm{GL}_n(\mathcal{O}_K)$ due to at least two reasons: irreducible k_E-representations of $\mathrm{GL}_n(\mathcal{O}_K)$ (hence of $\mathrm{GL}_n(k)$) are easy to classify due to section 2 in Part II of [Jan03]; at least classification of irreducible E-representations of $\mathrm{GL}_n(k)$ is well-known by Deligne–Lusztig theory in [DL76]. Moreover, we have

Theorem 2.2.1 (c.f. [Jan81] and Proposition 4.1.3 of [LLL16])**.** *For each sufficiently generic Deligne–Lusztig representation τ of $\mathrm{GL}_n(k)$ and each $\mathcal{O}_E$-lattice $\tau^\circ \subsetneq \tau$, the set of Jordan–Hölder factors*

$$\mathrm{JH}_{\mathrm{GL}_n(k)}\left(\tau^\circ \otimes_{\mathcal{O}_E} k_E\right)$$

is independent of the choice of τ° and has a purely combinatorial description via the affine Weyl group of $\mathrm{GL}_n/_k$.

Here the condition *sufficiently generic* means the following: when we parameterize a Deligne–Lusztig representation by a weight and an element of the Weyl group due to Jantzen (c.f. [Jan81] and Section 4 of [Her09]), this weight should be sufficiently far away from the boundary of the alcove it lies in.

The very first step to understand $\Pi(\overline{r})|_{\mathrm{GL}_n(\mathcal{O}_K)}$ is to characterize the socle

$$\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$$

which is already a highly non-trivial and deep problem and is historically a reformulation of the more classical weight part of Serre's conjecture if we take $F=\mathbf{Q}$ and $\mathcal{G}=\mathrm{GL}_2$. Conjectures on (partial) characterizations of $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ have been made by Buzzard–Diamond–Jarvis in [BDJ10] for the case $K=\mathbf{Q}_{p^f}$, $n=2$, by Herzig in [Her09] for $K=\mathbf{Q}_p$ and general n, and by Gee–Herzig–Savitt in [GHS] for certain more general cases. It is widely expected that there is a deep relation between the structure of $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ and the special fibers of various potentially semi-stable deformations rings of $\overline{\rho}$ which finally leads us to the so-called Breuil–Mézard–Emerton-Gee conjecture formulated in [BM02], [BM14] and [EG14].

The structure of $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ when $K=\mathbf{Q}_{p^f}$ was determined for $n=2$ by [GK14] and for $n=3$ by [EGH13], [HLM17], [MP17], [LMP] and [LLHLMa]. We set

$$\mathrm{K}(1):=\mathrm{Ker}\left(\mathrm{GL}_n(\mathcal{O}_K)\twoheadrightarrow\mathrm{GL}_n(k)\right)$$

and note that it is representation theoretically natural to study $\Pi(\overline{r})^{\mathrm{K}(1)}$ which obviously contains $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ as a subrepresentation. The representation $\Pi(\overline{r})^{\mathrm{K}(1)}$ is a very beginning point (already highly non-trivial) to reconstruct $\Pi(\overline{r})$ as a $\mathrm{GL}_n(K)$ representation, but unfortunately various evidences from [HW18], [LMS16] and [Le17] for $K=\mathbf{Q}_{p^f}$, $n=2$ and from work in preparation by the authors of [LLHLMa] $K=\mathbf{Q}_{p^f}$, $n=3$ suggest that $\Pi(\overline{r})^{\mathrm{K}(1)}$ is still not sufficient to determine $\overline{\rho}$ typically in the case when $\overline{\rho}$ is ordinary and maximally non-split (i.e. has no subquotient which is direct sum of two characters). We have the following example

Example 2.2.2. *We take $K=\mathbf{Q}_{p^f}$, $n=2$ and $\overline{\rho}$ is ordinary with the following form*

$$\begin{pmatrix}\overline{\chi}_2 & *\\ 0 & \overline{\chi}_1\end{pmatrix}.$$

Assume further that $\overline{\chi}_1\overline{\chi}_2^{-1}$ satisfies a weak generic assumption, then it follows from a standard computation of Euler–Poincaré characteristic of Galois cohomology that

$$\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbf{Q}_{p^f}}/\mathbf{Q}_{p^f})}(\overline{\chi}_2,\ \overline{\chi}_1)=f$$

and therefore $\overline{\rho}$ is determined by $\overline{\rho}^{\mathrm{ss}}$ together with $f-1$ invariants in $k_E\cup\{\infty\}$ up to isomorphism. If we assume moreover that $\overline{\rho}$ is Fontaine–Laffaille (which is again a generic assumption in this setting), then we can use Fontaine–Laffaille modules (c.f. [FL82]) to define a set of invariants in $k_E\cup\{\infty\}$ (c.f. Lemma 2.1.1 of [BD14]) that determine the isomorphism class of $\overline{\rho}$. It turns out that the structure of $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ usually cannot determine the value of these invariants. In general, suppose that all n-dimensional k_E-representation of $\mathrm{Gal}(\overline{K}/K)$ can be parameterized by some moduli space and such space has a natural stratification by locally closed subschemes given by certain explicit conditions generalizing splitness of extensions, then $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ is expected to contain only the information of which strata $\overline{\rho}$ lies in, rather than which point $\overline{rho}$ is in the strata.

We define the so-called Iwahori subgroup I (resp. pro-p Iwahori subgroup $I(1)$) of $\mathrm{GL}_n(K)$ as the pre-image of the set of upper-triangular (resp. unipotent) matrix in $\mathrm{GL}_n(k)$ under the surjection $\mathrm{GL}_n(\mathcal{O}_K)\twoheadrightarrow\mathrm{GL}_n(k)$. It is a standard fact in the study of p-adic reductive groups that the normalizer

of I inside $\mathrm{GL}_n(K)$ is generated by I and the following element

$$\Xi_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \varpi_K & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_n(K).$$

Since Ξ_n and $\mathrm{GL}_n(\mathcal{O}_K)$ generate $\mathrm{GL}_n(K)$, it is natural to expect that Ξ_n plays a key role in whatever possible methods to reconstruct a smooth irreducible representation of $\mathrm{GL}_n(K)$ from its restriction to $\mathrm{GL}_n(\mathcal{O}_K)$.

The first result to determine the values of *Fontaine–Laffaille invariants* (as sketched in Example 2.2.2) is [BD14] where they showed (in the case $K = \mathbf{Q}_{p^f}$ and $n = 2$) that $\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r})$ together with the action of Ξ_2 on $\Pi(\overline{r})^{I(1)}$ determine $\overline{\rho}$ up to isomorphism. The result in [BD14] has recently been generalized to $K = \mathbf{Q}_p$ and $n = 3$ by [HLM17] when $\overline{\rho}$ is ordinary and Fontaine–Laffaille and by [LMP] when $\overline{\rho}$ has a two dimensional irreducible sub-(or quotient-) representation. Both results for $n = 3$ consider the action of Ξ_3 on $\Pi(\overline{r})^{I(1)}$ and requires further 'technical generic assumption' on $\overline{\rho}$ which essentially ensures that

$$\mathrm{soc}_{\mathrm{GL}_n(\mathcal{O}_K)}\Pi(\overline{r}) \text{ has minimal length possible}$$

if we fix $\overline{\rho}^{\mathrm{ss}}$ and vary the extension parameters of $\overline{\rho}$. The first part of my thesis is to generalize the results in [HLM17] to $K = \mathbf{Q}_p$ but arbitrary n in Chapter 3. This is joint work with Chol Park.

Theorem 2.2.3. *Assume that $\Pi(\overline{r}) \neq 0$. Then the structure of $\Pi(\overline{r})$ as a admissible smooth k_E-representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ determines $\overline{\rho}$ up to isomorphism if $\overline{\rho}$ is Fontaine-Laffaille, ordinary and sufficiently generic.*

Note that the case $K = \mathbf{Q}_{p^f}$ and $n = 3$ has also been worked out recently by Enns in [En]. Another important result in this direction is Theorem 7.8 of [Sch15] where Scholze used a completely different and more geometric method to prove that $\Pi(\overline{r})$ always determines $\overline{\rho}$ for $n = 2$ and arbitrary K.

Now we start to give a more precise statement for Theorem 2.2.3 and refer to Chapter 3 for further details. We set $K = \mathbf{Q}_p$ from now on. We fix $\overline{r}$ such that

$$\Pi(\overline{r}) \neq 0$$

and assume that $\overline{\rho}$ is ordinary, then there exists a sequence of subrepresentations

$$0 \subsetneq \overline{\rho}_{1,1} \subsetneq \overline{\rho}_{1,2} \subsetneq \cdots \subsetneq \overline{\rho}_{1,n-1} \subsetneq \overline{\rho}_{1,n} = \overline{\rho}$$

such that

$$\overline{\chi}_1 := \overline{\rho}_{1,1} \text{ and } \overline{\chi}_i := \overline{\rho}_{1,i}/\overline{\rho}_{1,i-1}$$

are one-dimensional for all $2 \leq i \leq n$. Assume that $\overline{\chi}_{i-1}\overline{\chi}_i^{-1}$ satisfies some mild generic assumption similar to the one mentioned in Example 2.2.2, we have

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\chi}_{i-1}) = \mathrm{Ext}^2_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\chi}_{i-1}) = 0$$

and

$$\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\chi}_{i-1}) = 1$$

for each $2 \leq i \leq n$. Therefore by a simple devissage we can deduce that

$$\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)}(\overline{\chi}_i,\ \overline{\rho}_{1,i-1}) = i - 1$$

which roughly means that $\overline{\rho}_i$ is determined by $\overline{\chi}_i$, $\overline{\rho}_{1,i-1}$ together with $i-2$ extra invariants for each $2 \leq i \leq n$. Hence by an immediate induction we know that $\overline{\rho}$ is determined by

$$\overline{\rho}^{\mathrm{ss}} = \oplus_{i=1}^{n} \overline{\chi}_i$$

together with $\frac{(n-1)(n-2)}{2}$ invariants. According to a complete classification of k_E-characters of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ essentially due to class field theory, we know that there exists $\overline{a}_i \in \mathbf{Z}/(p-1)\mathbf{Z}$ such that

$$\overline{\chi}_i \omega^{-\overline{a}_i - i + 1}$$

is unramified for each $1 \leq i \leq n$ where ω is the reduction modulo p of the cyclotomic character

$$\varepsilon : \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p) \twoheadrightarrow \mathbf{Z}_p^{\times}.$$

It turns out that the condition

$$\overline{\rho} \text{ is Fontaine–Laffaille}$$

is essentially the same as the existing of the integers $a_i \in \mathbf{Z}$ whose images in $\mathbf{Z}/(p-1)\mathbf{Z}$ is $\overline{a}_i$ and satisfy

$$a_n > a_{n-1} > \cdots > a_1.$$

The theory in [FL82] associates a *Fontaine–Laffaille module* $\mathrm{FL}(\overline{\rho})$ with $\overline{\rho}$, and one can define a set of invariants $\mathrm{FL}_{i,j}(\overline{\rho})$ as certain explicit rational functions of the entries of the Frobenius matrix of the module $\mathrm{FL}(\overline{\rho})$ and prove that $\mathrm{FL}_{i,j}(\overline{\rho})$ depends only on the isomorphism class of $\mathrm{FL}(\overline{\rho})$, for each pair of integers (i,j) satisfying $1 \leq i < i+1 < j \leq n$. By elementary linear algebra we know that $\overline{\rho}$ is determined by $\overline{\rho}^{\mathrm{ss}}$ and the invariants $\mathrm{FL}_{i,j}(\overline{\rho})$, and thus Theorem 2.2.3 is reduced to the problem of capturing each invariant $\mathrm{FL}_{i,j}(\overline{\rho})$ inside $\Pi(\overline{r})$.

An important general philosophy is to study $\Pi(\overline{r})$ via $\Pi^{\mathrm{sm}}(r)$ for all kinds of lifts r of a fixed $\overline{r}$ such that the local Galois representation $\rho = r|_{\mathrm{Gal}(\overline{F_{w_0}}/F_{w_0})}$ satisfies some extra conditions. More precisely, for a given Deligne–Lusztig representation τ of $\mathrm{GL}_n(\mathbf{F}_p)$, there exists a tamely ramified inertia type $\mathrm{LL}^{-1}(\tau)$ (which is a representation of the inertia subgroup $I_{\mathbf{Q}_p} \subsetneq \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ that factors through the tame quotient) corresponding to τ via the inertial Langlands correspondence (c.f. Section 3 of [CEGGPS]). The notation $\mathrm{LL}^{-1}(\tau)$ comes from the compatibility between classical local Langlands correspondence and inertial local Langlands correspondence. Then we consider all lifts ρ of $\overline{\rho}$ with Hodge–Tate weights $\{0, 1, \cdots, n-1\}$ such that

$$\mathrm{WD}(\rho)|_{I_{\mathbf{Q}_p}} \cong \mathrm{LL}^{-1}(\tau). \tag{2.2.4}$$

We note that each such ρ corresponds to a E-point of a formal scheme given by a complete Noetherian local ring $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau),\{0,\cdots,n-1\}}$ which is a special case of the construction of potentially semi-stable deformation rings in [Kis08] for more general inertial types and Hodge–Tate weights.

For each pair of integers (i,j) such that $1 \leq i < i+1 < j \leq n$, we will choose a Deligne–Lusztig representation $\tau_{i,j}$ and then pick a Galois representation $r_{i,j} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E)$ such that

(i) $r_{i,j}$ is absolutely irreducible, conjugate self-dual and unramified at each finite place of F above a certain $v \in D(U^{v_0})$;

(ii) $r_{i,j}$ contains a Galois-invariant $\mathcal{O}_E$-lattice $r_{i,j}^{\circ}$ whose reduction modulo ϖ_E is $\overline{r}$;

(iii) $\rho_{i,j} := r_{i,j}|_{\mathrm{Gal}(\overline{F_{w_0}}/F_{w_0})}$ corresponds to a morphism $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau),\{0,\cdots,n-1\}} \to E$.

In particular, we have the following commutative diagram

$$\begin{array}{ccccc}
\widehat{\Pi}(r_{i,j}) & \hookleftarrow & \widehat{S}(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}] & \twoheadrightarrow & \Pi(\overline{r_{i,j}}) \\
\uparrow & & \uparrow & & \uparrow \\
\Pi^{\mathrm{sm}}(r_{i,j}) & \hookleftarrow & S(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}] & & \\
\uparrow & & \uparrow & & \\
\tau_{i,j} & \hookleftarrow & \tau^\circ_{i,j} & \twoheadrightarrow & \overline{\tau_{i,j}}
\end{array} \tag{2.2.5}$$

where

$$\tau^\circ_{i,j} := \tau_{i,j} \cap S(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}] \subsetneq \Pi^{\mathrm{sm}}(r_{i,j})$$

and

$$\overline{\tau_{i,j}} := \tau^\circ_{i,j} \otimes_{\mathcal{O}_E} k_E.$$

By taking $I(1)$-invariants, the diagram (2.2.5) induces another commutative diagram

$$\begin{array}{ccccc}
\Pi^{\mathrm{sm}}(r_{i,j})^{I(1)} & \hookleftarrow & S(U^{v_0},\mathcal{O}_E)[\mathfrak{p}_{r^\circ_{i,j}}]^{I(1)} & \twoheadrightarrow & \Pi(\overline{r_{i,j}})^{I(1)} \\
\uparrow & & \uparrow & & \uparrow \\
\tau_{i,j}^{I(1)} & \hookleftarrow & (\tau^\circ_{i,j})^{I(1)} & \twoheadrightarrow & \overline{\tau_{i,j}}^{I(1)}
\end{array} . \tag{2.2.6}$$

The fact that

$$\tau_{i,j}^{I(1)} \neq 0$$

implies that $\tau_{i,j}$ is a principal series representation of $\mathrm{GL}_n(\mathbf{F}_p)$, which means that $\tau_{i,j}$ comes from parabolic induction of a character of $T(\mathbf{F}_p)$. Each of the spaces in (2.2.6) carries an action of the Iwahori Hecke algebra which contains Ξ_n and $n-1$ operators $U_n^m \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Q}_p)]$ for $1 \leq m \leq n-1$ which are usually called U_p-operators. We use the notation P_m for the standard parabolic of GL_n containing the upper-triangular Borel and having Levi blocks of the form $\mathrm{GL}_m \times \mathrm{GL}_{n-m}$. We denote the unipotent radical of P_m by N_m. The following lemma summarizes two main properties of U_n^m.

Lemma 2.2.7. *We have*

$$(\Xi_n)^m \circ U_n^m \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)].$$

Given a smooth irreducible representation Π_m (resp. Π_{n-m}) of $\mathrm{GL}_m(\mathbf{Q}_p)$ (resp. of $\mathrm{GL}_{n-m}(\mathbf{Q}_p)$) with central character ω_{Π_m} (resp. $\omega_{\Pi_{n-m}}$). Then we have

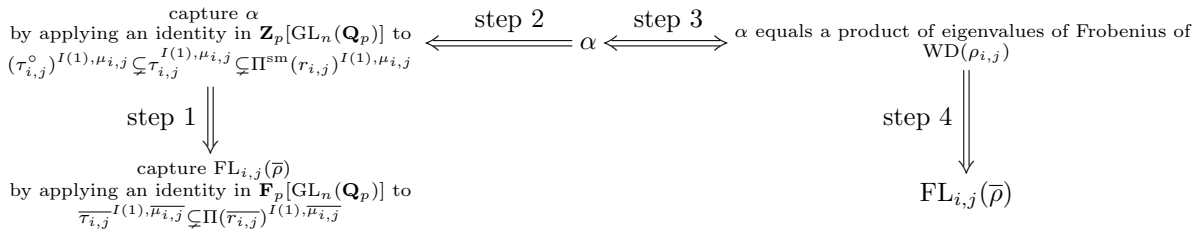
$$U_n^m = \omega_{\Pi_m}(p)^{-1}$$

on the image of

$$\Pi_m \otimes_E \Pi_{n-m} \hookrightarrow \left(\mathrm{Ind}_{P_m(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)} (\Pi_m \otimes_E \Pi_{n-m})\right)^{N_m(\mathbf{Z}_p)}.$$

We use the notation $\mu_{i,j}$ for a character $T(\mathbf{F}_p) \to \mathbf{Z}_p^\times$ (to be chosen) and $\overline{\mu_{i,j}} : T(\mathbf{F}_p) \to \mathbf{F}_p^\times$ for its reduction modulo p. We note that $T(\mathbf{F}_p)$-eigenspace $\Pi^{\mathrm{sm}}(r_{i,j})^{I(1),\mu_{i,j}}$ with eigencharacter $\mu_{i,j}$ is also an eigenspace of U_n^m and we denote the corresponding eigenvalue (of U_n^m) by α. Then we have

the following picture



(2.2.8)

which already appears in [BD14]. We emphasize that m is an integer satisfying $1 \leq m \leq n-1$ and is determined by the pair (i,j). The most standard part of picture 2.2.8 is step 3 which is an equality between the eigenvalue of U_n^m and a product of certain eigenvalues of Frobenius of $\mathrm{WD}(\rho_{i,j})$ and follows directly from classical local Langlands correspondence and the classical local-global compatibility (c.f. Theorem 2.1.1). Then step 4 which realizes the invariant $\mathrm{FL}_{i,j}(\overline{\rho})$ as reduction modulo ϖ_E of a product of Frobenius eigenvalues (up to a factor which is a power of p), is proven using technical computations in integral p-adic Hodge theory, most notably Breuil and Kisin modules. The step 4 is mostly done in Section 3.3. We need to emphasize a significant difference between step 3 and 4. Step 3 lies completely in characteristic zero with coefficient E, depends only on $\mathrm{WD}(\rho)$ and requires no extra condition on $\overline{\rho}$. On the other hand, step 4 relies on an extra technical condition on $\overline{\rho}$ which is going to be recalled in Definition 2.2.9.

Given a pair $(\overline{\rho}, \tau_{i,j})$, one can associate an element $\widetilde{w}(\overline{\rho}, \tau_{i,j})$ in the affine Weyl group $\widetilde{W}$ of GL_n via the theory of *shapes* essentially developed in [LLHLMa]. For each semi-simple Galois representation

$$\overline{\rho}_0 : \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p) \to \mathrm{GL}_n(k_E)$$

we define the following set

$$\Omega(\overline{\rho}_0, \tau_{i,j}) := \{\widetilde{w}(\overline{\rho}, \tau_{i,j}) \mid \overline{\rho}^{\mathrm{ss}} \cong \overline{\rho}_0\}.$$

We consider the length

$$\ell\left(\widetilde{w}(\overline{\rho}, \tau_{i,j})\right)$$

with respect to the standard Coxeter system in $\widetilde{W}$.

Definition 2.2.9. *We say that $\overline{\rho}$ is $\tau_{i,j}$-generic if $\widetilde{w}(\overline{\rho}, \tau_{i,j})$ has maximal length among all the elements in $\Omega(\overline{\rho}^{\mathrm{ss}}, \tau_{i,j})$.*

One can construct (via Fontaine–Laffaille modules) a k_E-scheme $M_{\overline{\rho}_0}$ whose k_E-points parameterize all Fontaine–Laffaille representation $\overline{\rho}$ with a fixed semi-simplification $\overline{\rho}^{\mathrm{ss}} \cong \overline{\rho}_0$. Then we have a natural stratification $\mathscr{S}$ of $M_{\overline{\rho}_0}$ indexed by the set $\Omega(\overline{\rho}_0, \tau_{i,j})$ (this should follow from some work in preparation by authors of [LLHLMa] on moduli of Kisin modules with tamely ramified descent data). It turns out that $M_{\overline{\rho}_0}$ is irreducible and there is a unique element of maximal length in $\Omega(\overline{\rho}_0, \tau_{i,j})$ which corresponds to the unique (non-empty) open strata in $\mathscr{S}$, which implies that $\tau_{i,j}$-generic is indeed a generic assumption. Concerning the terminology of [PQ18], the condition $\tau_{i,j}$-generic is essentially the same as saying that a subqutient of $\overline{\rho}$ determined by the pair (i,j) is *Fontaine–Laffaille generic*.

It follows from Lemma 2.2.7 that

$$U_n^m = \alpha_{i,j} \tag{2.2.10}$$

on $\Pi^{\mathrm{sm}}(r)^{I(1),\mu_{i,j}}$ for a certain constant $\alpha_{i,j} \in E^{\times}$ and we set

$$h_{i,j} := \mathrm{val}_p(\alpha_{i,j}),\ \widetilde{\alpha_{i,j}} := p^{-h_{i,j}}\alpha_{i,j} \in \mathcal{O}_E^{\times}$$

and let $\overline{\alpha_{i,j}}$ be the reduction modulo ϖ_E of $\widetilde{\alpha_{i,j}}$. The main output of step 3 and 4 in picture 2.2.8 is that

Proposition 2.2.11. *We have the equality*

$$\overline{\alpha_{i,j}} = \mathrm{FL}_{i,j}(\overline{\rho})^{-1} \in k_E^\times \tag{2.2.12}$$

if $\overline{\rho}$ is $\tau_{i,j}$-generic.

It remains to clarify step 1 and 2 in picture 2.2.8. We need two group operators

$$\widehat{S}^{i,j},\ \widehat{S}^{i,j,\prime} \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{F}_p)]$$

together with their reduction modulo p

$$S^{i,j},\ S^{i,j,\prime} \in \mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$$

which depends only on the choice of $\tau_{i,j}$ and $\mu_{i,j}$. Then the identity in $\mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Q}_p)]$ that we need has the form

$$\widehat{S}^{i,j,\prime} \circ (\Xi_n)^m \circ U_n^m = \widehat{c}_{i,j}\widehat{S}^{i,j} \tag{2.2.13}$$

on $\Pi^{\mathrm{sm}}(r)^{I(1),\mu_{i,j}}$ where $\widehat{c}_{i,j} \in E^\times$ is a constant depending only on the choice of $\tau_{i,j}$, $\mu_{i,j}$ and m. We emphasize that (2.2.13) can be computed inside $\mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$ because

$$(\Xi_n)^m \circ U_n^m \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$$

due to Lemma 2.2.7. Assume that

$$p^{-h_{i,j}}\widehat{c}_{i,j} \in \mathcal{O}_E^\times$$

and let $c_{i,j}$ be the reduction modulo ϖ_E of $p^{-h_{i,j}}\widehat{c}_{i,j}$. Then we can combine (2.2.13) with (2.2.10) and obtain that

$$\widehat{S}^{i,j,\prime} \circ (\Xi_n)^m = \alpha_{i,j}^{-1}\widehat{c}_{i,j}\widehat{S}^{i,j} \tag{2.2.14}$$

on $\Pi^{\mathrm{sm}}(r)^{I(1),\mu_{i,j}}$ whose reduction modulo ϖ_E is

$$S^{i,j,\prime} \circ (\Xi_n)^m = c_{i,j}\mathrm{FL}_{i,j}(\overline{\rho})S^{i,j} \tag{2.2.15}$$

on $\Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}}$ according to Proposition 2.2.11 and modulo an extra hypothesis:

Hypothesis 2.2.16. *We have*

$$\dim_{k_E}\Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}} = 1$$

and

$$\overline{\tau_{i,j}}^{I(1),\overline{\mu_{i,j}}} \xrightarrow{\sim} \Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}}.$$

One easily observes that Hypothesis 2.2.16 requires an optimal choice of the level U^{v_0}, otherwise we need to replace it by another hypothesis claiming that a certain module over the Hecke algebra $\mathbb{T}$ is free over its support. To remove the Hypothesis 2.2.16, one needs a strong version of the condition *$\tau_{i,j}$-generic* again. In fact, it follows from a computation of Kisin modules with tamely ramified descent data (generalizing the computation in [LLHLMa] and will be proven in [LLMPQ]) that there exists a non-empty open subscheme $M_{\overline{\rho}_0}^{\mathrm{sm}}$ of $M_{\overline{\rho}_0}$ contained in the $\tau_{i,j}$-generic open strata such that $\overline{\rho}$ is a k_E-point of $M_{\overline{\rho}_0}^{\mathrm{sm}}$ if and only if $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau_{i,j}),\{0,\cdots,n-1\}}$ is formally smooth. The very existence of $M_{\overline{\rho}_0}^{\mathrm{sm}}$ is an extremely strong restriction on the choice of $\tau_{i,j}$.

Definition 2.2.17. *We say that $\overline{\rho}$ is strongly $\tau_{i,j}$-generic if it is given by a k_E-point of $M_{\overline{\rho}_0}^{\mathrm{sm}}$.*

The following proposition (which is closely related to the Breuil–Mézard conjecture) will be proven in [LLMPQ] and essentially follows from a functor constructed by the celebrated Taylor–Wilies–Kisin patching method (c.f. Section 7.3 of [LLHLMa] for the case of $\mathrm{GL}_3(\mathbf{Q}_{p^f})$).

Proposition 2.2.18. *The Hypothesis 2.2.16 is true for a suitable choice of U^{v_0} if $\overline{\rho}$ is strongly $\tau_{i,j}$-generic.*

Now we need to understand why do the group operators $\widehat{S}^{i,j}$ and $\widehat{S}^{i,j,\prime}$ (satisfying (2.2.13)) exist and whether it is possible to *explicitly* construct them and compute the constant $\widehat{c}_{i,j}$. The existence of certain $\widehat{S}^{i,j}$ and $\widehat{S}^{i,j,\prime}$ that satisfy (2.2.13) is relatively formal and essentially follows from Hypothesis 2.2.16 but the group operators can be arbitrarily ugly and therefore we can say nothing about the constant $\widehat{c}_{i,j}$. On the other hand, the explicit construction of a single pair of such group operators is very technical and the most difficult part is to show that $S^{i,j}$ and $S^{i,j,\prime}$ are nonvanishing on $\Pi(\overline{r})^{I(1),\overline{\mu_{i,j}}}$. In Chapter 3, we indeed (up to modifications on notation) give an explicit construction of group operators in Section 3.4.2, prove the nonvanishing property mentioned above in Section 3.4.8 and then compute the constant $\widehat{c}_{i,j}$ in Section 3.4.5.

In all, we conclude that the identity (2.2.15) captures the invariant $\mathrm{FL}_{i,j}(\overline{\rho})$ inside $\Pi(\overline{r})$ if $\overline{\rho}$ is *strongly $\tau_{i,j}$-generic*, and thus $\Pi(\overline{r})$ determines $\overline{\rho}$ up to isomorphism if $\overline{\rho}$ is *strongly $\tau_{i,j}$-generic* for all pair (i,j) such that $1 \leq i < i+1 < j \leq n$.

There are various natural generalizations of Theorem 2.2.3 that have been largely done but not included in this thesis, and we will give a quick summary of them within three steps. We always assume in the following that $\overline{\rho}^{\mathrm{ss}}$ is *sufficiently generic*. The following generalizations are mostly work in preparation with Viet Bao Le Hung, Daniel Le, Stefano Morra and Chol Park.

The first step is to remove the condition *ordinary*. The new difficulty is divided into two parts: (i) one needs to choose a family of Fontaine–Laffaille invariants and a suitable τ for reach invariant such that the philosophy in Picture 2.2.8 works; (ii) one needs to explicitly construct certain pair of group operators $\widehat{S}$ and $\widehat{S}'$ for each invariant such that an analogue of (2.2.13) is true with an explicit constant $\widehat{c}$. Note that we need to treat much more different cases of $\overline{\rho}$ (compared to ordinary case) and each case requires different choices of Fontaine–Laffaille invariants and of τ. The second step is to remove certain condition *τ-generic* which is analogous to Definition 2.2.9. The difficulty of this step is transparent as we do not expect $\mathcal{R}_{\overline{\rho}}^{\mathrm{LL}^{-1}(\tau),\{0,\cdots,n-1\}}$ to be formally smooth without a strong generic assumption on $\overline{\rho}$. Hence Proposition 2.2.11, Hypothesis 2.2.16 and (2.2.15) completely fail in general. The crucial observation is that we can determine $\overline{\rho}$ using a set of invariants together with a set of modular Serre weights. The last step is to remove the condition *Fontaine–Laffaille*. The problem is that $\overline{\rho}$ usually has lifts with respect to much fewer possible choices of τ if we fix Hodge–Tate weights. The natural solution is to to consider lifts (of $\overline{\rho}$) with *higher Hodge–Tate weights*.

Remark 2.2.19. *It should be possible to generalize all the discussions above to the case $K = \mathbf{Q}_{p^f}$ after a combination with recent work by Enns in [En]. However, if K is ramified over $\mathbf{Q}_p$, the situation seems to be much more complicated as $\Pi(\overline{r})^{\mathrm{K}(1)}$ is usually too small and one may necessarily need wildly ramified inertial types to find sufficiently many invariants of $\overline{\rho}$ inside $\Pi(\overline{r})$. The only known result with no restriction on the ramification of K is by Scholze in [Sch15] for $n = 2$ and arbitrary K using a completely different and more geometric method. It is not known (at least to the author) how to generalize the method in [Sch15] to arbitrary n.*

2.3 Dilogarithm and higher $\mathscr{L}$-invariants for $\mathrm{GL}_3(\mathbf{Q}_p)$

Now we switch to the locally analytic representation $\Pi^{\mathrm{an}}(r)$ which is the main focus of the second part of my thesis. One can apply the theory of Verma modules (namely the theory of Bernstein–Gelfand–Gelfand category $\mathcal{O}$, c.f. [Hum08]) to study $\Pi^{\mathrm{an}}(r)$. Very little is known about the full classification of irreducible objects in the category $\mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_n(K),E}$ consisting of admissible locally analytic representations of $\mathrm{GL}_n(K)$. We consider the full subcategory $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$ (which is much better understood) of $\mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_n(K),E}$ generated by subquotients of locally analytic principal series representations $\left(\mathrm{Ind}_{B(K)}^{\mathrm{GL}_n(K)}\chi\right)^{\mathrm{an}}$ for arbitrary locally analytic characters χ of $T(K)$. When we say a full

subcategory is generated by a family of objects, we mean that it is the minimal full subcategory (with each object of finite length) that contains this family of objects and is closed under extensions and taking subquotients. We assume that r is unramified at each finite place of F lying above $D(U^{v_0})$ and moreover

$$\Pi^{\mathrm{alg}}(r) \neq 0. \tag{2.3.1}$$

Note that the condition (2.3.1) is the same as saying that r *is automorphic* and in particular ρ is potentially semi-stable. We define

$$\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$$

as the set of isomorphism classes of irreducible objects in $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$ and

$$\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r) \subseteq \mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$$

as the subset consisting of each irreducible object Π^{irr} such that the embedding

$$\Pi^{\mathrm{irr}} \hookrightarrow \Pi^{\mathrm{an}}(r)$$

exists. Moreover, we define the multiplicity

$$d_{\Pi^{\mathrm{irr}}} := \dim_E \mathrm{Hom}_{\mathrm{GL}_n(K)}\left(\Pi^{\mathrm{irr}},\ \Pi^{\mathrm{an}}(r)\right)$$

which is finite for each $\Pi^{\mathrm{irr}} \in \mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r)$ as $\Pi^{\mathrm{an}}(r)$ is admissible. When ρ is crystalline and has generic Frobenius eigenvalues, the set $\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r)$ is explicitly determined by certain combinatorial data given by the Hodge filtration of ρ due to [BHS17], under certain standard technical assumption. On the other hand, [BH18] constructs a certain subrepresentation of $\Pi^{\mathrm{an}}(r)$ which is maximal among those representations that satisfy a certain multiplicity one condition and belong to $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$. The problem one might be interested in is to explicitly determine ρ using the structure of possibly a subrepresentation of $\Pi^{\mathrm{an}}(r)$. If ρ is crystalline, evidences from Section 3.3 of [Bre17] and [BH18] suggest that it is still unclear how to construct a sufficiently large subrepresentation of $\Pi^{\mathrm{an}}(r)$ to determine ρ completely, especially when the Hodge filtration of ρ lies in a sufficiently generic position. The situation is quite different (and actually much better) if ρ is semi-stable and non-crystalline, at least when $n = 3$. We assume from now on that $F^+ = \mathbf{Q}$ (hence $K = \mathbf{Q}_p$ and $v_0 = p$) and that the (φ, N)-module $\underline{D}$ associated with $\mathrm{WD}(\rho)$ has the following form

$$\underline{D} = Ee_0 \oplus Ee_1 \oplus Ee_2 \tag{2.3.2}$$

where

$$N(e_0) = 0,\ N(e_1) = e_0,\ N(e_2) = e_1$$

and

$$\varphi(e_k) = \mathrm{ur}(\alpha)p^{k-n+1}e_k$$

for $0 \leq i \leq 2$ and a certain $\alpha \in E^\times$. It follows from classical local Langlands correspondence that

$$\mathrm{LL} \circ \mathrm{WD}(\rho) = \mathrm{St}_3^\infty \otimes_E (\mathrm{ur}(\alpha) \circ \det_3)$$

where St_3^∞ is the smooth Steinberg representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ defined as the unique quotient of the smooth principal series

$$\left(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)} 1\right)^\infty,$$

$\det_3$ is the determinant character $\mathrm{GL}_3(\mathbf{Q}_p) \twoheadrightarrow \mathbf{Q}_p^\times$ and $\mathrm{ur}(\alpha)$ is the unramified character of $\mathbf{Q}_p^\times$ which sends p to α. We use the notation $\underline{k} = (k_1, k_2, k_3) \in \mathbf{Z}^3$ with

$$k_1 > k_2 > k_3$$

for the set of Hodge–Tate weights of ρ. We set

$$\Pi^{\mathrm{alg}}(\underline{k},\underline{D}) := \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\otimes_E (\mathrm{ur}(\alpha)\circ\det_3)$$

where $\overline{L}(\lambda)$ is the finite dimensional algebraic representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ with highest weight

$$\lambda = (\lambda_1,\lambda_2,\lambda_3) = (k_1-2, k_2-1, k_3) \tag{2.3.3}$$

with respect to the upper-triangular Borel subgroup $\overline{B}(\mathbf{Q}_p)$. Then Theorem 2.1.1 implies that

$$\Pi^{\mathrm{alg}}(r) = \Pi^{\mathrm{alg}}(\underline{k},\underline{D})^{\oplus d}$$

for an integer $d = d_{\Pi^{\mathrm{alg}}(\underline{k},\underline{D})}$ depending on U^p and r. We say that ρ is *non-critical* (c.f. part (ii) of Remark 6.1.4 of [Bre17]) if the Hodge filtration of ρ corresponds to a point in the open cell of a flag variety, which implies the following equality

$$\mathrm{JH}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}(r) = \{\Pi^{\mathrm{alg}}(\underline{k},\underline{D})\}. \tag{2.3.4}$$

If ρ is non-criticial, then there exists a triple of invariants $(\mathscr{L}_{i,j})_{0\leq i<j\leq 2}\in E^3$ such that the (weakly admissible) Hodge filtration $\mathrm{Fil}^\bullet$ on $\underline{D}$ has the following form

$$\mathrm{Fil}^i\underline{D} = \begin{cases} \underline{D} & \text{if } i\leq k_3 \\ E(e_2+\mathscr{L}_{1,2}e_1+\mathscr{L}_{0,2}e_0)\oplus E(e_1+\mathscr{L}_{0,1}e_0) & \text{if } k_3+1\leq i\leq k_2 \\ E(e_2+\mathscr{L}_{1,2}e_1+\mathscr{L}_{0,2}e_0) & \text{if } k_2+1\leq i\leq k_1 \\ 0 & \text{if } i\geq k_1+1 \end{cases}. \tag{2.3.5}$$

One of the main purpose for pLL in the case (2.3.5) is to find all the invariants $\mathscr{L}_{i,j}$ for $0\leq i<j\leq 2$ inside $\Pi^{\mathrm{an}}(r)$. We usually use the terminology $\mathscr{L}$-invariants for these $\mathscr{L}_{i,j}$ and usually refer to simple $\mathscr{L}$-invariants for $\mathscr{L}_{0,1}$ and $\mathscr{L}_{1,2}$ and call $\mathscr{L}_{0,2}$ *the higher $\mathscr{L}$-invariant* of ρ. What makes this case (2.3.5) so special is that one actually expects to find all the invariants $\mathscr{L}_{i,j}$ inside a subrepresentation of $\Pi^{\mathrm{an}}(r)$ that belongs to $\mathrm{Rep}^{\mathrm{PS}}_{\mathrm{GL}_n(K),E}$. The case for $\mathrm{GL}_2(\mathbf{Q}_p)$ was firstly studied by Breuil in [Bre04] and [Bre10b] and was one of the starting points of pLL. In order to explicitly recover the unique *$\mathscr{L}$-invariant* from $\Pi^{\mathrm{an}}(r)$ if ρ is semi-stable non-crystalline of dimension two, one needs to consider the p-adic logarithm function to construct a basis for a certain Ext^1-group. A recent result by Ding generalizes [Bre04] and [Bre10b] to all the simple $\mathscr{L}$-invariants (which are generalizations of $\mathscr{L}_{0,1}$ and $\mathscr{L}_{1,2}$ here) of $\mathrm{GL}_n(K)$ for arbitrary K in [Ding18]. It remains to treat $\mathscr{L}_{0,2}$. In [Schr11], Schraen studied some locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ in [Schr11] to try to realize the filtered module $\mathrm{Fil}^\bullet\underline{D}$ inside the de Rham complex of the Drinfeld space with dimension two. One of the main technical ingredients in [Schr11] is the computation of various (analytic) group cohomologies which leads to the determination of several crucial Ext-groups between certain locally analytic representations. Schraen constructs an abstract object $\Sigma(\lambda,\underline{\mathscr{L}})'$ (c.f. Definition 5.19 of [Schr11]) in the derived category $\mathcal{D}\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$ of the abelian category $\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}$ consisting of abstract modules over the locally analytic distribution algebra $D(\mathrm{GL}_3(\mathbf{Q}_p),E)$. The motivation of this construction is to find a reasonable family of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ that depends on three invariants, as ρ depends on three invariants. In fact, Schraen firstly constructs a family of locally analytic representations $\Sigma(\lambda,\mathscr{L},\mathscr{L}')$ (c.f. (1.11) of [Schr11]) depending on two invariants. Then he observes that a third invariant lies inside the group

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p)}(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L},\mathscr{L}')) \tag{2.3.6}$$

rather than

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L},\mathscr{L}')),$$

and thus one has to replace a family of locally analytic representations with a family of complex $\Sigma(\lambda,\underline{\mathscr{L}})'$. Roughly speaking, one wishes to add $\overline{L}(\lambda)$ to obtain the third invariant but only succeeds

in the sense of an abstract complex in [Schr11]. A crucial observation by Breuil is that one can possibly extend each $\Sigma(\lambda, \mathscr{L}, \mathscr{L}')$ to a larger locally analytic representation that contains more invariants by adding some constituents other than $\overline{L}(\lambda)$. Along this line, Breuil constructed in [Bre17] two locally analytic representations $\Pi^1(\underline{k}, \underline{D})$ and $\Pi^2(\underline{k}, \underline{D})$ of $\mathrm{GL}_3(\mathbf{Q}_p)$ depending only on $\underline{k}$ and $\underline{D}$ such that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi^i(\underline{k}, \underline{D}),\ \Pi^{\mathrm{alg}}(\underline{k}, \underline{D})\right) = 3 \tag{2.3.7}$$

for $i = 1, 2$. To summary, we need the following technical conditions

Condition 2.3.8. *Assume that U^p and r satisfy the following*

(i) $\Pi^{\mathrm{alg}}(r) \neq 0$;

(ii) $\mathrm{WD}(\rho)$ *is given by the case $n = 3$ of (2.3.2);*

(iii) *ρ is non-critical;*

(iv) *the automorphic representation corresponding to r appears only once in the space of automorphic forms on $\mathcal{G}(\mathbb{A}^\infty_{F^+})$.*

Then it is shown in Theorem 6.2.1 of [Bre17] that for each fixed pair (U^p, r) satisfying Condition 2.3.8, there exists a unique locally analytic representation Π of $\mathrm{GL}_3(\mathbf{Q}_p)$ of the form

$$\Pi^{\mathrm{alg}}(\underline{k}, \underline{D}) \begin{array}{l} \nearrow \Pi^1(\underline{k}, \underline{D}) \\ \searrow \Pi^2(\underline{k}, \underline{D}) \end{array} \tag{2.3.9}$$

such that

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi \otimes_E \varepsilon^2 \circ \det_3,\ \Pi^{\mathrm{an}}(r)\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi^{\mathrm{alg}}(\underline{k}, \underline{D}) \otimes_E \varepsilon^2 \circ \det_3,\ \Pi^{\mathrm{an}}(r)\right). \tag{2.3.10}$$

Note that we use the notation $V \multimap W$ for a non-split extension of W by V. One crucial observation is that Π depends on four invariants according to (2.3.7) but ρ depends only on three invariants according to (2.3.5). In particular, we do not expect all locally analytic representations of the form (2.3.9) to be able to embed into $\Pi^{\mathrm{an}}(r)$ for certain U^p and r. A natural question is the following

Question 2.3.11. *Is there a purely representation theoretic criterion for a representation of the form (2.3.9) to embed into a certain $\Pi^{\mathrm{an}}(r)$?*

The key point is that $\overline{L}(\lambda)$ is exactly the obstruction we want in Question 2.3.11. More precisely, whether a locally analytic representation Π is included in a larger representation with $\overline{L}(\lambda)$ lying in the cosocle (actually with multiplicity two) gives a criterion for Π to embed into the completed cohomology. One very interesting insight of [Schr11] is that the p-adic dilogarithm function appears in the two dimensional space (2.3.6) (c.f. Corollary 5.17 of [Schr11]). Consequently, Breuil has suggested that there should be some explicit relation between [Schr11], [Bre17] and [BD18] especially in the sense that the p-adic dilogarithm function should be related to the unique higher $\mathscr{L}$-invariant $\mathscr{L}_{0,2}$ of ρ which should be analogues to the relation between the p-adic logarithm function and simple $\mathscr{L}$-invariants. We have arrived at the following question

Question 2.3.12. *What is the precise relation between the p-adic dilogarithm function and the unique higher $\mathscr{L}$-invariant $\mathscr{L}_{0,2}$ of ρ?*

The main ingredient of the second part of my thesis is to try to answer the two questions at the same time through the construction of a family of locally analytic representations $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ depending on three invariants $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ (c.f. Theorem 4.1.1 for an intuitive description of the representation). The superscript 'min, +' means that $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ is *slightly bigger than the minimal locally analytic representation of* $\mathrm{GL}_3(\mathbf{Q}_p)$ *which involves the p-adic dilogarithm function*. Note that λ and $\underline{k}$ determine each other by the formula (2.3.3). The main properties of $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ are summarized in the following three results.

Theorem 2.3.13. *For each fixed U^p and r satisfying Condition 2.3.8, there exists a unique choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ such that*

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det_3,\ \Pi^{\mathrm{an}}(r)\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Pi^{\mathrm{alg}}(\underline{k}, \underline{D}) \otimes_E \varepsilon^2 \circ \det_3,\ \Pi^{\mathrm{an}}(r)\right). \quad (2.3.14)$$

Remark 2.3.15. *We know that*

$$\mathscr{L}_1 = \mathscr{L}_{1,2},\ \mathscr{L}_2 = \mathscr{L}_{0,1} \quad (2.3.16)$$

due to our definition of $\mathscr{L}_1$ and $\mathscr{L}_2$, Theorem 1.2 of [Ding18] and certain p-adic Hodge theoretic arguments appearing in the $\mathrm{GL}_2(\mathbf{Q}_p)$ case (c.f. Remark 3.1 of [Ding18]). The statement of Theorem 2.3.13 implies that $\mathscr{L}_3$ is uniquely determined by U^p and r, but a priori not necessarily by ρ. It is strongly believed but not yet known how to prove (using purely representation theoretic methods) that there is an embedding

$$\Pi(D) \hookrightarrow \Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \quad (2.3.17)$$

for certain $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ uniquely determined by ρ where $\Pi(D)$ is the locally analytic representation constructed from ρ using pLL for $\mathrm{GL}_2(\mathbf{Q}_p)$ in (3.111) of [BD18]. If ρ is ordinary (namely has consecutive Hodge–Tate weights), it is shown in Theorem 1.1 of [BD18] that $\Pi(D)$ actually embeds into $\Pi^{\mathrm{an}}(r)$, which together with our Theorem 2.3.13 imply that (2.3.17) exists and $\mathscr{L}_3$ depends only on ρ.

The following is a positive answer to Question 2.3.11 (at least in one direction) and easily follows from Theorem 6.2.1 of [Bre17] and our Theorem 2.3.13.

Corollary 2.3.18. *We fix a representation Π of the form (2.3.9). If we have an embedding*

$$\Pi \otimes_E \varepsilon^2 \circ \det_3 \hookrightarrow \Pi^{\mathrm{an}}(r)$$

for a suitable choice of U^p and r, then we have an embedding

$$\Pi \hookrightarrow \Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E \mathrm{ur}(\alpha) \circ \det_3$$

for certain $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ uniquely determined by Π.

The following result is a natural consequence of the construction of $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$, more details can be found in Theorem 4.1.7.

Theorem 2.3.19. *There exists an explicit complex of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ whose strong dual realizes the object $\Sigma(\lambda, \underline{\mathscr{L}})'$ (up to changes on notation, c.f. (1.14) and (5.19) of [Schr11]).*

Chapter 3

Mod p local-global compatibility for $\mathrm{GL}_n(\mathbf{Q}_p)$ in the ordinary case

3.1 Introduction

It is believed that one can attach a smooth $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_n(K)$ (or a packet of such representations) to a continuous Galois representation $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ in a natural way, that is called mod p Langlands program for $\mathrm{GL}_n(K)$, where K is a finite extension of $\mathbf{Q}_p$. This conjecture is well-understood for $\mathrm{GL}_2(\mathbf{Q}_p)$ ([BL94], [Ber10], [Bre03a], [Bre03b], [Col10], [Pas13], [CDP], [Eme]). Beyond the $\mathrm{GL}_2(\mathbf{Q}_p)$-case, for instance $\mathrm{GL}_n(\mathbf{Q}_p)$ for $n > 2$ or even $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ for an unramified extension $\mathbf{Q}_{p^f}$ of $\mathbf{Q}_p$ of degree $f > 1$, the situation is still quite far from being understood. One of the main difficulties is that there is no classification of such smooth representations of $\mathrm{GL}_n(K)$ unless $K = \mathbf{Q}_p$ and $n = 2$: in particular, we barely understand the supercuspidal representations. Some of the difficulties in classifying the supercuspidal representations are illustrated in [BP12], [Hu10] and [Schr15].

Let F be a CM field in which p is unramified, and $\overline{r} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ an automorphic Galois representation. Although there is no precise statement of mod p Langlands correspondence for $\mathrm{GL}_n(K)$ unless $K = \mathbf{Q}_p$ and $n = 2$, one can define smooth representations $\Pi(\overline{r})$ of $\mathrm{GL}_n(F_w)$ in the spaces of mod p automorphic forms on a definite unitary group cut out by the maximal ideal of a Hecke algebra associated to $\overline{r}$, where w is a place of F above p. A precise definition of $\Pi(\overline{r})$ when p splits completely in F, which is our context, will be given in Section 3.1.4. (See also Section 3.5.6.) One wishes that $\Pi(\overline{r})$ is a candidate on the automorphic side corresponding to $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ for a mod p Langlands correspondence in the spirit of Emerton [Eme]. However, we barely understand the structure of $\Pi(\overline{r})$ as a representation of $\mathrm{GL}_n(F_w)$, though the ordinary part of $\Pi(\overline{r})$ is described in [BH15] when p splits completely in F and $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is ordinary. In particular, it is not known whether $\Pi(\overline{r})$ and $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ determine each other. But we have the following conjecture:

Conjecture 3.1.1. *The local Galois representation $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is determined by $\Pi(\overline{r})$.*

This conjecture is widely expected to be true by experts but not explicitly written down before. The case $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ was treated by Breuil–Diamond [BD14]. Herzig–Le–Morra [HLM17] considered the case $\mathrm{GL}_3(\mathbf{Q}_p)$ when $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is upper-triangular, while the case $\mathrm{GL}_3(\mathbf{Q}_p)$ when $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is an extension of a two dimensional irreducible representation by a character was considered by Le–Morra–Park [LMP]. A recent work of John Enns studied this conjecture for the group $\mathrm{GL}_3(\mathbf{Q}_{p^f})$. All of the results above are under certain generic assumptions on the tamely ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$.

From another point of view, to a admissible smooth $\overline{\mathbf{F}}_p$-representation Π of $\mathrm{GL}_n(K)$ for a finite extension K of $\mathbf{Q}_p$, Scholze [Sch15] attaches a admissible smooth $\overline{\mathbf{F}}_p$-representation $S(\Pi)$ of $D^\times$ for

a division algebra D over K with center K and invariant $\frac{1}{n}$, which also has a continuous action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$, via the mod p cohomology of the Lubin–Tate tower. Using this construction, it was possible for Scholze to prove Conjecture 3.1.1 in full generality for $\mathrm{GL}_2(K)$ (c.f. [Sch15], Theorem 1.5). On the other hand, the proof of Theorem 1.5 of [Sch15] does not tell us where the invariants that determine $S(\Pi)$ lie. We do not know if there is any relation between these two different methods.

The weight part of Serre's conjecture already gives part of the information of $\Pi(\overline{r})$: the local Serre weights of $\overline{r}$ at w determine the socle of $\Pi(\overline{r})|_{\mathrm{GL}_n(\mathcal{O}_{F_w})}$ at least up to possible multiplicities, where $\mathcal{O}_{F_w}$ is the ring of integers of F_w. If $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is semisimple, then it is believed that the Serre weights of $\overline{r}$ at w determine $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ up to twisting by unramified characters, but this is no longer the case if it is not semisimple: the Serre weights are not enough to determine the wildly ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$, so that we need to understand a deeper structure of $\Pi(\overline{r})$ than just its $\mathrm{GL}_n(\mathcal{O}_{F_w})$-socle.

In this paper, we show that Conjecture 3.1.1 is true when p splits completely in F and $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ is upper-triangular and sufficiently generic in a precise sense. Moreover, we describe the invariants in $\Pi(\overline{r})$ that determine the wildly ramified part of $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$. The generic assumptions on $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)}$ ensure that very few Serre weights of $\overline{r}$ at w will occur, which we call the weight elimination conjecture, Conjecture 3.1.11. The weight elimination results are significant for our method to prove Conjecture 3.1.1. But thanks to Bao V. Le Hung, this weight elimination conjecture is known to be true and will be proven in [LLMPQ]. We follow the basic strategy in [BD14] and [HLM17]: we define Fontaine–Laffaille parameters on the Galois side using Fontaine–Laffaille modules as well as automorphic parameters on the automorphic side using the actions of Jacobi sum operators, and then identify them via the classical local Langlands correspondence. However, there are many new difficulties that didn't occur in [BD14] or in [HLM17]. For instance, the classification of semi-linear algebraic objects of rank $n > 3$ on the Galois side is much more complicated. Moreover, failing of the multiplicity one property of the Jordan–Hölder factors of mod p reduction of Deligne–Lusztig representations of $\mathrm{GL}_n(\mathbf{Z}_p)$ for $n > 3$ implies that new ideas are required to show crucial non-vanishing of the automorphic parameters. In the rest of the introduction, we explain our ideas and results in more detail.

3.1.1 Local Galois side

Let E be a (sufficiently large) finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$, a uniformizer ϖ_E, and residue field $\mathbf{F}$, and let $I_{\mathbf{Q}_p}$ be the inertia subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and ω the fundamental character of niveau 1. We also let $\overline{\rho}_0 : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}_n(\mathbf{F})$ be a continuous (Fontaine-Laffaille) ordinary generic Galois representation. Namely, there exists a basis $\underline{e} := (e_{n-1}, e_{n-2}, \cdots, e_0)$ for $\overline{\rho}_0$ such that with respect to $\underline{e}$ the matrix form of $\overline{\rho}_0$ is written as follows:

$$\overline{\rho}_0|_{I_{\mathbf{Q}_p}} \cong \begin{pmatrix} \omega^{c_{n-1}+(n-1)} & *_{n-1} & * & \cdots & * & * \\ 0 & \omega^{c_{n-2}+(n-2)} & *_{n-2} & \cdots & * & * \\ 0 & 0 & \omega^{c_{n-3}+(n-3)} & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{c_1+1} & *_1 \\ 0 & 0 & 0 & \cdots & 0 & \omega^{c_0} \end{pmatrix} \tag{3.1.2}$$

for some integers c_i satisfying some genericity conditions (c.f. Definition 3.3.3). We also assume that $\overline{\rho}_0$ is maximally non-split, i.e., $*_i \neq 0$ for all $i \in \{1, 2, \cdots, n-1\}$.

Our goal on the Galois side is to show that the Frobenius eigenvalues of certain potentially crystalline lifts of $\overline{\rho}_0$ determine the Fontaine–Laffaille parameters of $\overline{\rho}_0$, which parameterizes the wildly ramified part of $\overline{\rho}_0$. When the unramified part and the tamely ramified part of $\overline{\rho}_0$ are fixed, we define the Fontaine–Laffaille parameters via the Fontaine–Laffaille modules corresponding to $\overline{\rho}_0$ (c.f. Definition 3.3.15). These parameters vary over the space of $\frac{(n-1)(n-2)}{2}$ copies of the projective line $\mathbb{P}^1(\mathbf{F})$,

and we write $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) \in \mathbb{P}^1(\mathbf{F})$ for each pair of integers (i_0, j_0) with $0 \le j_0 < j_0+1 < i_0 \le n-1$. For each such pair (i_0, j_0), the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ is determined by the subquotient $\overline{\rho}_{i_0,j_0}$ of $\overline{\rho}_0$ which is determined by the subset $(e_{i_0}, e_{i_0-1}, \cdots, e_{j_0})$ of $\underline{e}$ (c.f. (3.3.2)): in fact, we have the identity $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0})$ (c.f. Lemma 3.3.17).

Since potentially crystalline lifts of $\overline{\rho}_0$ are not Fontaine–Laffaille in general, we are no longer able to use Fontaine–Laffaille theory to study such lifts of $\overline{\rho}_0$; we use Breuil modules and strongly divisible modules for their lifts. It is obvious that any lift of $\overline{\rho}_0$ determines the Fontaine–Laffaille parameters, but it is not obvious how one can explicitly visualize the information that determines $\overline{\rho}_0$ in those lifts. Motivated by the automorphic side, we believe that for each pair (i_0, j_0) as above the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ is determined by a certain product of Frobenius eigenvalues of the potentially crystalline lifts of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), \cdots, -1, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$ where $\widetilde{\omega}$ is the Teichmüler lift of the fundamental character ω of niveau 1 and

$$k_i^{i_0,j_0} \equiv \begin{cases} c_{i_0} + i_0 - j_0 - 1 & \text{for } i = i_0; \\ c_{j_0} - (i_0 - j_0 - 1) & \text{for } i = j_0; \\ c_i & \text{otherwise} \end{cases} \tag{3.1.3}$$

modulo $(p-1)$. Here, c_i are the integers determining the tamely ramified part of $\overline{\rho}_0$ in (3.1.2) and our normalization of the Hodge–Tate weight of the cyclotomic character ε is -1.

Our main result on the Galois side is the following:

Theorem 3.1.4 (Theorem 3.3.44)**.** *Fix $i_0, j_0 \in \mathbf{Z}$ with $0 \le j_0 < j_0+1 < i_0 \le n-1$. Assume that $\overline{\rho}_0$ is generic (c.f. Definition 3.3.3) and that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic (c.f. Definition 3.3.18), and let $(\lambda_{n-1}^{i_0,j_0}, \lambda_{n-2}^{i_0,j_0}, \cdots, \lambda_0^{i_0,j_0}) \in (\mathcal{O}_E)^n$ be the Frobenius eigenvalues on the $(\widetilde{\omega}^{k_{n-1}^{i_0,j_0}}, \widetilde{\omega}^{k_{n-2}^{i_0,j_0}}, \cdots, \widetilde{\omega}^{k_0^{i_0,j_0}})$-isotypic components of $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\rho_0)$ where ρ_0 is a potentially crystalline lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, -1, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$.*

Then the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}$ associated to $\overline{\rho}_0$ is computed as follows:

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \left[1 : \overline{\left(\frac{p^{[(n-1)-\frac{i_0+j_0}{2}](i_0-j_0-1)}}{\prod_{i=j_0+1}^{i_0-1} \lambda_i^{i_0,j_0}}\right)}\right] \in \mathbb{P}^1(\mathbf{F}).$$

Note that by $\overline{\bullet} \in \mathbf{F}$ in the theorem above we mean the image of $\bullet \in \mathcal{O}_E$ under the natural surjection $\mathcal{O}_E \twoheadrightarrow \mathbf{F}$. We also note that $\overline{\rho}_{i_0,j_0}$ being Fontaine–Laffaille generic implies $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) \neq 0, \infty$ for all i_0, j_0 as in Theorem 3.1.4, but is a strictly stronger assumption if $i_0 - j_0 \ge 3$.

Let us briefly discuss our strategy for the proof of Theorem 3.1.4. Recall that the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ is defined in terms of the Fontaine–Laffaille module corresponding to $\overline{\rho}_0$. Thus we need to describe $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)$ by the data of the Breuil modules of inertial type $\bigoplus_{i=0}^{n-1} \omega^{k_i^{i_0,j_0}}$ corresponding to $\overline{\rho}_0$, and we do this via étale ϕ-modules, which requires classification of such Breuil modules. If the filtration of the Breuil modules is of a certain shape, then a certain product of the Frobenius eigenvalues of the Breuil modules determines a Fontaine–Laffaille parameter (c.f. Proposition 3.3.32). In order to get such a filtration, we need to assume that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic (c.f. Definition 3.3.18). Then we determine the structure of the filtration of the strongly divisible modules lifting the Breuil modules by direct computation, which immediately gives enough properties of Frobenius eigenvalues of the potentially crystalline representations we consider. But this whole process is subtle for general i_0, j_0. To resolve this issue we prove that any potentially crystalline lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$ has a potentially crystalline subquotient ρ_{i_0,j_0} of Hodge–Tate weights $\{-i_0, \cdots, -j_0\}$ and of Galois type $\bigoplus_{i=j_0}^{i_0} \widetilde{\omega}^{k_i^{i_0,j_0}}$ lifting $\overline{\rho}_{i_0,j_0}$. More precisely,

Theorem 3.1.5 (Corollary 3.3.41)**.** *Every potentially crystalline lift ρ_0 of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,0\}$ and Galois type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}^{k_i^{i_0,j_0}}$ is a successive extension*

$$\rho_0 \cong \begin{pmatrix} \rho_{n-1,n-1} & \cdots & * & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & \rho_{i_0+1,i_0+1} & * & * & \cdots & * \\ & & & \rho_{i_0,j_0} & * & \cdots & * \\ & & & & \rho_{j_0-1,j_0-1} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & \rho_{0,0} \end{pmatrix}$$

where

- *for $n-1\geq i>i_0$ and $j_0>i\geq 0$, $\rho_{i,i}$ is a 1-dimensional potentially crystalline lift of $\overline{\rho}_{i,i}$ with Hodge–Tate weight $-i$ and Galois type $\widetilde{\omega}^{k_i^{i_0,j_0}}$;*
- *ρ_{i_0,j_0} is a (i_0-j_0+1)-dimensional potentially crystalline lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0,-i_0+1,\cdots,-j_0\}$ and Galois type $\bigoplus_{i=j_0}^{i_0}\widetilde{\omega}^{k_i^{i_0,j_0}}$.*

Note that we actually prove the niveau f version of Theorem 3.1.5 since it adds only little more extra work (c.f. Corollary 3.3.41).

The representation $\rho_{i_0,j_0}\otimes\varepsilon^{-j_0}$ is a (i_0-j_0+1)-dimensional potentially crystalline lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-(i_0-j_0),-(i_0-j_0-1),\cdots,0\}$ and Galois type $\bigoplus_{i=j_0}^{i_0}\widetilde{\omega}^{k_i^{i_0,j_0}}$, so that, by Theorem 3.1.5, Theorem 3.1.4 reduces to the case $(i_0,j_0)=(n-1,0)$: we prove Theorem 3.1.4 when $(i_0,j_0)=(n-1,0)$, and then use the fact $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)=\mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0})$ to get the result for general i_0,j_0.

The Weil–Deligne representation $\mathrm{WD}(\rho_0)$ associated to ρ_0 (as in Theorem 3.1.4) contains those Frobenius eigenvalues of ρ_0. We then use the classical local Langlands correspondence for GL_n to transport the Frobenius eigenvalues of ρ_0 (and so the Fontaine–Laffaille parameters of $\overline{\rho}_0$ as well by Theorem 3.1.4) to the automorphic side (c.f. Corollary 3.3.46).

3.1.2 Local automorphic side

We start by introducing the Jacobi sum operators in characteristic p. Let T (resp. B) be the maximal torus (resp. the maximal Borel subgroup) consisting of diagonal matrices (resp. of upper-triangular matrices) of GL_n. We let $X(T):=\mathrm{Hom}(T,\mathbf{G}_m)$ be the group of characters of T and Φ^+ be the set of positive roots with respect to (B,T). We define $\epsilon_i\in X(T)$ as the projection of $T\cong\mathbf{G}_m^n$ onto the i-th factor. Then the elements $\{\epsilon_i\mid 1\leq i\leq n\}$ forms a $\mathbf{Z}$-basis for the free abelian group $X(T)$. We will use the notation $(d_1,d_2,\cdots,d_n)\in\mathbf{Z}^n$ for the element $\sum_{k=1}^n d_k\epsilon_k\in X(T)$. Note that the group of characters of the finite group $T(\mathbf{F}_p)\cong(\mathbf{F}_p^\times)^n$ can be identified with $X(T)/(p-1)X(T)$, and therefore we sometimes abuse the notation $(d_1,d_2,\cdots,d_n)$ for its image in $X(T)/(p-1)X(T)$. We define $\Delta:=\{\alpha_k:=\epsilon_k-\epsilon_{k+1}\mid 1\leq k\leq n-1\}\subset\Phi^+$ as the set of positive simple roots. Note that we write s_k for the reflection of the simple root α_k. For an element w in the Weyl group W, we define $\Phi_w^+=\{\alpha\in\Phi^+\mid w(\alpha)\in-\Phi^+\}\subseteq\Phi^+$ and $U_w=\prod_{\alpha\in\Phi_w^+}U_\alpha$, where U_α is a subgroup of U whose only non-zero off-diagonal entry corresponds to α. Note in particular that $\Phi^+=\Phi_{w_0}^+$, where w_0 is the longest element in W. For $w\in W$ and for a tuple of integers $\underline{k}=(k_\alpha)_{\alpha\in\Phi_w^+}\in\{0,1,\cdots,p-1\}^{|\Phi_w^+|}$, we define the Jacobi sum operator

$$S_{\underline{k},w}:=\sum_{A\in U_w(\mathbf{F}_p)}\left(\prod_{\alpha\in\Phi_w^+}A_\alpha^{k_\alpha}\right)A\cdot w\in\mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)] \tag{3.1.6}$$

where A_α is the entry of A corresponding to $\alpha \in \Phi_w^+$. In Section 3.4, we establish many technical results, both conceptual and computational, around these Jacobi sum operators. The use of these Jacobi sum operators can be traced back to at least [CL76], and are widely used for GL_2 in [BP12] and [Hu10] for instance. But systematic computation with these operators seems to be limited to GL_2 or GL_3. In this paper, we need to do some specific but technical computation on some special Jacobi sum operators for $\mathrm{GL}_n(\mathbf{F}_p)$, which is enough for our application to Theorem 3.1.15 below.

By the discussion on the local Galois side, our target on the local automorphic side is to capture the Frobenius eigenvalues coming from the local Galois side. By the classical local Langlands correspondence, the Frobenius eigenvalues of ρ_0 are transported to the unramified part of χ in the tamely ramified principal series $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}\chi$ corresponding to the Weil–Deligne representation $\mathrm{WD}(\rho_0)$ attached to ρ_0 in Theorem 3.1.4, and it is standard to use U_p-operators to capture the information in the unramified part of χ.

The normalizer of the Iwahori subgroup I in $\mathrm{GL}_n(\mathbf{Q}_p)$ is cyclic modulo I, and this cyclic quotient group is generated by an element $\Xi_n \in \mathrm{GL}_n(\mathbf{Q}_p)$ that is explicitly defined in (3.4.49). One of our goals is to translate the eigenvalue of U_p-operators into the action of Ξ_n on the space $(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}\chi)|_{\mathrm{GL}_n(\mathbf{Z}_p)}$. This is firstly done for $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ in [BD14], and then the method is generalized to $\mathrm{GL}_3(\mathbf{Q}_p)$ in the ordinary case by [HLM17]. Both [BD14] and [HLM17] need a pair of group algebra operators: for instance, group algebra operators $\widehat{S}, \widehat{S}' \in \mathbf{Q}_p[\mathrm{GL}_3(\mathbf{Q}_p)]$ are defined in [HLM17] and the authors prove an intertwining identity of the form $\widehat{S}' \cdot \Xi_3 = c\widehat{S}$ on a certain $I(1)$-fixed subspace of $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}\chi$ with χ assumed to be tamely ramified, where $I(1)$ is the maximal pro-p subgroup of I. Here, the constant $c \in \mathcal{O}_E^\times$ captures the eigenvalues of U_p-operators. This is the first technical point on the local automorphic side, and we generalize the results in [BD14] and [HLM17] by the following theorem.

For an n-tuple of integers $(a_{n-1}, a_{n-2}, \cdots, a_0) \in \mathbf{Z}^n$, we write $\mathcal{S}_n$ and $\mathcal{S}_n'$ for $S_{\underline{k}^1, w_0}$ with $\underline{k}^1 = (k_{i,j}^1)$ and $S_{\underline{k}^{1,\prime}, w_0}$ with $\underline{k}^{1,\prime} = (k_{i,j}^{1,\prime})$ respectively, where $k_{i,i+1}^1 = [a_0 - a_{n-i}]_1 + n - 2$, $k_{i,i+1}^{1,\prime} = [a_{n-i-1} - a_{n-1}]_1 + n - 2$ for $1 \le i \le n-1$, and $k_{i,j}^1 = k_{i,j}^{1,\prime} = 0$ otherwise. Here, (i,j) is the entry corresponding to α if $\alpha = \epsilon_i - \epsilon_j \in \Phi^+$ and by $[x]_1$ for $x \in \mathbf{Z}$ we mean the integer in $[0, p-1)$ such that $x \equiv [x]_1$ modulo $(p-1)$. We define $\widehat{\mathcal{S}}_n \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$ (resp. $\widehat{\mathcal{S}}_n' \in \mathbf{Z}_p[\mathrm{GL}_n(\mathbf{Z}_p)]$) by taking the Teichmüller lifts of the coefficients and the entries of the matrices of $\mathcal{S}_n \in \mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$ (resp. of $\mathcal{S}_n' \in \mathbf{F}_p[\mathrm{GL}_n(\mathbf{F}_p)]$).

We use the notation $\bullet$ for the composition of maps or group operators to distinguish from the notation $\circ$ for an $\mathcal{O}_E$-lattice inside a representation.

Theorem 3.1.7 (Theorem 3.4.71)**.** *Assume that the n-tuple of integers $(a_{n-1}, a_{n-2}, \cdots, a_0)$ is n-generic in the lowest alcove (c.f. Definition 3.4.5), and let*

$$\Pi_n = \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}(\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \ldots \otimes \chi_{n-2} \otimes \chi_{n-1} \otimes \chi_0)$$

be a tamely ramified principal series representation with the smooth characters $\chi_k : \mathbf{Q}_p^\times \to E^\times$ satisfying $\chi_k|_{\mathbf{Z}_p^\times} = \widetilde{\omega}^{a_k}$ for $0 \le k \le n-1$.

On the 1-dimensional subspace $\Pi_n^{I(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$ we have the identity:

$$\widehat{\mathcal{S}}_n' \bullet (\Xi_n)^{n-2} = p^{n-2}\kappa_n \left(\prod_{k=1}^{n-2} \chi_k(p)\right) \widehat{\mathcal{S}}_n \tag{3.1.8}$$

for $\kappa_n \in \mathbf{Z}_p^\times$ satisfying $\kappa_n \equiv \varepsilon^* \cdot \mathcal{P}_n(a_{n-1}, \cdots, a_0)$ mod (ϖ_E) where

$$\varepsilon^* = \prod_{k=1}^{n-2} (-1)^{a_0 - a_k}$$

and

$$\mathcal{P}_n(a_{n-1}, \cdots, a_0) = \prod_{k=1}^{n-2} \prod_{j=0}^{n-3} \frac{a_k - a_{n-1} + j}{a_0 - a_k + j} \in \mathbf{Z}_p^\times.$$

In fact, there are many identities similar to the one in (3.1.8) for each operator U_n^i for $1 \leq i \leq n-1$ (defined in (3.4.50)) which can be technically always reduced to Proposition 3.4.60, but it is clear from the proof of Theorem 3.1.7 in Section 3.4.5 that we need to choose U_n^{n-2} for the U_p-operator acting on $\Pi_n^{I(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$, motivated from the local Galois side via Theorem 3.1.4. The crucial point here is that the constant $p^{n-2}\kappa_n\left(\prod_{k=1}^{n-2}\chi_k(p)\right)$, which is closely related to $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)$ via Theorem 3.1.4 and classical local Langlands correspondence, should lie in $\mathcal{O}_E^\times$ for each Π_n appearing in our application of Theorem 3.1.7 to Theorem 3.1.15.

The next step is to consider the mod p reduction of the identity (3.1.8), which is effective to capture $p^{n-2}\prod_{k=1}^{n-2}\chi_k(p)$ modulo (ϖ_E) only if $\widehat{\mathcal{S}}_n\widehat{v} \not\equiv 0$ modulo (ϖ_E) for $\widehat{v} \in \Pi_n^{I(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$. It turns out that this non-vanishing property is very technical to prove for general $\mathrm{GL}_n(\mathbf{Q}_p)$. Before we state our non-vanishing result, we fix a little more notation: let

$$\begin{cases} \mu^* := (a_{n-1}-n+2, a_{n-2}, \cdots, a_1, a_0+n-2); \\ \mu_0 := (a_{n-1}, a_1, \cdots, a_{n-2}, a_0); \\ \mu_1 := (a_1, a_2, \cdots, a_{n-3}, a_{n-2}, a_{n-1}, a_0); \\ \mu_1' := (a_{n-1}, a_0, a_1, a_2, \cdots, a_{n-3}, a_{n-2}) \end{cases}$$

be four characters of $T(\mathbf{F}_p)$, and write π_0 (resp. $\widetilde{\pi}_0^\circ$) for the characteristic p principal series (resp. the characteristic 0 principal series) induced by the characters μ_0 (resp. by its Teichmüller lift $\widetilde{\mu}_0$). Note that we can attach an irreducible representation $F(\lambda)$ of $\mathrm{GL}_n(\mathbf{F}_p)$ to each $\lambda \in X(T)/(p-1)X(T)$ satisfying some regular conditions (c.f. the beginning of Section 3.4). If we assume that $(a_{n-1}, \cdots, a_0) \in \mathbf{Z}^n$ is n-generic in the lowest alcove, the characters μ^*, μ_0, μ_1 and μ_1' do satisfy the regular condition and thus we have four irreducible representations $F(\mu^*)$, $F(\mu_0)$, $F(\mu_1)$ and $F(\mu_1')$ of $\mathrm{GL}_n(\mathbf{F}_p)$. There is a unique (up to homothety) $\mathcal{O}_E$-lattice τ in $\widetilde{\pi}_0^\circ \otimes_{\mathcal{O}_E} E$ such that

$$\mathrm{soc}_{\mathrm{GL}_n(\mathbf{F}_p)}(\tau \otimes_{\mathcal{O}_E} \mathbf{F}) = F(\mu^*).$$

We are now ready to state the non-vanishing theorem.

Theorem 3.1.9 (Corollary 3.4.37)**.** *Assume that the n-tuple of integers $(a_{n-1}, a_{n-2}, \cdots, a_0)$ is $2n$-generic in the lowest alcove (c.f. Definition 3.4.5).*

Then we have

$$\mathcal{S}_n\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1}\right) \neq 0 \quad \textit{and} \quad \mathcal{S}_n'\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1'}\right) \neq 0.$$

The definition of μ_1, μ_1', μ_0 and μ^* is motivated by our application of Theorem 3.1.9 to Theorem 3.1.15 and is closely related to the Galois types we choose in Theorem 3.1.4. We emphasize that, technically speaking, it is crucial that $F(\mu^*)$ has multiplicity one in π_0. The proof of Theorem 3.1.9 is technical and makes full use of the results in Sections 3.4.1, 3.4.7, and 3.4.8.

3.1.3 Weight elimination and automorphy of a Serre weight

The weight part of Serre's conjecture is considered as a first step towards mod p Langlands program, since it gives a description of the socles of $\Pi(\overline{r})|_{\mathrm{GL}_n(\mathbf{Z}_p)}$ up to possible multiplicities. Substantial progress has been made for the groups $\mathrm{GL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a finite extension K of $\mathbf{Q}_p$ ([BDJ10], [Gee11], [GK14], [GLS14], [GLS15]). For groups in higher semisimple rank, we also have a detailed description. (See [EGH13], [HLM17], [LMP], [MP17], [LLHLMa] for GL_3; [Her09], [GG10], [BLGG], [LLL16], [GHS] for general n.)

Weight elimination results are significant for the proof of our main global application, Theorem 3.1.15. For the purpose of this introduction, we quickly review some notation. Let F^+ be the maximal totally real subfield of a CM field F, and assume that p splits completely in F. Fix a place w of F above p and set $v := w|_{F^+}$. We assume that $\overline{r}$ is automorphic: this means that there exist

a totally definite unitary group G_n defined over F^+ that is an outer form of GL_{n/F^+} and split at places above p, an integral model $\mathcal{G}_n$ of G_n such that $\mathcal{G}_n \times \mathcal{O}_{F^+_{v'}}$ is reductive if v' is a finite place of F^+ that splits in F, a compact open subgroup $U = \mathcal{G}_n(\mathcal{O}_{F^+_v}) \times U^v \subseteq \mathcal{G}_n(\mathcal{O}_{F^+_v}) \times G_n(\mathbf{A}^{\infty,v}_{F^+})$ that is sufficiently small and unramified above p, a Serre weight $V = \bigotimes_{v'|p} V_{v'}$ that is an irreducible smooth $\overline{\mathbf{F}}_p$-representation of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$, and a maximal ideal $\mathfrak{m}_{\overline{r}}$ associated to $\overline{r}$ in the Hecke algebra acting on the space $S(U,V)$ of mod p algebraic automorphic forms such that

$$S(U,V)[\mathfrak{m}_{\overline{r}}] \neq 0. \tag{3.1.10}$$

We write $W(\overline{r})$ for the set of Serre weights V satisfying (3.1.10) for some U, and $W_w(\overline{r})$ for the set of local Serre weights V_v, that is irreducible smooth representations of $\mathcal{G}_n(\mathcal{O}_{F^+_v}) \cong \mathrm{GL}_n(\mathcal{O}_{F_w}) \cong \mathrm{GL}_n(\mathbf{Z}_p)$, such that $V_v \otimes (\bigotimes_{v' \neq v} V_{v'}) \in W(\overline{r})$ for an irreducible smooth representation $\bigotimes_{v' \neq v} V_{v'}$ of $\prod_{v' \neq v} \mathcal{G}_n(\mathcal{O}_{F^+_{v'}})$. The local Serre weights V_v have an explicit description as representations of $\mathrm{GL}_n(\mathbf{F}_p)$: there exists a p-restricted (i.e. $0 \leq a_i - a_{i-1} \leq p-1$ for all $1 \leq i \leq n-1$) weight $\underline{a} := (a_{n-1}, a_{n-2}, \cdots, a_0) \in X(T)$ such that $F(\underline{a}) \cong V_v$ where $F(\underline{a})$ is the irreducible socle of the dual Weyl module associated to $\underline{a}$ (c.f. Section 3.5.2 as well as the beginning of Section 3.4).

Assume that $\overline{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_w)} \cong \overline{\rho}_0$, where $\overline{\rho}_0$ is defined as in (3.1.2). We define certain characters $\mu^{\square}$ and $\mu^{\square,i_1,j_1}$ of $T(\mathbf{F}_p)$ and a principal series

$$\pi_*^{i_1,j_1} = \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_n(\mathbf{F}_p)}(\mu^{\square,i_1,j_1})^{w_0}$$

at the beginning of Section 3.5.3. Our main conjecture for weight elimination is

Conjecture 3.1.11 (Conjecture 3.5.16)**.** *Assume that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic and that $\mu^{\square,i_1,j_1}$ is $2n$-generic. Then we have an inclusion*

$$W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee) \subseteq \{F(\mu^{\square})^\vee, F(\mu^{\square,i_1,j_1})^\vee\}. \tag{3.1.12}$$

We emphasize that the condition $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic is crucial in Conjecture 3.1.11. For example, if $n = 4$ and $(i_0, j_0) = (3, 0)$ and we assume merely $\mathrm{FL}_4^{3,0}(\overline{\rho}_0) \neq 0, \infty$ (which is strictly weaker than Fontaine–Laffaille generic), then we expect that an extra Serre weight can possibly appear in $W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee)$.

The Conjecture 3.1.11 is motivated by the proof of Theorem 3.1.4 and the theory of *shape* in [LLHLMa]. The special case $n = 3$ of Conjecture 3.1.11 was firstly proven in [HLM17] and can also be deduced from the computations of Galois deformation rings in [LLHLMa].

Remark 3.1.13. *In an earlier version of this paper, we prove Conjecture 3.1.11 for $n \leq 5$. But our method is rather elaborate to execute for general n. We are informed that Bao V. Le Hung can prove Conjecture 3.1.11 completely and we have decided to put it inside [LLMPQ]. Therefore, Conjecture 3.1.11 becomes a theorem based on the results in [LLMPQ].*

Finally, we also show the automorphy of the Serre weight $F(\mu^{\square})^\vee$. In other words,

$$F(\mu^{\square})^\vee \in W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee). \tag{3.1.14}$$

Showing the automorphy of a Serre weight, in general, is very subtle. But thanks to the work of [BLGG] we are able to show the automorphy of $F(\mu^{\square})^\vee$ by checking the existence of certain potentially diagonalizable crystalline lifts of $\overline{\rho}_0$ (c.f. Proposition 3.5.17).

3.1.4 Mod p local-global compatibility

We now state our main results on mod p local-global compatibility. As discussed at the beginning of this introduction, we prove that $\Pi(\overline{r})$ determines the ordinary representation $\overline{\rho}_0$. Moreover, we also

describe the invariants in $\Pi(\overline{r})$ that determine the wildly ramified parts of $\overline{\rho}_0$. We first recall the definition of $\Pi(\overline{r})$.

Keep the notation of the previous sections, and write $b_i = -c_{n-1-i}$ for all $0 \leq i \leq n-1$, with c_i as in (3.1.2). We fix a place w of F above p and write $v := w|_{F^+}$, and we let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be an irreducible automorphic representation, of a Serre weight $V \cong \bigotimes_{v'} V_{v'}$ (c.f. Section 3.1.3), with $\overline{r}|_{G_{F_w}} \cong \overline{\rho}_0$.

Let $V' := \bigotimes_{v' \neq v} V_{v'}$ and set $S(U^v, V') := \varinjlim S(U^v \cdot U_v, V')$ where the direct limit runs over compact open subgroups $U_v \subseteq \mathcal{G}_n(\mathcal{O}_{F_v^+})$. This space $S(U^v, V')$ has a natural smooth action of $G_n(F_v^+) \cong \mathrm{GL}_n(F_w) \cong \mathrm{GL}_n(\mathbf{Q}_p)$ by right translation as well as an action of a Hecke algebra that commutes with the action of $G_n(F_v^+)$. We define

$$\Pi(\overline{r}) := S(U^v, V')[\mathfrak{m}_{\overline{r}}]$$

where $\mathfrak{m}_{\overline{r}}$ is the maximal ideal of the Hecke algebra associated to $\overline{r}$. In the spirit of [Eme], this is a candidate on the automorphic side for a mod p Langlands correspondence corresponding to $\overline{\rho}_0$. Note that the definition of $\Pi(\overline{r})$ relies on U^v and V' as well as choice of a Hecke algebra, but we suppress them in the notation.

Fix $n-1 \geq i_0 > j_0 + 1 > j_0 \geq 0$, and define i_1 and j_1 by the equation $i_1 + i_0 = j_1 + j_0 = n-1$. Note that the following Jacobi sum operators

$$\mathcal{S}^{i_1,j_1}, \quad \mathcal{S}^{i_1,j_1,\prime}, \quad \mathcal{S}_1^{i_1,j_1}, \quad \mathcal{S}_1^{i_1,j_1,\prime} \in \mathbf{F}_p[\mathrm{GL}_{j_1-i_1+1}(\mathbf{F}_p)]$$

are defined at the beginning of Section 3.4.2.

Now we can state the main results in this paper.

Theorem 3.1.15 (Theorem 3.5.44)**.** *Fix a pair of integers (i_0, j_0) satisfying $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$, and let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be an irreducible automorphic representation with $\overline{r}|_{G_{F_w}} \cong \overline{\rho}_0$. Assume that*

- *$\mu^{\square,i_1,j_1}$ is $2n$-generic;*
- *$\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic.*

Assume further that

$$\{F(\mu^{\square})^{\vee}\} \subseteq W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^{\vee}) \subseteq \{F(\mu^{\square})^{\vee}, F(\mu^{\square,i_1,j_1})^{\vee}\}. \tag{3.1.16}$$

Then there exists a primitive vector (c.f. Definition 3.5.43) in $\Pi(\overline{r})^{I(1),\mu_1^{i_1,j_1}}$. Moreover, for each primitive vector $v^{i_1,j_1} \in \Pi(\overline{r})^{I(1),\mu_1^{i_1,j_1}}$, we have $\mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1} v^{i_1,j_1} \neq 0$ and

$$\mathcal{S}^{i_1,j_1,\prime} \bullet \mathcal{S}_1^{i_1,j_1,\prime} \bullet (\Xi_n)^{j_1-i_1-1} v^{i_1,j_1} = \varepsilon^{i_1,j_1} \mathcal{P}_{i_1,j_1}(b_{n-1}, \cdots, b_0) \cdot \mathrm{FL}_n^{i_0,j_0}(\overline{r}|_{G_{F_w}}) \cdot \mathcal{S}^{i_1,j_1} \bullet \mathcal{S}_1^{i_1,j_1} v^{i_1,j_1}$$

where

$$\varepsilon^{i_1,j_1} = \prod_{k=i_1+1}^{j_1-1} (-1)^{b_{i_1} - b_k - j_1 + i_1 + 1}$$

and

$$\mathcal{P}_{i_1,j_1}(b_{n-1}, \cdots, b_0) = \prod_{k=i_1+1}^{j_1-1} \prod_{j=1}^{j_1-i_1-1} \frac{b_k - b_{j_1} - j}{b_{i_1} - b_k - j} \in \mathbf{Z}_p^{\times}.$$

Note that the conditions in (3.1.16) can be removed under some standard Taylor–Wiles conditions (c.f. Remark 3.1.13 and (3.1.14)).

Theorem 3.1.15 relies on the choice of a principal series type (the niveau 1 Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$). But this choice is somehow the unique one that could possibly make our strategy of the proof of Theorem 3.1.15 work.

Be careful that we cannot apply Theorem 3.1.7 and Theorem 3.1.9 directly to our local global-compatibility for general (i_1, j_1). Instead, we need to generalize Theorem 3.1.9 (resp. Theorem 3.1.7) to Proposition 3.5.38 (resp. Proposition 3.5.25) .

Corollary 3.1.17. *Keep the notation of Theorem 3.1.15 and assume that each assumption in Theorem 3.1.15 holds for all (i_0, j_0) such that $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$. Assume further that a freeness assumption is true (c.f. Remark 3.5.47 and note that this assumption can be removed by results in [LLMPQ]).*

Then the structure of $\Pi(\overline{r})$ as a admissible smooth $\mathbf{F}$-representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ determines the Galois representation $\overline{\rho}_0$ up to isomorphism.

3.1.5 Notation

Much of the notation introduced in this section will also be (or have already been) introduced in the text, but we try to collect together various definitions here for ease of reading.

We let E be a (sufficiently large) extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$, a uniformizer ϖ_E, and residue field $\mathbf{F}$. We will use these rings E, $\mathcal{O}_E$, and $\mathbf{F}$ for the coefficients of our representations. We also let K be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_K$, a uniformizer ϖ, and residue field k. Let $W(k)$ be the ring of Witt vectors over k and write K_0 for $W(k)[\frac{1}{p}]$. (K_0 is the maximal absolutely unramified subextension of K.) In this paper, we are interested only in the fields K that are tamely ramified extension of $\mathbf{Q}_p$, in which case we let $e := [K : K_0] = p^f - 1$ where $f = [k : \mathbf{F}_p]$.

For a field F, we write G_F for $\mathrm{Gal}(\overline{F}/F)$ where $\overline{F}$ is a separable closure of F. For instance, we are mainly interested in $G_{\mathbf{Q}_p}$ as well as G_{K_0} in this paper. The choice of a uniformizer $\varpi \in K$ provides us with a map:

$$\widetilde{\omega}_{\varpi} : G_{\mathbf{Q}_p} \longrightarrow W(k) \ : \ g \longmapsto \frac{g(\varpi)}{\varpi}$$

whose reduction mod (ϖ) will be denoted as ω_{ϖ}. This map factors through $\mathrm{Gal}(K/\mathbf{Q}_p)$ and $\widetilde{\omega}_{\varpi}|_{G_{K_0}}$ becomes a homomorphism. Note that the choice of the embedding $\sigma_0 : k \hookrightarrow \mathbf{F}$ provides us with a fundamental character of niveau f, namely $\omega_f := \sigma_0 \circ \omega_{\varpi}|_{\mathrm{Gal}(K/K_0)}$, and we fix the embedding in this paper.

For $a \in k$, we write $\widetilde{a}$ for its Teichmüler lift in $W(k)$. We also use the notation $\lceil a \rceil$ for $\widetilde{a}$, in particular, in Section 3.4.5. When the notation for an element $\bullet$ in k is quite long, we prefer $\lceil \bullet \rceil$ to $\widetilde{\bullet}$. For instance, if $a, b, c, d \in k$ then we write

$$\lceil (a-b)(a-c)(a-d)(b-c)(b-d) \rceil \ \text{ for } \ (a-b)(a-c)\widetilde{(a-d)}(b-c)(b-d).$$

Note that $\widetilde{\omega}_{\varpi}$ is the Teichmüler lift of ω_{ϖ}.

We normalize the Hodge–Tate weight of the cyclotomic character ε to be -1. Our normalization on class field theory sends the geometric Frobenius to the uniformizers. If $a \in \mathbf{F}^{\times}$ or $a \in \mathcal{O}_E^{\times}$ then we write $\mathrm{U_a}$ for the unramified character sending the geometric Frobenius to a. We may regard a character of $G_{\mathbf{Q}_p}$ as a character of $\mathbf{Q}_p^{\times}$ via our normalization of class field theory.

As usual, we write S for the p-adic completion of $W(k)[u, \frac{u^{ie}}{i!}]_{i \in \mathbf{N}}$, and let $S_{\mathcal{O}_E} := S \otimes_{\mathbf{Z}_p} \mathcal{O}_E$ and $S_E := S_{\mathcal{O}_E} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. We also let $\overline{S}_{\mathbf{F}} := S_{\mathcal{O}_E}/(\varpi_E, \mathrm{Fil}^p S_{\mathcal{O}_E}) \cong (k \otimes_{\mathbf{F}_p} \mathbf{F})[u]/u^{ep}$. Choose a uniformizer ϖ of K and let $E(u) \in W(k)[u]$ be the monic minimal polynomial of ϖ. The group $\mathrm{Gal}(K/K_0)$ acts on S via the character $\widetilde{\omega}_{\varpi}$, and we write $(S_{\mathcal{O}_E})_{\widetilde{\omega}_{\varpi}^m}$ for the $\widetilde{\omega}_{\varpi}^m$-isotypical component of S for $m \in \mathbf{Z}$. We define $(\overline{S}_{\mathbf{F}})_{\omega_{\varpi}^m}$ in a similar fashion. If $\mathcal{O}_E$ or $\mathbf{F}$ are clear, we often omit them, i.e., we write $S_{\widetilde{\omega}_{\varpi}^m}$ and $\overline{S}_{\omega_{\varpi}^m}$ for $(S_{\mathcal{O}_E})_{\widetilde{\omega}_{\varpi}^m}$ and $(\overline{S}_{\mathbf{F}})_{\omega_{\varpi}^m}$ respectively. In particular, $\overline{S}_0 := \overline{S}_{\omega_{\varpi}^0} \cong (k \otimes_{\mathbf{F}_p} \mathbf{F})[u^e]/u^{ep}$ and

$$S_0 := S_{\widetilde{\omega}_{\varpi}^0} = \left\{ \sum_{i=0}^{\infty} a_i \frac{E(u)^i}{i!} \mid a_i \in W(k) \otimes_{\mathbf{Z}_p} \mathcal{O}_E \text{ and } a_i \to 0 \ p\text{-adically} \right\}.$$

The association $a \otimes b \mapsto (\sigma(a)b)_\sigma$ gives rise to an isomorphism $k \otimes_{\mathbf{F}_p} \mathbf{F} \cong \coprod_{\sigma: k \hookrightarrow \mathbf{F}} \mathbf{F}$, and we write e_σ for the idempotent element in $k \otimes_{\mathbf{F}_p} \mathbf{F}$ that corresponds to the idempotent element in $\coprod_{\sigma: k \hookrightarrow \mathbf{F}} \mathbf{F}$ whose only non-zero entry is 1 at the position of σ.

To lighten the notation, we often write G for $\mathrm{GL}_{n/\mathbf{Z}_p}$. (By G_n, we mean an outer form of GL_n defined in Section 3.5.1.) We let B be the Borel subgroup of G consisting of upper-triangular matrices of G, U the unipotent subgroup of B, and T the torus of diagonal matrices of GL_n. We also write B^- and U^- for the opposite Borel of B and the unipotent subgroup of B^-, respectively. Let Φ^+ denote the set of positive roots with respect to (B, T), and $\Delta = \{\alpha_k\}_{1 \le k \le n-1}$ the subset of positive simple roots. We also let W be the Weyl group of GL_n, which is often considered as a subgroup of GL_n, and let s_k be the simple reflection corresponding to α_k. We write w_0 for the longest Weyl element in W, and we hope that the reader is not confused with places w or w' of F.

We often write K for $\mathrm{GL}_n(\mathbf{Z}_p)$ for brevity. (Note that we use K for a tamely ramified extension of $\mathbf{Q}_p$ as well, and we hope that it does not confuse the reader.) We will often use the following three open compact subgroups of $\mathrm{GL}_n(\mathbf{Z}_p)$: if we let $\mathrm{red} : \mathrm{GL}_n(\mathbf{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbf{F}_p)$ be the natural mod p reduction map, then

$$K(1) := \mathrm{Ker}(\mathrm{red}) \subset I(1) := \mathrm{red}^{-1}(U(\mathbf{F}_p)) \subset I := \mathrm{red}^{-1}(B(\mathbf{F}_p)) \subset K.$$

If M is a free $\mathbf{F}$-module with a smooth action of K, then $T(\mathbf{F}_p)$ acts on the pro p Iwahori fixed subspace $M^{I(1)}$ via $I/I(1) \cong T(\mathbf{F}_p)$. We write $M^{I(1),\mu}$ for the eigenspace with respect to a character $\mu : T(\mathbf{F}_p) \to \mathbf{F}_p^\times$. $M^{I(1)}$ decomposes as

$$M^{I(1)} \cong \bigoplus M^{I(1),\mu}$$

as $T(\mathbf{F}_p)$-representations, where the direct sum runs over the characters μ of $T(\mathbf{F}_p)$. In the obvious similar fashion, we define $M^{I(1),\mu}$ when M is a free $\mathcal{O}_E$-module or a free E-module.

By $[m]_f$ for a rational number $m \in \mathbf{Z}[\frac{1}{p}] \subset \mathbf{Q}$ we mean the unique integer in $[0, e)$ congruent to m mod (e) via the natural surjection $\mathbf{Z}[\frac{1}{p}] \twoheadrightarrow \mathbf{Z}/e\mathbf{Z}$. By $\lfloor y \rfloor$ for $y \in \mathbf{R}$ we mean the floor function of y, i.e., the biggest integer less than or equal to y. For a set A, we write $|A|$ for the cardinality of A. If V is a finite-dimensional $\mathbf{F}$-representation of a group H, then we write $\mathrm{soc}_H V$ and $\mathrm{cosoc}_H V$ for the socle of V and the cosocle of V, respectively. If v is a non-zero vector in a free module over $\mathbf{F}$ (resp. over $\mathcal{O}_E$, resp. over E), then we write $\mathbf{F}[v]$ (resp. $\mathcal{O}_E[v]$, resp. $E[v]$) for the $\mathbf{F}$-line (resp. the $\mathcal{O}_E$-line, resp. the E-line) generated by v.

We write $\overline{x}$ for the image of $x \in \mathcal{O}_E$ under the natural surjection $\mathcal{O}_E \twoheadrightarrow \mathbf{F}$. We also have a natural surjection $\mathbb{P}^1(\mathcal{O}_E) \twoheadrightarrow \mathbb{P}^1(\mathbf{F})$ defined by letting $\overline{[x : y]} \in \mathbb{P}^1(\mathbf{F})$ be the image of $[x : y] \in \mathbb{P}^1(\mathcal{O}_E)$ where

$$\overline{[x : y]} = \begin{cases} [1 : \overline{(\frac{y}{x})}] & \text{if } \frac{y}{x} \in \mathcal{O}_E; \\ [\overline{(\frac{x}{y})} : 1] & \text{if } \frac{x}{y} \in \mathcal{O}_E. \end{cases}$$

We often write $\frac{y}{x}$ for $[x : y] \in \mathbb{P}^1(\mathbf{F})$ if $x \neq 0$.

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3.2 Integral p-adic Hodge theory

In this section, we do a quick review of some (integral) p-adic Hodge theory which will be needed later. We note that all of the results in this section are already known or easy generalization of known results. We closely follow [EGH13] as well as [HLM17] in this section.

3.2.1 Filtered (ϕ, N)-modules with descent data

In this section, we review potentially semi-stable representations and their corresponding linear algebra objects, admissible filtered (ϕ, N)-modules with descent data.

Let K be a finite extension of $\mathbf{Q}_p$, and K_0 the maximal unramified subfield of K, so that $K_0 = W(k) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ where k is the residue field of K. We fix the uniformizer $p \in \mathbf{Q}_p$, so that we fix an embedding $\mathbf{B}_{\mathrm{st}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$. We also let K' be a subextension of K with K/K' Galois, and write $\phi \in \mathrm{Gal}(K_0/\mathbf{Q}_p)$ for the arithmetic Frobenius.

A p-adic Galois representation $\rho : G_{K'} \to \mathrm{GL}_n(E)$ is potentially semi-stable if there is a finite extension L of K' such that $\rho|_{G_L}$ is semi-stable, i.e., $\mathrm{rank}_{L_0 \otimes E} \mathrm{D}_{\mathrm{st}}^{K'}(V) = \dim_E V$, where V is an underlying vector space of ρ and $\mathrm{D}_{\mathrm{st}}^{K'}(V) := (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_L}$. We often write $\mathrm{D}_{\mathrm{st}}^{K'}(\rho)$ for $\mathrm{D}_{\mathrm{st}}^{K'}(V)$. If K is the Galois closure of L over K', then $\rho|_{G_K}$ is semi-stable, provided that $\rho|_{G_L}$ is semi-stable.

Definition 3.2.1. *A filtered $(\phi, N, K/K', E)$-module of rank n is a free $K_0 \otimes E$-module D of rank n together with*

- *a $\phi \otimes 1$-automorphism ϕ on D;*
- *a nilpotent $K_0 \otimes E$-linear endomorphism N on D;*
- *a decreasing filtration $\{\mathrm{Fil}^i D_k\}_{i \in \mathbf{Z}}$ on $D_K = K \otimes_{K_0} D$ consisting of $K \otimes_{\mathbf{Q}_p} E$-submodules of D_K, which is exhaustive and separated;*
- *a K_0-semilinear, E-linear action of $\mathrm{Gal}(K/K')$ which commutes with ϕ and N and preserves the filtration on D_K.*

We say that D is *(weakly) admissible* if the underlying filtered $(\phi, N, K/K, E)$-module is weakly admissible in the sense of [Fon94]. The action of $\mathrm{Gal}(K/K')$ on D is often called *descent data* action. If V is potentially semi-stable, then $\mathrm{D}_{\mathrm{st}}^{K'}(V)$ is a typical example of an admissible filtered $(\phi, N, K/K', E)$-module of rank n.

Theorem 3.2.2 ([CF], Theorem 4.3)**.** *There is an equivalence of categories between the category of weakly admissible filtered $(\phi, N, K/K', E)$-modules of rank n and the category of n-dimensional potentially semi-stable E-representations of $G_{K'}$ that become semi-stable upon restriction to G_K.*

Note that Theorem 3.2.2 is proved in [CF] in the case $K = K'$, and that [Sav05] gives a generalization to the statement with non-trivial descent data.

If V is potentially semi-stable, then so is its dual $V^\vee$. We define $\mathrm{D}_{\mathrm{st}}^{*,K'}(V) := \mathrm{D}_{\mathrm{st}}^{K'}(V^\vee)$. Then $\mathrm{D}_{\mathrm{st}}^{*,K'}$ gives an anti-equivalence of categories from the category of n-dimensional potentially semi-stable E-representations of $G_{K'}$ that become semi-stable upon restriction to G_K to the category of weakly admissible filtered $(\phi, N, K/K', E)$-modules of rank n, with quasi-inverse

$$\mathrm{V}_{\mathrm{st}}^{*,K'}(D) := \mathrm{Hom}_{\phi,N}(D, \mathbf{B}_{\mathrm{st}}) \cap \mathrm{Hom}_{\mathrm{Fil}}(D_K, \mathbf{B}_{\mathrm{dR}}).$$

It will often be convenient to use covariant functors. We define an equivalence of categories: for each $r \in \mathbf{Z}$

$$\mathrm{V}_{\mathrm{st}}^{K',r}(D) := \mathrm{V}_{\mathrm{st}}^{*,K'}(D)^\vee \otimes \varepsilon^r.$$

The functor $\mathrm{D}_{\mathrm{st}}^{K',r}$ defined by $\mathrm{D}_{\mathrm{st}}^{K',r}(V) := \mathrm{D}_{\mathrm{st}}^{K'}(V \otimes \varepsilon^{-r})$ is a quasi-inverse of $\mathrm{V}_{\mathrm{st}}^{K',r}$.

For a given potentially semi-stable representation $\rho : G_{K'} \to \mathrm{GL}_n(E)$, one can attach a Weil–Deligne representation $\mathrm{WD}(\rho)$ to ρ, as in [CDT99], Appendix B.1. We refer to $\mathrm{WD}(\rho)|_{I_{\mathbf{Q}_p}}$ as to the *Galois type* associated to ρ. Note that $\mathrm{WD}(\rho)$ is defined via the filtered $(\phi, N, K/K', N)$-module $\mathrm{D}_{\mathrm{st}}^{K'}(\rho)$ and that $\mathrm{WD}(\rho)|_{I_{K'}} \cong \mathrm{WD}(\rho \otimes \varepsilon^r)|_{I_{K'}}$ for all $r \in \mathbf{Z}$.

Finally, we say that a potentially semi-stable representation ρ is *potentially crystalline* if the monodromy operator N on $\mathrm{D}_{\mathrm{st}}^{K'}(\rho)$ is trivial.

3.2.2 Strongly divisible modules with descent data

In this section, we review strongly divisible modules that correspond to Galois stable lattices in potentially semi-stable representations. We keep the notation of Section 3.2.1

From now on, we assume that K/K' is a tamely ramified Galois extension with ramification index $e(K/K')$. We fix a uniformizer $\varpi \in K$ with $\varpi^{e(K/K')} \in K'$. Let e be the absolute ramification index of K and $E(u) \in W(k)[u]$ the minimal polynomial of ϖ over K_0.

Let S be the p-adic completion of $W(k)[u, \frac{u^{ie}}{i!}]_{i\in\mathbf{N}}$. The ring S has additional structures:

- a continuous, ϕ-semilinear map $\phi : S \to S$ with $\phi(u) = u^p$ and $\phi(\frac{u^{ie}}{i!}) = \frac{u^{pie}}{i!}$;
- a continuous, $W(k)$-linear derivation of S with $N(u) = -u$ and $N(\frac{u^{ie}}{i!}) = -ie\frac{u^{ie}}{i!}$;
- a decreasing filtration $\{\mathrm{Fil}^i S\}_{i\in\mathbf{Z}_{\geq 0}}$ of S given by letting $\mathrm{Fil}^i S$ be the p-adic completion of the ideal $\sum_{j\geq i} \frac{E(u)^j}{j!} S$;
- a group action of $\mathrm{Gal}(K/K')$ on S defined for each $g \in \mathrm{Gal}(K/K')$ by the continuous ring isomorphism $\widehat{g} : S \to S$ with $\widehat{g}(w_i \frac{u^i}{[i/e]!}) = g(w_i) h_g^i \frac{u^i}{[i/e]!}$ for $w_i \in W(k)$, where $h_g \in W(k)$ satisfies $g(\varpi) = h_g \varpi$.

Note that ϕ and N satisfies $N\phi = p\phi N$ and that $\hat{g}(E(u)) = E(u)$ for all $g \in \mathrm{Gal}(K/K')$ since we assume $\varpi^{e(K/K')} \in K'$. We write ϕ_i for $\frac{1}{p^i}\phi$ on $\mathrm{Fil}^i S$. For $i \leq p-1$ we have $\phi(\mathrm{Fil}^i S) \subseteq p^i S$.

Let $S_{\mathcal{O}_E} := S \otimes_{\mathbf{Z}_p} \mathcal{O}_E$ and $S_E := S_{\mathcal{O}_E} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. We extend the definitions of ϕ, N, $\mathrm{Fil}^i S$, and the action of $\mathrm{Gal}(K/K')$ to $S_{\mathcal{O}_E}$ (resp. to S_E) $\mathcal{O}_E$-linearly (resp. E-linearly).

Definition 3.2.3. *Fix a positive integer $r < p-1$. A strongly divisible $\mathcal{O}_E$-module with descent data of weight r is a free $S_{\mathcal{O}_E}$-module $\widehat{\mathcal{M}}$ of finite rank together with*

- *a $S_{\mathcal{O}_E}$-submodule $\mathrm{Fil}^r \widehat{\mathcal{M}}$;*
- *additive maps $\phi, N : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$;*
- *$S_{\mathcal{O}_E}$-semilinear bijections $\widehat{g} : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ for each $g \in \mathrm{Gal}(K/K')$*

such that

- $\mathrm{Fil}^r S_{\mathcal{O}_E} \cdot \widehat{\mathcal{M}} \subseteq \mathrm{Fil}^r \widehat{\mathcal{M}}$;
- $\mathrm{Fil}^r \widehat{\mathcal{M}} \cap I\widehat{\mathcal{M}} = I\mathrm{Fil}^r \widehat{\mathcal{M}}$ *for all ideals I in $\mathcal{O}_E$;*
- $\phi(sx) = \phi(s)\phi(x)$ *for all $s \in S_{\mathcal{O}_E}$ and for all $x \in \widehat{\mathcal{M}}$;*
- $\phi(\mathrm{Fil}^r \widehat{\mathcal{M}})$ *is contained in $p^r \widehat{\mathcal{M}}$ and generates it over $S_{\mathcal{O}_E}$;*
- $N(sx) = N(s)x + sN(x)$ *for all $s \in S_{\mathcal{O}_E}$ and for all $x \in \widehat{\mathcal{M}}$;*
- $N\phi = p\phi N$;

- $E(u)N(\mathrm{Fil}^r\widehat{\mathcal{M}}) \subseteq \mathrm{Fil}^r\widehat{\mathcal{M}}$;
- *for all* $g \in \mathrm{Gal}(K/K')$ $\widehat{g}$ *commutes with* ϕ *and* N, *and preserves* $\mathrm{Fil}^r\widehat{\mathcal{M}}$;
- $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \cdot g_2}$ *for all* $g_1, g_2 \in \mathrm{Gal}(K/K')$.

We write $\mathcal{O}_E$-$\mathrm{Mod}^r_{\mathrm{dd}}$ for the category of strongly divisible $\mathcal{O}_E$-modules with descent data of weight r. It is easy to see that the map $\phi_r = \frac{1}{p^r}\phi : \mathrm{Fil}^r\widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ satisfies $cN\phi_r(x) = \phi_r(E(u)N(x))$ for all $x \in \mathrm{Fil}^r\widehat{\mathcal{M}}$ where $c := \frac{\phi(E(u))}{p} \in S^\times$.

For a strongly divisible $\mathcal{O}_E$-module $\widehat{\mathcal{M}}$ with descent data of weight r, we define a $G_{K'}$-module $\mathrm{T}^{*,K'}_{\mathrm{st}}(\widehat{\mathcal{M}})$ as follows (c.f. [EGH13], Section 3.1.):

$$\mathrm{T}^{*,K'}_{\mathrm{st}}(\widehat{\mathcal{M}}) := \mathrm{Hom}_{\mathrm{Fil}^r,\phi,N}(\widehat{\mathcal{M}}, \widehat{\mathbf{A}}_{\mathrm{st}}).$$

Proposition 3.2.4 ([EGH13], Proposition 3.1.4)**.** *The functor* $\mathrm{T}^{*,K'}_{\mathrm{st}}$ *provides an anti-equivalence of categories from the category* $\mathcal{O}_E$-$\mathrm{Mod}^r_{\mathrm{dd}}$ *to the category of* $G_{K'}$*-stable* $\mathcal{O}_E$*-lattices in finite-dimensional* E*-representations of* $G_{K'}$ *which become semi-stable over* K *with Hodge–Tate weights lying in* $[-r, 0]$, *when* $0 < r < p-1$.

Note that the case $K = K'$ and $E = \mathbf{Q}_p$ is proved by Liu [Liu08].

In this paper, we will be mainly interested in covariant functors $\mathrm{T}^{K',r}_{\mathrm{st}}$ from the category $\mathcal{O}_E$-$\mathrm{Mod}^r_{\mathrm{dd}}$ to the category $\mathrm{Rep}^{K-\mathrm{st},[-r,0]}_{\mathcal{O}_E} G_{K'}$ of $G_{K'}$-stable $\mathcal{O}_E$-lattices in finite-dimensional E-representations of $G_{K'}$ which become semi-stable over K with Hodge–Tate weights lying in $[-r, 0]$ defined by

$$\mathrm{T}^{K',r}_{\mathrm{st}}(\widehat{\mathcal{M}}) := \mathrm{T}^{*,K'}_{\mathrm{st}}(\widehat{\mathcal{M}})^\vee \otimes \varepsilon^r.$$

Let $\widehat{\mathcal{M}}$ in $\mathcal{O}_E$-$\mathrm{Mod}^r_{\mathrm{dd}}$, and define a free S_E-module $\mathcal{D} := \widehat{\mathcal{M}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. We extend ϕ and N on $\mathcal{D}$, and define a filtration on $\mathcal{D}$ as follows: $\mathrm{Fil}^r\mathcal{D} = \mathrm{Fil}^r\widehat{\mathcal{M}}[\frac{1}{p}]$ and

$$\mathrm{Fil}^i\mathcal{D} := \begin{cases} \mathcal{D} & \text{if } i \le 0; \\ \{x \in \mathcal{D} \mid E(u)^{r-i}x \in \mathrm{Fil}^r\mathcal{D}\} & \text{if } 0 \le i \le r; \\ \sum_{j=0}^{i-1}(\mathrm{Fil}^{i-j}S_{\mathbf{Q}_p})(\mathrm{Fil}^j\mathcal{D}) & \text{if } i > r, \text{ inductively.} \end{cases} \tag{3.2.5}$$

We let $D := \mathcal{D} \otimes_{S_{\mathbf{Q}_p}, s_0} K_0$ and $D_K := \mathcal{D} \otimes_{S_{\mathbf{Q}_p}, s_\varpi} K$, where $s_0 : S_{\mathbf{Q}_p} \to K_0$ and $s_\varpi : S_{\mathbf{Q}_p} \to K$ are defined by $u \mapsto 0$ and $u \mapsto \varpi$ respectively, which induce ϕ and N on D and the filtration on D_K by taking $s_\varpi(\mathrm{Fil}^i\mathcal{D})$. The K_0-vector space D also inherits an E-linear action and a semi-linear action of $\mathrm{Gal}(K/K')$. Then it turns out that D is a weakly admissible filtered $(\phi, N, K/K', E)$-module with $\mathrm{Fil}^{r+1}D = 0$. Moreover, there is a compatibility (c.f. [EGH13], Proof of Proposition 3.1.4.): if D corresponds to $\mathcal{D} = \widehat{\mathcal{M}}[\frac{1}{p}]$, then

$$\mathrm{T}^{K',r}_{\mathrm{st}}(\widehat{\mathcal{M}})[\frac{1}{p}] \cong \mathrm{V}^{K',r}_{\mathrm{st}}(D).$$

3.2.3 Breuil modules with descent data

In this section, we review Breuil modules with descent data. We keep the notation of Section 3.2.2, and assume further that $K' \subseteq K_0$.

We let $\overline{S} := S/(\varpi_E, \mathrm{Fil}^p S) \cong (k \otimes_{\mathbf{F}_p} \mathbf{F})[u]/u^{ep}$. It is easy to check that $\overline{S}$ inherits ϕ, N, the filtration of S, and the action of $\mathrm{Gal}(K/K')$.

Definition 3.2.6. *Fix a positive integer* $r < p-1$. *A* Breuil modules with descent data of weight r *is a free* $\overline{S}$*-module* $\mathcal{M}$ *of finite rank together with*

- *a* $\overline{S}$*-submodule* $\mathrm{Fil}^r\mathcal{M}$ *of* $\mathcal{M}$;

- *maps* $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ *and* $N : \mathcal{M} \to \mathcal{M}$*;*
- *additive bijections* $\widehat{g} : \mathcal{M} \to \mathcal{M}$ *for all* $g \in \mathrm{Gal}(K/K')$

such that

- $\mathrm{Fil}^r\mathcal{M}$ *contains* $u^{er}\mathcal{M}$*;*
- ϕ_r *is* $\mathbf{F}$*-linear and* ϕ*-semilinear (where* $\phi : k[u]/u^{ep} \to k[u]/u^{ep}$ *is the* p*-th power map) with image generating* $\mathcal{M}$ *as* $\overline{S}$*-module;*
- N *is* $k \otimes_{\mathbf{F}_p} \mathbf{F}$*-linear and satisfies*
  - $N(ux) = uN(x) - ux$ *for all* $x \in \mathcal{M}$*,*
  - $u^e N(\mathrm{Fil}^r\mathcal{M}) \subseteq \mathrm{Fil}^r\mathcal{M}$*, and*
  - $\phi_r(u^e N(x)) = cN(\phi_r(x))$ *for all* $x \in \mathrm{Fil}^r\mathcal{M}$*, where* $c \in (k[u]/u^{ep})^\times$ *is the image of* $\frac{1}{p}\phi(E(u))$ *under the natural map* $S \to k[u]/u^{ep}$*.*
- $\widehat{g}$ *preserves* $\mathrm{Fil}^r\mathcal{M}$ *and commutes with the* ϕ_r *and* N*, and the action satisfies* $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \cdot g_2}$ *for all* $g_1, g_2 \in \mathrm{Gal}(K/K')$*. Furthermore, if* $a \in k \otimes_{\mathbf{F}_p} \mathbf{F}$ *and* $m \in \mathcal{M}$ *then* $\widehat{g}(au^i m) = g(a)((\frac{g(\varpi)}{\varpi})^i \otimes 1)u^i\widehat{g}(m)$*.*

We write $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ for the category of Breuil modules with descent data of weight r. For $\mathcal{M} \in \mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$, we define a $G_{K'}$-module as follows (c.f. [EGH13], Section 3.2):

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{\mathrm{BrMod}}(\mathcal{M}, \widehat{\mathbf{A}}).$$

This gives an exact faithful contravariant functor from the category $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ to the category $\mathrm{Rep}_{\mathbf{F}}G_{K'}$ of finite dimensional $\mathbf{F}$-representations of $G_{K'}$. We also define a covariant functor as follows: for each $r \in \mathbf{Z}$

$$\mathrm{T}^r_{\mathrm{st}}(\mathcal{M}) := \mathrm{T}^*_{\mathrm{st}}(\mathcal{M})^\vee \otimes \omega^r,$$

in which we will be more interested in this paper.

If $\widehat{M}$ is a strongly divisible module with descent data, then

$$\mathcal{M} := \widehat{\mathcal{M}}/(\varpi_E, \mathrm{Fil}^p S)$$

is naturally an object in $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ ($\mathrm{Fil}^r\mathcal{M}$ is the image of $\mathrm{Fil}^r\widehat{\mathcal{M}}$ in $\mathcal{M}$, the map ϕ_r is induced by $\frac{1}{p^r}\phi|_{\mathrm{Fil}^r\widehat{\mathcal{M}}}$, and N and $\widehat{g}$ are those coming from $\widehat{\mathcal{M}}$). Moreover, there is a compatibility: if $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-Mod}^r_{\mathrm{dd}}$ and we let $\mathcal{M} = \widehat{\mathcal{M}}/(\varpi_E, \mathrm{Fil}^p S)$ then

$$\mathrm{T}^{K',r}_{\mathrm{st}}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbf{F} \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}).$$

(See [EGH13], Lemma 3.2.2 for detail.)

There is a notion of duality of Breuil modules, which will be convenient for our computation of Breuil modules as we will see later.

Definition 3.2.7. *Let* $\mathcal{M} \in \mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$*. We define* $\mathcal{M}^*$ *as follows:*

- $\mathcal{M}^* := \mathrm{Hom}_{k[u]/u^{ep}-\mathrm{Mod}}(\mathcal{M}, k[u]/u^{ep})$*;*
- $\mathrm{Fil}^r\mathcal{M}^* := \{f \in \mathcal{M}^* \mid f(\mathrm{Fil}^r\mathcal{M}) \subseteq u^{er}k[u]/u^{ep}\}$*;*
- $\phi_r(f)$ *is defined by* $\phi_r(f)(\phi_r(x)) = \phi_r(f(x))$ *for all* $x \in \mathrm{Fil}^r\mathcal{M}$ *and* $f \in \mathrm{Fil}^r\mathcal{M}^*$*, where* $\phi_r : u^{er}k[u]/u^{ep} \to k[u]/u^{ep}$ *is the unique semilinear map sending* u^{er} *to* c^r*;*

- $N(f) := N \circ f - f \circ N$, *where* $N : k[u]/u^{ep} \to k[u]/u^{ep}$ *is the unique* k*-linear derivation such that* $N(u) = -u$;
- $(\widehat{g}f)(x) = g(f(\widehat{g}^{-1}x))$ *for all* $x \in \mathcal{M}$ *and* $g \in \mathrm{Gal}(K/K')$, *where* $\mathrm{Gal}(K/K')$ *acts on* $k[u]/u^{ep}$ *by* $g(au^i) = g(a)(\frac{g(\varpi)}{\varpi})^i u^i$ *for* $a \in k$.

If $\mathcal{M}$ is an object of $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ then so is $\mathcal{M}^*$. Moreover, we have $\mathcal{M} \cong \mathcal{M}^{**}$ and

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}^*) \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}).$$

(c.f. [Car11]), Section 2.1.)

Finally, we review the notion of Breuil submodules developed mainly by [Car11]. See also [HLM17], Section 2.3.

Definition 3.2.8. *Let* $\mathcal{M}$ *be an object of* $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$. *A* Breuil submodule *of* $\mathcal{M}$ *is an* $\overline{S}$*-submodule* $\mathcal{N}$ *of* $\mathcal{M}$ *if* $\mathcal{N}$ *satisfies*

- $\mathcal{N}$ *is a* $k[u]/u^{ep}$*-direct summand of* $\mathcal{M}$;
- $N(\mathcal{N}) \subseteq \mathcal{N}$ *and* $\widehat{g}(\mathcal{N}) \subseteq \mathcal{N}$ *for all* $g \in \mathrm{Gal}(K/K')$;
- $\phi_r(\mathcal{N} \cap \mathrm{Fil}^r\mathcal{M}) \subseteq \mathcal{N}$.

If $\mathcal{N}$ is a Breuil submodule of $\mathcal{M}$, then $\mathcal{N}$ and $\mathcal{M}/\mathcal{N}$ are also objects of $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$. We now state a crucial result we will use later.

Proposition 3.2.9 ([HLM17], Proposition 2.3.5)**.** *Let* $\mathcal{M}$ *be an object in* $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$.
Then there is a natural inclusion preserving bijection

$$\Theta : \{\textit{Breuil submodules in } \mathcal{M}\} \to \{G_{K'}\textit{-subrepresentations of } \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})\}$$

sending $\mathcal{N} \subseteq \mathcal{M}$ *to the image of* $\mathrm{T}^r_{\mathrm{st}}(\mathcal{N}) \hookrightarrow \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})$. *Moreover, if* $\mathcal{M}_2 \subseteq \mathcal{M}_1$ *are Breuil submodules of* $\mathcal{M}$, *then* $\Theta(\mathcal{M}_1)/\Theta(\mathcal{M}_2) \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}_1/\mathcal{M}_2)$.

We will also need classification of Breuil modules of rank 1 as follows. We denote the Breuil modules in the following lemma by $\mathcal{M}(a, s, \lambda)$.

Lemma 3.2.10 ([MP17], Lemma 3.1)**.** *Let* $k := \mathbf{F}_{p^f}$, $e := p^f - 1$, $\varpi := \sqrt[e]{-p}$, *and* $K' = \mathbf{Q}_p$. *We also let* $\mathcal{M}$ *be a rank-one object in* $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$.
Then there exists a generator $m \in \mathcal{M}$ *such that:*

(i) $\mathcal{M} = \overline{S}_{\mathbf{F}} \cdot m$;

(ii) $\mathrm{Fil}^r\mathcal{M} = u^{s(p-1)}\mathcal{M}$ *where* $0 \le s \le \frac{re}{p-1}$;

(iii) $\varphi_r(u^{s(p-1)}m) = \lambda m$ *for some* $\lambda \in (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})^\times$;

(iv) $\widehat{g}(m) = (\omega_f(g)^a \otimes 1)m$ *for all* $g \in \mathrm{Gal}(K/K_0)$ *where* a *is an integer such that* $a + ps \equiv 0$ *mod* $(\frac{e}{p-1})$;

(v) $N(m) = 0$.

Moreover, one has

$$\mathrm{T}^r_{\mathrm{st}}(\mathcal{M})|_{I_{\mathbf{Q}_p}} = \omega_f^{a+ps}.$$

The following lemma will be used to determine if the Breuil modules violate the maximal non-splitness.

Lemma 3.2.11 ([MP17], Lemma 3.2)**.** *Let $k := \mathbf{F}_{p^f}$, $e := p^f - 1$, $\varpi := \sqrt[e]{-p}$, and $K' = \mathbf{Q}_p$. We also let $\mathcal{M}_x := \mathcal{M}(k_x, s_x, \lambda_x)$ and $\mathcal{M}_y := \mathcal{M}(k_y, s_y, \lambda_y)$ be rank-one objects in $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$. Assume that the integers $k_x, k_y, s_x, s_y \in \mathbf{Z}$ satisfy*

$$p(s_y - s_x) + [k_y - k_x]_f > 0. \tag{3.2.12}$$

Assume further that $f < p$ and let

$$0 \to \mathcal{M}_x \to \mathcal{M} \to \mathcal{M}_y \to 0$$

be an extension in $\mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$, with $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ being Fontaine–Laffaille.

If the exact sequence of $\overline{S}_{\mathbf{F}}$-modules

$$0 \to \mathrm{Fil}^r\mathcal{M}_x \to \mathrm{Fil}^r\mathcal{M} \to \mathrm{Fil}^r\mathcal{M}_y \to 0 \tag{3.2.13}$$

splits, then the $G_{\mathbf{Q}_p}$-representation $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ splits as a direct sum of two characters.

In particular, provided that $pk_y \not\equiv k_x$ modulo e and that $s_y(p-1) < re$ if $f > 1$, the representation $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ splits as a direct sum of two characters if the element $j_0 \in \mathbf{Z}$ uniquely defined by

$$j_0 e + [p^{-1}k_y - k_x]_f < s_x(p-1) \leq (j_0+1)e + [p^{-1}k_y - k_x]_f \tag{3.2.14}$$

satisfies

$$(r + j_0)e + [p^{-1}k_y - k_x]_f < (s_x + s_y)(p-1). \tag{3.2.15}$$

3.2.4 Linear algebra with descent data

In this section, we introduce the notion of framed basis for a Breuil module $\mathcal{M}$ and framed system of generators for $\mathrm{Fil}^r\mathcal{M}$. Throughout this section, we assume that $K_0 = K'$ and continue to assume that K is a tamely ramified Galois extension of K'. We also fix a positive integer $r < p-1$.

Definition 3.2.16. *Let $n \in \mathbf{N}$ and let $(k_{n-1}, k_{n-2}, \ldots, k_0) \in \mathbf{Z}^n$ be an n-tuple. A rank n Breuil module $\mathcal{M} \in \mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$* is of (inertial) type $\omega_\varpi^{k_{n-1}} \oplus \cdots \oplus \omega_\varpi^{k_0}$ *if $\mathcal{M}$ has an $\overline{S}$-basis $(e_{n-1}, \cdots, e_0)$ such that $\widehat{g}e_i = (\omega_\varpi^{k_i}(g) \otimes 1)e_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$. We call such a basis a* framed basis of $\mathcal{M}$.

We also say that $\underline{f} := (f_{n-1}, f_{n-2}, \ldots, f_0)$ is a framed system of generators of $\mathrm{Fil}^r\mathcal{M}$ *if $\underline{f}$ is a system of $\overline{S}$-generators for $\mathrm{Fil}^r\mathcal{M}$ and $\widehat{g}f_i = (\omega_\varpi^{p^{-1}k_i}(g) \otimes 1)f_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$.*

The existence of a framed basis and a framed system of generators for a given Breuil module $\mathcal{M} \in \mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$ is proved in [HLM17], Section 2.2.2.

Let $\mathcal{M} \in \mathbf{F}$-$\mathrm{BrMod}^r_{\mathrm{dd}}$ be of inertial type $\bigoplus_{i=0}^{n-1} \omega_\varpi^{k_i}$, and let $\underline{e} := (e_{n-1}, \ldots, e_0)$ be a framed basis for $\mathcal{M}$ and $\underline{f} := (f_{n-1}, \ldots, f_0)$ be a framed system of generators for $\mathrm{Fil}^r\mathcal{M}$. The *matrix of the filtration*, with respect to $\underline{e}, \underline{f}$, is the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) \in \mathrm{M}_n(\overline{S})$ such that

$$\underline{f} = \underline{e} \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}).$$

Similarly, we define the *matrix of the Frobenius* with respect to $\underline{e}$, $\underline{f}$ as the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n(\overline{S})$ characterized by

$$(\phi_r(f_{n-1}), \cdots, \phi_r(f_0)) = \underline{e} \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r).$$

As we require $\underline{e}$, $\underline{f}$ to be compatible with the framing, the entries in the matrix of the filtration satisfy the important additional properties:

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M})_{i,j} \in \overline{S}_{\omega_\varpi^{p^{f-1}k_j - k_i}}.$$

More precisely, $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M})_{i,j} = u^{[p^{f-1}k_j - k_i]_f} s_{i,j}$, where $s_{i,j} \in \overline{S}_{\omega_\varpi^0} = k \otimes_{\mathbf{F}_p} \mathbf{F}[u^e]/(u^{ep})$.

We can therefore introduce the subspace $\mathrm{M}_n^{\square}(\overline{S})$ of matrices with framed type $\overline{\tau} = \bigoplus_{i=0}^{n-1} \omega_f^{k_i}$ as

$$\mathrm{M}_n^{\square}(\overline{S}) := \left\{ V \in \mathrm{M}_n(\overline{S}) \mid V_{i,j} \in \overline{S}_{\omega_f^{k_j - k_i}} \text{ for all } 0 \leq i,j \leq n-1 \right\}.$$

Similarly, we define

$$\mathrm{M}_n^{\square,\prime}(\overline{S}) := \left\{ V \in \mathrm{M}_n(\overline{S}) \mid V_{i,j} \in \overline{S}_{\omega_f^{p^{-1}k_j - k_i}} \text{ for all } 0 \leq i,j \leq n-1 \right\}$$

and

$$\mathrm{M}_n^{\square,\prime\prime}(\overline{S}) := \left\{ V \in \mathrm{M}_n(\overline{S}) \mid V_{i,j} \in \overline{S}_{\omega_f^{p^{-1}(k_j - k_i)}} \text{ for all } 0 \leq i,j \leq n-1 \right\}.$$

We also define

$$\mathrm{GL}_n^{\bullet}(\overline{S}) := \mathrm{GL}_n(\overline{S}) \cap \mathrm{M}_n^{\bullet}(\overline{S})$$

for $\bullet \in \{\square\} \cup \{\square,\prime\} \cup \{\square,\prime\prime\}$.

As $\varphi_r(f_i)$ is a $\omega_f^{k_i}$-eigenvector for the action of $\mathrm{Gal}(K/K_0)$ we deduce that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M}) \in \mathrm{M}_n^{\square,\prime}(\overline{S}) \quad \text{and} \quad \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n^{\square}(\overline{S}).$$

Note that $\mathrm{M}_n^{\square}(\overline{S}) = \mathrm{M}_n^{\square,\prime}(\overline{S}) = \mathrm{M}_n^{\square,\prime\prime}(\overline{S})$ if the framed type $\overline{\tau}$ is of niveau 1.

We use similar terminologies for strongly divisible modules $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^r$.

Definition 3.2.17. *Let $n \in \mathbf{N}$ and let $(k_{n-1}, k_{n-2}, \ldots, k_0) \in \mathbf{Z}^n$ be an n-tuple. A rank n strongly divisible module $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^r$* is of (inertial) type $\widetilde{\omega}_{\varpi}^{k_{n-1}} \oplus \cdots \oplus \widetilde{\omega}_{\varpi}^{k_0}$ *if $\widehat{\mathcal{M}}$ has an $S_{\mathcal{O}_E}$-basis $\underline{\widehat{e}} := (\widehat{e}_{n-1}, \cdots, \widehat{e}_0)$ such that $g\widehat{e}_i = (\widetilde{\omega}_{\varpi}^{k_i}(g) \otimes 1)\widehat{e}_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$. We call such a basis a* framed basis for $\widehat{\mathcal{M}}$.

We also say that $\underline{\widehat{f}} := (\widehat{f}_{n-1}, \widehat{f}_{n-2}, \ldots, \widehat{f}_0)$ is a framed system of generators for $\mathrm{Fil}^r\widehat{\mathcal{M}}$ *if $\underline{\widehat{f}}$ is a system of S-generators for $\mathrm{Fil}^r\widehat{\mathcal{M}}/\mathrm{Fil}^r S \cdot \widehat{\mathcal{M}}$ and $g\widehat{f}_i = (\widetilde{\omega}_{\varpi}^{p^{-1}k_i}(g) \otimes 1)\widehat{f}_i$ for all i and all $g \in \mathrm{Gal}(K/K_0)$.*

One can readily check the existence of a framed basis for $\widehat{\mathcal{M}}$ and a framed system of generators for $\mathrm{Fil}^r\widehat{\mathcal{M}}$, by Nakayama Lemma. We also define

$$\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^r\widehat{\mathcal{M}}) \quad \text{and} \quad \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\phi_r)$$

each of whose entries satisfies

$$\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^r\widehat{\mathcal{M}})_{i,j} \in S_{\widetilde{\omega}_{\varpi}^{p^f - 1 k_j - k_i}} \quad \text{and} \quad \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\phi_r)_{i,j} \in S_{\widetilde{\omega}_{\varpi}^{k_j - k_i}},$$

in the similar fashion to Breuil modules. In particular,

$$\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^r\widehat{\mathcal{M}}) \in \mathrm{M}_n^{\square,\prime}(S) \quad \text{and} \quad \mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\varphi_r) \in \mathrm{GL}_n^{\square}(S)$$

where $\mathrm{M}_n^{\square,\prime}(S)$ and $\mathrm{GL}_n^{\square}(S)$ are defined in the similar way to Breuil modules. We also define $\mathrm{GL}_n^{\square,\prime\prime}(S)$ in the similar way to Breuil modules again.

The inertial types on a Breuil module $\mathcal{M}$ and on a strongly divisible modules are closely related to the Weil–Deligne representation associated to a potentially crystalline lift of $\mathrm{T}_{\mathrm{st}}^r(\mathcal{M})$.

Proposition 3.2.18 ([LMP], Proposition 2.12)**.** *Let $\widehat{\mathcal{M}}$ be an object in $\mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^r$ and let $\mathcal{M} := \widehat{\mathcal{M}} \otimes_S S/(\varpi_E, \mathrm{Fil}^p S)$ be the Breuil module corresponding to the mod p reduction of $\widehat{\mathcal{M}}$.*

If $\mathrm{T}_{\mathrm{st}}^{K_0,r}(\widehat{\mathcal{M}})[\frac{1}{p}]$ has Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$ for some integers k_i, then $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_{\varpi}^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1} \omega_{\varpi}^{k_i}$).

Finally, we need a technical result on change of basis of Breuil modules with descent data.

Lemma 3.2.19 ([HLM17], Lemma 2.2.8)**.** *Let $\mathcal{M} \in \mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ be of type $\bigoplus_{i=0}^{n-1} \omega_\varpi^{k_i}$, and let $\underline{e}$, $\underline{f}$ be a framed basis for $\mathcal{M}$ and a framed system of generators for $\mathrm{Fil}^r\mathcal{M}$ respectively. Write $V := \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) \in \mathrm{M}_n^{\square,\prime}(\overline{S})$ and $A := \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r) \in \mathrm{GL}_n^{\square}(\overline{S})$, and assume that there are invertible matrices $R \in \mathrm{GL}_n^{\square}(\overline{S})$ and $C \in \mathrm{GL}_n^{\square,\prime\prime}(\overline{S})$ such that*

$$R \cdot V \cdot C \equiv V' \bmod (u^{e(r+1)}),$$

for some $V' \in \mathrm{M}_n^{\square,\prime}(\overline{S})$.

Then $\underline{e}' := \underline{e} \cdot R^{-1}$ forms another framed basis for $\mathcal{M}$ and $\underline{f}' := \underline{e}' \cdot V'$ forms another framed system of generators for $\mathrm{Fil}^r\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e}',\underline{f}'}(\mathrm{Fil}^r\mathcal{M}) = V' \in \mathrm{M}_n^{\square,\prime}(\overline{S}) \quad \textit{and} \quad \mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_r) = R \cdot A \cdot \phi(C) \in \mathrm{GL}_n^{\square}(\overline{S}).$$

In particular, if $R^{-1} = A$ then $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_r) = \phi(C)$.

The statement of Lemma 3.2.19 is slightly more general than [HLM17], Lemma 2.2.8, but exactly the same argument works.

3.2.5 Fontaine–Laffaille modules

In this section, we briefly recall the theory of Fontaine–Laffaille modules over $\mathbf{F}$, and we continue to assume that $K_0 = K'$ and that K is a tamely ramified Galois extension of K'.

Definition 3.2.20. *A* Fontaine–Laffaille module *over $k \otimes_{\mathbf{F}_p} \mathbf{F}$ is the datum $(M, \mathrm{Fil}^\bullet M, \phi_\bullet)$ of*

- *a free $k \otimes_{\mathbf{F}_p} \mathbf{F}$-module M of finite rank;*
- *a decreasing, exhaustive and separated filtration $\{\mathrm{Fil}^j M\}_{j\in\mathbf{Z}}$ on M by $k \otimes_{\mathbf{F}_p} \mathbf{F}$-submodules;*
- *a ϕ-semilinear isomorphism $\phi_\bullet : \mathrm{gr}^\bullet M \to M$, where $\mathrm{gr}^\bullet M := \bigoplus_{j\in\mathbf{Z}} \frac{\mathrm{Fil}^j M}{\mathrm{Fil}^{j+1} M}$.*

We write $\mathbf{F}\text{-FLMod}_k$ for the category of Fontaine–Laffaille modules over $k \otimes_{\mathbf{F}_p} \mathbf{F}$, which is abelian. If the field k is clear from the context, we simply write $\mathbf{F}$-FLMod to lighten the notation.

Given a Fontaine–Laffaille module M, the set of its Hodge–Tate weights in the direction of $\sigma \in \mathrm{Gal}(k/\mathbf{F}_p)$ is defined as $\mathrm{HT}_\sigma := \left\{ i \in \mathbf{N} \mid e_\sigma \mathrm{Fil}^i M \neq e_\sigma \mathrm{Fil}^{i+1} M \right\}$. In the remainder of this paper we will be focused on Fontaine–Laffaille modules with *parallel* Hodge–Tate weights, i.e. we will assume that for all $i \in \mathbf{N}$, the submodules $\mathrm{Fil}^i M$ are free over $k \otimes_{\mathbf{F}_p} \mathbf{F}$.

Definition 3.2.21. *Let M be a Fontaine–Laffaille module with parallel Hodge–Tate weights. A $k \otimes_{\mathbf{F}_p} \mathbf{F}$ basis $\underline{f} = (f_0, f_1, \ldots, f_{n-1})$ on M is* compatible with the filtration *if for all $i \in \mathbf{Z}_{\geq 0}$ there exists $j_i \in \mathbf{Z}_{\geq 0}$ such that $\mathrm{Fil}^i M = \sum_{j=j_i}^n k \otimes_{\mathbf{F}_p} \mathbf{F} \cdot f_j$. In particular, the principal symbols $(\mathrm{gr}(f_0), \ldots, \mathrm{gr}(f_{n-1}))$ provide a $k \otimes_{\mathbf{F}_p} \mathbf{F}$ basis for $\mathrm{gr}^\bullet M$.*

Note that if the graded pieces of the Hodge filtration have rank at most one then any two compatible basis on M are related by a lower-triangular matrix in $\mathrm{GL}_n(k \otimes_{\mathbf{F}_p} \mathbf{F})$. Given a Fontaine–Laffaille module and a compatible basis $\underline{f}$, it is convenient to describe the Frobenius action via a matrix $\mathrm{Mat}_{\underline{f}}(\phi_\bullet) \in \mathrm{GL}_n(k \otimes_{\mathbf{F}_p} \mathbf{F})$, defined in the obvious way using the principal symbols $(\mathrm{gr}(f_0), \ldots, \mathrm{gr}(f_{n-1}))$ as a basis on $\mathrm{gr}^\bullet M$.

It is customary to write $\mathbf{F}\text{-FLMod}^{[0,p-2]}$ to denote the full subcategory of $\mathbf{F}$-FLMod formed by those modules M verifying $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{p-1} M = 0$ (it is again an abelian category). We have the following description of mod p Galois representations of G_{K_0} via Fontaine–Laffaille modules:

Proposition 3.2.22 ([FL82], Theorem 6.1)**.** *There is an exact fully faithful contravariant functor*

$$\mathrm{T}^*_{\mathrm{cris},K_0} : \mathbf{F}\text{-FLMod}_k^{[0,p-2]} \to \mathrm{Rep}_{\mathbf{F}}(G_{K_0})$$

which is moreover compatible with the restriction over unramified extensions: if L_0/K_0 is unramified with residue field l/k and if M is an object in $\mathbf{F}\text{-FLMod}_k^{[0,p-2]}$, then $l \otimes_k M$ is naturally regarded as an object in $\mathbf{F}\text{-FLMod}_l^{[0,p-2]}$ and

$$\mathrm{T}^*_{\mathrm{cris},L_0}(l \otimes_k M) \cong \mathrm{T}^*_{\mathrm{cris},K_0}(M)|_{G_{L_0}}.$$

We will often write $\mathrm{T}^*_{\mathrm{cris}}$ for $\mathrm{T}^*_{\mathrm{cris},K_0}$ if the base field K_0 is clear from the context.

Definition 3.2.23. *We say that $\overline{\rho} \in \mathrm{Rep}_{\mathbf{F}} G_{K_0}$ is* Fontaine–Laffaille *if $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}$ for some $M \in \mathbf{F}\text{-FLMod}^{[0,p-2]}$.*

3.2.6 Étale ϕ-modules

In this section, we review the theory of étale ϕ-modules, first introduced by Fontaine [Fon90], and its connection with Breuil modules and Fontaine–Laffaille modules. Throughout this section, we continue to assume that $K_0 = K'$ and that K is a tamely ramified Galois extension of K'.

Let $p_0 := -p$, and let $\underline{p}$ be identified with a sequence $(p_n)_n \in \left(\overline{\mathbf{Q}}_p\right)^{\mathbf{N}}$ verifying $p_n^p = p_{n-1}$ for all n. We also fix $\varpi := \sqrt[e]{-p} \in K$, and let $\varpi_0 = \varpi$. We fix a sequence $(\varpi_n)_n \in \left(\overline{\mathbf{Q}}_p\right)^{\mathbf{N}}$ such that $\varpi_n^e = p_n$ and $\varpi_n^p = \varpi_{n-1}$ for all $n \in \mathbf{N}$, and which is compatible with the norm maps $K(\varpi_{n+1}) \to K(\varpi_n)$ (c.f. [Bre14], Appendix A). By letting $K_\infty := \cup_{n\in\mathbf{N}} K(\varpi_n)$ and $(K_0)_\infty := \cup_{n\in\mathbf{N}} K_0(p_n)$, we have a canonical isomorphism $\mathrm{Gal}(K_\infty/(K_0)_\infty) \xrightarrow{\sim} \mathrm{Gal}(K/K_0)$ and we will identify ω_ϖ as a character of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$. The field of norms $k((\underline{\varpi}))$ associated to (K, ϖ) is then endowed with a residual action of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$, which is completely determined by $\widehat{g}(\underline{\varpi}) = \omega_\varpi(g)\underline{\varpi}$.

We define the category $\left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p}))\right)$-$\mathfrak{Mod}$ of étale $(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p})))$-modules as the category of free $\mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p}))$-modules of finite rank $\mathfrak{M}$ endowed with a semilinear map $\phi : \mathfrak{M} \to \mathfrak{M}$ with respect to the Frobenius on $k((\underline{p}))$ and inducing an isomorphism $\phi^*\mathfrak{M} \to \mathfrak{M}$ (with obvious morphisms between objects). We also define the category $(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{\varpi})))$-$\mathfrak{Mod}_{\mathrm{dd}}$ of étale $(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{\varpi})))$-modules with descent data: an object $\mathfrak{M}$ is defined as for the category $(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p})))$-$\mathfrak{Mod}$ but we moreover require that $\mathfrak{M}$ is endowed with a semilinear action of $\mathrm{Gal}(K_\infty/(K_0)_\infty)$ (semilinear with respect to the residual action on $\mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{\varpi}))$ where $\mathbf{F}$ is endowed with the trivial $\mathrm{Gal}(K_\infty/(K_0)_\infty)$-action) commuting with ϕ.

By work of Fontaine [Fon90], there are anti-equivalences

$$\left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p}))\right)\text{-}\mathfrak{Mod} \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{F}}(G_{(K_0)_\infty})$$

and

$$\left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{\varpi}))\right)\text{-}\mathfrak{Mod}_{\mathrm{dd}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{F}}(G_{(K_0)_\infty})$$

given by

$$\mathfrak{M} \longmapsto \mathrm{Hom}\left(\mathfrak{M}, k((\underline{p}))^{\mathrm{sep}}\right)$$

and

$$\mathfrak{M} \mapsto \mathrm{Hom}\left(\mathfrak{M}, k((\underline{\varpi}))^{\mathrm{sep}}\right)$$

respectively. See also [HLM17], Appendix A.2.

The following proposition summarizes the relation between the various categories and functors we introduced above.

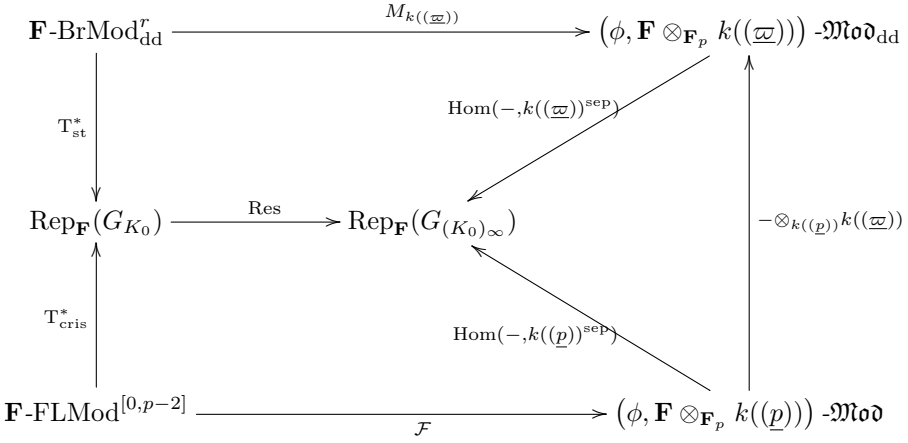
Proposition 3.2.24 ([HLM17], Proposition 2.2.9)**.** *There exist faithful functors*

$$M_{k((\underline{\varpi}))} : \mathbf{F}\text{-BrMod}^r_{\mathrm{dd}} \to \left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{\varpi}))\right) \text{-}\mathfrak{Mod}_{\mathrm{dd}}$$

and

$$\mathcal{F} : \mathbf{F}\text{-FLMod}^{[0,p-2]} \to \left(\phi, \mathbf{F} \otimes_{\mathbf{F}_p} k((\underline{p}))\right) \text{-}\mathfrak{Mod}$$

fitting in the following commutative diagram:



where the descent data is relative to K_0 and the functor $\mathrm{Res} \circ \mathrm{T}^*_{\mathrm{cris}}$ is fully faithful.

Note that the functors $M_{k((\underline{\varpi}))}$ and $\mathcal{F}$ are defined in [BD14]. (See also [HLM17], Appendix A). The following is an immediate consequence of Proposition 3.2.24, which is also stated in [LMP], Corollary 2.14.

Corollary 3.2.25. *Let $0 \le r \le p-2$, and let $\mathcal{M}$ (resp. M) be an object in $\mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ (resp. in $\mathbf{F}\text{-FLMod}^{[0,p-2]}$). Assume that $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ is Fontaine–Laffaille. If*

$$M_{k((\underline{\varpi}))}(\mathcal{M}) \cong \mathcal{F}(M) \otimes_{k((\underline{p}))} k((\underline{\varpi}))$$

then one has an isomorphism of G_{K_0}-representations

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) \cong \mathrm{T}^*_{\mathrm{cris}}(M).$$

The following two lemmas are very crucial in this paper, as we will see later, which describe the functors $M_{k((\underline{\varpi}))}$ and $\mathcal{F}$ respectively.

Lemma 3.2.26 ([HLM17], Lemma 2.2.6)**.** *Let $\mathcal{M}$ be a Breuil module of inertial type $\bigoplus_{i=0}^{n-1} \omega_{\varpi}^{k_i}$ with a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^r\mathcal{M}$, and write $\mathcal{M}^*$ for its dual as defined in Definition 3.2.7. Let $V = \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) \in \mathrm{M}_n^{\square,r}(\overline{S})$ and $A = \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) \in \mathrm{GL}_n^{\square}(\overline{S})$.*

Then there exists a basis $\mathfrak{e}$ for $M_{k((\underline{\varpi}))}(\mathcal{M}^*)$ with $\widehat{g}\cdot\mathfrak{e}_i = (\omega_{\varpi}^{-p^{-1}k_i}(g)\otimes 1)\mathfrak{e}_i$ for all $i \in \{0,1,\cdots,n-1\}$ and $g \in \mathrm{Gal}(K/K_0)$, such that the Frobenius ϕ on $M_{k((\underline{\varpi}))}(\mathcal{M}^*)$ is described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = \widehat{V}^t \left(\widehat{A}^{-1}\right)^t \in \mathrm{M}_n(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]])$$

where $\widehat{V}$, $\widehat{A}$ are lifts of V, A in $\mathrm{M}_n(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]])$ via the reduction morphism $\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]] \twoheadrightarrow \overline{S}$ induced by $\underline{\varpi} \mapsto u$ and $\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi)_{i,j} \in \left(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{\varpi}]]\right)_{\omega_{\varpi}^{p^{-1}k_i - k_j}}$.

Lemma 3.2.27 ([HLM17], Lemma 2.2.7). *Let $M \in \mathbf{F}\text{-FLMod}^{[0,p-2]}$ be a rank n Fontaine–Laffaille module with parallel Hodge–Tate weights $0 \le m_0 \le \cdots \le m_{n-1} \le p-2$ (counted with multiplicity). Let $\underline{e} = (e_0, \dots, e_{n-1})$ be a $k \otimes_{\mathbf{F}_p} \mathbf{F}$ basis for M, compatible with the Hodge filtration $\mathrm{Fil}^\bullet M$ and let $F \in \mathrm{M}_n(k \otimes_{\mathbf{F}_p} \mathbf{F})$ be the associated matrix of the Frobenius $\phi_\bullet : \mathrm{gr}^\bullet M \to M$.*

Then there exists a basis $\underline{\mathfrak{e}}$ for $\mathfrak{M} := \mathcal{F}(M)$ such that the Frobenius ϕ on $\mathfrak{M}$ is described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = \mathrm{Diag}\left(\underline{p}^{m_0}, \cdots, \underline{p}^{m_{n-1}}\right) \cdot F \in \mathrm{M}_n(\mathbf{F} \otimes_{\mathbf{F}_p} k[[\underline{p}]]).$$

3.3 Local Galois side

In this section, we study ordinary Galois representations and their potentially crystalline lifts. In particular, we prove that the Frobenius eigenvalues of certain potentially crystalline lifts preserve the information of the wildly ramified part of ordinary representations.

Throughout this section, we let f be a positive integer, $K' = \mathbf{Q}_p$, $e = p^f - 1$, and $K = \mathbf{Q}_{p^f}(\sqrt[e]{-p})$. We also fix $\varpi := \sqrt[e]{-p}$, and let $\overline{S} = (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})[u]/u^{ep}$ and $\overline{S}_0 := \overline{S}_{\omega_f^0} = (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})[u^e]/u^{ep} \subseteq \overline{S}$. Recall that by $[m]_f$ for a rational number $m \in \mathbf{Z}[\frac{1}{p}]$ we mean the unique integer in $[0, e)$ congruent to m mod (e).

We say that a representation $\overline{\rho}_0 : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(\mathbf{F})$ is *ordinary* if it is isomorphic to a representation whose image is contained in the Borel subgroup of upper-triangular matrices. Namely, an ordinary representation has a basis $\underline{e} := (e_{n-1}, e_{n-2}, \cdots, e_0)$ that gives rise to a matrix form as follows:

$$\overline{\rho}_0 \cong \begin{pmatrix} \mathrm{U}_{\mu_{n-1}}\omega^{c_{n-1}+(n-1)} & *_{n-1} & \cdots & * & * \\ 0 & \mathrm{U}_{\mu_{n-2}}\omega^{c_{n-2}+(n-2)} & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{U}_{\mu_1}\omega^{c_1+1} & *_1 \\ 0 & 0 & \cdots & 0 & \mathrm{U}_{\mu_0}\omega^{c_0} \end{pmatrix} \tag{3.3.1}$$

Here, U_μ is the unramified character sending the geometric Frobenius to $\mu \in \mathbf{F}^\times$ and c_i are integers. By $\overline{\rho}_0$, we always mean an n-dimensional ordinary representation that is written as in (3.3.1). For $n-1 \ge i \ge j \ge 0$, we write

$$\overline{\rho}_{i,j} \tag{3.3.2}$$

for the $(i-j+1)$-dimensional subquotient of $\overline{\rho}_0$ determined by the subset $(e_i, e_{i-1}, \cdots, e_j)$ of the basis $\underline{e}$. For instance, $\overline{\rho}_{i,i} = \mathrm{U}_{\mu_i}\omega^{c_i+i}$ and $\overline{\rho}_{n-1,0} = \overline{\rho}_0$.

An ordinary representation $G_{\mathbf{Q}_p} \to \mathrm{GL}_n(\mathbf{F})$ is *maximally non-split* if its socle filtration has length n. For instance, $\overline{\rho}_0$ in (3.3.1) is maximally non-split if and only if $*_i \neq 0$ for all $i = 1, 2, \cdots, n-1$. In this paper, we are interested in ordinary maximally non-split representations satisfying a certain genericity condition.

Definition 3.3.3. *We say that $\overline{\rho}_0$ is* generic *if*

$$c_{i+1} - c_i > n-1 \text{ for all } i \in \{0, 1, \cdots, n-2\} \text{ and } c_{n-1} - c_0 < (p-1) - (n-1).$$

We say that $\overline{\rho}_0$ is strongly generic *if $\overline{\rho}_0$ is generic and*

$$c_{n-1} - c_0 < (p-1) - (3n-5).$$

Note that this strongly generic condition implies $p > n^2 + 2(n-3)$.

We describe a rough shape of the Breuil modules with descent data from K to $K' = \mathbf{Q}_p$ corresponding to $\overline{\rho}_0$. Let r be a positive integer with $p-1 > r \ge n-1$, and let $\mathcal{M} \in \mathbf{F}\text{-BrMod}^r_{\mathrm{dd}}$ be a Breuil module of inertial type $\bigoplus_{i=0}^{n-1} \omega_f^{k_i}$ such that $\mathrm{T}^r_{\mathrm{st}}(\mathcal{M}) \cong \overline{\rho}_0$, for some $k_i \in \mathbf{Z}$. By Proposition 3.2.9, we

note that $\mathcal{M}$ is a successive extension of $\mathcal{M}_i$, where $\mathcal{M}_i := \mathcal{M}(k_i, r_i, \nu_i)$ (c.f. Lemma 3.2.10) is a rank one Breuil module of inertial type $\omega_f^{k_i}$ such that

$$\omega_f^{k_i+pr_i} \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}_i)|_{I_{\mathbf{Q}_p}} \cong \omega^{c_i+i} \tag{3.3.4}$$

for each $i \in \{0, 1, \cdots, n-1\}$. More precisely, there exist a framed basis $\underline{e} = (e_{n-1}, e_{n-2}, \cdots, e_0)$ for $\mathcal{M}$ and a framed system of generators $\underline{f} = (f_{n-1}, f_{n-2}, \cdots, f_0)$ for $\mathrm{Fil}^r\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}) = \begin{pmatrix} u^{r_{n-1}(p-1)} & u^{[p^{-1}k_{n-2}-k_{n-1}]_f}v_{n-1,n-2} & \cdots & u^{[p^{-1}k_0-k_{n-1}]_f}v_{n-1,0} \\ 0 & u^{r_{n-2}(p-1)} & \cdots & u^{[p^{-1}k_0-k_{n-2}]_f}v_{n-2,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{r_0(p-1)} \end{pmatrix}, \tag{3.3.5}$$

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} \nu_{n-1} & u^{[k_{n-2}-k_{n-1}]_f}w_{n-1,n-2} & \cdots & u^{[k_0-k_{n-1}]_f}w_{n-1,0} \\ 0 & \nu_{n-2} & \cdots & u^{[k_0-k_{n-2}]_f}w_{n-2,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu_0 \end{pmatrix}, \tag{3.3.6}$$

and

$$\mathrm{Mat}_{\underline{e}}(N) = \begin{pmatrix} 0 & u^{[k_{n-2}-k_{n-1}]_f}\gamma_{n-1,n-2} & \cdots & u^{[k_1-k_{n-1}]_f}\gamma_{n-1,1} & u^{[k_0-k_{n-1}]_f}\gamma_{n-1,0} \\ 0 & 0 & \cdots & u^{[k_1-k_{n-2}]_f}\gamma_{n-2,1} & u^{[k_0-k_{n-2}]_f}\gamma_{n-2,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & u^{[k_0-k_1]_f}\gamma_{1,0} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.3.7}$$

for some $\nu_i \in (\mathbf{F}_{p^f} \otimes_{\mathbf{F}_p} \mathbf{F})^\times$ and for some $v_{i,j}, w_{i,j}, \gamma_{i,j} \in \overline{S}_0$.

Fix $0 \le j \le i \le n-1$. We define the Breuil submodule

$$\mathcal{M}_{i,j} \tag{3.3.8}$$

that is a subquotient of $\mathcal{M}$ determined by the basis $(e_i, e_{i-1}, \cdots, e_j)$. For instance, $\mathcal{M}_{i,i} \cong \mathcal{M}_i$ for all $0 \le i \le n-1$. We note that $\mathrm{T}^r_{\mathrm{st}}(\mathcal{M}_{i,j}) \cong \overline{\rho}_{i,j}$ by Proposition 3.2.9.

We will keep these notation and assumptions for $\mathcal{M}$ throughout this paper.

3.3.1 Elimination of Galois types

In this section, we find out the possible Galois types of niveau 1 for potentially semi-stable lifts of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$.

We start this section with the following elementary lemma.

Lemma 3.3.9. *Let $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(E)$ be a potentially semi-stable representation with Hodge–Tate weights $\{-(n-1), ..., -2, -1, 0\}$ and of Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{k_i}$.*

Then

$$\det(\rho)|_{I_{\mathbf{Q}_p}} = \varepsilon^{\frac{n(n-1)}{2}} \cdot \widetilde{\omega}_f^{\sum_{i=0}^{n-1} k_i},$$

where ε is the cyclotomic character.

Proof. $\det(\rho)$ is a potentially crystalline character of $G_{\mathbf{Q}_p}$ with Hodge–Tate weight $-(\sum_{i=0}^{n-1} i)$ and of Galois type $\widetilde{\omega}_f^{\sum_{i=0}^{n-1} k_i}$, i.e., $\det(\rho) \cdot \widetilde{\omega}_f^{-\sum_{i=0}^{n-1} k_i}$ is a crystalline character with Hodge–Tate weight $-(\sum_{i=0}^{n-1} i) = -\frac{n(n-1)}{2}$ so that $\det(\rho)|_{I_{\mathbf{Q}_p}} \cdot \widetilde{\omega}_f^{-\sum_{i=0}^{n-1} k_i} \cong \varepsilon^{\frac{n(n-1)}{2}}$. □

We will only consider the Breuil modules $\mathcal{M}$ corresponding to the mod p reduction of the strongly divisible modules that corresponds to the Galois stable lattices in potentially semi-stable lifts of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, -1, 0\}$, so that we may assume that $r = n-1$, i.e., $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$.

Lemma 3.3.10. *Let $f = 1$. Assume that $\overline{\rho}_0$ is generic, and that $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ corresponds to the mod p reduction of a strongly divisible module $\widehat{\mathcal{M}}$ such that $\mathrm{T}_{\mathrm{st}}^{n-1}(\mathcal{M}) \cong \overline{\rho}_0$ and $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\widehat{\mathcal{M}})$ is a Galois stable lattice in a potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i}$ for some integers k_i.*

Then there exists a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, and $\mathrm{Mat}_{\underline{e}}(N)$ are as in (3.3.5), (3.3.6), and (3.3.7) respectively. Moreover, the (k_i, r_i) satisfy the following properties:

(i) $k_i \equiv c_i + i - r_i \pmod{e}$ *for all* $i \in \{0, 1, \cdots, n-1\}$;

(ii) $0 \leq r_i \leq n-1$ *for all* $i \in \{0, 1, \cdots, n-1\}$;

(iii) $\sum_{i=0}^{n-1} r_i = \frac{(n-1)n}{2}$.

Proof. Note that the inertial type of $\mathcal{M}$ is $\bigoplus_{i=0}^{n-1} \omega^{k_i}$ by Proposition 3.2.18. The first part of the Lemma is obvious from the discussion at the beginning of Section 3.3.

We now prove the second part of the Lemma. We may assume that the rank-one Breuil modules $\mathcal{M}_i$ are of weight $n-1$, so that $0 \leq r_i \leq n-1$ for $i = \{0, 1, ..., n-1\}$ by Lemma 3.2.10. By the equation (3.3.4), we have $k_i \equiv c_i + i - r_i \bmod (e)$, as $e = p-1$. By looking at the determinant of $\overline{\rho}_0$ we deduce the conditions

$$\omega^{\frac{n(n-1)}{2}+k_{n-1}+k_{n-2}+\cdots+k_0} = \det \mathrm{T}_{\mathrm{st}}^{n-1}(\mathcal{M})|_{I_{\mathbf{Q}_p}} = \det \overline{\rho}_0|_{I_{\mathbf{Q}_p}} = \omega^{c_{n-1}+c_{n-2}+\cdots+c_0+\frac{n(n-1)}{2}}$$

from Lemma 3.3.9, and hence we have $r_{n-1} + r_{n-2} + \cdots + r_0 = \frac{n(n-1)}{2}$ (as $p > n^2 + 2(n-3)$ due to the genericity of $\overline{\rho}_0$). □

One can further eliminate Galois types of niveau 1 if $\overline{\rho}_0$ is maximally non-split.

Proposition 3.3.11. *Keep the assumptions and notation of Lemma 3.3.10. If the tuple (k_i, r_i) further satisfy one of the following conditions*

- $r_i = n-1$ *for some* $i \in \{0, 1, 2, \cdots, n-2\}$;
- $r_i = 0$ *for some* $i \in \{1, 2, 3, \cdots, n-1\}$,

then $\overline{\rho}_0$ is not maximally non-split.

Proof. The main ingredient is Lemma 3.2.11. We fix $i \in \{0, 1, 2, \cdots, n-2\}$ and identify $x = i+1$ and $y = i$ and all the other following. From the results in Lemma 3.3.10, it is easy to compute that $[k_i - k_{i+1}]_1 = e - (c_{i+1} - c_i + 1) + (r_{i+1} - r_i)$. By the genericity conditions in Definition 3.3.3 and by part (ii) of Lemma 3.3.10, we see that $0 < [k_i - k_{i+1}]_1 < e$ so that if $r_i \geq r_{i+1}$ then the equation (3.2.12) in Lemma 3.2.11 holds.

If $r_{i+1}e \leq [k_i - k_{i+1}]_1$ and $r_i \geq r_{i+1}$, then $*_{i+1} = 0$ by Lemma 3.2.11. Since $0 < [k_i - k_{i+1}]_1 < e$, we have $r_{i+1}e \leq [k_i - k_{i+1}]_1$ if and only if $r_{i+1} = 0$, in which case $\overline{\rho}_0$ is not maximally non-split.

We now apply the second part of Lemma 3.2.11. It is easy to check that $j_0 = r_{i+1} - 1$. One can again readily check that the equation (3.2.15) is equivalent to $r_i = n-1$, in which case $*_{i+1} = 0$ so that $\overline{\rho}_0$ is not maximally non-split. □

Note that all of the Galois types that will appear later in this section will satisfy the conditions in Lemma 3.3.10, and Proposition 3.3.11 as well if we further assume that $\overline{\rho}_0$ is maximally non-split.

3.3.2 Fontaine–Laffaille parameters

In this section, we parameterize the wildly ramified part of generic and maximally non-split ordinary representations using Fontaine–Laffaille theory.

We start this section by recalling that if $\overline{\rho}_0$ is generic then $\overline{\rho}_0 \otimes \omega^{-c_0}$ is Fontaine–Laffaille (c.f. [GG10], Lemma 3.1.5), so that there is a Fontaine–Laffaille module M with Hodge–Tate weights $\{0, c_1 - c_0 + 1, \cdots, c_{n-1} - c_0 + (n-1)\}$ such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \otimes \omega^{-c_0}$ (if we assume that $\overline{\rho}_0$ is generic).

Lemma 3.3.12. *Assume that $\overline{\rho}_0$ is generic, and let $M \in \mathbf{F}\text{-FLMod}^{[0,p-2]}_{\mathbf{F}_p}$ be a Fontaine–Laffaille module such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \otimes \omega^{-c_0}$.*

Then there exists a basis $\underline{e} = (e_0, e_1, \cdots, e_{n-1})$ for M such that

$$\mathrm{Fil}^j M = \begin{cases} M & \text{if } j \leq 0; \\ \mathbf{F}(e_i, \cdots, e_{n-1}) & \text{if } c_{i-1} - c_0 + i - 1 < j \leq c_i - c_0 + i; \\ 0 & \text{if } c_{n-1} - c_0 + n - 1 < j. \end{cases}$$

and

$$\mathrm{Mat}_{\underline{e}}(\phi_\bullet) = \begin{pmatrix} \mu_0^{-1} & \alpha_{0,1} & \alpha_{0,2} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & \mu_1^{-1} & \alpha_{1,2} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ 0 & 0 & \mu_2^{-1} & \cdots & \alpha_{2,n-2} & \alpha_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{n-2}^{-1} & \alpha_{n-2,n-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu_{n-1}^{-1} \end{pmatrix} \tag{3.3.13}$$

where $\alpha_{i,j} \in \mathbf{F}$.

Note that the basis $\underline{e}$ on M in Lemma 3.3.12 is compatible with the filtration.

Proof. This is an immediate generalization of [HLM17], Lemma 2.1.7. □

For $i \geq j$, the subset $(e_j, \cdots, e_i)$ of $\underline{e}$ determines a subquotient $M_{i,j}$ of the Fontaine–Laffaille module M, which is also a Fontaine–Laffaille module with the filtration induced from $\mathrm{Fil}^s M$ in the obvious way and with Frobenius described as follows:

$$A_{i,j} := \begin{pmatrix} \mu_j^{-1} & \alpha_{j,j+1} & \cdots & \alpha_{j,i-1} & \alpha_{j,i} \\ 0 & \mu_{j+1}^{-1} & \cdots & \alpha_{j+1,i-1} & \alpha_{j+1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{i-1}^{-1} & \alpha_{i-1,i} \\ 0 & 0 & \cdots & 0 & \mu_i^{-1} \end{pmatrix}.$$

Note that $\mathrm{T}^*_{\mathrm{cris}}(M_{i,j}) \otimes \omega^{c_0} \cong \overline{\rho}_{i,j}$. We let $A'_{i,j}$ be the $(i-j) \times (i-j)$-submatrix of $A_{i,j}$ obtained by deleting the left-most column and the lowest row of $A_{i,j}$.

Lemma 3.3.14. *Keep the assumptions and notation of Lemma 3.3.12, and let $0 \leq j < j+1 < i \leq n-1$. Assume further that $\overline{\rho}_0$ is maximally non-split.*

If $\det A'_{i,j} \neq (-1)^{i-j+1} \mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1} \alpha_{j,i}$, then $[\alpha_{j,i} : \det A'_{i,j}] \in \mathbb{P}^1(\mathbf{F})$ does not depend on the choice of basis $\underline{e}$ compatible with the filtration.

Proof. This is an immediate generalization of [HLM17], Lemma 2.1.9. □

Definition 3.3.15. *Keep the assumptions and notation of Lemma 3.3.14, and assume further that $\overline{\rho}_0$ satisfies*

$$\det A'_{i,j} \neq (-1)^{i-j+1} \mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1} \alpha_{j,i} \tag{3.3.16}$$

for all $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$.

The Fontaine–Laffaille parameter associated to $\overline{\rho}_0$ *is defined as*

$$\mathrm{FL}_n(\overline{\rho}_0) := \left(\mathrm{FL}_n^{i,j}(\overline{\rho}_0)\right)_{i,j} \in [\mathbb{P}^1(\mathbf{F})]^{\frac{(n-2)(n-1)}{2}}$$

where

$$\mathrm{FL}_n^{i,j}(\overline{\rho}_0) := \left[\alpha_{j,i} : (-1)^{i-j+1} \cdot \det A'_{i,j}\right] \in \mathbb{P}^1(\mathbf{F})$$

for all $i, j \in \mathbf{Z}$ *such that* $0 \leq j < j+1 < i \leq n-1$.

We often write $\frac{y}{x}$ for $[x : y] \in \mathbb{P}^1(\mathbf{F})$ if $x \neq 0$. The conditions in (3.3.16) for i, j guarantee the well-definedness of $\mathrm{FL}_n^{i,j}(\overline{\rho}_0)$ in $\mathbb{P}^1(\mathbf{F})$. We also point out that $\mathrm{FL}_n^{i,j}(\overline{\rho}_0) \neq (-1)^{i-j}\mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1}$ in $\mathbb{P}^1(\mathbf{F})$.

One can define the inverses of the elements in $\mathbb{P}^1(\mathbf{F})$ in a natural way: for $[x_1 : x_2] \in \mathbb{P}^1(\mathbf{F})$, $[x_1 : x_2]^{-1} := [x_2 : x_1] \in \mathbb{P}^1(\mathbf{F})$.

Lemma 3.3.17. *Assume that* $\overline{\rho}_0$ *is generic. Then*

(i) $\overline{\rho}_0^\vee$ *is generic;*

(ii) *if* $\overline{\rho}_0$ *is strongly generic, then so is* $\overline{\rho}_0^\vee$*;*

(iii) *if* $\overline{\rho}_0$ *is maximally non-split, then so is* $\overline{\rho}_0^\vee$*;*

(iv) *if* $\overline{\rho}_0$ *is maximally non-split, then the conditions in (3.3.16) are stable under* $\overline{\rho}_0 \mapsto \overline{\rho}_0^\vee$.

Assume further that $\overline{\rho}_0$ *is maximally non-split and satisfies the conditions in (3.3.16).*

(v) *for all* $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$, $\mathrm{FL}_n^{i,j}(\overline{\rho}_0) = \mathrm{FL}_n^{i,j}(\overline{\rho}_0 \otimes \omega^b)$ *for any* $b \in \mathbf{Z}$*;*

(vi) *for all* $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$, $\mathrm{FL}_n^{i,j}(\overline{\rho}_0) = \mathrm{FL}_{i-j+1}^{i-j,0}(\overline{\rho}_{i,j})$*;*

(vii) *for all* $i, j \in \mathbf{Z}$ *with* $0 \leq j < j+1 < i \leq n-1$, $\mathrm{FL}_n^{i,j}(\overline{\rho}_0)^{-1} = \mathrm{FL}_n^{n-1-j,n-1-i}(\overline{\rho}_0^\vee)$.

Proof. (i), (ii) and (iii) are easy to check. We leave them for the reader.

The only effect on Fontaine–Laffaille module by twisting ω^b is shifting the jumps of the filtration. Thus (v) and (vi) are obvious.

For (iv) and (vii), one can check that the Frobenius of the Fontaine–Laffaille module associated to $\overline{\rho}_0^\vee$ is described by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \cdot [\mathrm{Mat}_{\underline{e}}(\phi_\bullet)^t]^{-1} \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $\mathrm{Mat}_{\underline{e}}(\phi_\bullet)$ is as in (3.3.13). Now one can check them by direct computation. □

We end this section by defining certain numerical conditions on Fontaine–Laffaille parameters. We consider the matrix $(1, n)w_0 \,\mathrm{Mat}_{\underline{e}}(\phi_\bullet)^t$, where $\mathrm{Mat}_{\underline{e}}(\phi_\bullet)$ is the upper-triangular matrix in (3.3.13). Here, w_0 is the longest element of the Weyl group W associated to T and $(1, n)$ is a permutation in W. Note that the anti-diagonal matrix displayed in the proof of Lemma 3.3.17 is w_0 seen as an element in $\mathrm{GL}_n(\mathbf{F})$. For $1 \leq i \leq n-1$ we let B_i be the square matrix of size i that is the left-bottom corner of $(1, n)w_0 \,\mathrm{Mat}_{\underline{e}}(\phi_\bullet)^t$.

Definition 3.3.18. *Keep the notation and assumptions of Definition 3.3.15. We say that* $\overline{\rho}_0$ *is* Fontaine–Laffaille generic *if moreover* $\det B_i \neq 0$ *for all* $1 \leq i \leq n-1$ *and* $\overline{\rho}_0$ *is strongly generic.*

We emphasize that by an ordinary representation $\overline{\rho}_0$ being Fontaine–Laffaille generic, we *always mean* that $\overline{\rho}_0$ satisfies the maximally non-splitness and the conditions in (3.3.16) as well as $\det B_i \neq 0$ for all $1 \leq i \leq n-1$ and the strongly generic assumption (c.f. Definition 3.3.3).

Although the Frobenius matrix of a Fontaine–Laffaille module depends on the choice of basis, it is easy to see that the non-vanishing of the determinants above is independent of the choice of basis compatible with the filtration. Note that the conditions in Definition 3.3.18 are necessary and sufficient conditions for

$$(1,n)w_0 \operatorname{Mat}_{\underline{e}}(\phi_\bullet)^t \in B(\mathbf{F})w_0B(\mathbf{F})$$

in the Bruhat decomposition, which will significantly reduce the size of the paper (c.f. Remark 3.3.19). We also note that

- $\det B_1 \neq 0$ if and only if $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq \infty$;
- $\det B_{n-1} \neq 0$ if and only if $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) \neq 0$.

Finally, we point out that the locus of Fontaine–Laffaille generic ordinary Galois representations $\overline{\rho}_0$ forms a (Zariski) open subset in $[\mathbb{P}^1(\mathbf{F})]^{\frac{(n-1)(n-2)}{2}}$.

Remark 3.3.19. *Definition 3.3.18 comes from the fact that the list of Serre weights of $\overline{\rho}_0$ is then minimal in the sense of Conjecture 3.5.16. It is very crucial in the proof of Theorem 3.5.44 as it is more difficult to track the Fontaine–Laffaille parameters on the automorphic side if we have too many Serre weights. Moreover, these conditions simplify our proof for Theorem 3.3.44.*

3.3.3 Breuil modules of certain inertial types of niveau 1

In this section, we classify the Breuil modules with certain inertial types, corresponding to the ordinary Galois representations $\overline{\rho}_0$ as in (3.3.1), and we also study their corresponding Fontaine–Laffaille parameters.

Throughout this section, we always assume that $\overline{\rho}_0$ is strongly generic. Since we are only interested in inertial types of niveau 1, we let $f=1$, $e=p-1$, and $\varpi = \sqrt[e]{-p}$. We define the following integers for $0 \leq i \leq n-1$:

$$r_i^{(0)} := \begin{cases} 1 & \text{if } i = n-1; \\ i & \text{if } 0 < i < n-1; \\ n-2 & \text{if } i = 0. \end{cases} \tag{3.3.20}$$

We also set

$$k_i^{(0)} := c_i + i - r_i^{(0)}$$

for all $i \in \{0, 1, \cdots, n-1\}$.

We first classify the Breuil modules of inertial types described as above.

Lemma 3.3.21. *Assume that $\overline{\rho}_0$ is strongly generic and that $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ corresponds to the mod p reduction of a strongly divisible modules $\widehat{\mathcal{M}}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}})$ is a Galois stable lattice in a potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{(0)}}$.*

Then $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ can be described as follows: there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that

$$\operatorname{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) = \begin{pmatrix} u^{r_{n-1}^{(0)}e} & \beta_{n-1,n-2}u^{r_{n-1}^{(0)}e-k_{n-1,n-2}^{(0)}} & \cdots & \beta_{n-1,0}u^{r_{n-1}^{(0)}e-k_{n-1,0}^{(0)}} \\ 0 & u^{r_{n-2}^{(0)}e} & \cdots & \beta_{n-2,0}u^{r_{n-2}^{(0)}e-k_{n-2,0}^{(0)}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{r_0^{(0)}e} \end{pmatrix}$$

and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \mathrm{Diag}\left(\nu_{n-1},\, \nu_{n-2},\, \cdots,\, \nu_0\right)$$

where $k_{i,j}^{(0)} := k_i^{(0)} - k_j^{(0)}$, $\nu_i \in \mathbf{F}^\times$ *and* $\beta_{i,j} \in \mathbf{F}$. *Moreover,*

$$\mathrm{Mat}_{\underline{e}}(N) = \left(\gamma_{i,j} \cdot u^{[k_j^{(0)} - k_i^{(0)}]_1}\right)$$

where $\gamma_{i,j} = 0$ *if* $i \le j$ *and* $\gamma_{i,j} \in u^{e[k_j^{(0)} - k_i^{(0)}]_1}\overline{S}_0$ *if* $i > j$.

Note that $\underline{e}$ and $\underline{f}$ in Lemma 3.3.21 are not necessarily the same as the ones in Lemma 3.3.10.

Proof. We keep the notation in (3.3.5), (3.3.6), and (3.3.7). That is, there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, $\mathrm{Mat}_{\underline{e}}(N)$ are given as in (3.3.5), (3.3.6), and (3.3.7) respectively. Since $k_i \equiv k_i^{(0)} \bmod (p-1)$, we have $r_i = r_i^{(0)}$ for all $i \in \{0, 1, \cdots, n-1\}$ by Lemma 3.3.10), following the notation of Lemma 3.3.10.

We start to prove the following claim: if $n-1 \ge i > j \ge 0$ then

$$e - (k_i^{(0)} - k_j^{(0)}) \ge n. \tag{3.3.22}$$

Indeed, by the strongly generic assumption, Definition 3.3.3

$$\begin{aligned} e - (k_i^{(0)} - k_j^{(0)}) &= (p-1) - (c_i + i - r_i^{(0)}) + (c_j + j - r_j^{(0)}) \\ &= (p-1) - (c_i - c_j) - (i-j) + (r_i^{(0)} - r_j^{(0)}) \\ &\ge (p-1) - (c_{n-1} - c_0) - (n-1-0) + (1-(n-2)) \\ &\ge 3n - 4 - 2n + 4 = n. \end{aligned}$$

Note that this claim will be often used during the proof later.

We now diagonalize $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ with some restriction on the powers of the entries of the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$. Let $V_0 = \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) \in \mathrm{M}_n^{\square}(\overline{S})$ and $A_0 = \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) \in \mathrm{GL}_n^{\square}(\overline{S})$. We also let $V_1 \in \mathrm{M}_n^{\square}(\overline{S})$ be the matrix obtained from V_0 by replacing $v_{i,j}$ by $v'_{i,j} \in \overline{S}_0$, and $B_1 \in \mathrm{GL}_n^{\square}(\overline{S})$ the matrix obtained from A_0 by replacing $w_{i,j}$ by $w'_{i,j} \in \overline{S}_0$. It is straightforward to check that $A_0 \cdot V_1 = V_0 \cdot B_1$ if and only if for all $i > j$

$$\begin{aligned} &\nu_i v'_{i,j} u^{[k_j^{(0)} - k_i^{(0)}]_1} + \sum_{s=j+1}^{i-1} w_{is} v'_{s,j} u^{[k_s^{(0)} - k_i^{(0)}]_1 + [k_j^{(0)} - k_s^{(0)}]_1} + w_{i,j} u^{r_j^{(0)} e + [k_j^{(0)} - k_i^{(0)}]_1} \\ &\quad = w'_{i,j} u^{r_i^{(0)} e + [k_j^{(0)} - k_i^{(0)}]_1} + \sum_{s=j+1}^{i-1} v_{i,s} w'_{s,j} u^{[k_s^{(0)} - k_i^{(0)}]_1 + [k_j^{(0)} - k_s^{(0)}]_1} + \nu_j v_{i,j} u^{[k_j^{(0)} - k_i^{(0)}]_1}. \end{aligned} \tag{3.3.23}$$

Note that the power of u in each term of (3.3.23) is congruent to $[k_j^{(0)} - k_i^{(0)}]_1$ modulo (e). It is immediate that for all $i > j$ there exist $v'_{i,j} \in \overline{S}_0$ and $w'_{i,j} \in \overline{S}_0$ satisfying the equation (3.3.23) with the following additional properties: for all $i > j$

$$\deg v'_{i,j} < r_i^{(0)} e. \tag{3.3.24}$$

Letting $\underline{e}' := \underline{e}A_0$, we have

$$\mathrm{Mat}_{\underline{e}',\underline{f}'}(\mathrm{Fil}^{n-1}\mathcal{M}) = V_1 \text{ and } \mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_{n-1}) = \phi(B_1)$$

where $\underline{f}' = \underline{e}'V_1$, by Lemma 3.2.19. Note that $\phi(B_1)$ is congruent to a diagonal matrix modulo (u^{ne}) by (3.3.22). We repeat this process one more time. We may assume that $w_{i,j} \in u^{ne}\overline{S}_0$, i.e., that

$A_0 \equiv B_1$ modulo (u^{ne}) where B_1 is assumed to be a diagonal matrix. It is obvious that there exists an upper-triangular matrix $V_1 = (v'_{i,j}u^{[p^{-1}k_j^{(0)}-k_i^{(0)}]_2})$ whose entries have bounded degrees as in (3.3.24), satisfying the equation $A_0V_1 \equiv V_0B_1$ modulo (u^{ne}). By Lemma 3.2.19, we get $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_{n-1})$ is diagonal. Hence, we may assume that $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ is diagonal and that $\deg v_{i,j}$ in $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ is bounded as in (3.3.24), and we do so. Moreover, this change of basis do not change the shape of $\mathrm{Mat}_{\underline{e}}(N)$, so that we also assume that $\mathrm{Mat}_{\underline{e}}(N)$ is still as in (3.3.7).

We now prove that

$$v_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1} = \beta_{i,j}u^{r_i^{(0)}e-(k_i^{(0)}-k_j^{(0)})} \tag{3.3.25}$$

for all $n-1 \geq i > j \geq 0$, where $\beta_{i,j} \in \mathbf{F}$. Note that this is immediate for $i = n-1$ and $i = 1$, since $r_i^{(0)} = 1$ if $i = n-1$ or $i = 1$. To prove (3.3.25), we induct on i. The case $i = 1$ is done as above. Fix $p_0 \in \{2, 3, \cdots, n-2\}$, and assume that (3.3.25) holds for all $i \in \{1, 2, \cdots, p_0 - 1\}$ and for all $j < i$. We consider the subquotient $\mathcal{M}_{p_0,0}$ of $\mathcal{M}$ defined in (3.3.8). By abuse of notation, we write $\underline{e} = (e_{p_0}, \cdots, e_0)$ for the induced framed basis for $\mathcal{M}_{p_0,0}$ and $\underline{f} = (f_{p_0}, \cdots, f_0)$ for the induced framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$.

We claim that for $p_0 \geq j \geq 0$

$$u^eN(f_j) \in \overline{S}_0u^ef_j + \sum_{t=j+1}^{p_0} \overline{S}_0u^{[k_j^{(0)}-k_t^{(0)}]_1}f_t.$$

Consider $N(f_j) = N(f_j - u^{r_j^{(0)}e}e_j) + N(u^{r_j^{(0)}e}e_j)$. It is easy to check that $N(f_j - u^{r_j^{(0)}e}e_j)$ and $N(u^{r_j^{(0)}e}e_j) + r_j^{(0)}ef_j$ are $\overline{S}$-linear combinations of $e_{n-1}, \cdots, e_{j+1}$, and they are, in fact, $\overline{S}_0$-linear combinations of $u^{[k_j^{(0)}-k_{n-1}^{(0)}]}e_{n-1}, \cdots, u^{[k_j^{(0)}-k_{j+1}^{(0)}]}e_{j+1}$ since they are $\omega^{k_j^{(0)}}$-invariant. Since $u^eN(f_j) \in \mathrm{Fil}^{n-1}\mathcal{M} \supset u^{(n-1)e}\mathcal{M}$ and $u^eN(f_j) + r_j^{(0)}eu^ef_j = [N(f_j - u^{r_j^{(0)}e}e_j)] + [N(u^{r_j^{(0)}e}e_j) + r_j^{(0)}ef_j]$, we conclude that

$$u^eN(f_j) + r_j^{(0)}eu^ef_j \in \sum_{t=j+1}^{p_0} \overline{S}_0u^{[k_j^{(0)}-k_t^{(0)}]_1}f_t,$$

which completes the claim.

Let $\mathrm{Mat}_{\underline{e},\underline{f}}(N|_{\mathcal{M}_{p_0,0}}) = \left(\gamma_{i,j} \cdot u^{[k_j^{(0)}-k_i^{(0)}]_1}\right)$ where $\gamma_{i,j} = 0$ if $i \leq j$ and $\gamma_{i,j} \in \overline{S}_0$ if $i > j$. We also claim that

$$\gamma_{i,j} \in u^{e[k_j^{(0)}-k_i^{(0)}]_1}\overline{S}_0$$

for $p_0 \geq i > j \geq 0$, which can be readily checked from the equation $cN\phi_{n-1}(f_j) = \phi_{n-1}(u^eN(f_j))$. (Note that $c = 1 \in \overline{S}$ as $E(u) = u^e + p$.) Indeed, we have

$$cN\phi_{n-1}(f_j) = N(\nu_je_j) = \nu_j \sum_{i=j+1}^{p_0} \gamma_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1}e_i.$$

On the other hand, since $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}|_{\mathcal{M}_{p_0,0}})$ is diagonal, the previous claim immediately implies that

$$\phi_r(u^eN(f_j)) \in \sum_{t=j+1}^{p_0} \overline{S}_0u^{p[k_j^{(0)}-k_t^{(0)}]_1}e_t.$$

Hence, we conclude the claim.

We now finish the proof of (3.3.25) by inducting on $p_0 - j$ as well. Write $v_{i,j} = \sum_{t=0}^{r_i^{(0)}-1} x_{i,j}^{(t)}u^{te}$ for $x_{i,j}^{(t)} \in \mathbf{F}$. We need to prove $x_{p_0,j}^{(t)} = 0$ for $t \in \{0, 1, \cdots, r_{p_0}^{(0)} - 2\}$. Assume first $j = p_0 - 1$, and we

compute $N(f_j)$ as follows:

$$N(f_{p_0-1}) = -\sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,p_0-1}^{(t)}[e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0} + \gamma_{p_0,p_0-1}u^{(r_{p_0-1}^{(0)}+1)e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0} - r_{p_0-1}^{(0)}eu^{r_{p_0-1}^{(0)}e}e_{p_0-1},$$

which immediately implies

$$N(f_{p_0-1}) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,p_0-1}^{(t)}[er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0} + \gamma_{p_0,p_0-1}u^{(r_{p_0-1}^{(0)}+1)e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. Since $\gamma_{p_0,p_0-1} \in u^{e[e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]}\overline{S}_0$ and $e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)}) \geq n$ by (3.3.22), we get

$$N(f_{p_0-1}) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,p_0-1}^{(t)}[er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$, so that

$$u^eN(f_{p_0-1}) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-2} x_{p_0,p_0-1}^{(t)}[er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})]u^{e(t+2)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$.

It is easy to check that

$$er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)}) \not\equiv 0 \tag{3.3.26}$$

modulo (p) for all $0 \leq t \leq r_{p_0}^{(0)}-2$. Indeed, $er_{p_0-1}^{(0)}-e(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)}) \equiv -r_{p_0-1}^{(0)}+(t+1)+(k_{p_0}^{(0)}-k_{p_0-1}^{(0)}) = (t+1)+(c_{p_0}-c_{p_0-1}+1)-r_{p_0}^{(0)}$ modulo (p). Since $0 \leq t \leq r_{p_0}^{(0)}-2$,

$$0 < (c_{p_0}-c_{p_0-1}+2)-r_{p_0}^{(0)} \leq (t+1)+(c_{p_0}-c_{p_0-1}+1)-r_{p_0}^{(0)} \leq (c_{p_0}-c_{p_0-1}-1) < p$$

by the strongly generic conditions, Definition 3.3.3. Hence, we conclude that $x_{p_0,p_0-1}^{(t)} = 0$ for all $0 \leq t \leq r_{p_0}^{(0)}-2$ since $u^eN(f_{p_0-1}) \in \mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. This completes the proof of (3.3.25) for $j = p_0-1$.

Assume that (3.3.25) holds for $i = p_0$ and $j \in \{p_0-1, p_0-2, \cdots, s+1\}$. We compute $N(f_s)$ for $p_0-1 > s \geq 0$ as follows: using the induction hypothesis on $i \in \{1,2,\cdots,p_0-1\}$

$$N(f_s) = -\sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,s}^{(t)}[e(t+1)-(k_{p_0}^{(0)}-k_s^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_s^{(0)})}e_{p_0} + \sum_{i=s+1}^{p_0-1}\beta_{i,s}u^{r_i^{(0)}e-(k_i^{(0)}-k_s^{(0)})}\left(\sum_{s=i+1}^{p_0}\gamma_{s,i}u^{e-(k_s^{(0)}-k_i^{(0)})}e_s - [r_i^{(0)}e-(k_i^{(0)}-k_s^{(0)})]e_i\right) + u^{r_s^{(0)}e}\sum_{i=s+1}^{p_0}\gamma_{i,s}u^{e-(k_i^{(0)}-k_s^{(0)})}e_i - r_s^{(0)}eu^{r_s^{(0)}e}e_s.$$

Since $\gamma_{i,j} \in u^{e[e-(k_i^{(0)}-k_j^{(0)})]}\overline{S}_0$, we have

$$N(f_s) \equiv -\sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,s}^{(t)}[e(t+1)-(k_{p_0}^{(0)}-k_s^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_s^{(0)})}e_{p_0} - \sum_{i=s+1}^{p_0-1}\beta_{i,s}[r_i^{(0)}e-(k_i^{(0)}-k_s^{(0)})]u^{r_i^{(0)}e-(k_i^{(0)}-k_s^{(0)})}e_i - r_s^{(0)}eu^{r_s^{(0)}e}e_s$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$, which immediately implies

$$N(f_s) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-1} x_{p_0,s}^{(t)}[r_s^{(0)}e-e(t+1)+(k_{p_0}^{(0)}-k_s^{(0)})]u^{e(t+1)-(k_{p_0}^{(0)}-k_s^{(0)})}e_{p_0} + \sum_{i=s+1}^{p_0-1}\beta_{i,s}[r_s^{(0)}e-r_i^{(0)}e+(k_i^{(0)}-k_s^{(0)})]u^{r_i^{(0)}e-(k_i^{(0)}-k_s^{(0)})}e_i$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. Now, from the induction hypothesis on $j \in \{p_0-1, p_0-2, \cdots, s+1\}$,

$$u^e\sum_{i=s+1}^{p_0-1}\beta_{i,s}[r_s^{(0)}e-r_i^{(0)}e+(k_i^{(0)}-k_s^{(0)})]u^{r_i^{(0)}e-(k_i^{(0)}-k_s^{(0)})}e_i \in \mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$$

and so we have

$$u^eN(f_s) \equiv \sum_{t=0}^{r_{p_0}^{(0)}-2} x_{p_0,s}^{(t)}[r_s^{(0)}e-e(t+1)+(k_{p_0}^{(0)}-k_s^{(0)})]u^{e(t+2)-(k_{p_0}^{(0)}-k_s^{(0)})}e_{p_0}$$

modulo $\mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$. By the same argument as (3.3.26), one can readily check that $r_s^{(0)}e-e(t+1)+(k_{p_0}^{(0)}-k_s^{(0)}) \not\equiv 0$ modulo (p) for all $0 \le t \le r_{p_0}^{(0)}-2$. Hence, we conclude that $x_{p_0,s}^{(t)}=0$ for all $0 \le t \le r_{p_0}^{(0)}-2$ as $u^eN(f_s) \in \mathrm{Fil}^{n-1}\mathcal{M}_{p_0,0}$, which completes the proof. □

Proposition 3.3.27. *Keep the assumptions and notation of Lemma 3.3.21. Assume further that $\overline{\rho}_0$ is maximally non-split and satisfies the conditions in (3.3.16).*

Then $\beta_{i,i-1} \in \mathbf{F}^\times$ for $i \in \{1,2,\cdots,n-1\}$ and we have the following identities: for $0 \le j < j+1 < i \le n-1$

$$\mathrm{FL}_n^{i,j}(\overline{\rho}_0) = \left[\beta_{i,j}\nu_{j+1}\cdots\nu_{i-1} : (-1)^{i-j+1}\det A'_{i,j}\right] \in \mathbb{P}^1(\mathbf{F})$$

where

$$A'_{i,j} = \begin{pmatrix} \beta_{j+1,j} & \beta_{j+2,j} & \beta_{j+3,j} & \cdots & \beta_{i-1,j} & \beta_{i,j} \\ 1 & \beta_{j+2,j+1} & \beta_{j+3,j+1} & \cdots & \beta_{i-1,j+1} & \beta_{i,j+1} \\ 0 & 1 & \beta_{j+3,j+2} & \cdots & \beta_{i-1,j+2} & \beta_{i,j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{i-1,i-2} & \beta_{i,i-2} \\ 0 & 0 & 0 & \cdots & 1 & \beta_{i,i-1} \end{pmatrix}.$$

Proof. We may assume $c_0=0$ by Lemma 3.3.17. We let $V := \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A := \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ be as in the statement of Lemma 3.3.21. By Lemma 3.2.26, the ϕ-module over $\mathbf{F}\otimes_{\mathbf{F}_p}\mathbf{F}_p((\varpi))$ defined by $\mathfrak{M} := M_{\mathbf{F}_p((\underline{\varpi}))}(\mathcal{M}^*)$ is described as follows:

$$\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = (U_{i,j})$$

where

$$U_{i,j} = \begin{cases} \nu_j^{-1} \cdot \underline{\varpi}^{r_j^{(0)} e} & \text{if } i = j; \\ 0 & \text{if } i > j; \\ \nu_j^{-1} \cdot \beta_{j,i} \cdot \underline{\varpi}^{r_j^{(0)} e - (k_j^{(0)} - k_i^{(0)})} & \text{if } i < j \end{cases}$$

in a framed basis $\underline{\mathfrak{e}} = (\mathfrak{e}_{n-1}, \mathfrak{e}_{n-2}, \cdots, \mathfrak{e}_0)$ with dual type $\omega^{-k_{n-1}^{(0)}} \oplus \omega^{-k_{n-2}^{(0)}} \cdots \oplus \omega^{-k_0^{(0)}}$.

By considering the change of basis $\underline{\mathfrak{e}}' = (\underline{\varpi}^{k_{n-1}^{(0)}} \mathfrak{e}_{n-1}, \underline{\varpi}^{k_{n-2}^{(0)}} \mathfrak{e}_{n-2}, \cdots, \underline{\varpi}^{k_0^{(0)}} \mathfrak{e}_0)$, $\mathrm{Mat}_{\underline{\mathfrak{e}}'}(\phi)$ is described as follows:

$$\mathrm{Mat}_{\underline{\mathfrak{e}}'}(\phi) = (V_{i,j})$$

where

$$V_{i,j} = \begin{cases} \nu_j^{-1} \cdot \underline{\varpi}^{e(k_j^{(0)} + r_j^{(0)})} & \text{if } i = j; \\ 0 & \text{if } i > j; \\ \nu_j^{-1} \cdot \beta_{j,i} \cdot \underline{\varpi}^{e(k_j^{(0)} + r_j^{(0)})} & \text{if } i < j. \end{cases}$$

Since $k_i^{(0)} = c_i + i - r_i^{(0)}$ for each $n - 1 \geq i \geq 0$, we easily see that the ϕ-module $\mathfrak{M}_0$ is the base change via $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{p})) \to \mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{\varpi}))$ of the ϕ-module $\mathfrak{M}_0$ over $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{p}))$ described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}''}(\phi) = \begin{pmatrix} \nu_{n-1}^{-1} \underline{p}^{c_{n-1}+(n-1)} & 0 & \cdots & 0 \\ \nu_{n-1}^{-1} \beta_{n-1,n-2} \underline{p}^{c_{n-1}+(n-1)} & \nu_{n-2}^{-1} \underline{p}^{c_{n-2}+(n-2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1}^{-1} \beta_{n-1,0} \underline{p}^{c_{n-1}+(n-1)} & \nu_{n-2}^{-1} \beta_{n-2,0} \underline{p}^{c_{n-2}+(n-2)} & \cdots & \nu_0^{-1} \underline{p}^{c_0} \end{pmatrix}$$

in an appropriate basis $\underline{\mathfrak{e}}'' = (\mathfrak{e}''_{n-1}, \mathfrak{e}''_{n-2}, \cdots, \mathfrak{e}''_0)$, which can be rewritten as

$$\mathrm{Mat}_{\underline{\mathfrak{e}}''}(\phi) = \underbrace{\begin{pmatrix} \nu_{n-1}^{-1} & 0 & \cdots & 0 \\ \nu_{n-1}^{-1} \beta_{n-1,n-2} & \nu_{n-2}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1}^{-1} \beta_{n-1,0} & \nu_{n-2}^{-1} \beta_{n-2,0} & \cdots & \nu_0^{-1} \end{pmatrix}}_{=:H'} \cdot \mathrm{Diag}\left(\underline{p}^{c_{n-1}+n-1}, \cdots, \underline{p}^{c_1+1}, \underline{p}^{c_0}\right).$$

By considering the change of basis $\underline{\mathfrak{e}}''' = \underline{\mathfrak{e}}'' \cdot H'$ and then reversing the order of the basis $\underline{\mathfrak{e}}'''$, the Frobenius ϕ of $\mathfrak{M}_0$ with respect to this new basis is described as follows:

$$\mathrm{Mat}(\phi) = \mathrm{Diag}\left(\underline{p}^{c_0}, \underline{p}^{c_1+1}, \cdots, \underline{p}^{c_{n-1}+(n-1)}\right) \underbrace{\begin{pmatrix} \nu_0^{-1} & \nu_1^{-1}\beta_{1,0} & \cdots & \nu_{n-1}^{-1}\beta_{n-1,0} \\ 0 & \nu_1^{-1} & \cdots & \nu_{n-1}^{-1}\beta_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \nu_{n-1}^{-1} \end{pmatrix}}_{=:H} \tag{3.3.28}$$

with respect to the new basis described as above.

The last displayed upper-triangular matrix H is the Frobenius of the Fontaine–Laffaille module M such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})$, by Lemma 3.2.27. Hence, we get the desired results (c.f. Definition 3.3.15). □

Remark 3.3.29. *We emphasize that the matrix H is the Frobenius of the Fontaine–Laffaille module M, with respect to a basis $(e_0, e_1, \cdots, e_{n-1})$ compatible with the filtration, such that $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0 \cong \mathrm{T}^r_{\mathrm{st}}(\mathcal{M})$, so that we can now apply the conditions in (3.3.16) as well as Definition 3.3.18 to the Breuil*

modules in Lemma 3.3.21. Moreover, H can be written as

$$H = \underbrace{\begin{pmatrix} 1 & \beta_{1,0} & \cdots & \beta_{n-1,0} \\ 0 & 1 & \cdots & \beta_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=:H'} \cdot \mathrm{Diag}\left(\nu_0^{-1}, \nu_1^{-1}, \cdots, \nu_{n-1}^{-1}\right),$$

so that we have $(1,n)w_0H^t \in B(\mathbf{F})w_0B(\mathbf{F})$ if and only if $(1,n)w_0(H')^t \in B(\mathbf{F})w_0B(\mathbf{F})$. Hence, $\overline{\rho}_0$ being Fontaine–Laffaille generic is a matter only of the entries of the filtration of the Breuil modules if the Breuil modules are written as in Lemma 3.3.21.

3.3.4 Fontaine–Laffaille parameters vs Frobenius eigenvalues

In this section, we study further the Breuil modules of Lemma 3.3.21. We show that if the filtration is of a certain shape then a certain product of Frobenius eigenvalues (of the Breuil modules) corresponds to the newest Fontaine–Laffaille parameter, $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)$. To get such a shape of the filtration, we assume further that $\overline{\rho}_0$ is Fontaine–Laffaille generic.

Lemma 3.3.30. *Keep the assumptions and notation of Lemma 3.3.21. Assume further that $\overline{\rho}_0$ is Fontaine–Laffaille generic (c.f. Definition 3.3.18).*

Then $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ can be described as follows: there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \mathrm{Diag}\left(\mu_{n-1},\, \mu_{n-2},\, \cdots,\, \mu_0\right)$$

and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) = (U_{i,j})$$

where

$$U_{i,j} = \begin{cases} u^{r_{n-1}^{(0)}e-(k_{n-1}^{(0)}-k_0^{(0)})} & \text{if } i = n-1 \text{ and } j = 0; \\ u^{r_i^{(0)}e} & \text{if } 0 < i = j < n-1; \\ x_{i,j} \cdot u^{r_i^{(0)}e-(k_i^{(0)}-k_j^{(0)})} & \text{if } n-1 > i > j; \\ u^{r_0^{(0)}e+(k_{n-1}^{(0)}-k_0^{(0)})} & \text{if } i = 0 \text{ and } j = n-1; \\ x_{0,j} \cdot u^{r_0^{(0)}e+(k_j^{(0)}-k_0^{(0)})} & \text{if } i = 0 \leq j < n-1; \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.31}$$

Here, $\mu_i \in \mathbf{F}^\times$ and $x_{i,j} \in \mathbf{F}$.

Moreover, we have the following identity:

$$\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) = \prod_{i=1}^{n-2} \mu_i^{-1}.$$

Due to the size of the matrix, we decide to describe the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ as (3.3.31). But for the reader we visualize the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ below, although it is less accurate:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & u^{r_{n-1}^{(0)}e-k_{n-1,0}^{(0)}} \\ 0 & u^{r_{n-2}^{(0)}} & \cdots & x_{n-2,1}u^{r_{n-2}^{(0)}e-k_{n-2,1}^{(0)}} & x_{n-2,0}u^{r_{n-2}^{(0)}e-k_{n-2,0}^{(0)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u^{r_1^{(0)}} & x_{1,0}u^{r_1^{(0)}e-k_{1,0}^{(0)}} \\ u^{r_0^{(0)}e+k_{n-1,0}^{(0)}} & x_{0,n-2}u^{r_0^{(0)}e+k_{n-2,0}^{(0)}} & \cdots & x_{0,1}u^{r_0^{(0)}e+k_{1,0}^{(0)}} & x_{0,0}u^{r_0^{(0)}e} \end{pmatrix}$$

where $k_{i,j}^{(0)} := k_i^{(0)} - k_j^{(0)}$.

Proof. Let $\underline{e}_0$ be a framed basis for $\mathcal{M}$ and $\underline{f}_0$ a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $V_0 := \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A_0 := \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\phi_{n-1})$ are given as in Lemma 3.3.21. So, in particular, V_0 is upper-triangular and A_0 is diagonal.

By Proposition 3.3.27, the upper-triangular matrix H in (3.3.28) is the Frobenius of the Fontaine–Laffaille module corresponding to $\overline{\rho}_0$, as in Definition 3.3.15. Since we assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic, we have $(1,n)w_0H^t \in B(\mathbf{F})w_0B(\mathbf{F})$ as discussed right after Definition 3.3.15, so that we have $w_0H^tw_0 \in (1,n)B(\mathbf{F})w_0B(\mathbf{F})w_0$. Equivalently, $w_0(H')^tw_0 \in (1,n)B(\mathbf{F})w_0B(\mathbf{F})w_0$ by Remark 3.3.29, where H' is defined in Remark 3.3.29. Hence, comparing V_0 with $w_0(H')^tw_0$, there exists a lower-triangular matrix $C \in \mathrm{GL}_n^{\square}(\overline{S})$ such that

$$V_0 \cdot C = V_1 := (U_{i,j})_{0\le i,j\le n-1}$$

where $U_{i,j}$ is described as in (3.3.31), since any matrix in $w_0B(\mathbf{F})w_0$ is lower-triangular. From the identity $V_0 \cdot C = V_1$, we have $V_1 = \mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A_1 := \mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\phi_{n-1}) = A_0 \cdot \phi(C)$ by Lemma 3.2.19, where $\underline{e}_1 := \underline{e}_0$ and $\underline{f}_1 := \underline{e}_1V_1$. If $i < j$, then $[k_j^{(0)} - k_i^{(0)}]_1 = k_j^{(0)} - k_i^{(0)} \ge n$ as $\overline{\rho}_0$ is strongly generic, so that A_1 is congruent to a diagonal matrix $B_2' \in \mathrm{GL}_n(\mathbf{F})$ modulo (u^{ne}) as $C = (c_{i,j} \cdot u^{[k_j^{(0)}-k_i^{(0)}]_1})$ is a lower-triangular and A_0 is diagonal.

Let V_2 be the matrix obtained from V_1 by replacing $x_{i,j}$ in (3.3.31) by $y_{i,j}$, and $B_2 = (b_{i,j})$ is the diagonal matrix defined by taking $b_{i,i} = b'_{i,i}$ if $1 \le i \le n-2$ and $b_{i,i} = b'_{n-1-i,n-1-i}$ otherwise, where $B_2' = (b'_{i,j})$. Then it is obvious that there exist $y_{i,j} \in \mathbf{F}$ such that

$$A_1 \cdot V_2 \equiv V_1 \cdot B_2$$

modulo (u^{ne}). Letting $\underline{e}_2 := \underline{e}_1 \cdot A_1$, we have $V_2 = \mathrm{Mat}_{\underline{e}_2,\underline{f}_2}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $\mathrm{Mat}_{\underline{e}_2,\underline{f}_2}(\phi_{n-1}) = \phi(B_2)$ by Lemma 3.2.19. Note that $A_2 := \mathrm{Mat}_{\underline{e}_2,\underline{f}_2}(\phi_{n-1})$ is diagonal. Hence, there exist a framed basis for $\mathcal{M}$ and a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ and $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ are described as in the statement.

We now prove the second part of the lemma. It is harmless to assume $c_0 = 0$ by Lemma 3.3.17. Let $V := \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A := \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ be as in the first part of the lemma. By Lemma 3.2.26, the ϕ-module over $\mathbf{F}\otimes_{\mathbf{F}_p}\mathbf{F}_p((\underline{\varpi}))$ defined by $\mathfrak{M} := M_{\mathbf{F}_p((\underline{\varpi}))}(\mathcal{M}^*)$ is described as follows: there exists a basis $\underline{\mathfrak{e}} = (\mathfrak{e}_{n-1}, \mathfrak{e}_{n-2}, \cdots, \mathfrak{e}_0)$, compatible with decent data, such that $\mathrm{Mat}_{\underline{\mathfrak{e}}}(\phi) = (\widehat{A}^{-1}\widehat{V})^t$ where $\widehat{V}^t$ and $(\widehat{A}^{-1})^t$ are computed as follows:

$$\widehat{V}^t = \begin{pmatrix} 0 & 0 & \cdots & 0 & \underline{\varpi}^{r_0^{(0)}e+k_{n-1,0}^{(0)}} \\ 0 & \underline{\varpi}^{r_{n-2}^{(0)}} & \cdots & 0 & x_{0,n-2}\underline{\varpi}^{r_0^{(0)}e+k_{n-2,0}^{(0)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{n-2,1}\underline{\varpi}^{r_{n-2}^{(0)}e-k_{n-2,1}^{(0)}} & \cdots & \underline{\varpi}^{r_1^{(0)}} & x_{0,1}\underline{\varpi}^{r_0^{(0)}e+k_{1,0}^{(0)}} \\ \underline{\varpi}^{r_{n-1}^{(0)}e-k_{n-1,0}^{(0)}} & x_{n-2,0}\underline{\varpi}^{r_{n-2}^{(0)}e-k_{n-2,0}^{(0)}} & \cdots & x_{1,0}\underline{\varpi}^{r_1^{(0)}e-k_{1,0}^{(0)}} & x_{0,0}\underline{\varpi}^{r_0^{(0)}e} \end{pmatrix}$$

and

$$\widehat{A}^{-1} = \mathrm{Diag}\left(\mu_{n-1}^{-1}, \mu_{n-2}^{-1}, \cdots, \mu_0^{-1}\right).$$

By considering the change of basis $\underline{\mathfrak{e}}' = (\underline{\varpi}^{k_{n-1}^{(0)}}\mathfrak{e}_{n-1}, \underline{\varpi}^{k_{n-2}^{(0)}}\mathfrak{e}_{n-2}, \cdots, \underline{\varpi}^{k_1^{(0)}}\mathfrak{e}_1, \underline{\varpi}^{k_0^{(0)}}\mathfrak{e}_0)$, we have

$$\mathrm{Mat}_{\underline{\mathfrak{e}}'}(\phi) = (\widehat{V}^t)' \cdot \mathrm{Diag}\left(\mu_{n-1}^{-1}, \mu_{n-2}^{-1}, \cdots, \mu_0^{-1}\right)$$

where

$$(\widehat{V}^t)' = \begin{pmatrix} 0 & 0 & \cdots & 0 & \underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \\ 0 & \underline{\varpi}^{e(k_{n-2}^{(0)}+r_{n-2}^{(0)})} & \cdots & 0 & x_{0,n-2}\underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{n-2,1}\underline{\varpi}^{e(k_{n-2}^{(0)}+r_{n-2}^{(0)})} & \cdots & \underline{\varpi}^{e(k_1^{(0)}+r_1^{(0)})} & x_{0,1}\underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \\ \underline{\varpi}^{e(k_{n-1}^{(0)}+r_{n-1}^{(0)})} & x_{n-2,0}\underline{\varpi}^{e(k_{n-2}^{(0)}+r_{n-2}^{(0)})} & \cdots & x_{1,0}\underline{\varpi}^{e(k_1^{(0)}+r_1^{(0)})} & x_{0,0}\underline{\varpi}^{e(k_0^{(0)}+r_0^{(0)})} \end{pmatrix}.$$

Since $k_j^{(0)} + r_j^{(0)} = c_j + j$ for all j, it is immediate that the ϕ-module $\mathfrak{M}$ over $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{\varpi}))$ is the base change via $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{p})) \to \mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{\varpi}))$ of the ϕ-module $\mathfrak{M}_0$ over $\mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{F}_p((\underline{p}))$ described by

$$\mathrm{Mat}_{\underline{\mathfrak{e}}''}(\phi) = F'' \cdot \mathrm{Diag}\left(\underline{p}^{c_{n-1}+n-1}, \underline{p}^{c_{n-2}+n-2}, \cdots, \underline{p}^{c_0}\right)$$

where

$$F'' = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu_0^{-1} \\ 0 & \mu_{n-2}^{-1} & 0 & \cdots & 0 & \mu_0^{-1}x_{0,n-2} \\ 0 & \mu_{n-2}^{-1}x_{n-2,n-3} & \mu_{n-3}^{-1} & \cdots & 0 & \mu_0^{-1}x_{0,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mu_{n-2}^{-1}x_{n-2,1} & \mu_{n-3}^{-1}x_{n-3,1} & \cdots & \mu_1^{-1} & \mu_0^{-1}x_{0,1} \\ \mu_{n-1}^{-1} & \mu_{n-2}^{-1}x_{n-2,0} & \mu_{n-3}^{-1}x_{n-3,0} & \cdots & \mu_1^{-1}x_{1,0} & \mu_0^{-1}x_{0,0} \end{pmatrix},$$

in an appropriate basis $\underline{\mathfrak{e}}''$.

Now, consider the change of basis $\underline{\mathfrak{e}}''' = \underline{\mathfrak{e}}'' \cdot F''$ and then reverse the order of the basis $\mathfrak{e}'''$. Then the matrix of the Frobenius ϕ for $\mathfrak{M}_0$ with respect to this new basis is given by

$$\mathrm{Diag}\left(\underline{p}^{c_0}, \underline{p}^{c_1+1}, \cdots, \underline{p}^{c_{n-1}+n-1}\right) \cdot F$$

where

$$F = \begin{pmatrix} \mu_0^{-1}x_{0,0} & \mu_1^{-1}x_{1,0} & \mu_2^{-1}x_{2,0} & \cdots & \mu_{n-2}^{-1}x_{n-2,0} & \mu_{n-1}^{-1} \\ \mu_0^{-1}x_{0,1} & \mu_1^{-1} & \mu_2^{-1}x_{2,1} & \cdots & \mu_{n-2}^{-1}x_{n-2,1} & 0 \\ \mu_0^{-1}x_{0,2} & 0 & \mu_2^{-1} & \cdots & \mu_{n-2}^{-1}x_{n-2,2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_0^{-1}x_{0,n-2} & 0 & 0 & \cdots & \mu_{n-2}^{-1} & 0 \\ \mu_0^{-1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By Lemma 3.2.27, there exists a Fontaine–Laffaille module M such that $\mathcal{F}(M) = \mathfrak{M}_0$ with Hodge–Tate weights $(c_0, c_1+1, \cdots, c_{n-1}+n-1)$ and $\mathrm{Mat}_{\underline{e}}(\phi_\bullet) = F$ for some basis $\underline{e}$ of M compatible with the Hodge filtration on M. On the other hand, since $\mathrm{T}^*_{\mathrm{cris}}(M) \cong \overline{\rho}_0$, there exists a basis $\underline{e}'$ of M compatible with the Hodge filtration on M such that

$$\mathrm{Mat}_{\underline{e}'}(\phi_\bullet) = \underbrace{\begin{pmatrix} w_0 & w_{0,1} & \cdots & w_{0,n-2} & w_{0,n-1} \\ 0 & w_1 & \cdots & w_{1,n-2} & w_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & w_{n-2} & w_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & w_{n-1} \end{pmatrix}}_{=:G}$$

where $w_{i,j} \in \mathbf{F}$ and $w_i \in \mathbf{F}^\times$ by Lemma 3.3.12. Since both $\underline{e}$ and $\underline{e}'$ are compatible with the Hodge filtration on M, there exists a unipotent lower-triangular $n \times n$-matrix U such that

$$U \cdot F = G.$$

Note that we have $w_{0,n-1} = \mu_{n-1}^{-1}$ by direct computation.

Let U' be the $(n-1)\times(n-1)$-matrix obtained from U by deleting the right-most column and the lowest row, and F' (resp. G') the $(n-1)\times(n-1)$-matrix obtained from F (resp. G) by deleting the left-most column and the lowest row. Then they still satisfy $G' = U'\cdot F'$ as U is a lower-triangular unipotent matrix, so that

$$\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) = [w_{0,n-1} : (-1)^n \det G'] = \left[\mu_{n-1}^{-1} : (-1)^n \det F'\right] = \left[1 : \prod_{i=1}^{n-2} \mu_i^{-1}\right],$$

which completes the proof. □

Proposition 3.3.32. *Keep the assumptions and notation of Lemma 3.3.30.*

Then $\mathcal{M} \in \mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ can be described as follows: there exist a framed basis $\underline{e}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & u^{e-(k_{n-1}^{(0)}-k_0^{(0)})} \\ 0 & u^{(n-2)e} & 0 & \cdots & 0 & 0 \\ 0 & 0 & u^{(n-3)e} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u^{e} & 0 \\ u^{(n-2)e+(k_{n-1}^{(0)}-k_0^{(0)})} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Moreover, if we let

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \left(\alpha_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1}\right)$$

for $\alpha_{i,i} \in \overline{S}_0^{\times}$ and $\alpha_{i,j} \in \overline{S}_0$ if $i \neq j$ then we have the following identity:

$$\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0) = \prod_{i=1}^{n-2}(\alpha_{i,i}^{(0)})^{-1} = \prod_{i=1}^{n-2}\mu_i^{-1}$$

where $\alpha_{i,j}^{(0)} \in \mathbf{F}$ is determined by $\alpha_{i,j}^{(0)} \equiv \alpha_{i,j}$ modulo (u^e).

Note that $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$ always belong to $\mathrm{GL}_n^{\square}(\overline{S})$ as $\underline{e}$ and $\underline{f}$ are framed.

Proof. We let $\underline{e}_0$ (resp. $\underline{e}_1$) be a framed basis for $\mathcal{M}$ and $\underline{f}_0$ (resp. $\underline{f}_1$) be a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $\mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\phi_{n-1})$ (resp. $\mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $\mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\phi_{n-1})$) are given as in the statement of Lemma 3.3.30 (resp. in the statement of Proposition 3.3.32). We also let $V_0 = \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\mathrm{Fil}^{n-1}\mathcal{M})$ and $A_0 = \mathrm{Mat}_{\underline{e}_0,\underline{f}_0}(\phi_{n-1})$ as well as $V_1 = \mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\mathrm{Fil}^{n-1})\mathcal{M}$ and $A_1 = \mathrm{Mat}_{\underline{e}_1,\underline{f}_1}(\phi_{n-1})$.

It is obvious that there exist $R = (r_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1})$ and $C = (c_{i,j}u^{[k_j^{(0)}-k_i^{(0)}]_1})$ in $\mathrm{GL}_n^{\square}(\overline{S})$ such that

$$R\cdot V_0\cdot C = V_1 \text{ and } \underline{e}_1 = \underline{e}_0 R^{-1}$$

for $r_{i,j}$ and $c_{i,j}$ in $\overline{S}_0$. From the first equation above, we immediately get the identities:

$$r_{n-1,n-1}^{(0)}\cdot c_{0,0}^{(0)} = 1 = r_{0,0}^{(0)}\cdot c_{n-1,n-1}^{(0)} \text{ and } r_{i,i}^{(0)}\cdot c_{i,i}^{(0)} = 1$$

for $0 < i < n-1$, where $r_{i,j}^{(0)} \in \mathbf{F}$ (resp. $c_{i,j}^{(0)} \in \mathbf{F}$) is determined by $r_{i,j}^{(0)} \equiv r_{i,j}$ modulo (u^e) (resp. $c_{i,j}^{(0)} \equiv c_{i,j}$ modulo (u^e)). By Lemma 3.2.19, we see that $A_1 = R\cdot A_0\cdot\phi(C)$.

Hence, if we let $A_1 = \left(\alpha_{i,j} u^{[k_j^{(0)} - k_i^{(0)}]_1}\right)$ then

$$r_{i,i}^{(0)} \cdot \mu_i \cdot c_{i,i}^{(0)} = \alpha_{i,i}^{(0)}$$

for each $0 < i < n-1$ since R and C are diagonal modulo (u), so that we have

$$\prod_{i=1}^{n-2} \mu_i = \prod_{i=1}^{n-2} \alpha_{i,i}^{(0)}$$

which completes its proof. □

Note that the matrix in the statement of Proposition 3.3.32 gives rise to the elementary divisors of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$.

3.3.5 Filtration of strongly divisible modules

In this section, we describe the filtration of the strongly divisible modules lifting the Breuil modules described in Proposition 3.3.32. Throughout this section, we keep the notation $r_i^{(0)}$ as in (3.3.20) as well as $k_i^{(0)}$.

We start to recall the following lemma, which is easy to prove but very useful.

Lemma 3.3.33. *Let $0 < f \leq n$ be an integer, and let $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-Mod}_{\mathrm{dd}}^{n-1}$ be a strongly divisible module corresponding to a lattice in a potentially semi-stable representation $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(E)$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type of niveau f such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbf{F} \cong \overline{\rho}_0$.*

If we let

$$X^{(i)} := \left(\frac{\mathrm{Fil}^{n-1}\widehat{\mathcal{M}} \cap \mathrm{Fil}^i S \cdot \widehat{\mathcal{M}}}{\mathrm{Fil}^{n-1} S \cdot \widehat{\mathcal{M}}} \right) \otimes_{\mathcal{O}_E} E$$

for $i \in \{0, 1, \cdots, n-1\}$, then for any character $\xi : \mathrm{Gal}(K/K_0) \to K^\times$ we have that the ξ-isotypical component $X_\xi^{(i)}$ of $X^{(i)}$ is a free $K_0 \otimes E$-module of finite rank

$$\mathrm{rank}_{K_0 \otimes_{\mathbf{Q}_p} E} X_\xi^{(i)} = \frac{n(n-1)}{2} - \frac{i(i+1)}{2}.$$

Moreover, multiplication by $u \in S$ induces an isomorphism $X_\xi^{(0)} \xrightarrow{\sim} X_{\xi\widetilde{\omega}}^{(0)}$.

Proof. Since ρ has Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$, by the analogue with E-coefficients of [Bre97], Proposition A.4, we deduce that

$$\mathrm{Fil}^{n-1}\mathcal{D} = \mathrm{Fil}^{n-1} S_E \widehat{f}_{n-1} \oplus \mathrm{Fil}^{n-2} S_E \widehat{f}_{n-2} \oplus \cdots \oplus \mathrm{Fil}^1 S_E \widehat{f}_1 \oplus S_E \widehat{f}_0$$

for some S_E-basis $\widehat{f}_0, \cdots, \widehat{f}_{n-1}$ of $\mathcal{D}$, where $\mathcal{D} := \widehat{\mathcal{M}}[\frac{1}{p}] \cong S_E \otimes_E \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(V)$, so that we also have

$$\mathrm{Fil}^{n-1}\mathcal{D} \cap \mathrm{Fil}^i S_E \mathcal{D} = \mathrm{Fil}^{n-1} S_E \widehat{f}_{n-1} \oplus \mathrm{Fil}^{n-2} S_E \widehat{f}_{n-2} \oplus \cdots \oplus \mathrm{Fil}^i S_E \widehat{f}_i \oplus \cdots \oplus \mathrm{Fil}^i S_E \widehat{f}_0.$$

Since $\rho \cong \mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} E$ is a $G_{\mathbf{Q}_p}$-representation, $\mathrm{Fil}^i(K \otimes_{K_0} \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\rho)) \cong K \otimes_{\mathbf{Q}_p} \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(\rho \otimes \varepsilon^{1-n})$, so that $X^{(i)} \cong \frac{\mathrm{Fil}^{n-1}\mathcal{D} \cap \mathrm{Fil}^i S_E \mathcal{D}}{\mathrm{Fil}^{n-1} S_E \mathcal{D}}$ is a free $K_0 \otimes_{\mathbf{Q}_p} E$-module. Since $\frac{S_E}{\mathrm{Fil}^{n-1} S_E} \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{e-1} (K_0 \otimes_{\mathbf{Q}_p} E) u^j E(u)^i$, we have $\mathrm{rank}_{K_0 \otimes_{\mathbf{Q}_p} E} X^{(i)} = \left[\frac{n(n-1)}{2} - \frac{i(i+1)}{2} \right] e$. We note that $\mathrm{Gal}(K/K_0)$ acts semisimply and that multiplication by u gives rise to an $K_0 \otimes_{\mathbf{Q}_p} E$-linear isomorphism on $S_E/\mathrm{Fil}^p S_E$ which cyclically permutes the isotypical components, which completes the proof. □

Note that Lemma 3.3.33 immediately implies that

$$\mathrm{rank}_{K_0\otimes_{\mathbf{Q}_p}E} X_\xi^{(i)} - \mathrm{rank}_{K_0\otimes_{\mathbf{Q}_p}E} X_\xi^{(i+1)} = i+1. \tag{3.3.34}$$

We will use this fact frequently to prove the main result, Proposition 3.3.36, in this subsection.

To describe the filtration of strongly divisible modules, we need to analyze the $\mathrm{Fil}^{n-1}\mathcal{M}$ of the Breuil modules $\mathcal{M}$ we consider.

Lemma 3.3.35. *Keep the notation and assumptions of Lemma 3.3.21.*

(i) *If u^a is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ then*

$$e-(k_{n-1}^{(0)}-k_0^{(0)}) \le a \le (n-2)e+(k_{n-1}^{(0)}-k_0^{(0)}).$$

Moreover, $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)\neq\infty$ (resp. $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)\neq 0$) if and only if $u^{e-(k_{n-1}^{(0)}-k_0^{(0)})}$ (resp. $u^{(n-2)e+(k_{n-1}^{(0)}-k_0^{(0)})}$) is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$.

(ii) *If we further assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic, then*

$$\{u^{(n-2)e+(k_{n-1}^{(0)}-k_0^{(0)})},\, u^{(n-2)e},\, u^{(n-3)e},\, \cdots,\, u^e,\, u^{e-(k_{n-1}^{(0)}-k_0^{(0)})}\}$$

are the elementary divisors of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$.

Proof. The first part of (i) is obvious since one can obtain the Smith normal form of $\mathrm{Mat}_{\underline{e},\underline{f}}\,\mathrm{Fil}^{n-1}\mathcal{M}$ by elementary row and column operations. By Proposition 3.3.27, we know that $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)\neq\infty$ if and only if $\beta_{n-1,0}\neq 0$. Since $u^{e-(k_{n-1}^{(0)}-k_0^{(0)})}$ has the minimal degree among the entries of $\mathrm{Mat}_{\underline{e},\underline{f}}\,\mathrm{Fil}^{n-1}\mathcal{M}$, we conclude the equivalence statement for $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)\neq\infty$ holds. The last part of (i) is immediate from the other equivalence statement, $\mathrm{FL}_n^{n-1,0}(\overline{\rho}_0)\neq\infty$ if and only if $\beta_{n-1,0}\neq 0$, by considering $\mathcal{M}^*$ and using Lemma 3.3.17, (vi).

Part (ii) is obvious from Proposition 3.3.32. □

Proposition 3.3.36. *Assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic and keep the notation $r_i^{(0)}$ as in (3.3.20) as well as $k_i^{(0)}$. Let $\widehat{\mathcal{M}}\in\mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^{n-1}$ be a strongly divisible module corresponding to a lattice in a potentially semi-stable representation $\rho: G_{\mathbf{Q}_p}\to\mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}^{k_i^{(0)}}$ and Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,0\}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}})\otimes_{\mathcal{O}_E}\mathbf{F}\cong\overline{\rho}_0$.*

Then there exists a framed basis $(\widehat{e}_{n-1},\widehat{e}_{n-2},\cdots,\widehat{e}_0)$ for $\widehat{\mathcal{M}}$ and a framed system of generators $(\widehat{f}_{n-1},\widehat{f}_{n-2},\cdots,\widehat{f}_0)$ for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$ modulo $\mathrm{Fil}^{n-1}S\cdot\widehat{\mathcal{M}}$ such that $\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$ is described as follows:

$$\begin{pmatrix} -\frac{p^{n-1}}{\alpha} & 0 & 0 & \cdots & 0 & u^{e-(k_{n-1}^{(0)}-k_0^{(0)})} \\ 0 & E(u)^{n-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & E(u)^{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E(u) & 0 \\ u^{k_{n-1}^{(0)}-k_0^{(0)}}\sum_{i=0}^{n-2}p^{n-2-i}E(u)^i & 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}$$

where $\alpha\in\mathcal{O}_E$ with $0<v_p(\alpha)<n-1$.

Proof. Note that we write the elements of $\widehat{\mathcal{M}}$ in terms of coordinates with respect to a framed basis $\underline{\widehat{e}}:=(\widehat{e}_{n-1},\widehat{e}_{n-2},\cdots,\widehat{e}_0)$. We let $\mathcal{M}:=\widehat{\mathcal{M}}\otimes_S\overline{S}$, which is a Breuil module of weight $n-1$ and of type $\bigoplus_{i=0}^{n-1}\omega^{k_i^{(0)}}$ by Proposition 3.2.18. Note also that $\mathcal{M}$ can be described as in Proposition 3.3.32,

and we assume that $\mathcal{M}$ has such a framed basis for $\mathcal{M}$ and such a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$.

Let

$$\widehat{f}_0 = \begin{pmatrix} u^{e-(k_{n-1}^{(0)}-k_0^{(0)})}\sum_{k=0}^{n-2} x_{n-1,k}E(u)^k \\ u^{e-(k_{n-2}^{(0)}-k_0^{(0)})}\sum_{k=0}^{n-2} x_{n-2,k}E(u)^k \\ \vdots \\ u^{e-(k_1^{(0)}-k_0^{(0)})}\sum_{k=0}^{n-2} x_{1,k}E(u)^k \\ \sum_{k=0}^{n-2} x_{0,k}E(u)^k \end{pmatrix} \in \left(\frac{\mathrm{Fil}^{n-1}\widehat{M}}{\mathrm{Fil}^{n-1}S\widehat{M}}\right)_{\widetilde{\omega}^{k_0^{(0)}}},$$

where $x_{i,j} \in \mathcal{O}_E$. The vector $\widehat{f}_0$ can be written as follows:

$$\widehat{f}_0 = u^{e-(k_{n-1}^{(0)}-k_0^{(0)})} \underbrace{\begin{pmatrix} \sum_{k=0}^{n-2} x_{n-1,k}E(u)^k \\ u^{(k_{n-1}^{(0)}-k_{n-2}^{(0)})}\sum_{k=0}^{n-2} x_{n-2,k}E(u)^k \\ \vdots \\ u^{(k_{n-1}^{(0)}-k_1^{(0)})}\sum_{k=0}^{n-2} x_{1,k}E(u)^k \\ u^{(k_{n-1}^{(0)}-k_0^{(0)})}\sum_{k=1}^{n-2} x_{0,k}[E(u)^k - p^k]/u^e \end{pmatrix}}_{=:\widehat{e}'_{n-1}} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_{0,0} + \sum_{k=1}^{n-2} x_{0,k}p^k \end{pmatrix}.$$

By (ii) of Lemma 3.3.35, we know that $u^{e-(k_{n-1}^{(0)}-k_0^{(0)})}$ is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ and all other elementary divisors have bigger powers, so that we may assume $v_p(x_{n-1,0}) = 0$. Since $\mathrm{Fil}^{n-1}\mathcal{M} \subseteq u^{e-(k_{n-1}^{(0)}-k_0^{(0)})}\mathcal{M}$, we must have $v_p(x_{0,0}) > 0$. So $\underline{\widehat{e}}_1 := (\widehat{e}'_{n-1}, \widehat{e}_{n-2}, \cdots, \widehat{e}_0)$ is a framed basis for $\widehat{\mathcal{M}}$ by Nakayama lemma and we have the following coordinates of $\widehat{f}_0$ with respect to $\underline{\widehat{e}}_1$:

$$\widehat{f}_0 = \begin{pmatrix} u^{e-(k_{n-1}^{(0)}-k_0^{(0)})} \\ 0 \\ \vdots \\ 0 \\ \alpha \end{pmatrix} \in \left(\frac{\mathrm{Fil}^{n-1}\widehat{M}}{\mathrm{Fil}^{n-1}S\widehat{M}}\right)_{\widetilde{\omega}^{k_0^{(0)}}}$$

for $\alpha \in \mathcal{O}_E$ with $v_p(\alpha) > 0$.

Since $u^{k_1^{(0)}-k_0^{(0)}}\widehat{f}_0 \in \left(\frac{\mathrm{Fil}^{n-1}\widehat{M}}{\mathrm{Fil}^{n-1}S\cdot\widehat{M}}\right)_{\widetilde{\omega}^{k_1^{(0)}}}$, there exists $\widehat{f}_1$ such that

$$\widehat{f}_1 = \begin{pmatrix} 0 \\ u^{e-(k_{n-2}^{(0)}-k_1^{(0)})}\sum_{k=0}^{n-2} y_{n-2,k}E(u)^k \\ \vdots \\ \sum_{k=0}^{n-2} y_{1,k}E(u)^k \\ u^{k_1^{(0)}-k_0^{(0)}}\sum_{k=0}^{n-2} y_{0,k}E(u)^k \end{pmatrix} \in \left(\frac{\mathrm{Fil}^{n-1}\widehat{M}}{\mathrm{Fil}^{n-1}S\widehat{M}}\right)_{\widetilde{\omega}^{k_1^{(0)}}},$$

where $y_{i,j} \in \mathcal{O}_E$. By Lemma 3.3.33, we have $y_{i,0} = 0$ for all i: otherwise, both $u^{k_1^{(0)}-k_0^{(0)}}\widehat{f}_0$ and $\widehat{f}_1$ belong to $X^{(0)}_{\widetilde{\omega}^{k_1^{(0)}}} - X^{(1)}_{\widetilde{\omega}^{k_1^{(0)}}}$ which violates (3.3.34). Since u^e is an elementary divisor of $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ by (ii) of Lemma 3.3.35, we may also assume $y_{1,1} = 1$. Hence, by the obvious change of basis we get $\widehat{f}_1$ as follows:

$$\widehat{f}_1 = E(u)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \left(\frac{\mathrm{Fil}^{n-1}\widehat{M}}{\mathrm{Fil}^{n-1}S\widehat{M}}\right)_{\widetilde{\omega}^{k_1^{(0)}}}.$$

By the same arguments, we get $\widehat{f}_i \in \left(\frac{\mathrm{Fil}^{n-1}\widehat{M}}{\mathrm{Fil}^{n-1}S\widehat{M}}\right)_{\widetilde{\omega}^{k_i^{(0)}}}$ for $i = 1, 2, \cdots, n-2$ as in the statement.

Note that the elements in the set

$$\{u^{k_{n-1}^{(0)}-k_0^{(0)}}\widehat{f}_0, E(u)u^{k_{n-1}^{(0)}-k_0^{(0)}}\widehat{f}_0, \cdots, E(u)^{n-2}u^{k_{n-1}^{(0)}-k_0^{(0)}}\widehat{f}_0\}$$
$$\cup \{u^{k_{n-1}^{(0)}-k_1^{(0)}}\widehat{f}_1, E(u)u^{k_{n-1}^{(0)}-k_1^{(0)}}\widehat{f}_1, \cdots, E(u)^{n-3}u^{k_{n-1}^{(0)}-k_1^{(0)}}\widehat{f}_1\}$$
$$\cup \cdots \cup \{u^{k_{n-1}^{(0)}-k_{n-2}^{(0)}}\widehat{f}_{n-2}\}$$

are linearly independent in $X^{(0)}_{\widetilde{\omega}^{k_{n-1}^{(0)}}}$ over E, so that the set forms a basis for $X^{(0)}_{\widetilde{\omega}^{k_{n-1}^{(0)}}}$ by Lemma 3.3.33. Hence, $\widehat{f}_{n-1}$ is a linear combination of those elements over E. We have

$$u^{k_{n-1}^{(0)}-k_0^{(0)}}\left(\sum_{i=0}^{n-2} p^{n-2-i}E(u)^i\right)\widehat{f}_0 = \begin{pmatrix} -p^{n-1} \\ 0 \\ \vdots \\ 0 \\ \alpha u^{k_{n-1}^{(0)}-k_0^{(0)}}\sum_{i=0}^{n-2} p^{n-2-i}E(u)^i \end{pmatrix}.$$

Hence, we may let

$$\widehat{f}_{n-1} := \frac{1}{\alpha}u^{k_{n-1}^{(0)}-k_0^{(0)}}\left(\sum_{i=0}^{n-2} p^{n-2-i}E(u)^i\right)\widehat{f}_0 \in \left(\frac{\mathrm{Fil}^{n-1}\widehat{M}}{\mathrm{Fil}^{n-1}S\widehat{M}}\right)_{\widetilde{\omega}^{k_{n-1}^{(0)}}}$$

since $u^{(n-2)e+(k_{n-1}^{(0)}-k_0^{(0)})}$ is an elementary divisor for $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ by (ii) of Lemma 3.3.35. Moreover, $v_p\left(\frac{p^{n-1}}{\alpha}\right) > 0$ since $\mathrm{Fil}^{n-1}\mathcal{M} \subseteq u^{e-(k_{n-1}^{(0)}-k_0^{(0)})}\mathcal{M} \subseteq u\mathcal{M}$ by Proposition 3.3.32.

It is obvious that the $\widehat{f}_i$ mod $(\varpi_E, \mathrm{Fil}^p S)$ generate $\mathcal{M}/\mathrm{Fil}^{n-1}\mathcal{M}$ for $\mathcal{M}$ written as in Proposition 3.3.27. By Nakayama Lemma, we conclude that the $\widehat{f}_i$ generate $\widehat{\mathcal{M}}/\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$, which completes the proof. □

Corollary 3.3.37. *Keep the notation and assumptions of Proposition 3.3.36, and let*

$$(\lambda_{n-1}, \lambda_{n-2}, \cdots, \lambda_0) \in (\mathcal{O}_E)^n$$

be the Frobenius eigenvalues on the $(\widetilde{\omega}^{k_{n-1}^{(0)}}, \widetilde{\omega}^{k_{n-2}^{(0)}}, \cdots, \widetilde{\omega}^{k_0^{(0)}})$*-isotypic component of* $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\rho)$*. Then*

$$v_p(\lambda_i) = \begin{cases} v_p(\alpha) & \text{if } i = n-1 \\ (n-1)-i & \text{if } n-1 > i > 0 \\ (n-1)-v_p(\alpha) & \text{if } i = 0. \end{cases}$$

Proof. The proof goes parallel to the proof of [HLM17], Corollary 2.4.11. □

3.3.6 Reducibility of certain lifts

In this section, we let $1 \leq f \leq n$ and $e = p^f - 1$, and we prove that every potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and certain prescribed Galois types $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ is reducible. We emphasize that we only assume that $\overline{\rho}_0$ is generic (c.f. Definition 3.3.3) for the results in this section.

Proposition 3.3.38. *Assume that* $\overline{\rho}_0$ *is generic, and let* $(k_{n-1}, k_{n-2}, \cdots, k_0)$ *be an* n*-tuple of integers. Assume further that* $k_0 \equiv (p^{f-1} + p^{f-2} + \cdots + p + 1)c_0$ *modulo* (e) *and that* k_i *are pairwise distinct modulo* (e)*.*

Then every potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,0\}$ and Galois types $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ is an extension of a 1-dimensional potentially semi-stable lift of $\overline{\rho}_{0,0}$ with Hodge–Tate weight 0 and Galois type $\widetilde{\omega}_f^{k_0}$ by an $(n-1)$-dimensional potentially semi-stable lift of $\overline{\rho}_{n-1,1}$ with Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,1\}$ and Galois types $\bigoplus_{i=1}^{n-1}\widetilde{\omega}_f^{k_i}$.

Note that if $f=1$ then the assumption that $\overline{\rho}_0$ is generic implies that k_i are pairwise distinct modulo (e) by Lemma 3.3.10. In fact, we believe that this is true for any $1\le f\le n$, but this requires extra works as we did in Lemma 3.3.10. Since we will need the results in this section only when $f=1$, we will add the assumption that k_i are pairwise distinct modulo (e) in the proposition.

Proof. Let $\widehat{\mathcal{M}}\in\mathcal{O}_E\text{-Mod}_{\mathrm{dd}}^{n-1}$ be a strongly divisible module corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho:G_{\mathbf{Q}_p}\to\mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ and Hodge–Tate weights $\{-(n-1),-(n-2),\cdots,0\}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}})\otimes_{\mathcal{O}_E}\mathbf{F}\cong\overline{\rho}_0$. We also let $\mathcal{M}$ be the Breuil module corresponding to the mod p reduction of $\widehat{\mathcal{M}}$. $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1}\omega_f^{k_i}$) by Proposition 3.2.18.

We let $\underline{f}=(f_{n-1},f_{n-2},\cdots,f_0)$ (resp. $\underline{\widehat{f}}=(\widehat{f}_{n-1},\widehat{f}_{n-2},\cdots,\widehat{f}_0)$) be a framed system of generators for $\mathrm{Fil}^{n-1}\mathcal{M}$ (resp. for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$). We also let $\underline{e}=(e_{n-1},e_{n-2},\cdots,e_0)$ (resp. $\underline{\widehat{e}}=(\widehat{e}_{n-1},\widehat{e}_{n-2},\cdots,\widehat{e}_0)$) be a framed basis for $\mathcal{M}$ (resp. for $\widehat{\mathcal{M}}$). If $x=a_{n-1}e_{n-1}+\cdots+a_0e_0\in\mathcal{M}$, we will write $[x]_{e_i}$ for a_i for $i\in\{0,1,\cdots,n-1\}$. We define $[x]_{\widehat{e}_i}$ for $x\in\widehat{\mathcal{M}}$ in the obvious similar way. We may assume that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, and $\mathrm{Mat}_{\underline{e}}(N)$ are written as in (3.3.5), (3.3.6), and (3.3.7) respectively, and we do so.

By the equation (3.3.4), we deduce $r_0\equiv 0$ modulo (e) from our assumption on k_0. Recall that $p>n^2+2(n-3)$ by the generic condition. Since $0\le r_0\le(n-1)(p^f-1)/(p-1)$ by (ii) of Lemma 3.2.10, we conclude that $r_0=0$. Thus, we may let f_0 satisfy that $[f_0]_{e_i}=0$ if $0<i\le n-1$ and $[f_0]_{e_0}=1$, so that we can also let

$$\widehat{f}_0=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}.$$

Hence, we can also assume that $[\widehat{f}_j]_{\widehat{e}_0}=0$ for $0<j\le n-1$. We let $V_0=\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})\in\mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E})$ and $A_0=\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\phi_{n-1})\in\mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$.

We construct a sequence of framed bases $\{\underline{\widehat{e}}^{(m)}\}$ for $\widehat{\mathcal{M}}$ by change of basis, satisfying that

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})\in\mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E})\ \text{ and }\ \mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\phi_{n-1})\in\mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$$

converge to certain desired forms as m goes to ∞. We let $V^{(m)}\in\mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E})$ and $A^{(m)}\in\mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$ for a non-negative integer m. We may write

$$(x_{n-1}^{(m+1)}u^{[k_{n-1}-k_0]_f},x_{n-2}^{(m+1)}u^{[k_{n-2}-k_0]_f},\cdots,x_{m+1}^{(1)}u^{[k_{m+1}-k_0]_f},x_0^{(m+1)})$$

for the last row of $(A^{(m)})^{-1}$, where $x_0^{(m+1)}\in(S_{\mathcal{O}_E}^{\times})_0$ and $x_j^{(m+1)}\in(S_{\mathcal{O}_E})_0$ for $0<j\le n-1$. We define an $n\times n$-matrix $R^{(m+1)}$ as follows:

$$R^{(m+1)}=\begin{pmatrix}1&0&\cdots&0&0\\ 0&1&\cdots&0&0\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&\cdots&1&0\\ \frac{x_{n-1}^{(m+1)}}{x_0^{(m+1)}}u^{[k_{n-1}-k_0]_f}&\frac{x_{n-2}^{(m+1)}}{x_0^{(m+1)}}u^{[k_{n-2}-k_0]_f}&\cdots&\frac{x_1^{(m+1)}}{x_0^{(m+1)}}u^{[k_1-k_0]_f}&1\end{pmatrix}.$$

We also define

$$C^{(m+1)}=\begin{pmatrix} 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ y_{n-1}^{(m+1)}u^{[p^{-1}(k_{n-1}-k_0)]_f} & y_{n-2}^{(m+1)}u^{[p^{-1}(k_{n-2}-k_0)]_f} & \cdots & y_1^{(m+1)}u^{[p^{-1}(k_1-k_0)]_f} & 1 \end{pmatrix}$$

by the equation

$$R^{(m+1)}\cdot V^{(m)}\cdot C^{(m+1)}=V^{(m)}$$

where $y_j^{(m+1)}\in(S_{\mathcal{O}_E})_0$ for $0<j\leq n-1$. Note that the existence of such a matrix $C^{(m+1)}$ is obvious, since $p^{-1}k_0\equiv k_0$ modulo (e) by our assumption on k_0 immediately implies $[p^{-1}(k_j-k_0)]_f\leq [k_s-k_0]_f+[p^{-1}k_j-k_s]_f$. We also note that $R^{(m+1)}\in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E})$ and $C^{(m+1)}\in \mathrm{GL}_n^{\square,\prime\prime}(S_{\mathcal{O}_E})$.

Let $V^{(m+1)}=V^{(m)}$ for all $m\geq 0$. Assume that $V^{(m)}=\mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})$ and $A^{(m)}=\mathrm{Mat}_{\underline{\widehat{e}}^{(m)},\underline{\widehat{f}}^{(m)}}(\phi_{n-1})$, with respect to a framed basis $\underline{\widehat{e}}^{(m)}$ and a framed system of generators $\underline{\widehat{f}}^{(m)}$. If we let $\underline{\widehat{e}}^{(m+1)}=\underline{\widehat{e}}^{(m)}\cdot(R^{(m+1)})^{-1}$, then

$$\begin{aligned}\phi_{n-1}(\underline{\widehat{e}}^{(m+1)}V^{(m+1)}) &= \phi_{n-1}(\underline{\widehat{e}}^{(m)}(R^{(m+1)})^{-1}V^{(m+1)})\\ &= \phi_{n-1}(\underline{\widehat{e}}^{(m)}V^{(m)}C^{(m+1)})\\ &= \underline{\widehat{e}}^{(m)}A^{(m)}\phi(C^{(m+1)})\\ &= \underline{\widehat{e}}^{(m+1)}R^{(m+1)}\cdot A^{(m)}\cdot\phi(C^{(m+1)}).\end{aligned}$$

Hence, we get

$$V^{(m+1)}=\mathrm{Mat}_{\underline{\widehat{e}}^{(m+1)},\underline{\widehat{f}}^{(m+1)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})\ \text{ and }\ R^{(m+1)}\cdot A^{(m)}\cdot\phi(C^{(m+1)})=\mathrm{Mat}_{\underline{\widehat{e}}^{(m+1)},\underline{\widehat{f}}^{(m+1)}}(\phi_{n-1}),$$

where $\underline{\widehat{f}}^{(m+1)}:=\underline{\widehat{e}}^{(m+1)}V^{(m+1)}$.

We compute the matrix product $A^{(m+1)}:=R^{(m+1)}\cdot A^{(m)}\cdot\phi(C^{(m+1)})$ as it follows. If we let $A^{(m)}=\left(\alpha_{i,j}^{(m)}u^{[k_j-k_i]_f}\right)_{0\leq i,j\leq n-1}$ for $\alpha_{i,j}^{(m)}\in(S_{\mathcal{O}_E})_0$ if $i\neq j$ and $\alpha_{i,i}^{(m)}\in(S_{\mathcal{O}_E}^{\times})_0$, then

$$A^{(m+1)}=\left(\alpha_{i,j}^{(m+1)}u^{[k_j-k_i]_f}\right)_{0\leq i,j\leq n-1}\in\mathrm{GL}_n^{\square}(S_{\mathcal{O}_E}) \tag{3.3.39}$$

where $\alpha_{i,j}^{(m+1)}u^{[k_j-k_i]_f}$ is described as follows:

$$\begin{cases}\alpha_{i,j}^{(m)}u^{[k_j-k_i]_f}+\alpha_{i,0}^{(m)}u^{[k_0-k_i]_f}\phi(y_j^{(m+1)})u^{p[p^{-1}(k_j-k_0)]_f} & \text{if } i>0 \text{ and } j>0;\\ \alpha_{i,0}^{(m)}u^{[k_0-k_i]_f} & \text{if } i>0 \text{ and } j=0;\\ \frac{1}{x_0^{(m+1)}}\phi(y_j^{(m+1)})u^{p[p^{-1}(k_j-k_0)]_f} & \text{if } i=0 \text{ and } j>0;\\ \frac{1}{x_0^{(m+1)}} & \text{if } i=0 \text{ and } j=0.\end{cases}$$

Let $V^{(0)}=V_0$ and $A^{(0)}=A_0$. We apply the algorithm above to $V^{(0)}$ and $A^{(0)}$. By the algorithm above, we have two matrices $V^{(m)}$ and $A^{(m)}$ for each $m\geq 0$. We claim that

$$\begin{cases}\alpha_{i,j}^{(m+1)}-\alpha_{i,j}^{(m)}\in u^{(1+p+\cdots+p^m)e}S_{\mathcal{O}_E} & \text{if } i>0 \text{ and } j>0;\\ \alpha_{i,j}^{(m+1)}=\alpha_{i,j}^{(m)} & \text{if } i>0 \text{ and } j=0;\\ \alpha_{i,j}^{(m+1)}\in u^{(1+p+\cdots+p^m)e}S_{\mathcal{O}_E} & \text{if } i=0 \text{ and } j>0;\\ \alpha_{i,j}^{(m+1)}-\alpha_{i,j}^{(m)}\in u^{(1+p+\cdots+p^{m-1})e}S_{\mathcal{O}_E} & \text{if } i=0 \text{ and } j=0.\end{cases}$$

It is obvious that the case $i > 0$ and $j = 0$ from the computation (3.3.39). For the case $i = 0$ and $j > 0$ we induct on m. Note that $p[p^{-1}(k_j - k_0)]_f - [k_j - k_0]_f = p([p^{-1}k_j]_f - k_0) - (k_j - k_0) \geq e$ if $j > 0$. From the computation (3.3.39) again, it is obvious that it is true for $m = 0$. Assume that it holds for m. This implies that $x_j^{(m+1)} \in u^{(1+p+\cdots+p^{m-1})e} S_{\mathcal{O}_E}$ for $0 < j \leq n-1$ and so $y_j^{(m+1)} \in u^{(1+p+\cdots+p^{m-1})e} S_{\mathcal{O}_E}$. Since $\phi(y_j^{(m+1)}) u^{p[p^{-1}(k_j-k_0)]_f - [k_j-k_0]_f} \in u^{(1+p+\cdots+p^m)e} S_{\mathcal{O}_E}$, by the computation (3.3.39) we conclude that the case $i = 0$ and $j > 0$ holds. The case $i > 0$ and $j > 0$ follows easily from the case $i = 0$ and $j > 0$, since $[p^{-1}(k_j - k_0)]_f + [k_0 - k_i]_f - [k_j - k_i]_f = p([p^{-1}k_j]_f - k_0) + e + k_0 - k_i - [k_j - k_i]_f \geq p[p^{-1}k_j]_f - k_j - (p-1)k_0 \geq e$. Finally, we check the case $i = 0$ and $j = 0$. We also induct on m for this case. It is obvious that it holds for $m = 0$. Note that $R^{(m+1)} \equiv I_n$ modulo $u^{(1+p+\cdots+p^{m-1})e} S_{\mathcal{O}_E}$. Since $A^{(m+1)} = R^{(m+1)} \cdot A^{(m)} \cdot \phi(C^{(m+1)})$, we conclude that the case $i = 0$ and $j = 0$ holds.

The previous claim says the limit of $A^{(m)}$ exists (entrywise), say $A^{(\infty)}$. By definition, we have $V^{(\infty)} = V^{(m)}$ for all $m \geq 0$. In other words, there exist a framed basis $\underline{\widehat{e}}^{(\infty)}$ for $\widehat{\mathcal{M}}$ and a framed system of generators $\underline{\widehat{f}}^{(\infty)}$ for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$ such that

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(\infty)}, \underline{\widehat{f}}^{(\infty)}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}) = V^{(\infty)} \in \mathrm{M}_n^{\square,\prime}(S_{\mathcal{O}_E})$$

and

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(\infty)}, \underline{\widehat{f}}^{(\infty)}}(\phi_{n-1}) = A^{(\infty)} \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E}).$$

Note that $(V^{(\infty)})_{i,j} = 0$ if either $i = 0$ and $j > 0$ or $i > 0$ and $j = 0$, and that $(A^{(\infty)})_{i,j} = 0$ if $i = 0$ and $j > 0$.

Since $\underline{\widehat{e}}^{(\infty)}$ is a framed basis for $\widehat{\mathcal{M}}$, we may write

$$\mathrm{Mat}_{\underline{\widehat{e}}^{(\infty)}}(N) = \left(\gamma_{i,j} u^{[k_j - k_i]_f}\right)_{0 \leq i,j \leq n-1} \in \mathrm{M}_n^{\square}(S_{\mathcal{O}_E})$$

for the matrix of the monodromy operator of $\widehat{\mathcal{M}}$ where $\gamma_{i,j} \in (S_{\mathcal{O}_E})_0$, and let

$$A^{(\infty)} = \left(\alpha_{i,j}^{(\infty)} u^{[k_j - k_i]_f}\right)_{0 \leq i,j \leq n-1} \in \mathrm{GL}_n^{\square}(S_{\mathcal{O}_E}).$$

We claim that $\gamma_{0,j} = 0$ for $n - 1 \geq j > 0$. Recall that $\alpha_{0,j}^{(\infty)} = 0$ for $j > 0$, and write $\underline{\widehat{f}}^{(\infty)} = (\widehat{f}_{n-1}^{(\infty)}, \widehat{f}_{n-2}^{(\infty)}, \cdots, \widehat{f}_0^{(\infty)})$ and $\underline{\widehat{e}}^{(\infty)} = (\widehat{e}_{n-1}^{(\infty)}, \widehat{e}_{n-2}^{(\infty)}, \cdots, \widehat{e}_0^{(\infty)})$. We also write

$$\widehat{f}_j^{(\infty)} = \sum_{i=1}^{n-1} \beta_{i,j}^{(\infty)} u^{[p^{-1}k_j - k_i]} \widehat{e}_i^{(\infty)}$$

where $\beta_{i,j}^{(\infty)} \in (S_{\mathcal{O}_E})_0$, for each $0 < j \leq n-1$. From the equation

$$[cN\phi_{n-1}(\widehat{f}_j^{(\infty)})]_{\widehat{e}_0^{(\infty)}} = [\phi_{n-1}(E(u)N(\widehat{f}_j^{(\infty)}))]_{\widehat{e}_0^{(\infty)}}$$

for $n - 1 \geq j > 0$, we have the identity

$$\sum_{i=1}^{n-1} \alpha_{i,j}^{(\infty)} u^{[k_j - k_i]_f + [k_i - k_0]_f} \gamma_{0,i} = p \sum_{i=1}^{n-1} \beta_{i,j}^{(\infty)} u^{p[p^{-1}k_j - k_i]_f + p[k_i - k_0]_f} \phi(\gamma_{0,i}) \alpha_{0,0}^{(\infty)} \tag{3.3.40}$$

for each $n - 1 \geq j > 0$. Choose an integer s such that $\mathrm{ord}_u(\gamma_{0,s} u^{[k_s - k_0]_f}) \leq \mathrm{ord}_u(\gamma_{0,i} u^{[k_i - k_0]_f})$ for all $n - 1 \geq i > 0$, and consider the identity (3.3.40) for $j = s$. Then the identity (3.3.40) induces

$$\alpha_{s,s}^{(\infty)} u^{[k_s - k_0]_f} \gamma_{0,s} \equiv 0$$

modulo $(u^{\mathrm{ord}_u(\gamma_{0,s})+[k_s-k_0]_f+1})$. Note that $\alpha_{s,s}^{(\infty)} \in S_{\mathcal{O}_E}^{\times}$, so that we get $\gamma_{0,s}=0$. Recursively, we conclude that $\gamma_{0,j}=0$ for all $0<j\leq n-1$.

Finally, it is now easy to check that $(\widehat{e}_{n-1}^{(\infty)}, \widehat{e}_{n-2}^{(\infty)}, \cdots, \widehat{e}_1^{(\infty)})$ determines a strongly divisible modules of rank $n-1$, that is a submodule of $\widehat{\mathcal{M}}$. This completes the proof. □

Corollary 3.3.41. *Fix a pair of integers (i_0, j_0) with $0\leq j_0\leq i_0\leq n-1$. Assume that $\overline{\rho}_0$ is generic, and let $(k_{n-1}, k_{n-2}, \cdots, k_0)$ be an n-tuple of integers. Assume further that*

$$k_i=(p^{f-1}+p^{f-2}+\cdots+p+1)c_i$$

for $i>i_0$ and for $i<j_0$ and that the k_i are pairwise distinct modulo (e).

Then every potentially semi-stable lift ρ of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois types $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ is a successive extension

$$\rho\cong\begin{pmatrix}\rho_{n-1,n-1} & \cdots & * & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & \rho_{i_0+1,i_0+1} & * & * & \cdots & * \\ & & & \rho_{i_0,j_0} & * & \cdots & * \\ & & & & \rho_{j_0-1,j_0-1} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & \rho_{0,0}\end{pmatrix}$$

where

- *$\rho_{i,i}$ is a 1-dimensional potentially semi-stable lift of $\overline{\rho}_{i,i}$ with Hodge–Tate weights $-i$ and Galois type $\widetilde{\omega}^{k_i^{i_0,j_0}}$ for $n-1\geq i>i_0$ and for $j_0>i\geq 0$;*
- *ρ_{i_0,j_0} is a (i_0-j_0+1)-dimensional potentially semi-stable lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0, -i_0+1, \cdots, -j_0\}$ and Galois types $\bigoplus_{i=j_0}^{i_0}\widetilde{\omega}^{k_i^{i_0,j_0}}$.*

Proof. Proposition 3.3.38 implies this corollary recursively. Let $\mathcal{M}\in\mathbf{F}\text{-BrMod}_{\mathrm{dd}}^{n-1}$ be a Breuil module corresponding to the mod p reduction of a strongly divisible module $\widehat{\mathcal{M}}\in\mathcal{O}_E\text{-Mod}_{\mathrm{dd}}^{n-1}$ corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho: G_{\mathbf{Q}_p}\to\mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ and Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}})\otimes_{\mathcal{O}_E}\mathbf{F}\cong\overline{\rho}_0$. Both $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1}\widetilde{\omega}_f^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1}\omega_f^{k_i}$) by Proposition 3.2.18. We may assume that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$, $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1})$, and $\mathrm{Mat}_{\underline{e}}(N)$ are written as in (3.3.5), (3.3.6), and (3.3.7) respectively, and we do so.

By the equation (3.3.4), it is easy to see that $r_i=(p^{f-1}+p^{f-2}+\cdots+p+1)i$ for $i>i_0$ and for $i<j_0$, by our assumption on k_i. By Proposition 3.3.38, there exists an $(n-1)$-dimensional subrepresentation $\rho'_{n-1,1}$ of ρ whose quotient is $\rho_{0,0}$ which is a potentially semi-stable lift of $\overline{\rho}_{0,0}$ with Hodge–Tate weight 0 and Galois type $\widetilde{\omega}_f^{k_0}$. Now consider $\rho'_{n-1,1}\otimes\varepsilon^{-1}$. Apply Proposition 3.3.38 to $\rho'_{n-1,1}\otimes\varepsilon^{-1}$. Recursively, one can readily check that ρ has subquotients $\rho_{i,i}$ for $0\leq i\leq j_0-1$. Considering $\rho^{\vee}\otimes\varepsilon^{n-1}$, one can also readily check that ρ has subquotients $\rho_{i,i}$ lifting $\overline{\rho}_{i,i}$ for $n-1\geq i\geq i_0+1$. □

The results in Corollary 3.3.41 reduce many of our computations for the main results on the Galois side.

3.3.7 Main results on the Galois side

In this section, we state and prove the main local results on the Galois side, that connects the Fontaine–Laffaille parameters of $\overline{\rho}_0$ with the Frobenius eigenvalues of certain potentially semi-stable lifts of $\overline{\rho}_0$.

Throughout this section, we assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic. We also fix $f = 1$ and $e = p - 1$.

Fix $i_0, j_0 \in \mathbf{Z}$ with $0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1$, and define the n-tuple of integers

$$(r_{n-1}^{i_0,j_0},\ r_{n-2}^{i_0,j_0},\ \cdots,\ r_0^{i_0,j_0})$$

as follows:

$$r_i^{i_0,j_0} := \begin{cases} i & \text{if } i_0 \neq i \neq j_0; \\ j_0 + 1 & \text{if } i = i_0; \\ i_0 - 1 & \text{if } i = j_0. \end{cases} \tag{3.3.42}$$

We note that if we replace n by $i_0 - j_0 + 1$ in the definition of $r_i^{(0)}$ in (3.3.20) then we have the identities:

$$r_{j_0+i}^{i_0,j_0} = j_0 + r_i^{(0)} \tag{3.3.43}$$

for all $0 \leq i \leq i_0 - j_0$. In particular, $r_i^{n-1,0} = r_i^{(0)}$ for all $0 \leq i \leq n - 1$.

From the equation $k_i^{i_0,j_0} \equiv c_i + i - r_i^{i_0,j_0} \bmod (e)$ (c.f. Lemma 3.3.10, (i)), this tuple immediately determines an n-tuple $(k_{n-1}^{i_0,j_0},\ k_{n-2}^{i_0,j_0},\ \cdots,\ k_0^{i_0,j_0})$ of integers mod (e), which will determine the Galois types of our representations. We set

$$k_i^{i_0,j_0} := c_i + i - r_i^{i_0,j_0}$$

for all $i \in \{0, 1, \cdots, n-1\}$.

The following is the main result on the Galois side.

Theorem 3.3.44. *Let i_0, j_0 be integers with $0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1$. Assume that $\overline{\rho}_0$ is generic and that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic. Let $(\lambda_{n-1}^{i_0,j_0}, \lambda_{n-2}^{i_0,j_0}, \cdots, \lambda_0^{i_0,j_0}) \in (\mathcal{O}_E)^n$ be the Frobenius eigenvalues on the $(\widetilde{\omega}^{k_{n-1}^{i_0,j_0}}, \widetilde{\omega}^{k_{n-2}^{i_0,j_0}}, \cdots, \widetilde{\omega}^{k_0^{i_0,j_0}})$-isotypic components of $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\rho_0)$ where ρ_0 is a potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, -1, 0\}$ and Galois types $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i^{i_0,j_0}}$.*

Then the Fontaine–Laffaille parameter $\mathrm{FL}_n^{i_0,j_0}$ associated to $\overline{\rho}_0$ is computed as follows:

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \overline{\left(\frac{p^{[(n-1)-\frac{i_0+j_0}{2}](i_0-j_0-1)}}{\prod_{i=j_0+1}^{i_0-1} \lambda_i^{i_0,j_0}} \right)} \in \mathbb{P}^1(\mathbf{F}).$$

We first prove Theorem 3.3.44 for the case $(i_0, j_0) = (n-1, 0)$ in the following proposition, which is the key first step to prove Theorem 3.3.44.

Proposition 3.3.45. *Keep the assumptions and notation of Theorem 3.3.44, and assume further $(i_0, j_0) = (n-1, 0)$. Then Theorem 3.3.44 holds.*

Recall that $(k_{n-1}^{n-1,0}, \cdots, k_0^{n-1,0})$ in Proposition 3.3.45 is the same as $(k_{n-1}^{(0)}, \cdots, k_0^{(0)})$ in (3.3.20). To lighten the notation, we let $k_i = k_i^{n-1,0}$ and $\lambda_i = \lambda_i^{n-1,0}$ during the proof of Proposition 3.3.45. We heavily use the results in Sections 3.3.3, 3.3.4 and 3.3.5 to prove this proposition.

Proof. Let $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-}\mathrm{Mod}_{\mathrm{dd}}^{n-1}$ be a strongly divisible module corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho_0 : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i}$ and Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ such that $\mathrm{T}_{\mathrm{st}}^{\mathbf{Q}_p,n-1}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbf{F} \cong \overline{\rho}_0$. We also let $\mathcal{M}$ be the Breuil module corresponding to the mod p reduction of $\widehat{\mathcal{M}}$. $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1} \widetilde{\omega}^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1} \omega^{k_i}$) by Proposition 3.2.18.

We let $\underline{\widehat{f}} = (\widehat{f}_{n-1}, \widehat{f}_{n-2}, \cdots, \widehat{f}_1, \widehat{f}_0)$ be a framed system of generators for $\mathrm{Fil}^{n-1}\widehat{\mathcal{M}}$, and $\underline{\widehat{e}} = (\widehat{e}_{n-1}, \widehat{e}_{n-2}, \cdots, \widehat{e}_1, \widehat{e}_0)$ be a framed basis for $\widehat{\mathcal{M}}$. We may assume that $\mathrm{Mat}_{\underline{\widehat{e}},\underline{\widehat{f}}}(\mathrm{Fil}^{n-1}\widehat{\mathcal{M}})$ is described as in Proposition 3.3.36, and we do so.

Define $\alpha_i \in \mathbf{F}^\times$ by the condition $\phi_{n-1}(\widehat{f}_i) \equiv \widetilde{\alpha}_i \widehat{e}_i$ modulo (ϖ_E, u) for all $i \in \{0, 1, \cdots, n-1\}$. There exists a framed basis $\underline{e} = (e_{n-1}, e_{n-2}, \cdots, e_0)$ for $\mathcal{M}$ and a framed system of generators $\underline{f} = (f_{n-1}, f_{n-2}, \cdots, f_0)$ for $\mathrm{Fil}^{n-1}\mathcal{M}$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^{n-1}\mathcal{M})$ is given as in Proposition 3.3.32 and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_{n-1}) = \left(\alpha_{i,j} u^{[k_j - k_i]_1}\right) \in \mathrm{GL}_n^{\square}(\overline{S})$$

for some $\alpha_{i,j} \in \overline{S}_0$ with $\alpha_{i,i} \equiv \alpha_i \bmod (u^e)$.

Recall that $\widehat{f}_i = E(u)^i \widehat{e}_i$ for $i \in \{1, 2, \cdots, n-2\}$ by Proposition 3.3.36. Write $\phi_{n-1}(\widehat{f}_j) = \sum_{i=0}^{n-1} \widehat{\alpha}_{i,j} u^{[k_j - k_i]_1} \widehat{e}_i$ for some $\widehat{\alpha}_{i,j} \in S_0$. Then we get

$$s_0(\widehat{\alpha}_{i,i}) \equiv \frac{p^i \lambda_i}{p^{n-1}} \pmod{\varpi_E}$$

for $i \in \{1, 2, \cdots, n-2\}$ since $\phi_{n-1} = \frac{1}{p^{n-1}}\phi$ for the Frobenius ϕ on $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\rho_0)$, so that we have

$$\prod_{i=1}^{n-2} \widetilde{\alpha}_i \equiv \prod_{i=1}^{n-2} \frac{\lambda_i}{p^{n-1-i}} \pmod{\varpi_E}.$$

(Note that $\frac{\lambda_i}{p^{n-1-i}} \in \mathcal{O}_E^\times$ by Corollary 3.3.37.) This completes the proof, by applying the results in Proposition 3.3.32. □

We now prove Theorem 3.3.44, which is the main result on the Galois side.

Proof of Theorem 3.3.44. Recall from the identities in (3.3.43) that

$$(r_{i_0}^{i_0,j_0}, r_{i_0-1}^{i_0,j_0}, \cdots, r_{j_0}^{i_0,j_0}) = j_0 + (1, n'-2, n'-3, \cdots, 1, n'-2)$$

where $n' := i_0 - j_0 + 1$. Recall also that ρ_0 has a subquotient ρ_{i_0,j_0} that is a potentially semi-stable lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0, -(i_0-1), ..., -j_0\}$ and of Galois type $\bigoplus_{i=j_0}^{i_0} k_i^{i_0,j_0}$, by Corollary 3.3.41.

It is immediate that $\rho'_{i_0,j_0} := \rho_{i_0,j_0} \varepsilon^{-j_0} \widetilde{\omega}^{j_0}$ is another potentially semi-stable lift of $\overline{\rho}_{i_0,j_0}$ with Hodge–Tate weights $\{-(i_0-j_0), -(i_0-j_0-1), ..., 0\}$ and of Galois type $\bigoplus_{i=j_0}^{i_0} \widetilde{\omega}^{k_i^{i_0,j_0}+j_0}$. We let $(\eta_{i_0}, \eta_{i_0-1}, \cdots, \eta_{j_0}) \in (\mathcal{O}_E)^{i_0-j_0+1}$ (resp. $(\delta_{i_0}, \delta_{i_0-1}, \cdots, \delta_{j_0}) \in (\mathcal{O}_E)^{i_0-j_0+1}$) be the Frobenius eigenvalues on the $(\widetilde{\omega}^{k_{i_0}^{i_0,j_0}}, \widetilde{\omega}^{k_{i_0-1}^{i_0,j_0}}, \cdots, \widetilde{\omega}^{k_{j_0}^{i_0,j_0}})$-isotypic component of $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, i_0-j_0}(\rho_{i_0,j_0})$ (resp. on the $(\widetilde{\omega}^{k_{i_0}^{i_0,j_0}+j_0}, \widetilde{\omega}^{k_{i_0-1}^{i_0,j_0}+j_0}, \cdots, \widetilde{\omega}^{k_{j_0}^{i_0,j_0}+j_0})$-isotypic component of $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, i_0-j_0}(\rho'_{i_0,j_0})$). Then we have

$$p^{-j_0}\delta_i = \eta_i$$

for all $i \in \{j_0, j_0+1, \cdots, i_0\}$ and, by Proposition 3.3.45,

$$\mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0}) = \overline{\left[\left(\prod_{i=j_0+1}^{i_0-1} \delta_i\right) : p^{\frac{(i_0-j_0)(i_0-j_0-1)}{2}}\right]} \in \mathbb{P}^1(\mathbf{F}).$$

But we also have that

$$p^{n-1-(i_0-j_0)}\eta_i = \lambda_i^{i_0,j_0}$$

for all $i \in \{j_0, j_0+1, \cdots, i_0\}$ by Corollary 3.3.41. Hence, we have $\delta_i = p^{-(n-1-i_0)}\lambda_i^{i_0,j_0}$ for all $i \in \{j_0, j_0+1, \cdots, i_0\}$ and we conclude that

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0}) = \overline{\left[\left(\prod_{i=j_0+1}^{i_0-1} \lambda_i^{i_0,j_0}\right) : p^{[(n-1)-\frac{i_0+j_0}{2}](i_0-j_0-1)}\right]} \in \mathbb{P}^1(\mathbf{F}).$$

(Note that $\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \mathrm{FL}_{i_0-j_0+1}^{i_0-j_0,0}(\overline{\rho}_{i_0,j_0})$ by Lemma 3.3.17.) □

In the following corollary, we prove that the Weil–Deligne representation $\mathrm{WD}(\rho_0)$ associated to ρ_0 still contains Fontaine–Laffaille parameters. As we will see later, we will transport this information to the automorphic side via local Langlands correspondence.

Corollary 3.3.46. *Keep the assumptions and notation of Theorem 3.3.44.*

Then ρ_0 is, in fact, potentially crystalline and

$$\mathrm{WD}(\rho_0)^{\mathrm{F-ss}} = \mathrm{WD}(\rho_0) \cong \bigoplus_{i=0}^{n-1} \Omega_i$$

where $\Omega_i : \mathbf{Q}_p^\times \to E^\times$ is defined by $\Omega_i := \mathrm{U}_{\lambda_i^{i_0,j_0}/p^{n-1}} \cdot \widetilde{\omega}^{k_i^{i_0,j_0}}$ for all $i \in \{0, 1, \cdots, n-1\}$. Moreover,

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \overline{\left(\frac{\prod_{i=j_0+1}^{i_0-1} \Omega_i^{-1}(p)}{p^{\frac{(i_0+j_0)(i_0-j_0-1)}{2}}} \right)} \in \mathbb{P}^1(\mathbf{F}).$$

Proof. Notice that ϕ is diagonal on $D := \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p}(\rho_0)$ with respect to a framed basis $\underline{e} := (e_{n-1}, \cdots, e_0)$ (which satisfies $ge_i = \widetilde{\omega}^{k_i^{i_0,j_0}}(g)e_i$ for all i and for all $g \in \mathrm{Gal}(K/\mathbf{Q}_p)$) since $\widetilde{\omega}^{k_i^{i_0,j_0}}$ are all distinct. Hence, we have $\mathrm{WD}(\rho_0) = \mathrm{WD}(\rho_0)^{\mathrm{F-ss}}$. Similarly, since the descent data action on D commutes with the monodromy operator N, it is immediate that $N = 0$.

From the definition of $\mathrm{WD}(\rho_0)$ (c.f. [CDT99]), the action of $W_{\mathbf{Q}_p}$ on D can be described as follows: let $\alpha(g) \in \mathbf{Z}$ be determined by $\bar{g} = \phi^{\alpha(g)}$, where ϕ is the arithmetic Frobenius in $G_{\mathbf{F}_p}$ and $\bar{g}$ is the image under the surjection $W_{\mathbf{Q}_p} \twoheadrightarrow \mathrm{Gal}(K/\mathbf{Q}_p)$. Then

$$\mathrm{WD}(\rho_0)(g) \cdot e_i = \left(\frac{\lambda_i^{i_0,j_0}}{p^{n-1}} \right)^{-\alpha(g)} \cdot \widetilde{\omega}^{k_i^{i_0,j_0}}(g) \cdot e_i$$

for all $i \in \{0, 1, \cdots, n-1\}$. (Recall that $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p, n-1}(\rho_0) = \mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p}(\rho_0 \otimes \varepsilon^{-(n-1)})$, so that the $\frac{\lambda_i^{i_0,j_0}}{p^{n-1}}$ are the Frobenius eigenvalues of the Frobenius on D.) Write Ω_i for the eigen-character with respect to e_i.

Recall that we identify the geometric Frobenius with the uniformizers in $\mathbf{Q}_p^\times$ (by our normalization of class field theory), so that $\Omega_i(p) = \frac{p^{n-1}}{\lambda_i^{i_0,j_0}}$ which completes the proof by applying Theorem 3.3.44. □

3.4 Local automorphic side

In this section, we establish several results concerning representation theory of GL_n, that will be applied to the proof of our main results on mod p local-global compatibility, Theorem 3.5.44. The main results in this section are the non-vanishing result, Corollary 3.4.37, as well as the intertwining identity in characteristic 0, Theorem 3.4.71.

We start this section by fixing some notation. Let $G := \mathrm{GL}_{n/\mathbf{Z}_p}$ and T be the maximal split torus consisting of diagonal matrices. We fix a Borel subgroup $B \subseteq G$ consisting of upper-triangular matrices, and let $U \subseteq B$ be the maximal unipotent subgroup. Let Φ^+ denote the set of positive roots with respect to (B, T), and $\Delta = \{\alpha_k\}_{1 \le k \le n-1}$ the subset of positive simple roots. Let $X(T)$ and $X^\vee(T)$ denote the abelian group of characters and cocharacters respectively. We often say a weight for an element in $X(T)$, and write $X(T)_+$ for the set of dominant weights. The set Φ^+ induces a partial order on $X(T)$: for $\lambda, \mu \in X(T)$ we say that $\lambda \le \mu$ if $\mu - \lambda \in \sum_{\alpha \in \Phi^+} \mathbf{Z}_{\ge 0}\alpha$. We will also write $\lambda < \mu$ if $\lambda \le \mu$ and $\lambda \ne \mu$.

We use the n-tuple of integers $\lambda = (d_1, d_2, \cdots, d_n)$ to denote the character associated to the weight $\sum_{k=1}^n d_k \epsilon_k$ of T where for each $1 \le i \le n$ ϵ_i is a weight of T defined by

$$\mathrm{diag}(x_1, x_2, \cdots, x_n) \overset{\epsilon_i}{\mapsto} x_i.$$

We will often use the following weight

$$\eta := (n-1, n-2, \cdots, 1, 0).$$

We let $\overline{G}$, $\overline{B}$, $\cdots$ be the base change to $\mathbf{F}_p$ of G, B, $\cdots$ respectively. The Weyl group of G, denoted by W, has a standard lifting inside G as the group of permutation matrix, likewise with $\overline{G}$. We denote the longest Weyl element by w_0 which is lifted to the antidiagonal permutation matrix in G or $\overline{G}$. We use the notation s_i for the simple reflection corresponding to $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$. We define the length $\ell(w)$ of $w \in W$ to be its minimal length of decomposition into product of s_i for $1 \leq i \leq n-1$. Given $A \in U(\mathbf{F}_p)$, we use A_α or equivalently $A_{i,j}$ to denote the (i,j)-entry of A, where $\alpha = \epsilon_i - \epsilon_j$ is the positive root corresponding to the pair (i,j) with $1 \leq i < j \leq n$.

Given a representation π of $G(\mathbf{F}_p)$, we use the notation π^μ for the $T(\mathbf{F}_p)$-eigenspace with the eigencharacter μ. Given an algebraic representation V of G or $\overline{G}$, we use the notation V_λ for the weight space of V associated to the weight λ. For any representation V of $\overline{G}$ or $G(\mathbf{F}_p)$ with coefficient in $\mathbf{F}_p$, we define

$$V_{\mathbf{F}} := V \otimes_{\mathbf{F}_p} \mathbf{F}$$

to be the extension of coefficient of V from $\mathbf{F}_p$ to $\mathbf{F}$. Similarly, we write $V_{\overline{\mathbf{F}}_p}$ for $V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$.

It is easy to observe that we can identify the character group of $T(\mathbf{F}_p)$ with $X(T)/(p-1)X(T)$. The natural action of the Weyl group W on T and thus on $T(\mathbf{F}_p)$ induces an action of W on the character group $X(T)$ and $X(T)/(p-1)X(T)$. We carefully distinguish the notation between them. We use the notation $w\lambda$ (resp. μ^w) for the action of W on $X(T)$ (resp. $X(T)/(p-1)X(T)$) satisfying

$$w\lambda(x) = \lambda(w^{-1}xw) \text{ for all } x \in T$$

and

$$\mu^w(x) = \mu(w^{-1}xw) \text{ for all } x \in T(\mathbf{F}_p).$$

As a result, without further comments, the notation $w\lambda$ is a weight but μ^w is just a character of $T(\mathbf{F}_p)$. There is another dot action of W on $X(T)$ defined by

$$w \cdot \lambda = w(\lambda + \eta) - \eta \text{ for all } \lambda \in X(T) \text{ and } w \in W.$$

The affine Weyl group $\widetilde{W}$ of G is defined as the semi-direct product of W and $X(T)$ with respect to the natural action of W on $X(T)$. We denote the image of $\lambda \in X(T)$ in $\widetilde{W}$ by t_λ. Then the dot action of W on $X(T)$ extends to the dot action of $\widetilde{W}$ on $X(T)$ through the following formula

$$\widetilde{w} \cdot \lambda = w \cdot (\lambda + p\mu)$$

if $\widetilde{w} = wt_\mu$. We use the notation $\lambda \uparrow \mu$ for $\lambda, \mu \in X(T)$ if $\lambda \leq \mu$ and $\lambda \in \widetilde{W} \cdot \mu$. We define a specific element of $\widetilde{W}$ by

$$\widetilde{w}_h := w_0 t_{-\eta}$$

following Section 4 of [LLL16].

We usually write K for $\mathrm{GL}_n(\mathbf{Z}_p)$ for brevity. We will also often use the following three open compact subgroups of $\mathrm{GL}_n(\mathbf{Z}_p)$: if we let $\mathrm{red} : \mathrm{GL}_n(\mathbf{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbf{F}_p)$ be the natural mod p reduction map, then

$$K(1) := \mathrm{Ker}(\mathrm{red}) \subset I(1) := \mathrm{red}^{-1}(U(\mathbf{F}_p)) \subset I := \mathrm{red}^{-1}(B(\mathbf{F}_p)) \subset K.$$

For each $\alpha \in \Phi^+$, there exists a subgroup U_α of G such that $xu_\alpha(t)x^{-1} = u_\alpha(\alpha(x)t)$ where $x \in T$ and $u_\alpha : \mathbb{G}_a \to U_\alpha$ is an isomorphism sending 1 to 1 in the entry corresponding to α. For each $\alpha \in \Phi^+$, we have the following equalities by [Jan03] II.1.19 (5) and (6):

$$u_\alpha(t) = \sum_{m \geq 0} t^m (X^{\mathrm{alg}}_{\alpha,m}). \tag{3.4.1}$$

where $X^{\mathrm{alg}}_{\alpha,m}$ is an element in the algebra of distributions on G supported at the origin $1 \in G$. The equation (3.4.1) is actually just the Taylor expansion with respect to t of the operation $u_\alpha(t)$ at the origin 1. We use the same notation $X^{\mathrm{alg}}_{\alpha,m}$ if G is replaced by $\overline{G}$.

We define the set of p-restricted weights as

$$X_1(T) := \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p-1 \text{ for all } \alpha \in \Delta\}$$

and the set of central weights as

$$X_0(T) := \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Delta\}.$$

We also define the set of p-regular weights as

$$X_1^{\mathrm{reg}}(T) := \{\lambda \in X(T) \mid 1 \leq \langle \lambda, \alpha^\vee \rangle \leq p-2 \text{ for all } \alpha \in \Delta\},$$

and in particular we have $X_1^{\mathrm{reg}}(T) \subsetneq X_1(T)$. We say that $\lambda \in X(T)$ *lies in the lowest p-restricted alcove* if

$$0 < \langle \lambda + \eta, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+. \tag{3.4.2}$$

We define a subset $\widetilde{W}^+$ of $\widetilde{W}$ as

$$\widetilde{W}^+ := \{\widetilde{w} \in \widetilde{W} \mid \widetilde{w} \cdot \lambda \in X(T)_+ \text{ for each } \lambda \text{ in the lowest } p\text{-restricted alcove}\}.$$

We define another subset $\widetilde{W}^{\mathrm{res}}$ of $\widetilde{W}$ as

$$\widetilde{W}^{\mathrm{res}} := \{\widetilde{w} \in \widetilde{W} \mid \widetilde{w} \cdot \lambda \in X_1(T) \text{ for each } \lambda \text{ in the lowest } p\text{-restricted alcove}\}. \tag{3.4.3}$$

In particular, we have the inclusion

$$\widetilde{W}^{\mathrm{res}} \subseteq \widetilde{W}^+.$$

For any weight $\lambda \in X(T)$, we let

$$H^0(\lambda) := \left(\mathrm{Ind}_{\overline{B}}^{\overline{G}} w_0\lambda\right)^{\mathrm{alg}}_{/\mathbf{F}_p}$$

be the associated dual Weyl module. Note by [Jan03], Proposition II.2.6 that $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T)_+$. Assume that $\lambda \in X(T)_+$, we write $F(\lambda) := \mathrm{soc}_{\overline{G}}(H^0(\lambda))$ for its irreducible socle as an algebraic representation (c.f. [Jan03] part II, section 2). When λ is running through $X_1(T)$, the $F(\lambda)$ exhaust all the irreducible representations of $G(\mathbf{F}_p)$. On the other hand, two weights $\lambda, \lambda' \in X_1(T)$ satisfies

$$F(\lambda) \cong F(\lambda')$$

as $G(\mathbf{F}_p)$-representation if and only if

$$\lambda - \lambda' \in (p-1)X_0(T).$$

If $\lambda \in X_1^{\mathrm{reg}}(T)$, then the structure of $F(\lambda)$ as a $G(\mathbf{F}_p)$-representation depends only on the image of λ in $X(T)/(p-1)X(T)$, namely as a character of $T(\mathbf{F}_p)$. Conversely, given a character μ of $T(\mathbf{F}_p)$ which lies in the image of

$$X_1^{\mathrm{reg}}(T) \to X(T)/(p-1)X(T),$$

its inverse image in $X_1^{\mathrm{reg}}(T)$ is uniquely determined up to a translation of $(p-1)X_0(T)$. In this case, we say that μ is p-regular. Whenever it is necessary for us to lift an element in $X(T)/(p-1)X(T)$ back into $X_1(T)$ (or maybe $X_1^{\mathrm{reg}}(T)$), we will clarify the choice of the lift.

Consider the standard Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} U_w wB = \bigsqcup_{w \in W} BwU_{w^{-1}}.$$

where U_w is the unique subgroup of U satisfying $BwB = U_w wB$ as schemes over $\mathbf{Z}_p$. The group U_w has a unique form of $\prod_{\alpha\in\Phi_w^+} U_\alpha$ for the subset Φ_w^+ of Φ^+ defined by $\Phi_w^+ = \{\alpha \in \Phi^+, w(\alpha) \in -\Phi^+\}$. (If $w = 1$, we understand $\prod_{\alpha\in\Phi_w^+} U_\alpha$ to be the trivial group 1.) We also have the following Bruhat decomposition:

$$G(\mathbf{F}_p) = \bigsqcup_{w\in W} B(\mathbf{F}_p)wB(\mathbf{F}_p) = \bigsqcup_{w\in W} U_w(\mathbf{F}_p)wB(\mathbf{F}_p) = \bigsqcup_{w\in W} B(\mathbf{F}_p)wU_{w^{-1}}(\mathbf{F}_p). \tag{3.4.4}$$

Given any integer x, recall that we use the notation $[x]_1$ to denote the only integer satisfying $0 \le [x]_1 \le p-2$ and $[x]_1 \equiv x \bmod (p-1)$. Given two non-negative integers m and k with $m \ge k$, we use the notation $c_{m,k}$ for the binomial number $\frac{m!}{(m-k)!k!}$. We use the notation $\bullet$ for composition of maps and, in particular, composition of elements in the group algebra $\mathbf{F}_p[G(\mathbf{F}_p)]$.

3.4.1 Jacobi sums in characteristic p

In this section we establish several fundamental properties of Jacobi sum operators on mod p principal series representations.

Definition 3.4.5. *A weight $\lambda \in X(T)$ is called* k-generic *for $k \in \mathbf{Z}_{>0}$ if for each $\alpha \in \Phi^+$ there exists $m_\alpha \in \mathbf{Z}$ such that*

$$m_\alpha p + k < \langle \lambda, \alpha^\vee \rangle < (m_\alpha + 1)p - k.$$

In particular, the n-tuple of integers $(a_{n-1}, \cdots, a_1, a_0)$ is called k-generic in the lowest alcove *if*

$$a_i - a_{i-1} > k \quad \forall\, 1 \le i \le n-1 \text{ and } a_{n-1} - a_0 < p - k.$$

Note that $(a_{n-1}, \cdots, a_0) - \eta$ lies the lowest restricted alcove in the sense of (3.4.2) if $(a_{n-1}, \cdots, a_0)$ is k-generic in the lowest alcove for some $k > 0$. Note also that the existence of a n-tuple of integers satisfying k-generic in the lowest alcove implies $p > n(k+1) - 1$.

We use the notation π for a general principal series representation:

$$\pi := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \mu_\pi = \{f : G(\mathbf{F}_p) \to \mathbf{F}_p \mid f(bg) = \mu_\pi(b) f(g) \quad \forall (b, g) \in B(\mathbf{F}_p) \times G(\mathbf{F}_p)\}$$

where μ_π is a mod p character of $T(\mathbf{F}_p)$. The action of $\mathrm{GL}_n(\mathbf{F}_p)$ on π is given by $(g \cdot f)(g') = f(g'g)$. We will assume throughout this article that μ_π is p-regular. By definition we have

$$\mathrm{cosoc}_{G(\mathbf{F}_p)}(\pi) = F(\mu_\pi) \text{ and } \mathrm{soc}_{G(\mathbf{F}_p)}(\pi) = F(\mu_\pi^{w_0}).$$

By Bruhat decomposition we can deduce that

$$\dim_{\mathbf{F}_p} \pi^{U(\mathbf{F}_p), \mu_\pi^w} = 1$$

for each $w \in W$. We denote by v_π a non-zero fixed vector in $\pi^{U(\mathbf{F}_p), \mu_\pi}$. We also consider the natural lift $\widetilde{\pi}^\circ$ of π defined as

$$\widetilde{\pi}^\circ := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \widetilde{\mu}_\pi = \{f : G(\mathbf{F}_p) \to \mathbf{Z}_p \mid f(bg) = \widetilde{\mu}_\pi(b) f(g) \quad \forall (b, g) \in B(\mathbf{F}_p) \times G(\mathbf{F}_p)\} \tag{3.4.6}$$

where $\widetilde{\mu}_\pi$ is the Teichmüller lift of μ_π.

Given $w \in W$ with $w \ne 1$ and $\underline{k} = (k_\alpha)_{\alpha\in\Phi_w^+} \in \{0, 1, \cdots, p-1\}^{|\Phi_w^+|}$, we define the Jacobi sum operators

$$S_{\underline{k},w} := \sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^+} A_\alpha^{k_\alpha} \right) A \cdot w \in \mathbf{F}_p[G(\mathbf{F}_p)]. \tag{3.4.7}$$

These Jacobi sum operators play a main role on the local automorphic side as a crucial computation tool. These operators already appeared in [CL76] for example.

For each $\alpha \in \Phi^+$ and each integer m satisfying $0 \leq m \leq p-2$, we define the operator

$$X_{\alpha,m} := \sum_{t \in \mathbf{F}_p} t^{p-1-m} u_\alpha(t) \in \mathbf{F}_p[U(\mathbf{F}_p)] \subseteq \mathbf{F}_p[G(\mathbf{F}_p)]. \tag{3.4.8}$$

The transition matrix between $\{u_\alpha(t) \mid t \in \mathbf{F}_p^\times\}$ and $\{X_{\alpha,m} \mid 0 \leq m \leq p-2\}$ is a Vandermonde matrix

$$\left(t^k\right)_{t \in \mathbf{F}_p^\times, 1 \leq k \leq p-1}$$

which has a non-zero determinant. Hence, we also have a converse formula

$$u_\alpha(t) = -\sum_{m=0}^{p-2} t^m X_{\alpha,m} \text{ for all } t \in \mathbf{F}_p. \tag{3.4.9}$$

By the equation (3.4.1), we note that we have the equality

$$X_{\alpha,m} = -\sum_{k \geq 0} X^{\mathrm{alg}}_{\alpha,m+(p-1)k}. \tag{3.4.10}$$

Lemma 3.4.11. *Fix $w \in W$ and $\alpha_0 = (i_0, j_0) \in \Phi^+_w$. Given a tuple of integers $\underline{k} = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{|\Phi^+_w|}$ satisfying*

$$k_{i_0,j} = 0 \text{ for all } (i_0, j) \in \Phi^+_w \text{ with } j \geq j_0 + 1, \tag{3.4.12}$$

we have

$$X_{\alpha_0,m} \bullet S_{\underline{k},w} = \begin{cases} (-1)^{m+1} c_{k_{\alpha_0},m} S_{\underline{k}',w} & \text{if } m \leq k_{\alpha_0} \\ 0 & \text{if } m > k_{\alpha_0} \end{cases}$$

where $\underline{k}' = (k'_\alpha)_{\alpha \in \Phi_w}$ satisfies

$$k'_\alpha = \begin{cases} k_{\alpha_0} - m & \text{if } \alpha = \alpha_0; \\ k_\alpha & \text{otherwise.} \end{cases}$$

Proof. We prove this lemma by direct computation.

$$\begin{aligned} X_{\alpha,m} \bullet S_{\underline{k},w} &= \sum_{t \in \mathbf{F}_p} t^{p-1-m} \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi^+_w} A_\alpha^{k_\alpha} \right) u_{\alpha_0}(t) A w \right) \\ &= \sum_{t \in \mathbf{F}_p} t^{p-1-m} \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi^+_w, \alpha \neq \alpha_0} A_\alpha^{k_\alpha} \right) (A_{\alpha_0} - t)^{k_{\alpha_0}} A w \right) \\ &= \sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi^+_w, \alpha \neq \alpha_0} A_\alpha^{k_\alpha} \right) \left(\sum_{t \in \mathbf{F}_p} t^{p-1-m} (A_{\alpha_0} - t)^{k_{\alpha_0}} \right) A w \end{aligned} \tag{3.4.13}$$

where the second equality follows from the change of variable $A \leftrightarrow u_{\alpha_0}(t) A$ and the assumption (3.4.12).

Notice that

$$\begin{aligned} \sum_{t \in \mathbf{F}_p} t^{p-1-m} (A_{\alpha_0} - t)^{k_{\alpha_0}} &= \sum_{t \in \mathbf{F}_p} t^{p-1-m} \left(\sum_{j=0}^{k_{\alpha_0}} (-1)^j c_{k_{\alpha_0},j} A_{k_{\alpha_0}}^{k_{\alpha_0}-j} t^j \right) \\ &= \sum_{j=0}^{k_{\alpha_0}} (-1)^j c_{k_{\alpha_0},j} A_{k_{\alpha_0}}^{k_{\alpha_0}-j} \left(\sum_{t \in \mathbf{F}_p} t^{p-1-m+j} \right), \end{aligned}$$

which can be easily seen to be

$$\begin{cases} (-1)^{m+1} c_{k_{\alpha_0},m} A_{k_{\alpha_0}}^{k_{\alpha_0}-m} & \text{if} \quad m \leq k_{\alpha_0} \\ 0 & \text{if} \quad m > k_{\alpha_0}. \end{cases} \tag{3.4.14}$$

The last computation (3.4.14) follows from the fact that

$$\sum_{t \in \mathbf{F}_p} t^\ell = \begin{cases} 0 & \text{if} \quad p-1 \nmid \ell; \\ -1 & \text{if} \quad p-1 \mid \ell \text{ and } \ell \neq 0. \end{cases} \tag{3.4.15}$$

Applying (3.4.14) back to (3.4.13) gives us the result. □

Lemma 3.4.16. *Fix $w \in W$ and $\alpha_0 = (i_0, j_0) \in \Phi_w^+$. Given a tuple of integers $\underline{k} = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{|\Phi_w^+|}$ satisfying*

$$k_{i_0,j} = 0 \text{ for all } (i_0, j) \in \Phi_w^+ \text{ with } j \geq j_0,$$

we have

$$u_{\alpha_0}(t) \bullet S_{\underline{k},w} = S_{\underline{k},w}.$$

Proof. By Lemma 3.4.11 we deduce that

$$X_{\alpha_0,m} \bullet S_{\underline{k},w} = \begin{cases} -S_{\underline{k},w} & \text{if} \quad m = 0 \\ 0 & \text{if} \quad 1 \leq m \leq p-2 \end{cases}$$

Therefore we conclude this lemma from (3.4.9). □

By the definition of principal series representations, we have the decomposition

$$\pi = \oplus_{w \in W} \pi_w \tag{3.4.17}$$

where $\pi_w \subset \pi|_{B(\mathbf{F}_p)}$ consists of the functions supported on a non-empty subset of the Bruhat cell

$$B(\mathbf{F}_p) w^{-1} B(\mathbf{F}_p) = B(\mathbf{F}_p) w^{-1} U_w(\mathbf{F}_p).$$

Proposition 3.4.18. *Fix a non-zero vector $v_\pi \in \pi^{U(\mathbf{F}_p), \mu_\pi}$. For each $w \in W$ with $w \neq 1$, the set*

$$\left\{ S_{\underline{k},w} v_\pi \mid \underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0, 1, \cdots, p-1\}^{|\Phi_w^+|} \right\}$$

forms a $T(\mathbf{F}_p)$-eigenbasis of π_w.

Proof. We have a decomposition $\pi_w = \oplus_{A \in U_w(\mathbf{F}_p)} \pi_{w,A}$ where $\pi_{w,A}$ is the subspace of π_w consisting of functions supported on $B(\mathbf{F}_p) w^{-1} A^{-1}$. It is easy to observe by the definition of parabolic induction that $\dim_{\mathbf{F}_p} \pi_{w,A} = 1$ and $\pi_{w,A}$ is generated by Awv_π.

We claim that the set of Jacobi sums with the Weyl element w, after being applied to v_π, differs from the set $\{Awv_\pi, A \in U_w(\mathbf{F}_p)\}$ by an invertible matrix. More precisely, for a fixed $w \in W$, the set of vectors

$$\{ S_{\underline{k},w} v_\pi \mid \underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0, 1, \cdots, p-1\}^{|\Phi_w^+|} \}$$

can be linearly represented by the set of vectors $\{Awv_\pi, A \in U_w(\mathbf{F}_p)\}$ through the matrix $\left(m_{\underline{k},A}\right)$ where

$$\underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0, 1, \cdots, p-1\}^{|\Phi_w^+|}, \qquad A \in U_w(\mathbf{F}_p)$$

and $m_{\underline{k},A} := \prod_{\alpha \in \Phi_w^+} A_\alpha^{k_\alpha}$. Note that this matrix is the $|\Phi_w^+|$-times tensor of the Vandermonde matrix

$$\left(\lambda^k\right)_{\lambda \in \mathbf{F}_p, 0 \leq k \leq p-1},$$

and therefore has a non-zero determinant. As a result, the matrix $\left(m_{\underline{k},A}\right)$ is invertible and $\{S_{\underline{k},w}v_\pi \mid 0 \le k_\alpha \le p-1 \quad \forall \alpha \in \Phi_w^+\}$ forms a basis of π_w.

The fact that this is a $T(\mathbf{F}_p)$-eigenbasis is immediate by the following calculation: if we let $x = \mathrm{diag}(x_1, x_2, \cdots, x_n)$

$$\begin{aligned}
x \bullet S_{\underline{k},w}v_\pi &= x \bullet \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+} A_\alpha^{k_\alpha} \right) A \ w \right) v_\pi \\
&= \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{(i,j) \in \Phi_w^+} A_{i,j}^{k_{i,j}} \right) xAx^{-1} \ w \right) \ \left(w^{-1}xw\right) \ v_\pi \\
&= \left(\sum_{B = xAx^{-1} \in U_w(\mathbf{F}_p)} \left(\prod_{(i,j) \in \Phi_w^+} (B_{i,j}x_jx_i^{-1})^{k_{i,j}} \right) B \ w \right) \ \left(w^{-1}xw\right) \ v_\pi \\
&= \mu_\pi(w^{-1}xw) \left(\prod_{(i,j) \in \Phi_w^+} (x_jx_i^{-1})^{k_{i,j}} \right) \left(\sum_{A \in U_w(\mathbf{F}_p)} \prod_{\alpha \in \Phi_w^+} A_\alpha^{k_\alpha} A \ w \right) v_\pi \\
&= (\mu_\pi^w \lambda)(x) S_{\underline{k},w}v_\pi,
\end{aligned}$$

where $\lambda(x) = \prod_{1 \le i < j \le n} (x_jx_i^{-1})^{k_{i,j}}$ and $B_{i,j} = A_{i,j}x_ix_j^{-1}$ for $1 \le i < j \le n$. □

We can further describe the action of $T(\mathbf{F}_p)$ on $S_{\underline{k},w}v_\pi$. By $\lfloor y \rfloor$ for $y \in \mathbf{R}$ we mean the floor function of y, i.e., the biggest integer less than or equal to y.

Lemma 3.4.19. *Let $\mu_\pi = (d_1, d_2, \cdots, d_{n-1}, d_n)$. If we write $(\ell_1, \ell_2 \cdots, \ell_{n-1}, \ell_n)$ for the $T(\mathbf{F}_p)$-eigencharacter of $S_{\underline{k},w}v_\pi$, then we have*

$$\ell_r \equiv d_{w^{-1}(r)} + \sum_{1 \le i < r} k_{i,r} - \sum_{r < j \le n} k_{r,j} \pmod{p-1}$$

for all $1 \le r \le n$, where $k_{i,j} = k_\alpha$ if $\alpha \in \Phi_w^+$ and (i,j) corresponds to α, and $k_{i,j} = 0$ otherwise.

In particular,

(i) *if $k_\alpha = 0$ for any $\alpha \in \Phi_w^+ \setminus \Delta$, then for all $1 \le r \le n$*

$$\ell_r \equiv d_{w^{-1}(r)} + (1 - \lfloor 1/r \rfloor)k_{r-1,r} - (1 - \lfloor 1/(n+1-r) \rfloor)k_{r,r+1} \pmod{p-1};$$

(ii) *if $w = w_0$ and $k_{i,j} = 0$ for any $2 \le i < j \le n$, then*

$$\ell_r \equiv \begin{cases} d_n - \sum_{j=2}^n k_{1,j} \pmod{p-1} & \text{if } r = 1; \\ d_{n+1-r} + k_{1,r} \pmod{p-1} & \text{if } 2 \le r \le n. \end{cases}$$

Proof. The first part of the Lemma is a direct calculation as shown at the end of the proof of Proposition 3.4.18. The second part follows trivially from the first part. □

Given any $w \in W$, we write $S_{\underline{0},w}$ for $S_{\underline{k},w}$ with $k_\alpha = 0$ for all $\alpha \in \Phi_w^+$.

Lemma 3.4.20. $\mathbf{F}_p[S_{\underline{0},w}v_\pi] = \pi^{U(\mathbf{F}_p),\mu_\pi^w}$.

Proof. Pick an arbitrary positive root α. If $\alpha \in \Phi_w^+$, then we have (since $u_\alpha(t) \in U_w(\mathbf{F}_p)$)

$$u_\alpha(t) \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right) = \left(\sum_{A \in U_w(\mathbf{F}_p)} A \right)$$

and therefore $u_\alpha(t)S_{\underline{0},w}v_\pi = S_{\underline{0},w}v_\pi$ for any $t \in \mathbf{F}_p$. On the other hand, if $\alpha \notin \Phi_w^+$, then

$$u_\alpha(t)\left(\sum_{A\in U_w(\mathbf{F}_p)} A\right) = \left(\sum_{A\in U_w(\mathbf{F}_p)} A\right) u'_\alpha(t)$$

and

$$u'_\alpha(t)wv_\pi = wu''_\alpha(t)v_\pi = wv_\pi$$

where $u'_\alpha(t) \in \prod_{\alpha\notin\Phi_w^+} \overline{U}_\alpha(\mathbf{F}_p)$ and $u''_\alpha(t) \in U(\mathbf{F}_p)$ are elements depending on x, w and α. Hence, $u_\alpha(t)S_{\underline{0},w}v_\pi = S_{\underline{0},w}v_\pi$ for any $t \in \mathbf{F}_p$ and any $\alpha \in \Phi^+$. So we conclude that $S_{\underline{0},w}v_\pi$ is $U(\mathbf{F}_p)$-invariant as $\{u_\alpha(t)\}_{\alpha\in\Phi^+,t\in\mathbf{F}_p}$ generate $U(\mathbf{F}_p)$.

Finally, we check that $x \cdot S_{\underline{0},w}v_\pi = \mu_\pi^w(x)S_{\underline{0},w}v_\pi$ for $x \in T(\mathbf{F}_p)$. But this is immediate from the following two easy computations:

$$x \bullet \left(\sum_{A\in U_w(\mathbf{F}_p)} A\right) = \left(\sum_{A\in U_w(\mathbf{F}_p)} A\right) \bullet x \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

and

$$xwv_\pi = w\left(w^{-1}xw\right)v_\pi = w\mu_\pi(w^{-1}xw)v_\pi = \mu_\pi^w(x)wv_\pi.$$

This completes the proof. □

Note that Proposition 3.4.18, Lemma 3.4.19, and Lemma 3.4.20 are very elementary and have essentially appeared in [CL76]. In this article, we formulate them and give quick proofs of them for the convenience.

Definition 3.4.21. *Given $\alpha, \alpha' \in \Phi^+$, we say that α is* strongly smaller than *α' with the notation*

$$\alpha \,\widetilde{\prec}\, \alpha'$$

if there exist $1 \le i \le j \le k \le n-1$ such that

$$\alpha = \sum_{r=i}^{j} \alpha_r \text{ and } \alpha' = \sum_{r=i}^{k} \alpha_r.$$

We call a subset Φ' of Φ^+ good *if it satisfies the following:*

(i) *if $\alpha, \alpha' \in \Phi'$ and $\alpha + \alpha' \in \Phi^+$, then $\alpha + \alpha' \in \Phi'$;*

(ii) *if $\alpha \in \Phi'$ and $\alpha \,\widetilde{\prec}\, \alpha'$, then $\alpha' \in \Phi'$.*

We associate a subgroup of U to Φ' by

$$U_{\Phi'} := \langle U_\alpha \mid \alpha \in \Phi' \rangle \tag{3.4.22}$$

and denote its reduction mod p by $\overline{U}_{\Phi'}$. We define U_1 to be the subgroup scheme of U generated by U_{α_r} for $2 \le r \le n-1$, and denote its reduction mod p by $\overline{U}_1$.

Example 3.4.23. *The following are two examples of good subsets of Φ^+, that will be important for us:*

$$\left\{\sum_{r=i}^{j} \alpha_r \mid 1 \le i < j \le n-1\right\} \text{ and } \left\{\sum_{r=i}^{j} \alpha_r \mid 2 \le i \le j \le n-1\right\}.$$

The subgroups of U associated with the two good subsets via (3.4.22) are $[U,U]$ and U_1 respectively.

We recall that we have defined $\pi_w \subsetneq \pi$ in (3.4.17) for each $w \in W$.

Proposition 3.4.24. *Let $\Phi' \subseteq \Phi^+$ be good. Pick an element $w \in W$ with $w \neq 1$. The following set of vectors*

$$\left\{ S_{\underline{k},w} v_\pi \mid \underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0,1,\cdots,p-1\}^{|\Phi_w^+|} \text{ with } k_\alpha = 0 \ \forall \alpha \in \Phi' \cap \Phi_w^+ \right\} \tag{3.4.25}$$

forms a basis of the subspace $\pi_w^{U_{\Phi'}(\mathbf{F}_p)}$ of π_w.

Proof. By Proposition 3.4.18, the Jacobi sums with the Weyl element w, after being applied to v_π, form a $T(\mathbf{F}_p)$-eigenbasis of π_w, and so we can firstly write any $U_{\Phi'}(\mathbf{F}_p)$-invariant vector v in π_w as a unique linear combination of Jacobi sums with the Weyl element w, namely

$$v = \sum_{\underline{k} \in \{0,\cdots,p-1\}^{|\Phi_w^+|}} C_{\underline{k},w} S_{\underline{k},w} v_\pi \text{ for some } C_{\underline{k},w} \in \mathbf{F}_p.$$

Assume that $C_{\underline{k},w} \neq 0$ for certain tuple of integers $\underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+}$ such that $k_\alpha > 0$ for some $\alpha \in \Phi' \cap \Phi_w^+$. We choose α_0 such that it is maximal with respect to the partial order $\widetilde{\prec}$ on Φ^+ for the property

$$C_{\underline{k},w} \neq 0, \qquad k_{\alpha_0} > 0, \qquad \text{and} \qquad \alpha_0 \in \Phi' \cap \Phi_w^+. \tag{3.4.26}$$

We may write v as follows:

$$v = \sum_{\substack{\underline{k} \in \{0,\cdots,p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} = 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi + \sum_{\substack{\underline{k} \in \{0,\cdots,p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} > 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi. \tag{3.4.27}$$

By the maximal assumption on α_0 we know that if $C_{\underline{k},w} \neq 0$ and $\alpha_0 \widetilde{\prec} \alpha$, then $k_\alpha = 0$. As a result, we deduce from Lemma 3.4.16 that

$$u_{\alpha_0}(t) \sum_{\substack{\underline{k} \in \{0,\cdots,p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} = 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi = \sum_{\substack{\underline{k} \in \{0,\cdots,p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} = 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi \tag{3.4.28}$$

for all $t \in \mathbf{F}_p$.

We define

$$\Phi_w^{\alpha_0,+} := \{\alpha \in \Phi_w^+ \mid \alpha_0 \widetilde{\prec} \alpha\} \ \text{ and } \ \Phi_w^{\alpha_0,-} := \Phi_w^+ \setminus \Phi_w^{\alpha_0,+},$$

and we use the notation

$$\underline{\ell} := (\ell_\alpha)_{\alpha \in \Phi_w^{\alpha_0,-}} \in \{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}$$

for a tuple of integers indexed by $\Phi_w^{\alpha_0,-}$. Given a tuple $\underline{\ell}$, we can define

$$\Lambda(\underline{\ell},\alpha_0) := \left\{ \underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0,\cdots,p-1\}^{|\Phi_w^+|} \;\middle|\; \begin{array}{ll} \cdot\ k_\alpha = 0 & \text{if } \alpha \in \Phi_w^{\alpha_0,+} \setminus \{\alpha_0\}; \\ \cdot\ k_\alpha > 0 & \text{if } \alpha = \alpha_0; \\ \cdot\ k_\alpha = \ell_\alpha & \text{if } \alpha \in \Phi_w^{\alpha_0,-} \end{array} \right\}.$$

Now we can define a polynomial

$$f_{(\underline{\ell},\alpha_0)}(x) = \sum_{\underline{k} \in \Lambda(\underline{\ell},\alpha_0)} C_{\underline{k},w} x^{k_{\alpha_0}} \in \mathbf{F}_p[x]$$

for each tuple of integers $\underline{\ell}$. By definition, we have

$$\sum_{\substack{\underline{k} \in \{0,\cdots,p-1\}^{|\Phi_w^+|} \\ k_{\alpha_0} > 0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi = \sum_{\underline{\ell} \in \{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi.$$

By the assumption on v we know that $u_{\alpha_0}(t)v = v$ for all $t \in \mathbf{F}_p$. Using (3.4.28) and (3.4.27) we have

$$u_{\alpha_0}(t) \sum_{\substack{\underline{k}\in\{0,\cdots,p-1\}^{|\Phi_w^+|}\\ k_{\alpha_0}>0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi = \sum_{\substack{\underline{k}\in\{0,\cdots,p-1\}^{|\Phi_w^+|}\\ k_{\alpha_0}>0}} C_{\underline{k},w} S_{\underline{k},w} v_\pi$$

and so

$$\begin{aligned}
&\sum_{\underline{\ell}\in\{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi \\
&\quad = u_{\alpha_0}(t) \sum_{\underline{\ell}\in\{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi \\
&\quad = \sum_{\underline{\ell}\in\{0,\cdots,p-1\}^{|\Phi_w^{\alpha_0,-}|}} \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0} - t) A \right) w v_\pi
\end{aligned}$$

where the last equality follows from a change of variable $A \leftrightarrow u_{\alpha_0}(t)A$.

By the linear independence of Jacobi sums from Proposition 3.4.18, we deduce an equality

$$\begin{aligned}
&\left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0}) A \right) w v_\pi \\
&\qquad = \left(\sum_{A\in U_w(\mathbf{F}_p)} \left(\prod_{\alpha\in\Phi_w^{\alpha_0,-}} A_\alpha^{\ell_\alpha} \right) f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0} - t) A \right) w v_\pi
\end{aligned}$$

for each fixed tuple $\underline{\ell}$.

Therefore, again by the linear independence of Jacobi sum operators in Proposition 3.4.18 we deduce that

$$f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0} - t) = f_{(\underline{\ell},\alpha_0)}(A_{\alpha_0})$$

for all $t \in \mathbf{F}_p$ and each $(\underline{\ell}, \alpha_0)$. We know that if $f \in \mathbf{F}_p[x]$ satisfies $\deg f \leq p-1$, $f(0) = 0$ and $f(x-t) = f(x)$ for all $t \in \mathbf{F}_p$ then $f = 0$. Thus we deduce that

$$f_{(\underline{\ell},\alpha_0)} = 0$$

for each tuple of integers $\underline{\ell}$, which is a contradiction to (3.4.26) and so we have $k_\alpha = 0$ for any $\alpha \in \Phi'$ for each tuple of integers $\underline{k}$ such that $C_{\underline{k},w} \neq 0$.

As a result, we have shown that each vector in $\pi_w^{U_{\Phi'}(\mathbf{F}_p)}$ can be written as certain linear combination of vectors in (3.4.25). On the other hand, by Proposition 3.4.18 we know that vectors in (3.4.25) are linear independent, and therefore they actually form a basis of $\pi_w^{U_{\Phi'}(\mathbf{F}_p)}$. □

Corollary 3.4.29. *Let $\mu_\pi = (d_1, \cdots, d_n)$ and fix a non-zero vector $v_\pi \in \pi^{U(\mathbf{F}_p),\mu_\pi}$. Given a weight $\mu = (\ell_1, \cdots, \ell_n) \in X_1(T)$ the space*

$$\pi_{w_0}^{[U(\mathbf{F}_p),U(\mathbf{F}_p)],\mu}$$

has a basis whose elements are of the form

$$S_{\underline{k},w_0} v_\pi$$

where $\underline{k} = (k_\alpha)$ satisfies

$$\ell_r \equiv d_{n+1-r} + (1 - \lfloor 1/r \rfloor) k_{r-1,r} - (1 - \lfloor 1/(n+1-r) \rfloor) k_{r,r+1} \ mod\ (p-1)$$

for all $1 \leq r \leq n$ and $k_\alpha = 0$ if $\alpha \in \Phi^+ \setminus \Delta$.

Proof. By a special case of Proposition 3.4.25 when $\Phi' = \{\sum_{r=i}^{j} \alpha_r \mid 1 \leq i < j \leq n-1\}$, we know that

$$\{S_{\underline{k},w_0} v_\pi \mid k_\alpha = 0 \text{ if } \alpha \in \Phi^+ \setminus \Delta\}$$

forms a basis of $\pi_{w_0}^{[U(\mathbf{F}_p),U(\mathbf{F}_p)]}$. On the other hand, we know from Proposition 3.4.18 that the above basis is actually an $T(\mathbf{F}_p)$-eigenbasis. Therefore the vectors in this basis with a fixed eigencharacter μ form a basis of the eigensubspace $\pi_{w_0}^{[U(\mathbf{F}_p),U(\mathbf{F}_p)],\mu}$. Finally, using (i) of the second part of Lemma 3.4.19 we conclude this lemma. □

Corollary 3.4.30. *Let $\mu_\pi = (d_1, d_2, \cdots, d_n)$ and fix a non-zero vector $v_\pi \in \pi^{U(\mathbf{F}_p),\mu_\pi}$. Given a weight $\mu = (\ell_1, \cdots, \ell_n) \in X_1(T)$, the space*

$$\pi_{w_0}^{U_1(\mathbf{F}_p),\mu}$$

has a basis whose elements are of the form

$$S_{\underline{k},w_0} v_\pi$$

where $\underline{k} = (k_{i,j})_{i,j}$ satisfies

$$k_{1,j} \equiv \ell_j - d_{n+1-j} \ mod\ (p-1)$$

for $2 \leq j \leq n$ and $k_{i,j} = 0$ for all $2 \leq i < j \leq n$.

Proof. By a special case of Proposition 3.4.25 when $\Phi' = \{\sum_{r=i}^{j} \alpha_r \mid 2 \leq i \leq j \leq n-1\}$, we know that

$$\{S_{\underline{k},w_0} v_\pi \mid k_{i,j} = 0 \text{ if } 2 \leq i < j \leq n\}$$

forms a basis of $\pi_{w_0}^{U_1(\mathbf{F}_p)}$. On the other hand, we know from Proposition 3.4.18 that the above basis is actually an $T(\mathbf{F}_p)$-eigenbasis. Therefore the vectors in this basis with a fixed eigencharacter μ form a basis of the eigensubspace $\pi_{w_0}^{U_1(\mathbf{F}_p),\mu}$. Finally, using (ii) of the second part of Lemma 3.4.19 we conclude this lemma. □

3.4.2 Main results in characteristic p

In this section, we state our main results on certain Jacobi sum operators in characteristic p. From now on we fix an n-tuple of integers $(a_{n-1}, \cdots, a_0)$ which is assumed to be n-generic in the lowest alcove (c.f. Definition 3.4.5).

We let

$$\begin{cases} \mu^* := (a_{n-1} - n + 2, a_{n-2}, a_{n-3}, \cdots, a_2, a_1, a_0 + n - 2); \\ \mu_1 := (a_1, a_2, \cdots, a_{n-3}, a_{n-2}, a_{n-1}, a_0); \\ \mu_1' := (a_{n-1}, a_0, a_1, a_2, \cdots, a_{n-3}, a_{n-2}); \\ \mu_0 := (a_{n-1}, a_1, a_2 \cdots, a_{n-3}, a_{n-2}, a_0); \end{cases} \tag{3.4.31}$$

and

$$\begin{cases} \pi_0 := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \mu_0; \\ \widetilde{\pi}_0^\circ := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \widetilde{\mu}_0; \end{cases} \tag{3.4.32}$$

where $\widetilde{\mu}_0$ is the Teichmüller lift of μ_0.

We let $\underline{k}^1 = (k_{i,j}^1)$, $\underline{k}^{1,\prime} = (k_{i,j}^{1,\prime})$ and $\underline{k}^0 = (k_{i,j}^0)$, where

$$\begin{cases} k_{i,i+1}^1 &= [a_0 - a_{n-i}]_1 + n - 2; \\ k_{i,i+1}^{1,\prime} &= [a_{n-i-1} - a_{n-1}]_1 + n - 2; \\ k_{i,i+1}^0 &= [a_0 - a_{n-1}]_1 + n - 2; \end{cases} \tag{3.4.33}$$

for $1 \leq i \leq n-1$ and $k^1_{i,j} = k^{1,\prime}_{i,j} = k^0_{i,j} = 0$ otherwise, and define two most important Jacobi sum operators $\mathcal{S}_n$ and $\mathcal{S}'_n$ to be

$$\mathcal{S}_n := S_{\underline{k}^1, w_0} \qquad \text{and} \qquad \mathcal{S}'_n := S_{\underline{k}^{1,\prime}, w_0}. \tag{3.4.34}$$

We also let V_1, V'_1, and V_0 denote the sub-representation of π_0 generated by

$$\mathcal{S}_n\left(\pi_0^{U(\mathbf{F}_p),\mu_1}\right),\ \mathcal{S}'_n\left(\pi_0^{U(\mathbf{F}_p),\mu'_1}\right),\ \text{and}\ \mathcal{S}_{\underline{k}^0, w_0}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right)$$

respectively.

The following theorem, which we usually call the *non-vanishing theorem*, is a technical heart on the local automorphic side. The proofs of this non-vanishing theorem as well as the next theorem, which we usually call the *multiplicity one theorem*, will occupy the following sections.

Theorem 3.4.35. *Assume that $(a_{n-1}, \cdots, a_0)$ is n-generic in the lowest alcove.*

Then we have

$$V_1 = V'_1 = V_0$$

and

$$F(\mu^*) \in \mathrm{JH}(V_0).$$

Proof. This is an immediate consequence of Corollary 3.4.67 and Theorem 3.4.137. □

We also have the following multiplicity one result.

Theorem 3.4.36. *Assume that $(a_{n-1}, \cdots, a_0)$ is $2n$-generic in the lowest alcove.*

Then $F(\mu^*)$ has multiplicity one in π_0.

Proof. This is a special case of Corollary 3.4.47: replace $\mu_\pi^{0,n-1}$ with μ^*. □

Corollary 3.4.37. *Assume that $(a_{n-1}, \cdots, a_0)$ is $2n$-generic in the lowest alcove and that τ is an $\mathcal{O}_E$-lattice in $\widetilde{\pi}_0^\circ \otimes_{\mathcal{O}_E} E$ such that*

$$\mathrm{soc}_{G(\mathbf{F}_p)}\left(\tau \otimes_{\mathcal{O}_E} \mathbf{F}\right) = F(\mu^*).$$

Then we have

$$\mathcal{S}_n\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1}\right) \neq 0 \quad \textit{and} \quad \mathcal{S}'_n\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu'_1}\right) \neq 0.$$

Proof. Such a τ is unique up to homothety by Theorem 3.4.36. By multiplying a suitable power of ϖ_E, we may assume that

$$\widetilde{\pi}_0^\circ \subsetneq \tau \text{ and } \widetilde{\pi}_0^\circ \nsubseteq \varpi\tau,$$

and thus we have a non-zero morphism

$$\pi_0 \to \tau \otimes_{\mathcal{O}_E} \mathbf{F}$$

whose image is the unique quotient of π_0 with socle $F(\mu^*)$. We now finish the proof by applying Theorem 3.4.35. □

Remark 3.4.38. *Theorem 3.4.35 and Corollary 3.4.37 can be generalized to the case when μ^* is replaced by any weight lying sufficiently deep in an arbitrary alcove. The crucial points here are the $[U(\mathbf{F}_p), U(\mathbf{F}_p)]$-invariance of $\mathcal{S}_n$ (resp. $\mathcal{S}'_n$) and that τ (in Corollary 3.4.37) is one of the simplest lattices of $\widetilde{\pi}_0^\circ \otimes_{\mathcal{O}_E} E$ apart from those coming from parabolic inductions from $B(\mathbf{F}_p)$.*

3.4.3 Summary of results on Deligne–Lusztig representations

In this section, we recall some standard facts on Deligne–Lusztig representations and fix the notation that will be used throughout this paper. We closely follow [Her09]. Throughout this article we will only focus the group $G(\mathbf{F}_p) = \mathrm{GL}_n(\mathbf{F}_p)$, which is the fixed point set of the standard (p-power) Frobenius F inside $\mathrm{GL}_n(\overline{\mathbf{F}}_p)$. We will identify a variety over $\overline{\mathbf{F}}_p$ with the set of its $\overline{\mathbf{F}}_p$-rational points for simplicity. Then our fixed maximal torus $\mathbb{T}$ is F-stable and split.

To each pair $(\mathbb{T}, \theta)$ consisting of an F-stable maximal torus $\mathbb{T}$ and a homomorphism $\theta : \mathbb{T}^F \to \overline{\mathbf{Q}}_p^\times$, Deligne–Lusztig [DL76] associate a virtual representation $R_{\mathbb{T}}^{\theta}$ of $\mathrm{GL}_n(\mathbf{F}_p)$. (We restrict ourself to $\mathrm{GL}_n(\mathbf{F}_p)$ although the result in [DL76] is much more general.) On the other hand, given a pair $(w, \mu) \in W \times X(T)$, one can construct a pair $(\mathbb{T}_w, \theta_{w,\mu})$ by the method in the third paragraph of [Her09], Section 4.1. Then we denote by $R_w(\mu)$ the representation corresponding to $R_{\mathbb{T}_w}^{\theta_{w,\mu}}$ after multiplying a sign. This is the so-called Jantzen parametrization in [Jan81] 3.1.

The representations $R_{\mathbb{T}}^{\theta}$ (resp. $R_w(\mu)$) can be isomorphic for different pairs $(\mathbb{T}, \theta)$ (resp. (w, μ)), and the explicit relation between is summarized in [Her09], Lemma 4.2. As each p-regular character $\mu \in X(T)/(p-1)X(T)$ of $T(\mathbf{F}_p)$ can be lift to an element in $X_1^{\mathrm{reg}}(T)$ which is unique up to $(p-1)X_0(T)$, the representation $R_w(\mu)$ is well defined for each $w \in W$ and such a μ.

We recall the notation $\Theta(\theta)$ for a cuspidal representation for $\mathrm{GL}_n(\mathbf{F}_p)$ from [Her06], Section 2.1 where θ is a *primitive* character of $\mathbf{F}_{p^n}^\times$ defined in [Her09], Section 4.2. We refer further discussion about the basic properties and references of $\Theta(\theta)$ to [Her06], Section 2.1. The relation between the notation $R_w(\mu)$ and the notation $\Theta(\theta)$ is summarized in [Her09], Lemma 4.7. In this paper, we will use the notation $\Theta_m(\theta_m)$ for a cuspidal representation for $\mathrm{GL}_m(\mathbf{F}_p)$ where θ_m is a primitive character of $\mathbf{F}_{p^m}^\times$.

We emphasize that, as a special case of [Her09], Lemma 4.7, we have the natural isomorphism

$$R_1(\mu) \cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \widetilde{\mu}$$

for a p-regular character μ of $T(\mathbf{F}_p)$, where $\widetilde{\mu}$ is the Teichmüler lift of μ.

3.4.4 Proof of Theorem 3.4.36

The main target of this section is to prove Theorem 3.4.36. In fact, we prove Corollary 3.4.47 which is a generalization of Theorem 3.4.36.

We recall some notation from [Jan03]. We use the notation $\overline{G}_r$ for the r-th Frobenius kernel defined in [Jan03] Chapter I.9 as kernel of r-th iteration of Frobenius morphism on the group scheme $\overline{G}$ over $\mathbf{F}_p$. We will consider the subgroup scheme $\overline{G}_r\overline{T}$, $\overline{G}_r\overline{B}$, $\overline{G}_r\overline{B}^-$ of $\overline{G}$ in the following. Note that our $\overline{B}$ (resp. $\overline{B}^-$) is denoted by B^+ (resp. B) in [Jan03] Chapter II. 9. We define

$$\begin{aligned} \widehat{Z}_r'(\lambda) &:= \mathrm{ind}_{\overline{B}^-}^{\overline{G}_r\overline{B}^-}\lambda; \\ \widehat{Z}_r(\lambda) &:= \mathrm{coind}_{\overline{B}}^{\overline{G}_r\overline{B}}\lambda \end{aligned}$$

where ind and coind are defined in I.3.3 and I.8.20 of [Jan03] respectively. By [Jan03] Proposition II.9.6 we know that there exists a simple $\overline{G}_r\overline{T}$-module $\widehat{L}_r(\lambda)$ satisfying

$$\mathrm{soc}_{\overline{G}_r}\left(\widehat{Z}_r'(\lambda)\right) \cong \widehat{L}_r(\lambda) \cong \mathrm{cosoc}_{\overline{G}_r}\left(\widehat{Z}_r(\lambda)\right).$$

The properties of $\widehat{Z}_r'(\lambda)$ and $\widehat{Z}_r(\lambda)$ are systematically summarized in [Jan03] II.9, and therefore we will frequently refer to results over there.

From now on we assume $r = 1$ in this section.

Now we recall several well-known results from [Jan81], [Jan84] and [Jan03]. We recall the definition of $\widetilde{W}^{res}$ from (3.4.3).

Theorem 3.4.39 ([Jan81], Satz 4.3)**.** *Assume that $\mu+\eta$ is in the lowest restricted alcove and $2n$-generic (Definition 3.4.5). Then we have*

$$\overline{R_w(\mu+\eta)} = \sum_{\substack{\widetilde{w}'\in\widetilde{W}^{\mathrm{res}}\\ \nu\in X(T)}} [\widehat{Z}_1(\mu-p\nu+p\eta):\widehat{L}_1(\widetilde{w}'\cdot\mu)]F(\widetilde{w}'\cdot(\mu+w\nu)).$$

Proposition 3.4.40. *Let $\lambda\in X(T)_+$. Suppose $\mu\in X(T)$ is maximal for $\mu\uparrow\lambda$ and $\mu\neq\lambda$. If $\mu\in X(T)_+$ and if $\mu\neq\lambda-p\alpha$ for all $\alpha\in\Phi^+$, then*

$$[H^0(\lambda):F(\mu)]=1.$$

Proof. This is the Corollary II 6.24 in [Jan03]. □

If M is an arbitrary $\overline{G}$-module, we use the notation $M^{[1]}$ for the Frobenius twist of M as defined in [Jan03], I.9.10.

Proposition 3.4.41 ([Jan03], Proposition II. 9.14)**.** *Let $\lambda\in X(T)_+$. Suppose each composition factor of $\widehat{Z}_1'(\lambda)$ has the form $\widehat{L}_1(\mu_0+p\mu_1)$ with $\mu_0\in X_1(T)$ and $\mu_1\in X(T)$ such that*

$$\langle\mu_1+\eta,\beta^\vee\rangle\geq 0$$

for all $\beta\in\Delta$. Then $H^0(\lambda)$ has a filtration with factors of the form $F(\mu_0)\otimes H^0(\mu_1)^{[1]}$. Each such module occurs as often as $\widehat{L}_1(\mu_0+p\mu_1)$ occurs in a composition series of $\widehat{Z}_1'(\lambda)$.

Remark 3.4.42. *Note that if μ_1 is in the lowest restricted alcove, then $F(\mu_0)\otimes H^0(\mu_1)^{[1]}=F(\mu)$.*

Lemma 3.4.43 ([Jan03], Lemma II. 9.18 (a))**.** *Let $\widehat{L}_1(\mu)$ be a composition factor of $\widehat{Z}_1'(\lambda)$, and write*

$$\lambda+\eta=p\lambda_1+\lambda_0 \text{ and } \mu=p\mu_1+\mu_0$$

with $\lambda_0,\mu_0\in X_1(T)$ and $\lambda_1,\mu_1\in X(T)$.

If

$$\langle\lambda,\alpha^\vee\rangle\geq n-2 \tag{3.4.44}$$

for all $\alpha\in\Phi^+$, then

$$\langle\mu_1+\eta,\beta^\vee\rangle\geq 0$$

for all $\beta\in\Phi^+$.

Proof. We only need to mention that $h_\alpha=n$ for all $\alpha\in\Phi^+$ and for our group $\overline{G}=\mathrm{GL}_{n/\mathbf{F}_p}$, where h_α is defined in [Jan03], II.9.18. □

We define an element $s_{\alpha,m}\in\widetilde{W}$ by

$$s_{\alpha,m}\cdot\lambda=s_\alpha\cdot\lambda+mp\alpha$$

for each $\alpha\in\Phi^+$ and $m\in\mathbf{Z}$.

Theorem 3.4.45. *Let $\lambda,\mu\in X(T)$ such that*

$$\mu=s_{\alpha,m}\cdot\lambda \quad\text{and}\quad mp<\langle\lambda+\eta,\alpha^\vee\rangle<(m+1)p. \tag{3.4.46}$$

Assume further that there exists $\nu\in X(T)$ such that $\lambda+p\nu$ satisfies the condition (3.4.44) and that ν and $\mu_1+\nu$ are in the lowest restricted alcove.

Then we have

$$[\widehat{Z}_1(\lambda):\widehat{L}_1(\mu)]=1.$$

Proof. The condition (3.4.46) ensures that for any fixed $\nu \in X(T)$, $\mu+p\nu$ is maximal for $\mu+p\nu \uparrow \lambda+p\nu$ and $\mu+p\nu \neq \lambda+p\nu$. Notice that we have

$$[\widehat{Z}_1(\lambda) : \widehat{L}_1(\mu)] = [\widehat{Z}'_1(\lambda) : \widehat{L}_1(\mu)]$$

by II 9.2(3) in [Jan03], as the character of a $\overline{G}_r\overline{T}$-module determine its Jordan–Hölder factors with multiplicities (or equivalently, determine the semisimplification of the $\overline{G}_r\overline{T}$-module).

By II 9.2(5) and II 9.6(6) in [Jan03] we have

$$[\widehat{Z}'_1(\lambda) : \widehat{L}_1(\mu)] = [\widehat{Z}'_1(\lambda) \otimes p\nu : \widehat{L}_1(\mu) \otimes p\nu] = [\widehat{Z}'_1(\lambda+p\nu) : \widehat{L}_1(\mu+p\nu)],$$

and thus we may assume that

$$\langle \lambda, \alpha^\vee \rangle \geq n-2$$

for all $\alpha \in \Phi^+$ by choosing appropriate ν (which exists by our assumption) and replacing λ by $\lambda+p\nu$ and μ by $\mu+p\nu$. Then by Lemma 3.4.43 we know that

$$\langle \mu'_1+\eta, \beta^\vee \rangle \geq 0$$

for any $\mu' = p\mu'_1+\mu'_0$ such that $\widehat{L}_1(\mu')$ is a factor of $\widehat{Z}'_1(\lambda)$.

Thus by Proposition 3.4.41, Proposition 3.4.40 and Remark 3.4.42 we know that

$$[\widehat{Z}'_1(\lambda) : \widehat{L}_1(\mu)] = [H^0(\lambda) : F(\mu_0) \otimes H^0(\mu_1)^{[1]}] = [H^0(\lambda) : F(\mu)] = 1$$

which finishes the proof. □

We pick an arbitrary principal series π and write

$$\mu_\pi = (d_1, \cdots, d_n)$$

For each pair of integers (i_1, j_1) satisfying $0 \leq i_1 < i_1+1 < j_1 \leq n-1$, we define

$$\mu_\pi^{i_1,j_1} := (d_1^{i_1,j_1}, \cdots, d_n^{i_1,j_1})$$

where

$$d_k^{i_1,j_1} = \begin{cases} d_k & \text{if } k \neq n-j_1 \text{ and } k \neq n-i_1; \\ d_{n-i_1}+j_1-i_1-1 & \text{if } k = n-i_1; \\ d_{n-j_1}-j_1+i_1+1 & \text{if } k = n-j_1. \end{cases}$$

Corollary 3.4.47. *Assume that μ_π is $2n$-generic in the lowest alcove (c.f. Definition 3.4.5). Then $F(\mu_\pi^{i_1,j_1})$ has multiplicity one in π, or equivalently in $\mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\mu_\pi^w$ for any $w \in W$.*

Proof. We notice at first that each $\mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}\mu_\pi^w$ has the same Jordan–Hölder factor as π with the same multiplicity as each of them is a mod p reduction of certain lattice of the same characteristic zero representation of $G(\mathbf{F}_p)$. We only need to apply Theorem 3.4.45 and Theorem 3.4.39 to these explicit examples. We will follow the notation of Theorem 3.4.39. We fix $w=1$ in Theorem 3.4.39 and take

$$\mu+\eta := \mu_\pi = \mu_\pi^{i_1,j_1} + (j_1-i_1-1)\left(\sum_{r=n-j_1}^{n-1-i_1} \alpha_r\right).$$

We are considering the multiplicity of $F(\mu_\pi^{i_1,j_1})$ in $\pi = R_1(\mu+\eta)$ and therefore we take $\widetilde{w}' := 1 \in \widetilde{W}^{\mathrm{res}}$ and

$$\nu := \eta - (j_1-i_1-1)\left(\sum_{r=n-j_1}^{n-1-i_1} \alpha_r\right).$$

By II. 9.2(4) and II.9.16 (4) in [Jan03] we know that

$$[\widehat{Z}_1(\mu - p\nu + p\eta) : \widehat{L}_1(\mu)] = [\widehat{Z}_1((n-j_1, n-i_1)\cdot(\mu - p\nu) + p\eta) : \widehat{L}_1(\mu)]. \tag{3.4.48}$$

We observe that

$$\begin{aligned}
&(n-j_1, n-i_1)\cdot(\mu - p\nu) + p\eta \\
&\quad = (n-j_1, n-i_1)\cdot\mu + p\left(\eta - (n-j_1, n-i_1)\eta - (j_1 - i_1 - 1)\left(\sum_{r=n-j_1}^{n-1-i_1}\alpha_r\right)\right) \\
&\quad = (n-j_1, n-i_1)\cdot\mu + p\left(\sum_{r=n-j_1}^{n-1-i_1}\alpha_r\right).
\end{aligned}$$

Therefore we have

$$p < \left\langle (n-j_1, n-i_1)\cdot(\mu - p\nu) + p\eta, \sum_{r=n-j_1}^{n-1-i_1}\alpha_r \right\rangle < 2p$$

and that

$$\mu = s_{\sum_{r=n-j_1}^{n-1-i_1}\alpha_r, p}\cdot((n-j_1, n-i_1)\cdot(\mu - p\nu) + p\eta).$$

Moreover, it is easy to see that

$$(n-j_1, n-i_1)\cdot(\mu - p\nu) + p\eta) + p\eta = (n-j_1, n-i_1)\cdot\mu + p\left(\sum_{r=n-j_1}^{n-1-i_1}\alpha_r\right) + p\eta$$

satisfies (3.4.44).

Hence, replacing the λ and μ in Theorem 3.4.45 by $(n-j_1, n-i_1)\cdot(\mu - p\nu) + p\eta$ and μ respectively, we conclude that

$$[\widehat{Z}_1((n-j_1, n-i_1)\cdot(\mu - p\nu) + p\eta) : \widehat{L}_1(\mu)] = 1$$

which finishes the proof by Theorem 3.4.39 and (3.4.48). □

3.4.5 Jacobi sums in characteristic 0

In this section, we establish an intertwining identity for lifts of Jacobi sums in characteristic 0 in Theorem 3.4.71, which is one of the main ingredients of the proof of Theorem 3.5.44. All of our calculations here are in the setting of $G(\mathbf{Q}_p) = \mathrm{GL}_n(\mathbf{Q}_p)$. We first fix some notation.

Let $A \in G(\mathbf{F}_p)$. By $\lceil A \rfloor$ we mean the matrix in $G(\mathbf{Q}_p)$ whose entries are the classical Teichmüller lifts of the entries of A. The map $A \mapsto \lceil A \rfloor$ is obviously not a group homomorphism but only a map between sets. On the other hand, we use the notation $\widetilde{\mu}$ for the Teichmüller lift of a character μ of $T(\mathbf{F}_p)$.

We denote the standard lifts of simple reflections in $G(\mathbf{Q}_p)$ by

$$s_i = \begin{pmatrix} \mathrm{Id}_{i-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & \mathrm{Id}_{n-i-1} \end{pmatrix}$$

for $1 \le i \le n-1$. We also use the following notation

$$t_i = \begin{pmatrix} p\mathrm{Id}_i & \\ & \mathrm{Id}_{n-i} \end{pmatrix}$$

for $1 \leq i \leq n$. Let

$$\Xi_n := w^* t_1, \tag{3.4.49}$$

where $w^* := s_{n-1} \bullet ... \bullet s_1$. We recall the Iwahori subgroup I and the pro-p Iwahori subgroup $I(1)$ from the beginning of Section 3.4. Note that the operator Ξ_n and the group I actually generates the normalizer of I inside $G(\mathbf{Q}_p)$. One easily sees that Ξ_n is nothing else than the following matrix:

$$\Xi_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ p & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in G(\mathbf{Q}_p).$$

For each $1 \leq i \leq n-1$, we consider the maximal parabolic subgroup P_i^- of G containing lower-triangular Borel subgroup B^- such that its Levi subgroup can be chosen to be $\mathrm{GL}_i \times \mathrm{GL}_{n-i}$ which embeds into G in the standard way. We denote the unipotent radical of P_i^- by N_i^-. Then we introduce

$$U_n^i = \sum_{A \in N_i^-(\mathbf{F}_p)} t_i^{-1} \lceil A \rceil \text{ for each } 1 \leq i \leq n-1. \tag{3.4.50}$$

Note that each $A \in N_i^-$ has the form

$$\begin{pmatrix} \mathrm{Id}_i & 0_{(n-i)\times i} \\ *_{i\times(n-i)} & \mathrm{Id}_{n-i} \end{pmatrix}.$$

for each $1 \leq i \leq n-1$.

For each $w \in W$ and each tuple $\underline{k} = (k_\alpha)_{\alpha \in \Phi_w^+} \in \{0, \ldots, p-1\}^{|\Phi_w^+|}$, we consider the following Jacobi sum

$$\widehat{\mathcal{S}}_{\underline{k},w} := \left(\sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil A \rceil \right) w \in \mathbf{Z}_p[G(\mathbf{Z}_p)].$$

In particular, we consider

$$\widehat{\mathcal{S}}_w := \left(\sum_{A \in U_w(\mathbf{F}_p)} \lceil A \rceil \right) w \in \mathbf{Z}_p[G(\mathbf{Z}_p)]$$

which is a characteristic zero lift of $S_{\underline{0},w}$. Recall the notation $\widetilde{\pi}^\circ$ from (3.4.6).

Lemma 3.4.51. *Assume that μ_π is n-generic (Definition 3.4.5). We have the equality*

$$\widehat{\mathcal{S}}_w \bullet \widehat{\mathcal{S}}_{w'} = p^{\frac{\ell(w)+\ell(w')-\ell(ww')}{2}} \widehat{\mathcal{S}}_{ww'}$$

on $(\widetilde{\pi}^\circ)^{I(1)}$ for all $w, w' \in W$.

Proof. One can quickly reduce the general case to the following two elementary equalities on $(\widetilde{\pi}^\circ)^{I(1)}$:

$$\widehat{\mathcal{S}}_w \bullet \widehat{\mathcal{S}}_{w'} = \widehat{\mathcal{S}}_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w') \tag{3.4.52}$$

and

$$\widehat{\mathcal{S}}_{s_r} \bullet \widehat{\mathcal{S}}_{s_r} = p \text{ for all } 1 \leq r \leq n-1. \tag{3.4.53}$$

The equality (3.4.52) follows directly from the definition of the Jacobi sum operators. The equality (3.4.53) follows from a simple Bruhat decomposition. In fact, we have for each $t \neq 0$

$$s_r u_{\alpha_r}(t) s_r = u_{\alpha_r}(t^{-1}) s_r \mathrm{diag}(1, \cdots, 1, t, -t^{-1}, 1, \cdots, 1) u_{\alpha_r}(t^{-1})$$

where the diagonal matrix has t at (r,r)-entry and $-t^{-1}$ at $(r+1, r+1)$-entry. Therefore for each $\widehat{v} \in (\widetilde{\pi}^\circ)^{I(1)}$ there exists an integer $n \leq \ell \leq p-n$ such that

$$\mathrm{diag}(1, \cdots, 1, t, -t^{-1}, 1, \cdots, 1)\widehat{v} = \pm \lceil t \rceil^\ell \widehat{v}$$

and thus

$$\begin{aligned} \widehat{\mathcal{S}}_{s_r} \bullet \widehat{\mathcal{S}}_{s_r} \widehat{v} &= \left(\sum_{t' \in \mathbf{F}_p} u_{\alpha_r}(t') \right) \left(\widehat{v} \pm \sum_{t \in \mathbf{F}_p'} \lceil t \rceil^\ell \lceil u_{\alpha_r}(t^{-1}) \rceil s_r \widehat{v} \right) \\ &= p\widehat{v} \pm \left(\sum_{t' \in \mathbf{F}_p, t \in \mathbf{F}_p} \lceil t \rceil^\ell \lceil u_{\alpha_r}(t' + t^{-1}) \rceil s_r \right) \widehat{v} \\ &= p\widehat{v}. \end{aligned}$$

This finishes the proof. □

Lemma 3.4.54. *We have the equality*

$$(\Xi_n)^k \bullet U_n^k = \widehat{\mathcal{S}}_{(w^*)^k}$$

Proof. This is immediate by definition. □

We quickly recall some standard facts about Jacobi sums and Gauss sums. We fix a primitive p-th root of unity $\xi \in E$ and set $\epsilon := \xi - 1$. For each pair of integers (a,b) with $0 \leq a, b \leq p-1$, we set

$$J(a,b) := \sum_{\lambda \in \mathbf{F}_p} \lceil \lambda \rceil^a \lceil 1 - \lambda \rceil^b. \tag{3.4.55}$$

We also set

$$G(a) := \sum_{\lambda \in \mathbf{F}_p} \lceil \lambda \rceil^a \xi^\lambda$$

for each integers a with $0 \leq a \leq p-1$. For example, we have $G(p-1) = -1$.

It is known by section 1.1, $GS3$ of [Lang] that if $a + b \not\equiv 0 \bmod (p-1)$, we have

$$J(a,b) = \frac{G(a)G(b)}{G(a+b)}. \tag{3.4.56}$$

It is also obvious from the definition that if $a, b, a+b \not\equiv 0 \bmod (p-1)$ then

$$J(b,a) = J(a,b) = (-1)^b J(b, [-a-b]_1) = (-1)^a J(a, [-a-b]_1). \tag{3.4.57}$$

By Stickelberger's theorem ([Lang] Section 1.2, Theorem 2.1), we know that

$$\mathrm{ord}_p(G(a)) = 1 - \frac{a}{p-1} \quad \text{and} \quad \frac{G(a)}{\epsilon^{p-1-a}} \equiv a! \pmod{p}. \tag{3.4.58}$$

Let $r \in \mathbf{Z}$ with $1 \leq r \leq n-1$ and $w \in W$. Given the data $\mu_\pi = (d_1, d_2, \cdots, d_n)$ and tuple $\underline{k} \in \{0, \ldots, p-1\}^{|\Phi_w^+|}$, we define a tuple

$$\underline{k}' \in \begin{cases} \{0, \ldots, p-1\}^{|\Phi_w^+|} & \text{if } \ell(ws_r) < \ell(w); \\ \{0, \ldots, p-1\}^{|\Phi_{ws_r}^+|} & \text{if } \ell(ws_r) > \ell(w) \end{cases} \tag{3.4.59}$$

by

$$k'_\alpha = \begin{cases} k_\alpha & \text{if } \alpha \in \Phi_w^+; \\ 0 & \text{if } \alpha = w\alpha_r \end{cases}$$

in the first case and

$$k'_\alpha = \begin{cases} [k_{w\alpha_r} - d_r + d_{r+1}]_1 & \text{if } \alpha = w\alpha_r; \\ k_\alpha & \text{if } \alpha \in \Phi_w^+ \text{ and } \alpha \neq w\alpha_r \end{cases}$$

in the second case.

Proposition 3.4.60. *Assume that $\mu_\pi = (d_1, d_2, \cdots, d_n)$ is n-generic and that*

$$k_\alpha = 0 \text{ for all } \alpha \in \Phi_w^+ \text{ with } w\alpha_r < \alpha.$$

Assume further that if $\ell(ws_r) < \ell(w)$ then $k_{w\alpha_r} \notin \{0, p-1, [d_r - d_{r+1}]_1\}$.

Then for each $1 \leq r \leq n-1$ we have

$$\widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r} = \begin{cases} \widehat{\mathcal{S}}_{\underline{k}',ws_r} & \text{if } \ell(ws_r) > \ell(w) \\ (-1)^{d_{r+1}} J(k_{w\alpha_r}, [d_{r+1} - d_r]_1) \widehat{\mathcal{S}}_{\underline{k}',w} & \text{if } \ell(ws_r) < \ell(w) \end{cases}$$

on $(\widetilde{\pi}^\circ)^{I(1),\widetilde{\mu}_\pi}$.

Proof. By definition we have

$$\widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r} = \sum_{A \in U(\mathbf{F}_p), t \in \mathbf{F}_p} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil A \rceil w \lceil u_{\alpha_r}(t) \rceil s_r.$$

We divide it into two cases:

(i) $\ell(ws_r) > \ell(w)$;

(ii) $\ell(ws_r) < \ell(w)$.

In case (i), we have the Bruhat decomposition

$$Awu_{\alpha_r}(t)s_r = Au_{w\alpha_r}(t)ws_r$$

and thus

$$\widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r} = \widehat{\mathcal{S}}_{\underline{k}',ws_r}.$$

In case (ii), we have the Bruhat decompositions: if $t = 0$

$$Awu_{\alpha_r}(0)s_r = A(ws_r) = A''ws_r u_{\alpha_r}(A_{w\alpha_r})$$

where A'' is the unipotent matrix that has the same entries as A except a zero at $w\alpha_r$-entry; if $t \neq 0$

$$Awu_{\alpha_r}(t)s_r = Au_{w\alpha_r}(t^{-1})w\mathrm{diag}(1, \cdots, t, -t^{-1}, \cdots, 1)u_{\alpha_r}(t^{-1}).$$

We fix a vector $\widehat{v}_\pi \in (\widetilde{\pi}^\circ)^{I(1),\widetilde{\mu}_\pi}$ whose mod p reduction is non-zero. Therefore, we have

$$\widehat{\mathcal{S}}_{\underline{k},w} \bullet \widehat{\mathcal{S}}_{s_r} \widehat{v}_\pi = (-1)^{d_{r+1}} \sum_{A \in U_w(\mathbf{F}_p), t \in \mathbf{F}_p^\times} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil t \rceil^{d_r - d_{r+1}} \lceil A \rceil u_{w\alpha_r}(t^{-1}) w \widehat{v}_\pi$$
$$+ \sum_{A \in U_w(\mathbf{F}_p)} \left(\prod_{\alpha \in \Phi_w^+} \lceil A_\alpha \rceil^{k_\alpha} \right) \lceil A \rceil ws_r \widehat{v}_\pi.$$

The summation $\sum_{A\in U_w(\mathbf{F}_p)}\left(\prod_{\alpha\in\Phi_w^+}\lceil A_\alpha\rceil^{k_\alpha}\right)Aws_r\widehat{v}_\pi$ can be rewritten as

$$\sum_{A''\in U_{ws_r}(\mathbf{F}_p)}\left(\prod_{\alpha\in\Phi_{ws_r}^+}\lceil A_\alpha\rceil^{k_\alpha}\right)\left(\sum_{A_{w\alpha_r}\in\mathbf{F}_p}\lceil A_{w\alpha_r}\rceil^{k_{w\alpha_r}}\right)A''ws_r\widehat{v}_\pi$$

which is 0 as we assume $0<k_{w\alpha_r}<p-1$. Hence, we have

$$\widehat{\mathcal{S}}_{\underline{k},w_0}\bullet\widehat{\mathcal{S}}_{s_r}\widehat{v}_\pi=(-1)^{d_{r+1}}\sum_{A\in U_w(\mathbf{F}_p),t\in\mathbf{F}_p^\times}\left(\prod_{\alpha\in\Phi_w^+}\lceil A_\alpha\rceil^{k_\alpha}\right)\lceil t\rceil^{d_r-d_{r+1}}\lceil Au_{\alpha_{w\alpha_r}}(t^{-1})\rceil w\widehat{v}_\pi.$$

On the other hand, after setting $A'=Au_{w\alpha_r}(t^{-1})$ we have

$$\begin{aligned}&\sum_{A\in U_w(\mathbf{F}_p),t\in\mathbf{F}_p^\times}\left(\prod_{\alpha\in\Phi_w^+}\lceil A_\alpha\rceil^{k_\alpha}\right)\lceil t\rceil^{d_r-d_{r+1}}\lceil Au_{w\alpha_r}(t^{-1})\rceil w\widehat{v}_\pi\\&\qquad=\sum_{A'\in U_w(\mathbf{F}_p),t\in\mathbf{F}_p^\times}\left(\prod_{\alpha\in\Phi_{ws_r}^+}\lceil A_\alpha\rceil^{k_\alpha}\right)\lceil(A'_{w\alpha_r}-t^{-1})\rceil^{k_{w\alpha_r}}\lceil t\rceil^{d_r-d_{r+1}}\lceil A'\rceil w\widehat{v}_\pi\end{aligned}\tag{3.4.61}$$

since $k_\alpha=0$ for all $w\alpha_r<\alpha$.

One can easily check that if $A'_{w\alpha_r}=0$ then

$$\sum_{t\in\mathbf{F}_p^\times}\lceil(A'_{w\alpha_r}-t^{-1})\rceil^{k_{w\alpha_r}}\lceil t\rceil^{d_r-d_{r+1}}=(-1)^{k_{w\alpha_r}}\sum_{t\in\mathbf{F}_p}\lceil t\rceil^{d_r-d_{r+1}-k_{w\alpha_r}}=0,$$

and if $A'_{w\alpha_r}\neq0$ then

$$\begin{aligned}&\sum_{t\in\mathbf{F}_p^\times}\lceil(A'_{w\alpha_r}-t^{-1})\rceil^{k_{w\alpha_r}}\lceil t\rceil^{d_r-d_{r+1}}\\&\qquad=\lceil A'_{w\alpha_r}\rceil^{k_{w\alpha_r}-d_r+d_{r+1}}\left(\sum_{t\in\mathbf{F}_p}\lceil(1-(A'_{w\alpha_r}t)^{-1})\rceil^{k_{w\alpha_r}}\lceil(A'_{w\alpha_r}t)^{-1})^{d_{r+1}-d_r}\rceil\right)\\&\qquad=J(k_{w\alpha_r},[d_{r+1}-d_r]_1)\lceil A'_{w\alpha_r}\rceil^{[k_{w\alpha_r}-d_{r+1}+d_r]_1}\end{aligned}$$

Combining these computations with (3.4.61) finishes the proof. □

Remark 3.4.62. *Proposition 3.4.60 is the technical heart of this section. It roughly says that $[U(\mathbf{F}_p),U(\mathbf{F}_p)]$-invariant vectors behave well under intertwining of principal series, which is essentially why the identities in Theorem 3.4.71 and Proposition 3.5.25 exist. On the other hand, it is crucial that the vector $\widehat{v}_\pi$ is invariant under $\lceil u_{\alpha_r}(t)\rceil$ for $t\in\mathbf{F}_p$.*

Recall the definition of μ_1 and μ_1' from (3.4.31). We recursively define sequences of elements in the Weyl group W by

$$\begin{cases}w_1=1,\ w_m=s_{n-m}w_{m-1};\\ w_1'=1,\ w_m'=s_mw_{m-1}'\end{cases}$$

for all $2\leq m\leq n-1$, where s_m are the reflection of the simple roots α_m. We define sequences of characters of $T(\mathbf{F}_p)$

$$\mu_m=\mu_1^{w_m}\ \text{ and }\ \mu_m'=(\mu_1')^{w_m'}$$

for all $1 \leq m \leq n-1$. We also define several families of Jacobi sums:

$$\widehat{\mathcal{S}}_{\underline{k}^m,w_0} \text{ and } \widehat{\mathcal{S}}_{\underline{k}^{m,\prime},w_0}$$

for all integers m with $1 \leq m \leq n-1$, where $\underline{k}^m = (k^m_{i,j})$ satisfies

$$k^m_{i,j} = \begin{cases} n-2+[a_0-a_{n-1}]_1 & \text{if } 1 \leq i = j-1 \leq m; \\ n-2+[a_0-a_{n-i}]_1 & \text{if } m+1 \leq i = j-1 \leq n-1; \\ 0 & \text{otherwise} \end{cases}$$

and $\underline{k}^{m,\prime} = (k^{m,\prime}_{i,j})$ satisfies

$$k^{m,\prime}_{i,j} = \begin{cases} n-2+[a_{n-i-1}-a_{n-1}]_1 & \text{if } 1 \leq i = j-1 \leq n-m-1; \\ n-2+[a_0-a_{n-1}]_1 & \text{if } n-m \leq i = j-1 \leq n-1; \\ 0 & \text{otherwise.} \end{cases}$$

We keep the notation in (3.4.32) and recall that $\underline{k}^0$ is defined in (3.4.33). We also define

$$\begin{cases} \kappa_n^{(1)} := (-1)^{\sum_{m=1}^{n-2} a_m} \prod_{m=1}^{n-2} J(n-2+[a_0-a_{n-m-1}]_1, [a_{n-m-1}-a_{n-1}]_1), \\ \kappa_n^{(2)} := (-1)^{(n-2)a_0} \prod_{m=1}^{n-2} J(n-2+[a_m-a_{n-1}]_1, [a_0-a_m]_1). \end{cases} \tag{3.4.63}$$

Proposition 3.4.64. *Assume that $(a_{n-1}, \cdots, a_0)$ is n-generic.*

Then we have

$$\widehat{\mathcal{S}}_{\underline{k}^1,w_0} \bullet \widehat{\mathcal{S}}_{w_{n-1}^{-1}} = \kappa_n^{(1)} \widehat{\mathcal{S}}_{\underline{k}^0,w_0} \quad \textit{and} \quad \widehat{\mathcal{S}}_{\underline{k}^{1,\prime},w_0} \bullet \widehat{\mathcal{S}}_{(w'_{n-1})^{-1}} = \kappa_n^{(2)} \widehat{\mathcal{S}}_{\underline{k}^0,w_0}$$

on the 1-dimensional space $(\widetilde{\pi}_0^\circ)^{I(1),\widetilde{\mu}_0}$ for all $1 \leq m \leq n-1$.

Proof. By the case $w = w_0$ of Proposition 3.4.60 and the fact that

$$k^m_{m+1,m+2} = n-2+[a_0-a_{n-m-1}]_1 \text{ and } k^{m,\prime}_{n-m-1,n-m} = n-2+[a_m-a_{n-1}]_1$$

we have

$$\widehat{\mathcal{S}}_{\underline{k}^m,w_0} \bullet \widehat{\mathcal{S}}_{s_{n-m-1}} = (-1)^{a_{n-m-1}} J(n-2+[a_0-a_{n-m-1}]_1, [a_{n-m-1}-a_{n-1}]_1) \widehat{\mathcal{S}}_{\underline{k}^{m+1},w_0}$$

and

$$\widehat{\mathcal{S}}_{\underline{k}^{m,\prime},w_0} \bullet \widehat{\mathcal{S}}_{s_{m+1}} = (-1)^{a_0} J(n-2+[a_m-a_{n-1}]_1, [a_0-a_m]_1) \widehat{\mathcal{S}}_{\underline{k}^{m+1,\prime},w_0}$$

on the 1-dimensional space $(\pi_0^\circ)^{I(1),\widetilde{\mu}_0}$ for all $1 \leq m \leq n-2$, and hence we finish the proof by induction on m. □

Lemma 3.4.65. *We have*

$$\begin{aligned} \kappa_n^{(1)} &\equiv (-1)^{\sum_{m=1}^{n-2} a_m} \left(\prod_{m=1}^{n-2} \tfrac{(n-2+[a_0-a_{n-m-1}]_1)!([a_{n-m-1}-a_{n-1}]_1)!}{(n-2+[a_0-a_{n-1}]_1)!} \right) \pmod{p} \\ \kappa_n^{(2)} &\equiv (-1)^{(n-2)a_0} \left(\prod_{m=1}^{n-2} \tfrac{(n-2+[a_m-a_{n-1}]_1)!([a_0-a_m]_1)!}{(n-2+[a_0-a_{n-1}]_1)!} \right) \pmod{p}. \end{aligned}$$

In particular,

$$\mathrm{ord}_p(\kappa_n^{(1)}) = \mathrm{ord}_p(\kappa_n^{(2)}) = 0.$$

Proof. This follows directly from (3.4.58), the definition of $\kappa_n^{(1)}$, $\kappa_n^{(2)}$, and the fact that $(a_{n-1}, \cdots, a_0)$ is n-generic. □

Corollary 3.4.66. *Assume that $(a_{n-1}, \cdots, a_0)$ is n-generic.*

Then we have

$$\widehat{\mathcal{S}}_{\underline{k}^1,w_0} = p^{n-2}\kappa_n^{(1)}\widehat{\mathcal{S}}_{\underline{k}^0,w_0} \bullet \widehat{\mathcal{S}}_{w_{n-1}} \quad \text{and} \quad \widehat{\mathcal{S}}_{\underline{k}^{1,\prime},w_0} = p^{n-2}\kappa_n^{(2)}\widehat{\mathcal{S}}_{\underline{k}^0,w_0} \bullet \widehat{\mathcal{S}}_{w'_{n-1}}$$

on the 1-dimensional space $(\widetilde{\pi}_0^\circ)^{I(1),\widetilde{\mu}_0}$ for all $1 \leq m \leq n-1$.

Proof. We have

$$\widehat{\mathcal{S}}_{w_{n-1}^{-1}} \bullet \widehat{\mathcal{S}}_{w_{n-1}} = p^{n-2} = \widehat{\mathcal{S}}_{(w'_{n-1})^{-1}} \bullet \widehat{\mathcal{S}}_{w'_{n-1}},$$

so that this follows from Proposition 3.4.64 and Lemma 3.4.51. □

Corollary 3.4.67. *We have the equality*

$$S_{\underline{k}^1,w_0}\left(\pi_0^{U(\mathbf{F}_p),\mu_1}\right) = S_{\underline{k}^{1,\prime},w_0}\left(\pi_0^{U(\mathbf{F}_p),\mu'_1}\right) = S_{\underline{k}^0,w_0}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right).$$

Proof. The follows directly from Proposition 3.4.60,

$$S_{\underline{0},w_{n-1}^{-1}}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right) = \pi_0^{U(\mathbf{F}_p),\mu_1} \quad \text{and} \quad S_{\underline{0},(w'_{n-1})^{-1}}\left(\pi_0^{U(\mathbf{F}_p),\mu_0}\right) = \pi_0^{U(\mathbf{F}_p),\mu'_1}$$

and Lemma 3.4.65 by taking mod p reduction. □

As in (3.4.34), we use the shorten notation

$$\widehat{\mathcal{S}}_n := \widehat{\mathcal{S}}_{\underline{k}^1,w_0} \quad \text{and} \quad \widehat{\mathcal{S}}'_n := \widehat{\mathcal{S}}_{\underline{k}^{1,\prime},w_0}.$$

To state the main result in this section, we also define

$$\mathcal{P}_n := \prod_{k=1}^{n-2}\prod_{j=1}^{n-2}\frac{[a_k - a_{n-1}]_1 + j}{[a_0 - a_k]_1 + j} = \prod_{k=1}^{n-2}\prod_{j=0}^{n-3}\frac{a_k - a_{n-1} + j}{a_0 - a_k + j} \in \mathbf{Z}_p^\times, \tag{3.4.68}$$

$$\varepsilon^* := \prod_{m=1}^{n-2}(-1)^{a_0 - a_m}, \tag{3.4.69}$$

and

$$\kappa_n := \kappa_n^{(2)}(\kappa_n^{(1)})^{-1}. \tag{3.4.70}$$

The main result of this section is the following theorem, which is a generalization of the case $n = 3$ in [HLM17], (3.2.1).

Theorem 3.4.71. *Let*

$$\Pi_n := \mathrm{Ind}_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}\chi$$

be a tamely ramified principal series representation where the $\chi = \chi_1 \otimes \cdots \otimes \chi_n : T(\mathbf{Q}_p) \to E^\times$ is a smooth character satisfying $\chi|_{T(\mathbf{Z}_p)} = \widetilde{\mu}_1$.

On the 1-dimensional subspace $\Pi_n^{I(1),\widetilde{\mu}_1}$ we have the identity:

$$\widehat{\mathcal{S}}'_n \bullet (\Xi_n)^{n-2} = p^{n-2}\kappa_n\left(\prod_{k=1}^{n-2}\chi_k(p)\right)\widehat{\mathcal{S}}_n$$

for some $\kappa_n \in \mathcal{O}_E^\times$ (defined in (3.4.70)) such that

$$\kappa_n \equiv \varepsilon^*\mathcal{P}_n(a_{n-1}, \cdots, a_0) \pmod{\varpi_E} \tag{3.4.72}$$

where $\varepsilon^* = \pm 1$ is a sign defined in (3.4.69) and $\mathcal{P}_n$ is a rational function defined in (3.4.68).

The following is a direct generalization of Lemma 3.2.5 in [HLM17].

Lemma 3.4.73. *We have the equality*

$$U_n^r = \left(\prod_{k=1}^{r} \chi_k(p)\right)^{-1}$$

on the 1*-dimensional space* $\Pi_n^{I(1),\widetilde{\mu}_1}$ *for each* $1 \leq r \leq n-1$.

Proof. The proof of this lemma is an immediate calculation which is parallel to that of [HLM17], Lemma 3.2.5. □

Proof of Theorem 3.4.71. Notice that

$$w'_{n-1}(w^*)^{n-2} = w_{n-1} \quad \text{and} \quad \ell(w'_{n-1}) + \ell((w^*)^{n-2}) = 3(n-2) = \ell(w_{n-1}) + 2(n-2),$$

so that by Lemma 3.4.51 we have

$$\widehat{\mathcal{S}}_{w'_{n-1}} \bullet \widehat{\mathcal{S}}_{(w^*)^{n-2}} = p^{n-2}\widehat{\mathcal{S}}_{w_{n-1}}. \tag{3.4.74}$$

By composing $\widehat{\mathcal{S}}_{\underline{k}^0,w_0}$ on both sides of (3.4.74), we deduce from Proposition 3.4.64 that

$$(\kappa_n^{(2)})^{-1}\widehat{\mathcal{S}}'_n \bullet \widehat{\mathcal{S}}_{(w^*)^{n-2}} = p^{n-2}(\kappa_n^{(1)})^{-1}\widehat{\mathcal{S}}_n$$

and thus

$$\widehat{\mathcal{S}}'_n \bullet \widehat{\mathcal{S}}_{(w^*)^{n-2}} = p^{n-2}\kappa_n\widehat{\mathcal{S}}_n \tag{3.4.75}$$

on the 1-dimensional subspace $\Pi_n^{I(1),\widetilde{\mu}_1}$. Now Lemma 3.4.54 together with Lemma 3.4.73 gives rise to the identity in the statement of this theorem.

Finally, one can readily check from Lemma 3.4.65 that

$$\begin{aligned}
\kappa_n &\equiv \kappa_n^{(2)}(\kappa_n^{(1)})^{-1} \\
&\equiv (-1)^{\sum_{m=1}^{n-2} a_0 - a_m} \prod_{m=1}^{n-2} \frac{(n-2+[a_0-a_{n-m-1}]_1)!([a_{n-m-1}-a_{n-1}]_1)!}{(n-2+[a_m-a_{n-1}]_1)!([a_0-a_m]_1)!} \\
&\equiv (-1)^{\sum_{m=1}^{n-2} a_0 - a_m} \prod_{m=1}^{n-2}\prod_{\ell=1}^{n-2} \frac{\ell+[a_0-a_m]_1}{\ell+[a_m-a_{n-1}]_1} \\
&\equiv \varepsilon^*\mathcal{P}_n \pmod{\varpi_E}.
\end{aligned}$$

Note that $\mathrm{ord}_p(\kappa_n) = 0$. This completes the proof. □

3.4.6 Special vectors in a dual Weyl module

Let R be a $\mathbf{F}_p$-algebra, and $A \in \overline{G}(R)$ a matrix. For $J_1, J_2 \subseteq \{1, 2, \cdots, n-1, n\}$, we write A_{J_1,J_2} for the submatrix of A consisting of the entries of A at the (i,j)-position for $i \in J_1$, $j \in J_2$. We define

$$J_0^i := \{1, 2, \cdots, i\} \subseteq \{1, \cdots, n\}$$

for each $1 \leq i \leq n$.

We fix a tuple of integers $(h_1, \cdots, h_s)$ for some $1 \leq s \leq n-1$ such that

$$1 \leq h_r \leq n-1 \text{ for all } 1 \leq r \leq s$$

and

$$\sum_{r=1}^{s} h_r = n-1$$

and we denote this tuple by $\underline{h}$. Then we can define $n-1$ positive roots $\beta_{\underline{h},i}$ for $1 \leq i \leq n-1$ as follows. Given an integer $2 \leq i \leq n-1$, there exists an unique integer $1 \leq r_0 \leq s-1$ such that

$$\sum_{r=1}^{r_0} h_r < i \leq \sum_{r=1}^{r_0+1} h_r,$$

and we use the notation

$$[i]_{\underline{h}} := \sum_{r=1}^{r_0} h_r.$$

Then we define

$$\beta_{\underline{h},i} := \sum_{k=1+[i]_{\underline{h}}}^{i} \alpha_k.$$

We always set

$$\beta_{\underline{h},1} = \alpha_1.$$

Then we define

$$\Phi^+_{\underline{h}} := \{\alpha \in \Phi^+ \mid \alpha \neq \beta_{\underline{h},i} \text{ for all } 1 \leq i \leq n-1\}$$

and notice that this set gives an unipotent group $U_{\underline{h}} \subsetneq U$ by setting

$$U_{\underline{h}} := \prod_{\alpha \in \Phi^+_{\underline{h}}} U_\alpha.$$

We emphasize that all $U_{\underline{h}}$ constructed here are *good* in the sense of Definition 3.4.21. In particular, if $s = n-1$ and $n_r = 1$ for $1 \leq r \leq n-1$ we recover $[U,U]$, and if $s = 1$ and $n_1 = n-1$ we recover U_1 (c.f. Example 3.4.23). We define $\overline{U}_{\underline{h}}$ as the reduction of $U_{\underline{h}}$ mod p.

Given a tuple $\underline{h}$ as above, we define the subsets $J^i_{\underline{h}} \subseteq \{1, \cdots, n\}$ for $1 \leq i \leq n-1$ as

$$J^i_{\underline{h}} := \{1, 2, \cdots, i+1\} \setminus \{\sum_{r=1}^{r_0'} h_r\} \text{ if } \sum_{r=1}^{r_0'} h_r \leq i < \sum_{r=1}^{r_0'+1} h_r$$

for some $1 \leq r_0' \leq s-1$ and we use the notation

$$[i]'_{\underline{h}} := \sum_{r=1}^{r_0'} h_r.$$

It is easy to see that $|J^i_{\underline{h}}| = i$ for $1 \leq i \leq n-1$.

We define

$$D_{\underline{h},i} := \det\left((w_0 A)_{J^i_0, J^i_{\underline{h}}}\right)$$

for all $1 \leq i \leq n-1$. We also set $D_n := \det(w_0 A)$. Hence, $D_{\underline{h},i}$ $(1 \leq i \leq n-1)$ and D_n are polynomials over the entries of A.

Given a weight $\lambda \in X_+(T)$, we now introduce an explicit model for the representation $H^0(\lambda)$, and then start some explicit calculation. Consider the space of polynomials on $\overline{G}_{/\mathbf{F}_p}$, which is denoted by $\mathcal{O}(\overline{G})$. The space $\mathcal{O}(\overline{G})$ has both a left action and a right action of $\overline{B}$ induced by right translation and left translation by $\overline{B}$ on $\overline{G}$ respectively. The fraction field of $\mathcal{O}(\overline{G})$ is denoted by $\mathcal{M}(\overline{G})$.

Consider the subspace

$$\mathcal{O}(\lambda) := \{f \in \mathcal{O}(\overline{G}) \mid f \cdot b = w_0\lambda(b)f \quad \forall b \in \overline{B}\},$$

which has a natural left $\overline{G}$-action by right translation. As the right action of $\overline{T}$ on $\mathcal{O}(\overline{G})$ is semisimple (and normalizes $\overline{U}$), we have a decomposition of algebraic representations of $\overline{G}$:

$$\mathcal{O}(\overline{G})^{\overline{U}} := \{f \in \mathcal{O}(\overline{G}) \mid f \cdot u = f \quad \forall u \in \overline{U}\} = \oplus_{\lambda \in X(\overline{T})}\mathcal{O}(\lambda). \tag{3.4.76}$$

It follows from the definition of the dual Weyl module as an algebraic induction that we have a natural isomorphism

$$H^0(\lambda) \cong \mathcal{O}(\lambda). \tag{3.4.77}$$

Note by [Jan03], Proposition II.2.6 that $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T)_+$.

We often write the weight λ explicitly as $(d_1, d_2, \cdots, d_n)$ where $d_i \in \mathbf{Z}$ for $1 \leq i \leq n$. We will restrict our attention to a p-restricted and dominant λ, i.e., $d_1 \geq d_2 \geq ... \geq d_n$ and $d_{i-1} - d_i < p$ for $2 \leq i \leq n$. We recall from the beginning of Section 3.4 the notation $(\cdot)_{\lambda'}$ for a weight space with respect to the weight λ'. We define Σ to be the set of $(n-1)$-tuple of integers $\underline{m} = (m_1, ..., m_{n-1})$ satisfying

$$0 \leq m_i \leq d_i - d_{i+1} \text{ for } 1 \leq i \leq n-1.$$

For each tuple $\underline{m}$, we can define a vector

$$v^{\mathrm{alg}}_{\underline{h},\underline{m}} := D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i}(D_{\underline{h},i})^{m_i}.$$

Proposition 3.4.78. *Let $\lambda = (d_1, d_2, \cdots, d_n) \in X_1(T)$. The set*

$$\{v^{\mathrm{alg}}_{\underline{h},\underline{m}} \mid \underline{m} \in \Sigma\} \tag{3.4.79}$$

forms a basis of $H^0(\lambda)^{\overline{U}_{\underline{h}}}$. Moreover, the weight of $v^{\mathrm{alg}}_{\underline{h},\underline{m}}$ is

$$\lambda - \left(\sum_{i=1}^{n-1} m_i \beta_{\underline{h},i}\right)$$

and thus each element in (3.4.79) has distinct weight.

Proof. We define

$${}^{\overline{U}_{\underline{h}}}\mathcal{O}(\overline{G})^{\overline{U}} := \{f \in \mathcal{O}(\overline{G}) \mid u_1 \cdot f = f \cdot u = f \quad \forall u \in \overline{U} \ \& \ \forall u_1 \in \overline{U}_{\underline{h}}\}$$

and

$${}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}} := \{f \in \mathcal{M}(\overline{G}) \mid u_1 \cdot f = f \cdot u = f \quad \forall u \in \overline{U} \ \& \ \forall u_1 \in \overline{U}_{\underline{h}}\}.$$

We consider a matrix A such that its entries $A_{i,j}$ are indefinite variables. Then we can formally do Bruhat decomposition

$$A = U_A w_0 T_{A,\underline{h}} U_{A,\underline{h}}$$

such that the entries of U_A, $T_{A,\underline{h}}$, $U_{A,\underline{h}}$ are rational functions of $A_{i,j}$ satisfying

$$(U_A)_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i > j, \end{cases}$$

$$(T_{A,\underline{h}})_{i,j} = \begin{cases} D_i(A) & \text{if } i = j; \\ D_{\underline{h},k}(A) & \text{if } (i,j) = \beta_{\underline{h},k}; \\ 0 & \text{otherwise }, \end{cases}$$

$$(U_{A,\underline{h}})_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i > j \text{ or } (i,j) = \beta_{\underline{h},k} \text{ for some } 1 \leq k \leq n-1. \end{cases}$$

For each rational function $f \in {}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}}$, we notice that f only depends on $T_{A,\underline{h}}$, which means that f is rational function of D_i for $1 \leq i \leq n$ and $D_{\underline{h},i}$ for $1 \leq i \leq n-1$. In other word, we have

$${}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}} = \mathbf{F}_p\left(D_1, \cdots, D_n, D_{\underline{h},1}, \cdots, D_{\underline{h},n-1}\right) \subseteq \mathcal{M}(\overline{G}).$$

Then we define

$${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{O}(\overline{G})^{\overline{U},\lambda} := \{f \in {}^{\overline{U}_{\underline{h}}}\mathcal{O}(\overline{G})^{\overline{U}} \mid x \cdot f = \lambda'(x)f, \text{ and } f \cdot x = \lambda(x)f \quad \forall x \in \overline{T}\}$$

and

$${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{M}(\overline{G})^{\overline{U},\lambda} := \{f \in {}^{\overline{U}_{\underline{h}}}\mathcal{M}(\overline{G})^{\overline{U}} \mid x \cdot f = \lambda'(x)f, \text{ and } f \cdot x = \lambda(x)f \quad \forall x \in \overline{T}\}.$$

Note that we have and an obvious inclusion

$${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{O}(\overline{G})^{\overline{U},\lambda} \subseteq {}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{M}(\overline{G})^{\overline{U},\lambda}.$$

We can also identify ${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{O}(\overline{G})^{\overline{U},\lambda}$ with $H^0(\lambda)^{\overline{U}_{\underline{h}}}_{\lambda'}$ via the isomorphism (3.4.77). By definition of D_i (resp. $D_{\underline{h},i}$) we know that they are $\overline{T}$-eigenvector with eigencharacter $\sum_{k=1}^{i}\epsilon_k$ (resp. $(\sum_{k=1}^{i+1}\epsilon_k) - \epsilon_{[i]'_{\underline{h}}}$) for $1 \leq i \leq n$ (resp. for $1 \leq i \leq n-1$). Therefore we observe that ${}^{\overline{U}_{\underline{h}},\lambda'}\mathcal{M}(\overline{G})^{\overline{U},\lambda}$ is one dimensional for any $\lambda, \lambda' \in X(T)$ and is spanned by

$$D_n^{d_n}\prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i}(D_{\underline{h},i})^{m_i}$$

where $\lambda = (d_1, \cdots, d_n)$ and

$$\lambda' = \lambda - \left(\sum_{i=1}^{n-1} m_i \beta_{\underline{h},i}\right).$$

As $\mathcal{O}(\overline{G})$ is a UFD and $D_i, D_{\underline{h},i}$ are irreducible, we deduce that

$$D_n^{d_n}\prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i}(D_{\underline{h},i})^{m_i} \in \mathcal{O}(\overline{G})$$

if and only if

$$0 \leq m_i \leq d_i - d_{i+1} \text{ for all } 1 \leq i \leq n-1$$

if and only if

$$H^0(\lambda)^{\overline{U}_{\underline{h}}}_{\lambda'} \neq 0$$

which finishes the proof. □

We consider the special case of Proposition 3.4.78 when $s = 1$ and $n_1 = n - 1$.

Corollary 3.4.80. *Let $\lambda = (d_1, d_2, \cdots, d_n) \in X_1(T)$. For $\lambda' \in X(T)$, we have*

$$\dim_{\mathbf{F}_p} H^0(\lambda)^{\overline{U}_1}_{\lambda'} \leq 1.$$

Moreover, the set of λ' such that the space above is nontrivial is described explicitly as follows: consider the set $\Sigma_{\{n-1\}}$ of $(n-1)$-tuple of integers $\underline{m} = (m_1, ..., m_{n-1})$ satisfying $m_i \leq d_i - d_{i+1}$ for $1 \leq i \leq n-1$, and let

$$v^{\text{alg}}_{\{n-1\},\underline{m}} := D_n^{d_n}\prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i}(D_{\{n-1\},i})^{m_i}.$$

Then the set

$$\{v^{\mathrm{alg}}_{\{n-1\},\underline{m}} \mid \underline{m} \in \Sigma_{\{n-1\}}\}$$

forms a basis of the space $H^0(\lambda)^{\overline{U}_1}$*, and the weight of the vector* $v^{\mathrm{alg}}_{\{n-1\},\underline{m}}$ *is*

$$(d_1 - \sum_{i=1}^{n-1} m_i, d_2 + m_1, ..., d_{n-1} + m_{n-2}, d_n + m_{n-1}).$$

Remark 3.4.81. *Corollary 3.4.80 essentially describes the decomposition of an irreducible algebraic representation of* GL_n *after restricting to a maximal Levi subgroup which is isomorphic to* $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$*. This classical result is crucial in the proof of Theorem 3.4.137.*

3.4.7 Some technical formula

In this section, we prove a technical formula that will be used in Section 3.4.8. The main result of this section is Proposition 3.4.101.

From now on, we assume that $(a_{n-1}, \cdots, a_0)$ is n-generic in the lowest alcove (c.f. Definition 3.4.5). We need to do some elementary calculation of Jacobi sums. For this purpose we need to define the following group operators for $2 \leq r \leq n-1$:

$$X_r^+ := \sum_{t \in \mathbf{F}_p} t^{p-2} u_{\sum_{i=r}^{n-1} \alpha_i}(t) \in \mathbf{F}_p[G(\mathbf{F}_p)],$$

and similarly

$$X_r^- := \sum_{t \in \mathbf{F}_p} t^{p-2} w_0 u_{\sum_{i=r}^{n-1} \alpha_i}(t) w_0 \in \mathbf{F}_p[G(\mathbf{F}_p)].$$

We notice that by definition we have the identification $X_r^+ = X_{\sum_{i=r}^{n-1} \alpha_i, 1}$, where $X_{\sum_{i=r}^{n-1} \alpha_i, 1}$ is defined in (3.4.8).

Lemma 3.4.82. *For a tuple of integers* $\underline{k} = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{|\Phi^+_{w_0}|}$*, we have*

$$X_r^+ \bullet S_{\underline{k}, w_0} = k_{r,n} S_{\underline{k}^{r,n,+}, w_0}$$

where $\underline{k}^{r,n,+} = (k_{i,j}^{r,n,+})$ *satisfies* $k_{r,n}^{r,n,+} = k_{r,n} - 1$*, and* $k_{i,j}^{r,n,+} = k_{i,j}$ *if* $(i,j) \neq (r,n)$*.*

Proof. This is just a special case of Lemma 3.4.11 when $\alpha_0 = \sum_{i=r}^n \alpha_i$ and $m = 1$. □

For the following lemma, we set

$$\mathbf{I} := \{(i_1, i_2, \cdots, i_s) \mid 1 \leq i_1 < i_2 < \cdots < i_s = n \text{ for some } 1 \leq s \leq n-1\}.$$

to lighten the notation.

Lemma 3.4.83. *Let* $X = (X_{i,j})_{1 \leq i,j \leq n}$ *be a matrix satisfying*

$$X_{i,j} = 0 \text{ if } 1 \leq j < i \leq n-1.$$

Then the determinant of X *is*

$$\det(X) = \sum_{(i_1, \cdots, i_s) \in \mathbf{I}} (-1)^{s-1} X_{n,i_1} \left(\prod_{j \neq i_k,\, 1 \leq k \leq s} X_{j,j} \right) \left(\prod_{k=1}^{s-1} X_{i_k, i_{k+1}} \right). \tag{3.4.84}$$

Proof. By definition of the determinant we know that

$$\det(X) = \sum_{w \in W} (-1)^{\ell(w)} \prod_{k=1}^{n} X_{k,w(k)}.$$

From the assumption on X, we know that each w that appears in the sum satisfies

$$w(k) < k \tag{3.4.85}$$

for all $2 \leq k \leq n-1$.

Assume that w has the decomposition into disjoint cycles

$$w = (i_1^1, i_2^1, \cdots, i_{n_1}^1) \cdots (i_1^m, i_2^m, \cdots, i_{n_m}^m) \tag{3.4.86}$$

where m is the number of disjoint cycles and $n_k \geq 2$ is the length for the k-th cycle appearing in the decomposition.

We observe that the largest integer in $\{i_j^k \mid 1 \leq j \leq n_k\}$ must be n for each $1 \leq k \leq m$ by condition (3.4.85). Therefore we must have $m = 1$ and we can assume without loss of generality that $i_{n_1}^1 = n$. It follows from the condition (3.4.85) that

$$i_j^1 < i_{j+1}^1$$

for all $1 \leq j \leq n_1 - 1$. Hence we can set

$$s := n_1, \quad i_1 := i_1^1, \cdots, i_s := i_{n_1}^1.$$

We observe that $\ell(w) = s - 1$ and the formula (3.4.84) follows. □

Recall from the beginning of Section 3.4.7 that we use the notation A_{J_1,J_2} for the submatrix of A consisting of the entries at the (i,j)-position with $i \in J_1, j \in J_2$, where J_1, J_2 are two subsets of $\{1, 2, \cdots, n\}$ with the same cardinality. For a pair of integers (m, r) with $1 \leq m \leq r-1 \leq n-2$, we let

$$J_0^{m,r} := \{1, 2, \cdots, r, n-m+1\}.$$

For a matrix $A \in U(\mathbf{F}_p)$, an element $t \in \mathbf{F}_p$, and a triple of integers (m, r, ℓ) satisfying $1 \leq m \leq r-1 \leq n-2$ and $1 \leq \ell \leq n-1$, we define some polynomials as follows:

$$\begin{cases} D_{m,r}(A,t) := \det\left(u_{\sum_{i=r}^{n-1} \alpha_i}(t) w_0 A w_0\right)_{J_0^{m,r}, J_0^{n-r+1}} & \text{when } 1 \leq m \leq r-1; \\ D_r^{(\ell)}(A,t) := \det\left(u_{\sum_{i=r}^{n-1} \alpha_i}(t) w_0 A w_0\right)_{J_0^{\ell}, J_0^{\ell}} & \text{when } 1 \leq \ell \leq n-r \end{cases} \tag{3.4.87}$$

We define the following subsets of $\mathbf{I}$: for each $1 \leq \ell \leq n-1$

$$\mathbf{I}_\ell := \{(i_1, i_2, \cdots, i_s) \in \mathbf{I} \mid n-\ell+1 \leq i_1 < i_2 < \cdots < i_s = n \text{ for some } 1 \leq s \leq \ell\}.$$

Note that we have natural inclusions

$$\mathbf{I}_\ell \subseteq \mathbf{I}_{\ell'} \subseteq \mathbf{I}$$

if $1 \leq \ell \leq \ell' \leq n-1$. In particular, $\mathbf{I}_1$ has a unique element (n). Similarly, for each $1 \leq \ell' \leq n-1$ we define

$$\mathbf{I}^{\ell'} := \{(i_1, i_2, \cdots, i_s) \mid 1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq n-\ell' < i_s = n \text{ for some } 1 \leq s \leq \ell'\},$$

and we set

$$\mathbf{I}_\ell^{\ell'} := \mathbf{I}_\ell \cap \mathbf{I}^{\ell'}$$

for all $1 \leq \ell' \leq \ell - 1 \leq n-2$. We often write $\underline{i} = (i_1, \cdots, i_s)$ for an arbitrary element of $\mathbf{I}$, and define the sign of $\underline{i}$ by

$$\varepsilon(\underline{i}) := (-1)^s.$$

We emphasize that all the matrices $\left(w_0 u_{\sum_{i=r}^{n-1} \alpha_i}(t) w_0 A w_0\right)_{J_1^{m,r}, J_2^{n-r+1}}$ for $1 \leq m \leq r-1$, and all the matrices $\left(w_0 u_{\sum_{i=r}^{n-1} \alpha_i}(t) w_0 A w_0\right)_{J_1^{\ell}, J_2^{\ell}}$ for $1 \leq \ell \leq n-r$, after multiplying a permutation matrix, satisfy the conditions on the matrix X in Lemma 3.4.83. Hence, by Lemma 3.4.83 we notice that

$$\begin{cases} D_{m,r}(A,t) = A_{m,r} + t f_{m,r}(A) & \text{when } 1 \leq m \leq r-1; \\ D_r^{(\ell)}(A,t) = 1 - t f_{r,n-\ell+1}(A) & \text{wehn } 1 \leq \ell \leq n-r; \end{cases} \tag{3.4.88}$$

where for all $1 \leq m \leq r-1$

$$f_{m,r}(A) := \sum_{\underline{i} \in \mathbf{I}_{n-r+1}} \left(\varepsilon(\underline{i}) A_{m,i_1} \prod_{j=2}^{s} A_{i_{j-1}, i_j} \right). \tag{3.4.89}$$

Let (m,r) be a tuple of integers with $1 \leq m \leq r-1 \leq n-2$. Given a tuple of integers $\underline{k} \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$, $\underline{i} = (i_1, i_2, \cdots, i_s) \in \mathbf{I}_{n-r+1}$, and an integer r' satisfying $1 \leq r' \leq r$, we define two tuples of integers

$$\underline{k}^{\underline{i},m,r} = (k_{i,j}^{\underline{i},m,r}) \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$$

and

$$\underline{k}^{\underline{i},m,r,r'} = (k_{i,j}^{\underline{i},m,r,r'}) \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$$

as follows:

$$k_{i,j}^{\underline{i},m,r} = \begin{cases} k_{m,r} - 1 & \text{if } (i,j) = (m,r) \text{ and } i_1 > r; \\ k_{m,i_1} + 1 & \text{if } (i,j) = (m,i_1) \text{ and } i_1 > r; \\ k_{m,r} & \text{if } (i,j) = (m,r) \text{ and } i_1 = r; \\ k_{i,j} + 1 & \text{if } (i,j) = (i_h, i_{h+1}) \text{ for } 1 \leq h \leq s-1; \\ k_{i,j} & \text{otherwise} \end{cases}$$

and

$$k_{i,j}^{\underline{i},m,r,r'} = \begin{cases} k_{r',n}^{\underline{i},m,r} - 1 & \text{if } (i,j) = (r',n); \\ k_{i,j}^{\underline{i},m,r} & \text{otherwise.} \end{cases}$$

Finally, we define one more tuple of integers $\underline{k}^{r,+} = (k_{i,j}^{r,+}) \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$ by

$$k_{i,j}^{r,+} := \begin{cases} k_{r,n} + 1 & \text{if } (i,j) = (r,n); \\ k_{i,j} & \text{otherwise.} \end{cases}$$

Lemma 3.4.90. *Fix two integers r and m such that $1 \leq m \leq r-1 \leq n-2$, and let $\underline{k} = (k_{i,j}) \in \{0,1,\cdots,p-1\}^{|\Phi^+_{w_0}|}$. Assume that $k_{i,j} = 0$ for $r+1 \leq j \leq n-1$ and that $k_{i,r} = 0$ for $i \neq m$, and assume further that*

$$a_{n-r} - a_1 + [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}]_1 + k_{m,r} < p.$$

Then we have

$$X_r^- \bullet S_{\underline{k},w_0}v_0 = k_{m,r}\sum_{\underline{i}\in\mathbf{I}_{n-r}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},m,r},w_0}v_0 + ([a_{n-r}-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1+k_{m,r})S_{\underline{k}^{r,+},w_0}v_0 - \sum_{\ell=2}^{n-r}(a_{n-r}-a_{\ell-1}+k_{m,r})\left(\sum_{\underline{i}\in\mathbf{I}_\ell\setminus\mathbf{I}_{\ell-1}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},r,n-\ell+1},w_0}v_0\right).$$

Proof. By the definition of X_r^-, we have

$$X_r^- \bullet S_{\underline{k},w_0}v_0 = \sum_{A\in U(\mathbf{F}_p),t\in\mathbf{F}_p}\left(t^{p-2}\left(\prod_{1\le i<j\le n}A_{i,j}^{k_{i,j}}\right)w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0\right)v_0. \tag{3.4.91}$$

For an element $w\in W$, we use E_w to denote the subset of $U(\mathbf{F}_p)\times\mathbf{F}_p$ consisting of all (A,t) such that

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0\in B(\mathbf{F}_p)wB(\mathbf{F}_p).$$

It is not difficult to see that if $E_w\neq\varnothing$ then $ww_0(i)=i$ for all $1\le i\le r-1$.

We define M_w to be

$$M_w := \sum_{(A,t)\in E_w}\left(t^{p-2}\left(\prod_{1\le i<j\le n}A_{i,j}^{k_{i,j}}\right)w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0\right)v_0.$$

By the definition of E_w, we deduce that there exist $A'\in U_w(\mathbf{F}_p)$, $A''\in U(\mathbf{F}_p)$, and $T\in T(\mathbf{F}_p)$ for each given $(A,t)\in E_w$ such that

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(t)w_0Aw_0 = A'wTA''. \tag{3.4.92}$$

Here, the entries of A', T and A'' are rational functions of t and the entries of A. We can rewrite (3.4.92) as

$$w_0u_{\sum_{h=r}^{n-1}\alpha_h}(-t)w_0A'w = Aw_0T^{-1}(T(A'')^{-1}T^{-1}) \tag{3.4.93}$$

In other words, the right side of (3.4.93) can also be viewed as the Bruhat decomposition of the left side. Therefore the entries of A, T, A'' can also be expressed as rational functions of the entries of A'.

For each $A'\in U_w(\mathbf{F}_p)$ and $w\in W$, we define

$$\begin{cases} D_{m,r}^w(A',t) := \det\left(\left(u_{\sum_{i=r}^{n-1}\alpha_i}(t)w_0A'w\right)_{J_0^{m,r},J_0^{n-r+1}}\right) & \text{when } 1\le m\le r-1;\\ D_r^{w,(\ell)}(A',t) := \det\left(\left(u_{\sum_{i=r}^{n-1}\alpha_i}(t)w_0A'w\right)_{J_0^{\ell},J_0^{\ell}}\right) & \text{when } 1\le \ell\le n-r. \end{cases} \tag{3.4.94}$$

Note that if $w=w_0$, then the definition in (3.4.94) specializes to (3.4.87). We notice that for a given matrix $A'\in U_w(\mathbf{F}_p)$, the equality (3.4.93) exists if and only if

$$D_r^{w,(\ell)}(A',-t)\neq 0 \text{ for all } 1\le\ell\le n-r. \tag{3.4.95}$$

On the other hand, we also notice that given a matrix $A\in U(\mathbf{F}_p)$, the equality (3.4.93) exists if and only if (3.4.95) holds.

By the Bruhat decomposition in (3.4.93), we have

$$T^{-1}=\mathrm{diag}\left(D_r^{w,(1)},\frac{D_r^{w,(2)}}{D_r^{w,(1)}},\cdots,\frac{D_r^{w,(n-r)}}{D_r^{w,(n-1-r)}},\frac{1}{D_r^{w,(n-r)}},1,\cdots,1\right) \tag{3.4.96}$$

in which we write $D_r^{w,(i)}$ for $D_r^{w,(i)}(A',-t)$ for brevity. We also have

$$A_{i,j}=\begin{cases} A'_{i,j} & \text{if } 1\le i<j\le n \text{ and } j\le r-1;\\ D_{m,r}^{w}(A',-t) & \text{if } (i,j)=(m,r);\\ \frac{A'_{i,n}}{D_r^{w,(1)}(A',-t)} & \text{if } 1\le i\le n-1 \text{ and } j=n. \end{cases} \tag{3.4.97}$$

We apply (3.4.92), (3.4.97) and (3.4.96) to M_w and get

$$M_w=\sum_{(A,t)\in E_w}\left(F(A',w,t)\left(\prod_{\substack{1\le i<j\le n\\ j\le r \text{ or } j=n}}(A'_{i,j})^{k_{i,j}}\right)A'w_0\right)v_0 \tag{3.4.98}$$

where

$$F(A',w,t):=t^{p-2}\left((D_{m,r}^{w})^{k_{m,r}}(D_r^{w,(1)})^{a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}}\prod_{s=2}^{n-r}(D_r^{w,(s)})^{a_s-a_{s-1}}\right)$$

in which we let $D_{m,r}^{w}:=D_{m,r}^{w}(A',-t)$ and $D_r^{w,(s)}:=D_r^{w,(s)}(A',-t)$ for brevity. We have discussed in (3.4.95) that $(A,t)\in E_w$ is equivalent to $(A',t)\in U_w(\mathbf{F}_p)\times\mathbf{F}_p$ satisfying the conditions in (3.4.95). As each $D_r^{w,(s)}(A',-t)$ appears in $F(A',w,t)$ with a positive power, we can automatically drop the condition (3.4.95) and reach

$$M_w=\sum_{(A,t)\in\overline{U}_w(\mathbf{F}_0)\times\mathbf{F}_p}\left(F(A',w,t)\left(\prod_{\substack{1\le i<j\le n\\ j\le r \text{ or } j=n}}(A'_{i,j})^{k_{i,j}}\right)A'w_0\right)v_0. \tag{3.4.99}$$

If $w\neq w_0$, then there exist a unique integer i_0 satisfying $r\le i_0\le n$ such that $ww_0(i_0)<i_0$ but $ww_0(i)=i$ for all $i_0+1\le i\le n$.

By applying Lemma 3.4.83 to $D_r^{w,(n+1-i_0)}(A',-t)$ (as $(u_{\sum_{i=r}^{n-1}\alpha_i}(t)w_0A'w)_{J_0^\ell,J_0^\ell}$ satisfy the condition of Lemma 3.4.83 after multiplying a permutation matrix), we deduce that

$$D_r^{w,(n+1-i_0)}(A',-t)=tf(A')$$

where $f(A')$ is certain polynomial of entries of A'.

Now we consider $F(A',w,t)$ as a polynomial of t. The minimal degree of monomials of t appearing in $F(A',w,t)$ is at least

$$p-2+a_{n+1-i_0}-a_{n-i_0}>p-1,$$

and the maximal degree of monomials of t appearing in $F(A',w,t)$ is

$$\begin{aligned} &p-2+k_{m,r}+[a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1+\sum_{s=2}^{n-r}a_s-a_{s-1}\\ &=p-2+k_{m,r}+[a_1-a_{n-1}-\sum_{i=1}^{n-1}k_{i,n}]_1+a_{n-r}-a_1\\ &<2(p-1). \end{aligned}$$

As a result, the degree of each monomials of t in $F(A', w, t)$ is not divisible by $p-1$. Hence, we conclude that

$$M_w = 0 \text{ for all } w \neq w_0$$

as we know that $\sum_{t\in\mathbf{F}_p} t^k \neq 0$ if and only if $p-1 \mid k$ and $k \neq 0$.

Finally, we calculate M_{w_0} explicitly using (3.4.99). Indeed, by applying (3.4.88), the monomials of t appearing in $F(A', w_0, t)$ is nothing else than

$$t^{p-1}\left(-k_{m,r}f_{m,r}(A') + [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}]_1 f_{r,n}(A') + \sum_{s=2}^{n-r}(a_s - a_{s-1})f_{r,n+1-s}(A')\right).$$

As $\sum_{t\in\mathbf{F}_p} t^{p-1} = -1$, we conclude that

$$X_r^- \bullet S_{\underline{k},w_0}v_0 = M_{w_0} = \sum_{A'\in U(\mathbf{F}_p)} \left(F_0(A')\left(\prod_{\substack{1\leq i<j\leq n\\ j\leq r \text{ or } j=n}} (A'_{i,j})^{k_{i,j}}\right) A'w_0\right) v_0 \tag{3.4.100}$$

where

$$F_0(A') := k_{m,r}f_{m,r}(A') - [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}]_1 f_{r,n}(A') - \sum_{s=2}^{n-r}(a_s - a_{s-1})f_{r,n+1-s}(A').$$

Recalling the explicit formula of $f_{m,r}$ and $f_{r,n+1-s}$ for $1 \leq s \leq n-r$ from (3.4.89) and then rewriting (3.4.100) as a sum of distinct monomials of entries of A' finishes the proof. □

Proposition 3.4.101. *Keep the assumptions and the notation of Lemma 3.4.90.*

Then we have

$$\begin{aligned} X_r^+ \bullet X_r^- \bullet S_{\underline{k},w_0}v_0 = k_{m,r}k_{r,n} &\sum_{\underline{i}\in\mathbf{I}_{n-r}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i},m,r},w_0}v_0 \\ &+ (k_{r,n}+1)\left([a_{n-r} - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}]_1 + k_{m,r}\right) S_{\underline{k}^{r,+},w_0}v_0 \\ &- k_{r,n}\sum_{\ell=2}^{n-r}(a_{n-r} - a_{\ell-1} + k_{m,r})\left(\sum_{\underline{i}\in\mathbf{I}_\ell\setminus\mathbf{I}_{\ell-1}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i},r,n-\ell+1},w_0}v_0\right). \end{aligned}$$

Proof. This is just a direct combination of Lemma 3.4.90 and Lemma 3.4.82. □

3.4.8 Proof of Theorem 3.4.35

The main target of this section is to prove Theorem 3.4.137. This theorem together with Corollary 3.4.67 immediately implies Theorem 3.4.35. We start this section by introducing some notation.

We first define a subset Λ_{w_0} of $\{0, \cdots, p-1\}^{|\Phi^+_{w_0}|}$ consisting of the tuples $\underline{k} = (k_{i,j})_{i,j}$ such that for each $1 \leq r \leq n-1$

$$\sum_{1\leq i\leq r<j\leq n} k_{i,j} = [a_0 - a_{n-1}]_1 + n - 2.$$

Note that the set Λ_{w_0} embeds into π_0 by sending $\underline{k}$ to $S_{\underline{k},w_0}v_0$. Moreover, this family of vectors $\{S_{\underline{k},w_0}v_0 \mid \underline{k} \in \Lambda_{w_0}\}$ shares the same eigencharacter by Lemma 3.4.19.

We define $\underline{k}^\sharp \in \Lambda_{w_0}$ where $\underline{k}^\sharp = (k^\sharp_{i,j})$ is defined by $k^\sharp_{1,n} = [a_0 - a_{n-1}]_1 + n - 2$ and $k^\sharp_{i,j} = 0$ otherwise. We define $V^\sharp$ to be the subrepresentation of π_0 generated by $S_{\underline{k}^\sharp,w_0}v_0$. We also need to

define several useful elements and subsets of Λ_{w_0}. For each $1 \leq r \leq n-1$, we define $\underline{k}^{\sharp,r} \in \Lambda_{w_0}$ where $\underline{k}^{\sharp,r} = (k_{i,j}^{\sharp,r})$ is defined by

$$k_{i,j}^{\sharp,r} := \begin{cases} n-2+[a_0-a_{n-1}]_1 & \text{if } 2 \leq j = i+1 \leq r; \\ n-2+[a_0-a_{n-1}]_1 & \text{if } (i,j) = (r,n); \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have

$$\underline{k}^{\sharp,1} = \underline{k}^{\sharp} \text{ and } \underline{k}^{\sharp,n-1} = \underline{k}^{0} \tag{3.4.102}$$

where $\underline{k}^0$ is defined in (3.4.33).

For each $1 \leq r \leq n-2$ and $0 \leq s \leq [a_0-a_{n-1}]_1+n-2$, we define a tuple $\underline{k}^{\sharp,r,s} \in \Lambda_{w_0}$ as follows:

$$k_{i,j}^{\sharp,r,s} = \begin{cases} n-2+[a_0-a_{n-1}] & \text{if } 2 \leq j = i+1 \leq r; \\ n-2+[a_0-a_{n-1}]_1-s & \text{if } (i,j) = (r,r+1); \\ s & \text{if } (i,j) = (r,n); \\ n-2+[a_0-a_{n-1}]_1-s & \text{if } (i,j) = (r+1,n); \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have

$$\underline{k}^{\sharp,r,0} = \underline{k}^{\sharp,r+1} \text{ and } \underline{k}^{\sharp,r,[a_0-a_{n-1}]_1+n-2} = \underline{k}^{\sharp,r} \tag{3.4.103}$$

for each $1 \leq r \leq n-2$.

We now introduce the rough idea of the proof of Theorem 3.4.137. The first obstacle to generalize the method of Proposition 3.1.2 in [HLM17] is that V_0 does not admit a structure as $\overline{G}$-representation in general. Our method to resolve this difficulty is to replace $S_{\underline{k}^0,w_0}v_0$ by $S_{\underline{k}^\sharp,w_0}v_0$. We prove in Proposition 3.4.133 that $V^\sharp$ admits a structure as $\overline{G}$-representation and actually can be identified with a dual Weyl module $H^0(\mu_0^{w_0})$. (The notation $\mu_0^{w_0}$ will be clear in the following.) Now it remains to prove that

$$S_{\underline{k}^\sharp,w_0}v_0 \in V_0 \tag{3.4.104}$$

to deduce Theorem 3.4.137. We will prove in Proposition 3.4.124 that

$$S_{\underline{k}^{\sharp,r,s-1},w_0}v_0 \in V_0 \Longrightarrow S_{\underline{k}^{\sharp,r,s},w_0}v_0 \in V_0$$

for each $1 \leq r \leq n-2$ and $1 \leq s \leq [a_0-a_{n-1}]_1+n-2$. As a result, we can thus pass from $S_{\underline{k}^0,w_0}v_0 \in V_0$ to $S_{\underline{k}^{\sharp,r},w_0}v_0 \in V_0$ for $r = n-2, n-3, \cdots, 1$. The identification $\underline{k}^\sharp = \underline{k}^{\sharp,1}$ thus gives us (3.4.104).

We firstly state three direct Corollaries of Proposition 3.4.101. It is easy to check that each tuple $\underline{k}$ appearing in the following Corollaries satisfies the assumption in Proposition 3.4.101.

Corollary 3.4.105. *For each $2 \leq r \leq n-1$ and $0 \leq s \leq [a_0-a_{n-1}]_1+n-3$, we have*

$$\begin{aligned} X_r^+ \bullet X_r^- \bullet S_{\underline{k}^{\sharp,r-1,s},w_0}v_0 &= ([a_0-a_{n-1}]_1+n-2-s)^2 \sum_{\underline{i} \in \mathbf{I}_{n-r}} \varepsilon(\underline{i}) S_{(\underline{k}^{\sharp,r-1,s})^{\underline{i},m,r,r},w_0}v_0 \\ &+ ([a_{n-r}-a_{n-1}]_1-s)([a_0-a_{n-1}]_1+n-1-s)S_{\underline{k}^{r-1,s},w_0}v_0 \\ &- ([a_0-a_{n-1}]_1+n-2-s)\sum_{\ell=2}^{n-r}(a_{n-r}-a_{\ell-1}+[a_0-a_{n-1}]_1+n-2-s) \\ &\qquad \cdot \left(\sum_{\underline{i} \in \mathbf{I}_\ell \backslash \mathbf{I}_{\ell-1}} \varepsilon(\underline{i}) S_{(\underline{k}^{\sharp,r-1,s})^{\underline{i},r,n-\ell+1,r},w_0}v_0 \right). \end{aligned}$$

Corollary 3.4.106. *Fix two integers r and m such that $1 \leq m \leq r-1 \leq n-2$, and let $\underline{k} = (k_{i,j})$ be a tuple of integers in Λ_{w_0} such that*

$$k_{i,j} = \begin{cases} 0 & \text{if } r+1 \leq j \leq n-1; \\ 0 & \text{if } i \neq m \text{ and } j = r; \\ 0 & \text{if } r+1 \leq i \leq n-1 \text{ and } j = n; \\ 1 & \text{if } (i,j) = (m,r); \\ 1 & \text{if } (i,j) = (r,n). \end{cases}$$

Then we have

$$X_r^+ \bullet X_r^- \bullet S_{\underline{k},w_0} v_0 = \sum_{\underline{i} \in \mathbf{I}_{n-r}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i},m,r,r},w_0} v_0 + 2(a_{n-r} - a_0 - n + 3) S_{\underline{k},w_0} v_0 \\ - \sum_{\ell=2}^{n-r} (a_{n-r} - a_{\ell-1} + 1) \left(\sum_{\underline{i} \in \mathbf{I}_\ell \backslash \mathbf{I}_{\ell-1}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i},r,n-\ell+1,r},w_0} v_0 \right).$$

Corollary 3.4.107. *Fix two integers r and m such that $1 \leq m \leq r-1 \leq n-2$, and let $\underline{k} = (k_{i,j})$ be a tuple of integers in Λ_{w_0} such that*

$$k_{i,j} = \begin{cases} 0 & \text{if } r \leq j \leq n-1; \\ 0 & \text{if } r \leq i \leq n-1 \text{ and } j = n. \end{cases}$$

Then we have

$$X_r^+ \bullet X_r^- \bullet S_{\underline{k},w_0} v_0 = (a_{n-r} - a_0 - n + 2) S_{\underline{k},w_0} v_0 \\ - \sum_{\ell=2}^{n-r} (a_{n-r} - a_{\ell-1} + 1) \left(\sum_{\underline{i} \in \mathbf{I}_\ell \backslash \mathbf{I}_{\ell-1}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i},r,n-\ell+1,r},w_0} v_0 \right).$$

We now define the following constants in $\mathbf{F}_p$:

$$\begin{cases} \mathbf{c}_\ell & := \prod_{k=1}^{\ell-1} (a_k - a_0 - n + 2 + k)^{2^{\ell-k-1}}; \\ \mathbf{c}'_\ell & := (a_\ell - a_0 - n + 3 + \ell) \mathbf{c}_\ell \end{cases}$$

for all $1 \leq \ell \leq n-1$ where we understand $\mathbf{c}_1$ to be 1. As the tuple $(a_{n-1}, \cdots, a_0)$ is n-generic in the lowest alcove, we notice that $\mathbf{c}_\ell \neq 0 \neq \mathbf{c}'_\ell$ for all $1 \leq \ell \leq n-1$. By definition of $\mathbf{c}_k$ and $\mathbf{c}'_k$ one can also easily check that

$$\prod_{k=1}^{\ell-1} (\mathbf{c}'_k - \mathbf{c}_k) = \mathbf{c}_\ell. \tag{3.4.108}$$

We also define inductively the constants: for each $1 \leq \ell \leq n-1$

$$\mathbf{d}_{\ell,\ell'} := \begin{cases} 2(a_\ell - a_0 - n + 3) & \text{if } \ell' = 0; \\ \mathbf{c}'_{\ell'} \mathbf{d}_{\ell,\ell'-1} - (a_\ell - a_{\ell'} + 1) \mathbf{c}_{\ell'} \prod_{k=1}^{\ell'-1} (\mathbf{c}'_k - \mathbf{c}_k) & \text{if } 1 \leq \ell' \leq \ell - 1. \end{cases}$$

We further define inductively a sequence of group operators $\mathcal{Z}_\ell$ as follows:

$$\mathcal{Z}_1 := \mathbf{d}_{1,0} \mathrm{Id} - X_{n-1}^+ \bullet X_{n-1}^- \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

and

$$\mathcal{Z}_\ell := \mathbf{d}_{\ell,\ell-1} \mathrm{Id} - \left(\mathcal{Z}_{\ell-1} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X_{n-\ell}^+ \bullet X_{n-\ell}^- \right) \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

for each $2 \leq \ell \leq n-2$.

Lemma 3.4.109. *For $1 \leq \ell \leq n-1$, we have the identity*

$$\mathbf{d}_{\ell,\ell-1} = (a_\ell - a_0 - n + 2)\left(\prod_{k=1}^{\ell-1} \mathbf{c}'_k\right) + \mathbf{c}'_\ell.$$

Proof. During the proof of this lemma, we will keep using the following obvious identity with two variables

$$ab = (a+1)(b-1) + a - b + 1 \tag{3.4.110}$$

By definition of $\mathbf{d}_{\ell,\ell-1}$ we know that

$$\mathbf{d}_{\ell,\ell-1} = 2(a_\ell - a_0 - n + 3)\prod_{k=1}^{\ell-1} \mathbf{c}'_k - \sum_{\ell'=1}^{\ell-1}\left((a_\ell - a_{\ell'} + 1)\mathbf{c}_{\ell'}\left(\prod_{k=1}^{\ell'-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=\ell'+1}^{\ell-1}\mathbf{c}'_k\right)\right)$$

and therefore

$$\begin{aligned}\mathbf{d}_{\ell,\ell-1} - (a_\ell - a_0 - n + 2)\left(\prod_{k=1}^{\ell-1}\mathbf{c}'_k\right) &= (a_\ell - a_0 - n + 4)\prod_{k=1}^{\ell-1}\mathbf{c}'_k \\ &\quad - \sum_{\ell'=1}^{\ell-1}\left((a_\ell - a_{\ell'} + 1)\mathbf{c}_{\ell'}\left(\prod_{k=1}^{\ell'-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=\ell'+1}^{\ell-1}\mathbf{c}'_k\right)\right).\end{aligned}$$

Now we prove inductively that for each $1 \leq j \leq \ell - 1$

$$\begin{aligned}\mathbf{d}_{\ell,\ell-1} - (a_\ell - a_0 - n + 2)\left(\prod_{k=1}^{\ell-1}\mathbf{c}'_k\right) &= (a_\ell - a_0 - n + 3 + j)\left(\prod_{k=1}^{j-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=j}^{\ell-1}\mathbf{c}'_k\right) \\ &\quad - \sum_{\ell'=j}^{\ell-1}\left((a_\ell - a_{\ell'} + 1)\mathbf{c}_{\ell'}\left(\prod_{k=1}^{\ell'-1}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=\ell'+1}^{\ell-1}\mathbf{c}'_k\right)\right).\end{aligned} \tag{3.4.111}$$

By the identity (3.4.110), one can easily deduce that

$$\begin{aligned}&(a_\ell - a_0 - n + 3 + j)\mathbf{c}'_j - (a_\ell - a_j + 1)\mathbf{c}_j \\ &\quad = \left[(a_\ell - a_0 - n + 3 + j)(a_j - a_0 - n + 3 + j) - (a_\ell - a_j + 1)\right]\mathbf{c}_j \\ &\quad = (a_\ell - a_0 - n + 4 + j)(a_j - a_0 - n + 2 + j)\mathbf{c}_j \\ &\quad = (a_\ell - a_0 - n + 4 + j)(\mathbf{c}'_j - \mathbf{c}_j).\end{aligned}$$

Hence, we get the identity:

$$\begin{aligned}&\left[(a_\ell - a_0 - n + 3 + j)\mathbf{c}'_j - (a_\ell - a_j + 1)\mathbf{c}_j\right]\left(\prod_{k=j+1}^{\ell-1}\mathbf{c}'_k\right)\left(\prod_{k=1}^{j-1}(\mathbf{c}'_k - \mathbf{c}_k)\right) \\ &\qquad = (a_\ell - a_0 - n + 4 + j)\left(\prod_{k=1}^{j}(\mathbf{c}'_k - \mathbf{c}_k)\right)\left(\prod_{k=j+1}^{\ell-1}\mathbf{c}'_k\right).\end{aligned} \tag{3.4.112}$$

Thus, if the equation (3.4.111) holds for j, we can deduce that it also holds for $j+1$. By taking $j = \ell - 1$ and using the equation (3.4.112) once more, we can deduce that

$$\mathbf{d}_{\ell,\ell-1} - (a_\ell - a_0 - n + 2)\left(\prod_{k=1}^{\ell-1}\mathbf{c}'_k\right) = (a_\ell - a_0 - n + 3 + \ell)\left(\prod_{k=1}^{\ell-1}(\mathbf{c}'_k - \mathbf{c}_k)\right).$$

Hence, by the equation (3.4.108), one finishes the proof. □

Proposition 3.4.113. *Fix two integers r and m such that $1 \leq m \leq r-1 \leq n-2$.*

(i) *Let $\underline{k} = (k_{i,j})$ be as in Corollary 3.4.106. Then we have*

$$\mathcal{Z}_{n-r} \bullet S_{\underline{k},w_0} = \mathbf{c}_{n-r} S_{\underline{k}',w_0} \tag{3.4.114}$$

where $\underline{k}' = (k'_{i,j})$ is defined as follows:

$$k'_{i,j} := \begin{cases} 0 & \text{if } (i,j) = (m,r) \text{ or } (i,j) = (r,n); \\ 1 & \text{if } (i,j) = (m,n); \\ k_{i,j} & \text{otherwise.} \end{cases}$$

(ii) *Let $\underline{k} = (k_{i,j})$ be as in Corollary 3.4.107. Then we have*

$$\mathcal{Z}_{n-r} \bullet S_{\underline{k},w_0} = \mathbf{c}'_{n-r} S_{\underline{k},w_0}. \tag{3.4.115}$$

We prove this proposition by a series of lemmas.

Lemma 3.4.116. *Proposition 3.4.113 is true for $r = n-1$.*

Proof. For part (i) of Proposition 3.4.113, by applying Corollary 3.4.106 to the case $r = n-1$ we deduce that

$$X^+_{n-1} \bullet X^-_{n-1} \bullet S_{\underline{k},w_0} v_0 = 2(a_1 - a_0 - n + 3) S_{\underline{k},w_0} v_0 - S_{\underline{k}^{\underline{i}_0,m,n-1,n-1},w_0} v_0$$

where $\underline{i}_0 = \{n-1, n\}$. Hence, part (i) of the proposition follows directly from the definition of $\mathcal{Z}_1$ and $\mathbf{c}_1$.

For part (ii) of Proposition 3.4.113, again by Corollary 3.4.107 to the case $r = n-1$ we deduce that

$$X^+_{n-1} \bullet X^-_{n-1} \bullet S_{\underline{k},w_0} v_0 = (a_1 - a_0 - n + 2) S_{\underline{k},w_0} v_0.$$

Then we have

$$\mathcal{Z}_1 \bullet S_{\underline{k},w_0} v_0 = (a_1 - a_0 - n + 4) S_{\underline{k},w_0} v_0$$

and part (ii) of the proposition follows directly from the definition of $\mathbf{c}'_1$. □

Lemma 3.4.117. *Let ℓ be an integer with $2 \leq \ell \leq n-1$. If Proposition 3.4.113 is true for $r \geq n-\ell+1$, then it is true for $r = n-\ell$.*

Proof. We prove part (ii) first. Assume that (3.4.115) holds for $r \geq n-\ell+1$. In fact, for a Jacobi sum $S_{\underline{k},w_0}$ satisfying the conditions in the (3.4.115) for $r = n-\ell$, we have

$$X^+_{n-\ell} \bullet X^-_{n-\ell} \bullet S_{\underline{k},w_0} v_0 = (a_\ell - a_0 - n + 2) S_{\underline{k},w_0} v_0$$

by Corollary 3.4.107. Then we can deduce

$$\mathcal{Z}_{\ell-1} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{n-\ell} \bullet X^-_{n-\ell} \bullet S_{\underline{k},w_0} v_0 = (a_\ell - a_0 - n + 2) \left(\prod_{s=1}^{\ell-1} \mathbf{c}'_s \right) S_{\underline{k},w_0} v_0$$

from the assumption of the Lemma. Hence, by definition of $\mathcal{Z}_\ell$, we have

$$\begin{aligned} \mathcal{Z}_\ell \bullet S_{\underline{k},w_0} v_0 &= \mathbf{d}_{\ell,\ell-1} S_{\underline{k},w_0} v_0 - \mathcal{Z}_{\ell-1} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{n-\ell} \bullet X^-_{n-\ell} \bullet S_{\underline{k},w_0} v_0 \\ &= \left(\mathbf{d}_{\ell,\ell-1} - (a_\ell - a_0 - n + 2) \left(\prod_{s=1}^{\ell-1} \mathbf{c}'_s \right) \right) S_{\underline{k},w_0} v_0 \\ &= \mathbf{c}'_\ell S_{\underline{k},w_0} v_0 \end{aligned}$$

where the last equality follows from Lemma 3.4.109.

Now we turn to part (i). Assume that (3.4.114) holds for $r \geq n-\ell+1$. We will prove inductively that for each ℓ' satisfying $1 \leq \ell' \leq \ell-1$, we have

$$\begin{aligned}\mathcal{Z}_{\ell'} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{n-\ell} \bullet X^-_{n-\ell} \bullet S_{\underline{k},w_0} v_0 \\ = \mathbf{d}_{\ell,\ell'} S_{\underline{k},w_0} v_0 + \left(\prod_{s=1}^{\ell'} (\mathbf{c}'_s - \mathbf{c}_s)\right) \left(\sum_{\underline{i} \in \mathbf{I}_\ell^{\ell'}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i}, m, n-\ell, n-\ell}, w_0} v_0\right) \\ + \left(\prod_{s=1}^{\ell'} (\mathbf{c}'_s - \mathbf{c}_s)\right) \left(\sum_{h=\ell'+1}^{\ell-1} (a_\ell - a_h + 1) \sum_{\underline{i} \in \mathbf{I}_h^{\ell'} \setminus \mathbf{I}_{h+1}^{\ell'}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i}, n-\ell, n-h, n-\ell}, w_0} v_0\right) \quad (3.4.118)\end{aligned}$$

We begin with studying some basic properties of the index sets $\mathbf{I}_h^{\ell'}$. First of all, the set $\mathbf{I}_{\ell'+1}^{\ell'} \setminus \mathbf{I}_{\ell'+2}^{\ell'}$ has a unique element, which is precisely $\underline{i} = \{n-\ell'-1, n\}$. Furthermore, there is a natural map of sets

$$\mathrm{res}_{\ell'} : \mathbf{I}_h^{\ell'} \to \mathbf{I}_h^{\ell'+1}$$

for all $\ell'+2 \leq h \leq \ell$ defined by eliminating the element $n-\ell'$ from $\underline{i} \in \mathbf{I}_h^{\ell'}$ if $n-\ell' \in \underline{i}$. In other words, for each $\underline{i} \in \mathbf{I}_h^{\ell'+1}$, we have

$$\mathrm{res}_{\ell'}^{-1}(\{\underline{i}\}) = \{\underline{i}, \underline{i} \cup \{n-\ell'\}\} \subseteq \mathbf{I}_h^{\ell'}.$$

We use the shorten notation

$$\underline{i}^{\ell'} := \underline{i} \cup \{n-\ell'\}$$

for each $\underline{i} \in \mathbf{I}_h^{\ell'+1}$. Note in particular that $\varepsilon(\underline{i}) = -\varepsilon(\underline{i}^{\ell'})$.

Given an arbitrary $\underline{i} \in \mathbf{I}_h^{\ell'+1}$ for $\ell'+2 \leq h \leq \ell-1$, then $S_{\underline{k}^{\underline{i}, n-\ell, n-h, n-\ell}, w_0}$ (resp. $S_{\underline{k}^{\underline{i}^{\ell'}, n-\ell, n-h, n-\ell}, w_0}$) satisfies the conditions before the equation (3.4.115) (resp. (3.4.114)). As a result, by the assumption that Proposition 3.4.113 is true for $r = n-\ell'-1$, we deduce that

$$\mathcal{Z}_{\ell'+1} \bullet \left(S_{\underline{k}^{\underline{i}, n-\ell, n-h, n-\ell}, w_0} v_0 - S_{\underline{k}^{\underline{i}^{\ell'}, n-\ell, n-h, n-\ell}, w_0} v_0\right) = \left(\mathbf{c}'_{\ell'+1} - \mathbf{c}_{\ell'+1}\right) S_{\underline{k}^{\underline{i}, n-\ell, n-h, n-\ell}, w_0} v_0. \quad (3.4.119)$$

Similarly, we have

$$\mathcal{Z}_{\ell'+1} \bullet \left(S_{\underline{k}^{\underline{i}, m, n-\ell, n-\ell}, w_0} v_0 - S_{\underline{k}^{\underline{i}^{\ell'}, m-\ell, n-\ell}, w_0} v_0\right) = \left(\mathbf{c}'_{\ell'+1} - \mathbf{c}_{\ell'+1}\right) S_{\underline{k}^{\underline{i}, m, n-\ell, n-\ell}, w_0} v_0 \quad (3.4.120)$$

for each $\underline{i} \in \mathbf{I}_\ell^{\ell'+1}$. We also have

$$\mathcal{Z}_{\ell'+1} \bullet S_{\underline{k}, w_0} v_0 = \mathbf{c}'_{\ell'+1} S_{\underline{k}, w_0} v_0 \quad (3.4.121)$$

by (3.4.115) for $r = n-\ell'-1$, and

$$\mathcal{Z}_{\ell'+1} \bullet S_{\underline{k}^{\underline{i}_0, n-\ell, n-\ell'-1, n-\ell}, w_0} v_0 = \mathbf{c}_{\ell'+1} S_{\underline{k}, w_0} v_0 \quad (3.4.122)$$

by (3.4.114) for $r = n-\ell'-1$ where $\underline{i}_0 = \{n-\ell'-1, n\}$.

Now assume that (3.4.118) is true for some $1 \leq \ell' \leq \ell-2$. Then by combing (3.4.119), (3.4.120), (3.4.121) and (3.4.122), we have

$$\begin{aligned}\mathcal{Z}_{\ell'+1} \bullet \cdots \bullet \mathcal{Z}_1 \bullet X^+_{n-\ell} \bullet X^-_{n-\ell} \bullet S_{\underline{k},w_0} v_0 \\ = \mathbf{d}_{\ell,\ell'} \mathcal{Z}_{\ell'+1} \bullet S_{\underline{k},w_0} v_0 + \left(\prod_{s=1}^{\ell'} (\mathbf{c}'_s - \mathbf{c}_s)\right) \mathcal{Z}_{\ell'+1} \bullet \left(\sum_{\underline{i} \in \mathbf{I}_\ell^{\ell'}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i}, m, n-\ell, n-\ell}, w_0} v_0\right) \\ + \left(\prod_{s=1}^{\ell'} (\mathbf{c}'_s - \mathbf{c}_s)\right) \mathcal{Z}_{\ell'+1} \bullet \left(\sum_{h=\ell'+1}^{\ell-1} (a_\ell - a_h + 1) \sum_{\underline{i} \in \mathbf{I}_h^{\ell'} \setminus \mathbf{I}_{h+1}^{\ell'}} \varepsilon(\underline{i}) S_{\underline{k}^{\underline{i}, n-\ell, n-h, n-\ell}, w_0} v_0\right)\end{aligned}$$

which is the same as

$$\mathbf{c}'_{\ell'}\mathbf{d}_{\ell,\ell'}S_{\underline{k},w_0}v_0 + \left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s - \mathbf{c}_s)\right)(X+Y+Z) \tag{3.4.123}$$

where

$$X = (a_\ell - a_{\ell'}+1)\mathcal{Z}_{\ell'+1}\bullet S_{\underline{k}^{\underline{i}_0,n-\ell,n-\ell'-1,n-\ell},w_0}v_0,$$

$$Y = \sum_{\underline{i}\in\mathbf{I}_\ell^{\ell'+1}}\varepsilon(\underline{i})\mathcal{Z}_{\ell'+1}\bullet\left(S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0 - S_{\underline{k}^{\underline{i}^{\ell'},m,n-\ell,n-\ell},w_0}v_0\right),$$

and

$$Z = \sum_{h=\ell'+2}^{\ell-1}(a_\ell - a_h+1)\sum_{\underline{i}\in\mathbf{I}_h^{\ell'+1}\setminus\mathbf{I}_{h+1}^{\ell'+1}}\varepsilon(\underline{i})\mathcal{Z}_{\ell'+1}\bullet\left(S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0 - S_{\underline{k}^{\underline{i}^{\ell'},n-\ell,n-h,n-\ell},w_0}v_0\right).$$

One can also readily check that (3.4.123) is also the same as

$$\begin{aligned}
&\left(\mathbf{c}'_{\ell'+1}\mathbf{d}_{\ell,\ell'} + \mathbf{c}_{\ell'+1}\left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s-\mathbf{c}_s)\right)(a_\ell - a_{\ell'}+1)\right)S_{\underline{k},w_0}v_0\\
&\quad + \left(\prod_{s=1}^{\ell'+1}(\mathbf{c}'_s-\mathbf{c}_s)\right)\left(\sum_{\underline{i}\in\mathbf{I}_\ell^{\ell'+1}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},m,n-\ell,n-\ell},w_0}v_0\right)\\
&\quad + \left(\prod_{s=1}^{\ell'+1}(\mathbf{c}'_s-\mathbf{c}_s)\right)\left(\sum_{h=\ell'+2}^{\ell-1}(a_\ell-a_h+1)\sum_{\underline{i}\in\mathbf{I}_h^{\ell'+1}\setminus\mathbf{I}_{h+1}^{\ell'+1}}\varepsilon(\underline{i})S_{\underline{k}^{\underline{i},n-\ell,n-h,n-\ell},w_0}v_0\right),
\end{aligned}$$

which finishes the proof of (3.4.118), as we have

$$\mathbf{d}_{\ell,\ell'+1} = \mathbf{c}'_{\ell'+1}\mathbf{d}_{\ell,\ell'} + \mathbf{c}_{\ell'+1}\left(\prod_{s=1}^{\ell'}(\mathbf{c}'_s-\mathbf{c}_s)\right)(a_\ell - a_{\ell'}+1)$$

by definition.

Note that (3.4.118) for each $1\le\ell'\le\ell-1$ then follows from Corollary 3.4.106 for $r=n-\ell$. Note that the case $\ell'=\ell-1$ for (3.4.118) is just the following

$$\mathcal{Z}_{\ell-1}\bullet\cdots\bullet\mathcal{Z}_1\bullet X^+_{n-\ell}\bullet X^-_{n-\ell}\bullet S_{\underline{k},w_0}v_0 = \mathbf{d}_{\ell,\ell-1}S_{\underline{k},w_0}v_0 - \left(\prod_{s=1}^{\ell-1}(\mathbf{c}'_s-\mathbf{c}_s)\right)S_{\underline{k}^{\underline{i}_1,m,n-\ell,n-\ell},w_0}v_0$$

where $\underline{i}_1=\{n\}$.

Finally, (3.4.114) for $r=n-\ell$ follows from the equation above together with the definition of $\mathcal{Z}_\ell$ and the identity (3.4.108). □

Proof of Proposition 3.4.113. It follows easily from Lemma 3.4.116 and Lemma 3.4.117. □

Proposition 3.4.124. *For each $1\le r\le n-2$ and $1\le s\le[a_0-a_{n-1}]_1+n-2$, if $S_{\underline{k}^\sharp,r,s-1},w_0v_0\in V_0$, then $S_{\underline{k}^\sharp,r,s},w_0v_0\in V_0$.*

Proof. By Proposition 3.4.113 and its proof, we can deduce the following equalities

$$\mathcal{Z}_{n-2-r}\bullet\cdots\bullet\mathcal{Z}_1\bullet S_{\underline{k}^\sharp,r,s-1},_{w_0}v_0=\left(\prod_{\ell=1}^{n-2-r}\mathbf{c}'_\ell\right)S_{\underline{k}^\sharp,r,s-1},_{w_0}v_0,$$

$$\mathcal{Z}_{n-2-r}\bullet\cdots\bullet\mathcal{Z}_1\bullet\left(\sum_{\underline{i}\in\mathbf{I}_{n-1-r}}\varepsilon(\underline{i})S_{(\underline{k}^\sharp,r,s-1})^{\underline{i},r,r+1,r+1},_{w_0}v_0\right)=-\left(\prod_{\ell=1}^{n-2-r}(\mathbf{c}'_\ell-\mathbf{c}_\ell)\right)S_{\underline{k}^\sharp,r,s},_{w_0}v_0,$$

and

$$\mathcal{Z}_{n-2-r}\bullet\cdots\bullet\mathcal{Z}_1\bullet\left(\sum_{\underline{i}\in\mathbf{I}_\ell\backslash\mathbf{I}_{\ell-1}}\varepsilon(\underline{i})S_{(\underline{k}^\sharp,r,s-1})^{\underline{i},r+1,n-\ell+1,r+1},_{w_0}v_0\right)$$
$$=\mathbf{c}_\ell\left(\prod_{h=1}^{\ell-1}(\mathbf{c}'_h-\mathbf{c}_h)\right)\left(\prod_{h=\ell+1}^{n-2-r}\mathbf{c}'_h\right)S_{\underline{k}^\sharp,r,s-1},_{w_0}v_0$$

for each $1\leq\ell\leq n-2-r$. Therefore by replacing (r,s) in Corollary 3.4.105 by $(r+1,s-1)$, we can deduce that

$$\begin{aligned}&\mathcal{Z}_{n-2-r}\bullet\cdots\bullet\mathcal{Z}_1\bullet X^+_{r+1}\bullet X^-_{r+1}\bullet S_{\underline{k}^\sharp,r,s-1},_{w_0}v_0\\&=-([a_0-a_{n-1}]_1+n-1-s)^2\left(\prod_{\ell=1}^{n-2-r}(\mathbf{c}'_\ell-\mathbf{c}_\ell)\right)S_{\underline{k}^\sharp,r,s},_{w_0}v_0+\mathbf{C}S_{\underline{k}^\sharp,r,s-1},_{w_0}v_0\\&=-([a_0-a_{n-1}]_1+n-1-s)^2\mathbf{c}_{n-1-r}S_{\underline{k}^\sharp,r,s},_{w_0}v_0+\mathbf{C}S_{\underline{k}^\sharp,r,s-1},_{w_0}v_0\end{aligned}$$

for certain constant $\mathbf{C}\in\mathbf{F}_p$. Note that we use the identity (3.4.108) for the last equality .

By our assumption, we know that $S_{\underline{k}^\sharp,r,s-1},_{w_0}v_0\in V_0$. Hence we can deduce

$$S_{\underline{k}^\sharp,r,s},_{w_0}v_0\in V_0$$

since $([a_0-a_{n-1}]_1+n-1-s)^2\mathbf{c}_{n-1-r}\neq 0$. □

Corollary 3.4.125. *We have $S_{\underline{k}^\sharp},_{w_0}v_0\in V_0$.*

Proof. By (3.4.103) and Proposition 3.4.124 we deduce that

$$S_{\underline{k}^\sharp,r}v_0\in V_0\Rightarrow S_{\underline{k}^\sharp,r-1}v_0\in V_0$$

for each $2\leq r\leq n-1$. Then by (3.4.102) and the definition of V_0, we finish the proof. □

We write β for $\sum_{r=1}^{n-1}\alpha_r$ to lighten the notation.

Lemma 3.4.126. *Given a Jacobi sum $S_{\underline{k},w_0}$, we have*

$$X_{\beta,k_{1,n}}\bullet S_{\underline{k},w_0}=(-1)^{k_{1,n}+1}S_{\underline{k}',w_0}$$

where $\underline{k}'=(k'_{i,j})$ satisfies $k'_{1,n}=0$ and $k'_{i,j}=k_{i,j}$ otherwise.

Proof. This is a special case of Lemma 3.4.11 when $\alpha_0=\beta$ and $m=k_{1,n}$. □

From now on, whenever we want to view the notation $\mu_0^{w_0}$ as a weight, namely to fix a lift of $\mu_0^{w_0}\in X(T)/(p-1)X(T)$ into $X_1^{\mathrm{reg}}(T)$, we always mean

$$\mu_0^{w_0}=(a_0+p-1,a_{n-2},\cdots,a_1,a_{n-1}-p+1)\in X(T).$$

In particular, we have

$$(1,n)\cdot\mu_0^{w_0}+p\beta=\mu^*.$$

We recall the operators $X^{\mathrm{alg}}_{\beta,k}$ from the beginning of Section 3.4.

Lemma 3.4.127. *For $1 \leq r \leq n-1$, we have the following equalities on $H^0(\mu_0^{w_0})_{\mu^*}$:*

$$X_{\beta,k} = -X_{\beta,k}^{\mathrm{alg}}$$

for all $1 \leq k \leq p-1$.

Proof. Note that we have

$$\mu_0^{w_0} - (\mu^* + k\beta) = ([a_0 - a_{n-1}]_1 + n - 2 - k, 0, \cdots, 0, k - ([a_0 - a_{n-1}]_1 + n - 2)).$$

Therefore $\mu_0^{w_0} - (\mu^* + k\beta) \notin \sum_{\alpha \in \Phi^+} \mathbf{Z}_{\geq 0}\alpha$ as long as $k > [a_0 - a_{n-1}]_1 + n - 2$. As $(a_{n-1}, \cdots, a_0)$ is assumed to be n-generic in the lowest alcove throughout this section, we deduce that

$$\mu_0^{w_0} - (\mu^* + k\beta) \notin \sum_{\alpha \in \Phi^+} \mathbf{Z}_{\geq 0}\alpha \text{ for all } k \geq p-1. \tag{3.4.128}$$

On the other hand, by the definition (3.4.1), the image of $X_{\beta,k}^{\mathrm{alg}}$ lies inside $H^0(\mu_0^{w_0})_{\mu^*+k\beta}$, which is zero by (3.4.128) assuming $k \geq p-1$. Hence we deduce that

$$X_{\beta,k}^{\mathrm{alg}} = 0 \text{ on } H^0(\mu_0^{w_0})_{\mu^*} \text{ for all } k \geq p-1.$$

Then the conclusion of this lemma follows from the equality (3.4.10). □

We have a natural embedding $H^0(\mu_0^{w_0}) \hookrightarrow \pi_0$ by the definition of algebraic induction and parabolic induction. Recall that we have defined U_1 in Example 3.4.23.

Lemma 3.4.129. *We have*

$$\mathbf{F}_p[S_{\underline{k}^\sharp, w_0} v_0] = H^0(\mu_0^{w_0})_{\mu^*}^{\overline{U}_1}.$$

In particular,

$$V^\sharp \subseteq H^0(\mu_0^{w_0}).$$

Proof. On one hand, by Corollary 3.4.80 we know that

$$\dim_{\mathbf{F}_p} H^0(\mu_0^{w_0})_{\mu^*}^{\overline{U}_1} = 1,$$

and this space is generated by $v_{\{n-1\}, \underline{m}^\sharp}^{\mathrm{alg}}$ where

$$\underline{m}^\sharp = (m_1^\sharp, \cdots, m_{n-1}^\sharp) := (0, \cdots, 0, [a_0 - a_{n-1}]_1 + n - 2). \tag{3.4.130}$$

We now need to identify the vector $v_{\{n-1\}, \underline{m}^\sharp}^{\mathrm{alg}}$ with certain linear combination of Jacobi sums. Note that by Corollary 3.4.80 we have

$$v_{\{n-1\}, \underline{m}^\sharp}^{\mathrm{alg}} = D_n^{a_{n-1}-p+1} D_{n-1}^{a_1 - a_0 - n + 2} (D_{\{n-1\}, n-1})^{[a_0 - a_{n-1}]_1 + n - 2} D_1^{[a_0 - a_{n-2}]_1} \prod_{i=2}^{n-2} D_i^{a_{n-i} - a_{n-i-1}}.$$

Given a matrix $A \in G(\mathbf{F}_p)$, then $D_i(A) \neq 0$ for all $1 \leq i \leq n-1$ if and only if

$$A \in B(\mathbf{F}_p) w_0 B(\mathbf{F}_p),$$

and thus the support of $v_{\{n-1\}, \underline{m}^\sharp}^{\mathrm{alg}}$ is contained in $B(\mathbf{F}_p) w_0 B(\mathbf{F}_p)$. As a result, by the proof of Proposition 3.4.18, we know that $v_{\{n-1\}, \underline{m}^\sharp}^{\mathrm{alg}}$ is a linear combination of vectors of the form

$$S_{\underline{k}, w_0} v_0.$$

As $v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}$ is $\overline{U}_1$-invariant, and in particular $U_1(\mathbf{F}_p)$-invariant, then by Proposition 3.4.30 we know that it is a linear combination of vectors of the form

$$S_{\underline{k},w_0}v_0 \tag{3.4.131}$$

such that $k_{1,n} = [a_0 - a_{n-1}]_1 + n - 2$, $k_{1,j} = 0$ or $p-1$ for $2 \leq j \leq n-1$ and $k_{i,j} = 0$ for all $2 \leq i < j \leq n$.

Finally, note that

$$u_\beta(t)\ v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} =$$
$$D_n^{a_{n-1}-p+1} D_{n-1}^{a_1-a_0-n+2} (D_{\{n-1\},n-1} + tD_{n-1})^{[a_0-a_{n-1}]_1+n-2} D_1^{[a_0-a_{n-2}]_1} \prod_{i=2}^{n-2} D_i^{a_{n-i}-a_{n-i-1}}$$

is a polynomial of t with degree $[a_0 - a_{n-1}] + n - 2$, we conclude that

$$X^{\mathrm{alg}}_{\beta,[a_0-a_{n-1}]_1+n-2}\ v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} = v^{\mathrm{alg}}_{\{n-1\},\underline{0}}$$

where $\underline{0}$ is the $(n-1)$-tuple with all entries zero.

By Lemma 3.4.127 and the fact that

$$\mathbf{F}_p[v^{\mathrm{alg}}_{\{n-1\},\underline{0}}] = \mathbf{F}_p[S_{\underline{0},w_0}v_0] = \pi_0^{U(\mathbf{F}_p),\mu_0^{w_0}},$$

we deduce that

$$X_{\beta,[a_0-a_{n-1}]_1+n-2}\ v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} = c' S_{\underline{0},w_0}v_0$$

for some non-zero constant c'. By Lemma 3.4.126 and the linear independence of Jacobi sums proved in Proposition 3.4.18, we know that only $S_{\underline{k}^\sharp,w_0}v_0$ can appear in the linear combination 3.4.131. In other words, we have shown that

$$v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp} = c'' S_{\underline{k}^\sharp,w_0}v_0$$

for some non-zero constant c'', and thus we finish the proof. □

Lemma 3.4.132. *The dual Weyl module $H^0(\mu_0^{w_0})$ is uniserial with length two with socle $F(\mu_0^{w_0})$ and cosocle $F(\mu^*)$.*

Proof. By [Jan03] Proposition II.2.2 we know that $\mathrm{soc}_{\overline{G}}\left(H^0(\mu_0^{w_0})\right)$ is irreducible and can be identified with $F(\mu_0^{w_0})$ (which is in fact the definition of $F(\mu_0^{w_0})$). Therefore it suffices to show that $H^0(\mu_0^{w_0})$ has only two Jordan–Hölder factor $F(\mu_0^{w_0})$ and $F(\mu^*)$, each of which has multiplicity one.

By [Jan03] II.2.13 (2) it is harmless for us to replace $H^0(\mu_0^{w_0})$ by the Weyl module $V(\mu_0^{w_0})$ (defined in [Jan03] II.2.13) and show that $V(\mu_0^{w_0})$ has only two Jordan–Hölder factor $F(\mu_0^{w_0})$ and $F(\mu^*)$ and each of them has multiplicity one. As

$$\begin{cases} p & < & \left\langle \mu_0^{w_0}, (\sum_{i=1}^{n-1}\alpha_i)^\vee \right\rangle & < & 2p; \\ 0 & < & \left\langle \mu_0^{w_0}, (\sum_{i=1}^{n-2}\alpha_i)^\vee \right\rangle & < & p; \\ 0 & < & \left\langle \mu_0^{w_0}, (\sum_{i=2}^{n-1}\alpha_i)^\vee \right\rangle & < & p, \end{cases}$$

we deduce that the only dominant alcove lying below the one $\mu_0^{w_0}$ lies in is the lowest p-restricted alcove. In particular, the only dominant weight which is linked to and strictly smaller than $\mu_0^{w_0}$ is μ^*.

By [Jan03] Proposition II. 8.19, we know the existence of a filtration of subrepresentation

$$V(\mu_0^{w_0}) \supseteq V_1(\mu_0^{w_0}) \supseteq \cdots$$

such that the following equality in Grothendieck group holds

$$\sum_{i>0} V_i(\mu_0^{w_0}) = F(\mu^*).$$

This equality implies that

$$V_1(\mu_0^{w_0}) = F(\mu^*)$$

and

$$V_i(\mu_0^{w_0}) = 0 \text{ for all } i \geq 2.$$

By [Jan03] II.8.19 (2) we also know that

$$V(\mu_0^{w_0})/V_1(\mu_0^{w_0}) \cong F(\mu_0^{w_0}),$$

and thus we have shown that

$$V(\mu_0^{w_0}) = F(\mu_0^{w_0}) + F(\mu^*)$$

in the Grothendieck group. □

Proposition 3.4.133. *We have* $V^\sharp = H^0(\mu_0^{w_0})$.

Proof. By Lemma 3.4.132, we have the natural surjection

$$H^0(\mu_0^{w_0}) \twoheadrightarrow F(\mu^*)$$

which induces a morphism

$$H^0(\mu_0^{w_0})_{\mu_*} \to F(\mu^*)_{\mu_*}.$$

Now we consider $H^0(\mu_0^{w_0})$ as a $\overline{L}_1$-representation where $L_1 \cong \mathbb{G}_m \times \mathrm{GL}_{n-1}$ is the standard Levi subgroup of G which contains U_1 as a maximal unipotent subgroup. For any $\lambda \in X_{L_1}(T)_+$ (c.f. (3.5.1)) we use the notation $H^0_{L_1}(\lambda)$ for the $\overline{L}_1$-dual Weyl module defined at the beginning of Section 3.4. The dual Weyl module $H^0(\mu_0^{w_0})$ is the mod p reduction of a lattice $V_{\mathbf{Z}_p}$ in the unique irreducible algebraic representation $V_{\mathbf{Q}_p}$ of G such that $\left(V^U_{\mathbf{Q}_p}\right)_{\mu_0^{w_0}} \neq 0$. As the category of finite dimensional algebraic representations of L_1 in characteristic 0 is semisimple, V decomposes into a direct sum of characteristic 0 irreducible representations of L_1. More precisely, we have the decomposition

$$V_{\mathbf{Q}_p}|_{L_1} = \bigoplus_{\substack{\lambda \in X_{L_1}(T)_+ \\ (V_{\mathbf{Q}_p})^{U_1}_\lambda \neq 0}} m_\lambda V_{L_1}(\lambda)$$

where $V_{L_1}(\lambda)$ is the unique (up to isomorphism) irreducible algebraic representation of L_1 such that $\left(V_{L_1}(\lambda)^{U_1}\right)_\lambda \neq 0$ and

$$m_\lambda := \dim_{\mathbf{Q}_p} \left(V^{U_1}_{\mathbf{Q}_p}\right)_\lambda.$$

Therefore in the Grothendieck group of algebraic representations of $\overline{L}_1$ over $\mathbf{F}_p$, we have

$$[H^0(\mu_0^{w_0})]|_{\overline{L}_1} = \bigoplus_{\substack{\lambda \in X_{L_1}(T)_+ \\ H^0(\mu_0^{w_0})^{U_1}_\lambda \neq 0}} m_\lambda [H^0_{L_1}(\lambda)] \tag{3.4.134}$$

as by Corollary 3.4.80 $H^0(\mu_0^{w_0})^{\overline{U}_1}$ is the mod p reduction of $V^{U_1}_{\mathbf{Z}_p}$ and $V^{U_1}_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = V^{U_1}_{\mathbf{Q}_p}$.

We say that

$$\mu^* \uparrow_{L_1} \lambda$$

if there exists $\widetilde{w} \in \widetilde{W}^{L_1}$ (see the beginning of Section 3.5) such that

$$\lambda = \widetilde{w} \cdot \mu^* \text{ and } \mu^* \leq \lambda.$$

Assume that there exists a $\lambda \in X_{L_1}(T)_+$ such that $\mu^* \uparrow_{L_1} \lambda$ and that $H^0(\mu_0^{w_0})^{\overline{U}_1}_{\lambda} \neq 0$. We denote by $v^{\mathrm{alg}}_{\{n-1\},\underline{m}}$ the vector in $H^0(\mu_0^{w_0})^{\overline{U}_1}_{\lambda} \neq 0$ given by Corollary 3.4.80. We note that by Corollary 3.4.80 the vector in $H^0(\mu_0^{w_0})^{\overline{U}_1}_{\mu^*}$ is $v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}$ (see (3.4.130)). As $\mu^* \uparrow_{L_1} \lambda$, we must firstly have $\sum_{i=1}^{n-1} m_i = [a_0 - a_{n-1}]_1 + n - 2$. By the last statement in Corollary 3.4.80, we have

$$\begin{aligned}\lambda &= \left(a_0 + p - 1 - \sum_{i=1}^{n-1} m_i, a_{n-2} + m_1, \cdots, a_1 + m_{n-2}, a_{n-1} - p + 1 + m_{n-1}\right) \\ &= (a_{n-1} - n + 2, a_{n-2} + m_1, \cdots, a_1 + m_{n-2}, a_{n-1} - p + 1 + m_{n-1}). \quad (3.4.135)\end{aligned}$$

Recall $\eta = (n-1, n-2, \cdots, 1, 0)$. We notice that $\mu^* - \eta$ lies in the lowest restricted $\overline{L}_1$-alcove in the sense that

$$0 < \langle \mu^*, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+_{L_1} \quad (3.4.136)$$

where $\Phi^+_{L_1}$ is the positive roots of L_1 defined at the beginning of Section 3.5.

As we assume that $(a_{n-1}, \cdots, a_0)$ is n-generic, it is easy to see the following

$$\begin{cases} a_{n-2} + m_1 - (a_{n-1} - p + 1 + m_{n-1}) \leq p + 1 + a_{n-2} - a_{n-1} + m_1 < 2p; \\ a_{n-2} + m_1 - (a_1 + m_{n-2}) \leq a_{n-2} + m_1 - a_1 \leq [a_0 - a_1]_1 < p; \\ a_{n-3} + m_2 - (a_{n-1} - p + 1 + m_{n-1}) \leq [a_{n-3} - a_{n-1}]_1 + m_2 \leq [a_{n-2} - a_{n-1}]_1 < p, \end{cases}$$

so that we know that $\lambda - \eta$ lies in either the lowest $\overline{L}_1$-alcove in the sense of (3.4.136) (if we replace μ^* by λ) or the p-restricted $\overline{L}_1$-alcove described by the conditions

$$\begin{cases} p & < & \left\langle \lambda, \left(\sum_{i=2}^{n-1} \alpha_i\right)^\vee \right\rangle & < & 2p \\ 0 & < & \left\langle \lambda, \left(\sum_{i=2}^{n-2} \alpha_i\right)^\vee \right\rangle & < & p \\ 0 & < & \left\langle \lambda, \left(\sum_{i=3}^{n-1} \alpha_i\right)^\vee \right\rangle & < & p \end{cases}$$

and

$$0 < \langle \lambda, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Delta_{L_1}$$

where $\Delta_{L_1} := \{\alpha_i \mid 2 \leq i \leq n-1\}$ is the positive simple roots in $\Phi^+_{L_1}$.

In the first case, if $\lambda - \eta$ lies in the lowest $\overline{L}_1$-alcove, as we assume that $\mu^* \uparrow_L \lambda$, we must have $\lambda = \mu^*$; in the second case, we must have

$$\lambda = (2, n) \cdot \mu^* + p\left(\sum_{i=2}^{n-1} \alpha_i\right) = (a_{n-1} - n + 2, a_0 + p, a_{n-3}, \cdots, a_1, a_{n-2} + n - 2 - p)$$

which means by (3.4.135) that

$$\underline{m} = (m_1, \cdots, m_{n-1}) = ([a_0 - a_{n-2}]_1 + 1, 0, \cdots, 0, a_{n-2} - a_{n-1} + n - 3).$$

This implies $a_{n-2} - a_{n-1} + n - 1 = m_{n-1} \geq 0$, which is a contradiction to the n-generic assumption on $(a_{n-1}, \cdots, a_0)$. Therefore we must have $\lambda = \mu^*$. Hence we deduce by (3.4.134) and the strong linkage principle [Jan03] II.2.12 (1) that $F^{L_1}(\mu^*)$ (see the beginning of Section 3.5 for notation) has multiplicity one in $\mathrm{JH}_{\overline{L}_1}(H^0(\mu_0^{w_0})|_{\overline{L}_1})$ and is actually a direct summand.

On the other hand, as $F^{L_1}(\mu^*)$ is obviously an $\overline{L}_1$-subrepresentation of $F(\mu^*)$, we know that the surjection of $\overline{G}$-representation $H^0(\mu_0^{w_0}) \twoheadrightarrow F(\mu^*)$ induces an isomorphism of $\overline{L}_1$-representation on the direct summand $F^{L_1}(\mu^*)$ on both sides with multiplicity one, by restriction from $\overline{G}$ to $\overline{L}_1$. In particular, we know that the map

$$H^0(\mu_0^{w_0})^{\overline{U}_1}_{\mu^*} \to F(\mu^*)_{\mu^*}$$

is a bijection, and therefore the composition

$$V^\sharp \hookrightarrow H^0(\mu_0^{w_0}) \twoheadrightarrow F(\mu^*)$$

is non-zero as

$$H^0(\mu_0^{w_0})^{\overline{U}_1}_{\mu^*} = \mathbf{F}_p[v^{\mathrm{alg}}_{\{n-1\},\underline{m}^\sharp}] = \mathbf{F}_p[S_{\underline{k}^\sharp,w_0}v_0]$$

by Lemma 3.4.129. Hence, we have a surjection

$$V^\sharp \twoheadrightarrow F(\mu^*).$$

Combining this surjection with the injection

$$V^\sharp \hookrightarrow H^0(\mu_0^{w_0}),$$

we finish the proof by Lemma 3.4.132. □

Theorem 3.4.137. *Assume that $(a_{n-1}, \cdots, a_0)$ is n-generic in the lowest alcove (c.f. Definition 3.4.5). Then $H^0(\mu_0^{w_0}) \subseteq V_0$. In particular, we have*

$$F(\mu^*) \in \mathrm{JH}(V_0).$$

Proof. The first inclusion is a direct consequence of Proposition 3.4.133 together with Corollary 3.4.125. The second inclusion follows from the first as we have $F(\mu^*) \in \mathrm{JH}(H^0(\mu_0^{w_0}))$. □

Before we end this section, we need several remarks to summarize the proof, and to clarify the necessity for all the constructions.

Remark 3.4.138. *If we assume that for all $2 \leq k \leq n-2$*

$$[a_0 - a_{n-1}]_1 + n - 2 < a_k - a_{k-1}, \tag{3.4.139}$$

then we can actually show that

$$S_{\underline{k}^0,w_0}v_0 \in H^0(\mu_0^{w_0})^{[\overline{U},\overline{U}]}_{\mu^*}$$

using Corollary 3.4.29 and the case $s = n-1$ of Proposition 3.4.78, and thus

$$V_0 = H^0(\mu_0^{w_0}).$$

Moreover, under the condition (3.4.139), we can even prove that the set

$$\{S_{\underline{k},w_0}v_0 \mid \underline{k} \in \Lambda_{w_0}\}$$

forms a basis for $H^0(\mu_0^{w_0})_{\mu^*}$.

On the other hand, if we have

$$[a_0 - a_{n-1}]_1 + n - 2 \geq a_k - a_{k-1}$$

for some $2 \leq k \leq n-2$, then we can show that

$$F(\mu_0^{s_k w_0}) \in \mathrm{JH}(V_0)$$

which means that the inclusion

$$H^0(\mu_0^{w_0}) \subseteq V_0$$

is actually strict.

In fact, through the proof of Proposition 3.4.124, the subrepresentation of π_0 generated by $S_{\underline{k}^{\sharp,r,s}}v_0$ is shrinking if r is fixed and s is growing. Therefore the subrepresentation of π_0 generated by $\overline{S}_{\underline{k}^{\sharp,r}}v_0$ shrinks as r decreases. Finally, we succeeded in shrinking from V_0 to $V^\sharp$ which can be identified with $H^0(\mu_0^{w_0})$.

Remark 3.4.140. *We need to emphasize that the choice of the operators X_r^+ and X_r^- for $2 \leq r \leq n-1$ are crucial. For example, the operator*

$$\sum_{t\in\mathbf{F}_p} t^{p-2} w_0 u_{\alpha_r}(t) w_0 \in \mathbf{F}_p[G(\mathbf{F}_p)]$$

for some $2 \leq r \leq n-2$ does not work in general. The reason is that, as one can check by explicit computation, applying such operator to $S_{\underline{k}w_0}v_0$ for some $\underline{k} \in \Lambda_{w_0}$ will generally give us a huge linear combination of Jacobi sum operators. From our point of view, it is basically impossible to compute such a huge linear combination explicitly and systematically. Instead, as stated in Proposition 3.4.101, our operators X_r^+ and X_r^- can be computed systematically, even though the computation is still complicated.

The motivation of the choice of operators X_r^+ and X_r^- can be roughly explained as follows. First of all, we need one 'weight raising operator' X^+ and one 'weight lowering operator' X^-. These are two operators lying in a subalgebra $\mathbf{F}_p\langle X^+, X^-\rangle$ of $\mathbf{F}_p[G(\mathbf{F}_p)]$ such that

$$\mathbf{F}_p\langle X^+, X^-\rangle \cong \mathbf{F}_p[\mathrm{GL}_2(\mathbf{F}_p)].$$

We start with the vector $S_{\underline{k},w_0}v_0$ for some $\underline{k} \in \Lambda_{w_0}$. We apply the operator X^- once and then X^+ once, the result is a vector with the same $T(\mathbf{F}_p)$-eigencharacter μ^*. We observe that $S_{\underline{k},w_0}v_0$ is in general not an eigenvector of the operator $X^+ \bullet X^-$ because the representation π_0, after restricting from $\mathbf{F}_p[G(\mathbf{F}_p)]$ to $\mathbf{F}_p\langle X^+, X^-\rangle$, is highly non-semisimple. The naive expectation is that we just take the difference

$$X^+ \bullet X^- \bullet S_{\underline{k},w_0}v_0 - cS_{\underline{k},w_0}v_0$$

for some constant $c \in \mathbf{F}_p$, and then repeat the procedure by applying some other operators similar to X^+ and X^-.

The case $n = 3$ is easy. In the case $n = 4$, the operator

$$\sum_{t\in\mathbf{F}_p} t^{p-2} w_0 u_{\alpha_2}(t) w_0 \in \mathbf{F}_p[\mathrm{GL}_4(\mathbf{F}_p)]$$

is not well behaved as we explained in this remark, and therefore we are forced to use our X_2^- to replace $\sum_{t\in\mathbf{F}_p} t^{p-2} w_0 u_{\alpha_2}(t) w_0$.

Now we consider the general case, and it is possible for us to carry on an induction step. We have a sequence of growing subgroups of $\overline{G}$

$$\overline{P}_{\{n-1\}} \subsetneq \overline{P}_{\{n-2,n-1\}} \subsetneq \cdots \subsetneq \overline{P}_{\{2,\cdots,n-1\}}$$

and

$$\overline{L}_{\{n-1\}} \subsetneq \overline{L}_{\{n-2,n-1\}} \subsetneq \cdots \subsetneq \overline{L}_{\{2,\cdots,n-1\}}$$

where $\overline{P}_{\{r,\cdots,n-1\}}$ is the standard parabolic subgroup corresponding to the simple roots α_k for $r \leq k \leq n-1$ and $\overline{L}_{\{r,\cdots,n-1\}}$ is its standard Levi subgroup. Technically speaking, constructing the vector $S_{\underline{k}^{\sharp,r+1},w_0}v_0$ (for some $1 \leq r \leq n-2$) from $S_{\underline{k}^0,w_0}v_0$ should be reduced to Corollary 3.4.125 when we

replace $\overline{G}$ by its Levi subgroup $\overline{L}_{\{r+1,\cdots,n-1\}}$. In other words, to construct $S_{\underline{k}^{\sharp,r+1},w_0}v_0$ from $S_{\underline{k}^0,w_0}v_0$ we only need the operators

$$X_k^+, X_k^- \in \mathbf{F}_p[\overline{L}_{\{r+2,\cdots,n-1\}}(\mathbf{F}_p)] \subsetneq \mathbf{F}_p[\overline{L}_{\{r+1,\cdots,n-1\}}(\mathbf{F}_p)]$$

for all $r+2 \leq k \leq n-1$.

In order to construct $S_{\underline{k}^{\sharp,r},w_0}v_0$ from $S_{\underline{k}^{\sharp,r+1},w_0}v_0$, we only need to prove Proposition 3.4.124. Then we summarize the proof of Proposition 3.4.124 as the following: for some $a \in \mathbf{F}_p^\times$ and $b \in \mathbf{F}_p$

$$X_{r+1}^+ \bullet X_{r+1}^- \bullet S_{\underline{k}^{\sharp,r,s-1},w_0}v_0 \equiv aS_{\underline{k}^{\sharp,r,s},w_0}v_0 + bS_{\underline{k}^{\sharp,r,s-1},w_0}v_0 + \textit{error terms}$$

and the error terms can be killed by combinations of the operators X_k^+, X_k^- for $r+2 \leq k \leq n-1$.

3.5 Mod p local-global compatibility

In this section, we state and prove our main results on mod p local-global compatibility, which is a global application of our local results of Sections 3.3 and 3.4. In the first two sections, we recall some necessary known results on algebraic automorphic forms and Serre weights, for which we closely follow [EGH13], [HLM17], and [BLGG].

We first fix some notation for the whole section. Let $P \supseteq B$ be an arbitrary standard parabolic subgroup and N its unipotent radical. We denote the opposite parabolic by $P^- := w_0Pw_0$ with corresponding unipotent radical $N^- := w_0Nw_0$. We fix a standard choice of Levi subgroup $L := P \cap P^- \subseteq G$. We denote the positive roots of L defined by the pair $(B \cap L, T)$ by Φ_L^+. We use

$$X_L(T)_+ := \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Phi_L^+\} \tag{3.5.1}$$

to denote the set of dominant weights with respect to the pair $(B \cap L, T)$. We denote the Weyl group of L by W^L and identify it with a subgroup of W. The longest Weyl element in W^L is denoted by w_0^L. We define the affine Weyl group $\widetilde{W}^L$ of L as the semi-direct product of W^L and $X(T)$ with respect to the natural action of W^L on $X(T)$. Therefore $\widetilde{W}^L$ has a natural embedding into $\widetilde{W}$. We define the subgroups $\overline{P}$, $\overline{L}$, $\cdots$ of $\overline{G}$ in the obvious similar fashion.

We also need to define several open compact subgroups of $L(\mathbf{Q}_p)$. We define

$$K^L := L(\mathbf{Z}_p),$$

and via the mod p reduction map

$$\mathrm{red}^L : K^L = L(\mathbf{Z}_p) \twoheadrightarrow L(\mathbf{F}_p)$$

we also define $K^L(1)$, $I^L(1)$, and I^L as follows:

$$\begin{array}{ccc} K^L(1) := (\mathrm{red}^L)^{-1}(1) & \subseteq & I^L(1) := (\mathrm{red}^L)^{-1}(U(\mathbf{F}_p) \cap L(\mathbf{F}_p)) \\ & \subseteq & I^L := (\mathrm{red}^L)^{-1}(B(\mathbf{F}_p) \cap L(\mathbf{F}_p)). \end{array} \tag{3.5.2}$$

For any dominant weight $\lambda \in X(T)_+$, we let

$$H_L^0(\lambda) := \left(\mathrm{Ind}_{\overline{B}\cap\overline{L}}^{\overline{L}} w_0^L\lambda\right)^{\mathrm{alg}}_{/\mathbf{F}_p}$$

be the associated dual Weyl module of L. We also write $F^L(\lambda) := \mathrm{soc}_{\overline{L}}\left(H_L^0(\lambda)\right)$ for its irreducible socle as an algebraic representation of $\overline{L}$. Through a similar argument presented at the beginning of Section 3.4, the notation $F^L(\lambda)$ is well defined as an irreducible representation of $L(\mathbf{F}_p)$ if $\lambda \in T(\mathbf{F}_p)$ is p-regular, namely lies in the image of $X_1^{\mathrm{reg}}(T) \to X(T)/(p-1)X(T)$. We will sometimes abuse the

notation $F^L(\lambda)$ for $F^L(\lambda)\otimes_{\mathbf{F}_p}\mathbf{F}$ or $F^L(\lambda)$ for $F^L(\lambda)\otimes_{\mathbf{F}_p}\overline{\mathbf{F}}_p$ in the literature. We will emphasize the abuse of the notation $F^L(\lambda)$ each time we do so.

We introduce some specific standard parabolic subgroups of G. Fix integers i_0 and j_0 such that $0\leq j_0<j_0+1<i_0\leq n-1$, and let i_1 and j_1 be the integers determined by the equation

$$i_0+i_1=j_0+j_1=n-1. \tag{3.5.3}$$

We let $P_{i_1,j_1}\supset B$ be the standard parabolic subgroup of $G=\mathrm{GL}_n$ corresponding to the subset $\{\alpha_k\mid j_0+1\leq k\leq i_0\}$ of Δ. By specifying the notation for general P to P_{i_1,j_1}, we can define $P^-_{i_1,j_1}$, L_{i_1,j_1}, N_{i_1,j_1} and $N^-_{i_1,j_1}$. We can naturally embeds $\mathrm{GL}_{j_1-i_1+1}$ into G with its image denoted by G_{i_1,j_1} such that $L_{i_1,j_1}=G_{i_1,j_1}T$:

$$\mathrm{GL}_{j_1-i_1+1}\xrightarrow{\sim}G_{i_1,j_1}\hookrightarrow L_{i_1,j_1}\hookrightarrow P_{i_1,j_1}\hookrightarrow G. \tag{3.5.4}$$

We define T_{i_1,j_1} to be the maximal tori of G_{i_1,j_1} that is contained in T, and define $X(T_{i_1,j_1})$ to be the character group of T_{i_1,j_1}. If i_1 and j_1 are clear from the context (or equivalently i_0 and j_0 are clear) then we often write P, P^- L, N, and N^- for P_{i_1,j_1}, $P^-_{i_1,j_1}$, L_{i_1,j_1}, N_{i_1,j_1}, and $N^-_{i_1,j_1}$, respectively.

3.5.1 The space of algebraic automorphic forms

Let $F/\mathbf{Q}$ be a CM field with maximal totally real subfield F^+. We write c for the generator of $\mathrm{Gal}(F/F^+)$, and let S_p^+ (resp. S_p) be the set of places of F^+ (resp. F) above p. For v (resp. w) a finite place of F^+ (resp. F) we write k_v (resp. k_w) for the residue field of F_v^+ (resp. F_w).

From now on, we assume that

- F/F^+ is unramified at all finite places;
- p splits completely in F.

Note that the first assumption above excludes $F^+=\mathbf{Q}$. We also note that the second assumption is not essential in this section, but it is harmless since we are only interested in $G_{\mathbf{Q}_p}$-representations in this paper. Every place v of F^+ above p further decomposes and we often write $v=ww^c$ in F.

There exists a reductive group G_{n/F^+} satisfying the following properties (c.f. [BLGG], Section 2):

- G_n is an outer form of GL_n with $G_{n/F}\cong\mathrm{GL}_{n/F}$,
- G_n is a quasi-split at any finite place of F^+;
- $G_n(F_v^+)\simeq U_n(\mathbf{R})$ for all $v|\infty$.

By [CHT08], Section 3.3, G_n admits an integral model $\mathcal{G}_n$ over $\mathcal{O}_{F^+}$ such that $\mathcal{G}_n\times_{\mathcal{O}_{F^+}}\mathcal{O}_{F_v^+}$ is reductive if v is a finite place of F^+ which splits in F. If v is such a place and w is a place of F above v, then we have an isomorphism

$$\iota_w:\mathcal{G}_n(\mathcal{O}_{F_v^+})\xrightarrow{\sim}\mathcal{G}_n(\mathcal{O}_{F_w})\xrightarrow{\sim}\mathrm{GL}_n(\mathcal{O}_{F_w}). \tag{3.5.5}$$

We fix this isomorphism for each such place v of F^+.

Define $F_p^+:=F^+\otimes_{\mathbf{Q}}\mathbf{Q}_p$ and $\mathcal{O}_{F^+,p}:=\mathcal{O}_{F^+}\otimes_{\mathbf{Z}}\mathbf{Z}_p$. If W is an $\mathcal{O}_E$-module endowed with an action of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$ and $U\subset G_n(\mathbf{A}_{F^+}^{\infty,p})\times\mathcal{G}_n(\mathcal{O}_{F^+,p})$ is a compact open subgroup, the space of algebraic automorphic forms on G_n of level U and coefficients in W, which is also an $\mathcal{O}_E$-module, is defined as follows:

$$S(U,W):=\left\{f:\,G_n(F^+)\backslash G_n(\mathbf{A}_{F^+}^{\infty})\to W\mid f(gu)=u_p^{-1}f(g)\ \forall\ g\in G_n(\mathbf{A}_{F^+}^{\infty}),u\in U\right\}$$

with the usual notation $u=u^pu_p$ for the elements in U.

We say that the level U is *sufficiently small* if

$$t^{-1}G_n(F^+)t \cap U$$

has finite order prime to p for all $t \in G_n(\mathbf{A}_{F^+}^{\infty})$. We say that U is *unramified* at a finite place v of F^+ if it has a decomposition

$$U = \mathcal{G}_n(\mathcal{O}_{F_v^+})U^v$$

for some compact open $U^v \subset G_n(\mathbf{A}_{F^+}^{\infty,v})$. If w is a finite place of F, then we say, by abuse of notation, that w is an unramified place for U or U is unramified at w if U is unramified at $w|_{F^+}$.

For a compact open subgroup U of $G_n(\mathbf{A}_{F^+}^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$, we let $\mathcal{P}_U$ denote the set consisting of finite places w of F such that

- $w|_{F^+}$ is split in F,
- $w \notin S_p$,
- U is unramified at w.

For a subset $\mathcal{P} \subseteq \mathcal{P}_U$ of finite complement and closed with respect to complex conjugation we write $\mathbf{T}^{\mathcal{P}} = \mathcal{O}_E[T_w^{(i)},\ w \in \mathcal{P},\ i \in \{0, 1, \cdots, n\}]$ for the universal Hecke algebra on $\mathcal{P}$, where the Hecke operator $T_w^{(i)}$ acts on $S(U, W)$ via the usual double coset operator

$$\iota_w^{-1}\left[\mathrm{GL}_n(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w\mathrm{Id}_i & 0\\ 0 & \mathrm{Id}_{n-i}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_w})\right]$$

where ϖ_w is a uniformizer of $\mathcal{O}_{F_w}$ and Id_i is the identity matrix of size i. The Hecke algebra $\mathbf{T}^{\mathcal{P}}$ naturally acts on $S(U, W)$.

Recall that we assume that p splits completely in F. Following [EGH13], Section 7.1 we consider the subset $(\mathbf{Z}_+^n)_0^{S_p}$ consisting of dominant weights $\underline{a} = (\underline{a}_w)_w$ where $\underline{a}_w = (a_{1,w}, a_{2,w}, \cdots, a_{n,w})$ satisfying

$$a_{i,w} + a_{n+1-i,w^c} = 0 \tag{3.5.6}$$

for all $w \in S_p$ and $1 \leq i \leq n$. We let

$$W_{\underline{a}_w} := M_{\underline{a}_w}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}} \mathcal{O}_E$$

where the $M_{\underline{a}_w}(\mathcal{O}_{F_w})$ is $\mathcal{O}_{F_w}$-specialization of the dual Weyl module associated to $\underline{a}_w$ (c.f. [EGH13], Section 4.1.1); by condition (3.5.6), one deduces an isomorphism of $\mathcal{G}_n(\mathcal{O}_{F_v^+})$-representations $W_{\underline{a}_w} \circ \iota_w \cong W_{\underline{a}_{w^c}} \circ \iota_{w^c}$. Therefore, by letting $W_{\underline{a}_v} := W_{\underline{a}_w} \circ \iota_w$ for any place $w|v$, the $\mathcal{O}_E$-representation of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$

$$W_{\underline{a}} := \bigotimes_{v|p} W_{\underline{a}_v}$$

is well-defined.

For a weight $\underline{a} \in (\mathbf{Z}_+^n)_0^{S_p}$, let us write $S_{\underline{a}}(\overline{\mathbf{Q}}_p)$ to denote the inductive limit of the spaces $S(U, W_{\underline{a}}) \otimes_{\mathcal{O}_E} \overline{\mathbf{Q}}_p$ over the compact open subgroups $U \subset G_n(\mathbf{A}_{F^+}^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$. (Note that the transition maps are induced, in a natural way, from the inclusions between levels U.) Then $S_{\underline{a}}(\overline{\mathbf{Q}}_p)$ has a natural left action of $G_n(\mathbf{A}_{F^+}^{\infty})$ induced by right translation of functions.

We briefly recall the relation between the space $\mathcal{A}$ of classical automorphic forms and the previous spaces of algebraic automorphic forms in the particular case which is relevant to us. Fix an isomorphism $\imath : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ for the rest of the paper. As we did for the $\mathcal{O}_{F_w}$-specialization of the dual Weyl modules, we define a finite dimensional $G_n(F^+ \otimes_{\mathbf{Q}} \mathbf{R})$-representation $\sigma_{\underline{a}} \cong \bigoplus_{v|\infty} \sigma_{\underline{a}_v}$ with $\mathbf{C}$-coefficients. (We refer to [EGH13], Section 7.1.4 for the precise definition of $\sigma_{\underline{a}}$.)

Lemma 3.5.7 ([EGH13], Lemma 7.1.6)**.** *The isomorphism $\imath : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ induces an isomorphism of smooth $G_n(\mathbf{A}_{F^+}^\infty)$-representations*

$$S_{\underline{a}}(\overline{\mathbf{Q}}_p) \otimes_{\overline{\mathbf{Q}}_p,\imath} \mathbf{C} \xrightarrow{\imath} \mathrm{Hom}_{G_n(F^+\otimes_{\mathbf{Q}}\mathbf{R})}(\sigma_{\underline{a}}^\vee, \mathcal{A})$$

for any $\underline{a} \in (\mathbf{Z}_+^n)_0^{S_p}$

The following theorem guarantees the existence of Galois representations attached to automorphic forms on the unitary group G_n. We let $|\ |^{\frac{1-n}{2}} : F^\times \to \mathbf{Q}_p^\times$ denote the unique square root of $|\ |^{1-n}$ whose composite with $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ takes positive values.

Theorem 3.5.8 ([EGH13], Theorem 7.2.1)**.** *Let Π be an irreducible $G_n(\mathbf{A}_{F^+}^\infty)$-subrepresentation of $S_{\underline{a}}(\overline{\mathbf{Q}}_p)$.*

Then there exists a continuous semisimple representation

$$r_\Pi : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

such that

(i) $r_\Pi^c \otimes \varepsilon^{n-1} \cong r_\Pi^\vee$;

(ii) *for each place w above p, the representation $r_\Pi|_{G_{F_w}}$ is de Rham with Hodge–Tate weights*

$$\mathrm{HT}(r_\Pi|_{G_{F_w}}) = \{a_{1,w} + (n-1), a_{2,w} + (n-2), \cdots, a_{n,w}\};$$

(iii) *if $w|p$ is a place of F and $v := w|_{F^+}$ splits in F, then*

$$\mathrm{WD}(r_\Pi|_{G_{F_w}})^{\mathrm{F-ss}} \cong \mathrm{rec}_w((\Pi_v \circ \iota_w^{-1}) \otimes |\cdot|^{\frac{1-n}{2}}).$$

We note that the fact that (iii) holds without semi-simplification on the automorphic side is one of the main results of [**?**]. We also note that property (iii) says that the restriction to G_{F_w} is compatible with the local Langlands correspondence at w, which is denoted by rec_w.

3.5.2 Serre weights and potentially crystalline lifts

In this section, we recall the relation of Serre weights and potentially crystalline lifts via (inertial) local Langlands correspondence.

Definition 3.5.9. *A* Serre weight *for $\mathcal{G}_n$ is an isomorphism class of an absolutely irreducible smooth $\overline{\mathbf{F}}_p$-representation V of $\mathcal{G}_n(\mathcal{O}_{F^+,p})$. If v is a place of F^+ above p, then a* Serre weight at v *is an isomorphism class of an absolutely irreducible $\overline{\mathbf{F}}_p$-smooth representation V_v of $\mathcal{G}_n(\mathcal{O}_{F_v^+})$. Finally, if w is a place of F above p, a* Serre weight at w *is an isomorphism class of an absolutely irreducible $\overline{\mathbf{F}}_p$-smooth representation V_w of $\mathrm{GL}_n(\mathcal{O}_{F_w})$.*

We will often say a Serre weight for a Serre weight for $\mathcal{G}_n$ if $\mathcal{G}_n$ is clear from the context. Note that if V_v is a Serre weight at v, there is an associated Serre weight at $w|v$ defined by $V_v \circ \iota_w^{-1}$.

As explained in [EGH13], Section 7.3, a Serre weight V admits an explicit description in terms of $\mathrm{GL}_n(k_w)$-representations. More precisely, let w be a place of F above p and write $v := w|_{F^+}$. For any n-tuple of integers $\underline{a}_w := (a_{1,w}, a_{2,w}, \cdots, a_{n,w}) \in \mathbf{Z}_+^n$, that is restricted (i.e., $0 \le a_{i,w} - a_{i+1,w} \le p-1$ for $i = 1, 2, \cdots, n-1$), we consider the Serre weight $F(\underline{a}_w) := F(a_{1,w}, a_{2,w}, \cdots, a_{n,w})$, as defined in [EGH13], Section 4.1.2. It is an irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_n(k_w)$ and of $\mathcal{G}_n(k_v)$ via the isomorphism ι_w. Note that $F(a_{1,w}, a_{2,w}, \cdots, a_{n,w})^\vee \circ \iota_{w^c} \cong F(a_{1,w}, a_{2,w}, \cdots, a_{n,w}) \circ \iota_w$ as $\mathcal{G}_n(k_v)$-representations, i.e. $F(\underline{a}_{w^c}) \circ \iota_{w^c} \cong F(\underline{a}_w) \circ \iota_w$ if $a_{i,w} + a_{n+1-i,w^c} = 0$ for all $1 \le i \le n$. Hence, if $\underline{a} = (\underline{a}_w)_w \in (\mathbf{Z}_+^n)_0^{S_p}$ that is restricted, then we can set $F_{\underline{a}_v} := F(\underline{a}_w) \circ \iota_w$ for $w|v$. We also set

$$F_{\underline{a}} := \bigotimes_{v|p} F_{\underline{a}_v}$$

which is a Serre weight for $\mathcal{G}_n(\mathcal{O}_{F^+,p})$. From [EGH13], Lemma 7.3.4 if V is a Serre weight for $\mathcal{G}_n$, there exists a restricted weight $\underline{a} = (\underline{a}_w)_w \in (\mathbf{Z}^n_+)^{S_p}_0$ such that V has a decomposition $V \cong \bigotimes_{v|p} V_v$ where the V_v are Serre weights at v satisfying $V_v \circ \iota_w^{-1} \cong F(\underline{a}_w)$.

Recall that we write $\mathbf{F}$ for the residue field of E.

Definition 3.5.10. *Let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be an absolutely irreducible continuous Galois representation and let V be a Serre weight for $\mathcal{G}_n$. We say that $\overline{r}$ is* automorphic of weight V *(or that V is* a Serre weight of $\overline{r}$*) if there exists a compact open subgroup U in $G_n(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$ unramified above p and a cofinite subset $\mathcal{P} \subseteq \mathcal{P}_U$ such that $\overline{r}$ is unramified at each place of $\mathcal{P}$ and*

$$S(U,V)_{\mathfrak{m}_{\overline{r}}} \neq 0$$

where $\mathfrak{m}_{\overline{r}}$ is the kernel of the system of Hecke eigenvalues $\overline{\alpha} : \mathbf{T}^{\mathcal{P}} \to \mathbf{F}$ associated to $\overline{r}$, i.e.

$$\det\left(1 - \overline{r}^{\vee}(\mathrm{Frob}_w)X\right) = \sum_{j=0}^{n} (-1)^j (\mathbf{N}_{F/\mathbf{Q}}(w))^{\binom{j}{2}} \overline{\alpha}(T_w^{(j)}) X^j$$

for all $w \in \mathcal{P}$.

We write $W(\overline{r})$ for the set of automorphic Serre weights of $\overline{r}$. Let w be a place of F above p and $v = w|_{F_p^+}$. We also write $W_w(\overline{r})$ for the set of Serre weights $F(\underline{a}_w)$ such that

$$(F(\underline{a}_w) \circ \iota_w) \otimes \left(\bigotimes_{v' \in S_p^+ \setminus \{v\}} V_{v'} \right) \in W(\overline{r})$$

where $V_{v'}$ are Serre weights of $\mathcal{G}_n(\mathcal{O}_{F_{v'}^+})$ for all $v' \in S_p^+ \setminus \{v\}$. We often write $W(\overline{r}|_{G_{F_w}})$ and $W_w(\overline{r}|_{G_{F_w}})$ for $W(\overline{r})$ and $W_w(\overline{r})$ respectively, when the given $\overline{r}|_{G_{F_w}}$ is clearly a restriction of an automorphic representation $\overline{r}$ to G_{F_w}.

Fix a place w of F above p and let $v = w|_{F_p^+}$. We also fix a compact open subgroup U of $G_n(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$ which is sufficiently small and unramified above p. We may write $U = \mathcal{G}_n(\mathcal{O}_{F_v^+}) \times U^v$. If W' is an $\mathcal{O}_E$-module with an action of $\prod_{v' \in S_p^+ \setminus \{v\}} \mathcal{G}_n(\mathcal{O}_{F_{v'}^+})$, we define

$$S(U^v, W') := \varinjlim_{U_v} S(U^v \cdot U_v, W')$$

where the limit runs over all compact open subgroups U_v of $\mathcal{G}_n(\mathcal{O}_{F_v^+})$, endowing W' with a trivial $\mathcal{G}_n(\mathcal{O}_{F_v^+})$-action. Note that $S(U^v, W')$ has a smooth action of $\mathcal{G}_n(F_v^+)$ (given by right translation) and hence of $\mathrm{GL}_n(F_w)$ via ι_w. We also note that $S(U^v, W')$ has an action of $\mathbf{T}^{\mathcal{P}}$ commuting with the smooth action of $\mathcal{G}_n(F_v^+)$, where $\mathcal{P}$ is a cofinite subset of $\mathcal{P}_U$.

Lemma 3.5.11 ([EGH13], Lemma 7.4.3)**.** *Let U be a compact open subgroup of $G_n(\mathbf{A}_F^{\infty,p}) \times \mathcal{G}_n(\mathcal{O}_{F^+,p})$ which is sufficiently small and unramified above p, and $\mathcal{P}$ a cofinite subset of $\mathcal{P}_U$. Fix a place w of F above p and let $v = w|_{F_p^+}$. Let $V \cong \bigotimes_{v' \in S_p^+} V_{v'}$ be a Serre weight for $\mathcal{G}_n$. Then there is a natural isomorphism of $\mathbf{T}^{\mathcal{P}}$-modules*

$$\mathrm{Hom}_{\mathcal{G}_n(\mathcal{O}_{F_v^+})}\left(V_v^{\vee},\ S(U^v, V')\right) \xrightarrow{\sim} S(U,V)$$

where $V' := \bigotimes_{v' \in S_p^+ \setminus \{v\}} V_{v'}$.

We now recall some formalism related to Deligne–Lusztig representations from Section 3.4.3. Let w be a place of F above p. For a positive integer m, let $k_{w,m}/k_w$ be an extension satisfying $[k_{w,m} :$

$k_w] = m$, and let $\mathbb{T}$ be a F-stable maximal torus in GL_{n/k_w} where F is the Frobenius morphism. We have an identification from [Her09], Lemma 4.7

$$\mathbb{T}(k_w) \xrightarrow{\sim} \prod_j k_{w,n_j}^\times$$

where $n \geq n_j > 0$ and $\sum_j n_j = n$; the isomorphism is unique up to $\prod_j \mathrm{Gal}(k_{w,n_j}/k_w)$-conjugacy. In particular, any character $\theta : \mathbb{T}(k_w) \to \overline{\mathbf{Q}}_p^\times$ can be written as $\theta = \otimes_j \theta_j$ where $\theta_j : k_{w,n_j}^\times \to \overline{\mathbf{Q}}_p^\times$.

Given a F-stable maximal torus $\mathbb{T}$ and a primitive character θ, we consider the Deligne-Lusztig representation $R_{\mathbb{T}}^\theta$ of $\mathrm{GL}_n(k_w)$ over $\overline{\mathbf{Q}}_p$ defined in Section 3.4.3. Recall from Section 3.4.3 that $\Theta(\theta_j)$ is cuspidal representation of $\mathrm{GL}_{n_j}(k_w)$ associated to the primitive character θ_j, we have

$$R_{\mathbb{T}}^\theta \cong (-1)^{n-r} \cdot \mathrm{Ind}_{P_{\underline{n}}(k_w)}^{\mathrm{GL}_n(k_w)}(\otimes_j \Theta(\theta_j))$$

where $P_{\underline{n}}$ is the standard parabolic subgroup containing the Levi $\prod_j \mathrm{GL}_{n_j}$ and r denotes the number of its Levi factors.

Let $F_{w,m} := W(k_{w,m})[\frac{1}{p}]$ for a positive integer m. We consider θ_j as a character on $\mathcal{O}_{F_{w,n_j}}^\times$ by inflation and we define the following Galois type $\mathrm{rec}(\theta) : I_{F_w} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ as follows:

$$\mathrm{rec}(\theta) := \bigoplus_{j=1}^{r} \left(\bigoplus_{\sigma \in \mathrm{Gal}(k_{w,n_j}/k_w)} \sigma\left(\theta_j \circ \mathrm{Art}_{F_{w,n_j}}^{-1}\right) \right)$$

where θ_j is a primitive character on $k_{w,n_j}^\times$ of niveau n_j for each $j = 1, \cdots, r$. Recall that $\mathrm{Art}_{F_{w,n_j}} : F_{w,n_j}^\times \to W_{F_{w,n_j}}^{ab}$ is the isomorphism of local class field theory, normalized by sending the uniformizers to the geometric Frobenius.

We quickly review inertial local Langlands correspondence.

Theorem 3.5.12 ([CEGGPS], Theorem 3.7 and [LLL16], Proposition 2.3.4)**.** *Let $\tau : I_{\mathbf{Q}_p} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a Galois type. Then there exists a finite dimensional irreducible smooth $\overline{\mathbf{Q}}_p$-representation $\sigma(\tau)$ of $\mathrm{GL}_n(\mathbf{Z}_p)$ such that if π is any irreducible smooth $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ then $\pi|_{\mathrm{GL}_n(\mathbf{Z}_p)}$ contains a unique copy of $\sigma(\tau)$ as a subrepresentation if and only if $\mathrm{rec}_{\mathbf{Q}_p}(\pi)|_{I_{\mathbf{Q}_p}} \cong \tau$ and $N = 0$ on $\mathrm{rec}_{\mathbf{Q}_p}(\pi)$.*

Moreover, if $\tau \cong \oplus_{j=1}^r \tau_j$ and the τ_j are pairwise distinct, then $\sigma(\tau) \cong R_{\mathbb{T}}^\theta$ and $\tau \cong \mathrm{rec}(\theta)$ for a maximal torus $\mathbb{T}$ in $\mathrm{GL}_{n/\mathbf{F}_p}$ and a primitive character $\theta : \mathbb{T}(\mathbf{F}_p) \to \overline{\mathbf{Q}}_p^\times$.

The following theorem provides a connection between Serre weights and potentially crystalline lifts, which will be useful for the main result, Theorem 3.5.44.

Theorem 3.5.13 ([LLL16], Proposition 4.2.5)**.** *Let w be a place of F above p, $\mathbb{T}$ a maximal torus in GL_{n/k_w}, $\theta = \bigotimes_{j=1}^r \theta_j : \mathbb{T}(k_w) \to \overline{\mathbf{Q}}_p^\times$ a primitive character such that θ_j are pairwise distinct, and V_w a Serre weight at w for a Galois representation $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$.*

Assume that V_w is a Jordan-Hölder constituent in the mod p reduction of the Deligne–Lusztig representation $R_{\mathbb{T}}^\theta$ of $\mathrm{GL}_n(k_w)$. Then $\overline{r}|_{G_{F_w}}$ has a potentially crystalline lift with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, 0\}$ and Galois type $\mathrm{rec}(\theta)$.

For a given automorphic Galois representation $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$, it is quite difficult to determine if a given Serre weight is a Serre weight of $\overline{r}$. Thanks to the work of [BLGG], we have the following theorem, in which we refer the reader to [BLGG] for the unfamiliar terminology.

Theorem 3.5.14 ([BLGG], Theorem 4.1.9)**.** *Assume that if n is even then so is $\frac{n[F^+:\mathbf{Q}]}{2}$, that $\zeta_p \notin F$, and that $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ is an absolutely irreducible representation with split ramification. Assume further that there is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ such that*

- $\overline{r} \simeq \overline{r}_{\Pi}$;
- *For each place $w|p$ of F, $r_{\Pi}|_{G_{F_w}}$ is potentially diagonalizable;*
- $\overline{r}(G_{F(\zeta_p)})$ *is adequate.*

If $\underline{a} = (\underline{a}_w)_w \in (\mathbf{Z}^n_+)^{S_p}_0$ and for each $w \in S_p$ $\overline{r}|_{G_{F_w}}$ has a potentially diagonalizable crystalline lift with Hodge–Tate weights $\{a_{1,w} + (n-1), a_{2,w} + (n-2), \cdots, a_{n-1,w} + 1, a_{n,w}\}$, then a Jordan–Hölder factor of $W_{\underline{a}} \otimes_{\mathbf{Z}_p} \mathbf{F}$ is a Serre weight of $\overline{r}$.

3.5.3 Weight elimination and automorphy of a Serre weight

In this section, we state our main Conjecture for weight elimination (Conjecture 3.5.16) which will be a crucial assumption in the proof of Theorem 3.5.44. This conjecture is now known by Bao V. Le Hung (c.f. [LLMPQ]). We also prove the automorphy of a certain obvious Serre weight under the assumptions of Taylor–Wiles type.

Throughout this section, we assume that $\overline{\rho}_0$ is always a restriction of an automorphic representation $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ to G_{F_w} for a fixed place w above p and is generic (c.f. Definition 3.3.3). Recall that for $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$ we have defined a tuple of integers $(r^{i_0,j_0}_{n-1}, \cdots, r^{i_0,j_0}_1, r^{i_0,j_0}_0)$ in (3.3.42), which determines the Galois types as in (3.1.3). In many cases, we will consider the dual of our Serre weights, so that we define a pair of integers (i_1, j_1) by the equation (3.5.3). We also let

$$b_k := -c_{n-1-k}$$

for all $0 \leq k \leq n-1$. We will keep the notation (i_1, j_1) and b_k for the rest of the paper.

For the rest of the this section, we are mainly interested in the following characters of $T(\mathbf{F}_p)$: let

$$\mu^{\square} := (b_{n-1}, \cdots, b_0)$$

and

$$\mu^{\square, i_1, j_1} := (y_{n-1}, y_{n-2}, \cdots, y_1, y_0)$$

where

$$y_j = \begin{cases} b_j & \text{if } j \notin \{j_1, i_1\}; \\ b_{i_1} - j_1 + i_1 + 1 & \text{if } j = j_1; \\ b_{j_1} + j_1 - i_1 - 1 & \text{if } j = i_1. \end{cases}$$

As $\overline{\rho}_0$ is generic, each of the characters above is p-regular and thus uniquely determines a p-restricted weight up to a twist in $(p-1)X_0(T)$, and, by abuse of notation, we write $\mu^{\square}$, $\mu^{\square, i_1, j_1}$ for those corresponding p-restricted weights, respectively. We will clarify the twist in $(p-1)X_0(T)$ whenever necessary. We also define a principal series representation

$$\pi^{i_1,j_1}_* := \mathrm{Ind}^{G(\mathbf{F}_p)}_{B(\mathbf{F}_p)}(\mu^{\square, i_1, j_1})^{w_0}. \tag{3.5.15}$$

We now state necessary results of weight elimination to our proof of the main results, Theorem 3.5.44, in this paper.

Conjecture 3.5.16. *Let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ be a continuous automorphic Galois representation with $\overline{r}|_{G_{F_w}} \cong \overline{\rho}_0$ as in (3.3.1). Fix a pair of integers (i_0, j_0) such that $0 \leq j_0 < j_0 + 1 < i_0 \leq n-1$, and assume that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic and that $\mu^{\square, i_1, j_1}$ is $2n$-generic.*

Then we have

$$W_w(\overline{r}) \cap \mathrm{JH}((\pi^{i_1,j_1}_*)^{\vee}) \subseteq \{F(\mu^{\square})^{\vee}, F(\mu^{\square, i_1, j_1})^{\vee}\}.$$

In an earlier version of this paper, we prove Conjecture 3.1.11 for $n \leq 5$. But our method is rather elaborate to execute for general n. We are informed that Bao V. Le Hung can prove Conjecture 3.1.11 completely. Therefore, Conjecture 3.1.11 becomes a theorem based on the results in the forthcoming paper [LLMPQ].

Finally, we prove the automorphy of the Serre weight $F(\mu^{\square})^{\vee}$.

Proposition 3.5.17. *Keep the assumptions and notation of Conjecture 3.5.16. Assume further that if n is even then so is $\frac{n[F^+:\mathbf{Q}]}{2}$, that $\zeta_p \notin F$, that $\overline{r} : G_F \to \mathrm{GL}_n(\mathbf{F})$ is an irreducible representation with split ramification, and that there is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ such that*

- *$\overline{r} \simeq \overline{r}_\Pi$;*
- *for each place $w'|p$ of F, $r_\Pi|_{G_{F_{w'}}}$ is potentially diagonalizable;*
- *$\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Then

$$\{F(\mu^\square)^\vee\} \subseteq W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee).$$

Proof. We prove that $F(\mu^\square)^\vee = F(c_{n-1}, c_{n-2}, \cdots, c_0) \in W_w(\overline{r})$ as well as $F(\mu^\square)^\vee \in \mathrm{JH}\left((\pi_*^{i_1,j_1})^\vee\right)$. Note that $(c_{n-1}, \cdots, c_0)$ is in the lowest alcove as $\overline{\rho}_0$ is generic, so that by Theorem 3.5.14 it is enough to show that $\overline{\rho}_0$ has a potentially diagonalizable crystalline lift with Hodge–Tate weights $\{c_{n-1} + (n-1), \cdots, c_1 + 1, c_0\}$. Since $\overline{\rho}_0$ is generic, by [BLGGT], Lemma 1.4.3 it is enough to show that $\overline{\rho}_0$ has an ordinary crystalline lift with those Hodge–Tate weights. The existence of such a crystalline lift is immediate by [GHLS], Proposition 2.1.10. On the other hand, we have $F(\mu^\square)^\vee \in \mathrm{JH}((\pi_*^{i_1,j_1})^\vee$ which is a direct corollary of Theorem 3.5.28. Therefore, we conclude that $F(\mu^\square)^\vee \in W_w(\overline{r}) \cap \mathrm{JH}\left((\pi_*^{i_1,j_1})^\vee\right)$. □

3.5.4 Some application of Morita theory

In this section, we will recall standard results from Morita theory to prove Corollary 3.5.23. We fix here an arbitrary finite group H and a finite dimensional irreducible E-representation V of H. We may assume that E is sufficiently large such that E (resp. its residual field $\mathbf{F}$) is a splitting field of V. By Proposition 16.16 in [CR90], we know that for any $\mathcal{O}_E$-lattice $V^\circ \subseteq V$, the set $\mathrm{JH}_{\mathbf{F}[H]}(V^\circ \otimes_{\mathcal{O}_E} \mathbf{F})$ depends only on V and is independent of the choice of V°, and thus we will use the notation $\mathrm{JH}_{\mathbf{F}[H]}(V)$ from now on. Let $\mathcal{C}$ be the category of all finitely generated $\mathcal{O}_E$-modules with an H-action which are isomorphic to subquotients of $\mathcal{O}_E$-lattices in $V^{\oplus k}$ for some $k \geq 1$. Then the irreducible objects of $\mathcal{C}$ are just elements of $\mathrm{JH}_{\mathbf{F}[H]}(V)$. If σ has multiplicity one in V, then there is an $\mathcal{O}_E$-lattice V^σ (unique up to homothety by following the proof of Lemma 4.4.1 of [EGS15] as it actually requires only the multiplicity one of σ in our notation) such that

$$\mathrm{cosoc}_H(V^\sigma \otimes_{\mathcal{O}_E} \mathbf{F}) = \sigma.$$

By considering an $\mathcal{O}_E$-lattice in the E-dual of V with the $\mathbf{F}$-dual of σ as cosocle and then taking $\mathcal{O}_E$-dual of this lattice, we reach another $\mathcal{O}_E$-lattice V_σ in V, which is the unique (up to homethety), such that

$$\mathrm{soc}_H(V_\sigma \otimes_{\mathcal{O}_E} \mathbf{F}) = \sigma.$$

By repeating the proof of Lemma 2.3.1, Lemma 2.3.2 and Proposition 2.3.3 in [Le15], we deduce the following.

Proposition 3.5.18. *If σ has multiplicity one in V, then the lattice V^σ is a projective object in $\mathcal{C}$.*

We need to emphasize that the proof of Proposition 2.3.3 in [Le15] requires only the multiplicity one of σ, although it is necessary for all Jordan–Hölder factors σ to have multiplicity one to have Proposition 2.3.4 in [Le15].

Corollary 3.5.19. *Let Σ be a subset of $\mathrm{JH}_{\mathbf{F}[H]}(V)$ such that each $\sigma \in \Sigma$ has multiplicity one in V. If an $\mathcal{O}_E$-lattice $V^\circ \subseteq V$ satisfies*

$$\mathrm{cosoc}_H(V^\circ \otimes_E \mathbf{F}) = \bigoplus_{\sigma\in\Sigma} \sigma \tag{3.5.20}$$

then we have a surjection

$$\bigoplus_{\sigma\in\Sigma} V^\sigma \twoheadrightarrow V^\circ. \tag{3.5.21}$$

Proof. By (3.5.20) we have a surjection

$$V^\circ \twoheadrightarrow \bigoplus_{\sigma\in\Sigma} \sigma.$$

By Proposition 3.5.18 we know that $\bigoplus_{\sigma\in\Sigma} V^\sigma$ is a projective object in $\mathcal{C}$. By the definition of V^σ we know that there is a surjection

$$\bigoplus_{\sigma\in\Sigma} V^\sigma \twoheadrightarrow \bigoplus_{\sigma\in\Sigma} \sigma$$

which can be lifted by projectiveness to (3.5.21). □

Note in particular that (3.5.21) implies automatically the surjection

$$\bigoplus_{\sigma\in\Sigma} V^\sigma \otimes_E \mathbf{F} \twoheadrightarrow V^\circ \otimes_{\mathcal{O}_E} \mathbf{F}. \tag{3.5.22}$$

Corollary 3.5.23. *Let Σ be a subset of $\mathrm{JH}_{\mathbf{F}[H]}(V)$ such that each $\sigma \in \Sigma$ has multiplicity one in V. If an $\mathcal{O}_E$-lattice $V^\circ \subseteq V$ satisfies*

$$\mathrm{soc}_H(V^\circ \otimes_E \mathbf{F}) = \bigoplus_{\sigma\in\Sigma} \sigma$$

then we have an injection

$$V^\circ \otimes_{\mathcal{O}_E} \mathbf{F} \hookrightarrow \bigoplus_{\sigma\in\Sigma} V_\sigma \otimes_E \mathbf{F}.$$

Proof. This is simply the $\mathbf{F}$-dual of (3.5.22). □

3.5.5 Generalization of Section 3.4

In this section, we fix a pair of integers (i_0, j_0) satisfying $0 \le j_0 < j_0 + 1 < i_0 \le n-1$, and determine (i_1, j_1) by the equation (3.5.3). We will use the shorten notation P (resp. N, L, $P^- \cdots$) for P_{i_1,j_1} (resp. N_{i_1,j_1}, L_{i_1,j_1}, $P^-_{i_1,j_1}$, $\cdots$) as introduced at the beginning of Section 3.5. Proposition 3.5.38 is crucial for the proof of Theorem 3.5.44. We assume throughout this section that $\mu^{\square,i_1,j_1}$ is $2n$-generic (c.f. Definition 3.4.5).

We start this section by defining some weights and Jacobi sum operators which will play a crucial role for our main results, Theorem 3.5.44. Let

$$\mu_1^{i_1,j_1} := (x^1_{n-1}, x^1_{n-2}, \cdots, x^1_1, x^1_0) \quad \text{and} \quad \mu_1^{i_1,j_1,\prime} := (x^{1,\prime}_{n-1}, x^{1,\prime}_{n-2}, \cdots, x^{1,\prime}_1, x^{1,\prime}_0)$$

where

$$x^1_j = \begin{cases} b_{n+i_1-j} & \text{if } n-j_1+i_1+1 \le j \le n-1; \\ b_{j+j_1-i_1-1} & \text{if } i_1+2 \le j \le n-j_1+i_1; \\ b_{j_1}+j_1-i_1-1 & \text{if } j = i_1+1; \\ b_{i_1}-j_1+i_1+1 & \text{if } j = i_1; \\ b_j & \text{if } 0 \le j \le i_1-1 \end{cases}$$

and

$$x_j^{1,\prime} = \begin{cases} b_{j_1-1-j} & \text{if } 0 \le j \le j_1 - i_1 - 2; \\ b_{j-j_1+i_1+1} & \text{if } j_1 - i_1 - 1 \le j \le j_1 - 2; \\ b_{j_1} + j_1 - i_1 - 1 & \text{if } j = j_1; \\ b_{i_1} - j_1 + i_1 + 1 & \text{if } j = j_1 - 1; \\ b_j & \text{if } j_1 + 1 \le j \le n-1. \end{cases}$$

We also fix certain two elements in the Weyl group W:

$$w_1^{i_1,j_1} := (s_{n-3-i_1} \cdots s_1)^{j_1-i_1-1} \in W \text{ and } w_1^{i_1,j_1,\prime} := (s_{n-j_1+2} \cdots s_{n-1})^{j_1-i_1-1} \in W,$$

and further define two more weights

$$\mu^{i_1,j_1} := (\mu_1^{i_1,j_1})^{w_1^{i_1,j_1}} \text{ and } \mu^{i_1,j_1,\prime} := (\mu_1^{i_1,j_1,\prime})^{w_1^{i_1,j_1,\prime}}.$$

More precisely, μ^{i_1,j_1} and $\mu^{i_1,j_1,\prime}$ can be written as follow:

$$\mu^{i_1,j_1} = (x_{n-1}, x_{n-2}, \cdots, x_1, x_0) \text{ and } \mu^{i_1,j_1,\prime} = (x'_{n-1}, x'_{n-2}, \cdots, x'_1, x'_0)$$

where

$$x_j = \begin{cases} b_j & \text{if } j > j_1 \text{ or } i_1 > j; \\ b_{j_1+i_1+1-j} & \text{if } j_1 \ge j > i_1 + 1; \\ b_{j_1} + j_1 - i_1 - 1 & \text{if } j = i_1 + 1; \\ b_{i_1} - j_1 + i_1 + 1 & \text{if } j = i_1 \end{cases}$$

and

$$x'_j = \begin{cases} b_j & \text{if } j > j_1 \text{ or } i_1 > j; \\ b_{j_1+i_1-1-j} & \text{if } j_1 - 1 > j \ge i_1; \\ b_{j_1} + j_1 - i_1 - 1 & \text{if } j = j_1; \\ b_{i_1} - j_1 + i_1 + 1 & \text{if } j = j_1 - 1. \end{cases}$$

Note that if we let

$$w^{i_1,j_1} := s_{n-j_1} \cdots s_{n-i_1-2} \in W^L \text{ and } w^{i_1,j_1,\prime} := s_{n-i_1-1} \cdots s_{n-j_1+1} \in W^L$$

then we have

$$(\mu^{i_1,j_1})^{w^{i_1,j_1}} = (\mu^{\square,i_1,j_1})^{w_0^L} = (\mu^{i_1,j_1,\prime})^{w^{i_1,j_1,\prime}}.$$

Recall that w_0^L is defined at the beginning of Section 3.5 and that $\mu^{\square,i_1,j_1}$ is defined in Section 3.5.3.

We now define certain mod p Jacobi sum operators:

$$\mathcal{S}_1^{i_1,j_1} := S_{\underline{0},w_1^{i_1,j_1}} \text{ and } \mathcal{S}_1^{i_1,j_1,\prime} := S_{\underline{0},w_1^{i_1,j_1,\prime}}.$$

We further define

$$\mathcal{S}^{i_1,j_1} := S_{\underline{k}^{i_1,j_1},w_0^L} \text{ and } \mathcal{S}^{i_1,j_1,\prime} := S_{\underline{k}^{i_1,j_1,\prime},w_0^L}$$

where $\underline{k}^{i_1,j_1} = (k_{i,j}^{i_1,j_1})_{i,j} \in \{0, \cdots, p-1\}^{|\Phi^+_{w_0^L}|}$ and $\underline{k}^{i_1,j_1,\prime} = (k_{i,j}^{i_1,j_1,\prime})_{i,j} \in \{0, \cdots, p-1\}^{|\Phi^+_{w_0^L}|}$ satisfy

$$k_{i,j}^{i_1,j_1} := \begin{cases} [b_{i_1} - b_{n-i}]_1 & \text{if } n - j_1 + 1 \le i = j - 1 \le n - i_1 - 1; \\ i_1 - j_1 + 1 + [b_{i_1} - b_{j_1}]_1 & \text{if } i = j - 1 = n - j_1; \\ 0 & \text{if } j \ge i + 2 \end{cases}$$

and

$$k_{i,j}^{i_1,j_1,\prime} := \begin{cases} [b_{n-1-i} - b_{j_1}]_1 & \text{if } n - j_1 \le i = j - 1 \le n - i_1 - 2; \\ i_1 - j_1 + 1 + [b_{i_1} - b_{j_1}]_1 & \text{if } i = j - 1 = n - i_1 - 1; \\ 0 & \text{if } j \ge i + 2. \end{cases}$$

We now consider characteristic 0 lifts of the mod p Jacobi sum operators above.

$$\widehat{\mathcal{S}}^{i_1,j_1} := \left(\sum_{A \in U_{w_0^L}(\mathbf{F}_p)} \left(\prod_{\ell=n-j_1}^{n-i_1-1} \lceil A_{\ell,\ell+1} \rceil^{k_{\ell,\ell+1}^{i_1,j_1}} \right) \lceil A \rceil \right) w_0^L$$

and

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime} := \left(\sum_{A \in U_{w_0^L}(\mathbf{F}_p)} \left(\prod_{\ell=n-j_1}^{n-i_1-1} \lceil A_{\ell,\ell+1} \rceil^{k_{\ell,\ell+1}^{i_1,j_1,\prime}} \right) \lceil A \rceil \right) w_0^L.$$

We also let

$$\widehat{\mathcal{S}}_0^{i_1,j_1} := \left(\sum_{A \in U_{w_0^L}(\mathbf{F}_p)} \left(\prod_{\ell=n-j_1}^{n-i_1-1} \lceil A_{\ell,\ell+1} \rceil^{k_{\ell,\ell+1}^{i_1,j_1,0}} \right) \lceil A \rceil \right) w_0^L$$

where $\underline{k}^{i_1,j_1,0} = (k_{i,j}^{i_1,j_1,0})_{i,j} \in \{0, \cdots, p-1\}^{|\Phi^+_{w_0^L}|}$ satisfies

$$k_{i,j}^{i_1,j_1,0} := \begin{cases} i_1 - j_1 + 1 + [b_{i_1} - b_{j_1}]_1 & \text{if } n - j_1 \le i = j - 1 \le n - i_1 - 1; \\ 0 & \text{if } j \ge i + 2. \end{cases} \tag{3.5.24}$$

Note that $\widehat{\mathcal{S}}^{i_1,j_1}$, $\widehat{\mathcal{S}}^{i_1,j_1,\prime}$, $\widehat{\mathcal{S}}_0^{i_1,j_1}$ are Teichmüler lifts of $\mathcal{S}^{i_1,j_1}$, $\mathcal{S}^{i_1,j_1,\prime}$, $S_{\underline{k}^{i_1,j_1,0},w_0^L}$, respectively. We will also consider the Teichmüler lifts of $\mathcal{S}_1^{i_1,j_1}$ and $\mathcal{S}_1^{i_1,j_1,\prime}$ as follows:

$$\widehat{\mathcal{S}}_1^{i_1,j_1} := \left(\sum_{A \in U_{w_1^{i_1,j_1}}(\mathbf{F}_p)} \lceil A \rceil \right) w_1^{i_1,j_1} \quad \text{and} \quad \widehat{\mathcal{S}}_1^{i_1,j_1,\prime} := \left(\sum_{A \in U_{w_1^{i_1,j_1,\prime}}(\mathbf{F}_p)} \lceil A \rceil \right) w_1^{i_1,j_1,\prime}.$$

We recall the operator $\Xi_n \in G(\mathbf{Q}_p)$ from (3.4.49). Note that $\widetilde{\mu}_1^{i_1,j_1} : T(\mathbf{F}_p) \to \mathcal{O}_E^\times$ is the Teichmüler lift of $\mu_1^{i_1,j_1}$. We also recall $\kappa_n^{(1)}$, $\kappa_n^{(2)}$ (c.f. (3.4.63)),κ_n (c.f. (3.4.70)), ε^* (c.f. (3.4.69)), and $\mathcal{P}_n$ (c.f. (3.4.68)), whose definitions are completely determined by fixing the data n and $(a_{n-1}, \cdots, a_0)$. We define $\kappa_{i_1,j_1}^{(1)}, \kappa_{i_1,j_1}^{(2)}, \kappa_{i_1,j_1} \in \mathbf{Z}_p^\times$, $\varepsilon^{i_1,j_1} = \pm 1$ and $\mathcal{P}_{i_1,j_1} \in \mathbf{Z}_p^\times$ by replacing n and $(a_{n-1}, \cdots, a_1, a_0)$ by $j_1 - i_1 + 1$ and $(b_{j_1} + j_1 - i_1 - 1, b_{j_1-1}, \cdots, b_{i_1+1}, b_{i_1} - j_1 + i_1 + 1)$ respectively with b_k as at the beginning of Section 3.5.3.

Proposition 3.5.25. *Assume that $\mu^{\square,i_1,j_1}$ is $2n$-generic. Let*

$$\Pi^{i_1,j_1} := \mathrm{Ind}_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)} \chi^{i_1,j_1}$$

be a tamely ramified principal series where $\chi^{i_1,j_1} = \chi_{n-1}^{i_1,j_1} \otimes \cdots \otimes \chi_0^{i_1,j_1} : T(\mathbf{Q}_p) \to E^\times$ is a smooth character satisfying $\chi \mid_{T(\mathbf{Z}_p)} \cong \widetilde{\mu}_1^{i_1,j_1}$. Then we have the identity

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime} \bullet \widehat{\mathcal{S}}_1^{i_1,j_1,\prime} \bullet (\Xi_n)^{j_1-i_1-1} = p^{(j_1-i_1-1)(i_1+1)} \kappa_{i_1,j_1} \left(\prod_{k=n-j_1+i_1+1}^{n-1} \chi_k^{i_1,j_1}(p) \right) \widehat{\mathcal{S}}^{i_1,j_1} \bullet \widehat{\mathcal{S}}_1^{i_1,j_1}$$

on the 1-dimensional space $(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}_1^{i_1,j_1}}$.

Proof. By Lemma 3.4.54 we know that

$$(\Xi_n)^{j_1-i_1-1} \bullet U_n^{j_1-i_1-1} = \widehat{\mathcal{S}}_{(w^*)^{j_1-i_1-1}}.$$

Then by Lemma 3.4.51 and the fact

$$\ell(w_1^{i_1,j_1,\prime}) + \ell((w^*)^{j_1-i_1-1}) = \ell(w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1}) + \ell(w_1^{i_1,j_1}) + 2(j_1-i_1-1)i_1$$

we deduce that

$$\widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}} \bullet \widehat{\mathcal{S}}_{(w^*)^{j_1-i_1-1}} = p^{(j_1-i_1-1)i_1}\widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1}} \bullet \widehat{\mathcal{S}}_{w_1^{i_1,j_1}}.$$

Therefore it remains to show that

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime} \bullet \widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1}} = p^{j_1-i_1-1}\kappa_{i_1,j_1}\widehat{\mathcal{S}}^{i_1,j_1}$$

on the 1-dimensional space

$$(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}^{i_1,j_1}} = \widehat{\mathcal{S}}_{w_1^{i_1,j_1}}\left((\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}_1^{i_1,j_1}}\right).$$

We observe by Lemma 3.4.51 that

$$\widehat{\mathcal{S}}_{w^{i_1,j_1,\prime}} \bullet \widehat{\mathcal{S}}_{w_1^{i_1,j_1,\prime}(w^*)^{j_1-i_1-1}(w_1^{i_1,j_1})^{-1}} = p^{j_1-i_1-1}\widehat{\mathcal{S}}_{w^{i_1,j_1}}$$

and therefore by composing $\widehat{\mathcal{S}}_0^{i_1,j_1}$ it remains to show that

$$\widehat{\mathcal{S}}_0^{i_1,j_1} \bullet \widehat{\mathcal{S}}_{w^{i_1,j_1,\prime}} = p^{j_1-i_1-1}(\kappa_{i_1,j_1}^{(2)})^{-1}\widehat{\mathcal{S}}^{i_1,j_1,\prime} \tag{3.5.26}$$

on $(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}^{i_1,j_1,\prime}}$ and

$$\widehat{\mathcal{S}}_0^{i_1,j_1} \bullet \widehat{\mathcal{S}}_{w^{i_1,j_1}} = p^{j_1-i_1-1}(\kappa_{i_1,j_1}^{(1)})^{-1}\widehat{\mathcal{S}}^{i_1,j_1} \tag{3.5.27}$$

on $(\Pi^{i_1,j_1})^{I(1),\widetilde{\mu}^{i_1,j_1}}$. But these can be checked by the same argument as in Corollary 3.4.66. □

We state here a generalization of the Theorem 3.4.36. Recall the definition of $\pi_*^{i_1,j_1}$ from (3.5.15).

Theorem 3.5.28. *The constituent $F(\mu^{\square})$ has multiplicity one in $\pi_*^{i_1,j_1}$.*

Proof. This is Corollary 3.4.47 if we replace $\mu_\pi^{i_1,j_1}$ by $\mu^{\square}$. □

We define a characteristic zero principal series

$$(\widetilde{\pi}_*^{i_1,j_1})^\circ := \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)}(\widetilde{\mu}^{\square,i_1,j_1})^{w_0}$$

which is an $\mathcal{O}_E$-lattice in $(\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E$.

Lemma 3.5.29. (i) *For $\mu \in \{\mu^{i_1,j_1}, \mu^{i_1,j_1,\prime}, \mu_1^{i_1,j_1}, \mu_1^{i_1,j_1,\prime}\}$, we have*

$$\dim_{\mathbf{F}_p}(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu} = 1.$$

(ii) *We have the following non-vanishing results:*

$$\mathcal{S}^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right) = \mathcal{S}^{i_1,j_1,\prime}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right) \neq 0.$$

(iii) *We also have the following non-vanishing results:*

$$\mathcal{S}_1^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}\right) = (\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$$

and

$$\mathcal{S}_1^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1,\prime}}\right) = (\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}.$$

Proof. The statement (i) is immediate by Bruhat decomposition (3.4.4).

Now we prove (ii). According to Lemma 3.4.51, (3.5.26) and (3.5.27) and Lemma 3.4.65, we deduce by mod p reduction with respect to the lattice $(\widetilde{\pi}_*^{i_1,j_1})^\circ$ that

$$\begin{aligned}\mathcal{S}^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right) &= \mathcal{S}^{i_1,j_1,\prime}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right)\\ &= S_{\underline{k}^{i_1,j_1,0},w_0^L}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0^L}}\right).\end{aligned}$$

If we abuse the notation $\underline{k}^{i_1,j_1,0}$ for the tuple in $\{0,\cdots,p-1\}^{|\Phi^+_{w_0}|}$ satisfying

$$\underline{k}_\alpha^{i_1,j_1,0}=0 \text{ for all } \alpha\notin\Phi^+_{w_0^L}$$

then by mod p reduction of first possibility of Proposition 3.4.60 we deduce that

$$S_{\underline{k}^{i_1,j_1,0},w_0^L}\bullet S_{\underline{0},w_0^Lw_0}=S_{\underline{k}^{i_1,j_1,0},w_0}$$

on the 1-dimensional subspace $(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}$. Thus we finish the proof of (ii) by

$$S_{\underline{k}^{i_1,j_1,0},w_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right)\neq 0$$

which follows from Proposition 3.4.18.

Finally we prove (iii). We only prove the first equality in (iii) as the same proof works for the second equality. By Lemma 3.4.20 we know that

$$S_{\underline{0},(w^{i_1,j_1})^{-1}w_0^Lw_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right)=(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$$

and

$$S_{\underline{0},(w^{i_1,j_1}w_1^{i_1,j_1})^{-1}w_0^Lw_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right)=(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}.$$

Therefore it remains to show that

$$\mathcal{S}_1^{i_1,j_1}\bullet S_{\underline{0},(w^{i_1,j_1}w_1^{i_1,j_1})^{-1}w_0^Lw_0}=S_{\underline{0},(w^{i_1,j_1})^{-1}w_0^Lw_0}$$

on the 1-dimensional subspace $(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}$, which follows from the mod p reduction of Lemma 3.4.51 and the fact that

$$\ell(w_1^{i_1,j_1})+\ell((w^{i_1,j_1}w_1^{i_1,j_1})^{-1}w_0^Lw_0)=\ell((w^{i_1,j_1})^{-1}w_0^Lw_0).$$

This completes the proof. □

We define V^{i_1,j_1} and $V^{i_1,j_1,\prime}$ to be the subrepresentations of $\pi_*^{i_1,j_1}$ generated by

$$\mathcal{S}^{i_1,j_1}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right)\quad\text{and}\quad\mathcal{S}^{i_1,j_1,\prime}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right)$$

respectively. Similarly, we define $V_0^{i_1,j_1}$ as the subrepresentation of $\pi_*^{i_1,j_1}$ generated by

$$S_{\underline{k}^{i_1,j_1,0}}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right).$$

Lemma 3.5.30. *We have*

$$V^{i_1,j_1}=V^{i_1,j_1,\prime}=V_0^{i_1,j_1}\tag{3.5.31}$$

and

$$F(\mu^\square)\in\mathrm{JH}(V_0^{i_1,j_1}).\tag{3.5.32}$$

Proof. The equality (3.5.31) follows directly from the proof of (ii) of Lemma 3.5.29.

We define a new tuple $\underline{k}^{i_1,j_1,0}=(k_{i,j}^{i_1,j_1,0})_{i,j}\in\{0,\cdots,p-1\}^{|\Phi^+_{w_0}|}$ defined by

$$k_{i,j}^{i_1,j_1,\sharp}:=\begin{cases} i_1-j_1+1+[b_{i_1}-b_{j_1}]_1 & \text{if } (i,j)=(n-j_1,n-i_1);\\ 0 & \text{otherwise.}\end{cases}\tag{3.5.33}$$

We also define $V^{i_1,j_1,\sharp}$ to be the subrepresentation of $\pi_*^{i_1,j_1}$ generated by

$$S_{\underline{k}^{i_1,j_1,\sharp},w_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right).$$

By Proposition 3.4.101 and the same method in the proof of Proposition 3.4.124 we deduce that

$$V^{i_1,j_1,\sharp}\subseteq V_0^{i_1,j_1}.\tag{3.5.34}$$

By abuse of notation we view $\mu^{\square,i_1,j_1}$ as a fixed weight in $X_1(T)$, and then there exists $\mu^{\square,\prime}\in X_+(T)$ such that

$$\mu^{\square,\prime}\equiv\mu^{\square}\pmod{(p-1)X(T)}\ \text{ and }\ \mu^{\square,\prime}=(n-i_1,n-j_1)\cdot\mu^{\square,i_1,j_1}+p\sum_{r=n-j_1}^{n-i_1-1}\alpha_r.$$

We define $\overline{U}_1^{i_1,j_1}$ to be the unipotent subgroup of $\overline{L}$ generated by $\overline{U}_{\alpha_r}$ for $n-j_1+1\leq r\leq n-i_1-1$ and then define

$$\overline{U}^{i_1,j_1}:=\overline{U}_1^{i_1,j_1}\cdot\overline{N}.$$

By a direct generalization of proof of Lemma 3.4.129, we can show that

$$S_{\underline{k}^{i_1,j_1,\sharp},w_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right)=H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}.$$

We define $V_{\mathrm{alg}}^{i_1,j_1}$ to be the $\overline{G}$-subrepresentation of $H^0(\mu^{\square,i_1,j_1})$ generated by $H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}$ and by definition we have

$$(V_{\mathrm{alg}}^{i_1,j_1})^{\overline{N}}\hookrightarrow H^0(\mu^{\square,i_1,j_1})^{\overline{N}}\ \text{ and }\ (V_{\mathrm{alg}}^{i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}=H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}.\tag{3.5.35}$$

We have natural identification (c.f. the beginning of Section 3.5 for definition of $H^0_L(\mu^{\square,i_1,j_1})$)

$$H^0(\mu^{\square,i_1,j_1})^{\overline{N}}\cong H^0_L(\mu^{\square,i_1,j_1})\ \text{ and }\ H^0(\mu^{\square,i_1,j_1})^{\overline{U}^{i_1,j_1}}\cong H^0_L(\mu^{\square,i_1,j_1})^{\overline{U}_1^{i_1,j_1}}.\tag{3.5.36}$$

By applying Lemma 3.4.132 and the proof of Proposition 3.4.133 to the Levi L, we deduce that $H^0_L(\mu^{\square,i_1,j_1})$ is uniserial of length two with socle $F^L(\mu^{\square,i_1,j_1})$ and cosocle $F^L(\mu^{\square,\prime})$ and that

$$H^0_L(\mu^{\square,i_1,j_1})^{\overline{U}_1^{i_1,j_1}}_{\mu^{\square,\prime}}\xrightarrow{\sim}F^L(\mu^{\square,\prime})_{\mu^{\square,\prime}}.\tag{3.5.37}$$

Combine (3.5.35), (3.5.36) and (3.5.37) we deduce the surjection of representations of $\overline{L}$

$$(V_{\mathrm{alg}}^{i_1,j_1})^{\overline{N}}\twoheadrightarrow F^L(\mu^{\square,\prime})\cong H^0_L(\mu^{\square,\prime})\cong H^0(\mu^{\square,\prime})^{\overline{N}}$$

and thus a non-zero morphism

$$(V_{\mathrm{alg}}^{i_1,j_1})\to H^0(\mu^{\square,\prime})\text{ and }(V_{\mathrm{alg}}^{i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}\xrightarrow{\sim}H^0(\mu^{\square,\prime})^{\overline{U}}_{\mu^{\square,\prime}}\xleftarrow{\sim}F(\mu^{\square,\prime})^{\overline{U}}_{\mu^{\square,\prime}}$$

by coinduction for algebraic representation from $\overline{P}$ to $\overline{G}$. In particular we know that

$$F(\mu^{\square,\prime})\in\mathrm{JH}_{\overline{G}}\left(V_{\mathrm{alg}}^{i_1,j_1}\right).$$

Now we restrict the action of $\overline{G}$ to $G(\mathbf{F}_p)$ and observe the injections

$$V^{i_1,j_1,\sharp} \hookrightarrow V_{\mathrm{alg}}^{i_1,j_1}|_{G(\mathbf{F}_p)} \quad \text{and} \quad F(\mu^{\square}) \hookrightarrow F(\mu^{\square,\prime})|_{G(\mathbf{F}_p)}$$

which induces

$$S_{\underline{k}^{i_1,j_1,\sharp},w_0}\left((\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right) = (V^{i_1,j_1,\sharp})^{U^{i_1,j_1}(\mathbf{F}_p),\mu^{\square}} = (V_{\mathrm{alg}}^{i_1,j_1})^{\overline{U}^{i_1,j_1}}_{\mu^{\square,\prime}}$$

and

$$F(\mu^{\square})^{U(\mathbf{F}_p),\mu^{\square}} = (F(\mu^{\square,\prime})|_{G(\mathbf{F}_p)})^{U(\mathbf{F}_p),\mu^{\square}} = F(\mu^{\square,\prime})_{\mu^{\square,\prime}}.$$

Hence we deduce that

$$F(\mu^{\square}) \in \mathrm{JH}_{G(\mathbf{F}_p)}\left(V^{i_1,j_1,\sharp}\right)$$

which together with (3.5.34) finishes the proof of (3.5.32). □

Proposition 3.5.38. *Let τ be an $\mathcal{O}_E$-lattice in $(\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E$ satisfying*

$$\mathrm{soc}_{G(\mathbf{F}_p)}(\tau \otimes_{\mathcal{O}_E} \mathbf{F}) \hookrightarrow F(\mu^{\square}) \oplus F(\mu^{\square,i_1,j_1}).$$

(i) *For $\mu \in \{\mu^{i_1,j_1}, \mu^{i_1,j_1,\prime}, \mu_1^{i_1,j_1}, \mu_1^{i_1,j_1,\prime}\}$, we have*

$$\dim_{\mathbf{F}}(\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu} = 1.$$

(ii) *We have the non-vanishing results for $\mathcal{S}^{i_1,j_1}$ and $\mathcal{S}^{i_1,j_1,\prime}$:*

$$\mathcal{S}^{i_1,j_1}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right) = \mathcal{S}^{i_1,j_1,\prime}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right) \neq 0.$$

(iii) *We also have the non-vanishing results for $\mathcal{S}_1^{i_1,j_1}$ and $\mathcal{S}_1^{i_1,j_1,\prime}$:*

$$\mathcal{S}_1^{i_1,j_1}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}\right) = (\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$$

and

$$\mathcal{S}_1^{i_1,j_1,\prime}\left((\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1,\prime}}\right) = (\tau \otimes_{\mathcal{O}_E} \mathbf{F})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}.$$

Proof. We can easily deduce (i) from

$$\dim_E((\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E)^{U(\mathbf{F}_p),\widetilde{\mu}^{i_1,j_1}} = \dim_E((\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E)^{U(\mathbf{F}_p),\widetilde{\mu}^{i_1,j_1,\prime}} = 1$$

and Frobenius reciprocity as $F(\mu^{i_1,j_1})$, $F(\mu^{i_1,j_1})$, $F(\mu^{i_1,j_1})$ and $F(\mu^{i_1,j_1})$ all have multiplicity one in $\tau \otimes_{\mathcal{O}_E} \mathbf{F}$.

We define $\pi_\flat^{i_1,j_1}$ as the mod p reduction of $(\widetilde{\pi}_*^{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} E$ with respect to the unique (up to homothety) $\mathcal{O}_E$-lattice such that

$$\mathrm{soc}_{G(\mathbf{F}_p)}\left(\pi_\flat^{i_1,j_1}\right) = F(\mu^{\square}).$$

Then we deduce from Corollary 3.5.23 that there exists an injection

$$\tau \otimes_{\mathcal{O}_E} \mathbf{F} \hookrightarrow \pi_*^{i_1,j_1} \oplus \pi_\flat^{i_1,j_1}$$

Note that we have

$$\left(\pi_*^{i_1,j_1} \oplus \pi_\flat^{i_1,j_1}\right)^{U(\mathbf{F}_p),\mu} = (\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu} \oplus (\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu} \tag{3.5.39}$$

for $\mu \in \{\mu^{i_1,j_1}, \mu^{i_1,j_1,\prime}, \mu_1^{i_1,j_1}, \mu_1^{i_1,j_1,\prime}\}$.

The equality of two spaces in (ii) is true because both of them can be identified with

$$S_{\underline{k}^{i_1,j_1,0},w_0}\left((\tau\otimes_{\mathcal{O}_E}\mathbf{F})^{U(\mathbf{F}_p),(\mu^{\square,i_1,j_1})^{w_0}}\right)$$

by the same argument as in the proof of (ii) of Lemma 3.5.29. Therefore we only need to show that $\mathcal{S}^{i_1,j_1}$ (resp. $\mathcal{S}^{i_1,j_1,\prime}$) gives rise to a bijection from $\left(\pi_*^{i_1,j_1}\oplus\pi_\flat^{i_1,j_1}\right)^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$ (resp. from $\left(\pi_*^{i_1,j_1}\oplus\pi_\flat^{i_1,j_1}\right)^{U(\mathbf{F}_p),\mu^{i_1,j_1}}$) to its image. According to (ii) of Lemma 3.5.29 and (3.5.39) we only need to show that

$$\mathcal{S}^{i_1,j_1}\left((\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1}}\right)\neq 0 \text{ and } \mathcal{S}^{i_1,j_1,\prime}\left((\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu^{i_1,j_1,\prime}}\right)\neq 0$$

which follows from Lemma 3.5.30 by definition of $\pi_\flat^{i_1,j_1}$.

We have a unique (up to scalar) non-zero morphism

$$\pi_*^{i_1,j_1}\to\pi_\flat^{i_1,j_1} \tag{3.5.40}$$

which by Lemma 3.5.30 induces isomorphisms

$$(\pi_*^{i_1,j_1})^{U(\mathbf{F}_p),\mu}\xrightarrow{\sim}(\pi_\flat^{i_1,j_1})^{U(\mathbf{F}_p),\mu}$$

for $\mu\in\{\mu^{i_1,j_1},\mu^{i_1,j_1,\prime}\}$, and hence (iii) follows from (iii) of Lemma 3.5.29 by considering the image of (iii) of Lemma 3.5.29 under (3.5.40) inside $\pi_\flat^{i_1,j_1}$. □

Corollary 3.5.41. *Let τ be an $\mathcal{O}_E$-lattice in $(\widetilde{\pi}_*^{i_1,j_1})^\circ\otimes_{\mathcal{O}_E}E$ satisfying*

$$\mathrm{soc}_{G(\mathbf{F}_p)}(\tau\otimes_{\mathcal{O}_E}\mathbf{F})\hookrightarrow F(\mu^\square)\oplus F(\mu^{\square,i_1,j_1}).$$

Then we have

$$0\neq\mathcal{S}^{i_1,j_1}\bullet\mathcal{S}_1^{i_1,j_1}\left((\tau\otimes_{\mathcal{O}_E}\mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}\right)=\mathcal{S}^{i_1,j_1,\prime}\bullet\mathcal{S}_1^{i_1,j_1,\prime}\left((\tau\otimes_{\mathcal{O}_E}\mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1,\prime}}\right).$$

3.5.6 Main results

In this section, we state and prove our main results on mod p local-global compatibility. Throughout this section, $\overline{\rho}_0$ is always assumed to be a restriction of a global representation $\overline{r}:G_F\to\mathrm{GL}_n(\mathbf{F})$ to G_{F_w} for a fixed place w of F above p. Let $v:=w|_{F^+}$, and assume further that $\overline{r}$ is automorphic of a Serre weight $V=\bigotimes_{v'}V_{v'}$ with $V_w:=V_v\circ\iota_w^{-1}\cong F(\mu^\square)^\vee$. We may write $V_{v'}\circ\iota_{w'}^{-1}\cong F(\underline{a}_{w'})^\vee$ for a dominant weight $\underline{a}_{w'}\in\mathbf{Z}_+^n$ where w' is a place of F above v', and define

$$V':=\bigotimes_{v'\neq v}V_{v'}\quad\text{and}\quad\widetilde{V}':=\bigotimes_{v'\neq v}W_{\underline{a}_{v'}}. \tag{3.5.42}$$

From now on, we also assume that *$\underline{a}_{w'}$ is in the lowest alcove for each place w' of F above p*, so that $V'\cong\widetilde{V}'\otimes_{\mathcal{O}_E}\mathbf{F}$. Let U be a compact open subgroup of $G_n(\mathbf{A}_F^{\infty,p})\times\mathcal{G}_n(\mathcal{O}_{F^+,p})$, which is sufficiently small and unramified above p, such that $S(U,V)[\mathfrak{m}_{\overline{r}}]\neq 0$ where $\mathfrak{m}_{\overline{r}}$ is the maximal ideal of $\mathbf{T}^{\mathcal{P}}$ attached to $\overline{r}$ for a cofinite subset $\mathcal{P}$ of $\mathcal{P}_U$.

We fix a pair of integers (i_0,j_0) such that $0\leq j_0<j_0+1<i_0\leq n-1$, and determine a pair inters (i_1,j_1) by the equation (3.5.3). We also define

$$\begin{cases} M & := S(U^v,\widetilde{V}')_{\mathfrak{m}_{\overline{r}}};\\ M^{i_1,j_1} & := S(U^v,\widetilde{V}')_{\mathfrak{m}_{\overline{r}}}^{I(1),\widetilde{\mu}_1^{i_1,j_1}}.\end{cases}$$

Note that M^{i_1,j_1} is a free $\mathcal{O}_E$-module of finite rank as M is a admissible smooth representation of $G(\mathbf{Q}_p)$ which is ϖ_E-torsion free. For any $\mathcal{O}_E$-algebra A, we write $M_A^{i_1,j_1}$ for $M^{i_1,j_1}\otimes_{\mathcal{O}_E} A$. We similarly define M_A.

Let $\mathbf{T}^{i_1,j_1}$ be the $\mathcal{O}_E$-module that is the image of $\mathbf{T}^{\mathcal{P}}$ in $\mathrm{End}_{\mathcal{O}_E}(M^{i_1,j_1})$. Then $\mathbf{T}^{i_1,j_1}$ is a local $\mathcal{O}_E$-algebras with the maximal ideal $\mathfrak{m}_{\overline{r}}$, where, by abuse of notation, we write $\mathfrak{m}_{\overline{r}}\subseteq\mathbf{T}$ for the image of $\mathfrak{m}_{\overline{r}}$ of $\mathbf{T}^{\mathcal{P}}$. As the level U is sufficiently small, by passing to a sufficiently large E as in the proof of Theorem 4.5.2 of [HLM17], we may assume that $\mathbf{T}_E^{i_1,j_1}\cong E^r$ for some $r>0$. For any $\mathcal{O}_E$-algebra A we write $\mathbf{T}_A^{i_1,j_1}$ for $\mathbf{T}^{i_1,j_1}\otimes_{\mathcal{O}_E}A$.

We have $M_E^{i_1,j_1}=\bigoplus_{\mathfrak{p}} M_E^{i_1,j_1}[\mathfrak{p}_E]$, where the sum runs over the minimal primes $\mathfrak{p}$ of $\mathbf{T}^{i_1,j_1}$ and $\mathfrak{p}_E:=\mathfrak{p}\mathbf{T}_E^{i_1,j_1}$. Note that for any such $\mathfrak{p}$ $\mathbf{T}_E^{i_1,j_1}/\mathfrak{p}_E\cong E$. By abuse of notation, we also write $\mathfrak{p}$ (resp. $\mathfrak{p}_E$) for its inverse image in $\mathbf{T}^{\mathcal{P}}$ (resp. $\mathbf{T}_E^{\mathcal{P}}$). We also note that for any such $\mathfrak{p}$ we have a surjection $M[\mathfrak{p}]\twoheadrightarrow M_{\mathbf{F}}[\mathfrak{m}_{\overline{r}}]$ as $\mathfrak{m}_{\overline{r}}=\mathfrak{p}+\varpi_E\mathbf{T}^{\mathcal{P}}$.

Definition 3.5.43. *A non-zero vector $v^{i_1,j_1}\in M_{\mathbf{F}}^{i_1,j_1}$ is said to be* primitive *if there exists a vector $\widehat{v}^{i_1,j_1}\in M^{i_1,j_1}[\mathfrak{p}]$ that lifts v^{i_1,j_1}, for certain minimal prime $\mathfrak{p}$ of $\mathbf{T}$.*

Note that the $G(\mathbf{Q}_p)$-subrepresentation of M_E generated by a lift $\widehat{v}^{i_1,j_1}$ of a primitive element v^{i_1,j_1} is irreducible and actually lies in $M_E[\mathfrak{p}_E]$.

Now we can state our main results in this paper. Recall that by $\overline{\rho}_0$ we always mean an n-dimensional ordinary representation of $G_{\mathbf{Q}_p}$ as described in (3.3.1).

Theorem 3.5.44. *Fix a pair of integers (i_0,j_0) satisfying $0\le j_0<j_0+1<i_0\le n-1$, and let (i_1,j_1) be a pair of integers such that $i_0+i_1=j_0+j_1=n-1$. We also let $\overline{r}:G_F\to\mathrm{GL}_n(\mathbf{F})$ be an irreducible automorphic representation with $\overline{r}|_{G_{F_w}}\cong\overline{\rho}_0$. Assume that*

- *$\mu^{\square,i_1,j_1}$ is $2n$-generic;*
- *$\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic.*

Assume further that

$$\{F(\mu^{\square})^\vee\}\subseteq W_w(\overline{r})\cap\mathrm{JH}((\pi_*^{i_1,j_1})^\vee)\subseteq\{F(\mu^{\square})^\vee,F(\mu^{\square,i_1,j_1})^\vee\}. \tag{3.5.45}$$

Then there exists a primitive vector in $S(U^v,V')[\mathfrak{m}_{\overline{r}}]^{I(1),\mu_1^{i_1,j_1}}$. Moreover, for each primitive vector $v^{i_1,j_1}\in S(U^v,V')[\mathfrak{m}_{\overline{r}}]^{I(1),\mu_1^{i_1,j_1}}$ we have $\mathcal{S}^{i_1,j_1}\bullet\mathcal{S}_1^{i_1,j_1}v^{i_1,j_1}\neq 0$ and

$$\mathcal{S}^{i_1,j_1,\prime}\bullet\mathcal{S}^{i_1,j_1,\prime}\bullet(\Xi_n)^{j_1-i_1-1}v^{i_1,j_1}=\varepsilon^{i_1,j_1}\mathcal{P}_{i_1,j_1}(b_{n-1},\cdots,b_0)\cdot\mathrm{FL}_n^{i_0,j_0}(\overline{r}|_{G_{F_w}})\cdot\mathcal{S}^{i_1,j_1}\bullet\mathcal{S}_1^{i_1,j_1}v^{i_1,j_1}$$

where

$$\varepsilon^{i_1,j_1}=\prod_{k=i_1+1}^{j_1-1}(-1)^{b_{i_1}-b_k-j_1+i_1+1}$$

and

$$\mathcal{P}_{i_1,j_1}(b_{n-1},\cdots,b_0)=\prod_{k=i_1+1}^{j_1-1}\prod_{j=1}^{j_1-i_1-1}\frac{b_k-b_{j_1}-j}{b_{i_1}-b_k-j}\in\mathbf{Z}_p^\times.$$

Remark 3.5.46. *The right inclusion of (3.5.45) is just Conjecture 3.5.16, which is now a theorem of Bao V. Le Hung (c.f. Remark 3.1.13 and [LLMPQ]). We also give an evidence for the left inclusion of (3.5.45) in Proposition 3.5.17 under some assumption of Taylor–Wiles type. As a result, the condition (3.5.45) can be removed under some standard Taylor–Wiles conditions.*

Remark 3.5.47. *If M^{i_1,j_1} is free as $\mathbf{T}^{i_1,j_1}$-module, then all vectors in $S(U^v,V')[\mathfrak{m}_{\overline{r}}]^{I(1),\mu_1^{i_1,j_1}}$ are primitive. As a result, one needs such a freeness result to remove the "primitive" condition. Under*

a stronger generic condition (compared to our Fontaine-Laffaille generic), it is possible to use results from [LLMPQ] to improve (3.5.45) *to an equality*

$$W_w(\overline{r}) \cap \mathrm{JH}((\pi_*^{i_1,j_1})^\vee) = \{F(\mu^\square)^\vee\}$$

in which case one is able to prove the freeness result mentioned above through the technique in Section 5 of [HLM17] under some standard global assumption. It is also possible to prove a freeness result over some enlarged Hecke algebra as in Section 5 of [HLM17], at least if $(i_1, j_1) = (0, n-1)$.

Proof. We firstly point out that $M^{i_1,j_1} \neq 0$, as $S(U, (F(\mu^\square)^\vee \circ \iota_w) \otimes V')_{\mathfrak{m}_{\overline{r}}} \neq 0$ and $F(\mu^\square)$ is a factor of $\overline{\mathrm{Ind}_I^K \widetilde{\mu}_1^{i_1,j_1}} = \mathrm{Ind}_{B(\mathbf{F}_p)}^{G(\mathbf{F}_p)} \mu_1^{i_1,j_1}$.

Picking an embedding $E \hookrightarrow \overline{\mathbf{Q}}_p$, as well as an isomorphism $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$, we see that

$$M_{\overline{\mathbf{Q}}_p}^{i_1,j_1} \cong \bigoplus_{\Pi} m(\Pi) \cdot \Pi_v^{I(1),\widetilde{\mu}_1^{i_1,j_1}} \otimes (\Pi^{\infty,v})^{U^v}, \tag{3.5.48}$$

where the sum runs over irreducible representations $\Pi \cong \Pi_\infty \otimes \Pi_v \otimes \Pi^{\infty,v}$ of $G_n(\mathbb{A}_{F^+})$ over $\overline{\mathbf{Q}}_p$ such that $\Pi \otimes_\iota \mathbf{C}$ is a cuspidal automorphic representation of multiplicity $m(\Pi) \in \mathbf{Z}_{>0}$ with $\Pi_\infty \otimes_\iota \mathbf{C}$ being determined by the algebraic representation $(\widetilde{V}')^\vee$ and with associated Galois representation r_Π lifting $\overline{r}^\vee$ (c.f. Lemma 3.5.7).

We write δ for the modulus character of $B(\mathbf{Q}_p)$:

$$\delta := |\ |^{n-1} \otimes |\ |^{n-2} \otimes \cdots \otimes |\ | \otimes 1$$

where $|\ |$ is the (unramfied) norm character sending p to p^{-1}. For any Π contributing to (3.5.48), we have

(i) $\Pi_v \cong \mathrm{Ind}_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}(\psi \otimes \delta)$ for some smooth character

$$\psi = \psi_{n-1} \otimes \psi_{n-2} \otimes \cdots \otimes \psi_1 \otimes \psi_0$$

of $T(\mathbf{Q}_p)$ such that $\psi|_{T(\mathbf{Z}_p)} = \widetilde{\mu}_1^{i_1,j_1}|_{T(\mathbf{Z}_p)}$, where ψ_k are the smooth characters of $\mathbf{Q}_p^\times$.

(ii) $r_\Pi^\vee|_{G_{F_w}}$ is a potentially crystalline lift of $\overline{r}$ with Hodge–Tate weights $\{-(n-1), -(n-2), \cdots, -1, 0\}$ and $\mathrm{WD}(r_\Pi^\vee|_{G_{F_w}})^{\mathrm{F-ss}} \cong \oplus_{k=0}^{n-1} \psi_k^{-1}$.

Here, part (i) follows from [EGH13], Propositions 2.4.1 and 7.4.4, and part (ii) follows from classical local-global compatibility (c.f. Theorem 3.5.8). Moreover, by Corollary 3.3.46, we have

$$\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0) = \frac{\overline{\prod_{k=j_0+1}^{i_0-1} \psi_{i_1+1+k}(p)}}{p^{\frac{(i_0+j_0)(i_0-j_0-1)}{2}}}. \tag{3.5.49}$$

(Note that we may identify ψ_{i_1+1+k} with Ω_k^{-1} for $j_0 < k < i_0$, where Ω_k is defined in Corollary 3.3.46.)

Now we pick an arbitrary primitive vector $v^{i_1,j_1} \in M_{\mathbf{F}}^{i_1,j_1}[\mathfrak{m}_{\overline{r}}]$ with a lift $\widehat{v}^{i_1,j_1} \in M^{i_1,j_1}[\mathfrak{p}]$. We set

$$\tau_E := \langle K\widehat{v}^{i_1,j_1} \rangle_E \subseteq M_E[\mathfrak{p}_E] \text{ and } \tau := \tau_E \cap M[\mathfrak{p}],$$

and thus τ is an $\mathcal{O}_E$-lattice in τ_E. Note that $M_E^{i_1,j_1}[\mathfrak{p}_E] \otimes_E \overline{\mathbf{Q}}_p$ is a direct summand of (3.5.48) where Π runs over a subset of automorphic representations in (3.5.48). The same argument as in the paragraph above (4.5.7) of [HLM17] using Cebotarev density shows us that the local component Π_v of each Π occurring in this direct summand does not depend on Π.

By the definition of τ, we obtain an injection

$$\tau \otimes_{\mathcal{O}_E} \mathbf{F} \hookrightarrow (M[\mathfrak{p}]) \otimes_{\mathcal{O}_E} \mathbf{F} = M_{\mathbf{F}}[\mathfrak{m}_{\overline{r}}] \tag{3.5.50}$$

as $\mathfrak{p}+\varpi_E\mathbf{T}^{\mathcal{P}}=\mathfrak{m}_{\overline{r}}$. By the assumption (3.5.45) (c.f. Conjecture 3.5.16), we deduce that

$$\mathrm{JH}\left(\mathrm{soc}_{G(\mathbf{F}_p)}\left(M_{\mathbf{F}}[\mathfrak{m}_{\overline{r}}]\right)\right)\subseteq\{F(\mu^{\square}),F(\mu^{\square,i_1,j_1})\}$$

and therefore by (3.5.50) we have

$$\mathrm{JH}\left(\mathrm{soc}_{G(\mathbf{F}_p)}\left(\tau\otimes_{\mathcal{O}_E}\mathbf{F}\right)\right)\subseteq\{F(\mu^{\square}),F(\mu^{\square,i_1,j_1})\}.$$

Hence by Corollary 3.5.41 we know that

$$\mathcal{S}^{i_1,j_1}\bullet\mathcal{S}_1^{i_1,j_1}(\tau\otimes_{\mathcal{O}_E}\mathbf{F})^{U(\mathbf{F}_p),\mu_1^{i_1,j_1}}\neq 0. \tag{3.5.51}$$

On the other hand, we have the following equality by Proposition 3.5.25

$$\widehat{\mathcal{S}}^{i_1,j_1,\prime}\bullet\widehat{\mathcal{S}}_1^{i_1,j_1,\prime}\bullet(\Xi_n)^{j_1-i_1-1}\widehat{v}^{i_1,j_1}=\kappa_{i_1,j_1}\left(\frac{\prod_{k=j_0+1}^{i_0-1}\psi_{i_1+1+k}(p)}{p^{\frac{(i_0+j_0)(i_0-j_0-1)}{2}}}\right)\widehat{\mathcal{S}}^{i_1,j_1}\bullet\widehat{\mathcal{S}}_1^{i_1,j_1}\widehat{v}^{i_1,j_1}. \tag{3.5.52}$$

By taking mod p reduction of (3.5.52) we deduce from (3.5.49) that

$$\mathcal{S}^{i_1,j_1,\prime}\bullet\mathcal{S}^{i_1,j_1,\prime}\bullet(\Xi_n)^{j_1-i_1-1}v^{i_1,j_1}=\varepsilon^{i_1,j_1}\mathcal{P}_{i_1,j_1}(b_{n-1},\cdots,b_0)\cdot\mathrm{FL}_n^{i_0,j_0}(\overline{r}|_{G_{F_w}})\cdot\mathcal{S}^{i_1,j_1}\bullet\mathcal{S}_1^{i_1,j_1}v^{i_1,j_1}.$$

This equation together with (3.5.51) finishes the proof. □

Corollary 3.5.53. *Keep the notation of Theorem 3.5.44 and assume that each assumption in Theorem 3.5.44 holds for all (i_0,j_0) such that $0\leq j_0<j_0+1<i_0\leq n-1$. Assume further that M^{i_1,j_1} is free over $\mathbf{T}^{i_1,j_1}$ for all pair (i_1,j_1) (c.f. Remark 3.5.47).*

Then the structure of $S(U^v,V')[\mathfrak{m}_{\overline{r}}]$ as a admissible smooth $\mathbf{F}$-representation of $G(\mathbf{Q}_p)$ determines $\overline{\rho}_0$ up to isomorphism.

Proof. We follow the notation in Section 3.4 of [BH15]. As $\overline{\rho}_0$ is ordinary, we can view it as a morphism

$$\overline{\rho}_0:\quad G_{\mathbf{Q}_p}\to\widehat{B}(\mathbf{F})\subseteq\widehat{G}(\mathbf{F})$$

where $\widehat{B}$ (resp. $\widehat{G}$) is the dual group of B (resp. G). The local class field theory gives us a bijection between smooth characters of $\mathbf{Q}_p^\times$ and the smooth characters of the Weil group of $\mathbf{Q}_p$ in characteristic zero. This bijection restricts to a bijection between smooth characters of $\mathbf{Q}_p^\times$ and smooth characters of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ both with values in $\mathcal{O}_E^\times$. Taking mod p reduction and then taking products we reach a bijection between smooth $\mathbf{F}$-characters of $T(\mathbf{Q}_p)$ and $\mathrm{Hom}\left(\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p),\widehat{T}(\mathbf{F})\right)$. We can therefore define $\chi_{\overline{\rho}_0}$ as the character of $T(\mathbf{Q}_p)$ corresponding to the composition

$$\widehat{\chi}_{\overline{\rho}_0}:\quad\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)\to\widehat{B}(\mathbf{F})\twoheadrightarrow\widehat{T}(\mathbf{F}).$$

In [BH15], a closed subgroup $C_{\overline{\rho}_0}\subseteq B$ (at the beginning of section 3.2) and a subset $W_{\overline{\rho}_0}$ ((2) before Lemma 2.3.6) of W is defined.

As we are assuming that $\overline{\rho}_0$ is maximally non-split, we observe that $C_{\overline{\rho}_0}=B$ and $W_{\overline{\rho}_0}=\{1\}$ in our case. Therefore by the definition of $\Pi^{ord}(\overline{\rho}_0)$ in [BH15] before Definition 3.4.3, we know that it is indecomposable with socle

$$\mathrm{Ind}_{B^-(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}\chi_{\overline{\rho}_0}\cdot(\omega^{-1}\circ\theta)$$

where $\theta\in X(T)$ is a twist character defined after Conjecture 3.1.2 in [BH15] which can be chosen to be η in our notation. Then as a Corollary of Theorem 4.4.7 in [BH15], we deduce that $S(U^v,V')[\mathfrak{m}_{\overline{r}}]$ determines $\chi_{\overline{\rho}_0}$ and hence $\widehat{\chi}_{\overline{\rho}_0}$.

Now, we know that $\overline{\rho}_0$ is determined by the Fontaine–Laffaille parameters $\{\mathrm{FL}_n^{i_0,j_0}(\overline{\rho}_0)\in\mathbb{P}^1(\mathbf{F})\mid 0\leq i_0<i_0+1<j_0\leq n-1\}$ and $\widehat{\chi}_{\overline{\rho}_0}$, up to isomorphism. Our conclusion thus follows from Theorem 3.5.44 and Remark 3.5.47. □

Chapter 4

Dilogarithm and higher $\mathscr{L}$-invariants for $\mathrm{GL}_3(\mathbf{Q}_p)$

4.1 Introduction

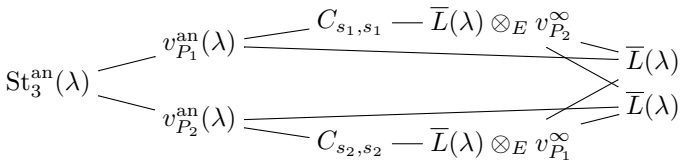
Let p be a prime number and F an imaginary quadratic extension of $\mathbf{Q}$ such that p splits in F. We fix a unitary algebraic group G over $\mathbf{Q}$ which becomes GL_n over F and such that $G(\mathbf{R})$ is compact and G is split at all places of F above p. Then to each finite extension E of $\mathbf{Q}_p$ and to each prime-to-p level U^p in $G(\mathbb{A}_{\mathbf{Q}}^{\infty,p})$, one can associate the Banach space of p-adic automorphic forms $\widehat{S}(U^p, E)$. One can also associate with U^p a set of finite places $D(U^p)$ of $\mathbf{Q}$ and a Hecke algebra $\mathbb{T}(U^p)$ which is the polynomial algebra freely generated by Hecke operators at places of F lying above $D(U^p)$. In particular, the commutative algebra $\mathbb{T}(U^p)$ acts on $\widehat{S}(U^p, E)$ and commutes with the action of $G(\mathbf{Q}_p) \cong \mathrm{GL}_n(\mathbf{Q}_p)$ coming from translations on $G(\mathbb{A}_{\mathbf{Q}}^{\infty})$.

If $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E)$ is a continuous irreducible representation, one considers the associated Hecke isotypic subspace $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]$, which is a continuous admissible representation of $G(\mathbf{Q}_p) \cong \mathrm{GL}_n(\mathbf{Q}_p)$ over E, or its locally $\mathbf{Q}_p$-analytic vectors $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]^{\mathrm{an}}$, which is an admissible locally $\mathbf{Q}_p$-analytic representation of $\mathrm{GL}_n(\mathbf{Q}_p)$. We fix w_p a place of F above p and it is widely wished that $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]$ (and its subspace $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]^{\mathrm{an}}$ as well) determines and depends only on $\rho_p := \rho|_{\mathrm{Gal}(\overline{F_{w_p}}/F_{w_p})}$. The case $n = 2$ is well-known essentially due to various results in [Col10], [Eme]. The case $n \geq 3$ is much more difficult and only some partial results are known. We are particularly interested in the case when the subspace of locally algebraic vectors $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]^{\mathrm{alg}} \subsetneq \widehat{S}(U^p, E)[\mathfrak{m}_\rho]$ is non-zero, which implies that ρ_p is potentially semi-stable. Certain cases when $n = 3$ and ρ_p is semi-stable and non-crystalline have been studied in [Bre17] and [BD18]. We are going to continue their work and obtain some interesting relation between results in [Bre17], [BD18] and previous results in [Schr11] which involve the p-adic dilogarithm function.

We use the notation $\lambda \in X(T)_+$ for a weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ (of the diagonal split torus T of GL_3) which is dominant with respect to the upper-triangular Borel subgroup $\overline{B}$ and hence satisfies $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Given two locally analytic representations V, W of $\mathrm{GL}_3(\mathbf{Q}_p)$, we use the shorten notation $V \text{ —— } W$ (resp. the shorten notation V – – W) for a locally analytic representation determined by a non-zero (resp. possibly zero) element in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p)}(W,\ V)$.

Theorem 4.1.1. *[Proposition 4.6.8, Proposition 4.6.29] For each choice of $\lambda \in X(T)_+$ and $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in$*

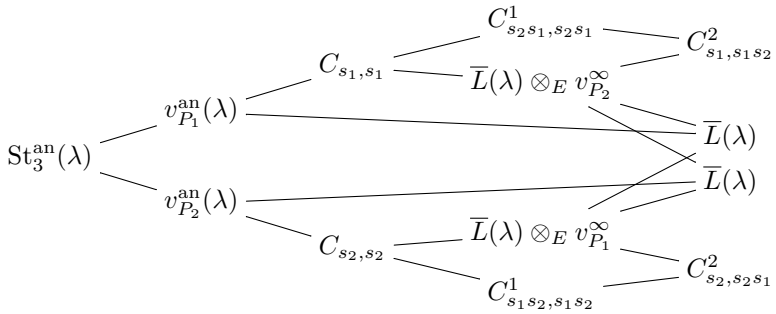
E, there exists a locally analytic representation $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of $\mathrm{GL}_3(\mathbf{Q}_p)$ of the form:



(4.1.2)

where $\mathrm{St}_3^{\mathrm{an}}(\lambda)$, $v_{P_1}^{\mathrm{an}}(\lambda)$, $v_{P_2}^{\mathrm{an}}(\lambda)$, $\overline{L}(\lambda)$ and $C_{w',w}^{*}$ for $w, w' \in \{s_1, s_2, s_1s_2, s_2s_1\}$ and $* \in \{\varnothing, 1, 2\}$ are various explicit locally analytic representations defined in Section 4.2.3. Moreover, different choices of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ give non-isomorphic representations.

We will see in Lemma 4.6.47 and (4.6.55) that $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ is *the minimal locally analytic representation that involves p-adic dilogarithm*, hence explains the notation 'min'. We also construct a locally analytic representation $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form



which contains and is uniquely determined by $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$.

Theorem 4.1.3. *[Theorem 4.7.5] Assume that $p \geq 5$ and $n = 3$. Assume moreover that*

(i) *ρ is unramified at all finite places of F above $D(U^p)$;*

(ii) *$\widehat{S}(U^p, E)[\mathfrak{m}_\rho]^{\mathrm{alg}} \neq 0$;*

(iii) *ρ_p is semi-stable with Hodge–Tate weights $\{k_1 > k_2 > k_3\}$ such that $N^2 \neq 0$;*

(iv) *ρ_p is non-critical in the sense of Remark 6.1.4 of [Bre17];*

(v) *only one automorphic representation contributes to $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]^{\mathrm{alg}}$.*

Then there exists a unique choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ such that $\widehat{S}(U^p, E)[\mathfrak{m}_\rho]^{\mathrm{an}}$ contains (copies of) the locally analytic representation

$$\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (k_1 - 2, k_2 - 1, k_3)$ and $\alpha \in E^\times$ is determined by the Weil–Deligne representation $\mathrm{WD}(\rho_p)$ associated with ρ_p. Moreover, we have

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \\ &\quad \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\overline{L}(\lambda) \otimes_E \mathrm{St}_3^{\infty} \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right). \end{aligned}$$

(4.1.4)

The assumptions of our Theorem 4.1.3 are the same as that of Theorem 1.3 of [Bre17]. We do not attempt to obtain any explicit relation between $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ and ρ_p, which is similar in flavor to Theorem 1.3 of [Bre17]. On the other hand, Theorem 7.52 of [BD18] does care about the explicit relation between invariants of the locally analytic representation associated with ρ_p, under further technical assumptions such as ρ_p is ordinary with consecutive Hodge–Tate weights and has an irreducible mod p reduction but without assuming our condition (v). The improvement of our Theorem 4.1.3 upon Theorem 1.3 of [Bre17] will be explained in Section 4.1.2. One can naturally wish that there is a common refinement or generalization of our Theorem 4.1.3 and Theorem 7.52 of [BD18] by removing as many technical assumptions as possible.

Remark 4.1.5. *It is actually possible to construct a locally analytic representation* $\Sigma^{\max}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ *of* $\mathrm{GL}_3(\mathbf{Q}_p)$ *containing* $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ *which is characterized by the fact that it is maximal (for inclusion) among the locally analytic representations* V *satisfying the following conditions:*

(i) $\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V) = V^{\mathrm{alg}} = \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty$*;*

(ii) *each constituent of* V *is a subquotient of a locally analytic principal series*

where V^{alg} *is the subspace of locally algebraic vectors in* V*. Moreover, one can use an immediate generalization of the arguments in the proof of Theorem 4.1.3 (and thus of Theorem 1.1 of [Bre17]) to show that*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\max}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \\ \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right).\end{aligned} \tag{4.1.6}$$

We can also show that

$$\Sigma^{\max}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda) \otimes_E \mathrm{St}_3$$

is independent of the choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$*, which is compatible with the fact that*

$$\Sigma^{\min,*}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda) \otimes_E \mathrm{St}_3$$

is independent of the choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ *for each* $* \in \{\varnothing, +\}$ *as mentioned in Remark 4.6.58. However, the full construction of* $\Sigma^{\max}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ *is lengthy and technical and thus we decided not to put it in the present article.*

4.1.1 Derived object and dilogarithm

We consider the bounded derived category

$$\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

associated with the abelian category $\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}$ of abstract modules over the algebra $D(\mathrm{GL}_3(\mathbf{Q}_p), E)$ of locally $\mathbf{Q}_p$-analytic distributions on $\mathrm{GL}_3(\mathbf{Q}_p)$. An object

$$\Sigma(\lambda, \underline{\mathscr{L}})' \in \mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

(one should not confuse this notation $\Sigma(\lambda, \underline{\mathscr{L}})'$ borrowed directly from [Schr11] with our notation $\Sigma^+(\lambda, \underline{\mathscr{L}})$ before Lemma 4.6.18) has been constructed in [Schr11] and plays a key role in Theorem 1.2 of [Schr11]. An interesting feature of [Schr11] is the appearance of the p-adic dilogarithm function in the construction of $\Sigma(\lambda, \underline{\mathscr{L}})'$ in Definition 5.19 of [Schr11]. Roughly, the object $\Sigma(\lambda, \underline{\mathscr{L}})'$ was constructed from the choice of an element in $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$ together with general

formal arguments in triangulated categories (c.f. Proposition 3.2 of [Schr11]). In particular, $\Sigma(\lambda, \underline{\mathscr{L}})'$ fits into the following distinguished triangle:

$$F'_\lambda \longrightarrow \Sigma(\lambda, \underline{\mathscr{L}})' \longrightarrow \Sigma(\lambda, \mathscr{L}, \mathscr{L}')'[-1] \xrightarrow{+1}$$

as illustrated in (5.99) of [Schr11]. However, it was not clear in [Schr11] whether there is an explicit complex $[C_\bullet]$ of locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ such that the object

$$\mathcal{D}' \in \mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

associated with $\left[C'_{-\bullet}\right]$ satisfies

$$\mathcal{D}' \cong \Sigma(\lambda, \underline{\mathscr{L}})' \in \mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right).$$

Although our notation are slightly different from [Schr11] in the sense that the notation $\Sigma(\lambda, \mathscr{L}, \mathscr{L}')$ (resp. the notation F_λ) is replaced with $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ (resp. with $\overline{L}(\lambda)$), we show that

Theorem 4.1.7. *[Proposition 4.6.36, (4.2.28) and Lemma 4.2.37] The complex*

$$\left[\left(\ \overline{L}(\lambda) \otimes_E v^\infty_{P_{3-i}} \longrightarrow \overline{L}(\lambda)\ \right)' \longrightarrow \Sigma_i^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)'\right] \tag{4.1.8}$$

maps to the object $\Sigma(\lambda, \underline{\mathscr{L}})'$ in the derived category where $\overline{L}(\lambda) \otimes_E v^\infty_{P_{3-i}} \longrightarrow \overline{L}(\lambda)$ is the unique non-split extension of $\overline{L}(\lambda)$ by $\overline{L}(\lambda) \otimes_E v^\infty_{P_{3-i}}$ thanks to Proposition 4.4.1, $\Sigma_i^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ is the locally analytic subrepresentation of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form

$$\begin{array}{ccccc} & & v_{P_i}^{\mathrm{an}}(\lambda) & \nearrow & C_{s_i,s_i} \longrightarrow \overline{L}(\lambda) \otimes_E v^\infty_{P_{3-i}} \\ \mathrm{St}_3^{\mathrm{an}}(\lambda) & \nearrow & & \searrow & \overline{L}(\lambda) \\ & \searrow & v_{P_{3-i}}^{\mathrm{an}}(\lambda) & \nearrow & C_{s_{3-i},s_{3-i}} \end{array}$$

and the invariants $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ are determined by the formula

$$\mathscr{L}_1 = -\mathscr{L}',\ \mathscr{L}_2 = -\mathscr{L},\ \mathscr{L}_3 = \gamma(\mathscr{L}'' - \frac{1}{2}\mathscr{L}\mathscr{L}')$$

with the constant $\gamma \in E^\times$ defined in Lemma 4.2.34.

Remark 4.1.9. *Strictly speaking, the complex (4.1.8) realizes an object in $\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$ characterized by an element in*

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$$

due to formal arguments from Proposition 3.2 of [Schr11]. However, we can prove that there is a canonical isomorphism

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$$

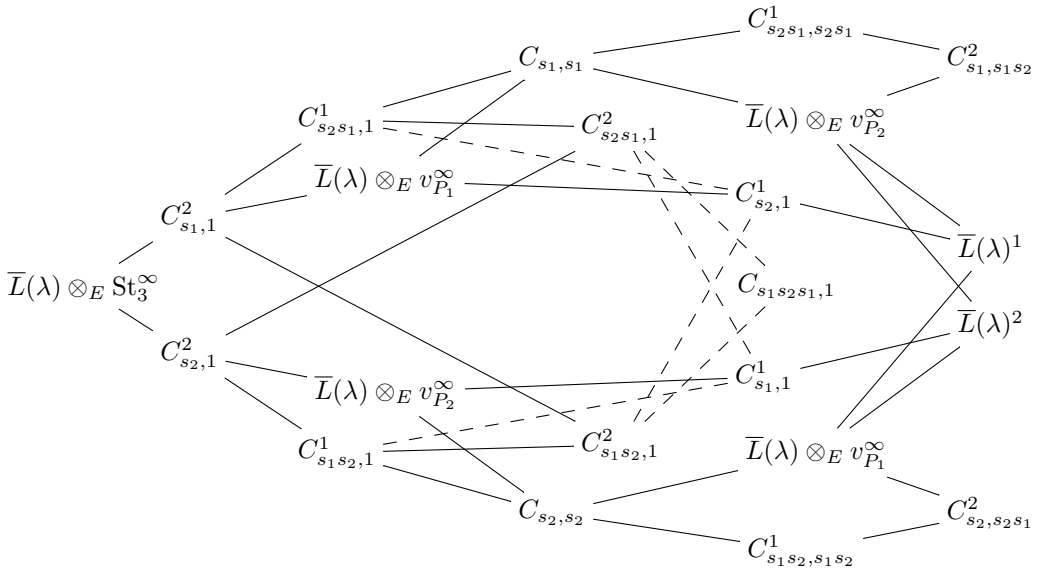
and hence we can equally say that (4.1.8) realizes $\Sigma(\lambda, \underline{\mathscr{L}})'$ for a suitable normalization of notation as $\Sigma(\lambda, \underline{\mathscr{L}})$ has been constructed by choosing a non-zero element in $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}, \mathscr{L}')\right)$ via Proposition 3.2 of [Schr11]. Note that we have

$$\Sigma(\lambda, \mathscr{L}, \mathscr{L}') \cong \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$$

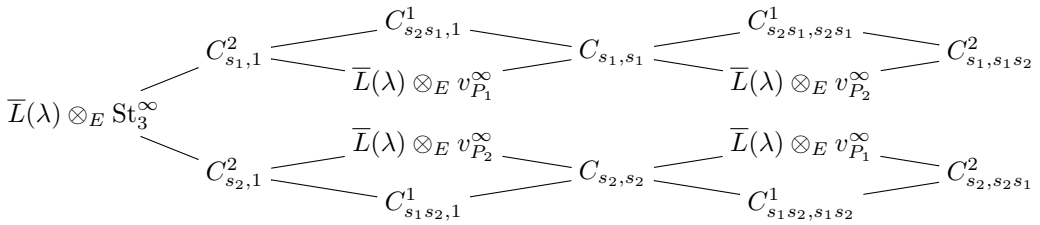
by (4.2.27).

4.1.2 Higher $\mathscr{L}$-invariants for $\mathrm{GL}_3(\mathbf{Q}_p)$

It follows from (4.6.55) and (4.6.57) that $\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ can be described more precisely by the following picture:



and therefore contains a unique subrepresentation of the form



which is denoted by

$$\overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3 \begin{array}{c} \nearrow \Pi^1(\underline{k},\underline{D}) \\ \searrow \Pi^2(\underline{k},\underline{D}) \end{array} \tag{4.1.10}$$

in Theorem 1.1 of [Bre17]. It follows from Theorem 1.2 of [Bre17] that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\Pi^i(\underline{k},\underline{D}),\ \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3\right)=3$$

for $i=1,2$, and therefore a locally analytic representation of the form (4.1.10) depends on four invariants. On the other hand, by a computation of extensions of rank one (φ,Γ)-modules we know that ρ_p depends on three invariants. As a result, Theorem 1.1 of [Bre17] predicts that not all representations of the form (4.1.10) can be embedded into $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ for a certain pair of U^p and ρ_p. This is actually the case as we show that

Theorem 4.1.11. *[Corollary 4.7.17] If a locally analytic representation Π of the form (4.1.10) can be embedded into $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ for a certain pair of U^p and ρ_p, then it can be embedded into*

$$\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

for a unique choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ determined by Π.

4.1.3 Sketch of content

Section 4.2 recalls various well-known facts around locally analytic representations and our notation for a family of specific irreducible subquotients of locally analytic principal series to be used in the rest of the article. We emphasize that our definition of various Ext-groups follows [Bre17] closely and the only difference is that we use the dual notation compared to that of [Bre17]. We also recall the p-adic dilogarithm function from Section 5.3 of [Schr11] which is part of the main motivation of this article to relate [Schr11] with [Bre17] and [BD18].

Section 4.3 proves a crucial fact (Proposition 4.3.14) on the non-existence of locally analytic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ of a certain specific form using arguments involving infinitesimal characters of locally analytic representations. We learn such arguments essentially from Y. Ding.

Section 4.4 is a collection of various computational results necessary for the applications in Section 4.6. These computations essentially make use of the formula in Section 5.2 and 5.3 of [Bre17].

Section 4.5 serves as the preparation of Section 4.6 for the construction of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$. It makes full use of the computational results from Section 4.4 to compute the dimension of various more complicated Ext-groups to be crucially used in various important long exact sequences in Section 4.6(c.f. Lemma 4.6.1 and Proposition 4.6.8).

Section 4.6 finishes the construction of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ as well as $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$. Moreover, the construction of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ leads naturally to the construction of an explicit complex as in Theorem 4.1.7 that realizes the derived object $\Sigma(\lambda, \underline{\mathscr{L}})$ constructed in [Schr11].

Section 4.7 finishes the proof of Theorem 4.7.5 by directly mimicking arguments from the proof of Theorem 6.2.1 of [Bre17]. In particular, we give a purely representation theoretic criterion for a representation of the form (4.1.10) to embed into completed cohomology as mentioned in Theorem 4.1.11.

4.1.4 Acknowledgement

The author expresses his gratefulness to Christophe Breuil for introducing the problem of relating [Schr11] with [Bre17] and [BD18] and especially for his interest on the role played by the p-adic dilogarithm function. The author also benefited a lot from countless discussions with Y. Ding especially for Section 4.3 of this article. Finally, the author thanks B. Schraen for his beautiful thesis which improved the author's understanding on the subject.

4.2 Preliminary

4.2.1 Locally analytic representations

In this section, we recall the definition of some well-known objects in the theory of locally analytic representations of p-adic reductive groups.

We fix a locally $\mathbf{Q}_p$-analytic group H and denote the algebra of locally $\mathbf{Q}_p$-analytic distribution with coefficient E on H by $\mathcal{D}(H, E)$, which is defined as the strong dual of the locally convex E-vector space $C^{\mathrm{an}}(H, E)$ consisting of locally $\mathbf{Q}_p$-analytic functions on H. We use the notation $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ (resp. $\mathrm{Rep}^{\infty}_{H,E}$) for the abelian category consisting of admissible locally $\mathbf{Q}_p$-analytic representations of H (resp. admissible smooth representations of H) with coefficient E. Therefore taking strong dual induces a faithful contravariant functor from $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ to the abelian category $\mathrm{Mod}_{\mathcal{D}(H,E)}$ of abstract modules over $\mathcal{D}(H, E)$. The E-vector space $\mathrm{Ext}^i_{\mathcal{D}(H,E)}(M_1, M_2)$ is well-defined for any two objects $M_1, M_2 \in \mathrm{Mod}_{\mathcal{D}(H,E)}$, and therefore we define

$$\mathrm{Ext}^i_H(\Pi_1, \Pi_2) := \mathrm{Ext}^i_{\mathcal{D}(H,E)}(\Pi_2', \Pi_1')$$

for any two objects $\Pi_1, \Pi_2 \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$ where $\cdot'$ is the notation for strong dual. We also define the cohomology of an object $M \in \mathrm{Mod}_{\mathcal{D}(H,E)}$ by

$$H^i(H, M) := \mathrm{Ext}^i_{\mathcal{D}(H,E)}(1, M)$$

where 1 is the strong dual of the trivial representation of H. If H' is a closed locally $\mathbf{Q}_p$-analytic normal subgroup of H, then H/H' is also a locally $\mathbf{Q}_p$-analytic group. It follows from the fact

$$D(H,E)\otimes_{D(H',E)}E\cong D(H/H',E)$$

(see Section 5.1 of [Bre17] for example) that $H^i(H',M)$ admits a structure of $\mathcal{D}(H/H',E)$-module for each $M\in\mathrm{Mod}_{\mathcal{D}(H,E)}$. We define the H'-homology of $\Pi\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$ as the object (if it exists up to isomorphism) $H_i(H',\Pi)\in\mathrm{Rep}^{\mathrm{la}}_{H/H',E}$ such that

$$H_i(H',\Pi)'\cong H^i(H',\Pi').$$

We emphasize that $H_i(H',\Pi)$ is well defined in the sense above only after we know its existence. We fix a subgroup Z of the center of the group H, then the algebra $\mathcal{D}(Z,E)$ consisting of locally $\mathbf{Q}_p$-analytic distribution with coefficient E on Z is naturally contained in the center of $\mathcal{D}(H,E)$. For each locally $\mathbf{Q}_p$-analytic E-character χ of Z, we can define the abelian subcategory $\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$ consisting of all the objects in $\mathrm{Mod}_{\mathcal{D}(H,E)}$ on which $\mathcal{D}(Z,E)$ acts by χ'. Then we consider the functors $\mathrm{Ext}^i_{\mathcal{D}(H,E)}(-,-)$ defined as $\mathrm{Ext}^i_{\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}}(-,-)$ which are extensions inside the abelian category $\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$. Consequently we can define

$$\mathrm{Ext}^i_{H,\chi}(\Pi_1,\Pi_2):=\mathrm{Ext}^i_{\mathcal{D}(H,E),\chi'}(\Pi_2',\Pi_1')$$

for any two objects $\Pi_1,\Pi_2\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$ such that $\Pi_1',\Pi_2'\in\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$. In particular, if Z is the center of H and acts on $\Pi\in\mathrm{Rep}^{\mathrm{la}}_{H,E}$ via the character χ, then $\Pi'\in\mathrm{Mod}_{\mathcal{D}(H,E),\chi'}$, and we usually say that Π admits a central character χ.

Assume now H is the set of $\mathbf{Q}_p$-points of a split reductive group over $\mathbf{Q}_p$. We recall the category $\mathcal{O}$ together with its subcategory $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ for each parabolic subgroup $P\subseteq H$ from Section 9.3 of [Hum08] or [OS15]. The construction by Orlik–Strauch in [OS15] gives us a functor

$$\mathcal{F}_P^H:\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}\times\mathrm{Rep}^{\infty}_{L,E}\to\mathrm{Rep}^{\mathrm{la}}_{H,E}$$

for each parabolic subgroup $P\subseteq H$ with Levi quotient L. We use the notation $\mathrm{Rep}^{\mathcal{OS}}_{H,E}$ for the abelian full subcategory of $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ generated by the irreducible objects inside the image of $\mathcal{F}_P^H$ when P varies over all possible parabolic subgroups of H. Here we say a full subcategory is generated by a family of objects if it is the minimal full subcategory that contains these objects and is stable under extensions and taking subquotients. In particular, all objects in $\mathrm{Rep}^{\mathcal{OS}}_{H,E}$ have finite length.

4.2.2 Formal properties

In this section, we recall and prove some general formal properties of locally analytic representations of p-adic reductive groups.

We fix a split p-adic reductive group H and a parabolic subgroup P of H. We use the notation N for the unipotent radical of P and fix a Levi subgroup L of P.

Lemma 4.2.1. *We have a spectral sequece*

$$\mathrm{Ext}^j_{L,*}\left(H_k(N,\ \Pi_1),\ \Pi_2\right)\Rightarrow\mathrm{Ext}^{j+k}_{H,*}\left(\Pi_1,\ \mathrm{Ind}_P^H\left(\Pi_2\right)^{\mathrm{an}}\right).$$

which implies an isomorphism

$$\mathrm{Hom}_{L,*}\left(H_0(N,\ \Pi_1),\ \Pi_2\right)\xrightarrow{\sim}\mathrm{Hom}_{H,*}\left(\Pi_1,\ \mathrm{Ind}_P^H\left(\Pi_2\right)^{\mathrm{an}}\right)$$

and a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{L,*}\left(H_0(N,\ \Pi_1),\ \Pi_2\right)\hookrightarrow\mathrm{Ext}^1_{H,*}&\left(\Pi_1,\ \mathrm{Ind}_P^H\left(\Pi_2\right)^{\mathrm{an}}\right)\\&\to\mathrm{Hom}_{L,*}\left(H_1(N,\ \Pi_1),\ \Pi_2\right)\to\mathrm{Ext}^2_{L,*}\left(H_0(N,\ \Pi_1),\ \Pi_2\right)\end{aligned}$$

for each $\Pi_1 \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$, $\Pi_2 \in \mathrm{Rep}^{\mathrm{la}}_{L,E}$ *satisfying the (*FIN*) condition in Section 6 of [ST05],* $* \in \{\varnothing, \chi\}$ *where* χ *is a locally analytic character of the center of* H*.*

Proof. This follows directly from our definition of Ext^k and H_k in Section 4.2.1 for $k \geq 0$, the original dual version in (44) and (45) of [Bre17]. □

We fix a Borel subgroup $B \subseteq H$ together with its opposite Borel subgroup $\overline{B}$. We fix an irreducible object $M \in \mathcal{O}^{\overline{\mathfrak{b}}}_{\mathrm{alg}}$. We choose a parabolic subgroup $P \subseteq H$ such that P is maximal among all the parabolic subgroups $Q \subseteq H$ such that $M \in \mathcal{O}^{\overline{\mathfrak{q}}}_{\mathrm{alg}}$ where $\overline{\mathfrak{q}}$ is the Lie algebra of the opposite parabolic subgroup $\overline{Q}$ associated with Q. We fix a smooth irreducible representation π^∞ of L and a smooth character δ of H. Then we know that [OS15] constructed an irreducible locally analytic representation

$$\mathcal{F}_P^H(M,\ \pi^\infty)$$

of H.

Lemma 4.2.2. *The functor*

$$-\otimes_E \delta$$

induces an equivalence of category from $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ *to itself. Moreover, the restriction of* $-\otimes_E \delta$ *to the subcategory* $\mathrm{Rep}^{\mathcal{OS}}_{H,E}$ *is again an equivalence of category to itself and satisfies*

$$\mathcal{F}_P^H(M,\ \pi^\infty)\otimes_E \delta \cong \mathcal{F}_P^H(M,\ \pi^\infty \otimes_E \delta|_L) \tag{4.2.3}$$

for each irreducible object $\mathcal{F}_P^H(M,\ \pi^\infty) \in \mathrm{Rep}^{\mathcal{OS}}_{H,E}$*.*

Proof. The functor $-\otimes_E\delta$ is clearly an equivalence of category from $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ to itself with quasi-inverse given by

$$-\otimes_E \delta^{-1}.$$

It is sufficient to prove the formula (4.2.3) to finish the proof. First of all, we notice by formal reason (equivalence of category) that $\mathcal{F}_P^H(M,\ \pi^\infty)\otimes_E\delta$ is an irreducible object in $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ since $\mathcal{F}_P^H(M,\ \pi^\infty)$ is. We use the notation $\overline{\mathfrak{n}}$ (resp. $\mathfrak{h}$) for the Lie algebra associated with the unipotent radical $\overline{N}$ of the opposite parabolic subgroup $\overline{P}$ of P (resp. for the Lie algebra associated with H). We define M_L as the (finite dimensional) algebraic representation of L whose dual is isomorphic to $M^{\overline{\mathfrak{n}}}$ as a representation of $\mathfrak{l}$ and note that we have a surjection

$$U(\mathfrak{h})\otimes_{U(\overline{\mathfrak{p}})} M^{\overline{\mathfrak{n}}} \twoheadrightarrow M$$

where $U(\mathfrak{h})$ is the universal enveloping algebra of $\mathfrak{h}$. We observe that N acts trivially on δ, and therefore we have

$$H_0\left(N,\ \mathcal{F}_P^H(M,\ \pi^\infty)\otimes_E\delta\right) \cong H_0\left(N,\ \mathcal{F}_P^H(M,\ \pi^\infty)\right)\otimes_E\delta|_L \twoheadrightarrow M_L\otimes_E\pi^\infty\otimes_E\delta|_L$$

which induces by Lemma 4.2.1 a non-zero morphism

$$\mathcal{F}_P^H(M,\ \pi^\infty)\otimes_E\delta \to \mathrm{Ind}_P^H\left(M_L\otimes_E\pi^\infty\otimes_E\delta|_L\right)^{\mathrm{an}} \cong \mathcal{F}_P^H(U(\mathfrak{h})\otimes_{U(\overline{\mathfrak{p}})} M^{\overline{\mathfrak{n}}},\ \pi^\infty\otimes_E\delta|_L). \tag{4.2.4}$$

We finish the proof by the fact that $\mathcal{F}_P^H(M,\ \pi^\infty)\otimes_E\delta$ is irreducible and that

$$\mathcal{F}_P^H(M,\ \pi^\infty\otimes_E\delta|_L) \cong \mathrm{soc}_H\left(\mathcal{F}_P^H(U(\mathfrak{h})\otimes_{U(\overline{\mathfrak{p}})} M^{\overline{\mathfrak{n}}},\ \pi^\infty\otimes_E\delta|_L)\right).$$

due to Corollary 3.3 of [Bre16]. □

We fix a finite length locally analytic representation $V \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$ equipped with a increasing filtration of subrepresentations $\{\mathrm{Fil}_k V\}_{0\leq k\leq m}$ such that

$$\mathrm{Fil}_0(V)=0,\ \mathrm{Fil}_m(V)=V \text{ and } \mathrm{gr}_{k+1}V := \mathrm{Fil}_{k+1}V/\mathrm{Fil}_k V \neq 0 \text{ for all } 0\leq k\leq m-1.$$

Note that the assumption above automatically implies that

$$\ell(V)\geq m$$

where $\ell(V)$ is the length of V.

Proposition 4.2.5. *Assume that W is another object of $\mathrm{Rep}^{\mathrm{la}}_{H,E}$ and χ is a locally analytic character of the center of H.*

(i) *If $\mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{gr}_k V)=0$ for each $1\leq k\leq m$, then we have*

$$\mathrm{Ext}^1_{H,\chi}(W,\ V)=0.$$

(ii) *If there exists $1\leq k_0\leq m$ such that $\mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{gr}_k V)=0$ for each $1\leq k\neq k_0\leq m$ and $\dim_E \mathrm{Ext}^1_{H,\chi}\left(W,\ \mathrm{gr}_{k_0}V\right)=1$, then we have*

$$\dim_E \mathrm{Ext}^1_{H,\chi}(W,\ V)\leq 1;$$

if moreover $\mathrm{Ext}^2_{H,\chi}(W,\ \mathrm{gr}_k V)=0$ for each $1\leq k\leq k_0-1$ and $\mathrm{Hom}_{H,\chi}(W,\ \mathrm{gr}_k V)=0$ for each $k_0+1\leq k\leq m$, then we have

$$\dim_E \mathrm{Ext}^1_{H,\chi}(W,V)=1.$$

Proof. The short exact sequence $\mathrm{Fil}_k V\hookrightarrow \mathrm{Fil}_{k+1}V\twoheadrightarrow \mathrm{gr}_{k+1}V$ induces a long exact sequence

$$\mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_k V)\to \mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_{k+1}V)\to \mathrm{Ext}^1_{H,\chi}\left(W,\ \mathrm{gr}_{k+1}V\right)$$

which implies

$$\dim_E \mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_{k+1}V)\leq \dim_E \mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_k V)+\dim_E \mathrm{Ext}^1_{H,\chi}\left(W,\ \mathrm{gr}_{k+1}V\right).$$

Therefore we finish the proof of part (i) and the first claim of part (ii) by induction on k and the fact that $\mathrm{gr}_1 V=\mathrm{Fil}_1 V$.

It remains to show the second claim of part (ii). The same method as in the proof of part (i) shows that

$$\mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_{k_0-1}V)=\mathrm{Ext}^2_{H,\chi}(W,\ \mathrm{Fil}_{k_0-1}V)=0 \tag{4.2.6}$$

and

$$\mathrm{Ext}^1_{H,\chi}(W,\ V/\mathrm{Fil}_{k_0}V)=\mathrm{Hom}_{H,\chi}(W,\ V/\mathrm{Fil}_{k_0}V)=0 \tag{4.2.7}$$

The short exact sequence $\mathrm{Fil}_{k_0-1}V\hookrightarrow \mathrm{Fil}_{k_0}V\twoheadrightarrow \mathrm{gr}_{k_0}V$ induces the long exact sequence

$$\mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_{k_0-1}V)\ \to\ \mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_{k_0}V)\ \to\ \mathrm{Ext}^1_{H,\chi}\left(W,\ \mathrm{gr}_{k_0}V\right)\ \to\ \mathrm{Ext}^2_{H,\chi}(W,\ \mathrm{Fil}_{k_0-1}V)$$

which implies that

$$\dim_E \mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_{k_0}V)=1 \tag{4.2.8}$$

by (4.2.6). The short exact sequence $\mathrm{Fil}_{k_0}V\hookrightarrow V\twoheadrightarrow V/\mathrm{Fil}_{k_0}V$ induces the long exact sequence

$$\mathrm{Hom}_{H,\chi}(W,\ V/\mathrm{Fil}_{k_0}V)\ \to\ \mathrm{Ext}^1_{H,\chi}(W,\ \mathrm{Fil}_{k_0}V)\ \to\ \mathrm{Ext}^1_{H,\chi}(W,\ V)\ \to\ \mathrm{Ext}^1_{H,\chi}(W,\ V/\mathrm{Fil}_{k_0}V)$$

which finishes the proof by combining (4.2.7) and (4.2.8). □

4.2.3 Some notation

In this section, we are going to recall some standard notation for the p-adic reductive groups $\mathrm{GL}_2(\mathbf{Q}_p)$ and $\mathrm{GL}_3(\mathbf{Q}_p)$ as well as notation for some locally analytic representations of these groups.

We denote the lower-triangular Borel subgroup (resp. the diagonal maximal split torus) of $\mathrm{GL}_{2/\mathbf{Q}_p}$ by B_2 (resp. by T_2) and the unipotent radical of B_2 by N_{GL_2}. We use the notation s for the non-trivial element in the Weyl group of GL_2. We fix a weight $\nu \in X(T_2)$ of GL_2 of the following form

$$\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$$

which corresponds to an algebraic character of $T_2(\mathbf{Q}_p)$

$$\delta_{T_2,\nu} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^{\nu_1} b^{\nu_2}.$$

We denote the upper-triangular Borel subgroup by $\overline{B_2}$. If ν is dominant with respect to $\overline{B_2}$, namely if $\nu_1 \geq \nu_2$, we use the notation $\overline{L}_{\mathrm{GL}_2}(\nu)$ (resp. $L_{\mathrm{GL}_2}(-\nu)$) for the irreducible algebraic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ with highest weight ν (resp. $-\nu$) with respect to the positive roots determined by $\overline{B_2}$ (resp. B_2). In particular, $\overline{L}_{\mathrm{GL}_2}(\nu)$ and $L_{\mathrm{GL}_2}(-\nu)$ are the dual of each other. We use the shorten notation

$$I_{B_2}^{\mathrm{GL}_2}(\chi_{T_2}) := \left(\mathrm{Ind}_{B_2(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \chi_{T_2}\right)^{\mathrm{an}}$$

for any locally analytic character χ_{T_2} of $T_2(\mathbf{Q}_p)$ and set

$$i_{B_2}^{\mathrm{GL}_2}(\chi_{T_2}) := \left(\mathrm{Ind}_{B_2(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \chi_{T_2}^{\infty}\right)^{\infty} \otimes_E \overline{L}_{\mathrm{GL}_2}(\nu)$$

if $\chi_{T_2} = \delta_{T_2,\nu} \otimes_E \chi_{T_2}^{\infty}$ is locally algebraic where $\chi_{T_2}^{\infty}$ is a smooth character of $T_2(\mathbf{Q}_p)$. Then we define the locally analytic Steinberg representation as well as the smooth Steinberg representation for $\mathrm{GL}_2(\mathbf{Q}_p)$ as follows

$$\mathrm{St}_2^{\mathrm{an}}(\nu) := I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\mu})/\overline{L}_{\mathrm{GL}_2}(\nu),\ \mathrm{St}_2^{\infty} := i_{B_2}^{\mathrm{GL}_2}(1_{T_2})/1_2$$

where 1_2 (resp. 1_{T_2}) is the trivial representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ (resp. of $T_2(\mathbf{Q}_p)$).

We denote the lower-triangular Borel subgroup (resp. the diagonal maximal split torus) of $\mathrm{GL}_{3/\mathbf{Q}_p}$ by B (resp. by T) and the unipotent radical of B by N. We fix a weight $\lambda \in X(T)$ of GL_3 of the following form

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3,$$

which corresponds to an algebraic character of $T(\mathbf{Q}_p)$

$$\delta_{T,\lambda} := \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mapsto a^{\lambda_1} b^{\lambda_2} c^{\lambda_3}.$$

We denote the center of GL_3 by Z and notice that $Z(\mathbf{Q}_p) \cong \mathbf{Q}_p^{\times}$. Hence the restriction of $\delta_{T,\lambda}$ to $Z(\mathbf{Q}_p)$ gives an algebraic character of $Z(\mathbf{Q}_p)$:

$$\delta_{Z,\lambda} := \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mapsto a^{\lambda_1+\lambda_2+\lambda_3}.$$

We use the shorten notation

$$\mathrm{Ext}_{*,\lambda}^{i}(-,-) := \mathrm{Ext}_{*,\delta_{Z,\lambda}}^{i}(-,-)$$

for $* \in \{T(\mathbf{Q}_p), L_1(\mathbf{Q}_p), L_2(\mathbf{Q}_p), \mathrm{GL}_3(\mathbf{Q}_p)\}$. In particular, the notation

$$\mathrm{Ext}^i_{*,0}(-,-)$$

means (higher) extensions with the trivial central character. We denote the upper-triangular Borel subgroup of GL_3 by $\overline{B}$. If λ is dominant with respect to $\overline{B}$, namely if $\lambda_1 \geq \lambda_2 \geq \lambda_3$, we use the notation $\overline{L}(\lambda)$ (resp. $L(-\lambda)$) for the irreducible algebraic representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ with highest weight λ (resp. $-\lambda$) with respect to the positive roots determined by $\overline{B}$ (resp. B). In particular, $\overline{L}(\lambda)$ and $L(-\lambda)$ are dual of each other. We use the notation $P_1 := \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$ and $P_2 := \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$ for the two standard maximal parabolic subgroups of GL_3 with unipotent radical N_1 and N_2 respectively, and the notation $\overline{P_i}$ for the opposite parabolic subgroup of P_i for $i = 1, 2$. We set

$$L_i := P_i \cap \overline{P_i}$$

and set s_i for the simple reflection in the Weyl group of L_i for each $i = 1, 2$. In particular, the Weyl group W of GL_3 can be lifted to a subgroup of GL_3 with the following elements

$$\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}.$$

The group W acts on $X(T)$ via the dot action

$$w \cdot \lambda := w(\lambda + (2, 1, 0)) - (2, 1, 0).$$

We will usually use the shorten notation N_i (c.f. Section 4.4) for its set of $\mathbf{Q}_p$-points $N_i(\mathbf{Q}_p)$ if it does not cause any ambiguity. We use the notation $M(-\lambda)$ for the Verma module in $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$ with highest weight $-\lambda$ (with respect to B) and simple quotient $L(-\lambda)$ for each $\lambda \in X(T)$ (not necessarily dominant). Similarly, we use the notation $M_i(-\lambda)$ for the parabolic Verma module in $\mathcal{O}^{\mathfrak{p}_i}_{\mathrm{alg}}$ with highest weight $-\lambda$ with respect to B (c.f. Section 9.4 of [Hum08]). We define $\overline{L}_i(\lambda)$ as the irreducible algebraic representation of $L_i(\mathbf{Q}_p)$ with a highest weight λ dominant with respect to $\overline{B} \cap L_i$. For example, if $\lambda \in X(T)_+$, then we know that λ, $s_i \cdot \lambda$ and $s_i s_{3-i} \cdot \lambda$ are dominant with respect to $\overline{B} \cap L_{3-i}$ for $i = 1, 2$. We use the following notation for various parabolic inductions

$$I_B^{\mathrm{GL}_3}(\chi) := \left(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}\chi\right)^{\mathrm{an}},\ I_{P_i}^{\mathrm{GL}_3}(\pi_i) := \left(\mathrm{Ind}_{P_i(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}\pi_i\right)^{\mathrm{an}}$$

if χ is an arbitrary locally analytic character of $T(\mathbf{Q}_p)$ and π_i is an arbitrary locally analytic representation of $L_i(\mathbf{Q}_p)$ for each $i = 1, 2$. Moreover, we use the notation

$$i_B^{\mathrm{GL}_3}(\chi) := \left(\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}\chi^\infty\right)^\infty \otimes_E \overline{L}(\lambda),\ i_{P_i}^{\mathrm{GL}_3}(\pi_i) := \left(\mathrm{Ind}_{P_i(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}\pi_i^\infty\right)^\infty \otimes_E \overline{L}(\lambda)$$

for $i = 1, 2$ if $\chi = \delta_{T,\lambda} \otimes_E \chi^\infty$ and $\pi_i = \overline{L}_i(\lambda) \otimes_E \pi_i^\infty$ are locally algebraic where χ^∞ (resp. π_i^∞) is a smooth representation of $T(\mathbf{Q}_p)$ (resp. of $L_i(\mathbf{Q}_p)$). We will also use similar notation for parabolic induction to Levi subgroups such as $I^{L_i}_{B\cap L_i}$ and $i^{L_i}_{B\cap L_i}$ for $i = 1, 2$. Then we define the locally analytic (generalized) Steinberg representation as well as the smooth (generalized) Steinberg representation for $\mathrm{GL}_3(\mathbf{Q}_p)$ by

$$\mathrm{St}_3^{\mathrm{an}}(\lambda) := I_B^{\mathrm{GL}_3}(\delta_{T,\lambda})/\left(I_{P_1}^{\mathrm{GL}_3}(\overline{L}_1(\lambda)) + I_{P_2}^{\mathrm{GL}_3}(\overline{L}_2(\lambda))\right),\ \mathrm{St}_3^\infty := i_B^{\mathrm{GL}_3}(1)/\left(i_{P_1}^{\mathrm{GL}_3}(1_{L_1}) + i_{P_2}^{\mathrm{GL}_3}(1_{L_2})\right)$$

and

$$v_{P_i}^{\mathrm{an}}(\lambda) := I_{P_i}^{\mathrm{GL}_3}(\overline{L}_i(\lambda))/\overline{L}(\lambda),\ v_{P_i}^\infty := i_{P_i}^{\mathrm{GL}_3}(1_{L_i})/1_3$$

where 1_3 (resp. 1_{L_i}) is the trivial representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ (resp. of $L_i(\mathbf{Q}_p)$ for each $i=1,2$). We define the following irreducible smooth representations of $L_1(\mathbf{Q}_p)$:

$$\begin{array}{lcc} \pi_{1,1}^\infty & := & \mathrm{St}_2^\infty \otimes_E 1 \\ \pi_{1,2}^\infty & := & i_{B_2}^{\mathrm{GL}_2}\left(1\otimes_E |\cdot|^{-1}\right)\otimes_E |\cdot| \\ \pi_{1,3}^\infty & := & \left(\mathrm{St}_2^\infty \otimes_E (|\cdot|^{-1}\circ \det_2)\right)\otimes_E |\cdot|^2 \end{array}$$

and the following smooth representations of $L_2(\mathbf{Q}_p)$:

$$\begin{array}{lcc} \pi_{2,1}^\infty & := & 1\otimes_E \mathrm{St}_2^\infty \\ \pi_{2,2}^\infty & := & |\cdot|^{-1}\otimes_E i_{B_2}^{\mathrm{GL}_2}(|\cdot|\otimes_E 1) \\ \pi_{2,3}^\infty & := & |\cdot|^{-2}\otimes_E (\mathrm{St}_2^\infty \otimes_E (|\cdot|\circ \det_2)) \end{array}$$

Consequently, we can define the following locally analytic representations for $i=1,2$:

$$\begin{array}{lcclcc} C^1_{s_i,1} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_i\cdot\lambda),\ 1_{L_{3-i}}\right) & C^2_{s_i,1} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_i\cdot\lambda),\ \pi^\infty_{i,1}\right) \\ C^1_{s_is_{3-i},1} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_is_{3-i}\cdot\lambda),\ 1_{L_{3-i}}\right) & C^2_{s_is_{3-i},1} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_is_{3-i}\cdot\lambda),\ \pi^\infty_{i,1}\right) \\ C_{s_i,s_i} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_i\cdot\lambda),\ \pi^\infty_{i,2}\right) & C_{s_is_{3-i},s_i} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_is_{3-i}\cdot\lambda),\ \pi^\infty_{i,2}\right) \\ C^1_{s_i,s_is_{3-i}} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_i\cdot\lambda),\ \mathfrak{d}^\infty_{P_{3-i}}\right) & C^2_{s_1,1} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_i\cdot\lambda),\ \pi^\infty_{i,3}\right) \\ C^1_{s_is_{3-i},s_is_{3-i}} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_is_{3-i}\cdot\lambda),\ \mathfrak{d}^\infty_{P_{3-i}}\right) & C^2_{s_is_{3-i},1} & := & \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}\left(L(-s_is_{3-i}\cdot\lambda),\ \pi^\infty_{i,3}\right) \end{array} \tag{4.2.9}$$

where

$$\mathfrak{d}^\infty_{P_1} := |\cdot|^{-1}\circ\det_2\otimes_E|\cdot|^2 \text{ and } \mathfrak{d}^\infty_{P_2} := |\cdot|^{-2}\otimes_E|\cdot|\circ\det_2.$$

We also define

$$C_{s_1s_2s_1,w} := \mathcal{F}^{\mathrm{GL}_3}_B\left(L(-s_1s_2s_1\cdot\lambda),\ \chi^\infty_w\right) \tag{4.2.10}$$

for each $w\in W$ where

$$\begin{array}{lll} \chi_1^\infty := 1_T & \chi_{s_1}^\infty := |\cdot|^{-1}\otimes_E|\cdot|\otimes_E 1 & \chi_{s_2}^\infty := 1\otimes_E|\cdot|^{-1}\otimes_E|\cdot| \\ \chi_{s_1s_2}^\infty := |\cdot|^{-2}\otimes_E|\cdot|\otimes_E|\cdot| & \chi_{s_2s_1}^\infty := |\cdot|^{-1}\otimes_E|\cdot|^{-1}\otimes_E|\cdot|^2 & \chi_{s_1s_2s_1}^\infty := |\cdot|^{-2}\otimes_E 1\otimes_E|\cdot|^2 \end{array}$$

As one can write out $\mathcal{O}^{\overline{\mathfrak{b}}}_{\mathrm{alg}}$ explicitly for each parabolic subgroup $P\subseteq \mathrm{GL}_3$, we notice that the representations considered in (4.2.9) and (4.2.10) are all irreducible objects inside $\mathrm{Rep}^{\mathcal{OS}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$ according to the main theorem of [OS15]. We use the notation Ω for the set whose elements are listed as the following:

$$\begin{array}{cccc} \overline{L}(\lambda) & \overline{L}(\lambda)\otimes_E v_{P_1}^\infty & \overline{L}(\lambda)\otimes_E v_{P_2}^\infty & \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \\ C^1_{s_1,1} & C^2_{s_1,1} & C^1_{s_2,1} & C^2_{s_2,1} \\ C^1_{s_1s_2,1} & C^2_{s_1s_2,1} & C^1_{s_2s_1,1} & C^2_{s_2s_1,1} \\ C^1_{s_1,s_1s_2} & C^2_{s_1,s_1s_2} & C^1_{s_2,s_2s_1} & C^2_{s_2,s_2s_1} \\ C^1_{s_1s_2,s_1s_2} & C^2_{s_1s_2,s_1s_2} & C^1_{s_2s_1,s_2s_1} & C^2_{s_2s_1,s_2s_1} \\ C_{s_1,s_1} & C_{s_1s_2,s_1} & C_{s_2,s_2} & C_{s_2s_1,s_2} \\ C_{s_1s_2s_1,w} & w\in W & & \end{array} \tag{4.2.11}$$

Remark 4.2.12. *It is actually possible to show that Ω is the set of (isomorphism classes of) irreducible objects of the block inside $\mathrm{Rep}^{\mathcal{OS}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$ containing the object $\overline{L}(\lambda)$.*

Lemma 4.2.13. *The representation $v_{P_i}^{\mathrm{an}}(\lambda)$ fits into a non-split extension*

$$\overline{L}(\lambda)\otimes_E v_{P_i}^\infty \hookrightarrow v_{P_i}^{\mathrm{an}}(\lambda) \twoheadrightarrow C^1_{s_{3-i},1} \tag{4.2.14}$$

for $i = 1, 2$. On the other hand, the representation $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ has the following form:

$$\begin{array}{ccccccc} & & C^2_{s_1,1} & - & C^1_{s_2s_1,1} & - & C^2_{s_2s_1,1} & & \\ & \diagup & & \diagdown\diagup & & \diagup\diagdown & & \diagdown & \\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty} & & & & & & & & C_{s_1s_2s_1,1}\ . \\ & \diagdown & & \diagup\diagdown & & \diagdown\diagup & & \diagup & \\ & & C^2_{s_2,1} & - & C^1_{s_1s_2,1} & - & C^2_{s_1s_2,1} & & \end{array} \tag{4.2.15}$$

Proof. The non-split short exact sequence follows directly from (3.62) of [BD18]. It follows easily from the definition of $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ and the main theorem of [OS15] that

$$\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathrm{St}_3^{\mathrm{an}}(\lambda)\right) = \{\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ C^2_{s_1,1},\ C^2_{s_2,1},\ C^1_{s_2s_1,1},\ C^1_{s_1s_2,1},\ C^2_{s_2s_1,1},\ C^2_{s_1s_2,1},\ C_{s_1s_2s_1,1}\}$$

and each Jordan–Hölder factor occurs with multiplicity one. It follows from Section 5.2 of [Bre17] that

$$H_0\left(N_i,\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)\right) = \overline{L}_i(-s_{3-i}s_i\cdot\lambda)\otimes_E i_{B\cap L_i}^{L_i}(1_T)$$

which together with

$$\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)\right) = \{C^1_{s_{3-i}s_i,1},\ C^2_{s_{3-i}s_i,1}\}$$

imply that $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)$ fits into a non-split extension

$$C^1_{s_{3-i}s_i,1} \hookrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right) \twoheadrightarrow C^2_{s_{3-i}s_i,1} \tag{4.2.16}$$

for $i = 1, 2$. We also observe from Section 5.2 and 5.3 of [Bre17] that

$$H_2\left(N_{3-i},\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)\right) \ncong H_2(N_{3-i},\ C^2_{s_{3-i},1}) \oplus H_2(N_{3-i},\ C^2_{s_{3-i}s_i,1})$$

which together with

$$\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)\right) = \{C^2_{s_{3-i},1},\ C^2_{s_{3-i}s_i,1}\}$$

imply that $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)$ fits into a non-split extension

$$C^2_{s_{3-i},1} \hookrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right) \twoheadrightarrow C^2_{s_{3-i}s_i,1} \tag{4.2.17}$$

for $i = 1, 2$. We notice that both $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(L(-s_{3-i}s_i\cdot\lambda),\ i_{B\cap L_i}^{L_i}(1_T)\right)$ and $\mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-s_{3-i}\cdot\lambda),\ \pi_{i,1}^{\infty}\right)$ are subquotients of $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ by various properties of the functors $\mathcal{F}_{P_i}^{\mathrm{GL}_3}$ (c.f. main theorem of [OS15]) and the definition of $\mathrm{St}_3^{\mathrm{an}}(\lambda)$. We finish the proof by combining (4.2.16) and (4.2.17) with the results before Remark 3.38 of [BD18]. □

Remark 4.2.18. *It is actually possible to show that all the possibly non-split extensions indicated in (4.2.15) are non-split, although they are essentially unrelated to the p-adic dilogarithm function.*

4.2.4 *p*-adic logarithm and dilogarithm

In this section, we recall p-adic logarithm and dilogarithm function as well as their representation theoretic interpretations.

We recall the p-adic logarithm function $\log_0 : \mathbf{Q}_p^{\times} \to \mathbf{Q}_p$ defined by the power series

$$\log_0(1-z) := -\sum_{k=0}^{\infty}\frac{z^k}{k}$$

on the open subgroup $1+p\mathbb{Z}_p$ of $\mathbb{Z}_p^\times$ and then extended to $\mathbf{Q}_p^\times$ by the condition $\log_0(p)=\log_0(\zeta)=0$ for each root of unity ζ. We also recall the p-adic valuation function $\mathrm{val}_p : \mathbf{Q}_p^\times \to \mathbb{Z}$ satisfying $|\cdot| = p^{-\mathrm{val}_p(\cdot)}$ (and in particular $\mathrm{val}_p(p)=1$). We notice that

$$\{\log_0,\ \mathrm{val}_p\}$$

forms a basis of the two dimensional E-vector space

$$\mathrm{Hom}_{\mathrm{cont}}\left(\mathbf{Q}_p^\times,\ E\right).$$

We define $\log_{\mathscr{L}} := \log_0 - \mathscr{L}\mathrm{val}_p$ for each $\mathscr{L}\in E$ and consider the following two dimensional locally analytic representation of $\mathbf{Q}_p^\times$

$$V_{\mathscr{L}} : \mathbf{Q}_p^\times \to B_2(E),\ a \mapsto \begin{pmatrix} 1 & \log_{\mathscr{L}}(a) \\ 0 & 1 \end{pmatrix}$$

and therefore

$$\mathrm{soc}_{\mathbf{Q}_p^\times}(V_{\mathscr{L}}) = \mathrm{cosoc}_{\mathbf{Q}_p^\times}(V_{\mathscr{L}}) = 1 \tag{4.2.19}$$

where 1 is the notation for the trivial character of $\mathbf{Q}_p^\times$. We notice that

$$\mathrm{Ext}^1_{\mathbf{Q}_p^\times}(1,1) \cong \mathrm{Hom}_{\mathrm{cont}}\left(\mathbf{Q}_p^\times,\ E\right),$$

by a standard fact in (continuous) group cohomology and therefore the set $\{V_{\mathscr{L}} \mid \mathscr{L}\in E\}$ exhausts (up to isomorphism) all different two dimensional locally analytic non-smooth E-representations of $\mathbf{Q}_p^\times$ satisfying (4.2.19). We observe that $V_{\mathscr{L}}$ can be viewed as a representation of $T_2(\mathbf{Q}_p)\cong \mathbf{Q}_p^\times\times\mathbf{Q}_p^\times$ by composing with the map

$$T_2(\mathbf{Q}_p)\to\mathbf{Q}_p^\times :\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^{-1}b. \tag{4.2.20}$$

As a result, we can consider the parabolic induction

$$I_{B_2}^{\mathrm{GL}_2}\left(V_{\mathscr{L}}\otimes_E \delta_{T_2,\nu}\right)$$

which naturally fits into an exact sequence

$$I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu}) \hookrightarrow I_{B_2}^{\mathrm{GL}_2}\left(V_{\mathscr{L}}\otimes_E \delta_{T_2,\nu}\right) \twoheadrightarrow I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu}). \tag{4.2.21}$$

Then we define $\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})$ as the subrepresentation of $I_{B_2}^{\mathrm{GL}_2}\left(V_{\mathscr{L}}\otimes_E \delta_{T_2,\nu}\right)/\overline{L}_{\mathrm{GL}_2}(\nu)$ with cosocle $\overline{L}_{\mathrm{GL}_2}(\nu)$. It follows from (the proof of) Theorem 3.14 of [BD18] that $\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})$ has the form

$$\mathrm{St}_2^{\mathrm{an}}(\nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu) \tag{4.2.22}$$

and the set $\{\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L}) \mid \mathscr{L}\in E\}$ exhausts (up to isomorphism) all different locally analytic E-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ of the form (4.2.22) that do not contain

$$\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)$$

as a subrepresentation. We have the embeddings

$$\iota_i : \mathrm{GL}_2 \hookrightarrow L_i$$

for $i=1,2$ by identifying GL_2 with a Levi block of L_i, which induce the embeddings

$$\iota_{T,i} : T_2 \hookrightarrow T$$

by restricting ι_i to $T_2 \subsetneq \mathrm{GL}_2$. We use the notation $\iota_{T,i}(V_{\mathscr{L}})$ for the locally analytic representation of $T(\mathbf{Q}_p) \cong (\mathbf{Q}_p^\times)^3$ which is $V_{\mathscr{L}}$ after restricting to T_2 via $\iota_{T,i}$ and is trivial after restricting to the other copy of $\mathbf{Q}_p^\times$. By a direct analogue of $\Sigma_{\mathrm{GL}_2}(\nu, \mathscr{L})$, we can construct $\Sigma_{L_i}(\lambda, \mathscr{L})$ as the subrepresentation of $I_{B\cap L_i}^{L_i}\left(\iota_{T,i}(V_{\mathscr{L}}) \otimes_E \delta_{T,\lambda}\right)/\overline{L}_i(\lambda)$ with cosocle $\overline{L}_i(\lambda)$. In fact, if we have $\lambda|_{T_2,\iota_{T,i}} = \nu$, then we obviously know that $\Sigma_{L_i}(\lambda, \mathscr{L})|_{\mathrm{GL}_2,\iota_i} \cong \Sigma_{\mathrm{GL}_2}(\nu, \mathscr{L})$ where the notation $(\cdot)|_{*,\star}$ means the restriction of $\cdot$ to $*$ via the embedding $\star$. We observe that the parabolic induction $I_{P_i}^{\mathrm{GL}_3}\left(\Sigma_{L_i}(\lambda, \mathscr{L})\right)$ fits into the exact sequence

$$v_{P_{3-i}}^{\mathrm{an}}(\lambda) \text{ —— } \mathrm{St}_3^{\mathrm{an}}(\lambda) \hookrightarrow I_{P_i}^{\mathrm{GL}_3}\left(\Sigma_{L_i}(\lambda, \mathscr{L})\right) \twoheadrightarrow \overline{L}(\lambda) \text{ —— } v_{P_i}^{\mathrm{an}}(\lambda)\ .$$

According to Proposition 5.6 of [Schr11] for example, we know that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) = 0$$

and thus we can define $\Sigma_i(\lambda, \mathscr{L})$ as the unique quotient of $I_{P_i}^{\mathrm{GL}_3}\left(\Sigma_{L_i}(\lambda, \mathscr{L})\right)$ that fits into the exact sequence

$$\mathrm{St}_3^{\mathrm{an}}(\lambda) \hookrightarrow \Sigma_i(\lambda, \mathscr{L}) \twoheadrightarrow v_{P_i}^{\mathrm{an}}(\lambda).$$

The constructions of $\Sigma_i(\lambda, \mathscr{L})$ above actually induce canonical isomorphisms

$$\mathrm{Hom}_{\mathrm{cont}}\left(\mathbf{Q}_p^\times,\ E\right) \cong \mathrm{Ext}^1_{\mathbf{Q}_p^\times}(1,\ 1) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_i}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) \tag{4.2.23}$$

for $i = 1, 2$. We denote the image of $\log_0$ (resp, of val_p) in

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_i}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right)$$

by $b_{i,\log_0}$ (resp. by b_{i,val_p}). We use the notation 1_T for the trivial character of $T(\mathbf{Q}_p)$. We use the same notation $b_{i,\log_0}$ and b_{i,val_p} for the image of $\log_0$ and val_p respectively under the embedding

$$\mathrm{Ext}^1_{\mathbf{Q}_p^\times}(1,\ 1) \hookrightarrow \mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T,\ 1_T)$$

induced by the maps

$$T(\mathbf{Q}_p) \xrightarrow{p_i} T_2(\mathbf{Q}_p) \xrightarrow{(4.2.20)} \mathbf{Q}_p^\times$$

where p_i is the section of $\iota_{T,i}$ which is compatible with the projection $L_i \twoheadrightarrow \mathrm{GL}_2$. Recall the elements $c_{i,\log}, c_{i,\mathrm{val}} \in \mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T, 1_T)$ constructed after (5.24) of [Schr11] and observe that

$$\begin{cases} c_{1,\log} = b_{1,\log_0} + 2b_{2,\log_0}, & c_{1,\mathrm{val}} = b_{1,\mathrm{val}_p} + 2b_{2,\mathrm{val}_p} \\ c_{2,\log} = 2b_{1,\log_0} + b_{2,\log_0}, & c_{2,\mathrm{val}} = 2b_{1,\mathrm{val}_p} + b_{2,\mathrm{val}_p} \end{cases}\ . \tag{4.2.24}$$

We notice that there exists canonical surjections

$$\mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T,\ 1_T) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_i}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) \tag{4.2.25}$$

with kernel spanned by $\{c_{i,\log},\ c_{i,\mathrm{val}}\}$, according to (5.70) and (5.71) of [Schr11]. Therefore the relation (4.2.24) reduces via the surjection (4.2.25) to

$$c_{3-i,\log} = -3b_{i,\log_0},\ c_{3-i,\mathrm{val}} = -3b_{i,\mathrm{val}_p} \tag{4.2.26}$$

inside the quotient $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_i}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right)$. We define $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ as the amalgamate sum of $\Sigma_1(\lambda, \mathscr{L}_1)$ and $\Sigma_2(\lambda, \mathscr{L}_2)$ over $\mathrm{St}_3^{\mathrm{an}}(\lambda)$, for each $\mathscr{L}_1, \mathscr{L}_2 \in E$. Consequently, $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ has the following form

$$\begin{array}{ccc} & & v_{P_1}^{\mathrm{an}}(\lambda) \\ \mathrm{St}_3^{\mathrm{an}}(\lambda) & \nearrow & \\ & \searrow & \\ & & v_{P_2}^{\mathrm{an}}(\lambda) \end{array}$$

and we have

$$\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2) \cong \Sigma(\lambda, \mathscr{L}, \mathscr{L}') \tag{4.2.27}$$

if

$$\mathscr{L}_1 = -\mathscr{L}', \mathscr{L}_2 = -\mathscr{L} \in E, \tag{4.2.28}$$

where $\Sigma(\lambda, \mathscr{L}, \mathscr{L}')$ is the locally analytic representation defined in Definition 5.12 of [Schr11] using the element

$$(c_{2,\log} + \mathscr{L}' c_{2,\mathrm{val}},\ c_{1,\log} + \mathscr{L} c_{1,\mathrm{val}})$$

in

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(v_{P_1}^{\mathrm{an}}(\lambda) \oplus v_{P_2}^{\mathrm{an}}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right).$$

Remark 4.2.29. *The appearance of a sign in (4.2.28) (which is an issue of normalization) is essentially due to Remark 3.1 of [Ding18], which implies that our invariants $\mathscr{L}_1$ and $\mathscr{L}_2$ can be identified with Fontaine–Mazur $\mathscr{L}$-invariants of the corresponding Galois representation via local-global compatibility.*

We have a canonical morphism by (5.26) of [Schr11]

$$\kappa : \mathrm{Ext}^2_{T(\mathbf{Q}_p),0}(1_T, 1_T) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right). \tag{4.2.30}$$

Note that we also have

$$\mathrm{Ext}^2_{T(\mathbf{Q}_p),0}(1_T, 1_T) \cong \wedge^2\left(\mathrm{Ext}^1_{T(\mathbf{Q}_p),0}(1_T, 1_T)\right)$$

by (5.24) of [Schr11] and thus the set

$$\{b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}, b_{1,\log_0} \wedge b_{2,\mathrm{val}_p}, b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}, b_{1,\log_0} \wedge b_{2,\log_0}, b_{1,\mathrm{val}_p} \wedge b_{1,\log_0}, b_{2,\mathrm{val}_p} \wedge b_{2,\log_0}\}$$

forms a basis of $\mathrm{Ext}^2_{T(\mathbf{Q}_p),0}(1_T,\ 1_T)$. It follows from (5.27) of [Schr11] and (4.2.24) that the set

$$\{\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\log_0} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}), \kappa(b_{1,\log_0} \wedge b_{2,\log_0})\}$$

forms a basis of the image of (4.2.30).

We recall the p-adic dilogarithm function $li_2 : \mathbf{Q}_p \setminus \{0, 1\} \to \mathbf{Q}_p$ defined by Coleman in [Cole82] and we consider the function

$$D_{\mathscr{L}}(z) := li_2(z) + \frac{1}{2}\log_{\mathscr{L}}(z)\log_{\mathscr{L}}(1-z)$$

as in (5.34) of [Schr11]. We also define

$$d(z) := \log_{\mathscr{L}}(1-z)\mathrm{val}_p(z) - \log_{\mathscr{L}}(z)\mathrm{val}_p(1-z)$$

as in (5.36) of [Schr11] which is also a locally analytic function over $\mathbf{Q}_p \setminus \{0, 1\}$ and is independent of the choice of $\mathscr{L} \in E$. Note by our definition that

$$D_{\mathscr{L}} - D_0 = \frac{\mathscr{L}}{2} d.$$

It follows from Theorem 7.2 of [Schr11] that $\{D_0, d\}$ can be identified with a basis of

$$\mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p),0}\left(1,\ \mathrm{St}_2^{\mathrm{an}}\right)$$

(c.f. (5.38) of [Schr11]) which naturally embeds into $\mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(1,\ \mathrm{St}_2^{\mathrm{an}}\right)$. Then the map $\iota_i : \mathrm{GL}_2 \hookrightarrow L_i$ induces the isomorphisms

$$\mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(1_2,\ \mathrm{St}_2^{\mathrm{an}}\right) \xleftarrow{\sim} \mathrm{Ext}^2_{L_i(\mathbf{Q}_p),0}\left(1_{L_i},\ \mathrm{St}_2^{\mathrm{an}}\right) \xleftarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}\left(1_3,\ I_{P_i}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}})\right) \tag{4.2.31}$$

where $L_i(\mathbf{Q}_p)$ acts on $\mathrm{St}_2^{\mathrm{an}}$ via the projection p_i. We abuse the notation for the composition

$$\iota_i : \mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(1_2,\ \mathrm{St}_2^{\mathrm{an}}\right) \xleftarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}\left(1_3,\ I_{P_i}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}})\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}\left(1_3,\ \mathrm{St}_3^{\mathrm{an}}\right) \tag{4.2.32}$$

given by (4.2.31) and the surjection

$$I_{P_i}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}}) \twoheadrightarrow \mathrm{St}_3^{\mathrm{an}}.$$

Finally there is canonical isomorphism

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),0}\left(1_3,\ \mathrm{St}_3^{\mathrm{an}}\right) \cong \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right)$$

by (5.20) of [Schr11].

Lemma 4.2.33. *We have*

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) = 5$$

and the set

$$\{\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\log_0} \wedge b_{2,\mathrm{val}_p}), \kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}), \kappa(b_{1,\log_0} \wedge b_{2,\log_0}), \iota_i(D_0)\}$$

forms a basis of $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right)$ *for* $i = 1, 2$.

Proof. This follows directly from (5.57) of [Schr11] and (4.2.24). □

Lemma 4.2.34. *There exists* $\gamma \in E^\times$ *such that*

$$\iota_1(d) = \iota_2(d) = \gamma\left(\kappa(b_{1,\log_0} \wedge b_{2,\mathrm{val}_p} + b_{1,\mathrm{val}_p} \wedge b_{2,\log_0}\right).$$

Proof. This follows directly from Lemma 5.8 of [Schr11] and (4.2.24) if we take

$$\gamma := -3\alpha$$

where $\alpha \in E^\times$ is the constant in the statement of Lemma 5.8 of [Schr11]. □

Lemma 4.2.35. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right) = 1 \text{ and } \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right) = 2.$$

Moreover, the image of

$$\{\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}), \iota_i(D_0)\}$$

under

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$$

forms a basis of $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$ *for* $i = 1$ *or* 2.

Proof. This follows directly from Corollary 5.17 of [Schr11] and (4.2.24). □

We recall from (5.55) of [Schr11] that

$$c_0 := \alpha^{-1}\iota_1(D_0) - \frac{1}{2}\kappa(c_{1,\log} \wedge c_{2,\log}) \tag{4.2.36}$$

where α is defined in Lemma 5.8 of [Schr11].

Lemma 4.2.37. *Assume that $\mathscr{L}_3 \in E$ satisfies the equality*

$$E\left(\iota_1(D_0) + \mathscr{L}_3\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p})\right) = E\left(c_0 + \mathscr{L}''\kappa(c_{1,\mathrm{val}} \wedge c_{2,\mathrm{val}})\right) \subsetneq \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right). \quad (4.2.38)$$

Then we have

$$\mathscr{L}_3 = \gamma(\mathscr{L}'' - \frac{1}{2}\mathscr{L}_1\mathscr{L}_2) = \gamma(\mathscr{L}'' - \frac{1}{2}\mathscr{L}\mathscr{L}').$$

Proof. All the equalities in this lemma are understood to be inside

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$$

without causing ambiguity. It follows from our assumption (4.2.38) that

$$\iota_1(D_0) + \mathscr{L}_3\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}) = \alpha\left(c_0 + \mathscr{L}''\kappa(c_{1,\mathrm{val}} \wedge c_{2,\mathrm{val}})\right)$$

which together with (4.2.36) imply that

$$\mathscr{L}_3\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}) = \frac{\alpha}{2}\kappa(c_{1,\log} \wedge c_{2,\log}) + \alpha\mathscr{L}''\kappa(c_{1,\mathrm{val}} \wedge c_{2,\mathrm{val}}). \quad (4.2.39)$$

We know that

$$\kappa(c_{1,\log} \wedge c_{2,\log}) = \mathscr{L}\mathscr{L}'\kappa(c_{1,\mathrm{val}} \wedge c_{2,\mathrm{val}}) \quad (4.2.40)$$

from the proof of Corollary 5.17 of [Schr11] and that

$$\kappa(c_{1,\mathrm{val}} \wedge c_{2,\mathrm{val}}) = -3\kappa(b_{1,\mathrm{val}_p} \wedge b_{2,\mathrm{val}_p}) \quad (4.2.41)$$

from (4.2.24). Therefore we finish the proof by combining (4.2.39), (4.2.40) and (4.2.41) with (4.2.28) and the equality $\gamma = -3\alpha$ from Lemma 4.2.34. □

Remark 4.2.42. *We emphasize that we do not know whether*

$$E\iota_1(D_0) = E\iota_2(D_0)$$

in $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}_3^{\mathrm{an}}(\lambda)\right)$ *or not, which is of independent interest.*

4.3 A key result for $\mathrm{GL}_2(\mathbf{Q}_p)$

In this section, we are going to prove Proposition 4.3.14 which will be a crucial ingredient for the proof of Lemma 4.5.8 and Proposition 4.6.8. We usually identify $\mathrm{GL}_2(\mathbf{Q}_p)$ with a Levi factor of a maximal parabolic of GL_3 when we apply the results from this section.

We use the following shorten notation

$$I(\nu) := I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu}),\ \widetilde{I}(\nu) := I_{B_2}^{\mathrm{GL}_2}(\delta_{T_2,\nu} \otimes_E (|\cdot|^{-1} \otimes_E |\cdot|))$$

for each weight $\nu \in X(T_2)$.

Lemma 4.3.1. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\widetilde{I}(s\cdot\nu),\ \Sigma_{\mathrm{GL}_2}(\nu, \mathscr{L})\right) = 1.$$

Proof. This is essentially contained in the proof of Theorem 3.14 of [BD18]. In fact, we know that

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\widetilde{I}(s\cdot\nu),\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } I(s\cdot\nu)\right) &= 0\\ \mathrm{Ext}^2_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\widetilde{I}(s\cdot\nu),\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } I(s\cdot\nu)\right) &= 0\end{aligned}$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\widetilde{I}(s\cdot\nu),\ \overline{L}_{\mathrm{GL}_2}(\nu)) = 1$$

which finish the proof by a simple devissage induced by the short exact sequence

$$\left(\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } I(s\cdot\nu)\ \right) \hookrightarrow \Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L}) \twoheadrightarrow \overline{L}_{\mathrm{GL}_2}(\nu).$$

□

We fix a split p-adic reductive group H and have a natural embedding

$$U(\mathfrak{h}) \hookrightarrow D(H,E)_{\{1\}} \hookrightarrow D(H,E)$$

where $D(H,E)_{\{1\}}$ is the closed subalgebra of $D(H,E)$ consisting of distributions supported at the identity element (c.f. [Koh07]). The embedding above induces another embedding

$$Z(U(\mathfrak{h})) \hookrightarrow Z(D(H,E)) \tag{4.3.2}$$

by the main result of [Koh07] where $Z(\cdot)$ is the notation for the center of a non-commutative algebra. We say that $\Pi \in \mathrm{Rep}^{\mathrm{la}}_{\mathrm{GL}_2(\mathbf{Q}_p),E}$ has an infinitesimal character if $Z(U(\mathfrak{h}))$ acts on Π' via a character.

Lemma 4.3.3. *If $V, W \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$ have both the same central character and the same infinitesimal character and satisfy*

$$\mathrm{Hom}_H\,(V,\ W) = 0,$$

then any non-split extension of the form W —— V has both the same central character and the same infinitesimal character as the one for V and W.

Proof. This is a direct analogue of Lemma 3.1 in [BD18] and follows essentially from the fact that both $D(Z(H),E)$ and $Z(U(\mathfrak{h}))$ are subalgebras of $Z(D(H,E))$ by [Koh07]. □

We fix a Borel subgroup $B_H \subseteq H$ as well as its opposite Borel subgroup $\overline{B_H}$. We consider the split maximal torus $T_H := B_H \cap \overline{B_H}$ and use the notation N_H (resp. $\overline{N_H}$) for the unipotent radical of B_H (resp. of $\overline{B_H}$). We use the notation $J_{\overline{B_H}}(\cdot)$ for Emertion's Jacquet functor.

Lemma 4.3.4. *If $V \in \mathrm{Rep}^{\mathrm{la}}_{H,E}$ has an infinitesimal character, then $U(\mathfrak{t}_\mathfrak{h})^{W_H}$ (as a subalgebra of $U(\mathfrak{t}_\mathfrak{h})$) acts on $J_{\overline{B_H}}(V)$ via a character where W_H is the Weyl group of H.*

Proof. We know by our assumption that $Z(U(\mathfrak{h}))$ acts on V' (and hence on V as well) via a character. We note from (4.3.2) that $Z(U(\mathfrak{h}))$ commutes with $D(\overline{N_H},E) \subseteq D(H,E)$ and thus the action of $Z(U(\mathfrak{h}))$ on V commutes with that of $\overline{N_H}$, which implies that $Z(U(\mathfrak{h}))$ acts on $V^{\overline{N_H}^\circ}$ via a character for each open compact subgroup $\overline{N_H}^\circ \subseteq \overline{N_H}$. We use the notation

$$\theta : Z(U(\mathfrak{h})) \xrightarrow{\sim} U(\mathfrak{t}_\mathfrak{h})^{W_H}$$

for the Harish-Chandra isomorphism (c.f. Section 1.7 of [Hum08]) and the notation j_1 and j_2 for the embeddings

$$j_1 : Z(U(\mathfrak{h})) \hookrightarrow U(\mathfrak{h}) \text{ and } j_2 : U(\mathfrak{t}_\mathfrak{h}) \hookrightarrow U(\mathfrak{h}).$$

We choose an arbitrary Verma module $M_H(\lambda_H)$ with highest weight λ_H, namely we have

$$M_H(\lambda) := U(\mathfrak{h}) \otimes_{U(\overline{\mathfrak{b}_H})} \lambda_H.$$

We use the notation $M_H(\lambda_H)_\mu$ for the subspace of $M_H(\lambda)$ with $\mathfrak{t}_\mathfrak{h}$-weight μ and note that

$$\dim_E M_H(\lambda_H)_{\lambda_H} = 1.$$

We easily observe that

$$Z(U(\mathfrak{h})) \cdot M_H(\lambda_H)_{\lambda_H} = M_H(\lambda_H)_{\lambda_H} \text{ and } U(\mathfrak{t}_\mathfrak{h}) \cdot M_H(\lambda_H)_{\lambda_H} = M_H(\lambda_H)_{\lambda_H}. \tag{4.3.5}$$

It is well-known that the the direct sum decomposition

$$\mathfrak{h} = \mathfrak{n}_H \oplus \mathfrak{t}_\mathfrak{h} \oplus \overline{\mathfrak{n}_H} \tag{4.3.6}$$

induces a tensor decomposition of E-vector space

$$U(\mathfrak{h}) = U(\mathfrak{n}_H) \otimes_E U(\mathfrak{t}_\mathfrak{h}) \otimes_E U(\overline{\mathfrak{n}_H}). \tag{4.3.7}$$

Hence we can write each element in $U(\mathfrak{h})$ as a polynomial with variables indexed by a standard basis of $\mathfrak{h}$ that is compatible with (4.3.6). It follows from the definition of θ as the restriction to $Z(U(\mathfrak{h}))$ of the projection $U(\mathfrak{h}) \twoheadrightarrow U(\mathfrak{t}_\mathfrak{h})$ (coming from (4.3.7)) that

$$j_1(z) - j_2 \circ \theta(z) \in U(\mathfrak{h}) \cdot \overline{\mathfrak{n}_H} + \mathfrak{n}_H \cdot U(\mathfrak{h})$$

for each $z \in Z(U(\mathfrak{h}))$. If a monomial f in the decomposition (4.3.7) of $j_1(z) - j_2 \circ \theta(z)$ belongs to

$$\mathfrak{n}_H \cdot U(\mathfrak{n}_H) \cdot U(\mathfrak{t}_\mathfrak{h}),$$

then we have

$$f \cdot M_H(\lambda_H)_{\lambda_H} \subseteq M_H(\lambda_H)_\mu$$

for some $\mu \neq \lambda_H$, which contradicts the fact (4.3.5). Hence we conclude that

$$j_1(z) - j_2 \circ \theta(z) \in U(\mathfrak{h}) \cdot \overline{\mathfrak{n}_H}$$

and in particular

$$j_1(z) = j_2 \circ \theta(z)$$

on $V^{\overline{N_H}^\circ}$ for each $z \in Z(U(\mathfrak{h}))$. Hence we deduce that $U(\mathfrak{t}_\mathfrak{h})^{W_H}$ acts on $V^{\overline{N_H}^\circ}$ via a character. We note by the definition of $J_{\overline{B_H}}$ (c.f. [Eme06I]) that we have a T_H^+-equivariant embedding

$$J_{\overline{B_H}}(V) \hookrightarrow V^{\overline{N_H}^\circ} \tag{4.3.8}$$

where T_H^+ is a certain submonoid of T_H containing an open compact subgroup. As a result, (4.3.8) is also $U(\mathfrak{t}_\mathfrak{h})$-equivariant and thus $U(\mathfrak{t}_\mathfrak{h})^{W_H}$ acts on $J_{\overline{B_H}}(V)$ via a character which finishes the proof. □

We set $H = \mathrm{GL}_2(\mathbf{Q}_p)$, $B_H = B_2$ and $\overline{B_H} = \overline{B_2}$ in the rest of this section. The idea of the following lemma which is closely related to Lemma 3.20 of [BD18], owes very much to Y.Ding.

Lemma 4.3.9. *A locally analytic representation of either the form*

$$\overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \text{ —— } I(s \cdot \nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \tag{4.3.10}$$

or the form

$$\overline{L}_{\mathrm{GL}_2}(\nu) \text{ —— } \widetilde{I}(s \cdot \nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu) \otimes_E \mathrm{St}_2^\infty \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu) \tag{4.3.11}$$

does not have an infinitesimal character.

Proof. Assume that a representation V of the form (4.3.10) has an infinitesimal character. Note that V can be represented by an element in the space $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty, \Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L}))$ for certain $\mathscr{L}\in E$. We consider the upper-triangular Borel subgroup $\overline{B_2}$ and the diagonal split torus T_2. Then by the proof of Lemma 3.20 of [BD18] we know that the Jacquet functor $J_{\overline{B_2}}$ (c.f. [Eme06I] for the definition) induces a injection

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty,\ \Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})\right) \hookrightarrow \mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}\left(\delta_{T_2,\nu}\otimes_E(|\cdot|\otimes_E|\cdot|^{-1}),\ \delta_{T_2,\nu}\otimes_E(|\cdot|\otimes_E|\cdot|^{-1})\right). \quad (4.3.12)$$

By twisting $\delta_{T_2,-\nu}\otimes_E(|\cdot|^{-1}\otimes_E|\cdot|)$ we have an isomorphism

$$\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}\left(\delta_{T_2,\nu}\otimes_E(|\cdot|\otimes_E|\cdot|^{-1}),\ \delta_{T_2,\nu}\otimes_E(|\cdot|\otimes_E|\cdot|^{-1})\right)\cong \mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1_{T_2},\ 1_{T_2}). \quad (4.3.13)$$

It follows from Lemma 3.20 of [BD18] (up to changes on notation) that the image of the composition of (4.3.13) and (4.3.12) is a certain two dimensional subspace $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1,1)_{\mathscr{L}}$ of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1,1)$ depending on $\mathscr{L}$. More precisely, if we use the notation ϵ_1, ϵ_2 for the two charaters

$$\epsilon_1: T_2(\mathbf{Q}_p)\to\mathbf{Q}_p^\times,\ \begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\mapsto a \text{ and } \epsilon_2: T_2(\mathbf{Q}_p)\to\mathbf{Q}_p^\times,\ \begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\mapsto b,$$

then the set

$$\{\log_0\circ\epsilon_1, \mathrm{val}_p\circ\epsilon_1, \log_0\circ\epsilon_2, \mathrm{val}_p\circ\epsilon_2\}$$

forms a basis of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1,1)$, and the subspace $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1,1)_{\mathscr{L}}$ has a basis

$$\{\log_0\circ\epsilon_1+\log_0\circ\epsilon_2, \mathrm{val}_p\circ\epsilon_1+\mathrm{val}_p\circ\epsilon_2, \log_0\circ\epsilon_1-\log_0\circ\epsilon_2+\mathscr{L}(\mathrm{val}_p\circ\epsilon_1-\mathrm{val}_p\circ\epsilon_2)\}.$$

It follows from Lemma 4.3.4 that $U(\mathfrak{t}_2)^{W_{\mathrm{GL}_2}}$ acts on $J_{\overline{B_2}}(V)$ via a character where W_{GL_2} is the notation for the Weyl group of $\mathrm{GL}_2(\mathbf{Q}_p)$. Therefore we deduce by a twisting that the the subspace of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1,1)$ corresponding to $J_{\overline{B_2}}(V)$ is killed by $U(\mathfrak{t}_2)^{W_{\mathrm{GL}_2}}$. We notice that the subspace M of $\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1,1)$ killed by $U(\mathfrak{t}_2)^{W_{\mathrm{GL}_2}}$ is two dimensional with basis

$$\{\mathrm{val}_p\circ\epsilon_1, \mathrm{val}_p\circ\epsilon_2\}$$

and we have

$$M\cap\mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(1,1)_{\mathscr{L}} = E(\mathrm{val}_p\circ\epsilon_1+\mathrm{val}_p\circ\epsilon_2).$$

However, the representation given by the line $E(\mathrm{val}_p\circ\epsilon_1+\mathrm{val}_p\circ\epsilon_2)$ has a subrepresentation of the form

$$\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E\mathrm{St}_2^\infty \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E\mathrm{St}_2^\infty$$

which is a contradiction.

The proof of the second statement is a direct analogue as we observe that $J_{\overline{B_2}}$ also induces the following embedding

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(\overline{L}_{\mathrm{GL}_2}(\nu),\ \ \overline{L}_{\mathrm{GL}_2}(\nu) \text{ —— } \widetilde{I}(s\cdot\nu) \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E\mathrm{St}_2^\infty \text{ —— } \overline{L}_{\mathrm{GL}_2}(\nu)\right) \hookrightarrow \mathrm{Ext}^1_{T_2(\mathbf{Q}_p)}(\delta_{T_2,\nu},\ \delta_{T_2,\nu}).$$

□

We define $\Sigma_2^+(\nu,\mathscr{L})$ as the unique (up to isomorphism) non-split extension of $\widetilde{I}(s\cdot\nu)$ by $\Sigma_{\mathrm{GL}_2}(\nu,\mathscr{L})$ given by Lemma 4.3.1.

Proposition 4.3.14. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)\ ,\ \Sigma_2^+(\nu,\mathscr{L})\Big)=0.$$

Proof. Assume on the contrary that V is a representation given by a certain non-zero element inside

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)\ ,\ \Sigma_2^+(\nu,\mathscr{L})\Big).$$

We deduce that V has both a central character and an infinitesimal character from Lemma 4.3.3 and the fact

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)\ ,\ \Sigma_2^+(\nu,\mathscr{L})\Big)=0.$$

Note that we have

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty,\ I(s\cdot\nu))=\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\overline{L}_{\mathrm{GL}_2}(\nu),\ \widetilde{I}(s\cdot\nu))=0,$$

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\big(\overline{L}_{\mathrm{GL}_2}(\nu),\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty\big)=1$$

and

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\big(\overline{L}_{\mathrm{GL}_2}(\nu),\ I(s\cdot\nu)\big)=1$$

by a combination of Lemma 3.13 of [BD18] with Lemma 4.2.1, and thus V has a subrepresentation of one of the three following forms

(i) $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$;

(ii) $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ —— $I(s\cdot\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$;

(iii) $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ —— $I(s\cdot\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)$ —— $\widetilde{I}(s\cdot\nu)$ —— $\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty$ — $\overline{L}_{\mathrm{GL}_2}(\nu)$.

In the first case, we know from Proposition 4.7 of [Schr11] and the main result of [Or05] that

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p),\nu}\big(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty,\ \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty\big)=0$$

and therefore this case is impossible due to the existence of central character for V (and hence for its subrepresentations). In the second case, we deduce from Lemma 4.3.9 a contradiction as V has an infinitesimal character. In the third case, we thus know that V has a quotient representation of the form

$$\overline{L}_{\mathrm{GL}_2}(\nu) \text{—— } \widetilde{I}(s\cdot\nu) \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)$$

which can not have an infinitesimal character due to Lemma 4.3.9, a contradiction again. Hence we finish the proof. □

Remark 4.3.15. *Note that the argument in Proposition 4.3.14 actually implies that*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)\ ,\ \ I(s\cdot\nu) \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu) \text{—— } \widetilde{I}(s\cdot\nu)\Big)=0$$

and we can show by the same method that

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}\Big(\overline{L}_{\mathrm{GL}_2}(\nu) \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty\ ,\ \ \widetilde{I}(s\cdot\nu) \text{—— } \overline{L}_{\mathrm{GL}_2}(\nu)\otimes_E \mathrm{St}_2^\infty \text{—} I(s\cdot\nu)\Big)=0.$$

4.4 Computations of Ext I

In this section, we are going to compute various Ext-groups based on known results on group cohomology in Section 5.2 and 5.3 of [Bre17].

Proposition 4.4.1. *The following spaces are one dimensional*

$$\begin{array}{ll}
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right) & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty},\ \overline{L}(\lambda)\right)\\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right) & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty},\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right)\\
\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ \overline{L}(\lambda)\right) & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right)\\
\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right) & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)
\end{array}$$

for $i=1,2$. Moreover, we have

$$\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1,\ V_2\right)=0$$

in all the other cases where $1\leq k\leq 2$ and $V_1,V_2\in\{\overline{L}(\lambda),\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\}$.

Proof. This follows from a special case of Proposition 4.7 of [Schr11] together with the main result of [Or05]. □

Lemma 4.4.2. *We have*

$$\begin{aligned}
\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\text{ —— }\overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right) &= 0\\
\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\text{ —— }\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\ ,\ \overline{L}(\lambda)\right) &= 0\\
\mathrm{Ext}^k_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\text{ —— }\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\ ,\ \overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty}\right) &= 0
\end{aligned}$$

for $i=1,2$ and $k=1,2$.

Proof. It is sufficient to prove that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\text{ —— }\overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right)=0 \tag{4.4.3}$$

and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\text{ —— }\overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right)=0 \tag{4.4.4}$$

as the other cases are similar. We observe that (4.4.3) is equivalent to the non-existence of a uniserial representation of the form

$$\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\text{ —— }\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\text{ —— }\overline{L}(\lambda)$$

which is again equivalent to the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\text{ —— }\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\ \right)=0 \tag{4.4.5}$$

according to the fact

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right)=0$$

due to Proposition 4.4.1. The short exact sequence

$$\left(\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\text{ —— }\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\ \right)\hookrightarrow \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-\lambda),\ \pi_{1,3}^{\infty}\right)\twoheadrightarrow C^2_{s_{3-i},s_{3-i}s_i}$$

induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \text{ —— } \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_i}^{\mathrm{GL}_3}\left(M_i(-\lambda),\ \pi_{i,3}^\infty\right)\right).$$

Therefore (4.4.5) follows from Lemma 4.2.1 and the facts that

$$\mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(H_0(N_i,\overline{L}(\lambda)),\ \overline{L}_i(\lambda)\otimes_E \pi_{i,3}^\infty\right) = \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(H_1(N_i,\overline{L}(\lambda)),\ \overline{L}_i(\lambda)\otimes_E \pi_{i,3}^\infty\right) = 0.$$

On the other hand, the short exact sequence

$$\overline{L}(\lambda)\otimes_E v_{P_i}^\infty \hookrightarrow \left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ —— } \overline{L}(\lambda)\ \right) \twoheadrightarrow \overline{L}(\lambda)$$

induces a long exact sequence

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ —— } \overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)\\
&\quad\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)\\
&\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ —— } \overline{L}(\lambda)\ ,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)
\end{aligned}$$

and thus we cab deduce (4.4.4) from Proposition 4.4.1 and (4.4.3). □

We define W_0 as the unique locally algebraic representation of length three satisfying

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(W_0) = \overline{L}(\lambda)\otimes_E \left(v_{P_1}^\infty \oplus v_{P_2}^\infty\right) \text{ and } \mathrm{cosoc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(W_0) = \overline{L}(\lambda).$$

We also define the (unique up to isomorphism) locally algebraic representation of the form

$$W_i :=\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty \text{ —— } \overline{L}(\lambda)$$

for each $i = 1, 2$

Lemma 4.4.6. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 1$$

and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 0.$$

Proof. The short exact sequence

$$\overline{L}(\lambda)\otimes_E v_{P_1}^\infty \hookrightarrow W_0 \twoheadrightarrow W_2$$

induces a long exact sequence

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)\\
&\quad\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)\\
&\qquad\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)
\end{aligned}$$

which finishes the proof by Proposition 4.4.1, (4.4.3) and (4.4.4). □

Recall that we have introduced a set Ω consisting of irreducible locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$ in (4.2.11). We define the following subsets of Ω:

$$\begin{aligned}
\Omega_1\left(\overline{L}(\lambda)\right) &:= \{\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ C^1_{s_1,1},\ C^1_{s_2,1}\}\\
\Omega_1\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right) &:= \{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ C^2_{s_1,1},\ C_{s_2,s_2},\ C^1_{s_1,s_1s_2}\}\\
\Omega_1\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right) &:= \{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ C^2_{s_2,1},\ C_{s_1,s_1},\ C^1_{s_2,s_2s_1}\}\\
\Omega_1\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right) &:= \{\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ C^2_{s_1,s_1s_2},\ C^2_{s_2,s_2s_1}\}\\
\Omega_2\left(\overline{L}(\lambda)\right) &:= \{\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty},\ C^2_{s_1,1},\ C^2_{s_2,1},\ C^1_{s_1s_2,1},\ C^1_{s_2s_1,1}\}\\
\Omega_2\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right) &:= \{\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ C^1_{s_1,1},\ C^2_{s_1,s_1s_2},\ C^2_{s_1s_2,1},\ C_{s_2s_1,s_2}\}\\
\Omega_2\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right) &:= \{\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ C^1_{s_2,1},\ C^2_{s_2,s_2s_1},\ C^2_{s_2s_1,1},\ C_{s_1s_2,s_1}\}\\
\Omega_2\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right) &:= \{\overline{L}(\lambda),\ C^1_{s_1,s_1s_2},\ C^1_{s_2,s_2s_1},\ C^2_{s_1s_2,s_1s_2},\ C^2_{s_2s_1,s_2s_1}\}
\end{aligned}$$

Proposition 4.4.7. *We have an explicit formula for*

$$H_k\left(N_i,\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(M,\ \pi_j^{\infty})\right)$$

for each admissible smooth representation π_j^{∞} of $L_j(\mathbf{Q}_p)$, each

$$M\in\{L(-\lambda),\ M_j(-\lambda), L(-s_{3-j}\cdot\lambda),\ M_j(-s_{3-j}\cdot\lambda),\ L(-s_{3-j}s_j\cdot\lambda)\}$$

and each $0\le k\le 2$, $i,j=1,2$.

Proof. This follows directly from Section 5.2 and 5.3 of [Bre17]. □

Lemma 4.4.8. *For*

$$V_0\in\{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\},$$

we have

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_0,\ V\right)=1$$

if $V\in\Omega_1(V_0)$ and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_0,\ V\right)=0$$

if $V\in\Omega\setminus\Omega_1(V_0)$.

Proof. We only prove the statements for $V_0=\overline{L}(\lambda)$ as other cases are similar. If

$$V\in\{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\}$$

then the conclusion follows from Proposition 4.4.1. If

$$V=\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^{\infty})$$

for a smooth irreducible representation π_j^{∞} and $j=1$ or 2, then it follows from Lemma 4.2.1 that

$$\begin{aligned}
\mathrm{Ext}^1_{L_j(\mathbf{Q}_p),\lambda}\left(H_0(N_j,\ \overline{L}(\lambda)),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^{\infty}\right)&\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\\
&\to \mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(H_1(N_j,\ \overline{L}(\lambda)),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^{\infty}\right)\\
&\to \mathrm{Ext}^2_{L_j(\mathbf{Q}_p),\lambda}\left(H_0(N_j,\ \overline{L}(\lambda)),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^{\infty}\right). \quad (4.4.9)
\end{aligned}$$

It follows from Proposition 4.4.7 and (4.4.9) that

$$\begin{aligned}\mathrm{Ext}^1_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right)&\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\\&\to\mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}\cdot\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right).\end{aligned}\tag{4.4.10}$$

We notice that $Z(L_j(\mathbf{Q}_p))$ acts via different characters on $\overline{L}_j(\lambda)$, $\overline{L}_j(s_{3-j}\cdot\lambda)$ and $\overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty$, and thus we have the equalities

$$\begin{aligned}\mathrm{Ext}^1_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right)&=0\\\mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}\cdot\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right)&=0\end{aligned}$$

which imply that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty)\right)=0\tag{4.4.11}$$

for each π_j^∞ and $j=1,2$. If

$$V=\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)$$

for a smooth irreducible representation π_j^∞ and $j=1$ or 2, then the short exact sequence

$$\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\hookrightarrow\mathcal{F}_{P_j}^{\mathrm{GL}_3}(M_j(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\twoheadrightarrow\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty)$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}&\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(M_j(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\right)\\&\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty)\right)\end{aligned}$$

which implies an isomorphism

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(M_j(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\right)\tag{4.4.12}$$

by (4.4.11). It follows from Proposition 4.4.7 and Lemma 4.2.1 that

$$\begin{aligned}&\mathrm{Ext}^1_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\\&\to\mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}\cdot\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right)\to\mathrm{Ext}^2_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right).\end{aligned}\tag{4.4.13}$$

As $Z(L_j(\mathbf{Q}_p))$ acts via different characters on $\overline{L}_j(\lambda)$ and $\overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty$, we have the equalities

$$\begin{aligned}\mathrm{Ext}^1_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right)&=0\\\mathrm{Ext}^2_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right)&=0\end{aligned}$$

which imply that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)\xrightarrow{\sim}\mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}\cdot\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right).\tag{4.4.14}$$

It is then obvious that

$$\mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}\cdot\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right)=0$$

for each smooth irreducible $\pi_j^\infty\neq 1_{L_j}$, and therefore

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}\cdot\lambda),\ 1_{L_j})\right)=1$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}\cdot\lambda),\ 1_{L_j})\right)=0$$

for each smooth irreducible $\pi_j^\infty \neq 1_{L_j}$. Finally, similar methods together with Proposition 4.4.7 also show that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{B}^{\mathrm{GL}_3}(L(-s_1s_2s_1\cdot\lambda),\ \chi_w^\infty)\right)=0$$

for each $w\in W$. □

We define

$$\Omega^-:=\Omega\setminus\{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\}.$$

Then we define the following subsets of Ω^- for $i=1,2$:

$$\begin{aligned}
\Omega_1\left(C^1_{s_i,1}\right) &:= \{C^1_{s_is_{3-i},1},\ C^2_{s_{3-i}s_i,1},\ C^2_{s_i,1},\ C^1_{s_i,1}\}\\
\Omega_1\left(C^2_{s_i,1}\right) &:= \{C^2_{s_is_{3-i},1},\ C_{s_{3-i}s_i,s_{3-i}},\ C^1_{s_i,1},\ C^2_{s_i,1}\}\\
\Omega_1\begin{pmatrix}C^1_{s_i,s_is_{3-i}}\end{pmatrix} &:= \{C^1_{s_is_{3-i},s_is_{3-i}},\ C_{s_{3-i}s_i,s_{3-i}},\ C^2_{s_i,s_is_{3-i}},\ C^1_{s_i,s_is_{3-i}}\}\\
\Omega_1\begin{pmatrix}C^2_{s_i,s_is_{3-i}}\end{pmatrix} &:= \{C^2_{s_is_{3-i},s_is_{3-i}},\ C^1_{s_{3-i}s_i,s_{3-i}s_i},\ C^1_{s_i,s_is_{3-i}},\ C^2_{s_i,s_is_{3-i}}\}\\
\Omega_1\left(C_{s_i,s_i}\right) &:= \{C_{s_is_{3-i},s_i},\ C^1_{s_{3-i}s_i,1},\ C^2_{s_{3-i}s_i,s_{3-i}s_i},\ C_{s_i,s_i}\}
\end{aligned}$$

Lemma 4.4.15. *For*

$$V_0\in\{C^1_{s_i,1},\ C^2_{s_i,1},\ C^1_{s_i,s_is_{3-i}},\ C^2_{s_i,s_is_{3-i}},\ C_{s_i,s_i}\mid i=1,2\},$$

we have

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=1$$

if $V\in\Omega_1(V_0)$ *and*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=0$$

if $V\in\Omega^-\setminus\Omega_1(V_0)$.

Proof. The proof is very similar to that of Lemma 4.4.15. □

Lemma 4.4.16. *For*

$$V_0\in\{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\},$$

we have

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=1$$

if $V\in\Omega_2(V_0)$ *and*

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(V_0,\ V)=0$$

if $V\in\Omega\setminus\Omega_2(V_0)$.

Proof. We only prove the statements for $V_0=\overline{L}(\lambda)$ as other cases are similar. If

$$V\in\{\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_1}^\infty,\ \overline{L}(\lambda)\otimes_E v_{P_2}^\infty,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\}$$

then the conclusion follows from Proposition 4.4.1. We notice that $Z(L_j(\mathbf{Q}_p))$ acts via different characters on $\overline{L}_j(\lambda)$, $\overline{L}_j(s_{3-j}\cdot\lambda)$ and $\overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty$, and thus we have

$$\begin{aligned}
\mathrm{Ext}^2_{L_j(\mathbb{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right) &= 0\\
\mathrm{Ext}^1_{L_j(\mathbb{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}\cdot\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right) &= 0\\
\mathrm{Ext}^3_{L_j(\mathbb{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right) &= 0
\end{aligned}\tag{4.4.17}$$

On the other hand, we notice that

$$\mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}s_j\cdot\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\otimes_E\pi_j^\infty\right)=0 \tag{4.4.18}$$

for each smooth irreducible $\pi_j^\infty\neq 1_{L_j}$ and

$$\dim_E\mathrm{Hom}_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}s_j\cdot\lambda),\ \overline{L}_j(s_{3-j}s_j\cdot\lambda)\right)=1. \tag{4.4.19}$$

We combine (4.4.17), (4.4.18) and (4.4.19) with Lemma 4.2.1 and Proposition 4.4.7 and deduce that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty)\right)=0 \tag{4.4.20}$$

for each smooth irreducible $\pi_j^\infty\neq 1_{L_j}$ and

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ 1_{L_j})\right)=1 \tag{4.4.21}$$

which finishes the proof if

$$V=\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty).$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Ext}^2_{L_j(\mathbb{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right) &&=0\\
&\mathrm{Hom}_{L_j(\mathbb{Q}_p),\lambda}\left(\overline{L}_j(s_{3-j}s_j\cdot\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right) &&=0\\
&\mathrm{Ext}^3_{L_j(\mathbb{Q}_p),\lambda}\left(\overline{L}_j(\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right) &&=0
\end{aligned} \tag{4.4.22}$$

On the other hand, we have

$$\mathrm{Ext}^1_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-i}\cdot\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_j^\infty\right)=0 \tag{4.4.23}$$

for each smooth irreducible $\pi_j^\infty\neq\pi_{j,1}^\infty$ and

$$\dim_E\mathrm{Ext}^1_{L_j(\mathbf{Q}_p),\lambda}\left(\overline{L}_j(s_{3-i}\cdot\lambda),\ \overline{L}_j(s_{3-j}\cdot\lambda)\otimes_E\pi_{j,1}^\infty\right)=1. \tag{4.4.24}$$

We combine (4.4.22), (4.4.23) and (4.4.24) with Lemma 4.2.1 and Proposition 4.4.7 and deduce that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(M_j(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\right)=0 \tag{4.4.25}$$

for each smooth irreducible $\pi_j^\infty\neq\pi_{j,1}^\infty$ and

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(M_j(-s_{3-j}\cdot\lambda),\ \pi_{j,1}^\infty)\right)=1. \tag{4.4.26}$$

The short exact sequence

$$\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\hookrightarrow\mathcal{F}_{P_j}^{\mathrm{GL}_3}(M_j(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\twoheadrightarrow\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty)$$

induces a long exact sequence

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty)\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\right)\\
&\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(M_j(-s_{3-j}\cdot\lambda),\ \pi_j^\infty)\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}s_j\cdot\lambda),\ \pi_j^\infty)\right)
\end{aligned}$$

which finishes the proof if

$$V=\mathcal{F}_{P_j}^{\mathrm{GL}_3}(L(-s_{3-j}\cdot\lambda),\ \pi_j^\infty).$$

Finally, similar methods together with Proposition 4.4.7 also show that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathcal{F}_B^{\mathrm{GL}_3}(L(-s_1s_2s_1\cdot\lambda),\ \chi_w^\infty)\right)=0$$

for each $w\in W$. □

Lemma 4.4.27. *We have*

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty} \text{---} \overline{L}(\lambda)\ ,\ C^2_{s_i,1}\right) &&= 0\\
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty} \text{---} \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\ ,\ C^1_{s_i,s_is_{3-i}}\right) &&= 0\\
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \text{---} \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\ ,\ C^1_{s_i,1}\right) &&= 0\\
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty} \text{---} \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\ ,\ C^2_{s_i,s_is_{3-i}}\right) &&= 0
\end{aligned}$$

for $i = 1, 2$.

Proof. We only prove the first vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ C^2_{s_i,1}\right) = 0 \tag{4.4.28}$$

as the other cases are similar. The embedding

$$C^2_{s_i,1} \hookrightarrow \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^{\infty}_{3-i,1})$$

induces an embedding

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ C^2_{s_i,1}\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^{\infty}_{3-i,1})\right). \tag{4.4.29}$$

It follows from Proposition 4.4.7 that

$$\begin{aligned}
H_0(N_{3-i},\ W_i) &= \overline{L}_{3-i}(\lambda)\otimes_E \left(i^{L_{3-i}}_{B\cap L_{3-i}}(\chi^{\infty}_{s_{3-i}})\oplus \mathfrak{d}^{\infty}_{P_{3-i}}\right)\\
H_1(N_{3-i},\ W_i) &= \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E \left(i^{L_{3-i}}_{B\cap L_{3-i}}(\chi^{\infty}_{s_{3-i}})\oplus \mathfrak{d}^{\infty}_{P_{3-i}}\right)
\end{aligned}\quad . \tag{4.4.30}$$

We notice that $Z(L_{3-i}(\mathbf{Q}_p))$ acts on $\overline{L}_{3-i}(\lambda)$ and $\overline{L}_{3-i}(s_i\cdot\lambda)$ via different characters and that

$$\mathrm{Hom}_{L_{3-i}(\mathbf{Q}_p),\lambda}\left(i^{L_{3-i}}_{B\cap L_{3-i}}(\chi^{\infty}_{s_{3-i}}),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E \pi^{\infty}_{3-i,1}\right) = 0.$$

Therefore we deduce from (4.4.30) the equalities

$$\begin{aligned}
&\mathrm{Ext}^1_{L_{3-i}(\mathbb{Q}_p),\lambda}\left(H_0(N_{3-i},\ W_i),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E \pi^{\infty}_{3-i,1}\right) &&= 0\\
&\mathrm{Hom}_{L_{3-i}(\mathbb{Q}_p),\lambda}\left(H_1(N_{3-i},\ W_i),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E \pi^{\infty}_{3-i,1}\right) &&= 0
\end{aligned}$$

which imply by Lemma 4.2.1 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^{\infty}_{3-i,1})\right) = 0.$$

Hence we finish the proof of (4.4.28) by the embedding (4.4.29). □

Lemma 4.4.31. *We have for* $i = 1, 2$*:*

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty} \text{---} C_{s_i,s_i}\ ,\ C^2_{s_i,1}\right) &&= 0\\
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty} \text{---} C^2_{s_i,s_is_{3-i}}\ ,\ C_{s_i,s_i}\right) &&= 0\\
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \text{---} C^1_{s_i,s_is_{3-i}}\ ,\ C^1_{s_i,1}\right) &&= 0\\
&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\ C^2_{s_i,1}\ ,\ C^2_{s_i,s_is_{3-i}}\right) &&= 0
\end{aligned}$$

Proof. We only prove that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty} \text{ —— } C_{s_i,s_i}\ ,\ C^2_{s_i,1}\Big)=0 \tag{4.4.32}$$

as the other cases are similar. The surjection

$$\mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2}) \twoheadrightarrow \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty} \text{ —— } C_{s_i,s_i}$$

and the embedding

$$C^2_{s_i,1}\hookrightarrow \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^{\infty}_{3-i,1})$$

induce an embedding

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty} \text{ —— } C_{s_i,s_i}\ ,\ C^2_{s_i,1}\Big)\\ &\qquad\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(\mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2}),\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^{\infty}_{3-i,1})\Big).\end{aligned} \tag{4.4.33}$$

It follows from Proposition 4.4.7 that

$$H_0(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2}))=\big(\overline{L}_{3-i}(\lambda)\oplus\overline{L}_{3-i}(s_i\cdot\lambda)\big)\otimes_E\pi^{\infty}_{3-i,2}$$

and

$$\begin{aligned}&H_1(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2}))\\ &\quad=\big(\overline{L}_{3-i}(s_i\cdot\lambda)\oplus\overline{L}_{3-i}(s_is_{3-i}\cdot\lambda)\big)\otimes_E\pi^{\infty}_{3-i,2}\oplus I^{L_{3-i}}_{B\cap L_{3-i}}(\delta_{s_i\cdot\lambda})\oplus I^{L_{3-i}}_{B\cap L_{3-i}}\big(\delta_{s_i\cdot\lambda}\otimes_E\chi^{\infty}_{s_1s_2s_1}\big).\end{aligned}$$

We notice that $Z(L_{3-i}(\mathbf{Q}_p))$ acts on each direct summand of $H_k(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2}))$ $(k=0,1)$ via a different character, and the only direct summand that produces the same character as $\overline{L}_{3-i}(s_i\cdot\lambda)\otimes\pi^{\infty}_{3-i,1}$ is $I^{L_{3-i}}_{B\cap L_{3-i}}(\delta_{s_i\cdot\lambda})$. However, we know that

$$\mathrm{cosoc}_{L_{3-i}(\mathbf{Q}_p),\lambda}\Big(I^{L_{3-i}}_{B\cap L_{3-i}}(\delta_{s_i\cdot\lambda})\Big)=I^{L_{3-i}}_{B\cap L_{3-i}}\big(\delta_{s_{3-i}s_i\cdot\lambda}\big)$$

and thus

$$\mathrm{Hom}_{L_{3-i}(\mathbf{Q}_p),\lambda}\Big(I^{L_{3-i}}_{B\cap L_{3-i}}\big(\delta_{s_{3-i}s_i\cdot\lambda}\big)\ ,\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes\pi^{\infty}_{3-i,1}\Big)=0.$$

As a result, we deduce the equalities

$$\begin{aligned}\mathrm{Ext}^1_{L_{3-i}(\mathbb{Q}_p),\lambda}\Big(H_0(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2})),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E\pi^{\infty}_{3-i,1}\Big)&=0\\ \mathrm{Hom}_{L_{3-i}(\mathbb{Q}_p),\lambda}\Big(H_1(N_{3-i},\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2})),\ \overline{L}_{3-i}(s_i\cdot\lambda)\otimes_E\pi^{\infty}_{3-i,1}\Big)&=0\end{aligned}$$

which imply by Lemma 4.2.1 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\Big(\mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-\lambda),\ \pi^{\infty}_{3-i,2}),\ \mathcal{F}^{\mathrm{GL}_3}_{P_{3-i}}(M_{3-i}(-s_i\cdot\lambda),\ \pi^{\infty}_{3-i,1})\Big)=0.$$

Hence we finish the proof of (4.4.32) by the embedding (4.4.33). □

Lemma 4.4.34. *There exists a unique representation of the form*

$$\begin{array}{ccccc} & & C^1_{s_{3-i}s_i,1} & & \\ & \nearrow & & \searrow & \\ C^2_{s_i,1} & & & & C_{s_i,s_i} \\ & \searrow & & \nearrow & \\ & & \overline{L}(\lambda)\otimes_E v^{\infty}_{P_i} & & \end{array}$$

or of the form

$$\begin{array}{ccccc} & & C^1_{s_{3-i}s_i,s_{3-i}s_i} & & \\ & \diagup & & \diagdown & \\ C_{s_i,s_i} & & & & C^2_{s_i,s_is_{3-i}}\ . \\ & \diagdown & & \diagup & \\ & & \overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}} & & \end{array}$$

Proof. We only prove the first statement as the second is similar. It follows from Proposition 4.4.2 of [Bre17] that there exists a unique representation of the form

$$\begin{array}{ccccc} & & C^1_{s_{3-i}s_i,1} & & \\ & \diagup & & \cdots & \\ C^2_{s_i,1} & & & & C_{s_i,s_i} \\ & \diagdown & & \diagup & \\ & & \overline{L}(\lambda)\otimes_E v^\infty_{P_i} & & \end{array}$$

but it is not proven there whether its quotient

$$C^1_{s_{3-i}s_i,1} \text{ - - - } C_{s_i,s_i} \tag{4.4.35}$$

is split or not. However, If (4.4.35) is split, then there exists a representation of the form

$$C^2_{s_i,1} \text{ — } \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \text{ — } C_{s_i,s_i}$$

which contradicts the first vanishing in Lemma 4.4.31, and thus we finish the proof. □

Remark 4.4.36. *Our method used in Lemma 4.4.31 and in Lemma 4.4.34 is different from the one due to Y.Ding mentioned in part (ii) of Remark 4.4.3 of [Bre17]. It is not difficult to observe that*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\begin{array}{ccccc} & & & & C^1_{s_{3-i}s_i,1} \\ C_{s_i,s_i}, & & C^2_{s_i,1} & \diagup & \\ & & & \diagdown & \\ & & & & \overline{L}(\lambda)\otimes_E v^\infty_{P_i} \end{array}\right) = 1 \tag{4.4.37}$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\begin{array}{ccccc} & & & & C^1_{s_{3-i}s_i,s_{3-i}s_i} \\ C^2_{s_i,s_is_{3-i}}, & & C_{s_i,s_i} & \diagup & \\ & & & \diagdown & \\ & & & & \overline{L}(\lambda)\otimes_E v^\infty_{P_{3-i}} \end{array}\right) = 1 \tag{4.4.38}$$

for $i = 1, 2$. Similar methods as those used in Proposition 4.4.2 of [Bre17], in Lemma 4.4.31 and in Lemma 4.4.34 also imply the existence of a unique representation of the form

$$\begin{array}{ccccc} & & C_{s_{3-i}s_i,s_{3-i}} & & \\ & \diagup & & \diagdown & \\ C^1_{s_i,1} & & & & C^1_{s_i,s_is_{3-i}} \\ & \diagdown & & \diagup & \\ & & \overline{L}(\lambda) & & \end{array}$$

or of the form

$$\begin{array}{ccccc} & & C_{s_{3-i}s_i,s_{3-i}} & & \\ & \diagup & & \diagdown & \\ C^2_{s_i,s_is_{3-i}} & & & & C^2_{s_i,1}\ . \\ & \diagdown & & \diagup & \\ & & \overline{L}(\lambda)\otimes_E \mathrm{St}^\infty_3 & & \end{array}$$

4.5 Computations of Ext II

In this section, we are going to establish several computational results (most notably Lemma 4.5.8) which have crucial applications in Section 4.7. We firstly recall the definition of $\Sigma_i(\lambda, \mathscr{L})$ for $i = 1, 2$ and $\mathscr{L} \in E$ right before (4.2.23).

Lemma 4.5.1. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_i,s_i},\ \Sigma_i(\lambda, \mathscr{L}_i)\right) = 1$$

for $i = 1, 2$.

Proof. We only prove that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(C_{s_1,s_1},\ \Sigma_1(\lambda, \mathscr{L}_1)\right) = 1 \tag{4.5.2}$$

as the other equality is similar. We note that $\Sigma_1(\lambda, \mathscr{L}_1)$ admits a subrepresentation of the form

$$W := \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \text{ — } C^2_{s_1,1} \begin{array}{l} \diagup\ C^1_{s_2s_1,1} \\ \diagdown\ \overline{L}(\lambda) \otimes_E v^\infty_{P_1} \end{array}$$

due to Lemma 3.34, Lemma 3.37 and Remark 3.38 of [BD18]. Therefore $\Sigma_1(\lambda, \mathscr{L}_1))$ admits a filtration such that W appears as one term of the filtration and the only reducible graded piece is

$$V_1 := C^2_{s_1,1} \begin{array}{l} \diagup\ C^1_{s_2s_1,1} \\ \diagdown\ \overline{L}(\lambda) \otimes_E v^\infty_{P_1} \end{array}.$$

It follows from Lemma 4.4.1 and Proposition 4.2.1 of [Bre17] as well as our Lemma 4.4.15 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(C_{s_1,s_1},\ V\right) = 0 \tag{4.5.3}$$

for all graded pieces V such that $V \neq V_1$. On the other hand, we have

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(C_{s_1,s_1},\ V_1\right) = 1 \tag{4.5.4}$$

due to (4.4.37) and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(C_{s_1,s_1},\ \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty\right) = 0 \tag{4.5.5}$$

by Proposition 4.6.1 of [Bre17]. Hence we finish the proof by combining (4.5.3), (4.5.4), (4.5.5) and part (ii) of Proposition 4.2.5. □

Lemma 4.5.6. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \otimes_E v^\infty_{P_{3-i}},\ \Sigma_i^+(\lambda, \mathscr{L}_i)\right) = 3$$

for $i = 1, 2$.

Proof. By symmetry, it suffices to prove that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \otimes_E v^\infty_{P_2},\ \Sigma_1^+(\lambda, \mathscr{L}_1\right) = 3.$$

This follows immediately from Lemma 3.42 of [Bre17] as our $\Sigma_1^+(\lambda, \mathscr{L}_1)$ can be identified with the locally analytic representation $\widetilde{\Pi}^1(\lambda, \psi)$ defined before (3.76) of [Bre17] up to changes on notation. □

We define $\Sigma_1^+(\lambda,\mathscr{L}_1)$ (resp. $\Sigma_2^+(\lambda,\mathscr{L}_2)$) as the unique non-split extension given by a non-zero element in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(C_{s_1,s_1},\ \Sigma_1(\lambda,\mathscr{L}_1))$ (resp. in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(C_{s_2,s_2},\ \Sigma_2(\lambda,\mathscr{L}_2))$). Hence we may consider the amalgamate sum of $\Sigma_1^+(\lambda,\mathscr{L}_1)$ and $\Sigma_2^+(\lambda,\mathscr{L}_2)$ over $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ and denote it by $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$. In particular, $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ has the following form

$$\begin{array}{ccccc} & & v_{P_1}^{\mathrm{an}}(\lambda) & \text{——} & C_{s_1,s_1} \\ \mathrm{St}_3^{\mathrm{an}}(\lambda) & \diagup & & & \\ & \diagdown & & & \\ & & v_{P_2}^{\mathrm{an}}(\lambda) & \text{——} & C_{s_2,s_2} \end{array}.$$

Lemma 4.5.7. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

for $i=1,2$.

Proof. The short exact sequence

$$\Sigma_2^+(\lambda,\mathscr{L}_2)\hookrightarrow \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow\ v_{P_1}^{\mathrm{an}}(\lambda)\text{——}C_{s_1,s_1}$$

induces the following long exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda)\text{——}C_{s_1,s_1}\right)&\\ \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma_2^+(\lambda,\mathscr{L}_2)\right)&\\ \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)&\\ \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda)\text{——}C_{s_1,s_1}\right)&.\end{aligned}$$

As a result, we can deduce

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

from Lemma 4.5.6 and the facts

$$\dim_E \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda)\text{——}C_{s_1,s_1}\right)=1$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \ v_{P_1}^{\mathrm{an}}(\lambda)\text{——}C_{s_1,s_1}\right)=0$$

by Proposition 4.4.1 and Lemma 4.4.8. The proof for

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

is similar. □

Lemma 4.5.8. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_i,\ \Sigma_i^+(\lambda,\mathscr{L}_i))=0$$

and in particular

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_i,\ \Sigma_i(\lambda,\mathscr{L}_i))=0$$

for $i=1,2$.

Proof. We only need to show the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_2,\ \Sigma_1^+(\lambda,\mathscr{L}_1))=0$$

as the others are similar or easier. We define $\nu:=\lambda_{T_2,\iota_{T,1}}$ (which is the restriction of λ from T to T_2 via the embedding $\iota_{T,1}:T_2\hookrightarrow T$) and view $\Sigma^+_{\mathrm{GL}_2}(\nu,\mathscr{L}_1)$ (which is defined before Proposition 4.3.14) as a locally analytic representation of $L_1(\mathbf{Q}_p)$ via the projection $p_1:L_1\twoheadrightarrow \mathrm{GL}_2$ and denote it by $\Sigma^+_{L_1}(\lambda,\mathscr{L}_1)$. We note by definition by of $\Sigma_1(\lambda,\mathscr{L}_1)$ that we have an isomorphism

$$\Sigma_1(\lambda,\mathscr{L}_1)\xrightarrow{\sim} I_{P_1}^{\mathrm{GL}_3}\left(\Sigma_{L_1}(\lambda,\mathscr{L}_1)\right)/\left(\ v_{P_2}^{\mathrm{an}}(\lambda)\longrightarrow \overline{L}(\lambda)\ \right).$$

Therefore we can deduce from the short exact sequence

$$\Sigma^+_{\mathrm{GL}_2}(\nu,\mathscr{L}_1)\hookrightarrow \Sigma^+_{\mathrm{GL}_2}(\nu,\mathscr{L}_1)\twoheadrightarrow \widetilde{I}(s\cdot\nu)$$

and the fact (up to viewing $\widetilde{I}(s\cdot\nu)$ as a locally analytic representation of $L_1(\mathbf{Q}_p)$ via the projection p_1)

$$C_{s_1,s_1}\cong \mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(I_{P_1}^{\mathrm{GL}_3}\left(\widetilde{I}(s\cdot\nu)\right)\right)$$

that we have an injection

$$\Sigma_1^+(\lambda,\mathscr{L}_1)\hookrightarrow I_{P_1}^{\mathrm{GL}_3}\left(\Sigma^+_{L_1}(\lambda,\mathscr{L}_1)\right)/\left(\ v_{P_2}^{\mathrm{an}}(\lambda)\longrightarrow \overline{L}(\lambda)\ \right)$$

which induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ V\right)\tag{4.5.9}$$

where we use the shorten notation

$$V:=I_{P_1}^{\mathrm{GL}_3}\left(\Sigma^+_{L_1}(\lambda,\mathscr{L}_1)\right)/\left(\ v_{P_2}^{\mathrm{an}}(\lambda)\longrightarrow \overline{L}(\lambda)\ \right).$$

Note that we have an exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}&\left(W_2,\ I_{P_1}^{\mathrm{GL}_3}\left(\Sigma^+_{L_1}(\lambda,\mathscr{L}_1)\right)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ V\right)\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \ v_{P_2}^{\mathrm{an}}(\lambda)\longrightarrow \overline{L}(\lambda)\ \right)\end{aligned}\tag{4.5.10}$$

It follows from Proposition 4.4.7 that

$$\begin{array}{rcl} H_0(N_1,\ W_2) & = & \overline{L}_1(\lambda)\otimes_E i_{B\cap L_1}^{L_1}(\chi_{s_1}^\infty)\\ H_1(N_1,\ W_2) & = & \overline{L}_1(s_2\cdot\lambda)\otimes_E\otimes_E i_{B\cap L_1}^{L_1}(\chi_{s_1}^\infty)\end{array}\quad .$$

Therefore we observe that

$$\mathrm{Hom}_{L_1(\mathbf{Q}_p),\lambda}\left(H_1(N_1,\ W_2),\ \Sigma^+_{L_1}(\lambda,\mathscr{L}_1)\right)=0$$

from the action of $Z(L_1(\mathbf{Q}_p))$ and

$$\mathrm{Ext}^1_{L_1(\mathbf{Q}_p),\lambda}\left(H_0(N_1,\ W_2),\ \Sigma^+_{L_1}(\lambda,\mathscr{L}_1)\right)=0$$

according to Proposition 4.3.14 and the natural identification

$$\mathrm{Ext}^1_{L_1(\mathbf{Q}_p),\lambda}(-,-)\cong \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(-,-).$$

As a result, we deduce

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ I_{P_1}^{\mathrm{GL}_3}\left(\Sigma^+_{L_1}(\lambda,\mathscr{L}_1)\right)\right)=0 \tag{4.5.11}$$

from Lemma 4.2.1. We know that

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(W_2,\ \ v^{\mathrm{an}}_{P_2}(\lambda)\text{ —— }\overline{L}(\lambda)\ \right)=0 \tag{4.5.12}$$

due to Proposition 4.4.1, Lemma 4.4.16 and a simple devissage, and thus we finish the proof by (4.5.9), (4.5.10), (4.5.11) and (4.5.12). □

Lemma 4.5.13. *We have*

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+_i(\lambda,\mathscr{L}_i)\right)=3 \tag{4.5.14}$$

for each $i=1,2$,

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2 \tag{4.5.15}$$

and

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=1. \tag{4.5.16}$$

Proof. The equalities (4.5.15) and (4.5.16) follow directly from Lemma 4.2.35 and the fact that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C_{s_i,s_i}\right)=\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C_{s_i,s_i}\right)=0 \tag{4.5.17}$$

by Lemma 4.4.8 and Lemma 4.4.16 using a long exact sequence induced from the short exact sequence

$$\Sigma_i(\lambda,\mathscr{L}_i)\hookrightarrow\Sigma^+_i(\lambda,\mathscr{L}_i)\twoheadrightarrow C_{s_i,s_i}.$$

Due to a similar argument using (4.5.17), we only need to show that

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right)=3 \tag{4.5.18}$$

to finish the proof of (4.5.14). The short exact sequence

$$\mathrm{St}^{\mathrm{an}}_3(\lambda)\hookrightarrow\Sigma_i(\lambda,\mathscr{L}_i)\twoheadrightarrow v^{\mathrm{an}}_{P_i}(\lambda)$$

induces a long exact sequence

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right)\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right)\\&\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}^{\mathrm{an}}_3(\lambda)\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right).\end{aligned} \tag{4.5.19}$$

We know that

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \mathrm{St}^{\mathrm{an}}_3(\lambda)\right)=5$$

by Lemma 4.2.33. It follows from Proposition 4.4.1, Lemma 4.4.8, Lemma 4.4.16 and a simple devissage that

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right)=2 \tag{4.5.20}$$

and

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right)=0. \tag{4.5.21}$$

Hence it remains to show that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda,\mathscr{L}_i)\right)=0 \tag{4.5.22}$$

to deduce (4.5.18) from (4.5.19). The short exact sequence

$$v_{P_{3-i}}^{\mathrm{an}}(\lambda) \text{---} \overline{L}(\lambda) \hookrightarrow I_{P_i}^{\mathrm{GL}_3}\left(\Sigma_{L_i}(\lambda, \mathscr{L}_i)\right) \twoheadrightarrow \Sigma_i(\lambda, \mathscr{L}_i)$$

induces

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}&\left(\overline{L}(\lambda),\ \ v_{P_{3-i}}^{\mathrm{an}}(\lambda) \text{---} \overline{L}(\lambda)\ \right)\\ &\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ I_{P_i}^{\mathrm{GL}_3}\left(\Sigma_{L_i}(\lambda, \mathscr{L}_i)\right)\right) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i(\lambda, \mathscr{L}_i)\right)\end{aligned}$$

by the vanishing

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v_{P_{3-i}}^{\mathrm{an}}(\lambda) \text{---} \overline{L}(\lambda)\ \right) = 0$$

using Proposition 4.4.1 and Lemma 4.4.16. Therefore we only need to show that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ v_{P_{3-i}}^{\mathrm{an}}(\lambda) \text{---} \overline{L}(\lambda)\ \right) = 1 \tag{4.5.23}$$

and

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ I_{P_i}^{\mathrm{GL}_3}\left(\Sigma_{L_i}(\lambda, \mathscr{L}_i)\right)\right) = 1. \tag{4.5.24}$$

The equality (4.5.24) follows from Lemma 4.2.1 and the facts

$$\dim_E \mathrm{Ext}^1_{L_i(\mathbf{Q}_p),\lambda}\left(H_0(N_i,\ \overline{L}(\lambda)),\ \Sigma_{L_i}(\lambda, \mathscr{L}_i)\right) = 1,\ \mathrm{Hom}_{L_i(\mathbf{Q}_p),\lambda}\left(H_1(N_i,\ \overline{L}(\lambda)),\ \Sigma_{L_i}(\lambda, \mathscr{L}_i)\right) = 0$$

where the first equality essentially follows from Lemma 3.14 of [BD18] and the second equality follows from checking the action of $Z(L_i(\mathbf{Q}_p))$. On the other hand, (4.5.23) follows from (4.5.20) and Proposition 4.4.1 by an easy devissage. Hence we finish the proof. □

Proposition 4.5.25. *The short exact sequence*

$$\overline{L}(\lambda) \otimes_E v_{P_i}^{\infty} \hookrightarrow W_i \twoheadrightarrow \overline{L}(\lambda)$$

induces the following isomorphisms

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \otimes_E v_{P_{3-i}}^{\infty},\ \Sigma_i^+(\lambda, \mathscr{L}_i)\right) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i^+(\lambda, \mathscr{L}_i)\right) \tag{4.5.26}$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \otimes_E v_{P_{3-i}}^{\infty},\ \Sigma^+(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right) \tag{4.5.27}$$

for $i = 1, 2$.

Proof. The vanishing from Lemma 4.5.8 implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda) \otimes_E v_{P_{3-i}}^{\infty},\ \Sigma_i^+(\lambda, \mathscr{L}_i)\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_i^+(\lambda, \mathscr{L}_i)\right)$$

is an injection and hence an isomorphism as both spaces have dimension three according to Lemma 4.5.6 and Lemma 4.5.13. The proof of (4.5.27) is similar. We emphasize that both (4.5.26) and (4.5.27) can be interpreted as the isomorphism given by the cup product with the one dimensional space

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda) \otimes_E v_{P_{3-i}}^{\infty}\right).$$

□

We define

$$\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2):=\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \text{ and } \Sigma_i^\flat(\lambda,\mathscr{L}_i):=\Sigma_i(\lambda,\mathscr{L}_i)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty$$

for $i=1,2$.

Lemma 4.5.28. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=1.$$

Proof. We define $\Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as the subrepresentation of $\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ that fits into the following short exact sequence

$$\Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow C^1_{s_2,1}\oplus C^1_{s_1,1}, \tag{4.5.29}$$

(c.f. (4.2.9) for the definition of $C^1_{s_2,1}$, $C^1_{s_1,1}$, $C^2_{s_2,1}$ and $C^2_{s_1,1}$) and then define $\Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as the subrepresentation of $\Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ that fits into

$$\Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow \Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow \left(\begin{array}{c} C^2_{s_1,1}\\ \overline{L}(\lambda)\otimes_E v_{P_1}^\infty\end{array}\right)\oplus\left(\begin{array}{c} C^2_{s_2,1}\\ \overline{L}(\lambda)\otimes_E v_{P_2}^\infty\end{array}\right). \tag{4.5.30}$$

It follows from Lemma 4.4.8 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ V)=0$$

for each $V\in \mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ and therefore

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\flat,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0 \tag{4.5.31}$$

by part (i) of Proposition 4.2.5. On the other hand, we know from Lemma 4.4.8 and Lemma 4.4.27 that there is no uniserial representation of the form

$$C^2_{s_i,1}\text{ — }\overline{L}(\lambda)\otimes_E v_{P_2}^\infty\text{ — }\overline{L}(\lambda)$$

which implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \begin{array}{c} C^2_{s_i,1}\\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\end{array}\right)=0 \tag{4.5.32}$$

for $i=1,2$. Hence we deduce from (4.5.30), (4.5.31), (4.5.32) and Proposition 4.2.5 that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\flat,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0. \tag{4.5.33}$$

Therefore (4.5.29) induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C^1_{s_2,1}\oplus C^1_{s_1,1}\right). \tag{4.5.34}$$

Assume first that (4.5.34) is a surjection, then we pick a representation W represented by a non-zero element in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ lying in the preimage of $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C^1_{s_2,1}\right)$ under (4.5.34). We note that there is a short exact sequence

$$\Sigma_1^\flat(\lambda,\mathscr{L}_1)\hookrightarrow \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow v_{P_2}^{\mathrm{an}}(\lambda).$$

We observe that $\overline{L}(\lambda)$ lies above neither $C^1_{s_1,1}$ nor $\overline{L}(\lambda)\otimes_E v_{P_2}^\infty$ inside W by our definition and (4.5.32), and thus W is mapped to zero under the map

$$f:\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v_{P_2}^{\mathrm{an}}(\lambda)\right)$$

which means that W comes from an element in

$$\mathrm{Ker}(f) = \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right)$$

and in particular

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) \neq 0 \tag{4.5.35}$$

The short exact sequence

$$\overline{L}(\lambda)\otimes_E v_{P_2}^\infty \hookrightarrow W_2 \twoheadrightarrow \overline{L}(\lambda)$$

induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right). \tag{4.5.36}$$

On the other hand, the short exact sequence

$$\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty \hookrightarrow \Sigma_1(\lambda,\mathscr{L}_1) \twoheadrightarrow \Sigma_1^\flat(\lambda,\mathscr{L}_1) \tag{4.5.37}$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1(\lambda,\mathscr{L}_1)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) \to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right)\end{aligned}$$

which implies

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1(\lambda,\mathscr{L}_1)\right) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) \tag{4.5.38}$$

as we have

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 0$$

from Lemma 4.4.2. We combine Lemma 4.5.8, (4.5.36) and (4.5.38) and deduce that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^\flat(\lambda,\mathscr{L}_1)\right) = 0$$

which contradicts (4.5.35). In all, we have thus shown that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) < \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C^1_{s_2,1}\oplus C^1_{s_1,1}\right) = 2 \tag{4.5.39}$$

by combining Lemma 4.4.8. Finally, the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) = 0$$

from Proposition 4.4.1 implies an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

which finishes the proof by combining Lemma 4.2.35 and (4.5.39). □

Lemma 4.5.40. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) = 2.$$

Proof. The short exact sequence

$$\Sigma_i^\flat(\lambda,\mathscr{L}_i)\hookrightarrow\Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow v_{P_{3-i}}^{\mathrm{an}}(\lambda)$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right)&\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma_i^\flat(\lambda,\mathscr{L}_i)\right)\\&\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right).\end{aligned}\tag{4.5.41}$$

It is easy to observe that

$$\dim_E\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right)=1$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ v_{P_{3-i}}^{\mathrm{an}}(\lambda)\right)=0$$

from Proposition 4.4.1 and Lemma 4.4.8. We can actually observe from Lemma 4.4.8 that the only $V\in\mathrm{JH}_{\mathrm{GL}_3(\mathbf{Q}_p)}(\Sigma_i^\flat(\lambda,\mathscr{L}_i))$ such that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ V\right)\neq0$$

is $V=C^2_{s_{3-i},1}$ and

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ C^2_{s_{3-i},1}\right)=1.$$

Hence we deduce that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma_i^\flat(\lambda,\mathscr{L}_i)\right)\leq1$$

and therefore

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^\infty,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0\tag{4.5.42}$$

for $i=1,2$. The short exact sequence

$$\overline{L}(\lambda)\otimes_E\left(v_{P_1}^\infty\oplus v_{P_2}^\infty\right)\hookrightarrow W_0\twoheadrightarrow\overline{L}(\lambda)$$

induces

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)&\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\&\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E\left(v_{P_1}^\infty\oplus v_{P_2}^\infty\right),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{aligned}$$

which implies

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\tag{4.5.43}$$

by (4.5.42). Finally, the short exact sequence (4.5.37) induces

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)&\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\&\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^\flat(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)\end{aligned}$$

which finishes the proof by

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)=1\text{ and }\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\right)=0$$

from Lemma 4.4.6, and by Lemma 4.5.28 as well as (4.5.43). □

Lemma 4.5.44. *We have the inequality*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),\lambda}\left(W_0,\ \ v^{\mathrm{an}}_{P_i}(\lambda) \text{ —— } C_{s_i,s_i}\ \right) \leq 2$$

for $i = 1, 2$.

Proof. We know that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_j},\ C^1_{s_i,1}\right) = \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_j},\ \overline{L}(\lambda)\otimes_E v^{\infty}_{P_i}\right) = 0$$

for $i, j = 1, 2$ from Proposition 4.4.1 and Lemma 4.4.8, and thus

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_j},\ v^{\mathrm{an}}_{P_i}(\lambda)\right) = 0$$

for $i, j = 1, 2$ which together with (4.5.20) imply that

$$\begin{aligned}\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ v^{\mathrm{an}}_{P_i}(\lambda)\right) &\leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_i,\ v^{\mathrm{an}}_{P_i}(\lambda)\right)\\ &\leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ v^{\mathrm{an}}_{P_i}(\lambda)\right) - \dim_E \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_i},\ v^{\mathrm{an}}_{P_i}(\lambda)\right)\\ &= 2 - 1 = 1. \quad (4.5.45)\end{aligned}$$

On the other hand, note that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ C_{s_i,s_i}\right) = \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_i},\ C_{s_i,s_i}\right) = 0$$

by Lemma 4.4.8 and thus we have

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_i,s_i}\right) \leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_{3-i}},\ C_{s_i,s_i}\right) = 1 \quad (4.5.46)$$

where the last equality follows again from Lemma 4.4.8. We finish the proof by combining (4.5.45) and (4.5.46) with the inequality

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \ v^{\mathrm{an}}_{P_i}(\lambda) \text{ —— } C_{s_i,s_i}\ \right)\\ &\qquad \leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ v^{\mathrm{an}}_{P_i}(\lambda)\right) + \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_i,s_i}\right).\end{aligned}$$

□

4.6 Key exact sequences

Lemma 4.6.1. *We have the inequality*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \leq 3.$$

Proof. The short exact sequence

$$\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2) \hookrightarrow \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2) \twoheadrightarrow C_{s_1,s_1}\oplus C_{s_2,s_2}$$

induces the exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}&\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\oplus C_{s_2,s_2}\right). \quad (4.6.2)\end{aligned}$$

We know that

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\oplus C_{s_2,s_2}\right)\\&\qquad=\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\right)+\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_2,s_2}\right)=1+1=2\end{aligned}$$

by Lemma 4.4.8 and Lemma 4.4.16. We also know that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

by Lemma 4.5.40, and thus we obtain the following inequality:

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\&\leq \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)+\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ C_{s_1,s_1}\oplus C_{s_2,s_2}\right)=2+2=4.\end{aligned}\tag{4.6.3}$$

Assume first that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=4.\tag{4.6.4}$$

The short exact sequence

$$\Sigma_1^+(\lambda,\mathscr{L}_1)\hookrightarrow \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\twoheadrightarrow \left(\ v_{P_2}^{\mathrm{an}}(\lambda)\text{ —— }C_{s_2,s_2}\ \right)$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \ v_{P_2}^{\mathrm{an}}(\lambda)\text{ —— }C_{s_2,s_2}\ \right)\end{aligned}\tag{4.6.5}$$

which implies

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\geq 2\tag{4.6.6}$$

by (4.6.4) and Lemma 4.5.44. We observe that $\Sigma_1^+(\lambda,\mathscr{L}_1)$ admits a filtration whose only reducible graded piece is

$$C^2_{s_1,1}\text{ —— }\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}$$

and thus it follows from Lemma 4.4.8 and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ C^2_{s_1,1}-\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)=0$$

(coming from Proposition 4.4.1, Lemma 4.4.8 together with a simple devissage) that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ V\right)=0$$

for all graded pieces of such a filtration except the subrepresentation $\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}$. Hence we deduce by part (ii) of Proposition 4.2.5 an isomorphism of one dimensional spaces

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}\right)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right).\tag{4.6.7}$$

Then the short exact sequence

$$\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\hookrightarrow W_0\twoheadrightarrow W_2$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\\ \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty},\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\end{aligned}$$

which together with (4.6.6) and (4.6.7) implies that

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^+(\lambda,\mathscr{L}_1)\right)\geq 1$$

which contradicts Lemma 4.5.8. Hence we finish the proof. □

Proposition 4.6.8. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=3.$$

Proof. The short exact sequence

$$\overline{L}(\lambda)\otimes_E\left(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty}\right)\hookrightarrow W_0\twoheadrightarrow\overline{L}(\lambda)$$

induces a long exact sequence

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\&\quad\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E\left(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty}\right),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{aligned}\tag{4.6.9}$$

and thus we have

$$\begin{aligned}&\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(W_0,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\&\geq\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))+\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda)\otimes_E\left(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty}\right),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\&\qquad-\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2))=1+4-2=3\end{aligned}\tag{4.6.10}$$

due to Lemma 4.5.7 and Lemma 4.5.13, which finishes the proof by combining with Lemma 4.6.1. □

We define $\Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ as the unique non-split extension of $\overline{L}(\lambda)$ by $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ (c.f. Lemma 4.2.35) and then set $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ to be the amalgamate sum of $\Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ and $\Sigma^{+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ over $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$. Hence $\Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ has the form

$$\begin{array}{ccccc}&&v_{P_1}^{\mathrm{an}}(\lambda)&&\\&\nearrow&&\searrow&\\\mathrm{St}_3^{\mathrm{an}}(\lambda)&&&&\overline{L}(\lambda)\\&\searrow&&\nearrow&\\&&v_{P_2}^{\mathrm{an}}(\lambda)&&\end{array}$$

and $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ has the form

$$\begin{array}{ccccc}&&&&C_{s_1,s_1}\\&&v_{P_1}^{\mathrm{an}}(\lambda)&\nearrow&\\&\nearrow&&\searrow&\overline{L}(\lambda)\\\mathrm{St}_3^{\mathrm{an}}(\lambda)&&&\nearrow&\\&\searrow&v_{P_2}^{\mathrm{an}}(\lambda)&&\\&&&\searrow&C_{s_2,s_2}\end{array}\ .$$

It follows from Lemma 4.2.35, Proposition 4.4.1, (4.5.17) and an easy devissage that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0.\tag{4.6.11}$$

Then we set

$$\Sigma^{*,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2):=\Sigma^{*}(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty}$$

for $*=\{+\},\{\sharp\}$ and $\{\sharp,+\}$. It follows from Lemma 4.5.28, (4.5.17) and an easy devissage that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0.\tag{4.6.12}$$

Lemma 4.6.13. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0$$

and

$$\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=\dim_E\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2$$

Proof. It follows from (4.5.17) that we only need to show that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0 \text{ and } \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2.$$

These results follow from combining the long exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\right)&\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\right)\\ &\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ &\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\right),\end{aligned}$$

with Lemma 4.2.35 and the equalities

$$\begin{array}{rrcl}\dim_E & \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)) & = & 1\\ & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)) & = & 0\\ & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)) & = & 0\end{array}$$

due to Proposition 4.4.1. □

Lemma 4.6.14. *We have*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))=0$$

and

$$\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))=\dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\geq 1.$$

Proof. It follows from (4.5.17) that we only need to show the equalities

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))=0 \text{ and } \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\geq 1,$$

which follow from combining (4.6.12), Lemma 4.6.13 and the long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty)&\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2))\\ &\to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty)\\ &\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^\sharp(\lambda,\mathscr{L}_1,\mathscr{L}_2))\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2))\end{aligned}\tag{4.6.15}$$

with the equalities

$$\begin{array}{rrcl} & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty) & = & 0\\ \dim_E & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty) & = & 1\end{array}$$

due to Proposition 4.4.1. □

We use the shorten notation $\underline{\mathscr{L}}:=(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_1',\mathscr{L}_2')$ for a tuple of four elements in E. We recall from Proposition 4.5.25 an isomorphism of two dimensional spaces

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\xrightarrow{\sim}\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right).\tag{4.6.16}$$

We emphasize that the isomorphism (4.6.16) can be naturally interpreted as the cup product map

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^\infty,\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\ \cup\ &\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^\infty\right)\\ &\to \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{aligned}\tag{4.6.17}$$

where $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right)$ is one dimensional by Proposition 4.4.1. We recall from the proof of Lemma 4.5.13 that there is a canonical isomorphism

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\xrightarrow{\sim}\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

which together with Lemma 4.2.35 implies that $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ admits a basis of the form

$$\{\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p}),\iota_1(D_0)\},$$

and therefore the element

$$\iota_1(D_0)+\mathscr{L}\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$$

generates a line in $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ for each $\mathscr{L}\in E$. We define $\Sigma_i^+(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_i')$ as the representation represent by the preimage of

$$\iota_1(D_0)+\mathscr{L}_i'\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$$

in

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty},\ \Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

via (4.6.16) for $i=1,2$. Then we define $\Sigma^+(\lambda,\underline{\mathscr{L}})$ as the amalgamate sum of $\Sigma_1^+(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_1')$ and $\Sigma_2^+(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_2')$ over $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$, and therefore $\Sigma^+(\lambda,\underline{\mathscr{L}})$ has the form

$$\begin{array}{ccccc} & & v_{P_1}^{\mathrm{an}}(\lambda) & \text{---} & C_{s_1,s_1}\text{ — }\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\\ \mathrm{St}_3^{\mathrm{an}}(\lambda) & \diagup & & & \\ & \diagdown & & & \\ & & v_{P_2}^{\mathrm{an}}(\lambda) & \text{---} & C_{s_2,s_2}\text{ — }\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\end{array}.$$

We define $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$ as the amalgamate sum of $\Sigma^+(\lambda,\underline{\mathscr{L}})$ and $\Sigma^{\sharp}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ over $\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)$, and thus $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$ has the form

$$\begin{array}{ccccc} & & & & C_{s_1,s_1}\text{ — }\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\\ & & v_{P_1}^{\mathrm{an}}(\lambda) & \diagup & \\ \mathrm{St}_3^{\mathrm{an}}(\lambda) & \diagup & & \diagdown & \overline{L}(\lambda)\\ & \diagdown & & \diagup & \\ & & v_{P_2}^{\mathrm{an}}(\lambda) & \diagdown & \\ & & & & C_{s_2,s_2}\text{ — }\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\end{array}.$$

We also need the quotients

$$\Sigma^{+,\flat}(\lambda,\underline{\mathscr{L}}):=\Sigma^+(\lambda,\underline{\mathscr{L}})/\overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty},\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}}):=\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})/\overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty}.$$

Lemma 4.6.18. *We have the inequality*

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right)\leq 1.$$

Proof. The short exact sequence

$$\Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\twoheadrightarrow\overline{L}(\lambda)\otimes_E\left(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty}\right)$$

induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\left(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty}\right)\right)\tag{4.6.19}$$

by Lemma 4.6.14. Note that we have

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\left(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty}\right)\right)=2$$

by Proposition 4.4.1. Assume first that (4.6.19) is a surjection, and thus we can pick a representation W represented by a non-zero element lying in the preimage of $\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}$ under (4.6.19). We observe that the very existence of W implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\neq 0. \tag{4.6.20}$$

We define

$$\Sigma_i^{+,\flat}(\lambda,\mathscr{L}_i):=\Sigma_i^{+}(\lambda,\mathscr{L}_i)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty}$$

and thus we have an embedding

$$\Sigma_i^{+,\flat}(\lambda,\mathscr{L}_i)\hookrightarrow \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$$

for each $i=1,2$. We notice that the quotient $\Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)$ fits into a short exact sequence

$$\left(\ v_{P_2}^{\mathrm{an}}(\lambda)\text{ —— }\overline{L}(\lambda)\ \right)\hookrightarrow \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\twoheadrightarrow C_{s_2,s_2}.$$

Hence it remains to show the equality

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \ v_{P_2}^{\mathrm{an}}(\lambda)\text{ —— }\overline{L}(\lambda)\ \right)=0 \tag{4.6.21}$$

and the equality

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ C_{s_2,s_2}\right)=0 \tag{4.6.22}$$

to finish the proof of

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)/\Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\right)=0. \tag{4.6.23}$$

The vanishing (4.6.22) follows from Lemma 4.4.8 and part (i) of Proposition 4.2.5. It follows from Proposition 4.4.1, Lemma 4.4.8 and a simple devissage that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty},\ C^1_{s_1,1}\right)=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ C^1_{s_1,1}\text{ —— }\overline{L}(\lambda)\ \right)=0. \tag{4.6.24}$$

Hence if

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \ C^1_{s_1,1}\text{ —— }\overline{L}(\lambda)\ \right)\neq 0$$

then there exists a uniserial representation of the form

$$C^1_{s_1,1}\text{ —— }\overline{L}(\lambda)\text{ — }\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}$$

which contradicts (4.6.24) and Lemma 4.4.27. As a result, we have shown that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \ C^1_{s_1,1}\text{ —— }\overline{L}(\lambda)\ \right)=0$$

which together with Proposition 4.4.1 and part (i) of Proposition 4.2.5 implies (4.6.21) and hence (4.6.23) as well concerning (4.6.22). Therefore we can combine (4.6.23) with Lemma 4.5.8 and conclude that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=0$$

which contradicts (4.6.20). Consequently, the injection (4.6.19) must be strict and we finish the proof. □

According to Lemma 4.6.14, the short exact sequence

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\twoheadrightarrow\overline{L}(\lambda)\otimes_E(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty})$$

induces a long exact sequence:

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right)\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty})\right)\\ \xrightarrow{f}\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{aligned}\tag{4.6.25}$$

Proposition 4.6.26. *We have*

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}}))=1$$

and the image of f is not contained in the image of the natural injection

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty}\right)\hookrightarrow\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right).$$

Proof. We use the shorten notation for the two dimensional space

$$M:=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E(v_{P_2}^{\infty}\oplus v_{P_1}^{\infty})\right).$$

We actually have the following commutative diagram

$$\begin{array}{ccccc}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right) & \xrightarrow{i} & M & \xrightarrow{f} & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\ \downarrow h & & \| & & \downarrow k\\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right) & \xrightarrow{j} & M & \xrightarrow{g} & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\end{array}\tag{4.6.27}$$

where the middle vertical map is just an equality. We know that h is injective by the vanishing

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty}\right)=0$$

and k has a one dimensional image by (4.6.15). Both i and j are injective due to (4.6.11) and (4.6.12). Therefore by a simple diagram chasing we have

$$\begin{aligned}\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right)\\ =\dim_E M-\dim_E\mathrm{Im}(g)\geq\dim_E M-\dim_E\mathrm{Im}(k)=2-1=1\end{aligned}$$

by Lemma 4.6.14 and therefore

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+,\flat}(\lambda,\underline{\mathscr{L}})\right)=1$$

by Lemma 4.6.18. Moreover, the map g has a one dimensional image and hence $k\circ f$ has one dimensional image, meaning that the image of f has dimension one or two and is not contained in $\mathrm{Ker}(k)$, which is exactly the image of

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E\mathrm{St}_3^{\infty}\right)\to\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\tag{4.6.28}$$

by (4.6.15). In fact, the restriction of f to the direct summand $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right)$ is given by the cup product map with a non-zero element in the line of

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty},\ \Sigma^{+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

given by the preimage of

$$E\left(\iota_1(D_0)+\mathscr{L}_i'\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})\right)$$

via (4.6.16) by our definition of $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$ and it is obvious that $\iota_1(D_0)+\mathscr{L}_i'\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$ does not lie in the image of (4.6.28) which is exactly the line $E\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$. □

Proposition 4.6.29. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right)=1$$

if and only if $\mathscr{L}_1'=\mathscr{L}_2'=\mathscr{L}_3$ *for a certain* $\mathscr{L}_3\in E$.

Proof. It follows from (4.6.25) that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})\right)=1$$

if and only if the image of f is one dimensional. Then we notice by the interpretation of f as cup product in Proposition 4.6.26 that the image of

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}\right)$$

under f is the line of

$$\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

generated by

$$\iota_1(D_0)+\mathscr{L}_i'\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p})$$

for each $i=1,2$. Therefore the image of f is one dimensional if and only if the two lines for $i=1,2$ coincide which means that

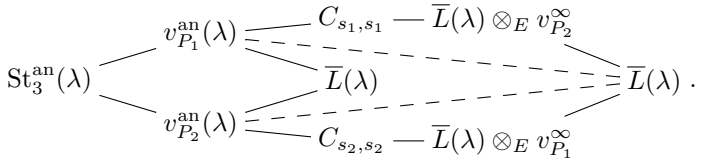
$$\mathscr{L}_1'=\mathscr{L}_2'=\mathscr{L}_3$$

for a certain $\mathscr{L}_3\in E$. □

We use the notation $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ for the representation $\Sigma^{\sharp,+}(\lambda,\underline{\mathscr{L}})$ when

$$\underline{\mathscr{L}}=(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_3).$$

We define $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ as the unique representation (up to isomorphism) given by a non-zero element in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)$ according to Proposition 4.6.29. Therefore by our definition $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ has the following form



(4.6.30)

It follows from Proposition 4.4.1, Proposition 4.6.29, the definition of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and an easy devissage that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)=0. \tag{4.6.31}$$

Remark 4.6.32. *The definition of the invariant* $\mathscr{L}_3\in E$ *of* $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ *obviously relies on the choice of a special p-adic dilogarithm function* D_0 *which is non-canonical. This is similar to the definition of the invariants* $\mathscr{L}_1,\mathscr{L}_2\in E$ *which relies on the choice of a special p-adic logarithm function* $\log_0$.

Lemma 4.6.33. *We have*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)=2.$$

Moreover, if V *is a locally analytic representation determined by a line*

$$M_V\subsetneq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

satisfying

$$M_V \neq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right),$$

then there exists a unique $\mathscr{L}_3 \in E$ *such that*

$$V \cong \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3).$$

Proof. The short exact sequence

$$\overline{L}(\lambda)\otimes_E\left(v_{P_1}^\infty \oplus v_{P_2}^\infty\right) \hookrightarrow W_0 \twoheadrightarrow \overline{L}(\lambda)$$

together with Lemma 4.6.13 induce a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ V^+\right) & \xrightarrow{g_1} & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^+\right) & \xrightarrow{k_1} & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^+\right) \\
h_1\downarrow & & h_2\uparrow & & h_3\downarrow \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ V^{\sharp,+}\right) & \xrightarrow{g_2} & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^{\sharp,+}\right) & \xrightarrow{k_2} & \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{\sharp,+}\right)
\end{array} \tag{4.6.34}$$

where we use shorten notation V_i^{alg} for $\overline{L}(\lambda)\otimes_E v_{P_i}^\infty$, V^+ for $\Sigma^+(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ and $V^{\sharp,+}$ for $\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)$ to save space. We observe that g_2 is an injection due to Lemma 4.6.13, k_1 is a surjection by the proof of Proposition 4.6.8, h_3 is an isomorphism by Proposition 4.4.1 and an easy devissage and finally h_2 is an injection. Assume that h_2 is not surjective, then any representation given by a non-zero element in $\mathrm{Coker}(h_2)$ admits a quotient of the form

$$C^1_{s_i,1} \text{ —— } \overline{L}(\lambda) \text{ —— } V_i^{\mathrm{alg}} \tag{4.6.35}$$

for $i=1$ or 2 due to Lemma 4.4.8. However, it follows from Lemma 4.4.27 that there is no uniserial representation of the form (4.6.35), which implies that h_2 is indeed an isomorphism, and hence k_2 is surjective by a diagram chasing. Therefore we conclude that

$$\begin{aligned}
&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ V^{\sharp,+}\right) \\
&\quad= \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^{\sharp,+}\right) - \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{\sharp,+}\right) \\
&\quad= \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_1^{\mathrm{alg}}\oplus V_2^{\mathrm{alg}},\ V^{+}\right) - \dim_E \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{+}\right) = 4-2=2.
\end{aligned}$$

The final claim on the existence of a unique $\mathscr{L}_3$ follows from Proposition 4.6.29, our definition of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ and the observation that the restriction of k_2 to the direct summand

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_i^{\mathrm{alg}},\ V^{\sharp,+}\right)$$

induces isomorphisms

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(V_i^{\mathrm{alg}},\ V^{\sharp,+}\right) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V^{\sharp,+}\right)$$

which can be interpreted as the cup product morphism with the one dimensional space

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V_i^{\mathrm{alg}}\right)$$

for $i=1,2$. □

We define $\Sigma_i^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ as the subrepresentation of $\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ that fits into the short exact sequence

$$\Sigma_i^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\twoheadrightarrow\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}$$

for each $i=1,2$. We use the notation $\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'$ for the object in the derived category $\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$ associated with the complex

$$\left[W'_{3-i}\longrightarrow\Sigma_i^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\right].$$

Proposition 4.6.36. *The object*

$$\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\in\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

fits into the distinguished triangle

$$\overline{L}(\lambda)'\longrightarrow\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1]\xrightarrow{+1}\tag{4.6.37}$$

for each $i=1,2$. Moreover, the element in

$$\begin{aligned}&\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\&\quad\xrightarrow{\sim}\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)\\&\quad\cong\mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)}\left(\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-2],\ \overline{L}(\lambda)'\right)\end{aligned}\tag{4.6.38}$$

associated with the distinguished triangle (4.6.37) is

$$\iota_1(D_0)+\mathscr{L}_3\kappa(b_{1,\mathrm{val}_p}\wedge b_{2,\mathrm{val}_p}).\tag{4.6.39}$$

Proof. It follows from Proposition 3.2 of [Schr11] that there is a unique (up to isomorphism) object

$$\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\in\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

that fits into a distinguished triangle

$$\overline{L}(\lambda)'\longrightarrow\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\longrightarrow\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)[-1]\xrightarrow{+1}\tag{4.6.40}$$

such that the element in $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$ associated with (4.6.40) via (4.6.38) is (4.6.39). It follows from **TR2** (c.f. Section 10.2.1 of [Wei94]) that

$$\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1]\longrightarrow\overline{L}(\lambda)'[1]\xrightarrow{+1}\tag{4.6.41}$$

is another distinguished triangle. The isomorphism (4.6.16) can be reinterpreted as the isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)}\left(\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1],\ \left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty}\right)'\right)\\&\quad\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)}\left(\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1],\ \overline{L}(\lambda)'[1]\right)\end{aligned}\tag{4.6.42}$$

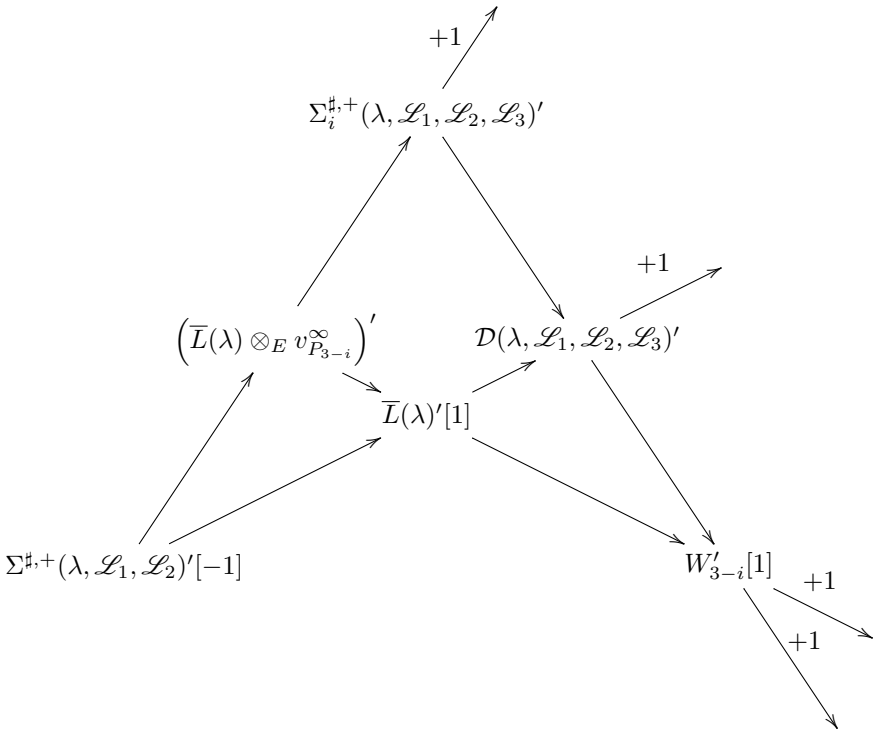
induced by the composition with $\mathrm{Hom}_{\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)}\left(\left(\overline{L}(\lambda)\otimes_E v_{P_{3-i}}^{\infty}\right)',\ \overline{L}(\lambda)'[1]\right)$. As a result, each morphism

$$\Sigma^{\natural,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1]\to\overline{L}(\lambda)'[1]$$

uniquely factors through a composition

$$\Sigma^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)'[-1]\to\left(\overline{L}(\lambda)\otimes_E v^{\infty}_{P_{3-i}}\right)'\to\overline{L}(\lambda)'[1]$$

which induces a commutative diagram with four distinguished triangles



(4.6.43)

by **TR4**. Hence we deduce that

$$\Sigma_i^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow W'_{3-i}[1]\xrightarrow{+1}$$

or equivalently

$$W'_{3-i}\longrightarrow\Sigma_i^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\xrightarrow{+1}$$

is a distinguished triangle. On the other hand, it is easy to see that $\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'$ fits into the distinguished triangle

$$W'_{3-i}\longrightarrow\Sigma_i^{\sharp,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\longrightarrow\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\xrightarrow{+1}$$

and thus we conclude that

$$\mathcal{D}_i(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\cong\mathcal{D}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)'\in\mathcal{D}^b\left(\mathrm{Mod}_{D(\mathrm{GL}_3(\mathbf{Q}_p),E)}\right)$$

by the uniqueness in Proposition 3.2 of [Schr11]. Hence we finish the proof. □

We define $\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ as the unique subrepresentation of $\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ of the form

$$\mathrm{St}_3^{\mathrm{an}}(\lambda)\begin{array}{l}\diagup\ v_{P_1}^{\mathrm{an}}(\lambda)\text{ —— }C_{s_1,s_1}\text{ — }\overline{L}(\lambda)\otimes_E v^{\infty}_{P_2}\\ \diagdown\ v_{P_2}^{\mathrm{an}}(\lambda)\text{ —— }C_{s_2,s_2}\text{ — }\overline{L}(\lambda)\otimes_E v^{\infty}_{P_1}\end{array}$$

that fits into the short exact sequence

$$\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow\Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\twoheadrightarrow\overline{L}(\lambda)^{\oplus 2}\tag{4.6.44}$$

and $\Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ as the unique subrepresentation of $\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ of the form

$$\mathrm{St}_3^{\mathrm{an}}(\lambda)\begin{array}{l}\diagup\ \overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\ \text{——}\ C_{s_1,s_1}\\ \diagdown\ \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\ \text{——}\ C_{s_2,s_2}\end{array}$$

that fits into the short exact sequence

$$\Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\twoheadrightarrow\left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)\oplus\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right)\oplus C^1_{s_2,1}\oplus C^1_{s_1,1}.\tag{4.6.45}$$

The short exact sequence (4.6.44) induces a long exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)^{\oplus 2}\right)&\hookrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)\\&\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)\to\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda)^{\oplus 2}\right)\end{aligned}$$

which easily implies that

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)=2$$

by Proposition 4.4.1 and (4.6.31). On the other hand, we notice that $\Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ admits a filtration whose only reducible graded piece is

$$C^1_{s_i,1}\text{——}\overline{L}(\lambda)\otimes_E v_{P_i}^{\infty}$$

and

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V\right)=0$$

for all graded pieces V of such a filtration by Lemma 4.4.8 and Lemma 4.4.27, which implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,--}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)=0.$$

Therefore (4.6.45) induces an injection of a two dimensional space into a four dimensional space

$$\begin{aligned}M^{\min}:=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}&\left(\overline{L}(\lambda),\ \Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\right)\\\hookrightarrow M^+:=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}&\left(\overline{L}(\lambda),\ \ \left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)\oplus\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right)\oplus C^1_{s_2,1}\oplus C^1_{s_1,1}\right).\end{aligned}\tag{4.6.46}$$

It follows from the definition of $\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ that we have embeddings

$$\Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{+}(\lambda,\mathscr{L}_1,\mathscr{L}_2)\hookrightarrow\Sigma^{\min,-}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

which allow us to identify

$$M^-:=\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma(\lambda,\mathscr{L}_1,\mathscr{L}_2)\right)$$

with a line in $M^{\min}$. We use the number $1,2,3,4$ to index the four representations $\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}$, $\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}$, $C^1_{s_2,1}$ and $C^1_{s_1,1}$ respectively, and we use the notation M_I for each subset $I\subseteq\{1,2,3,4\}$ to denote the corresponding subspace of M^+ with dimension the cardinality of I. For example, $M_{\{1,2\}}$ denotes the two dimensional subspace

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \left(\overline{L}(\lambda)\otimes_E v_{P_1}^{\infty}\right)\oplus\left(\overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\right)\right)$$

of M^+.

Lemma 4.6.47. *We have the following characterizations of $M^{\min}$ inside M^+:*

$$M^{\min} \cap M_{\{i,j\}} = 0 \text{ for } \{i,j\} \neq \{3,4\},$$

$$M^{\min} \cap M_{\{1,3,4\}} = M^{\min} \cap M_{\{2,3,4\}} = M^{\min} \cap M_{\{3,4\}} = M^-,$$

and

$$M^{\min} = (M^{\min} \cap M_{\{1,2,3\}}) \oplus (M^{\min} \cap M_{\{1,2,4\}}).$$

Proof. As $C^1_{s_1,1}$ and $C^1_{s_2,1}$ are in the cosocle of $\Sigma(\lambda, \mathscr{L}_1, \mathscr{L}_2)$, it is immediate that

$$M^- \subseteq M_{\{3,4\}}.$$

It follows from (4.6.30) that

$$M^{\min} \not\subseteq M_{\{3,4\}}$$

and thus $M^{\min} \cap M_{\{3,4\}}$ is one dimensional which must coincide with M^-. The proof of Lemma 4.6.1 implies that $M \not\subseteq M_{\{i,3,4\}}$ for $i = 1, 2$ and therefore $M \cap M_{\{i,3,4\}}$ is one dimensional, which implies that

$$M^{\min} \cap M_{\{i,3,4\}} = M^-$$

by the inclusion

$$M^{\min} \cap M_{\{3,4\}} \subseteq M^{\min} \cap M_{\{i,3,4\}}$$

for $i = 1, 2$. We observe (c.f. Lemma 4.5.8) that

$$M^- \cap M_{\{3\}} = M^- \cap M_{\{4\}} = 0$$

and thus

$$M^{\min} \cap M_{\{i,j\}} = M^- \cap M_{\{i,j\}} = 0$$

for each $\{i,j\} \neq \{3,4\}, \{1,2\}$. We define $\Sigma^{\min,-,\prime}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ as the unique subrepresentation of $\Sigma^{\min,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ that fits into the short exact sequence

$$\Sigma^{\min,-,\prime}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \hookrightarrow \Sigma^{\min,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \twoheadrightarrow C^1_{s_1,1} \oplus C^1_{s_2,1} \oplus C_{s_1 s_2 s_1, 1}$$

and then define

$$\Sigma^{\min,-,\prime,\flat}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) := \Sigma^{\min,-,\prime}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty.$$

It is obvious that $M^{\min} \cap M_{\{1,2\}} \neq 0$ if and only if

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-,\prime}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\right) \neq 0$$

which implies that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma^{\min,-,\prime,\flat}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\right) \neq 0 \tag{4.6.48}$$

as

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty\right) = 0$$

due to Proposition 4.4.1. We notice that we have a direct sum decomposition

$$\Sigma^{\min,-,\prime,\flat}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = V_1 \oplus V_2$$

where V_i is a representation of the form

$$\begin{array}{ccccccc} & & C^1_{s_{3-i}s_i,1} & & C^2_{s_{3-i}s_i,1} & & \\ C^2_{s_i,1} & \nearrow & & \searrow & & \searrow & \overline{L}(\lambda) \otimes_E v^\infty_{P_{3-i}} \\ & \searrow & & \nearrow & C_{s_i,s_i} & \nearrow & \\ & & \overline{L}(\lambda) \otimes_E v^\infty_{P_i} & & & & \end{array}.$$

Switching V_1 and V_2 if necessary, we can assume by (4.6.48) that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V_1\right) \neq 0.$$

On the other hand, we have an embedding

$$V_1 \hookrightarrow\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{---} \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}$$

which induces an embedding

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ V_1\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{---} \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\ \right)$$

and in particular

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{---} \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\ \right) \neq 0. \tag{4.6.49}$$

The short exact sequences

$$\overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty} \hookrightarrow \Sigma_1(\lambda,\mathscr{L}_1) \twoheadrightarrow \Sigma_1^{\flat}(\lambda,\mathscr{L}_1),\ \overline{L}(\lambda)\otimes_E \mathrm{St}_3^{\infty} \hookrightarrow \Sigma_1^{+}(\lambda,\mathscr{L}_1) \twoheadrightarrow \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)$$

induce isomorphisms

$$\begin{aligned} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1(\lambda,\mathscr{L}_1)\right) \ &\xrightarrow{\sim}\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{\flat}(\lambda,\mathscr{L}_1)\right) \\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{+}(\lambda,\mathscr{L}_1)\right) \ &\xrightarrow{\sim}\ \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\right) \end{aligned} \tag{4.6.50}$$

by Lemma 4.4.2. Hence we deduce that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^{\flat}(\lambda,\mathscr{L}_1)\right) = \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\right) = 0 \tag{4.6.51}$$

from Lemma 4.5.8 and (4.6.50). The surjection $W_2 \twoheadrightarrow \overline{L}(\lambda)$ induces an embedding

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{\flat}(\lambda,\mathscr{L}_1)\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_2,\ \Sigma_1^{\flat}(\lambda,\mathscr{L}_1)\right)$$

which together with (4.6.51) imply that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{\flat}(\lambda,\mathscr{L}_1)\right) = 0$$

and hence

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1)\right) = 0 \tag{4.6.52}$$

by (4.5.17) and an easy devissage. It follows from (4.6.51) and (4.6.52) that

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(\overline{L}(\lambda),\ \ \Sigma_1^{+,\flat}(\lambda,\mathscr{L}_1) \text{---} \overline{L}(\lambda)\otimes_E v_{P_2}^{\infty}\ \right) = 0$$

which contradicts (4.6.49). A a result, we have shown that

$$M^{\min}\cap M_{\{1,2\}} = 0.$$

As $M^- \not\subseteq M_{\{1,2,i\}}$ for $i = 3, 4$, we deduce that both $M^{\min}\cap M_{\{1,2,3\}}$ and $M^{\min}\cap M_{\{1,2,4\}}$ are one dimensional. On the other hand, since we know that

$$(M^{\min}\cap M_{\{1,2,3\}})\cap(M^{\min}\cap M_{\{1,2,4\}}) = M^{\min}\cap M_{\{1,2\}} = 0,$$

we deduce the following direct sum decomposition

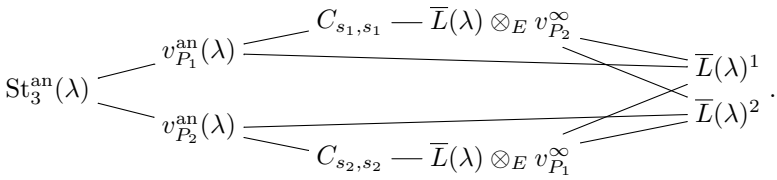
$$M^{\min} = (M^{\min}\cap M_{\{1,2,3\}})\oplus(M^{\min}\cap M_{\{1,2,4\}}).$$

□

We use the notation $\overline{L}(\lambda)^i$ for copy of $\overline{L}(\lambda)$ inside $\overline{L}(\lambda)^{\oplus 2}$ corresponding to the one dimensional space $M^{\min} \cap M_{\{1,2,i+2\}}$ inside $M^{\min}$, and therefore we have a surjection

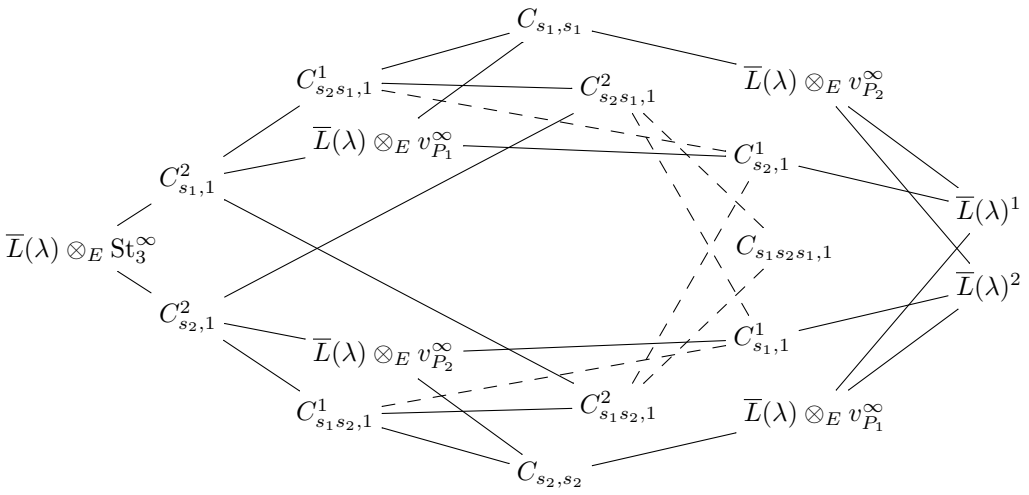
$$\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \twoheadrightarrow \left(C^1_{s_2,1} \text{ —— } \overline{L}(\lambda)^1 \right) \oplus \left(C^1_{s_1,1} \text{ —— } \overline{L}(\lambda)^2 \right). \tag{4.6.53}$$

As a result, the representation $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ has the following form:



. (4.6.54)

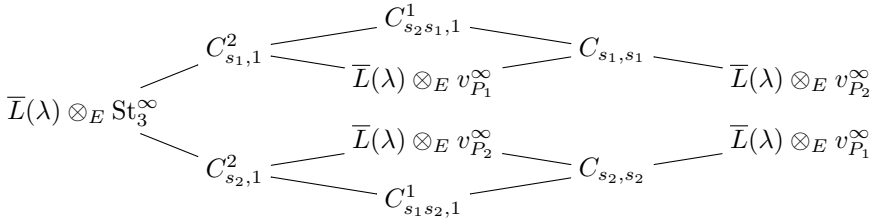
If we clarify the internal structure of $\mathrm{St}^{\mathrm{an}}_3(\lambda)$, $v^{\mathrm{an}}_{P_1}(\lambda)$ and $v^{\mathrm{an}}_{P_2}(\lambda)$ using Lemma 4.2.13, then $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ has the following form:



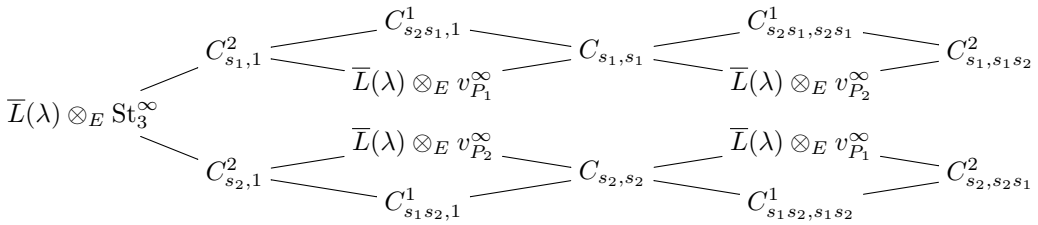
. (4.6.55)

Remark 4.6.56. *It is actually possible to show that all the possibly split extensions illustrated in (4.6.55) are non-split. However, the proof is quite technical and not related to the p-adic dilogarithm function, and thus we decided not to include the proof here.*

We observe that $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ admits a unique subrepresentation $\Sigma^{\mathrm{Ext}^1,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form



which can be uniquely extend to a representation $\Sigma^{\mathrm{Ext}^1}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form:



(4.6.57)

according to Section 4.4 and 4.6 of [Bre17] together with our Lemma 4.4.34. Finally, we define $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ as the amalgamate sum of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ and $\Sigma^{\mathrm{Ext}^1}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ over $\Sigma^{\mathrm{Ext}^1,-}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$.

Remark 4.6.58. *It is actually possible to prove (by several technical computations of* Ext*-groups) that the quotient*

$$\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty$$

and the quotient

$$\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\overline{L}(\lambda)\otimes_E \mathrm{St}_3^\infty$$

are independent of the choices of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$.

4.7 Local-global compatibility

We are going to borrow most of the notation and assumptions from Section 6 of [Bre17]. We fix embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, an imaginary quadratic CM extension F of $\mathbf{Q}$ and a unitary group $G/\mathbf{Q}$ attached to the extension $F/\mathbf{Q}$ such that $G\times_{\mathbf{Q}} F \cong \mathrm{GL}_3$ and $G(\mathbf{R})$ is compact. If ℓ is a finite place of $\mathbf{Q}$ which splits completely in F, we have isomorphisms $\iota_{G,w} : G(\mathbf{Q}_\ell) \xrightarrow{\sim} G(F_w) \cong \mathrm{GL}_3(F_w)$ for each finite place w of F over ℓ. We assume that p splits completely in F, and we fix a finite place w_0 of F dividing p and therefore $G(\mathbf{Q}_p) \cong G(F_{w_0}) \cong \mathrm{GL}_3(\mathbf{Q}_p)$.

We fix an open compact subgroup $U^p \subsetneq G(\mathbb{A}_{\mathbf{Q}}^{\infty,p})$ of the form $U^p = \prod_{\ell\neq p} U_\ell$ where U_ℓ is an open compact subgroup of $G(\mathbf{Q}_\ell)$. For each finite extension E of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}_p}$, we consider the following $\mathcal{O}_E$-lattice inside a p-adic Banach space:

$$\widehat{S}(U^p, \mathcal{O}_E) := \{f : G(\mathbf{Q})\backslash G(\mathbb{A}_{\mathbf{Q}}^\infty)/U^p \to \mathcal{O}_E,\ f \text{ continuous}\} \tag{4.7.1}$$

and note that $\widehat{S}(U^p, E) := \widehat{S}(U^p, \mathcal{O}_E)\otimes_{\mathcal{O}_E} E$. The right translation of $G(\mathbf{Q}_p)$ on $G(\mathbf{Q})\backslash G(\mathbb{A}_{\mathbf{Q}}^\infty)/U^p$ induces a p-adic continuous action of $G(\mathbf{Q}_p)$ on $\widehat{S}(U^p, \mathcal{O}_E)$ which makes $\widehat{S}(U^p, E)$ an admissible Banach representation of $G(\mathbf{Q}_p)$ in the sense of [ST02]. We use the notation $\widehat{S}(U^p, E)^{\mathrm{alg}} \subseteq \widehat{S}(U^p, E)^{\mathrm{an}}$ following Section 6 of [Bre17] for the subspaces of locally $\mathbf{Q}_p$-algebraic vectors and locally $\mathbf{Q}_p$-analytic vectors inside $\widehat{S}(U^p, E)$ respectively. Moreover, we have the following decomposition:

$$\widehat{S}(U^p, E)^{\mathrm{alg}}\otimes_E \overline{\mathbf{Q}_p} \cong \bigoplus_\pi (\pi_f^{v_0})^{U_p}\otimes_{\overline{\mathbf{Q}}} (\pi_{v_0}\otimes_{\overline{\mathbf{Q}}} W_p) \tag{4.7.2}$$

where the direct sum is over the automorphic representations π of $G(\mathbb{A}_{\mathbf{Q}})$ over $\mathbf{C}$ and W_p is the $\mathbf{Q}_p$-algebraic representation of $G(\mathbf{Q}_p)$ over $\overline{\mathbf{Q}_p}$ associated with the algebraic representation π_∞ of $G(\mathbf{R})$ over $\mathbf{C}$ via ι_p and ι_∞. In particular, each distinct π appears with multiplicity one (c.f. the paragraph after (55) of [Bre17] for further references).

We use the notation $D(U^p)$ for the set of finite places ℓ of $\mathbf{Q}$ that are different from p, split completely in F and such that U_ℓ is a maximal open compact subgroup of $G(\mathbf{Q}_\ell)$. Then we consider

the commutative polynomial algebra $\mathbb{T}(U^p) := E[T_w^{(j)}]$ generated by the variables $T_w^{(j)}$ indexed by $j \in \{1, \cdots, n\}$ and w a finite place of F over a place ℓ of $\mathbf{Q}$ such that $\ell \in D(U^p)$. The algebra $\mathbb{T}(U^p)$ acts on $\widehat{S}(U^p, E)$, $\widehat{S}(U^p, E)^{\mathrm{alg}}$ and $\widehat{S}(U^p, E)^{\mathrm{an}}$ via the usual double coset operators. The action of $\mathbb{T}(U^p)$ commutes with that of $G(\mathbf{Q}_p)$.

We fix now $\alpha \in E^\times$, hence a Deligne–Fontaine module $\underline{D}$ over $\mathbf{Q}_p = F_{w_0}$ of rank three of the form

$$\underline{D} = Ee_2 \oplus Ee_1 \oplus Ee_0, \text{ with } \begin{cases} \varphi(e_2) = \alpha e_2 \\ \varphi(e_1) = p^{-1}\alpha e_1 \\ \varphi(e_0) = p^{-2}\alpha e_0 \end{cases} \text{ and } \begin{cases} N(e_2) = e_1 \\ N(e_1) = e_0 \\ N(e_0) = 0 \end{cases} \tag{4.7.3}$$

and finally a tuple of Hodge–Tate weights $\underline{k} = (k_1 > k_2 > k_3)$. If $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_3(E)$ is an absolute irreducible continuous representation which is unramified at each finite place w lying over a finite place $\ell \in D(U^p)$, we can associate to ρ a maximal ideal $\mathfrak{m}_\rho \subseteq \mathbb{T}(U^p)$ with residual field E by the usual method described in the middle paragraph on Page 58 of [Bre17]. We use the notation $\star_{\mathfrak{m}_\rho}$ for spaces of localization and $\star[\mathfrak{m}_\rho]$ for torsion subspaces where $\star \in \{\widehat{S}(U^p, E), \widehat{S}(U^p, E)^{\mathrm{alg}}, \widehat{S}(U^p, E)^{\mathrm{an}}\}$.

We assume that there exists U^p and ρ such that

(i) ρ is absolutely irreducible and unramified at each finite place w of F over a place ℓ of $\mathbf{Q}$ satisfying $\ell \in D(U^p)$;

(ii) $\widehat{S}(U^p, E)^{\mathrm{alg}}[\mathfrak{m}_\rho] \neq 0$ (hence ρ is automorphic and $\rho_{w_0} := \rho|_{\mathrm{Gal}(\overline{F_{w_0}}/F_{w_0})}$ is potentially semi-stable);

(iii) ρ_{w_0} has Hodge–Tate weights $\underline{k}$ and gives the Deligne–Fontaine module $\underline{D}$.

By identifying $\widehat{S}(U^p, E)^{\mathrm{alg}}$ with a representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ via ι_{G,w_0}, we have the following isomorphism up to normalization from (4.7.2) and [Ca14]:

$$\widehat{S}(U^{v_0}, E)^{\mathrm{alg}}[\mathfrak{m}_\rho] \cong \left(\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det\right)^{\oplus d(U^p, \rho)} \tag{4.7.4}$$

for all (U^p, ρ) satisfying the conditions (i), (ii) and (iii), where $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (k_1 - 2, k_2 - 1, k_3)$ and $d(U^p, \rho) \geq 1$ is an integer depending only on U^p and ρ.

Theorem 4.7.5. *We consider $U^p = \prod_{\ell \neq p} U_\ell$ and $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_3(E)$ such that*

(i) *ρ is absolutely irreducible and unramified at each finite place w of F lying above $D(U^p)$;*

(ii) *$\widehat{S}(U^p, E)^{\mathrm{alg}}[\mathfrak{m}_\rho] \neq 0$;*

(iii) *ρ has Hodge–Tate weights $\underline{k}$ and gives the Deligne–Fontaine module $\underline{D}$ as in (4.7.3);*

(iv) *the filtration on $\underline{D}$ is non-critical in the sense of (ii) of Remark 6.1.4 of [Bre17];*

(v) *only one automorphic representation π contributes to $\widehat{S}(U^p, E)^{\mathrm{alg}}[\mathfrak{m}_\rho]$.*

Then there exists a unique choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ such that:

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right). \end{aligned} \tag{4.7.6}$$

We recall several useful results from [Bre17] and [BH18].

Proposition 4.7.7. *Suppose that $U^p = \prod_{\ell \neq p} U_\ell$ is a sufficiently small open compact subgroup of $G(\mathbb{A}_{\mathbf{Q}}^{\infty,p})$, $\widehat{S}(U^p, E)^{\mathrm{an}} \hookrightarrow \Pi \twoheadrightarrow \Pi_1$ a short exact sequence of admissible locally analytic representations of $\mathrm{GL}_3(\mathbf{Q}_p)$, $\chi : T(\mathbf{Q}_p) \to E^\times$ a locally analytic character and $\eta : U(\mathfrak{t}) \to E$ its derived character, then we have $T(\mathbf{Q}_p)^+$-equivariant short exact sequences of finite dimensional E-spaces*

$$(\widehat{S}(U^p, E)^{\mathrm{an}})^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t} = \eta] \hookrightarrow \Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t} = \eta] \twoheadrightarrow \Pi_1^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t} = \eta]$$

and

$$(\widehat{S}(U^p, E)^{\mathrm{an}})^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t} = \eta]_\chi \hookrightarrow \Pi^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t} = \eta]_\chi \twoheadrightarrow \Pi_1^{\overline{N}(\mathbb{Z}_p)}[\mathfrak{t} = \eta]_\chi$$

where $T(\mathbf{Q}_p)^+$ is a submonoid of $T(\mathbf{Q}_p)$ defined by

$$T(\mathbf{Q}_p)^+ := \{t \in T(\mathbf{Q}_p) \mid t\overline{N}(\mathbb{Z}_p)t^{-1} \subseteq \overline{N}(\mathbb{Z}_p)\}.$$

Proof. This is Proposition 6.3.3 of [Bre17] and Proposition 4.1 of [BH18]. □

Proposition 4.7.8. *We fix U^p and ρ as in Theorem 4.7.5. For a locally analytic character $\chi : T(\mathbf{Q}_p) \to E^\times$, we have*

$$\mathrm{Hom}_{T(\mathbf{Q}_p)^+}\left(\chi \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ (\widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho])^{\overline{N}(\mathbb{Z}_p)}\right) \neq 0$$

if and only if $\chi = \delta_\lambda$.

Proof. This is Proposition 6.3.4 of [Bre17]. □

We recall the notation $i_B^{\mathrm{GL}_3}(\chi_w^\infty)$ for a smooth principal series for each $w \in W$ from Section 4.2.3. Given three locally analytic representations V_i for $i = 1, 2, 3$ and two surjections $V_1 \twoheadrightarrow V_2$ and $V_3 \twoheadrightarrow V_2$, we use the notation $V_1 \times_{V_2} V_3$ for the representation given by the fiber product of V_1 and V_3 over V_2 with natural surjections $V_1 \times_{V_2} V_3 \twoheadrightarrow V_1$ and $V_1 \times_{V_2} V_3 \twoheadrightarrow V_3$. We also use the shorten notation V^{alg} for the maximally locally algebraic subrepresentation of a locally analytic representation V. We recall that U^p is *sufficiently small* if there exists $\ell \neq p$ such that U_ℓ has no non-trivial element with finite order.

Proposition 4.7.9. *We fix U^p and ρ as in Theorem 4.7.5 and assume moreover that U^p is a sufficiently small open compact subgroup of $G(\mathbb{A}_{\mathbf{Q}}^{\infty,p})$. We also fix a non-split short exact sequence $V_1 \hookrightarrow V_2 \twoheadrightarrow V_3$ of finite length representations inside the category $\mathrm{Rep}^{\mathcal{OS}}_{\mathrm{GL}_3(\mathbf{Q}_p),E}$ such that $V_1 \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det$ embeds into $\widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]$. We conclude that:*

(i) *if V_3 is irreducible and not locally algebraic, then we have an embedding*

$$V_2 \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det \hookrightarrow \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho];$$

(ii) *if there is a surjection*

$$\overline{L}(\lambda) \otimes_E i_B^{\mathrm{GL}_3}(\chi_w^\infty) \twoheadrightarrow V_3$$

for a certain $w \in W$, then there exists a certain quotient V_4 of $V_2 \times_{V_3} \left(\overline{L}(\lambda) \otimes_E i_B^{\mathrm{GL}_3}(\chi_w^\infty)\right)$ satisfying

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V_4) = V_4^{\mathrm{alg}} = \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty$$

such that we have an embedding

$$V_4 \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det \hookrightarrow \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho].$$

Proof. This is an immediate generalization (or rather formalization) of Section 6.4 of [Bre17]. More precisely, part (i) (resp. (ii)) generalizes the Étape 1 (resp. the Étape 2) of Section 6.4 of [Bre17]. □

proof of Theorem 4.7.5. We may assume that $\alpha = 1$ for simplicity of notation thanks to Lemma 4.2.2. According to the Étape 1 and 2 of Section 6.2 of [Bre17], we may assume without loss of generality that U^p is sufficiently small and it is sufficient to show that there exists a unique choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ such that

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \neq 0. \tag{4.7.10}$$

We borrow the notation $\Pi^i(\underline{k}, \underline{D})$ from Theorem 6.2.1 of [Bre17]. We observe from (4.6.55) that $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ contains a unique subrepresentation $\Sigma^{\mathrm{Ext}^1}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of the form

$$\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \begin{array}{l} \nearrow\ \Pi^1(\underline{k}, \underline{D}) \\ \\ \searrow\ \Pi^2(\underline{k}, \underline{D}) \end{array}. \tag{4.7.11}$$

Moreover, $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ is uniquely determined by $\Sigma^{\mathrm{Ext}^1}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ up to isomorphism. It is known by Étape 3 of Section 6.2 of [Bre17] that there is at most one choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ such that

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\mathrm{Ext}^1}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \neq 0,$$

and thus there is at most one choice of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ such that (4.7.10) holds. As a result, it remains to show the existence of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ that satisfies (4.7.10). We notice that $\Sigma^{\min,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ admits an increasing filtration $\mathrm{Fil}_\bullet$ satisfying the following conditions

(i) the representations $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ and $\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)$ (c.f. their definition after Proposition 4.6.8 and Proposition 4.6.29) appear as two consecutive terms of the filtration;

(ii) each graded piece is either locally algebraic or irreducible.

As a result, the only reducible graded pieces of this filtration is the quotient

$$\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2) \cong W_0.$$

Then we can prove the existence of $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in E$ satisfying (4.7.10) by reducing to the isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathrm{Fil}_{k+1}\Sigma^{\max}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \\ &\quad \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\mathrm{Fil}_k\Sigma^{\max}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \end{aligned} \tag{4.7.12}$$

for each $k \in \mathbb{Z}$. If

$$\mathrm{Gr}_k := \mathrm{Fil}_{k+1}\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)/\mathrm{Fil}_k\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$$

is not locally algebraic, then (4.7.12) is true in this case by part (i) of Proposition 4.7.9. The only locally algebraic graded pieces of the filtration except $\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty$ are $\overline{L}(\lambda) \otimes_E v_{P_1}^\infty$, $\overline{L}(\lambda) \otimes_E v_{P_2}^\infty$ and W_0. The isomorphism (4.7.12) when the graded piece Gr_k equals $\overline{L}(\lambda) \otimes_E v_{P_1}^\infty$ or $\overline{L}(\lambda) \otimes_E v_{P_2}^\infty$ has been treated in Étape 2 of Section 6.4 of [Bre17]. As a result, it remains to show that

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \\ &\quad \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2) \otimes_E (\mathrm{ur}(\alpha) \otimes_E \varepsilon^2) \circ \det,\ \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \end{aligned} \tag{4.7.13}$$

to finish the proof of Theorem 4.7.5. It follows from results in Section 5.3 of [Bre17] (c.f. (53) of [Bre17]) that $i_B^{\mathrm{GL}_3}(\chi^\infty_{s_1s_2s_1})$ has the form

$$\begin{array}{ccccc} & & v_{P_1}^\infty & & \\ \mathrm{St}_3^\infty & \nearrow & & \searrow & 1_3 \\ & \searrow & & \nearrow & \\ & & v_{P_2}^\infty & & \end{array}$$

and thus there is a surjection

$$\overline{L}(\lambda) \otimes_E i_B^{\mathrm{GL}_3}(\chi^\infty_{s_1s_2s_1}) \twoheadrightarrow W_0.$$

According to part (ii) of Proposition 4.7.9, we only need to show that any quotient V of

$$V^\diamond := \Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \times_{W_0} \left(\overline{L}(\lambda) \otimes_E i_B^{\mathrm{GL}_3}(\chi^\infty_{s_1s_2s_1})\right)$$

such that

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V) = V^{\mathrm{alg}} = \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \tag{4.7.14}$$

must have the form

$$\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3')$$

for certain $\mathscr{L}_3' \in E$. We recall from Proposition 4.6.29 and our definition of $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ afterwards that $\Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ fits into a short exact sequence

$$\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2) \hookrightarrow \Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \twoheadrightarrow W_0 \tag{4.7.15}$$

and thus $V^\diamond$ fits (by definition of fiber product) into a short exact sequence

$$\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2) \hookrightarrow V^\diamond \twoheadrightarrow i_B^{\mathrm{GL}_3}(\chi^\infty_{s_1s_2s_1}) \tag{4.7.16}$$

and in particular

$$\mathrm{soc}_{\mathrm{GL}_3(\mathbf{Q}_p)}(V^\diamond) = \left(\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty\right)^{\oplus 2}.$$

Hence the condition (4.7.14) implies that V fits into a short exact sequence

$$\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \xrightarrow{j} V^\diamond \twoheadrightarrow V$$

and that

$$j\left(\overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty\right) \cap \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2) = 0 \subseteq V^\diamond$$

which induces an injection

$$\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2) \hookrightarrow V.$$

Therefore V fits into a short exact sequence

$$\Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2) \hookrightarrow V \twoheadrightarrow W_0$$

and thus corresponds to a line M_V inside

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right)$$

which is two dimensional by Lemma 4.6.33. Moreover, the condition (4.7.14) implies that M_V is different from the line

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \overline{L}(\lambda) \otimes_E \mathrm{St}_3^\infty\right) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbf{Q}_p),\lambda}\left(W_0,\ \Sigma^{\sharp,+}(\lambda, \mathscr{L}_1, \mathscr{L}_2)\right).$$

Hence it follows from Lemma 4.6.33 that there exists $\mathscr{L}_3' \in E$ such that

$$V \cong \Sigma^{\min}(\lambda, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3').$$

□

Corollary 4.7.17. *If a locally analytic representation* Π *of the form (4.7.11) is contained in* $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ *for a certain* U^p *and* ρ *as in Theorem 4.7.5, then there exists* $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ *uniquely determined by* Π *such that*

$$\Pi\hookrightarrow\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3).$$

Proof. We fix U^p and ρ such that the embedding

$$\Pi\hookrightarrow\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho] \tag{4.7.18}$$

exists. Then (4.7.18) restricts to an embedding

$$\overline{L}(\lambda)\otimes_E\mathrm{St}_3^\infty\hookrightarrow\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$$

which extends to an embedding

$$\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho] \tag{4.7.19}$$

for a unique choice of $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ according to Theorem 4.7.5. The embedding (4.7.19) induces by restriction an embedding

$$\Sigma^{\mathrm{Ext}^1}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\hookrightarrow\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$$

and therefore we have

$$\Pi\cong\Sigma^{\mathrm{Ext}^1}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

by Theorem 6.2.1 of [Bre17]. In particular, we deduce an embedding

$$\Pi\hookrightarrow\Sigma^{\min,+}(\lambda,\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$$

for certain invariants $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in E$ determined by Π. □

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