Groupes hyperboliques et logique du premier ordre

Simon André

THÈSE DE L'UNIVERSITÉ DE RENNES 1

Spécialité : mathématiques

Préparée par

Simon ANDRÉ

à l'institut de recherche mathématique de Rennes

Groupes hyperboliques et logique du premier ordre

Thèse soutenue à Rennes le 15 juillet 2019 devant le jury composé de :

Montserrat CASALS-RUIZ
Chargée de recherche Ikerbasque, université du Pays basque, *examinatrice*

Rémi COULON
Chargé de recherche au CNRS, université de Rennes 1, *examinateur*

Vincent GUIRARDEL
Professeur, université de Rennes 1, *directeur de thèse*

Frédéric PAULIN
Professeur, université Paris-Sud, *rapporteur*

Françoise POINT
Directrice de recherche au FNRS, universités Paris 7 et de Mons, *examinatrice*

Zlil SELA
Professeur, université de Jérusalem, *rapporteur*

au vu des rapports de **Frédéric PAULIN** et **Zlil SELA**

Remerciements

Mes premiers remerciements sont naturellement adressés à Vincent Guirardel, mon directeur de thèse. Vincent s'est toujours montré très disponible et bienveillant. Il n'a eu de cesse de répondre avec enthousiasme à mes questions et de m'aider à mettre de l'ordre dans mes idées, tout en m'en insufflant de nouvelles. Il a relu avec rigueur mes différents écrits. Je le remercie pour la confiance et la liberté qu'il m'a accordées. J'ai beaucoup appris à son contact ces trois dernières années, et je tiens à lui témoigner ici ma profonde gratitude.

Je suis particulièrement reconnaissant envers Frédéric Paulin pour le temps qu'il a consacré à relire ma thèse avec une très grande minutie. Ses nombreuses remarques et questions auront permis d'améliorer nettement la qualité du texte.

Je suis honoré que Zlil Sela ait accepté d'être rapporteur de ma thèse, et de faire le déplacement jusqu'à Rennes. Ses travaux sont une grande source d'inspiration, en même temps qu'un défi.

Merci à Montserrat Casals-Ruiz, Rémi Coulon et Françoise Point qui se sont intéressés à mes recherches et ont gentiment accepté de faire partie de mon jury.

Je remercie les différentes personnes qui ont porté un intérêt à mon travail, notamment celles qui m'ont invité à donner des exposés dans leur séminaire.

C'est avec plaisir que j'ai travaillé au sein du laboratoire de mathématiques de Rennes. Je remercie toutes les personnes sympathiques que j'ai eu l'occasion d'y côtoyer et qui ont contribué, d'une façon ou d'une autre, au bon déroulement de cette thèse.

Enfin, je remercie mes proches, tout particulièrement mes parents, pour leur soutien précieux.

Résumé

Deux groupes sont dits élémentairement équivalents s'ils satisfont les mêmes énoncés du premier ordre dans le langage des groupes, c'est-à-dire les mêmes énoncés mathématiques dont les variables représentent uniquement des éléments d'un groupe. Aux environs de l'année 1945, A. Tarski posa la question suivante, connue désormais comme le *problème de Tarski* : les groupes libres non abéliens sont-ils élémentairement équivalents ? Une réponse positive à cette fameuse question fut apportée plus d'un demi-siècle plus tard par Z. Sela dans [**Sel06b**], et en parallèle par O. Kharlampovich et A. Myasnikov dans [**KM06**], comme le point d'orgue de deux volumineuses séries de travaux. Dans la foulée, Sela généralisa aux groupes hyperboliques sans torsion, dont les groupes libres sont des représentants emblématiques, les méthodes de nature géométrique qu'il avait précédemment introduites à l'occasion de son travail sur le problème de Tarski (voir [**Sel09**]). Son approche a mis au jour de profondes connexions entre la théorie des modèles (l'étude des structures mathématiques du point de vue de la logique) et la théorie géométrique des groupes (l'étude des groupes de type fini en tant qu'objets géométriques à part entière), qui se sont révélées très fructueuses et ont permis d'établir nombre de résultats nouveaux durant la décennie passée. Citons notamment les articles de C. Perin et R. Sklinos [**Per13**] et [**PS12**], qui ont inspiré les deux premiers chapitres du présent manuscrit.

Les résultats rassemblés ici s'inscrivent dans cette lignée, en s'en démarquant toutefois dans la mesure où ils traitent de la théorie du premier ordre des groupes hyperboliques *en présence de torsion*, au contraire de tous les travaux mentionnés précédemment dans lesquels les auteurs supposent les groupes sans torsion, c'est-à-dire sans éléments d'ordre fini non triviaux. La torsion est à l'origine de phénomènes nouveaux qui sont au cœur des travaux exposés dans cette thèse. Le lecteur trouvera dans la suite les trois résultats que voici.

- Tout groupe de type fini qui est élémentairement équivalent à un groupe hyperbolique est lui-même hyperbolique. Cela généralise un résultat dû à Sela en l'absence de torsion (voir [**Sel09**], théorème 7.10). On démontre des résultats apparentés pour les sous-groupes des groupes hyperboliques, ainsi que pour les groupes hyperboliques et cubulables.
- Les groupes virtuellement libres sont presque homogènes. On renvoie à la partie 4.3 pour une définition. Il s'agit d'une généralisation d'un résultat dû à Perin et Sklinos (voir [**PS12**]), et indépendamment à A. Ould Houcine (voir [**OH11**]), selon lequel les groupes libres sont homogènes (voir définition 4.1). On conjecture par ailleurs que les groupes virtuellement libres ne sont pas homogènes, et on donne des arguments en faveur de cette hypothèse. On démontre également qu'un élément générique d'un groupe hyperbolique (au sens des marches aléatoires) est déterminé à automorphisme près par les énoncés du premier ordre qu'il satisfait.
- Enfin, on donne une classification complète des groupes virtuellement libres de type fini du point de l'équivalence élémentaire à deux quantificateurs $\forall\exists$. Autrement dit, on donne des conditions nécessaires et suffisantes pour que deux groupes virtuellement libres de type fini satisfassent exactement les mêmes énoncés de la forme $\forall \boldsymbol{x} \exists \boldsymbol{y} \psi(\boldsymbol{x}, \boldsymbol{y})$, où $\boldsymbol{x}$ et $\boldsymbol{y}$ désignent des uplets de variables, et ψ désigne une formule sans quantificateur en ces variables.

L'introduction, rédigée en français, offre un aperçu de ces résultats. Le corps du manuscrit est composé de trois chapitres, rédigés en anglais, qui correspondent aux trois points ci-dessus. Les chapitres sont largement indépendants les uns des autres et peuvent être lus dans un ordre quelconque.

Abstract

Two groups are said to be elementarily equivalent if they satisfy the same first-order sentences in the language of groups, that is the same mathematical statements whose variables are only interpreted as elements of a group. Around 1945, A. Tarski asked the following question: are non-abelian free groups elementarily equivalent? An affirmative answer to this famous Tarski's problem was given by Z. Sela in [**Sel06b**] and by O. Kharlampovich and A. Myasnikov in [**KM06**], as the culmination of two voluminous series of papers. Then, Sela gave a classification of all finitely generated groups elementarily equivalent to a given torsion-free hyperbolic group (see [**Sel09**]). The results contained in the present document fall into this context and deal with first-order theories of hyperbolic groups with torsion. In this thesis, the reader will find the following three results.

- A finitely generated group elementarily equivalent to a hyperbolic group is itself a hyperbolic group. This result generalizes a result proved by Sela for torsion-free hyperbolic groups in [**Sel09**]. In addition, we prove similar results for subgroups of hyperbolic groups, and for hyperbolic and cubulable groups.
- Virtually free groups are almost homogeneous, meaning that elements are almost determined up to automorphism by their type, i.e. the first-order formulas they satisfy. This result is a generalization of a theorem proved by C. Perin and R. Sklinos in [**PS12**], and independently by A. Ould Houcine in [**OH11**], claiming that free groups are homogeneous. Moreover, we conjecture that virtually free groups are not homogeneous, and we give arguments in favour of this hypothesis. We also prove that a generic element in a hyperbolic group (in the sense of random walks) is determined by its type, up to automorphism.
- We give a complete classification of finitely generated virtually free groups up to $\forall\exists$-elementary equivalence. In other words, we give necessary and sufficient conditions under which two finitely generated virtually free groups satisfy exactly the same first-order sentences of the form $\forall \boldsymbol{x}\exists \boldsymbol{y}\psi(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are two tuples of variables, and ψ is a quantifier-free formula in these variables.

This dissertation is composed of a global introduction to the results announced above, in French, followed by three chapters in English corresponding to the previous three points. All chapters are independent of each other.

Table des matières - Table of contents

Introduction

1. Un peu de théorie des modèles

Un *énoncé du premier ordre* dans le langage des groupes est une suite finie de symboles de la forme

$$\varphi : Q_1 x_1 \cdots Q_n x_n \ \psi(x_1, \ldots, x_n),$$

où les Q_i sont des quantificateurs $\forall$ ou $\exists$, les x_i sont des symboles de variables dites quantifiées, et $\psi(x_1, \ldots, x_n)$ est une combinaison booléenne d'équations et d'inéquations en ces variables, autrement dit une suite finie de symboles de la forme

$$\left\{\begin{array}{c} w_{1,1}(x_1, \ldots, x_n) \underset{\neq}{=} 1 \\ \vdots \\ w_{1,m_1}(x_1, \ldots, x_n) \underset{\neq}{=} 1 \end{array}\right. \vee \left\{\begin{array}{c} w_{2,1}(x_1, \ldots, x_n) \underset{\neq}{=} 1 \\ \vdots \\ w_{2,m_2}(x_1, \ldots, x_n) \underset{\neq}{=} 1 \end{array}\right. \vee \ldots$$

où les $w_{k,\ell}$ désignent des mots en les variables x_i et leurs inverses. Parfois, on a besoin de faire intervenir des variables non quantifiées, dites libres ; l'énoncé est alors plutôt appelé une *formule*. Par ailleurs, on peut souhaiter mettre l'accent sur la forme des énoncés considérés : on parlera d'énoncé existentiel pour désigner un énoncé qui ne fait intervenir que le quantificateur $\exists$, ou encore d'énoncé $\forall\exists$ pour désigner un énoncé du type

$$\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \ \psi(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

où ψ est une formule sans quantificateur à $n+m$ variables libres.

Étant donné un groupe G, on décrète maintenant que les symboles de variables de l'énoncé φ incarnent des éléments du groupe G, que le symbole 1 incarne l'élément neutre, et que les mots en les variables représentent des produits au sein du groupe. L'énoncé φ acquiert alors une signification dans le groupe G. En particulier, on peut lui attribuer une valeur de vérité. On dira que l'énoncé est satisfait, ou n'est pas satisfait, par le groupe G. On notera $G \models \varphi$ si l'énoncé φ est satisfait par G. Par exemple, l'énoncé $\forall x \forall y \ (xy = yx)$ est satisfait par le groupe G si et seulement si ce groupe est abélien.

Insistons sur le fait que les symboles de variables sont interprétés uniquement comme des éléments du groupe G, jamais comme des entiers, comme des sous-groupes, ou encore des morphismes. Par exemple, la suite de symboles $\forall x \neq 1,\ \forall n \in \mathbb{N},\ x^n \neq 1$ n'est pas un énoncé du premier ordre dans le langage des groupes. On peut en fait démontrer qu'il n'existe pas d'énoncé du premier ordre qui capture la propriété d'être sans torsion, c'est-à-dire qui soit vérifié par les groupes sans torsion, et uniquement par eux ; en revanche, on peut exprimer cette propriété à l'aide d'une infinité d'énoncés.

Le pouvoir d'expression de la logique du premier ordre, dans le langage des groupes, est donc à première vue très restreint. Notons par exemple qu'il est impossible de parler de partie génératrice d'un groupe en général, même finie : les produits qu'il nous faudrait considérer seraient en effet de longueur non bornée, ce qui n'est pas possible dans le langage en question. La question suivante est naturelle.

Question 1.1. Que peut-on dire à propos d'un groupe G donné au moyen d'énoncés du premier ordre ? Autrement dit, quelles propriétés de ce groupe peut-on capturer avec des énoncés du premier ordre ?

Pour préciser cette question, introduisons deux nouvelles définitions. La *théorie du premier ordre* d'un groupe G est l'ensemble des énoncés du premier ordre, dans le langage des groupes, qui sont satisfaits par le groupe G. On note cet ensemble $\mathrm{Th}(G)$. De même, on utilisera les notations $\mathrm{Th}_{\exists}(G)$, $\mathrm{Th}_{\forall}(G)$ et $\mathrm{Th}_{\forall\exists}(G)$ pour désigner l'ensemble des énoncés existentiels, universels ou $\forall\exists$ satisfaits par le groupe G. On dit que deux groupes G et Γ sont *élémentairement équivalents* si $\mathrm{Th}(G) = \mathrm{Th}(\Gamma)$, et on utilise la notation $G \equiv \Gamma$. On dit que que G et Γ sont $\forall\exists$-*équivalents* dans le cas où $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$. Nous pouvons maintenant reformuler la question précédente.

Question 1.2. Si le groupe G a une certaine propriété $\mathcal{P}$, et si $G \equiv \Gamma$, le groupe Γ possède-t-il également la propriété $\mathcal{P}$?

On peut affiner la question précédente en supposant par exemple seulement l'égalité des théories universelles de G et Γ, ou encore l'égalité des théories $\forall\exists$. C'est ce que nous ferons notamment dans le premier chapitre relatif à la préservation de l'hyperbolicité.

1.1. Premiers exemples.

Commençons par un exemple très simple. On dit qu'un groupe G satisfait une loi s'il existe un mot $w(x_1, \ldots, x_n)$ non trivial tel que, pour tout $(g_1 \ldots, g_n) \in G^n$, l'égalité $w(g_1, \ldots, g_n) = 1$ a lieu, autrement dit si G satisfait l'énoncé universel

$$\forall x_1 \cdots \forall x_n \; w(x_1, \ldots, x_n) = 1.$$

Si Γ est un groupe tel que $\mathrm{Th}_{\forall}(\Gamma) = \mathrm{Th}_{\forall}(G)$, alors Γ satisfait les mêmes lois que G. Par exemple, être abélien, nilpotent de classe c, résoluble de classe c, ou encore de torsion bornée par un entier donné, sont des lois. L'exemple suivant est légèrement plus subtil, car il fait intervenir une infinité d'énoncés.

Exemple 1.3. Si G est un groupe sans torsion, et si $\mathrm{Th}_{\forall}(\Gamma) = \mathrm{Th}_{\forall}(G)$, alors Γ est lui aussi sans torsion.

En effet, pour chaque entier $k \geq 2$, on peut exprimer au moyen d'un énoncé universel φ_k le fait que G ne contient aucun élément d'ordre k. La collection infinie de tous les énoncés φ_k pour $k \geq 2$ capture donc la propriété d'être sans torsion. Cependant, il est impossible d'exprimer cette propriété en utilisant un énoncé unique, comme en atteste l'exemple 1.8. On retiendra donc que certaines propriétés, qui ne sont pas finiment exprimables, le sont néanmoins au moyen d'un ensemble infini d'énoncés.

Naïvement, on peut se demander si la classe d'isomorphisme d'un groupe est préservée par équivalence élémentaire (la question 1.2 posée plus haut serait dès lors caduque). Nous allons voir que c'est le cas si, et seulement si, le groupe en question est fini.

Exemple 1.4. Tout groupe fini est caractérisé à isomorphisme près par sa théorie du premier ordre.

Preuve. Soit G un groupe fini. On peut coder le fait que G est d'ordre fini, disons n, ainsi que sa loi de multiplication. Le lecteur pourra vérifier que l'énoncé existentiel suivant convient :

$$\exists x_1 \cdots \exists x_n \forall y \bigwedge_{i,j=1}^{n} x_i x_j = x_{f(i,j)} \;\wedge\; \bigwedge_{i \neq j} x_i \neq x_j \;\wedge\; \bigvee_{i=1}^{n} y = x_i,$$

où f est l'application de $[\![1, n]\!]^2$ vers $[\![1, n]\!]$ induite par la muliplication de G (ayant fixé une numérotation des éléments). □

Ainsi, toute l'information au sujet d'un groupe fini est encodée dans sa théorie du premier ordre, et même dans un énoncé unique. Un argument très simple montre que cela n'est plus vrai pour les groupes infinis. Faisons l'observation suivante : puisqu'il n'y a qu'un nombre dénombrable d'énoncés du premier ordre dans le langage des groupes, la théorie du premier ordre d'un groupe G peut être vue comme un élément de $\{0,1\}^{\mathbb{N}}$. Il y a donc au plus un continuum de théories du premier ordre. D'un autre côté, il y a bien plus d'un continuum de groupes (à isomorphisme près) puisque deux groupes libres $F(S)$ et $F(T)$ sont isomorphes si et seulement si les ensembles S et T sont de même cardinal, et on sait qu'il existe bien plus d'un continuum de cardinaux d'ensembles. Voici un exemple peut-être plus explicite : considérons la famille $(G_t)_{t\in\mathbb{R}}$ des groupes de type fini librement indécomposables, et posons $H_P = *_{t\in P}G_t$ pour tout $P \subset \mathcal{P}(\mathbb{R})$; les H_P sont deux à deux non isomorphes, et la famille $(H_P)_{P\in\mathcal{P}(\mathbb{R})}$ convient.

Cet argument simple montre qu'il existe nécessairement deux groupes infinis qui, bien que non isomorphes, sont indiscernables du point de vue de la logique du premier ordre. Pire, il montre que l'ensemble des théories du premier ordre est minuscule en comparaison de la collection de tous les groupes, à isomorphisme près. En fait, le théorème de Löwenheim et Skolem 1.11 affirme qu'étant donné un groupe infini, on peut toujours trouver un groupe non isomorphe qui lui est élémentairement équivalent. Dès lors qu'un groupe G est infini, la question 1.2 est donc intéressante.

Avant de donner des exemples un peu plus élaborés de propriétés qui sont ou ne sont pas préservées par équivalence élémentaire, nous allons introduire quelques outils de théorie des modèles.

1.2. Quelques outils de théorie des modèles.

Dans tout ce qui suit, le langage considéré est toujours implicitement celui de la théorie des groupes, mais les résultats restent valables dans un cadre beaucoup plus général. On renvoie à [**Mar02**] pour une introduction à la théorie des modèles.

Définition 1.5. Un *ultrafiltre* sur $\mathbb{N}$ est une mesure finiment additive de masse totale 1, définie sur $\mathcal{P}(\mathbb{N})$ et à valeurs dans $\{0,1\}$. Autrement dit, c'est une application

$$\omega : \mathcal{P}(\mathbb{N}) \to \{0,1\}$$

telle que $\omega(\mathbb{N}) = 1$ et, pour toutes parties A et B de $\mathbb{N}$ telles que $A\cap B = \varnothing$,

$$\omega(A\cup B) = \omega(A) + \omega(B).$$

Un ultrafiltre est dit *non principal* si ce n'est pas une masse de Dirac, c'est-à-dire si tout ensemble fini est de masse nulle.

Soit ω un ultrafiltre, que nous supposerons non principal. On dit qu'une propriété $P(n)$ dépendant de l'entier n est vraie pour presque tout n (relativement à ω) si

$$\omega(\{n\in\mathbb{N} \mid P(n)\}) = 1.$$

Étant donné une famille de groupes $(G_n)_{n\in\mathbb{N}}$, l'égalité presque partout est une relation d'équivalence naturelle sur le produit $\prod_{n\in\mathbb{N}} G_n$. Nous la noterons $\sim_\omega$.

Définition 1.6. L'*ultraproduit* de la famille (G_n) relativement à l'ultrafiltre ω est le groupe quotient

$$\left(\prod_{n\in\mathbb{N}} G_n\right) / \sim_\omega .$$

En général, l'ultraproduit est un groupe très gros : si les G_n sont dénombrables, de cardinaux non bornés par une constante (finie), et si ω est non principal, alors l'ultraproduit $(\prod_{n\in\mathbb{N}} G_n)/\sim_\omega$ a la puissance du continu (voir par exemple [**Bou16**]).

Si tous les G_n sont le même groupe G, l'ultraproduit est appelée l'*ultrapuissance* de G relativement à ω. Ce groupe est noté G^ω. D'après la remarqe précédente, l'ultrapuissance d'un groupe infini relativement à un ultrafiltre non principal n'est jamais dénombrable.

Le principal intérêt des ultraproduits réside dans le théorème de Loś, que l'on peut démontrer assez simplement par récurrence sur la complexité des formules.

THÉORÈME 1.7 (Łoś). *Soit ω un ultrafiltre non principal. Considérons $\varphi(x_1, \ldots, x_k)$ une formule du premier ordre à k variables libres, et k éléments $(g_{1,n})_{n\in\mathbb{N}}, \ldots, (g_{k,n})_{n\in\mathbb{N}}$ de $\prod_{n\in\mathbb{N}} G_n$. L'ultraproduit G^ω satisfait $\varphi((g_{1,n})_n, \ldots, (g_{k,n})_n)$ si et seulement si*

$$\omega\left(\{n \in \mathbb{N} \mid G_n \models \varphi(g_{1,n}, \ldots, g_{k,n})\}\right) = 1.$$

En particulier, pour tout groupe G, le groupe G^ω est élémentairement équivalent à G.

Voici une première application du théorème de Łoś.

EXEMPLE 1.8. Il est impossible d'exprimer la propriété d'être sans torsion en utilisant un énoncé unique. Par exemple, pour tout ultrafiltre non principal, l'ultraproduit des $\mathbb{Z}/p\mathbb{Z}$, avec p parcourant l'ensemble des nombres premiers, est un groupe sans torsion.

Le théorème de Łoś permet de démontrer facilement l'un des théorèmes fondamentaux de la théorie des modèles : le théorème de compacité de la logique du premier ordre. On dit qu'une théorie T (i.e. un ensemble d'énoncés) est *satisfaisable* si elle possède un modèle, c'est-à-dire s'il existe un groupe qui satisfait tous les énoncés de T.

THÉORÈME 1.9 (Compacité). *Une théorie T est satisfaisable si et seulement si tout sous-ensemble fini de T est satisfaisable.*

Preuve. Soit T une théorie. Comme le langage de la théorie des groupes est dénombrable, la théorie T est dénombrable. On peut donc écrire T comme la réunion dénombrable croissante de sous-théories finies T_n, avec $n \in \mathbb{N}$. Par hypothèse, chaque T_n a un modèle G_n. Considérons un ultraproduit $G = (\prod G_n)/\omega$ où ω est un ultrafiltre non principal, et vérifions que G est bien un modèle de T. Soit $\varphi \in T$. Il existe n_0 tel que $\varphi \in T_n$, i.e. $G_n \models \varphi$, pour tout $n \geq n_0$. Donc $\omega(\{n \mid T_n \models \varphi\}) = 1$, d'où $G \models \varphi$ d'après le théorème de Łoś. □

Voici une conséquence immédiate du théorème de compacité : une théorie qui possède des modèles finis arbitrairement grands possède un modèle infini. On en déduit par exemple qu'il n'est pas possible d'axiomatiser la classe des groupes finis, ou encore celle des groupes de torsion (non bornée).

Une autre conséquence importante du théorème de Łoś est le théorème de Löwenheim-Skolem, dans sa version dite ascendante : si une théorie a un modèle infini, alors elle a des modèles infinis arbitrairement grands. Avant de prouver ce résultat, donnons une preuve de la version dite descendante du théorème de Löwenheim-Skolem.

THÉORÈME 1.10 (Version descendante du théorème de Löwenheim-Skolem). *Si une théorie T a un modèle infini G, elle a un modèle infini dénombrable Γ. Plus généralement, pour tout cardinal infini κ, si $|G| \geq \kappa$, on peut construire un modèle Γ de T de cardinal κ. En outre, on peut choisir Γ élémentairement plongé dans G.*

Preuve. On se contente de donner la preuve dans le cas où $\kappa = \aleph_0$. Considérons un sous-ensemble infini dénombrable de G, noté A. Pour tout entier $n \geq 0$, pour toute formule du premier ordre à $n+1$ variables libres $\varphi(x, x_1, \ldots, x_n)$, pour tout n-uplet $(a_1, \ldots, a_n) \in A^n$, si $G \models \exists x \varphi(x, a_1, \ldots, a_n)$, on peut choisir (grâce à l'axiome du choix) un élément $x_{\varphi,a_1,\ldots,a_n}$ tel que

$$G \models \varphi(x_{\varphi,a_1,\ldots,a_n}, a_1, \ldots, a_n).$$

Notons $E(A)$ l'ensemble des $x_{\varphi,a_1,\ldots,a_n}$. Pour tout entier $p \geq 2$, notons $E^p(A)$ l'ensemble $E(E^{p-1}(A))$. L'ensemble $E(A)$ contient A puisque pour tout $a \in A$, $G \models \exists x(x = a)$. Plus généralement, la suite $(E^p(A))_p$ est croissante pour l'inclusion. Posons

$$\Gamma = \bigcup_{p\in\mathbb{N}} E^p(A).$$

Pour tout ensemble $X \subset G$ dénombrable, l'ensemble $E(X)$ est dénombrable puisque l'ensemble des formules du premier ordre est dénombrable (le langage étant dénombrable).

Ainsi, pour tout entier p, $E^p(A)$ est dénombrable. L'ensemble Γ est donc lui aussi infini dénombrable. En outre, Γ est bien un sous-groupe de G. En effet, $1 \in E(A) \subset \Gamma$ puisque $G \models \exists x(x = 1)$. Si $g \in \Gamma$, il existe p tel que $g \in E^p(A)$, donc $g^{-1} \in E^{p+1}(A)$ puisque $G \models \exists x(x = g^{-1})$. De même, Γ est stable par produit.

Pour pour tout $(g_1, \dots, g_n) \in \Gamma^n$, et pour toute formule à $n+1$ variables libres $\varphi(x, x_1, \dots, x_n)$, on a l'équivalence suivante :

$$\Gamma \models \exists x \varphi(x, g_1, \dots, g_n) \text{ si et seulement si } G \models \exists x \varphi(x, g_1, \dots, g_n).$$

L'implication $\Rightarrow$ est triviale. Réciproquement, si $G \models \exists x \varphi(x, g_1, \dots, g_n)$, alors il existe un entier p tel que $g_1, \dots, g_n \in E^p(A)$, et puisque $G \models \exists x \varphi(x, g_1, \dots, g_n)$, l'élément $x_{\varphi, g_1, \dots, g_n}$ appartient à $E^{p+1}(A)$, ce qui conclut. L'inclusion de Γ dans G est donc élémentaire grâce au test de Tarski-Vaught (voir [**Mar02**], Proposition 2.3.5). □

Voici maintenant la version dite ascendante du théorème, qui est un simple corollaire du théorème descendant combiné au théorème de compacité de la logique du premier ordre.

Théorème 1.11 (Version ascendante du théorème de Löwenheim-Skolem). *Une théorie qui a un modèle infini G a un modèle Γ de cardinal κ, pour tout $\kappa \geq |G|$. De plus, on peut choisir Γ de sorte que G soit élémentairement plongé dans Γ.*

Preuve. Soit G un groupe infini. Soit T une théorie qui a pour modèle G dans le langage L de la théorie des groupes. Ajoutons des symboles $(c_i)_{i \in I}$ avec I de cardinal κ au langage L, pour obtenir un langage L'. Créons une théorie T' dans le langage L' en ajoutant à T les énoncés $c_i \neq c_j$ si $i \neq j$. Comme tout sous-ensemble fini de T' ne fait intervenir qu'un nombre fini des nouveaux symboles c_i, et comme G est infini, on peut interpréter ces symboles comme des éléments de G. Ainsi, G est un modèle de tout sous-ensemble fini de T'. En vertu du théorème de compacité, T' a un modèle, qui est de cardinal $\geq \kappa$. D'après le théorème de Löwenheim-Skolem descendant, T' a un modèle Γ de cardinal exactement κ. □

Mentionnons pour finir un théorème beaucoup plus difficile.

Théorème 1.12 (Keisler-Shelah). *Si G et Γ ont la même théorie du premier ordre, il existe un ultrafiltre non principal ω tel que G^ω et Γ^ω sont isomorphes.*

Notons que la réciproque du théorème de Keisler-Shelah est une conséquence immédiate du théorème de Loś.

Nous allons utiliser les quelques outils introduits ci-dessus pour donner des exemples de propriétés des groupes qui sont préservées ou non par équivalence élémentaire.

1.3. Plus d'exemples.

Theorem 1.1 (Malcev). *La linéarité est préservée par équivalence élémentaire.*

Preuve. Soit Γ un sous-groupe de $\mathrm{GL}_n(K)$, où K est un anneau (resp. un corps). Soit G un groupe ayant la même théorie du premier ordre que Γ. D'après le théorème 1.12, il existe un ultrafiltre non principal ω tel que G^ω et Γ^ω sont isomorphes. Donc G^ω se plonge dans $\mathrm{GL}_n(K)^\omega \simeq \mathrm{GL}_n(K^\omega)$. L'ultraproduit K^ω est un anneau, et un corps si K en est un. Or G se plonge dans G^ω via $g \mapsto (g)_{n \in \mathbb{N}}$, ce qui conclut. □

Donnons maintenant quelques propriétés qui ne sont pas préservées par équivalence élémentaire (voir détails plus bas) : être de type fini, être dénombrable, être de torsion non bornée, être libre, avoir un sous-groupe isomorphe à $\mathbb{Z}$, à F_2, à $\mathbb{Z}^2$, être moyennable, avoir la propriété (T).

D'après le théorème de Löwenheim et Skolem, la logique du premier ordre est incapable de contrôler le cardinal des modèles d'une théorie du premier ordre, à moins que cette théorie possède un modèle fini (donc unique). En particulier, la propriété d'être de type fini n'est pas une propriété du premier ordre. Voici un exemple concret : Szmielew a donné en 1955 une classification complète des groupes abéliens à équivalence élémentaire près, et a notamment démontré que les groupes $\mathbb{Z}$ et $\mathbb{Z} \times \mathbb{Q}$ sont élémentairement équivalents. Par conséquent, même parmi les groupes dénombrables, la propriété d'être de type fini n'est pas préservée par équivalence élémentaire. Notons en outre que le groupe des entiers relatifs est hyperbolique, tandis que $\mathbb{Z} \times \mathbb{Q}$ ne l'est pas. Nous y reviendrons plus loin lorsque nous démontrerons que l'hyperbolicité est préservée par équivalence élémentaire au sein des groupes de type fini.

En fait, bon nombre de propriétés intéressantes en théorie des groupes ne sont pas préservées par équivalence élémentaire.

Exemple 1.13. Si G est un groupe de torsion tel qu'il n'existe pas de borne sur l'ordre des éléments de torsion, alors il existe un groupe Γ qui contient un élément d'ordre infini et qui a la même théorie du premier ordre que G.

On peut prendre pour Γ un ultraproduit G^ω, où ω désigne un ultrafiltre non principal. Un tel groupe est en effet élémentairement équivalent à G en vertu du théorème de Loś 1.7, et possède clairement un élément d'ordre infini. Autrement dit, contenir $\mathbb{Z}$ comme sous-groupe n'est pas une propriété du premier ordre. Il en est de même si l'on remplace $\mathbb{Z}$ par $\mathbb{Z}^2$ puisque $\mathbb{Z}$ et $\mathbb{Z} \times \mathbb{Q}$ sont élémentairement équivalents. Voici un autre exemple : si ω est un ultrafiltre non principal, alors $(F_2)^\omega$ contient $\mathbb{Z}^2$ (on peut écrire $F_2 = \langle x, y \rangle$ et considérer par exemple les deux suites (x^n) et (x^{n^2})). De même, contenir F_2 n'est pas une propriété du premier ordre, comme le montre l'exemple qui suit.

Exemple 1.14. Soit G un groupe qui ne contient pas F_2 et qui ne satisfait aucune loi (par exemple le groupe de Thompson). Il existe un groupe Γ qui contient F_2 et qui a la même théorie du premier ordre que G.

Encore une fois, on peut prendre pour Γ un ultraproduit G^ω, où ω désigne un ultrafiltre non principal. Comme G ne satisfait pas de loi, le groupe G^ω contient F_2 (voir par exemple [**DK18**], lemme 10.42), et il est élémentairement équivalent à G par le théorème de Loś 1.7. Dans l'exemple précédent, on peut même prendre un groupe G moyennable qui ne satisfait pas de loi, par exemple la somme directe de tous les groupes finis

$$G := \bigoplus_{k \in \mathbb{N}} G_k.$$

Vérifions que ce groupe G convient. Par l'absurde, supposons qu'il satisfasse une loi $w(x_1, \ldots, x_n)$. Comme le groupe libre F_2 ne satisfait quant à lui aucune loi, il existe des éléments $\gamma_1, \ldots, \gamma_n$ de F_2 tels que $w(\gamma_1, \ldots, \gamma_n) \neq 1$. Puisque F_2 est résiduellement fini, il existe un morphisme ϕ de F_2 vers un groupe fini H tel que $\phi(w(\gamma_1, \ldots, \gamma_n)) \neq 1$. Or H se plonge dans G, par définition de G, ce qui fournit la contradiction attendue. Par ailleurs, le groupe G est moyennable comme union d'une suite emboîtée de groupes finis. Enfin, le groupe G^ω n'est pas moyennable puisqu'il contient F_2. Ainsi, la moyennabilité n'est pas une propriété du premier ordre.

À l'extrême opposé du spectre, la propriété (T) de Kazhdan n'est pas préservée par équivalence élémentaire, même au sein des groupes de type fini. Pour le démontrer, considérons un groupe hyperbolique sans torsion Γ possédant la propriété (T), par exemple un réseau uniforme dans $\mathrm{Sp}(1, n)$ avec $n \geq 2$ (on renvoie au chapitre 3 de [**BdlHV08**] pour une preuve de la propriété (T) de $\mathrm{Sp}(1, n)$, $n \geq 2$). D'après [**Sel09**], les groupes Γ et $\Gamma * \mathbb{Z}$ ont la même théorie du premier ordre. Or, $\Gamma * \mathbb{Z}$ n'a pas la propriété (T), puisqu'il n'a pas la propriété (FA) de Serre.

Comme nous l'avons vu, la plupart des propriétés susceptibles d'intéresser les géomètres des groupes ne sont pas préservées par équivalence élémentaire, notamment parce que tout groupe de type fini a la même théorie du premier ordre qu'un groupe indénombrable, en conséquence du théorème de Loś. Du point de vue de la théorie géométrique des groupes, il est naturel de focaliser notre attention sur les groupes de type fini, et de nous intéresser à l'équivalence élémentaire au sein de cette classe de groupes. C'est ce que nous ferons dorénavant.

1.4. Au sein des groupes de type fini.

Exemple 1.15. Si un groupe G de type fini a la même théorie du premier ordre que $\mathbb{Z}$, alors G est isomorphe à $\mathbb{Z}$.

Preuve. Ayant la même théorie du premier ordre que $\mathbb{Z}$, le groupe G est abélien et sans torsion. Or, G est de type fini. Il est donc isomorphe à $\mathbb{Z}^m$ pour un certain entier $m \geq 1$. Démontrons que $m = 1$, c'est-à-dire que le rang est exprimable au premier ordre. Considérons pour cela l'énoncé suivant, qui est satisfait par le groupe $\mathbb{Z}$:

$$\theta : \forall x_1 \forall x_2 \forall x_3 \exists x_4 \ (x_1 + x_2 = 2x_4) \vee (x_1 + x_3 = 2x_4) \vee (x_2 + x_3 = 2x_4).$$

Cet énoncé exprime le fait que le quotient $\mathbb{Z}/2\mathbb{Z}$ est d'ordre au plus 2, ce qui distingue $\mathbb{Z}$ de $\mathbb{Z}^n$ pour $n \geq 2$. Ceci achève la preuve. $\square$

L'exemple précédent reste valable si l'on remplace $\mathbb{Z}$ par $\mathbb{Z}^n$, pour $n \geq 1$: de la même façon, on peut exprimer que le quotient $\mathbb{Z}^n/2\mathbb{Z}^n$ est d'ordre au plus 2^n. L'énoncé utilisé est de la forme $\forall\exists$, et on ne peut pas faire mieux puisque les $\mathbb{Z}^n$ partagent tous la même théorie universelle. Le lecteur pourra vérifier que l'exemple précédent reste valide si l'on remplace $\mathbb{Z}^n$ par n'importe quel groupe abélien de type fini. Cette propriété est appelée *rigidité du premier ordre* (**[ALM17]**, définition 1.2).

Définition 1.16. Un groupe Γ de type fini a la propriété de rigidité du premier ordre si tout groupe de type fini G élémentairement équivalent à Γ est isomorphe à Γ.

Un résultat d'Oger **[Oge83]** affirme que deux groupes nilpotents de type fini G et Γ sont élémentairement équivalents si et seulement si $G \times \mathbb{Z} \simeq \Gamma \times \mathbb{Z}$. Par exemple, les groupes virtuellement cycliques

$$\Gamma = \langle c, t \mid c^{25} = 1, \ tct^{-1} = c^{11} \rangle \quad \text{et} \quad G = \langle c, z \mid c^{25} = 1, \ zcz^{-1} = c^{6} \rangle$$

ne sont pas isomorphes, mais ils sont élémentairement équivalents car les groupes

$$\Gamma \times \mathbb{Z} = \langle \Gamma, s \mid [s, c] = [s, t] = 1 \rangle \quad \text{et} \quad G \times \mathbb{Z} = \langle G, y \mid [y, c] = [y, z] = 1 \rangle$$

sont quant à eux isomorphes. En effet, puisque la matrice

$$\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}$$

est inversible, le morphisme $\varphi : \Gamma \times \mathbb{Z} \to G \times \mathbb{Z}$ qui fixe c et envoie t sur $z^2 y$ et s sur $z^5 y^2$ est bijectif. Ces deux groupes illustrent un phénomène qui joue un rôle de premier plan dans notre classification des groupes virtuellement libres de type fini du point de vue de l'égalité des théories $\forall\exists$. Plus loin, nous prouverons l'égalité $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$ en utilisant une méthode qui se généralise bien à certains groupes virtuellement libres plus généraux.

Les groupes nilpotents de type fini n'ont donc pas la propriété de rigidité du premier ordre, mais presque : l'ensemble des groupes de type fini élémentairement équivalents à un tel groupe est fini (voir **[ALM17]**, remarque 6.2). Dans **[ALM17]**, Avni, Lubotzky et Meiri démontrent que de nombreux réseaux non uniformes de rang supérieur ont cette propriété de rigidité. C'est le cas par exemple de $\mathrm{SL}_n(\mathbb{Z})$ pour $n \geq 3$. On renvoie à la partie 4.1.1 pour quelques remarques supplémentaires à ce sujet. Notons que la situation est complètement différente dans les groupes hyperboliques : d'après **[Sel09]**, si Γ est un groupe hyperbolique non élémentaire sans torsion, le groupe $\Gamma * F_n$ a la même théorie du premier ordre que Γ, pour tout entier naturel n. Il existe donc une infinité de groupes de type fini qui ont la même théorie que Γ.

Il est également intéressant de noter que tout groupe de type fini $G = \langle g_1, \ldots, g_n \rangle$ qui a la même théorie universelle qu'un groupe de type fini virtuellement nilpotent Γ est lui-même virtuellement nilpotent. En effet, considérons un sous-groupe N de Γ nilpotent de classe c et d'indice fini p. Observons que le groupe libre $F_n = F(x_1, \ldots, x_n)$ n'a qu'un nombre fini q de sous-groupes finis d'indice $\leq p$. Pour chacun d'eux, on peut donner une partie génératrice $\{w_{i,1}(x_1, \ldots, x_n), \ldots, w_{i,k_i}(x_1, \ldots, x_n)\}$ avec $1 \leq i \leq q$. Pour tout $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, le sous-groupe $H = \langle \gamma_1, \ldots, \gamma_n \rangle$ satisfait $[H : H \cap N] \leq p$. Donc $H \cap N$ est nilpotent d'indice $\leq p$ dans H, de classe de nilpotence $\leq c$. Par conséquent, il existe un entier $i \in [\![1, q]\!]$ tel que $\langle w_{i,1}(\gamma_1, \ldots, \gamma_n), \ldots, w_{i,k_i}(\gamma_1, \ldots, \gamma_n) \rangle$ est nilpotent de classe $\leq c$. Or, un groupe engendré par une partie Y est nilpotent de classe $\leq c$ si et seulement si, pour tout $(y_1, \ldots, y_{c+1}) \in Y^{c+1}$, on a $[[\cdots[[y_1, y_2], y_3] \cdots y_c] y_{c+1}] = 1$. L'énoncé

$$\forall \gamma_1 \cdots \forall \gamma_n \bigvee_{i=1}^{q} \langle w_{i,1}(\gamma_1, \ldots, \gamma_n), \ldots, w_{i,k_i}(\gamma_1, \ldots, \gamma_n) \rangle \text{ est nilpotent de classe } \leq c$$

peut donc être traduit par un énoncé universel satisfait par Γ. Cet énoncé est donc encore satisfait par G, qui est par conséquent virtuellement nilpotent de classe $\leq c$.

Ainsi, en vertu d'un célèbre théorème de Gromov, la croissance polynomiale est une propriété du premier ordre (au sein des groupes de type fini). Nous ignorons ce qu'il en est de la croissance exponentielle ou intermédiaire.

Venons-en maintenant aux groupes qui nous intéressent plus particulièrement, à savoir les groupes hyperboliques.

2. Le problème de Tarski

Nous avons vu plus haut que l'on peut exprimer le rang d'un groupe abélien de type fini au moyen d'un énoncé du premier ordre (parmi les groupes de type fini). Ce phénomène reste vrai pour les groupes nilpotents ou résolubles libres de type fini : deux tels groupes sont élémentairement équivalents si et seulement s'ils sont isomorphes, d'après [**RSS86**]. Plus précisément, on peut distinguer les groupes nilpotents libres de type fini avec des énoncés $\forall\exists$, et les groupes résolubles libres de type fini avec des énoncés $\forall\exists\forall$. La question suivante, posée par Tarski, revient donc à demander ce qu'il advient de ce résultat dans le cas non commutatif.

Question 2.1 (Tarski). Les groupes libres sont-ils élémentairement équivalents ?

On trouve la première trace écrite du problème de Tarski, semble-t-il, dans un article de Vaught paru en 1955 [**Vau55**], dans lequel l'auteur démontre que les plongements naturels entre groupes libres de rang infini sont élémentaires (en particulier, ces groupes sont élémentairement équivalents).[1] Il est ensuite fait mention du problème dans [**TV58**]. La tradition orale fait toutefois remonter la question à la décennie précédente (autour de 1945, lit-on souvent). Mentionnons qu'à la même époque, Tarski formula un second problème concernant la décidabilité de la théorie du premier ordre des groupes libres : existe-t-il un algorithme qui décide si un énoncé du premier ordre dans le langage des groupes appartient à la théorie des groupes libres ? Rappelons que la théorie du premier ordre d'un groupe abélien est décidable (Szmielew 1955), tandis que la théorie des groupes finis est indécidable (Malcev 1961). Kharlampovich et Myasnikov ont proposé une preuve de la décidabilité de la théorie des groupes libres. Dans ce qui suit, nous nous intéressons uniquement au problème de Tarski sur l'équivalence élémentaire des groupes libres (de type fini).

Les premiers progrès significatifs sont accomplis par Merzlyakov. Dans [**Mer66**], il établit que les groupes libres ont la même théorie positive, autrement dit qu'ils satisfont les mêmes énoncés du premier ordre sans inéquations. Il introduit pour cela une notion de

[1] On ne parlait pas d'équivalence élémentaire à l'époque, mais d'équivalence arithmétique.

solution formelle qui sera plus tard généralisée par Sela et par Kharlampovich et Myasnikov dans le cas où les inéquations sont autorisées, et jouera un rôle essentiel dans la solution finale du problème de Tarski. Quelques années plus tard, Sacerdote généralise le résultats de Merzlyakov et prouve dans [Sac73a] que tous les groupes qui se décomposent non trivialement sous forme d'un produit libre ont la même théorie positive, à l'exception du groupe diédral infini $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. La même année, Sacerdote prouve dans [Sac73b] que les groupes libres ont la même théorie $\forall\exists$. Les approches de Merzlyakov et Sacerdote reposent de façon cruciale sur la théorie de la petite simplification, développée par divers auteurs depuis le milieu du siècle dernier, s'inspirant des travaux de Dehn sur les groupes de surfaces (voir par exemple [Tar49], [Gre60], [Lyn66], [Sch68], et [Cou] pour une présentation moderne). La petite simplification joue également un rôle clé dans notre classification des groupes virtuellement libres du point de vue de l'égalité des théories $\forall\exists$ (on généralise notamment le lemme principal de [Sac73b] dans le cadre des groupes hyperboliques avec torsion). On renvoie le lecteur au chapitre 3.

Dans une direction différente, des progrès essentiels sont accomplis par Makanin en 1982. Il démontre dans [Mak82] que lorsqu'un système d'équations a une solution dans un groupe libre, ce système a une solution de longueur bornée, avec une borne calculable qui dépend uniquement des coefficients du système. Il en tire un algorithme qui décide si un système d'équations dans un groupe libre a une solution. Par la suite, il démontre que les théories positive et existentielle d'un groupe libre sont décidables (voir [Mak84]). Razborov va plus loin et décrit l'ensemble des solutions d'un système d'équations dans un groupe libre, dans [Raz84], en généralisant les méthodes de Makanin. On comprend aisément que cette étude des solutions d'un système d'équations dans un groupe libre est une étape essentielle à la compréhension de la théorie du premier ordre des groupes libres, puisque les systèmes d'équations sont les briques élémentaires constituant les énoncés du premier ordre dans le langage des groupes.

L'algorithme de Makanin-Razborov sera reformulé et généralisé par Sela au cas où le groupe libre étudié est remplacé par un groupe hyperbolique sans torsion. Rappelons qu'un groupe est dit *hyperbolique* (au sens de Gromov) s'il agit géométriquement (c'est-à-dire proprement discontinûment, cocompactement, par isométries) sur un espace métrique *δ-hyperbolique* (au sens de Gromov), i.e. un espace métrique propre géodésique dont les triangles géodésiques sont δ-fins : tout côté d'un triangle géodésique est inclus dans la réunion des δ-voisinages des deux autres côtés de ce même triangle, la constante δ étant bien entendu uniforme. Les exemples les plus fondamentaux de groupes hyperboliques sont les groupes libres et les groupes fondamentaux de variétés hyperboliques fermées.

Expliquons brièvement comment les arbres réels entrent en scène. Considérons un groupe $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n) = 1\rangle$ de présentation finie, où $\Sigma(s_1, \ldots, s_n) = 1$ désigne un système fini d'équations en n variables, autrement dit une conjonction d'égalités de la forme $w(s_1, \ldots, s_n) = 1$, et donnons-nous un groupe hyperbolique Γ. Faisons l'observation suivante : l'ensemble $\mathrm{Hom}(G, \Gamma)$ des morphismes de groupes de G vers Γ est en correspondance bijective avec l'ensemble des n-uplets $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ tels que $\Sigma(\gamma_1, \ldots, \gamma_n) = 1$. Ainsi, comprendre l'ensemble des solutions de ce système d'équations dans Γ revient à comprendre l'ensemble $\mathrm{Hom}(G, \Gamma)$. Or, par un procédé dû à Bestvina et Paulin (voir [Bes88] et [Pau88]), certaines suites de morphismes $(\varphi_n : G \to \Gamma)_{n\in\mathbb{N}}$ donnent lieu à une action isométrique de G sur un arbre réel. Grosso modo, il s'agit de renormaliser l'action de G induite par φ_n sur le graphe de Cayley de Γ à l'aide d'un facteur de normalisation bien choisi, et de remarquer que ce graphe de Cayley, vu de très loin, ressemble de plus en plus à un arbre. À la limite, on obtient un arbre réel T, muni d'une action de G. Pour un peu plus de détails sur ce procédé, on renvoie le lecteur aux préliminaires du chapitre 3, ou aux articles de référence susmentionnés.

La *machine de Rips* pour les actions de groupes sur les arbres réels, inspirée des outils développés par Makanin pour résoudre les équations dans F_n, permet ensuite d'analyser finement l'action de G sur T (voir [**GLP94**], [**RS94**], [**BF95**], [**Sel97**], [**Gui08**]). La technique dite du *raccourcissement*, introduite par Sela, combinée à sa théorie du scindement JSJ (voir [**GL17**] pour une exposition de cette théorie dans un cadre très général), fournit une description complète de l'ensemble $\mathrm{Hom}(G,\Gamma)$ que l'on nomme *diagramme de Makanin-Razborov*. Nous ne pousserons pas plus avant les explications ici. Pour de plus amples détails dans le cas où Γ est un groupe libre, le lecteur intéressé pourra consulter [**CG05**] ou encore [**Pau04**]. La technique du raccourcissement a par la suite été généralisée par Reinfeldt et Weidmann dans le cas où Γ est un groupe hyperbolique avec torsion dans [**RW14**]. Nous utiliserons abondamment tous ces résultats dans la suite du manuscrit (voir par exemple le théorème 3.6 un peu plus bas).

Une solution complète au problème de Tarski a finalement été apportée par Sela dans [**Sel06b**] et par Kharlampovich et Myasnikov dans [**KM06**].

Théorème 2.2. *Les groupes libres non abéliens sont élémentairement équivalents.*

Outre la généralisation de l'algorithme de Makanin et Razborov, la solution de Sela comprend une étude de la théorie $\forall\exists$ des groupes libres (voir [**Sel04**]), combinée à une élimination des quantificateurs : il montre que tout énoncé du premier ordre est équivalent à un énoncé $\forall\exists$ (voir [**Sel05**] et [**Sel06a**]). Sela a ensuite largement étendu ses résultats aux groupes hyperboliques sans torsion, ce qui nous conduit à la partie suivante.

3. Hyperbolicité et équivalence élémentaire

L'un des résultats majeurs obtenus par Sela dans sa série de travaux sur le problème de Tarski est une classification des groupes de type fini élémentairement équivalents à un groupe hyperbolique sans torsion donné. La corollaire suivant est non moins remarquable.

Théorème 3.1 (Sela). *Tout groupe de type fini qui est élémentairement équivalent à un groupe hyperbolique sans torsion est lui-même un groupe hyperbolique sans torsion.*

Ce résultat se révèle particulièrement frappant si l'on songe que l'hyperbolicité d'un groupe est une propriété purement géométrique, qui n'a à première vue aucune raison de se laisser capturer par des énoncés du premier ordre. Dans ce contexte, une question légitime est celle de l'influence de la torsion sur la préservation de l'hyperbolicité par équivalence élémentaire. Notre premier résultat montre que la restriction aux groupes sans torsion est superflue.

Théorème 3.2. *Tout groupe de type fini qui est élémentairement équivalent à un groupe hyperbolique est lui-même hyperbolique. Plus précisément, si Γ est un groupe hyperbolique et G est un groupe de type fini tels que $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$, alors G est hyperbolique.*

Par des techniques similaires, on démontre que la propriété d'être un sous-groupe d'un groupe hyperbolique est préservée par équivalence élémentaire, parmi les groupes de type fini. Plus précisément, on démontre le théorème suivant.

Théorème 3.3. *Soient Γ un sous-groupe d'un groupe hyperbolique, et G un groupe de type fini. Si $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, alors G se plonge lui aussi dans un groupe hyperbolique.*

Un groupe est dit *cubulable* s'il agit proprement discontinûment cocompactement par isométries sur un complexe cubique CAT(0) localement fini. La théorie des groupes qui sont à la fois hyperboliques et cubulables a été développée par Wise, Haglund et d'autres au cours des vingt dernières années. Ces groupes possèdent des propriétés tout à fait remarquables. Ils sont au cœur de la récente preuve par Agol [**Ago13**] de la conjecture dite *virtuellement Haken*, l'une des plus importantes questions ouvertes concernant la topologie des variétés de dimension 3, formulée en 1968. Agol démontre notamment que les groupes hyperboliques et cubulables sont virtuellement spéciaux, donc linéaires.

Dans le premier chapitre, on démontre le résultat suivant.

THÉORÈME 3.4. *Soient* Γ *un groupe hyperbolique et* G *un groupe de type fini. Supposons que* $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$*. Alors* Γ *est cubulable si et seulement si* G *est cubulable.*

Rappelons qu'un sous-groupe H d'un groupe de type fini G est dit *quasi-convexe* s'il est quasi-convexe en tant que sous-ensemble du graphe de Cayley X de G (pour une partie génératrice finie de G donnée), autrement dit s'il existe une constante C telle que toute géodésique de X reliant deux points de H est contenue dans le C-voisinage de H.

On dit qu'un groupe de type fini G est *localement quasi-convexe* si tous ses sous-groupes de type fini sont quasi-convexes. Dans le cas où G est hyperbolique et localement quasi-convexe, tous ses sous-groupes de type fini sont hyperboliques, puisqu'un sous-groupe quasi-convexe d'un groupe hyperbolique est encore hyperbolique, et on dit alors que G est *localement hyperbolique*. Soulignons qu'un groupe hyperbolique n'est pas localement hyperbolique en général, comme l'illustre le fameux exemple dû à Rips (voir [**Rip82**]) d'un sous-groupe d'un groupe hyperbolique qui est de type fini sans pour autant être hyperbolique (*a fortiori*, ce sous-groupe n'est pas quasi-convexe).

La classe des groupes qui sont à la fois hyperboliques et localement quasi-convexes (donc localement hyperboliques) est riche ; elle contient nombre de groupes intéressants : les groupes virtuellement libres, les groupes fondamentaux de variétés hyperboliques connexes fermées de dimension 2 ou 3, de nombreux groupes à petite simplification (voir [**MW05**] et [**MW08**]), ou encore certains groupes admettant une présentation finie avec une seule relation. Par exemple, si S est un ensemble fini, et si w est un élément non trivial du groupe libre $F(S)$, il existe un entier r_0 tel que, pour tout entier $r \geq r_0$, le groupe $G = \langle S \mid w^r = 1\rangle$ est hyperbolique, cubulable et localement quasi-convexe ; plus précisément, G est hyperbolique pour tout $r \geq 2$, cubulable pour tout $r \geq 4$ et localement quasi-convexe pour tout $r \geq 3|w|_S$, voir [**MW05**].

On démontre une variante des théorèmes 3.2 et 3.4 dans le cas où le groupe Γ est localement hyperbolique.

THÉORÈME 3.5. *Soient* Γ *un groupe localement hyperbolique et* G *un groupe de type fini. Si* $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$*, alors* G *est localement hyperbolique. De plus,* Γ *est cubulable si et seulement si* G *est cubulable.*

3.1. Esquisse d'une preuve.

Nous nous proposons ici de donner une ébauche des preuves des théorèmes 3.2, 3.3 et 3.5. Elles reposent de façon essentielle sur la technique du raccourcissement inventée par Sela dans le cadre des groupes hyperboliques sans torsion, puis étendue aux groupes hyperboliques en toute généralité par Reinfeldt et Weidmann. En voici une version qui nous sera suffisante pour le moment.

THÉORÈME 3.6 (Sela, Reinfeldt-Weidmann). *Soit* Γ *un groupe hyperbolique, soit* G *un groupe de type fini, à un bout. Il existe un sous-ensemble fini de* $F \subset G \setminus \{1\}$ *avec la propriété suivante : pour tout morphisme non injectif* $\phi \in \mathrm{Hom}(G, \Gamma)$*, il existe un automorphisme* σ *de* G *tel que* $\ker(\phi \circ \sigma) \cap F \neq \varnothing$*.*

Le lecteur intéressé trouvera une formulation plus générale du résultat énoncé ci-dessus dans le chapitre 1, qui traite aussi du cas où ϕ est injectif. Dans un premier temps, nous allons esquisser une preuve des théorèmes 3.2 et 3.3 dans le cas particulier où G est un groupe de présentation finie, à un bout, avec $\mathrm{Out}(G)$ trivial. Nous commenterons ensuite cette preuve et tâcherons de mettre en évidence les difficultés principales qui surviennent lorsque l'on veut l'étendre au cas général.

PROPOSITION 3.7. *Soit* Γ *un groupe hyperbolique, soit* G *un groupe de présentation finie, à un bout. Supposons que* $\mathrm{Th}_{\forall}(\Gamma) \subset \mathrm{Th}_{\forall}(G)$*. Si* $\mathrm{Out}(G)$ *est trivial, le groupe* G *se plonge dans* Γ*. Si, de plus,* Γ *est à un bout et* $\mathrm{Out}(\Gamma)$ *est trivial, les groupes* G *et* Γ *sont isomorphes.*

Preuve. Soit $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n)\rangle$ une présentation finie de G, où Σ désigne un système fini d'équations en n variables, autrement dit une conjonction d'égalités de la forme $w(s_1, \ldots, s_n) = 1$, où $w(s_1, \ldots, s_n)$ désigne un élément du groupe libre $F(s_1, \ldots, s_n)$. Observons que l'ensemble $\mathrm{Hom}(G, \Gamma)$ est en correspondance bijective avec l'ensemble des n-uplets $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ tels que $\Sigma(\gamma_1, \ldots, \gamma_n) = 1$. Notons $g_1, \ldots, g_p$ les éléments du sous-ensemble fini F de $G \setminus \{1\}$ donné par le théorème 3.6. Tout élément g_k est un mot $w_k(s_1, \ldots, s_n)$. Raisonnons par l'absurde et supposons que le groupe G ne se plonge pas dans Γ. Alors tout morphisme de G vers Γ est nécessairement non injectif, donc tue l'un des g_k en vertu du théorème 3.6. En d'autres termes, le groupe Γ satisfait l'énoncé universel suivant :

$$\forall x_1 \ldots \forall x_n \ (\Sigma(x_1, \ldots, x_n) = 1) \Rightarrow ((w_1(x_1, \ldots, x_n) = 1) \vee \ldots \vee (w_k(x_1, \ldots, x_n) = 1)).$$

Puisque $\mathrm{Th}_\forall(\Gamma) \subset \mathrm{Th}_\forall(G)$, cet énoncé est vrai dans G aussi. En particulier, en prenant $x_1 = s_1, \ldots, x_n = s_n$, l'énoncé nous dit que l'un des éléments g_k de F est le neutre de G, ce qui est faux. Par conséquent, nous venons de démontrer que G se plonge dans Γ.

Pour le moment, rien ne nous permet de conclure quant à l'hyperbolicité de G (en effet, rappelons qu'un sous-groupe de présentation finie d'un groupe hyperbolique n'a aucune raison d'être hyperbolique en général, voir par exemple [**Bra99**]). Cependant, si l'on suppose en outre que Γ est à un bout et que son groupe d'automorphismes extérieurs est trivial, alors on peut démontrer de la même façon que dans le paragraphe qui précède que Γ se plonge dans G (après avoir observé que le théorème 3.6 reste vrai pour les sous-groupes des groupes hyperboliques). Comme les groupes hyperboliques à un bout sont co-hopfiens (voir ci-dessous), les groupes Γ et G sont isomorphes. Cela achève la démonstration du cas particulier 3.7. □

Le caractère co-hopfien d'un groupe hyperbolique à un bout a été établi par Sela dans [**Sel97**] dans le cas sans torsion, puis par Moioli dans sa thèse [**Moi13**] dans le cas général, utilisant un résultat de [**Del95**]. On donne ci-dessous une preuve dans le cas particulier où $\mathrm{Out}(G)$ est fini.

Théorème 3.8. *Tout groupe hyperbolique G à un bout, avec* $\mathrm{Out}(G)$ *fini, est co-hopfien.*

Remarque 3.9. Le groupe des automorphismes extérieurs d'un groupe hyperbolique sans torsion est fini si et seulement si ce groupe ne se scinde pas sur un groupe cyclique maximal (voir [**Pau91**]). Ce résultat n'est plus vrai en présence de torsion. Dans [**GL15**] (théorème 7.14), Guirardel et Levitt démontrent qu'un groupe hyperbolique a un groupe d'automorphismes extérieurs fini si et seulement s'il se scinde sur un groupe virtuellement cyclique à centre infini qui est maximal parmi les groupes virtuellement cycliques à centre infini.

Preuve. Comme $\mathrm{Out}(G)$ est fini, il n'y a qu'un nombre fini d'injections de G dans lui-même, à post-conjugaison près par un automorphisme intérieur (voir théorème 2.5). Par conséquent, étant donné un monomorphisme $\phi : G \hookrightarrow G$, il existe deux entiers $k, n \geq 1$ et un élément $g \in G$ tels que $\phi^{k+n} = \mathrm{ad}(g) \circ \phi^n$. Soit h un élément de G. On a

$$\phi^n(\phi^{k+n}(h)) = \phi^k(\phi^n(\phi^n(h))) = g\phi^{2n}(h)g^{-1}, \text{ et}$$
$$\phi^n(\phi^{k+n}(h)) = \phi^n(g\phi^n(h)g^{-1}) = \phi^n(g)\phi^{2n}(h)(\phi^n(g))^{-1}.$$

Ainsi, $g^{-1}\phi^n(g)$ appartient au centralisateur dans G de $\phi^{2n}(G)$. Puisque $\phi^{2n}(G)$ est non élémentaire, son centralisateur dans G est fini.

Si G est sans torsion, le centralisateur de $\phi^{2n}(G)$ est trivial, d'où $\phi^n(g) = g$. On a donc $\phi^{k+n} = \mathrm{ad}(g) \circ \phi^n = \phi^n \circ \mathrm{ad}(g)$, d'où $\mathrm{Im}(\phi^n) = \mathrm{Im}(\phi^{k+n})$, ce qui prouve que ϕ est bijectif.

S'il y a de la torsion, l'élément $g^{-1}\phi^n(g) = a$ est d'ordre fini, disons p. Rappelons que ϕ^k coïncide avec $\mathrm{ad}(g)$ sur $\phi^n(G)$. Par conséquent, ϕ^{kp} coïncide avec $\mathrm{ad}(g^p)$ sur $\phi^n(G)$. Or, $g^p = (\phi^n(g)a^{-1})^p = (\phi^n(g))^p a^{-p}$ puisque a centralise $\phi^n(G)$, donc commute avec $\phi^n(g)$.

Ainsi, $g^p = (\phi^n(g))^p = \phi^n(g^p)$. On a donc démontré que ϕ^{kp} coïncide avec $\mathrm{ad}(\phi^n(g^p))$ sur $\phi^n(G)$, ce qui conclut quant à la bijectivité de ϕ. $\square$

Dans la preuve de la proposition 3.7, l'hypothèse que G est de présentation finie est trop restrictive, mais elle peut être relâchée. En effet, les groupes hyperboliques sont connus pour être équationnellement noethériens ([**Sel09**] et [**RW14**]), ce qui signifie qu'ils ont la propriété remarquable suivante : tout système infini d'équations Σ (en un nombre uniformément fini de variables) est équivalent à un sous-système fini de Σ. Cela nous permet de supposer sans perte de généralité que le groupe G est seulement de type fini. Considérons en effet une présentation de G de la forme $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n) = 1\rangle$, où $\Sigma(s_1, \ldots, s_n) = 1$ est un système infini d'équations en les variables $s_1, \ldots, s_n$. Pour tout entier naturel $i \geq 1$, désignons par Σ_i le système d'équations composé des i premières équations de Σ, et posons $G_i = \langle s_1, \ldots, s_n \mid \Sigma_i(s_1, \ldots, s_n) = 1\rangle$. Puisque le groupe Γ est équationnellement noethérien, les solutions dans Γ^n du système d'équations $\Sigma(s_1, \ldots, s_n) = 1$ sont exactement les solutions du système $\Sigma_i(s_1, \ldots, s_n) = 1$, dès que l'entier i est suffisamment grand. Les ensembles $\mathrm{Hom}(G, \Gamma)$ et $\mathrm{Hom}(G_i, \Gamma)$ sont donc en bijection.

Remarque 3.10. Si G est linéaire, on peut voir facilement qu'il est équationnellement noethérien. Considérons un système infini d'équations, noté $\Sigma(x_1, \ldots, x_n) = 1$. Pour tout entier $k \geq 1$, notons Σ_k le sous-système fini constitué des k premières équations de Σ. Si G est un sous-groupe de $\mathrm{GL}_N(K)$ où K est un corps, toute équation $w_j(x_1, \ldots, x_n) = 1$ peut être vue comme la donnée de N^2 polynômes $P_{j,1}, \ldots, P_{j,N^2}$ de $K[X_1, \ldots, X_{nN^2}]$ (où les inconnues X_i correspondent aux coefficients de n matrices de $\mathrm{GL}_N(K)$). L'ensemble des solutions dans G de Σ_k est $G \cap Z(I_k)$, où I_k est l'idéal de $K[X_1, \ldots, X_{nN^2}]$ engendré par les $P_{j,1}, \ldots, P_{j,N^2}$, avec $1 \leq j \leq k$. Puisque K est un corps, l'anneau $K[X_1, \ldots, X_{nN^2}]$ est noethérien. Comme la suite (I_k) est croissante, elle est stationnaire, donc la suite $(Z(I_k))$ l'est également, ce qui conclut. En fait, la plupart des groupes hyperboliques auxquels on peut penser sont linéaires, donc équationnellement noethériens : les groupes libres, les groupes fondamentaux de variétés hyperboliques, plus généralement les groupes hyperboliques et cubulables grâce à [**Ago13**]. En fait, on connait très peu de groupes hyperboliques non linéaires. Les seuls exemples, à notre connaissance, sont des quotients à petite simplification de $\mathrm{Sp}(1, n)$ (voir [**Kap05**], partie 8). Dans [**OW11**], les auteurs démontrent que les groupes qui sont à la fois hyperboliques et cubulables sont génériques parmi les groupes de présentation finie, en un certain sens.

Remarque 3.11. Le groupe $BS(2, 4) = \langle a, t \mid ta^2t^{-1} = a^4\rangle$ n'est pas équationnellement noethérien. Par récurrence immédiate, on a $t^k a^2 t^{-k} = a^{2^{k+1}}$ pour tout entier $k \geq 0$. Posons $y_k = a^{2^{k+1}}$. Pour tout $i \leq k$, l'élément $t^{-i} y_k t^i$ est donc une puissance de a, donc commute avec a. Mais $t^{-(k+1)} y_k t^{k+1} = t^{-1} a^2 t$ ne commute pas avec a. Notons Σ le système infini d'équations $w_i(x, y, z) = [x^{-i} y_k x^i, z] = 1$ avec $i \geq 1$. Pour tout k, (t, y_k, a) est solution de Σ_k, mais pas de Σ_{k+1}. Cela achève la preuve. Soulignons que le groupe $BS(2, 4)$ est un exemple prototypique de groupe non hyperbolique : dans un groupe hyperbolique sans torsion, la relation $tat^{-1} = a^2$ implique $a = 1$.

Notons que la preuve de 3.7 s'adapte facilement au cas où $\mathrm{Out}(G)$ est fini : on peut en effet écrire
$$\mathrm{Aut}(G) = \bigsqcup_{1 \leq i \leq \ell} \varphi_i \mathrm{Int}(G)$$
et poser
$$F' := \bigsqcup_{1 \leq i \leq \ell} \varphi_i(F).$$
Qu'en est-il si le groupe $\mathrm{Out}(G)$ est infini ? Dans ce cas, la preuve ne fonctionne plus, car on ne peut plus exprimer l'intégralité de l'énoncé du théorème 3.6 au moyen d'un énoncé du premier ordre, comme nous l'avons fait dans la preuve de 3.7 exposée plus haut. On touche

aux limites du pouvoir d'expression de la logique du premier ordre : il est impuissant à décrire la précomposition d'un morphisme par un automorphisme. Néanmoins, on peut exprimer un fragment du théorème 3.6, à la manière de Perin dans [**Per11**], et démontrer la dichotomie suivante (dont nous ne détaillerons pas la preuve ici).

Proposition 3.12. *Soit Γ un groupe hyperbolique, soit G un groupe de type fini à un bout tel que* $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. *Alors,*

- *soit G se plonge dans Γ,*
- *soit G se scinde en un graphe de groupes particulier, que nous appellerons un* quasi-étage*, au-dessus d'un certain groupe G_1.*

Remarque 3.13. Dans cette thèse, le graphe sous-jacent à un graphe de groupes est toujours supposé connexe.

Les quasi-étages, dont nous donnerons une définition un peu plus loin, sont une généralisation des étages hyperboliques. Nous allons dans un premier temps ébaucher des preuves des théorèmes 3.2 et 3.3 en faisant l'hypothèse que les groupes étudiés sont sans torsion.

3.1.1. *En l'absence de torsion.*

Les *tours hyperboliques* ont été introduites par Sela dans [**Sel01**] (voir la définition 6.1) sous le nom de tours ω-résiduellement libres (on trouve la notion analogue de groupes NTQ introduite dans [**KM05**] par Kharlampovich et Myasnikov). Les tours sont des actrices majeures dans la preuve par Sela du problème de Tarski sur l'équivalence élémentaire des groupes libres [**Sel06b**]. Sela les utilise également dans [**Sel09**] pour classifier tous les groupes de type fini qui ont la même théorie du premier ordre qu'un groupe hyperbolique donné. Grosso modo, une tour hyperbolique est obtenue par empilement successif d'étages hyperboliques, et un *étage hyperbolique* est un groupe obtenu en collant une surface à bord sur un autre groupe, de telle sorte qu'il existe une rétraction du gros groupe sur le petit. Donnons une définition plus formelle.

Définition 3.14. Soit Σ une surface compacte connexe à bord (de caractéristique d'Euler ≤ -2, ou un tore percé). Soit S le groupe fondamental de Σ, et soient $B_1, \ldots, B_n$ les groupes de bord (bien définis à conjugaison près). Soit H un groupe et soient $h_1, \ldots, h_n$ des élements de H d'ordre infini. On considère un graphe de groupes à deux groupes de sommets S et H, et n arêtes reliant les deux sommets et identifiant B_i avec $\langle h_i \rangle$ pour tout $1 \leq i \leq n$. Appelons G le groupe fondamental de ce graphe de groupes. On dit que G est un étage hyperbolique sur H s'il existe une rétraction $r : G \to H$ (telle que $r(S)$ est non abélien). Voir Figure 1 ci-après.

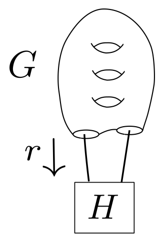


Figure 1. Le groupe G est un étage hyperbolique sur H.

On dit aussi que G est un étage hyperbolique sur H dans le cas où il est de la forme $G = H * S$ ou $G = H * F_n$, où S est un groupe de surface hyperbolique fermée de caractéristique d'Euler ≤ -2, et F_n est le groupe libre de rang n. On dit que G est une tour hyperbolique sur H s'il existe une suite finie $(G_k)_{1\leq k\leq n}$ telle que $G_1 = G$, $G_n = H$, et G_k est un étage hyperbolique sur G_{k+1} pour tout $1 \leq k \leq n-1$.

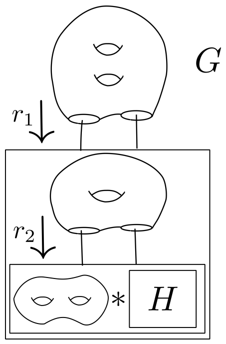


FIGURE 2. Le groupe G est une tour hyperbolique sur H.

REMARQUE 3.15. Il existe en fait une notion légèrement plus générale introduite par Perin dans l'erratum [**Per13**] sous le nom d'*étage (ou tour) hyperbolique étendu*, pour corriger un défaut de la définition originale. Cependant, nous ne détaillerons pas cette subtilité ici.

Soit G un groupe, et soit H un sous-groupe de G. Supposons que G est une tour hyperbolique sur H. L'observation qui suit est essentielle : si G est hyperbolique, alors H est hyperbolique puisque l'hyperbolicité d'un groupe est préservée par rétraction ; réciproquement, si H est hyperbolique, il découle du théorème de combinaison de Bestvina et Feighn [**BF92**] que G est hyperbolique. Par conséquent, si un groupe G est une tour hyperbolique sur un sous-groupe H, l'un des deux groupes est hyperboliques si et seulement si l'autre l'est. De façon tout à fait analogue, en utilisant un théorème de combinaison dû à Hsu et Wise [**HW15**], on peut démontrer que si G est une tour hyperbolique sur H, l'un des deux groupes est hyperbolique et cubulable si et seulement si l'autre l'est. En fait, tout ceci est encore vrai si l'on considère la propriété de se plonger dans un groupe hyperbolique plutôt que celle d'être hyperbolique.

En utilisant la dichotomie de la proposition 3.12, nous allons tout d'abord esquisser une preuve du théorème 3.3 dans le cas où G est sans torsion. En l'absence d'éléments d'ordre fini, un quasi-étage est peu ou prou équivalent à un étage hyperbolique (en un sens précisé dans le premier chapitre, voir proposition 5.5). Notamment, on peut démontrer que si nos groupes sont sans torsion, la proposition 3.12 reste vraie si l'on remplace « quasi-étage » par « étage hyperbolique ». En particulier, avec les notations de la proposition 3.12, si G est sans torsion, on peut supposer que G_1 est un sous-groupe de G, et qu'il existe une rétraction de ce dernier sur G_1. L'existence de cette rétraction jouera un rôle très important dans le preuve de la proposition 3.16 ci-dessous (où les groupes sont supposés sans torsion), mais ne sera plus vraie en présence de torsion, comme nous le verrons plus loin. Nous supposerons également que G est à un bout, hypothèse peu coûteuse qui peut être facilement relâchée en considérant un scindement de Grushko de G, ce que nous épargnerons au lecteur ici.

PROPOSITION 3.16 (Cas particulier du théorème 3.3). *Soit Γ un groupe hyperbolique sans torsion, et soit G un groupe de type fini à un bout tel que $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Alors le groupe G se plonge dans un groupe hyperbolique.*

Rappelons qu'un groupe de type fini est dit Γ-*limite* si sa théorie existentielle est contenue dans celle du groupe hyperbolique Γ.

Le théorème de la chaîne descendante stipule que si $(G_n)_{n\in\mathbb{N}}$ est une suite de groupes Γ-limites, et si $(\phi_n : G_n \to G_{n+1})_{n\in\mathbb{N}}$ est une suite de morphismes surjectifs, alors ϕ_n est un isomorphisme à partir d'un certain rang. Il a été démontré par Sela pour Γ sans torsion, puis généralisé par Reinfeldt et Weidmann si Γ a des éléments d'ordre fini non triviaux.

Preuve. L'idée consiste à itérer la dichotomie 3.12. Si la seconde alternative de la dichotomie 3.12 a lieu, alors G est un étage hyperbolique au-dessus d'un groupe G_1 (voir le paragraphe avant l'énoncé de la proposition 3.16). En particulier, G_1 est un sous-groupe propre de G, et il existe une rétraction de G sur G_1. En outre, G_1 a la propriété essentielle que voici, qui est une simple conséquence du théorème de combinaison de Bestvina et Feighn [**BF92**] : si G_1 se plonge dans un groupe hyperbolique, alors G se plonge dans un groupe hyperbolique. Afin de démontrer que G se plonge dans un groupe hyperbolique, on veut itérer l'argument précédent avec G_1 dans le rôle de G. Cependant, $\mathrm{Th}_{\forall\exists}(\Gamma)$ n'a aucune raison *a priori* d'être contenu dans $\mathrm{Th}_{\forall\exists}(G_1)$; de plus, G_1 n'est pas un groupe à un bout en général. Nous pouvons néanmoins raffiner la proposition 3.12 et démontrer que le groupe G_1 satisfait exactement la même dichotomie. En itérant cet argument, on obtient une suite de groupes $(G_n)_{n\in\mathbb{N}}$ telle que, pour tout entier n, le groupe G_n est un étage hyperbolique sur G_{n+1}. En particulier, G_{n+1} est un sous-groupe strict de G_n, et G_n se rétracte sur G_{n+1}. Il découle du théorème de la chaîne descendante sur les groupes Γ-limites que cette suite de groupes est nécessairement finie. Notons G' le dernier groupe obtenu. Par définition, il se plonge dans Γ. Comme G est une tour sur G', il se plonge également dans un groupe hyperbolique, ce qui conclut. Notons que le théorème de la chaîne descendante s'applique bien ici puisque G_{n+1} est un quotient de G_n, et $\mathrm{Th}_{\exists}(G_n)$ est contenu dans $\mathrm{Th}_{\exists}(\Gamma)$ (en effet, $\mathrm{Th}_{\exists}(G_n)$ est contenu dans $\mathrm{Th}_{\exists}(G)$ car G_n est un sous-groupe de G, et $\mathrm{Th}_{\exists}(G)$ est contenu dans $\mathrm{Th}_{\exists}(\Gamma)$ par hypothèse). $\square$

Soulignons que si Γ est localement hyperbolique (voir la définition et les exemples dans le paragraphe qui précède l'énoncé du théorème 3.5), il découle de la proposition ci-dessus que tout groupe G de type fini à un bout tel que $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$ est hyperbolique, ce qui prouve le théorème 3.5 dans ce cas précis.

Si l'on ne sait pas que Γ est localement hyperbolique, nous avons besoin de l'inclusion réciproque $\mathrm{Th}_{\forall\exists}(G) \subset \mathrm{Th}_{\forall\exists}(\Gamma)$ pour démontrer le théorème 3.2, comme le montre la proposition ci-dessous.

PROPOSITION 3.17 (Cas particulier du théorème 3.2). *Soit Γ un groupe hyperbolique (à un bout) sans torsion, et soit G un groupe de type fini (à un bout). Supposons que $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Alors le groupe G est hyperbolique.*

Preuve. On va simplement donner les grandes lignes. Reprenons le fil de la preuve de la proposition 3.16 précédente. On a démontré que G est une tour hyperbolique sur un groupe G' qui se plonge dans Γ. On peut procéder de la même façon en échangeant Γ et G, et démontrer que Γ est une tour hyperbolique sur un groupe Γ' qui se plonge dans G. Soulignons toutefois qu'il n'est pas évident *a priori* que l'on puisse faire jouer le rôle de Γ à G, puisqu'on ne sait pas que G est hyperbolique. En fait, la condition importante dont on a besoin est seulement que G se plonge dans un groupe hyperbolique, ce que l'on vient de démontrer (on renvoie au chapitre 1 pour les détails). En raffinant ces arguments, on peut démontrer que G' se plonge dans Γ', et que Γ' se plonge dans G'. Comme Γ est hyperbolique, Γ' l'est aussi. Pour simplifier, supposons que ce groupe est à un bout, donc co-hopfien. La composition des monomorphismes $\Gamma' \hookrightarrow G' \hookrightarrow \Gamma'$ est donc un automorphisme. Il s'ensuit que le plongement $G' \hookrightarrow \Gamma'$ est surjectif, donc bijectif. Le groupe G' est donc hyperbolique, en tant que tour hyperbolique sur un groupe hyperbolique. $\square$

Le théorème de la chaîne descendante est un résultat difficile qui repose sur la théorie des actions sur les arbres réels et la technique du raccourcissement. On peut toutefois en donner une preuve élémentaire dans le cas où le groupe est linéaire (voir théorème 3.19 ci-dessous). Pour cela, nous aurons besoin de la définition suivante.

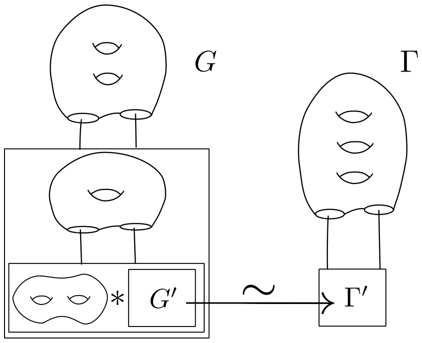


FIGURE 3. Illustration synthétique de la preuve précédente.

DÉFINITION 3.18. Soient G et Γ deux groupes de type fini. Une suite de morphismes $(\varphi_n : G \to \Gamma)_{n\in\mathbb{N}}$ est dite discriminante si, pour tout élément $g \in G$ non trivial, il existe un entier $n_0 \geq 0$ tel que $\varphi_n(g) \neq 1$ pour tout $n \geq n_0$.

On peut démontrer que si G est de présentation finie, ou si Γ est équationnellement noethérien (par exemple hyperbolique), il existe une suite discriminante $(\varphi_n : G \to \Gamma)_{n\in\mathbb{N}}$ si et seulement si $\mathrm{Th}_\exists(G) \subset \mathrm{Th}_\exists(\Gamma)$ (on renvoie à la proposition 2.1 du chapitre 1).

THÉORÈME 3.19 (Voir [**BMR99**]). *Soit Γ un sous-groupe de $\mathrm{GL}_N(K)$, où K désigne un corps. Considérons une suite $G_1 \twoheadrightarrow G_2 \twoheadrightarrow G_3 \twoheadrightarrow \cdots$ de quotients de groupes de type fini. Si les G_k sont pleinement résiduellement Γ, alors les épimorphismes sont des isomorphismes à partir d'un certain rang.*

Preuve. Notons $S = \{s_1, \ldots, s_n\}$ une partie génératrice finie du groupe G_1 et, pour tout entier $k \geq 2$, notons $\{s_1^{(k)}, \ldots, s_n^{(k)}\}$ l'image de S par la surjection de G_1 sur G_k obtenue en composant toutes les surjections de la suite. Notons $\langle s_1^{(k)}, \ldots, s_n^{(k)} \mid R_k\rangle$ une présentation de G_k, avec $R_k \subset R_{k+1}$, et considérons le sous-ensemble V_k de $\mathrm{M}_N(K)^n$ constitué des n-uplets $(M_1, \ldots, M_n)$ tels que $r(M_1, \ldots, M_n) = I_N$ pour toute relation $r \in R_k$. Chaque relation $r \in R_k$ peut être vue comme la donnée de N^2 polynômes $P_{r,1}, \ldots, P_{r,N^2} \in K[X_1, \ldots, X_{nN^2}]$ (où les inconnues X_i correspondent aux coefficients des n matrices $M_1, \ldots, M_n$). On peut écrire $V_k = Z(I_k)$, où I_k désigne l'idéal de $K[X_1, \ldots, X_{nN^2}]$ engendré par les $P_{r,i}$ pour $r \in R_k$ et $1 \leq i \leq nN^2$. Comme G_{k+1} est un quotient de G_k pour tout entier k, la suite d'idéaux (I_k) de $K[X_1, \ldots, X_{nN^2}]$ est croissante pour l'inclusion. Puisque l'anneau $K[X_1, \ldots, X_{nN^2}]$ est noethérien, il existe un entier ℓ tel que $I_k = I_\ell$, donc $Z(I_k) = Z(I_\ell)$, i.e. $V_k = V_\ell$ pour tout $k \geq \ell$. Par l'absurde, si G_k n'est pas isomorphe à G_ℓ, il existe une relation $r \in R_k \setminus R_\ell$ telle que $g = r(s_1^{(\ell)}, \ldots, s_n^{(\ell)}) \neq 1$. Notons $(\varphi_p : G_\ell \to \Gamma)_{p\in\mathbb{N}}$ une suite discriminante (voir définition 3.18). Pour p assez grand, on a $\varphi_p(g) \neq 1$, i.e. $r(M_1, \ldots, M_n) \neq 1$ avec $M_i = \varphi_p(s_i^{(\ell)})$ pour tout $1 \leq i \leq n$, d'où $(M_1, \ldots, M_n) \in V_k \setminus V_\ell$, une contradiction. □

3.1.2. *En présence de torsion.*

Nous venons d'esquisser une preuve des théorèmes 3.3 et 3.2 sous l'hypothèse que G est sans torsion (et à un bout). En présence de torsion, les étages et tours hyperboliques ne sont plus adaptés ; il nous faut les généraliser. La définition de quasi-étage que nous proposons est plus technique que celle d'un étage hyperbolique, pour des raisons qui semblent intrinsèques à la présence de torsion. On a besoin d'une notion de *graphe de groupes centré*, définie ci-dessous (on renvoie au chapitre 1 pour les définitions de sommet QH, et de groupe de bord étendu ou de groupe de singularité conique étendu). On note $\overline{\mathcal{Z}}$ la classe des groupes qui sont finis ou virtuellement cycliques à centre infini.

DÉFINITION 3.20. Un graphe de groupes sur $\overline{\mathcal{Z}}$, avec au moins deux sommets, est dit centré si les conditions suivantes sont satisfaites.

- Le graphe sous-jacent est biparti, avec un sommet QH v tel que tout sommet différent de v est adjacent à v;
- Tout groupe d'arête G_e d'une arête e incidente à v coïncide avec un groupe de bord étendu ou avec un groupe de singularité conique étendu de G_v;
- Pour tout groupe de bord étendu B, il existe une unique arête e incidente à v telle que G_e est conjugué à B dans G_v;
- Si un élément d'ordre infini fixe un segment de longueur ≥ 2 dans l'arbre de Bass-Serre du scindement, alors ce segment est de longueur exactement égale à 2 et ses extrémités sont des translatés du sommet v.

Le sommet v est appelé le sommet central.

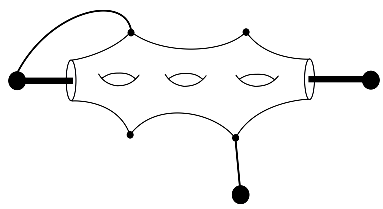


FIGURE 4. Un graphe de groupes centré. Les arêtes dont le stabilisateur est infini sont en gras.

Par exemple, un étage hyperbolique est un graphe de groupes centré (où le groupe de sommet central est un simple groupe de surface hyperbolique). Nous avons besoin de généraliser cette notion pour tenir compte de la torsion.

DÉFINITION 3.21. Soient G et H deux groupes de type fini, Δ_G un scindement centré de G, Δ_H un scindement de H sur des groupes finis, r un morphisme de G vers H et j un morphisme de H vers G. Soit V_G l'ensemble des sommets de Δ_G, et v le sommet central. Soit V_H l'ensemble des sommets de Δ_H. Soit $V_H = V_H^1 \sqcup V_H^2$ une partition de V_H et $s : V_G \setminus \{v\} \to V_H^1$ une bijection telles que les conditions suivantes sont satisfaites.

- $j \circ r$ coïncide avec id_G à conjugaison près sur chaque groupe de sommet G_w avec $w \neq v$, et sur tous les sous-groupes finis de G ;
- pour tout w dans $V_G \setminus \{v\}$, r envoie G_w isomorphiquement sur $h_w H_{s(w)} h_w^{-1}$, pour un certain élément $h_w \in H$;
- pour tout $u \in V_H^2$, H_u est fini et j est injective sur H_u.

On dit que $(G, H, \Delta_G, \Delta_H, r, j)$ est un *quasi-étage*. Si, de plus, il existe un sous-groupe à un bout $A \subset G$ tel que $A \cap \ker(r) \neq \{1\}$, le quasi-étage est dit *strict*.

En général, le contexte sera clair et on évitera la notation $(G, H, \Delta_G, \Delta_H, r, j)$, au profit d'une notation plus légère comme (G, H, r, j).

REMARQUE 3.22. Le lecteur pourrait s'étonner que l'on ne suppose pas que l'image du groupe de sommet central est non (virtuellement) abélienne. En fait, il s'avère que cette hypothèse technique que l'on trouve dans la définition d'un étage hyperbolique ne nous est d'aucune utilité pour démontrer que l'hyperbolicité est préservée par équivalence élémentaire.

Le lien entre cette définition et la définition d'un étage hyperbolique est plus apparent si l'on voit G et H comme sous-groupes d'un groupe G' obtenu à partir de G en faisant des extensions HNN sur des groupes finis, tel qu'il existe un épimorphisme $\rho : G' \twoheadrightarrow H$ satisfaisant $\rho_{|G} = r$ et $\rho_{|H} = \mathrm{id}_H$. Ce groupe G' peut être vu comme le groupe fondamental

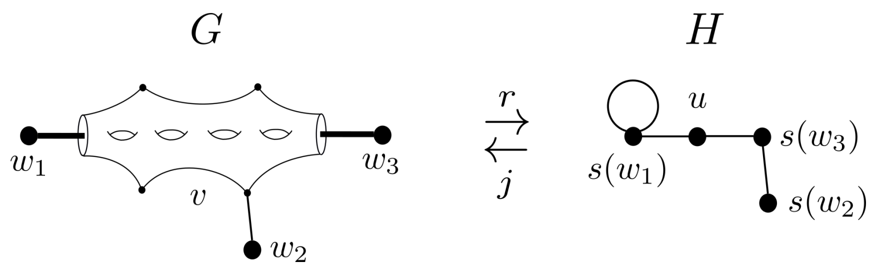


FIGURE 5. Le groupe G est un quasi-étage sur H au sens de la définition précédente. Ici $V_H^2 = \{u\}$ et $V_H^1 = \{s(w_1), s(w_2), s(w_3)\}$. Le sommet u a un stabilisateur fini. Les arêtes dont le stabilisateur est infini apparaissent en gras sur le dessin.

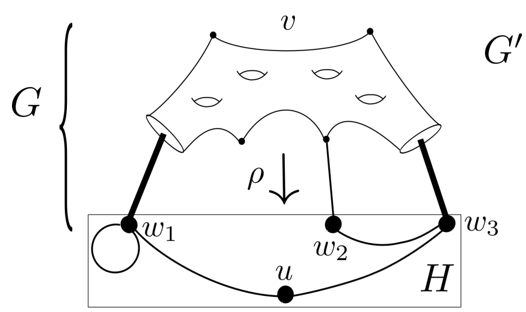


FIGURE 6. On construit un groupe G' en identifiant w et $s(w)$ pour tout $w \in V_G \setminus \{v\}$.

d'un graphe de groupes Λ obtenu à partir de Δ_G et Δ_H en identifiant chaque $w \in V_G \setminus \{v\}$ avec $s(w) \in V_H^1$. Cette construction est illustrée ci-dessous.

Malheureusement, en présence d'éléments de torsion, les quasi-étages ne possèdent pas d'aussi bonnes propriétés que les étages hyperboliques. Pour le comprendre, reprenons les notations utilisées dans la preuve de la proposition 3.16 plus haut, en supposant cette fois qu'il y a de la torsion. Alors, pour tout entier n, le groupe G_n obtenu en itérant la dichotomie de la proposition 3.12 se scinde en un graphe de groupes au-dessus de groupes finis, dont tous les groupes de sommets se plongent dans G, mais G_n ne se plonge pas lui-même dans G en général. Par conséquent, $\mathrm{Th}_\exists(G_n)$ n'a aucune raison d'être contenu dans $\mathrm{Th}_\exists(\Gamma)$ *a priori*, comme l'illustre l'exemple 3.23 ci-après. De plus, G_{n+1} n'est pas un quotient de G_n. Ces pathologies rendent plus délicate la preuve de la terminaison de la construction itérative décrite dans les paragraphes précédents, puisqu'on ne peut plus utiliser de façon directe le théorème de la chaîne descendante. On résout une partie de ces problèmes en démontrant que la classe des groupes hyperboliques se plonge dans celle des groupes hyperboliques torsion-saturés (voir définition 3.24 et théorème 3.25 ci-dessous), introduite dans le chapitre 1.

3.2. Un phénomène nouveau.

En l'absence de torsion, faire une extension HNN au-dessus d'un groupe fini revient à faire un produit libre avec $\mathbb{Z}$, et tout groupe hyperbolique sans torsion G a la même théorie universelle que $G * \mathbb{Z}$ (voir proposition 2.22). En fait, ils ont la même théorie du premier ordre, d'après [**Sel09**]. Par opposition, en présence de torsion, faire une extension HNN au-dessus d'un sous-groupe fini, même trivial, peut modifier la théorie universelle du groupe hyperbolique considéré. Voici une manifestation simple de ce phénomène.

EXEMPLE 3.23. Posons $G = F_2 \times \mathbb{Z}/2\mathbb{Z}$. L'énoncé $\forall x \forall y (x^2 = 1) \Rightarrow (xy = yx)$ est satisfait par G, mais pas par $G * \mathbb{Z}$.

Cet exemple montre qu'en général, la classe des groupes qui ont la même théorie universelle qu'un groupe hyperbolique donné n'est pas préservée par extensions HNN et produits amalgamés au-dessus d'un groupe fini. On résout ce problème en démontrant le résultat suivant, qui est crucial pour prouver que notre construction (esquissée plus haut) s'arrête en un nombre fini d'itérations.

Définition 3.24. Un groupe hyperbolique Γ est dit torsion-saturé si la classe des groupes Γ-limites est préservée par extensions HNN et produits amalgamés au-dessus d'un groupe fini.

Théorème 3.25. *Tout groupe hyperbolique Γ se plonge dans un groupe hyperbolique torsion-saturé $\overline{\Gamma}$.*

La construction du groupe hyperbolique torsion-saturé $\overline{\Gamma}$ consiste à faire suffisamment d'extensions HNN sur des groupes finis, de sorte que les deux conditions suivantes soient satisfaites :

- tous les sous-groupes finis isomorphes de $\overline{\Gamma}$ sont conjugués ;
- tout automorphisme d'un sous-groupes fini de $\overline{\Gamma}$ est induit par un automorphisme intérieur de $\overline{\Gamma}$.

On renvoie à la partie 4 du chapitre 1 pour les détails de la construction et des propriétés satisfaites par le groupe $\overline{\Gamma}$. Le point crucial est que la théorie universelle de ce groupe $\overline{\Gamma}$ n'est pas modifiée lorsque l'on fait une extension HNN sur un groupe fini. Dans ce groupe, une partie des difficultés techniques auxquelles nous étions confrontés avec le groupe Γ s'évanouissent.

Dans le chapitre 3, nous aurons de nouveau affaire aux extensions HNN au-dessus de sous-groupes finis, mais nous devrons les comprendre de manière beaucoup plus fine que dans le cas présent. Nous caractériserons complètement celles qui ne modifient pas la théorie $\forall\exists$ du groupe hyperbolique Γ, en utilisant entre autres des techniques de petite simplification. Le lecteur est invité à consulter le chapitre 3, plus particulièrement le théorème 5.5.

4. Homogénéité dans les groupes

En quel sens peut-on dire que deux éléments d'un groupe G sont indiscernables ? Une réponse possible à cette question, sans doute la plus naturelle du point de vue de l'algèbre, est la suivante : deux éléments u et v peuvent être considérés comme indiscernables s'ils sont dans la même orbite sous l'action du groupe des automorphismes de G.

Du point de vue de la logique du premier ordre, on préférera la réponse qui suit : les éléments u et v sont indistinguables si, pour tout énoncé du premier ordre $\theta(x)$ à une variable libre, l'énoncé $\theta(u)$ est satisfait par G si et seulement si l'énoncé $\theta(v)$ est satisfait par G ; on dit alors que u et v ont le même *type* (définition introduite par Morley au début des années 1960, voir [**Las98**]), et l'on note $\mathrm{tp}(u) = \mathrm{tp}(v)$. On généralise cette définition aux n-uplets $(u_1, \ldots, u_n)$ d'éléments de G, pour n'importe quel entier $n \geq 1$, en considérant les énoncés à n variables libres $\theta(x_1, \ldots, x_n)$ tels que $\theta(u_1, \ldots, u_n)$ est satisfait par G.

Naturellement, lorsque u et v sont dans la même orbite sous l'action de $\mathrm{Aut}(G)$, ils ont le même type. Autrement dit, deux n-uplets qui sont indiscernables du point de vue de l'algèbre le sont à plus forte raison du point de vue de la logique du premier ordre. En revanche, la réciproque est fausse en général, comme nous le verrons plus loin (voir 4.15). Un groupe G pour lequel la réciproque a lieu est dit *homogène*.

Définition 4.1. Un groupe G est dit homogène si, pour tout entier $n \geq 1$ et toute paire (u, v) de n-uplets d'éléments de G de même type, il existe un automorphisme σ de G tel que $\sigma(u) = v$.

Au sein des groupes homogènes, on peut dégager une hiérarchie fondée sur le nombre d'alternance de quantificateurs dans les énoncés. Plus précisément,

- on dira qu'un groupe G est fortement homogène, si deux n-uplets d'éléments de G qui satisfont les mêmes énoncés sans quantificateur à n variables libres sont dans la même orbite, pour tout entier $n \geq 1$.
- On dira que G est $\exists$-homogène, ou existentiellement homogène, si deux n-uplets qui satisfont les mêmes énoncés $\exists$ à n variables libres sont dans la même orbite.
- De même, on parlera de groupes $\forall\exists$-homogènes.

Toutefois, la définition de l'homogénéité 4.1 donnée ci-dessus ne permet pas de rendre compte des phénomènes en jeu dans les groupes virtuellement libres. Nous démontrons en effet qu'il existe un groupe virtuellement libre qui n'est pas $\exists\forall\exists$-homogène, et conjecturons que ce groupe n'est pas homogène. Nous prouvons cependant que les groupes virtuellement libres satisfont une propriété un peu plus faible que l'homogénéité, que nous nommons la presque homogénéité.

Définition 4.2. Un groupe G est dit *presque homogène* s'il existe un entier $N \geq 1$ tel que, pour tout entier $n \geq 1$ et tout n-uplet $u \in G^n$, le nombre d'orbites de n-uplets qui ont le même type que u est majoré par N. Notons que G est homogène si et seulement si on peut prendre $N = 1$ dans cette définition.

Notons qu'il existe des groupes hyperboliques qui ne sont pas presque homogènes, comme par exemple le groupe fondamental de la surface hyperbolique fermée orientable de genre 3 (voir plus loin).

Soulignons que certains groupes virtuellement libres sont bel et bien homogènes. C'est le cas notamment des groupes libres (voir [**PS12**] et [**OH11**]) et des groupes virtuellement libres qui sont co-hopfiens (il s'agit d'un travail en cours). Une question naturelle se pose.

Question 4.3. Dans quelle mesure un groupe hyperbolique qui n'est pas homogène est-il loin d'être homogène ? Peut-on estimer son défaut d'homogénéité ?

La presque homogénéité est une façon possible d'approcher cette question. On en propose une autre, d'une nature différente : un élément générique au sens des marches aléatoires dans un groupe hyperbolique est déterminé par son type, au sens de la définition 4.4 ci-après. Plus précisément, soit G un groupe hyperbolique et soit μ une mesure de probabilité sur G, de support fini engendrant le groupe G. On appelle élément aléatoire de longueur n un élément de G obtenu par une marche aléatoire de longueur n engendrée par μ. On démontre que la probabilité qu'un tel élément soit déterminé par son type tend vers 1 quand la longueur n de la marche tend vers l'infini.

Définition 4.4. On dit qu'un k-uplet $u \in G^k$ est *déterminé par son type* s'il existe une unique orbite sous l'action de $\mathrm{Aut}(G)$ de k-uplets qui ont le même type que u.

Dans cette partie introductive, on donne tout d'abord un petit aperçu de l'homogénéité dans les groupes, en mettant l'accent sur les groupes de type fini. On donne différents exemples, parfois nouveaux à notre connaissance, de groupes fortement ou existentiellement homogènes. On présente ensuite le théorème dû à Perin et Sklinos [**PS12**], et indépendamment à Ould Houcine [**OH11**], selon lequel les groupes libres sont $\forall\exists$-homogènes. Enfin, nous expliquons dans ses grandes lignes la preuve de la presque homogénéité des groupes virtuellement libres, et démontrons qu'il existe un groupe virtuellement libre qui n'est pas $\exists\forall\exists$-homogène. Le lecteur intéressé trouvera des preuves détaillées dans le chapitre 2.

4.1. Premiers exemples de groupes homogènes.

Exemple 4.5. Le groupe additif d'un corps est fortement homogène puisque son groupe d'automorphismes agit transitivement sur l'ensemble des éléments non triviaux.

Exemple 4.6. Tout groupe fini est $\exists$-homogène.

Exemple 4.7. Le groupe $(\mathbb{Z}, +)$ est $\exists$-homogène.

Preuve. Soient $u = (u_1, \ldots, u_n)$ et $v = (v_1, \ldots, v_n)$ deux n-uplets d'entiers. Supposons qu'ils ont le même type. En particulier, pour tout $1 \leq i \leq n$, les entiers u_i et v_i ont le même type. Comme la divisibilité par un entier d est exprimable au premier ordre par un énoncé existentiel, et comme deux entiers qui ont les mêmes diviseurs sont égaux au signe près, on a $u_i = \varepsilon_i v_i$ pour tout $1 \leq i \leq n$, avec $\varepsilon_i = \pm 1$. Remarquons ensuite que, pour tout couple $(i, j) \in [\![1, n]\!]^2$, les entiers $u_i + u_j$ et $v_i + v_j$ ont le même type. Il en découle facilement que $\varepsilon_i = \varepsilon_j$. Ainsi, $u = \pm v$, ce qui conclut. $\square$

Plus généralement, tout groupe abélien de type fini est $\exists$-homogène.

Définition 4.8 ([**OH11**], définition 1.4). Un groupe G est dit fortement co-hopfien s'il existe un ensemble fini $F \subset G \setminus \{1\}$ avec la propriété suivante : tout endomorphisme ϕ de G tel que $\ker(\phi) \cap F = \varnothing$ est bijectif.

Notons que la propriété d'être fortement co-hopfien est très contraignante, et fait défaut à la plupart des groupes infinis. Notamment, elle n'est pas satisfaite par les groupes libres, ni par la majorité des groupes virtuellement libres. Le principal intérêt de cette notion vient du petit lemme suivant.

Lemme 4.9 ([**OH11**], lemme 3.5). *Tout groupe de présentation finie fortement co-hopfien est $\exists$-homogène.*

Preuve. Soit $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n) = 1 \rangle$ un groupe de présentation finie fortement co-hopfien. Rappelons que l'ensemble $\mathrm{Hom}(G, G)$ est en correspondance bijective avec les $\boldsymbol{g} \in G^n$ tels que $\Sigma(\boldsymbol{g}) = 1$. Comme G est fortement co-hopfien, la bijectivité d'un morphisme $\varphi_{\boldsymbol{g}} : s_i \mapsto g_i$ est exprimable à l'aide d'un système fini d'inéquations. $\square$

On recense ci-dessous quelques exemples notables de groupes satisfaisant les hypothèses du lemme précédent.

- Dans un travail en cours, on démontre que tout groupe virtuellement libre qui est co-hopfien, et dont le groupe des automorphismes extérieur est fini, est fortement co-hopfien. C'est le cas notamment du groupe $\mathrm{GL}_2(\mathbb{Z}) \simeq \mathrm{Out}(F_2) \simeq D_6 *_{D_2} D_4$.
- Les groupes hyperboliques à un bout sont co-hopfiens, et il découle de l'argument de raccourcissement de Sela que tout groupe hyperbolique à un bout G tel que $\mathrm{Out}(G)$ est fini est fortement co-hopfien. En particulier, tout groupe hyperbolique rigide est fortement co-hopfien, comme noté dans [**OH11**]. Par exemple, le groupe fondamental d'une variété hyperbolique fermée de dimension plus grande que 3 est fortement co-hopfien.
- Les groupes $\mathrm{SL}_n(\mathbb{Z})$ et $\mathrm{GL}_n(\mathbb{Z})$ sont fortement co-hopfiens pour $n \geq 3$. C'est le cas, plus généralement, de nombreux réseaux de rang supérieur, comme corollaire des résultats de rigidité de Margulis.
- Dans [**BV00**], Bridson et Vogtmann démontrent que $\mathrm{Out}(\mathrm{Out}(F_n))$ est trivial pour n plus grand que 3. En adaptant leur argument, on peut prouver que le groupe $\mathrm{Out}(F_n)$ est fortement co-hopfien pour $n \geq 3$.
- Le groupe $\mathrm{Aut}(F_n)$ est fortement co-hopfien pour $n \geq 2$, en conséquence d'un résultat de Khramtsov [**Khr05**].
- Il découle d'un théorème de Aramayona et Souto [**AS12**] que tout groupe modulaire $\mathrm{Mod}(\Sigma)$ d'une surface fermée orientable Σ de genre ≥ 4 est fortement co-hopfien.

Enfin, rappelons qu'un groupe est dit *premier* s'il admet un plongement élémentaire dans tout modèle de sa théorie. Il découle immédiatement de la caractérisation suivante que tout groupe dénombrable premier est homogène : un groupe G dénombrable est premier si et seulement si, pour tout entier $n \geq 1$, les orbites de G^n sous l'action de $\mathrm{Aut}(G)$ sont définissables (voir par exemple [**Mar02**]). Dans [**Oge06**], Oger donne une caractérisation des groupes premiers parmi les groupes virtuellement nilpotents de type fini. Dans [**Las13**], Lasserre caractérise les groupes premiers parmi les groupes polycycliques-par-finis de type fini. Ces résultats fournissent donc de nombreux exemples de groupes homogènes.

4.1.1. *Une digression sur les groupes fortement co-hopfiens.*

On peut démontrer (voir [**OH11**], lemme 3.5) qu'un groupe G qui est de présentation finie et fortement co-hopfien est premier. Les groupes énumérés précédemment sont donc des exemples de groupes premiers. Dans [**ALM17**], les auteurs démontrent que les réseaux non uniformes de rang supérieur ont une propriété plus forte (au sein des groupes de type fini), la *rigidité du premier ordre* : tout groupe de type fini élémentairement équivalent à un tel groupe lui est isomorphe. On sait en revanche depuis les travaux de Sela que les groupes hyperboliques non élémentaires (sans torsion) n'ont pas cette propriété. En effet, si Γ est un tel groupe, $\Gamma * \mathbb{Z}$ a la même théorie du premier ordre que Γ.

Les groupes $\mathrm{Out}(F_n)$ et $\mathrm{Mod}(\Sigma)$ possèdent de fortes similarités avec le groupe $\mathrm{SL}_n(\mathbb{Z})$ pour $n \geq 3$ (voir par exemple [**BV06**]) ; néanmoins, ils s'apparentent aussi aux groupes hyperboliques, en tant que groupes acylindriquement hyperboliques (voir par exemple [**MM99**], [**Bow06**], [**DGO17**], [**Osi16**]). La question suivante est donc intrigante.

Question 4.10. Les groupes $\mathrm{Out}(F_n)$ et $\mathrm{Mod}(\Sigma)$ possèdent-ils la propriété de rigidité du premier ordre ? En d'autres termes, ces groupes se comportent-ils, du point de vue de la logique du premier ordre, plutôt comme $\mathrm{SL}_n(\mathbb{Z})$ ($n \geq 3$) ou plutôt comme un groupe hyperbolique ?

La preuve de [**ALM17**] de la rigidité du premier ordre de $\mathrm{SL}_n(\mathbb{Z})$ ne semble pas s'adapter au cas de $\mathrm{Out}(F_n)$ et $\mathrm{Mod}(\Sigma)$. Par ailleurs, les techniques développées par Sela pour étudier la théorie du premier ordre des groupes hyperboliques ne sont pas directement applicables aux groupes acylindriquement hyperboliques. Par exemple, comme l'action d'un groupe acylindriquement hyperbolique Γ sur l'espace hyperbolique associé n'est pas cocompacte et proprement discontinue, une suite d'endomorphismes de Γ deux à deux non conjugués ne donne pas toujours lieu à une action sur un arbre réel.

En fait, si un groupe de type fini G possède la même théorie existentielle qu'un groupe de présentation finie fortement co-hopfien et équationnellement noethérien Γ, alors G se rétracte sur Γ, i.e. G est de la forme $N \rtimes \Gamma$. De plus, si G et Γ ont la même théorie élémentaire, on peut supposer que Γ est élémentairement plongé dans G. Notons que l'on ne sait pas si $\mathrm{Out}(F_n)$ est équationnellement noethérien (notamment, il n'est pas linéaire si $n \geq 4$). Groves et Hull annoncent dans [**GH17a**] que $\mathrm{Mod}(\Sigma)$ a cette propriété.

4.2. Les groupes libres sont homogènes.

En 2003, Nies observe dans [**Nie03**] que le groupe libre F_2 est $\exists$-homogène. Toutefois, sa preuve exploite une particularité du groupe F_2 et ne se généralise pas aux groupes libres de rang supérieur ou égal à 3. Plus tard, Pillay prouve dans [**Pil09**] le résultat suivant : dans un groupe libre de type fini, tout élément qui a le même type qu'un élément primitif est lui-même primitif (et appartient donc à la même orbite sous l'action du groupe d'automorphismes). La preuve de l'homogénéité en toute généralité, beaucoup plus difficile, est due à Perin et Sklinos [**PS12**], et indépendamment à Ould Houcine [**OH11**]. Notons que les deux articles démontrent en fait que les groupes libres sont $\forall\exists$-homogènes. Les deux approches, quoique très différentes, utilisent d'une façon ou d'une autre le lemme facile 4.11 ci-dessous. La preuve de Perin-Sklinos est très géométrique et repose sur des techniques introduites par Sela pour résoudre le problème de Tarski. Enfin, mentionnons qu'une description complète des groupes hyperboliques sans torsion à un bout qui sont homogènes est donnée par Dente Byron dans sa thèse, dans un travail commun avec Perin.

Lemme 4.11. *Soit F_n un groupe libre. Soient u et v des k-uplets d'éléments de F_n. Les deux assertions suivantes sont équivalentes.*

(1) Il existe un automorphisme de F_n qui envoie u sur v.

(2) *Il existe un monomorphisme de F_n qui envoie u sur v, et un monomorphisme de F_n qui envoie v sur u.*

Preuve. Supposons qu'il existe un monomorphisme ϕ de F_n qui envoie u sur v, et un monomorphisme ψ de F_n qui envoie v sur u. Notons G_u (resp. G_v) le plus petit facteur libre de G contenant u (resp. v). On a $\phi(G_u) \subset G_v$ et $\psi(G_v) \subset G_u$, donc l'endomorphisme $\psi \circ \phi$ induit par restriction un monomorphisme de G_u fixant u. Il découle du théorème 2.37, qui généralise le fait que les groupes hyperboliques à un bout sont co-hopfiens, que la restriction de $\psi \circ \phi$ à G_u est un automorphisme de G_u. Ainsi, ϕ induit un isomorphisme de G_u vers G_v. On peut écrire $F_n = G_u * H_u = G_v * H_v$ où H_u et H_v sont deux groupes libres de même rang. Notons α un isomorphisme de H_u vers H_v, et définissons un automorphisme σ de F_n par $\sigma_{|G_u} = \phi_{|G_u}$ et $\sigma_{|H_u} = \alpha$. Cet automorphisme envoie u sur v, ce qui conclut. □

Nous verrons dans la partie suivante que ce résultat tombe en défaut dans les groupes virtuellement libres. En effet, la preuve donnée ci-dessus repose de façon essentielle sur la possibilité d'étendre à notre guise le morphisme $\phi_{|G_u}$ en un automorphisme de F_n, ce qui n'est pas possible de façon systématique lorsque l'on a affaire à un produit amalgamé ou une extension HNN au-dessus d'un groupe fini non trivial.

Pour l'heure, nous allons utiliser le lemme précédent pour démontrer que le groupe libre F_2 de rang 2 est existentiellement homogène.

THÉORÈME 4.12 (Nies). *Le groupe libre F_2 est $\exists$-homogène.*

Preuve. Notons $\{a, b\}$ une partie génératrice de F_2. Soient u et v deux k-uplets d'éléments de F_2. Supposons qu'ils ont le même type existentiel. Pour alléger les notations, on va prendre $k = 1$, le cas général étant identique. Il existe un mot $w(a, b)$ en les générateurs a et b tel que $u = w(a, b)$. Considérons l'énoncé $\theta(z) : \exists x \exists y \ z = w(x, y) \wedge [x, y] \neq 1$ à une variable libre. L'énoncé $\theta(u)$ est satisfait par F_2, pour $x = a$ et $y = b$. Puisque u et v ont le même type existentiel, l'énoncé $\theta(v)$ est satisfait par F_2 également. Notons x et y les deux éléments de F_2 fournis par $\theta(v)$, et définissons un endomorphisme ϕ de F_2 par $\phi(a) = x$ et $\phi(b) = y$. On a $\phi(u) = v$. De plus, ϕ est injectif. En effet, l'image de ϕ est un groupe libre non abélien de rang inférieur ou égal à 2, donc de rang 2. Comme le groupe F_2 est hopfien, le morphisme ϕ est bien injectif. Par symétrie des rôles joués par u et v, il existe également un monomorphisme de F_2 qui envoie v sur u. Le lemme 4.11 achève la preuve. □

On voit tout de suite que cette preuve ne se généralise pas aux groupes F_n avec $n \geq 3$. Comment en effet exprimer l'injectivité d'un morphisme au moyen d'un énoncé du premier ordre ? C'est là que réside toute la difficulté. Notons que les preuves de [**PS12**] et [**OH11**] nécessitent l'utilisation d'énoncés $\forall\exists$, à la différence de la preuve ci-dessus. La question suivante demeure ouverte.

QUESTION 4.13. Les groupes libres sont-ils $\exists$-homogènes ?

REMARQUE 4.14. Plus loin, nous donnerons des exemples de groupes virtuellement libres qui ne sont pas $\exists$-homogènes. On utilisera pour cela le fait que le lemme 4.11 ne tient plus dans les groupes virtuellement libres en général. Cette approche ne peut donc pas être utilisée pour démontrer que les groupes libres ne sont pas $\exists$-homogènes.

Avant d'aller plus loin dans la preuve de l'homogénéité des groupes libres, mentionnons quelques autres résultats d'homogénéité et de non homogénéité dus à Perin-Sklinos [**PS12**], Louder-Perin-Sklinos [**LPS13**] et Ould Houcine [**OH11**].

THÉORÈME 4.15 (Perin-Sklinos). *Le groupe fondamental d'une surface hyperbolique connexe fermée de caractéristique d'Euler ≤ -3 n'est pas homogène.*

REMARQUE 4.16. La même preuve que celle de Perin-Sklinos établit que ce groupe n'est pas presque homogène.

Il est intéressant de noter que les groupes fondamentaux des surfaces hyperboliques fermées de caractéristique d'Euler ≤ -3 sont tous élémentairement équivalents aux groupes libres, qui eux sont homogènes. L'homogénéité n'est donc pas une propriété du premier ordre.

Notons également la différence avec les groupes fondamentaux de variétés hyperboliques fermées de dimension ≥ 3 qui, eux, sont homogènes (voir partie précédente).

La preuve du théorème précédent donnée dans [**PS12**] est jolie, et on peut l'expliquer en quelques mots. Considérons la surface orientable fermée de genre 3, notée Σ_3. Notons Σ la sous-surface de Σ_3 obtenue en coupant Σ_3 en deux surfaces à bord identiques. Il s'agit d'une surface orientable de genre 1 qui compte deux composantes de bord ; son groupe fondamental est donc isomorphe au groupe libre F_3. Considérons un lacet simple γ sur Σ. Tout automorphisme de $S_3 = \pi_1(\Sigma_3)$ transforme γ en un lacet simple, puisque, d'après le théorème de Dehn-Nielsen-Baer, $\mathrm{Out}(S_3)$ est isomorphe au groupe modulaire $\mathrm{Mod}(\Sigma_3)$, qui est défini comme le quotient du groupe des homéomorphismes de Σ_3 par le sous-groupe distingué formé par les homéomorphismes isotopes à l'identité ; autrement dit, $\mathrm{Mod}(\Sigma_3)$ est le groupe des classes d'isotopie d'homéomorphismes de Σ_3. D'autre part, un argument de comptage des lacets simples et des automorphismes du groupe libre montre qu'il existe un automorphisme σ de F_3 tel que $\sigma(\gamma)$ n'est pas un lacet simple. Or, d'après Sela, le sous-groupe $\pi_1(\Sigma) \simeq F_3$ de $S_3 = \pi_1(\Sigma_3)$ est élémentairement plongé dans S_3. Les éléments γ et $\sigma(\gamma)$, qui ont le même type dans $\pi_1(\Sigma)$, ont donc le même type dans S_3, mais il n'existe pas d'automorphisme de S_3 envoyant γ sur $\sigma(\gamma)$, ce qui conclut.

THÉORÈME 4.17 (Louder-Perin-Sklinos). *Considérons Σ_2 la surface connexe orientable fermée de genre 2. Son groupe fondamental G est homogène. En revanche, le groupe $G * \mathbb{Z}$ n'est pas homogène.*

Ce résultat est intéressant à deux égards. D'abord, il montre que l'homogénéité n'est pas préservée par produit libre (rappelons en effet que le groupe $\mathbb{Z}$ est homogène, voir 4.7). Ensuite, il met en évidence le fait que l'homogénéité n'est pas préservée par passage à un sous-groupe d'indice fini, puisque le groupe fondamental $\pi_1(\Sigma_3)$ de la surface orientable fermée de genre 3 est d'indice 2 dans le groupe $\pi_1(\Sigma_2)$, la surface S_3 étant un revêtement d'ordre 2 de la surface Σ_2. On retiendra que l'homogénéité se comporte mal vis-à-vis des opérations algébriques.

Mentionnons enfin un résultat dû à Ould Houcine, qui étend l'homogénéité existentielle du groupe libre F_2.

THÉORÈME 4.18 (Ould Houcine). *Tout groupe hyperbolique sans torsion engendré par deux éléments est $\exists$-homogène.*

4.3. Les groupes virtuellement libres sont presque homogènes.

Rappelons qu'un groupe est dit virtuellement libre s'il a un sous-groupe libre d'indice fini.

EXEMPLE 4.19. Le groupe $G = \mathrm{SL}_2(\mathbb{Z})$ est virtuellement libre.

Preuve. En utilisant son action sur le plan hyperbolique $\mathbb{H}^2$, on peut démontrer que le groupe $\mathrm{SL}_2(\mathbb{Z})$ admet la présentation

$$\langle a, b \mid a^4 = b^6 = 1,\ a^2 = b^3 \rangle \simeq \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}.$$

Son abélianisé est donc isomorphe à $\mathbb{Z}/12\mathbb{Z}$. Par conséquent, le sous-groupe dérivé $D(G)$ de G est d'indice 12. On peut démontrer que ce groupe est sans torsion. Il agit donc librement sur l'arbre de Bass-Serre du scindement de G en produit amalgamé. Ainsi, c'est un groupe libre. Plus précisément, on peut démontrer que $D(G)$ est isomorphe au groupe libre F_2 de rang 2. $\square$

Plus généralement, si A et B sont deux groupes virtuellement libres, et si C est un sous-groupe fini de A et B, les groupes $A *_C B$ et $A *_C$ sont encore virtuellement libres. Autrement dit, tout graphe de groupes finis est virtuellement libre. Un théorème classique affirme que la réciproque est vraie (voir par exemple [**SW79**] théorème 7.3). Ainsi, un groupe de type fini est virtuellement libre si et seulement s'il se décompose en un graphe fini de groupes finis, c'est-à-dire s'il agit cocompactement sur un arbre simplicial avec des stabilisateurs de sommet finis.

Comme nous l'avons déjà mentionné, l'homogénéité se comporte mal vis-à-vis des produits libres, à plus forte raison vis-à-vis des produits amalgamés et des extensions HNN sur un groupe fini. Par conséquent, il y a peu d'espoir de démontrer l'homogénéité d'un groupe virtuellement libre directement à partir de sa décomposition en graphe de groupes finis. De même, l'homogénéité se comporte mal par extension par un groupe fini, et il n'y a donc aucune raison apparente que l'on puisse déduire l'homogénéité des groupes virtuellement libres de celle des groupes libres. En fait, on conjecture que les groupes virtuellement libres ne sont pas homogènes en général. On démontre un résultat dans cette direction.

Théorème 4.20. *Il existe un groupe virtuellement libre qui n'est pas $\exists\forall\exists$-homogène.*

En dépit de ce défaut d'homogénéité supposé, on prouve que les groupes virtuellement libres sont presque homogènes au sens de la définition 4.2 donnée plus haut.

Théorème 4.21. *Les groupes virtuellement libres sont presque homogènes.*

On démontre par ailleurs le théorème suivant.

Théorème 4.22. *Soit G un groupe hyperbolique, et soit μ une mesure de probabilité sur G dont le support est fini et engendre G. On appelle élément aléatoire de longueur n un élément de G obtenu par une marche aléatoire de longueur n engendrée par μ. On démontre que la probabilité qu'un tel élément soit déterminé par son type (au sens de la définition 4.4) tend vers 1 quand la longueur n de la marche tend vers l'infini. Plus précisément, on peut démontrer qu'un élément générique au sens des marches aléatoires est déterminé par son type existentiel.*

Remarque 4.23. Voici un autre résultat de généricité qu'il peut être intéressant de mentionner : un groupe aléatoire (en un sens que l'on peut rendre précis) est $\exists$-homogène. En effet, il est bien connu qu'un tel groupe G est hyperbolique. En outre, G a la propriété (FA) de Serre d'après [**DGP11**], et on a déjà dit plus haut qu'un groupe hyperbolique rigide est $\exists$-homogène, car fortement co-hopfien (ce que l'on prouve en utilisant la technique du raccourcissement de Sela).

Nous allons donner un schéma de preuve du théorème 4.21. Dans un premier temps, expliquons brièvement pourquoi le groupe $G = \mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ est $\exists$-homogène. La preuve peut être décomposée en deux étapes (on renvoie au chapitre 2 pour les détails). On se donne deux uplets u et v d'éléments de G, et on suppose qu'ils ont le même type (existentiel suffit, dans ce cas précis).

- *Étape 1.* En utilisant la logique du premier ordre, on peut démontrer qu'il existe un monomorphisme $\phi : G \hookrightarrow G$ tel que $\phi(u) = v$ (on peut démontrer qu'il suffit d'exprimer via un énoncé du premier ordre l'existence d'un morphisme $G \hookrightarrow G$ qui envoie u sur v et qui est injectif sur les groupes de sommets (qui sont finis), ce qui ne pose pas de difficulté). Par symétrie, on montre qu'il existe un monomorphisme $\psi : G \hookrightarrow G$ tel que $\psi(v) = u$.
- *Étape 2.* On utilise les morphismes ϕ et ψ pour construire un automorphisme de G envoyant u sur v. Il y a deux cas : soit G est à un bout relativement à u, donc co-hopfien relativement à u, auquel cas le monomorphisme $\psi \circ \phi$ est un automorphisme, puisqu'il fixe u ; soit u est elliptique dans l'arbre de Bass-Serre du scindement de $\mathrm{SL}_2(\mathbb{Z})$ au-dessus de $\mathbb{Z}/2\mathbb{Z}$, auquel cas on peut sans difficulté modifier le morphisme ϕ pour le rendre bijectif.

La preuve esquissée ci-dessus fait office de modèle pour comprendre le cas général. Nous allons voir que la première étape reste essentiellement valide, tandis que la seconde tombe en défaut. Nous contournerons cet obstacle en exploitant un résultat de cocompacité de l'ensemble des scindements JSJ au-dessus des groupes finis.

4.3.1. *Généralisation de l'étape 1.*

LEMME 4.24. *Soit G un groupe virtuellement libre de type fini. Soient u et v deux uplets d'éléments de G de même taille. Notons G_u un groupe de sommet contenant u d'un scindement JSJ de G au-dessus de ses sous-groupes finis relativement au groupe $\langle u\rangle$. Si u et v ont le même type, il existe un endomorphisme de G qui envoie u sur v et dont la restriction au sous-groupe G_u est injective.*

REMARK 4.1. On peut voir que le groupe G_u est unique dès que le groupe $\langle u\rangle$ est infini.

Comme dans le cas de $\mathrm{SL}_2(\mathbb{Z})$, la preuve de ce point repose fondamentalement sur la logique du premier ordre. Elle nécessite toutefois l'emploi d'énoncés logiques plus élaborés, de la forme $\forall\exists$. Nous allons exposer brièvement les grandes idées de la preuve du lemme ci-dessus dans le cas où G est un groupe libre. On s'inspire pour cela de [**PS12**]. Si l'on autorise la torsion, la preuve esquissée ci-dessous reste valide en remplaçant les étages hyperboliques par des quasi-étages, tels que nous les avons définis dans la partie précédente.

Nous aurons besoin d'une version relative de l'argument de raccourcissement de Sela.

THÉORÈME 4.25. *Soit F_n un groupe libre. Soit u un uplet d'éléments de G non tous triviaux. Notons G_u le facteur à un bout contenant u d'une décomposition de Grushko de G relativement au sous-groupe $\langle u\rangle$. Il existe un ensemble fini $F \subset G_u \setminus \{1\}$ tel que, pour tout morphisme $\phi : F_n \to F_n$ envoyant u sur v et non injectif en restriction à G_u, il existe un automorphisme σ de G_u qui est une conjugaison sur les groupes de sommets rigides du scindement JSJ cyclique de G_u relativement à $\langle u\rangle$, qui s'étend en un automorphisme de F_n, tel que* $\ker(\phi\circ\sigma)\cap F \neq \varnothing$.

Soient u et v deux k-uplets d'éléments de F_n. Supposons qu'ils ont le même type $\forall\exists$. Par l'absurde, supposons qu'il n'existe pas de monomorphisme de F_n envoyant u sur v. Alors il n'existe pas d'endomorphisme de F_n qui envoie u sur v et dont la restriction à G_u soit injective. D'après le théorème ci-dessus, pour tout endomorphisme ϕ de F_n envoyant u sur v, il existe donc un automorphisme σ de G_u qui est une conjugaison sur les groupes de sommets rigides du scindement JSJ cyclique de G_u relativement à $\langle u\rangle$ et tel que $\ker(\phi\circ\sigma)\cap F \neq \varnothing$. Cela n'est pas exprimable à l'aide d'un énoncé du premier ordre en général, à cause de la précomposition par un automorphisme. Toutefois, il est possible d'exprimer via un énoncé $\forall\exists$, que nous désignerons par $\theta(v)$, un résultat un peu moins fort : pour tout endomorphisme ϕ de F_n envoyant u sur v, il existe un endomorphisme ϕ' de F_n (à savoir $\phi\circ\sigma$) qui envoie u sur v, tue un élément de F, et est *relié* à ϕ au sens suivant : ϕ et ϕ' coïncident à conjugaison près sur les groupes de sommet rigides du scindement JSJ cyclique de G_u relativement à u. Comme u et v ont le même type $\forall\exists$, l'énoncé $\theta(u)$ est vrai dans F_n : pour tout endomorphisme ϕ de F_n fixant u, il existe un endomorphisme ϕ' de F_n qui fixe u, tue un élément de F, et est relié à ϕ. En particulier, en prenant pour $\phi = \mathrm{id}_{F_n}$, on obtient un endomorphisme particulier de F_n, appelé prérétraction non injective. On peut ensuite utiliser cette prérétraction non injective pour décomposer F_n sous la forme d'un étage hyperbolique étendu ; de là, on tire une absurdité.

Pour de plus amples détails, on renvoie le lecteur à la proposition 4.1 du chapitre 2.

4.3.2. *Pas de généralisation de l'étape 2.*

Si G est un groupe libre, l'étape 2 de la preuve de l'homogénéité de $\mathrm{SL}_2(\mathbb{Z})$ reste parfaitement valable. En effet, en vertu du lemme 4.11, l'existence de deux monomorphismes de F_n envoyant u sur v et vice-versa garantit l'existence d'un automorphisme de F_n envoyant u sur v.

En revanche, si G est un groupe virtuellement libre quelconque, la seconde étape ne fonctionne plus. Voici deux exemples qui montrent que l'existence des deux morphismes injectifs ϕ et ψ n'est pas suffisante pour garantir l'existence d'un isomorphisme envoyant u sur v.

Exemple 4.26. Considérons les deux groupes virtuellement cycliques suivants, de la forme $\mathbb{Z}/25\mathbb{Z} \rtimes \mathbb{Z}$, que nous avons déjà eu l'occasion de croiser dans la première partie de l'introduction :

$$U = \langle u, t \mid u^{25} = 1,\ tut^{-1} = u^{11} \rangle \quad \text{et} \quad V = \langle v, z \mid v^{25} = 1,\ zvz^{-1} = v^{6} \rangle.$$

Posons $G = U * V$. On définit un endomorphisme ϕ de G par $\phi(u) = v$, $\phi(v) = u$, $\phi(z) = t^2$ et $\phi(t) = z^3$. Ce morphisme est bien défini puisque $6^2 = 11 \mod 25$ et $11^3 = 6 \mod 25$, et on peut démontrer qu'il est injectif. On peut poser $\psi = \phi$ pour faire le lien avec les notations utilisées plus haut. Pourtant, il n'existe pas d'automorphisme de G envoyant u sur v. En effet, un tel automorphisme induirait un isomorphisme entre le normalisateur de u dans G, à savoir U, et le normalisateur de v, à savoir V. Or, comme vu précédemment, ces deux groupes ne sont pas isomorphes (bien qu'élémentairement équivalents !).

Voici un second exemple.

Exemple 4.27. Considérons $C = (\mathbb{Z}/2\mathbb{Z})^4$ et posons $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$. Soit $(e_i\ e_j)$ l'élément de $\mathrm{Aut}(C) = \mathrm{GL}_4(\mathbb{F}_2)$ qui permute e_i et e_j tout en fixant e_k pour $k \notin \{i,j\}$. Définissons un morphisme f de $(\mathbb{Z}/2\mathbb{Z})^2 = \langle u \rangle \times \langle v \rangle$ vers $\mathrm{Aut}(C)$ par $f(u) = (e_1\ e_2)$ et $f(v) = (e_3\ e_4)$. Posons $A = C \rtimes_f (\langle x \rangle \times \langle y \rangle)$. Ensuite, définissons un morphisme h de $\mathbb{Z}/3\mathbb{Z} = \langle z \rangle$ vers $\mathrm{Aut}(C)$ par $h(z) = (e_1\ e_2\ e_3)$ (autrement dit $h(z)$ fixe e_4 et permute cycliquement e_1, e_2, e_3 dans l'ordre indiqué), et posons $B = C \rtimes_h \langle z \rangle$. Enfin, définissons $G = A *_C B$. Considérons maintenant l'automorphisme ϕ de A défini comme suit:

$$\begin{aligned} \phi(u) &= v, \\ \phi(v) &= u, \\ \phi_{|C} &= (e_1\ e_3)(e_2\ e_4). \end{aligned}$$

Posons $x = z^{-1}uvz \in G$. On vérifie sans encombre que ϕ et $\mathrm{ad}(x)$ coïncident sur C. On peut donc définir un endomorphisme de G, toujours noté ϕ, en posant $\phi_{|B} = \mathrm{ad}(x)$ et $\phi_{|A} = \phi$. On peut démontrer que ce morphisme est injectif, mais qu'il n'existe pas d'automorphisme de G envoyant u sur v. On renvoie au chapitre 2 pour les preuves de ces affirmations.

Notons en particulier que dans les exemples précédents, on a l'égalité $\mathrm{tp}_{\exists}(u) = \mathrm{tp}_{\exists}(v)$. Grâce aux techniques développées dans le troisième chapitre, on peut même démontrer que $\mathrm{tp}_{\exists\forall\exists}(u) = \mathrm{tp}_{\exists\forall\exists}(v)$ dans le second exemple, et on conjecture que les deux éléments ont le même type. Nous donnerons quelques détails un peu plus bas.

4.3.3. *Compacité de l'espace de déformation.*

Comme nous venons de le voir, la preuve de l'homogénéité de F_n ou de $\mathrm{SL}_2(\mathbb{Z})$ ne se généralise à un groupe virtuellement libre G quelconque. Rappelons que le lemme 4.24 assure l'existence d'un endomorphisme ϕ de G qui envoie u sur v et dont la restriction au sous-groupe à un bout maximal G_u contenant $\langle u \rangle$ est injective. Par symétrie, il existe un endomorphisme ψ de G qui envoie v sur u et dont la restriction au sous-groupe à un bout maximal G_v contenant $\langle v \rangle$ est injective. La propriété co-Hopf relative nous garantit que le morphisme $\psi \circ \phi$ induit un automorphisme du facteur G_u, donc que le morphisme $\phi_{|G_u} : G_u \to G_v$ est un isomorphisme. On aimerait pouvoir étendre cet isomorphisme en un automorphisme de G, comme on l'a fait dans le cas de F_n, mais rien ne le permet *a priori*. Pour surmonter ce blocage, on a recours a un résultat de cocompacité sous l'action de $\mathrm{Aut}(G)$ de l'espace de déformation $\mathcal{D}(G)$ composé des décompositions JSJ de G sur ses sous-groupes finis, modulo isométrie équivariante.

Proposition 4.28. *Soit G un groupe virtuellement libre. Il existe un nombre fini d'arbres $T_1, \ldots, T_n$ de $\mathcal{D}(G)$ tels que, pour tout arbre non-redondant $T \in \mathcal{D}(G)$, il existe un automorphisme σ de G et un entier $1 \leq k \leq n$ tels que $T = T_k^\sigma$.*

On renvoie à la proposition 2.9 du chapitre 2 pour une preuve. Ce résultat a pour corollaire que le nombre de sous-groupes finis de G qui sont groupes de sommets dans un scindement de G sur des groupes finis est fini modulo $\mathrm{Aut}(G)$. De ce fait, si l'on se donne suffisamment d'éléments $u_1, \ldots, u_n$ de G (ou des uplets) qui ont le même type $\forall\exists$, on en trouvera nécessairement deux, disons u_1 et u_2, tels que les sous-groupes à un bout maximaux G_{u_1} et G_{u_2} contenant respectivement u_1 et u_2 sont dans la même orbite sous $\mathrm{Aut}(G)$. On peut donc supposer que $G_{u_1} = G_{u_2}$. Enfin, puisque G n'a qu'un nombre fini de classes de conjugaison de sous-groupes finis, l'isomorphisme $\phi : G_{u_1} \to G_{u_2}$ donné par le lemme 4.24, qui est donc un automorphisme de G_{u_1}, n'a qu'un nombre fini de façons possibles de ne pas s'étendre. La finitude dans le théorème 4.21 vient donc de la finitude de $\mathcal{D}(G)/\mathrm{Aut}(G)$ combinée à la finitude du nombre de classes de conjugaison de sous-groupes finis de G.

4.4. Un groupe virtuellement libre qui n'est pas $\exists\forall\exists$-homogène.

Rappelons qu'un plongement d'un groupe G dans un groupe Γ est dit élémentaire si, pour toute formule du premier ordre $\theta(x_1, \ldots, x_n)$ à n variables libres, pour tout $(g_1, \ldots, g_n) \in G^n$, on a l'équivalence $G \models \theta(g_1, \ldots, g_n) \Leftrightarrow \Gamma \models \theta(g_1, \ldots, g_n)$. On définit de même un plongement $\exists\forall\exists$-élémentaire, en ne considérant que les formules de la forme $\exists\forall\exists$.

Le théorème suivant est un cas particulier d'un résultat démontré dans le troisième chapitre de cette thèse.

Théorème 4.29. *Soit $G = A *_C B$ un groupe virtuellement libre, avec A, B finis et C distingué dans A et dans B, donc dans G. Soit α un automorphisme de C induit par conjugaison par un élément de G. Posons $\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C} = \alpha \rangle$. L'inclusion de G dans Γ est $\exists\forall\exists$-élémentaire.*

Corollaire 4.30. *Soient $G = A *_C B$ et u, v les objets donnés dans l'exemple 4.27. On a $\mathrm{tp}_{\exists\forall\exists}(u) = \mathrm{tp}_{\exists\forall\exists}(v)$. En particulier, G n'est pas $\exists\forall\exists$-homogène.*

Preuve du corollaire. Le monomorphisme ϕ de G construit dans l'exemple 4.27 coïncide sur C avec $\mathrm{ad}(x)$. Posons $\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C} = \mathrm{ad}(x)_{|C} \rangle$. Grâce au théorème précédent, G est $\exists\forall\exists$-élémentairement plongé dans Γ. On définit un endomorphisme θ de Γ en posant $\theta_{|A} = \phi_{|A}$, $\theta_{|B} = \mathrm{ad}(t)_{|C}$ et $\theta(t) = t$. Ce morphisme est bien défini puisque ϕ et $\mathrm{ad}(t)$ coïncident sur C. De plus, il s'agit d'un automorphisme puisque son image contient $\theta(A) = A$, $\theta(B) = B^t$ et $\theta(t) = t$, donc $A \cup B \cup \{t\}$ qui engendre Γ ; cela démontre que θ est surjectif, donc bijectif puisque Γ a la propriété de Hopf, en tant que groupe virtuellement libre. Notons en outre que $\theta(u) = v$ puisque $u, v \in A$ et θ coïncide avec ϕ sur A. On a donc démontré qu'il existe un automorphisme de Γ qui envoie u sur v. En particulier, u et v ont le même type dans Γ. Or, le $\exists\forall\exists$-type est préservé par plongement $\exists\forall\exists$-élémentaire. Le groupe G étant $\exists\forall\exists$-plongé dans Γ d'après le théorème précédent, nous avons $\mathrm{tp}^G_{\exists\forall\exists}(u) = \mathrm{tp}^G_{\exists\forall\exists}(v)$. $\square$

On conjecture que les éléments u et v ci-dessus ont le même type dans G. En fait, on fait la conjecture plus forte suivante : dans le théorème 4.29 (ainsi que sa version générale du chapitre 3), l'inclusion de G dans Γ est élémentaire.

Remarque 4.31. Si la conjecture précédente est vraie, alors le groupe G n'est pas homogène. Ce serait à notre connaissance le premier exemple d'un groupe qui est à la fois virtuellement homogène et non homogène. Rappelons au passage que l'homogénéité est connue pour n'être pas héritée par les sous-groupes d'indice fini. Par exemple, $\pi_1(\Sigma_2)$

est homogène d'après [LPS13], tandis que $\pi_1(\Sigma_3)$ ne l'est pas. Plus simplement, il existe un groupe triangulaire qui contient $\pi_1(\Sigma_3)$ comme sous-groupe d'indice fini, or un groupe triangulaire est rigide, donc homogène.

Remarque 4.32. Deux élements u et v d'un groupe G ont le même ordre si et seulement si G se plonge dans un groupe Γ dans lequel u et v sont conjugués (si les éléments ont le même ordre, il suffit en effet de poser $\Gamma = \langle G, t \mid tut^{-1} = v\rangle$). On peut voir cette petite observation comme un cas très particulier du résultat classique suivant de théorie des modèles (qui ne se limite pas aux groupes) : deux éléments u et v ont le même type dans un groupe G si et seulement s'il existe un surgroupe Γ de G dans lequel G est élémentairement plongé et tel que les deux élements u et v sont dans la même orbite sous $\mathrm{Aut}(\Gamma)$. Cependant, le groupe Γ est en général très gros (non dénombrable), et peu compréhensible du point de vue de la géométrie des groupes. Cela contraste avec le corollaire 4.30, où le groupe Γ est obtenu en faisant une simple extension HNN au-dessus d'un groupe fini. Plus généralement, si G est hyperbolique et non homogène (par exemple $\pi_1(\Sigma_3)$), on peut se demander s'il existe un groupe homogène qui ne soit pas trop gros, et dans lequel G soit élementairement plongé.

5. Le problème de Tarski pour les groupes virtuellement libres

Le problème de Tarski originel concerne l'équivalence élémentaire au sein de la famille des groupes libres. Par extension, on appelle problème de Tarski dans une classe de groupes $\mathcal{C}$ le problème de la classification des groupes de $\mathcal{C}$ à équivalence élémentaire près. Les groupes abéliens ont par exemple été complètement classés à équivalence élémentaire près par Szmielew dans [Szm55].

Dans la classe des groupes finis-par-nilpotents de type fini, Oger a démontré dans [Oge91] que $G \equiv \Gamma$ si et seulement si $G \times \mathbb{Z} \simeq \Gamma \times \mathbb{Z}$. L'implication $\Leftarrow$ est en fait vraie pour des groupes quelconques d'après [Oge83]. Nous avons déjà eu l'occasion de rencontrer à deux reprises dans cette introduction les deux groupes virtuellement libres

$$\Gamma = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{11}\rangle \quad \text{et} \quad G = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{6}\rangle,$$

qui sont élémentairement équivalents d'après le résultat susmentionné, bien que n'étant pas isomorphes. Nous y reviendrons plus loin.

Rappelons pour finir que Sela et en parallèle Kharlampovich et Myasnikov ont démontré que les groupes libres non abéliens sont élémentairement équivalents, répondant ainsi positivement à la vieille question de Tarski. Quelques années plus tard, Sela a complètement classé les groupes hyperboliques sans torsion à équivalence élémentaire près. Les résultats des deux groupes d'auteurs sont le point culminant de deux longues séries de travaux qui sont inspirées, tout en les dépassant de beaucoup, des techniques développées par Merzlyakov dans [Mer66] et par Sacerdote dans [Sac73b]. Dans ce dernier article, l'auteur démontre que les groupes libres ont la même théorie $\forall\exists$.

Dans le chapitre 3 du présent manuscrit, nous établissons une classification complète des groupes virtuellement libres de type fini modulo égalité des théories $\forall\exists$, généralisant ainsi le résultat de Sacerdote de 1973. Nous espérons qu'il s'agisse là d'une première étape vers une classification des groupes virtuellement libres à équivalence élémentaire près (voir à ce propos la conjecture 5.1 proposée plus bas).

Notre approche repose notamment sur la théorie de la petite simplification, ainsi que sur la théorie des actions de groupes sur les arbres réels et sur la technique du raccourcissement, déjà essentielles dans les deux premiers chapitres.

En fait, certains de nos résultats restent valables dans le cadre plus général des groupes hyperboliques avec torsion. L'un des théorèmes principaux du troisième chapitre est en effet une condition nécessaire et suffisante pour qu'un groupe hyperbolique G ait la même théorie $\forall\exists$ que $G*_C$, où C est un groupe fini (ce résultat devrait jouer un rôle important dans une éventuelle classification des groupes hyperboliques à équivalence $\forall\exists$ près). Il est intéressant de mettre ce résultat en parallèle des résultats démontrés dans le chapitre 1 sur la torsion-saturation d'un groupe hyperbolique (voir la définition 3.24 et le théorème 3.25). Rappelons que nous avons introduit cette notion pour ne pas avoir à résoudre le problème suivant, qui est ouvert à notre connaissance : déterminer des conditions nécessaires et suffisantes sous lesquelles $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(G*_C)$, avec les mêmes notations que ci-avant. Par exemple, il n'existe pas à ce jour de classification des groupes virtuellement libres modulo l'égalité de leurs théories existentielles.

Soulignons par ailleurs que nous démontrons dans certains cas des résultats plus forts que la simple équivalence $\forall\exists$, à savoir l'existence de plongements $\exists\forall\exists$-élémentaires de G dans $G*_C$, avec G et C comme ci-dessus. Cela ouvre la voie à d'intéressantes applications à l'homogénéité des groupes virtuellement libres et hyperboliques avec torsion (ou plutôt à la non-homogénéité, comme illustré dans la partie 4.4 précédente), que nous n'avons pas encore explorées.

Tout le long de cette partie, et tout le long du chapitre 3, les groupes virtuellement libres sont supposés de type fini, et nous ne répéterons plus cette hypothèse.

5.1. Invariants de la théorie $\forall\exists$ et extensions HNN légales.

Notre quête d'une classification $\forall\exists$ des groupes virtuellement libres commence par la recherche d'invariants de la théorie $\forall\exists$.

Rappelons que les seuls sous-groupes définissables dans un groupe hyperbolique sans torsion sont le groupe trivial, le groupe entier et les sous-groupes cycliques (voir [**KM13**] et [**PPST12**]).[2] La torsion apporte son lot de sous-groupes définissables nouveaux (sous-groupes finis et normalisateurs de sous-groupes finis notamment[3]), et donc de phénomènes inexplorés.

L'exemple le plus simple auquel on puisse penser est celui d'un sous-groupe fini distingué maximal (qui existe toujours et est unique, dans un groupe hyperbolique). Ce sous-groupe C est définissable sans constante à l'aide d'un énoncé $\exists\forall$ dont l'interprétation en langage naturel est « il existe $n := |C|$ éléments $x_1, \ldots, x_n$ qui engendrent un sous-groupe fini isomorphe à C et tels que, pour tout élément g du groupe ambiant G, l'un des n produits gx_i est d'ordre infini, et g induit par conjugaison une permutation des x_i.» Notons que pour vérifier qu'un élément est d'ordre infini, il suffit de vérifier que sa puissance $K!$-ème est non triviale, où K désigne l'ordre maximal d'un élément d'ordre fini.

Si deux groupes hyperboliques G et Γ sont tels que $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$, ils ont donc un sous-groupe fini normal maximal isomorphe, que nous noterons C. Si l'on suppose en plus, pour simplifier, que $\mathrm{Th}(G) = \mathrm{Th}(\Gamma)$, il s'ensuit que $\mathrm{Th}(G/C) = \mathrm{Th}(\Gamma/C)$. Toutefois, une partie de l'information s'est envolée lors du passage au quotient. Que l'on songe par exemple à $(\mathbb{Z}/2\mathbb{Z})^2 \times F_n$ et à un produit semi-direct non direct $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes F_n$: ces deux groupes n'ont évidemment pas la même théorie $\forall\exists$, puisque l'on peut exprimer à l'aide d'un énoncé $\exists\forall$ que le sous-groupe fini normal maximal du second groupe n'est pas central. Le sous-groupe

$$\mathrm{Aut}_G(C) := \{\sigma \in \mathrm{Aut}(C) \mid \exists g \in N_G(C),\ \sigma = \mathrm{ad}(g)_{|C}\}$$

de $\mathrm{Aut}(C)$ semble donc jouer un rôle de première importance. Par exemple, si l'on reprend nos deux groupes modèles

$$\Gamma = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{11}\rangle \quad \text{et} \quad G = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{6}\rangle,$$

[2] Remarquons que l'équivalence élémentaire des groupes libres non abéliens a la conséquence suivante : le sous-groupe dérivé d'un groupe libre non abélien (de type fini) n'est pas définissable, car dans le cas contraire l'équivalence élémentaire de F_n et F_m impliquerait celle de $F_n/D(F_n) = \mathbb{Z}^n$ et $F_m/D(F_m) = \mathbb{Z}^m$, ce qui est faux dès que $n \neq m$ comme nous l'avons vu plus haut.

[3] Déterminer les sous-groupes définissables d'un groupe hyperbolique avec torsion est d'ailleurs une question ouverte digne d'intérêt.

on a bien $\mathrm{Aut}_G(\langle c\rangle) \simeq \mathrm{Aut}_\Gamma(\langle c\rangle) \simeq \mathbb{Z}/5\mathbb{Z}$ puisque 11 et 6 sont d'ordre 5 dans le groupe des inversibles $(\mathbb{Z}/25\mathbb{Z})^\times \simeq \mathbb{Z}/20\mathbb{Z}$.

Faisons maintenant l'hypothèse supplémentaire que G est une extension finie

$$1 \to C \to G \to F_n \to 1$$

et que Γ est virtuellement libre. La propriété, satisfaite par G, de posséder un sous-groupe distingué fini isomorphe à C qui contient tout sous-groupe fini est exprimable à l'aide d'un énoncé $\exists\forall$. Puisque Γ et G ont la même théorie $\exists\forall$, le groupe Γ possède un sous-groupe C' fini distingué, isomorphe à C, tel que tout sous-groupe fini de Γ est contenu dans C'. Notons T l'arbre de Bass-Serre d'un scindement JSJ de Γ au-dessus de ses sous-groupes finis. Comme le groupe C' est distingué dans Γ et fixe un sommet de T, et comme Γ agit transitivement sur l'ensemble des sommets de T, le sous-groupe C' fixe l'arbre T point par point. Par conséquent, l'action de Γ sur T passe au quotient en une action de Γ/C' sur T. Puisque tout sous-groupe fini de Γ est inclus dans C', cette dernière action est libre, ce qui montre que le groupe Γ/C' est un groupe libre en vertu d'un résultat classique. Ainsi, le groupe Γ est également une extension finie

$$1 \to C' \to \Gamma \to F_{n'} \to 1,$$

avec $C' \simeq C$. Nous verrons que la théorie $\forall\exists$ de G est en fait entièrement déterminée au sein des groupes virtuellement libres par la classe d'isomorphisme de C et par la classe de conjugaison du groupe $\mathrm{Aut}_G(C)$ dans $\mathrm{Aut}(C)$.

Considérons un exemple légèrement plus élaboré : supposons dorénavant que G est une extension finie

$$1 \to C \to G \to A_1 * \cdots A_p * F_n \to 1$$

avec A_i fini pour chaque $i \in [\![1,p]\!]$. On peut démontrer que la théorie $\forall\exists$ de G est caractérisée par la donnée suivante : C, $\mathrm{Aut}_G(C)$, et la collection $(G_1,C),\ldots,(G_p,C)$ où G_i désigne une préimage de A_i dans G. En particulier, notons que l'entier p est préservé par équivalence $\forall\exists$. C'est une conséquence d'un phénomène très général : si G est un groupe quelconque, le nombre $n_1(G)$ de classes de conjugaison de sous-groupes finis de G (éventuellement infini) est un invariant $\forall\exists$, au sens où $n_1(G) = n_1(\Gamma)$ dès lors que $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$. En effet, étant donné $k \in \mathbb{N}$, il est possible d'exprimer au moyen d'un énoncé $\exists\forall$ que G a au moins k classes de conjugaisons de sous-groupes finis.

Dans le cas d'un groupe virtuellement libre quelconque, la situation n'est bien sûr pas aussi simple que dans le cas précédent, mais les phénomènes entraperçus ci-avant auront encore leur importance. En notant $C_1,\ldots,C_\ell$ des représentants des classes de conjugaison de sous-groupes finis de G, on définit $n_2(G)$ comme la somme des cardinaux des $\mathrm{Aut}_G(C_k)$ pour $1 \leq k \leq \ell$. Il s'agit d'un deuxième invariant $\forall\exists$.

5.1.1. *Extensions HNN légales.*

Comme chaque groupe virtuellement libre de type fini peut-être obtenu à partir de groupes finis par une itération finie de produits libres amalgamés et d'extensions HNN sur des groupes finis, nous devons répondre à la question suivante.

Question 5.1. Peut-on caractériser les produits amalgamés et les extensions HNN sur un groupe fini qui n'affectent pas la théorie $\forall\exists$ d'un groupe virtuellement libre ?

Pour répondre à cette question, commençons par démontrer un petit lemme qui est une conséquence immédiate de la discussion précédente.

Lemme 5.2. *Soit G un groupe ayant un nombre fini de classes de conjugaisons de sous-groupes finis. Soient C_1 et C_2 deux sous-groupes finis isomorphes de G, et $\alpha : C_1 \to C_2$ un isomorphisme. Si $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G*_\alpha)$, alors C_1 et C_2 sont conjugués dans G, et l'isomorphisme α est induit par un automorphisme intérieur de G. On peut donc réécrire l'extension HNN sous la forme $\langle G, t \mid [t,c] = 1, \forall c \in C\rangle$, où l'on a posé $C = C_1$.*

Preuve. Comme $n_1(G) = n_1(G*_\alpha)$, les deux groupes C_1 et C_2 sont nécessairement conjugués. On peut supposer sans perte de généralité que $C := C_1 = C_2$. Si l'on note t la lettre stable associée à α, l'égalité $n_2(G) = n_2(G*_\alpha)$ implique l'existence d'un élément $g \in G$ tel que $tct^{-1} = gcg^{-1}$ pour tout $c \in C$. Quitte à remplacer t par $g^{-1}t$, on peut donc mettre l'extension HNN sous la forme annoncée. □

Nous sommes donc ramenés à comprendre les extensions HNN de la forme

$$\Gamma = \langle G, t \mid [t, c] = 1, \forall c \in C\rangle.$$

Intéressons-nous à l'effet d'une telle extension HNN sur le normalisateur de C dans G. On rappelle que pour un groupe G hyperbolique, on a la dichotomie suivante :

(1) $N_G(C)$ est fini,
(2) ou $N_G(C)$ est virtuellement cyclique infini,
(3) ou $N_G(C)$ contient le groupe libre F_2 (auquel cas on dit qu'il est non élémentaire[4]).

Observons que la propriété d'être virtuellement cyclique infini est exprimable au moyen d'un énoncé universel, pour les sous-groupes d'un groupe hyperbolique G donné : $H \subset G$ est virtuellement cyclique si et seulement si $[h^{K!}, h'^{K!}] = 1$ pour tout $h, h' \in H$, où K désigne l'ordre maximal d'un sous-groupe fini de G. Par conséquent, le nombre $n_3(G)$ de classes de conjugaison de sous-groupes finis C de G tel que $N_G(C)$ est virtuellement cyclique infini est un invariant de la théorie $\forall\exists$. Or, si $N_G(C)$ est fini, alors $N_\Gamma(C)$ est virtuellement cyclique infini, et si $N_G(C)$ est virtuellement cyclique infini, alors $N_\Gamma(C)$ est non élémentaire. Ce changement de catégorie du normalisateur de C est détecté par le fait que $n_3(G) \neq n_3(\Gamma)$. Par conséquent, si $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$, le normalisateur $N_G(C)$ est nécessairement non élémentaire. Cette condition ne suffit cependant pas à garantir la réciproque, comme l'illustre l'exemple suivant.

EXEMPLE 5.3. Soit $G = F_2 \times \mathbb{Z}/2\mathbb{Z}$. L'énoncé universel $\forall x \forall y\ (x^2 = 1) \Rightarrow (xy = yx)$ est vrai dans G, mais pas dans $\Gamma = G*_{\{1\}} = G * \mathbb{Z}$. *A fortiori*, G et Γ n'ont pas la même théorie $\forall\exists$.

Plus généralement, le fait que $N_G(C)$ normalise un sous-groupe fini C' d'ordre $\leq k$ qui contient C strictement s'exprime via un énoncé $\exists\forall$. Une condition nécessaire à l'égalité des théories $\forall\exists$ de Γ et de G est donc $E_G(N_G(C)) = C$, où $E_G(N_G(C))$ désigne le sous-groupe fini maximal de G normalisé par $N_G(C)$ (voir **[Os93]** et la partie 2.3 du chapitre 3 pour plus de détails), ce que l'on peut encore formuler à l'aide d'un invariant $n_4(G)$, le nombre de classes de conjugaison de sous-groupes finis C de G tel que $N_G(C)$ est non élémentaire et $E_G(N_G(C)) = C$. Cela nous amène à la définition suivante.

DÉFINITION 5.4. Soit G un groupe hyperbolique non élémentaire, et soient C_1, C_2 deux sous-groupes finis de G. Supposons que C_1 et C_2 sont isomorphes, et soit $\alpha : C_1 \to C_2$ un isomorphisme. L'extension HNN $G*_\alpha = \langle G, t \mid \mathrm{ad}(t)_{|C_1} = \alpha\rangle$ est dite *légale* si les trois conditions suivantes sont satisfaites.

(1) Il existe un élément $g \in G$ tel que $gC_1g^{-1} = C_2$ et $\mathrm{ad}(g)_{|C_1} = \alpha$.
(2) $N_G(C_1)$ est non-élémentaire.
(3) $E_G(N_G(C_1)) = C_1$.

On dit qu'un groupe Γ est une *grande extension légale* de G s'il se décompose sous la forme d'une extension HNN légale $\Gamma = H*_\alpha$ avec $H \simeq G$.

L'exemple 5.3 est une illustration typique d'une extension non légale. En effet, la troisième condition de la définition précédente est clairement violée.

Comme expliqué ci-dessus, si l'extension HNN $G*_\alpha$ n'est pas légale au sens de la définition précédente, alors G et $G*_\alpha$ ont des théories $\forall\exists$ distinctes. Inversement, nous prouvons que la légalité d'une extension HNN $G*_\alpha$ sur des groupes finis est une condition suffisante

[4]Ce qui est sans rapport avec l'élémentarité au sens de la logique du premier ordre.

pour assurer l'égalité de $\mathrm{Th}_{\forall\exists}(G*_{\alpha})$ et $\mathrm{Th}_{\forall\exists}(G)$. On le démontre en généralisant le lemme clé de [**Sac73b**], en utilisant les techniques introduites par Sela, notamment la technique du raccourcissement, qui a été généralisée aux groupes hyperboliques avec présence éventuelle de torsion par Reinfeldt et Weidmann dans [**RW14**]. Nous renvoyons également le lecteur à [**Hei18**] pour quelques résultats sur les énoncés $\forall\exists$ dans les groupes hyperboliques avec torsion, à savoir une généralisation des solutions formelles de Merzlyakov.

Théorème 5.5. *Soit G un groupe hyperbolique non élémentaire. Soit $G*_{\alpha}$ une extension HNN au-dessus d'un sous-groupe fini. Alors $\mathrm{Th}_{\forall\exists}(G*_{\alpha}) = \mathrm{Th}_{\forall\exists}(G)$ si et seulement si $G*_{\alpha}$ est une extensions légale de G au sens de la définition 5.4.*

Voici deux illustrations du théorème ci-dessus.

Exemple 5.6. Soit G un groupe virtuellement libre sans sous-groupe fini distingué non trivial, par exemple $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Alors $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}\left(G*_{\{1\}}\right)$. De façon plus générale, un groupe hyperbolique G a la même théorie $\forall\exists$ que $G*_{\{1\}}$ si et seulement si G n'a pas de sous-groupe fini normal non trivial. Il convient de noter que dans le cas où G est sans torsion, Sela a prouvé que G et $G*_{\{1\}}$ ont la même théorie du premier ordre, comme cas particulier de sa classification des groupes hyperboliques sans torsion à équivalence élémentaire près.

Exemple 5.7. Soit $G = F_2 \times \mathbb{Z}/2\mathbb{Z}$. Le groupe $\Gamma = G*_{\mathbb{Z}/2\mathbb{Z}} = G *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}) = F_3 \times \mathbb{Z}/2\mathbb{Z}$ a la même théorie $\forall\exists$ que G.

Pour résumer, nous avons répondu à la question 5.1 dans le cas d'une extension HNN sur un groupe fini. Qu'en est-il d'un produit amalgamé sur un groupe fini ? Rappelons que le nombre de classes de conjugaison de sous-groupes finis est un invariant $\forall\exists$. Par conséquent, si un groupe virtuellement libre G se décompose sous la forme $G = A *_C B$ sur un groupe fini C, alors G et A, B ont des théories $\forall\exists$ distinctes dès lors que A ou B contient un sous-groupe fini non contenu dans C, i.e. n'est pas isomorphe à $C \rtimes F_n$. Dans ce cas, le produit amalgamé peut être écrit sous la forme d'une extension HNN multiple. Pour répondre à la question 5.1, seul le cas où $G = A*_C$ avec C fini est donc à considérer, ce qui est déjà fait.

5.2. Une autre forme d'extension légale.

Le théorème 5.5 ne rend pas compte de l'intégralité des phénomènes qui peuvent être à la source de l'égalité des théories $\forall\exists$ de deux groupes virtuellement libres. Rappelons en effet que les deux groupes virtuellement $\mathbb{Z}$ suivants sont élémentairement équivalents :

$$\Gamma = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{11} \rangle \quad \text{et} \quad G = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{6} \rangle.$$

Ces groupes sont élémentaires, mais le même phénomène peut se produire dans des groupes virtuellement libres non élémentaires : si G est un groupe hyperbolique, nous montrerons qu'il est possible de remplacer sous certaines conditions un sous-groupe virtuellement infini cyclique $N \subset G$ par un autre sous-groupe virtuellement infini cyclique N' tel que $\mathrm{Th}_{\forall\exists}(N) = \mathrm{Th}_{\forall\exists}(N')$, sans modifier la théorie $\forall\exists$ de G. Nous renvoyons le lecteur à la définition 5.9 et au théorème 5.10 ci-dessous pour un énoncé précis.

Notre principal résultat, le théorème 5.11, affirme que ces deux formes d'opérations (le fait d'effectuer une extension HNN légale sur un groupe fini, ou de remplacer un sous-groupe virtuellement $\mathbb{Z}$ particulier par un autre groupe virtuellement $\mathbb{Z}$ qui a la même théorie $\forall\exists$), sont les seules dont nous avons besoin pour classifier les groupes virtuellement libres modulo égalité de leurs théories $\forall\exists$. Le théorème 5.16 donne trois autres caractérisations de l'équivalence $\forall\exists$ parmi les groupes virtuellement libres.

Dans toute la suite, l'entier K_G désigne le plus gros cardinal d'un sous-groupe fini de G. Pour tout sous-groupe virtuellement cyclique $N \subset G$, considérons le sous-ensemble définissable $D(N) = \{g^{2K_G!} \mid g \in N\}$. On peut démontrer que $D(N)$ est un sous-groupe de

N. Notons que le quotient $N/D(N)$ est fini. Nous verrons que ce groupe fini est déterminé par la théorie $\forall\exists$ de N.

DÉFINITION 5.8. Soient N et N' deux groupes virtuellement cycliques dont le plus gros sous-groupe fini est d'ordre plus petit que K_G, et soit φ un morphisme de N vers N'. On dit que φ est *spécial* s'il satisfait les trois conditions suivantes :

- φ est injectif ;
- si C_1 et C_2 sont deux sous-groupes finis non conjugués de N, alors $\varphi(C_1)$ et $\varphi(C_2)$ ne sont pas conjugués dans N' ;
- le morphisme induit $\overline{\varphi} : N/D(N) \to N'/D(N')$ est injectif.

DÉFINITION 5.9. Soit G un groupe. Supposons que G se scinde sous la forme $A*_C B$ ou $A*_C$ sur un sous-groupe fini C dont le normalisateur N dans G est virtuellement cyclique et non elliptique dans le scindement. Soit N' un groupe virtuellement cyclique dont le plus gros sous-groupe fini est d'ordre plus petit que K_G. Supposons qu'il existe deux plongements spéciaux $\iota : N \hookrightarrow N'$ et $\iota' : N' \hookrightarrow N$. Le produit amalgamé

$$\Gamma = G *_N N' = \langle G, N' \mid g = \iota(g),\ \forall g \in N \rangle$$

est qualifié de *petite extension légale* de G.

On démontre le résultat suivant.

THÉORÈME 5.10. *Soit G un groupe hyperbolique, et soit $\Gamma = G *_N N'$ une petite extension légale de G au sens de la définition 5.9. Alors* $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$.

5.3. Classification des groupes virtuellement libres à équivalence $\forall\exists$ près.

Soit G un groupe hyperbolique. On dit qu'un groupe Γ est une *extension légale multiple* de G s'il existe une suite finie de groupes $G = G_0 \subset G_1 \subset \cdots \subset G_n \simeq \Gamma$ où G_{i+1} est une extension légale petite ou grande de G_i au sens des définitions 5.4 et 5.9, pour chaque entier $0 \leq i \leq n-1$. Voici le principal résultat du chapitre 3.

THÉORÈME 5.11. *Deux groupes virtuellement libres G et G' ont la même théorie $\forall\exists$ si et seulement si il existe deux extensions légales multiples Γ et Γ' de G et G' respectivement telles que $\Gamma \simeq \Gamma'$.*

EXEMPLE 5.12. Un groupe virtuellement libre G a la même théorie $\forall\exists$ que $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ si et seulement si G se scinde sous la forme $(\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z} \times F_n)$, avec F_n libre de rang $n \geq 0$.

En outre, on donne trois caractérisations supplémentaires de l'équivalence $\forall\exists$ au sein de la classes des groupes virtuellement libres. Mais avant cela, il nous faut généraliser la définition 5.8.

DÉFINITION 5.13. Soient G et G' deux groupes (qu'on ne suppose pas virtuellement cycliques) tels que $K_G \geq K_{G'}$. Un morphisme $\varphi : G \to G'$ est dit *spécial* s'il satisfait les trois conditions suivantes :

- φ est injectif sur les sous-groupes finis ;
- si C_1 et C_2 sont deux sous-groupes finis non conjugués de G, alors $\varphi(C_1)$ et $\varphi(C_2)$ ne sont pas conjugués dans G' ;
- si C est un sous-groupe fini de G dont le normalisateur est virtuellement cyclique infini maximal, alors la restriction $\varphi_{|N_G(C)} : N_G(C) \to N_{G'}(\varphi(C))$ est un morphisme spécial au sens de la définition 5.8 (en particulier, $\varphi_{|N_G(C)}$ est injectif).

REMARQUE 5.14. Notons que la définition 5.13 coïncide avec la définition 5.8 si G est virtuellement cyclique.

Définition 5.15. Soient G et G' deux groupes. Un morphisme spécial $\varphi : G \to G'$ est dit *fortement spécial* si, pour tout sous-groupe fini C de G, la condition suivante est satisfaite : si le normalisateur $N_G(C)$ est non élémentaire, alors $N_{G'}(\varphi(C))$ est lui aussi non élémentaire, et $\varphi(E_G(N_G(C))) = E_{G'}(N_{G'}(\varphi(C)))$.

On associe à tout groupe virtuellement libre G un énoncé $\zeta_G \in \mathrm{Th}_{\exists\forall}(G)$ (défini dans la partie 4 du troisième chapitre) de sorte que le théorème suivant soit vrai.

Théorème 5.16. *Soient G et G' deux groupes virtuellement libres de type fini. Les cinq assertions que voici sont équivalentes.*

(1) $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G')$.
(2) $G' \models \zeta_G$ *et* $G' \models \zeta_{G'}$.
(3) Il existe deux suites discriminantes (voir définition 3.18) de morphismes spéciaux $(\varphi_n : G \to G')_{n\in\mathbb{N}}$ *et* $(\varphi'_n : G' \to G)_{n\in\mathbb{N}}$ *(voir définition 5.13).*
(4) Il existe deux morphismes fortement spéciaux $\varphi : G \to G'$ *et* $\varphi' : G' \to G$ *(voir définition 5.15).*
(5) Il existe deux extensions légales multiples Γ *et* Γ' *de* G *et* G' *respectivement telles que* $\Gamma \simeq \Gamma'$.

Remarque 5.17. Un résultat classique (et facile) affirme que deux groupes de type fini G et G' ont la même théorie existentielle si et seulement s'il existe deux suites discriminantes $(\varphi_n : G \to G')_{n\in\mathbb{N}}$ et $(\varphi'_n : G' \to G)_{n\in\mathbb{N}}$. On peut comparer ce résultats à la troisième assertion ci-dessus : de ce point de vue, la seule différence entre les théories existentielle et $\forall\exists$ est qu'il est impossible de parler des classes de conjugaisons de sous-groupes avec un seul quantificateur, alors que cela est possible avec deux quantificateurs.

Sela et Kharlampovich-Myasnikov ont démontré que tout énoncé du premier ordre de la théorie d'un groupe libre de type fini est équivalent à un énoncé $\forall\exists$. Cette élimination des quantificateurs a ensuite été généralisée par Sela aux groupes hyperboliques sans torsion. Il nous semble donc raisonnable de faire la conjecture suivante, qui généralise le fameux problème de Tarski sur l'équivalence élémentaire des groupes libres (de type fini).

Conjecture 5.1. *Deux groupes virtuellement libres ont la même théorie $\forall\exists$ si et seulement s'ils sont élémentairement équivalents.*

Remarque 5.18. En conséquence des travaux de Sela sur la théorie du premier ordre des groupes hyperboliques sans torsion (voir [**Sel09**]), la conjecture ci-dessus est vraie si l'on remplace « virtuellement libre par « hyperbolique sans torsion ». De plus, grâce au résultat de Sela sur la théorie du premier ordre des produits libres, on sait que la conjecture est vraie si les deux groupes virtuellement libres en question sont des produits libres de groupes finis avec un groupe libre.

5.3.1. *Une remarque sur la classification des groupes hyperboliques sans torsion.*

Dans [**Sel09**], Sela associe (canoniquement) à tout groupe hyperbolique non élémentaire sans torsion G un groupe $C(G)$ bien défini à isomorphisme près qui a la même théorie que G, qu'il appelle le cœur de G. Cela lui permet de donner une classification complète des groupes hyperboliques sans torsion à équivalence élémentaire près.

Théorème 5.19 (Sela). *Deux groupes hyperboliques non élémentaires sans torsion ont la même théorie du premier ordre si et seulement si leurs cœurs sont isomorphes.*

Le groupe $C(G)$ est un sous-groupe de G (non canoniquement plongé en général), et même un rétract de G. En présence de torsion, une telle classification s'avère impossible, car on ne peut plus définir un cœur qui soit un rétract du groupe ambiant. Si l'on reprend à nouveau les deux groupes virtuellement cycliques

$$\Gamma = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{11}\rangle \quad \text{et} \quad G = \langle c, t \mid c^{25} = 1,\ tct^{-1} = c^{6}\rangle,$$

on peut facilement vérifier que tout rétract de G est fini ou égal à G (de même pour Γ). De ce fait, G et Γ ne peuvent pas avoir un rétract commun K tel que $\mathrm{Th}(G) = \mathrm{Th}(\Gamma) = \mathrm{Th}(K)$. Cela justifie peut-être aussi d'une certaine façon pourquoi, dans les quasi-tours du chapitre 1, on n'a pas de rétraction du gros groupe sur le petit en général.

CHAPTER 1

Hyperbolicity and cubulability are preserved under elementary equivalence

Abstract. The following properties are preserved under elementary equivalence, among finitely generated groups: being hyperbolic (possibly with torsion), being hyperbolic and cubulable, and being a subgroup of a hyperbolic group. In other words, if a finitely generated group G has the same first-order theory as a group possessing one of the previous property, then G enjoys this property as well.

1. Introduction

Around 1945, Tarski asked whether all non-abelian free groups have the same first-order theory. This famous question remained open for six decades and was finally answered in the affirmative by Sela in [**Sel06b**], and by Kharlampovich and Myasnikov in [**KM06**]. Sela then generalized his work in [**Sel09**] and classified all finitely generated groups with the same first-order theory as a given torsion-free hyperbolic group. A consequence of Sela's classification is the following striking theorem (see [**Sel09**], Theorem 7.10).

THEOREM 1.1 (Sela). *A finitely generated group with the same first-order theory as a torsion-free hyperbolic group is itself torsion-free hyperbolic.*

This result is particularly remarkable in view of the fact that hyperbolicity is defined in a purely geometric way. It is natural to ask whether Theorem 1.1 remains true if we allow torsion. We answer this question affirmatively.

THEOREM 1.2. *A finitely generated group with the same first-order theory as a hyperbolic group is itself hyperbolic.*

In fact, the previous theorem is still valid if one only assumes that the groups satisfy the same $\forall\exists$ sentences, that is the same sentences of the form $\forall \boldsymbol{x}\exists \boldsymbol{y}\psi(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are two tuples of variables, and ψ is a quantifier-free formula in these variables. The set of such sentences satisfied by a group G is called the $\forall\exists$ theory of G, denoted by $\mathrm{Th}_{\forall\exists}(G)$. Similarly, we denote by $\mathrm{Th}_{\forall}(G)$ the universal theory of G, i.e. the set of first-order sentences of the form $\forall \boldsymbol{x}\psi(\boldsymbol{x})$ that are true in G.

It is worth noting that, in Theorems 1.1 and 1.2, it is not sufficient to assume that the groups have the same universal theory. For instance, the class of finitely generated groups with the same universal theory as the free group $\boldsymbol{F}_2$ coincides with the well-known class of non-abelian limit groups (see [**Sel01**]), also known as finitely generated fully residually free non-abelian groups (see [**Rem89**]), and this class contains some non-hyperbolic groups, such as $\boldsymbol{F}_2*\mathbb{Z}^2$. Note also that it is impossible to remove the assumption of finite generation. For example, Szmielew proved in [**Szm55**] that $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Q}$ have the same first-order theory. Moreover, every hyperbolic group has the same first-order theory as groups of arbitrarily large cardinalities, by the Löwenheim–Skolem theorem.

We also show that the property of being a subgroup of a hyperbolic group is preserved under elementary equivalence, among finitely generated groups. More precisely, we prove the following theorem.

THEOREM 1.3. *Let Γ be a group, possibly non-finitely-generated, that can be embedded into a hyperbolic group, and let G be a finitely generated group. If $\mathrm{Th}_{\forall\exists}(\Gamma)\subset\mathrm{Th}_{\forall\exists}(G)$, then G embeds into a hyperbolic group.*

A hyperbolic group is said to be *locally hyperbolic* if its finitely generated subgroups are hyperbolic. The class of locally hyperbolic groups is rich and includes for instance virtually free groups, fundamental groups of compact hyperbolic manifolds of dimension 2 or 3, lots of small cancellation groups (see [**MW05**] and [**MW08**]), and lots of one-relator groups. For example, if S is a finite set and w is a non-trivial element in the free group $F(S)$, there exists an integer r_0 such that, for every integer $r\geq r_0$, the group $G=\langle S\mid w^r=1\rangle$ is locally hyperbolic and cubulable; more precisely, G is hyperbolic for $r \geq 2$, cubulable for $r \geq 4$ and locally hyperbolic for $r \geq 3|w|_S$, see [**MW05**].

In Theorem 1.3, in the case where the group Γ embeds into a finitely generated locally hyperbolic group, then we prove that G embeds into a locally hyperbolic group as well. In this case, the group G is itself hyperbolic.

THEOREM 1.4. *Let Γ be a group, possibly non-finitely-generated, that can be embedded into a finitely generated locally hyperbolic group, and let G be a finitely generated group. If* $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$*, then G is hyperbolic.*

Recall that a group is called cubulable if it admits a proper and cocompact action by isometries on a locally finite CAT(0) cube complex (see [**Sag14**] for an introduction). Groups that are both hyperbolic and cubulable have remarkable properties; they play a crucial role in the proof of the virtually Haken conjecture (see [**Ago13**]). By using results of Agol, Haglund and Wise, we prove the following result.

THEOREM 1.5. *Let Γ be a hyperbolic group and G a finitely generated group. Suppose that* $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$*. Then Γ is cubulable if and only if G is cubulable.*

Strategy. We now describe the strategy of proof of Theorems 1.2 and 1.3. The proofs of these results rely crucially on the *shortening argument* proved by Sela for torsion-free hyperbolic groups and subsequently generalized by Reinfeldt and Weidmann to hyperbolic groups with torsion.

THEOREM 1.6 (Sela, Reinfeldt-Weidmann). *Let Γ be a hyperbolic group, and let G be a one-ended finitely generated group. There exists a finite set $F \subset G \setminus \{1\}$ such that, for every non-injective homomorphism $\phi \in \mathrm{Hom}(G, \Gamma)$, there exists an automorphism σ of G such that $\ker(\phi \circ \sigma) \cap F \neq \varnothing$.*

We refer the reader to Theorem 2.2 for a more general version of the result stated above.

First, we shall outline a proof of Theorems 1.2 and 1.3 in the particular case where G is one-ended, finitely presented, and $\mathrm{Out}(G)$ is trivial.

CLAIM 1.7. *Let Γ be a hyperbolic group, and let G be a one-ended finitely presented group. Suppose that* $\mathrm{Th}_{\forall}(\Gamma) \subset \mathrm{Th}_{\forall}(G)$*. If* $\mathrm{Out}(G)$ *is trivial, then G embeds into Γ. Suppose moreover that Γ is one-ended and* $\mathrm{Out}(\Gamma)$ *is trivial. Then Γ and G are isomorphic.*

Let $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n) = 1 \rangle$ be a finite presentation of G, where Σ stands for a finite system of equations in n variables. Note that there is a one-to-one correspondence between $\mathrm{Hom}(G, \Gamma)$ and the set of n-tuples $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ such that $\Sigma(\gamma_1, \ldots, \gamma_n) = 1$. Let $F = \{g_1, \ldots, g_p\}$ be the finite set given by Theorem 1.6. Every element g_k can be written as a word $w_k(s_1, \ldots, s_n)$. Assume towards a contradiction that G does not embed into Γ. Then every homomorphism from G to Γ is non-injective, so kills an element of F by Theorem 1.6. In other words, the group Γ satisfies the following first-order sentence:

$$\forall x_1 \ldots \forall x_n \; (\Sigma(x_1, \ldots, x_n) = 1) \Rightarrow ((w_1(x_1, \ldots, x_n) = 1) \vee \ldots \vee (w_k(x_1, \ldots, x_n) = 1)).$$

Since $\mathrm{Th}_{\forall}(\Gamma) \subset \mathrm{Th}_{\forall}(G)$, this sentence is true in G as well. Taking $x_1 = s_1, \ldots, x_n = s_n$, the previous sentence means that an element of F is trivial. This is a contradiction. Thus, G embeds into Γ.

At this stage, we cannot conclude that G is hyperbolic. However, if Γ is one-ended and $\mathrm{Out}(\Gamma)$ is trivial, then we can prove in the same way as in the previous paragraph that Γ embeds into G by observing that Theorem 1.6 is still valid for subgroups of hyperbolic groups (see Corollary 2.4). Since one-ended hyperbolic groups are co-Hopfian (see [**Sel97**] for torsion-free hyperbolic groups and [**Moi13**], based on [**Del95**], for hyperbolic groups with torsion), the groups Γ and G are isomorphic. This concludes the proof of Claim 1.7.

In the proof of the claim, the hypothesis that G is finitely presented can be replaced by the hypothesis that G is finitely generated, using the fact that hyperbolic groups are equationally noetherian ([**Sel09**] and [**RW14**]), meaning that every infinite system of equations in finitely many variables Σ is equivalent to a finite subsystem of Σ.

Note that the proof of Claim 1.7 adapts easily to the case where $\mathrm{Out}(G)$ is finite. If $\mathrm{Out}(G)$ is infinite, the proof does not work, because we cannot fully express Theorem 1.6 by means of first-order logic as we did in the proof of the claim, since the expressive power of first-order logic is not sufficient to describe precomposition by an automorphism. However, by employing a strategy comparable with that used by Perin in [**Per11**], one can express some fragments of Theorem 1.6 and establish the following dichotomy.

PROPOSITION 1.8. *Let Γ be a hyperbolic group and G a one-ended finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Then,*

- *either G embeds into Γ,*
- *or G splits as a particular graph of groups called a* quasi-floor *over a group G_1 (see Definition 5.2).*

Quasi-floors generalize *hyperbolic floors* of *hyperbolic towers* in the sense of Sela and Perin (see Section 5 for further details). Without torsion, a quasi-floor is equivalent to a hyperbolic floor (see Proposition 5.5).

By using Proposition 1.8, we shall sketch a proof of (a particular case of) Theorem 1.3. Let Γ be a hyperbolic group, and let G be a one-ended finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. We aim to prove that G can be embedded into a hyperbolic group.

For now, suppose that the group G is torsion-free. If the second alternative of the dichotomy 1.8 holds, then G is a hyperbolic floor over a group G_1. In particular, G_1 is a proper subgroup of G, and there exists a retraction from G onto G_1. Moreover, G_1 satisfies the following key property, which is an easy consequence of the combination theorem of Bestvina and Feighn [**BF92**]: if G_1 embeds into Γ, then G embeds into a hyperbolic group. In order to prove that G embeds into a hyperbolic group, one wants to iterate the previous argument with G_1 instead of G. However, $\mathrm{Th}_{\forall\exists}(\Gamma)$ is not contained in $\mathrm{Th}_{\forall\exists}(G_1)$ and G_1 is not one-ended in general. We can nevertheless refine Proposition 1.8 and prove that G_1 satisfies the same dichotomy as G. By iterating this argument, we get a sequence of groups $(G_n)_{n\in\mathbb{N}}$ such that, for every integer n, the group G_n is a hyperbolic floor over G_{n+1}. In particular, G_{n+1} is a proper subgroup of G_n, and G_n retracts onto G_{n+1}. It follows from the descending chain condition 2.8 for Γ-limit groups that this sequence of groups is necessarily finite. Note that the descending chain condition applies since G_{n+1} is a quotient of G_n and $\mathrm{Th}_{\exists}(G_n)$ is contained in $\mathrm{Th}_{\exists}(\Gamma)$ (indeed, $\mathrm{Th}_{\exists}(G_n)$ is contained in $\mathrm{Th}_{\exists}(G)$ because G_n is a subgroup of G, and $\mathrm{Th}_{\exists}(G)$ is contained in $\mathrm{Th}_{\exists}(\Gamma)$ since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$). Let G_m denote the last group of the sequence. The group G is a hyperbolic tower over G_m since, for every integer n, the group G_n is a hyperbolic floor over G_{n+1}. As a consequence of the dichotomy, G_m embeds into the hyperbolic group Γ. It follows that G embeds into a hyperbolic group. This concludes the proof of Theorem 1.3 in the torsion-free case (assuming G is one-ended).

Unfortunately, quasi-floors do not have the same good features as hyperbolic floors. For every integer n, the group G_n (obtained as above by iterating the dichotomy 1.8) splits over finite groups as a graph of groups all of whose vertex groups embed into G, but G_n is not a subgroup of G in general. Consequently, $\mathrm{Th}_{\exists}(G_n)$ is not contained in $\mathrm{Th}_{\exists}(\Gamma)$ *a priori*, as shown by Example 1.9 below. Moreover, G_{n+1} is not a quotient of G_n. These pathologies make more complicated the termination of the iterative construction described in the previous paragraph, since we cannot use the descending condition directly. We solve part of these problems by proving that all hyperbolic groups can be embedded into torsion-saturated hyperbolic groups (see Theorem 1.10 below), a class of groups we introduce in Section 4.

A new phenomenon arising from the presence of torsion. In the case of torsion-free groups, performing a HNN extension over a finite group is the same as doing a free product with $\mathbb{Z}$, and every torsion-free non-elementary hyperbolic group G has the same universal theory as $G * \mathbb{Z}$ (see Proposition 2.22), and even the same first-order theory, as a consequence of the work of Sela [**Sel09**]. By contrast, in the presence of torsion, performing a HNN extension over a finite subgroup, even trivial, may modify the universal theory of a hyperbolic group. Let us consider the following simple example.

EXAMPLE 1.9. Let $G = F_2 \times \mathbb{Z}/2\mathbb{Z}$. Then the sentence $\forall x \forall y\ (x^2 = 1) \Rightarrow (xy = yx)$ is satisfied by G, but not by $G * \mathbb{Z}$.

This example shows that, in general, the class of groups with the same universal theory as a given hyperbolic group with torsion is not closed under HNN extensions and amalgamated products over finite groups. In Section 4, we deal with this problem by proving the following result, which is crucial for proving that the iterative construction outlined above eventually terminates.

THEOREM 1.10. *Every hyperbolic group Γ can be embedded into a torsion-saturated hyperbolic group $\overline{\Gamma}$, i.e. a hyperbolic group such that the class of $\overline{\Gamma}$-limit groups is closed under amalgamated free products and HNN extensions over finite groups.*

The hyperbolic group $\overline{\Gamma}$ is obtained from Γ by performing finitely many HNN extensions over finite groups (see Theorem 4.8).

Acknowledgements. I am very grateful to Vincent Guirardel for his help, and for the time he spent carefully reading previous versions of this paper. I would also like to thank Chloé Perin for a talk she gave at "Session états de la recherche de la SMF", where she presented a proof of Theorem 1.1; it was an important source of inspiration. Lastly, I warmly thank the referees for their numerous and valuable remarks and suggestions.

2. Preliminaries

In this section we recall some facts and definitions about the elementary theory of groups, Γ-limit groups, K-CSA groups, JSJ decompositions. We also prove some results that are useful in the sequel, and whose proofs are independent from the main body of the paper.

2.1. Notation.

Let $n \geq 1$ be an integer. We use the notation $[\![1, n]\!]$ to denote the set of integers k such that $1 \leq k \leq n$.

Let G be a group, and g an element of G. We write $\mathrm{ad}(g)$ for the inner automorphism $x \in G \mapsto gxg^{-1}$.

2.2. The elementary theory of groups.

For detailed background, we refer the reader to [**Mar02**]. A *first-order formula* in the language of groups is a finite formula using the following symbols: $\forall$, $\exists$, $=$, $\wedge$, $\vee$, $\Rightarrow$, $\neq$, 1 (standing for the identity element), $^{-1}$ (standing for the inverse), $\cdot$ (standing for the group multiplication) and variables $x, y, g, z \ldots$ which are to be interpreted as elements of a group. A variable is *free* if it is not bound by any quantifier $\forall$ or $\exists$. A *first-order sentence* is a first-order formula without free variables.

Given a formula $\psi(x_1, \ldots, x_n)$ with $n \geq 0$ free variables, and n elements $g_1, \ldots, g_n$ of a group G, we say that $\psi(g_1, \ldots, g_n)$ is satisfied by G if its interpretation is true in G. This is denoted by $G \models \psi(g_1, \ldots, g_n)$. The *elementary theory* of a group G, denoted by $\mathrm{Th}(G)$, is the collection of all sentences which are true in G. The *universal-existential* theory of G, denoted by $\mathrm{Th}_{\forall\exists}(G)$, is the collection of sentences true in G of the form

$$\forall x_1 \ldots \forall x_m \exists y_1 \ldots \exists y_n\ \psi(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

where $m, n \geq 1$ and ψ is a quantifier-free formula with $m+n$ free variables. In the same way, we define the *universal theory* of G, denoted by $\mathrm{Th}_{\forall}(G)$, and its *existential theory* $\mathrm{Th}_{\exists}(G)$. Note that $\mathrm{Th}_{\exists}(G)$ is contained in $\mathrm{Th}_{\exists}(\Gamma)$ if and only if $\mathrm{Th}_{\forall}(\Gamma)$ is contained in $\mathrm{Th}_{\forall}(G)$.

2.3. Γ-limit groups.

Let Γ and G be two groups. We say that G is *fully residually* Γ if, for every finite subset $F \subset G$, there exists a homomorphism $\phi : G \to \Gamma$ whose restriction to F is injective. If G is countable, then G is fully residually Γ if and only if there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of homomorphisms from G to Γ such that, for every non-trivial element $g \in G$, $\phi_n(g)$ is non-trivial for every n large enough. Such a sequence is called a *discriminating sequence.*

Γ-*limit groups* have been introduced by Sela in [**Sel09**] in order to study $\mathrm{Hom}(G, \Gamma)$, where Γ stands for a torsion-free hyperbolic group, and G stands for a finitely generated group. Sela proved that the class of Γ-limit groups coincides with the class of finitely generated groups that are fully residually Γ ([**Sel09**], Proposition 1.18). Then, Reinfeldt and Weidmann generalized this result in the case where Γ is hyperbolic possibly with torsion, in [**RW14**].

The following easy proposition builds a bridge between group theory and first-order logic. Recall that a group Γ is termed *equationally noetherian* if every infinite system of equations in uniformly finitely many variables Σ is equivalent to a finite subsystem of Σ.

Proposition 2.1. *Let G and Γ be finitely generated groups. Suppose that G is finitely presented or that Γ is equationally noetherian. Then G is a Γ-limit group if and only if $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$.*

Proof. Suppose that G is fully residually Γ. Let $(\phi_n : G \to \Gamma)_{n\in\mathbb{N}}$ be a discriminating sequence. Let $\Sigma(x_1, \ldots, x_m) = 1 \wedge \psi(x_1, \ldots, x_m) \neq 1$ be a system of equations and inequations. Suppose that this system has a solution $(g_1, \ldots, g_m)$ in G^m. Then, for n large enough, $\psi(\phi_n(g_1), \ldots, \phi_n(g_m)) = \phi_n(\psi(g_1, \ldots, g_m))$ is non-trivial. Thus, $(\phi_n(g_1), \ldots, \phi_n(g_m))$ is a solution of the previous system in Γ^m.

Conversely, suppose that $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$. Let $G = \langle s_1, \ldots, s_k \mid \Sigma(s_1, \ldots, s_k) = 1\rangle$ be a presentation of G, where Σ stands for a system of equations in k variables, possibly infinite. If G is finitely presentable, one can suppose that the system Σ is finite. If Γ is equationally noetherian, Σ is equivalent in Γ to a finite subsystem of Σ, so one can assume without loss of generality that Σ is finite. Let $S = \{g_1, \ldots, g_r\}$ be a finite subset of $G \setminus \{1\}$. Every element g_i can be written as a word $w_i(s_1, \ldots, s_k)$. We shall find a homomorphism $\phi : G \to \Gamma$ such that $\ker(\phi) \cap S = \varnothing$. Observe that the following sentence is satisfied by G, since the identity of G kills no element of S:

$$\theta : \exists x_1 \cdots \exists x_k \ \Sigma(x_1, \ldots, x_k) = 1 \wedge \bigwedge_{1\leq i\leq r} w_i(x_1, \ldots, x_k) \neq 1.$$

Since $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$, this sentence is satisfies by Γ as well. Let $\gamma_1, \ldots, \gamma_k$ be the elements of Γ given by the interpretation of the sentence θ in Γ. One defines a homomorphism $\phi : G \to \Gamma$ by sending s_i to γ_i. This homomorphism satisfies $\phi(g_i) = w_i(\gamma_1, \ldots, \gamma_k) \neq 1$, for every $1 \leq i \leq r$. This concludes the proof. □

The proofs of our main results rely essentially on the *shortening argument*, proved by Sela for torsion-free hyperbolic groups in [**Sel09**] and later generalized by Reinfeldt and Weidmann to the case of hyperbolic groups possibly with torsion in [**RW14**].

Given a hyperbolic group Γ and a Γ-limit group G, the *modular group* $\mathrm{Mod}(G)$ is a subgroup of $\mathrm{Aut}(G)$ that will be defined in Section 2.6.5, by means of the JSJ decomposition of G over virtually cyclic groups (with infinite center).

Theorem 2.2 ([**Sel09**] and [**RW14**] Theorem 4.2). *Let Γ be a hyperbolic group and let G be a one-ended finitely generated group. There exist non trivial elements $g_1, \ldots, g_k$*

of G such that, for every non-injective homomorphism $\phi : G \to \Gamma$, there exist a modular automorphism $\sigma \in \mathrm{Mod}(G)$ and an integer $1 \leq \ell \leq k$ such that $\phi \circ \sigma(g_\ell) = 1$.

REMARK 2.3. In the case where G is not a Γ-limit group, the result is obvious.

One easily sees that Theorem 2.2 remains true if one only assumes that Γ is a subgroup of a hyperbolic group.

COROLLARY 2.4. *Let Γ be a group that embeds into a hyperbolic group, and let G be a one-ended finitely generated group. There exist non-trivial elements $g_1, \ldots, g_k$ of G such that, for every non-injective homomorphism $\phi : G \to \Gamma$, there exist a modular automorphism $\sigma \in \mathrm{Mod}(G)$ and an integer $1 \leq \ell \leq k$ such that $\phi \circ \sigma(g_\ell) = 1$.*

PROOF. Denote by i an embedding of Γ into a hyperbolic group Ω. By the previous theorem, there exist non-trivial elements $g_1, \ldots, g_k$ of G such that every non-injective homomorphism from G to Ω kills some element g_ℓ, up to precomposition by a modular automorphism of G. If $\phi : G \to \Gamma$ is a non-injective homomorphism, then $i \circ \phi$ is non-injective as well, so $i \circ \phi \circ \sigma(g_\ell) = 1$ for some $1 \leq \ell \leq k$ and some $\sigma \in \mathrm{Mod}(G)$. Since i is injective, we have $\phi \circ \sigma(g_\ell) = 1$. $\square$

The following result is not explicitly stated in [**RW14**]. We refer the reader to the second chapter of this thesis for a proof (see Theorem 2.34).

THEOREM 2.5 (Sela, Reinfeldt-Weidmann). *Let Γ be a hyperbolic group and let G be a one-ended finitely generated group. Suppose that G embeds into Γ. Then there exists a finite set $\{\phi_1, \ldots, \phi_\ell\}$ of monomorphisms from G into Γ such that, for every monomorphism $\phi : G \to \Gamma$, there exist a modular automorphism $\sigma \in \mathrm{Mod}(G)$, an integer $1 \leq \ell \leq k$ and an element $\gamma \in \Gamma$ such that*

$$\phi = \mathrm{ad}(\gamma) \circ \phi_\ell \circ \sigma.$$

REMARK 2.6. In contrast with Theorem 2.2, Theorem 2.5 does not extend to the case where Γ is a subgroup of a hyperbolic group.

THEOREM 2.7 ([**Sel09**] and [**RW14**] Corollary 6.13). *Hyperbolic groups are equationally noetherian.*

The following result is known as the *descending chain condition* for Γ-limit groups, with Γ hyperbolic. It is an easy consequence of the equational noetherianity of hyperbolic groups.

THEOREM 2.8 ([**Sel09**] and [**RW14**] Corollary 5.3). *Let Γ be a hyperbolic group. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of Γ-limit groups. If $(\phi_n : G_n \twoheadrightarrow G_{n+1})_{n\in\mathbb{N}}$ is a sequence of epimorphisms, then ϕ_n is an isomorphism for n sufficiently large.*

PROOF. Let $G_0 = \langle s_1, \ldots, s_k \mid R\rangle$ and $G_n = \langle s_{1,n}, \ldots, s_{k,n} \mid R_n\rangle$ for every integer n, where $s_{i,n} = \phi_n \circ \cdots \circ \phi_0(s_i)$. Since Γ is equationally noetherian, there exists an integer n_0 such that, for every integer $n \geq n_0$, the set of solutions of R_n in Γ coincides with the set of solutions of R_{n_0} in Γ. Assume towards a contradiction that there exists an integer $n > n_0$ such that $R_n(s_{1,n_0}, \ldots, s_{k,n_0}) \neq 1$. Let $(\rho_i : G_n \to \Gamma)_{i\in\mathbb{N}}$ be a discriminating sequence. For i large enough, $R_n(\rho_i(s_{1,n_0}), \ldots, \rho_i(s_{k,n_0})) \neq 1$. Hence $R_{n_0}(\rho_i(s_{1,n_0}), \ldots, \rho_i(s_{k,n_0})) \neq 1$, so $R_{n_0}(s_{1,n_0}, \ldots, s_{k,n_0}) \neq 1$. This is a contradiction. This shows that the kernel of ϕ_n is trivial for all integers $n \geq n_0$. $\square$

THEOREM 2.9 ([**Sel09**] and [**RW14**] Proposition 6.2). *Let Γ be a hyperbolic group and G a Γ-limit group. Then every abelian subgroup of G is finitely generated.*

2.4. K-CSA groups.

A group is said to be CSA if its maximal abelian subgroups are malnormal. It is well-known that every torsion-free hyperbolic group is CSA. However, it is not true anymore in the presence of torsion. To overcome this problem, Guirardel and Levitt defined K-CSA groups in [GL17] (Definition 9.7).

Definition 2.10. Let $K > 0$. A group G is called a K-CSA group if the following conditions hold:

- every finite subgroup of G has order bounded by above by K (hence, an element g has infinite order if and only if $g^{K!} \neq e$);
- every element g of infinite order is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(g)$. Moreover $M(g)$ is K-virtually torsion-free abelian (i.e. $M(g)$ has a torsion-free abelian subgroup of index less than K);
- $M(g)$ is equal to its normalizer.

We recall some useful facts about K-CSA groups.

Proposition 2.11. *Every hyperbolic group is K-CSA for some $K > 0$.*

Proof. Let G be a hyperbolic group. It is well-known that there exists a constant K_1 such that all finite subgroups of G have order bounded from above by K_1 (see for instance [Bra00]).

Let g be an element of G of infinite order. Let g^+ and g^- denote the attracting and repelling fixed points of g on the boundary $\partial_\infty G$ of G. Recall that the stabilizer $S(g)$ of the pair of points $\{g^+, g^-\}$ is virtually cyclic. We claim that $S(g)$ is the unique maximal virtually abelian subgroup of G containing g. Let h be an element of G. If the subgroup $\langle g, h\rangle$ is virtually abelian, then it is virtually cyclic, so there exists an integer $n \geq 1$ such that g^n and $(hgh^{-1})^n$ commute. So we have $\mathrm{Fix}(g) = \mathrm{Fix}(hgh^{-1}) = h(\mathrm{Fix}(g))$. Hence h belongs to $S(g)$, which proves that $S(g)$ is the unique maximal virtually abelian subgroup of G containing g. Moreover, $S(g)$ is its own normalizer: if $S(g) = hS(g)h^{-1}$, then $\mathrm{Fix}(g) = \mathrm{Fix}(hgh^{-1})$, so h belongs to $S(g)$.

Last, we shall prove that $S(g)$ is K_2-virtually cyclic for some constant K_2. Let us observe that there are only finitely many isomorphism classes of virtually cyclic groups all of whose finite subgroups have order bounded from above by K_1. Indeed, every virtually cyclic group is either a semi-direct product $C \rtimes \mathbb{Z}$ with C finite, or an amalgamated free product $A *_C B$ with A, B and C finite. Up to isomorphism, there are only finitely many such groups satisfying $|A|, |B|, |C| \leq K_1$). As a consequence, there is a constant K_2 such that every virtually abelian subgroup of G is K_2-virtually cyclic. Therefore, the group G is $\max(K_1, K_2)$-CSA. ☐

Proposition 2.12 ([GL17] Lemma 9.8). *Let G be a K-CSA group.*

1. *If $g, h \in G$ have infinite order, the following conditions are equivalent:*
   - *(a) $M(g) = M(h)$.*
   - *(b) $g^{K!}$ and $h^{K!}$ commute.*
   - *(c) $\langle g, h\rangle$ is virtually abelian.*
2. *Let H be an infinite virtually abelian subgroup of G. Then H is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(H)$. This group is almost malnormal: if $gM(H)g^{-1} \cap M(H)$ is infinite, then g belongs to $M(H)$. Moreover, for every element h of H of infinite order, $M(H) = M(h)$.*

Proposition 2.13. *Let G be a K-CSA group and g an element of G of infinite order. The subgroup $M(g)$ is definable without quantifiers with respect to g. In other words, there exists a first-order formula $\psi_K(x, y)$ without quantifiers such that*

$$M(g) = \{h \in G \mid \psi_K(h, g)\}.$$

Proof. First, let us remark that

$$M(g) = \{h \in G \mid \langle g, h\rangle \text{ is K-virtually torsion-free abelian}\}.$$

Indeed, if $\langle g, h\rangle$ is K-virtually abelian, then $\langle g, h\rangle \subset M(g)$ by maximality of $M(g)$. Conversely, if $h \in M(g)$ then $\langle g, h\rangle$ is a subgroup of $M(g)$, which is K-virtually torsion-free abelian, so $\langle g, h\rangle$ is K-virtually torsion-free abelian. We now prove that there exists a first-order formula $\psi_K(x, y)$ with two free variables such that $\langle g, h\rangle$ is K-virtually torsion-free abelian if and only if $\psi_K(g, h)$ is true in G. Let $\pi : F_2 = \langle x, y\rangle \to G$ be the epimorphism sending x to g and y to h. If A is a subgroup of $\langle g, h\rangle$ of index less than K, there exists a subgroup B of $\langle x, y\rangle$ of index less than K such that $A = \pi(B)$. Denote by $H_1, \dots, H_n$ the n subgroups of F_2, of index $\leq K$. For each $1 \leq i \leq n$, let $(w_{i,j}(x, y))_{1\leq j\leq n_i}$ be a finite generating set of H_i. We can define $\psi_K(g, h)$ by

$$\psi_K(g, h) = \bigvee_{i=1}^{n} \bigwedge_{k=1}^{n_i} \bigwedge_{\ell=1}^{n_i} [w_{i,k}(g, h), w_{i,\ell}(g, h)] = 1.$$

□

Proposition 2.14 ([GL17] Proposition 9.9). *The property K-CSA is defined by a set of universal formulas.*

Since every hyperbolic group is K-CSA for some $K > 0$ (see Proposition 2.11), the following proposition holds.

Proposition 2.15 ([GL17] Corollary 9.10). *Let Ω be a hyperbolic group. There exists a constant $K > 0$ such that every group satisfying $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Omega)$ is K-CSA.*

We now prove that a group with the same $\forall\exists$-theory as a subgroup of a hyperbolic group does not contain $\mathbb{Z}^2$ as a subgroup (see Corollary 2.18 below). First, let us recall the following well-known result.

Lemma 2.16. *There exists a $\forall\exists$-sentence ψ such that, if G is a finitely generated torsion-free abelian group, $G \models \psi$ if and only if G is cyclic.*

Proof. Since $\mathbb{Z}^n/2\mathbb{Z}^n$ has 2^n elements, the pigeonhole principle shows that the following $\forall\exists$-sentence is satisfied by $\mathbb{Z}^n$ if and only if $n \leq 1$:

$$\forall x_1 \forall x_2 \forall x_3 \exists x_4 \ (x_1 = x_2 x_4^2) \vee (x_1 = x_3 x_4^2) \vee (x_2 = x_3 x_4^2).$$

□

Proposition 2.17. *Let Ω be a hyperbolic group, let Γ be a subgroup of Ω and let G be a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. If g is an element of G of infinite order, then $M(g)$ is virtually cyclic.*

Proof. First, note that $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma) \subset \mathrm{Th}_{\exists}(\Omega)$. By Proposition 2.15, the groups Ω, Γ and G are K-CSA for some K. Let g be an element of G of infinite order. Since the group $M(g)$ is K-virtually torsion-free abelian, it has a normal torsion-free abelian subgroup N of index dividing $K!$. For every element h of $M(g)$ the element $h^{K!}$ belongs to N. Denote by $N(g)$ the subgroup of $M(g)$ generated by $\{h^{K!} \mid h \in M(g)\}$. It is a subgroup of N, so it is torsion-free abelian.

It is enough to show that $N(g)$ is cyclic, then we will be able to conclude that $M(g)$ is virtually cyclic. Indeed, if $N(g)$ is cyclic, so is $\langle x^{K!}, y^{K!}\rangle$ for every $x, y \in M(g)$. As a consequence there is no pair of elements of $M(g)$ generating a subgroup isomorphic to $\mathbb{Z}^2$. Since $M(g)$ is virtually abelian and finitely generated by Theorem 2.9, it is virtually cyclic.

For every integer $\ell \geq 1$, let $N_\ell(g) = \{h_1^{K!} \cdots h_\ell^{K!} \mid h_1, \dots, h_\ell \in M(g)\}$. Since $N(g)$ is finitely generated, there exist an integer r and some elements $g_1, \dots, g_r$ of $M(g)$ such that

$N(g)$ is generated by $\{g_1^{K!}, \ldots, g_r^{K!}\}$. We claim that $N(g) = N_r(g)$. In order to see this, remark that, since $N(g)$ is abelian, every element $h \in N(g)$ can be written as follows:

$$h = \left(g_1^{K!}\right)^{n_1} \cdots \left(g_r^{K!}\right)^{n_r} = (g_1^{n_1})^{K!} \cdots (g_r^{n_r})^{K!},$$

where $n_1, \ldots, n_r$ lie in $\mathbb{Z}$. This proves that $N(g) \subset N_r(g)$, and the reverse inclusion is immediate. Then, recall that there exists a first-order formula without quantifiers $\psi(x, y)$ such that $M(g) = \{h \in G \mid \psi(h, g)\}$ (see Proposition 2.13). Hence

$$N_r(g) = \left\{h \in G \mid \exists h_1 \ldots \exists h_r \ \left(h = h_1^{K!} \cdots h_r^{K!} \wedge \psi(h_1, g) \wedge \ldots \wedge \psi(h_r, g)\right)\right\}.$$

It remains to prove that $N_r(g)$ is cyclic. Recall that, by Lemma 2.16, a finitely generated torsion-free abelian group is cyclic if and only if it satisfies $\forall x_1 \forall x_2 \forall x_3 \exists x_4 \theta(x_1, x_2, x_3, x_4)$, where $\theta(x_1, x_2, x_3, x_4) : (x_1 = x_2 x_4^2) \vee (x_1 = x_3 x_4^2) \vee (x_2 = x_3 x_4^2)$. Since Ω is a hyperbolic group and $\Gamma \subset \Omega$, every torsion-free abelian subgroup of Γ is cyclic, so satisfies the previous sentence. We can write a $\forall\exists$-sentence χ_r satisfied by Γ, with the following interpretation: for every element γ of Γ of infinite order, $N_r(\gamma)$ is cyclic. Below is the sentence χ_r, where $\boldsymbol{h}_i$ stands for $(h_{i,1}, \ldots, h_{i,r})$ and $x_i := h_{i,1}^{K!} \cdots h_{i,r}^{K!}$.

$$\chi_r : \forall \gamma \forall \boldsymbol{h}_1 \forall \boldsymbol{h}_2 \forall \boldsymbol{h}_3 \exists \boldsymbol{h}_4 \left(\bigwedge_{i=1}^{3} \bigwedge_{j=1}^{r} \psi(h_{i,j}, \gamma) \wedge \left(\gamma^{K!} \neq 1\right)\right) \Rightarrow \left(\bigwedge_{j=1}^{r} \psi(h_{4,j}, \gamma) \wedge \theta(x_1, x_2, x_3, x_4)\right).$$

Since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, the sentence χ_r is true in G as well. It follows that $N_r(g)$ is cyclic. This concludes the proof. $\square$

Corollary 2.18. *Let Ω be a hyperbolic group, let Γ be a subgroup of Ω and let G be a finitely generated group. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Then every abelian subgroup of G is virtually cyclic.*

Proof. First, note that $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Omega)$. By Proposition 2.15, the group G is K-CSA for some K. Let H be an infinite abelian subgroup of G. By Proposition 2.12, H is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(H)$, and $M(H) = M(h)$ for every element h of H of infinite order (note that such an element exists since H is abelian, finitely generated (according to Lemma 2.9) and infinite). According to Proposition 2.17 above, $M(h)$ is virtually cyclic, hence H is virtually cyclic. $\square$

2.5. Generalized Baumslag's lemma.

If g is an element of infinite order of a hyperbolic group G, we denote by g^+ and g^- the attracting and repelling fixed points of g on the boundary $\partial_\infty G$ of G. The following proposition generalizes a criterion proved by Baumslag in the case of free groups (see [**Bau62**] Proposition 1 or [**Bau67**] Lemma 7). This result is very useful to show that some sequences of homomorphisms taking values in a hyperbolic group are discriminating. A proof in the hyperbolic case can be found in [**Os93**], Lemma 2.4. We include a proof for completeness. We begin with a preliminary lemma.

Lemma 2.19. *Let G be a hyperbolic group and S a finite generating set of G. Let g, h, x be elements of G such that g and h have infinite order. For every $p \geq 1$, denote by α_p a geodesic path between g^p and g^{p-1} in $\mathrm{Cay}(G, S)$, and β_p a geodesic path between xh^{p-1} and xh^p in $\mathrm{Cay}(G, S)$. Denote by γ a path joining 1 and x. If $g^+ \neq x \cdot h^+$, then there exist two constants λ and k such that for every integers $p \geq 1$ and $q \geq 1$, $w_{p,q} = \alpha_p \cdot \cdots \cdot \alpha_1 \cdot \gamma \cdot \beta_1 \cdot \cdots \cdot \beta_q$ is a (λ, k)-quasi-geodesic.*

PROOF. It is well-known that $p \in \mathbb{Z} \mapsto g^p$ and $p \in \mathbb{Z} \mapsto h^p$ are quasi-geodesics, i.e. there exist constants λ_g, k_g and λ_h, k_h such that for every p, $\alpha_p \cdots \alpha_1$ is a (λ_g, k_g)-quasi-geodesic joining g^p and 1, and $\beta_1 \cdots \beta_p$ is a (λ_h, k_h)-quasi-geodesic joining x and xh^p. An easy computation gives

$$\mathrm{length}(w_{p,q}) \leq \max(\lambda_g, \lambda_h)(d(1, g^p) + d(1, xh^q)) + (k_g + k_h + \mathrm{length}(\gamma) + \lambda_h d(1, x)).$$

Since $g^+ \neq x \cdot h^+$, there exists a constant C such that the Gromov product $(g^p, xh^q)_1$ is less than C, that is

$$d(1, g^p) + d(1, xh^q) - d(g^p, xh^q) \leq 2C.$$

It follows that $\mathrm{length}(w_{p,q}) \leq \lambda d(g^p, xh^q) + k$ for some constants λ and k. □

One can now prove the following criterion generalizing that of Baumslag.

PROPOSITION 2.20. *Let $a_0, a_1, \ldots, a_m, c_1, \ldots, c_m$ be elements of a hyperbolic group. Let $w(p_1, \ldots, p_m) = a_0 c_1^{p_1} a_1 c_2^{p_2} a_2 \cdots a_{m-1} c_m^{p_m} a_m$ with $p_1, \ldots, p_m \geq 0$. Suppose that the following two conditions hold:*

(1) every c_i has infinite order;
(2) for every $i \in [\![1, m-1]\!]$, $c_i^+ \neq a_i \cdot c_{i+1}^+$.

Then there exists a constant C such that $w(p_1, \ldots, p_m) \neq 1$ for every $p_1, \ldots, p_m \geq C$.

PROOF. We will prove that there exist constants λ, k satisfying the following property: for every integer n, there exists an integer $p(n)$ such that for every $p_1, \ldots, p_m \geq p(n)$, $w(p_1, \ldots, p_m)$ (viewed as a path in $\mathrm{Cay}(G, S)$, for a given finite generating set S) is a local (λ, k, n)-quasi-geodesic, i.e. every subword of $w(p_1, \ldots, p_m)$ whose length is less than n is a (λ, k)-quasi-geodesic. Then, it will follow from [**CDP90**] Theorem 1.4 that there exist three constants L, λ' and k' such that for every $p_1, \ldots, p_m \geq p(\lceil L \rceil)$, $w(p_1, \ldots, p_m)$ is a (λ', k')-quasi-geodesic. Hence:

$$d(1, w(p_1, \ldots, p_m)) \geq 1/\lambda'(\mathrm{length}(w(p_1, \ldots, p_m)) - k').$$

Moreover $\lim\limits_{p_1, \ldots, p_m \to +\infty} \mathrm{length}(w(p_1, \ldots, p_m)) = +\infty$, so $d(1, w(p_1, \ldots, p_m)) \geq 1$ for every $p_1, \ldots, p_m$ large enough. As a consequence, $w(p_1, \ldots, p_m) \neq 1$ for every $p_1, \ldots, p_m$ large enough.

To prove the existence of constants λ and k, let's look at subwords of $w(p_1, \ldots, p_m)$. For every n, every subword of $w(p_1, \ldots, p_m)$ whose length is less that n is a subword of a word of the form c_i^k or $c_i^k a_i c_{i+1}^k$ where $i \in [\![1, m-1]\!]$.

Now, it suffices to show that the words above are quasi-geodesic with constants that do not depend on k. This follows from Lemma 2.19. □

If G is a hyperbolic group, each element g of infinite order is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(g)$, namely the stabilizer of the pair of points $P = \{g^+, g^-\}$. An element $h \in G$ belongs to $M(g)$ if and only if $h(P) \cap P \neq \varnothing$. The following straightforward consequence of Proposition 2.20 is easier to use in practice.

COROLLARY 2.21. *Let $a_0, a_1, \ldots, a_m$ and c be elements of a hyperbolic group. Let (ε_i) in $\{-1, +1\}^m$. Let $w(p) = a_0 c^{\varepsilon_1 p} a_1 c^{\varepsilon_2 p} a_2 \cdots a_{m-1} c^{\varepsilon_m p} a_m$ with $p \geq 0$. Suppose that the following two conditions hold:*

(1) c has infinite order;
(2) for every $i \in [\![1, m-1]\!]$, $a_i \notin M(c)$.

Then there exists a constant C such that $w(p) \neq 1$ for $p \geq C$.

Here is an interesting application of Baumslag's lemma.

PROPOSITION 2.22. *Let G be a non-elementary torsion-free hyperbolic group. We have*

$$\mathrm{Th}_\forall(G * \mathbb{Z}) = \mathrm{Th}_\forall(G).$$

Proof. Fix a finite generating set of $G * \langle t\rangle$, and let B_n denote the ball of radius n in $G * \langle t\rangle$ for this generating set. We shall use Baumslag's lemma in order to find a homomorphism ϕ_n from $G * \langle t\rangle$ to G that does not kill any non-trivial element of B_n. As a consequence, the sequence of homomorphisms $(\phi_n)_{n\in\mathbb{N}}$ will be discriminating, hence $\mathrm{Th}_\forall(G*\mathbb{Z}) \subset \mathrm{Th}_\forall(G)$. Note that the reverse inclusion $\mathrm{Th}_\forall(G*\mathbb{Z}) \supset \mathrm{Th}_\forall(G)$ is obvious since $G \subset G * \mathbb{Z}$.

First, note that every non-trivial element x of B_n can be written as a reduced word $x = g_{1,x}t^{n_{1,x}}g_{2,x}t^{n_{2,x}}\cdots t^{n_{k_x,x}}g_{k_x+1,x}$, with $n_{i,x} \neq 0$ and $g_{i,x} \in G\setminus\{1\}$, except $g_{1,x}$ and $g_{k_x+1,x}$ that may be trivial. Let $A = \bigcup_{x\in B_n}\{g_{1,x},\ldots,g_{k_x,x}\}$. Since the group G is torsion-free hyperbolic, two elements commute if and only if they fix a common point on the boundary of G. By hypothesis, G is non-elementary, so $\{g^+ \mid g \in G\setminus\{1\}\}$ is infinite. Moreover, every non-trivial element of G fixes exactly two distinct points on the boundary, and A is a finite set, so one can find an element $g \in G$ that does not fix any point on the boundary that is fixed by a non-trivial element of A. Hence, g does not commute with any non-trivial element of A. It follows from Corollary 2.21 that the homomorphism ϕ_n from $G * \langle t\rangle$ to G whose restriction to G is the identity and that sends t to a sufficiently big power of g does not kill any non-trivial element of B_n. Hence, the sequence $(\phi_n)_{n\in\mathbb{N}}$ is discriminating. □

Note that Proposition 2.22 above is false if G is not assumed to be torsion-free, as illustrated by Example 1.9 in the introduction. Section 4 is devoted to this problem.

2.6. JSJ decompositions of finitely generated groups.

JSJ decompositions first appeared in 3-dimensional topology with the theory of the characteristic submanifold by Jaco and Shalen (see [**JS79**]) and by Johannson (see [**Joh79**]). The terminology JSJ was popularized by Sela. He adapted the topological ideas from [**JS79**] and [**Joh79**] to torsion-free hyperbolic groups in [**Sel97**] (see also [**RS97**] and [**Bow98**]). Constructions of JSJ decompositions were given in more general settings by many authors. In [**GL17**], Guirardel and Levitt give a simple general definition of JSJ decompositions and explain how to construct JSJ decompositions in a wide range of contexts. Here below we summarize briefly their definitions and constructions. We refer the reader to [**GL17**] for details.

We consider a finitely generated group G and a family $\mathcal{A}$ of subgroups of G, closed under conjugating and taking subgroups, and we aim to understand decompositions of G as a graph of groups with edge groups belonging to $\mathcal{A}$. In other words, we consider trees with an action of the group G, and we require that edge stabilizers be in $\mathcal{A}$. We call such a tree an $\mathcal{A}$-tree.

Definition 2.23. An $\mathcal{A}$-tree is *universally elliptic* if its edge stabilizers are elliptic in every $\mathcal{A}$-tree.

Definition 2.24. Let G be a finitely generated group, and let $\mathcal{A}$ be a class of subgroups of G closed under conjugating and taking subgroups. Let T and T' be two $\mathcal{A}$-trees. We say that T *dominates* T' if there is a G-equivariant map $T \to T'$; equivalently, any group which is elliptic in T is also elliptic in T'.

Definition 2.25. A *JSJ decomposition* of G over $\mathcal{A}$ is an $\mathcal{A}$-tree T such that:

- T is universally elliptic;
- T dominates any other universally elliptic tree T'.

We also call the quotient graph of groups T/G a JSJ decomposition, or a JSJ splitting.

The second condition in the definition above is a maximality condition expressing that vertex stabilizers of T are as small as possible (they are elliptic in every universally elliptic tree).

2.6.1. *Existence of JSJ decompositions.*

Note that JSJ decompositions do not always exist: Dunwoody constructed in [**Dun93**] a finitely generated group that has no JSJ decomposition over finite groups. In this thesis, we are mainly concerned with the following JSJ decompositions:

(1) JSJ decompositions of a finitely generated group G over finite subgroups of G. The existence of these decompositions is guaranteed as soon as there exists a constant K such that every finite subgroup of G has order less than K (see [**Lin83**]). However, such a decomposition is not unique in general. See Section 2.6.6 for more details.
(2) JSJ decompositions of a one-ended finitely generated group G over virtually $\mathbb{Z}$ subgroups of G. Such decompositions exist for one-ended hyperbolic groups, and more generally for one-ended finitely generated K-CSA groups that do not contain $\mathbb{Z}^2$. For a proof of the existence, we refer the reader to the first point of Theorem 9.14 of [**GL17**], which is an immediate consequence of Corollary 9.1.
(3) JSJ decompositions of a one-ended finitely generated group G over the family $\mathcal{A} = \mathcal{Z} \cup \mathcal{F}$, where $\mathcal{Z}$ is the family consisting of virtually $\mathbb{Z}$ subgroups of G with infinite center, and $\mathcal{F}$ is the family of finite subgroups of G. Such decompositions exist for one-ended hyperbolic groups, and more generally for one-ended finitely generated K-CSA groups that do not contain $\mathbb{Z}^2$. Again, the proof of the existence is an immediate consequence of Corollary 9.1. of [**GL17**]. Note that edge groups belongs to $\mathcal{Z} \subset \mathcal{A}$ since G is assumed to be one-ended, but we need to consider the family $\mathcal{A}$ since $\mathcal{Z}$ is not closed under taking subgroups.
(4) JSJ decompositions of a one-ended finitely generated K-CSA group over virtually abelian groups (see [**GL17**], first point of Theorem 9.14).

2.6.2. *Uniqueness.*

Let G be a one-ended finitely generated as above (cases 2 to 4). In [**GL17**], the authors associate to each JSJ decomposition T of G over $\mathcal{A}$ a tree called the *tree of cylinders* of T, denoted by T_c (see [**GL17**] Definition 7.2). They prove that T_c is a JSJ decomposition over $\mathcal{A}$, and that T_c does not depend on the initial JSJ decomposition T: if T' is another JSJ decomposition over $\mathcal{A}$, then $T_c = T'_c$ (meaning that there exists a G-equivariant isomorphism between them). This tree T_c is called the *canonical* JSJ decomposition of G over $\mathcal{A}$ (see the third point of Lemma 7.3 in [**GL17**] for details). If we are in the third case above (i.e. $\mathcal{A} = \mathcal{Z} \cup \mathcal{F}$), we refer to the canonical decomposition as the $\mathcal{Z}$-JSJ splitting of G (we use the symbol $\mathcal{Z}$ instead of $\mathcal{A}$ to emphasize that edges groups belong to $\mathcal{Z}$).

In particular, since $\mathcal{A}$ is preserved under automorphisms of G, the following holds: for every automorphism α of G, there is a unique α-equivariant automorphism $H_\alpha : T \to T$. Hence, one gets an action of $\mathrm{Aut}(G)$ on T, and this action can be used to study $\mathrm{Aut}(G)$. For a one-ended torsion-free hyperbolic group G, this idea is due to Rips and Sela (see [**RS94**], [**Sel97**] and [**Lev05**]).

2.6.3. *Description of the vertex groups.*

Once we know that JSJ decompositions exist, the essential feature of the JSJ theory is the description of their flexible vertices, as defined below.

Definition 2.26. Let G be a finitely generated group, and let $\mathcal{A}$ be a class of subgroups of G closed under conjugating and taking subgroups. Let T be a JSJ decomposition of G over $\mathcal{A}$. A vertex group G_v of T is said to be *rigid* if it is elliptic in every splitting of G over $\mathcal{A}$. If G_v fails to be elliptic in some splitting of G over $\mathcal{A}$, it is said to be *flexible*.

Before giving a description of flexible vertex groups of the $\mathcal{Z}$-JSJ splitting, we shall recall some basic facts about hyperbolic 2-dimensional orbifolds.

A compact connected 2-dimensional orbifold with boundary $\mathcal{O}$ is said to be *hyperbolic* if it is equipped with a hyperbolic metric with totally geodesic boundary. It is the quotient of a closed convex subset $C \subset \mathbb{H}^2$ by a proper discontinuous group of isometries $G_{\mathcal{O}} \subset \mathrm{Isom}(\mathbb{H}^2)$. We denote by $p : C \to \mathcal{O}$ the quotient map. By definition, the orbifold fundamental group$\pi_1(\mathcal{O})$ of $\mathcal{O}$ is $G_{\mathcal{O}}$. We may also view $\mathcal{O}$ as the quotient of a compact orientable hyperbolic surface with geodesic boundary by a finite group of isometries. A point of $\mathcal{O}$ is *singular* if its preimages in C have non-trivial stabilizer. A *mirror* is the image by p of a component of the fixed point set of an orientation-reversing element of $G_{\mathcal{O}}$ in C. Singular points not contained in mirrors are *conical points*; the stabilizer of the preimage in $\mathbb{H}^2$ of a conical point is a finite cyclic group consisting of orientation-preserving maps (rotations). The orbifold $\mathcal{O}$ is said to be *conical* if it has no mirror.

Definition 2.27. A group G is called a *finite-by-orbifold group* if it is an extension

$$1 \to F \to G \to \pi_1(\mathcal{O}) \to 1$$

where $\mathcal{O}$ is a compact connected hyperbolic conical 2-orbifold, possibly with totally geodesic boundary, and F is an arbitrary finite group called the *fiber*. We call an *extended boundary subgroup* of G the preimage in G of a boundary subgroup of the orbifold fundamental group $\pi_1(\mathcal{O})$ (for an indifferent choice of regular base point). We define in the same way *extended conical subgroups*.

Definition 2.28. A vertex v of a graph of groups is said to be quadratically hanging (denoted by QH) if its stabilizer G_v is a finite-by-orbifold group $1 \to F \to G \to \pi_1(\mathcal{O}) \to 1$ such that $\mathcal{O}$ has non-empty boundary, and such that any incident edge group is finite or contained in an extended boundary subgroup of G. We also say that G_v is QH.

The following proposition is crucial (see Section 6 of [**GL17**], Theorem 6.5 and the paragraph below Remark 9.29).

Proposition 2.29. *Let G be a one-ended finitely generated K-*CSA *group that does not contain $\mathbb{Z}^2$. The flexible vertex groups of its $\mathcal{Z}$-JSJ splitting are QH.*

2.6.4. *Properties of the $\mathcal{Z}$-JSJ decomposition.*

Proposition 2.30 below summarizes the properties of the $\mathcal{Z}$-JSJ decomposition that will be useful in the sequel.

Proposition 2.30. *Let G be a one-ended finitely generated K-*CSA *group that does not contain $\mathbb{Z}^2$ as a subgroup. Let T be its $\mathcal{Z}$-JSJ decomposition.*

- *The tree T is bipartite: every edge joins a vertex carrying a maximal virtually cyclic group to a vertex carrying a non-virtually-cyclic group.*
- *The action of G on T is acylindrical in the following strong sense: if an element $g \in G$ of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its midpoint has virtually cyclic stabilizer.*
- *Let v be a vertex of T, and let e, e' be two distinct edges incident to v. If G_v is not virtually cyclic, then the group $\langle G_e, G_{e'} \rangle$ is not virtually cyclic.*
- *There are two kinds of vertices of T carrying a non-virtually cyclic group: rigid ones, and QH ones. If v is a QH vertex of T, every edge group G_e of an edge e incident to v coincides with an extended boundary subgroup of G_v. Moreover, given any extended boundary subgroup B of G_v, there exists a unique incident edge e such that $G_e = B$.*

Proof. The first three points follow from the fact that T is defined as the tree of cylinders of a JSJ decomposition of G with edge groups in $\mathcal{Z}$ (see [**GL17**] Definition 7.2 for a definition of the tree of cylinders).

The fact that vertices of T carrying a non-virtually cyclic group are rigid or QH is a consequence of Proposition 2.29, according to which flexible vertex groups are QH.

By definition of a QH vertex group, every edge group G_e of an edge e incident to v is contained in an extended boundary subgroup B of G_v. This inclusion is in fact an equality because G_e is virtually cyclic maximal in G_v (by construction of the tree of cylinders).

Given any extended boundary subgroup B of G_v, there exists an incident edge e such that $G_e = B$, according to the first point of Proposition 5.20 in [**GL17**]. Moreover, e is unique. Indeed, if there is another edge e' incident to v and different from e such that $G_{e'} = B$, then we have a segment of length 2 with infinite stabilizer (equal to B) and whose midpoint is not virtually cyclic, which contradicts the second point above. $\square$

The following result will be constantly used in the sequel.

Proposition 2.31. *Let Ω be a hyperbolic group and Γ a subgroup of Ω. Let G be a finitely generated group, and H a one-ended finitely generated subgroup of G. If* $\mathrm{Th}_{\forall\exists}(\Gamma)$ *is contained in* $\mathrm{Th}_{\forall\exists}(G)$*, then H is K-CSA for some constant $K > 0$ and does not contain $\mathbb{Z}^2$ as a subgroup. Consequently, H has a $\mathcal{Z}$-JSJ splitting.*

Proof. By Proposition 2.15, the group G is K-CSA for some constant K. Moreover, by Proposition 2.18, every abelian subgroup of G is virtually cyclic. Since H is a subgroup of G, it is K-CSA and every abelian subgroup of H is virtually cyclic. Hence, H has a $\mathcal{Z}$-JSJ splitting. $\square$

2.6.5. *The modular group.*

Definition 2.32. Let G be a one-ended finitely generated K-CSA group that does not contain $\mathbb{Z}^2$ as a subgroup. The modular group $\mathrm{Mod}(G)$ of G is the subgroup of $\mathrm{Aut}(G)$ composed of all automorphisms that act by conjugation on non-QH vertex groups of the $\mathcal{Z}$-JSJ splitting, and on finite subgroups of G, and that act trivially on the underlying graph of the $\mathcal{Z}$-JSJ splitting.

2.6.6. *JSJ splittings over finite groups.*

Let G be a finitely generated group. Under the hypothesis that there exists a constant K such that every finite subgroup of G has order less than K, Linnell proved in [**Lin83**] that G splits as a finite graph of groups with finite edge groups and all of whose vertex groups are finite or one-ended. We call such a splitting a $\mathcal{F}$-JSJ splitting of G, where $\mathcal{F}$ stands for "finite" (in [**GL17**], such a splitting is called a Stallings-Dunwoody splitting of G).

Contrary to the $\mathcal{Z}$-JSJ splitting, there is no canonical $\mathcal{F}$-JSJ splitting, but the conjugacy classes of one-ended vertex groups do not depend on the splitting. Moreover, the conjugacy classes of finite vertex groups are the same in all reduced $\mathcal{F}$-JSJ splittings of G. A one-ended subgroup of G that appears as a vertex group of a $\mathcal{F}$-JSJ splitting is called a *one-ended factor* of G. Note that if G is virtually free, then there is no one-ended factors, all vertex stabilizers are finite.

If G is a Γ-limit group, where Γ is hyperbolic, one can prove that there exists a uniform bound on the order of finite subgroups of G (see [**RW14**] Lemma 1.18). As a consequence, G has a $\mathcal{F}$-JSJ splitting.

Definition 2.33. Let G be a finitely generated group whose finite subgroup have bounded order. Let G' be a group. A homomorphism $\phi : G \to G'$ is said to be factor-injective if ϕ is injective in restriction to every one-ended factor of G that is not finite-by-orbifold.

2.7. Preliminaries on orbifolds.

In the sequel, all orbifolds are compact, connected, 2-dimensional, hyperbolic and conical, i.e. without mirrors.

2.7.1. *Cutting an orbifold into elliptic components.*

Definition 2.34. A set $\mathcal{C}$ of simple closed curves on a conical orbifold is said to be essential if its elements are non null-homotopic, two-sided, non boundary-parallel, pairwise non parallel, and represent elements of infinite order (in other words, no curve of $\mathcal{C}$ bounds a disk with a single singularity (or cone point)).

Proposition 2.35. *Let $\mathcal{O}$ be a hyperbolic orbifold. Suppose that $S = \pi_1(\mathcal{O})$ acts minimally on a simplicial tree T in such a way that its boundary elements are elliptic. Then there exists an essential set $\mathcal{C}$ of curves on $\mathcal{O}$, and a surjective S-equivariant map $f : T_{\mathcal{C}} \to T$, where $T_{\mathcal{C}}$ stands for the Bass-Serre tree associated with the splitting of S dual to $\mathcal{C}$. In other words:*

- *every element of S corresponding to a loop of $\mathcal{C}$ fixes an edge of T;*
- *every fundamental group of a connected component of $\mathcal{O} \setminus \mathcal{C}$ is elliptic.*

The proposition above is proved in [**MS84**] for surfaces (see Theorem 3.2.6), and the generalization to compact hyperbolic 2-orbifolds of conical type is straightforward since conical subgroups are elliptic in T. Then, Proposition 2.35 extends to finite-by-orbifold groups through the following observation: if G is a finite extension $F \hookrightarrow G \twoheadrightarrow \pi_1(\mathcal{O})$ acting minimally on a tree T, then the action factors through $G/F \simeq \pi_1(\mathcal{O})$, because F acts as the identity on T. Indeed, since F is finite, it fixes a point x of T. Since F is normal, the non-empty subtree of T pointwise-fixed by F is invariant under the action of G. Since this action is minimal, F fixes T pointwise.

2.7.2. *Non-pinching homomorphisms.*

In this section, we establish conditions ensuring that a homomorphism between two finite-by-orbifold groups is an isomorphism.

Definition 2.36. Let G and G' be conical finite-by-orbifold groups (see Definition 2.27). A homomorphism from G to G' is called a morphism of finite-by-orbifold groups if it sends each extended boundary subgroup injectively into an extended boundary subgroup, and if it is injective on finite subgroups.

Definition 2.37. Let G be a conical finite-by-orbifold group $F \hookrightarrow G \overset{q}{\twoheadrightarrow} \pi_1(\mathcal{O})$. Let ϕ be a homomorphism from G to a group G'. Let α be an essential curve on $\mathcal{O}$, let $\boldsymbol{\alpha}$ be a corresponding element of the fundamental orbifold group of $\mathcal{O}$ (for a given choice of base point) and let $C_{\boldsymbol{\alpha}} = q^{-1}(\langle \boldsymbol{\alpha} \rangle) \simeq F \rtimes \mathbb{Z}$. The curve α is said to be pinched by ϕ if $\phi(C_{\boldsymbol{\alpha}})$ is finite. The homomorphism ϕ is said to be non-pinching if it does not pinch any two-sided simple loop. Otherwise, ϕ is said to be pinching.

Lemma 2.38. *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds. Let G and G' be their orbifold fundamental groups. Let $\phi : G \to G'$ be a non-pinching morphism of orbifold groups. Suppose that $\phi(G)$ is not contained in a conical or boundary subgroup of G'. Then $\phi(G)$ has infinite index in G'.*

Proof. Suppose towards a contradiction that $\phi(G)$ has infinite index in G'. Then $\phi(G)$ is the fundamental group of an infinite degree orbifold covering $\overline{\mathcal{O}'}$ of $\mathcal{O}'$. This covering is a geometrically finite hyperbolic conical orbifold of infinite volume. Let K be its convex core. The inclusion $K \subset \overline{\mathcal{O}'}$ induces an isomorphism between the orbifold fundamental groups of K and $\overline{\mathcal{O}'}$. Since $\overline{\mathcal{O}'}$ is of infinite volume and has no cusp end, it has at least one funnel end, so K has at least one boundary component that is not a preimage in $\overline{\mathcal{O}'}$ of a boundary component of $\mathcal{O}'$. Therefore, $\phi(G)$ can be written as a free product

$$\phi(G) = \langle c_1 \rangle * \cdots * \langle c_p \rangle * \langle b_1 \rangle * \cdots * \langle b_q \rangle * F_{r-1},$$

where $c_1, \ldots, c_p$ are conical elements of $\phi(G)$, $b_1, \cdots, b_q$ denote boundary elements of $\phi(G)$ conjugate to boundary elements of G', and $r \geq 1$ is the number of funnel ends of $\overline{\mathcal{O}'}$. Note that this splitting is non-trivial because, by assumption, $\phi(G)$ is not contained in a boundary subgroup or in a conical subgroup of G'. Let T be the Bass-Serre tree of this splitting. The group G acts on T via ϕ. Moreover, by definition of a morphism of orbifold groups, every boundary element of G is elliptic in T for this action. Then it follows from Proposition 2.35 that there exists a simple loop on $\mathcal{O}$ pinched by ϕ. This is a contradiction. $\square$

An Euler characteristic on a class of groups $\mathcal{G}$ closed under taking subgroups of finite index is a function $\chi : \mathcal{G} \to \mathbb{R}$ satisfying the following condition: if $G \in \mathcal{G}$ and H is a subgroup of finite index in G, then $\chi(H) = [G : H]\chi(G)$. Euler characteristics have been defined on various classes of groups, by many authors (see for instance [**Syk05**]).

In the sequel, we consider the class $\mathcal{G}$ composed of groups G of the form

$$1 \to F \to G \to A_1 * \cdots * A_m * F_n \to 1$$

where $F, A_1, \ldots, A_m$ are finite and F_n is the free group of rank $n \geq 0$. Let us remark that finite extensions of fundamental groups of conical hyperbolic orbifolds with non-empty boundary belong to $\mathcal{G}$. In [**Syk05**] Corollary 3.6, Sykiotis proved that the function χ defined below (Definition 2.39) is an Euler characteristic on $\mathcal{G}$.

Definition 2.39. Let $G \in \mathcal{G}$, and let Δ be a $\mathcal{F}$-JSJ splitting of G. Denote by $V(\Delta)$ the set of vertices of Δ, and by $E(\Delta)$ its set of edges. We define $\chi(\Delta)$ as follows:

$$\chi(\Delta) = \sum_{e \in E(\Delta)} \frac{1}{|G_e|} - \sum_{v \in V(\Delta)} \frac{1}{|G_v|}.$$

This number does not depend on the choice of the $\mathcal{F}$-JSJ splitting Δ of G. The Euler characteristic of G is defined as $\chi(G) = \chi(\Delta)$, for any $\mathcal{F}$-JSJ splitting Δ of G.

Lemma 2.40. *Let $G, G' \in \mathcal{G}$. Let Δ and Δ' be $\mathcal{F}$-JSJ splittings of G and G' respectively. Suppose that the edge groups of Δ and Δ' are trivial. Let $\phi : G \twoheadrightarrow G'$ be an epimorphism that is injective on finite subgroups of G. Then $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is injective.*

Proof. If Δ is reduced to a point, then G is finite, so ϕ is injective by assumption. From now on, we will suppose that Δ has at least two vertices. Let T, T' be Bass-Serre trees of Δ, Δ' respectively. We build a ϕ-equivariant map $f : T \to T'$ in the following way: for every vertex v of T, since $\phi(G_v)$ is finite (because G_v is finite), it fixes a vertex v' of T'. Moreover, v' is unique since ϕ is injective on finite subgroups, and edge groups of Δ' are trivial. We let $f(v) = v'$. Next, if e is an edge of T, with endpoints v and w, there exists a unique path e' from $f(v)$ to $f(w)$ in T'. We let $f(e) = e'$. Let us denote by d' the natural distance function on T'. Up to subdividing the edges of T, we can assume that, for every adjacent vertices $v, w \in T$, $d'(f(v), f(w)) \in \{0, 1\}$. We will prove that ϕ can be written as a composition $i \circ \pi_n \circ \cdots \circ \pi_0$, with $n \geq 0$, $\pi_0 = \mathrm{id}$ and i injective, such that $\chi(\pi_\ell \circ \cdots \circ \pi_0(G)) \geq \chi(\pi_{\ell+1} \circ \cdots \circ \pi_0(G))$ for every $0 \leq \ell < n$ (if $n > 0$), with equality if and only if $\pi_{\ell+1} = \mathrm{id}$.

Step 1. Assume that there exist two adjacent vertices v, w such that $d'(f(v), f(w)) = 0$. Let e be the edge between v and w. We collapse e in T, as well as all its translates under the action of G. Collapsing e gives rise to a new G-tree S with a new vertex x labelled by $G_x = \langle G_v, G_w \rangle$ if v and w are not in the same orbit, or $G_x = \langle G_v, g \rangle$ if $w = g \cdot v$. Let N be the kernel of the restriction of ϕ to G_x and let $\langle\langle N \rangle\rangle$ denote the subgroup of G normally generated by N. Let $G_1 = G/\langle\langle N \rangle\rangle$, let us denote by $\pi_1 : G \twoheadrightarrow G_1$ the associated epimorphism, and let $\phi_1 : G_1 \to G$ be the unique morphism such that $\phi = \phi_1 \circ \pi_1$. The group G_1 splits as a graph of groups Δ_1 obtained from the splitting S/G of G by replacing the vertex group G_x by $G_x/N \simeq \phi(G_x)$. Note that G_x/N is finite since $\phi(G_x)$ fixes the vertex $f(v) = f(w)$ of T'. Hence, Δ_1 is a $\mathcal{F}$-JSJ splitting of G_1. Let us compare $\chi(G_1)$ with $\chi(G)$. It is not hard to see that

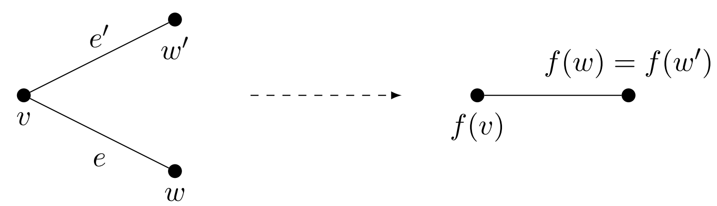
$$\chi(G) - \chi(G_1) = \begin{cases} 1 - \dfrac{1}{|G_v|} - \dfrac{1}{|G_w|} + \dfrac{1}{|G_x/N|} & \text{if } v \text{ and } w \text{ are not in the same orbit} \\ 1 - \dfrac{1}{|G_v|} + \dfrac{1}{|G_x/N|} & \text{if } v \text{ and } w \text{ are in the same orbit} \end{cases}.$$

Hence, if v and w are in the same orbit, it is clear that $\chi(G) > \chi(G_1)$. If v and w are not in the same orbit, there are four distinct cases: if $|G_v| \geq 2$ and $|G_w| \geq 2$, it is clear that $\chi(G) > \chi(G_1)$; if $|G_v| = 1$ and $|G_w| \geq 2$ (respectively $|G_w| = 1$ and $|G_v| \geq 2$), then $G_x = G_w$ (respectively $G_x = G_v$) and $\chi(G) \geq \chi(G_1)$ with equality if and only if $N = \{1\}$, i.e. $\pi_1 = \mathrm{id}$; if $|G_v| = |G_w| = 1$, then $G_x = N = \{1\}$, i.e. $\pi_1 = \mathrm{id}$.

Let T_1 be the Bass-Serre tree of the splitting Δ_1 of G_1. We build a ϕ_1-equivariant map $f_1 : T_1 \to T'$ as above. If f_1 collapses some edge, we repeat the previous operation. Since T has only finitely many orbits of edges under the action of G, the procedure terminates after finitely many steps.

Up to replacing G by the last group of the sequence of quotients of G built above, we can assume without loss of generality that f sends adjacent vertices to adjacent vertices.

Step 2. Assume that f folds some pair of edges, as pictured below.



Let us fold e and e' together in T, as well as all their translates under the action of G. Note that e and e' are not in the same G_v-orbit since T' has trivial edge stabilizers and ϕ is injective on G_v. Folding e and e' together gives rise to a new G-tree S with a new vertex x labelled by $G_x = \langle G_w, G'_w \rangle$ if w and w' are not in the same orbit, or $G_x = \langle G_w, g \rangle$ if $w' = g \cdot w$.

Let N be the kernel of the restriction of ϕ to G_x and let $\langle\langle N \rangle\rangle$ denote the subgroup of G normally generated by N. Let $G_1 = G/\langle\langle N \rangle\rangle$, let us denote by $\pi_1 : G \twoheadrightarrow G_1$ the associated epimorphism, and let $\phi_1 : G_1 \to G$ be the unique morphism such that $\phi = \phi_1 \circ \pi_1$. The group G_1 splits as a graph of groups Δ_1 obtained from the splitting S/G of G by replacing the vertex group G_x by $G_x/N \simeq \phi(G_x)$. Note that G_x/N is finite since $\phi(G_x)$ fixes a vertex of T'. Hence, Δ_1 is a $\mathcal{F}$-JSJ splitting of G_1. Let us compare $\chi(G_1)$ with $\chi(G)$. As in the first step, we can see that $\chi(G) > \chi(G_1)$. Again, we can repeat this operation only finitely many times since T has only finitely many orbits of edges under the action of G. Let T_1 denote the Bass-Serre tree of Δ_1. At the end of the process, with obvious notations, we get a ϕ_n-equivariant map $f_n : T_n \to T'$ that is locally injective, thus injective. It remains to prove that ϕ_n is injective: if $\phi_n(g) = 1$, then for every vertex v of T_n, $f_n(gv) = f_n(v)$, so $gv = v$. Since G_n acts on T_n with trivial edge stabilizers, we get $g = 1$. $\square$

Proposition 2.41 below follows easily from Lemmas 2.38 and 2.40.

PROPOSITION 2.41. *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds, with non-empty boundary. Denote by G and G' their orbifold fundamental groups. If $\phi : G \to G'$ is a non-pinching morphism of orbifold groups such that $\phi(G)$ is not contained in a conical or boundary subgroup of G', then $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is an isomorphism.*

PROOF. By Lemma 2.38, $d := [G' : \phi(G)]$ is finite. Since χ is an Euler characteristic (see [**Syk05**]), $\chi(\phi(G)) = d\chi(G')$. By Lemma 2.40, $\chi(G) \geq \chi(\phi(G))$, with equality if and only if ϕ is injective. It follows that $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is an isomorphism. $\square$

We conclude this section by generalizing Proposition 2.41 to finite extensions of conical hyperbolic orbifold groups. First, we need the following lemma.

LEMMA 2.42. *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds. Let G and G' be two finite extensions $F \hookrightarrow G \twoheadrightarrow \pi_1(\mathcal{O})$ and $F' \hookrightarrow G' \twoheadrightarrow \pi_1(\mathcal{O}')$. If $\phi : G \to G'$ is a homomorphism whose restriction to F is injective and whose image is infinite, then $\phi(F) \subset F'$. As a consequence, ϕ induces a homomorphism σ from $\pi_1(\mathcal{O})$ to $\pi_1(\mathcal{O}')$.*

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[d, dashed] & S \arrow[r, "\pi"] \arrow[d, "\phi"] & \pi_1(\mathcal{O}) \arrow[d, dashed, "\sigma"] \\
F' \arrow[r, hook] & S' \arrow[r, "\pi'"] & \pi_1(\mathcal{O}')
\end{tikzcd}$$

PROOF. First, we make the following observation: if A is a finite subgroup of G' which is not contained in F', then the normalizer $N_{G'}(A)$ of A in G' is finite. Indeed, if A is not contained in F', then $\pi(A)$ is a non-trivial finite subgroup of $\mathcal{O}'$, so its normalizer is a finite cyclic group (since we can see $\pi(A)$ as an elliptic subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{H}^2$). Now, if $\phi(F)$ is not contained in F', then $\phi(G) \subset N_{G'}(\phi(F))$, which is finite. This is a contradiction. $\square$

PROPOSITION 2.43. *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds, with non-empty boundary. Let G and G' be two finite extensions $F \hookrightarrow G \twoheadrightarrow \pi_1(\mathcal{O})$ and $F' \hookrightarrow G' \twoheadrightarrow \pi_1(\mathcal{O}')$. Let $\phi : G \to G'$ be a non-pinching morphism of finite-by-orbifold groups such that $\phi(G)$ is not contained in an extended conical or boundary subgroup of G'. Then $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is an isomorphism.*

PROOF. First, let us observe that $\chi(\pi_1(\mathcal{O})) = |F|\chi(G)$ and $\chi(\pi_1(\mathcal{O}')) = |F'|\chi(G')$. By Lemma 2.42, ϕ induces a non-pinching homomorphism of orbifolds $\sigma : \pi_1(\mathcal{O}) \to \pi_1(\mathcal{O}')$ whose image is not contained in a boundary or conical subgroup of $\pi_1(\mathcal{O}')$. According to Proposition 2.41, $\chi(\pi_1(\mathcal{O})) \geq \chi(\pi_1(\mathcal{O}'))$. But $\phi(F) \subset F'$, and ϕ is injective in restriction to F, so $|F'| \geq |F|$. As a consequence, $\chi(G) \geq \chi(G')$. Moreover, if $\chi(G) = \chi(G')$, then $\chi(\pi_1(\mathcal{O})) = \chi(\pi_1(\mathcal{O}'))$, so $\sigma : \pi_1(\mathcal{O}) \to \pi_1(\mathcal{O}')$ is an isomorphism by Proposition 2.41, and $|F| = |F'|$. Thus, $\phi_{|F} : F \to F'$ is an isomorphism, so ϕ is an isomorphism. $\square$

3. How to extract information from the JSJ using first-order logic

3.1. A particular case.

In the introduction, we proved a particular case of Theorems 1.2 and 1.3 under the hypotheses that G and Γ are one-ended and have finitely many outer automorphisms, namely Claim 1.7. We shall now consider another particular case of these theorems, which is little more general, and much more instructive. First of all, note that we cannot express the full statement of the shortening argument 2.4 in first-order logic, since precomposition by a modular automorphism is not expressible by a first-order formula in general. To deal with this problem, let us consider the following corollary of the shortening argument 2.4, which follows immediately from the definition of the modular group.

COROLLARY 3.1. *Let Γ be a group that embeds into a hyperbolic group, and let G be a one-ended finitely generated group. There exists a finite set $F \subset G \setminus \{1\}$ with the following property: for every non-injective homomorphism $\phi : G \to \Gamma$, there exists a homomorphism $\phi' : G \to \Gamma$ such that*

- $\ker(\phi') \cap F \neq \varnothing$;
- *for every non-QH vertex group G_v of the $\mathcal{Z}$-JSJ splitting of G, there exists an element $\gamma \in \Gamma$ such that $\phi' = \mathrm{ad}(\gamma) \circ \phi$ in restriction to G_v;*
- *for every finite group F of G, there exists an element $\gamma \in \Gamma$ such that $\phi' = \mathrm{ad}(\gamma)\circ\phi$ in restriction to F.*

We say that ϕ and ϕ' are JSJ-related (see Definition 3.4 below).

Here is a particular case of Theorems 1.2 and 1.3.

PROPOSITION 3.2. *Let Γ be a hyperbolic group, and let G be a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Suppose that G is one-ended, and that there is no QH vertex in the $\mathcal{Z}$-JSJ decomposition of G. Then G embeds into Γ. Suppose moreover that Γ is one-ended and that there is no QH vertex in the $\mathcal{Z}$-JSJ decomposition of Γ. Then Γ and G are isomorphic.*

REMARK 3.3. This result generalizes Claim 1.7, because the existence of a QH vertex group in the $\mathcal{Z}$-JSJ decomposition of G (or Γ) gives rise to infinitely many outer automorphisms.

PROOF. First, we prove that G embeds into Γ. Argue by contradiction and suppose that every homomorphism from G to Γ is non-injective. By Corollary 3.1, there exists a finite set $F \subset G\backslash\{1\}$ such that, for every homomorphism $\phi : G \to \Gamma$, there exists $\phi' : G \to \Gamma$ that kills an element of F and that coincides with ϕ up to conjugacy on every vertex group of the canonical $\mathcal{Z}$-JSJ splitting Δ of G. One easily sees that this fact can be expressed by a $\forall\exists$-sentence satisfied by Γ (see Lemma 3.7 for details). Since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, this sentence is also satisfied by G, and its interpretation in G yields the following: for every endomorphism $\phi : G \to G$, there exists an endomorphism $\phi' : G \to G$ that kills an element of F and that coincides with ϕ up to conjugacy on every vertex group of the canonical $\mathcal{Z}$-JSJ splitting Δ of G. Taking $\phi = \mathrm{id}_G$, we get an endomorphism of G that kills an element of F and that is inner in restriction to every vertex group. But we will prove later on that such an endomorphism is necessarily injective (see Proposition 7.1). This contradicts the fact that ϕ' kills an element of F. Hence, we have shown that G embeds into Γ. Now, using Corollary 3.1, we prove in the same way that Γ embeds into G. As a one-ended hyperbolic group, Γ is co-Hopfian, so $G \simeq \Gamma$. $\square$

New difficulties arise when Γ and G are not supposed to be one-ended, or when the $\mathcal{Z}$-JSJ splittings of Γ and G contain QH vertices. However, the example above highlights the crucial role played by endomorphims of G that coincide up to conjugacy with the identity of G on non-QH vertex groups. This example also brings out the key idea to obtain these special homomorphisms, due to Sela-Perin, that consists in expressing a consequence of the shortening argument 2.2 by a $\forall\exists$-sentence that Γ satisfies (assuming G is one-ended). Since Γ and G have the same $\forall\exists$-theory, G satisfies this sentence as well, and its interpretation in G endows us with a special endomorphism of G. This example leads us to the definition of related homomorphisms.

3.2. Related homomorphisms and preretractions.

The following definition is similar (but slightly different) to Definition 5.9 and Definition 5.15 of [**Per11**].

DEFINITION 3.4 (Related homomorphisms). Let G be a one-ended finitely generated K-CSA group that does not contain $\mathbb{Z}^2$, and let G' be a group. Let Δ be the $\mathcal{Z}$-JSJ splitting of G. Let f and f' be two homomorphisms from G to G'. We say that ϕ and ϕ' are *$\mathcal{Z}$-JSJ-related* or *Δ-related* if the following two conditions hold:

- for every non-QH vertex v of Δ, there exists an element $g_v \in G'$ such that
$$\phi'_{|G_v} = \mathrm{ad}(g_v) \circ \phi_{|G_v};$$
- for every finite subgroup F of G, there exists an element $g \in G'$ such that
$$\phi'_{|F} = \mathrm{ad}(g) \circ \phi_{|F}.$$

Note that being JSJ-related is an equivalence relation on $\mathrm{Hom}(G, G')$.

REMARK 3.5. The second condition above can be reformulated as follows: for every QH vertex v, and for every finite subgroup F of G_v, there exists an element $g \in G'$ such that $\phi_{|F} = \mathrm{ad}(g) \circ \phi'_{|F}$. Indeed, every finite group has Serre's property (FA), so every finite subgroup F of G is contained in a conjugate of some vertex group G_w of Δ. Furthermore, if w is a non-QH vertex, it follows from the first condition that there exists an element $g \in G'$ such that $\phi_{|F} = \mathrm{ad}(g) \circ \phi'_{|F}$.

DEFINITION 3.6 (Preretraction). Let G be a finitely generated K-CSA group that does not contain $\mathbb{Z}^2$, and let H be a one-ended subgroup of G. Let Δ be the $\mathcal{Z}$-JSJ decomposition of H. A *$\mathcal{Z}$-JSJ-preretraction* or *Δ-preretraction* from H to G is a homomorphism $H \to G$ that is JSJ-related to the inclusion of H into G in the sense of Definition 3.4. A JSJ-preretraction is said to be non-degenerate if it sends each QH subgroup of H isomorphically to a conjugate of itself.

The following lemma shows that being JSJ-related can be expressed in first-order logic.

LEMMA 3.7 (compare with [**Per11**] Lemma 5.18). *Let G be a finitely generated one-ended K-CSA group that does not contain $\mathbb{Z}^2$. Let $\{g_1, \ldots, g_n\}$ be a generating set of G. Let G' be a group. There exists an existential formula $\psi(x_1, \ldots, x_{2n})$ with $2n$ free variables such that, for every $\phi, \phi' \in \mathrm{Hom}(G, G')$, ϕ and ϕ' are JSJ-related if and only if G' satisfies $\psi\left(\phi(g_1), \ldots, \phi(g_n), \phi'(g_1), \ldots, \phi'(g_n)\right)$.*

PROOF. Let Δ be the $\mathcal{Z}$-JSJ splitting of G. First, remark that there exist finitely many (say $p \geq 1$) conjugacy classes of finite subgroups of QH vertex groups of Δ (indeed, a QH vertex group possesses finitely many conjugacy classes of finite subgroups, and Δ has finitely many vertices). Denote by $F_1, \ldots, F_p$ a system of representatives of those conjugacy classes. Denote by $R_1, \ldots, R_m$ the non-QH vertex groups of Δ. Remark that these groups are finitely generated since G and the edge groups of Δ are finitely generated. Denote by $\{A_i\}_{1 \leq i \leq p+m}$ the union of $\{F_i\}_{1 \leq i \leq p}$ and $\{R_i\}_{1 \leq i \leq m}$. For every $i \in [\![1, m+p]\!]$, let $\{a_{i,1}, \ldots, a_{i,k_i}\}$ be a finite generating set of A_i. For every $i \in [\![1, m+p]\!]$ and $j \in [\![1, k_i]\!]$, there exists a word $w_{i,j}$ in n letters such that $a_{i,j} = w_{i,j}(g_1, \ldots, g_n)$. Let

$$\psi(x_1, \ldots, x_{2n}) : \exists u_1 \ldots \exists u_{m+p} \bigwedge_{i=1}^{m+p} \bigwedge_{j=1}^{k_i} w_{i,j}(x_1, \ldots, x_n) = u_i w_{i,j}(x_{n+1}, \ldots, x_{2n}) {u_i}^{-1}.$$

Since $\phi(a_{i,j}) = w_{i,j}(\phi(g_1), \ldots, \phi(g_n))$ and $\phi'(a_{i,j}) = w_{i,j}\left(\phi'(g_1), \ldots, \phi'(g_n)\right)$ for every $i \in [\![1, m+p]\!]$ and $j \in [\![1, k_i]\!]$, the homomorphisms ϕ and ϕ' are JSJ-related if and only if the sentence $\psi\left(\phi(g_1), \ldots, \phi(g_n), \phi'(g_1), \ldots, \phi'(g_n)\right)$ is satisfied by G'. $\square$

3.3. Centered graph of groups.

We need to define relatedness in a more general context. In order to deal with groups that are not assumed to be one-ended, we define below the notion of a centered graph of groups. We denote by $\overline{\mathcal{Z}}$ the class of groups that are either finite or virtually cyclic with infinite center.

DEFINITION 3.8 (Centered graph of groups). A graph of groups over $\overline{\mathcal{Z}}$, with at least two vertices, is said to be *centered* if the following conditions hold:

- the underlying graph is bipartite, with a QH vertex v such that every vertex different from v is adjacent to v; moreover, the finite-by-orbifold group G_v is conical;
- for every edge e incident to v, the edge group G_e coincides with an extended boundary subgroup or with an extended conical subgroup of G_v (see Definition 2.27);
- given any extended boundary subgroup B, there exists a unique edge e incident to v such that G_e is conjugate to B in G_v;

- if an element of infinite order fixes a segment of length ≥ 2 in the Bass-Serre tree of the splitting, then this segment has length exactly 2 and its endpoints are translates of v.

The vertex v is called the central vertex. See Figure 2.6 below.

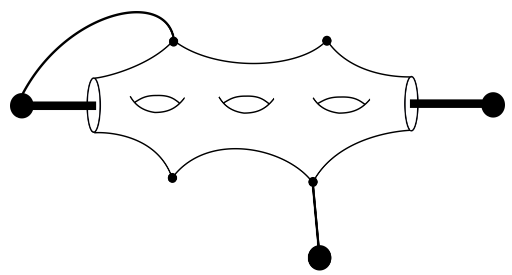


FIGURE 1. A centered graph of groups. Edges with infinite stabilizer are depicted in bold.

Recall that a subgroup H of a group G is called almost malnormal if $H \cap gHg^{-1}$ is finite for every g in $G \setminus H$. In particular, every finite subgroup of G is malnormal.

LEMMA 3.9. *Edge groups in a centered graph of groups are almost malnormal in the central vertex group.*

PROOF. Let G be a group with a centered splitting Δ_G. Let v be the central vertex and let e be an edge of Δ_G incident to v. By the second condition of Definition 3.8, G_e is an extended conical subgroup or an extended boundary subgroup of the finite-by-orbifold group G_v. In the first case, G_e is finite, so it is clearly malnormal. In the second case, G_e is virtually cyclic infinite maximal in G_v, so its fixed-point set $P \subset \partial_\infty G_v$ is a pair of points, and we have $\mathrm{Stab}_{G_v}(P) = G_e$. Hence, if $g \in G_v$ is such that $G_e \cap gG_eg^{-1}$ is infinite, then gG_eg^{-1} fixes P. It follows that g fixes P, i.e. that g belongs to G_e. $\square$

We will need the following two classical results.

PROPOSITION 3.10 ([**BF92**], pages 99 and 100).
Let $G = A *_C B$ be an amalgamated product such that A and B are hyperbolic and C is virtually cyclic (possibly finite) and almost malnormal in A or B. Then G is hyperbolic.
Let $G = A*_C$ be an HNN extension such that A is hyperbolic, C is virtually cyclic (possibly finite), and the two copies C_1 and C_2 of C form an almost malnormal family of subgroups, i.e. at least one of them is almost malnormal in A, and $C_1 \cap gC_2g^{-1}$ is finite for every $g \in G$. Then G is hyperbolic.

PROPOSITION 3.11 ([**Bow98**], Proposition 1.2). *If a hyperbolic group splits over quasi-convex subgroups, then every vertex group is quasi-convex (hence hyperbolic).*

The following result is an immediate consequence of the previous two propositions.

PROPOSITION 3.12. *Let G be a group that splits as a centered graph of groups, with central vertex v. Then the following assertions are equivalent.*

(1) G is hyperbolic
(2) For every vertex $w \neq v$, G_w is hyperbolic.

PROOF. The implication (2) $\Rightarrow$ (1) is an immediate consequence of the combination theorem of Bestvina and Feighn (see Proposition 3.10 above) together with Lemma 3.9. The implication (1) $\Rightarrow$ (2) follows from Proposition 3.11. $\square$

We define relatedness for centered splittings (compare with JSJ-relatedness, Definition 3.4), and preretractions (compare with JSJ-prereactions, Definition 3.6).

Definition 3.13 (Related homomorphisms). Let G and G' be two groups. Let Δ be a centered splitting of G, with central vertex v. Let ϕ and ϕ' be two homomorphisms from G to G'. We say that ϕ and ϕ' are Δ-*related* if the following two conditions hold:

- for every vertex $w \neq v$, there exists an element $g_w \in G'$ such that
$$\phi'_{|G_w} = \mathrm{ad}(g_w) \circ \phi_{|G_w};$$
- for every finite subgroup F of G, there exists an element $g \in G'$ such that
$$\phi'_{|F} = \mathrm{ad}(g) \circ \phi_{|F}.$$

Note that being Δ-related is an equivalence relation on $\mathrm{Hom}(G, G')$.

Definition 3.14. Let G be a group, and let Δ be a centered splitting of G. Let v be the central vertex of Δ. An endomorphism ϕ of G is called a Δ-*preretraction* if it is Δ-related to the identity of G in the sense of Definition 3.13. A preretraction is said to be *non-degenerate* if it sends G_v isomorphically to a conjugate of itself.

4. Torsion-saturated groups

To deal with certain pathologies arising from torsion (see Example 1.9), we prove in the current section that every hyperbolic group G embeds into a hyperbolic group $\overline{G}$ such that the class of $\overline{G}$-limit groups is closed under HNN extensions and amalgamated free products over finite groups.

Definition 4.1. We say that a hyperbolic group G is torsion-saturated if the following two conditions hold.

1. For every isomorphism $\alpha : F_1 \to F_2$ between finite subgroups F_1, F_2 of G, there exists an element $g \in G$ such that $gxg^{-1} = \alpha(x)$ for every $x \in F_1$.
2. For every finite subgroup F of G, there exists an infinite subset $\{g_1, g_2, \ldots\}$ of G such that g_n has infinite order, $M(g_n) = \langle g_n\rangle \times F$ for every n, and the intersection $M(g_n) \cap M(g_m)$ is equal to F whenever $n \neq m$.

We shall use the following lemma.

Lemma 4.2. *Let G be a hyperbolic group, and let F be a finite subgroup of G. Suppose that there exist two elements $a, b \in G$ generating a free group of rank two, and such that $M(a) = \langle a\rangle \times F$ and $M(b) = \langle b\rangle \times F$. Then there exists an infinite subset $\{g_1, g_2, \ldots\}$ of G such that g_n has infinite order, $M(g_n) = \langle g_n\rangle \times F$ for every n, and $M(g_n) \cap M(g_m) = F$ whenever $n \neq m$.*

Proof. Note that $H = N_G(F)$ is non-elementary since it contains the free group $\langle a, b\rangle \simeq F_2$. By Proposition 1 in [**Os93**], there exists a unique maximal finite subgroup of G normalized by H, denoted by $E_G(H)$. By Lemma 3.8 in [**Os93**], there exists an infinite subset $\{g_1, g_2, \ldots\}$ of H such that g_n has infinite order, $M(g_n) = \langle g_n\rangle \times E_G(H)$ for every n, and $M(g_n) \cap M(g_m) = E_G(H)$ whenever $n \neq m$. In order to conclude, we just have to prove that $E_G(H) = F$. First, observe that F is obviously contained in $E_G(H)$. Then, note that $E_G(H)$ is normalized by a. Since $E_G(H)$ is finite, a^r centralizes $E_G(H)$ for some integer $r \geq 1$. It follows that $E_G(H)$ preserves the pair of points $\{a^+, a^-\}$ fixed by a in the boundary of G. But the stablizer of this pair of points is equal to $M(a)$ (by maximality of $M(a)$), which is equal to $\langle a\rangle \times F$ by hypothesis. Thus, $E_G(H) \subset F$, so $E_G(H) = F$. ☐

We shall see that every hyperbolic group G embeds into a torsion-saturated hyperbolic group $\overline{G}$ obtained from G by performing finitely many HNN extensions over finite groups (see Theorem 4.8 below). The main interest of torsion-saturated groups resides in the following result.

Theorem 4.3. *Let G be a torsion-satured hyperbolic group. Then the class of G-limit groups is closed under HNN extensions and amalgamated free products over finite groups.*

Before proving Theorem 4.3, we need some preliminary results.

Lemma 4.4. *Let G be a hyperbolic group and let H be a G-limit group. Let $\alpha : F_1 \to F_2$ be an isomorphism between finite subgroups F_1, F_2 of H. Then there exists an isomorphism $\beta : F_1' \to F_2'$ between finite subgroups F_1', F_2' of G such that $H*_\alpha$ is a $G*_\beta$-limit group.*

Lemma 4.5. *Let G be a hyperbolic group. Suppose that the first condition of Definition 4.1 holds. Let A, B be G-limit groups. Let F be a finite group that embeds into A and B. Then there exists a finite subgroup F' of G and an HNN extension*

$$G' = \langle G, t \mid [t, x] = 1 \ \forall x \in F \rangle$$

of G such that $A *_F B$ is a G'-limit group.

Lemma 4.6. *Let G be a hyperbolic group, and let $\alpha : F_1 \to F_2$ be an isomorphism between finite subgroups F_1, F_2 of G. Suppose that the following two conditions hold:*

(1) there exists an element $g \in G$ such that $gxg^{-1} = \alpha(x)$ for every $x \in F_1$;
(2) there exists an infinite subset $E = \{g_1, g_2, \ldots\} \subset G$ such that g_n has infinite order, $M(g_n) = \langle g_n \rangle \times F_1$ for every n, and $M(g_n) \neq M(g_m)$ whenever $n \neq m$.

Then $G*_\alpha$ is a G-limit group.

Proof of Lemma 4.4. Let $(\phi_n)_{n\in\mathbb{N}}$ be a discriminating sequence of homomorphisms from H to G. Without loss of generality, we can suppose that ϕ_n is injective in restriction to F_1 and F_2. We will construct a discriminating sequence $(\rho_n)_{n\in\mathbb{N}}$ of homomorphisms from $H*_\alpha$ to $G*_\beta$, for some β that will be defined below. Since G has finitely many conjugacy classes of finite subgroups, there exist two finite subgroups F_1', F_2' of G such that, up to extracting a subsequence, $\phi_n(F_1)$ is conjugate to F_1' and $\phi_n(F_2)$ is conjugate to F_2' for every n. Up to composing ϕ_n by an inner automorphism of G, one can assume that $\phi_n(F_1) = F_1'$ and $\mathrm{ad}(g_n) \circ \phi_n(F_2) = F_2'$ for some $g_n \in G$. Denote by β_n the isomorphism from F_1' to F_2' making the following diagram commute:

$$\begin{tikzcd} F_1 \arrow[r, "\phi_{n|F_1}"] \arrow[d, "\alpha"] & F_1' \arrow[d, "\beta_n"] \\ F_2 \arrow[r, "\mathrm{ad}(g_n)\circ\phi_{n|F_2}"] & F_2' \end{tikzcd}.$$

Since $\mathrm{Isom}(F_1', F_2')$ is finite, there exists an isomorphism β between F_1' and F_2' such that (up to extracting a subsequence) $\beta_n = \beta$ for every n. Let t and u be the stable letters of $H*_\alpha$ and $G*_\beta$, i.e. $txt^{-1} = \alpha(x)$ for all $x \in F_1$ and $uyu^{-1} = \beta(y)$ for all $y \in F_1'$. For every n, we define a map ρ_n from $H*_\alpha$ to $G*_\beta$ as follows:

$$\rho_n(x) = \begin{cases} \phi_n(x) & \text{if } x \in H \\ g_n^{-1}u & \text{if } x = t \end{cases}.$$

The map ρ_n clearly extends to a homomorphism since the diagram commutes, and we claim that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is discriminating. Let x be a non-trivial element of $H*_\alpha$. If x lies in H, it is obvious that $\rho_n(x)$ is non-trivial for every n large enough since $(\phi_n)_{n\in\mathbb{N}}$ is discriminating. Assume now that $x \notin H$. Then x can be written in reduced form as $x = h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} h_2 \cdots t^{\varepsilon_p} h_{p+1}$ with $p > 0$, $h_i \in H$, $\varepsilon_i = \pm 1$, $h_i \notin F_1$ if $\varepsilon_i = -\varepsilon_{i+1} = 1$ and $h_i \notin F_2$ if $\varepsilon_i = -\varepsilon_{i+1} = -1$. One has:

$$\rho_n(x) = \phi_n(h_0)(g_n u)^{\varepsilon_1}\phi_n(h_1)(g_n u)^{\varepsilon_2}\phi_n(h_2)\cdots(g_n u)^{\varepsilon_p}\phi_n(h_{p+1}).$$

One has to prove that $\rho_n(x) \neq 1$ for every n large enough. In order to apply Britton's lemma (see [**Bri63**]), one verifies that, for every subword of $\rho_n(x)$ (written as above) of the form uvu^{-1} with v not involving u, the element v does not lie in F_1', and that for every subword of $\rho_n(x)$ of the form $u^{-1}vu$ with v not involving u, the element v does not lie in F_2'.

- If uvu^{-1} is a subword of w with v not involving u, then v is of the form $\phi_n(h_i)$ with $h_i \notin F_1$, and $\phi_n(h_i) \notin F_1'$ for every n large enough because $(\phi_n)_{n\in\mathbb{N}}$ is discriminating.
- Similarly, if $u^{-1}vu$ is a subword of w with v not involving u, then v is of the form $g_n\phi_n(h_i)g_n^{-1}$ with $h_i \notin F_2$, and $g_n\phi_n(h_i)g_n^{-1} \notin F_2' = g_n\phi_n(F_2)g_n^{-1}$ for every n large enough because the sequence $(\mathrm{ad}(g_n) \circ \phi_n)_{n\in\mathbb{N}}$ is discriminating.

Hence, it follows from Britton's lemma that $\rho_n(x) \neq 1$ for every n large enough. □

Remark 4.7. Note that in the previous proof, the hypothesis that G is hyperbolic is only used in order to ensure that G has finitely many classes of finite subgroups.

Proof of Lemma 4.5. Let $(\phi_n)_{n\in\mathbb{N}}$ be a discriminating sequence from A to G, and let $(\psi_n)_{n\in\mathbb{N}}$ be a discriminating sequence from B to G. Up to extracting a subsequence, we can assume that ϕ_n and ψ_n are injective on F. Hence, $\phi_n(F)$ and $\psi_n(F)$ are isomorphic to F, for every n.

Since G has only finitely many conjugacy classes of finite subgroups, we can assume that there exists a finite subgroup F' of G such that $\phi_n(F) = F'$ for every n, up to extracting a subsequence of $(\phi_n)_{n\in\mathbb{N}}$ and precomposing ϕ_n by an inner automorphism.

By hypothesis, there exists an element $g_n \in G$ making the following diagram commute:

$$\begin{tikzcd} & \phi_n(F) \arrow[dd, "\mathrm{ad}(g_n)\,\cdot"] \\ F \arrow[ur, "\phi_n"] \arrow[dr, "\psi_n"'] & \\ & \psi_n(F) \end{tikzcd}$$

Let $\chi_n = \mathrm{ad}(g_n^{-1}) \circ \psi_n$. Then χ_n and ϕ_n coincide on F.

Let $G*_{F'} = \langle G, t \mid [t,x] = 1\ \forall x \in F'\rangle$ be the HNN extension of G over the identity of F'. For every n, since ϕ_n and χ_n coincide on F, we define a homomorphism ρ_n from $A *_F B$ to $G*_{F'}$ as follows:

$$\rho_n(x) = \begin{cases} \phi_n(x) & \text{if } x \in A \\ \mathrm{ad}(t) \circ \chi_n(x) & \text{if } x \in B \end{cases}.$$

We claim that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is discriminating, i.e. that for every non-trivial element $x \in A*_F B$, we have $\rho_n(x) \neq 1$ for every n large enough. If $x \in F\setminus\{1\}$, then $\rho_n(x) = \phi_n(x)$, so $\rho_n(x)$ is non-trivial for every n large enough since $(\phi_n)_{n\in\mathbb{N}}$ is discriminating. Assume now that $x \notin F$. Then x can be written in a reduced form $a_1b_1a_2b_2\cdots a_kb_k$ with $a_i \in A\setminus F$ and $b_i \in B \setminus F$ (except maybe a_1 and b_k). The following holds:

$$\rho_n(x) = \phi_n(a_1)t\chi_n(b_1)t^{-1}\phi_n(a_2)t\chi_n(b_2)t^{-1}\cdots\phi_n(a_k)t\chi_n(b_k)t^{-1}.$$

Since $(\phi_n)_{n\in\mathbb{N}}$ and $(\chi_n)_{n\in\mathbb{N}}$ are discriminating, $\phi_n(a_i) \notin F'$ and $\chi_n(b_i) \notin F'$ for every n large enough (except maybe $\phi_n(a_1)$ and $\chi_n(b_k)$). Hence, it follows from Britton's lemma that $\rho_n(x) \neq 1$ for every n large enough. □

Proof of Lemma 4.6. Let $G = \langle S \mid R\rangle$ be a presentation of G, and let

$$G*_\alpha = \langle S, t \mid R,\ txt^{-1} = \alpha(x)\ \forall x \in F_1\rangle$$

be a presentation of $G*_\alpha$. By hypothesis, there exists an element $g \in G$ such that $gxg^{-1} = \alpha(x)$ for every $x \in F_1$. Up to replacing t by $g^{-1}t$, we can assume that $F_1 = F_2$ and that α is the identity of F_1. The presentation of $G*_\alpha$ becomes $\langle S, t \mid R,\ [t,x] = 1\ \forall x \in F_1\rangle$.

For every $n \in \mathbb{N}$, and for every integer p, we define a map $\phi_{p,n}$ from $G*_{\alpha}$ to G by

$$\phi_{p,n} : \begin{cases} z \mapsto z & \text{if } z \in G \\ t \mapsto g_n^p \end{cases}.$$

The map $\phi_{p,n}$ clearly extends to a homomorphism since $\mathrm{ad}(t)$ and $\mathrm{ad}(g_n^p)$ coincide on F_1 for every p, because $M(g_n) = \langle g_n \rangle \times F_1$ by hypothesis.

Denote by B_m the ball of radius m in $G*_{\alpha}$ (for a given generating set). We shall prove the existence of two sequences $(n_m)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(p_{n_m})_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\phi_{p_{n_m},n_m}(x) \neq 1$ for every $x \in B_m \setminus \{1\}$, for every m. Let $x \in B_m \setminus \{1\}$. If x lies in G, $\phi_{p,n}(x) = x \neq 1$ for all p and n. Assume now that x does not belong to G. Then x can be written in a reduced form as

$$x = y_0 t^{\varepsilon_1} y_1 t^{\varepsilon_2} \cdots t^{\varepsilon_k} y_k$$

with $k > 0$, $\varepsilon_i = \pm 1$, $y_i \notin F_1$ if $\varepsilon_i = -\varepsilon_{i+1}$. We claim that for p and n sufficiently large, the homomorphism $\phi_{p,n}$ satisfies $\phi_{p,n}(x) \neq 1$. In order to prove this, we will use Baumslag's lemma (see Corollary 2.21) with $c = g_n$. We have

$$\phi_{p,n}(x) = y_0 g_n^{\varepsilon_1 p} y_1 g_n^{\varepsilon_2 p} \cdots g_n^{\varepsilon_k p} y_k.$$

First, let us rewrite $\phi_{p,n}(x)$ under a more convenient form. Let $i \in [\![1, k-1]\!]$, and suppose that $\varepsilon_i = \varepsilon_{i+1} = 1$, and that y_i lies in F_1. Then, since $[g_n, F_1] = 1$, we replace the subword $g_n^p y_i g_n^p y_{i+1}$ by $g_n^{2p} y_i y_{i+1}$. In the case where $\varepsilon_{i+2} = -1$, note that $y_i y_{i+1}$ does not belong to F_1, since $y_i \in F_1$ and $y_{i+1} \notin F_1$. In the case where $\varepsilon_{i+2} = 1$ and y_{i+1} belongs to F_1, we repeat the previous operation, and so on. Similarly, if $\varepsilon_i = \varepsilon_{i+1} = -1$ and y_i lies in F_1, we replace the subword $g_n^{-p} y_i g_n^{-p} y_{i+1}$ of $\phi_{p,n}$ by $g_n^{-2p} y_i y_{i+1}$, and so on. At the end of this process, we have

$$\phi_{p,n}(x) = z_0 g_n^{\varepsilon_1 n_1 p} z_1 g_n^{\varepsilon_2 n_2 p} \cdots g_n^{\varepsilon_\ell n_\ell p} z_\ell,$$

with $n_1, \ldots, n_\ell \in \mathbb{N}^*$, and $z_i \notin F_1$ for every $i \in [\![1, \ell-1]\!]$.

We can now use Baumslag's lemma (see Corollary 2.21) with $c = g_n$. We claim that there exists $n(x)$ such that z_i does not belong to $M(g_n)$ for every $n \geq n(x)$ and for every $i \in [\![1, \ell-1]\!]$. Indeed, suppose that z_i lies in $M(g_n) = \langle g_n \rangle \times F_1$, for some n. Since z_i does not belong to F_1, it has infinite order, so $M(z_i)$ is well-defined and $M(z_i) = M(g_n)$. Then the claim follows from the fact that $M(g_m) \cap M(g_n) = F_1$ if $n \neq m$, and $z_i \notin F_1$.

Let $n_m := \max\{n(x) \mid x \in B_m\}$. By Corollary 2.21, there exists an integer p_{n_m} such that $\ker(\phi_{p_{n_m},n_m}) \cap B_m = \{1\}$. This concludes the proof. ☐

We are ready to prove Theorem 4.3.

Proof. Let H be a G-limit group. Let $\alpha : F_1 \to F_2$ be an isomorphism between two finite subgroups F_1, F_2 of H. We shall prove that the HNN extension $H*_{\alpha}$ is a G-limit group. According to Lemma 4.4 above, there exists an isomorphism $\beta : F'_1 \to F'_2$ between finite subgroups F'_1, F'_2 of G such that $H*_{\alpha}$ is a $G*_{\beta}$-limit group. But $G*_{\beta}$ is a G-limit group thanks to Lemma 4.6 above, hence $H*_{\alpha}$ is a G-limit group. It remains to consider the case of an amalgamated free product. Let A, B be G-limit groups. Let F be a finite group that embeds into A and B. According to Lemma 4.5 below, there exists a finite subgroup F' of G such that $A *_F B$ is a $G*_{F'}$-limit group, and $G*_{F'}$ is a G-limit group by Lemma 4.6, hence $A *_F B$ is a G-limit group as well. ☐

We conclude this section by proving that every hyperbolic group embeds into a torsion-saturated hyperbolic group.

Theorem 4.8. *Every hyperbolic group embeds into a torsion-saturated hyperbolic group.*

Proof. Let G be a hyperbolic group. Denote by $F_1, \ldots, F_m$ a system of representatives of the conjugacy classes of finite subgroups of G. Define $\theta : [\![1, m]\!]^2 \to \{0, 1\}$ by $\theta(i, j) = 1$ if and only if F_i and F_j are isomorphic. For every $(i, j) \in \theta^{-1}(1)$, let $n_{i,j} = |\mathrm{Isom}(F_i, F_j)|$

and $\mathrm{Isom}(F_i, F_j) = \{\alpha_{i,j,1}, \ldots, \alpha_{i,j,n_{i,j}}\}$. If $i = j$, one can assume that $\alpha_{i,i,1}$ is the identity of F_i. Let us define $\overline{G}$ by adding new generators and relations to G, as follows:

$$\overline{G} = \langle G, \{s_{i,j,k}, t_{i,j,k}\}_{(i,j)\in\theta^{-1}(1)}^{1\leq k\leq n_{i,j}} \mid \mathrm{ad}(s_{i,j,k})_{|F_i} = \mathrm{ad}(t_{i,j,k})_{|F_i} = \alpha_{i,j,k}\rangle.$$

This group is hyperbolic by the combination theorem of Bestvina and Feighn (see [**BF92**] pages 99 and 100, or Proposition 3.10 above); indeed, $\overline{G}$ is obtained iteratively from the hyperbolic group G by performing HNN extensions over finite groups, and two finite subgroups always form an almost malnormal family. Let us prove that $\overline{G}$ is torsion-saturated. Let F_1 and F_2 be two finite subgroups of $\overline{G}$. Since finite groups have property (FA), there exist two elements g_1 and g_2 of $\overline{G}$ such that $g_1F_1g_1^{-1}$ and $g_2F_2g_2^{-1}$ are contained in G. Without loss of generality, we now assume that $F_1, F_2 \subset G$. By definition of $\overline{G}$, for every isomorphism $\alpha : F_1 \to F_2$ between finite subgroups of $\overline{G}$, there exists an element g of $\overline{G}$ such that $gxg^{-1} = \alpha(x)$, for all $x \in F_1$. Hence the first condition of Definition 4.1 is satisfied by $\overline{G}$. It remains to verify that the second condition holds. Let F be a finite subgroup of $\overline{G}$. We can assume that $F = F_i$ for some $1 \leq i \leq m$. Let a and b be the two generators of $\overline{G}$ corresponding to the identity of F_i, i.e. $a := s_{i,i,1}$ and $b := t_{i,i,1}$. The group $\langle a, b\rangle$ is free, and $M(a) = \langle a\rangle \times F$, $M(b) = \langle b\rangle \times F$. This concludes the proof, thanks to Lemma 4.2. ☐

5. Quasi-floors and quasi-towers

5.1. Definitions.

Hyperbolic towers were introduced by Sela in [**Sel01**] (Definition 6.1) under the name of non-elementary hyperbolic ω-residually free towers (see the paragraph before Proposition 6 in [**Sel06b**], and Definition 6.1 of [**Sel01**]). We also refer the reader to Kharlampovich and Myasnikov's NTQ groups.

Sela used hyperbolic towers in [**Sel06b**] to solve Tarski's problem about the elementary equivalence of free groups, and in [**Sel09**] to classify all finitely generated groups with the same first-order theory as a given torsion-free hyperbolic group. A hyperbolic tower is a group obtained by successive addition of hyperbolic floors. We refer the reader to [**Per11**] and [**Per13**]. Here is a slightly different.

Definition 5.1. Let G be a torsion-free finitely generated group, Δ_G a centered splitting of G with exactly one vertex w other than the central vertex v, and r a retraction from G onto $H = G_w$. We say that (G, H, Δ_G, r) is a weak hyperbolic floor or, more simply, that the group G is a weak hyperbolic floor over H. We say that G is a *hyperbolic floor* over H if, in addition, $r(G_v)$ is non-abelian and the underlying surface of G_v has topological Euler characteristic at most -2 or is a punctured torus (i.e. is not a punctured Klein bottle, a twice punctured projective plane or a pair of pants).

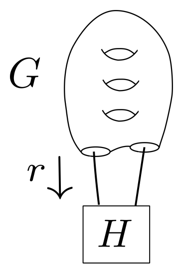


Figure 2. The group G is a (weak) hyperbolic floor over H.

This definition is suitable for dealing with hyperbolic groups without torsion. In order to handle torsion, we need new definitions.

Definition 5.2. Let G and H be two finitely generated groups, Δ_G a centered splitting of G, Δ_H a splitting of H with finite edge groups, r a homomorphism from G to H and j a homomorphism from H to G. Let V_G be the set of vertices of Δ_G, and v the central vertex. Let V_H be the set of vertices of Δ_H. Let $V_H = V_H^1 \sqcup V_H^2$ be a partition of V_H and let $s : V_G \setminus \{v\} \to V_H^1$ be a bijection such that the following conditions hold:

- $j \circ r$ is a Δ_G-preretraction, i.e. $j \circ r$ coincides up to conjugacy with id_G on every vertex group G_w with $w \neq v$, and on every finite subgroup of G;
- for every w in $V_G \setminus \{v\}$, r maps G_w isomorphically to $h_w H_{s(w)} h_w^{-1}$ for some $h_w \in H$;
- for every $u \in V_H^2$, H_u is finite and j is injective on H_u.

We say that $(G, H, \Delta_G, \Delta_H, r, j)$ is a *quasi-floor*. If, moreover, there exists a one-ended subgroup A of G such that $A \cap \ker(r) \neq \{1\}$, the quasi-floor is said to be *strict*.

For reasons of brevity, one sometimes wants to avoid the notation $(G, H, \Delta_G, \Delta_H, r, j)$. More simply, one writes (G, H, r, j), and one says that G is a (strict) quasi-floor over H (with respect to r and j).

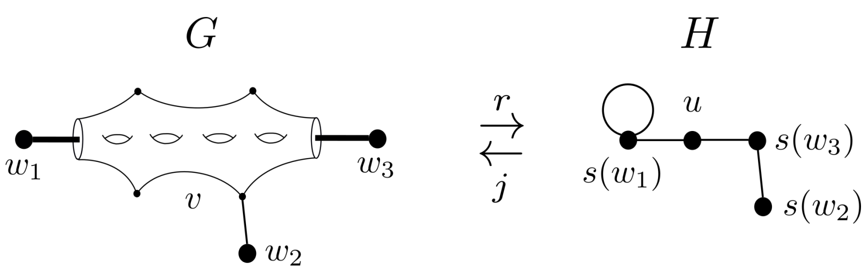


Figure 3. The group G is a quasi-floor over H. Here $V_H^2 = \{u\}$ and $V_H^1 = \{s(w_1), s(w_2), s(w_3)\}$. The vertex u is labelled by a finite group. Edges with infinite stabilizer are depicted in bold.

Lemma 5.3. *We use the same notations as in Definition 5.2 above.*

(1) The homomorphism $r \circ j$ is inner on every vertex group H_u with $u \in V_H^1$ (in particular on every one-ended subgroup of H).
(2) The homomorphism j is injective on one-ended subgroups of H.

Proof. Let $u \in V_H^1$. By definition, there exists a vertex $w = s^{-1}(u)$ of Δ_G such that $r(G_w) = xH_u x^{-1}$ for some $x \in H$. Moreover, there exists an element $g \in G$ such that $j \circ r = \mathrm{ad}(g)$ on G_w. Let h be an element of H_u, and set $h' = xhx^{-1}$. Note that h' belongs to $r(G_w)$. Since r induces an isomorphism from G_w to $r(G_w)$, we can write $r \circ j(h') = r \circ j(r(r^{-1}(h')))$. But $j \circ r(r^{-1}(h')) = gr^{-1}(h')g^{-1}$, so $r \circ j(h') = r(g)h'r(g)^{-1}$. It follows that $r \circ j(h) = yhy^{-1}$ with $y = (r \circ j(x))^{-1} r(g)x$. Hence, $r \circ j = \mathrm{ad}(y)$ on H_u. This proves the first assertion.

Let T be the Bass-Serre tree of the splitting Δ_H of H, and let A be a one-ended subgroup of H. Since edge groups of T are finite, the subgroup A fixes a vertex of T. Hence, A is contained in a conjugate of H_u for some $u \in V_H$. According to the previous paragraph, the homomorphism $r \circ j$ coincides with an inner automorphism on H_u. In particular, it is injective on A. Hence, j is injective on A. This proves the second point. ☐

Remark 5.4. It should be noted that in Definition 5.2, no assumption is made about the image of the QH subgroup G_v, nor about the orbifold Euler characteristic of the underlying conical orbifold of G_v, whereas in the definition of a hyperbolic floor 5.1, $r(G_v)$ is assumed to be non-abelian and the underlying topological surface of G_v has Euler characteristic at most -2, or is a punctured torus. In fact, these technical hypotheses are not necessary in order to prove that hyperbolicity is preserved under elementary equivalence.

Sometimes it is convenient to think of G and H as subgroups of a bigger group G' obtained from G by performing amalgamated products and HNN extensions over finite groups, with the property that there exists an epimorphism $\rho : G' \twoheadrightarrow H$ satisfying $\rho_{|G} = r$ and $\rho_{|H} = \mathrm{id}_H$. This group G' can be built as follows. Let $w_1, \ldots, w_d$ denote the vertices of Δ_G different from the central vertex v. First, let

$$\widehat{G} = \langle G, H \mid g = r(g), \forall g \in G_{w_1} \rangle$$

which is an amalgamated product $G *_{G_{w_1}} H$, and let $\widehat{r} : \widehat{G} \twoheadrightarrow H$ be the retraction defined by $\widehat{r}_{|G} = r$ and $\widehat{r}_{|H} = \mathrm{id}_H$. Then, let us define an overgroup G' of $\widehat{G}$ and a retraction $\rho : G' \twoheadrightarrow H$ as follows:

$$G' = \langle \widehat{G}, t_2, \ldots, t_d \mid t_k g t_k^{-1} = r(g), \forall g \in G_{w_k}, \forall k \in [\![1, d]\!] \rangle,$$

$\rho_{|\widehat{G}} = \widehat{r}$ and $\rho(t_k) = 1$ for every $2 \leq k \leq d$. The group G' can be viewed in an equivalent way as the fundamental group of the graph of groups Λ obtained from Δ_G and Δ_H by identifying, for every vertex $w \in V_G \setminus \{v\}$, the vertex group G_w with $H_{s(w)}$ via the isomorphism $\mathrm{ad}(h_w^{-1}) \circ r_{|G_w} : G_w \to H_{s(w)}$, with the same notations as in Definition 5.2. Figure 4 below illustrates this construction (to be compared with Figure 3).

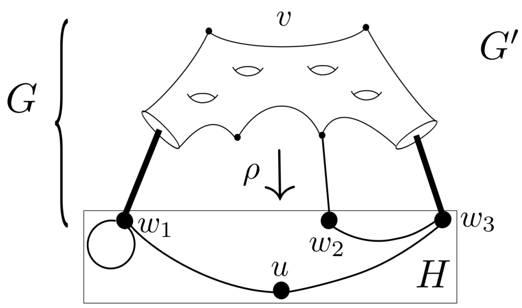


FIGURE 4. We construct a group G' by identifying w with $s(w)$ for every $w \in V_G \setminus \{v\}$. Edges with infinite stabilizer are depicted in bold.

In the case where G is torsion-free, the group G' defined above is a free product of G with a free group of rank n depending on $|V_G \setminus \{v\}|$ and $\mathrm{rk}(\pi_1(\mathcal{G}))$, where $\mathcal{G}$ stands for the underlying graph of Δ_H. It follows from this observation that, in the torsion-free case, the definition of a weak hyperbolic floor 5.1 and the definition of a quasi-floor 5.2 are equivalent, up to replacing G by $G * \boldsymbol{F}_n$. This equivalence is explained in details in the following statement.

PROPOSITION 5.5. *Let G be a torsion-free group. If (G, H, Δ_G, r) is a weak hyperbolic floor, then $(G, H, \Delta_G, \Delta_H, r, i)$ is a quasi-floor, where i stands for the inclusion of H into G, and Δ_H stands for the splitting of H reduced to a point. Conversely, if $(G, H, \Delta_G, \Delta_H, r, j)$ is a quasi-floor, then there exist a free group $\boldsymbol{F}_n$ and a retraction $\rho : G' \twoheadrightarrow H$ where $G' = G * \boldsymbol{F}_n$, such that $\rho_{|G} = r$ and $(G', H, \Delta_{G'}, \rho)$ is a weak hyperbolic floor, where $\Delta_{G'}$ is the splitting of G' obtained from the graph Λ defined above by collapsing to a point the graph Δ_H, viewed as a subgraph of Λ.*

REMARK 5.6. Let Γ be a torsion-free hyperbolic group and let G be a Γ-limit group. It is worth noting that $G * \boldsymbol{F}_n$ is a Γ-limit group as well, since Γ and $\Gamma * \boldsymbol{F}_n$ have the same existential theory by Proposition 2.22, and even the same theory, by [**Sel06b**]. Hence, replacing G by $G * \boldsymbol{F}_n$ does not affect the first-order theory, in the torsion-free case.

It is important to emphasize that, maybe, our construction of a quasi-floor in Section 7 might be modified to ensure that r is an epimorphism, and that j is a monomorphism. However, it seemed to us that this could give rise to a number of new technical difficulties.

These complications will be avoided by using Theorem 4.8 stating that every hyperbolic group embeds into a torsion-saturated hyperbolic group (see Definition 4.1).

Here below is an example of a quasi-floor in the presence of torsion.

EXAMPLE 5.7. Let $A = (\langle x\rangle \times F(x_1,x_2)) * (\langle y\rangle \times F(x_3,x_4))$, with x of order 6 and y of order 10, where $F(x_i,x_j)$ stands for the free group on two generators x_i and x_j. The group A admits the following presentation:

$$A = \langle x, y, x_1, x_2, x_3, x_4 \mid [x,x_1] = [x,x_2] = [y,x_3] = [y,x_4] = x^6 = y^{10} = 1\rangle.$$

Let Σ be the orientable surface of genus two with two boundary components, and let $S = \pi_1(\Sigma)$. Call $\langle b_1\rangle$ and $\langle b_2\rangle$ its two boundary subgroups (up to conjugation). Let $B = \langle z\rangle \times S$ with z of order 2. The group B admits the following presentation:

$$B = \langle z, s_1, s_2, s_3, s_4, b_1, b_2 \mid [s_3,s_4][s_1,s_2]b_1b_2 = [z,b_i] = [z,s_j] = z^2 = 1\rangle.$$

Let us define a graph of groups Δ_G with two vertices labelled by A and B, and two edges linking these vertices, identifying the extended boundary subgroup $\langle z\rangle \times \langle b_1\rangle$ of B with the subgroup $\langle x^3\rangle \times \langle [x_1,x_2]\rangle$ of A by $z \mapsto x^3, b_1 \mapsto [x_2,x_1]$, and the extended boundary subgroup $\langle z\rangle \times \langle b_2\rangle$ of B with the subgroup $\langle y^5\rangle \times \langle [x_3,x_4]\rangle$ of A by $z \mapsto y^5, b_2 \mapsto [x_4,x_3]$. Call G the fundamental group of this graph of groups Δ_G.

First, note that G cannot be a quasi-floor over A. To see that, remark that each element of G of order 2 commutes with an element of order 3 and with an element of order 5, whereas there are two conjugacy classes of elements of order 2 in A: those commuting with an element of order 3, and those commuting with an element of order 5. Hence, there cannot exist any homomorphism $G \to A$ that is injective on finite subgroups.

Let $H = \langle A, t \mid tx^3t^{-1} = y^5\rangle$, let $j : H \to G$ denote the monomorphism which is the identity on A and on t, and let Δ_H be the splitting of H with exactly one vertex v such that $H_v = A$ and exactly one edge identifying $\langle x^3\rangle$ with $\langle y^5\rangle$.

Here are two finite presentations of G and H:

$$\begin{aligned} G &= \langle A, B, t \mid z = x^3,\ b_1 = [x_2,x_1],\ tb_2t^{-1} = [x_4,x_3],\ tzt^{-1} = y^5\rangle \\ &= \left\langle x, y, x_1, x_2, x_3, x_4, s_1, s_2, s_3, s_4, t \;\middle|\; \begin{array}{c} tx^3t^{-1} = y^5,\ x^6 = y^{10} = 1, \\ [s_3,s_4][s_1,s_2][x_2,x_1]t^{-1}[x_4,x_3]t = 1, \\ [x,x_1] = [x,x_2] = [y,x_3] = [y,x_4] = 1 \end{array} \right\rangle, \end{aligned}$$

$$\begin{aligned} H &= \left\langle x, y, x_1, x_2, x_3, x_4, t \;\middle|\; \begin{array}{c} tx^3t^{-1} = y^5,\ x^6 = y^{10} = 1, \\ \\ [x,x_1] = [x,x_2] = [y,x_3] = [y,x_4] = 1 \end{array} \right\rangle \\ &= \langle A, t \mid tx^3t^{-1} = y^5\rangle. \end{aligned}$$

The group G retracts onto H via the epimorphism $r : G \to H$ defined as follows:

$$r : \begin{cases} a \mapsto a & \text{if } a \in \{x, y, x_1, x_2, x_3, x_4, t\} \\ s_i \mapsto x_i & \text{if } 1 \leq i \leq 2 \\ s_i \mapsto t^{-1}x_it^1 & \text{if } 3 \leq i \leq 4 \end{cases}.$$

Hence, $(G, H, \Delta_G, \Delta_H, r, j)$ is a quasi-floor. Indeed, $j \circ r$ coincides with the identity on A, and r maps $A \subset G$ isomorphically to $A \subset H$.

See Figure 5 below.

We now define quasi-towers, which are obtained by successive addition of quasi-floors.

DEFINITION 5.8. A *quasi-tower* is a finite sequence of quasi-floors

$$(G_k, H_k, \Delta_{G_k}, \Delta_{H_k}, r_k, j_k)_{1\leq k\leq n}$$

such that $G_{k+1} = H_k$ for every $1 \leq k \leq n-1$. Note that $\Delta_{G_{k+1}}$ and Δ_{H_k} are two distinct splittings of $G_{k+1} = H_k$.

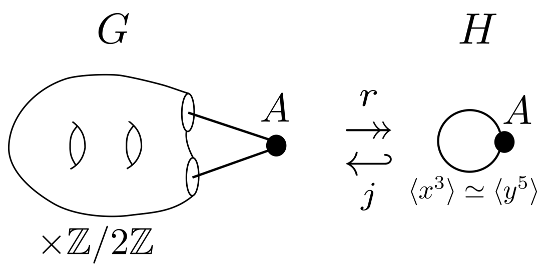


FIGURE 5. G is a quasi-floor over H, but not over A.

If, moreover, every quasi-floor is strict, then the quasi-tower is said to be strict. The integer n is called the *height* of the quasi-tower. The homomorphisms $r_k \circ \cdots \circ r_1$ and $j_1 \circ \cdots \circ j_k$ are denoted by $\overline{r}_k$ and $\overline{j}_k$ respectively.

Let $G := G_1, H := H_n, r := \overline{r}_n$ and $j := \overline{j}_n$. For reasons of brevity, one sometimes writes (G, H, r, j) instead of $(G_k, H_k, \Delta_{G_k}, \Delta_{H_k}, r_k, j_k)_{1\leq k\leq n}$.

We end this subsection with two definitions that are comparable to Sela's elementary prototype (Definition 7.3 of [**Sel09**]) and Sela's elementary core (Definition 7.5 of [**Sel09**]).

DEFINITION 5.9. A *quasi-prototype* is a finitely generated group G that is not a strict quasi-floor over any group H.

EXAMPLE 5.10. A one-ended hyperbolic group whose $\mathcal{Z}$-JSJ splitting does not contain any QH vertex is a quasi-prototype.

DEFINITION 5.11. Let G be a finitely generated group. If G is not a quasi-prototype, a *quasi-core* of G is a group C satisfying the following two conditions:

- G is a strict quasi-tower over C.
- C is a quasi-prototype.

If G is a quasi-prototype, we define G as the only quasi-core of G.

REMARK 5.12. Note that if a quasi-core exists, it is not unique *a priori*.

5.2. Inheritance of hyperbolicity.

Here is an easy but crucial proposition.

PROPOSITION 5.13. *Let G and H be two groups. Suppose that G is a quasi-tower over H. The following hold:*

- *G is hyperbolic if and only if H is hyperbolic.*
- *G embeds into a hyperbolic group if and only if H embeds into a hyperbolic group.*
- *G is hyperbolic and cubulable if and only if H is hyperbolic and cubulable.*

The proof of the third claim is postponed to Section 6.4 (see Proposition 6.21).

PROOF. We shall prove the proposition in the case where G is a quasi-floor over H, the general case follows immediately by induction. Let Δ_G and Δ_H be the splittings of G and H associated with the quasi-floor structure. By definition, Δ_G is a centered graph of groups. Let V_G be its set of vertices, and let v be the central vertex. Let $V_H = V_H^1 \sqcup V_H^2$ be a partition of the set of vertices V_H of Δ_H as in Definition 5.2. By definition, there exists a bijection $s : V_G \setminus \{v\} \to V_H^1$ such that $G_w \simeq H_{s(w)}$ for every $w \in V_G \setminus \{v\}$.

We prove the first claim. Suppose that H is hyperbolic. Then, by Proposition 3.11, H_u is hyperbolic for every vertex u of Δ_H. As a consequence, the vertex groups of Δ_G are hyperbolic. It follows from Proposition 3.12 that G is hyperbolic. Conversely, if G is hyperbolic, then the vertex groups of Δ_G are hyperbolic by Proposition 3.12. Thus, the vertex groups of Δ_H are hyperbolic. Since Δ_H has finite edge groups, the combination theorem of Bestvina and Feighn (see Proposition 3.10) applies and shows that H is hyperbolic.

Now, let us prove the second claim (see Figure 6 below). Suppose that H embeds into a hyperbolic group Γ. In particular, each H_u embeds into Γ. As a consequence, each G_w embeds into Γ, for $w \in V_G \setminus \{v\}$. We construct a graph of groups Δ_G^Γ from Δ_G by replacing each vertex group G_w by Γ. Call Ω the fundamental group of Δ_G^Γ, and let us observe that Δ_G^Γ is a centered splitting of Ω. Since Γ is hyperbolic, Proposition 3.12 tells us that Ω is hyperbolic. In addition, it is clear that G embeds into Ω. Conversely, if G embeds into a hyperbolic group, we prove in the same way that H embeds into a hyperbolic group. $\square$

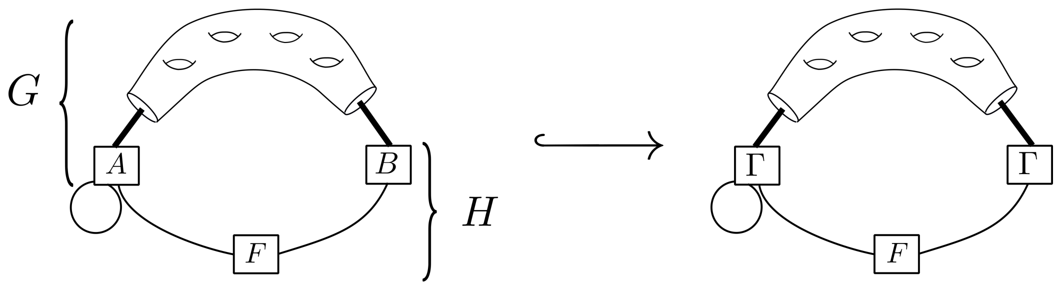


FIGURE 6. Second point of Proposition 5.13: the group G is a quasi-floor over H; if H embeds into a hyperbolic group Γ, then G embeds into the hyperbolic group obtained by replacing A and B by Γ. Edges with infinite stabilizer are depicted in bold.

5.3. Every Γ-limit group has a quasi-core.

This subsection is devoted to a proof of the following result.

PROPOSITION 5.14. *Let Γ be a hyperbolic group, and G a Γ-limit group. Then G has a quasi-core C. Moreover, G is hyperbolic if and only if C is hyperbolic.*

The second part of the previous proposition is an immediate consequence of Proposition 5.13. It remains to prove the first part, that is the following proposition.

PROPOSITION 5.15. *If Γ is a hyperbolic group, then every Γ-limit group has a quasi-core.*

In other words, the previous proposition claims that a Γ-limit group is either a quasi-prototype, or a strict quasi-tower over a quasi-prototype. Proposition 5.15 is an easy consequence of the following lemma, which will be proved below.

LEMMA 5.16. *There does not exist any infinite sequence $(G_n)_{n\in\mathbb{N}}$ of finitely generated groups such that G_0 is a Γ-limit group and, for every integer n, G_n is a strict quasi-floor over G_{n+1}.*

Before proving this lemma, let us make some observations. First, note that if Γ is a torsion-free hyperbolic group, it follows from the descending chain condition (see Theorem 2.8) that there does not exist any infinite sequence $(G_n)_{n\in\mathbb{N}}$ such that G_0 is a Γ-limit group and G_n is a hyperbolic floor over G_{n+1} (in the sense of Sela).

In the presence of torsion, however, Definition 5.2 has two drawbacks that seem to be obstacles to the use of the descending chain condition: if G is a Γ-limit group, and if G is a quasi-floor over H, then in general

(1) H is not a Γ-limit group,
(2) H is not a quotient of G.

We remedy the first issue (see Proposition 5.17 and 5.18) by using the fact that every hyperbolic group embeds into a torsion-saturated hyperbolic group (see Theorem 4.8).

PROPOSITION 5.17. *Let Γ be a hyperbolic group, and G a Γ-limit group. Let $\overline{\Gamma}$ be a torsion-saturated hyperbolic group containing Γ. Suppose that G is a quasi-floor over a finitely generated group H. Then H is a $\overline{\Gamma}$-limit group.*

PROOF. Let Δ_H be the splitting of H over finite groups associated to the structure of a quasi-floor. It follows from the definition of a quasi-floor that every vertex group of Δ_H embeds into G, so is a Γ-limit group. By Theorem 4.3, the class of $\overline{\Gamma}$-limit groups is closed under HNN extensions and amalgamated free products over finite groups, so H is a $\overline{\Gamma}$-limit group. □

The following corollary is immediate.

COROLLARY 5.18. *Let Γ be a hyperbolic group. Let $\overline{\Gamma}$ be a torsion-saturated hyperbolic group containing Γ. Let $(G_n)_{n\geq 0}$ be a sequence of finitely generated groups such that G_0 is a Γ-limit group and G_n is a quasi-floor over G_{n+1} for every n. Then every G_n is a $\overline{\Gamma}$-limit group.*

Then, the following proposition remedies the lack of surjectivity in the definition of a quasi-floor (see the second issue listed above), under the assumption that each quasi-floor is a Γ-limit group.

PROPOSITION 5.19. *Let Γ be a hyperbolic group. There does not exist any infinite sequence $(G_n)_{n\in\mathbb{N}}$ of Γ-limit groups such that, for every integer n, G_n is a strict quasi-floor over G_{n+1}.*

PROOF. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of Γ-limit groups. Suppose that G_n is a quasi-floor over G_{n+1}, and let $r_n : G_n \to G_{n+1}$ and $j_n : G_{n+1} \to G_n$ be the associated homomorphisms, for every integer n. We shall prove that there exists an integer n_0 such that, for every $n \geq n_0$, the quasi-floor (G_n, G_{n+1}, r_n, j_n) is not strict.

Since the homomorphism r_n is not assumed to be surjective, one cannot apply the descending chain condition (see Theorem 2.8) to the sequence of homomorphisms $(r_n : G_n \to G_{n+1})_{n\in\mathbb{N}}$. Let $G'_0 = G_0$ and $G'_{n+1} = r_n(G'_n)$ for every integer n. The descending chain condition, applied to the sequence of epimorphisms $(r_{n|G'_n} : G'_n \twoheadrightarrow G'_{n+1})_{n\in\mathbb{N}}$, ensures that $r_{n|G'_n}$ is injective for every n large enough.

In order to conclude the proof, we shall verify that the restriction of r_n to G'_n is not injective if G_n is a strict quasi-floor over G_{n+1}.

If G_n is a strict quasi-floor over G_{n+1}, there exists by definition a one-ended subgroup A_n of G_n such that $A_n \cap \ker(r_n) \neq \{1\}$. Set $A'_n = r_{n-1} \circ \cdots \circ r_0 \circ j_0 \circ \cdots \circ j_n(A_n)$. It is a subgroup of $G'_n = r_{n-1} \circ \cdots \circ r_0(G_0)$. In order to prove that r_n is non-injective in restriction to G'_n, it is enough to prove that r_n is non-injective in restriction to A'_n. Indeed, let us prove that $A'_n = gA_ng^{-1}$ for some element $g \in G_n$. It is enough to conclude because we know that there is a non-trivial element x in $A_n \cap \ker(r_n)$, hence gxg^{-1} is a non-trivial element of $A'_n \cap \ker(r_n)$.

Let us prove that A'_n and A_n are conjugate in G_n. Set $B_n = j_0 \circ \cdots \circ j_{n-1}(A_n)$. Since $j_1 \circ \cdots \circ j_{n-1}(A_n)$ is a one-ended subgroup of G_1, it follows from Lemma 5.3 that $r_0 \circ j_0(j_1 \circ \cdots \circ j_{n-1}(A_n)) = r_0(B_n)$ is conjugate to $j_1 \circ \cdots \circ j_{n-1}(A_n)$. Thus, $r_1 \circ r_0(B_n)$ is conjugate to $r_1 \circ j_1 \cdots \circ j_{n-1}(A_n)$, which is conjugate to $j_2 \circ \cdots \circ j_{n-1}(A_n)$ by Lemma 5.3, since $j_2 \circ \cdots \circ j_{n-1}(A_n)$ is a one-ended subgroup of G_2. An immediate induction shows that $r_{n-1} \circ \cdots \circ r_0(B_n) = gA_ng^{-1}$ for some $g \in G_n$. Hence, $A'_n = gA_ng^{-1}$. □

We can now prove Lemma 5.16, which implies Proposition 5.15.

PROOF. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finitely generated groups such that G_0 is a Γ-limit group and, for every integer n, G_n is a strict quasi-floor over G_{n+1}. We aim to prove that this sequence is finite. By Corollary 5.18, every G_n is a $\overline{\Gamma}$-limit group, where $\overline{\Gamma}$ stands for a torsion-saturated hyperbolic group containing Γ. Then it follows from Proposition 5.19 that the sequence $(G_n)_{n\in\mathbb{N}}$ is finite. □

5.4. Quasi-towers and relatedness.

In this section, we collect some lemmas that will be useful in the proofs of our main theorems.

LEMMA 5.20. *Let $(G, H, \Delta_G, \Delta_H, r, j)$ be a quasi-floor. Let A be a one-ended finitely generated K-CSA group that does not contain $\mathbb{Z}^2$, and let Δ_A be its $\mathcal{Z}$-JSJ splitting. Then,*

(1) for every monomorphism $\phi : A \hookrightarrow G$, $j \circ r \circ \phi$ and ϕ are Δ_A-related;
(2) for every monomorphism $\phi : A \hookrightarrow H$, $r \circ j \circ \phi$ and ϕ are Δ_A-related.

PROOF. We shall prove the first assertion. We have to prove that ϕ and $j \circ r \circ \phi$ coincide, up to conjugacy by an element of G, on every non-QH vertex group of Δ_A, as well as on every finite subgroup of A. The condition concerning finite subgroups is obvious, since $j \circ r$ is inner on each finite subgroup of G. Now, let R be an infinite non-QH vertex group of Δ_A. We shall prove that $\phi(R)$ fixes a non-QH vertex of the centered splitting Δ_G. Since R is non-QH, it is rigid (see Proposition 2.29). Hence, R is elliptic in every tree on which A acts with edge stabilizers in $\mathcal{Z}$ or finite. Therefore, $\phi(R)$ is elliptic in the Bass-Serre tree T of Δ_G (indeed, the preimage under ϕ of every $\mathcal{Z}$-subgroup (resp. finite subgroup) of G is a $\mathcal{Z}$-subgroup (resp. finite subgroup) of A, because ϕ is injective). We make the following observation: let S be a finite-by-orbifold group, and let B be a subgroup of S. If B is elliptic in every splitting of S over $\mathcal{Z}$, then B is finite or lies in an extended boundary subgroup (see [**GL17**] Corollary 5.24 (5)). As a consequence, if R is rigid, $\phi(R)$ lies in a conjugate of a non-central vertex group of Δ_G. In the case where R is virtually cyclic, then $\phi(R)$ may possibly lie in a boundary subgroup of the central QH vertex group of Δ_G. In any case, $\phi(R)$ lies in a conjugate of a non-central vertex group of Δ_G. Hence, since $j \circ r$ is Δ_G-related to the identity of G, there exists an element g in G such that

$$j \circ r \circ \phi_{|R} = \mathrm{ad}(g) \circ \phi_{|R}.$$

This finishes the proof of the first assertion.

We now prove the second assertion. By Lemma 5.3, $r \circ j$ is inner on every one-ended vertex group of Δ_H. Since $\phi(A) \simeq A$ is one-ended, it is contained in a one-ended vertex group of Δ_H. Thus, $r \circ j$ is inner on $\phi(A)$. □

Lemma 5.21 below follows from Lemma 5.20 by induction.

LEMMA 5.21. *Let $(G_k, H_k, \Delta_{G_k}, \Delta_{H_k}, r_k, j_k)_{1\leq k\leq n}$ be a quasi-tower of height $n \geq 1$. Let $G = G_1$ and $H = G_n$. Let A be a one-ended finitely generated K-CSA group that does not contain $\mathbb{Z}^2$, and let Δ_A be its $\mathcal{Z}$-JSJ splitting. Then,*

(1) for every monomorphism $\phi : A \hookrightarrow G$, if $\overline{r}_{n-1} \circ \phi$ is injective, then $j \circ r \circ \phi$ and ϕ are Δ_A-related;
(2) for every monomorphism $\phi : A \hookrightarrow H$, $r \circ j \circ \phi$ and ϕ are Δ_A-related.

PROOF. Set $\overline{r}_0 = \mathrm{id}_G$. Let us prove the first point by induction on the integer n. According to Lemma 5.20, the claim is true if $n = 1$. Now, suppose that the claim is true for a given integer n, and let us prove that it is true for $n + 1$. Let $\phi : A \hookrightarrow G$ be a monomorphism such that $\overline{r}_n \circ \phi$ is injective, where $\overline{r}_n = r_n \circ \cdots \circ r_1$. We claim that $j \circ r \circ \phi$ and ϕ are Δ_A-related, where $j = \overline{j}_{n+1} = j_1 \circ \cdots \circ j_{n+1}$ and $r = \overline{r}_{n+1} = r_{n+1} \circ \overline{r_n}$. By induction hypothesis, the morphisms $j_1 \circ \cdots \circ j_n \circ r_n \circ \cdots \circ r_1 \circ \phi$ and ϕ are Δ_A-related. Let R be a rigid vertex group of Δ_A. Since $\overline{r}_n \circ \phi = r_n \circ \cdots \circ r_1 \circ \phi$ is injective, the group $\overline{r}_n \circ \phi(R)$ is a rigid subgroup of G_n. As a consequence, it is contained in a non-central vertex group of the centered splitting Δ_{G_n} of G_n. By definition of a quasi-floor, the restriction of the morphism $j_{n+1} \circ r_{n+1}$ to $\overline{r}_n \circ \phi(R)$ is a conjugacy. Hence, $j_{n+1} \circ r_{n+1} \circ \overline{r}_n \circ \phi(R)$ is conjugate to $\overline{r}_n \circ \phi(R)$. It follows that $\overline{j}_n \circ j_{n+1} \circ r_{n+1} \circ \overline{r}_n \circ \phi(R) = j \circ r \circ \phi(R)$ is conjugate to $\overline{j}_n \circ \overline{r}_n \circ \phi(R)$, which is conjugate to $\phi(R)$ by induction hypothesis. This proves that $j \circ r \circ \phi$ and ϕ are Δ_A-related and concludes the proof of the first proof. The proof of the second point is similar and is left to the reader. $\square$

By combining Lemma 5.21 with the shortening argument 3.1, we get the following result.

LEMMA 5.22. *Let (Γ, Γ', r, j) be a quasi-tower. Suppose that Γ embeds into a hyperbolic group. Let H be a one-ended finitely generated K-CSA group that does not contain $\mathbb{Z}^2$. Let Δ_H be its $\mathcal{Z}$-JSJ splitting. Assume that H is not a finite-by-orbifold group. There exists a finite set $F \subset H \setminus \{1\}$ such that, for every homomorphism $\phi : H \to \Gamma$,*

- *either $r \circ \phi : H \to \Gamma'$ is injective,*
- *or ϕ is Δ_H-related to a homomorphism $\phi' : H \to \Gamma$ that kills an element of F.*

PROOF. Let $F \subset H \setminus \{1\}$ be the finite set given by Corollary 3.1 applied to $\mathrm{Hom}(H, \Gamma)$. Let ϕ be a homomorphism from H to Γ. Suppose that the homomorphism $r \circ \phi : H \to \Gamma'$ is non-injective, and let $n \geq 1$ be the smallest integer such that $\overline{r}_n \circ \phi$ is non-injective. Since $\overline{r}_{n-1} \circ \phi$ is injective, $\overline{j}_n \circ \overline{r}_n \circ \phi$ and ϕ are Δ_H-related, by the first assertion of Lemma 5.21. Thanks to Corollary 3.1, there exists a homomorphism ϕ' that kills an element of F and that is Δ-related to $\overline{j}_n \circ \overline{r}_n \circ \phi$, which is a non-injective homomorphism from H to Γ. By transitivity of the Δ-relatedness, the morphism ϕ' is Δ-related to ϕ. $\square$

We need a variant of Lemma 5.22 in the case where H is not one-ended. Recall that a homomorphism is said to be factor-injective if it is injective in restriction to every one-ended factor that is not finite-by-orbifold (see Definition 2.33). The proof of the following result is similar to that of Lemma 5.22 above.

LEMMA 5.23. *Let (Γ, Γ', r, j) be a quasi-tower. Suppose that Γ embeds into a hyperbolic group. Let H be a finitely generated K-CSA group that does not contain $\mathbb{Z}^2$. Let $H_1, \ldots, H_n$ be its one-ended factors (for a fixed $\mathcal{F}$-JSJ splitting of H) that are not finite-by-orbifold. Let Δ_k be the $\mathcal{Z}$-JSJ splitting of H_k, for $1 \leq k \leq n$. There exists a finite set $F \subset H \setminus \{1\}$ such that, for every homomorphism $\phi : H \to \Gamma$,*

- *either $r \circ \phi : H \to \Gamma'$ is factor-injective,*
- *or there exists an integer $1 \leq k \leq n$ such that $\phi_{|H_k}$ is Δ_k-related to a homomorphism $\phi' : H_k \to \Gamma$ that kills an element of F.*

6. Proofs of Theorems 1.2 to 1.5

In the current section, we prove our main theorems by admitting a technical result, namely Proposition 6.3, whose proof is postponed to Section 7 for the sake of clarity.

6.1. How to build a quasi-floor using first-order logic.

PROPOSITION 6.1. *Let (Γ, Γ', r, j) and (G, G', ρ, η) be two quasi-towers. Suppose that Γ embeds into a hyperbolic group and that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Let H be a one-ended factor of G'. Suppose that H is not a finite-by-orbifold group. Then one of the following holds:*

- *there exists a morphism $\phi : H \to \Gamma$ such that $r \circ \phi$ is injective,*
- *or there exists a non-injective preretraction $H \to G'$.*

Before proving this result, let us justify why H and its canonical $\mathcal{Z}$-JSJ splitting are well-defined. Let Ω denote a hyperbolic group in which Γ embeds. By [**Bra00**], there exists a constant K such that all finite subgroups of Ω have order less than K. Since $\Gamma \subset \Omega$ and $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$, and since the morphism $\eta : G' \to G$ is injective on finite subgroups of G', every finite subgroup of G' has order less than K. As a consequence, G' has a $\mathcal{F}$-JSJ decomposition (see [**Lin83**] and 2.6.6). Hence, one-ended factors of G' and in particular H

are well-defined. Then, by the second assertion of Lemma 5.3, the morphism η is injective on H, because H is one-ended. Proposition 2.31 therefore guarantees that the canonical $\mathcal{Z}$-JSJ splitting of H exists. We now prove Proposition 6.1.

Proof. Let Δ_H denote the $\mathcal{Z}$-JSJ splitting of H. Suppose that, for every homomorphism $\phi : H \to \Gamma$, the homomorphism $r \circ \phi : H \to \Gamma'$ is not injective in restriction to H. We will prove that there exists a non-injective Δ_H-preretraction from H to G'.

Step 1. By Lemma 5.22, there exists a finite set $F \subset H \setminus \{1\}$ such that every homomorphism $\phi : H \to \Gamma$ is Δ_H-related to a homomorphism $\phi' : H \to \Gamma$ that kills an element of F. Set $F = \{h_1, \ldots, h_\ell\}$.

Step 2. We claim that there exists a non-injective homomorphism $p : H \to G$ that is Δ_H-related to $\eta_{|H}$. The proof of this claim consists in expressing by a $\forall\exists$-sentence the statement of Step 1, and by interpreting this sentence in G. Note that H is finitely generated, because G' is finitely generated by definition of a quasi-tower. Let $H = \langle s_1, \ldots, s_n \mid w_1, w_2, \ldots \rangle$ be a (possibly infinite) presentation of H. Let $W_i = \{w_1, \ldots, w_i\}$ for every $i \geq 0$. Denote by H_i the finitely presented group $\langle s_1, \ldots, s_n \mid W_i \rangle$. By hypothesis, Γ embeds into a hyperbolic group Ω. As a hyperbolic group, Ω is equationally noetherian (see [**RW14**], Corollary 6.13). Let us note that Γ and G are Ω-limit groups, because $\mathrm{Th}_\exists(G) \subset \mathrm{Th}_\exists(\Gamma) \subset \mathrm{Th}_\exists(\Omega)$. This implies that Γ and G are equationally noetherian (see [**OH07**] Corollary 2.10). As a consequence, the sets $\mathrm{Hom}(H, \Gamma)$ and $\mathrm{Hom}(H, G)$ are respectively in bijection with $\mathrm{Hom}(H_i, \Gamma)$ and $\mathrm{Hom}(H_i, G)$ for i large enough (see Lemma 5.2 of [**RW14**]). Hence, there exists an integer i such that $\mathrm{Hom}(H, \Gamma)$ (resp. $\mathrm{Hom}(H, G)$) is in bijection with the n-tuples in Γ^n (resp. G^n) solutions of the following system of equations, denoted by $\Sigma_i(x_1, \ldots, x_n)$:

$$\Sigma_i(x_1, \ldots, x_n) : \begin{cases} w_1(x_1, \ldots, x_n) = 1 \\ \qquad\vdots \\ w_i(x_1, \ldots, x_n) = 1 \end{cases}.$$

Let ϕ and ϕ' be homomorphisms from H to Γ. Recall that there exists an existential formula $\psi(x_1, \ldots, x_{2n})$ with $2n$ free variables such that

$$\Gamma \models \phi\left(\phi(s_1), \ldots, \phi(s_n), \phi'(s_1), \ldots, \phi'(s_n)\right)$$

if and only if ϕ and ϕ' are Δ_H-related (see Lemma 3.7).

We can write a $\forall\exists$-first-order sentence μ, satisfied by Γ, whose interpretation in Γ is the statement of Step 1: for every homomorphism $\phi : H \to \Gamma$, there exists a homomorphism $\phi' : H \to \Gamma$ that is Δ_H-related to ϕ and that kills some h_ℓ. The sentence μ is the following:

$$\mu : \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_n \left(\Sigma_i(x_1, \ldots, x_n) = 1 \Rightarrow \left(\begin{array}{c} \Sigma_i(y_1, \ldots, y_n) = 1 \\ \wedge\ \psi(x_1, \ldots, x_n, y_1, \ldots, y_n) \\ \wedge \displaystyle\bigvee_{\ell \in [\![1,k]\!]} h_\ell(y_1, \ldots, y_n) = 1 \end{array} \right) \right).$$

In this form, it is not totally clear that μ is a $\forall\exists$-sentence. Recall that $\psi(\boldsymbol{x}, \boldsymbol{y})$ is of the form $\exists \boldsymbol{u}\ \theta(\boldsymbol{x}, \boldsymbol{y})$ where θ is a quantifier-free formula and $\boldsymbol{x}$, $\boldsymbol{y}$ are two n-tuples of variables. Hence, μ can be rewritten in the following manner:

$$\mu : \forall \boldsymbol{x} \exists \boldsymbol{y} \exists \boldsymbol{u}\ \Sigma_i(\boldsymbol{x}) \neq 1 \vee \Sigma_i(\boldsymbol{y}) = 1 \wedge\ \theta(\boldsymbol{x}, \boldsymbol{y}) \wedge \bigvee_{\ell \in [\![1,k]\!]} h_\ell(\boldsymbol{y}) = 1.$$

Since $\mu \in \mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, the sentence μ is true in G as well. The interpretation of μ in G is the following: for every homomorphism $\phi : H \to G$, there exists a homomorphism $\phi' : H \to G$ that is Δ_H-related to ϕ and that kills some h_ℓ. By taking $\phi = \eta_{|H}$, we get a non-injective homomorphism $p : H \to G$ Δ_H-related to $\eta_{|H}$.

Conclusion. Since $p : H \to G$ is Δ_H-related to $\eta_{|H}$, the homomorphism $\rho \circ p : H \to G'$ is Δ_H-related to $\rho \circ \eta_{|H}$. Indeed, the fact that p and $\eta_{|H}$ coincide up to conjugacy on finite subgroups of H and on vertex groups of Δ_H remains true for $\rho \circ p$ and $\rho \circ \eta_{|H}$. Moreover, $\rho \circ \eta_{|H}$ is Δ_H-related to the inclusion of H into G' by the second assertion of Lemma 5.21. This concludes the proof. $\square$

In the same way, one can prove the following result, which generalizes Proposition 6.1.

Proposition 6.2. *Let (Γ, Γ', r, j) and (G, G', ρ, η) be two quasi-towers. Suppose that Γ embeds into a hyperbolic group and that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Let $G'_1, \ldots, G'_n$ be the one-ended factors of G' that are not finite-by-orbifold groups. Then one of the following assertion holds:*

- *there exists a morphism $\phi : G' \to \Gamma$ such that $r \circ \phi$ is factor-injective (i.e. injective on each G'_i),*
- *or there exists an integer $1 \leq k \leq n$ and a non-injective preretraction $G'_k \to G'$.*

In Section 7, we shall prove the following proposition, whose proof is quite technical.

Proposition 6.3. *Let G be a finitely generated K-CSA group that does not contain $\mathbb{Z}^2$. Let A be a one-ended factor of G. Suppose that A is not a finite-by-orbifold group. If there exists a non-injective preretraction $p : A \to G$, then there exist a finitely generated group H and two morphisms $r : G \to H$ and $j : H \to G$ such that (G, H, r, j) is a strict quasi-floor.*

Together, Propositions 6.1 and 6.3 endow us with a machinery to build quasi-floors.

Proposition 6.4. *Let (Γ, Γ', r, j) be a quasi-tower. Suppose that Γ embeds into a hyperbolic group. Let G be a finitely generated group. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. In particular, G has a quasi-core G'. If H is a one-ended factor of G', then H embeds into Γ'.*

Similarly, by putting together Propositions 6.2 and 6.3, one gets the following more general statement.

Proposition 6.5. *Let (Γ, Γ', r, j) be a quasi-tower. Suppose that Γ embeds into a hyperbolic group. Let G be a finitely generated group. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. In particular, G has a quasi-core G'. Then there exists a factor-injective homomorphism from G' to Γ'.*

6.2. Proofs of Theorems 1.3 and 1.4.

Theorem 6.6. *Let Γ be a group that embeds into a hyperbolic group Ω, and let G be a finitely generated group. If $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, then G embeds into a hyperbolic group.*

Proof. Since G is a Ω-limit group, it possesses a quasi-core G' (by Proposition 5.15). By [**Bra00**], there exists an integer K such that all finite subgroups of Ω have order $\leq K$. As a consequence, all finite subgroups of G' have order $\leq K$. Therefore, by [**Lin83**], G' has a $\mathcal{F}$-JSJ splitting. We claim that all one-ended factors of G' are hyperbolic. Let H be such a one-ended factor. If H is a finite-by-orbifold group, then it is hyperbolic by definition. If H is not a finite-by-orbifold group, it follows from Proposition 6.4 that H embeds into Γ, hence into Ω. Now, let $H_1, \ldots, H_p$ be the one-ended factors of G' (well-defined up to conjugation) that are not finite-by-orbifold groups, and let Λ be the graph of groups obtained by replacing each H_k in a $\mathcal{F}$-JSJ splitting of G' by Ω. Let Ω' be the fundamental group of Λ. It is clear that G' embeds into Ω'. In addition, vertex groups of Λ being hyperbolic and edge groups of Λ being finite, the group Ω' is hyperbolic thanks to the combination theorem of Bestvina and Feighn (see 3.10). Therefore, by Proposition 5.13, G embeds into a hyperbolic group Ω''. $\square$

We claim that if Ω is locally hyperbolic then, with the same notations as in the previous proof, the groups Ω' and Ω'' are locally hyperbolic as well. We need the following lemmas.

LEMMA 6.7. *Let $G = A *_C B$ be an amalgamated product such that A and B are locally hyperbolic and C is virtually cyclic (possibly finite) and almost malnormal in A or B. Then G is locally hyperbolic.*

PROOF. Let T be the Bass-Serre tree of the splitting $G = A *_C B$. Let H be a finitely generated subgroup of G, and let $T_H \subset T$ be the minimal subtree invariant under the action of H. In order to prove that H is hyperbolic, let us consider the graph of groups T_H/H. First, note that every vertex group H_v of T_H is a finitely generated subgroup of a conjugate of A or B. Since A and B are locally hyperbolic, H_v is hyperbolic. Moreover, we claim that if $e = [v, w]$ is an edge of T_H, then the edge group H_e is almost malnormal in H_v or in H_w. Up to exchanging v and w, one can suppose that H_v is contained in a conjugate of A and that H_w is contained in a conjugate of B. The edge group H_e is contained in a conjugate of C. By assumption, we know that C is almost malnormal in A or B. Suppose for instance that C is almost malnormal in A, and let us prove that H_e is almost malnormal in H_v. Let $g \in G$ be an element such that $H_v \subset gAg^{-1}$ and $H_e \subset gCg^{-1}$. If $hH_eh^{-1} \cap H_e$ is infinite for some $h \in H_v$, then $hgCg^{-1}h^{-1} \cap gCg^{-1}$ is infinite, so h belongs to gCg^{-1}. Hence, h belongs to $gCg^{-1} \cap H_v = H_e$. This shows that H_e is almost malnormal in H_v. Now, by the combination theorem of Bestvina and Feighn (see Proposition 3.10), the group H is hyperbolic. □

In the same way, one can prove the following lemma.

LEMMA 6.8. *Let $G = A*_C$ be an HNN extension such that A is locally hyperbolic, C is virtually cyclic (possibly finite), and the two copies C_1 and C_2 of C form an almost malnormal family of subgroups, i.e. at least one of them is almost malnormal in A, and $C_1 \cap gC_2g^{-1}$ is finite for every $g \in G$. Then G is locally hyperbolic.*

As a consequence, the following theorem holds.

THEOREM 6.9. *Let Ω be a locally hyperbolic group, let Γ be a subgroup of Ω and let G be a finitely generated group. If $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, then G is a locally hyperbolic group.*

REMARK 6.10. We stress that the theorem above can be derived more directly from Sela's shortening argument, without using quasi-towers. In [**Sel01**], Sela proved (Corollary 4.4) that a limit group is hyperbolic if and only if it does not contain $\mathbb{Z}^2$. This result is also proved by Kharlampovich and Myasnikov in [**KM98**] (Corollary 4). It follows that a finitely generated group G satisfying $\mathrm{Th}_{\forall\exists}(\boldsymbol{F}_2) \subset \mathrm{Th}_{\forall\exists}(G)$ is hyperbolic, thanks to Corollary 2.18. In fact, Sela's proof shows that G is locally hyperbolic, and his proof remains valid if we replace $\boldsymbol{F}_2$ by any locally hyperbolic group.

6.3. Proof of Theorem 1.2.

We will prove Theorem 1.2 using the following result, whose proof is postponed.

PROPOSITION 6.11. *Let Γ be a hyperbolic group and G a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Let G' be a quasi-core of G and Γ' a quasi-core of Γ. Then every one-ended factor of G' that is not a finite-by-orbifold group is isomorphic to a one-ended factor of Γ'. Therefore, G' is hyperbolic.*

More precisely, we prove the following result, which immediately implies Theorem 1.2.

THEOREM 6.12. *Let Γ be a hyperbolic group and G a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Then G is hyperbolic.*

PROOF. Let Γ' be a quasi-core of Γ, and let G' be a quasi-core of G. By Proposition 5.13 about inheritance of hyperbolicity, the group Γ' is hyperbolic. Therefore, by Proposition

3.11, all one-ended factors of Γ' are hyperbolic. Then Proposition 6.11 above shows that all one-ended factors of G' are hyperbolic, which implies that G is hyperbolic according to the combination theorem of Bestvina and Feighn (see Proposition 3.10). Finally, Proposition 5.13 proves that G is hyperbolic, as a quasi-tower over a hyperbolic group. $\square$

The rest of this subsection is devoted to a proof of Proposition 6.11. Let $G'_1, \ldots, G'_n$ be the one-ended factors (well-defined up to conjugacy) of G' that are not finite-by-orbifold groups. Let $\Gamma'_1, \ldots, \Gamma'_m$ be the one-ended factors of Γ' that are not finite-by-orbifold groups. By Proposition 6.5, there exist a factor-injective homomorphism ϕ from G' to Γ', and a factor-injective homomorphism ψ from Γ' to G'. For every integer $1 \leq i \leq n$, the group $\phi(G'_i)$ is contained in a conjugate of Γ'_j for some $1 \leq j \leq m$. Hence, one can associate to ϕ a map $\tau(\phi) : [\![1, n]\!] \to [\![1, m]\!]$ such that for every $1 \leq i \leq n$, the group $\phi(G'_i)$ is contained in a conjugate of $\Gamma'_{\tau(\phi)(i)}$. Likewise, one can associate to ψ a map $\sigma(\psi) : [\![1, m]\!] \to [\![1, n]\!]$ such that for every $1 \leq j \leq m$, $\psi(\Gamma'_j)$ is contained in a conjugate of $G'_{\sigma(\psi)(j)}$. In order to prove Proposition 6.11, it suffices to show that we can choose $\tau(\phi)$ and $\sigma(\psi)$ so that $\sigma(\psi) \circ \tau(\phi)$ is a permutation of $[\![1, n]\!]$, since one-ended hyperbolic groups are co-Hopfian (see [**Sel97**] for torsion-free hyperbolic groups and [**Moi13**], based on [**Del95**], for hyperbolic groups with torsion).

LEMMA 6.13. *Let Γ be a hyperbolic group and G a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Let G' be a quasi-core of G and Γ' a quasi-core of Γ. We keep the same notations as above. There exists a factor-injective homomorphism ϕ from G' to Γ' such that $\tau(\phi)$ satisfies the following condition: for every $i \in [\![1, n]\!]$, if $\phi(G'_i)$ is equal to $\Gamma'_{\tau(\phi)(i)}$ up to conjugacy by an element of Γ', and if $\Gamma'_{\tau(\phi)(i)}$ is hyperbolic, then i is the unique preimage of $\tau(\phi)(i)$ by $\tau(\phi)$.*

PROOF. Assume towards a contradiction that the lemma is false.

Step 1. We claim that there exist some finite subsets $X_1 \subset G'_1, \ldots, X_n \subset G'_n$ containing only elements of infinite order, and a finite set $F \subset G' \setminus \{1\}$ such that, for every homomorphism ϕ from G' to Γ, one of the following claims is true:

- there exists an integer $k \in [\![1, n]\!]$ such that $\phi_{|G'_k}$ is Δ_k-related to a homomorphism $\phi' : G'_k \to \Gamma$ that kills an element of F, where Δ_k stands for the $\mathcal{Z}$-JSJ splitting of G'_k,
- or there exist an element $\gamma \in \Gamma$ and two elements $x_k \in X_k$ and $x_\ell \in X_\ell$ (with $k \neq \ell$) such that $\phi(x_k) = \gamma\phi(x_\ell)\gamma^{-1}$.

Proof of Step 1. Let F be the set given by Lemma 5.23. We shall define the sets $X_1, \ldots, X_n$. Let I be the subset of $[\![1, n]\!]$ consisting of the integers i such that G'_i is hyperbolic. In the case where I is empty, let $X_1 = \cdots = X_n = \varnothing$. Now, assume that I is non-empty. For each $k \in [\![1, n]\!]$, since G'_k is one-ended and is not a finite-by-orbifold group, there exists at least one infinite non-QH vertex group A_k in the $\mathcal{Z}$-JSJ splitting Δ_k of G'_k. Fix an element of infinite order $x_k \in A_k$. For each $\ell \in [\![1, n]\!] \setminus \{k\}$, if G'_ℓ is not hyperbolic, let $X_{k,\ell} := \varnothing$; if G'_ℓ is hyperbolic, let $Y_{k,\ell} := \{f(x_k) \mid f \in \mathrm{Mono}(G'_k, G'_\ell)\}$ and let $X_{k,\ell}$ be a set of representatives for the orbits of $Y_{k,\ell}$ under the action of G'_ℓ by conjugation. We claim that the set $X_{k,\ell}$ is finite (possibly empty). Indeed, thanks to Sela's shortening argument 2.5, there exist finitely many monomorphisms $f_1, \ldots, f_r : G'_k \hookrightarrow G'_\ell$ such that, for every monomorphism $f : G'_k \hookrightarrow G'_\ell$, there exist an integer $1 \leq p \leq r$ and a modular automorphism σ of G'_k such that f and $f_p \circ \sigma$ coincide up to conjugacy. Since σ coincides with a conjugay on each rigid subgroup of G'_k, the elements $f(x_k)$ and $f_p(x_k)$ are in the same orbit under the action of G'_ℓ by conjugation, which proves that $X_{k,\ell}$ is finite. Finally, we define

$$X_k := \{x_k\} \cup \bigcup_{1 \leq \ell \neq k \leq n} X_{k,\ell}.$$

Now, we will prove that these sets have the expected property. Since Γ' is a quasi-core of Γ, there exists a quasi-tower (Γ, Γ', r, j). By Lemma 5.23, for every homomorphism $\phi : G' \to \Gamma$,

- either there exists an integer $1 \leq k \leq n$ such that the restriction $\phi_{|G'_k}$ is Δ_k-related to a homomorphism $\phi' : G'_k \to \Gamma$ that kills an element of F,
- or $\theta := r \circ \phi$ is factor-injective.

As explained in the paragraph before Lemma 6.13, one can associate to θ a map $\tau(\theta) : [\![1, n]\!] \to [\![1, m]\!]$ such that for every $1 \leq i \leq n$, the group $\theta(G'_i)$ is contained in a conjugate of $\Gamma'_{\tau(\theta)(i)}$. In the event that the second alternative above holds, since we have assumed that Lemma 6.13 is false (towards a contradiction), there exist two integers $k \neq \ell$ such that $\tau(\theta)(k) = \tau(\theta)(\ell)$ and $\theta(G'_\ell) = \Gamma'_{\tau(\theta)(\ell)}$ up to conjugacy, and such that $\Gamma'_{\tau(\theta)(\ell)}$ is hyperbolic. Thus, $\theta(G'_k) \subset \mathrm{ad}(\gamma)(\theta(G'_\ell)) = \mathrm{ad}(\gamma')(\Gamma'_{\tau(\theta)(\ell)})$ for some $\gamma, \gamma' \in \Gamma'$, and G'_ℓ is hyperbolic. We have

$$\left(\theta_{|G'_\ell}\right)^{-1} \circ \mathrm{ad}(\gamma^{-1}) \circ \left(\theta_{|G'_k}\right) \in \mathrm{Mono}(G'_k, G'_\ell).$$

Hence, by definition of X_k and X_ℓ, there exist $x_k \in X_k$ and $x_\ell \in X_\ell$ such that $\rho(x_k) = \mathrm{ad}(\gamma)(\theta(x_\ell))$. So $j \circ \theta(x_k) = \mathrm{ad}(\gamma'')j \circ \theta(x_\ell)$ for some $\gamma'' \in \Gamma$. But $j \circ \theta = j \circ r \circ \phi$ is Δ_k-related to ϕ thanks to Proposition 5.20. Therefore, $\phi(x_k) = \mathrm{ad}(\gamma''') \circ \phi(x_\ell)$ for some $\gamma''' \in \Gamma$. This concludes the proof of the first step.

Step 2. The statement of Step 1 is expressible by a $\forall\exists$-sentence, denoted by μ, which is true in Γ; more precisely, the first point of Step 1 is expressible as in the second step of the proof of Proposition 6.1) (see Step 2), and the second point is easily expressible using the fact that X_k and X_ℓ are finite. Since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, μ is true in G as well. Thus, for every $\phi \in \mathrm{Hom}(G', G)$, one of the following claims is true:

- there exist a vertex group G'_k together with a homomorphism $\phi' : G'_k \to G$ which is Δ_k-related to $\phi_{|G'_k}$ and kills an element of F,
- or there exist an element $g \in G$ and two elements $x_k \in X_k$ and $x_\ell \in X_\ell$ (with $k \neq \ell$) such that $\phi(x_k) = g\phi(x_\ell)g^{-1}$, with x_k of infinite order.

By definition, G is a quasi-tower over G'. Let $\rho : G \to G'$ and $\eta : G' \to G$ be the two homomorphisms associated with this structure of a quasi-tower. Taking $\phi := \eta$, the second claim above is false. Otherwise, the intersection $\eta(G'_k) \cap g\eta(G'_\ell)g^{-1}$ is infinite, so $\rho(\eta(G'_k)) \cap \rho(g)\rho(\eta(G'_\ell))\rho(g)^{-1}$ is infinite, since ρ is injective in restriction to $\eta(G'_k)$. But $\rho \circ \eta$ is inner on G'_k and on G'_ℓ. Hence, there exists an element $h \in G'$ such that $G'_k \cap hG'_\ell h^{-1}$ is infinite. This is a contradiction, since G'_k and G'_ℓ are two different vertex groups of a $\mathcal{F}$-JSJ splitting of G'.

As a consequence, the first claim is necessarily true (for $\phi := \eta$). There exist a one-ended factor G'_k together with a homomorphism $\phi' : G'_k \to G$ that is Δ_k-related to $\eta_{|G'_k}$ and kills an element of F. Thus $\rho \circ \phi'$ is Δ_k-related to $\rho \circ \eta_{|G'_k}$ and kills an element of F. Let i denote the inclusion of G'_k into G'. By Lemma 5.20, $\rho \circ \eta_{|G'_k} = \rho \circ \eta_{|G'_k} \circ i$ is Δ_k-related to i. Hence, $\rho \circ \phi' : G'_k \to G'$ is Δ_k-related to i and kills an element of F. In other words, $\rho \circ \phi'$ is a non-injective preretraction. It follows from Proposition 6.3 that G' is a strict quasi-floor. This is a contradiction, since G' is a quasi-core by hypothesis. □

Before proving Proposition 6.11, we need the following definition.

Definition 6.14. Let X be a set and let $f : X \to X$ be a map. An element $x \in X$ is said to be periodic if there exists an integer $k \geq 1$ such that $f^k(x) = x$.

We now prove Proposition 6.11.

Proof. We keep the same notations as above. Since Γ is hyperbolic, Γ' is hyperbolic by Proposition 5.13, so $\Gamma'_1, \ldots, \Gamma'_m$ are hyperbolic by Proposition 3.11. Let $\phi : G' \to \Gamma'$

and $\psi : \Gamma' \to G'$ be the factor-injective homomorphisms given by Lemma 6.13. If $i \in [\![1,n]\!]$ is periodic under $\sigma(\psi) \circ \tau(\phi)$, then i is the unique preimage of $\tau(\phi)(i)$ by the map $\tau(\phi)$. Indeed, if i is periodic, then ϕ induces an isomorphism between G'_i and $\phi(G'_i) = \Gamma'_{\tau(\phi)(i)}$, because one-ended hyperbolic groups are co-Hopfian. It follows from Lemma 6.13 that i is the unique preimage of $\tau(\phi)(i)$ by the map $\tau(\phi)$. Similarly, if $j \in [\![1,m]\!]$ is periodic under $\tau(\phi) \circ \sigma(\psi)$, then j is the unique preimage of $\sigma(\psi)(j)$ by the map $\sigma(\psi)$. The conclusion follows immediately from Lemma 6.15 below. $\square$

LEMMA 6.15. *Let $n, m \geq 1$ be two integers. Let $\tau : [\![1,n]\!] \to [\![1,m]\!]$ and $\sigma : [\![1,m]\!] \to [\![1,n]\!]$ be two maps. Suppose that the following two conditions hold.*

(1) For every $i \in [\![1,n]\!]$, if i is periodic under $\sigma \circ \tau$, then i is the unique preimage of $\tau(i)$ by τ.
(2) For every $j \in [\![1,m]\!]$, if j is periodic under $\tau \circ \sigma$, then j is the unique preimage of $\sigma(j)$ by σ.

Then $n = m$ and τ, σ are bijective.

PROOF. We claim that every $i \in [\![1,n]\!]$ is periodic under $\sigma \circ \tau$. Assume towards a contradiction that i is not periodic under $\sigma \circ \tau$. Let $k \geq 1$ be the smallest integer such that $(\sigma \circ \tau)^k(i)$ is periodic under $\sigma \circ \tau$. Let $j = \tau \circ (\sigma \circ \tau)^{k-1}(i)$. If j is periodic under $\tau \circ \sigma$, there exists an integer $p \in [\![1,n]\!]$ periodic under $\sigma \circ \tau$ such that $\tau(p) = j$. But $q = (\sigma \circ \tau)^{k-1}(i)$ is not periodic under $\sigma \circ \tau$ by definition of k, so $q \neq p$ (since p is periodic and q is not periodic), and $\tau(q) = j = \tau(p)$. This contradicts the second condition. If j is not periodic under $\tau \circ \sigma$, we find a contradiction with the first condition, in the same way. Hence, every element $i \in [\![1,n]\!]$ is periodic. It follows from condition (1) that τ is injective. Likewise, σ is injective. Thus, $n = m$ and τ, σ are bijective. $\square$

6.4. Proof of Theorem 1.5.

In this subsection, we prove the following results.

THEOREM 6.16. *Let Γ be a hyperbolic group and let G be a finitely generated group. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Then Γ is cubulable if and only if G is cubulable.*

COROLLARY 6.17. *Let Γ and G be finitely generated groups. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Then Γ is hyperbolic and cubulable if and only if G is hyperbolic and cubulable.*

Note that Corollary 6.17 follows immediately from Theorem 6.16 and from the fact that hyperbolicity is preserved under $\forall\exists$-equivalence among finitely generated groups (see Proposition 6.11).

We refer the reader to [**Sag14**] for a definition of CAT(0) and special cube complexes. A group G is said to be *cubulable* (resp. *special*) if it acts properly and cocompactly by isometries on a locally finite CAT(0) (resp. special) cube complex, and *virtually special* if it has a finite-index subgroup which is special.

Let us recall Wise's quasiconvex hierarchy theorem for hyperbolic cubulable groups.

THEOREM 6.18 ([**Wis12**], Theorems 13.1 and 13.3). *Let G be a hyperbolic group. Suppose that G splits as a graph of groups all of whose vertex groups are virtually special cubulable and quasiconvex in G. Then G is virtually special cubulable.*

Using results of Haglund and Wise, Agol proved the following celebrated theorem.

THEOREM 6.19 ([**Ago13**], Theorem 1.1). *Every hyperbolic and cubulable group is virtually special.*

We will also need the following proposition, claiming that cubulability is inherited by quasi-convex subgroups.

PROPOSITION 6.20 ([**Hag08**], Corollary 2.29). *Let G be a hyperbolic and cubulable group. If H is a quasi-convex subgroup of G, then H is cubulable.*

Before proving Theorem 6.16, we establish the following preliminary result.

PROPOSITION 6.21. *Let $(G, H, \Delta_G, \Delta_H)$ be a quasi-tower. Suppose that G is hyperbolic. Then G is cubulable if and only if H is cubulable.*

PROOF. We prove the proposition in the case where G is a quasi-floor over H, and the general case follows immediately by induction on the height of the quasi-tower.

Let V_G and V_H denote the sets of vertices of Δ_G and Δ_H. Let v be the central vertex of Δ_G. By definition of a quasi-floor, there exists a subset $V_H^1 \subset V_H$ and a bijection $s : V_G \setminus \{v\} \to V_H^1$ such that $G_w \simeq H_{s(w)}$ for every $w \in V_G \setminus \{v\}$.

Suppose that G is cubulable. Then, by Propositions 3.11 and 6.20, all vertex groups of Δ_G are hyperbolic and cubulable. By Agol's theorem 6.19, all vertex groups of Δ_G are therefore virtually special. As a consequence, every vertex group H_u with $u \in V_H^1$ is virtually special cubulable (see the bijection s above). Moreover, recall that $V_H = V_H^1 \sqcup V_H^2$, with H_u finite for every $u \in V_H^2$. Hence, all vertex groups of Δ_H are virtually special cubulable. The group H being hyperbolic thanks to Proposition 5.13 (since G is hyperbolic), one can now apply Wise's quasiconvex hierarchy theorem 6.18 to conclude that H is virtually special cubulable, using the fact that vertex groups of Δ_H are quasi-convex in H, by Proposition 3.11.

Conversely, if H is cubulable, we prove in exactly the same way that G is cubulable. □

We now prove Theorem 6.16.

PROOF. Let Γ be a hyperbolic group and let G be a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Suppose that Γ is cubulable. We will prove that G is cubulable. Let us denote by G' a quasi-core of G and by Γ' a quasi-core of Γ. According to Proposition 6.21, Γ' is cubulable, so each vertex group of a $\mathcal{F}$-JSJ splitting of Γ' is cubulable, by Proposition 6.20 (and the trivial case of finite groups). Moreover, according to Proposition 6.11, every vertex group of a $\mathcal{F}$-JSJ splitting of G' that is not a finite-by-orbifold group is isomorphic to a vertex group of a $\mathcal{F}$-JSJ splitting of Γ'. Consequently, by Theorem 6.18, the group G' is cubulable. Then, it follows from Proposition 6.21 above that G is cubulable. Symmetrically, if G is cubulable, then so is Γ (because G is hyperbolic, by Theorem 6.12). □

7. From a preretraction to a quasi-floor: proof of Proposition 6.3

In the previous section, we proved our main theorems by admitting Proposition 6.3, which claims that one can build a strict quasi-floor from a non-injective preretraction. This section is devoted to a proof of Proposition 6.3.

7.1. A preliminary proposition.

PROPOSITION 7.1. *Let G be a one-ended finitely generated K-CSA group that does not contain $\mathbb{Z}^2$. Let Δ denote the $\mathcal{Z}$-JSJ splitting of G, and let $p : G \to G$ be a non-degenerate Δ-preretraction. Then p is injective.*

Recall that a Δ-preretraction p is said to be non-degenerate if it sends each QH subgroup of G isomorphically to a conjugate of itself (see Definition 3.6).

PROOF. Let T be the Bass-Serre tree of Δ. We denote by V the set of vertices of T. First of all, let us recall some properties of Δ that will be useful in the sequel (see Section 2.6.4).

(1) The graph Δ is bipartite, with every edge joining a vertex carrying a virtually cyclic group to a vertex carrying a non-virtually-cyclic group.
(2) Let v be a vertex of T, and let e, e' be two distinct edges incident to v. If G_v is not virtually cyclic, then the group $\langle G_e, G_{e'}\rangle$ is not virtually cyclic.

(3) The action of G on T is 2-acylindrical: if an element $g \in G$ fixes a segment of length > 2 in T, then g has finite order.

If Δ is reduced to a point, then p is obviously injective. From now on, we will suppose that Δ has at least two vertices.

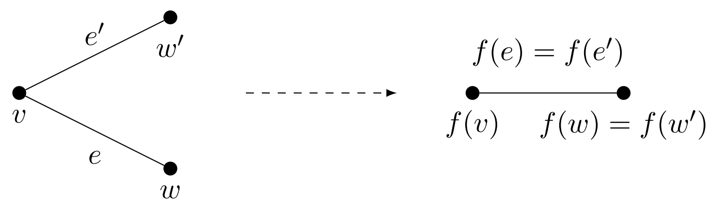
As a first step, we build a p-equivariant map $f : T \to T$. Let $v_1, \ldots, v_n$ be some representatives of the orbits of vertices. For every $1 \leq k \leq n$, there exists $g_k \in G$ such that $p(G_{v_k}) = g_k G_{v_k} g_k^{-1}$. We let $f(v_k) = g_k v_k$, so that $p(G_{v_k}) = G_{f(v_k)}$. Then we define f on each vertex of T by p-equivariance. Next, we define f on the edges of T in the following way: if e is an edge of T, with endpoints v and w, there exists a unique injective path e' from $f(v)$ to $f(w)$ in T. We let $f(e) = e'$.

Now we will prove that f is injective, which allows to conclude that p is injective. Indeed, if $p(g) = 1$ for some $g \in G$, then for every vertex $v \in V$ one has $p(g) \cdot f(v) = f(g \cdot v) = f(v)$, thus $g \cdot v = v$ for every v. Since the action of G on Δ is 2-acylindrical, g has finite order. Moreover the restriction of p to every element of finite order is injective (by definition of Δ-relatedness), so $g = 1$, which proves that p is injective.

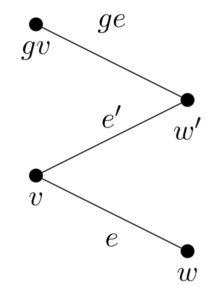
We now prove that f is injective. The proof will proceed in two steps: first, one shows that f sends adjacent vertices on adjacent vertices, then one proves that there are no foldings.

The map f sends adjacent vertices to adjacent vertices: let's consider two adjacent vertices v and w of T. One has $d(f(v), f(w)) \leq 2$, because the action of G on T is 2-acylindrical and p is injective on edge groups, which are virtually cyclic and infinite. Since the tree is bipartite and since f sends a vertex of T to a vertex of the same type, the integers $d(f(v), f(w))$ and $d(v, w) = 1$ have the same parity. Hence $d(f(v), f(w)) = 1$.

There are no foldings: let v be a vertex of T, let w and w' be two distinct vertices adjacent to v. Denote by e and e' the edges between v and w, and between v and w' respectively. Argue by contradiction and suppose that $f(w) = f(w')$. Then $f(e) = f(e')$ since there are no circuits in a tree.



If G_v is not virtually cyclic, then $\langle G_e, G_{e'} \rangle$ is not virtually cyclic (see the second property of Δ recalled above), thus $p(\langle G_e, G_{e'} \rangle)$ is not virtually cyclic since $\langle G_e, G_{e'} \rangle$ is contained in G_v, and p is injective on G_v. But $p(\langle G_e, G_{e'} \rangle)$ is contained in the stabilizer of $f(e)$, which is virtually cyclic. This is a contradiction. Hence G_v is virtually cyclic. There exists an element $g \in G$ such that $w' = g \cdot w$. Since f is p-equivariant, $p(g) \cdot f(w) = f(w)$, i.e. $p(g) \in G_{f(w)} = p(G_w)$. As a consequence, there exists an element $h \in G_w$ such that $p(g) = p(h)$. Up to multiplying g by the inverse of h, one can assume that $p(g) = 1$. Then g does not fix a point of T, because p is injective on vertex groups and $g \neq 1$. It follows that g is hyperbolic, with translation length equal to 2.



The group $\langle G_{e'}, G_{ge}\rangle$ is not virtually cyclic since $G_{w'}$ is not virtually cyclic. It follows that $p(\langle G_{e'}, G_{ge}\rangle)$ is not virtually cyclic (indeed, p is injective on $G_{w'}$). On the other hand, $p(\langle G_{e'}, G_{ge}\rangle)$ is equal to $p(\langle G_{e'}, G_{e}\rangle)$ because $p(g) = 1$. Thus $p(\langle G_{e'}, G_{e}\rangle)$, which is contained in the stabilizer of $f(e)$, is not virtually cyclic. This is a contradiction. $\square$

7.2. Building a quasi-floor from a non-injective preretraction.

Definition 7.2 (Maximal pinched set, pinched quotient, pinched decomposition). Let G be a group that splits as a centered splitting Δ_G, with central vertex v. The stabilizer G_v of v is a conical finite-by-orbifold group $F \hookrightarrow G_v \twoheadrightarrow \pi_1(\mathcal{O})$. Denote by q the epimorphism from G_v onto $\pi_1(\mathcal{O})$. Let G' be a group, and let $p : G \to G'$ be a homomorphism. Let S be an essential set of curves on $\mathcal{O}$ (see Definition 2.34). Suppose that each element of S is pinched by p (meaning that $p\left(q^{-1}(\alpha)\right)$ is finite for every $\alpha \in S$, with $q^{-1}(\alpha)$ well-defined up to conjugacy), and that S is maximal for this property. The set S is called a maximal pinched set for p. Note that S may be empty.

For every $\alpha \in S$, $q^{-1}(\alpha)$ (well-defined up to conjugacy) is a virtually cyclic subgroup of G_v, isomorphic to $F \rtimes \mathbb{Z}$. Let $N_\alpha = \ker\left(p_{|q^{-1}(\alpha)}\right)$, and let N be the subgroup of G normally generated by $\{N_\alpha\}_{\alpha\in S}$. The quotient group $Q = G/N$ is called the pinched quotient of G associated with S. Let $\pi : G \twoheadrightarrow Q$ be the quotient epimorphism. Since each N_α has finite index in $q^{-1}(\alpha)$, and since p is injective on finite subgroups, killing N gives rise to new QH vertices with new conical points, and new edges whose stabilizers coincide with the new conical groups (see Figure 7 below). The group Q splits naturally as a graph of groups Δ_Q obtained by replacing the vertex v in Δ_G by the splitting of $\pi(G_v)$ over finite groups obtained by killing N (see Figure 7 below). Δ_Q is called the pinched decomposition of Q.

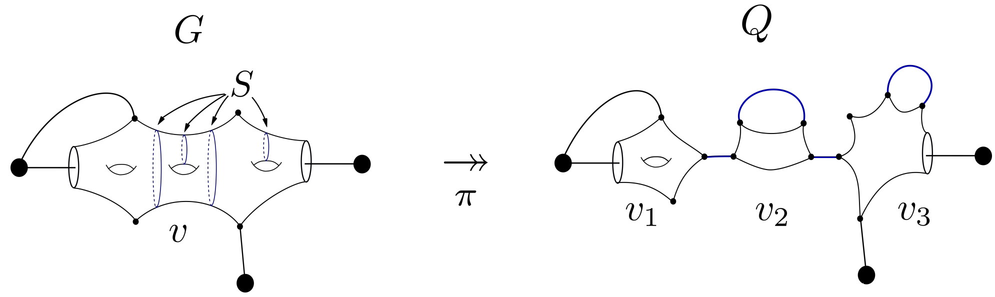


Figure 7. For convenience, on this figure, F is trivial. For each $\alpha \in S$, there exists a smallest integer $n \geq 1$ such that $p(\alpha^n) = 1$. Killing $\langle \alpha^n \rangle$ gives rise to a conical point of order n, and a new edge group equal to the conical group. The new QH vertices coming from v are denoted by v_1, v_2, v_3.

We keep the same notations. Suppose that there exists an endomorphism p of G that is Δ_G-related to the identity of G. Let S be a maximal pinched set for p, let Q be the pinched quotient and let Δ_Q be its pinched decomposition. Denote by $v_1, \ldots, v_n$ the new vertices coming from v (see Figure 7 above). Let $\pi : G \twoheadrightarrow Q$ be the quotient epimorphism.

There exists a unique homomorphism $\phi : Q \to G$ such that $p = \phi \circ \pi$. This homomorphism is non-pinching in restriction to Q_{v_i} for $1 \leq i \leq n$, since S is assumed to be maximal. We will need a lemma.

LEMMA 7.3. *We keep the same notations. Assume that p does not send G_v isomorphically to a conjugate of itself, and denote by $\mathcal{F}$ the set of edges of Δ_Q with finite stabilizer (note that $\mathcal{F}$ contains all the new edges arising from the pinching). Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of Δ_Q coming from v, and let W be the set of vertices of Δ_Q that do not belong to V. Let Y be a connected component of $\Delta_Q \setminus \mathcal{F}$, and let Q_Y be the fundamental group of Y. Then, the following assertions hold:*

- *$\phi(Q_Y)$ is elliptic in the Bass-Serre tree of Δ_G;*
- *Y contains at most one vertex w of W and if it does, then $\phi(Q_Y) = \phi(Q_w)$.*

PROOF. **Step 1.** First, we will prove that $\phi(Q_{v_k})$ is elliptic in the Bass-Serre tree T of Δ_G, for every $k \in [\![1, n]\!]$.

Denote by O_k the underlying orbifold of Q_{v_k}. We will use Proposition 2.35 in order to split Q_{v_k} as a graph of groups all of whose vertex groups are elliptic in T via ϕ. If C is an extended boundary subgroup of Q_{v_k}, C is of the form $\pi(C')$ where C' stands for an extended boundary subgroup of G_v, so $\phi(C) = p(C')$ is elliptic in T by definition of Δ_G-relatedness. Consequently, by Proposition 2.35, there exists an essential set $\mathcal{C}_k$ of simple loops on $\mathcal{O}_k$ such that, if X is a connected component of $\mathcal{O}_k \setminus \mathcal{C}_k$, and if H is the preimage of the orbifold fundamental group $\pi_1(X)$ in Q_{v_k}, then $\phi(H)$ is elliptic in the Bass-Serre tree T.

Let $\Delta_Q(\mathcal{C}_k)$ be the splitting of Q obtained by replacing v_k in Δ_Q by the splitting of Q_{v_k} dual to $\mathcal{C}_k$. First, note that if $\mathcal{C}_k$ is empty, then $\phi(Q_{v_k})$ is obviously elliptic in T. Assume now that $\mathcal{C}_k$ is non-empty and denote by $v_{k,1}, \ldots, v_{k,m}$ the new vertices coming from v_k. Let T' be the Bass-Serre tree of $\Delta_Q(\mathcal{C}_k)$. Let $w_{k,i}, w_{k,j} \in T'$ be two representatives of $v_{k,i}, v_{k,j} \in \Delta_Q(\mathcal{C}_k)$ that are adjacent in T' and linked by an edge with infinite stabilizer. By the previous paragraph, there exists a non-empty subset $I \subset T$ pointwise-fixed by $\phi\left(Q_{w_{k,i}}\right)$, and a non-empty subset $J \subset T$ pointwise-fixed by $\phi\left(Q_{w_{k,j}}\right)$. Let $x \in I$ and $y \in J$ be such that $d(x, y) = d(I, J)$, where d is the natural metric on T. By definition of a centered splitting, if an element of G of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its endpoints are translates of the central vertex v. Therefore, since ϕ is non-pinching on Q_{v_k}, the distance $d(x, y)$ between x and y belongs to $\{0, 1, 2\}$, otherwise ϕ would pinch a common boundary element of $Q_{w_{k,i}})$ and $Q_{w_{k,j}}$. In addition, if the equality $d(x, y) = 2$ holds, then x and y are translates of v.

Claim. We will prove that $d(x, y) = 0$.

First, suppose that $d(x, y) = 2$. Then we can assume without loss of generality that $x = v$, i.e. $\phi\left(Q_{w_{k,i}}\right) \subset G_v$. Since ϕ is non-pinching on $Q_{w_{k,i}}$, the group $\phi\left(Q_{w_{k,i}}\right)$ is infinite, so it is not contained in an extended conical subgroup of G_v. Let us suppose towards a contradiction that $\phi\left(Q_{w_{k,i}}\right)$ is not contained in an extended boundary subgroup of G_v. Then, it follows from Proposition 2.43 that $\chi(Q_{w_{k,i}}) \geq \chi(G_v)$ with equality if and only if f induces an isomorphism from $Q_{w_{k,i}}$ to G_v. But $\chi(Q_{w_{k,i}}) < \chi(G_v)$ since the complexity decreases as soon as we cut along a loop or pinch a loop and since $Q_{w_{k,i}}$ is not isomorphic to G_v. This is a contradiction. Therefore, $\phi\left(Q_{w_{k,i}}\right)$ is necessarily contained in an extended boundary subgroup of G_v. Then $\phi\left(Q_{w_{k,i}}\right)$ fixes a point z in T such that $d(x, z) = 1$. As a consequence, $d(z, y) = 1$ or $d(z, y) = 3$. This last case is impossible since an element of G of infinite order fixes a segment of length ≤ 2 in T. This $d(z, y) = 1$, and this contradicts the definition of x since x, which is at distance 2 from y, is the closest point to y fixed by $\phi(Q_{w_{k,i}})$.

Now, suppose that $d(x, y) = 1$. Since Δ_G is bipartite, one can assume, up to composing ϕ by an inner automorphism and permuting x and y, that $x = v$. If $\phi\left(Q_{w_{k,i}}\right)$ is not contained in an extended boundary subgroup of G_v, we get a contradiction thanks to Proposition 2.43, as above. Thus $\phi\left(Q_{w_{k,i}}\right)$ is contained in an extended boundary subgroup of G_v. So $\phi\left(Q_{w_{k,i}}\right)$ has a fixed point z in T such that $d(x,z)=1$, so $d(z,y)=0$ or $d(z,y)=2$. This last case is impossible since z and y are not translates of v and, by definition of a centered splitting, if an element of G of infinite order fixes a segment of length 2 in T, then its endpoints are translates of the central vertex v. As a consequence, $d(z,y)=0$, and this contradicts the definition of x.

Hence, we have proved that $d(x,y)=0$. This concludes the proof of the claim.

We are now ready to complete the proof of the first step. Thanks to the previous claim, we know that $\phi(Q_{w_{k,i}})$ and $\phi(Q_{w_{k,j}})$ have a common fixed point in the Bass-Serre tree T of Δ_G. We will now deduce that $\phi(Q_{v_k})$ is elliptic in T. Let T'' be the tree obtained from T' by collapsing the adjacent vertices $w_{k,i}$ and $w_{k,j}$ to a point, and let u be the new vertex. The group $\phi(Q_u)$ is elliptic in T. One can repeat the previous proof with the set $S'=\{u\}\cup\{v_{k,\ell}\mid \ell\neq i,j\}$ instead of $S=\{v_{k,1},\ldots,v_{k,m}\}$. Since $|S'|=|S|-1<|S|$, a straightforward iteration proves that $\phi(Q_{v_k})$ is elliptic in T. This concludes the proof of the first step.

Step 2. Let Y be a connected component of $\Delta_Q\setminus\mathcal{F}$, and let Q_Y be the fundamental group of Y. Note that Y contains at least one vertex v_k coming from the central vertex v. We distinguish two cases:

(1) either Y does not contain a vertex w different from the vertices v_i coming from the central vertex v. Then Y is reduced to v_k, and we have proved above that $\phi(Q_{v_k})$ is elliptic in the Bass-Serre tree T of Δ_G.
(2) Or Y contains a vertex w different from the vertices v_i. We will prove that w is the only vertex of Y different from the v_i, and that $\phi(Q_{v_k})$ is contained in $\phi(Q_w)$. This will prove that $\phi(Q_Y)=\phi(Q_w)$.

Suppose we are in the situation described in the second point above. The vertices w and v_k are linked by an edge with infinite stabilizer. Let T'' be the Bass-Serre tree of Δ_Q. For convenience, we still denote by w and v_k two adjacent representatives of w and v_k in T'' linked by an edge with infinite stabilizer. We shall prove that $\phi(Q_{v_k})$ is contained in $\phi(Q_w)$.

We have already proved the existence of a subset $I\subset T$ pointwise-fixed by $\phi(Q_{v_k})$. Since $p_{|G_w}$ is inner, $\phi(Q_w)=p(G_w)$ fixes a vertex $y=gw$ of T, and we have $\phi(Q_w)=G_y=gG_wg^{-1}$. Let x be a point of I such that $d(x,y)=d(I,y)$. Recall that, by definition of a centered splitting, if an element of G of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its endpoints are translates of the central vertex v. Therefore, since y is not a translate of v, we have $d(x,y)\leq 1$.

Suppose towards a contradiction that $d(x,y)=1$. Then we can assume without loss of generality that $x=v$. Hence, ϕ induces a non-pinching morphism of finite-by-orbifold groups from Q_{v_k} to G_v. We claim that $\phi\left(Q_{v_k}\right)$ is contained in an extended boundary subgroup of G_v. First, observe that the group $\phi\left(Q_{v_k}\right)$ is infinite, since ϕ is non-pinching. Thus $\phi\left(Q_{v_k}\right)$ is not contained in an extended conical subgroup of G_v. Assume towards a contradiction that $\phi(Q_{v_k})$ is not contained in an extended boundary subgroup of G_v. Then, it follows from Proposition 2.43 that $\chi(Q_{v_k})\geq\chi(G_v)$, with equality if and only if ϕ is an isomorphism. But the complexity decreases as soon as we cut along a loop or pinch a loop, therefore $\chi(G_v)=\chi(Q_{v_k})$ and p sends G_v isomorphically to a conjugate of itself. This contradicts the hypothesis of the lemma, and proves that $\phi(Q_{v_k})$ is contained in an extended boundary subgroup of G_v. Consequently, $\phi(Q_{v_k})$ fixes a point z in T such that $d(x,z)=1$. As a consequence, $z=y$ or $d(z,y)=2$. This last case is impossible since y and z are not translates of v, so $z=y$ and this contradicts the definition of x. Hence, we have proved that $\phi(Q_{v_k})$ fixes the vertex y, whose stabilizer is $\phi(Q_w)$. Thus $\phi(Q_{v_k})\subset\phi(Q_w)$.

In order to prove that $\phi(Q_Y) = \phi(Q_w)$, it is enough to prove that w is the only vertex of Y different from the vertices v_i. Assume towards a contradiction that there is another vertex $w_2 \in Y$ with this property. One can suppose without loss of generality that $d(w, w_2) = 2$ and that w and w_2 are both linked to v_k by an edge with infinite stabilizer (up to replacing v_k by another vertex $v_i \in Y$), because Y is bipartite: the vertices arising from v on the one hand, and the vertices not arising from v on the other hand. We have already shown that $\phi(Q_{v_k}) \subset \phi(Q_{w_1})$ and $\phi(Q_{v_k}) \subset \phi(Q_{w_2})$. Remark, in addition, that Q_{v_k} has an extended boundary subgroup C such that $\phi(C)$ is infinite. Hence, $\phi(Q_{w_1}) \cap \phi(Q_{w_2})$ is infinite. By definition of a centered splitting, if an element of G of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its endpoints are translates of the central vertex v. But w and w_2 are not translates of v. This proves that $w_1 = w_2$ and completes the proof of the lemma. □

PROPOSITION 7.4. *Let G be a finitely generated group. Suppose that G is not a finite-by-orbifold group. Let Δ_G be a centered splitting of G, with central vertex v. Suppose that there exists a non-injective degenerate Δ_G-preretraction $p : G \to G$. Suppose moreover that G has a one-ended subgroup A such that $p_{|A}$ is non-injective. Then, there exist a group H, a splitting Δ_H of H and two morphisms $r : G \to H$ and $j : H \to G$ such that $(G, H, \Delta_G, \Delta_H, r, j)$ is a strict quasi-floor.*

Recall that a Δ_G-preretraction p is said to be degenerate if it does not send the central vertex group G_v isomorphically to a conjugate of itself (see Definition 3.14).

PROOF.
A. The non-pinching case.

Suppose that p is non-pinching on G_v. By hypothesis, p does not send G_v isomorphically to a conjugate of itself, so it follows from Lemma 7.3 that Δ_G has only one vertex w different from v, and that $p(G_v) \subset p(G_w)$. Since p is inner on $H := G_w$, there exists an element $g \in G$ such that $r := \mathrm{ad}(g) \circ p$ is a retraction from G onto H. Let Δ_H be the splitting of H reduced to a point, and let j denote the inclusion of H into G. The tuple $(G, H, \Delta_G, \Delta_H, r, j)$ is a quasi-floor. Moreover, this quasi-floor is strict since, by hypothesis, there exists a one-ended subgroup A of G such that $p_{|A}$, and therefore $r_{|A}$, are non-injective.

B. The pinching case.

Step 1: pinching a maximal set of simple loops.

Let S be a maximal pinched set for (p, Δ_G), let Q be the pinched quotient and let Δ_Q be its pinched decomposition. Denote by $v_1, \ldots, v_n$ the new vertices coming from v (see Figure 8 below). Let $\pi : G \twoheadrightarrow Q$ be the quotient epimorphism. There exists a unique homomorphism $\phi : Q \to G$ such that $p = \phi \circ \pi$. This homomorphism is non-pinching since S is assumed to be maximal.

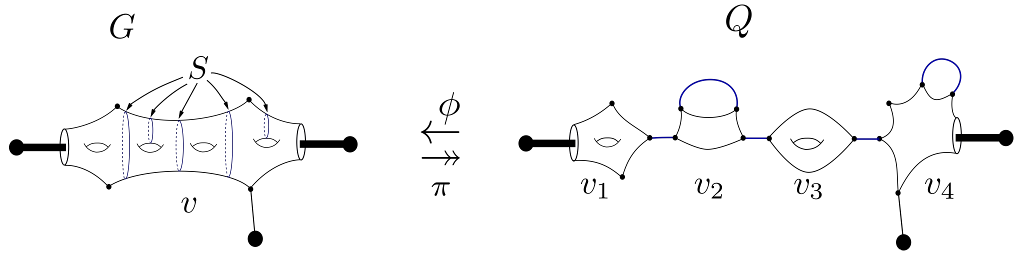


FIGURE 8. Step 1. Edges with infinite stabilizer are depicted in bold.

Let $\mathcal{F}$ be the set of edges of Δ_Q with finite stabilizer. For every $k \in [\![1, n]\!]$, we denote by Y_k the connected component of $\Delta_Q \setminus \mathcal{F}$ containing v_k, and we denote by Q_{Y_k}

its fundamental group. By Lemma 7.3, $\phi(Q_{Y_k})$ is elliptic in the Bass-Serre tree of the splitting Δ_G (in particular, $\phi(Q_{v_k})$ is elliptic). Hence, there exists a vertex w of Δ_G such that $\phi(Q_{Y_k})$ is contained in a conjugate of G_w. If w is unique, we define $w_k := w$. If w is not unique, then we can assume that $w \neq v$, and we define $w_k := w$.

We will construct the group H together with its splitting Δ_H by eliminating the new vertices $v_1, \ldots, v_n$ coming from the central vertex v. We will illustrate each step of the construction in the case of the example pictured above (Figure 8).

Step 2: eliminating orbifolds with non-empty boundary.

Let v_k be a vertex such that the underlying orbifold of Q_{v_k} has non-empty boundary. Then Y_k contains a vertex w different from the vertices $v_1, \ldots, v_n$. By Lemma 7.3, w is the only vertex of Y_k with this property, and we have $\phi(Q_{Y_k}) = \phi\left(Q_w\right) = gG_wg^{-1}$ for some $g \in G$. Since w is unique, we have $w = w_k$ (when viewed as a vertex of Δ_G).

Therefore, the quotient $Q' = Q/N$ of Q by the subgroup N normally generated by

$$\left\{ \ker\left(\phi_{|Q_{Y_k}}\right) \mid \text{the underlying orbifold of } Q_{v_k} \text{ has non-empty boundary} \right\}$$

splits naturally as a graph of groups $\Delta_{Q'}$ obtained by replacing in Δ_Q each subgraph Y_k as above by a new vertex labelled by $\phi(Q_{Y_k}) = \phi(Q_{w_k}) = \phi \circ \pi\left(G_{w_k}\right) = p\left(G_{w_k}\right) = gG_{w_k}g^{-1}$ for some $g \in G$. With a little abuse of notation, this new vertex is still denoted by w_k (see Figure 9 below). Note that the graph of groups $\Delta_{Q'}$ has finite edge groups. Let $\pi' : Q \twoheadrightarrow Q'$ be the quotient epimorphism. There exists a unique homomorphism $\phi' : Q' \to G$ such that $\phi = \phi' \circ \pi'$, hence $p = \phi' \circ \pi' \circ \pi$.

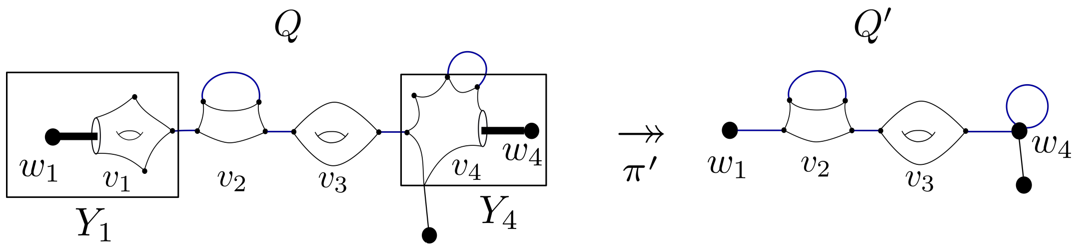


FIGURE 9. Step 2. Edges with infinite stabilizer are depicted in bold. Note that, by construction, $\Delta_{Q'}$ has finite edge groups.

Step 3: eliminating vertices v_k such that $w_k = v$.

Let $k \in [\![1, n]\!]$ be such that $w_k = v$. Note that, by definition of w_k, $\phi'(Q'_{v_k}) = \phi(Q_{v_k})$ is not contained in an extended boundary subgroup of G_v. Since ϕ' is non-pinching on Q'_{v_k} (by maximality of S), and since the complexity of Q'_{v_k} is strictly less than the complexity of G_v, it follows from Proposition 2.43 that $\phi'(Q'_{v_k})$ is contained in an extended conical subgroup of gG_vg^{-1}; in particular, $\phi'(Q'_{v_k})$ is a finite group. As in the previous step, we replace the vertex v_k by a new vertex labelled by $\phi'(Q'_{v_k})$. This new vertex is called x_k (see Figure 10 below). We perform the previous operation for each $k \in [\![1, n]\!]$ such that $w_k = v$. Let Λ be the resulting graph of groups. Call Q'' its fundamental group, and let $\Delta_{Q''} := \Lambda$. Let $\pi'' : Q' \twoheadrightarrow Q''$ be the quotient epimorphism. There exists a unique homomorphism $\phi'' : Q'' \to G$ such that $\phi' = \phi'' \circ \pi''$, hence $p = \phi'' \circ (\pi'' \circ \pi' \circ \pi)$.

Step 4: eliminating the remaining QH vertices.

Denote by V the set of vertices of $\Delta_{Q''}$ coming from the initial pinching of G_v and that have not been treated yet. For each $v_k \in V$, recall that w_k stands for a vertex of Δ_G such that $\phi(Q_{v_k})$ is contained in a conjugate of G_{w_k}. Note that w_k is not the central vertex v of Δ_G (see Step 3).

To complete the proof of Proposition 7.4, it is convenient to adopt a topological point of view. Let X_G be a $K(G, 1)$ obtained as a graph of spaces using, for each vertex or edge w of

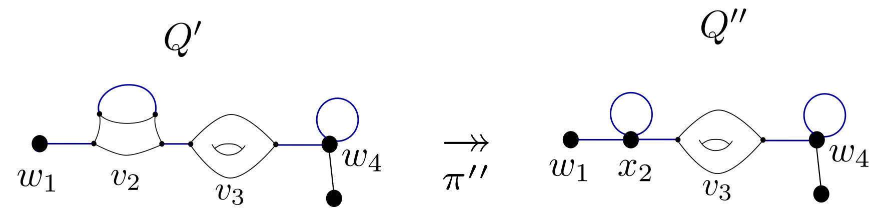


FIGURE 10. Step 3. In this example, $w_2 = v$. The vertex v_2 is replaced by a vertex x_2 labelled by the finite group $\phi'\left(Q'_{v_2}\right)$.

Δ_G, a $K(G_w, 1)$ denoted by X_G^w (see [SW79]). Let $X_{Q''}$ be a $K(Q'', 1)$ obtained in the same way. There exists a morphism of CW-complexes $f : X_{Q''} \to X_G$ inducing $\phi'' : Q'' \to G$ at the level of fundamental groups and such that $f\left(X_{Q''}^{v_k}\right) \subset f\left(X_{Q''}^{w_k}\right)$ for each remaining vertex v_k coming from v, and f induces an homeomophism between $X_{Q''}^w$ and X_G^w for each vertex w that does not come from v. We define an equivalence relation $\sim$ on $X_{Q''}$ by $x \sim y$ if $x = y$, or if $x \in X_{Q''}^{v_k}$, $y \in X_{Q''}^{w_k}$ and $f(x) = f(y)$. Let $g : X_{Q''} \twoheadrightarrow (X_{Q''}/\sim)$ be the quotient map. There exists a unique continuous function $h : (X_{Q''}/\sim) \to X_G$ such that $f = h \circ g$. Hence $\phi'' = h_* \circ g_*$. Note that the homomorphism g_* is not surjective in general. Call H the fundamental group of $X_{Q''}/\sim$, let $j = h_*$ and $r = g_* \circ \pi'' \circ \pi' \circ \pi$, so that $p = j \circ r$. Note that $X_{Q''}/\sim$ naturally has the structure of a graph of spaces, and denote by Δ_H the corresponding splitting of H. We claim that G is a strict quasi-floor over H (see below).

Let us explain the topological construction above from an algebraic point of view in the case of our example. The only remaining vertex coming from the central vertex v is v_3 (see Figure 10). Up to replacing Q'' by $Q''/\langle\langle \ker(\varphi)\rangle\rangle$, where φ stands for the restriction of ϕ'' to the stabilizer Q''_{v_3} of v_3 in Q'', we can assume that ϕ'' is injective on Q''_{v_3}. We know that $\phi''(Q''_{v_3})$ is contained in $gG_{w_3}g^{-1}$ for some $g \in G$, where w_3 is the vertex associated with v_3 defined in Step 1. Moreover, ϕ'' sends Q''_{w_3} isomorphically onto $G_{w_3}^h$ for some $h \in G$. As a consequence, $i := (\phi'')^{-1} \circ \mathrm{ad}(hg^{-1}) \circ \phi'' : {Q''}_{v_3} \to {Q''}_{w_3}$ is a monomorphism. We add an edge e to the graph of groups $\Delta_{Q''}$ between v_3 and w_3 identifying ${Q''}_{v_3}$ with its image $i(Q''_{v_3})$ in Q''_{w_3} (see Figure 11 below). Call H the fundamental group of this graph of groups. In other words, we obtain H by adding a new generator t to Q'', as well as the relation $\mathrm{ad}(t)(x) = i(x)$ for every $x \in Q''_{v_3}$. Last, we collapse the edge e, and we call Δ_H the resulting splitting of H (see Figure 12 below). We define $r : G \to H$ as the composition of $\pi'' \circ \pi' \circ \pi : G \to Q''$ with the natural homomorphism from Q'' to H. Note that r is not surjective in general. Then, we define a morphism j from H to G that extends $\phi'' : Q'' \to G$ by sending t to hg^{-1}. Since $p = \phi'' \circ (\pi'' \circ \pi' \circ \pi)$, we have $p = j \circ r$.

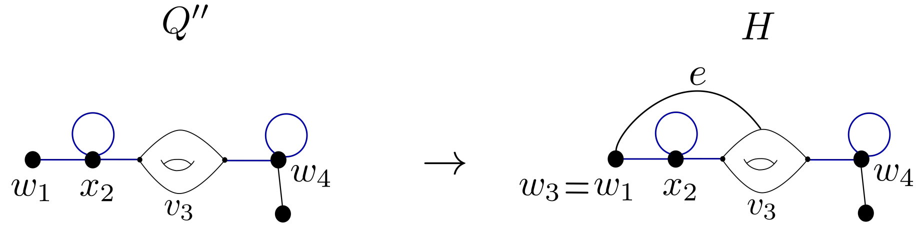


FIGURE 11. In this example, $w_3 = w_1$. We define H as the fundamental group of the graph of groups obtained by adding an edge e to the graph $\Delta_{Q''}$, identifying Q''_{v_3} with $i(Q''_{v_3}) \subset Q''_{w_3}$. The natural homomorphism from Q'' to H is not surjective in general.

It remains to verify that $(G, H, \Delta_G, \Delta_H, r, j)$ is a strict quasi-floor.

- $j \circ r = p$ is Δ_G-related to the identity of G by definition of p.

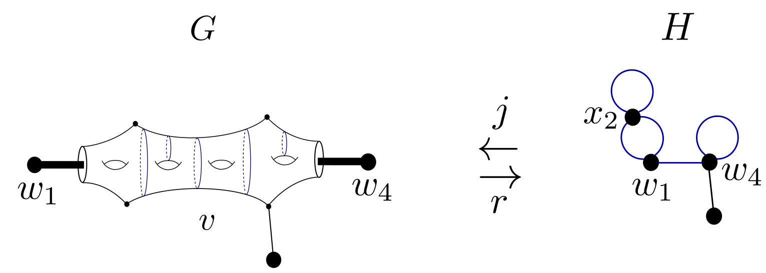


FIGURE 12. After collapsing the edge e, we get the desired splitting Δ_H of H.

- Let V_G be the set of vertices of Δ_G, and let V_H be the set of vertices of Δ_H. Recall that v stands for the central vertex of Δ_G. By construction of H and Δ_H, the homomorphism $r : G \to H$ induces a bijection s between $V_G \setminus \{v\}$ and a subset $V_1 \subset V_H$ such that $r(G_w) = H_{s(w)}$, for every $w \in V_G \setminus \{v\}$.
- Let $V_2 = V_H \setminus V_1$. By construction (see Step 3 above), for every $w \in V_2$, the vertex group H_w is finite and j is injective on H_w.
- By hypothesis, there exists a one-ended subgroup A of G such that $p_{|A}$ is non-injective, with $p = j \circ r$. We claim that r is not injective on A. Assume towards a contradiction that r is injective on A. Then, $r(A)$ is a one-ended subgroup of H. Since j is injective on one-ended subgroups of H, j is injective on $r(A)$. Therefore, $p = j \circ r$ is injective on A. This is a contradiction. Hence, we have $A \cap \ker(r) \neq \{1\}$, which proves that the quasi-floor is strict.

□

Before proving Proposition 6.3, we need an easy lemma.

LEMMA 7.5. *Let G be a group endowed with a splitting over finite groups. Let T denote the Bass-Serre tree associated with this splitting. Let H be a group endowed with a splitting over infinite groups, and let S be the associated Bass-Serre tree. If $p : H \to G$ is a homomorphism injective on edge groups of S, and such that $p(H_v)$ is elliptic in T for every vertex v of S, then $p(H)$ is elliptic in T.*

PROOF. Consider two adjacent vertices v and w in S. Let H_v and H_w be their stabilizers. The group $H_v \cap H_w$ is infinite by hypothesis. Moreover, p is injective on edge groups, thus $p(H_v \cap H_w)$ is infinite. Hence $p(H_v) \cap p(H_w)$ is infinite. Since edge groups of T are finite, $p(H_v)$ and $p(H_w)$ fix necessarily the same unique vertex x of T. As a consequence, for each vertex v of S, $p(H_v)$ fixes x.

Now, let h be an element of H, and let v be any vertex of S. By the previous paragraph, $p(H_v)$ and $p(H_{hv}) = p(h)p(H_v)p(h)^{-1}$ are both contained in G_x. Thus, $p(H_v)$ is contained in the intersection $G_x \cap p(h)^{-1}G_x p(h)$. Since H_v is infinite and p is injective on H_v, the intersection $G_x \cap p(h)^{-1}G_x p(h)$ is infinite as well. Moreover, edge groups of S are finite. It follows that the vertex groups G_x and $p(h)^{-1}G_x p(h)$ are equal. Thus, $p(h)$ belongs to G_x. Therefore, $p(H)$ is contained in G_x. □

We now prove the main result of this section, Proposition 6.3.

PROPOSITION 7.6. *Let G be a finitely generated K-CSA group that does not contain $\mathbb{Z}^2$. Let A be a one-ended factor of G. Suppose that A is not a finite-by-orbifold group. If there exists a non-injective preretraction $p : A \to G$, then there exist a finitely generated group H and two morphisms $r : G \to H$ and $j : H \to G$ such that (G, H, r, j) is a strict quasi-floor.*

PROOF. Let Δ be the $\mathcal{Z}$-JSJ splitting of A, which exists since A is K-CSA and does not contain $\mathbb{Z}^2$. Let Λ be a $\mathcal{F}$-JSJ splitting of G containing a vertex v_A with stabilizer A. Let T be the Bass-Serre tree of Λ.

Step 1. We shall prove that there exists a QH vertex v of Δ such that A_v is not sent isomorphically to a conjugate of itself by p.

If w is a non-QH vertex of Δ, then $p(A_w)$ is conjugate to A_w by definition of a Δ-preretraction. In particular, $p(A_w)$ is elliptic in T. Suppose for the sake of contradiction that each stabilizer A_v of a QH vertex v of Δ is sent isomorphically to a conjugate of itself by p. Then $p(A_v)$ is elliptic in T, for every QH vertex v. Therefore, it follows from Lemma 7.5 above that $p(A)$ is elliptic in T, because p is injective on edge groups of Δ (indeed, each edge group of Δ is contained in a non-QH vertex group of Δ), and T has finite edge groups. Moreover, since p is inner on non-QH vertices of Δ, $p(A)$ is contained in gAg^{-1} for some $g \in G$ (note that there exists at least one non-QH vertex since A is not finite-by-orbifold by hypothesis). Up to composing p by the conjugation by g^{-1}, one can thus assume that p is an endomorphism of A. Now, by Proposition 7.1, p is injective. This is a contradiction. Hence, we have proved that there exists a QH vertex v of Δ such that A_v is not sent isomorphically to a conjugate of itself by p.

Step 2. We shall complete the proof using Proposition 7.4. For this purpose, we shall construct a centered splitting of G, together with an endomorphism of G satisfying the hypotheses of Proposition 7.4.

Note that every stabilizer G_e of an edge e of Λ incident to v_A is elliptic in the splitting Δ of A, as a finite group. First, we refine Λ by replacing the vertex v_A by the splitting Δ of A. The resulting splitting of A is denoted by Λ_2. With a little abuse of notation, we still denote by v the vertex of Λ_2 corresponding to the QH vertex v of Δ defined in the previous step. Then, we collapse to a point every connected component of the complement of $\mathrm{star}(v)$ in Λ_2 (where $\mathrm{star}(v)$ stands for the subgraph of Λ_2 composed of v and all its incident edges); the stabilizer of such a point in the new graph of groups is the fundamental group of the corresponding connected component, viewed as a graph of groups. This new graph of groups, denoted by Λ_3, is non-trivial, since A is not finite-by-orbifold (by hypothesis). Thus, Λ_3 is a centered splitting of G, with central vertex v.

The homomorphism $p : A \to G$ is well-defined on G_v because $G_v = A_v$ is contained in A. Moreover, p restricts to a conjugation on the group of each edge e of Λ_3 incident to v. Indeed, either e is an edge coming from Δ or G_e is a finite subgroup of A; in both cases, $p_{|G_e}$ is a conjugation since p is Δ-related to the inclusion of A into G.

Now, we claim that there exists an endomorphism $q : G \to G$ that coincides with p on $G_v = A_v$ and that coincides with a conjugation on every vertex group of Λ_3 different from G_v. We proceed by induction on the number of edges of the graph of groups Λ_3. It is enough to prove the claim in the case where Λ_3 has only one edge. If $G = A_v *_C B$ with $p_{|C} = \mathrm{ad}(g)$, one defines $q : G \to G$ by $q_{|A_v} = p$ and $q_{|B} = \mathrm{ad}(g)$. If $G = A_v *_C = \langle A_v, t \mid tct^{-1} = \alpha(c), \forall c \in C\rangle$ with $p_{|C} = \mathrm{ad}(g_1)$ and $p_{|\alpha(C)} = \mathrm{ad}(g_2)$, one defines $q : G \to G$ by $q_{|A_v} = p$ and $q(t) = g_2^{-1}tg_1$.

The endomorphism q defined above is Λ_3-related to the identity of G (in the sense of Definition 3.13), by construction. Moreover, q does not send G_v isomorphically to a conjugate of itself, by Step 1. In other words, q is a degenerate Λ_3-preretraction of G. In order to apply Proposition 7.4 to the group G with the splitting Λ_3 and the degenerate Λ_3-preretraction q, we will prove that the restriction of q to the one-ended subgroup A of G is non-injective. In the case where q kills an element of $A_v \subset A$, this is obvious. Now, let us suppose that the restriction of q to A_v is injective. Then q is *a fortiori* non-pinching on A_v, and we claim that $q(A_v)$ is elliptic in T. First, let us observe that the images by q of the extended boundary subgroups of the finite-by-orbifold group A_v are elliptic in T, since q restricts to a conjugacy on these subgroups. Then, by Proposition 2.35 and the paragraph below Proposition 2.35, one can cut the underlying orbifold of A_v into connected components that are elliptic in T via q; but edge groups of T are finite, and q is non-pinching on A_v, so $q(A_v)$ is elliptic in T by Lemma 7.5. Again by Lemma 7.5, $q(A)$ is contained in A (up to conjugacy), so q induces an endomorphism of A. Now, we are ready to find a non-trivial element in $\ker(q) \cap A$. Let Δ' be the splitting of A obtained by collapsing every connected component of the complement of $\mathrm{star}(v)$ in the $\mathcal{Z}$-JSJ splitting Δ of A. With abuse of notation, we still denote by v the vertex of Δ' coming from the vertex v of Δ. The splitting Δ' is centered, with central vertex v, and $q_{|A}$ is an endomorphism of A that does not send A_v isomorphically to a conjugate of itself, by Step 1 and since $q_{|A_v} = p_{|A_v}$. Moreover, $q_{|A}$ is Δ'-related to the identity of A (in the sense of Definition 3.13); indeed, if w is a vertex of Δ' different from v, there exists a vertex $\tilde{w} \in \Lambda_3$ such that A_w is contained in $G_{\tilde{w}}$, and q restricts to a conjugation on $G_{\tilde{w}}$ since q is Λ-related to the identity of G, by construction. So it follows from Lemma 7.3 that Δ' has only one vertex w different from v, and that $q(A_v) \subset q(A_w)$. Since q is inner on A_w, there exists an element $a \in A$ such that $\mathrm{ad}(a) \circ q$ is a retraction from A onto A_w. Let x be an element of A_v that does not belong to A_w. Let $y = \mathrm{ad}(a) \circ q(x)$; we have seen that y lies in A_w, so $\mathrm{ad}(a) \circ q(y) = y$. Hence, $\mathrm{ad}(a) \circ q(xy^{-1}) = 1$, with $xy^{-1} \in A \setminus \{1\}$. We have proved that the restriction of q to A is non-injective. Now, it follows from Proposition 7.4 that there exist a finitely generated group H and two morphisms $r : G \to H$ and $j : H \to G$ such that (G, H, r, j) is a strict quasi-floor. This concludes the proof of the proposition. $\square$

CHAPTER 2

Virtually free groups are almost homogeneous

Abstract. Free groups are known to be homogeneous, meaning that finite tuples of elements which satisfy the same first-order properties are in the same orbit under the action of the automorphism group. We show that virtually free groups have a slightly weaker property, which we call uniform almost-homogeneity: the set of k-tuples which satisfy the same first-order properties as a given k-tuple $\boldsymbol{u}$ is the union of a finite number of $\mathrm{Aut}(G)$-orbits, and this number is bounded independently from $\boldsymbol{u}$ and k. Moreover, we prove that there exists a virtually free group which is not $\exists$-homogeneous. We also prove that all hyperbolic groups are homogeneous in a probabilistic sense.

1. Introduction

Let G be a group and let $k \geq 1$ be an integer. We say that two tuples $\boldsymbol{u} = (u_1, \ldots, u_k)$ and $\boldsymbol{v} = (v_1, \ldots, v_k)$ in G^k have the same *type* if, given any first-order formula $\theta(x_1, \ldots, x_k)$ with k free variables, the statement $\theta(\boldsymbol{u})$ is true in G if and only if the statement $\theta(\boldsymbol{v})$ is true in G. We use the notation $\mathrm{tp}(\boldsymbol{u}) = \mathrm{tp}(\boldsymbol{v})$. Roughly speaking, two tuples have the same type if they are indistinguishable from the point of view of first-order logic. Obviously, two tuples that are in the same orbit under the action of the automorphism group of G have the same type. The group G is termed *homogeneous* if the converse holds.

In [**Nie03**], Nies proved that the free group $\boldsymbol{F}_2$ on two generators is $\exists$-homogeneous. Then, Perin and Sklinos, and independently Ould Houcine, proved that the free groups are homogeneous (see [**PS12**] and [**OH11**]). Moreover, Perin and Sklinos showed in [**PS12**] that the fundamental group of a closed orientable surface of genus ≥ 2 is not homogeneous, using a deep result of Sela. More recently, Byron and Perin gave a complete characterization of torsion-free homogeneous hyperbolic groups, in terms of their JSJ decomposition (see [**DBP19**]).

In this chapter, we are mainly concerned with homogeneity in the class of finitely generated *virtually free groups*, i.e. finitely generated groups with a free subgroup of finite index. Note that there is no *a priori* relation between homogeneity in a group and homogeneity in a finite index subgroup or in a finite extension. In the sequel, all virtually free groups are assumed to be finitely generated.

We prove that all virtually free groups satisfy a slightly weaker form of homogeneity: a group G is *almost-homogeneous* if for any $k \geq 1$ and $\boldsymbol{u} \in G^k$, there exists an integer $N \geq 1$ such that the set of k-tuples of elements of G having the same type as $\boldsymbol{u}$ is the union of N orbits under the action of $\mathrm{Aut}(G)$, and G is *uniformly almost-homogeneous* if N can be chosen independently from $\boldsymbol{u}$ and k.

THEOREM 1.1. *Virtually free groups are uniformly almost-homogeneous.*

REMARK 1.2. In fact, virtually free groups are uniformly $\forall\exists$-almost-homogeneous (see Definition 2.3).

It is worth noting that the work of Perin and Sklinos (see [**PS12**]) shows in fact that the fundamental group of an orientable hyperbolic closed surface is not almost-homogeneous.

In some cases, Theorem 1.1 above can be strengthened. For instance, in a work in progress, we prove that co-Hopfian virtually free groups are homogeneous. As an example, the group $\mathrm{GL}_2(\mathbb{Z})$ is homogeneous. We don't know whether or not virtually free groups are homogeneous in general. However, we construct in Section 5 a virtually free group that is not $\exists$-homogeneous. We build this example by exploiting a phenomenom that is specific to torsion (see paragraph 1 below for further details). In fact, we can even prove that this virtually free group is not $\exists\forall\exists$-homogeneous, using techniques introduced in Chapter 3, and we conjecture that this group is not homogeneous. We refer the reader to Section 4.4 for more details.

Conjecture 1.3. *There exists a non-homogeneous virtually free group.*

An important ingredient of the proof of Theorem 1.1 above is the so-called *shortening argument*, which generalizes to one-ended hyperbolic groups relative to a given subgroup, in the presence of torsion, using results of Guirardel [**Gui08**] and Reinfeldt-Weidmann [**RW14**] (generalizing previous work of Rips and Sela, see [**RS94**]). In particular, the relative co-Hopf property holds for hyperbolic groups with torsion.

Theorem 1.4. *Let G be a hyperbolic group and let H be a finitely generated subgroup of G. Assume that G is one-ended relative to H (see Definition 2.6). Then G is co-Hopfian relative to H, i.e. every monomorphism of G whose restriction to H is the identity is an automorphism of G.*

Remark 1.5. For G torsion-free and H trivial, this result was proved by Sela in [**Sel97**]. For G torsion-free and H infinite, it was proved by Perin in her PhD thesis [**Per08**] (see also [**Per11**] and [**PS12**]). If G has torsion and H is finite, this theorem is due to Moioli (see his PhD thesis [**Moi13**]).

$\exists$-homogeneity. In [**Nie03**], Nies proved that the free group $\boldsymbol{F}_2$ is $\exists$-homogeneous. By contrast, for $n \geq 3$, it is not known whether the free group $\boldsymbol{F}_n$ is $\exists$-homogeneous for $n \geq 3$. In Section 5, we will give an example of a virtually free group that is not $\exists$-homogeneous (and we conjecture that this group is not homogeneous). More precisely, we will prove the following result.

Proposition 1.6. *There exist a virtually free group $G = A *_C B$, with A, B finite, and two elements $x, y \in G$ such that:*

- *there exists a monomorphism $G \hookrightarrow G$ that interchanges x and y (in particular, x and y have the same existential type);*
- *no automorphism of G maps x to y.*

This is a new phenomenon, which does not appear in free groups, as shown by the proposition below (see [**OH11**] Lemma 3.7, or Proposition 5.1 below).

Proposition 1.7. *Let x and y be two elements of the free group $\boldsymbol{F}_n$. The two following statements are equivalent.*

(1) There exists a monomorphism $G \hookrightarrow G$ that sends x to y, and a monomorphism $G \hookrightarrow G$ that sends y to x.
(2) There exists an automorphism of G that sends x to y.

The reason why the previous result holds in free groups is that every isomorphism between the free factors of $\boldsymbol{F}_n$ containing x and y respectively is the restriction of an ambient automorphism of $\boldsymbol{F}_n$. This fact fails in virtually free groups in general.

Generic homogeneity. It is natural to wonder to what extent a given non-homogeneous hyperbolic group is far from being homogeneous. We shall prove that, in the sense of random walks, the $\mathrm{Aut}(G)$-orbit of a tuple in a hyperbolic group G is determined by first-order logic. More precisely, a random tuple in a hyperbolic group has the property that if it has the same type as another tuple, then these two tuples are in the same orbit under the action of the automorphism group. We introduce the following definition.

Definition 1.8. Let G be a group. A k-tuple $u \in G^k$ is said to be *type-determined* if it has the following property: for every k-tuple $v \in G^k$, if u and v have the same type, then they are in the same $\mathrm{Aut}(G)$-orbit.

Let G be a finitely generated group and let μ be a probability measure on G whose support is finite and generates G. An element of G arising from a random walk on G of length n generated by μ is called a *random element* of length n. We define a *random k-tuple* of length n as a k-tuple of random elements of length n arising from k independant random walks. In Section 7, we prove the following result.

THEOREM 1.9. *Fix an integer $k \geq 1$. In a hyperbolic group, the probability that a random k-tuple of length n is type-determined tends to one as n tends to infinity.*

Theorem 1.9 is a consequence of the two following results. Say a tuple u is *rigid* if the group G does not split non-trivially over a virtually cyclic group (finite or infinite) relative to $\langle u \rangle$ (see Definition 2.6).

PROPOSITION 1.10. *Fix an integer $k \geq 1$. In a hyperbolic group, the probability that a random k-tuple of length n is rigid tends to one as n tends to infinity.*

PROPOSITION 1.11. *Fix an integer $k \geq 1$. Let G be a hyperbolic group, and let $u \in G^k$. If u is rigid, then u is type-determined.*

REMARK 1.12. In fact, we shall prove the following stronger result: if $u \in G^k$ is rigid, then it is $\exists$-*type-determined*, meaning that any k-tuple v with the same $\exists$-type as u belongs to the same $\mathrm{Aut}(G)$-orbit.

The first proposition above is proved by Guirardel and Levitt in [**GL19**] (first point of Corollary 6.5), based on a result of Maher and Sisto proved in [**MS17**]. The second proposition will be proved in Section 7.

Strategy for proving almost-homogeneity. In Section 3, we prove that the group $\mathrm{SL}_2(\mathbb{Z})$ is homogeneous (more precisely, it is $\exists$-homogeneous). This example will serve as a model for proving almost-homogeneity of virtually free groups. Recall that $\mathrm{SL}_2(\mathbb{Z})$ splits as $\mathrm{SL}_2(\mathbb{Z}) = A *_C B$ where $A \simeq \mathbb{Z}/4\mathbb{Z}$, $B \simeq \mathbb{Z}/6\mathbb{Z}$ and $C \simeq \mathbb{Z}/2\mathbb{Z}$. In particular, it is a virtually free group. Let us give a brief outline of the proof of the homogeneity of $G = \mathrm{SL}_2(\mathbb{Z})$. Let $u, v \in G^k$ be two k-tuples. Suppose that u and v have the same type.

Step 1. By means of first-order logic, we prove that there exists a monomorphism $\phi : G \hookrightarrow G$ sending u to v, and a monomorphism $\psi : G \hookrightarrow G$ sending v to u.

Step 2. We modify (if necessary) the monomorphism ϕ in order to get an automorphism σ of G that maps u to v.

We shall use a similar approach to prove that virtually free groups are uniformly almost-homogeneous. In Section 4, we generalize the method used by Perin and Sklinos to prove the homogeneity of free groups, and we prove the following result (which can be compared with Step 1 above).

PROPOSITION 1.13. *Let G be a virtually free group, let $k \geq 1$ be an integer and let $u, v \in G^k$. Let U be the maximal one-ended subgroup of G relative to $\langle u \rangle$, and let V be the maximal one-ended subgroup of G relative to $\langle v \rangle$. If u and v have the same type, then*

- *there exists an endomorphism ϕ of G that maps u to v and whose restriction to U is injective,*
- *there exists an endomorphism ψ of G that maps v to u and whose restriction to V is injective.*

In fact, it is enough to suppose that u and v have the same $\forall\exists$-type.

REMARK 1.14. Let us emphasize that Proposition 1.13 together with Proposition 1.7 imply that finitely generated free groups are homogeneous. Indeed, for free groups, the existence of the endomorphisms ϕ and ψ above is equivalent to the existence of an automorphism of G sending u to v, by Proposition 1.7.

In the proof of the homogeneity of $\boldsymbol{F}_n$ and $\mathrm{SL}_2(\mathbb{Z})$, the existence of these endomorphisms ϕ and ψ implies the existence of an automorphism of G that sends u to v. Unfortunately, the counterexample 1.6 above shows that this step fails in the general case. In Section 6, we circumvent this problem and prove that virtually free groups are uniformly almost-homogeneous. The key ingredient is the cocompactness of the Stallings deformation space of a virtually free group (see Proposition 2.9 for a precise statement).

Acknowledgements. It is a pleasure to express my gratitude to my advisor Vincent Guirardel for his precious help and his many comments on previous versions of this paper.

2. Preliminaries

2.1. First-order logic.

For detailed background on first-order logic, we refer the reader to [**Mar02**].

A *first-order formula* in the language of groups is a finite formula using the following symbols: $\forall$, $\exists$, $=$, $\wedge$, $\vee$, $\Rightarrow$, $\neq$, 1 (standing for the identity element), $^{-1}$ (standing for the inverse), $\cdot$ (standing for the group multiplication) and variables $x, y, g, z \dots$ which are to be interpreted as elements of a group. A variable is *free* if it is not bound by any quantifier $\forall$ or $\exists$. A *sentence* is a formula without free variables. A $\forall\exists$*-formula* is a formula of the form $\theta(\boldsymbol{x}) : \forall \boldsymbol{y} \exists \boldsymbol{z} \varphi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. An *existential formula* (or $\exists$*-formula*) is a formula in which the symbol $\forall$ does not appear.

Given a formula $\theta(\boldsymbol{x})$ with $k \geq 0$ free variables, and a k-tuple $\boldsymbol{u}$ of elements of a group G, we say that $\boldsymbol{u}$ *satisfies* $\theta(\boldsymbol{x})$ if the statement $\theta(\boldsymbol{u})$ is true in G.

Definition 2.1. Let G be a group. We say that two k-tuples $\boldsymbol{u}$ and $\boldsymbol{v}$ of elements of G have the same *type* if, for every first-order formula $\theta(\boldsymbol{x})$ with k free variables, the statement $\theta(\boldsymbol{u})$ is true in G if and only if the statement $\theta(\boldsymbol{v})$ is true in G. We use the notation $\mathrm{tp}(\boldsymbol{u}) = \mathrm{tp}(\boldsymbol{v})$. In the same way, we say that two tuples have the same $\exists$*-type* (resp. $\forall\exists$*-type*) if they satisfy the same $\exists$-formulas (resp. $\forall\exists$-formulas). We use the notation $\mathrm{tp}_{\exists}(\boldsymbol{u}) = \mathrm{tp}_{\exists}(\boldsymbol{v})$ (resp. $\mathrm{tp}_{\forall\exists}(\boldsymbol{u}) = \mathrm{tp}_{\forall\exists}(\boldsymbol{v})$).

Definition 2.2. Let G be a group. We say that G is *homogeneous* if, for every integer k and for all k-tuples $\boldsymbol{u}$ and $\boldsymbol{v}$ having the same type, there exists an automorphism σ of G sending $\boldsymbol{u}$ to $\boldsymbol{v}$. In the same way, we define $\exists$*-homogeneity* by considering only $\exists$-formulas, and $\forall\exists$*-homogeneity* by considering only $\forall\exists$-formulas.

Definition 2.3. Let G be a group. We say that G is *almost-homogeneous* if for every integer $k \geq 1$ and for every k-tuple $\boldsymbol{u} \in G^k$,

$$|\{v \in G^k \mid \mathrm{tp}(\boldsymbol{u}) = \mathrm{tp}(\boldsymbol{v})\}/\mathrm{Aut}(G)| < \infty.$$

We say that G is *uniformly almost-homogeneous* if there exists an integer $N \geq 1$ such that for every integer $k \geq 1$ and every k-tuple $\boldsymbol{u} \in G^k$,

$$|\{v \in G^k \mid \mathrm{tp}(\boldsymbol{u}) = \mathrm{tp}(\boldsymbol{v})\}/\mathrm{Aut}(G)| \leq N.$$

In the same way, we define (uniform) $\exists$*-almost-homogeneity* by considering only $\exists$-formulas, and (uniform) $\forall\exists$*-almost-homogeneity* by considering only $\forall\exists$-formulas.

Remark 2.4. Note that a group is homogeneous if and only if it is uniformly almost-homogeneous with $N = 1$.

2.2. Properties relative to a subgroup.

Let G be a finitely generated group, and let $\mathcal{H}$ be a collection of subgroups of G.

Definition 2.5. An *action* of the pair $(G, \mathcal{H})$ on a tree T is an action of G on T such that every $H \in \mathcal{H}$ fixes a point of T. The tree T (or the quotient graph of groups by G) is called a *splitting of* $(G, \mathcal{H})$, or a *splitting of* G *relative to* $\mathcal{H}$. The action is said to be *trivial* if G fixes a point of T.

Definition 2.6. We say that G is *one-ended relative to* $\mathcal{H}$ if G does not split as an amalgamated product $A *_C B$ or as an HNN extension $A*_C$ such that C is finite and every $H \in \mathcal{H}$ is contained in a conjugate of A or B. In other words, G is one-ended relative to $\mathcal{H}$ if any action of the pair $(G, \mathcal{H})$ on a tree with finite edge stabilizers is trivial.

We say that G *does not split non-trivially relative to* $\mathcal{H}$ if any action of $(G, \mathcal{H})$ on a tree is trivial (in particular, G is one-ended relative to $\mathcal{H}$).

Definition 2.7. The group G is said to be *co-Hopfian relative to* a subgroup $H \subset G$ if every monomorphism $\phi : G \hookrightarrow G$ that coincides with the identity on H is an automorphism of G.

2.3. Virtually free groups and Stallings splittings.

Let G be a finitely generated group. Under the hypothesis that there exists a constant K such that every finite subgroup of G has order less than K, Linnell proved in [**Lin83**] that G splits as a finite graph of groups with finite edge groups and all of whose vertex groups are finite or one-ended. This result applies in particular to hyperbolic groups. Such a splitting is called a *Stallings splitting (or tree)* of G. A Stallings splitting is not unique in general, but the conjugacy classes of one-ended vertex groups are the same in all Stallings splittings of G, and the conjugacy classes of finite vertex groups are the same in all reduced Stallings splittings of G. A one-ended subgroup of G that appears as a vertex group of a Stallings splitting is called a *one-ended factor* of G.

A classical result asserts that a finitely generated group G is virtually free if and only if it splits as a finite graph of finite groups. In other words, G is virtually free if and only if there is no one-ended vertex group in its Stallings splittings.

The *Stallings deformation space* of G, denoted by $\mathcal{D}(G)$, is the set of Stallings trees of G up to equivariant isometry.

Given a collection $\mathcal{H}$ of finitely generated subgroups of G, Linnel's proof can be adapted in order to prove the following result: the pair $(G, \mathcal{H})$ splits as a finite graph of groups with finite edge groups such that each vertex group G_v is finite or one-ended relative to the collection $\mathcal{H}_v = \{H \subset G_v \mid \exists H' \in \mathcal{H}, \exists g \in G, H = gH'g^{-1}\}$ (see Definition 2.6). Such a splitting is called a *Stallings splitting of G relative to $\mathcal{H}$*. Note that if $\mathcal{H} = \{H\}$ with H infinite, there exists a unique vertex group containing H, called the *one-ended factor of G relative to $\mathcal{H}$* (or H).

Let T be a Stallings tree of G, and let $e = [v, w]$ be an edge of T. Suppose that $G_v = G_e$ and that v and w are in distinct orbits. Collapsing every edge in the orbit of e to a point produces a new Stallings tree T' of G. We say that T' is obtained from T by an *elementary collapse*, and that T is obtained from T' by an *elementary expansion*. This elementary expansion of T' introduces a new vertex group G_v by using the isomorphism $G_w \simeq G_w *_{G_v} G_v$. These two operations are called *elementary deformations*. A *slide move* is defined as an elementary expansion followed by an elementary collapse. A Stallings tree T of G is said to be *reduced* if there is no edge of the form $e = [v, w]$ with $G_v = G_e$, i.e. if one cannot perform any elementary collapse in T. Two Stallings trees of G are connected by a sequence of elementary deformations. Two reduced Stallings trees of G are connected by a sequence of slide moves. A vertex of T is called *redundant* if it has degree 2. The tree T is called *non-redundant* if every vertex is non-redundant.

The following result is classical in the Bass-Serre theory of group actions on trees, see for instance Lemmas 2.20 and 2.22 in [**DG11**], and Definition 2.19 in [**DG11**] (definition of an isomorphism of graphs of groups).

Proposition 2.8. *Let T and T' be two Stallings trees of G. The two following assertions are equivalent.*

1. *The quotient graphs of groups T/G and T'/G are isomorphic.*
2. *There exist an automorphism σ of G and a σ-equivariant isometry $f : T \to T'$.*

In the latter case, we use the notation $T' = T^\sigma$. This is not ambiguous since we consider elements in $\mathcal{D}(G)$ up to equivariant isometry.

We shall need the following proposition that claims, in a sense, that $\mathcal{D}(G)$ is cocompact under the action of $\mathrm{Aut}(G)$.

Proposition 2.9. *Let G be a virtually free group. There exist finitely many trees $T_1, \ldots, T_n$ in $\mathcal{D}(G)$ such that, for every non-redundant tree $T \in \mathcal{D}(G)$, there exist an automorphism σ of G and an integer $1 \leq k \leq n$ such that $T = T_k^\sigma$.*

Before proving this proposition, we need two lemmas.

LEMMA 2.10. *Let G be a virtually free group. There exist finitely many reduced trees $T_1, \dots, T_n$ in $\mathcal{D}(G)$ such that, for every reduced tree $T \in \mathcal{D}(G)$, there exists an integer $1 \leq k \leq n$ such that $T = T_k^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$.*

PROOF. Given two reduced Stallings trees T and T' of G, one can pass from T to T' by a sequence of slide moves. Consequently, all reduced Stallings trees of G have the same number of orbits of vertices, say p, and the same number of orbits of edges, say q. Moreover, all reduced Stallings trees have the same vertex groups. Let r be the maximal order of such a vertex group. Now, observe that there are only finitely many isomorphism classes of graphs of finite groups with p vertices, q edges, and whose vertex groups have order $\leq r$. By Proposition 2.8, it is sufficient to conclude. $\square$

LEMMA 2.11. *Let G be a virtually free group, and let T be a reduced Stallings tree of G. There are only finitely many non-redundant Stallings trees that can be obtained from T by a sequence of elementary expansions.*

PROOF. We claim that for any sequence of Stallings trees $(T_k)_{k\in\mathbb{N}}$ such that $T_0 = T$ and such that T_{k+1} is obtained from T_k by an elementary expansion, the tree T_k is redundant for k sufficiently large. This is enough to conclude. Indeed, given any Stallings tree T' of G, there are only finitely many Stallings trees that can be obtained from T' by an elementary expansion, so the lemma follows from the previous claim combined with König's lemma.

We now prove the claim. Let us prove that for any integer $m \geq 1$, one can find an integer k such that T_k/G contains a vertex path of length $\geq m$ in which each vertex has degree 2. Let V_k denote the set of vertices of T_k/G. For each vertex $v \in V_k$, we denote by $\deg(v)$ the degree of v in T_k/G. We make the following observation: the sum $S_k = \sum_{v\in V_k}(\deg(v) - 2)$ is constant along the sequence (T_k), see Figure 1 below.

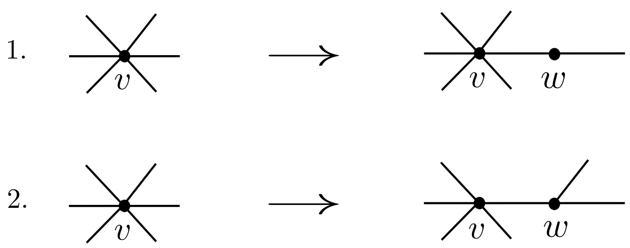


FIGURE 1. The vertex v has degree 6 in T_k/G. In the first case, the elementary expansion gives rise to a vertex w of degree 2 in T_{k+1}/G. In the second case, the elementary expansion gives rise to a vertex w of degree 3 in T_{k+1}/G, and the degree of v decreases to 5. In both cases, the sums (6-2)+(2-2) and (5-2)+(3-2) are equal to 4.

Moreover, the number of vertices of degree 1 in T_k/G is bounded by the number of conjugacy classes of maximal finite subgroups of G. It follows that the number of vertices of degree ≥ 3 in T_k/G is bounded independently from k. Therefore, for any $m \geq 1$, there exists an integer k and a vertex path $[v_1, v_2, \dots, v_m]$ in T_k/G such that, for every $\ell \in [\![1, m]\!]$, v_ℓ has degree 2 in T_k/G and $G_{v_{\ell+1}} \subset G_{v_\ell}$. Now, taking for m the maximal order of a finite subgroup of G, one has necessarily $G_{v_\ell} = G_{v_{\ell+1}}$ for some $\ell \in [\![1, m]\!]$. Hence, each preimage of v_ℓ in T_k has degree 2, so T_k is redundant. $\square$

We can now prove Proposition 2.9.

PROOF. Let $T_1, \dots, T_n$ be the reduced Stallings trees of G given by Lemma 2.10. For each T_k in this finite set, Lemma 2.11 provides us with a finite collection of non-redundant trees $T_{k,1}, \dots, T_{k,r_k}$. Let $\mathcal{T}$ be the union of these sets $\{T_{k,1}, \dots, T_{k,r_k}\}$, for k from 1 to n.

Let T be a non-redundant tree in $\mathcal{D}(G)$. We claim that T belongs to the orbit of an element of $\mathcal{T}$ under the action of $\mathrm{Aut}(G)$. First, note that T collapses onto a reduced Stallings tree T'. In other words, T is obtained from T' by a sequence of elementary expansions. By Lemma 2.10, $T' = T_k^\sigma$ for some automorphism σ of G and some $k \in [\![1, n]\!]$. Thanks to Lemma 2.11, $T = T_{k,i}^\sigma$ for some $i \in [\![1, r_k]\!]$. □

2.4. The JSJ decomposition and the modular group.

JSJ decompositions first appeared in 3-dimensional topology with the theory of the characteristic submanifold by Jaco and Shalen (see [**JS79**]) and by Johannson (see [**Joh79**]). Sela then adapted the topological ideas from [**JS79**] and [**Joh79**] to torsion-free hyperbolic groups in [**Sel97**] (see also [**RS97**] and [**Bow98**]). Constructions of JSJ decompositions were given in more general settings by many authors.

2.4.1. *The canonical JSJ splitting of a hyperbolic group over $\mathcal{Z}$ relative to a finitely generated subgroup.*

We denote by $\mathcal{Z}$ the class of groups that are either finite or virtually cyclic with infinite center. Let G be a hyperbolic group, and let H be a finitely generated subgroup of G. Suppose that G is one-ended relative to H (see Definition 2.6). In [**GL17**], Guirardel and Levitt construct a splitting of G relative to H (see Definition 2.5) called the canonical JSJ splitting of G over $\mathcal{Z}$ relative to H. In the sequel, we refer to this decomposition as the *$\mathcal{Z}$-JSJ splitting of G relative to H*. This tree T enjoys particularly nice properties and is a powerful tool for studying the pair (G, H). Before giving a description of T, let us recall some basic facts about hyperbolic 2-dimensional orbifolds.

A compact connected 2-dimensional orbifold with boundary $\mathcal{O}$ is said to be *hyperbolic* if it is equipped with a hyperbolic metric with totally geodesic boundary. It is the quotient of a closed convex subset $C \subset \mathbb{H}^2$ by a proper discontinuous group of isometries $G_{\mathcal{O}} \subset \mathrm{Isom}(\mathbb{H}^2)$. We denote by $p : C \to \mathcal{O}$ the quotient map. By definition, the orbifold fundamental group$\pi_1(\mathcal{O})$ of $\mathcal{O}$ is $G_{\mathcal{O}}$. We may also view $\mathcal{O}$ as the quotient of a compact orientable hyperbolic surface with geodesic boundary by a finite group of isometries. A point of $\mathcal{O}$ is *singular* if its preimages in C have non-trivial stabilizer. A *mirror* is the image by p of a component of the fixed point set of an orientation-reversing element of $G_{\mathcal{O}}$ in C. Singular points not contained in mirrors are *conical points*; the stabilizer of the preimage in $\mathbb{H}^2$ of a conical point is a finite cyclic group consisting of orientation-preserving maps (rotations). The orbifold $\mathcal{O}$ is said to be *conical* if it has no mirror.

Definition 2.12. A group G is called a *finite-by-orbifold group* if it is an extension

$$1 \to F \to G \to \pi_1(\mathcal{O}) \to 1$$

where $\mathcal{O}$ is a compact connected hyperbolic conical 2-orbifold, possibly with totally geodesic boundary, and F is an arbitrary finite group called the *fiber*. We call an *extended boundary subgroup* of G the preimage in G of a boundary subgroup of the orbifold fundamental group $\pi_1(\mathcal{O})$ (for an indifferent choice of regular base point). We define in the same way *extended conical subgroups.*

Definition 2.13. A vertex v of a graph of groups is said to be *quadratically hanging* (denoted by QH) if its stabilizer G_v is a finite-by-orbifold group $1 \to F \to G \to \pi_1(\mathcal{O}) \to 1$ such that $\mathcal{O}$ has non-empty boundary, and such that any incident edge group is finite or contained in an extended boundary subgroup of G. We also say that G_v is QH.

Definition 2.14. Let G be a hyperbolic group, and let H be a finitely generated subgroup of G. Let T be the $\mathcal{Z}$-JSJ decomposition of G relative to H. A vertex group G_v of T is said to be *rigid* if it is elliptic in every splitting of G over $\mathcal{Z}$ relative to H.

The following proposition is crucial (see Section 6 of [**GL17**], Theorem 6.5 and the paragraph below Remark 9.29). We keep the same notations as in the previous definition.

Proposition 2.15. *If G_v is not rigid, i.e. if it fails to be elliptic in some splitting of G over $\mathcal{Z}$ relative to H, then G_v is quadratically hanging.*

Proposition 2.16 below summarizes the properties of the $\mathcal{Z}$-JSJ decomposition relative to H that will be useful in the sequel.

Proposition 2.16. *Let G be a hyperbolic group, and let H be a finitely generated subgroup of G. Suppose that G is one-ended relative to H. Let T be its $\mathcal{Z}$-JSJ decomposition.*

- *The tree T is bipartite: every edge joins a vertex carrying a maximal virtually cyclic group to a vertex carrying a non-virtually-cyclic group.*
- *The action of G on T is acylindrical in the following strong sense: if an element $g \in G$ of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its midpoint has virtually cyclic stabilizer.*
- *Let v be a vertex of T, and let e, e' be two distinct edges incident to v. If G_v is not virtually cyclic, then the group $\langle G_e, G_{e'} \rangle$ is not virtually cyclic.*
- *There are two kinds of vertices of T carrying a non-virtually cyclic group: rigid ones, and QH ones. If v is a QH vertex of T, every edge group G_e of an edge e incident to v coincides with an extended boundary subgroup of G_v. Moreover, given any extended boundary subgroup B of G_v, there exists a unique incident edge e such that $G_e = B$.*

2.4.2. *The modular group.*

Definition 2.17. Let G be a hyperbolic group and let H be a finitely generated subgroup of G. Suppose that G is one-ended relative to H. We denote by $\mathrm{Aut}_H(G)$ the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms whose restriction to H is the conjugacy by an element of G.

The *modular group* $\mathrm{Mod}_H(G)$ of G relative to H is the subgroup of $\mathrm{Aut}_H(G)$ consisting of all automorphisms σ satisfying the following conditions:

- the restriction of σ to each non-QH vertex group of the $\mathcal{Z}$-JSJ splitting of G relative to H is the conjugacy by an element of G,
- the restriction of σ to each finite subgroup of G is the conjugacy by an element of G,
- σ acts trivially on the underlying graph of the $\mathcal{Z}$-JSJ splitting relative to H.

Remark 2.18. Rather than defining $\mathrm{Mod}_H(G)$ as above by imposing conditions on the action on vertex groups, one could define it by giving generators: twists around edges, and certain automorphisms of vertex groups (see for example [**Per11**] and [**RW14**]). These two definitions yield the same subgroup of $\mathrm{Aut}_H(G)$, up to finite index. We refer the reader to [**GL15**] Section 5 for further discussion about this issue.

The following result is an immediate consequence of Theorem 4.6 in [**GL15**].

Theorem 2.19. *The modular group $\mathrm{Mod}_H(G)$ has finite index in $\mathrm{Aut}_H(G)$.*

Proof. According to Theorem 4.6 in [**GL15**], the modular group $\mathrm{Mod}_H(G)/\mathrm{Inn}(G)$ contains a subgroup, denoted by $\mathrm{Out}^1(G, \{H\}^{(t)})$ in [**GL15**], whose index is finite in the group $\mathrm{Aut}_H(G)/\mathrm{Inn}(G)$, denoted by $\mathrm{Out}^0(G, \{H\}^{(t)})$ in [**GL15**]. $\square$

2.5. Related homomorphisms and preretractions.

In the sequel, we denote by $\mathrm{ad}(g)$ the inner automorphism $h \mapsto ghg^{-1}$.

Definition 2.20 (Related homomorphisms). Let G be a hyperbolic group and let H be a finitely generated subgroup of G. Assume that G is one-ended relative to H. Let G' be a group. Let Δ be the $\mathcal{Z}$-JSJ splitting of G relative to H. Let ϕ and ϕ' be two homomorphisms from G to G'. We say that ϕ and ϕ' are *$\mathcal{Z}$-JSJ-related* or Δ-related if the two following conditions hold:

- for every non-QH vertex v of Δ, there exists an element $g_v \in G'$ such that
$$\phi'_{|G_v} = \mathrm{ad}(g_v) \circ \phi_{|G_v};$$
- for every finite subgroup F of G, there exists an element $g \in G'$ such that
$$\phi'_{|F} = \mathrm{ad}(g) \circ \phi_{|F}.$$

DEFINITION 2.21 (Preretraction). Let G be a hyperbolic group, and let H be a subgroup of G. Assume that G is one-ended relative to H. Let Δ be the $\mathcal{Z}$-JSJ splitting of G relative to H. A *$\mathcal{Z}$-JSJ-preretraction* or *Δ-preretraction* of G is an endomorphism of G that is Δ-related to the identity map. More generally, if G is a subgroup of a group G', a preretraction from G to G' is a homomorphism that is Δ-related to the inclusion of G into G'.

The following easy lemma shows that being Δ-related can be expressed in first-order logic.

LEMMA 2.22. *Let G be a hyperbolic group and let H be a finitely generated subgroup of G. Assume that G is one-ended relative to H. Let G' be a group. Let Δ be the canonical JSJ splitting of G over $\mathcal{Z}$ relative to H. Let $\{g_1, \ldots, g_n\}$ be a generating set of G. There exists an existential formula $\theta(x_1, \ldots, x_{2n})$ with $2n$ free variables such that, for every $\phi, \phi' \in \mathrm{Hom}(G, G')$, ϕ and ϕ' are Δ-related if and only if G' satisfies $\theta(\phi(g_1), \ldots, \phi(g_n), \phi'(g_1), \ldots, \phi'(g_n))$.*

PROOF. First, remark that there exist finitely many (say $p \geq 1$) conjugacy classes of finite subgroups of QH vertex groups of Δ (indeed, a QH vertex group possesses finitely many conjugacy classes of finite subgroups, and Δ has finitely many vertices). Denote by $F_1, \ldots, F_p$ a system of representatives of those conjugacy classes. Denote by $R_1, \ldots, R_m$ the non-QH vertex groups of Δ. Remark that these groups are finitely generated since G and the edge groups of Δ are finitely generated. Denote by $\{A_i\}_{1 \leq i \leq p+m}$ the union of $\{F_i\}_{1 \leq i \leq p}$ and $\{R_i\}_{1 \leq i \leq m}$. For every $i \in [\![1, m+p]\!]$, let $\{a_{i,1}, \ldots, a_{i,k_i}\}$ be a finite generating set of A_i. For every $i \in [\![1, m+p]\!]$ and $j \in [\![1, k_i]\!]$, there exists a word $w_{i,j}$ in n letters such that $a_{i,j} = w_{i,j}(g_1, \ldots, g_n)$. Let
$$\theta(x_1, \ldots, x_{2n}) : \exists u_1 \ldots \exists u_m \bigwedge_{i=1}^{m+p} \bigwedge_{j=1}^{k_i} w_{i,j}(x_1, \ldots, x_n) = u_i w_{i,j}(x_{n+1}, \ldots, x_{2n}) {u_i}^{-1}.$$
Since $\phi(a_{i,j}) = w_{i,j}(\phi(g_1), \ldots, \phi(g_n))$ and $\phi'(a_{i,j}) = w_{i,j}(\phi'(g_1), \ldots, \phi'(g_n))$ for every $i \in [\![1, m+p]\!]$ and $j \in [\![1, k_i]\!]$, the homomorphisms ϕ and ϕ' are Δ-related if and only if the sentence $\theta(\phi(g_1), \ldots, \phi(g_n), \phi'(g_1), \ldots, \phi'(g_n))$ is satisfied by G'. $\square$

The proof of the following lemma is identical to that of Proposition 7.1 in Chapter 1.

LEMMA 2.23. *Let G be a hyperbolic group. Suppose that G is one-ended relative to a subgroup H. Let Δ be the $\mathcal{Z}$-JSJ splitting of G relative to H. Let ϕ be a preretraction of G. If ϕ sends every QH group isomorphically to a conjugate of itself, then ϕ is injective.*

2.6. Centered graph of groups.

DEFINITION 2.24 (Centered graph of groups). A graph of groups over $\mathcal{Z}$, with at least two vertices, is said to be centered if the following conditions hold:

- the underlying graph is bipartite, with a QH vertex v such that every vertex different from v is adjacent to v;
- every stabilizer G_e of an edge incident to v coincides with an extended boundary subgroup or with an extended conical subgroup of G_v (see Definition 2.12);
- given any extended boundary subgroup B, there exists a unique edge e incident to v such that G_e is conjugate to B in G_v;

- if an element of infinite order fixes a segment of length ≥ 2 in the Bass-Serre tree of the splitting, then this segment has length exactly 2 and its endpoints are translates of v.

The vertex v is called the central vertex.

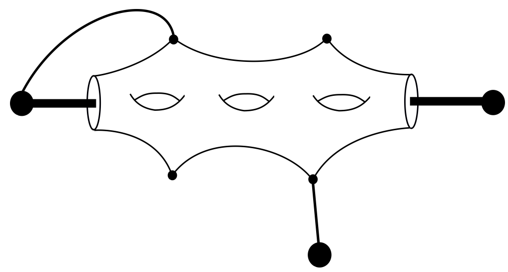


FIGURE 2. A centered graph of groups. Edges with infinite stabilizer are depicted in bold.

REMARK 2.25. Infinite edge groups in a centered graph of groups are of the form $F \rtimes \mathbb{Z}$, with F the maximal normal finite subgroup of the finite-by-orbifold group G_v. Indeed, an infinite edge group G_e coincides with an extended boundary subgroup of G_v, by the second condition in the previous definition.

DEFINITION 2.26 (Related homomorphisms). Let G and G' be two groups. Suppose that G possesses a centered splitting Δ, with central vertex v. Let ϕ and ϕ' be two homomorphisms from G to G'. We say that ϕ and ϕ' are Δ-related if the two following conditions hold:

- for every vertex $w \neq v$, there exists an element $g_w \in G'$ such that
$$\phi'_{|G_w} = \mathrm{ad}(g_w) \circ \phi_{|G_w};$$
- for every finite subgroup F of G, there exists an element $g \in G'$ such that
$$\phi'_{|F} = \mathrm{ad}(g) \circ \phi_{|F}.$$

DEFINITION 2.27 (Preretraction). Let G be a hyperbolic group, and let Δ be a centered splitting of G. Let v be the central vertex of Δ. An endomorphism ϕ of G is called a preretraction if it is Δ-related to the identity of G in the sense of the previous definition. A preretraction is said to be non-degenerate if if does not send G_v isomorphically to a conjugate of itself.

A set of disjoint simple closed curves S on a conical 2-orbifold is said to be *essential* if its elements are non null-homotopic, two-sided, non boundary-parallel, pairwise non parallel, and represent elements of infinite order (in other words, no curve of S circles a singularity). Let G be a group with a centered splitting Δ, with central vertex v. The stabilizer G_v of v is a conical finite-by-orbifold group
$$F \hookrightarrow G_v \overset{q}{\twoheadrightarrow} \pi_1(\mathcal{O}).$$
Let ϕ be an endomorphism of G. Let S be an essential set of curves on $\mathcal{O}$. An element $\alpha \in S$ is said to be *pinched* by ϕ if $\phi\left(q^{-1}(\alpha)\right)$ is finite. The set S is called a *maximal pinched set* for ϕ if each element of S is pinched by ϕ, and if S is maximal for this property. Note that S may be empty.

The proof of the following lemma is identical to that of Lemma 7.3 in Chapter 1.

LEMMA 2.28. *Let G be a hyperbolic group, and let Δ be a centered splitting of G. Let v be the central vertex of Δ and let ϕ be a non-degenerate Δ-preretraction of G. Let S be a maximal pinched set for ϕ and let $\Delta(S)$ be the splitting of G obtained from Δ by replacing* *the vertex v by the splitting of G_v dual to S. Let $v_1, \ldots, v_n$ be the vertices of $\Delta(S)$ coming from the splitting of G_v. We denote by $\mathcal{E}$ the set of edges of $\Delta(S)$ corresponding to the maximal pinched set S, and we denote by $\mathcal{F}$ the set of edges of $\Delta(S)$ whose stabilizer is finite. Let W be a connected component of $\Delta(S) \setminus (\mathcal{E} \cup \mathcal{F})$, and let G_W be its fundamental group. Then $\phi(G_W)$ is elliptic in the Bass-Serre tree T_Δ of Δ. Moreover, W contains at most one vertex w different from the vertices v_k coming from the central vertex v. If it contains one, then $\phi(G_W) = \phi(G_w)$. See Figure 3 below.*

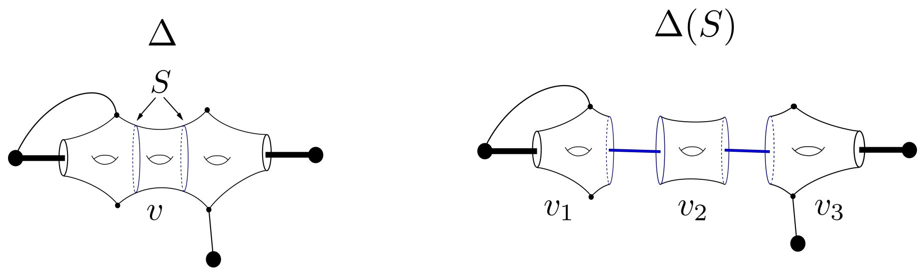


FIGURE 3. On the left, the centered splitting Δ of G. On the right, the splitting $\Delta(S)$ of G obtained from Δ by replacing the central vertex v by the splitting of G_v dual to S.

2.7. The shortening argument.

Let Γ and G be two hyperbolic groups. A sequence of homomorphisms $(\phi_n : G \to \Gamma)_{n\in\mathbb{N}}$ is said to be *stable* if, for any $g \in G$, either $\phi_n(g) = 1$ for almost all n or $\phi_n(g) \neq 1$ for almost all n. The *stable kernel* of the sequence is defined as $\{g \in G \mid \phi_n(g) = 1 \text{ for almost all } n\}$.

Let S denote a finite generating set of G, and let (X, d) denote the Cayley graph of Γ (for a given finite generating set). For any $\phi \in \mathrm{Hom}(G, \Gamma)$, we define the *length* of ϕ as $\ell(\phi) = \max_{s\in S} d(1, \phi(s))$.

Let H be a finitely generated subgroup of G. Assume that G is one-ended relative to H (see Definition 2.6). The *shortening argument* (Proposition 2.33 below) asserts that, given a stable sequence of pairwise distinct homomorphisms $(\phi_n : G \to \Gamma)_{n\in\mathbb{N}}$ such that, for every n, ϕ_n coincides with ϕ_0 on H up to conjugacy by an element of Γ,

- either the sequence has non-trivial stable kernel,
- or one can shorten ϕ_n for every n large enough, meaning that there exists a modular automorphism $\sigma_n \in \mathrm{Aut}_H(G)$ and an element $\gamma_n \in \Gamma$ such that

$$\ell(\mathrm{ad}(\gamma_n) \circ \phi_n \circ \sigma_n) < \ell(\phi_n).$$

This result has strong consequences on the structure of $\mathrm{Hom}(G, \Gamma)$ (see Theorems 2.34 and 2.38 below). In particular, when $G = \Gamma$, the shortening argument implies that G is co-Hopfian relative to H (see Definition 2.7).

Let $\psi : H \to \Gamma$ be a homomorphism. In the sequel, we denote by $E(\psi)$ the subset of $\mathrm{Hom}(G, \Gamma)$ composed of all homomorphisms which coincide with ψ on H up to conjugation by an element of Γ. The group $\Gamma \times \mathrm{Aut}_H(G)$ acts on $E(\psi)$ by $(\gamma, \sigma) \cdot \phi = \mathrm{ad}(\gamma) \circ \phi \circ \sigma$. We say that a homomorphism $\phi \in E(\psi)$ is *short* (with respect to ψ and H) if it has minimal length among its orbit under the action of $\Gamma \times \mathrm{Aut}_H(G)$.

Recall that an action of the pair (G, H) on a tree T is an action of G on T such that H fixes a point. In the sequel, if $X \subset T$, $\mathrm{Stab}(X)$ denotes the pointwise stabilizer of X in G. An arc I is said to be *unstable* is there exists a non-degenerate arc $J \subset I$ such that $\mathrm{Stab}(I) \subsetneq \mathrm{Stab}(J)$. We shall need the following theorem, proved by Guirardel, which enables us to decompose actions on real trees into tractable building blocks, under certain conditions. We refer the reader to [**Gui08**] for the definition of a graph of actions.

THEOREM 2.29 ([**Gui08**] Theorem 5.1). *Let G be a finitely generated group. Let H be a subgroup of G. Consider a minimal and non-trivial action of (G,H) on an $\mathbb{R}$-tree T by isometries. Assume that*

1. *T satisfies the ascending chain condition: for any decreasing sequence of arcs $I_1 \supset I_2 \supset \ldots$ whose lengths converge to 0, the sequence of their pointwise stabilizers $\mathrm{Stab}(I_1) \subset \mathrm{Stab}(I_2) \subset \ldots$ stabilizes.*
2. *For any unstable arc $I \subset T$,*
   (a) *$\mathrm{Stab}(I)$ is finitely generated,*
   (b) *$\forall g \in G$, $\mathrm{Stab}(I)^g \subset \mathrm{Stab}(I) \Rightarrow \mathrm{Stab}(I)^g = \mathrm{Stab}(I)$.*

Then either (G,H) splits over the stabilizer of an unstable arc, or over the stabilizer of an infinite tripod, or T has a decomposition into a graph of actions where each vertex action is either

1. *simplicial: $G_v \curvearrowright Y_v$ is a simplicial action on a simplicial tree;*
2. *of Seifert type: the vertex action $G_v \curvearrowright Y_v$ has kernel N_v, and the faithful action $G_v/N_v \curvearrowright Y_v$ is dual to an arational measured foliation on a closed conical 2-orbifold with boundary;*
3. *axial: Y_v is a line, and the image of G_v in $\mathrm{Isom}(Y_v)$ is a finitely generated group acting with dense orbits on Y_v.*

We will need the two following results.

THEOREM 2.30. *Let G be a finitely generated group and let H be a subgroup of G. Suppose that (G,H) acts on a real tree (T,d) that decomposes as a graph of actions $\mathcal{G}$. Let us denote by $\Delta_{\mathcal{G}}$ the corresponding splitting of G. Fix a point $x \in T$. Let S be a finite generating set for G. There exists an element $\sigma \in \mathrm{Aut}_H(G)$ such that for every $s \in S$, the following holds:*

- *if the geodesic segment $[x, s\cdot x]$ intersects non-trivially a component of Seifert type or an axial component, then $d(x,\sigma(s)\cdot x) < d(x, s\cdot x)$;*
- *if not, then $\sigma(s) = s$.*

For a proof of the result above, we refer the reader to [**Per08**] for the torsion-free case (Theorems 5.12 and 5.17) and [**RW14**] for the general case (Theorems 4.8 and 4.15). Note that in [**Per08**], Theorem 2.30 is stated for $\mathrm{Mod}(\Delta_{\mathcal{G}})$ (see Definition 4.15 in [**Per08**]) instead of $\mathrm{Aut}_H(G)$, and in [**RW14**], Theorem 2.30 is stated for $\mathrm{Mod}(G)$ (see Corollary 3.18 in [**RW14**] for a definition). However, one easily sees that $\mathrm{Mod}(\Delta_{\mathcal{G}})$ and $\mathrm{Mod}(G)$ are subgroups of $\mathrm{Aut}_H(G)$, because H fixes a point in T.

THEOREM 2.31. *Let ω be a non-principal ultrafilter. Let G be a finitely generated group and let S be a finite generating set for G. Let Γ be a hyperbolic group, and denote by (X,d) the Cayley graph of Γ (for a given generating set). Let $(\phi_n : G \to \Gamma)_{n\in\mathbb{N}}$ be a sequence of homomorphisms, and let $\lambda_n = \max_{s\in S} d(1,\phi_n(s))$. Assume that the sequence of pointed metric spaces $(X, d_n = d/\lambda_n, 1)_{n\in\mathbb{N}}$ ω-converges to a pointed real tree (T, d_ω, x), and that this tree decomposes as a graph of actions $\mathcal{G}$. Let $\Delta_{\mathcal{G}}$ be the splitting of G associated with $\mathcal{G}$, and let $\Lambda_{\mathcal{G}}$ be the splitting of G obtained from $\Delta_{\mathcal{G}}$ by replacing each vertex v of simplicial type by the corresponding splitting of G_v. The edges coming from this refinement are called simplicial. For every simplicial edge e, there exists a Dehn twist σ around e, together with a sequence of integers $(m_n)_n$, such that for every $s \in S$*

- *if the geodesic segment $[x, s\cdot x]$ contains e, then $d_n(1, \phi_n\circ\sigma^{m_n}(s)) < d_n(1,\phi_n(s))$ for every n large enough, and*
- *if the geodesic segment $[x, s\cdot x]$ does not contain e, then $d_n(1,\phi_n\circ\sigma^{m_n}(s)) = d_n(1,\phi_n(s))$ for every n.*

For a proof of the previous result, we refer the reader to [**Per08**] Theorem 5.22 for the torsion-free case, and [**RW14**] Corollary 4.20 for the general case.

REMARK 2.32. If a finitely generated subgroup H of G fixes a point in T, then all Dehn twists around simplicial edges belong to $\mathrm{Aut}_H(G)$.

PROPOSITION 2.33. *Let G and Γ be hyperbolic groups. Let H be a finitely generated subgroup of G. Suppose that G is one-ended relative to H. Let $\psi : H \hookrightarrow \Gamma$ be a monomorphism. Let $(\phi_n : G \to \Gamma)_{n\in\mathbb{N}} \in E(\psi)^{\mathbb{N}}$ be a stable sequence of distinct short homomorphisms. Then the stable kernel of the sequence is non-trivial.*

PROOF. Assume towards a contradiction that the stable kernel is trivial. We shall find a contradiction as follows: first, we prove that the group G acts fixed-point-freely on a real tree T. Secondly, Theorem 2.29 provides us with a decomposition of T into a graph of actions. Last, we can prove that the homomorphisms ϕ_n can be shortened, using the above results of [**RW14**]. This is a contradiction since the ϕ_n are supposed to be short.

Let S be a finite generating set of G. Let (X, d) be the Cayley graph of Γ (for a given finite generating set of Γ). Since the homomorphisms ϕ_n are short, the point $1 \in X$ minimizes the function $x \in X \mapsto \max_{s\in S} d(x, \phi_n(s) \cdot x) \in \mathbb{N}$, for each n. Let $\lambda_n = \max_{s\in S} d(1, \phi_n(s) \cdot 1)$.

Let us observe that λ_n goes to infinity as n goes to infinity. Indeed, the homomorphisms ϕ_n are pairwise distinct, and for every integer R the set

$$\{\phi \in \mathrm{Hom}(G, \Gamma) \mid \phi(S) \subset B_\Gamma(1, R)\}$$

is finite, since ϕ is determined by the restriction $\phi_{|S}$, and $\#B_\Gamma(1, R) < +\infty$.

Let ω be a non-principal ultrafilter. The sequence of metric spaces $((X, d_n := d/\lambda_n))_{n\in\mathbb{N}}$ ω-converges to a real tree $(T = X_\omega, d_\omega)$. Let $x = [(1)_{n\in\mathbb{N}}] \in T$. We claim that G acts fixed-point-freely on T. Towards a contradiction, suppose that G fixes a point $y = [(y_n)_{n\in\mathbb{N}}] \in T$. Since the generating set S of G is finite, there exists an element $s \in S$ such that $\omega(\{n \mid d(1, \phi_n(s) \cdot 1) \leq d(y_n, \phi_n(s) \cdot y_n)\}) = 1$. Therefore, $d_\omega(y, s \cdot y) \geq d_\omega(x, s \cdot x) = 1$. This is a contradiction since y is supposed to be fixed by G.

The group H fixes a point of T. Indeed, let $\{h_1, \ldots, h_p\}$ be a finite generated set for H. Since the sequence $(\phi_n(h_i))_{n\in\mathbb{N}}$ is constant up to conjugation, the translation length $\ell_n(\phi_n(h_i))$ of $\phi_n(h_i)$ acting on (X, d_n) goes to 0 as n goes to infinity. Hence, h_i acts on the limit tree T with translation length equal to 0. If follows that h_i fixes a point of T, for every $1 \leq i \leq p$. Similarly, $h_i h_j$ fixes a point of T for every $1 \leq i, j \leq p$. If follows from Serre's lemma that H fixes a point of T.

The group G acts hyperbolically on T, so there exists a (unique) minimal G-invariant subtree of T, namely the union of axes of translation of hyperbolic elements of G. Hence, one can assume that the action of G on T is minimal and non-trivial. By Remark 3.4 in [**Pau97**], or Theorem 1.16 in [**RW14**], the action of G on the limit tree T has the following properties:

(1) the stabilizer of any non-degenerate tripod is finite;
(2) the stabilizer of any non-degenerate arc is virtually cyclic with infinite center;
(3) the stabilizer of any unstable arc is finite.

In particular, the tree T satisfies the ascending chain condition of Theorem 2.29 since the stabilizer of any arc is virtually cyclic, and any ascending sequence of virtually cyclic subgroups of a hyperbolic group stabilizes.

Then, it follows from Theorem 2.29 that either (G, H) splits over the stabilizer of an unstable arc, or over the stabilizer of an infinite tripod, or T has a decomposition into a graph of actions. Since G is one-ended relative to H, and since the stabilizer of an unstable arc or of an infinite tripod is finite, it follows that T has a decomposition into a graph of actions.

Now, it follows from Theorems 2.30 and 2.31 that there exists a sequence of automorphisms $(\sigma_n)_n \in \mathrm{Aut}_H(G)^{\mathbb{N}}$ such that $\phi_n \circ \sigma_n$ is shorter than ϕ_n for n large enough. This is a contradiction since the ϕ_n are assumed to be short. □

The following theorem is a (relative) finiteness result for monomorphisms between two hyperbolic groups.

THEOREM 2.34. *Let* Γ *and* G *be hyperbolic groups, and let* H *be a finitely generated subgroup of* G. *Suppose that* G *is one-ended relative to* H. *Assume in addition that* G *embeds into* Γ, *and let* $\psi : G \hookrightarrow \Gamma$ *be a monomorphism. Then there exists a finite set* $\{i_1, \ldots, i_\ell\} \subset \mathrm{Hom}(G, \Gamma)$ *of monomorphisms that coincide with* ψ *on* H *such that, for every monomorphism* $\phi : G \hookrightarrow \Gamma$ *that coincides with* ψ *on* H, *there exist an automorphism* $\sigma \in \mathrm{Aut}(G)$ *such that* $\sigma_{|H} = \mathrm{id}_H$, *an integer* $1 \leq \ell \leq k$ *and an element* $\gamma \in \Gamma$ *such that*

$$\phi = \mathrm{ad}(\gamma) \circ i_\ell \circ \sigma.$$

REMARK 2.35. Note that the element γ in the previous theorem belongs to $C_\Gamma(\psi(H))$. Indeed, for every $h \in H$, $\phi(h) = \psi(h) = \mathrm{ad}(\gamma)(i_\ell(\sigma(h))) = \mathrm{ad}(\gamma)(i_\ell(h)) = \mathrm{ad}(\gamma)(\psi(h))$.

PROOF. Assume towards a contradiction that there exists a sequence of monomorphisms $(\phi_n : G \hookrightarrow \Gamma)_{n\in\mathbb{N}}$ such that for every n, ϕ_n coincides with ψ on H, and for every $n \neq m$, for every $\gamma \in \Gamma$, for every $\sigma \in \mathrm{Aut}(G)$ whose restriction to H is the identity, $\phi_n \neq \mathrm{ad}(\gamma) \circ \phi_m \circ \sigma$. Then, for every $n \neq m$, for every $\gamma \in \Gamma$, for every $\sigma \in \mathrm{Aut}_H(G)$, $\phi_n \neq \mathrm{ad}(\gamma) \circ \phi_m \circ \sigma$. Indeed, if σ belongs to $\mathrm{Aut}_H(G)$, there exists an element $g \in G$ and an automorphism α of G whose restriction to H is the identity and such that $\sigma = \mathrm{ad}(g)\circ\alpha$, so $\mathrm{ad}(\gamma) \circ \phi_m \circ \sigma = \mathrm{ad}(\gamma\phi_m(g)) \circ \phi_m \circ \alpha$. Hence, one can assume that the ϕ_n are short, pairwise distinct and injective. But the stable kernel of the sequence $(\phi_n)_{n\in\mathbb{N}}$ is trivial since each ϕ_n is injective. This contradicts Proposition 2.33. □

COROLLARY 2.36. *Let* G *be a hyperbolic group, let* H *be a finitely generated subgroup of* G. *Assume that* H *is infinite, and that* G *is one-ended relative to* H. *Then every monomorphism of* G *whose restriction to* H *is the identity is an automorphism of* G.

PROOF. Let ϕ be a monomorphism of G whose restriction to H is the identity.

First, assume that the subgroup H is non-elementary. Then $C_G(H)$ is finite. Since ϕ^n is a monomorphism for every integer n, it follows from Theorem 2.34 that there exist two integers $n > m$ such that $\phi^n = \mathrm{ad}(g) \circ \phi^m \circ \sigma$ for some $g \in C_G(H)$ and $\sigma \in \mathrm{Aut}(G)$ whose restriction to H is the identity. We have $\phi(C_G(H)) \subset C_G(H)$. Thus, since $C_G(H)$ is finite, ϕ induces a bijection of $C_G(H)$. As a consequence, there exists an element $k \in C_G(H)$ such that $g = \phi^m(k)$, so $\phi^n = \phi^m \circ \mathrm{ad}(k) \circ \sigma$. Hence $\phi^{n-m} = \mathrm{ad}(k) \circ \sigma$ is an automorphism, so ϕ is an automorphism.

Now, assume that H is infinite virtually cyclic. It fixes a pair of points $\{x^-, x^+\}$ on the boundary of G. The stabilizer S of the pair of points $\{x^-, x^+\}$ is virtually cyclic and contains H and $C_G(H)$, which is infinite. Note that H and $C_G(H)$ have finite index in S, so $Z(H) = H \cap C_G(H)$ has finite index as well. In particular, this group is normal and has finite index in $C_G(H)$. Let

$$p = [C_G(H) : Z(H)] \quad \text{and} \quad C_G(H) = \bigcup_{1\leq i\leq p} u_i Z(H).$$

By applying Theorem 2.34 to the powers of ϕ, and extracting a subsequence, there exists an increasing sequence of integers $(\ell_n)_{n\in\mathbb{N}}$ such that, for every n, there exist $g_n \in C_G(H)$ and an automorphism σ_n of G whose restriction to H is the identity, such that $\phi^{\ell_{n+1}} = \mathrm{ad}(g_n) \circ \phi^{\ell_n} \circ \sigma_n$. Up to taking a subsequence, one can assume that there exists an integer $i \in [\![1, p]\!]$ such that g_n belongs to $u_i Z(H)$ for each n. We have

$$\phi^{\ell_p} = \mathrm{ad}(g_{p-1} \cdots g_0) \circ \phi^{\ell_0} \circ \sigma_0 \circ \cdots \circ \sigma_{p-1},$$

with $g_{p-1} \cdots g_0$ belonging to $(u_i Z(H))^p = Z(H)$. Since ϕ coincides with the identity on H and $g_{p-1} \cdots g_0$ belongs to $Z(H) \subset H$, we have $g_{p-1} \cdots g_0 = \phi^{\ell_0}(g_{p-1} \cdots g_0)$, so

$$\phi^{\ell_p - \ell_0} = \mathrm{ad}(g_{p-1} \cdots g_0) \circ \sigma_0 \circ \cdots \circ \sigma_{p-1}.$$

Hence ϕ is an automorphism. □

Zlil Sela proved in [**Sel97**] that torsion-free one-ended hyperbolic groups are co-Hopfian. Later, Christophe Moioli generalized this result in the presence of torsion in his PhD thesis (see [**Moi13**]). By combining this result with Corollary 2.36, we get the following theorem (note that a finitely generated group G is one-ended relative to a finite subgroup H if and only if G is one-ended).

THEOREM 2.37. *Let G be a hyperbolic group, let H be a finitely generated subgroup of G. Assume that G is one-ended relative to H. Then every monomorphism of G whose restriction to H is the identity is an automorphism of G.*

We conclude this section by proving the following finiteness result for non-injective homomorphisms between two hyperbolic groups, relative to a subgroup.

THEOREM 2.38. *Let Γ and G be hyperbolic groups, and let H be a finitely generated subgroup of G that embeds into Γ. Let $\psi : H \hookrightarrow \Gamma$ be a monomorphism. Assume that G is one-ended relative to H. Then there exists a finite subset $F \subset G \setminus \{1\}$ such that, for any non-injective homomorphism $\phi : G \to \Gamma$ that coincides with ψ on H up to conjugation, there exists an automorphism $\sigma \in \mathrm{Mod}_H(G)$ such that $\ker(\phi \circ \sigma) \cap F \neq \varnothing$.*

PROOF. Since $\mathrm{Mod}_H(G)$ has finite index in $\mathrm{Aut}_H(G)$ (see Theorem 2.19), it suffices to prove the theorem with $\mathrm{Aut}_H(G)$ instead of $\mathrm{Mod}_H(G)$.

Let B_n denote the set of elements of G whose length is less that n, for a given finite generating set of G. Assume towards a contradiction that there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of homomorphisms from G to Γ such that, for every integer n, the three following conditions hold:

(1) ϕ_n coincides with ψ on H up to conjugation;
(2) ϕ_n is non-injective;
(3) for each $\sigma \in \mathrm{Aut}_H(G)$, $\phi_n \circ \sigma$ is injective in restriction to B_n.

Up to precomposing each ϕ_n by an element of $\mathrm{Aut}_H(G)$, and postcomposing by an inner automorphism, one can assume that ϕ_n is short. Moreover, up to passing to a subsequence, one can assume that the ϕ_n are pairwise distinct. But the stable kernel of the sequence $(\phi_n)_{n\in\mathbb{N}}$ is trivial, contradicting Proposition 2.33. $\square$

3. An example: the group $\mathrm{SL}_2(\mathbb{Z})$

In this section, we shall prove that the group $G = \mathrm{SL}_2(\mathbb{Z})$ is $\exists$-homogeneous. Recall that G splits as $G = A *_C B$ with $A \simeq \mathbb{Z}/4\mathbb{Z}$, $B \simeq \mathbb{Z}/6\mathbb{Z}$ and $C \simeq \mathbb{Z}/2\mathbb{Z}$. Let u and v be two tuples in G^k (for some $k \geq 1$) with the same existential type. We shall prove that there exists an automorphism of G sending u to v.

We shall decompose the proof into two steps. First, we shall prove that there exist a monomorphism ϕ of G such that $\phi(u) = v$, and a monomorphism ψ of G such that $\psi(v) = u$. Then, we shall establish the existence of an automorphism of G sending u to v.

3.1. First step. We claim that there exists a monomorphism ϕ of G such that $\phi(u) = v$. Let a denote a generator of A, and let b denote a generator of B. A finite presentation of G is given by $\langle a, b \mid \Sigma(a,b) = 1\rangle$, where $\Sigma(a,b) : (a^4 = 1) \wedge (b^6 = 1) \wedge (a^2 = b^3)$. Let $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ with $u_i, v_i \in G$ for every $1 \leq i \leq k$. Each element u_i can be written as a word $w_i(a,b)$. The group G satisfies the following sentence $\theta(u)$:

$$\theta(u) : \exists x \exists y\ \Sigma(x,y) = 1 \bigwedge_{1\leq i\leq k} u_i = w_i(x,y) \bigwedge_{1\leq \ell\leq 3} x^\ell \neq 1 \bigwedge_{1\leq \ell\leq 5} y^\ell \neq 1.$$

Indeed, taking $x = a$ and $y = b$, the statement is obvious. Then, since u and v have the same existential type, the sentence $\theta(v)$ is satisfied by G as well. This sentence asserts the existence of a tuple $(x,y) \in G^2$ which is solution to the system of equations and inequations

$$\Sigma(x,y) = 1 \bigwedge_{1\leq i\leq k} v_i = w_i(x,y) \bigwedge_{1\leq \ell\leq 3} x^\ell \neq 1 \bigwedge_{1\leq \ell\leq 5} y^\ell \neq 1.$$

In particular, x has order 4 and y has order 6. Since $\Sigma(x,y)=1$, the function $\phi : G \to G$ defined by $\phi(a)=x$ and $\phi(b)=y$ is an endomorphism of G. For every $1 \leq i \leq k$, one has $v_i = w_i(x,y) = w_i(\phi(a),\phi(b)) = \phi(w_i(a,b)) = \phi(u_i)$. Hence, ϕ sends u to v. It remains to prove that ϕ is injective.

First of all, note that ϕ is injective in restriction to A and B. As a consequence, ϕ sends A isomorphically to a conjugate gAg^{-1} of A, and ϕ sends B isomorphically to a conjugate hBh^{-1} of B. Let T be the Bass-Serre tree of the splitting $G = A *_C B$, let v_A be a vertex fixed by A and v_B a vertex fixed by B. The group C fixes the edge $[v_A, v_B]$, so $\phi(C)$ fixes the path $[g \cdot v_A, h \cdot w_B]$. But C is normal in G, so it is the stabilizer of any edge of T; in particular, C is the stabilizer of the path $[g \cdot v_A, h \cdot w_B]$. Thus, $\phi(C)$ is contained in C. But $\phi(C)$ has order 2, so $\phi(C)=C$.

Up to composing ϕ by $\mathrm{ad}(g^{-1})$, one can assume that $g=1$. Hence, one has $\phi(A)=A$ and $\phi(B) = hBh^{-1}$. If h belongs to A or B, then ϕ is clearly surjective, so it is an automorphism of G since G is Hopfian. If h does not belong to $A \cup B$, then one easily checks that ϕ is injective.

We have proved that there exists a monomorphism $\phi : G \hookrightarrow G$ which sends u to v. Likewise, there exists a monomorphism $\psi : G \hookrightarrow G$ sending v to u.

3.2. Second step. We shall modify the monomorphism ϕ (if necessary) in order to get an automorphism of G sending u to v. First of all, let us prove the following lemma.

LEMMA 3.1. *The group G splits non-trivially over a finite subgroup relative to $\langle u \rangle$ if and only if $\langle u \rangle$ has finite order.*

PROOF. Let T denote the Bass-Serre tree associated with the splitting $A *_C B$ of G. If $\langle u \rangle$ has finite order, then T gives a splitting of G relative to $\langle u \rangle$.

Conversely, assume that G splits non-trivially over a finite subgroup relative to $\langle u \rangle$. First, observe that G does not admit any epimorphism onto $\mathbb{Z}$ since its abelianization is finite, so G cannot split as an HNN extension. Consequently, G splits as $U *_W V$ with W finite and strictly contained in U and V, and $\langle u \rangle \leq U$. Let T'_0 be the Bass-Serre tree of this splitting. Note that U and V are virtually free and finitely generated, since G is virtually free and W is finitely generated. Let T_U and T_V be two reduced Stallings trees of U and V respectively, and let T' be the Stallings tree of G obtained by refining T'_0 with T_U and T_V. Up to forgetting the possibly redundant vertices of T', one can assume that T' is non-redundant. Since T is the unique reduced Stallings tree of G, the tree T' collapses onto T. Moreover, every expansion of T is redundant. Thus, $T'=T$. It follows that U is conjugate to A or B. In particular, $\langle u \rangle$ has finite order. □

We shall now prove that there exists an automorphism of G which sends u to v. There are two cases.

First case. If G is one-ended relative to $\langle u \rangle$ (i.e. if $\langle u \rangle$ has infinite order, by Lemma 3.1), then G is co-Hopfian relative to $\langle u \rangle$ (cf. Theorem 2.37). Therefore, the monomorphism $\psi \circ \phi$ is an automorphism of G, since it fixes $\langle u \rangle$. As a consequence, ϕ and ψ are two automorphisms.

Second case. If G is not one-ended relative to $\langle u \rangle$ (i.e. if $\langle u \rangle$ has finite order, by Lemma 3.1), then ϕ is not an automorphism *a priori*. We claim that it is possible to modify ϕ in such a way as to obtain an automorphism of G which maps u to v. First, observe that there exist two elements g and h of G such that $\phi(A) = gAg^{-1}$ and $\phi(B) = hBh^{-1}$, because $\phi(A)$ has order 4 and $\phi(B)$ has order 6. Recall that the subgroup C is central in G. In particular, g and h centralize C. Hence, one can define an endomorphism α of G by setting $\alpha_{|A} = \mathrm{ad}(g^{-1}) \circ \phi_{|A}$ and $\alpha_{|B} = \mathrm{ad}(h^{-1}) \circ \phi_{|B}$. This homomorphism is surjective since its image contains A and B. As G is Hopfian (as a virtually free group), α is an automorphism of G. Last, let us observe that α sends u to a conjugate of v. Indeed, since the group $\langle u \rangle$ has finite order, it is contained in a conjugate of A or B; hence, there exists an element $\gamma \in G$ such that $\alpha(u) = \gamma\phi(u)\gamma^{-1} = \gamma v\gamma^{-1}$. Therefore, the automorphism $\mathrm{ad}(\gamma^{-1}) \circ \alpha$ sends u to v.

REMARK 3.2. Note that $\mathrm{SL}_n(\mathbb{Z})$ is $\exists$-homogeneous for $n \geq 3$ as well. Indeed, it can be derived from Margulis superrigidity that any non-trivial endomorphism of $\mathrm{SL}_n(\mathbb{Z})$ (with $n \geq 3$) is an automorphism.

3.3. Generalization. In the next section 4, we shall prove that the first step in the previous proof remains valid for any virtually free group G. More precisely, if u and v have the same type, then there is an endomorphism of G which sends u to v and whose restriction to the maximal one-ended subgroup containing $\langle u\rangle$ is injective (see Proposition 4.1). As above, the proof will rely on first-order logic, but it will require $\forall\exists$-sentences in general (instead of $\exists$-sentences).

However, the second step does not work anymore in general: in Section 5, we give a counterexample.

4. Isomorphism between relative one-ended factors

The proposition below should be compared with the first step in the previous section.

PROPOSITION 4.1. *Let G be a virtually free group, let $k \geq 1$ be an integer and let $u, v \in G^k$. Suppose that u and v have the same $\forall\exists$-type. Suppose moreover that the subgroup $\langle u\rangle$ of G is infinite. Consequently, $\langle v\rangle$ is infinite. Let U be the one-ended factor of G relative to $\langle u\rangle$, and let V be the one-ended factor of G relative to $\langle v\rangle$ (see Section 2.3). The following assertions hold:*

- *there exists an endomorphism ϕ of G which sends u to v and whose restriction to U is injective,*
- *there exists an endomorphism ψ of G which sends v to u and whose restriction to V is injective.*

The following corollary is easy.

COROLLARY 4.2. *Let G be a virtually free group, let k be an integer and let $u, v \in G^k$. Suppose that u and v have same $\forall\exists$-type and that the subgroup $\langle u\rangle$ of G is infinite. Let U be the one-ended factor of G relative to $\langle u\rangle$, and let V be the one-ended factor of G relative to $\langle v\rangle$. There exists an endomorphism of G that maps u to v and induces an automorphism between U and V.*

In order to prove the corollary, we need the following classical result.

PROPOSITION 4.3 ([**Bow98**], Proposition 1.2). *If a hyperbolic group splits over quasi-convex subgroups, then every vertex group is quasi-convex (hence hyperbolic).*

Here below is a proof of Corollary 4.2.

PROOF. First, note that U is hyperbolic, thanks to Proposition 4.3 recalled above. Let ϕ and ψ be the endomorphisms of G given by Proposition 4.1. Since $\phi(u) = v$, we have $\phi(U) \subset V$. Likewise, $\psi(V) \subset U$. Then, observe that the monomorphism $\psi \circ \phi_{|U} : U \hookrightarrow U$ coincides with the identity on $\langle u\rangle$. Therefore, since U is hyperbolic and one-ended relative to $\langle u\rangle$ by definition, Theorem 2.37 applies and tells us that $\psi \circ \phi_{|U}$ is an automorphism of U. Hence, ϕ induces an isomorphism between U and V. ☐

Before proving Proposition 4.1, we need some preliminary results. For now, let us admit the following lemma that will be useful in the sequel.

LEMMA 4.4. *Let G be a virtually free group, and let Δ be a centered splitting of G. Then G does not admit any non-degenerate Δ-preretraction.*

We shall also need the following easy result.

LEMMA 4.5. *Let G be a group endowed with a splitting over finite groups. Let T denote the Bass-Serre tree associated with this splitting. Let H be a group endowed with a splitting over infinite groups, and let S be the associated Bass-Serre tree. If $p : H \to G$ is a homomorphism injective on edge groups of S, and such that $p(H_v)$ is elliptic in T for every vertex v of S, then $p(H)$ is elliptic in T.*

PROOF. Consider two adjacent vertices v and w in S. Let H_v and H_w be their stabilizers. The group $H_v \cap H_w$ is infinite by hypothesis. Moreover, p is injective on edge groups, thus $p(H_v \cap H_w)$ is infinite. Hence $p(H_v) \cap p(H_w)$ is infinite. Since edge groups of T are finite, $p(H_v)$ and $p(H_w)$ fix necessarily the same unique vertex x of T. As a consequence, for each vertex v of S, $p(H_v)$ fixes x.

Now, let h be an element of H, and let v be any vertex of S. By the previous paragraph, $p(H_v)$ and $p(H_{hv}) = p(h)p(H_v)p(h)^{-1}$ are both contained in G_x. Thus, $p(H_v)$ is contained in the intersection $G_x \cap p(h)^{-1}G_x p(h)$. Since H_v is infinite and p is injective on H_v, the intersection $G_x \cap p(h)^{-1}G_x p(h)$ is infinite as well. Moreover, edge groups of S are finite. It follows that the vertex groups G_x and $p(h)^{-1}G_x p(h)$ are equal. Thus, $p(h)$ belongs to G_x. Therefore, $p(H)$ is contained in G_x. □

We shall now prove Proposition 4.1.

PROOF. Recall that the subgroup $\langle u\rangle$ of G is assumed to be infinite, and that U denotes the one-ended factor of G relative to $\langle u\rangle$. Assume towards a contradiction that the proposition is false, i.e. (interchanging u and v if necessary) that every endomorphism ϕ of G which sends u to v is not injective on U. We will build a centered splitting Δ of G, together with a non-degenerate Δ-preretraction, and this will be a contradiction, according to Lemma 4.4 above.

Here are the different steps of our proof.

- *Step 1.* Note that U is a hyperbolic group, thanks to Proposition 4.3. In addition, U is one-ended relative to $\langle u\rangle$ by definition. Hence, the $\mathcal{Z}$-JSJ splitting of U relative to u, denoted by Λ, is well-defined. The first step of our proof consists in building a non-injective Λ-preretraction $p : U \to G$ (see Definition 2.21).
- *Step 2.* We then prove that there is a QH vertex in Λ, denoted by x, such that p does not send U_x isomorphically to a conjugate of itself.
- *Step 3.* Last, we build a centered splitting Δ of G, whose central vertex is the vertex x previously found. Then, we define an endomorphism ϕ of G as follows: ϕ coincides with p on the central vertex group $G_x = U_x$, and ϕ coincides with the identity map anywhere else. This endomorphism is a non-degenerate Δ-preretraction.

We now go into the details.

Step 1. We build a non-injective Λ-preretraction $p : U \to G$.

Since the elements u and v have the same $\forall\exists$-type, one can define a monomorphism $i : \langle u\rangle \hookrightarrow G$ by $i(u) = v$. Let $F = \{w_1, \ldots, w_\ell\} \subset U\backslash\{1\}$ be the finite set given by Theorem 2.38, such that every non-injective homomorphism from U to G that coincides with i on $\langle u\rangle$, i.e. that maps u to v, kills an element of F, up to precomposition by a modular automorphism of U relative to $\langle u\rangle$. Since we have assumed that Proposition 4.1 is false, the following holds, up to interchanging u and v: every endomorphism ϕ of G that sends u to v is non-injective in restriction to U. As a consequence, for every endomorphism ϕ of G that sends u to v, there exists an automorphism $\sigma \in \mathrm{Mod}_{\langle u\rangle}(U)$ such that the homomorphism $\psi = \phi_{|U} \circ \sigma : U \to G$ kills an element of F. Note that this homomorphism is Λ-related to $\phi_{|U}$ (see Definition 2.20). Hence, the following assertion holds: for every endomorphism ϕ of G that sends u to v, there exists a homomorphism $\psi : U \to G$ that is Λ-related to $\phi_{|U}$ and kills an element of F. We shall see that this statement can be expressed by means of a first-order formula $\mu(\boldsymbol{z})$, with k free variables, satisfied by v.

Let $G = \langle s_1, \ldots, s_n \mid \Sigma(s_1, \ldots, s_n)\rangle$ be a finite presentation of G. Observe that there is a one-to-one correspondance between the set of endomorphisms of G and the set of solutions in G^n of the system of equations $\Sigma(x_1, \ldots, x_n) = 1$. Let $U = \langle t_1, \ldots, t_p \mid \Pi(t_1, \ldots, t_p)\rangle$ be a finite presentation of U. Similarly, there is a one-to-one correspondance between $\mathrm{Hom}(U, G)$ and the set of solutions in G^p of the system of equations $\Pi(x_1, \ldots, x_p) = 1$. According to Lemma 2.22, there exists an existential formula $\theta(x_1, \ldots, x_{2p})$ with $2p$ free variables such that $\phi, \phi' \in \mathrm{Hom}(U, G)$ are Δ-related if and only if the statement $\theta\left(\phi(t_1), \ldots, \phi(t_p), \phi'(t_1), \ldots, \phi'(t_p)\right)$ is true in G. Each element u_i can be seen as a word $u_i(s_1, \ldots, s_n)$ on the generators $s_1, \ldots, s_n$ of G. Likewise, each generator t_i of $U \subset G$ can be seen as word $t_i(s_1, \ldots, s_n)$ on the generators of G, and each element $w_i \in F$ can be seen as a word $w_i(t_1, \ldots, t_p)$ on the generators of U. We define the formula $\mu(\boldsymbol{z})$ as follows:

$$\mu(\boldsymbol{z}) : \forall x_1 \ldots \forall x_n \ \left((\Sigma(x_1, \ldots, x_n) = 1) \wedge \bigwedge_{i \in [\![1,k]\!]} z_i = u_i(x_1, \ldots, x_n)\right)$$

$$\Rightarrow \exists y_1, \ldots \exists y_p \left(\begin{array}{c} \Pi(y_1, \ldots, y_p) = 1 \\ \wedge \displaystyle\bigvee_{i \in [\![1,\ell]\!]} w_i(y_1, \ldots, y_p) = 1 \\ \wedge\ \theta(t_1(x_1, \ldots, x_n), \ldots, t_p(x_1, \ldots, x_n), y_1, \ldots, y_p) \end{array} \right).$$

The statement $\mu(v)$ is true in G. Indeed, this statement tells us that for every endomorphism ϕ of G which maps u to v (defined by setting $\phi(s_i) = x_i$ for every $1 \leq i \leq n$), there exists a homomorphism $\psi : U \to G$, defined by setting $\psi(t_i) = y_i$ for every $1 \leq i \leq p$, such that $\ker(\psi) \cap F$ is non-empty and ψ and $\phi_{|U}$ are Λ-related. Since u and v have the same $\forall\exists$-type, the statement $\mu(u)$ is true in G as well. In other words, for every endomorphism ϕ of G that sends u to itself, the homomorphism $\phi_{|U}$ is Λ-related to a homomorphism $\psi : U \to G$ that kills an element of F. Now, taking for ϕ the identity of G, we get a non-injective Λ-preretraction $p : U \to G$.

Note that $\mu(\boldsymbol{z})$ is a $\forall\exists$-formula. Consequently, it is enough to assume that u and v have the same $\forall\exists$-type.

Step 2. We claim that there exists a QH vertex x of Λ such that U_x is not sent isomorphically to a conjugate of itself by p.

Assume towards a contradiction that the claim above is false, i.e. that each stabilizer U_x of a QH vertex x of Λ is sent isomorphically to a conjugate of itself by p. We claim that $p(U)$ is contained in a conjugate of U. Let Δ_u be a Stallings splitting of G relative to $\langle u\rangle$. First, let us verify the hypotheses of Lemma 4.5 in this situation:

(1) by definition, Δ_u is a splitting of G over finite groups, and Λ is a splitting of U over infinite groups;
(2) p is injective on edge groups of Λ (as a Λ-preretraction);
(3) if x is a QH vertex of Λ, then $p(U_x)$ is conjugate to U_x by assumption. In particular, $p(U_x)$ is contained in a conjugate of U in G. As a consequence, $p(U_x)$ is elliptic in the Bass-Serre tree T_u of Δ_u. If x is a non-QH vertex of Λ, then $p(U_x)$ is conjugate to U_x by definition of a Λ-preretraction. In particular, $p(U_x)$ is elliptic in T_u.

By Lemma 4.5, $p(U)$ is elliptic in T_u. It remains to prove that $p(U)$ is contained in a conjugate of U. Observe that U is not finite-by-(closed orbifold), as a virtually free group. Therefore, there exists at least one non-QH vertex x in Λ. Moreover, since p is inner on non-QH vertices of Λ, there exists an element $g \in G$ such that $p(U_x) = gU_xg^{-1}$. Hence, $p(U) \cap gUg^{-1}$ is infinite, which proves that $p(U)$ is contained in gUg^{-1} since edge groups of Δ_u are finite.

Now, up to composing p by the conjugation by g^{-1}, one can thus assume that p is an endomorphism of U. By Lemma 2.23, p is injective. This is a contradiction since p is non-injective (see Step 1). Hence, we have proved that there exists a QH vertex x of Λ such that U_x is not sent isomorphically to a conjugate of itself by p.

Step 3. It remains to construt a centered splitting of G (see Definition 2.24). First, we refine Δ_u by replacing the vertex y fixed by U by the JSJ splitting Λ of U relative to u (which is possible since edge groups of Δ_u adjacent to y are finite, ans thus are elliptic in Λ). With a little abuse of notation, we still denote by x the vertex of Δ_u corresponding to the QH vertex x of Λ defined in the previous step. Then, we collapse to a point every connected component of the complement of $\mathrm{star}(x)$ in Δ_u (where $\mathrm{star}(x)$ stands for the subgraph of Δ_u constituted of x and all its incident edges). The resulting graph of groups, denoted by Δ, is non-trivial. One easily sees that Δ is a centered splitting of G, with central vertex x.

The homomorphism $p : U \to G$ is well-defined on G_x because $G_x = U_x$ is contained in U. Moreover, p restricts to a conjugation on each stabilizer of an edge e of Δ incident to x. Indeed, either e is an edge coming from Λ, either G_e is a finite subgroup of U; in each case, $p_{|G_e}$ is a conjugation since p is Λ-related to the inclusion of U into G. Now, one can define an endomorphism $\phi : G \to G$ that coincides with p on $G_x = U_x$ and coincides with a conjugation on every vertex group G_y of Δ_u, with $y \neq x$. By induction on the number of edges of Δ_u, it is enough to define ϕ in the case where Δ_u has only one edge. If $G = U_x *_C B$ with $p_{|C} = \mathrm{ad}(g)$, one defines $\phi : G \to G$ by $\phi_{|U_x} = p$ and $\phi_{|B} = \mathrm{ad}(g)$. If $G = U_x *_C = \langle U_x, t \mid tct^{-1} = \alpha(c), \forall c \in C \rangle$ with $p_{|C} = \mathrm{ad}(g_1)$ and $p_{|\alpha(C)} = \mathrm{ad}(g_2)$, one defines $\phi : G \to G$ by $\phi_{|U_x} = p$ and $\phi(t) = g_2^{-1} t g_1$.

The endomorphism ϕ defined above is Δ-related to the identity of G (in the sense of Definition 2.26), and ϕ does not send G_x isomorphically to a conjugate of itself, by Step 2. Hence, ϕ is a non-degenerate Δ-preretraction (see Definition 2.27). This contradicts Lemma 4.4 and completes the proof of Proposition 4.1. □

It remains to prove Lemma 4.4.

4.1. Proof of Lemma 4.4. We need two preliminary results.

LEMMA 4.6. *Let G be a virtually free group, and let Δ be a centered splitting of G. If G admits a non-degenerate Δ-preretraction, then it has a finitely generated subgroup H with the following property: there exists a non-trivial minimal splitting Γ of H over virtually cyclic groups with infinite center such that, for every vertex x of Γ, the vertex group H_x does not split non-trivially over a finite group relative to the stabilizers of edges incident to x in Γ.*

Recall that a finite graph of groups Γ is said to be minimal if there is no proper subtree of the Bass-Serre tree T of Γ invariant under the action of G. Equivalently, Γ is minimal if and only if T has no vertex of degree equal to one, i.e. if Γ has no vertex v of degree equal to one such that $G_v = G_e$, where e denotes the unique edge incident to v in Γ.

PROOF. Let v denote the central vertex of Δ. Let ϕ be a non-degenerate Δ-preretraction. Let Δ' denote a splitting of G obtained from Δ by replacing each vertex $w \neq v$ by a Stallings splitting Λ_w of G_w relative to the stabilizers of edges incident to w in Δ (see Section 2.3 for the definition of the Stallings splitting relative to a collection of subgroups), and each edge $e = [v, w]$ of Δ by an edge $e' = [v, w']$ with w' a vertex of Λ_w such that $G_e \subset G_{w'}$. Note that such a vertex w' exists by definition of the relative Stallings splitting, but may not be unique if G_e is finite. If w' is not unique, we can define the edge e' by choosing any vertex w' such that $G_e \subset G_{w'}$. This choice is innocuous and does not affect the rest of the proof. Figure 4 below illustrates a possible construction of Δ'. Let $\mathcal{F}'$ be the set of edges e of Δ' such that G_e is finite. Let Γ denote the connected component of v in $\Delta' \setminus \mathcal{F}'$ (see Figure 4 below), and let H be the fundamental group of Γ.

By construction, for every vertex $x \neq v$ of Γ, the vertex group H_x is one-ended relative to the incident edge groups. Moreover, since $H_v = G_v$ is a finite-by-orbifold group, it does

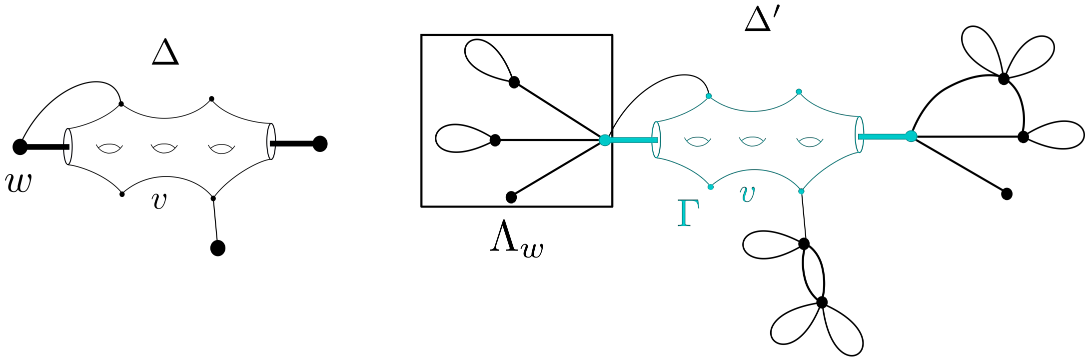


FIGURE 4. The graphs of groups Δ, Δ' and Γ. The latter is pictured in blue as a subgraph of Δ'. Edges with infinite stabilizer are depicted in bold.

not split non-trivially over a finite group relative to its extended boundary subgroups. By Remark 2.25, infinite edge groups of Γ are virtually cyclic with infinite center. In order to completes the proof of the lemma, it remains to prove that Γ is minimal. It is enough to prove that for every vertex $x \neq v$ of Γ such that there is exactly one edge e between x and v in Γ, the stabilizer H_e of the edge e is strictly contained in H_x.

Let x be such a vertex of Γ, and let e denote the unique edge between x and v. By construction, there exists a vertex w of Δ such that x is a vertex of the Stallings splitting Λ_w of G_w relative to the stabilizers of edges incident to w in Δ.

Let S be a maximal pinched set for ϕ and let $\Delta(S)$ be the splitting of G obtained from Δ by replacing the central vertex v by the splitting of G_v dual to S. Let $v_1, \ldots, v_n$ be the new vertices coming from v (see Figure 5 below).

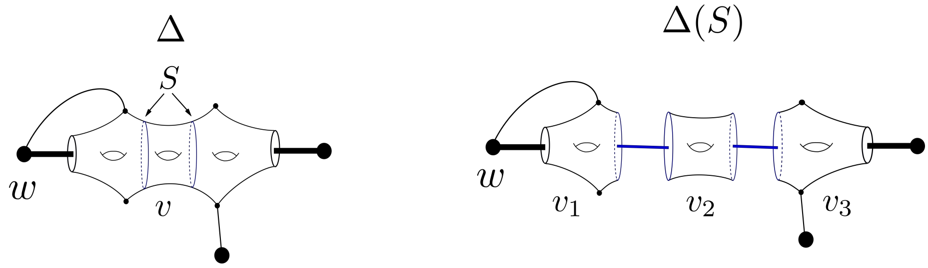


FIGURE 5. On the left, the centered splitting Δ of G; edges with infinite stabilizer are depicted in bold. On the right, the splitting $\Delta(S)$ of G is obtained from Δ by replacing the central vertex v by the splitting of G_v dual to S.

We denote by $\mathcal{S}$ the set of edges of $\Delta(S)$ corresponding to the maximal pinched set S, and we denote by $\mathcal{F}$ the set of edges of $\Delta(S)$ whose stabilizer is finite. Let W be the connected component of $\Delta(S) \setminus (\mathcal{S} \cup \mathcal{F})$ that contains the vertex w, and let G_W denote one of its fundamental groups (see Figure 6 below). By Lemma 2.28, W contains exactly two vertices: the vertex w and a vertex v_k coming from v. Moreover, $\phi(G_W) = \phi(G_w)$.

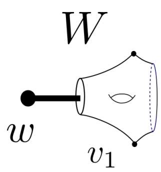


FIGURE 6. W is the connected component of $\Delta(S) \setminus (\mathcal{S} \cup \mathcal{F})$ that contains the vertex w.

By definition of Δ-relatedness, there exists an element $g \in G$ such that the restriction of ϕ to G_w coincides with the inner automorphism $\mathrm{ad}(g)$. Consequently, up to composing ϕ with $\mathrm{ad}(g^{-1})$, we can assume that ϕ coincides with the identity map on G_w (i.e. that $\phi_{|G_W} : G_W \twoheadrightarrow G_w$ is a retraction).

Let us refine the graph of groups W by replacing the vertex w by the Stallings splitting Λ_w of G_w relative to the stabilizers of edges incident to w. Let Λ_W denote this new graph of groups (see Figure 7 below).

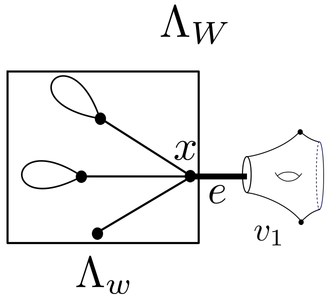


FIGURE 7. Λ_W is the splitting of G_W obtained from W by replacing w by the splitting Λ_w of G_w. The endomorphism ϕ of G induces a retraction $\phi_{|G_W}$ from G_W onto G_w.

Assume towards a contradiction that $G_x = G_e$. Recall that the stabilizer G_{v_k} of v_k is an extension $F \hookrightarrow G_{v_k} \twoheadrightarrow \pi_1(\mathcal{O}_k)$ where $\mathcal{O}_k$ is a conical compact hyperbolic 2-orbifold, and F is a finite group called the fiber. The image of G_e in $\pi_1(\mathcal{O}_k)$ is an infinite cyclic group $\langle\gamma\rangle$. Let $\widetilde{\gamma}$ be a preimage of γ in G_{v_k}. One has $G_x = G_e = F \rtimes \langle\widetilde{\gamma}\rangle$. We claim that G_w surjects onto $\langle\widetilde{\gamma}\rangle$. For every edge ε incident to x in Λ_w, the stabilizer G_ε is a subgroup of F since F is normal in G_x. Let $V(\Lambda_w)$ denote the set of vertices of Λ_w, and let N be the subgroup of G_w normally generated by $\{G_y \mid y \in V(\Lambda_w),\ y \neq x\} \cup F$. The quotient group G_w/N retracts onto $\langle\widetilde{\gamma}\rangle$, so G_w retracts onto $\langle\widetilde{\gamma}\rangle$. Let $r : G_w \twoheadrightarrow \langle\widetilde{\gamma}\rangle$ denote a retraction. Let n denote the genus of the orbifold $\mathcal{O}_k$, let $c_1, \ldots, c_p$ denote the conical elements, and let $\gamma_1, \ldots, \gamma_q$ denote the generators of the boundary subgroups of $\mathcal{O}_k$ different from γ.

If the orbifold $\mathcal{O}_k$ is orientable, there exist $2n$ elements $a_1, b_1, \ldots, a_n, b_n$ in $\pi_1(\mathcal{O}_k)$ such that the following relation holds in $\pi_1(\mathcal{O}_k)$:

$$\gamma = [a_1, b_1] \cdots [a_n, b_n] c_1 \cdots c_p \gamma_1 \cdots \gamma_q.$$

For every element α in $\pi_1(\mathcal{O}_k)$, let us denote by $\widetilde{\alpha}$ a preimage of α in G_{v_k}. There exists an element $z \in F$ such that the following relation holds in G_{v_k}:

$$\widetilde{\gamma} = [\widetilde{a_1}, \widetilde{b_1}] \cdots [\widetilde{a_n}, \widetilde{b_n}] \widetilde{c_1} \cdots \widetilde{c_p} \widetilde{\gamma_1} \cdots \widetilde{\gamma_q} z.$$

One has $r \circ \phi(z) = 1$ and $r \circ \phi(\widetilde{c_i}) = 1$ for $1 \leq i \leq p$, because z and $\widetilde{c_i}$ have finite order in G_{v_k}, and $\langle\widetilde{\gamma}\rangle$ is torsion-free. Likewise, $r \circ \phi(\widetilde{\gamma_i}) = 1$, for $1 \leq i \leq q$, because $\widetilde{\gamma_i}$ is pinched by ϕ. Moreover, since $\langle\widetilde{\gamma}\rangle$ is abelian, the image of the commutator $[\widetilde{a_i}, \widetilde{b_i}]$ by $r \circ \phi$ is trivial, for every $1 \leq i \leq n$. As a consequence, the element $r \circ \phi(\widetilde{\gamma})$ is trivial. But $r \circ \phi(\widetilde{\gamma}) = \widetilde{\gamma}$ since $r \circ \phi : G_W \twoheadrightarrow G_x = G_e = \langle\widetilde{\gamma}\rangle$ is a retraction. This is a contradiction.

If the orbifold $\mathcal{O}_k$ is non-orientable, there exist n elements $a_1, \ldots, a_n$ such that the following relation holds in $\pi_1(\mathcal{O}_k)$:

$$\gamma = a_1^2 \cdots a_n^2 c_1 \cdots c_p \gamma_1 \cdots \gamma_q.$$

There exists an element $z \in F$ such that the following relation holds in G_{v_k}:

$$\widetilde{\gamma} = \widetilde{a_1}^2 \cdots \widetilde{a_n}^2 \widetilde{c_1} \cdots \widetilde{c_p} \widetilde{\gamma_1} \cdots \widetilde{\gamma_q} z.$$

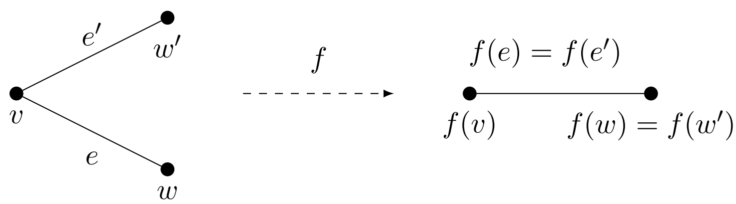
It follows that $r \circ \phi(\widetilde{\gamma}) = (r \circ \phi(\widetilde{a_1} \cdots \widetilde{a_n}))^2 = \widetilde{\gamma}$. Let $r' : \widetilde{\gamma} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ be the epimorphism obtained by killing the squares. Since $r \circ \phi(\widetilde{\gamma})$ is a square, one has $r' \circ r \circ \phi(\widetilde{\gamma}) = 1$. But $r' \circ r \circ \phi(\widetilde{\gamma})$ has order two, since $r \circ \phi(\widetilde{\gamma}) = \widetilde{\gamma}$. This is a contradiction. □

Recall that every virtually cyclic group G with infinite center splits as $F \hookrightarrow G \twoheadrightarrow \mathbb{Z}$, with F finite. In particular, the group G cannot be generated by two finite subgroups, since its abelianization G^{ab} maps onto $\mathbb{Z}$ as well. Note also that any infinite subgroup of G is virtually cyclic with infinite center. The following result is adapted from [**Hor17**], Lemma 6.11.

LEMMA 4.7. *Let G be a virtually free group equipped with a non-trivial minimal splitting Γ over virtually cyclic groups with infinite center. There exists a vertex x of Γ such that G_x splits non-trivially over a finite group relative to the stabilizers of edges incident to x in Γ.*

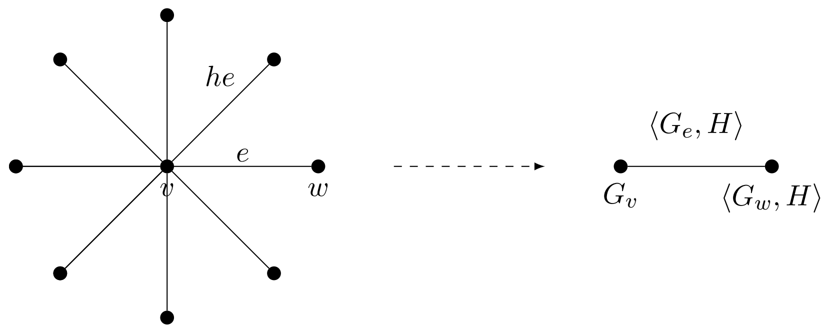
Before proving the lemma above, we need a definition. Let T, T' be two simplicial G-trees. A G-equivariant map $f : T \to T'$ is a *fold* if one of the two following situations occurs.

(1) Either there exist two edges $e = [v, w]$ and $e' = [v, w']$ in T, incident to a common vertex v in T and belonging to different G-orbits, such that T' is obtained from T by G-equivariantly identifying e and e', and $f : T \to T'$ is the quotient map (see Figure below). The fold $f : T \to T'$ is determined by the orbit of the pair of edges (e, e') identified by f.



There are two distinct subcases:
  (a) if w and w' belong to distinct G-orbits, then one has $G_{f(w)} = \langle G_w, G_{w'} \rangle$ and $G_{f(e)} = \langle G_e, G_{e'} \rangle$;
  (b) if $w' = gw$ and e, e' belong to distinct G-orbits, then $G_{f(w)} = \langle G_w, g \rangle$ and $G_{f(e)} = \langle G_e, G_{e'} \rangle$.

(2) Or there exist an edge $e = [v, w]$ in T and a subgroup H of G_v not contained in G_e such that T' is obtained from T by G-equivariantly identifying e and he for every $h \in H$, and $f : T \to T'$ is the quotient map. We have $G_{f(w)} = \langle G_w, H \rangle$ and $G_{f(e)} = \langle G_e, H \rangle$. The fold $f : T \to T'$ is determined by the orbit of the pair (e, H).



REMARK 4.8. Let $e = [v, w]$ and $e' = [v, w']$ be two edges of T incident to a common vertex v. Let $f : T \to T'$ be a fold such that $f(e) = f(e')$. Suppose that G acts on the quotient T' without inversion. If $e' = ge$, then g fixes the vertex v. Hence, there exists a subgroup H of G_v containing g such that f is defined by the pair (e, H).

We are ready to prove Lemma 4.7.

PROOF. Let T be a Stallings tree of G, and let T' be the Bass-Serre tree of Γ. All vertex stabilizers of T are finite, so elliptic in T'. First, we define an equivariant map $f : T \to T'$ that sends each edge of T to a point of T' or to a path of edges in T'. Let $v_1, \ldots, v_n$ be some representatives of the orbits of vertices of T. For every $1 \leq k \leq n$, there exists a vertex v'_k of T' such that $G_{v_k} \subset G_{v'_k}$. We let $f(v_k) = v'_k$. Then we define f on each vertex of T by equivariance. Next, we define f on the edges of T in the following way: if e is an edge of T, with endpoints v and w, there exists a unique injective path e' from $f(v)$ to $f(w)$ in T'. We let $f(e) = e'$.

We claim that there exists a finite sequence of G-trees $(T_k)_{0 \leq k \leq n+1}$ with $n \geq 0$ and $T_0 = T$, such that

- T_{k+1} is obtained from T_k by a fold f_k,
- T_{n+1} is equivariantly isometric to T' (we note $T_{n+1} = T'$),
- and the fold f_n identifies two distinct adjacent edges $e = [v, w]$ and $e' = [v, w']$ of T_n, with G_e finite.

Before proving the claim, we explain how to derive the lemma from this claim. We shall find a non-trivial splitting of the vertex group $G_{f_n(w)}$ over a finite group, relative to the stabilizers of edges incident to $f_n(w)$ in T'.

(1) If w and w' are not in the same G-orbit, the stabilizer of $f_n(w) \in T'$ splits as $G_w *_{G_e} \langle G_{w'}, G_e \rangle$. This splitting enjoys the following property: the stabilizer of any edge incident to $f_n(w) \in T'$ is contained, up to conjugacy, in one of the two factors G_w or $\langle G_{w'}, G_e \rangle$. In other words, this splitting is relative to the stabilizers of the edges incident to $f_n(w)$ in T'. Indeed, the stabilizer $\langle G_e, G_{e'} \rangle$ of the edge $f_n(e)$ incident to $f_n(w)$ is contained in $\langle G_{w'}, G_e \rangle$, and if ε is an edge of T_n incident to w or to w', then the edge $f_n(\varepsilon)$ is incident to $f_n(w)$ in T' and its stabilizer is contained in one of the two factors of $G_{f_n(w)}$. Moreover, this splitting is non-trivial (see below).

(2) If $w' = gw$ and $e' = ge$, then $G_{f_n(w)}$ splits as $\langle G_e, g \rangle *_{G_e} G_w$. As above, one can see that this splitting is relative to the stabilizers of the edges incident to $f_n(w)$ in T'. Moreover, it is non-trivial (see below).

(3) If $w' = gw$ but e and e' are not in the same G-orbit, then $G_{f_n(w)}$ splits as an HNN-extension $G_{w'} *_{G_e \cap G_{e'}} = \langle G_{w'}, G_e \rangle *_{G_e}$. These splittings are non-trivial (as HNN-extensions) and the second is relative to the stabilizers of the edges incident to $f_n(w)$ in T'.

In cases (1) and (2), it remains to prove that the splitting of $G_{f_n(w)}$ obtained above is non-trivial. It is enough to prove that G_e is strictly contained in G_w. Assume towards a contradiction that $G_e = G_w$, and let us consider an edge $\varepsilon \neq e$ adjacent to w in T_n. Since $G_w = G_e$ is finite, G_ε is finite as well. But there are at most two orbits of edges with finite stabilizer in T_n since f_n is the last fold of the sequence, namely the orbit of e and (possibly) the orbit of e'. If $\varepsilon = ge$, then either g fixes w, so g belongs to $G_w = G_e$, which contradicts the fact that $\varepsilon \neq e$; either g is hyperbolic with translation length equal to 1, so $w = gv$. Thus, G_v is finite. This is impossible because $G_v = G_{f_n(v)}$ is a vertex group of T', and edge groups of T' are infinite. If $\varepsilon = ge'$, then g is hyperbolic with translation length equal to 2, so $w = gw'$. In particular, $G_{w'}$ and $G_{e'}$ are finite. By hypothesis, the edge group $G_{f_n(e)} = \langle G_e, G_{e'} \rangle$ is virtually cyclic with infinite center, so it maps onto $\mathbb{Z}$. But G_e and $G_{e'}$ are finite, so $\langle G_e, G_{e'} \rangle$ has finite abelianization. This is a contradiction. Hence, G_e is strictly contained in G_w.

It remains to prove the claim. By [**Sta83**], 3.3, we know that the map $f : T \to T'$ can be decomposed into a sequence of G-equivariant edge folds $(f_k : T_k \twoheadrightarrow T_{k+1})_{0 \leq k \leq n}$, with $T_0 = T$ and $T_{n+1} = T'$. But in general there is no reason why the last fold $f_n : T_n \to T'$ should involve an edge with finite stabilizer. We shall prove that it is always possible to find a sequence satisfying this condition, performing the folds in a certain order. Adapting the terminology of [Hor17], we say that a fold is

- of type 1 if it identifies two edges e and e' belonging to distinct G-orbits, and both e and e' have infinite stabilizer, and
- of type 2 if it identifies two edges e and e' belonging to distinct G-orbits, and e or e' has finite stabilizer, and
- of type 3 if it identifies two edges belonging to the same G-orbit.

Since the number of orbits of edges decreases when performing a fold of type 1 or 2, we can assume that along the folding sequence, we only perform a fold of type 2 if no fold of type 1 is possible, and we only perform a fold of type 3 if no fold of type 2 is possible.

For every $k \in [\![1, n]\!]$, let $\phi_k : T_k \to T'$ be the unique map such that $f = \phi_k \circ f_k \circ \cdots \circ f_0$. We shall construct by induction a sequence of folds with the following *maximality property*: for every edge e_k of T_k with infinite stabilizer, the stabilizer of $\phi_k(e_k) \subset T'$ is equal to G_{e_k}. In particular, if f_k identifies two adjacent edges e_k and e'_k of T_k with infinite stabilizers, then e_k and e'_k belong to distinct G-orbits and $G_{e_k} = G_{e'_k}$.

The maximality property obviously holds for T_0, because all stabilizers of edges in T_0 are finite. Now, suppose that there exists a sequence $(f_i)_{0\leq i\leq k}$ satisfying the maximality property. If $T_k \neq T'$, there exists a fold $f_k : T_k \twoheadrightarrow T_{k+1}$. We can assume that f_k is of type $t \in \{1, 2, 3\}$ with t as small as possible. Let e_k and e'_k be two adjacent edges of T_k that are identified by f_k. Let $e = \phi_k(e_k) = \phi_k(e'_k) \in T'$. There are three distinct cases.

(1) If e_k or e'_k has infinite stabilizer, then $G_{e_k} = G_e$ or $G_{e'_k} = G_e$ according to our induction assumption, so $G_{f_k(e_k)} = G_e$ in both cases. Hence, T_{k+1} has the maximality property.
(2) If e_k and e'_k have finite stabilizers and $G_{f_k(e_k)}$ is finite, then T_{k+1} has the maximality property.
(3) If e_k and e'_k have finite stabilizers and $G_{f_k(e_k)}$ is infinite, let v_k be the common endpoint of e_k and e'_k. Note that $e'_k = ge_k$ for some $g \in G_{v_k}$ of infinite order, otherwise the stabilizer of $f(e_k)$ would be equal to $\langle G_{e_k}, G_{e'_k}\rangle$ or $\langle G_{e_k}, H\rangle$ for some finite group H, but these groups are finite since G_e is virtually cyclic with infinite center, and this is a contradiction. The subgroup $\langle g\rangle$ has finite index in G_e and fixes the vertex v_k. It follows that G_e is elliptic in T_k as well. Let x be the point the closest to v_k that is fixed by G_e in T_k. Assume towards a contradiction that $x \neq v_k$. Every edge ε in the segment $[x, v_k] \subset T_k$ has infinite stabilizer since g fixes $[x, v_k]$. Moreover, the stabilizer of ε is strictly contained in G_e, by definition of x. In addition, $\phi_k(\varepsilon) \neq e$ by the maximality property for T_k. Consequently, f_k is non-injective on $[x, v_k]$. Thus, there exist two adjacent edges in $[x, v_k]$ with infinite stabilizers that are identified by f_k. It follows from the maximality property for T_k that these two edges have the same infinite stabilizer, and belong to distinct G-orbits. Hence, we could perform a fold of type 1 in the tree T_k. This contradicts the priority order, since f_k is not a fold of type 1 (because G_{e_k} is finite). So we have proved that G_e fixes v_k. In addition, note that G_{e_k} is strictly contained in G_e. As a consequence, we can replace $f_k : T_k \twoheadrightarrow T_{k+1}$ by the fold identifying e_k with he_k for every $h \in G_e$. This new tree T_{k+1} has the maximality property.

Now, let $(f_k)_{k\in\mathbb{N}}$ be a sequence of folds, with $T_0 = T$, which respects the priority order (type 1 before type 2 before type 3) and the maximality property. Let T_n be the last tree along the folding sequence that contains an edge with finite stabilizer. We claim that $T_{n+1} = T'$. Assume towards a contradiction that $T_{n+1} \neq T'$, and let us prove that we could perform a fold of type 1 in T_n, contradicting the order of priority in the sequence of folds.

Let us observe that G acts without inversion on T_{n+1}. Indeed, G acts without inversion on T', and T' is obtained from T_{n+1} by a sequence of folds.

Since the fold f_n involves an edge with finite stabilizer, it is not of type 1. If f_n is of type 2, it is defined by a pair of adjacent edges $(e_n = [x,y], e_n' = [x,y'])$ in T_n. If f_n is of type 3, it is defined by a pair $(e_n = [x,y], K)$ where K is a subgroup of G_x (see Remark 4.8). Up to exchanging e_n and e_n' (in the case where f_n is of type 2), one can assume that the stabilizer of e_n is finite.

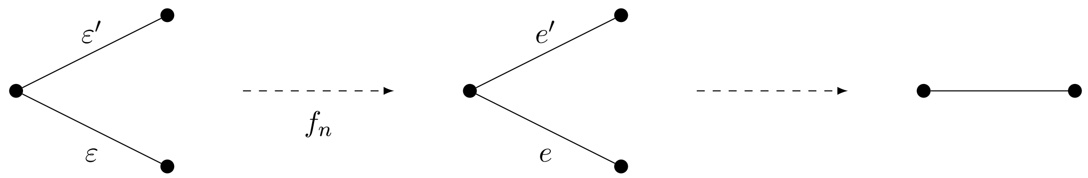
All possible folds in T_{n+1} identify two edges $e = [f_n(v), f_n(w)]$ and $e' = [f_n(v), f_n(w')]$ in distinct G-orbits such that $H := G_e = G_{e'}$ is infinite (by the maximality property). Let ε be an edge in the preimage of e by f_n, and let ε' be an edge in the preimage of e' by f_n.

First, let us prove that the group H fixes an extremity of ε (and similarly an extremity of ε'). If G_ε is infinite, then the maximality property implies that $G_\varepsilon = G_e = H$. If G_ε is finite, then one can assume without loss of generality that $\varepsilon = e_n$.

- If f_n is of type 2, then $H = G_e = \langle G_{e_n}, G_{e_n'} \rangle = G_{e_n'}$ by the maximality property. Since x is an endpoint of the edge e_n', the group $H = G_{e_n'}$ fixes x.
- If f_n is of type 3, then $H = G_e = \langle G_{e_n}, K \rangle$. But G_{e_n} and K are subgroups of G_x by definition. Thus, H fixes x.

Hence, the group H fixes an extremity of both ε and ε', denoted respectively by v and v'. If ε and ε' were disjoint, then H would fix the segment between v and v'. Up to replacing v (resp. v') by the other endpoint of ε (resp. ε') if necessary, one can suppose that v and v' are sent on the same point by f_n (because $e = f_n(\varepsilon)$ and $e' = f_n(\varepsilon')$ are adjacent in T_{n+1}). As a consequence, there are two adjacent edges a and a' in the segment $[v, v']$ such that $f_n(a) = f_n(a')$. Since H is infinite and fixes a and a', these two edges belong to distinct G-orbits, and $G_a = G_{a'}$, according to the maximality property. This is a contradiction, because f_n is of type 2 or 3 since it involves an edge with finite stabilizer.

Therefore, the edges ε and ε' are adjacent in T_n. Moreover, they belong to distinct G-orbits since e and e' belong to distinct G-orbits. At least one of the edges ε and ε', say ε, has finite stabilizer, otherwise we could perform a fold of type 1 in T_n identifying ε and ε', and this would contradict the order of priority in the folding sequence.



Let us observe in addition that it is possible to fold ε and ε' in T_n, and that this fold is of type 2 since ε and ε' are not in the same orbit. As a consequence, the fold f_n is necessarily of type 2 (if it were of type 3, we should have folded ε and ε' before, according to the priority order).

Since G_ε is finite, one can assume without loss of generality that $\varepsilon = e_n$. Thus, one has $G_e = \langle G_{e_n}, G_{e_n'} \rangle$. It follows from the maximality property that $G_{e_n'} = G_e = H$. The edges $\varepsilon = e_n$ and ε' being adjacent in T_n, the edges e_n' and ε' are adjacent as well. Moreover, they are identified in T', since $f_n(e_n') = e$ and $f_n(\varepsilon') = e'$ are identified in T_{n+1} (in particular in T'). In addition, note that e_n' and ε' lie in distinct G-orbits since $f_n(e_n') = e$ and $f_n(\varepsilon') = e'$ lie in distinct G-orbits. Thus, we could have performed a fold of type 1 in T_n by identifying e_n' and ε', a contradiction. □

We can now prove Lemma 4.4. Recall that this lemma claims that if G is a virtually free group with a centered splitting Δ, then G has no non-degenerate Δ-preretraction. Assume towards a contradiction that G has a non-degenerate Δ-preretraction. Then by Lemma 4.6 above, G has a subgroup H with the following property: there exists a non-trivial minimal splitting Γ of H over virtually cyclic groups with infinite center such that, for every vertex x of Γ, the vertex group H_x does not split non-trivially over a finite group relative to the stabilizers of edges incident to x in Γ. But H is virtually free, as a subgroup of the virtually free group G, so Lemma 4.7 above tells us that there exists a vertex x of Γ such that H_x splits non-trivially over a finite group relative to the stabilizers of edges incident to x in Γ. This is a contradiction.

5. A non-$\exists$-homogeneous virtually free group

In this section, we give an example of a virtually free group which is not $\exists$-homogeneous. By the way, this example shows that the second step in the proof of the homogeneity of $\mathrm{SL}_2(\mathbb{Z})$ fails in general.

More precisely, we shall construct a virtually free group $G = A *_C B$, with A, B finite, and two elements $x, y \in G$ such that:

- there exists a monomorphism $G \hookrightarrow G$ which interchanges x and y (in particular, $\mathrm{tp}_\exists(x) = \mathrm{tp}_\exists(y)$);
- there is no automorphism of G which sends x to y.

This is a new phenomenon, that does not occur in free groups, as shown by the following proposition (see **[OH11]** Lemma 3.7).

Proposition 5.1. *Let x and y be two elements of the free group $\boldsymbol{F}_n$. The following two statements are equivalent.*

(1) There exists a monomorphism $\boldsymbol{F}_n \hookrightarrow \boldsymbol{F}_n$ that sends x to y, and a monomorphism $\boldsymbol{F}_n \hookrightarrow \boldsymbol{F}_n$ that sends y to x.
(2) There exists an automorphism of $\boldsymbol{F}_n$ that maps x to y.

Here is a proof of Proposition 5.1 above.

Proof. If x or y is trivial, then $x = y$ and the result is obvious. Now, let us assume that x and y have infinite order. Let H_x be the one-ended factor of $\boldsymbol{F}_n$ relative to x (see Section 2.3) and let H_y be the one-ended factor of $\boldsymbol{F}_n$ relative to y. Let $\phi : \boldsymbol{F}_n \hookrightarrow \boldsymbol{F}_n$ be a monomorphism which sends x to y, and let $\psi : \boldsymbol{F}_n \hookrightarrow \boldsymbol{F}_n$ a monomorphism which sends y to x. One easily sees that $\phi(H_x) \subset H_y$ and $\psi(H_y) \subset H_x$. The monomorphism $(\psi \circ \phi)_{|H_x}$ maps H_x into itself and fixes x. Moreover, H_x is one-ended relative to $\langle x \rangle$ by definition. Hence, the relative co-Hopf property 2.37 applies and tells us that $(\psi \circ \phi)_{|H_x}$ is an automorphism of H_x. As a consequence, ϕ induces an isomorphism from H_x to H_y. It remains to extend ϕ to an automorphism of $\boldsymbol{F}_n$. Write $\boldsymbol{F}_n = H_x * K_x = H_y * K_y$. The groups K_x and K_y are free and have the same rank, so there is an isomorphism $\alpha : K_x \to K_y$. One defines an automorphism ψ of $\boldsymbol{F}_n$ that sends x to y by $\psi_{|H_x} = \phi_{|H_x}$ and $\psi_{|K_x} = \alpha_{|K_x}$. $\square$

Remark 5.2. It follows from the previous proposition that the free group $\boldsymbol{F}_2 = \langle a, b \rangle$ is $\exists$-homogeneous. Indeed, if x and y have the same existential type, one can express by means of an existential sentence that there exists an endomorphism ϕ of $\boldsymbol{F}_2$ that sends x to y and that does not kill the commutator $[a, b]$. The subgroup $\phi(\boldsymbol{F}_2)$ of $\boldsymbol{F}_2$ is generated by the two elements $\phi(a)$ and $\phi(b)$ that do not commute, so $\phi(\boldsymbol{F}_2)$ is isomorphic to $\boldsymbol{F}_2$ and ϕ is injective. Likewise, there exists a monomorphism $\psi : \boldsymbol{F}_2 \hookrightarrow \boldsymbol{F}_2$ that sends y to x. Now, the previous proposition implies that there exists an automorphism of $\boldsymbol{F}_2$ that maps x to y.

The following proposition shows that the elements x and y must have finite order in the counterexample.

Proposition 5.3. *Let $G = A *_C B$, where A and B are finite. Suppose that A has no subgroup isomorphic to B, and that B has no subgroup isomorphic to A. Let x and y be two elements of G which have the same $\exists$-type. If x has infinite order, then there exists an automorphism of G that maps x to y.*

Proof. As in the proof of the homogeneity of $\mathrm{SL}_2(\mathbb{Z})$, one can prove that there exist a monomorphism ϕ of G that maps x to y, and a monomorphism ψ of G that maps y to

x. Moreover, by Lemma 3.1, the group G is one-ended relative to x. Since x has infinite order, the element y has infinite order as well, so G is one-ended relative to y. Theorem 2.37 tells us that G is co-hopfian relative to x. Therefore, the monomorphism $\psi \circ \phi$ is an automorphism of G, since it fixes x. As a consequence, ϕ and ψ are two automorphisms. $\square$

5.1. Definition of the group we seek. In Section 3, we proved that the group $\mathrm{SL}_2(\mathbb{Z})$ is $\exists$-homogeneous. Similarly, one can prove the following proposition that give sufficient conditions under which a group of the form $G = A *_C B$, with A, B finite and $C \lhd G$, is $\exists$-homogeneous.

PROPOSITION 5.4. *Let $G = A *_C B$, where A and B are finite, and $C \lhd G$. Suppose that A has no subgroup isomorphic to B, and that B has no subgroup isomorphic to A. Assume moreover that, for every $g \in G$, there are two elements $a \in A$ and $b \in B$ such that $\mathrm{ad}(g)_{|C} = \mathrm{ad}(ba)_{|C}$. Then the group G is $\exists$-homogeneous.*

For instance, if C is cyclic, or more generally if $\mathrm{Out}(C)$ is abelian, the group G is $\exists$-homogeneous.

PROOF. Let x and y be two elements of G which have the same $\exists$-type. If x has infinite order, then it follows from Proposition 5.3 above that there exists an automorphism of G that maps x to y. Now, suppose that x has finite order. As in the proof of the homogeneity of $\mathrm{SL}_2(\mathbb{Z})$, one can prove that there exist a monomorphism ϕ of G that maps x to y and a monomorphism ψ of G that maps y to x. We claim that it is possible to modify ϕ in such a way as to obtain an automorphism of G which sends x to y. First, since A has no subgroup isomorphic to B, and B has no subgroup isomorphic to A by assumption, there exist two elements g and h of G such that $\phi(A) = gAg^{-1}$ and $\phi(B) = hBh^{-1}$. Up to composing ϕ by $\mathrm{ad}(h^{-1})$, one may suppose that the element h is trivial. By the second assumption, there are two elements $a \in A$ and $b \in B$ such that $\mathrm{ad}(bag^{-1})$ coincides with the identity map on $C = \phi(C)$. Thus, one can define an endomorphism α of G by setting $\alpha_{|A} = \mathrm{ad}(bag^{-1}) \circ \phi$ and $\alpha_{|B} = \phi$. This homomorphism is surjective since its image contains A and B (indeed, $\alpha(A) = A^b$ and $\alpha(B) = B$). The group G being Hopfian (as a virtually free group), the epimorphism α is an automorphism of G. Moreover, since x has finite order, it is contained in a conjugate of A or B. But α coincides with ϕ up to conjugacy on all conjugates of A and B. It follows that α maps x to a conjugate y^γ of y. Thus, the automorphism $\mathrm{ad}(\gamma^{-1}) \circ \alpha$ maps x to y. $\square$

In order to construct a non-$\exists$-homogeneous group G of the form $G = A *_C B$, where A and B are finite, it is necessary to violate the conditions given by Proposition 5.4 above. Note in addition that the elements x and y we are looking for must have finite order, by Proposition 5.3. We are now ready to define the group G together with the elements x and y.

Set $C = (\mathbb{Z}/2\mathbb{Z})^4$ and $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$. Let $(e_i\ e_j)$ denote the element of $\mathrm{Aut}(C) = \mathrm{GL}_4(\mathbb{F}_2)$ that interchanges e_i and e_j while leaving fixed e_k for $k \notin \{i,j\}$. Let us define a homomorphism f from $(\mathbb{Z}/2\mathbb{Z})^2 = \langle x \rangle \times \langle y \rangle$ to $\mathrm{Aut}(C)$ by setting $f(x) = (e_1\ e_2)$ and $f(y) = (e_3\ e_4)$. Let $A = C \rtimes_f (\langle x \rangle \times \langle y \rangle)$. Then, let us define a homomorphism h from $\mathbb{Z}/3\mathbb{Z} = \langle z \rangle$ to $\mathrm{Aut}(C)$ by setting $h(z) = (e_1\ e_2\ e_3)$, and let $B = C \rtimes_h \langle z \rangle$. Finally, let $G = A *_C B$.

5.2. A monomorphism that interchanges x and y. First, let us consider the following endomorphism ψ of A:

$$\psi(x) = y,$$
$$\psi(y) = x,$$
$$\psi_{|C} = (e_1\ e_3)(e_2\ e_4).$$

One easily sees that ψ is well-defined. In addition, its image contains the generating set $C \cup \{x, y\}$ of A. Thus, ψ is surjective. Since A is finite, ψ is an automorphism of A.

Let $u = z^{-1}xyz \in G$. One easily checks that ψ and $\mathrm{ad}(u)$ coincide on C, so one can define an endomorphism ϕ of G by setting $\phi_{|B} = \mathrm{ad}(u)$ and $\phi_{|A} = \psi$. Note that $\phi(C) = C$ since ψ preserves C. Since ϕ induces an automorphism ψ of A, and restricts to a conjugation on B, it sends any reduced normal form to a reduced normal form. It follows that ϕ is injective.

5.3. There does not exist any automorphism which sends x to y. Observe that the subgroup $\langle x, z\rangle$ of G generated by x and z fixes e_4. This subgroup acts by conjugation on C as the permutation group S_3, whereas the subgroup $\langle y, z\rangle$ of G acts on C as S_4. Assume towards a contradiction that there exists an automorphism σ of G sending x to y. The splitting $A *_C B$ is the only reduced Stallings splitting of G. Since the Stallings deformation space of G is invariant under the automorphisms of G, and since any automorphism maps a reduced tree to a reduced tree, the Bass-Serre tree T of $A *_C B$ is invariant under $\mathrm{Aut}(G)$. Hence, there exists a σ-equivariant isometry $f : T \to T$. Up to composing σ by an inner automorphism, since G acts transitively on the set of edges of T, one can assume that f preserves the edge $[v, w]$ such that $G_v = A$ and $G_w = B$. Thus, $\sigma(A) = A$, $\sigma(B) = B$ and $\sigma(C) = C$. Consequently, $\sigma(z) = z^{\pm 1}c$ for some $c \in C$. It follows that the action of $\langle \sigma(z), y\rangle$ on C by conjugation is the same as the action of $\langle z, y\rangle$, which acts as the permutation group S_4. But $\sigma(\langle z, x\rangle) = \langle \sigma(z), \sigma(x)\rangle = \langle \sigma(z), y\rangle$ acts as S_3 on C. This is a contradiction.

6. Uniform almost-homogeneity in virtually free groups

In the current section, we prove our main result.

THEOREM 6.1. *Virtually free groups are uniformly almost-homogeneous.*

Let us recall Corollary 4.2, which will play a key role in the proof of the theorem above.

COROLLARY 6.2. *Let G be a virtually free group, let k be an integer and let $u, v \in G^k$. Suppose that u and v have same $\forall\exists$-type and that the subgroup $\langle u\rangle$ of G is infinite. Let U be the one-ended factor of G relative to $\langle u\rangle$, and let V be the one-ended factor of G relative to $\langle v\rangle$. There exists an endomorphism of G that maps u to v and induces an automorphism between U and V.*

Let us prove the theorem.

PROOF. Let G be a virtually free group, let $k \geq 1$ be an integer and let u be a k-tuple of elements of G. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of k-tuples of elements of G such that $u_0 = u$ and $\mathrm{tp}(u) = \mathrm{tp}(u_n)$ for every n. We shall prove that there exist two distinct integers $n \neq m$ and an automorphism σ of G such that $\sigma(u_n) = u_m$. More precisely, we shall prove that there exists an integer $N \geq 1$ which does not depend on u, $(u_n)_{n\in\mathbb{N}}$ and k, such that $|\{u_n\}_{n\in\mathbb{N}}/\mathrm{Aut}(G)| \leq N$.

For every integer n, let T_n denote the Bass-Serre tree associated with a reduced Stallings splitting of G relative to $\langle u_n\rangle$. We distinguish two cases.

Case 1. If the subgroup $\langle u\rangle$ of G is infinite, then $\langle u_n\rangle$ is infinite as well, for every integer n, because u_n and u have the same $\forall\exists$-type. Let U_n denote the unique vertex group of T_n that contains $\langle u_n\rangle$ (in other words, U_n is the one-ended factor of G relative to $\langle u_n\rangle$). By Corollary 6.2, there exists an endomorphism ϕ_n of G that sends $u = u_0$ to u_n and that induces an isomorphism from $U := U_0$ to U_n.

Case 2. If the subgroup $\langle u\rangle$ of G is finite, then $\langle u_n\rangle$ is finite as well, for every integer n. It follows that $\langle u_n\rangle$ fixes a point of T_n. However, this point is not necessarily unique. Let U_0 be a finite subgroup of G that contains $\langle u\rangle$ and whose order is maximal among finite subgroups containing $\langle u\rangle$. Note that U_0 is a vertex group of T_0. Since u and u_n

have the same $\forall\exists$-type, there exists an endomorphism ϕ_n of G that maps u to u_n and that is injective on finite subgroups of G. Let $U_n = \phi_n(U)$. The morphism ϕ_n induces an isomorphism from $U := U_0$ to U_n. This subgroup U_n is a finite subgroup of G that contains $\langle u_n \rangle$ and whose order is maximal among finite subgroups containing $\langle u_n \rangle$. Hence, U_n is a vertex group of T_n.

We will prove that there are only finitely many U_n modulo $\mathrm{Aut}(G)$ (note that this result is obvious in the second case, since G has only finitely many conjugacy classes of finite subgroups). For each n, there exists a non-redundant tree S_n in the Stallings deformation space of G, together with a collapse $\pi_n : S_n \twoheadrightarrow T_n$. Indeed, let S_n be the tree obtained from T_n by replacing the vertex fixed by U_n by a reduced Stallings splitting of U_n. This tree S_n is a Stallings splitting of G, and it collapses onto T_n. Up to forgetting the vertices of degree 2, one can assume that S_n is non-redundant.

By Proposition 2.9, the Stallings deformation space of G is cocompact in the following sense: there exist finitely many trees $X_1, \ldots, X_p$ in the deformation space such that, for every non-redundant tree S in the Stallings deformation space of G, we have $S = X_i^\sigma$ for some $1 \leq i \leq p$ and some $\sigma \in \mathrm{Aut}(G)$. As a consequence, up to extracting a subsequence from $(u_n)_{n\in\mathbb{N}}$, one can suppose that there exists a tree S in the deformation space such that for every n, there exists an automorphism $\sigma_n \in \mathrm{Aut}(G)$ such that $S_n = S^{\sigma_n}$. But there are only finitely many ways in which we can collapse S, since there are only finitely many orbits of edges under the action of G. Hence, up to extracting a subsequence from $(u_n)_{n\in\mathbb{N}}$ once again, one can suppose that there exists a splitting T of G such that $T_n = T^{\sigma_n}$ for every n. Therefore, there exist two distinct integers n and m such that $U_n = \sigma_n \circ \sigma_m^{-1}(U_m)$.

Up to extracting a subsequence from $(u_n)_{n\in\mathbb{N}}$, and up to replacing each u_n by a k-tuple belonging to the same orbit under $\mathrm{Aut}(G)$, one can now assume that each $\langle u_n \rangle$ lies in the same relative one-ended factor, for instance U. Therefore, for each n, the homomorphism ϕ_n induces an automorphism α_n of U that sends u to u_n. Hence, $\alpha_m \circ \alpha_n^{-1}$ is an automorphism of U that sends u_n to u_m.

Since the group U has finitely many conjugacy classes of finite subgroups, there exist two distinct integers $n \neq m$ such that the restriction of $\alpha_m \circ \alpha_n^{-1}$ to any finite subgroup of U is a conjugation by some element of G. As a consequence, $\alpha_m \circ \alpha_n^{-1}$ is a conjugation on each stabilizer of an edge incident to the vertex fixed by U in T_u. Thus $\alpha_m \circ \alpha_n^{-1}$ extends to an automorphism α of G.

We have proved that $|\{u_n\}_{n\in\mathbb{N}}/\mathrm{Aut}(G)|$ is bounded from above by a constant that does not depend on u, $(u_n)_{n\in\mathbb{N}}$ and k. ☐

7. Generic homogeneity in hyperbolic groups

It is natural to wonder to what extent a non-homogeneous hyperbolic group is far from being homogeneous. In the current section, we will prove that, in a probabilistic sense, the deficiency of homogeneity is negligible. To that end, let us introduce the following definition.

DEFINITION 7.1. Let G be a group. Let k be an integer ≥ 1. A k-tuple u of elements of G is said to be *type-determined* if $|\{v \in G^k \mid \mathrm{tp}(v) = \mathrm{tp}(u)\}/\mathrm{Aut}(G)| = 1$.

Let μ be a probability measure on a finitely generated group G whose support is finite and generates G as a semigroup. An element of G arising from a random walk on G of length n generated by μ is called a *random element* of length n. We define a *random k-tuple* of length n as a k-tuple of random elements of length n arising from k independant random walks.

We will prove the following result.

THEOREM 7.2. *Let k be an integer ≥ 1. Let G be a hyperbolic group. Let μ be a probability measure on G whose support is finite and generates G as a semigroup. The* *probability that a random k-tuple of length n is $\exists$-type-determined tends to one as n tends to infinity.*

First, we prove a weaker version of Theorem 7.2 for virtually free groups (see Theorem 7.3 below). In this case, the result is an easy consequence of Proposition 4.1 combined with a result of Maher and Sisto [**MS17**].

7.1. Generic homogeneity in virtually free groups.

THEOREM 7.3. *Let k be an integer ≥ 1. Let G be a virtually free group. There exists an element $g \in G$ with the following property: for every probability measure μ on G whose support is finite, generates G as a semigroup and contains g, the probability that a random k-tuple of length n is $\forall\exists$-type-determined tends to one as n tends to infinity.*

The proof of Theorem 7.3 can be divided into two parts. Let $u \in G^k$.

Step 1. If G is one-ended relative to $\langle u \rangle$, then u is $\forall\exists$-type-determined (see Proposition 7.4). This is an easy consequence of Proposition 4.1.

Step 2. The probability that G is one-ended relative to the subgroup generated by a random k-tuple of length n tends to one as n tends to infinity (see Proposition 7.8).

PROPOSITION 7.4. *Let G be a virtually free group and let u be a finite tuple of elements of G. If G is one-ended relative to $\langle u \rangle$, then u is $\forall\exists$-type-determined.*

PROOF. If G is finite, the result is obvious. Assume now that G is infinite. As a consequence, the subgroup $\langle u \rangle$ of G is infinite, since G is one-ended relative to $\langle u \rangle$. Let v be a tuple of elements of G such that $\mathrm{tp}_{\forall\exists}(u) = \mathrm{tp}_{\forall\exists}(v)$. Since G is one-ended relative to $\langle u \rangle$, it follows from Proposition 4.1 that there exists a monomorphism $\phi : G \hookrightarrow G$ that sends u to v. Let V be the one-ended factor of G relative to $\langle v \rangle$. By Proposition 4.1, there exists an endomorphism $\psi : G \to G$ that sends v to u and whose restriction to V is injective. Moreover, $\psi(G)$ is contained in V. As a consequence, $\psi \circ \phi$ is a monomorphism of G that fixes u. The group G being co-Hopfian relative to $\langle u \rangle$ (see Theorem 2.37), $\psi \circ \phi$ is an automorphism of G. Thus, ϕ is an automorphism of G that maps u to v. $\square$

Before proving the second part, we need some preliminary results. Let G be a virtually free group, and let T be a Stallings tree of G. We say that two elements g and g' of G non-elliptic in T have a *p-match* if their axes γ and γ' in T have translates whose intersection has length greater than p. We shall use the following result, which is a particular case of Proposition 10 of [**MS17**].

PROPOSITION 7.5. *Let G be a virtually free group, let T be a Stallings tree of G. Let μ be a probability distribution on G such that $\mathrm{supp}(\mu)$ generates G as a semigroup. Let $g \in G$ be an element of infinite order which lies in the support à μ, and let γ denote the axis of g. A random element of length n is hyperbolic with large probability, and its translation length goes to infinity as n goes to infinity. We denote by γ_n the axis of such a random element of length n. For every integer $p \geq 0$, the probability that γ and γ_n have a p-match tends to 1 as n tends to infinity.*

We keep the same notations. Let v be a vertex of T. Let A_v be the set of vertices of T adjacent to v. Let $g \in G$ be an element of infinite order. The *Whitehead graph* $\mathrm{Wh}_T(g, v)$ is the labeled graph defined as follows: its vertex set is A_v, and two vertices v_1 and v_2 in A_v are joined by an edge if the axis of a conjugate of g contains the segment $[v_1, v_2]$. Whitehead graphs were first introduced by Whitehead in the context of free groups to give a criterion for characterizing primitive elements. Our presentation is inspired from [**GH17b**], in which the authors extend Whitehead criterion to free products of groups.

The following proposition gives a sufficient condition on an element $g \in G$ for being one-ended, in terms of Whitehead graphs defined above. We say that g *fills* T if, for every vertex $v \in T$, the graph $\mathrm{Wh}_T(g, v)$ is complete.

PROPOSITION 7.6. *Let G be a virtually free group, let T be a Stallings tree of G, and let g be an element of G of infinite order. If g fills T, then G is one-ended relative to $\langle g \rangle$.*

PROOF. Towards a contradiction, suppose that G has a splitting of the form $A *_C B$ or $A*_C$ with C finite, such that $g \in A$. Let T_A be the A-invariant minimal subtree of T. It is a Stallings tree of A. Let T_B be a Stallings tree of B. Let S be the Stallings tree of G obtained by replacing the vertex fixed by A by T_A in the Bass-Serre tree of $A *_C B$ or $A*_C$, and by replacing the vertex fixed by B by T_B. One can define a G-equivariant cellular map $f : S \to T$ which coincides with the identity map on T_A. The axis γ of g in T is contained in T_A, thus it avoids T_B. Consequently, if f is an isometry, the axis $f(\gamma)$ avoids $f(T_B)$. This is a contradiction since g fills T by assumption. Thus, f is not an isometry, so it folds two edges or collapses an edge.

First case. Assume that f maps two adjacent edges $[v, w]$ and $[v, w']$ of S to the same edge $[f(v), f(w) = f(w')]$ in T. Let A_w (resp. $A_{w'}$) denote the set of vertices of S which are adjacent to w (resp. to w'). Since the restriction of f to the axis γ of g is an isometry, this axis cannot contain the segment $[w, w']$. Likewise, none of the translates $h \cdot \gamma$ of γ (with $h \in G$) contain $[w, w']$. As a consequence, none of the translates of $f(\gamma)$ contains a segment $[f(x), f(x')]$ with $x \in A_w$ and $x' \in A_{w'}$. Hence, the Whitehead graph $\mathrm{Wh}_T(g, f(w) = f(w'))$ is not complete. This is a contradiction.

Second case. Let $[v, w]$ be an edge of S collapsed by f. Let A_v be the set of vertices of S that are adjacent to v. We define A_w in the same manner. Since f is an isometry on γ, none of the translates of γ contains the edge $[v, w]$, thus none of the translates of $f(\gamma)$ contains the segment $[f(x), f(y)]$ with $x \in A_v$ and $y \in A_w$. So the graph $\mathrm{Wh}_T(g, f(v) = f(w))$ is not complete. This is a contradiction. $\square$

We are ready to prove the second step of the proof of Theorem 7.3. We need the following easy lemma.

LEMMA 7.7. *Let G be a virtually free group, let T be a Stallings tree of G. There exists an element $g \in G$ of infinite order that fills T.*

PROOF. Let $v_1, \ldots, v_n$ be some representatives of the orbits of vertices of T. For every v_i, let $e_{i,1}, \ldots, e_{i,m_i}$ be some representatives of the orbits of edges adjacent to v_i. We construct iteratively a geodesic segment γ in T as follows:

- first, we take $\gamma = e_{1,1} = [v_1, w_1]$;
- then, we choose a geodesic segment α from w_1 to a translate hv_1 of v_1 in such a way that α does not contain $e_{1,1}$, and we redefine γ as the concatenation of γ and α and $he_{1,2}$;
- we repeat the previous operation until γ contains a translate of each $e_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m_1$.

Any element $g \in G$ of infinite order whose axis contains a translate of γ fills T. $\square$

PROPOSITION 7.8. *Let G be a virtually free group, let T be a Stallings tree of G, and let g be an element of infinite order that fills T (which exists according to the previous lemma). Let μ be a probability distribution on G such that $\mathrm{supp}(\mu)$ is finite, generates G as a semigroup and contains g. Then, the probability that G is one-ended relative to a random element of length n tends to one as n tends to infinity.*

PROOF. Let L be the translation length of g on T. By Proposition 7.5, there exists a constant $C \geq 0$ such that, for every $p \geq \max(C, L + 1)$, the probability that g and a random element g_n of length n have a p-match tends to 1 as n tends to infinity. Moreover, if g and g_n have a p-match, the element g_n fills T since $p \geq L + 1$. Now, it follows from Proposition 7.6 that g_n is one-ended. $\square$

7.2. Generic homogeneity in the general case. A tuple u of elements of G is termed *rigid* if the group G does not split non-trivially over a virtually cyclic group (finite or infinite) relative to $\langle u\rangle$. In [**GL19**], Guirardel and Levitt prove the following result, which claims that a hyperbolic group has no non-trivial splitting over a virtually cyclic group (finite or infinite) relative to a random tuple, i.e. that a random tuple is rigid. Their proof relies on Proposition 10 in [**MS17**].

PROPOSITION 7.9. *Fix an integer $k\geq 1$. In (G,μ) as above with G a hyperbolic group, the probability that a random k-tuple of length n is rigid tends to one as n tends to infinity.*

Theorem 7.2 is an immediate corollary of the previous result, combined with the following proposition.

PROPOSITION 7.10. *Fix an integer $k\geq 1$. Let G a hyperbolic group, and let $u=(u_1,\ldots,u_k)$ be a k-tuple of element of G. If u is rigid, then u is type-determined. In fact, the following stronger property holds: any k-tuple v with the same $\exists$-type as u belongs to the same* $\mathrm{Aut}(G)$*-orbit.*

PROOF. Since u is rigid, the JSJ splitting of G relative to $\langle u\rangle$ is trivial and the modular group $\mathrm{Mod}_{\langle u\rangle}(G)$ is trivial as well. It follows from Theorem 2.38 that there exists a finite subset $F=\{w_1,\ldots,w_\ell\}\subset G\setminus\{1\}$ such that any non-injective endomorphism ϕ of G which fixes u kills an element of F. We claim that this statement is expressible by means of an existential formula $\theta(\boldsymbol{y})$ with k free variables satisfied by u.

Let $G=\langle s_1,\ldots,s_n\mid\Sigma(s_1,\ldots,s_n)\rangle$ be a finite presentation of G. Let $v=(v_1,\ldots,v_k)$ be a k-tuple of elements of G such that $\mathrm{tp}_{\exists}(v)=\mathrm{tp}_{\exists}(u)$. Observe that there is a one-to-one correspondance between the set of endomorphisms of G and the set of solutions in G^n of the system of equations $\Sigma(x_1,\ldots,x_n)=1$. Each element u_i can be written as a word $u_i(s_1,\ldots,s_n)$, and each element w_i can be written as a word $w_i(s_1,\ldots,s_n)$. We define the formula $\theta(\boldsymbol{y})$ as follows:

$$\theta(\boldsymbol{y}):\exists x_1\ldots\exists x_n\ \Sigma(x_1,\ldots,x_n)=1\bigwedge_{i=1}^{k}y_i=u_i(x_1,\ldots,x_n)\bigwedge_{i=1}^{\ell}w_i(x_1,\ldots,x_n)\neq 1.$$

Since the identity of G fixes u and does not kill any element of F, the statement $\theta(u)$ holds in G (by setting $x_i=s_i$ for every $1\leq i\leq n$). Since u and v have the same existential type, $\theta(v)$ is satisfied by G as well. Hence, one can define an endomorphism ϕ of G which maps u to v by sending s_i to x_i for every $1\leq i\leq n$. This endomorphism is injective by Theorem 2.38, so it is an automorphism by Theorem 2.37. ☐

CHAPTER 3

On Tarski's problem for virtually free groups

Abstract. We give a complete classification of finitely generated virtually free groups up to $\forall\exists$-elementary equivalence.

1. Introduction

The problem of classifying algebraic structures up to elementary equivalence emerged in the middle of the twentieth century. Around 1945, Tarski asked whether all non-abelian finitely generated free groups are elementarily equivalent. Two decades later, Merzlyakov made an important step forward by proving that free groups have the same *positive theory*, i.e. satisfy the same first-order sentences that do not involve inequalities (see [**Mer66**]). Sacerdote subsequently generalized Merzlyakov's result and proved in [**Sac73a**] that all groups that split as a non-trivial free product have the same positive theory, except the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$. In the same year, Sacerdote proved in [**Sac73b**] that free groups have the same $\forall\exists$-*theory*, meaning that they satisfy the same sentences of the form $\forall \boldsymbol{x}\exists \boldsymbol{y}\psi(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are two tuples of variables, and ψ is a quantifier-free formula in these variables. Merzlyakov and Sacerdote's proofs rely heavily on small cancellation theory, as developed by several authors since the middle of the twentieth century, taking inspiration from Dehn's work on surface groups (see for instance [**Tar49**], [**Gre60**], [**Lyn66**] and [**Sch68**]).

Another major breakthrough towards the resolution of Tarski's problem was the study of systems of equations defined over a free group, due to Makanin and Razborov (see [**Mak82**], [**Mak84**] and [**Raz84**]).

A positive answer to Tarski's question was eventually given by Sela in [**Sel06b**] and by Kharlampovich and Myasnikov in [**KM06**], as the culmination of two voluminous series of papers.

Sela further generalized his work and classified torsion-free hyperbolic groups up to elementary equivalence. His solution involves a study of the $\forall\exists$-theory of a given hyperbolic group (see [**Sel04**] and [**Sel09**]), combined with a quantifier elimination procedure down to $\forall\exists$-sentences (see [**Sel05**] and [**Sel06a**]). The theory of group actions on real trees plays a crucial role in his approach (see for instance [**GLP94**], [**RS94**], [**BF95**], [**Sel97**]).

In this chapter, we give a complete classification of finitely generated virtually free groups up to $\forall\exists$-elementary equivalence, i.e. we give necessary and sufficient conditions for two finitely generated virtually free groups G and G' to have the same $\forall\exists$-theory, denoted by $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G')$. Recall that a group is said to be *virtually free* if it has a free subgroup of finite index. For instance, it is well-known that $\mathrm{SL}_2(\mathbb{Z})$ has a subgroup of index 12 isomorphic to the free group F_2.

Among virtually free groups, a wide variety of behaviours can be observed from the point of view of first-order logic. Here is an interesting illustration: on the one hand, all non-abelian free groups are elementarily equivalent (see [**Sel06b**] and [**KM06**]), while at the other extreme, it can be proved that two co-Hopfian virtually free groups are elementarily equivalent if and only if they are isomorphic. Recall that a group is said to be *co-Hopfian* if every monomorphism from this group into itself is bijective. One example of a co-Hopfian virtually free group is $\mathrm{GL}_2(\mathbb{Z})$. Between these two extremes, the picture is much more varied, and our goal in this chapter is to give a description of it.

In fact, in the class of virtually cyclic groups, we already have a glimpse of the unexpected influence of torsion on the first-order theory, as shown by the following example.

Example 1.1. Consider the following two $\mathbb{Z}/25\mathbb{Z}$-by-$\mathbb{Z}$ groups:

$$N = \langle a,t \mid a^{25}=1,\ tat^{-1}=a^6\rangle \quad \text{and} \quad N' = \langle a',t' \mid a'^{25}=1,\ t'a't'^{-1}=a'^{11}\rangle.$$

These groups are non-isomorphic, but $N\times\mathbb{Z}$ and $N'\times\mathbb{Z}$ are isomorphic. It follows from a theorem of Oger (see [**Oge83**]) that N and N' are elementarily equivalent.

This example is a particular manifestation of a more general phenomenon that plays an important role in our classification. Here below is an informal version of our main result; see Theorem 1.17 for a precise statement.

Main result (see Theorem 1.17). *Two finitely generated virtually free groups G and G' are $\forall\exists$-elementarily equivalent if and only if there exist two isomorphic groups $\Gamma \supset G$ and $\Gamma' \supset G'$ obtained respectively from G and G' by performing a finite sequence of specific HNN extensions over finite groups (called* legal large extensions*) or replacements of virtually cyclic subgroups by virtually cyclic overgroups (called* legal small extensions*).*

Moreover, Theorem 1.23 gives three other characterizations of $\forall\exists$-elementary equivalence among virtually free groups. In fact, it is worth noting that some of our results are proved in the more general context of hyperbolic groups (see in particular Theorem 1.9 and Theorem 1.14), and we except that they will be useful in a future classification of hyperbolic groups (possibly with torsion) up to elementary equivalence.

In addition, in some cases, we establish results stronger than $\forall\exists$-elementary equivalence, namely the existence of elementary embeddings (or rather $\exists\forall\exists$-elementary embeddings, see Defintion 2.4). We refer the reader to Theorem 1.10.

Before stating precise results, we need to introduce some definitions. Throughout the paper, all virtually free groups are assumed to be finitely generated, and we shall not repeat this assumption anymore.

Legal large extensions. By [**KPS73**] (see also [**SW79**] Theorem 7.3), a finitely generated group is virtually free if and only if it splits as a finite graph of finite groups, i.e. acts cocompactly by isometries on a simplicial tree with finite vertex stabilizers. Hence, every finitely generated virtually free group can be obtained from finite groups by iterating amalgamated free products and HNN extensions over finite groups. As a consequence, one of the basic questions we have to answer is the following: *how amalgamated free products and HNN extensions over finite groups do affect the $\forall\exists$-theory of a virtually free group, or more generally of a hyperbolic group?*

It can easily be seen that the number of conjugacy classes of finite subgroups of a given group is determined by its $\forall\exists$-theory. Thus, if a virtually free group G splits as $G = A*_C B$ over a finite group C, then G and A, B have distinct $\forall\exists$-theories provided that A or B is not isomorphic to $C \rtimes F_n$, in which case the amalgamated product can be written as a multiple HNN extension. Hence, we can restrict our attention to the case where $G = A*_C$, with C finite, which is more subtle: sometimes, the $\forall\exists$-theory is preserved when performing an HNN extension over finite groups, as shown by the following example.

Example 1.2. Let G be a virtually free group (and more generally a hyperbolic group) without non-trivial normal finite subgroup, for instance F_2 or $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Then, by Theorem 1.9 below, we have $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}\left(G*_{\{1\}}\right)$.

But sometimes, performing an HNN extension over finite groups modifies the $\forall\exists$-theory of a (non-elementary) virtually free group (and even its universal theory).

Example 1.3. Let $G = F_2 \times \mathbb{Z}/2\mathbb{Z}$. The universal sentence $\forall x \forall y \ (x^2 = 1) \Rightarrow (xy = yx)$ is satisfied by G, but not by $G*_{\{1\}} = G * \mathbb{Z}$. *A fortiori*, G and $G*_{\{1\}}$ do not have the same $\forall\exists$-theory. More generally, if G is hyperbolic and if the normalizer $N_G(C)$ of a finite subgroup $C \subset G$ normalizes a finite subgroup C' that contains C strictly, then $G*_C$ and G have different $\forall\exists$-theories.

This raises the following problem.

Problem 1.4. *Given a hyperbolic group G, characterize the HNN extensions $G*_\alpha$ over finite groups that do not perturb the $\forall\exists$-theory of G, i.e. such that $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G*_\alpha)$.*

In order to solve this problem (whose solution is given by Theorem 1.9 below), let us consider an isomorphism $\alpha : C_1 \to C_2$ between two finite subgroups of the hyperbolic group G, and suppose that G and the HNN extension $G*_\alpha = \langle G, t \mid \alpha(c) = tct^{-1},\ \forall c \in C\rangle$ have the same $\forall\exists$-theory. Let us derive some easy consequences from this assumption.

First, note that G must be non-elementary. Indeed, a hyperbolic group is finite if and only if it satisfies the first-order sentence $\forall x\ (x^N = 1)$ for some integer $N \geq 1$, and virtually cyclic if and only if it satisfies $\forall x \forall y\ ([x^N, y^N] = 1)$ for some integer $N \geq 1$.

Then, observe that C_1 and C_2 are necessarily conjugate in G, because the number of conjugacy classes of finite subgroups is an invariant of the $\forall\exists$-theory. Therefore, one can assume without loss of generality that $C_1 = C_2 := C$.

In addition, denoting by $\mathrm{Aut}_G(C)$ the subgroup

$$\{\sigma \in \mathrm{Aut}(C) \mid \exists g \in N_G(C),\ \sigma = \mathrm{ad}(g)_{|C}\}$$

of $\mathrm{Aut}(C)$, where $\mathrm{ad}(g)$ denotes the inner automorphism $x \mapsto gxg^{-1}$ and $N_G(C)$ denotes the normalizer of C, it can be observed that we have $|\mathrm{Aut}_G(C)| = |\mathrm{Aut}_{G*_\alpha}(C)|$. We refer the reader to Proposition 5.4 for further details. This means that there exists an element $g \in G$ such that $\mathrm{ad}(g)_{|C} = \alpha$.

Before giving two other consequences of the equality $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G*_\alpha)$, let us recall the following result, proved by Olshanskiy in [**Os93**].

PROPOSITION 1.5. *Let G be a non-elementary hyperbolic group, and let H be a non-elementary subgroup of G. There exists a unique maximal finite subgroup of G normalized by H. This group is denoted by $E_G(H)$.*

One can prove (see Proposition 5.4) that the equality $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G*_\alpha)$ implies that the normalizer $N_G(C)$ of C in G is non-elementary, and that C is the unique maximal finite subgroup of G normalized by $N_G(C)$, i.e. that $E_G(N_G(C)) = C$, as illustrated by Example 1.3 above.

This leads us to the following definition.

DEFINITION 1.6. Let G be a non-elementary hyperbolic group, and let C_1, C_2 be two finite subgroups of G. Suppose that C_1 and C_2 are isomorphic, and let $\alpha : C_1 \to C_2$ be an isomorphism. The HNN extension $G*_\alpha = \langle G, t \mid \mathrm{ad}(t)_{|C_1} = \alpha\rangle$ is said to be *legal* if the following three conditions hold.

(1) There exists an element $g \in G$ such that $gC_1g^{-1} = C_2$ and $\mathrm{ad}(g)_{|C_1} = \alpha$.
(2) $N_G(C_1)$ is non-elementary.
(3) $E_G(N_G(C_1)) = C_1$.

A group Γ is said to be a *legal large extension* of G if it splits as a legal HNN extension $\Gamma = H*_\alpha$ with $H \simeq G$. Sometimes we need to keep track of the order m of the finite group over which the HNN extension is performed, and we say that Γ is a *m-legal large extension* of G

REMARK 1.7. Up to replacing t by $g^{-1}t$ in the presentation above, one can assume without loss of generality that the presentation has the following form: $\langle G, t \mid \mathrm{ad}(t)_{|C_1} = \mathrm{id}_{C_1}\rangle$.

Example 1.3 is a typical illustration of a non-legal extension. Indeed, the third condition of the previous definition is clearly violated. By contrast, Example 1.2 (that is $\mathrm{PSL}_2(\mathbb{Z})*_{\{1\}}$) is a legal large extension. Here is another example of a legal large extension (to be compared to Example 1.3).

EXAMPLE 1.8. Let $G = F_2 \times \mathbb{Z}/2\mathbb{Z}$. The HNN extension $G*_{\mathbb{Z}/2\mathbb{Z}} = G *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z})$ is legal.

If G and $G*_\alpha$ have the same $\forall\exists$-theories, the previous discussion shows that $G*_\alpha$ is a legal large extension of G (see Proposition 5.4). One of our main results is that the converse also holds: if $G*_\alpha$ is a legal large extension of G, then we have $\mathrm{Th}_{\forall\exists}(G*_\alpha) = \mathrm{Th}_{\forall\exists}(G)$.

THEOREM 1.9. *Let G be a non-elementary hyperbolic group, and let $G*_\alpha$ be an HNN extension over finite groups. Then,* $\mathrm{Th}_{\forall\exists}(G*_\alpha) = \mathrm{Th}_{\forall\exists}(G)$ *if and only if $G*_\alpha$ is a legal large extension of G in the sense of Definition 1.6.*

The proof of this result relies on a generalization of the key lemma of [**Sac73b**], using techniques introduced by Sela for torsion-free hyperbolic groups and extended by Reinfeldt and Weidmann to hyperbolic groups with torsion in [**RW14**], in particular the *shortening argument*. We also refer the reader to [**Hei18**] for some results about $\forall\exists$-sentences in hyperbolic groups (possibly with torsion), namely a generalization of Merzlyakov's formal solutions.

In fact, we shall prove the following result, which is stronger than Theorem 1.9.

THEOREM 1.10. *Let G be a non-elementary hyperbolic group, and let $G*_\alpha$ be an HNN extension over finite groups. Then, the inclusion of G into $G*_\alpha$ is a $\exists\forall\exists$-elementary embedding (see Definition 2.4) if and only if $G*_\alpha$ is a legal large extension of G.*

Legal small extensions. Perhaps more surprisingly, another phenomenon of a different nature plays a crucial role in our classification of virtually free groups up to $\forall\exists$-elementary equivalence, as illustrated by Example 1.1. This phenomenon is not limited to infinite virtually cyclic groups: more generally, if G is a hyperbolic group, we will prove that one can replace a virtually cyclic subgroup $N \subset G$ by a virtually cyclic overbgroup $N' \supset N$ without modifying the $\forall\exists$-theory of G, as soon as certain additional technical conditions are satisfied (in particular, N has to be the normalizer of a finite subgroup of G). Before giving a precise statement (Theorem 1.14 below), we need some definitions.

DEFINITION 1.11. Given an infinite virtually cyclic group N and an integer p, we denote by $D_p(N)$ the definable subset $D_p(N) = \{n^p \mid n \in N\}$.

Let K_N be the maximal order of a finite subgroup of N. One can easily prove that for every integer $K \geq K_N$, the set $D_{2K!}(N)$ is a normal subgroup of N (see Lemma 6.5). Note that the quotient group $N/D_{2K!}(N)$ is finite. This finite group is determined by the $\forall\exists$-theory of N.

DEFINITION 1.12. Let N and N' be two infinite virtually cyclic groups. Let K_N and $K_{N'}$ denote the maximal order of a finite subgroup of N and N' respectively, and let $K \geq \max(K_N, K_{N'})$ be an integer. A homomorphism $\varphi : N \to N'$ is said to be *K-nice* if it satisfies the following three properties.

- φ is injective.
- If C_1 and C_2 are two non-conjugate finite subgroups of N, then $\varphi(C_1)$ and $\varphi(C_2)$ are non-conjugate in N'.
- The induced homomorphism $\overline{\varphi} : N/D_{2K!}(N) \to N'/D_{2K!}(N)$ is injective.

DEFINITION 1.13. Let G be a hyperbolic group. Let K_G denote the maximal order of a finite subgroup of G. Suppose that G splits as $A *_C B$ or $A*_C$ over a finite subgroup C whose normalizer N is infinite virtually cyclic and non-elliptic in the splitting. Let N' be a virtually cyclic group such that $K_{N'} \leq K_G$ and let $\iota : N \hookrightarrow N'$ be a K_G-nice embedding (in the sense of Definition 1.12 above). The amalgamated product

$$G' = G *_N N' = \langle G, N' \mid g = \iota(g),\ \forall g \in N \rangle$$

is called a *legal small extension* of G if there exists a K_G-nice embedding $\iota' : N' \hookrightarrow N$. Sometimes we need to keep track of the cardinality m of the edge group C, and we say that Γ is a m-legal extension of G.

For instance, the two virtually cyclic groups of Example 1.1 are legal small extensions of each other: in this example, C is the cyclic group $\langle a \rangle \simeq \mathbb{Z}/25\mathbb{Z}$, and one can define $\iota : N \hookrightarrow N'$ by $\iota : a \mapsto a', t \mapsto t'^3$ and $\iota' : N' \hookrightarrow N$ by $\iota' : a' \mapsto a, t' \mapsto t^2$.

We will prove the following result.

THEOREM 1.14. *Let G be a hyperbolic group that splits as $A*_C B$ or $A*_C$ over a finite subgroup C whose normalizer N is infinite virtually cyclic and non-elliptic in the splitting. Let K_G denote the maximal order of a finite subgroup of G. Let N' be a virtually cyclic group such that $K_{N'} \leq K_G$, and let $\iota : N \hookrightarrow N'$ be a K_G-nice embedding. The amalgamated product*

$$G' = G *_N N' = \langle G, N' \mid g = \iota(g),\ \forall g \in N\rangle$$

is a legal small extension in the sense of Definition 1.13 if and only if $\mathrm{Th}_{\forall\exists}(G') = \mathrm{Th}_{\forall\exists}(G)$.

REMARK 1.15. In general, the group G is not $\exists\forall$-elementarily embedded into G' (note the difference with Theorem 1.10, which generalizes Theorem 1.9). For instance, in Example 1.1, the element $a \in N$ satisfies the following $\exists\forall$-formula $\theta(a)$:

$$\theta(a) : \exists t \forall u\ (tat^{-1} = a^6) \wedge (t \neq u^3),$$

while it can be easily seen that any monomorphism $\iota : N \hookrightarrow N'$ maps a to a'^p for some integer p satisfying $\gcd(p, 25) = 1$, and that $\theta(a'^p)$ is false is N'.

A remark about the terminology. If a group G' is a legal large or small extension of a hyperbolic group G, then G' can be written as an amalgamated free product $G' = G*_N N'$, where N is the normalizer of a finite subgroup C of G, and N' is an overgroup of N in which C is normal. The terminology "large" or "small" refers to the size of N and N': if the legal extension is large, N and N' are non-elementary, and if the extension is small, N and N' are infinite virtually cyclic.

Classification of virtually free groups up to $\forall\exists$-elementary equivalence. Our main result, Theorem 1.17, asserts that the two kinds of extensions defined above are the only ones we need in order to classify virtually free groups up to $\forall\exists$-equivalence.

DEFINITION 1.16. Let G be a hyperbolic group. A group Γ is called a *multiple legal extension* of G if there exists a finite sequence of groups $G = G_0 \subset G_1 \subset \cdots \subset G_n \simeq \Gamma$ where G_{i+1} is a legal (large or small) extension of G_i in the sense of Definitions 1.6 or 1.13, for every integer $0 \leq i \leq n-1$.

Here is our main result (see also Theorem 1.23).

THEOREM 1.17. *Two finitely generated virtually free groups G and G' have the same $\forall\exists$-theory if and only if there exist two multiple legal extensions Γ and Γ' of G and G' respectively, such that $\Gamma \simeq \Gamma'$.*

EXAMPLE 1.18. One can deduce from Theorem 1.17 that a virtually free group G has the same $\forall\exists$-theory as $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ if and only if G splits as

$$(\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z} \times F_n),$$

where F_n denotes the free group of rank $n \geq 0$.

Theorem 1.23, which extends Theorem 1.17 above, gives three other characterizations of $\forall\exists$-equivalence among virtually free groups. Before stating this result, we need to generalize the definition of a nice homomorphism (see Definition 1.12).

DEFINITION 1.19. Let G and G' be two hyperbolic groups. Let K_G (resp. $K_{G'}$) denote the maximal order of a finite subgroup of G (resp. G'). Suppose that $K_G \geq K_{G'}$. A homomorphism $\varphi : G \to G'$ is said to be *special* if it satisfies the following three properties.

- It is injective on finite subgroups.
- If C_1 and C_2 are two non-conjugate finite subgroups of G, then $\varphi(C_1)$ and $\varphi(C_2)$ are non-conjugate in G'.

- If C is a finite subgroup of G whose normalizer is infinite virtually cyclic maximal, then $N_{G'}(\varphi(C))$ is infinite virtually cyclic, and the restriction
$$\varphi_{|N_G(C)} : N_G(C) \to N_{G'}(\varphi(C))$$
is K_G-nice in the sense of Definition 1.12 (in particular, $\varphi_{|N_G(C)}$ is injective).

REMARK 1.20. Note that if G and G' have the same universal theory, then $K_G = K_{G'}$.

REMARK 1.21. If G and G' are infinite virtually cyclic, then a homomorphism $\varphi : G \to G'$ is special if and only if it is K_G-nice.

DEFINITION 1.22. Let G and G' be two hyperbolic groups. A special homomorphism $\varphi : G \to G'$ is said to be *strongly special* if the following holds: for every finite subgroup C of G, if the normalizer $N_G(C)$ of C in G is not virtually cyclic, then $N_{G'}(\varphi(C))$ is not virtually cyclic and $\varphi(E_G(N_G(C))) = E_{G'}(N_{G'}(\varphi(C)))$.

Recall that a sequence of homomorphisms $(\varphi_n : G \to G')_{n\in\mathbb{N}}$ is said to be *discriminating* if the following holds: for every $g \in G \setminus \{1\}$, $\varphi_n(g)$ is non-trivial for every integer n sufficiently large.

We associate to every virtually free group G a sentence $\zeta_G \in \mathrm{Th}_{\exists\forall}(G)$ (see Section 4) such that the following result holds.

THEOREM 1.23. *Let G and G' be two finitely generated virtually free groups. The following five assertions are equivalent.*

(1) $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G')$.
(2) $G' \models \zeta_G$ *and* $G \models \zeta_{G'}$.
(3) There exist two discriminating sequences $(\varphi_n : G \to G')_{n\in\mathbb{N}}$ *and* $(\varphi'_n : G' \to G)_{n\in\mathbb{N}}$ *of special homomorphisms .*
(4) There exists two strongly special homomorphisms $\varphi : G \to G'$ *and* $\varphi' : G' \to G$.
(5) There exist two multiple legal extensions Γ *and* Γ' *of* G *and* G' *respectively, such that* $\Gamma \simeq \Gamma'$.

REMARK 1.24. A classical and easy result claims that two finitely presented groups G and G' have the same existential theory if and only if there exist two discriminating sequences of homomorphisms $(\varphi_n : G \to G')_{n\in\mathbb{N}}$ and $(\varphi'_n : G' \to G)_{n\in\mathbb{N}}$ (see for instance [**And18a**], Proposition 2.1). This should be compared with the third assertion above: from this perspective, as a consequence of Theorem 1.23, the only difference between the existential and $\forall\exists$ theories is that one cannot talk about the conjugacy classes of finite subgroups with only one quantifier, whereas it is possible with two quantifiers.

It seems reasonable to make the following conjecture, which generalizes the famous Tarski's problem about the elementary equivalence of non-abelian free groups (see [**Sel06b**] and [**KM06**]).

CONJECTURE 1.25. *Two virtually free groups have the same $\forall\exists$-theory if and only if they are elementarily equivalent.*

REMARK 1.26. As a consequence of Sela's work on the first-order theory of hyperbolic groups without torsion (see [**Sel09**]), the above conjecture is known to be true if one replaces "virtually free" by "hyperbolic without torsion". Moreover, thanks to Sela's result on the first-order theory of free products, the conjecture is known to be true if the two virtually free groups in question are free products of finite groups with a free group.

1.1. Outline of the proof of Theorem 1.23. We shall prove the following series of implications.

Note that (1) ⇒ (2) is obvious. Implications (1) ⇒ (3), (2) ⇒ (4) and (3) ⇒ (4) consist mainly in proving that our definitions are expressible by means of $\forall\exists$-sentences.

$$\begin{array}{ccccc} & & (2) & & \\ (1) & \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} & & \begin{smallmatrix}\searrow \\ \nearrow\end{smallmatrix} & (4) \Rightarrow (5) \Rightarrow (1) \\ & & (3) & & \end{array}$$

The proof of (5) $\Rightarrow$ (1) is a consequence of Theorems 1.9 and 1.14 (see Section 7). The proof of Theorem 1.9 (as well as of Theorem 1.10) consists in revisiting and generalizing the key lemma of Sacerdote's paper [**Sac73b**] dating from 1973, using some of the tools developed since then by Sela and others (in particular, the theory of group actions on real trees, the shortening argument and test sequences). The proof of Theorem 1.14 makes also important use of these techniques, but involves more technicalities.

We prove (4) $\Rightarrow$ (5) in three steps: first, we assume that all edge groups in reduced Stallings splittings of G and G' are equal. Then, we deal with the case where all edge groups have the same cardinality, by using a construction called the tree of cylinders, introduced by Guirardel and Levitt. In the general case, different cardinalities of edge groups may coexist in reduced Stallings splittings of G and G'. The proof is by induction on the number of edges in these splittings. By carefully collapsing certain edges, we can assume that there is only one cardinality of edge groups, and the proof boils down to the second case above.

2. Preliminaries

2.1. First-order logic. For detailed background, we refer the reader to [**Mar02**].

Definition 2.1. A *first-order formula* in the language of groups is a finite formula using the following symbols: $\forall$, $\exists$, $=$, $\wedge$, $\vee$, $\Rightarrow$, $\neq$, 1 (standing for the identity element), $^{-1}$ (standing for the inverse), $\cdot$ (standing for the group multiplication) and variables $x, y, g, z \ldots$ which are to be interpreted as elements of a group. A variable is *free* if it is not bound by any quantifier $\forall$ or $\exists$. A *sentence* is a formula without free variables.

Definition 2.2. Given a formula $\psi(x_1, \ldots, x_n)$ with $n \geq 0$ free variables, and n elements $g_1, \ldots, g_n$ of a group G, we say that $\psi(g_1, \ldots, g_n)$ is *satisfied* by G if its interpretation is true in G. This is denoted by $G \models \psi(g_1, \ldots, g_n)$. For brevity, we use the notation $\psi(\boldsymbol{x})$ where $\boldsymbol{x}$ denotes a tuple of variables.

Definition 2.3. The *elementary theory* of a group G, denoted by $\mathrm{Th}(G)$, is the collection of all sentences that are true in G. The *universal-existential theory* of G, denoted by $\mathrm{Th}_{\forall\exists}(G)$, is the collection of sentences true in G of the form

$$\forall x_1 \ldots \forall x_m \exists y_1 \ldots \exists y_n \ \psi(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

where $m, n \geq 1$ and ψ is a quantifier-free formula with $m + n$ free variables. In the same way, we define the *universal theory* of G, denoted by $\mathrm{Th}_{\forall}(G)$, its *existential theory* $\mathrm{Th}_{\exists}(G)$, etc. We say that two groups G and G' are *elementarily equivalent* (resp. $\forall\exists$-*elementarily equivalent*) if $\mathrm{Th}(G) = \mathrm{Th}(G')$ (resp. $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G')$).

To keep track of quantifiers and make the first-order formulas more readable, we will often use notations such as $\mathrm{Formula}_1^{\forall}(\boldsymbol{x})$ for universal formulas, $\mathrm{Formula}_2^{\exists}(\boldsymbol{x})$ for existential formulas, and so on.

Definition 2.4. Let G and Γ be two groups. An *elementary embedding* of G into Γ is a map $i : G \to \Gamma$ such that, for every first-order formula $\theta(x_1, \ldots, x_n)$ with n free variables and for every tuple $(g_1, \ldots, g_n) \in G^n$, the group G satisfies $\theta(g_1, \ldots, g_n)$ if and only if Γ satisfies $\theta(i(g_1), \ldots, i(g_n))$. We define $\exists\forall$-*elementary embeddings* and $\exists\forall\exists$-*elementary embeddings* in the same way, by considering only $\exists\forall$-formulas and $\exists\forall\exists$-formulas respectively, i.e. first-order formulas of the form $\exists \boldsymbol{x} \forall \boldsymbol{y} \psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ and $\exists \boldsymbol{x} \forall \boldsymbol{y} \exists \boldsymbol{z} \psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$ respectively, where $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{t}$ are tuples of variables, and ψ is a quantifier-free formula in these variables.

2.2. Virtually cyclic groups. Recall that an infinite virtually cyclic group G can be written as an extension of exactly one of the following two forms:

$$1 \to C \to G \to \mathbb{Z} \to 1 \quad \text{or} \quad 1 \to C \to G \to D_\infty \to 1,$$

where C is finite and $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ denotes the infinite dihedral group. In the first case, G has infinite center and splits as $G = C \rtimes \mathbb{Z}$. We say that G is of *cyclic type*. In the second case, G has finite center, and we say that G is of *dihedral type*. It splits as an amalgamated free product $A *_C B$ with $[A : C] = [B : C] = 2$.

We need to describe under which conditions the normalizer of a finite edge group in a splitting is a virtually cyclic group.

Lemma 2.5. *Let G be a group. Suppose that G splits as an amalgamated free product $G = A *_C B$ over a finite group C, and that $N_G(C)$ is not contained in a conjugate of A or B. Then $N_G(C)$ is infinite virtually cyclic if and only if C has index 2 in $N_A(C)$ and in $N_B(C)$. In this case, $N_G(C)$ is of dihedral type, equal to $N_A(C) *_C N_B(C)$.*

Lemma 2.6. *Let G be a group. Suppose that G splits as an HNN extension $G = A*_C$ over a finite group C. Let C_1 and C_2 denote the two copies of C in A and t be the stable letter associated with the HNN extension. Suppose that $N_G(C)$ is not contained in a conjugate of A. Then $N_G(C)$ is infinite virtually cyclic if and only if one of the following two cases holds.*

(1) *If C_1 and C_2 are conjugate in A and $N_A(C_1) = C_1$, then the normalizer $N_G(C_1)$ is of cyclic type, equal to $C_1 \rtimes \langle at\rangle$, where a denotes an element of A such that $aC_2a^{-1} = C_1$.*

(2) *If C_1 and $C_2 = t^{-1}C_1t$ are non-conjugate in G and C_1 has index 2 in $N_A(C_1)$ and $N_{tAt^{-1}}(C_1)$, then the normalizer $N_G(C_1)$ is of dihedral type, equal to*

$$N_A(C_1) *_{C_1} N_{tAt^{-1}}(C_1).$$

2.3. Maximal infinite virtually cyclic subgroups. Let G be a hyperbolic group. If $g \in G$ has infinite order, we denote by g^+ and g^- the attracting and repellings fixed points of g on the boundary $\partial_\infty G$ of G. The stabilizer of the pair $\{g^+, g^-\}$ is the unique maximal virtually cyclic subgroup of G containing g. We denote this subgroup by $M(g)$. If h and g are two elements of infinite order, either $M(h) = M(g)$ or $M(h) \cap M(g)$ is finite; in the latter case, the subgroup $\langle h, g\rangle$ is non-elementary. The following easy lemma shows that $M(g)$ is definable by means of a quantifier-free formula.

Lemma 2.7. *Let g be an element of G of infinite order. Let K denote the maximum order of an element of G of finite order.*

(1) *For every element $h \in G$, we have*

$$h \text{ belongs to } M(g) \ \Leftrightarrow\ [g^{K!}, hg^{K!}h^{-1}] = 1.$$

(2) *For every element $h \in G$ of infinite order, we have*

$$h \text{ belongs to } M(g) \ \Leftrightarrow\ [g^{K!}, h^{K!}] = 1.$$

Proof. We only prove the first point, the proof of the second point is similar. If h belongs to $M(g)$, then hgh^{-1} belongs to $M(g)$. Therefore, $g^{K!}$ and $(hgh^{-1})^{K!}$ commute, since $M(g)$ has a cyclic subgroup of index $\leq K$. Conversely, if $g^{K!}$ and $hg^{K!}h^{-1}$ commute, $hg^{K!}h^{-1}$ fixes the pair of points $\{g^+, g^-\}$, so h fixes $\{g^+, g^-\}$ as well. Thus, h belongs to $M(g)$. □

Corollary 2.8. *Let g, h be two elements of G of infinite order. The subgroup $\langle g, h\rangle$ is elementary if and only if $[g^{K!}, h^{K!}] = 1$.*

Recall that if H is a non-elementary subgroup of G, there exists a unique maximal finite subgroup $E_G(H)$ of G normalized by H. The following fact is proved by Olshanskiy in [**Os93**].

PROPOSITION 2.9 ([**Os93**] Proposition 1). *The finite subgroup $E_G(H)$ admits the following description:*

$$E_G(H) = \bigcap_{h \in H^0} M(h)$$

where H^0 denotes the set of elements of H of infinite order.

2.4. Small cancellation condition. Let G be a hyperbolic group, let (X, d) be a Cayley graph of G, and let δ be its hyperbolicity constant. Let g be an element of G of infinite order. We define the *translation length* of g as

$$||g|| = \inf_{x \in X} d(x, gx).$$

The *quasi-axis* of g, denoted by $A(g)$, is the union of all geodesics joining g^- and g^+. By Lemma 2.26 in [**Cou13**] (see also Remark below Definition 2.8, *loc. cit.*), the quasi-axis $A(g)$ is 11δ-quasi-convex. If g' is another element of G of infinite order, $\Delta(g, g')$ is defined as follows:

$$\Delta(g, g') = \operatorname{diam}\left(A(g)^{+100\delta} \cap A(g')^{+100\delta}\right) \in \mathbb{N} \cup \{\infty\},$$

where $A(g)^{+100\delta}$ is the 100δ-neighbourhood of $A(g)$ in (X, d), and $A(g')^{+100\delta}$ is defined similarly. It is well-known that there exists a constant $N(g) \geq 0$ such that every element $h \in G$ satisfying $\Delta(g, hgh^{-1}) \geq N(g)$ belongs to $M(g)$.

The small-cancellation condition defined below will play a crucial role in the proof of the implication (5) $\Rightarrow$ (1) of Theorem 1.23 (for further details, see Section 5).

DEFINITION 2.10. Let G be a hyperbolic group. Let $\varepsilon > 0$. An element g of infinite order satisfies the ε-small cancellation condition if the following holds: for every $h \in G$, if

$$\Delta(g, hgh^{-1}) > \varepsilon ||g||,$$

then h and g commute (so h belongs to $M(g)$). In particular, g is central in $M(g)$.

2.5. Actions on real trees. Recall that a *real tree* is a geodesic metric space in which every triangle is a tripod. A group action by isometries on a real tree is *minimal* if it has no proper invariant subtree. Note that if an action of a finitely generated group on a real tree has no global fixed point, then there is a unique invariant minimal subtree, which is the union of all translation axes. A subtree T' of a real tree T is said to be *non-degenerate* if it contains more than one point.

2.5.1. *Stable and superstable actions.* General results about group actions on real trees involve hypotheses on infinite sequences of nested arc stabilizers (see for instance Theorem 2.13 and Theorem 2.19 below). An action of a group on a real tree is said to be *stable* in the sense of Bestvina and Feighn (see [**BF95**]) if the pointwise stabilizer of any arc eventually stabilizes when this arc gets smaller and smaller. Here is a formal definition.

DEFINITION 2.11. Let T be a real tree. A non-degenerate subtree T' of T is *stable* if, for every non-degenerate subtree $T'' \subset T'$, the pointwise stabilizers of T'' and T' coincide. Otherwise, T' is called *unstable*. An action on a real tree is *stable* if any non-degenerate arc contains a non-degenerate stable subarc.

In [**Gui08**], Guirardel introduced the notion of a *superstable* action on a real tree.

DEFINITION 2.12. An action on a real tree is *M-superstable* if every arc whose pointwise stabilizer has order $> M$ is stable.

Let T be a real tree, and let x be a point of T. A *direction* at x is a connected component of $T \setminus \{x\}$. We say that x is a *branch point* if there are at least three directions at x. The following result is a work in preparation by Guirardel and Levitt (improving [**Gui01**]).

THEOREM 2.13. *Let L be a finitely generated group acting on a real tree T. Suppose that the action is M-superstable, with finitely generated arc stabilizers. Then every point stabilizer is finitely generated, the number of orbit of branch points in T is finite, the number of orbit of directions at branch points in T is finite.*

2.5.2. *The Bestvina-Paulin method.* In the sequel, ω denotes a non-principal ultrafilter, i.e. a finitely additive probability measure $\omega : \mathcal{P}(\mathbb{N}) \to \{0,1\}$ such that $\omega(F) = 0$ whenever $F \subset \mathbb{N}$ is finite.

Let G be a hyperbolic group, and let (X, d) be a Cayley graph of G. Let G' be a finitely generated group, equipped with a finite generating set S. Let $(\varphi_n : G' \to G)_{n\in\mathbb{N}}$ be a sequence of homomorphisms. We define the displacement of φ_n as

$$\lambda = \max_{s\in S} d(1, \varphi_n(s)).$$

Suppose that $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ tends to infinity. Let d_n denote the modified metric d/λ_n on X. The following result is sometimes called the Bestvina-Paulin method in reference to [**Bes88**] and [**Pau88**].

THEOREM 2.14. *The ultralimit (X_ω, d_ω) of the metric spaces $(X, d_n)_{n\in\mathbb{N}}$ is a real tree endowed with an action of G', and there exists a unique minimal G'-invariant non-degenerate subtree $T \subset X_\omega$. Moreover, some subsequence of the sequence $((X, d_n))_{n\in\mathbb{N}}$ converges to T in the Gromov-Hausdorff topology.*

2.5.3. *The Rips machine.* Under certain conditions, group actions on real trees can be analysed using the so-called Rips machine, which enables us to decompose the action into tractable building blocks. We shall use the following version of the Rips machine, proved by Guirardel in [**Gui08**] (Theorem 5.1). See also [**GLP94**], [**RS94**], [**BF95**], [**Sel97**].

Given a group G and a family $\mathcal{H}$ of subgroups of G, an action of the pair $(G, \mathcal{H})$ on a tree T is an action of G on T such that each $H \in \mathcal{H}$ fixes a point.

THEOREM 2.15. *Let G be a finitely generated group. Consider a minimal and non-trivial action of $(G, \mathcal{H})$ on an $\mathbb{R}$-tree T by isometries. Assume that*

- *(i) T satisfies the ascending chain condition: for any decreasing sequence of non-degenerate arcs $I_1 \supset I_2 \supset \dots$ whose lengths converge to 0, the sequence of their pointwise stabilizers* $\mathrm{Stab}(I_1) \subset \mathrm{Stab}(I_2) \subset \dots$ *stabilizes.*
- *(ii) For any unstable arc $I \subset T$,*
  - *(a)* $\mathrm{Stab}(I)$ *is finitely generated,*
  - *(b)* $\forall g \in G$, $g\mathrm{Stab}(I)g^{-1} \subset \mathrm{Stab}(I) \Rightarrow g\mathrm{Stab}(I)g^{-1} = \mathrm{Stab}(I)$.

Then either $(G, \mathcal{H})$ splits over the pointwise stabilizer of an unstable arc, or over the pointwise stabilizer of a non-degenerate tripod whose normalizer contains F_2, or T has a decomposition into a graph of actions where each vertex action $G_v \curvearrowright Y_v$ is either

- *(1) simplicial: $G_v \curvearrowright Y_v$ is a simplicial action on a simplicial tree;*
- *(2) of Seifert type: the vertex action $G_v \curvearrowright Y_v$ has kernel N_v, and the faithful action $G_v/N_v \curvearrowright Y_v$ is dual to an arational measured foliation on a compact conical 2-orbifold with boundary;*
- *(3) axial: Y_v is a line, and the image of G_v in* $\mathrm{Isom}(Y_v)$ *is a finitely generated group acting with dense orbits on Y_v.*

2.5.4. *Transverse covering.* We will use the following definitions (see [**Gui04**], Definitions 4.6 and 4.8).

DEFINITION 2.16. Let T be a real tree endowed with an action of a group G, and let $(Y_j)_{i\in J}$ be a G-invariant family of non-degenerate closed subtrees of T. We say that $(Y_j)_{j\in J}$ is a transverse covering of T if the following two conditions hold.

- *Transverse intersection:* if $Y_i \cap Y_j$ contains more than one point, then $Y_i = Y_j$.
- *Finiteness condition:* every arc of T is covered by finitely many Y_j.

Definition 2.17. Let T be a real tree, and let $(Y_j)_{j\in J}$ be a transverse covering of T. The *skeleton* of this transverse covering is the bipartite simplicial tree S defined as follows:

(1) $V(S) = V_0(S) \sqcup V_1(S)$ where $V_1(S) = \{Y_j \mid j \in J\}$ and $V_0(S)$ is the set of points $x \in T$ that belong to at least two distinct subtrees Y_i and Y_j. The stabilizer of a vertex of S is the global stabilizer of the corresponding subtree of T.

(2) There is an edge $\varepsilon = (Y_j, x)$ between $Y_j \in V_1(S)$ and $x \in V_0(S)$ if and only if x, viewed as a point of T, belongs to Y_j, viewed as a subtree of T. The stabilizer of ε is $G_{Y_i} \cap G_x$.

Moreover, the action of G on S is minimal provided that the action of G on T is minimal (see [**Gui04**] Lemma 4.9).

2.6. G-limit groups and the shortening argument. Let G be a hyperbolic group, and let G' be a finitely generated group. A sequence of homomorphisms $(\varphi_n : G' \to G)_{n\in\mathbb{N}}$ is termed stable if, for every element $x \in G'$, either $\varphi_n(x)$ is trivial for every n large enough, or $\varphi_n(x)$ is non-trivial for every n large enough. The stable kernel of the sequence is defined as follows:

$$\ker((\varphi_n)_{n\in\mathbb{N}}) = \{x \in G' \mid \varphi_n(x) = 1 \text{ for every } n \text{ large enough}\}.$$

The quotient $L = G'/\ker((\varphi_n)_{n\in\mathbb{N}})$ is called the G-limit group associated with the sequence (φ_n). This group acts on the tree T given by Theorem 2.14. The class of G-limit groups admits several equivalent descriptions.

Theorem 2.18. *Let G be a hyperbolic group, and let L be a finitely generated group. The following three assertions are equivalent.*

- *L is a G-limit group.*
- *L is fully residually G, meaning that there exists a sequence of homomorphisms $(\varphi_n) \in \mathrm{Hom}(L, G)^{\mathbb{N}}$ such that, for every non-trivial element $x \in L$, $\varphi_n(x)$ is non-trivial for every n large enough. Such a sequence is said to be discriminating.*
- $\mathrm{Th}_{\exists}(L) \subset \mathrm{Th}_{\exists}(G)$.

The third point justifies why G-limit groups play a crucial role in Sela's resolution of the Tarski's problem [**Sel06b**], and more generally in his classification of torsion-free hyperbolic groups up to elementary equivalence [**Sel09**].

Given ω and $(\varphi_n)_{n\in\mathbb{N}}$, the action of the limit group on the real tree given by Theorem 2.14 has nice properties. The following result generalizes a theorem proved by Sela for torsion-free hyperbolic groups.

Theorem 2.19 ([**RW14**], Theorem 1.16). *Let G be a hyperbolic group, and let L be a G-limit group. Let T be the corresponding real tree. The following hold, for the action of L on T:*

- *the pointwise stabilizer of any non-degenerate tripod is finite;*
- *the pointwise stabilizer of any non-degenerate arc is finitely generated and finite-by-abelian;*
- *the pointwise stabilizer of any unstable arc is finite.*

Remark 2.20. Note that the action of L on T is M-superstable, where M denotes the maximal order of a finite subgroup of L (which is bounded from above by the maximal order of a finite subgroup of G). In particular, Theorem 2.13 is applicable, so the number of orbit of branch points in T for the action of L is finite. This fact will be useful later. Note also that the tree T satisfies the ascending chain condition of Theorem 2.15 since any ascending sequence of finite-by-abelian subgroups of a hyperbolic group stabilizes.

In order to study the set $\mathrm{Hom}(L, G)$, Sela introduced a technique called the shortening argument, later generalized to hyperbolic groups possibly with torsion by Reinfeldt and Weidmann in [**RW14**]. In this chapter, we need a relative version of this argument.

Definition 2.21. Let G be a hyperbolic group and let H be a finitely generated subgroup of G. Suppose that G is one-ended relative to H. We denote by $\mathrm{Aut}_H(G)$ the subgroup of $\mathrm{Aut}(U)$ consisting of all automorphisms σ such that the following two conditions hold:

(1) $\sigma_{|H} = \mathrm{id}_{|H}$;
(2) for every finite subgroup F of G, there exists an element $g \in G$ such that $\sigma_{|F} = \mathrm{ad}(g)_{|F}$.

Definition 2.22. Let G and Γ be two hyperbolic groups, and let H be a finitely generated subgroup of G. Suppose that G is one-ended relative to H and that there exists a monomorphism $i : H \hookrightarrow \Gamma$. Let S be a finite generating set of G. A homomorphism $\varphi : G \to \Gamma$ such that $\varphi_{|H} = \mathrm{ad}(\gamma) \circ i$ for some $\gamma \in \Gamma$ is said to be *short* if its length $\ell(\varphi) := \max_{s \in S} d(1, \varphi(s))$ is minimal among the lengths of homomorphisms in the orbit of φ under the action of $\mathrm{Aut}_H(G) \times \mathrm{Inn}(\Gamma)$.

Proposition 2.23. *We keep the same notations as in the previous definition. Let $(\phi_n : G \to \Gamma)_{n \in \mathbb{N}}$ be a stable sequence of distinct short homomorphisms. Then the stable kernel of the sequence is non-trivial.*

Here below are two very useful consequences of the shortening argument whose proofs can be found in [**RW14**] (see [**Sel09**] for the torsion-free case).

Theorem 2.24 (Descending chain condition for G-limit groups). *Let G be a hyperbolic group. Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of G-limit groups. If $(\varphi_n : L_n \to L_{n+1})_{n \in \mathbb{N}}$ is a sequence of epimorphisms, then φ_n is an isomorphism for all n sufficiently large.*

Definition 2.25. A group L is said to be *equationally noetherian* if, for any system of equations in finitely many variables Σ, there exists a finite subsystem Σ_0 of Σ such that $\mathrm{Sol}(\Sigma) = \mathrm{Sol}(\Sigma_0)$ in L.

Theorem 2.26. *Let G be a hyperbolic group, and let L be a G-limit group. Then L is equationally noetherian.*

2.7. Tree of cylinders. Let $k \geq 1$ be an integer, let G be a finitely generated group, and let T be a splitting of G over finite groups of order exactly k. Recall that the deformation space of T is the set of G-trees which can be obtained from T by some collapse and expansion moves, or equivalently, which have the same elliptic subgroups as T. In [**GL11**], Guirardel and Levitt construct a tree that only depends on the deformation space of T. This tree is called the tree of cylinders of T, denoted by T_c. This tree will play a crucial role in our proof of the implication (4) $\Rightarrow$ (5) of Theorem 1.23 (see Section 8). We summarize below the construction of the tree of cylinders T_c.

First, we define an equivalence relation $\sim$ on the set of edges of T. We declare two edges e and e' to be equivalent if $G_e = G_{e'}$. Since all edge stabilizers have the same order, the union of all edges having a given stabilizer C is a subtree Y_C, called a cylinder of T. In other words, Y_C is the subset of T pointwise fixed by C. Two distinct cylinders meet in at most one point. The *tree of cylinders* T_c of T is the bipartite tree with set of vertices $V_0(T_c) \sqcup V_1(T_c)$ such that $V_0(T_c)$ is the set of vertices x of T which belong to at least two cylinders, $V_1(T_c)$ is the set of cylinders of T, and there is an edge $\varepsilon = (x, Y_C)$ between $x \in V_0(T_c)$ and $Y_C \in V_1(T_c)$ in T_c if and only if $x \in Y_C$. In other words, one obtains T_c from T by replacing each cylinder Y_C by the cone on its boundary (defined as the set of vertices of Y_C belonging to some other cylinder). If Y_C belongs to $V_1(T_c)$, the vertex group G_{Y_C} is the global stabilizer of Y_C in T, i.e. the normalizer of C in G.

2.8. JSJ splittings over finite groups. A splitting T of a group G is said to be *reduced* if there is no edge of T of the form $e = [v, w]$ with $G_v = G_e$.

Let $m \geq 1$ be an integer. A finitely generated group is termed $(\leq m)$-*rigid* if it does not split non-trivially over a finite subgroup of order $\leq m$.

Let G be a finitely generated virtually free group. Let $\mathcal{F}$ be the set of finite subgroups of G, and let $\mathcal{F}_m$ be the set of finite subgroups of G of order $\leq m$. A splitting of G over $\mathcal{F}$ all of whose vertex stabilizers are finite is called a *Stallings splitting* of G, and a splitting of G over $\mathcal{F}_m$ all of whose vertex stabilizers are m-rigid is called a *m-JSJ splitting* of G. A m-JSJ splitting is not unique, but the conjugacy classes of vertex and edge groups do not depend on the choice of a reduced m-JSJ splitting. A vertex group of a reduced m-JSJ splitting is called a *m-factor*.

Note that a Stallings splitting is a m-JSJ splitting whenever $m \geq M$, where M denotes the maximal order of a finite subgroup of G. Note also that one gets a m-JSJ splitting from any m'-splitting of G, with $m' > m$, by collapsing all edges whose stabilizer has order $> m$.

2.9. An extension lemma.

LEMMA 2.27. *Let G and G' be two groups. Let $\varphi : G \to G'$ be a homomorphism. Consider a splitting of G as a graph of groups Λ. For every vertex v of Λ, let $\widehat{G}_v$ be an overgroup of G_v, and let $\widehat{\varphi}_v : \widehat{G}_v \to G'$ be a homomorphism. Let $\widehat{G}$ denote the group obtained from G by replacing each G_v by $\widehat{G}_v$ in Λ. For every vertex $v \in \Lambda$ and every edge e incident to v, suppose that there exists an element $g'_{e,v} \in G'$ such that*

$$(\widehat{\varphi}_v)_{|G_e} = \mathrm{ad}(g'_{e,v}) \circ \varphi_{|G_e}.$$

Then, there exists a homomorphism $\widehat{\varphi} : \widehat{G} \to G'$ such that, for every vertex v of Λ,

$$\widehat{\varphi}_{|\widehat{G}_v} = \mathrm{ad}(g'_v) \circ \widehat{\varphi}_v \text{ for some } g'_v \in G'.$$

PROOF. We proceed by induction on the number of edges of the graph of groups Λ. It is enough to prove the lemma in the case where Λ has only one edge.

First case. Suppose that $G = G_v *_C G_w$. By hypothesis, there exist two elements $g'_1, g'_2 \in G'$ such that

$$(\widehat{\varphi}_v)_{|C} = \mathrm{ad}(g'_1) \circ \varphi_{|C} \quad \text{and} \quad (\widehat{\varphi}_w)_{|C} = \mathrm{ad}(g'_2) \circ \varphi_{|C}.$$

One can define $\widehat{\varphi} : \widehat{G} \to G'$ by

$$\widehat{\varphi}_{|\widehat{G}_v} = \mathrm{ad}({g'_1}^{-1}) \circ \widehat{\varphi}_v \quad \text{and} \quad \widehat{\varphi}_{|\widehat{G}_w} = \mathrm{ad}({g'_2}^{-1}) \circ \widehat{\varphi}_w.$$

Second case. Suppose that

$$G = G_v *_C = \langle G_v, t \mid tct^{-1} = \alpha(c), \ \forall c \in C \rangle.$$

By hypothesis, there exist two elements $g'_1, g'_2 \in G'$ such that

$$(\widehat{\varphi}_v)_{|C} = \mathrm{ad}(g'_1) \circ \varphi_{|C} \quad \text{and} \quad (\widehat{\varphi}_v)_{|tCt^{-1}} = \mathrm{ad}(g'_2) \circ \varphi_{|tCt^{-1}}.$$

One can define $\widehat{\varphi} : \widehat{G} \to G'$ by

$$\widehat{\varphi}_{|\widehat{G}_v} = \widehat{\varphi}_v \quad \text{and} \quad \widehat{\varphi}(t) = g'_2 \varphi(t) {g'_1}^{-1}.$$

□

3. Proof of (1) ⇒ (3)

DEFINITION 3.1. For any hyperbolic group G, we denote by K_G the maximal order of a finite subgroup of G.

PROPOSITION 3.2. *Let $G = \langle s_1, \ldots, s_p \rangle$ be a hyperbolic group. There exists a universal formula*

$$\mathrm{Special}^{\forall}(x_1, \ldots, x_p)$$

such that, for every hyperbolic group G' satisfying $K_{G'} \leq K_G$, and for every tuple $\boldsymbol{g'} = (g'_1, \ldots, g'_p)$ of elements of G', the following two assertions are equivalent:

(1) *the group G' satisfies* $\mathrm{Special}^{\forall}(\boldsymbol{g'})$*;*
(2) *the map $\varphi_{\boldsymbol{g'}} : \{s_1, \ldots, s_p\} \to G'$ defined by $s_i \mapsto g'_i$ for $1 \leq i \leq p$ extends to a special (see Definition 1.19) homomorphism from G to G'.*

Proof. Let $G = \langle s_1, \ldots, s_p \mid \Sigma(s_1, \ldots, s_p) = 1\rangle$ be a finite presentation of G, where $\Sigma(s_1, \ldots, s_p) = 1$ denotes a finite system of equations in p variables. Let $\{C_1, \ldots, C_q\}$ be a set of representatives of the conjugacy classes of finite subgroups of G, and let I be the subset of $[\![1, q]\!]$ such that $N_G(C_i)$ is infinite virtually cyclic maximal if and only if $i \in I$.

Observe that there is a one-to-one correspondence between the set of homomorphisms $\mathrm{Hom}(G, G')$ and the set $\mathrm{Sol}_{G'}(\Sigma) = \{\boldsymbol{g'} \in G'^p \mid \Sigma(\boldsymbol{g'}) = 1\}$.

For $1 \leq i \leq q$, the injectivity of the homomorphism $\varphi_{\boldsymbol{g'}} : \boldsymbol{s} \to \boldsymbol{g'}$ on C_i can be expressed by a quantifier-free formula $\mathrm{Inj}^1_i(\boldsymbol{g'})$.

For $i \in I$, the injectivity of $\varphi_{\boldsymbol{g'}}$ on N_i can be expressed by a quantifier-free formula $\mathrm{Inj}^2_i(\boldsymbol{g'})$. Indeed, it is enough to say that $\varphi_{\boldsymbol{g'}}$ is injective on finite subgroups and that $\varphi_{\boldsymbol{g'}}(N_i)$ has infinite order. The latter point is easily expressible, since an element g' of G' has infinite order if and only if $g'^{K_G!} \neq 1$, because $K_G \geq K_{G'}$.

If $1 \leq i \neq j \leq q$, the assertion that $\varphi_{\boldsymbol{g'}}(C_i)$ and $\varphi_{\boldsymbol{g'}}(C_j)$ are non-conjugate translates into a universal formula $\mathrm{NonConj}^{\forall}_{i,j}(\boldsymbol{g'})$.

If i belongs to I, since the quotient $N_i/D_{2K_G!}(N_i)$ is finite, one can choose a finite collection of representatives $g_1, \ldots, g_r \in N_i$ of the cosets of $D_{2K_G!}(N_i)$. Each element g_k can be written as a word $g_k(\boldsymbol{s})$, and the injectivity of the induced homomorphism

$$\overline{\varphi_{\boldsymbol{g'}}} : N_i/D_{2K_G!}(N_i) \to N_{G'}(\varphi_{\boldsymbol{g'}}(C_i))/D_{2K_G!}(N_{G'}(\varphi_{\boldsymbol{g'}}(C_i))),$$

well-defined since $K_{G'} \leq K_G$, translates into a universal formula $\mathrm{InjD}^{\forall}_i(\boldsymbol{g'})$ expressing the fact that $g_k^{-1}(\boldsymbol{g'})g_\ell(\boldsymbol{g'})$ does not belong to $D_{2K_G!}(N_{G'}(\varphi_{\boldsymbol{g'}}(C_i)))$ if $k \neq \ell$.

Now, let us define the formula $\mathrm{Special}^{\forall}(\boldsymbol{x})$ by

$$\mathrm{Special}^{\forall}(\boldsymbol{x}) : (\Sigma(\boldsymbol{x}) = 1) \wedge \bigwedge_{i=1}^{q} \bigwedge_{j \neq i} \mathrm{NonConj}^{\forall}_{i,j}(\boldsymbol{x}) \wedge \bigwedge_{i \in I} (\mathrm{Inj}^1_i(\boldsymbol{g'}) \wedge \mathrm{Inj}^2_i(\boldsymbol{g'}) \wedge \mathrm{InjD}^{\forall}_i(\boldsymbol{x})).$$

For any $\boldsymbol{g'} \in G'^p$, the group G' satisfies $\mathrm{Special}^{\forall}(\boldsymbol{g'})$ if and only if the homomorphism $\varphi_{\boldsymbol{g'}} : G \to G' : \boldsymbol{s} \mapsto \boldsymbol{g'}$ is special. □

The following result is a slight variation of the previous proposition.

Proposition 3.3. *Let $G = \langle s_1, \ldots, s_p\rangle$ be a hyperbolic group. For every integer $n \geq 1$, there exists a universal formula*

$$\mathrm{Special}^{\forall}_n(x_1, \ldots, x_p)$$

such that, for every hyperbolic group G' satisfying $K_{G'} \leq K_G$, and for every tuple $\boldsymbol{g'} = (g'_1, \ldots, g'_p)$ of elements of G', the following two assertions are equivalent:

(1) *the group G' satisfies* $\mathrm{Special}^{\forall}_n(\boldsymbol{g'})$*;*
(2) *the map $\varphi_{\boldsymbol{g'}} : \{s_1, \ldots, s_p\} \to G'$ defined by $s_i \mapsto g'_i$ for $1 \leq i \leq p$ extends to a special (see Definition 1.19) homomorphism from G to G' injective on the ball of radius n in G, with respect to $\{s_1, \ldots, s_p\}$.*

Proof. We keep the same notations as in the proof of the previous proposition.

The injectivity of the homomorphism $\varphi_{\boldsymbol{g'}} : \boldsymbol{s} = (s_1, \ldots, s_p) \mapsto \boldsymbol{g'} \in \mathrm{Sol}_{G'}(\Sigma)$ on the ball of radius n centered at 1 in G for the generating set $\{s_1, \ldots, s_p\}$ is expressible by means of a finite system of inequations in p variables $B_n(\boldsymbol{g'}) \neq 1$.

The universal sentence $\mathrm{Special}^{\forall}_n(\boldsymbol{x})$ defined by

$$\mathrm{Special}^{\forall}_n(\boldsymbol{x}) : \mathrm{Special}^{\forall}(\boldsymbol{x}) \wedge (B_n(\boldsymbol{x}) \neq 1)$$

has a witness $\boldsymbol{g'} \in G'^p$ if and only if the homomorphism $\varphi_{\boldsymbol{g'}} : G \to G' : \boldsymbol{s} \mapsto \boldsymbol{g'}$ is special and injective on the ball of radius n in G. □

Corollary 3.4. *Let G and G' be two hyperbolic groups. If* $\mathrm{Th}_{\exists\forall}(G) \subset \mathrm{Th}_{\exists\forall}(G')$, *then there exists a discriminating sequence of special homomorphisms* $(\varphi_n : G \to G')_{n\in\mathbb{N}}$.

Proof. Note that for every integer $n \geq 1$, the group G satisfies the $\exists\forall$-sentence

$$\zeta_n : \exists \boldsymbol{x}\ \mathrm{Special}_n^{\forall}(\boldsymbol{x}),$$

since the identity of G is a special homomorphism. Then, $\mathrm{Th}_{\exists\forall}(G)$ being contained in $\mathrm{Th}_{\exists\forall}(G')$, the group G' satisfies the sentence ζ_n as well. Since G and G' have the same existential theory, we have $K_G = K_{G'}$, hence Proposition 3.3 above applies and tells us that there exists a special homomorphism $\varphi_n : G \to G'$ injective on B_n. □

The following corollary is immediate.

Corollary 3.5 (Implication (1) $\Rightarrow$ (3) of Theorem 1.23). *Let G and G' be two hyperbolic groups. If* $\mathrm{Th}_{\exists\forall}(G) = \mathrm{Th}_{\exists\forall}(G')$, *then there exists two discriminating sequences of special homomorphisms* $(\varphi_n : G \to G')_{n\in\mathbb{N}}$ *and* $(\varphi'_n : G' \to G)_{n\in\mathbb{N}}$.

4. Definition of ζ_G and proofs of (2) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (4)

4.1. Preliminary results.

Definition 4.1. Let G be a hyperbolic group. A *G-chain* is a tuple $(g_1, \ldots, g_c)$ of elements of G of infinite order such that the inclusions

$$M(g_1) \supset (M(g_1) \cap M(g_2)) \supset \cdots \supset (M(g_1) \cap \cdots \cap M(g_c))$$

are all strict.

Definition 4.2. Let G be a hyperbolic group, and let H be a non-elementary subgroup of G. The *complexity* $c(H)$ of H is the maximal size of a G-chain of elements of H.

Remark 4.3. If (g_1, g_2) is a G-chain, then $M(g_1) \cap M(g_2)$ is finite. It follows that $c(H) < \infty$.

The following lemma is an immediate consequence of the fact that $E_G(H) = \bigcap_{h\in H^0} M(h)$, where H^0 denotes the set of elements of H of infinite order.

Lemma 4.4. *If* $(h_1, \ldots, h_{c(H)}) \in H^{c(H)}$ *is a G-chain of length $c(H)$, then*

$$E_G(H) = \bigcap_{i=1}^{c(H)} M(h_i).$$

Lemma 4.5. *Let $N \geq 1$ and $K \geq 1$ be two integers. There exists an existential formula* $\mathrm{Chain}_N^{\exists}(\boldsymbol{x})$ *with N free variables such that, for any hyperbolic group G all of whose finite subgroups have order $\leq K$, a tuple $\boldsymbol{g} \in G^N$ is a G-chain if and only if* $G \models \mathrm{Chain}_N^{\exists}(\boldsymbol{g})$.

Proof. Let $\boldsymbol{g} \in G^N$. The fact that every g_k has infinite order translates into $g_k^{K!} \neq 1$. Recall that $M(g_k) = \{g \in G \mid [g_k^{K!}, g g_k^{K!} g^{-1}] = 1\}$. The N-tuple $\boldsymbol{g}$ is a chain if and only if, for every $1 \leq k < N$, there exists an element x_k in $\bigcap_{i=1}^{k} M(g_i)$ that does not belong to $M(g_{k+1})$. This condition is clearly expressible by means of an existential sentence $\mathrm{Chain}_N^{\exists}(\boldsymbol{g})$. □

Strongly special homomorphisms are not definable by means of a $\forall\exists$-sentence. For that reason, we introduce below a weaker definition.

Definition 4.6. Let G and G' be hyperbolic groups. A special homomorphism $\varphi : G \to G'$ is said to be *pre-strongly special* if, for every finite subgroup C of G, the following conditions hold.

1. If the normalizer $N_G(C)$ is virtually cyclic, then φ is injective in restriction to $N_G(C)$.

(2) If the normalizer $N_G(C)$ is not virtually cyclic, then
   (a) the normalizer $N_{G'}(\varphi(C))$ is not virtually cyclic, and
   (b) there exists a G-chain $(h_1, \ldots, h_c)$, with $h_i \in N_G(C)$ and $c := c(N_G(C))$ (see Definition 4.2), such that $(\varphi(h_1), \ldots, \varphi(h_c))$ is a G'-chain.

In the case where two pre-strongly special homomorphisms $\varphi : G \to G'$ and $\varphi' : G' \to G$ exist simultaneously, then these two homomorphisms are in fact strongly special, as shown by the following lemma.

LEMMA 4.7. *Let G and G' be hyperbolic groups. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be pre-strongly special homomorphisms. Then φ and φ' are strongly special.*

PROOF. Let C be a finite subgroup of G such that $N_G(C)$ is non-elementary. By definition of a pre-strongly special homomorphism, there exists a G-chain $(h_1, \ldots, h_c)$, with $h_i \in N_G(C)$ and $c := c(N_G(C))$, such that $(\varphi(h_1), \ldots, \varphi(h_c))$ is a G'-chain of elements of $N_{G'}(\varphi(C))$. As a consequence, the maximal size of a G'-chain of elements of $N_{G'}(\varphi(C))$ is $\geq c$, so

$$c = c(N_G(C)) \leq c(N_{G'}(\varphi(C))).$$

Similarly, for every finite subgroup C' of G', the following inequality holds:

$$c(N_{G'}(C')) \leq c(N_G(\varphi'(C'))).$$

Now, note that the homomorphisms φ and φ' induce two bijections between the conjugacy classes of finite subgroups of G and G', because φ and φ' are special. It follows that the previous inequalities are in fact equalities. In particular, $(\varphi(h_1), \ldots, \varphi(h_c))$ is a maximal G'-chain of elements of $N_{G'}(\varphi(C))$. So we have

$$E := E_G(N_G(C)) = \bigcap_{i=1}^{c} M(h_i) \quad \text{and} \quad E' := E_{G'}(N_{G'}(\varphi(C))) = \bigcap_{i=1}^{c} M(\varphi(h_i)).$$

It follows that

$$\varphi(E) \subset \bigcap_{i=1}^{c} \varphi(M(h_i)) \subset \bigcap_{i=1}^{c} M(\varphi(h_i)) = E'.$$

By symmetry, these inclusions are in fact equalities. □

4.2. Definition of ζ_G. Recall that the universal formula $\mathrm{Special}^{\forall}(\boldsymbol{x})$ is defined in Proposition 3.2.

PROPOSITION 4.8. *Let $G = \langle s_1, \ldots, s_p \rangle$ be a hyperbolic group. There exists an existential formula*

$$\mathrm{PreStrong}^{\exists}(x_1, \ldots, x_p)$$

such that, for every hyperbolic group G' satisfying $K_{G'} \leq K_G$, and for every tuple $\boldsymbol{g'} = (g'_1, \ldots, g'_p)$ of elements of G', the following two assertions are equivalent:

(1) the group G' satisfies $\mathrm{Special}^{\forall}(\boldsymbol{g'}) \wedge \mathrm{PreStrong}^{\exists}(\boldsymbol{g'})$;
(2) the map $\varphi_{\boldsymbol{g'}} : \{s_1, \ldots, s_p\} \to G'$ defined by $s_i \mapsto g'_i$ for $1 \leq i \leq p$ extends to a pre-strongly special homomorphism from G to G'.

PROOF. Let $G = \langle s_1, \ldots, s_p \mid \Sigma(s_1, \ldots, s_p) = 1 \rangle$ be a finite presentation of G, where $\Sigma(s_1, \ldots, s_p) = 1$ denotes a finite system of equations in p variables. Let $\{C_1, \ldots, C_q\}$ be a set of representatives of the conjugacy classes of finite subgroups of G. Let I be the subset of $[\![1, q]\!]$ such that $N_i = N_G(C_i)$ is virtually cyclic if and only if $i \in I$.

For $i \notin I$, there exists a quantifier-free formula $\mathrm{NonVC}_i(\boldsymbol{x})$ such that, for every $\boldsymbol{g'}$ in $\mathrm{Sol}_{G'}(\Sigma)$, the group $N_{G'}(\varphi_{\boldsymbol{g'}}(C_i))$ is not virtually cyclic if and only if $G' \models \mathrm{NonVC}_i(\boldsymbol{g'})$. This uses the fact that $K_{G'} \leq K_G$, which implies that two elements g'_1, g'_2 of G' of infinite order generate a non virtually cyclic subgroup if and only if $[{g'_1}^{K_G!}, {g'_2}^{K_G!}] \neq 1$ (see Corollary 2.8).

For $i \notin I$, let $E_i := E_G(N_G(C_i))$ and $c_i := c(N_G(E_i))$ (see Definition 4.1). Consider $\boldsymbol{h}_i \in N_G(E_i)^{c_i}$ a G-chain. This chain can be written as a c_i-tuple of words $\boldsymbol{h}_i(\boldsymbol{s})$.

We define the existential formula $\mathrm{PreStrong}^{\exists}(x_1, \ldots, x_p)$ as follows:

$$\mathrm{PreStrong}^{\exists}(\boldsymbol{x}) : \bigwedge_{i \notin I} \left(\mathrm{NonVC}_i(\boldsymbol{x}) \wedge \mathrm{Chain}_i^{\exists}(\boldsymbol{h}_i(\boldsymbol{x}))\right),$$

where $\mathrm{Chain}_i^{\exists}(\boldsymbol{h}_i)$ denotes the $\exists$-formula given by Lemma 4.5.

For any $\boldsymbol{g}' \in G'^p$, the group G' satisfies $\mathrm{Special}^{\forall}(\boldsymbol{g}') \wedge \mathrm{PreStrong}^{\exists}(\boldsymbol{g}')$ if and only if the homomorphism $\varphi_{\boldsymbol{g}'} : G \to G' : \boldsymbol{s} \mapsto \boldsymbol{g}'$ is pre-strongly special. $\square$

Corollary 4.9 below, which follows immediately from Proposition 4.8 above, tells us that the existence of a pre-strongly special homomorphism from G to a hyperbolic group G' such that $K_{G'} \leq K_G$ is captured by a single $\exists\forall$-sentence ζ_G that does not depend on G'.

Corollary 4.9. *Let G be a hyperbolic group. Let us define the $\exists\forall$-sentence ζ_G by*

$$\zeta_G : \exists \boldsymbol{x}\ \mathrm{Special}^{\forall}(\boldsymbol{x}) \wedge \mathrm{PreStrong}^{\exists}(\boldsymbol{x}).$$

Note that this sentence is satisfied by G since the identity of G is a pre-strongly special homomorphism. For every hyperbolic group G' such that $K_{G'} \leq K_G$, the following two assertions are equivalent:

(1) the group G' satisfies ζ_G;
(2) there exists a pre-strongly special homomorphism from G to G'.

4.3. Proof of (2) $\Leftrightarrow$ (4)**.** The equivalence (2) $\Leftrightarrow$ (4) of Theorem 1.23 follows immediately from Corollary 4.9 together with Lemma 4.7.

Corollary 4.10 (Equivalence (2) $\Leftrightarrow$ (4) of Theorem 1.23). *Let G and G' be hyperbolic groups. There exist strongly special homomorphisms $\varphi : G \to G'$ and $\varphi' : G' \to G$ if and only if $G' \models \zeta_G$ and $G \models \zeta_{G'}$.*

Proof. Suppose that the morphisms φ and φ' exist. Since they are injective on finite groups, we have $K_G = K_{G'}$. Hence Corollary 4.9 applies and guarantees that G satisfies $\zeta_{G'}$ and G' satisfies ζ_G.

Conversely, suppose that G satisfies $\zeta_{G'}$ and G' satisfies ζ_G. Up to exchanging G and G', one can assume without loss of generality that $K_{G'} \leq K_G$. By Corollary 4.9, there exists a pre-strongly special homomorphism $\varphi : G \to G'$. As a consequence, since φ is injective on finite groups, we have $K_G = K_{G'}$. Then again by Corollary 4.9, there exists a pre-strongly special homomorphism $\varphi' : G' \to G$. $\square$

4.4. Proof of (3) $\Rightarrow$ (4)**.**

Proposition 4.11 ((3) $\Rightarrow$ (4)). *Let G and G' be hyperbolic groups. Let $(\varphi_n : G \to G')_{n \in \mathbb{N}}$ and $(\varphi'_n : G' \to G)_{n \in \mathbb{N}}$ be two discriminating sequences of special homomorphisms. Then, for n large enough, $\varphi_n : G \to G'$ and $\varphi'_n : G' \to G$ are strongly special.*

Note that $K_G = K_{G'}$, with G and G' as above. Indeed, φ_n and φ'_n are injective on finite groups for n large enough. So Proposition 4.11 is an immediate consequence of the following result combined with Lemma 4.7.

Proposition 4.12. *Let G and G' be two hyperbolic groups, and let $(\varphi_n : G \to G')_{n \in \mathbb{N}}$ be a discriminating sequence of special homomorphisms. Suppose that $K_G = K_{G'}$. Then φ_n is pre-strongly special, for n large enough.*

Proof. We keep the same notations as in the proof of Proposition 4.9 above. Note that the existential formula $\mathrm{PreStrong}^{\exists}(\boldsymbol{x})$ is satisfied by the generating set $\boldsymbol{s} = \boldsymbol{x}$ of G. So, for n large enough, the statement $\mathrm{PreStrong}^{\exists}(\varphi_n(\boldsymbol{s}))$ is satisfied by G' since the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is discriminating. Moreover, the morphism φ_n being special, the tuple $\varphi_n(\boldsymbol{s})$ satisfies the universal formula $\mathrm{Special}^{\forall}(\boldsymbol{x})$. Thus, the group G' satisfies the statement $\mathrm{Special}^{\forall}(\varphi_n(\boldsymbol{s}))\wedge\mathrm{PreStrong}^{\exists}(\varphi_n(\boldsymbol{s}))$. It follows from Proposition 4.8 that φ_n is pre-strongly special. $\square$

REMARK 4.13. Under the hypotheses of Proposition 4.12 above, φ_n is not necessarily strongly special for n large enough. More precisely, $\varphi_n(E_G(N_G(C)))$ is not necessarily equal to $E_{G'}(N_{G'}(\varphi_n(C)))$ for n large enough, as shown by the following example. Let

$$G = \langle c \mid c^4 = 1\rangle \times F_2 \quad\text{and}\quad G' = \langle G, t \mid [t, c^2] = 1\rangle.$$

Note that $E_G(N_G(c^2)) = E_G(G) = \langle c\rangle$, and that $E_{G'}(N_{G'}(c^2)) = E_{G'}(G') = \langle c^2\rangle$. Thus, the inclusion φ of G into G' satisfies

$$\varphi(E_G(N_G(c^2))) \not\subset E_{G'}(N_{G'}(\varphi(c^2))).$$

Note that, in this example, there is no discriminating sequence $(\varphi'_n : G' \to G)$ since the commutator $[t, c]$ is killed by any homomorphism $\varphi' : G' \to G$.

5. Proof of Theorems 1.9 and 1.10

In this section, we prove Theorem 1.9 and Theorem 1.10. The proof of these theorems consists mainly in revisiting and generalizing the key lemma of Sacerdote's paper [**Sac73b**] dating from 1973, using some of the tools developed since then by Sela and others (theory of group actions on real trees, shortening argument, test sequences).

5.1. Legal large extensions. Let G be a hyperbolic group, and let C_1, C_2 be two finite subgroups of G. Suppose that C_1 and C_2 are isomorphic, and let $\alpha : C_1 \to C_2$ be an isomorphism. We want to find necessary and sufficient conditions under which $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G*_\alpha)$. As observed in the introduction, if G and $G*_\alpha$ have the same $\forall\exists$-theory, then G is non-elementary. Indeed, a hyperbolic group is finite if and only if it satisfies the first-order sentence $\forall x\ (x^N = 1)$ for some integer $N \geq 1$, and virtually cyclic if and only if it satisfies $\forall x\forall y\ ([x^N, y^N] = 1)$ for some integer $N \geq 1$. We therefore restrict attention to non-elementary hyperbolic groups. We will prove the following result.

THEOREM 5.1 (see Theorems 1.9 and 1.10). *Let G be a non-elementary hyperbolic group, and let C_1 and C_2 be two finite isomorphic subgroups of G. Let $\alpha : C_1 \to C_2$ be an isomorphism. Let us consider the HNN extension $\Gamma = G*_\alpha = \langle G, t \mid \mathrm{ad}(t)_{|C_1} = \alpha\rangle$. The following three assertions are equivalent.*

(1) The inclusion of G into Γ is a $\exists\forall\exists$-elementary embedding (see Definition 2.4).
(2) $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$.
(3) The group $\Gamma = G*_\alpha$ is a legal large extension of G in the sense of Definition 1.6, i.e. there exists an element $g \in G$ such that $gC_1g^{-1} = C_2$ and $\mathrm{ad}(g)_{|C_1} = \alpha$, the normalizer $N_G(C_1)$ is non-elementary, and $E_G(N_G(C_1)) = C_1$.

The implication (1) $\Rightarrow$ (2) of Theorem 5.1 is obvious. In order to prove the implication (2) $\Rightarrow$ (3), the first step consists in finding some $\forall\exists$-invariants of hyperbolic groups, i.e. some numbers that are preserved by $\forall\exists$-equivalence among hyperbolic groups.

DEFINITION 5.2. Let G be a hyperbolic group. We associate to G the following five integers:

- the number $n_1(G)$ of conjugacy classes of finite subgroups of G,
- the sum $n_2(G)$ of $|\mathrm{Aut}_G(C_k)|$ for $1 \leq k \leq n_1(G)$, where the C_k are representatives of the conjugacy classes of finite subgroups of G, and

$$\mathrm{Aut}_G(C_k) = \{\alpha \in \mathrm{Aut}(C_k) \mid \exists g \in N_G(C_k),\ \mathrm{ad}(g)_{|C} = \alpha\},$$

- the number $n_3(G)$ of conjugacy classes of finite subgroups C of G such that $N_G(C)$ is infinite virtually cyclic,

- the number $n_4(G)$ of conjugacy classes of finite subgroups C of G such that $N_G(C)$ is non-elementary,
- the number $n_5(G)$ of conjugacy classes of finite subgroups C of G such that $N_G(C)$ is non-elementary and $E(N_G(C)) \neq C$.

One can easily see that these numbers are preserved by elementary equivalence. However, proving that they are preserved by $\forall\exists$-equivalence is a little bit more tedious.

LEMMA 5.3. *Let G and G' be two hyperbolic groups. Suppose that $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G')$. Then $n_i(G) = n_i(G')$, for $1 \leq i \leq 5$.*

As usual, we denote by K_G the maximal order of a finite subgroup of G. Since G and G' have the same existential theory, we have $K_G = K_{G'}$.

PROOF. Let $n \geq 1$ be an integer. If $n_1(G) \geq n$, then the following $\exists\forall$-sentence, written in natural language for convenience of the reader and denoted by $\theta_{1,n}$, is satisfied by G: there exist n finite subgroups $F_1, \ldots, F_n$ of G such that, for every $g \in G$ and $1 \leq i \neq j \leq n$, the groups gF_ig^{-1} and F_j are distinct. Since G and G' have the same $\exists\forall$-theory, the sentence $\theta_{1,n}$ is satisfied by G' as well. As a consequence, $n_1(G') \geq n$. It follows that $n_1(G') \geq n_1(G)$. By symmetry, we have $n_1(G) = n_1(G')$.

In the rest of the proof, we give similar sentences $\theta_{2,n}, \ldots, \theta_{5,n}$ such that the following series of implications hold: $n_i(G) \geq n \Rightarrow G$ satisfies $\theta_{i,n} \Rightarrow G'$ satisfies $\theta_{i,n} \Rightarrow n_i(G') \geq n$.

If $n_2(G) \geq n$, then G satisfies $\theta_{2,n}$: there exist ℓ finite subgroups $F_1, \ldots, F_\ell$ of G and a finite subset $\{g_{i,j}\}_{1\leq i\leq \ell,\ 1\leq j\leq n_i}$ of G, with $n_1 + \cdots + n_\ell = n$, such that for every $g \in G$ and $1 \leq p \neq q \leq \ell$, the groups gF_pg^{-1} and F_q are distinct, and for every $1 \leq i \leq \ell$, we have:

- for every $1 \leq j \leq n_i$, the element $g_{i,j}$ normalizes F_i,
- and for every $1 \leq j \neq k \leq n_i$, the automorphisms $\mathrm{ad}(g_j)_{|F_i}$ and $\mathrm{ad}(g_k)_{|F_i}$ of F_i are distinct.

If $n_3(G) \geq n$, then G satisfies $\theta_{3,n}$: there exist n finite subgroups $F_1, \ldots, F_n$ of G such that, for every $g \in G$ and $1 \leq i \neq j \leq n$, the groups gF_ig^{-1} and F_j are distinct, and for every $1 \leq i \leq n$ and $g, h \in N_G(F_i)$, we have $[g^{K_G!}, h^{K_G!}] = 1$.

If $n_4(G) \geq n$, then G satisfies $\theta_{4,n}$: there exist n finite subgroups $F_1, \ldots, F_n$ of G and a finite subset $\{g_{i,j}\}_{1\leq i\leq n,\ 1\leq j\leq 2}$ of G such that, for every $g \in G$ and $1 \leq p \neq q \leq n$, the groups gF_pg^{-1} and F_q are distinct, and for every $1 \leq i \leq n$, we have:

- $g_{i,1}$ and $g_{i,2}$ normalize F_i,
- and $[g_{i,1}^{K_G!}, g_{i,2}^{K_G!}] \neq 1$.

If $n_5(G) \geq n$, then G satisfies $\theta_{5,n}$: there exist $2n$ finite subgroups $F_1, \ldots, F_n$ and $F'_1, \ldots, F'_n$ of G and a finite subset $\{g_{i,j}\}_{1\leq i\leq n,\ 1\leq j\leq 2}$ of G such that, for every $g \in G$ and $1 \leq p \neq q \leq n$, the groups gF_pg^{-1} and F_q are distinct, and for every $1 \leq i \leq n$, we have:

- $g_{i,1}$ and $g_{i,2}$ normalize F_i,
- $[g_{i,1}^{K_G!}, g_{i,2}^{K_G!}] \neq 1$,
- for every $h \in G$, if h normalizes F_i then h normalizes F'_i,
- and F_i is strictly contained in F'_i.

This concludes the proof of the lemma. □

As an application, we prove the implication (2) $\Rightarrow$ (3) of Theorem 5.1.

PROPOSITION 5.4 (Implication (2) $\Rightarrow$ (3) of Theorem 5.1). *We keep the same notations as in Theorem 5.1. If $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$, then Γ is a legal large extension of G.*

PROOF. Thanks to the previous lemma, we know that $n_i(G) = n_i(\Gamma)$ for $1 \leq i \leq 5$. Recall that $\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C_1} = \alpha \rangle$, where $\alpha : C_1 \to C_2$ is an isomorphism. The equality

$n_1(G) = n_1(\Gamma)$ implies that $gC_1g^{-1} = C_2$ for some $g \in G$. It follows that $g^{-1}t$ induces an automorphism of C_1.

Then, observe that for every finite subgroup F of G, we have $|\mathrm{Aut}_\Gamma(F)| \geq |\mathrm{Aut}_G(F)|$. Thus, the equality $n_2(G) = n_2(\Gamma)$ guarantees that $|\mathrm{Aut}_\Gamma(C_1)| = |\mathrm{Aut}_G(C_1)|$. Hence, since $\mathrm{ad}(g^{-1}t)_{|C_1}$ belongs to $\mathrm{Aut}_\Gamma(C_1)$, there exists an element $g' \in N_G(C_1)$ such that $\mathrm{ad}(g')_{|C_1} = \mathrm{ad}(g^{-1}t)_{|C_1}$. Therefore, $gg' \in G$ induces by conjugacy the same automorphism of C_1 as t, which proves that the first condition of Definition 1.6 holds.

The equalities $n_3(G) = n_3(\Gamma)$ and $n_4(G) = n_4(\Gamma)$ ensure that $N_G(C_1)$ is non-elementary. Indeed, if $N_G(C_1)$ were finite, then $N_\Gamma(C_1)$ would be infinite virtually cyclic and $n_3(\Gamma)$ would be at least $n_3(G)+1$; similarly, if $N_G(C_1)$ were infinite virtually cyclic, then $N_\Gamma(C_1)$ would be non-elementary and $n_4(\Gamma) \geq n_4(G)+1$. Hence, the second condition of Definition 1.6 is satisfied.

Lastly, it follows from the fact that $n_5(G) = n_5(\Gamma)$ that $E_G(N_G(C_1)) = C_1$, otherwise $n_5(\Gamma) \geq n_5(G) + 1$, since $E_\Gamma(N_\Gamma(C_1)) = C_1$. Thus, the third condition of Definition 1.6 holds. As a conclusion, Γ is a legal large extension of G. $\square$

We shall now prove the difficult part of Theorem 5.1, namely the following result.

PROPOSITION 5.5 (Implication (3) $\Rightarrow$ (1) of Theorem 5.1). *We keep the same notations as in Theorem 5.1. If Γ is a legal large extension of G, then the inclusion of G into Γ is a $\exists\forall\exists$-elementary embedding.*

First, we define a notion of test sequences adapted to our context.

5.2. Test sequences.

DEFINITION 5.6. Let G be a non-elementary hyperbolic group, and let

$$\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C} = \mathrm{id}_C \rangle$$

be a legal extension of G over a finite subroup C. Let $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ be a sequence of homomorphisms. For every integer n, let $t_n := \varphi_n(t)$. The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is called a test sequence if the following conditions hold:

(1) for every n, the morphism φ_n is a retraction, i.e. $\varphi_n(g) = g$ for every $g \in G$;
(2) the translation length $||t_n||$ of t_n goes to infinity when n goes to infinity;
(3) there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 such that, for every n, the element t_n satisfies the ε_n-small cancellation condition (see Definition 2.10), and $M(t_n) = \langle t_n \rangle \times C$. Therefore, the image of t_n in $M(t_n)/C$ has not root.

REMARK 5.7. Note that any subsequence of a test sequence is a test sequence as well.

The following easy lemma will be useful in the sequel.

LEMMA 5.8. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a test sequence. For every infinite subset $A \subset \mathbb{N}$, we have*

$$\bigcap_{n\in A} M(t_n) = C.$$

PROOF. Suppose that g belongs to $M(t_n)$ for every $n \in A$. Then, there exists an integer k_n and an element $c_n \in C$ such that $g = t_n^{k_n}c_n$, for every $n \in A$. Now, observe that k_n must be equal to 0 for every n large enough, otherwise (up to extracting a subsequence) $||t_n^{k_n}||$ goes to infinity, and so does the constant $||g||$, which is a contradiction. It follows that g belongs to C. $\square$

The following result is well-known, however we are not aware of a reference in the literature.

LEMMA 5.9. *Let G be a δ-hyperbolic group, and let C be a finite subgroup of G. Then the centralizer $C_G(C)$ of C is quasi-convex in G.*

Proof. Let $c \in C$, and let $h \in C_G(c)$. Let x be a vertex on the geodesic $[1, h]$ in the Cayley graph of G, for a given finite generating set S of G. Let d_S be the induced metric on G. Since the elements c and $x^{-1}cx$ are conjugate, there exists an element y such that $x^{-1}cx = ycy^{-1}$ and $d_S(1, y) \leq 2d_S(1, c) + R(\delta, |S|) =: K$ according to Proposition 2.3 in [**BH05**], where $R(\delta, |S|)$ is a constant that only depends on δ and $|S|$. Observe that xy centralizes c, and that $d_S(x, xy) = d_S(1, y) \leq K$. This shows that $C_G(c)$ is K-quasi-convex in G. Now, recall that the intersection of two quasi-convex subgroups of a hyperbolic group is still quasi-convex (see for instance [**Sho91**]). Hence, since C is finite, the intersection $C_G(C) = \cap_{c \in C} C_G(c)$ is quasi-convex in G. □

We now build a test sequence.

Proposition 5.10. *Let G be a non-elementary hyperbolic group, and let*

$$\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C} = \mathrm{id}_C \rangle$$

be a legal extension of G over a finite subroup C. Then, there exists a test sequence $(\varphi_n : \Gamma \to G)_{n \in \mathbb{N}}$.

Proof. We proceed in two stages.

(1) First, we define a subgroup $\langle a, b \rangle$ of the centralizer $C_G(C)$ of C in G that is free of rank 2 and quasi-convex in G.
(2) Then, we build a sequence of elements $(t_n)_{n \in \mathbb{N}}$ that satisfies the $(1/n)$ -small cancellation condition in the free group $\langle a, b \rangle$, and check that, for every n, the retraction $\varphi_n : \Gamma \twoheadrightarrow G : t \mapsto t_n$ is well-defined and has the expected properties.

Recall that $E_G(N_G(C)) = C$. So, by [**Os93**] Lemma 3.3, there exists an element $a \in C_G(C)$ of infinite order such that $M(a) = \langle a \rangle \times C$ in G. Recall that $N_G(C)$ is non-elementary, by definition of a legal large extension. As a finite-index subgroup of $N_G(C)$, the centralizer $C_G(C)$ is non-elementary as well. By [**Cha12**] Corollary I.1.9, there exists an element $b \in C_G(C)$ of infinite order such that the subgroup of $C_G(C)$ generated by $\{a, b, C\}$ is quasi-convex in $C_G(C)$ and isomorphic to $\langle a, C \rangle *_C \langle b, C \rangle = C \times (\langle a \rangle * \langle b \rangle)$. Hence, the subgroup $\langle a, b \rangle$ is quasi-convex in $C_G(C)$, and free of rank 2. By Lemma 5.9, the centralizer $C_G(C)$ of C is quasi-convex in G. Thus, the free group $\langle a, b \rangle$ is quasi-convex in G as well. For any integer $n \geq 0$, we set $t_n = a^n b a^{n+1} b \cdots a^{2n} b$. Let $\varphi_n : \Gamma \twoheadrightarrow G$ be the retraction defined by $\varphi_n(t) = t_n$ and $\varphi_n(g) = g$ for all $g \in G$, which is well-defined since t_n centralizes C. We will prove that $(\varphi_n)_{n \in \mathbb{N}}$ is a test sequence. Since the definition of a test sequence is clearly invariant by a change of choice of a finite generating set of G, let us consider a convenient finite generating set S of G that contains the two elements a and b introduced above. Let (X, d) denote the Cayley graph of G for this generating set. Let τ_n be the path of X that links 1 to t_n and is labeled with the word t_n in a and b, and consider the bi-infinite path $\overline{\tau}_n = \cup_{k \in \mathbb{Z}} t_n^k \tau_n$. A standard argument shows that $\overline{\tau}_n$ is a quasi-geodesic in (X, d), for some constants that do not depend on n. Consequently, $\overline{\tau}_n$ lies in the λ-neighborhood of $A(t_n)$ for some constant $\lambda \geq 0$ independent from n. Similarly, let α be the edge of X linking 1 to a, let $\overline{\alpha}$ denote the quasi-geodesic $\overline{\alpha} = \cup_{k \in \mathbb{Z}} a^k \alpha$ and let μ be a constant such that $\overline{\alpha}$ lies in the μ-neighborhood of $A(a)$.

Let d' denote the metric in the free group $\langle a, b \rangle$ for the generating set $\{a, b\}$. Since t_n is cyclically reduced in $\langle a, b \rangle$, $||t_n||_{d'} = d'(1, t_n) \sim (3/2)n^2$. Since $\langle a, b \rangle$ is quasi-convex in G by construction, there is a constant $R > 0$ such that $||t_n|| \geq Rn^2$ for all n large enough.

It remains to prove the third condition of Definition 5.6. Classically, since the element a has infinite order, there exists a constant $N \geq 0$ such that, for every element $g \in G$, if $\Delta(a, gag^{-1}) \geq N$, then g belongs to $M(a) = \langle a \rangle \times C$ (see paragraph 2.4). Let n_0 be an integer such that Rn_0 is large compared to $N' = N + 204\delta + 2\lambda + 2\mu$. We will show that for every $n \geq n_0$, the element t_n satisfies the $(1/n)$-small cancellation condition. Let n be an integer greater than n_0. Consider an element $g \in G$ such that $\Delta(t_n, gt_ng^{-1}) \geq ||t_n||/n$. We will show that g belongs to the subgroup $\langle t_n \rangle \times C$.

We first show that g belongs to the subgroup $\langle a, b\rangle \times C$. Since

$$\Delta(t_n, gt_ng^{-1}) \geq ||t_n||/n \geq Rn \geq Rn_0 >> N',$$

we can choose two subpaths ν_n and μ_n of $\overline{\tau}_n$ and $g\overline{\tau}_n$ respectively, of length N' and labeled by $a^{N'}$, such that $\mathrm{diam}((\nu_n)^{+(100\delta+\lambda)} \cap (\mu_n)^{+(100\delta+\lambda)}) \geq N'$. Denoting by x_n and y_n the initial points of ν_n and μ_n respectively, we have $\mathrm{diam}(x_n\overline{\alpha}^{+(100\delta+\lambda)} \cap y_n\overline{\alpha}^{+(100\delta+\lambda)}) \geq N'$. It follows that $\mathrm{diam}(A(a)^{+(100\delta+\lambda+\mu)} \cap x_n^{-1}y_nA(a)^{+(100\delta+\lambda+\mu)}) \geq N'$. By Lemma 2.13 in [**Cou13**], we have:

$$\begin{aligned}\Delta(a, x_n^{-1}y_nax_n^{-1}y_n{}^{-1}) &\geq \mathrm{diam}(A(a)^{+(100\delta+\lambda+\mu)} \cap x_n^{-1}y_nA(a)^{+(100\delta+\lambda+\mu)}) - (204\delta + 2\lambda + 2\mu)\\ &\geq N' - (204\delta + 2\lambda + 2\mu) = N.\end{aligned}$$

So $x_n^{-1}y_n$ belongs to $M(a) = \langle a\rangle \times C$. Now, observe that x_n is a word in a and b since it is on the quasi-geodesic $\overline{\tau}_n$. Similarly, y_n can be written as $y_n = gz_n$ with z_n a word in a and b. It follows that g belongs to the subgroup $\langle a, b\rangle \times C$.

Up to replacing g with gc for some $c \in C$, we can now assume that g belongs to the free group $\langle a, b\rangle$. This does not affect the condition $\Delta(t_n, gt_ng^{-1}) \geq ||t_n||/n$; indeed, $gct_ngc^{-1} = gt_ng^{-1}$, since t_n centralizes C. Recall in addition that the group $\langle a, b\rangle$ is quasi-isometrically embedded into G. Denoting by $A > 0$ and $B \geq 0$ two constants such that

$$\frac{1}{A}d'(x, y) - B \leq d(x, y) \leq Ad'(x, y) + B$$

for all $x, y \in \langle a, b\rangle \times C$, we can verify that the following inequality holds, in the Cayley graph Y of the free group $\langle a, b\rangle$ equipped with the distance d':

$$\mathrm{diam}\left((\overline{\tau}_n)^{+(A(100\delta+B)+1)} \cap (g\overline{\tau}_n)^{+(A(100\delta+B)+1)}\right) \geq d'(1, t_n)/(2A^2n).$$

Since the Cayley graph of $\langle a, b\rangle$ is a tree, this inequality tells us that the axes of t_n and gt_ng^{-1} have an overlap of length larger than $4n - 2$ in this tree.

To conclude, let us observe that two distinct cyclic conjugates of $a^nba^{n+1}b \cdots a^{2n}b$ have at most their first $4n - 2$ letters in common. Thus, if the axes of t_n and gt_ng^{-1} have a common subsegment in Y of length $> 4n - 2$, then t_n and gt_ng^{-1} have the same axis, so t_n and g have a common root. Now, observe that t_n has no root. It follows that g is a power of t_n, which concludes the proof. □

Let G be a non-elementary hyperbolic group. Consider $\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C} = \mathrm{id}_C\rangle$ a legal extension of G, and $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ a test sequence. Let Γ' be a finitely generated overgroup of Γ. Suppose that each φ_n extends to a homomorphism $\widehat{\varphi}_n : \Gamma' \to G$. Let L be the quotient of Γ' by the stable kernel of the sequence $(\widehat{\varphi}_n)_{n\in\mathbb{N}}$, and $r : \Gamma' \twoheadrightarrow L$ the associated epimorphism. Since G is equationally noetherian (according to [**Sel09**] and [**RW14**] Corollary 6.13), there exists (for n sufficiently large) a unique homomorphism $\rho_n : L \to G$ such that $\varphi_n = \rho_n \circ r$. Let $\lambda_n = \max_{s\in S} d(1, \rho_n(s))$ be the displacement of ρ_n, where S is a finite generating set of L containing the image of t in L. Let (X, d) denote a Cayley graph of G, and consider the rescaled metric $d_n = d/\lambda_n$. Last, let ω be a non-principal ultrafilter and let (X_ω, d_ω) be the ultralimit of $((X, d_n))_{n\in\mathbb{N}}$. By Theorem 2.14, X_ω is a real tree and there exists a unique minimal L-invariant non-degenerate subtree $T_L \subset X_\omega$. Moreover, some subsequence of the sequence $((X, d/\lambda_n))_{n\in\mathbb{N}}$ converges to T_L in the Gromov-Hausdorff topology.

Lemma 5.11. *If Γ does not fix a point in T_L, then the minimal subtree T_Γ is isometric to the Bass-Serre tree T' of the splitting $G*_C$, up to rescaling the metric on T'.*

Proof. Suppose that Γ does not fix a point of T_L, and let us prove that T_Γ is isometric to the Bass-Serre tree T' of the splitting $G*_C$. Observe that G is elliptic in T_L, by the first assumption of Definition 5.6. More precisely, the point $x := (1)_\omega \in T_L$ is fixed by G. Note that $||t_n||/\lambda_n$ does not approach 0 as n goes to infinity, otherwise Γ, which is generated by G and t, would be elliptic in T_L. Hence, t acts hyperbolically on T_Γ. In addition, the translation axis of t contains x. Therefore, up to rescaling the metric on T_Γ, there exists a simplicial map $f : T' \to T_\Gamma$ that is isometric in restriction to the axis of t. In order to prove that f is an isometric embedding, let us prove that there is no folding. Note that the surjectivity is automatically satisfied because T_Γ is minimal. Assume towards a contradiction that there is a folding at the vertex $v \in T'$ fixed by G. Let w and w' denote two vertices of T' adjacent to v such that $f([v,w]) \cap f([v,w'])$ is non-degenerate. One can assume without loss of generality that $w = tv$ (up to translating w by an element of G and replacing t with t^{-1}). Since tv and $t^{-1}v$ are on the axis of t, and since f is isometric on this axis, $f([v,tv]) \cap f([v,t^{-1}v]) = \{f(v)\}$. Therefore, the vertex w' is of the form gtv or $gt^{-1}v$, for some element $g \in G$ that does not belong to C (indeed, if $g \in C$, then $gtv = tv$ and $gt^{-1}v = t^{-1}v$). Thus, the axes of t and gtg^{-1} have an overlap I of length > 0 in the limit tree T. It follows that

$$\Delta(t_n, \rho_n(g)t_n\rho_n(g)^{-1}) \geq \eta\lambda_n$$

for every n large enough, for some $\eta > 0$. But $\lambda_n/||t_n||$ is greater than 1, and $(\varepsilon_n)_{n\in\mathbb{N}}$ approaches 0 when n goes to infinity. Thus, for n large enough, we have:

$$\Delta(t_n, \rho_n(g)t_n\rho_n(g)^{-1}) \geq \eta\lambda_n \geq \varepsilon_n||t_n||.$$

Then, since t_n satisfies the ε_n-small cancellation condition, the element $\rho_n(g) = g$ belongs to $M(t_n)$. By Lemma 5.8, g belongs to C, which is a contradiction. Hence, T_Γ is isometric to T'. □

Corollary 5.12. *Every test sequence is discriminating.*

Proof. Let G be a non-elementary hyperbolic group. Consider $\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C} = \mathrm{id}_C\rangle$ a legal extension of G, and $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ a test sequence. By taking $\Gamma' = L = \Gamma$ in the previous lemma, T_Γ is isometric to the Bass-Serre tree T' of the splitting $\Gamma = G*_C$. Let $\gamma \in \Gamma$ be a non-trivial element.

If γ belongs to a conjugate of G, then $\rho_n(\gamma) \neq 1$ for every n since ρ_n is the identity on G.

If γ does not belong to a conjugate of G, i.e. if γ is not elliptic in the splitting T', then it acts hyperbolically on T, because T' and T are isometric. Thus, $\rho_n(\gamma)$ is non-trivial for infinitely many n (otherwise γ would be elliptic). It remains to prove that $\rho_n(\gamma)$ is non-trivial for every n large enough. Assume towards a contradiction that some infinite subsequence $(\rho_{f(n)})$ kills γ for every n. Applying the previous argument to $(\rho_{f(n)})_{n\in\mathbb{N}}$ instead of $(\rho_n)_{n\in\mathbb{N}}$, we get a contradiction. Hence, the sequence of morphisms $(\rho_n)_{n\in\mathbb{N}}$ is discriminating. □

Corollary 5.13. *With the same notations and the same hypotheses as in Lemma 5.11, T_Γ is transverse to its translates, i.e. for every $h \in L \setminus \Gamma$, $hT_\Gamma \cap T_\Gamma$ is at most one point. In addition, if e is an edge of T_Γ, there are only finitely many branch points on e in T_L.*

Proof. Let h be an element of L such that $hT_\Gamma \cap T_\Gamma$ is non-degenerate. Since T_Γ is isometric to the Bass-Serre tree of the splitting $G*_C$ of Γ, we can find two elements $u, v \in \Gamma$ such that the axes of utu^{-1} and $h(vtv^{-1})h^{-1}$ have a non-trivial overlap in the limit tree T_L, so

$$\Delta(t_n, \rho_n(u^{-1}hv)t_n\rho_n(u^{-1}hv)^{-1}) \geq \varepsilon_n||t_n||$$

for n large enough. Hence, $\rho_n(u^{-1}hv)$ belongs to $M(t_n) = C \times \langle t_n\rangle$. So, for every n, there is an element $c_n \in C$ and an integer p_n (possibly zero if $u^{-1}hv$ has finite order) such that $\rho_n(u^{-1}hv) = c_n t_n^{p_n} = \rho_n(c_n t^{p_n})$. Since C is finite, we can pass to a subsequence and assume that $c_n = c$ for all n. On the other hand, since t acts hyperbolically on T_Γ, the integer p_n is bounded by a constant that does not depend on n. Otherwise, up to extracting, $||\rho_n(t)||/||\rho_n(u^{-1}hv)||$ tends to 0. Hence, since $\rho_n(u^{-1}hv)/\lambda_n$ is bounded,

$||\rho_n(t)||/\lambda_n$ tends to 0, contradicting that t is hyperbolic. Up to extracting, one can assume that $p_n = p$ for all n. So we have $\rho_n(u^{-1}hv) = \rho_n(ct^p)$ for all n. The sequence $(\rho_n)_{n\in\mathbb{N}}$ being discriminating, $h = uct^pv^{-1} \in \Gamma$, since $u, v \in \Gamma$.

Last, let e be an edge of T_Γ. Let us prove that there are only finitely many branch points on e in T_L. Let M denote the maximal order of a finite subgroup of G. Note that M is also the maximal order of a finite subgroup of L. By Theorem 2.19, due to Reinfeldt and Weidmann, the action of L on T_L is M-superstable with finitely generated arc stabilizers. By Theorem 2.13, due to Guirardel and Levitt, the number of orbits of directions at branch points in T_L is finite. Assume towards a contradiction that there are infinitely many branch points on e. Then there exist two non-degenerate subsegments I and J in e, with $I \cap J = \varnothing$, and an element $g \in G$ such that $gI = J$. But we proved that T_Γ is transverse to its translates, so g belongs to Γ. Since T_Γ is isometric to the Bass-Serre tree of $G*_C$, it follows that g fixes e. This is a contradiction. ☐

5.3. Generalized Sacerdote's lemma. We are now ready to prove a generalization of the main lemma of [**Sac73b**].

PROPOSITION 5.14. *Let Γ be a legal large extension of G. Let $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ be a test sequence. Let $\boldsymbol{g}$ be a tuple of elements of G. Let $\Sigma(\boldsymbol{x},\boldsymbol{y},\boldsymbol{g}) = 1 \wedge \Psi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{g}) \neq 1$ be a conjunction of equations and inequations in the p-tuple $\boldsymbol{x}$ and the q-tuple $\boldsymbol{y}$. Let $\boldsymbol{\gamma}$ be a p-tuple of elements of Γ. Suppose that G satisfies the following condition: for every n, there exists a q-tuple $\boldsymbol{g}_n \in G^q$ such that*

$$\Sigma(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n,\boldsymbol{g}) = 1 \ \wedge \ \Psi(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n,\boldsymbol{g}) \neq 1.$$

Then there exists a retraction r from $\Gamma_\Sigma := \langle \Gamma, \boldsymbol{y} \mid \Sigma(\boldsymbol{\gamma},\boldsymbol{y},\boldsymbol{g}) = 1\rangle$ onto Γ such that all components of the tuple $r(\Psi(\boldsymbol{\gamma},\boldsymbol{y},\boldsymbol{g}))$ are non-trivial. In particular, the q-tuple $\boldsymbol{\gamma}' := r(\boldsymbol{y}) \in \Gamma^q$ satisfies

$$\Sigma(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{g}) = 1 \ \wedge \ \Psi(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{g}) \neq 1.$$

Before proving this result, whose proof is quite technical, we will use it to deduce that the inclusion of G into Γ is a $\exists\forall\exists$-elementary embedding. We begin by proving a corollary to Proposition 5.14 above, which allows us to deal with disjunctions of systems of equations and inequations.

COROLLARY 5.15. *Let Γ be a legal large extension of G, and let $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ be a test sequence. Let g be a tuple of elements of G. Let*

$$\bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x},\boldsymbol{y},\boldsymbol{g}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x},\boldsymbol{y},\boldsymbol{g}) \neq 1)$$

be a disjunction of systems of equations and inequations in the p-tuple $\boldsymbol{x}$ and the q-tuple $\boldsymbol{y}$. Let $\boldsymbol{\gamma}$ be a p-tuple of elements of Γ. Suppose that G satisfies the following condition: for every integer n, there exists a q-tuple $\boldsymbol{g}_n \in G^q$ such that

$$\bigvee_{k=1}^{N} (\Sigma_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n,\boldsymbol{g}) = 1 \ \wedge \ \psi_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n,\boldsymbol{g}) \neq 1).$$

Then there exists a q-tuple $\boldsymbol{\gamma}'$ such that

$$\bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{g}) = 1 \ \wedge \ \Psi_k(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{g}) \neq 1).$$

PROOF. Up to extracting a subsequence of (φ_n) (which is still a test sequence), one can assume that there exists an integer $1 \leq k \leq N$ such that, for every n, there exists $\boldsymbol{g}_n \in G^q$ such that $\Sigma_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n,\boldsymbol{g}) = 1$ and $\Psi_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n,\boldsymbol{g}) \neq 1$. Proposition 5.14 applies and establishes the existence of a tuple $\boldsymbol{\gamma}' \in \Gamma^q$ satisfying $\Sigma_k(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{g}) = 1$ and $\Psi_k(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{g}) \neq 1$, which concludes the proof. ☐

We deduce Proposition 5.5 from Corollary 5.15.

Proof of Proposition 5.5. Let $\theta(\boldsymbol{t})$ be a $\exists\forall\exists$-formula with m free variables. This formula has the following form:

$$\theta(\boldsymbol{t}) : \exists \boldsymbol{x} \forall \boldsymbol{y} \exists \boldsymbol{z} \bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) \neq 1).$$

Let $\boldsymbol{g}$ be a tuple of elements of G such that $G \models \theta(\boldsymbol{g})$, and let us prove that $\Gamma \models \theta(\boldsymbol{g})$. There exists a tuple of elements of G, denoted by $\boldsymbol{x}$, such that

$$G \models \forall \boldsymbol{y} \exists \boldsymbol{z} \bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{g}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{g}) \neq 1). \tag{1}$$

Let us prove that

$$\Gamma \models \forall \boldsymbol{y} \exists \boldsymbol{z} \bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{g}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{g}) \neq 1). \tag{2}$$

Let $\boldsymbol{y}$ be a tuple of elements of Γ. By (1), for every integer n, there exists a tuple $\boldsymbol{z}_n$ of elements of G such that

$$G \models \bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x}, \varphi_n(\boldsymbol{y}), \boldsymbol{z}_n, \boldsymbol{g}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x}, \varphi_n(\boldsymbol{y}), \boldsymbol{z}_n, \boldsymbol{g}) \neq 1). \tag{3}$$

By Corollary 5.15, there exists a tuple $\boldsymbol{z}$ of elements of Γ such that

$$\Gamma \models \bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{g}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{g}) \neq 1). \tag{4}$$

□

We now prove Proposition 5.14, that is the generalized Sacerdote's lemma.

Proof of Proposition 5.14. Let $\boldsymbol{\gamma} \in \Gamma^p$. Suppose that, for every integer n, there exists $\boldsymbol{g}_n \in G^q$ such that

$$\Sigma(\varphi_n(\boldsymbol{\gamma}), \boldsymbol{g}_n, \boldsymbol{g}) = 1 \ \wedge \ \Psi(\varphi_n(\boldsymbol{\gamma}), \boldsymbol{g}_n, \boldsymbol{g}) \neq 1.$$

Let $\Gamma_\Sigma = \langle \Gamma, \boldsymbol{y} \mid \Sigma(\boldsymbol{\gamma}, \boldsymbol{y}, \boldsymbol{g}) = 1 \rangle$. Let i denote the natural morphism from Γ to Γ_Σ. By hypothesis, for every n, there exists a homomorphism $\widehat{\varphi}_n : \Gamma_\Sigma \to G$ mapping $\boldsymbol{y}$ to $\boldsymbol{g}_n$ such that $\widehat{\varphi}_n \circ i = \varphi_n$. Note that the test sequence $(\varphi_n)_{n\in\mathbb{N}}$ is discriminating by Corollary 5.12. As a consequence, the homomorphism i is injective. From now on, we omit mentioning the morphism i.

We shall construct a retraction $r : \Gamma_\Sigma \twoheadrightarrow \Gamma$ that does not kill any component of the tuple $\Psi(\boldsymbol{\gamma}, \boldsymbol{y}, \boldsymbol{g})$. Let $L = \Gamma_\Sigma / \ker((\widehat{\varphi}_n)_{n\in\mathbb{N}})$ and let $\pi : \Gamma_\Sigma \twoheadrightarrow L$ be the associated epimorphism. As a G-limit group, L is equationally noetherian (see [**RW14**] Corollary 6.13). It follows that there exists a unique homomorphism $\rho_n : L \to G$ such that $\widehat{\varphi}_n = \rho_n \circ \pi$.

Since the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is discriminating, we have $\widehat{\varphi}_n(x) = \rho_n(\pi(x)) \neq 1$ for every $x \in \Gamma$ and every n large enough. In addition, by construction, the morphism $\widehat{\varphi}_n$ does not kill any component of $\Psi(\varphi_n(\boldsymbol{\gamma}), \boldsymbol{g}_n, \boldsymbol{g})$. Thus, the homomorphism $\pi : \Gamma \to L$ is injective and does not kill any component of $\Psi(\varphi_n(\boldsymbol{\gamma}), \boldsymbol{g}_n, \boldsymbol{g})$. In the sequel, we identify Γ and $\pi(\Gamma)$.

In order to construct r, we will construct a discriminating sequence of retractions $(r_n : L \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$. Then, we will conclude by taking $r := r_n \circ \pi$ for n sufficiently large.

Let (X, d) be a Cayley graph of G. Let us consider a Stallings splitting Λ of L relative to Γ, and let U be the one-ended factor that contains Γ. Let S be a generating set of U. Recall that $\mathrm{Aut}_\Gamma(U)$ is the subgroup of $\mathrm{Aut}(U)$ consisting of all automorphisms σ satisfying the following two conditions:

(1) $\sigma_{|\Gamma} = \mathrm{id}_{|\Gamma}$;

(2) for every finite subgroup F of U, there exists an element $u \in U$ such that $\sigma_{|F} = \mathrm{ad}(u)_{|F}$.

Recall that a homomorphism $\varphi : U \to G$ is said to be *short* if its length $\ell(\varphi) := \max_{s\in S} d(1, \varphi(s))$ is minimal among the lengths of homomorphisms in the orbit of φ under the action of $\mathrm{Aut}_\Gamma(U) \times \mathrm{Inn}(G)$. Since $||t_n||$ goes to infinity, there exists a sequence of automorphisms $(\sigma_n)_{n\in\mathbb{N}} \in \mathrm{Aut}_\Gamma(U)^{\mathbb{N}}$ and a sequence of elements $(x_n)_{n\in\mathbb{N}} \in G^n$ such that the homomorphisms $\mathrm{ad}(x_n)\circ\rho_n\circ\sigma_n$ are short, pairwise distinct, and such that the sequence $(\mathrm{ad}(x_n)\circ\rho_n\circ\sigma_n)_{n\in\mathbb{N}}$ is stable (see paragraph 2.5.1), up to extracting a subsequence. Since ρ_n coincides with the identity on G, we have $\mathrm{ad}(x_n)\circ\rho_n\circ\sigma_n = \rho_n\circ\mathrm{ad}(x_n)\circ\sigma_n$ and, up to replacing σ_n by $\mathrm{ad}(x_n)\circ\sigma_n$, we can forget the postconjugation by x_n and assume that $\sigma_{n|\Gamma}$ is a conjugation by an element of G.

We claim that σ_n extends to an automorphism of L, still denoted by σ_n. By the second condition above, σ_n is a conjugacy on finite subgroups of U. We proceed by induction on the number of edges of Λ (the Stallings splitting of L relative to Γ used previously in order to define U). It is enough to prove the claim in the case where Λ has only one edge.

If $L = U *_C B$ with $\sigma_{n|C} = \mathrm{ad}(u)$, one defines $\sigma_n : L \to G$ by $\sigma_{n|U} = \sigma_n$ and $\sigma_{n|B} = \mathrm{ad}(u)$.

If $L = U*_C = \langle U, t \mid tct^{-1} = \alpha(c), \forall c \in C\rangle$ with $\sigma_{n|C} = \mathrm{ad}(u_1)$ and $\sigma_{n|\alpha(C)} = \mathrm{ad}(u_2)$, one defines $\sigma_n : L \to G$ by $\sigma_{n|U} = \sigma_n$ and $\sigma_n(t) = u_2^{-1}tu_1$.

In order to complete the proof of the generalized Sacerdote's lemma, we will use the following result.

Lemma 5.16. *We keep the same notations as above. Let T be the limit tree of the sequence of metric spaces $(X, d/\ell(\rho_n\circ\sigma_n))_{n\in\mathbb{N}}$. The following dichotomy holds:*

- *either Γ does not fix a point of T, in which case there exists a discriminating sequence of retractions $(r_n : L \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$,*
- *or Γ is elliptic, and there exist a proper quotient L_1 of L, an embedding $\Gamma \hookrightarrow L_1$ allowing us to identify Γ with a subgroup of L_1, and two discriminating sequences $(\rho_n^1 : L_1 \to G)_{n\in\mathbb{N}}$ and $(\theta_n^1 : L \twoheadrightarrow L_1)_{n\in\mathbb{N}}$ such that the following three conditions are satisfied:*
  (1) $\rho_n\circ\sigma_n = \rho_n^1\circ\theta_n^1$;
  (2) ρ_n^1 coincides with ρ_n on Γ; in particular, $(\rho^1_{n|\Gamma} : \Gamma \to G)_{n\in\mathbb{N}}$ is a test sequence.
  (3) There exists an element $g_n \in G$ such that σ_n and θ_n^1 coincides with $\mathrm{ad}(g_n)$ on Γ.

Before proving this lemma, we will use it to conclude the proof of Proposition 5.14. If Γ does not fix a point of T, we are done. If Γ fixes a point of T, by iterating Lemma 5.16, we get a sequence of proper quotients

$$L_0 = L \twoheadrightarrow L_1 \twoheadrightarrow \cdots L_i \twoheadrightarrow \cdots$$

such that, for every integer $i \geq 1$, there exist two discriminating sequences of morphisms $(\rho_n^i : L_i \to G)_{n\in\mathbb{N}}$ and $(\theta_n^i : L_{i-1} \twoheadrightarrow L_i)_{n\in\mathbb{N}}$ such that $\rho_n^{i-1}\circ\sigma_n = \rho_n^i\circ\theta_n^i$, ρ_n^i coincides with ρ_n^{i-1} on Γ, and there exists an element $g_n \in G$ such that σ_n and θ_n^i coincides with $\mathrm{ad}(g_n)$ on Γ.

By the descending chain condition 2.24, the iteration eventually terminates. Let L_k be the last quotient of the series. By Lemma 5.16, there exists a discriminating sequence of retractions $(r_n : L_k \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$. For every finite set $F \subset L$, one can find some integers $n_1, \ldots, n_k$ such that the morphism $r_{n_k}\circ\theta^k_{n_k}\circ\cdots\circ\theta^1_{n_1} : L \twoheadrightarrow \Gamma$ is injective on F. Moreover, since every θ_{n_i} is a conjugation on Γ by an element of G, there exists an element $v_n \in G$ such that $\mathrm{ad}(v_n)\circ r_{n_k}\circ\theta^k_{n_k}\circ\cdots\circ\theta^1_{n_1} : L \twoheadrightarrow \Gamma$ is a retraction. This concludes the proof of Lemma 5.14. □

It remains to prove Lemma 5.16.

Proof of Lemma 5.16. Recall that U denotes the one-ended factor of L relative to Γ. We distinguish two cases.

First case. Suppose that Γ fixes a point of T. Let us prove that the stable kernel of the sequence $(\rho_n \circ \sigma_n)_{n\in\mathbb{N}}$ is non-trivial. Assume towards a contradiction that the stable kernel is trivial. Then, by Theorem 1.16 of [**RW14**], the action of (U, Γ) on the limit tree T has the following properties:

- the pointwise stabilizer of any non-degenerate tripod is finite;
- the pointwise stabilizer of any non-degenerate arc is finitely generated and finite-by-abelian;
- the pointwise stabilizer of any unstable arc is finite.

In particular, the tree T satisfies the ascending chain condition of Theorem 2.15 since any ascending sequence of finitely generated and finite-by-abelian subgroups of a hyperbolic group stabilizes.

Then, it follows from Theorem 2.15 that either (U, Γ) splits over the stabilizer of an unstable arc, or over the stabilizer of an infinite tripod, or T has a decomposition into a graph of actions. Since U is one-ended relative to Γ, and since the stabilizer of an unstable arc or of an infinite tripod is finite, it follows that T has a decomposition into a graph of actions.

Now, it follows from Theorem 2.23 that there exists a sequence of automorphisms $(\alpha_n)_{n\in\mathbb{N}} \in \mathrm{Aut}_\Gamma(U)^{\mathbb{N}}$ such that $(\rho_n \circ \sigma_n) \circ \alpha_n$ is shorter than $\rho_n \circ \sigma_n$ for n large enough. This is a contradiction since the morphisms $\rho_n \circ \sigma_n$ are assumed to be short. Hence, the stable kernel of the sequence $(\rho_n \circ \sigma_n)_{n\in\mathbb{N}}$ is non-trivial.

As in the proof of Lemma 5.14 above, since σ_n coincides with an inner automorphism on each finite subgroup of U (by definition of $\mathrm{Aut}_\Gamma(U)$), it extends to an automorphism of L, still denoted by σ_n. Let $L_1 := L/\ker((\rho_n \circ \sigma_n)_{n\in\mathbb{N}})$ and let $\pi_1 : L \twoheadrightarrow L_1$ be the corresponding epimorphism. Observe that π_1 is injective on Γ, allowing to identify Γ with a subgroup of L_1.

As a G-limit group, L_1 is equationally noetherian (see [**RW14**] Corollary 6.13). It follows that, for every integer n, there exists a unique homomorphism $\tau_n^1 : L_1 \to G$ such that $\rho_n \circ \sigma_n = \tau_n^1 \circ \pi_1$. There exists an element $g_n \in G$ such that σ_n coincides with $\mathrm{ad}(g_n)$ on Γ. Hence, since ρ_n coincides with the identity on G, we can write $\rho_n \circ \sigma_n = \mathrm{ad}(g_n) \circ \rho_n^1 \circ \pi_1$ in such a way that $\rho_n^1 := \mathrm{ad}(g_n^{-1}) \circ \tau_n^1$ coincides with the identity on G. For every n, let $\theta_n^1 = \pi_1 \circ (\sigma_n)^{-1}$, so that $\rho_n = \mathrm{ad}(g_n) \circ \rho_n^1 \circ \theta_n^1$. The sequence $(\theta_n^1 : L \to L_1)_{n\in\mathbb{N}}$ is therefore discriminating, and every homomorphism θ_n^1 coincides with $\mathrm{ad}(g_n^{-1})$ on Γ.

Second case. Suppose that Γ is not elliptic in T. We will construct a discriminating sequence of retractions $(r_n : L \to \Gamma)_{n\in\mathbb{N}}$. Let $T_\Gamma \subset T$ be the minimal invariant subtree of Γ. By Lemma 5.11, we may assume up to rescaling that T_Γ is isometric to the Bass-Serre tree of the splitting $\Gamma = G *_C$.

Let $\sim$ be the relation on T defined by $x \sim y$ if $[x, y] \cap uT_\Gamma$ contains at most one point, for every element $u \in U$. Note that $\sim$ is an equivalence relation. Let $(Y_j)_{j\in J}$ denote the equivalence classes that are not reduced to a point. Each Y_j is a subtree of T. Let us prove that $(Y_j)_{j\in J} \cup \{uT_\Gamma \mid u \in U/\Gamma\}$ is a transverse covering of T, in the sense of Definition 2.16.

- *Transverse intersection.* For every $i \neq j$, the intersection $Y_i \cap Y_j$ is clearly empty. For every i and $u \in U$, $Y_i \cap uT_\Gamma$ contains at most one point by definition. For every $u, u' \in U$ such that $u'u^{-1} \notin \Gamma$, $|uT_\Gamma \cap u'T_\Gamma| \leq 1$ thanks to Lemma 5.13.
- *Finiteness condition.* Let x and y be two points of T. By Lemma 5.13, there exists a constant $\varepsilon > 0$ such that, for every $u \in U$, if the intersection $[x, y] \cap uT_\Gamma$ is non-degenerate, the length of $[x, y] \cap uT_\Gamma$ is bounded from below by ε. Consequently, the arc $[x, y]$ is covered by at most $\lfloor d(x, y)/\varepsilon \rfloor$ translates of T_Γ and at most $\lfloor d(x, y)/\varepsilon \rfloor + 1$ distinct subtrees Y_j.

Hence, the collection $(Y_j)_{j\in J} \cup \{uT_\Gamma \mid u \in U\}$ is a transverse covering of T. One can construct what Guirardel calls the skeleton of this transverse covering (see Definition 2.17), denoted by T_c. Since the action of U on T is minimal, the same holds for the action of U on T_c, according to Lemma 4.9 of [**Gui04**]. The question is now to understand the decomposition $\Delta_c = T_c/U$ of U as a graph of groups.

We begin with a description of the stabilizer in U of an edge e of T_Γ. Let u be an element of U that fixes e. Then e is contained in $T_\Gamma \cap uT_\Gamma$, so u belongs to Γ, thanks to Lemma 5.13. It follows that u belongs to C, because the stabilizer of e in Γ is equal to C (indeed, recall that T_Γ is isometric to the Bass-Serre tree of the splitting $\Gamma = G*_C$, by Lemma 5.11). Thus, the stabilizer of e in U is equal to C.

We now prove that if one of the subtrees of the covering other than T_Γ intersects T_Γ in a point, then this point is necessarily one of the extremities of a translate of the edge $e \in T_\Gamma$. Assume towards a contradiction that Y_j or uT_Γ with $u \notin \Gamma$ intersects T_Γ in a point x that is not one of the extremities of e. Then, T_c contains an edge $\varepsilon = (x, T_\Gamma)$ whose stabilizer is $\mathrm{Stab}(x)\cap\Gamma$ (where $\mathrm{Stab}(x)$ denotes the stabilizer of x in U), which is equal to C by the previous paragraph. So the splitting Δ_c of U is a non-trivial splitting over the finite subgroup C, relative to Γ. This is impossible since U is one-ended relative to Γ, by definition of U. Hence, if $Y_j \cap T_\Gamma = \{x\}$ or $uT_\Gamma \cap T_\Gamma = \{x\}$ with $u \notin \Gamma$, then the point x is one of the extremities of e in T_Γ. As a consequence, $\mathrm{Stab}(x)$ is a conjugate of G in Γ, and every edge adjacent to T_Γ in T_c is of the form $(\gamma x, T_\Gamma) = \gamma\varepsilon$ with $\varepsilon = (x, T_\Gamma)$.

Therefore, ε is the only edge adjacent to T_Γ in the quotient graph Δ_c. Its stabilizer is G. By collapsing all edges of Δ_c except ε, one gets a splitting of U of the following form: $U = \Gamma *_G U'$ for some subgroup $U' \subset U$. This splitting can be written as

$$U = U'*_C = \langle U', t \mid [t,c] = 1,\ \forall c \in C\rangle.$$

Since every finite subgroup of U is conjugate to a finite subgroup of U', the group L splits as $L = U''*_C = \langle U'', t \mid [t,c] = 1,\ \forall c \in C\rangle$ with $G \subset U' \subset U''$. One now defines a retraction $r_n : L \to \Gamma$ by $r_n(t) = t$ and $r_{n|U''} = \rho_{n|U''}$, well-defined since ρ_n coincides with the identity on G.

To conclude, let us prove that the sequence $(r_n)_{n\in\mathbb{N}}$ is discriminating. Let ℓ be a non-trivial element of L. This element can be written in reduced form, with respect to the HNN extension $L = U''*_C$, as $\ell = u_0 t^{\varepsilon_1} u_1 t^{\varepsilon_2} u_2 \cdots t^{\varepsilon_p} u_{p+1}$, with $u_i \in U''$. For every i, if $\varepsilon_i = -\varepsilon_{i+1}$, then $u_i \notin C$. Thus $r_n(u_i) \notin C$ for every n large enough (otherwise, up to extracting a subsequence, one can assume that $r_n(u_i) = u_n(c)$ for every n, so $u_i = c$ (since $(r_{n|U''})$ is discriminating), which is impossible). Hence, for every n large enough, $r_n(\ell) = \rho_n^k(u_0)t^{\varepsilon_1}\rho_n^k(u_1)t^{\varepsilon_2}\rho_n^k(u_2)\cdots t^{\varepsilon_p}\rho_n^k(u_{p+1})$ is non-trivial. □

6. Proof of Theorem 1.14

We will prove the following result.

Theorem 6.1 (see Theorem 1.14). *Let G be a hyperbolic group that splits as $A *_C B$ or $A*_C$ over a finite subgroup C whose normalizer N is infinite virtually cyclic and non-elliptic in the splitting. Let K_G be the maximal order of a finite subgroup of G. Let N' be a virtually cyclic group such that $K_{N'} \leq K_G$, and let $\iota : N \hookrightarrow N'$ be a K_G-nice embedding (see Definition 1.12). Let us define G' by*

$$G' = G *_N N' = \langle G, N' \mid g = \iota(g),\ \forall g \in N\rangle.$$

The following two assertions are equivalent.

1. *The group G' is a legal small extension of G in the sense of Definition 1.13, i.e. there exists a K_G-nice embedding $\iota' : N' \hookrightarrow N$.*
2. $\mathrm{Th}_{\forall\exists}(G') = \mathrm{Th}_{\forall\exists}(G)$.

REMARK 6.2. As observed in the introduction, G is not $\exists\forall$-elementarily embedded into G' in general.

First, we prove the easy direction.

PROPOSITION 6.3. *We keep the same notations as in Theorem 6.1. If* $\mathrm{Th}_{\forall\exists}(G') = \mathrm{Th}_{\forall\exists}(G)$, *then G' is a legal small extension of G.*

PROOF. According to the implication (1) $\Rightarrow$ (4) of Theorem 1.23, there exists a strongly special morphism $\varphi' : G' \to G \subset G'$. Since φ' maps non-conjugate finite subgroups to non-conjugate finite subgroups, there exists an integer $n \geq 1$ such that φ'^n maps C to $g'Cg'^{-1}$ for some $g' \in G'$. Hence we have $\varphi'^n(N') \subset N_G(g'Cg'^{-1}) = g'Ng'^{-1}$, since $\varphi'(G')$ is contained in G. Let us define ι' by $\iota' := (\mathrm{ad}(g'^{-1}) \circ \varphi'^n)_{|N'}$. This morphism is a K_G-nice embedding, because φ' is strongly special. $\square$

In the rest of this section, we prove that the converse also holds.

THEOREM 6.4. *We keep the same notations as in Theorem 6.1. If G' is a legal small extension of G, then* $\mathrm{Th}_{\forall\exists}(G') = \mathrm{Th}_{\forall\exists}(G)$.

Recall that an infinite virtually cyclic group N can be written as an extension of exactly one of the following two forms:

$$1 \to C \to N \to \mathbb{Z} \to 1 \quad \text{or} \quad 1 \to C \to N \to D_\infty \to 1,$$

where C is finite and $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ denotes the infinite dihedral group. In the first case, N splits as $N = C \rtimes \langle t \rangle$, where t denotes an element of infinite order. In the second case, N splits as an amalgamated free product $\langle C, a\rangle *_C \langle C, b\rangle$ where a and b have order 2 in N/C. We choose such elements a and b and we define t by $t = ba$. Note that the image of t in N/C generates N^+/C, where N^+ denotes the kernel of the epimorphism $N \twoheadrightarrow D_\infty \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. In other words, $N^+ \simeq C \rtimes \langle t \rangle$.

In the sequel, we say that two elements $g, g' \in N$ are equal modulo C, and we write $g' = g \mod C$, if $g^{-1}g'$ belongs to C.

Recall that for every integer p, we denote by $D_p(N)$ the definable subset $\{n^p \mid n \in N\}$.

LEMMA 6.5. *Let N be a virtually cyclic group. Let C be the maximal finite normal subgroup of N, and let m be the order of* $\mathrm{ad}(t)_{|C}$ *in* $\mathrm{Aut}(C)$. *Then* $D_{2m|C|}(N) = \langle t^{2m|C|}\rangle$.

REMARK 6.6. In particular, $D_{2m|C|}(N)$ is a subgroup of N. Moreover, it is central in N.

PROOF. Let g be an element of N. The element g^2 can be written as ct^{2r} for some $c \in C$ and $r \in \mathbb{Z}$, so $g^{2m} = c't^{2mr}$ for some $c' \in C$. By definition of m, t^m commutes with c', so $(g^{2m})^{|C|} = (c')^{|C|}t^{2m|C|} = t^{2m|C|}$. As a consequence, $D_{2m|C|}(N)$ is contained in $\langle t^{2m|C|}\rangle$. The other inclusion is obvious. $\square$

REMARK 6.7. Note that the action of $\mathrm{Aut}(C)$ on $C \setminus \{1\}$ gives an embedding from $\mathrm{Aut}(C)$ into the symmetric group $\mathfrak{S}_{|C|-1}$. It follows that $|C|\mathrm{Aut}(C)$ divides $|C|!$. Hence, $2m|C|$ divides $2|C|!$.

In the sequel, G denotes a hyperbolic group and K denotes the maximal order of a finite subgroup of G. For every virtually cyclic infinite subgroup N of G, we define the subgroup $D(N)$ of N by $D(N) := D_{2K!}(N)$. The following result in an immediate consequence of Lemma 6.5 (see the remark above).

COROLLARY 6.8. *If C is a finite subgroup of G whose normalizer N is virtually cyclic infinite, then $D(N) = \langle t^{2K!}\rangle$ (for any element t chosen as above).*

Recall that the normalizer of a finite edge group in a splitting is an infinite virtually cyclic group if and only if one of the situations described below arises.

Lemma 6.9. *Let G be a group. Suppose that G splits as an amalgamated free product $G = A *_C B$ over a finite group C, and that $N_G(C)$ is not contained in a conjugate of A or B. Then $N_G(C)$ is virtually cyclic if and only if C has index 2 in $N_A(C)$ and in $N_B(C)$. In this case, $N_G(C)$ is of dihedral type, equal to $N_A(C) *_C N_B(C)$.*

Lemma 6.10. *Let G be a group. Suppose that G splits as an HNN extension $G = A*_C$ over a finite group C. Let C_1 and C_2 denote the two copies of C in A and t be the stable letter associated with the HNN extension. Suppose that $N_G(C)$ is not contained in a conjugate of A. Then $N_G(C)$ is virtually cyclic if and only if one of the following two cases holds.*

(1) *If C_1 and C_2 are conjugate in A and $N_A(C_1) = C_1$, then the normalizer $N_G(C_1)$ is of cyclic type, equal to $C_1 \rtimes \langle at\rangle$, where a denotes an element of A such that $aC_2a^{-1} = C_1$.*
(2) *If C_1 and $C_2 = t^{-1}C_1t$ are non-conjugate in G and C_1 has index 2 in $N_A(C_1)$ and $N_{tAt^{-1}}(C_1)$, then the normalizer $N_G(C_1)$ is of dihedral type, equal to*
$$N_A(C_1) *_{C_1} N_{tAt^{-1}}(C_1).$$

6.1. Small test sequences.

Definition 6.11 (Twist). Let G be a group. Suppose that G splits as $A *_C B$ or $A*_C$ over a finite subgroup C whose normalizer N is virtually cyclic infinite and non-elliptic in the splitting. Let δ be an element of $D(N)$.

- If $G = A*_C = \langle A, t \mid tct^{-1} = \theta(c), \forall c \in C\rangle$ where $\theta \in \mathrm{Aut}(C)$, the *twist* τ_δ is the endomorphism of G that coincides with the identity on A and that maps the stable letter t to $t\delta$.
- If $G = A *_C B$, the twist τ_δ is the endomorphism of G that coincides with the identity on A, and that coincides with $\mathrm{ad}(\delta)$ on B.

Remark 6.12. Note that τ_δ is well-defined in both cases because δ centralizes C, as en element of $D(N)$. Moreover, in both cases, τ_δ is a monomorphism: in the first case, it suffices to observe that $t\delta$ has infinite order, which is true since t has infinite order and δ is a power of $t^{2K!}$; in the second case, the injectivity is automatic thanks to Britton's lemma. In addition, in the second case, note that τ_δ maps $t = ba$ to $\delta b\delta^{-1}a = t^rbt^{-r}a = (ba)^rb(ba)^{-r}a = (ba)^{2r+1} = t^{2r+1} = t\delta^2$ for some multiple r of $2K!$.

By analogy with test sequences defined in the previous section, we introduce below the notion of a *small test sequence*, designed for legal small extensions.

Lemma 6.13. *Let N and N' be two virtually cyclic infinite groups. Let C and C' be the maximal normal finite subgroups of N and N' respectively. Suppose that there exist two embeddings $\iota : N \hookrightarrow N'$ and $\iota' : N' \hookrightarrow N$. Then $\iota(C) = C'$ and $\iota'(C') = C$.*

Proof. Note that N and N' are both of cyclic type or of dihedral type, since a virtually cyclic group of dihedral type does not embed into a virtually cyclic group of cyclic type.

First case. Suppose that N and N' are of cyclic type. Then $N = C \rtimes \mathbb{Z}$ and $N' = C' \rtimes \mathbb{Z}$. It follows that $\iota(C) \subset C'$ and $\iota'(C') \subset C$. Hence C and C' have the same cardinality, and we have $\iota(C) = C'$ and $\iota'(C') = C$.

Second case. Suppose that N and N' are of dihedral type. There exist two elements $a, b \in N$ such that $N = \langle C, a, b \mid a^2 \in C, b^2 \in C, aC = Ca, bC = Cb\rangle$. Note that all finite subgroups of N that are not contained in C are of the form $n\langle C_1, a\rangle n^{-1}$ or $n\langle C_1, b\rangle n^{-1}$ with $C_1 \subset C$ and $n \in N$. In addition, note that the normalizers of a and b in N are finite. Thus, the normalizer of the finite groups $\langle C_1, a\rangle$ and $\langle C_1, b\rangle$ are finite. Then, observe that the normalizer of $\iota'(C')$ is equal to $\iota'(N')$, which is infinite since ι' is injective and N' is infinite. It follows that $\iota'(C')$ is contained in C. Likewise, $\iota(C)$ is contained in C'. Hence C and C' have the same cardinality, and we have $\iota(C) = C'$ and $\iota'(C') = C$. □

Let G be a hyperbolic group that splits over a finite group C whose normalizer N is virtually cyclic infinite and non-elliptic in the splitting. Let

$$\Gamma = G *_N N' = \langle G, N' \mid g = \iota(g),\ \forall g \in N\rangle$$

be a legal small extension of G, where N' is virtually cyclic and $\iota : N \hookrightarrow N'$ is K-nice. Let C and C' be the maximal normal finite subgroups of N and N' respectively. By the previous lemma, $\iota(C)$ is equal to C'. As a consequence, in Γ, we have $C = C'$. We make the following two observations.

(1) The group Γ splits as $A' *_C B'$ or $A' *_C$. The normalizer $N_\Gamma(C)$ of C in Γ is equal to N'. This group is virtually cyclic infinite and non-elliptic in the previous splitting of Γ over C.
(2) The maximal order of a finite subgroup is the same for Γ and G. Indeed, every finite subgroup F of Γ is contained in a conjugate of G or in a conjugate of N'. In the second case, F embeds into N since there exists an embedding from N' into N. Hence, in both cases, F embeds into G. As a consequence, K_Γ is equal to K_G and there is no ambiguity about the notation $D(N')$

Thanks to these two observations, the Dehn twist τ_δ is well-defined, as an endomorphism of Γ, for any element $\delta \in D(N')$. We are now ready to define small test sequences.

Definition 6.14 (Small test sequence). Let G be a hyperbolic group. Suppose that G splits over a finite group C whose normalizer N is virtually cyclic infinite and non-elliptic in the splitting. Let

$$\Gamma = G *_N N' = \langle G, N' \mid g = \iota(g),\ \forall g \in N\rangle$$

be a legal small extension of G, where N' is virtually cyclic and $\iota : N \hookrightarrow N'$ is K-nice. A sequence of homomorphisms $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ is called a small test sequence if there exist a strictly increasing sequence of prime numbers $(p_n)_{n\in\mathbb{N}}$ and a sequence $(\delta_n)_{n\in\mathbb{N}} \in D(N')^{\mathbb{N}}$ such that $\varphi_n = \tau_{\delta_n}$ (viewed as an endomorphism of Γ) and $[N : \varphi_n(N')] = p_n$, for every integer n.

The following lemma shows that small test sequences exist as soon as $\iota : N \hookrightarrow N'$ is not surjective.

Lemma 6.15. *Let G be a hyperbolic group. Suppose that G splits over a finite group C whose normalizer $N = N_G(C)$ is virtually cyclic infinite and non-elliptic in the splitting. Let $\Gamma = G *_N N'$ be a legal small extension of G. Suppose that N is a strict subgroup of N' in Γ. Then, there exists a small test sequence $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$.*

Proof. By definiton of a legal small extension, there exist two K-nice embeddings $\iota : N \hookrightarrow N'$ and $\iota' : N' \hookrightarrow N$. Let C and C' be the maximal normal finite subgroups of N and N' respectively. In Γ, we have the identification $\iota(n) = n$ for every $n \in N$. In particular, C and C' are identified. In the sequel, we do not mention ι anymore. We distinguish two cases.

First case. Suppose that N is virtually cyclic of cyclic type. Since ι' is special, there exists an integer $m \geq 1$ such that ι'^m coincides with the identity on C and induces the identity of $N'/D(N')$. Up to replacing ι' by ι'^m, one can assume without loss of generality that ι' coincides with the identity on C and induces the identity of $N'/D(N')$.

Let t and z denote two elements such that such that $N = N_G(C) = C \rtimes \langle t\rangle$ and $N' = N_\Gamma(C) = C \rtimes \langle z\rangle$. Recall that $D(N') = \langle z^{2K!}\rangle$, by Corollary 6.8. Since ι' induces the identity of $N'/D(N')$, we have $\iota'(z) = z^{1+2K!q}$ for some integer q. Note that q is non-zero because N is a proper subgroup of N' by assumption. Let k and ℓ denote two integers such that $t = z^\ell \mod C$ and $\iota'(z) = t^k \mod C$. It follows that $k\ell = 1 + 2K!q$. In particular, $\gcd(k, 2K!) = 1$. By the Dirichlet prime number theorem, there exists a strictly increasing sequence of integers $(\lambda_n)_{n\in\mathbb{N}}$ such that $p_n := k + 2K!\lambda_n$ is prime for every integer n. Let $\delta = t^{2K!}$, and let us define $\iota'_n : N' \to N$ by $\iota'_n(z) = \iota'(z)\delta^{\lambda_n}$ and $\iota'_n(c) = \iota'(c) = c$ for every $c \in C$. This homomorphism is well-defined: if $zcz^{-1} = c'$, then $\iota'_n(zcz^{-1}) = \iota'(z)\delta^{\lambda_n}\iota'(c)(\iota'(z)\delta^{\lambda_n})^{-1} = \iota'(z)\iota'(c)\iota'(z)^{-1} = \iota'(czc^{-1}) = \iota'(c') = \iota'_n(c')$, because δ belongs to the center $Z(N)$ of N. An easy calculation gives $\iota'_n(z) = t^{p_n} \mod C$. As a consequence $p_n = [N : \iota'_n(N')]$.

Last, by considering the decompositions $G = A *_C N$ and $\Gamma = A *_C N'$, one can define a homomorphism $\varphi_n : \Gamma \to G$ that coincides with the identity on A and with ι'_n on N' (well-defined since ι'_n coincides with the identity on C). Note that $\delta_n = z^{-1}\varphi_n(z) = z^{-1}t^{p_n} = z^{\ell p_n - 1} = z^{\ell k + 2K!\ell\lambda_n - 1} = z^{2K!(\ell\lambda_n - q)}$ belongs to $D(N')$. Hence, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is a small test sequence in the sense of Definition 6.14.

Second case. Suppose that N is virtually cyclic of dihedral type. It splits as $N = \langle C, a\rangle *_C \langle C, b\rangle$ with a, b of order 2 modulo C. There exists two elements a' and b' of order 2 modulo C such that $N' = \langle C, a'\rangle *_C \langle C, b'\rangle$. Up to exchanging a' and b', one can suppose without loss of generality that a is a conjugate of a' in N' (modulo C) and that b is a conjugate of b' (modulo C); indeed, the inclusion of N into N' maps non-conjugate finite subgroups to non-conjugate finite subgroups (ι is K-nice).

Since ι' is K-nice, $\iota'(a')$ and $\iota'(b')$ are not conjugate modulo C. Hence, there exists an integer $j \in \{1, 2\}$ such that $(\iota' \circ \iota)^j$ maps a' to a conjugate gag^{-1} of a, with $g \in G$, and b' to a conjugate of b. Up to replacing ι' by $\mathrm{ad}(g^{-1}) \circ \iota'^j$, one can assume without loss of generality that $\iota'(a') = a$. Then, note that there exists an integer $m \geq 1$ such that ι'^m coincides with the identity on C and induces the identity of $N'/D(N')$. Up to replacing ι' by ι'^m, one can assume without loss of generality that ι' coincides with the identity on C and induces the identity of $N'/D(N')$.

Let us define z by $z = b'a' \in N' \subset \Gamma$. Note that $\iota'(z) = z^{1+2K!q}$ for some integer q, because ι' induces the identity of $N'/D(N')$ and $D(N') = \langle z^{2K!}\rangle$, by Corollary 6.8. The integer q is non-zero since N is a strict subgroup of N' by assumption. Let k and ℓ denote two integers such that $t = z^\ell \mod C$ and $\iota'(z) = t^k \mod C$. It follows that $k\ell = 1 + 2K!q$. In particular, $\gcd(k, 2K!) = 1$. By the Dirichlet prime number theorem, there exists a strictly increasing sequence of integers $(\lambda_n)_{n\in\mathbb{N}}$ such that $p_n := k + 2K!\lambda_n$ is prime for every integer n. Let $\delta = t^{K!}$, and let us define $\iota'_n : N' \to N$ by $\iota'_n = \iota'$ on $\langle C, a'\rangle$ and $\iota'_n = \mathrm{ad}(\delta^{\lambda_n}) \circ \iota'$ on $\langle C, b'\rangle$. This homomorphism is well-defined since δ centralizes C and N' splits as $N' = \langle C, a'\rangle *_C \langle C, b'\rangle$. Since $\iota'(a') = a$ and $t = ba$, the following series of equalities holds (modulo C):

$$\begin{aligned}
\iota'_n(z) &= \delta^{\lambda_n}\iota'(b')\delta^{-\lambda_n}\iota'(a')\\
&= \delta^{\lambda_n}\iota'(b')(ba)^{-\lambda_n K!}a\\
&= \delta^{\lambda_n}\iota'(b')(ab)^{\lambda_n K!}a\\
&= \delta^{\lambda_n}\iota'(b')a(ba)^{\lambda_n K}\\
&= \delta^{\lambda_n}\iota'(b'a')\delta^{\lambda_n}\\
&= \iota'(z)\delta^{2\lambda_n}\\
&= t^{p_n}.
\end{aligned}$$

It follows that $p_n = [N : \iota'_n(N')]$.

Last, as in the first case, ι'_n extends to a homomorphism $\varphi_n : \Gamma \to G$ that coincides with the identity on A and with an inner automorphism on B. □

6.2. Main result. We will prove the difficult part of Theorem 6.1, that is the following result.

THEOREM 6.16. *Let G be a hyperbolic group. Suppose that G splits over a finite subgroup C whose normalizer N is virtually cyclic and non-elliptic in the splitting. Let $\Gamma = G *_N N'$ be a legal small extension of G. Then $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$.*

The proof of this theorem relies on the following lemma, which can be viewed as an analogue of Proposition 5.14.

LEMMA 6.17. *We keep the same notations as in the statement of Theorem 6.16. Let $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ be a small test sequence (whose existence is guaranteed by Lemma 6.15). Let $\Sigma(\boldsymbol{x},\boldsymbol{y}) = 1 \wedge \Psi(\boldsymbol{x},\boldsymbol{y}) \neq 1$ be a system of equations and inequations, where $\boldsymbol{x}$ denotes a p-tuple of variables and $\boldsymbol{y}$ denotes a q-tuple of variables. Let $\boldsymbol{\gamma} \in \Gamma^p$. Suppose that G satisfies the following condition: for every integer n, there exists $\boldsymbol{g}_n \in G^q$ such that*

$$\Sigma(\varphi_n(\boldsymbol{\gamma}), \boldsymbol{g}_n) = 1 \ \wedge \ \Psi(\varphi_n(\boldsymbol{\gamma}), \boldsymbol{g}_n) \neq 1.$$

Then there exists a retraction r from $\Gamma_{\Sigma,\boldsymbol{\gamma}} = \langle \Gamma, \boldsymbol{y} \mid \Sigma(\boldsymbol{\gamma},\boldsymbol{y}) = 1\rangle$ onto Γ such that each component of the tuple $r(\Psi(\boldsymbol{\gamma},\boldsymbol{y}))$ is non-trivial. In particular, the q-tuple $\boldsymbol{\gamma}' := r(\boldsymbol{y}) \in \Gamma^q$ satisfies

$$\Sigma(\boldsymbol{\gamma},\boldsymbol{\gamma}') = 1 \ \wedge \ \Psi(\boldsymbol{\gamma},\boldsymbol{\gamma}') \neq 1.$$

6.2.1. *Proof of Lemma 6.17 in a particular case.* We first prove Lemma 6.17 in the case where G and Γ are virtually cyclic of cyclic type. Let $G = C \rtimes \langle t\rangle$ and $\Gamma = C \rtimes \langle z\rangle$. The main difference compared to the general case is that we do not have to deal with actions on real trees here.

PROOF. For every n, the morphism φ_n extends to a homomorphism $\widehat{\varphi}_n : \Gamma_{\Sigma,\boldsymbol{\gamma}} \to G$ mapping $\boldsymbol{y}$ to $\boldsymbol{g}_n$. We shall construct a retraction $r : \Gamma_{\Sigma,\boldsymbol{\gamma}} \twoheadrightarrow \Gamma$ that does not kill any component of the tuple $\Psi(\boldsymbol{\gamma},\boldsymbol{y})$. Up to extracting a subsequence, one can suppose that the sequence $(\widehat{\varphi}_n)_{n\in\mathbb{N}}$ is stable. Let $K = \ker((\widehat{\varphi}_n)_{n\in\mathbb{N}})$ be the stable kernel of the sequence, let $L = \Gamma_{\Sigma,\boldsymbol{\gamma}}/K$ and $\pi : \Gamma_{\Sigma,\boldsymbol{\gamma}} \twoheadrightarrow L$ be the associated epimorphism. As a G-limit group, L is equationally noetherian (see [**RW14**] Corollary 6.13). It follows that there exists a unique homomorphism $\rho_n : L \to G$ such that $\widehat{\varphi}_n = \rho_n \circ \pi$, for n sufficiently large.

By Remark 6.12, every ρ_n is injective. As a consequence, the homomorphism $\pi : \Gamma \to L$ is injective, and every component of the tuple $\pi(\Psi(\boldsymbol{\gamma},\boldsymbol{y})) \in L$ is non-trivial. From now on, Γ is viewed as a subgroup of L and we do not mention the monomorphism $\pi : \Gamma \hookrightarrow L$ anymore.

In order to construct r, we will construct a discriminating sequence of retractions $(r_n : L \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$. Then, we will conclude by taking $r := r_n \circ \pi$ for n sufficiently large. Note that ρ_n coincides with $\widehat{\varphi}_n$ on Γ; in particular, $(\rho_{n|\Gamma} : \Gamma \to G)_{n\in\mathbb{N}}$ is a small test sequence.

Note that L is finitely generated, as a quotient of the finitely generated group $\Gamma_{\Sigma,\boldsymbol{\gamma}}$. In addition, the sequence $(\rho_n)_{n\in\mathbb{N}}$ is discriminating. Therefore, L is an extension

$$1 \to C \to L \overset{\chi}{\twoheadrightarrow} V \simeq \mathbb{Z}^m \to 1.$$

First, we prove that $\chi(z)$ has no root in V. This element can be written as $\chi(z) = v^j$ for some element $v \in V$ with no root, and some integer $j \neq 0$. We will prove that $j = \pm 1$. Each homomorphism $\rho_n : L \to G$ induces a homomorphism $\overline{\rho}_n : V \to G/C \simeq \langle t\rangle$. For every integer n, we have $\overline{\rho}_n(v)^j = t^{p_n}$. It follows that j divides p_n, for every n. Since $(p_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of prime numbers, $j = \pm 1$.

In order to define the retraction $r_n : L \twoheadrightarrow \Gamma$, we use a presentation of L. Let $(v_1, v_2, \ldots, v_m)$ be a basis of V, with $v_1 = v$. For $1 \leq i \leq m$, let z_i be a preimage of v_i in L. One can suppose that $z_1 = z$. Each element z_i induces an automorphism ζ_i of C, and each commutator $[z_i, z_j]$ is equal to an element $c_{i,j} \in C$. Here is a presentation of L:

$$\langle C, z_1, \ldots, z_m \mid \mathrm{ad}(z_i)_{|C} = \zeta_i, \ [z_i, z_j] = c_{i,j}\rangle.$$

Let $\gamma = z^{K!}$. We denote by τ_n the endomorphism of Γ defined by $\tau_n = \mathrm{id}$ on C and $\tau_n(z) = z\gamma^n$ (well-defined since γ centralizes C). Let us define $r_n : L \to \Gamma$ by $r_n = \mathrm{id}$ on $\langle C, z\rangle$ and $r_n(z_i) = \tau_n \circ \rho_n(z_i)$ for $2 \leq i \leq m$. In order to verify that r_n is well-defined, it suffices to show that $[r_n(z), r_n(z_i)] = r_n(c_{1,i})$, since r_n coincides with the identity on C.

Recall that $\rho_n(z) = z\delta_n$ with $\delta_n \in D(\Gamma)$. As a consequence, $\tau_n \circ \rho_n(z) = zz^{R_n}$ with z^{R_n} in the center of Γ. Therefore,

$$\tau_n \circ \rho_n([z, z_i]) = [zz^{R_n}, \tau_n \circ \rho_n(z_i)] = [z, \tau_n \circ \rho_n(z_i)] = c_{1,i}.$$

Hence, $[r_n(z), r_n(z_i)] = r_n(c_{1,i})$, so r_n is well-defined.

It remains to prove that the sequence $(r_n)_{n\in\mathbb{N}}$ is discriminating. Let $x \in L$ be a non-trivial element, and let us prove that $r_n(x)$ is non-trivial for every n sufficiently large. The element x can be written as $x = cz^{q_1}z_2^{q_2} \cdots z_m^{q_m}$ with $c \in C$ and $q_i \in \mathbb{Z}$. If x lies in Γ, then $r_n(x) = x \neq 1$. Else, if x does not belong to Γ, then $y = z_2^{q_2} \cdots z_m^{q_m}$ has infinite order (otherwise, y would belong to C, so x would belong to Γ). Since the sequence $(\rho_n)_{n\in\mathbb{N}}$ is discriminating, $\rho_n(y)$ has infinite order for every n large enough, so $\rho_n(y) = z^{\ell_n} \mod C$ with $\ell_n \neq 0$. Thus $\tau_n \circ \rho_n(y) = (zz^{nK!})^{\ell_n} = z^{\ell_n(1+nK!)}$. For $n > |q_1|$, $|\ell_n(1 + nK!)| > |q_1|$, so $r_n(x)$ is non-trivial. As a conclusion, the sequence of retractions $(r_n : L \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$ is discriminating. $\square$

6.2.2. *Proof of Lemma 6.17 in the general case.* We now prove Lemma 6.17.

Proof. For every n, the map φ_n extends to a homomorphism $\widehat{\varphi}_n : \Gamma_{\Sigma,\boldsymbol{\gamma}} \to G$ mapping $\boldsymbol{y}$ to $\boldsymbol{g}_n$. We shall construct a retraction $r : \Gamma_{\Sigma,\boldsymbol{\gamma}} \twoheadrightarrow \Gamma$ that does not kill any component of the tuple $\Psi(\boldsymbol{\gamma}, \boldsymbol{y})$. Up to exctracting a subsequence, one can suppose that the sequence $(\widehat{\varphi}_n)_{n\in\mathbb{N}}$ is stable. Let $K = \ker((\widehat{\varphi}_n)_{n\in\mathbb{N}})$ be the stable kernel of the sequence, let $L = \Gamma_{\Sigma,\boldsymbol{\gamma}}/K$ and $\pi : \Gamma_{\Sigma,\boldsymbol{\gamma}} \twoheadrightarrow L$ be the associated epimorphism. As a G-limit group, L is equationally noetherian (see [**RW14**] Corollary 6.13). It follows that there exists a unique homomorphism $\rho_n : L \to G$ such that $\widehat{\varphi}_n = \rho_n \circ \pi$, for n sufficiently large.

By Remark 6.12, every ρ_n is injective. It follows that the homomorphism $\pi : \Gamma \to L$ is injective, and every component of the tuple $\pi(\Psi(\boldsymbol{\gamma}, \boldsymbol{y})) \in L$ is non-trivial. From now on, Γ is viewed as a subgroup of L and we do not mention the monomorphism $\pi : \Gamma \hookrightarrow L$ anymore.

In order to construct r, we will construct a discriminating sequence of retractions $(r_n : L \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$. Then, we will conclude by taking $r := r_n \circ \pi$ for n sufficiently large. Note that ρ_n coincides with $\widehat{\varphi}_n$ on Γ; in particular, $(\rho_{n|\Gamma} : \Gamma \to G)_{n\in\mathbb{N}}$ is a small test sequence.

Let (X, d) be a Cayley graph of G. Let us consider a Stallings splitting Λ of L relative to Γ, and let U be the one-ended factor that contains Γ. Let S be a generating set of U. Recall that $\mathrm{Aut}_\Gamma(U)$ is the subgroup of $\mathrm{Aut}(U)$ consisting of all automorphisms σ satisfying the following two conditions:

(1) $\sigma_{|\Gamma} = \mathrm{id}_{|\Gamma}$;
(2) for every finite subgroup F of U, there exists an element $u \in U$ such that $\sigma_{|F} = \mathrm{ad}(u)_{|F}$.

Recall that a homomorphism $\varphi : U \to G$ is said to be *short* if its length $\ell(\varphi) := \max_{s\in S} d(1, \varphi(s))$ is minimal among the lengths of homomorphisms in the orbit of φ under the action of $\mathrm{Aut}_\Gamma(U) \times \mathrm{Inn}(G)$.

Since $||t_n||$ goes to infinity, there exists a sequence of automorphisms $(\sigma_n)_{n\in\mathbb{N}} \in \mathrm{Aut}_\Gamma(U)^{\mathbb{N}}$ and a sequence of elements $(x_n)_{n\in\mathbb{N}} \in G^n$ such that the homomorphisms $\mathrm{ad}(x_n)\circ\rho_n\circ\sigma_n$ are short, pairwise distinct, and such that the sequence $(\mathrm{ad}(x_n)\circ\rho_n\circ\sigma_n)_{n\in\mathbb{N}}$ is stable (see paragraph 2.5.1), up to extracting a subsequence. Since ρ_n coincides with the identity on G, we have $\mathrm{ad}(x_n) \circ \rho_n \circ \sigma_n = \rho_n \circ \mathrm{ad}(x_n) \circ \sigma_n$ and, up to replacing σ_n by $\mathrm{ad}(x_n)\circ\sigma_n$, we can forget the postconjugation by x_n and assume that $\sigma_{n|\Gamma}$ is a conjugation by an element of G.

We claim that σ_n extends to an automorphism of L, still denoted by σ_n. By the second condition above, σ_n is a conjugacy on finite subgroups of U. We proceed by induction on the number of edges of Λ (the Stallings splitting of L relative to Γ used previously in order to define U). It is enough to prove the claim in the case where Λ has only one edge.

If $L = U *_C B$ with $\sigma_{n|C} = \mathrm{ad}(u)$, one defines $\sigma_n : L \to G$ by $\sigma_{n|U} = \sigma_n$ and $\sigma_{n|B} = \mathrm{ad}(u)$.

If $L = U*_C = \langle U, t \mid tct^{-1} = \alpha(c), \forall c \in C\rangle$ with $\sigma_{n|C} = \mathrm{ad}(u_1)$ and $\sigma_{n|\alpha(C)} = \mathrm{ad}(u_2)$, one defines $\sigma_n : L \to G$ by $\sigma_{n|U} = \sigma_n$ and $\sigma_n(t) = u_2^{-1}tu_1$.

In order to complete the proof of Lemma 6.17, we will use the following lemma.

Lemma 6.18. *We keep the same notations as above. Let T be the limit tree of the sequence of metric spaces $(X, d/\ell(\rho_n \circ \sigma_n))_{n\in\mathbb{N}}$. The following dichotomy holds:*

- *either Γ does not fix a point of T, in which case there exists a discriminating sequence of retractions $(r_n : L \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$,*
- *or Γ is elliptic, and there exist a proper quotient L_1 of L, an embedding $\Gamma \hookrightarrow L_1$ allowing us to identify Γ with a subgroup of L_1, and two discriminating sequences $(\rho_n^1 : L_1 \to G)_{n\in\mathbb{N}}$ and $(\theta_n^1 : L \twoheadrightarrow L_1)_{n\in\mathbb{N}}$ such that the following three conditions are satisfied:*
  - *(1) $\rho_n \circ \sigma_n = \rho_n^1 \circ \theta_n^1$;*
  - *(2) ρ_n^1 coincides with ρ_n on Γ; in particular, $(\rho^1_{n|\Gamma} : \Gamma \to G)_{n\in\mathbb{N}}$ is a test sequence.*
  - *(3) There exists an element $g_n \in G$ such that σ_n and θ_n^1 coincides with $\mathrm{ad}(g_n)$ on Γ.*

Before proving this lemma, we will use it to conclude the proof of Lemma 6.17. If Γ does not fix a point of T, we are done. If Γ fixes a point of T, by iterating Lemma 6.18, we get a sequence of proper quotients

$$L_0 = L \twoheadrightarrow L_1 \twoheadrightarrow \cdots L_i \twoheadrightarrow \cdots$$

such that, for every integer $i \geq 1$, there exist two discriminating sequences of morphisms $(\rho_n^i : L_i \to G)_{n\in\mathbb{N}}$ and $(\theta_n^i : L_{i-1} \twoheadrightarrow L_i)_{n\in\mathbb{N}}$ such that $\rho_n^{i-1} \circ \sigma_n = \rho_n^i \circ \theta_n^i$, ρ_n^i coincides with ρ_n^{i-1} on Γ, and there exists an element $g_n \in G$ such that σ_n and θ_n^i coincides with $\mathrm{ad}(g_n)$ on Γ.

By the descending chain condition, the iteration eventually terminates. Let L_k be the last quotient of the series. By Lemma 6.18, there exists a discriminating sequence of retractions $(r_n : L_k \twoheadrightarrow \Gamma)_{n\in\mathbb{N}}$. For every finite set $F \subset L$, one can find some integers $n_1, \ldots, n_k$ such that the morphism $r_{n_k} \circ \theta_{n_k}^k \circ \cdots \circ \theta_{n_1}^1 : L \twoheadrightarrow \Gamma$ is injective on F. Moreover, since every θ_{n_i} is a conjugation on Γ by an element of G, there exists an element $v_n \in G$ such that $\mathrm{ad}(v_n) \circ r_{n_k} \circ \theta_{n_k}^k \circ \cdots \circ \theta_{n_1}^1 : L \twoheadrightarrow \Gamma$ is a retraction. This concludes the proof of Lemma 6.17 $\square$

It remains to prove Lemma 6.18.

Proof. Recall that U denotes the one-ended factor of L relative to Γ. We distinguish two cases.

First case. Suppose that Γ fixes a point of T. Let us prove that the stable kernel of the sequence $(\rho_n \circ \sigma_n)_{n\in\mathbb{N}}$ is non-trivial. Assume towards a contradiction that the stable kernel is trivial. Then, by Theorem 1.16 of [**RW14**], the action of (U, Γ) on the limit tree T has the following properties:

- the pointwise stabilizer of any non-degenerate tripod is finite;
- the pointwise stabilizer of any non-degenerate arc is finitely generated and finite-by-abelian;
- the pointwise stabilizer of any unstable arc is finite.

In particular, the tree T satisfies the ascending chain condition of Theorem 2.15 since any ascending sequence of finitely generated and finite-by-abelian subgroups of a hyperbolic group stabilizes.

Then, it follows from Theorem 2.15 that either (U, Γ) splits over the stabilizer of an unstable arc, or over the stabilizer of an infinite tripod, or T has a decomposition into a graph of actions. Since U is one-ended relative to Γ, and since the stabilizer of an unstable arc or of an infinite tripod is finite, it follows that T has a decomposition into a graph of actions.

Now, it follows from Theorem 2.23 that there exists a sequence of automorphisms $(\alpha_n)_{n\in\mathbb{N}} \in \mathrm{Aut}_\Gamma(U)^{\mathbb{N}}$ such that $(\rho_n \circ \sigma_n) \circ \alpha_n$ is shorter than $\rho_n \circ \sigma_n$ for n large enough. This is a contradiction since the morphisms $\rho_n \circ \sigma_n$ are assumed to be short. Hence, the stable kernel of the sequence $(\rho_n \circ \sigma_n)_{n\in\mathbb{N}}$ is non-trivial.

As in the proof of Lemma 6.17 above, since σ_n coincides with an inner automorphism on each finite subgroup of U (by definition of $\mathrm{Aut}_\Gamma(U)$), it extends to an automorphism of L, still denoted by σ_n. Let $L_1 := L/\ker((\rho_n \circ \sigma_n)_{n\in\mathbb{N}})$ and let $\pi_1 : L \twoheadrightarrow L_1$ be the corresponding epimorphism. Observe that π_1 is injective on Γ, allowing to identify Γ with a subgroup of L_1.

As a G-limit group, L_1 is equationally noetherian (see [**RW14**] Corollary 6.13). It follows that, for every integer n, there exists a unique homomorphism $\tau_n^1 : L_1 \to G$ such that $\rho_n \circ \sigma_n = \tau_n^1 \circ \pi_1$. There exists an element $g_n \in G$ such that σ_n coincides with $\mathrm{ad}(g_n)$ on Γ. Hence, since ρ_n coincides with the identity on G, we can write $\rho_n \circ \sigma_n = \mathrm{ad}(g_n) \circ \rho_n^1 \circ \pi_1$ in such a way that $\rho_n^1 := \mathrm{ad}(g_n^{-1}) \circ \tau_n^1$ coincides with the identity on G. For every n, let $\theta_n^1 = \pi_1 \circ (\sigma_n)^{-1}$, so that $\rho_n = \mathrm{ad}(g_n) \circ \rho_n^1 \circ \theta_n^1$. The sequence $(\theta_n^1 : L \to L_1)_{n\in\mathbb{N}}$ is therefore discriminating, and every homomorphism θ_n^1 coincides with $\mathrm{ad}(g_n^{-1})$ on Γ.

Second case. Suppose that Γ does not fix a point of T. In the sequel, the letter z denotes an element of $N' = N_\Gamma(C)$ defined as follows.

- If $\Gamma = A*_C$ and the copies of C in A are conjugate, one can suppose without without loss of generality that there are equal. The letter z denotes the stable letter of the HNN extension. We have $N_\Gamma(C) = C \rtimes \langle z\rangle$.
- If $\Gamma = A *_C B$, we define z by $z = ba$ where b and a are such that $N_B(C) = C \rtimes \langle b\rangle$ et $N_A(C) = C \rtimes \langle a\rangle$. Note that b and a have order 2 modulo C in $N_B(C)$ and $N_A(C)$ respectively.
- If $\Gamma = A*_C$ and the two copies of C in A are not conjugate, let s be the stable letter of the HNN extension, and let $a \in A$ and $b \in sAs^{-1}$ be two elements such that $N_A(C) = C \rtimes \langle a\rangle$ and $N_{sAs^{-1}}(C) = C \rtimes \langle b\rangle$. Then we define z by $z = ba$.

In the same way, we define $t \in N = N_G(C)$.

Since Γ is not elliptic in T, and since A and B are elliptic, z acts hyperbolically on T. Let d denote its axis. We will proceed in two steps.

- **First step.** We prove that the one-ended factor U relative to Γ can be decomposed as a graph of groups Δ in which S is a vertex group, where S denotes the global stabilizer of the axis d of z in T.
- **Second step.** We construct a discriminating sequence of retractions $(r_n : L \to \Gamma)$.

First step. We shall prove that U splits as a graph of groups in which S is a vertex group. First, we prove that the translates of d are transverse. Let u be an element of U such that $ud \cap d$ contains a segment I which is not reduced to a point. Let us prove that $ud = d$. There exists a constant R_t such that the following holds: every element $g \in G$ such that $A(t)^{+\delta} \cap gA(t)^{+\delta}$ has a diameter larger than R_t belongs to $M(t)$, where $A(t)$ refers to the quasi-axis of t (see Section 2.4). Taking $g = \rho_n(u)$, we have $\rho_n(u) \in M(t) = N_G(C)$ for n large enough. Therefore, since $\rho_n(z)$ belongs to $\langle t\rangle$, one of the following two possibilities occurs, for n sufficiently large: either $\rho_n([u, z])$ belongs to C, or $\rho_n((uz)^2)$ belongs to C. Since $C = \rho_n(C)$, and since the sequence (ρ_n) is discriminating, we deduce that $[u, z]$ or $(uz)^2$ belongs to C. Since C fixes d pointwise, u fixes d pointwise in the first case; in the second case, u acts on d by reversing the orientation. As a conclusion, $ud = d$. Hence, the translates of the axis d are transverse.

Recall that S is the global stabilizer of d in U. The following facts can be proved by using the discriminating sequence $(\rho_{n|S} : S \to N_G(C) = M(t))$:

(1) C is the maximal finite normal subgroup of S,
(2) $S = N_U(C)$,
(3) S is virtually abelian. To be more precise, S/C is abelian in the case where $N_G(C)$ is of cyclic type, and S/C has an abelian subgroup of index 2 in the case where $N_G(C)$ is of dihedral type (see Lemma 6.19 below).

By [**RW14**] Theorem 6.3, the group S is finitely generated. We are ready to prove that U splits as a graph of groups in which S is a vertex group.

First case. Suppose that the action of S on d is discrete. We define an equivalence relation $\sim$ on T by $x \sim y$ if the intersection $[x, y] \cap ud$ contains at most one point, for any element $u \in U$. Let $(Y_j)_{j\in J}$ denote the equivalence classes that are not reduced to a point. Note that each Y_j is a closed subtree of T. Let us verify that the family $(Y_j)_{j\in J} \cup \{ud \mid \in U\}$ is a transverse covering of T (see Definition 2.16).

- *Transverse intersection.* By definition, $Y_i \cap Y_j = \varnothing$ for every $i \in I$, and $|Y_i \cap ud| \leq 1$ for every $i \in I$ and every $u \in U$. In addition, d is transverse to its translates.
- *Finiteness condition.* Let x and y be two points of T. Since the action of S on d is discrete, there exists a constant $\varepsilon > 0$ such that the following holds: for any $u \in U$, if the intersection $[x, y] \cap ud$ is non-degenerate, then the length of $[x, y] \cap ud$ is bounded from below by ε. Therefore, the arc $[x, y]$ is covered by at most $\lfloor d'(x, y)/\varepsilon \rfloor$ translates of the axis d and at most $\lfloor d'(x, y)/\varepsilon \rfloor + 1$ distinct subtrees Y_j, where d' denotes the metric on T.

Thus, the family $(Y_j)_{j\in J} \cup \{ud \mid u \in U\}$ is a transverse covering of the tree T. We can now build the skeleton of this transverse covering in the sense of Guirardel (see Definition 2.17), denoted by T'. Since the action of U on T is minimal, so is the action of U on T', according to Lemma 4.9 of [**Gui04**]. One of the vertex group of T' is equal to S, by construction. This concludes the proof of the first case.

Second case. Suppose that the action of S on d has dense orbits. The previous argument no longer works (because ε does not exist). Note that S is virtually $\mathbb{Z}^n$ with $n \geq 2$.

Since the discriminating sequence (ρ_n) coincides with inner automorphisms on A and B, these groups are elliptic in T. We will apply Theorem 2.15 to the pair $(U, \{A, B\})$ (resp. (U, A)) in order to decompose T as a graph of actions. This is enough to conclude because d is one of the components of the graph of actions. Indeed, the action of S on d is indecomposable (see Definition 1.17 in [**Gui08**]), so d is contained in one of the components of the graph of actions by Lemma 1.18 of [**Gui08**]. Let $\mathcal{C}$ denote this component. Note that each component given by Theorem 2.15 is either indecomposable or simplicial. Since $S \curvearrowright d$ has dense orbits, $\mathcal{C}$ is not simplicial. Thus, $\mathcal{C}$ is indecomposable. The axis $d \subset \mathcal{C}$ being transverse to its translates, d is necessarily equal to $\mathcal{C}$.

Assume towards a contradiction that Theorem 2.15 does not give a splitting of T as a graph of actions. Then, still by Theorem 2.15, the pair (U, A) (resp. $(U, \{A, B\})$) splits over a finite subgroup E which is either the pointwise stabilizer of an unstable arc, or the pointwise stabilizer of an infinite tripod whose normalizer contains the free group F_2. Let Y be the Bass-Serre tree of this splitting. We will prove that Γ is elliptic in Y, contradicting the fact that U is one-ended relative to Γ. Let Y_Γ be the minimal subtree of Γ. Let Z be the tree of cylinders of the splitting $A*_C$ or $A *_C B$ of Γ. Its vertex groups are A and N in the first case, and A, B and N in the second case. In the cyclic case, there is only one edge group in Z, namely C; in the dihedral case, there are at most two edges groups C_1 and C_2 that contain C with index 2. We claim that Z dominates Y_Γ, i.e. that its vertex groups are elliptic in Y_Γ. Since Y_Γ is a splitting relative to the pair $\{A, B\}$, we juste have to prove that N is elliptic in Y_Γ. Since $N \subset S$, it is enough to observe that S is elliptic in Y, as a one-ended group (because it is virtually $\mathbb{Z}^n$ with $n \geq 2$). Thus, the tree of cylinders Z dominates Y_Γ. As a consequence, E contains C up to conjugation. Since $N_U(C) = S$ is virtually abelian, the normalizer $N_U(E)$ of E is virtually abelian as well. It follows that E is not the pointwise stabilizer of an infinite tripod whose normalizer contains the free group F_2. So E is the stabilizer of an unstable arc $I \subset T$. Therefore, there exists a subarc $I' \subset I$ whose stabilizer E' satisfies $E' \supset E$ and $E' \neq E$. The set of fixed points of C is exactly d. Since $E \supset C$, $\mathrm{Fix}(E)$ is contained in $d = \mathrm{Fix}(C)$, and since d is transverse to its translates, $\mathrm{Fix}(E) = d$ or $\mathrm{Fix}(E)$ is one point of d. This second possibility cannot happen since $\mathrm{Fix}(E)$ contains I. Hence, E is contained in S, so $E = S_0$ (the pointwise stabilizer of d). The same argument shows that $E' = S_0 = E$, a contradiction. As a conclusion, T splits as a graph of actions one component of which is d. So U splits as a graph of groups in which S is a vertex group. This completes the second case.

It remains to construct a discriminating sequence of retractions $(r_n : L \to \Gamma)$. We distinguish two cases, depending on the type of $N_G(C)$. In the sequel, we denote by Δ a splitting of U as a graph of groups in which S is a vertex group.

Second step. Construction of the discriminating sequence of retractions $(r_n : L \to \Gamma)$.

A. The cyclic case. Suppose that $N_G(C)$ is of cyclic type. Then S is an extension

$$1 \to C \to S \overset{\chi}{\twoheadrightarrow} V \simeq \mathbb{Z}^m \to 1,$$

for some integer m. We claim that $\chi(z)$ has no root in V, i.e. that $\langle \chi(z) \rangle$ is cyclic maximal. This element can be written as $\chi(z) = v^j$ for some element $v \in V$ with no proper root, and some integer $j \neq 0$. We will prove that $j = \pm 1$. Each homomorphism $\rho_{n|S} : S \to G$ induces a homomorphism $\overline{\rho}_n : V \to G/C \simeq \langle t \rangle$. For every integer n, we have $\overline{\rho}_n(v)^j = t^{p_n}$. It follows that j divides p_n, for every n. Since (p_n) is a strictly increasing sequence of prime numbers, $j = \pm 1$.

In order to define the retraction $r_n : L \twoheadrightarrow \Gamma$, we need a presentation of S. Let $(v_1, v_2, \ldots, v_m)$ be a basis of V, with $v_1 = v$. For $1 \leq i \leq m$, let z_i be a preimage of v_i in L. One can suppose that $z_1 = z$. Each element z_i induces an automorphism ζ_i of C, and each commutator $[z_i, z_j]$ is equal to an element $c_{i,j} \in C$. Here is a presentation of S:

$$\langle C, z_1, \ldots, z_m \mid \mathrm{ad}(z_i)_{|C} = \zeta_i,\ [z_i, z_j] = c_{i,j} \rangle.$$

Let v denote the vertex of Δ whose stabilizer is S. Note that any edge adjacent to v in Δ has a stabilizer contained in $S_0 = \langle C, z_2, \ldots, z_m \rangle$, because any stabilizer of a point of d is contained in S_0. Let us refine Δ by replacing the vertex v with the decomposition $S = S *_{S_0} S_0$ of S. More precisely,

- the v vertex is replaced by a graph of groups with two vertices denoted by v_0 and v_1, linked by a single edge e, such that the stabilizers of v_0 and e are equal to S_0 and the stabilizer of v_1 is equal to S;
- we replace each edge $\varepsilon = [v, w]$ Δ with an edge $[v_0, w]$ if $w \neq v$, and with an edge $[v_0, v_0]$ if $w = v$, which is always possible since U_ε is contained in S_0.

We denote by Δ' this new decomposition of U as a graph of groups. Then, let us consider a JSJ splitting of L relative to U over finite groups, and let Λ be the splitting of L obtained from this JSJ splitting by replacing the vertex fixed by U with the splitting Δ' of U. We keep the notations v_0 and v_1 for the vertices of Λ corresponding to the vertices v_0 and v_1 of Δ'. Since all finite subgroups of S are contained in S_0, we can assume without loss of generality (up to performing some slides of edges) that there is a only one edge e adjacent to v_1 in Λ, namely the one that connects v_1 to v_0. Now, we collapse all the edges of Λ other than e, and we get a decomposition of L as an amalgamated product $L = L_0 *_{S_0} S$, which can also be viewed as an HNN extension $L = L_0 *_{S_0}$ with stable letter z.

We are now ready to define the retraction $r_n : L \to \Gamma$. First, we define r_n on S. In order to do this, we follow the same strategy as in the proof of the particular case discussed above. Let $\gamma = z^{K!}$. We denote by τ_n the endomorphism of Γ defined by $\tau_n = \mathrm{id}$ on C and $\tau_n(z) = z\gamma^n$ (well-defined since γ centralizes C). Let us define $r_n : S \to \Gamma$ by $r_n = \mathrm{id}$ on $\langle C, z\rangle$ and $r_n(z_i) = \tau_{f(n)} \circ \rho_n(z_i)$ for $2 \leq i \leq m$ (where $f : \mathbb{N} \to \mathbb{N}$ denotes a strictly increasing function that will be specified later). In order to verify that r_n is well-defined, it suffices to show that $[r_n(z), r_n(z_i)] = r_n(c_{1,i})$, since r_n coincides with the identity on C. Recall that $\rho_n(z) = z\delta_n$ with $\delta_n \in D(\Gamma)$. As a consequence, $\tau_{f(n)} \circ \rho_n(z) = zz^{R_n}$ with z^{R_n} in the center of Γ. Therefore,

$$\tau_{f(n)} \circ \rho_n([z, z_i]) = [zz^{R_n}, \tau_{f(n)} \circ \rho_n(z_i)] = [z, \tau_{f(n)} \circ \rho_n(z_i)] = c_{1,i}.$$

Hence, $[r_n(z), r_n(z_i)] = r_n(c_{1,i})$, so r_n is well-defined.

Since $L = S{*}_{S_0}L_0$ and since r_n coincides with $\tau_{f(n)} \circ \rho_n$ on S_0, the morphism $r_n : S \to \Gamma$ extends to a morphism from L to Γ that coincides with r_n on S and with $\tau_{f(n)} \circ \rho_n$ on L_0. This new morphism is still denoted by r_n. Note that r_n is a retraction from L onto Γ. Indeed, $r_n(z) = z$ and $r_n = \tau_{f(n)} \circ \rho_n = \mathrm{id}$ on A, and $\Gamma = \langle z, A\rangle$.

To conclude, we will prove that the sequence (r_n) is discriminating, provided that the function $f : \mathbb{N} \to \mathbb{N}$ is properly chosen. Let $F \subset L \setminus \{1\}$ be a finite set. For simplicity, one can assume that $F = \{x\}$, the proof being identical in the case where F has more than one element.

If x belongs to L_0, since the sequence (ρ_n) is discriminating, we have $\rho_n(x) \neq 1$ for n large. Since $\tau_{f(n)} : \Gamma \to \Gamma$ is injective, we have $\tau_{f(n)} \circ \rho_n(x) = r_n(x) \neq 1$. Now, suppose that x does not belong to L_0. Then x can be written as $x = y_1 z^{n_1} y_2 z^{n_2} \cdots y_k z^{n_k}$ with $k \geq 1$, $y_i \in L \setminus S_0$ and $n_i \neq 0$ for all $1 \leq i \leq k$ (except possibly n_k, which can be zero). Note that S_0 is the normalizer of C in L_0. It follows from this observation that for n sufficiently large, $\rho_n(y_i)$ does not belong to $N_\Gamma(C) = \langle C, z\rangle$. We can therefore write $\rho_n(y_i)$ as a product $a_{i,1}z^{\ell_{i,1}}a_{i,2}z^{\ell_{i,2}} \cdots a_{i,q_i}z^{\ell_{i,q_i}}$ with $q_i \geq 1$, $a_{i,j} \in A \setminus C$ for all $1 \leq j \leq q_i$ and $\ell_{i,j} \neq 0$ for all $1 \leq j \leq q_i - 1$ (the integer ℓ_{i,q_i} can be zero). So $r_n(y_i)$ can be written as a reduced form as follows

$$r_n(y_i) = \tau_{f(n)} \circ \rho_n(y_i) = a_{i,1}z^{\ell_{i,1}f(n)}a_{i,2}z^{\ell_{i,2}f(n)} \cdots a_{i,q_i}z^{\ell_{i,q_i}f(n)}.$$

To prove that $r_n(x)$ is non-trivial, it suffices to prove that the word obtained by concatenating the reduced forms of $r_n(y_i)$ and z^{n_i} is still in reduced form in the splitting $\Gamma = A{*}_C$ with stable letter z, i.e. is of the form $r_n(x) = u_1 z^{k_1} u_2 z^{k_2} \cdots u_M z^{k_M}$ with $M \geq 2$, $k_i \neq 0$ and $u \in A \setminus C$ (except maybe $k_M = 0$ and $u_1 \in C$). Let's look at the subword at the junction of $r_n(y_i)$, z^{n_i} and $r_n(y_{i+1})$, for $1 \leq i < k$. This subword is of the form $a_{i,q_i}z^{\ell_{i,q_i}f(n)}z^{n_i}a_{i+1,1}$. Since $i < k$, the integer n_i is not zero. We distinguish two possibilities: either $\ell_{i,q_i} = 0$, in which case there is nothing to be done, or $\ell_{i,q_i} \neq 0$, in which case we can choose $f(n)$ large enough so that $\ell_{i,q_i}f(n) + n_i \neq 0$.

As a conclusion, $f(n)$ can be chosen sufficiently large so that $r_n(x) \neq 1$, and the same proof is still valid if one replaces the singleton $\{x\}$ by a finite subset of $L \setminus \{1\}$. Hence, the sequence of retractions $(r_n : L \twoheadrightarrow \Gamma)$ is discriminating.

B. The dihedral case. Suppose that $N_G(C)$ is of dihedral type. Let us begin with an easy lemma about D_∞-limit groups.

LEMMA 6.19. *Let L be a group. Suppose that* $\mathrm{Th}_\forall(D_\infty) \subset \mathrm{Th}_\forall(L)$. *Then, either L is torsion-free abelian, or L is a semi-direct product $Z \rtimes \langle w\rangle$ with Z torsion-free abelian, and w an element of order 2 acting by* $-\mathrm{id}$ *on Z.*

PROOF. The universal sentence $\forall x \forall y\ ((x^2 \neq 1) \wedge (y^2 \neq 1) \wedge (xy \neq 1)) \Rightarrow ((xy)^2 \neq 1)$, which is satisfies by D_∞, express the fact that the $\forall$-definable subset of D_∞ composed of all elements of order > 2 is a subgroup (if one adds 1). Moreover, one can express by means of universal sentences that this subgroup is torsion-free abelian, and has index at most 2. Therefore, any group L such that $\mathrm{Th}_\forall(D_\infty) \subset \mathrm{Th}_\forall(L)$ contains a torsion-free abelian subgroup of index at most 2. It follows that if L is not torsion-free abelian, then it splits as a semi-direct product $Z \rtimes \langle w \rangle$ with Z torsion-free abelian, and w of order 2. Last, the action of w on Z is described by the following universal sentence, satisfied by D_∞:

$$\forall x \forall y \ ((x \neq 1) \wedge (x^2 = 1) \wedge (y^2 \neq 1)) \Rightarrow (xyx^{-1} = y^{-1}).$$

This concludes the proof of Lemma 6.19. $\square$

Since $N_G(C)$ is of dihedral type, it follows from the lemma above that S is an extension

$$1 \to C \to S \overset{\chi}{\twoheadrightarrow} V \rtimes \mathbb{Z}_2 \to 1,$$

with $V \simeq \mathbb{Z}^m$ such that $\chi(z) \in V$, for some integer m. We claim that $\chi(z)$ has no root in V. This element can be written as $\chi(z) = v^j$ for some element $v \in V$ with no proper root, and some integer $j \neq 0$. We will prove that $j = \pm 1$. Each homomorphism $\rho_{n|S} : S \to G$ induces a homomorphism $\overline{\rho}_n : V \to \langle t \rangle$. For every integer $n \geq n_0$, we have $\overline{\rho}_n(v)^j = t^{p_n}$. It follows that j divides p_n, for every $n \geq n_0$. Since (p_n) is a strictly increasing sequence of prime numbers, $j = \pm 1$.

Let $\{z_1, z_2, \ldots, z_r\} \subset S$ be a finite set, with $z_1 = z$, such that $\{\chi(z_1), \ldots, \chi(z_r)\}$ is a basis of V. Before we construct the retraction $r_n : L \twoheadrightarrow \Gamma$, we need a splitting of L. Let v denote the vertex of Δ whose stabilizer is S. We refine Δ by replacing v by the decomposition $S = S_1 *_{S_0} S_2$ where $S_0 = \langle C, z_2, \ldots, z_r \rangle$, $S_1 = S_0 \rtimes \langle a \rangle$ and $S_2 = S_0 \rtimes \langle b \rangle$ with $b = za$ (where a and b have been defined above). Note that any edge adjacent to v in Δ has a stabilizer contained in a conjugate of S_1 or S_2 in S. We refine Δ by replacing v by this splitting of S. Let Δ' denote this new decomposition of U as a graph of groups. Then, consider a JSJ decomposition of L relative to U over finite groups, and let Λ denote the decomposition of L obtained from this JSJ splitting by replacing the vertex fixed by U with the splitting Δ' of U. Then, collapsing edges if necessary, we obtain a decomposition of L of the form

$$L = L_1 *_{S_1} *S *_{S_2} L_2 \quad \text{our} \quad L = (L_1 *_{S_1} *S) *_{S_2} .$$

First case: $L = L_1 *_{S_1} *S *_{S_2} L_2$. The group L can also be decomposed as $L = L_1 *_{S_0} L_2$. We will suppose that $\Gamma = A *_C B$, with C is strictly contained in A and in B. The proof is similar in the case where $\Gamma = A*_C$ (left to the reader).

Since A and B are elliptic in the decomposition $L = L_1 *_{S_0} L_2$, and since z has a translation length equal to 2, one can suppose without loss of generality that $A \subset L_1$ et $B \subset L_2$.

Recall that the morphism ρ_n coincides with the identity on A and with the inner automorphism $\mathrm{ad}(\delta_n)$ on B, with $\delta_n \in D(N_\Gamma(C)) = \langle z^{2K!} \rangle$. Set $\delta = z^{2K!}$, so that $\delta_n = \delta^{\lambda_n} = z^{2K!\lambda_n}$ for a certain integer $\lambda_n \neq 0$ (which goes to infinity as n goes to infinity.

We denote by τ_n the endomorphism of Γ defined by $\tau_n = \mathrm{id}$ on A and $\tau_n = \mathrm{ad}(\delta^n)$ on B (well-defined since δ centralizes C). Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function that will be specified later. The homomorphism $\tau_{f(n)} \circ \rho_n : L \to \Gamma$ coincides with $\mathrm{ad}(\tau_{f(n)}(\delta_n)\delta^{f(n)})$ on B. Let $\gamma_n = \tau_{f(n)}(\delta_n)\delta^{f(n)}$. An easy calculation shows that $\gamma_n = z^{\mu_n}$ where $\mu_n = 2K!\lambda_n(1 + 4K!f(n)) + 2K!f(n)$.

Let us define r_n on L by $r_n = \tau_{f(n)} \circ \rho_n$ on L_1 and $r_n = \mathrm{ad}(\gamma_n^{-1}) \circ \tau_{f(n)} \circ \rho_n$ on L_2. Note that r_n is a retraction from L onto $\Gamma = A *_C B$ (well-defined since γ_n centralizes S_0). Let us summarize the properties of r_n.

$$\begin{aligned}
r_n(a) &= a\\
r_n(b) &= b\\
r_n(z) &= z\\
r_{n|S_0} &= \tau_{f(n)} \circ \rho_{n|S_0}\\
r_{n|L_1} &= \tau_{f(n)} \circ \rho_{n|L_1}\\
r_{n|L_2} &= \mathrm{ad}(\gamma_n^{-1}) \circ \tau_{f(n)} \circ \rho_{n|S_0}
\end{aligned}$$

To conclude, we will prove that the sequence $(r_n : L \to \Gamma)$ is discriminating provided that the sequence $(f(n))$ is properly chosen. Let us consider the decompositions

$$L = L_1 *_{S_1} *S *_{S_2} L_2 \qquad \text{and} \qquad \Gamma = A *_{C_1} *N *_{C_2} B,$$

where C_1 and C_2 are overgroups of C of index 2. More precisely, $C_1 = \langle C, a\rangle$ and $C_2 = \langle C, b\rangle$ where a and b denote two elements of order 2 modulo C such that $z = ba$.

The sequences $(r_{n|L_1})$ and $(r_{n|L_2})$ are both discriminating since the sequence (ρ_n) is discriminatig and since the homomorphisms $\tau_{f(n)}$ are injective. We claim that the sequence $(r_{n|S})$ is discriminating as well. Let $F \subset S \setminus \{1\}$ be a finite set. For simplicity, we suppose that $F = \{s\}$, the proof being identical in the case where F has more than one element. This element s can be written as $s = z^k s_0 a^\varepsilon \mod C$ with $k \in \mathbb{Z}$, $s_0 \in S_0$ and $\varepsilon \in \{0, 1\}$, where a is an element of order 2 modulo C such that $S_1 = \langle S_0, a\rangle$. We distinguish two cases. If $\varepsilon = 1$, then $szs^{-1} = z^{-1} \mod C$, and this relation is preserved by r_n, so $r_n(s)$ is non-trivial. If $\varepsilon = 0$, either s_0 has finite order, in which case there is nothing to be done, or s_0 has infinite order, in which case $\rho_n(s_0)$ has infinite order for n large enough. Then $\rho_n(s_0) = z^{\ell_n} \mod C$ with $\ell_n \neq 0$, so $r_n(s_0) = \tau_{f(n)} \circ \rho_n(s_0) = z^{\ell_n(1+4K!f(n))} \mod C$. For $f(n)$ large enough, the element $r_n(s)$ has infinite order. Hence, the sequence $(r_{n|S})$ is discriminating. It remains to prove that (r_n) is discriminating.

Let $x \in L$ be a non-trivial element. We will prove that $r_n(x)$ is non-trivial for n large enough, by appropriately choosing the sequence of integers $(f(n))$. If x belongs to a conjugate of one of the vertex groups of the graph of groups $L = L_1 *_{S_1} *S *_{S_2} L_2$, then $r_n(x) \neq 1$ by the previous paragraph. From now on, we assume that x is not elliptic in this decomposition. Let us write x as a non-trivial reduced word $x_1 s_1 x_2 s_2 x_3 \cdots$ in the graph of groups $L = L_1 *_{S_1} *S *_{S_2} L_2$, with $x_i \in L_1$ or L_2 and $s_i \in S$. By definition of a reduced word, the following three conditions hold.

- x_i does not belong to $S_1 = \langle S_0, a\rangle$ or $S_2 = \langle S_0, b\rangle$.
- If x_i and x_{i+1} are both in L_1, then s_i does not belong to S_1.
- If x_i and x_{i+1} are both in L_2, then s_i does not belong to S_2.

We will prove that $r_n(x)$ can be written as a non-trivial reduced word in the graph of groups $\Gamma = A *_{C_1} N *_{C_2} B$, which implies that $r_n(x) \neq 0$. Note first that $r_n(s_i)$ belongs to N since $S = N_L(C)$. Each element $r_n(x_i)$ can be decomposed as a reduced word w_i in the graph of groups $\Gamma = A *_{C_1} N *_{C_2} B$. Let us consider the concatenation of these reduced words $w = w_1 r_n(s_1) w_2 r_n(s_2) w_3 \cdots$. We will prove that w is (almost) a reduced word in the decomposition $\Gamma = A *_{C_1} N *_{C_2} B$. The subwords of x (viewed as a non-trivial reduced word in $L_1 *_{S_1} *S *_{S_2} L_2$) are of one of the following three forms.

- Case I: $x_i s_i x_{i+1}$ with $x_i, x_{i+1} \in L_1$ (or L_2).
- Case II: $x_i s_i x_{i+1}$ with $x_i \in L_1$ and $x_{i+1} \in L_2$ (or $x_i \in L_2$ and $x_{i+1} \in L_1$).
- Case III: $s_i x_i s_{i+1}$ with $x_i \in L_1$ (or L_2).

In each case, we will see that the corresponding subword $w_i r_n(s_i) w_{i+1}$ or $r_n(s_i) w_i r_n(s_{i+1})$ of w is (almost) reduced.

Case I. Let $x_i s_i x_j$ be a subword of x with $s_i \in S$ and $x_i, x_j \in L_1$, where $j = i + 1$ (the case $x_i, x_j \in L_2$ is identical). Since x is reduced, s_i does not belong to $S_1 = \langle S_0, a\rangle$, so $s_i = z^k s_0 a^\varepsilon \mod C$ with $k \neq 0$, $s_0 \in S_0$ and $\varepsilon \in \{0,1\}$. We have

$$r_n(s_i) = z^{k+\ell_n(1+4K!f(n))} a^\varepsilon \mod C,$$

where ℓ_n denotes the integer such that $\rho_n(s_0) = z^{\ell_n}$ (modulo C).

Then, let us decompose $r_n(x_i)$ and $r_n(x_j)$ as reduced words in $\Gamma = A *_{C_1} N *_{C_2} B$, where $C_1 = \langle C, a\rangle$ et $C_2 = \langle c, b\rangle$. First, we decompose $\rho_n(x_i)$ as a reduced word whose first and last letters belong to N. This word ends with $y_{i,n}n_{i,n}$ where $n_{i,n} \in N$ and $y_{i,n} \in A \setminus C_1$ or $y_{i,n} \in B \setminus C_2$. The element $n_{i,n}$ can be written as $n_{i,n} = z^{k_{i,n}} a^{\varepsilon_{i,n}} \mod C$ with $k_{i,n} \in \mathbb{Z}$ and $\varepsilon_{i,n} \in \{0,1\}$. We decompose $\rho_n(x_j)$ in the same way. The corresponding reduced word begin with $n_{j,n}y_{j,n}$, where $n_{j,n} \in N$ and $y_{j,n} \in A \setminus C_1$ ou $B \setminus C_2$. Again, $n_{j,n}$ can be written as $n_{j,n} = z^{k_{j,n}} a^{\varepsilon_{j,n}} \mod C$ where $k_{j,n} \in \mathbb{Z}$ and $\varepsilon_{j,n} \in \{0,1\}$. So we have

$$\tau_{f(n)}(n_{i,n}) = z^{k_{i,n}(1+4K!f(n))} a^{\varepsilon_{i,n}} \mod C \quad \text{and} \quad \tau_{f(n)}(n_{j,n}) = z^{k_{j,n}(1+4K!f(n))} a^{\varepsilon_{j,n}} \mod C.$$

Subcase i. Suppose that $y_{i,n}$ and $y_{j,n}$ are both in A. At the junction of the concatenation of the reduced forms of $r_n(x_i)$, $r_n(s_i)$ and $r_n(x_j)$, we see the subword

$$y_{i,n}\omega_n y_{j,n}, \text{ with } \omega_n = \tau_{f(n)}(n_{i,n}) r_n(s_i) \tau_{f(n)}(n_{j,n}).$$

We have to prove that one can choose $f(n)$ so that ω_n does not belong to $C_1 = \langle C, a\rangle$. Any easy calculation shows that

$$\omega = z^{R_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}},$$

where R_n is of the form

$$R_n = \pm k + (1 + 4K!f(n))D$$

for some integer D. Since the integer k is not zero, one can always choose $f(n)$ in such a way that R_n is non-zero. This concludes the first subcase.

Subcase ii. Suppose that $y_{i,n}$ and $y_{j,n}$ are both in B. Then

$$\tau_{f(n)}(y_{i,n}) = \delta^{f(n)} y_{i,n} \delta^{-f(n)} \quad \text{and} \quad \tau_{f(n)}(y_{j,n}) = \delta^{f(n)} y_{j,n} \delta^{-f(n)}.$$

By concatenating $r_n(x_i)$, $r_n(s_i)$ and $r_n(x_j)$, we see the subword

$$y_{i,n}\delta^{-f(n)} z^{R_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}} \delta^{f(n)} y_{j,n} = y_{i,n} z^{R'_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}} y_{j,n},$$

where

$$R'_n = R_n - 2K!f(n)(1 + (-1)^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}+1}).$$

Let us prove that the subword $z^{R'_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}}$ does not belong to $C_2 = \langle C, b\rangle$. First, observe that this subword belongs to C_2 if and only if one of the following two conditions holds:

- $R'_n = 0$ and $\varepsilon_{i,n} + \varepsilon + \varepsilon_{j,n} = 0 \mod 2$, in which case $z^{R'_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}} = 1 \mod C$, or
- $R'_n = 1$ and $\varepsilon_{i,n} + \varepsilon + \varepsilon_{j,n} = 1 \mod 2$, in which case $z^{R'_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}} = b \mod C$.

In the first case $R'_n = R_n$ and we saw in the first subcase that $f(n)$ can be chosen large enough so that $R_n \neq 0$ (because $k \neq 0$). In the second case,

$$\begin{aligned} R'_n &= R_n - 4K!f(n) \\ &= \pm k + (1 + 4K!f(n))(k_{i,n} \pm \ell_n \pm k_{j,n}) - 4K!f(n) \end{aligned}$$

so

$$R'_n = 1 \Leftrightarrow \pm k + (1 + 4K!f(n))(k_{i,n} \pm \ell_n \pm k_{j,n} - 1) = 0.$$

But k is non-zero, so $f(n)$ can be chosen sufficiently large so that the previous equality does not hold.

Subcase iii. If $y_{i,n} \in A$ and $y_{j,n} \in B$, or $y_{i,n} \in B$ and $y_{j,n} \in A$, there is nothing to be done.

Case II. Let us consider the subword $x_i s_i x_j$ of x (viewed as word) with $x_i \in L_1$, $s_i \in S$ and $x_j \in L_2$ (the case where $x_i \in L_2$ and $x_j \in L_1$ can be tackled in exactly the same way), where $j = i+1$. The element s_i can be decomposed as $s_i = z^k s_0 a^{\varepsilon} \mod C$ with $k \in \mathbb{Z}$ (note that k may be zero here), $s_0 \in S_0$ and $\varepsilon \in \{0,1\}$. So we have

$$r_n(s_i) = z^{k+\ell_n(1+4K!f(n))} a^{\varepsilon} \mod C,$$

where ℓ_n is the integer such that $\rho_n(s_0) = z^{\ell_n}$. Then, as above, we decompose $r_n(x_i)$ and $r_n(x_j)$ as reduced words in the decomposition $\Gamma = A *_{C_1} N *_{C_2} B$. With the same notations, we have

$$\tau_{f(n)}(n_{i,n}) = z^{k_{i,n}(1+4K!f(n))} a^{\varepsilon_{i,n}} \mod C \quad \text{et} \quad \tau_{f(n)}(n_{j,n}) = z^{k_{j,n}(1+4K!f(n))} a^{\varepsilon_{j,n}} \mod C.$$

As in Case I, there are three subcases because $y_{i,n}$ and $y_{j,n}$ may be in A or in B. We will suppose that $y_{i,n} \in A$ and $y_{j,n} \in A$. The reader can check that the three other cases can be solved in the same manner.

By concatenating the words corresponding to $r_n(x_i)$, $r_n(s_i)$ and $r_n(x_j)$, we see the subword

$$y_{i,n}\tau_{f(n)}(n_{i,n}) r_n(s_i) z^{-\mu_n} \tau_{f(n)}(n_{j,n}) y_{j,n} = y_{i,n} z^{R_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}} y_{j,n},$$

where

$$\begin{aligned} R_n &= k_{i,n}(1+4K!f(n)) + (-1)^{\varepsilon_{i,n}}(k+\ell_n(1+4K!f(n))) + (-1)^{\varepsilon_{i,n}+\varepsilon}(k_{j,n}(1+4K!f(n)) - \mu_n) \\ &= \pm k + (1+4K!f(n))(k_{i,n} \pm \ell_n \pm k_{j,n}) \pm \mu_n \\ &= \pm k + 2K!f(n)(2\alpha_n \pm 4K!\lambda_n \pm 1) + (\alpha_n \pm 2K!\lambda_n). \end{aligned}$$

where $\alpha_n = k_{i,n} \pm \ell_n \pm k_{j,n}$ and $\mu_n = (1+4K!f(n))2K!\lambda_n + 2K!f(n)$. The integer μ_n comes from the fact that $r_n = \mathrm{ad}(z^{-\mu_n}) \circ \tau_{f(n)} \circ \rho_n$ on L_2). We claim that $f(n)$ can be chosen in such a way that $z^{R_n} a^{\varepsilon_{i,n}+\varepsilon+\varepsilon_{j,n}} \notin C_1 = \langle C, a\rangle$. Indeed, for $f(n)$ large enough, $R_n \neq 0$ since $2\alpha_n \pm 4K!\lambda_n \pm 1$ is odd so non-zero.

Case III. Consider a subword $s_i x_i s_j$ with $s_i, s_j \in S$ and $x_i \in L_1$ (or $x_i \in L_2$), where $j = i+1$.

Since the word representing x is reduced, x_i does not belong to S, so $\rho_n(x_i)$ does not belong to N. Hence, the decomposition of $\rho_n(x)$ as a reduced word in $\Gamma = A *_{C_1} N *_{C_2} B$ is of the form $n_1 y_1 \cdots y_r n_r$ with $r \geq 1$, $y_k \in A \setminus C$ or $B \setminus C$, and $n_1, n_r \in N$ (maybe zero), so there is nothing to be done. $\square$

We will use Lemma 6.17 to prove Theorem 6.16.

6.2.3. *Proof of Theorem 6.16.* Let G be a hyperbolic group. Suppose that G splits over a finite subgroup C whose normalizer N is virtually cyclic. Let $\Gamma = G *_N N'$ be a legal small extension of G. We shall prove that $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(\Gamma)$. By Lemma 6.20 below, it suffices to prove that $\mathrm{Th}_{\forall\exists}(G) \subset \mathrm{Th}_{\forall\exists}(\Gamma)$.

LEMMA 6.20. *Let G be a hyperbolic group. Suppose that G splits over a finite subgroup C whose normalizer N is virtually cyclic. Let $\Gamma = G *_N N'$ be a legal small extension of G. Then G is a legal small extension of Γ (viewed as abstract groups).*

PROOF. There exists an injective twist $\varphi : \Gamma \to G \subset \Gamma$ (one can take any homomorphism of the small test sequence $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ whose existence was proved above). Let $N'' = N_{\varphi(\Gamma)}(C)$. The group G can be decomposed as $G = \varphi(\Gamma) *_{N''} N$. The inclusion of N'' into N is legal, an there exists a legal embedding of N into N'' (for example $\iota' \circ \iota$, with the same notations as above). $\square$

Before proving Theorem 6.16, we prove that Lemma 6.17 remains true, more generally, if one replaces the system of equations and inequations by a boolean combination of equations and inequations.

Corollary 6.21. *Let Γ be a legal small extension of G, and let $(\varphi_n : \Gamma \to G)_{n\in\mathbb{N}}$ be a small test sequence. Let*

$$\bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x},\boldsymbol{y}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x},\boldsymbol{y}) \neq 1)$$

be a disjunction of systems of equations and inequations, where $\boldsymbol{x}$ is a p-tuple of variables and $\boldsymbol{y}$ is a q-tuple of variables. Let $\boldsymbol{\gamma} \in \Gamma^p$. Suppose that G satisfies the following condition: for every integer n, there exists $\boldsymbol{g}_n \in G^q$ such that

$$\bigvee_{k=1}^{N} (\Sigma_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n) = 1 \ \wedge \ \psi_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n) \neq 1).$$

Then there exists $\boldsymbol{\gamma}' \in \Gamma^q$ such that

$$\bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{\gamma},\boldsymbol{\gamma}') = 1 \ \wedge \ \Psi_k(\boldsymbol{\gamma},\boldsymbol{\gamma}') \neq 1).$$

Proof. Up to extracting a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ (which is still a test sequence), one can assume that there exists an integer $1 \leq k \leq N$ such that, for every integer n, there exits $\boldsymbol{g}_n \in G^q$ such that $\Sigma_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n) = 1$ and $\Psi_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n) \neq 1$. Proposition 6.17 applies and asserts the existence of a tuple $\boldsymbol{\gamma}' \in \Gamma^q$ satisfying $\Sigma_k(\boldsymbol{\gamma},\boldsymbol{\gamma}') = 1$ and $\Psi_k(\boldsymbol{\gamma},\boldsymbol{\gamma}') \neq 1$, which concludes the proof. □

We now prove Theorem 6.16.

Proof. According to Lemma 6.20, it suffices to prove that $\mathrm{Th}_{\forall\exists}(G) \subset \mathrm{Th}_{\forall\exists}(\Gamma)$. Let θ be a $\forall\exists$-sentence such that $G \models \theta$. Let us prove that $\Gamma \models \theta$. The sentence θ has the following form:

$$\theta : \forall \boldsymbol{x} \exists \boldsymbol{y} \bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{x},\boldsymbol{y}) = 1 \ \wedge \ \Psi_k(\boldsymbol{x},\boldsymbol{y}) \neq 1),$$

where $\boldsymbol{x}$ is a p-tuple of variables and $\boldsymbol{y}$ is a q-tuple of variables. Let $\boldsymbol{\gamma}$ a p-tuple of elements of Γ. For every integer n, there exists $\boldsymbol{g}_n \in G^q$ such that

$$G \models \bigvee_{k=1}^{N} (\Sigma_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n) = 1 \ \wedge \ \Psi_k(\varphi_n(\boldsymbol{\gamma}),\boldsymbol{g}_n) \neq 1).$$

By Lemma 6.21, there exists $\boldsymbol{\gamma}' \in \Gamma^q$ such that

$$\Gamma \models \bigvee_{k=1}^{N} (\Sigma_k(\boldsymbol{\gamma},\boldsymbol{\gamma}') = 1 \ \wedge \ \Psi_k(\boldsymbol{\gamma},\boldsymbol{\gamma}') \neq 1).$$

Hence, $\Gamma \models \theta$. □

7. Proof of (5) ⇒ (1)

We are now ready to prove the implication (5) ⇒ (1) of Theorem 1.23. In fact, we prove a stronger result, since we only assume that G and G' are hyperbolic.

Theorem 7.1 (Implication (5) ⇒ (1) of Theorem 1.23). *Let G, G' be two hyperbolic groups. Suppose that there exist two multiple legal extensions Γ and Γ' of G and G' respectively, such that $\Gamma \simeq \Gamma'$. Then $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G')$.*

Proof. By definition of a multiple legal extension, there exists a finite sequence of groups $G = G_0 \subset G_1 \subset \cdots \subset G_n \simeq \Gamma$ where G_{i+1} is a legal large or small extension of G_i in the sense of Definitions 1.6 or 1.13, for every integer $0 \leq i \leq n-1$. According to Theorems 1.9 and 1.14, we have $\mathrm{Th}_{\forall\exists}(G_i) = \mathrm{Th}_{\forall\exists}(G_{i+1})$, for every $0 \leq i \leq n-1$. Thus, G

and Γ have the same $\forall\exists$-theory. Similarly, G' and Γ' have the same $\forall\exists$-theory. But Γ and Γ' are isomorphic, so $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(\Gamma')$. Hence, G and G' have the same $\forall\exists$-theory. $\square$

8. Proof of (4) $\Rightarrow$ (5)

This section is dedicated to a proof of the implication (4) $\Rightarrow$ (5) of Theorem 1.23, that is the following result.

PROPOSITION 8.1 (Implication (4) $\Rightarrow$ (5) of Theorem 1.23). *Let G and G' be two virtually free groups. Suppose that there exist two strongly special homomorphisms $\varphi : G \to G'$ and $\varphi' : G' \to G$. Then, there exist two multiple legal extensions Γ and Γ' of G and G' respectively, such that $\Gamma \simeq \Gamma'$.*

While in the previous sections we stated and proved results in the general context of hyperbolic groups (with torsion), the present section is specific to virtually free groups. However, we believe that the construction of multiple legal extensions should play a role in a classification of hyperbolic groups up to elementary equivalence as well.

There are, in brief, three increasing levels of complexity in the proof of Proposition 8.1 above.

(1) We assume that all edge groups in reduced Stallings splittings of G and G' are equal. In other words, we suppose that there is only one cylinder in these splittings (see paragraph 2.7 for the definition of a cylinder). We refer the reader to Corollary 8.37.
(2) We assume that all edge groups in reduced Stallings splittings T and T' of G and G' have the same cardinality. The importance of the previous point appears when one considers the trees of cylinders T_c and T'_c of T and T' (see paragraph 2.7 for the definition of the tree of cylinders). See Proposition 8.43.
(3) In the general case, different cardinalities of edge groups may coexist in reduced Stallings splittings of G and G'. The proof of Proposition 8.1 is by induction on the number of edges in reduced Stallings splittings of G and G'. By carefully collapsing certain edges, we can assume that there is only one cardinality of edge groups in the splittings we consider, and we can use the same techniques as in the second point above.

As usual, we denote by K_G the maximal order of a finite subgroup of G. Note that, under the hypotheses of Proposition 8.1, the integers K_G and $K_{G'}$ are equal, because strongly special morphisms are injective on finite subgroups. We define $K := K_G = K_{G'} \geq 1$. Note that this integer is preserved by legal extensions. In the sequel, we assume that the maximal order of a finite subgroup in all virtually free groups we consider is at most equal to K.

Before proving Proposition 8.1, we need to introduce new definitions and to prove some lemmas.

8.1. Preliminaries.

8.1.1. *Strongly special pair of homomorphisms.*

DEFINITION 8.2. Let G and G' be two groups. Given two morphisms $\varphi, \varphi' \in \mathrm{Hom}(G, G')$, we use the notation $\varphi \sim \varphi'$ if, for every finite subgroup C of G, there exists an element $g' \in G'$ such that $\varphi' = \mathrm{ad}(g') \circ \varphi$ on C.

DEFINITION 8.3. Let G and G' be two groups. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be two homomorphisms. The pair (φ, φ') is said to be strongly special if the following two conditions hold.

(1) φ and φ are strongly special.
(2) $\varphi' \circ \varphi \sim \mathrm{id}_G$ and $\varphi \circ \varphi' \sim \mathrm{id}_{G'}$.

Note that if $\varphi : G \to G'$ and $\varphi : G' \to G$ are both strongly special, then $\varphi' \circ \varphi$ is strongly special. As a consequence, taking $\psi = \varphi$ and $\psi' = \varphi' \circ (\varphi \circ \varphi')^k$ for a suitable k, one gets the following result.

LEMMA 8.4. *Let G and G' be two groups. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be two strongly special homomorphisms. Suppose that G and G' have finitely many conjugacy classes of finite subgroups. Then there exists a strongly special pair (ψ, ψ').*

According to the previous lemma, in order to prove Proposition 8.1, it suffices to prove the following result.

PROPOSITION 8.5. *Let G and G' be two virtually free groups. Suppose that there exists a strongly special pair of homomorphisms $(\varphi : G \to G', \varphi' : G' \to G)$. Then there exist two multiple legal extensions Γ and Γ' of G and G' respectively, such that $\Gamma \simeq \Gamma'$.*

8.1.2. *Smallest order of an edge group in a reduced Stallings splitting.*

DEFINITION 8.6. Given an infinite virtually free group G, we denote by $m(G)$ the smallest order of an edge group in a reduced Stallings splitting of G. Note that this integer does not depend on a particular reduced Stallings splitting of G, because conjugacy classes of edge groups are the same in all reduced Stallings splittings of G, since one can pass from a reduced Stallings splitting to another by a sequence of slide moves. If G and G' are two virtually free groups, we define $m_{G,G'} = \min(m(G), m(G'))$.

8.1.3. *m-splittings.*

DEFINITION 8.7. Let $m \geq 1$ be an integer. Let G be a virtually free group. A *m-splitting* of G as a graph of groups is a non-trivial splitting of G over subgroups of order exactly m.

LEMMA 8.8. *Let $m \geq 1$ be an integer. Let G be a virtually free group, and let T be a reduced m-splitting of G. Suppose that T has a vertex group of order exactly m. Then, G is finite-by-free.*

PROOF. Suppose that there exists a vertex v of T such that $|G_v| = m$. Since T/G is a reduced splitting of G over edge groups of order m, the existence of a vertex group of order exactly m implies that the underlying graph of the graph of groups T/G has only one vertex, i.e. is a bouquet of circles. Hence, all edge groups and vertex groups of T are equal to G_v. As a consequence, G_v is the unique maximal finite normal subroup of G, and the quotient group G/G_v is the fundamental group of a bouquet of circles, i.e. a free group. Hence, the group G is G_v-by-free, with G_v finite. □

8.1.4. *Strongly ($> m$)-special homomorphisms.* We need to slightly weaken the definitions of a strongly special homomorphism and of a strongly special pair of homomorphisms. The following definitions are suitable for proofs by induction.

DEFINITION 8.9. Let G and G' be virtually free groups. Let $m \geq 0$ be an integer. A homomorphism $\varphi : G \to G'$ is said to be a *strongly ($> m$)-special homomorphism* if it satisfies the following four properties:

(1) φ is injective on finite subgroups;
(2) if C_1 and C_2 are two non-conjugate finite subgroups of G of order $> m$, then $\varphi(C_1)$ and $\varphi(C_2)$ are non-conjugate in G';
(3) if C is a finite subgroup of G of order $> m$ whose normalizer $N_G(C)$ is non-elementary, then the normalizer $N_{G'}(\varphi(C))$ is non-elementary and
$$\varphi(E_G(N_G(C))) = E_{G'}(N_{G'}(\varphi(C))).$$
In particular, if the finite group $E_G(N_G(C))$ is equal to C, then
$$E_{G'}(N_{G'}(\varphi(C))) = \varphi(C);$$

(4) if C is a finite subgroup of G of order $> m$ whose normalizer is virtually cyclic infinite maximal, then $N_{G'}(\varphi(C))$ is virtually cyclic infinite maximal, and the restriction $\varphi_{|N_G(C)} : N_G(C) \to N_{G'}(\varphi(C))$ is K-nice, with K the maximal order of a finite subgroup of G (see Definition 1.12).

We define *strongly* ($\geq m$)*-special homomorphisms* in the same way.

Remark 8.10. A homomorphism is strongly special (see Definition 1.22) if and only if it is strongly (> 0)-special.

Definition 8.11. Let G and G' be virtually free groups. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be two homomorphisms. The pair (φ, φ') is said to be a *strongly* ($> m$)*-special pair* if the following three conditions hold:

(1) φ and φ' are strongly ($> m$)-special (see Definition 8.9);
(2) $\varphi' \circ \varphi \sim \mathrm{id}_G$, which means that for every finite subgroup $C \subset G$, there exists an element $g \in G$ such that $\varphi' \circ \varphi$ and $\mathrm{ad}(g)$ coincide on C;
(3) $\varphi \circ \varphi' \sim \mathrm{id}_{G'}$.

We define *strongly* ($\geq m$)*-special pairs* in the same way.

8.1.5. *Preservation of specialness.* The following lemma shows that the property of being strongly ($\geq m$)-special is preserved by composition.

Lemma 8.12 (Composition of strongly ($\geq m$)-special homomorphisms). *Let G, G' and G'' be virtually free groups. Let $m \geq 1$ be an integer. Let $\varphi : G \to G'$ and $\varphi' : G' \to G''$ be homomorphisms. Suppose that φ and φ' are strongly ($\geq m$)-special. Then $\varphi' \circ \varphi$ is strongly ($\geq m$)-special.*

Remark 8.13. We also prove that if φ and φ' satisfy the first three conditions of Definition 8.9, then $\varphi' \circ \varphi$ satisfies the first three conditions of Definition 8.9.

Proof. There are four conditions that need to be verified.

Condition 1. Since φ and φ' are injective on finite subgroups of G and G', $\varphi' \circ \varphi$ is injective on finite subgroups of G as well.

Condition 2. If C_1 and C_2 are two non-conjugate finite subgroups of G of order $\geq m$, then $\varphi(C_1)$ and $\varphi(C_2)$ are two non-conjugate subgroups of G' of order $\geq m$, hence $\varphi' \circ \varphi(C_1)$ and $\varphi' \circ \varphi(C_2)$ are non-conjugate in G''.

Condition 3. If C is a finite subgroup of G of order $\geq m$ whose normalizer in G is non-elementary, then $N_{G'}(\varphi(C))$ is non-elementary and $\varphi(E_G(N_G(C)))$ is equal to $E_{G'}(N_{G'}(\varphi(C)))$, since φ is strongly ($\geq m$)-special. Note that $|\varphi(C)| = |C| \geq m$. Hence, since φ' is strongly ($\geq m$)-special, the group $N_{G''}(\varphi'(\varphi(C)))$ is non-elementary and the following equality holds: $\varphi'(E_{G'}(N_{G'}(\varphi(C)))) = E_{G''}(N_{G''}(\varphi'(\varphi(C))))$.

Condition 4. If C is a finite subgroup of G of order $\geq m$ whose normalizer in G is virtually cyclic infinite maximal, then $N_{G'}(\varphi(C))$ is virtually cyclic infinite maximal, and the restriction of φ to $N_G(C)$ is K-nice (see Definition 1.12), because φ is strongly ($\geq m$)-special. Since φ' is strongly ($\geq m$)-special, the group $N_{G''}(\varphi'(\varphi(C)))$ is virtually cyclic maximal, and the restriction of φ' to $N_{G'}(\varphi(C))$ is K-nice. Hence, the restriction of $\varphi' \circ \varphi$ to $N_G(C)$ is K-nice, as the composition of K-nice homomorphisms. $\square$

We now prove that the first three conditions of Definition 8.9 are preserved by the equivalence relation $\sim$ (see Definition 8.2).

Lemma 8.14. *Let G and G' be two virtually free groups, and let $\varphi, \psi : G \to G'$ be two homomorphisms. Suppose that $\psi \sim \varphi$. If φ satisfies conditions 1 to 3 of Definition 8.9, then ψ satisfies conditions 1 to 3 of Definition 8.9.*

PROOF. There are three conditions that need to be verified.

Condition 1. By definition of $\sim$, the homomorphisms φ and ψ coincide up to conjugacy on finite subgroups. Since φ is injective on finite subgroups of G, the homomorphism ψ is injective on finite subgroups as well.

Condition 2. If C_1 and C_2 are non-conjugate finite subgroups of G of order $\geq m$, then $\varphi(C_1)$ and $\varphi(C_2)$ are non-conjugate in G', because φ is $(\geq m)$-special. In addition $\psi(C_1)$ is conjugate to $\varphi(C_1)$ and $\psi(C_2)$ is conjugate to $\varphi(C_2)$. Hence, $\psi(C_1)$ and $\psi(C_2)$ are non-conjugate in G.

Condition 3. If C is a finite subgroup of G of order $\geq m$ whose normalizer in G is non-elementary, then $N_{G'}(\varphi(C))$ is non-elementary and $\varphi(E_G(N_G(C)))$ is equal to $E_{G'}(N_{G'}(\varphi(C)))$, since φ is strongly $(\geq m)$-special. The group $\psi(C)$ being conjugate to $\varphi(C)$, its normalizer is non-elementary and the following equality holds: $\psi(E_G(N_G(C))) = E_{G'}(N_{G'}(\psi(C)))$. □

8.1.6. *Preliminary lemmas about legal extensions.*

LEMMA 8.15. *Let G be a non-elementary virtually free group. Let C be a finite subgroup of G. Suppose that the group $\Gamma = \langle G, t \mid \mathrm{ad}(t)_{|C} = \mathrm{id}_C\rangle$ is a legal large extension of G. Let i denote the inclusion of G into Γ, and let $r : \Gamma \twoheadrightarrow G$ denote the retraction defined by $r_{|G} = \mathrm{id}_G$ and $r(t) = 1$. Then r and i are strongly special, and $i \circ r \sim \mathrm{id}_\Gamma$.*

PROOF. Let T be the Bass-Serre tree of the splitting $\Gamma = G*_{\mathrm{id}_C}$, and let v denote a vertex of T fixed by G. Note that this vertex is unique, because the infinite vertex group G is not equal to the adjacent edge group C, which is finite.

First, let us prove that the retraction r is strongly special. There are four conditions that need to be verified.

Condition 1. Since every finite subgroup of Γ is conjugate to a subgroup of G, the retraction r is injective on finite subgroups of Γ.

Condition 2. Let A and B be two non-conjugate finite subgroups of Γ. One can suppose without loss of generality that they are contained in G. Therefore, $r(A) = A$ and $r(B) = B$ are non-conjugate in G.

Condition 3. Now, let A be a finite subgroup of Γ whose normalizer $N := N_\Gamma(A)$ is non-elementary. One can suppose without loss of generality that A is contained in G. We distinguish two cases.

First case. If A is not contained in a conjugate of C in G, then v is the unique fixed point of A in T. It follows that $N_\Gamma(A)$ fixes the vertex v. Hence, $N_\Gamma(A) = N_\Gamma(A) \cap G = N_G(A)$. This shows in particular that $N_G(r(A))$ is non-elementary. Moreover, note that $E_\Gamma(N_\Gamma(A)) \supset A$ only fixes v. This implies that $E_\Gamma(N_\Gamma(A)) = E_G(N_\Gamma(A)) = E_G(N_G(A))$. Hence, we have $r(E_\Gamma(N_\Gamma(A))) = E_G(N_G(r(A)))$.

Second case. Otherwise, one can suppose without loss of generality that A is contained in C. Then $N_G(C)$ has a subgroup of finite index that centralizes A. Since $N_G(C)$ is non-elementary by definition of a large extension, the normalizer $N_G(A) = N_G(r(A))$ is non-elementary as well. Now, let us prove that $r(E_\Gamma(N_\Gamma(A))) = E_G(N_G(r(A)))$. First, we prove that $E_\Gamma(N_G(A))$ is contained in C. Let us observe that there exists an integer $n \geq 1$ such that, for every $g \in N_G(C)$, the element g^n normalizes (and even centralizes) A. Then, recall that $E_\Gamma(N_G(A))$ is equal to the intersection of all $M(g) \subset \Gamma$ where g runs through the set $N_G(A)^0$ consisting of all elements of $N_G(A)$ of infinite order. Hence, we have:

$$E_\Gamma(N_G(A)) = \bigcap_{g \in N_G(A)^0} M(g) \subset \bigcap_{g \in N_G(C)^0} M(g^n) \overset{(*)}{=} \bigcap_{g \in N_G(C)^0} M(g) = E_\Gamma(N_G(C)) = C.$$

Note that the equality $(*)$ follows from the fact that $M(g^n) = M(g)$ for any non-trivial integer n. We have proved that $E_\Gamma(N_G(A))$ is contained in C, hence in G. This shows that the following equality holds:

$$E_\Gamma(N_G(A)) = E_G(N_G(A)). \tag{5}$$

In addition, recall that the stable letter t centralizes C. In particular, t centralizes $E_\Gamma(N_G(A))$. Moreover $N_G(A)$ normalizes $E_\Gamma(N_G(A))$, by definition of $E_\Gamma(N_G(A))$. Thus, the group $N_\Gamma(A) = \langle N_G(A), t\rangle$ normalizes $E_\Gamma(N_G(A))$, which implies that $E_\Gamma(N_G(A))$ is contained in $E_\Gamma(N_\Gamma(A))$, by definition of $E_\Gamma(N_\Gamma(A))$. The reverse inclusion is obvious since $N_G(A)$ is contained in $N_\Gamma(A)$. Hence, we have

$$E_\Gamma(N_\Gamma(A)) = E_\Gamma(N_G(A)). \tag{6}$$

By combining the two equalities (5) and (6), we get the equality $E_\Gamma(N_\Gamma(A)) = E_G(N_G(A))$, i.e. $r(E_\Gamma(N_\Gamma(A)) = E_G(N_G(r(A)))$, which concludes.

Condition 4. Let A be a finite subgroup of Γ whose normalizer is virtually $\mathbb{Z}$ maximal. One can suppose without loss of generality that A is contained in G. Note that A is not contained in a conjugate of C in G, otherwise $N_G(A)$ would be non-elementary (see above). Thus, the vertex v is the unique fixed point of A in T. Therefore, $N_\Gamma(A)$ and $N_G(A)$ are equal, which implies that $N_G(A)$ is virtually cyclic maximal in Γ. In addition, r coincides with the identity on $N_\Gamma(A)$; in particular, it is K-nice.

We have proved that the retraction r is strongly special. Now, let us prove that the inclusion $i : G \subset \Gamma$ is strongly special.

Condition 1. The inclusion is injective on finite subgroups.

Condition 2. Let A and B be two finite subgroups of G. If $B = \gamma A\gamma^{-1}$ for a certain element $\gamma \in \Gamma$, then $B = r(\gamma)Ar(\gamma)^{-1}$.

Condition 3. Let A be a finite subgroup of G whose normalizer $N_G(A)$ is non-elementary. Then $N_\Gamma(A)$ is non-elementary. We have to prove that $E_\Gamma(N_\Gamma(A))$ and $E_G(N_G(A))$ are equal. We distinguish two cases. If A is not contained in a conjugate of C in G, then $N_\Gamma(A) = N_G(A)$ and $E_\Gamma(N_\Gamma(A)) = E_G(N_G(A))$ (same proof as above). Otherwise, one can suppose without loss of generality that A is contained in C. Then one can prove that $E_\Gamma(N_G(A))$ is contained in C and deduce that $E_\Gamma(N_\Gamma(A)) = E_G(N_G(A))$ (same proof as above).

Condition 4. Let A be a finite subgroup of G whose normalizer is virtually $\mathbb{Z}$ maximal. Note that A is not contained in a conjugate of C in G, otherwise $N_G(A)$ would be non-elementary. Thus, the vertex v is the unique fixed point of A in T. Therefore, $N_\Gamma(A)$ fixes v, so $N_\Gamma(A)$ and $N_G(A)$ are equal. Let M be the maximal virtually cyclic subgroup of Γ containing $N_\Gamma(A)$. Since $N_\Gamma(A)$ has finite index in M and fixes the vertex v, the group M fixes the vertex v as well. Thus M is contained in G. Since $N_G(A)$ is maximal and contained in M, it is equal to M. This proves that $N_\Gamma(A)$ is virtually cyclic infinite maximal. In addition, the restriction of the inclusion i to $N_G(A)$ is K-nice.

We have proved that i is strongly special. It remains to prove that $i \circ r \sim \mathrm{id}_\Gamma$. If A is a finite subgroup of Γ, then there exists an element $\gamma \in \Gamma$ such that $\gamma A\gamma^{-1}$ is contained in G. Consequently, $i \circ r$ coincides with $\mathrm{ad}(r(\gamma)^{-1}\gamma)$ on A. □

We need an analogous result for legal small extensions. First, we prove an easy lemma.

LEMMA 8.16. *Let G be a virtually free group. Suppose that $\Gamma = G *_N N'$ is a legal small extension of G. By definition, there exists a nice emebedding $j : N' \hookrightarrow N$. This homomorphism extends to a monomorphism $j : \Gamma \hookrightarrow G$.*

PROOF. By definition, G splits as $A *_C B$ or $A*_C$ over a finite subgroup C whose normalizer is N. Moreover, N is assumed to be non-elliptic in the splitting. The corresponding tree of cylinders gives a splitting Λ of G as a graph of groups whose vertices are A, N and B (only in the first case), and whose edge groups are equal to C or contain C as a finite subgroup of index 2. Since j coincides on each finite subgroup of $N \subset N'$ with an inner automorphism, it extends to a homomorphism $j : \Gamma \to G$ that coincides with inner automorphisms on A and B (see Lemma 2.27). Let Λ' be the splitting of Γ obtained from Λ be replacing N by N'. One can see that j maps a non-trivial reduced word in the splitting Λ' of Γ to a non-trivial reduced word in the splitting Λ of G. This shows that j is injective. □

LEMMA 8.17. *Let G be a non-elementary virtually free group. Let Γ be a legal small extension of G. Let i denote the inclusion of G into Γ and let $j : \Gamma \hookrightarrow G$ denote a monomorphism as in Lemma 8.16. Then i and j are strongly special. Moreover, $i \circ j \sim \mathrm{id}_\Gamma$.*

PROOF. Let T be the Bass-Serre tree of the splitting $\Gamma = G*_{\mathrm{id}_C}$, and let v denote a vertex of T fixed by G. Note that this vertex is unique, because the vertex group G is not equal to the adjacent edge group C.

By symmetry, it is enough to prove that j is strongly special. There are four conditions that need to be verified.

Condition 1. By definition, j is injective on finite subgroups of Γ.

Condition 2. Since $i \circ j$ maps every finite subgroup of Γ to a conjugate of itself, j maps non-conjugate finite subgroups of Γ to non-conjugate finite subgroups of Γ.

Condition 3. Let A be a finite subgroup of Γ. One can suppose without loss of generality that $i \circ j(A) = A$. Let $N := N_\Gamma(A)$. The inclusions $j(N) \subset N_G(j(A))$ and $i(N_G(j(A))) \subset N$ show that N is non-elementary (respectively virtually $\mathbb{Z}$) if and only if $N_G(j(A))$ is non-elementary (respectively virtually $\mathbb{Z}$).

Suppose that N is non-elementary and let $E := E_\Gamma(N)$. Since E is finite, one can suppose without loss of generality that $i \circ j(E) = E$. We claim that $j(E)$ is equal $E_G(N_G(j(A)))$. Recall that a Γ-chain is a tuple $(\gamma_1, \dots, \gamma_c)$ of elements of Γ of infinite order such that the inclusions

$$M(\gamma_1) \supset (M(\gamma_1) \cap M(\gamma_2)) \supset \cdots \supset (M(\gamma_1) \cap \cdots \cap M(\gamma_c)) \tag{7}$$

are all strict (see Definition 4.1), and that the complexity $c := c(N)$ of N is the maximal size of a Γ-chain of elements of N (see Definition 4.2). Let $(\gamma_1, \dots, \gamma_c)$ be a Γ-chain of elements of N. By injectivity of j, each $j(\gamma_k)$ has infinite order, for $1 \le k \le c$. Moreover, the sequence of proper inclusions 7 is mapped by j to a sequence of proper inclusions, which proves that the tuple $(j(\gamma_1), \dots, j(\gamma_c))$ is a G-chain of elements of $N_G(j(A))$. In particular, we have $c(N) \le c(N_G(j(A)))$. Symmetrically, $c(N_G(j(A))) \le c(N)$. Therefore, the complexities $c(N)$ and $c(N_G(j(A)))$ are the same. This implies that $E_G(N_G(j(A)))$ is equal to the intersection $\cap_{1 \le k \le c} M(j(\gamma_k))$ (see Lemma 4.4). Hence, the following holds:

$$j(E) = j\Big(\bigcap_{1 \le k \le c} M(\gamma_k)\Big) \subset \bigcap_{1 \le k \le c} j(M(\gamma_k)) \subset \bigcap_{1 \le k \le c} M(j(\gamma_k)) = E_G(N_G(j(A))).$$

Symmetrically, $i(E_G(N_G(j(A)))$ is contained in E. Hence, $j(E) = E_G(N_G(j(A)))$.

Condition 4. Last, if $N = N_\Gamma(A)$ is virtually $\mathbb{Z}$ maximal, then $N_G(j(A))$ is virtually $\mathbb{Z}$ maximal. In addition, $j_{|N}$ is K-nice. □

The following lemma shows that the relation $\sim$ is preserved by left of right composition with any homomorphism.

LEMMA 8.18. *Let G, G' and G'' be virtually free groups. Let $\varphi : G \to G'$ and $\psi : G \to G'$ be two homomorphisms. Suppose that $\varphi \sim \psi$. Then, the following hold:*

- *if $\rho : G' \to G''$ is a homomorphism, then $\rho \circ \varphi \sim \rho \circ \psi$;*
- *if $\rho : G'' \to G$ is a homomorphism, then $\varphi \circ \rho \sim \psi \circ \rho$.*

PROOF. Let $\rho : G' \to G''$ be a homomorphism, and let C be a finite subgroup of G. By hypothesis, there exists an element $g' \in G'$ such that $\varphi_{|C} = \mathrm{ad}(g') \circ \psi_{|C}$. By left composing this equality by ρ, one gets $\rho \circ \varphi_{|C} = \mathrm{ad}(\rho(g')) \circ \rho \circ \psi_{|C}$.

Let $\rho : G'' \to G$ be a homomorphism, and let C be a finite subgroup of G''. By hypothesis, since $\rho(C)$ is finite, there exists an element $g' \in G'$ such that $\varphi_{|\rho(C)} = \mathrm{ad}(g') \circ \psi_{|\rho(C)}$, that is $\varphi \circ \rho_{|C} = \mathrm{ad}(g') \circ \psi \circ \rho_{|C}$. □

We now prove a lemma that will be crucial in the proof of Proposition 8.44.

LEMMA 8.19. *Let G and G' be two non-elementary virtually free groups, and let Γ be a legal extension of G. Let $m \geq 1$ be an integer, let $\varphi : G \to G'$ be a homomorphism satisfying conditions 1 to 3 of Definition 8.9, and let $\psi : \Gamma \to G'$ be a homomorphism. If $\psi_{|G} \sim \varphi$, then ψ satisfies conditions 1 to 3 of Definition 8.9.*

PROOF. First, let us suppose that Γ is a legal large extension of G. Let i denote the inclusion of G into Γ, and let $r : \Gamma \twoheadrightarrow G$ be the retraction as in Lemma 8.15. First, note that $\psi_{|G} = \psi \circ i$. Therefore $\psi \circ i \sim \varphi$. By the second point of Lemma 8.18, we have $\psi \circ i \circ r \sim \varphi \circ r$. Moreover, $i \circ r$ is equivalent (in the sense of $\sim$) to the identity of Γ according to Lemma 8.15, so $\psi \circ i \circ r$ is equivalent to ψ, by the first point of Lemma 8.18. As a consequence, ψ is equivalent to $\varphi \circ r$. Recall that r is strongly special thanks to Lemma 8.15. In particular, r is strongly $(\geq m)$-special, so r satisfies conditions 1 to 3 of Definition 8.9. In addition, φ satisfies conditions 1 to 3 of Definition 8.9 by assumption. It follows from Remark 8.13 below Lemma 8.12 that $\varphi \circ r$ satisfies conditions 1 to 3 of Definition 8.9. Hence, by Lemma 8.14, the morphism ψ satisfies conditions 1 to 3 of Definition 8.9.

Now, suppose that Γ is a legal small extension of G. Let i denote the inclusion of G into Γ, and let $j : \Gamma \hookrightarrow G$ denote a monomorphism as in Lemma 8.16. First, note that $\psi_{|G} = \psi \circ i$. Therefore $\psi \circ i \sim \varphi$. By the second point of Lemma 8.18, we have $\psi \circ i \circ j \sim \varphi \circ j$. Moreover, $i \circ j \sim \mathrm{id}_\Gamma$, so $\psi \circ i \circ j \sim \psi$. As a consequence, $\psi \sim \varphi \circ j$. Since φ is satisfies conditions 1 to 3 of Definition 8.9, and since j is strongly special (in particular strongly $(\geq m)$-special) thanks to Lemma 8.17, it follows from Remark 8.13 below Lemma 8.12 that $\varphi \circ j$ satisfies conditions 1 to 3 of Definition 8.9. Hence, by Lemma 8.14, the homomorphism ψ satisfies the first three conditions of Definition 8.9. □

The following lemma is an immediate consequence of Lemma 8.19 above.

LEMMA 8.20 (Extension of a strongly $(\geq m)$-special homomorphism). *Let G and G' be two non-elementary virtually free groups, and let Γ be a legal extension of G. Let $m \geq 1$ be an integer, let $\varphi : G \to G'$ be a strongly $(\geq m)$-special homomorphism and $\psi : \Gamma \to G'$ a homomorphism. If $\psi_{|G} \sim \varphi$ and if ψ satisfies the fourth condition of Definition 8.9, then ψ is strongly $(\geq m)$-special.*

The following lemma shows that strongly $(\geq m)$-special homomorphisms behave nicely with respect to legal extensions of the target group.

LEMMA 8.21. *Let G and G' be two non-elementary virtually free groups, and let Γ' be a legal (large or small) extension of G'. Let i denote the inclusion of G into Γ. Let $m \geq 1$ be an integer and let $\varphi : G \to G'$ be a strongly $(\geq m)$-special homomorphism. Then $i \circ \varphi : G \to \Gamma'$ is strongly $(\geq m)$-special.*

PROOF. By Lemmas 8.15 and 8.17, the inclusion i is strongly special, in particular strongly $(\geq m)$-special. Then, by Lemma 8.12, $i \circ \varphi$ is strongly $(\geq m)$-special. □

LEMMA 8.22 (Restriction of a strongly $(\geq m)$-special pair). *Let G and G' be two virtually free groups. Suppose that they are not finite-by-free. Let m denote the integer $m_{G,G'}$. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be two homomorphisms such that (φ, φ') is a strongly $(\geq m)$-special pair. Let H be a m-factor of G. Suppose that $\varphi(H)$ is contained in a m-factor H' of G', and that $\varphi'(H') \subset H$. Then the pair $(\varphi_{|H}, \varphi'_{|H'})$ is strongly $(> m)$-special.*

PROOF. Let T be a reduced m-JSJ tree of G. First, we will prove the following preliminary observations:

(1) if C is a finite subgroup of H of order $> m$, then $N_G(C) = N_H(C)$ and
$$E_G(N_G(C)) = E_H(N_H(C));$$
(2) if C_1 and $C_2 = gC_1g^{-1}$ are two subgroups of H of order $> m$ with $g \in G$, then g belongs to H.

By definition of a m-factor, there exists a vertex $v \in T$ such that $H = G_v$. The vertex v is the unique vertex of T fixed by C, because $|C| > m$ and edge groups of T have order m. This implies that $N_G(C)$ fixes v, i.e. that $N_G(C)$ is contained in H. Hence, $N_G(C) = N_H(C)$. For the same reason, $E_G(N_G(C)) \supset C$ is contained in H, which proves that $E_G(N_G(C)) = E_H(N_H(C))$. The proof of the second point is similar.

Now, let us prove that $\varphi_{|H} : H \to H'$ is strongly $(> m)$-special. There are four points that need to be satisfied.

Condition 1. The restriction $\varphi_{|H}$ is injective on finite subgroups of H, because φ is injective on finite subgroups of G.

Condition 2. Let C_1 and C_2 be two non-conjugate finite subgroups of H of order $> m$. By the second preliminary observation, these groups are non-conjugate in G. Since φ is $(\geq m)$-special, $\varphi(C_1)$ and $\varphi(C_2)$ are non-conjugate in G'. Thus, $\varphi(C_1)$ and $\varphi(C_2)$ are non-conjugate in H'.

Condition 3. If C is a finite subgroup of H of order $> m$ whose normalizer $N_H(C)$ is non-elementary, then $N_G(C)$ is non-elementary. This implies that $N_{G'}(\varphi(C))$ is non-elementary, because φ is strongly $(\geq m)$-special. Moreover, we have $N_{G'}(\varphi(C)) = N_{H'}(\varphi(C))$ according to the first preliminary observation. Hence, $N_{H'}(\varphi(C))$ is non-elementary. In addition, we have
$$\varphi(E_G(N_G(C))) = E_{G'}(N_{G'}(\varphi(C))),$$
since φ is strongly $(\geq m)$-special. It follows that
$$\varphi(E_H(N_H(C))) = E_{H'}(N_{H'}(\varphi(C))),$$
because, by the first preliminary observation,
$$E_G(N_G(C)) = E_H(N_H(C)) \quad \text{and} \quad E_{G'}(N_{G'}(\varphi(C))) = E_{H'}(N_{H'}(\varphi(C))).$$

Condition 4. Let C be a finite subgroup of H of order $> m$ such that $N_H(C)$ is virtually cyclic infinite maximal. The morphism φ being strongly $(\geq m)$-special, the normalizer of $\varphi(C)$ in G' is virtually cyclic infinite maximal and the restriction of φ to $N_G(C)$ is K-nice in the sense of Definition 1.12. Since $N_H(C) = N_G(C)$ and $N_{H'}(\varphi(C)) = N_{G'}(\varphi(C))$, the restriction of φ to $N_H(C)$ is K-nice.

We have proved that $\varphi_{|H}$ is strongly $(> m)$-special. Since the same arguments remain valid with φ' instead of φ, the restriction $\varphi'_{|H'}$ is strongly $(> m)$-special as well.

It remains to prove that the pair $(\varphi_{|H}, \varphi'_{|H'})$ is strongly $(> m)$-special. To that end, let us consider a finite subgroup C of H of order $> m$. Since $\varphi' \circ \varphi \sim \mathrm{id}_G$, there exists an element $g \in G$ such that $\varphi' \circ \varphi(C) = gCg^{-1}$. Since $\varphi' \circ \varphi(H)$ is contained in H by assumption, the groups C and gCg^{-1} belong to H. By the preliminary observation, g belongs to H. Hence, $\varphi' \circ \varphi$ maps every finite subgroup of H of order $> m$ to a conjugate of itself in H. Symmetrically, $\varphi \circ \varphi'$ maps every finite subgroup of H' of order $> m$ to a conjugate of itself in H'. □

8.1.7. *Legal $(\geq m)$-extensions.*

Definition 8.23. Let $m \geq 1$ be an integer. Let Γ be a virtually free group, and let G be a subgroup of Γ. We say that Γ is a *multiple legal $(\geq m)$-extension* of G if there exist nested subgroups $G = G_1 \subset G_2 \subset \cdots \subset G_n = \Gamma$ and integers $(k_i)_{1 \leq i \leq n-1}$ such that $k_i \geq m$ and G_{i+1} is a legal large or small k_i-extension of G_i (see Definitions 1.6 and 1.13) for every $1 \leq i \leq n-1$. If $n = 2$, we simply say that Γ is a *legal $(\geq m)$-extension* of G. In the same way, we define *multiple legal* $(> m)$*-extensions* and *multiple legal* m*-extensions* of G if $k_i > m$ or $k_i = m$ respectively.

LEMMA 8.24. *Let $m \geq 1$ be an integer. Let G be a virtually free group, and let Δ be a m-splitting of G as a graph of groups. Let $k \geq m$ be an integer, let v be a vertex of Δ and let $\widehat{G}_v$ be a legal large k-extension of G_v. Then the group $\widehat{G}$ obtained from G by replacing G_v by $\widehat{G}_v$ in Δ is a legal large k-extension of G.*

PROOF. By definition of a legal large k-extension, there is a subgroup C of G_v of order k such that $\widehat{G}_v = \langle G_v, t \mid [t,c] = 1, \forall c \in C\rangle$, with $N_{G_v}(C)$ non-elementary and $E_{G_v}(N_{G_v}(C)) = C$. Thus, the normalizer of C in G is non-elementary, and we only have to prove that $E := E_G(N_G(C)) = C$. Note that the inclusion $C \subset E$ always holds, because E is the unique maximal finite subgroup of G normalized by $N_G(C)$. It remains to prove that $E \subset C$.

Let T be the Bass-Serre tree of Δ, endowed with the action of G. We shall still denote by v a lift in T of the vertex v of Δ. As a first step, we shall prove that $E_G(N_{G_v}(C))$ fixes the vertex $v \in T$. Assume towards a contradiction that $E_G(N_{G_v}(C))$ does not fix v. Then the inclusion $C \subset E_G(N_{G_v}(C))$ is strict. This shows in particular that $E_G(N_{G_v}(C))$ has order $> |C| \geq m$. Since $E_G(N_{G_v}(C))$ is finite, it fixes a vertex $w \neq v$ of T, and this vertex is unique because $E_G(N_{G_v}(C))$ has order $> m$. It follows that $N_{G_v}(C)$ fixes w as well. Hence, $N_{G_v}(C)$ is contained in the finite group $G_w \cap G_v$, contradicting the fact that N_{G_v} is non-elementary. We have proved that $E_G(N_{G_v}(C))$ fixes v. As a consequence, the following equality holds:

$$E_{G_v}(N_{G_v}(C)) = E_G(N_{G_v}(C)). \tag{8}$$

Now, let us assume towards a contradiction that C is strictly contained in E. Since E is finite, it fixes a vertex w of T. Moreover, this vertex is unique since $|E| > |C| = k \geq m$. It follows that $N_G(E)$ is contained in G_w. But $N_G(C)$ is contained in $N_G(E)$ by definition of E, hence $N_G(C)$ fixes w. Since $N_{G_v}(C)$ is infinite, it fixes only v in T, which proves that $w = v$. As a consequence $N_G(C) = N_{G_v}(C)$, and therefore

$$E_G(N_G(C)) = E_G(N_{G_v}(C)). \tag{9}$$

By combining equations (8) and (9), we get $E_G(N_G(C)) = E_{G_v}(N_{G_v}(C))$, i.e. $E = C$, contradicting the assumption that C is strictly contained in E. As a conclusion, we have proved that $E = C$. This proves that $\widehat{G}$ is a legal large k-extension of G. □

We need an analogous result for small extensions.

LEMMA 8.25. *Let $m \geq 1$ be an integer. Let G be a virtually free group, and let Δ be a m-splitting of G as a graph of groups. Let v be a vertex of Δ. Let Δ_v be a one edge splitting of G_v over a finite group C of order $k \geq m$ whose normalizer $N_{G_v}(C)$ is virtually cyclic and non-elliptic in Δ_v. Let $\widehat{G}_v$ be a legal small k-extension of G_v. If $N_{G_v}(C) = N_G(C)$, then the group $\widehat{G}$ obtained from G by replacing G_v by $\widehat{G}_v$ in Δ is a legal small k-extension of G.*

PROOF. We only need to verify that C is an edge group in a splitting of G in which $N_G(C)$ is non-elliptic. First, note that for any edge e of Δ incident to v, the edge group G_e is contained in a conjugate of A or B, as a finite group. Let Δ' be the splitting of G obtained from Δ by replacing the vertex v by the splitting Δ_v of G_v, and let ε be the new edge of Δ' coming from Δ_v. By collapsing all edges of Δ' different from ε, we get a one edge splitting of G over C in which $N_G(C)$ is non-elliptic. □

REMARK 8.26. Let T be the Bass-Serre tree of Δ. We still denote by v a lift in T of the vertex v of Δ. Let us observe that the equality $N_{G_v}(C) = N_G(C)$ holds if C is not an edge group of Δ, i.e. if v is the unique vertex of T fixed by C, because in this case

$N_G(C)$ fixes v as well, which implies that $N_{G_v}(C) = N_G(C) \cap G_v = N_G(C)$. For instance, if $k = |C| > m$, then v is the unique vertex of T fixed by C.

The following lemma allows us to iterate Lemmas 8.24 and 8.25 above.

LEMMA 8.27. *Let $m \geq 1$ be an integer. Let G be a virtually free group, and let Δ be a m-splitting of G as a graph of groups. Let v and w be two distinct vertices of Δ. Let $\widehat{G}_v$ and $\widehat{G}_w$ be two legal $(\geq m)$-extensions of G_v and G_w. If the extension $\widehat{G}_v$ (respectively G_w) is small, suppose that $N_{G_v}(C) = N_G(C)$ (respectively $N_{G_w}(C) = N_G(C)$), where C is the edge group of the one edge splitting of G_v (respectively G_w) associated with the small extension. Then the group $\widehat{G}$ obtained from G by replacing G_v by $\widehat{G}_v$ and G_w by $\widehat{G}_w$ in Δ is a multiple legal $(\geq m)$-extension of G.*

PROOF. Let Γ be the group obtained from G by replacing G_v by $\widehat{G}_v$. By Lemmas 8.24 and 8.25, Γ is a legal $(\geq m)$-extension of G. We claim that $\widehat{G}$ is a legal $(\geq m)$-extension of Γ. Note that $\Gamma_w = G_w$. We distinguish two cases. If $\widehat{G}_w$ is a large extension of G_w, there is no condition to be checked and Lemma 8.24 claims that $\widehat{G}$ is a legal $(\geq m)$-extension of Γ. If $\widehat{G}_w$ is a small extension of G_w, the group $N_\Gamma(C)$ is equal to $N_G(C)$, which is equal to $N_{G_w}(C)$ by assumption. Since $\Gamma_w = G_w$, we have $N_\Gamma(C) = N_{\Gamma_w}(C)$. Hence Lemma 8.25 applies and guarantees that $\widehat{G}$ is a legal $(\geq m)$-extension of Γ. □

By iterating Lemma 8.27, we get the following result.

COROLLARY 8.28. *Let $m \geq 1$ be an integer. Let G be a virtually free group and let Δ be a m-splitting of G as a graph of groups. For every vertex v of Δ, let $\widehat{G}_v$ be a multiple legal $(\geq m)$-extension of G_v. If the extension $\widehat{G}_v$ is small, suppose that the edge group of the one edge splitting of G_v associated with the small extension has order $> m$. Then the group $\widehat{G}$ obtained from G by replacing every G_v by $\widehat{G}_v$ in Δ is a multiple legal $(\geq m)$-extension of G.*

8.1.8. *Property $\mathcal{P}_m$.*

DEFINITION 8.29. Let G and G' be two virtually free groups. Let m denote the integer $m_{G,G'}$. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be two homomorphisms. We say that the tuple $(G, G', \varphi, \varphi')$ has property $\mathcal{P}_m$ if the following two conditions hold:

(1) the pair (φ, φ') is strongly $(\geq m)$-special;
(2) $\varphi' \circ \varphi$ maps each m-factor of G isomorphically to a conjugate of itself, and $\varphi \circ \varphi'$ maps each m-factor of G' isomorphically to a conjugate of itself.

REMARK 8.30. Let $G_1, \ldots, G_p$ and $G'_1, \ldots, G'_{p'}$ be two sets of representatives of the m-factors of G and G' respectively. If $(G, G', \varphi, \varphi')$ has $\mathcal{P}_m$, then $p = p'$ and, up to renumbering $G_1, \ldots, G_p$, the homomorphism φ maps each G_i isomorphically to a conjugate of G'_i, and φ maps each G'_i isomorphically to a conjugate of G_i. Indeed, let T be a reduced m-JSJ splitting of G, and let H be a vertex group of T, that is a m-factor of G. Since $\varphi' \circ \varphi$ is injective on H, the morphism φ is injective on H as well. Hence $\varphi(H) \simeq H$ is $(\leq m)$-rigid. As a consequence, $\varphi(H)$ is contained in a vertex group H' of T'. We claim that φ induces an isomorphism from H to H'. First, note that $\varphi'(H')$ is contained in a vertex group K of T, for the same reason as above. Therefore, $\varphi' \circ \varphi(H)$ is contained in K. Moreover, we know that $\varphi' \circ \varphi(H) = H^g$ for some $g \in G$. It follows that H^g is contained in K. Since T is reduced and since H^g and K are two vertex groups of T, we have $H^g = K$. Likewise, $\varphi'(H^g) = H'^{g'}$ for some $g' \in G'$. Hence, the following series of inclusions holds:

$$H \overset{\varphi}{\hookrightarrow} H' \overset{\varphi'}{\hookrightarrow} H^g \overset{\varphi}{\hookrightarrow} H'^{g'}.$$

Recall that the homomorphism $\varphi \circ \varphi'$ is surjective from H' onto $H'^{g'}$. Thus, φ induces an isomorphism from H^g to $H'^{g'}$, and it follows that φ induces an isomorphism from H to H'.

8.1.9. *Expansions.*

DEFINITION 8.31. Let G and G' be two virtually free groups. Let m denote the integer $m_{G,G'}$. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be two homomorphisms. Let Γ and Γ be two virtually free groups containing G and G' respectively, and let $\psi : \Gamma \to \Gamma'$ and $\psi' : \Gamma' \to \Gamma$ be two homomorphisms. We say that $(\Gamma, \Gamma', \psi, \psi')$ is a *$(\geq m)$-expansion* of $(G, G', \varphi, \varphi')$ if the following conditions are satisfied:

(1) Γ and Γ' are two multiple legal $(\geq m)$-extensions of G and G';
(2) $\psi_{|G} \sim \varphi$ and $\psi'_{|G'} \sim \varphi'$;

In the same way, we define *$(> m)$-expansions*.

REMARK 8.32. Let $\mathcal{U} = (G, G', \varphi, \varphi')$ be a tuple as in Definition 8.31 above. If $\mathcal{U}_1$ is a $(\geq m)$-expansion of $\mathcal{U}$, and if $\mathcal{U}_2$ is a $(\geq m)$-expansion of $\mathcal{U}_1$, then $\mathcal{U}_2$ is a $(\geq m)$-expansion of $\mathcal{U}$.

8.2. Finite extensions of free products. In this section, we are concerned with the case where there is only one cylinder in the trees we consider, which means that all edge groups are equal. This particular case will play a crucial role in the proof of Proposition 8.44, which uses extensively the trees of cylinders.

Given a group G and a subgroup $C \subset G$, we denote by $\mathrm{Aut}_G(C)$ the following subgroup of $\mathrm{Aut}(C)$:

$$\mathrm{Aut}_G(C) = \{\sigma \in \mathrm{Aut}(C) \mid \exists g \in N_G(C),\ \mathrm{ad}(g)_{|C} = \sigma\}.$$

LEMMA 8.33. *Let G and G' be two non-elementary hyperbolic groups. Let T and T' be two simplicial trees endowed with actions of G and G' respectively. Suppose that all edge groups of T and T' are equal, and let C and C' denote these edge groups. Suppose in addition that T and T' have the same number of orbits of vertex groups, say p. Let $G_1, \ldots, G_p$ and $G'_1, \ldots, G'_p$ be some representatives of the vertex groups of T and T'. Suppose that the following two conditions hold:*

- *there exists a strongly $(\geq |C|)$-special homomorphism $\varphi : G \to G'$ that maps each subgroup G_i isomorphically to a conjugate of G'_i,*
- *and there exists a strongly $(\geq |C'|)$-special homomorphism $\varphi' : G' \to G$ that maps each subgroup G'_i isomorphically to a conjugate of G_i.*

Then C and C' have the same order m, and there exists a $(\geq m)$-expansion $(\widehat{G}, \widehat{G}', \widehat{\varphi}, \widehat{\varphi}')$ of $(G, G', \varphi, \varphi')$ such that $\widehat{\varphi}$ and $\widehat{\varphi}'$ are bijective and satisfy the following two conditions:

- *for every $1 \leq i \leq p$, there exists an element $g'_i \in \widehat{G}'$ such that*
$$\varphi_{|G_i} = \mathrm{ad}(g'_i) \circ \widehat{\varphi}_{|G_i},$$
where φ is viewed as a homomorphism from G to $\widehat{G}'$;
- *for every $1 \leq i \leq p$, there exists an element $g_i \in \widehat{G}$ such that*
$$\varphi'_{|G'_i} = \mathrm{ad}(g_i) \circ \widehat{\varphi}'_{|G'_i},$$
where φ' is viewed as a homomorphism from G' to $\widehat{G}$.

REMARK 8.34. It is worth noticing that in the particular case where T and T' are m-JSJ splittings of G and G' respectively (which is equivalent to say that the vertex groups of T and T' are m-rigid), the conclusion of the lemma can be reformulated as follows: there exists a $(\geq m)$-expansion $(\widehat{G}, \widehat{G}', \widehat{\varphi}, \widehat{\varphi}')$ of $(G, G', \varphi, \varphi')$ with property $\mathcal{P}_m$. However, the lemma is stated in a more general context because, at some point in the proof (more precisely in the third case of the second step of the proof of Proposition 8.42), we will be considering some splittings that are not m-JSJ splittings.

REMARK 8.35. The groups $\widehat{G}$ and $\widehat{G}'$ constructed in the proof below are obtained by performing legal large extensions over C and C' only.

Proof. Note that C and C' are normal in G and G'. As a first step, we shall prove that $\varphi(C) = C'$ and $\varphi'(C') = C$. Since φ is strongly $(\geq |C|)$-special, and since $N_G(C) = G$ is non-elementary, we have

$$\varphi(E_G(N_G(C))) = E_{G'}(N_{G'}(\varphi(C))).$$

The left-hand side of this equation is equal to $\varphi(C)$. Indeed, $N_G(C) = G$ and $E_G(G) = C$ since C is the unique maximal finite normal subgroup of G. The right-hand side of the equation contains C'. Indeed, since C' is a normal subgroup of G', it is in particular normalized by $N_{G'}(\varphi(C)) \subset G'$; hence, C' is contained in the unique maximal finite subgroup of G' normalized by $N_{G'}(\varphi(C))$, namely $E_{G'}(N_{G'}(\varphi(C)))$. We have proved that C' is contained in $\varphi(C)$. Likewise, we have $C \subset \varphi'(C')$.

Since the homomorphisms φ and φ' are injective on finite subgroups, the inclusions $C' \subset \varphi(C)$ and $C \subset \varphi'(C')$ show that $\varphi(C) = C'$ and $\varphi'(C') = C$. More precisely, φ induces an isomorphism from C to C', and φ' induces an isomorphism from C' to C. In particular, the finite groups C and C' have the same order denoted by m.

We will now define the groups $\widehat{G}$ and $\widehat{G}'$. First, let us define a homomorphism $\overline{\varphi}$: $\mathrm{Aut}_G(C) \to \mathrm{Aut}_{G'}(C')$ as follows: for every $\theta = \mathrm{ad}(g)_{|C} \in \mathrm{Aut}_G(C)$, set

$$\overline{\varphi}(\theta) = \varphi_{|C} \circ \theta \circ (\varphi_{|C})^{-1} = \mathrm{ad}(\varphi(g))_{|C'} \text{ independent from the choice of } g.$$

Note that this homomorphism is injective. Indeed, if there exists an element $c \in C$ such that $gcg^{-1} \neq c$, then $\varphi(g)\varphi(c)\varphi(g)^{-1} \neq \varphi(c)$, because gcg^{-1} belongs to C and φ is injective on C. Likewise, φ' induces a monomorphism $\mathrm{Aut}_{G'}(C') \hookrightarrow \mathrm{Aut}_G(C)$. As a consequence, $\mathrm{Aut}_G(C)$ and $\mathrm{Aut}_{G'}(C')$ have the same order, say ℓ. Set $\mathrm{Aut}_G(C) = \{\theta_1, \ldots, \theta_\ell\}$ and $\mathrm{Aut}_{G'}(C') = \{\theta'_1, \ldots, \theta'_\ell\}$ with $\theta'_i = \overline{\varphi}(\theta_i)$ for every $1 \leq i \leq \ell$.

Note that the groups G and G' split respectively as

$$1 \to C \to G \to Q = Q_1 * \cdots * Q_p * F_k \to 1 \quad \text{and} \quad 1 \to C' \to G' \to Q' = Q'_1 * \cdots * Q'_p * F_{k'} \to 1$$

where Q_i is the image in Q of a conjugate of G_i, and Q'_i is the image in Q' of a conjugate of G'_i. Up to replacing G_i by a conjugate of itself, one can suppose that G_i is the preimage of Q_i in G, for every $1 \leq i \leq p$. Likewise, one can suppose that G'_i is the preimage of Q'_i for every $1 \leq i \leq p$. Let $\{x_1, \ldots, x_k\}$ be a generating set of the free group F_k, and let t_i be a preimage of x_i in G, for every $1 \leq i \leq k$. Each element $t_i \in G$ induces by conjugacy an automorphism $\theta_{s(i)} \in \mathrm{Aut}_G(C)$. Let H denote the subgroup $G_1 *_C \cdots *_C G_p \subset G$, preimage of $Q_1 * \cdots * Q_p$ in G. We define t'_i, $\theta'_{s'(i)}$ and H' in the same manner.

The group G admits the following finite presentation:

$$G = \langle H, t_1, \ldots, t_k \mid \mathrm{ad}(t_i)_{|C} = \theta_{s(i)} \ \forall i \in [\![1, k]\!] \rangle.$$

Similarly, the group G' has a finite presentation of the form

$$G' = \langle H', t'_1, \ldots, t'_{k'} \mid \mathrm{ad}(t'_i)_{|C'} = \theta'_{s'(i)} \ \forall i \in [\![1, k']\!] \rangle.$$

Let $n = \max(k, k') - k$ and $n' = \max(k, k') - k'$, so that $n + k = n' + k'$. Let us define the overgroups $\widehat{G}$ and $\widehat{G}$ of G and G' as follows:

$$\widehat{G} = \left\langle \ G, t_{k+1}, \ldots, t_{k+\ell}, \ldots, t_{k+\ell+n} \ \middle| \begin{array}{ll} \mathrm{ad}(t_i)_{|C} = \theta_i & \forall i \in [\![k+1, k+\ell]\!] \\ \mathrm{ad}(t_i)_{|C} = \mathrm{id}_C & \forall i \geq k+\ell+1 \end{array} \right\rangle,$$

$$\widehat{G}' = \left\langle \ G', t'_{k+1}, \ldots, t'_{k'+\ell}, \ldots, t'_{k'+\ell+n'} \ \middle| \begin{array}{ll} \mathrm{ad}(t'_i)_{|C'} = \theta'_i & \forall i \in [\![k'+1, k'+\ell]\!] \\ \mathrm{ad}(t'_i)_{|C'} = \mathrm{id}_{C'} & \forall i \geq k'+\ell+1 \end{array} \right\rangle.$$

Note that $\widehat{G}$ is a multiple legal large m-extension of G. We can see that by defining a finite sequence of groups $(\widehat{G}_q)_{0 \leq q \leq \ell+n}$ by $\widehat{G}_0 = G$ and $\widehat{G}_{q+1} = \langle \widehat{G}_q, t_{k+q+1} \rangle$ for $0 \leq q < \ell+n$, and by observing that $\widehat{G} = \widehat{G}_{\ell+n}$, and that $\widehat{G}_{q+1}$ is a legal $|C|$-extension of $\widehat{G}_q$ for every q, because $\mathrm{ad}(t_{k+q+1})_{|C}$ belongs to $\mathrm{Aut}_G(C)$. In the same way, the group $\widehat{G}'$ is a legal large $|C'|$-extension of G'.

We will now construct the isomorphisms $\widehat{\varphi} : \widehat{G} \to \widehat{G}'$ and $\widehat{\varphi}' : \widehat{G}' \to \widehat{G}$ satisfying the expected conditions. Let $N := \ell + k + n = \ell + k' + n'$. Up to renumbering the elements t_i, one can assume that $\mathrm{ad}(t_i)_{|C} = \theta_i$ for every $1 \leq i \leq \ell$. Then, for every $i \geq \ell+1$, there exists an integer $1 \leq j \leq \ell$ such that $\mathrm{ad}(t_i)_{|C} = \mathrm{ad}(t_j)_{|C}$, because $\mathrm{Aut}_G(C) = \{\theta_1, \ldots, \theta_\ell\}$ and $\mathrm{ad}(t_i)_{|C}$ belongs to $\mathrm{Aut}_G(C)$. Hence, up to replacing t_i by $t_j^{-1}t_i$, one can assume without loss of generality that $\theta_i = \mathrm{id}_C$. Now, $\widehat{G}$ has the following presentation:

$$\widehat{G} = \left\langle \begin{array}{c|cc} H, t_1, \ldots, t_N & \mathrm{ad}(t_i)_{|C} = \theta_i & \forall i \in [\![1, \ell]\!] \\ & \mathrm{ad}(t_i)_{|C} = \mathrm{id}_C & \forall i \geq \ell + 1 \end{array} \right\rangle.$$

Likewise, $\widehat{G}'$ has a presentation of the following form:

$$\widehat{G}' = \left\langle \begin{array}{c|cc} H', t'_1, \ldots, t'_N & \mathrm{ad}(t'_i)_{|C'} = \theta'_i & \forall i \in [\![1, \ell]\!] \\ & \mathrm{ad}(t'_i)_{|C'} = \mathrm{id}_{C'} & \forall i \geq \ell + 1 \end{array} \right\rangle.$$

We are now ready to define $\widehat{\varphi}$ and $\widehat{\varphi}'$. By assumption, $\varphi(G_i) = g'_i G'_i {g'_i}^{-1}$ for some $g'_i \in G'$. Since $\mathrm{Aut}_{G'}(C') = \{\mathrm{ad}(t'_j)_{|C'}, 1 \leq j \leq \ell\}$, there exists an integer $\sigma(i) \in [\![1, \ell]\!]$ such that $\mathrm{ad}(g'_i)_{|C'} = \mathrm{ad}(t'_{\sigma(i)})_{|C'}$. Recall that $H = G_1 *_C \cdots *_C G_p$. First, let us define a homomorphism $\psi : H \to \widehat{G}'$ by

$$\psi_{|G_i} = \mathrm{ad}\left(t'_{\sigma(i)}{g'_i}^{-1}\right) \circ \varphi_{|G_i}$$

for every $1 \leq i \leq p$. This homomorphism is well-defined since the element $t'_{\sigma(i)}{g'_i}^{-1}$ of G' centralizes C'.

Then, let us define $\widehat{\varphi} : \widehat{G} \to \widehat{G}'$ by $\widehat{\varphi}_{|H} = \psi$ and $\widehat{\varphi}(t_i) = t'_i$ for every $1 \leq i \leq N$. This homomorphism is well-defined because $\mathrm{ad}(t_i)_{|C} = \theta_i$ and $\mathrm{ad}(t'_i)_{|C'} = \varphi_C \circ \theta_i \circ (\varphi_C)^{-1} = \theta'_i$.

Last, note that

$$\widehat{\varphi}(G_i) = t'_{\sigma(i)} G'_i {t'_{\sigma(i)}}^{-1} = \widehat{\varphi}\left(t_{\sigma(i)}\right) G'_i \widehat{\varphi}\left(t_{\sigma(i)}\right)^{-1},$$

i.e.

$$G'_i = \widehat{\varphi}\left(t_{\sigma(i)}^{-1} G_i t_{\sigma(i)}\right).$$

Hence, the image of $\widehat{\varphi}$ contains G'_i, for every $1 \leq i \leq p$ and $t'_i = \widehat{\varphi}(t_i)$ for every $1 \leq i \leq N$. As a consequence, since $\widehat{G}'$ is generated by $G'_1, \ldots, G'_p, t'_1, \ldots, t'_N$, the homomorphism $\widehat{\varphi}$ is surjective. Likewise, there exists an epimorphism $\widehat{\varphi}' : \widehat{G}' \twoheadrightarrow \widehat{G}$ that coincides with φ' on each G'_i up to conjugacy. Since hyperbolic groups are Hopfian, the epimorphism $\widehat{\varphi}' \circ \widehat{\varphi} : \widehat{G} \twoheadrightarrow \widehat{G}$ is an automorphism. Hence, $\widehat{\varphi}$ and $\widehat{\varphi}'$ are two isomorphisms. □

REMARK 8.36. This remark will be useful for proving that there exists an algorithm that takes as input two finite presentations of virtually free groups, and decides whether these groups have the same $\forall\exists$-theory or not. We keep the same notations as in the proof above. Let r be the rank of G (that is the smallest cardinality of a generating set for G), and let r' be the rank of G'. Note that $k \leq r$ and $k' \leq r'$. We constructed the groups $\widehat{G}$ and $\widehat{G}'$ from G and G' by performing less than $\max(k, k') + \ell \leq \max(r, r') + |C|!$ legal large extensions.

We now consider reduced Stallings splittings of virtually free groups G and G'. If all edge groups are equal, then Lemma 8.33 applies. Here below are two consequences of Lemma 8.33 in this context. These results will be useful in the proof of the general case of the implication (4) ⇒ (5) of Theorem 1.23 (see Proposition 8.44).

COROLLARY 8.37. *Let $Q = Q_1 * \cdots * Q_p * F_k$ and $Q' = Q'_1 * \cdots * Q'_{p'} * F_{k'}$ be two free products of finite groups with a free group. Suppose that Q and Q' are non-elementary. Let G and G' be two finite extensions*

$$1 \to C \to G \to Q \to 1 \quad \textit{and} \quad 1 \to C' \to G' \to Q' \to 1.$$

Suppose that there exist two homomorphisms $\varphi : G \to G'$ and $\varphi' : G' \to G$ such that the pair (φ, φ') is strongly $(\geq m_{G,G'})$-special. Then C and C' have the same order $m_{G,G'}$, and there exists a $(\geq m_{G,G'})$-expansion $(\Gamma, \Gamma', \psi, \psi')$ of $(G, G', \varphi, \varphi')$ such that $\psi : \Gamma \to \Gamma'$ and $\psi' : \Gamma' \to \Gamma$ are bijective.

PROOF. For every $1 \leq i \leq p$, let G_i be a preimage of Q_i in G and for every $1 \leq i \leq p'$, let G'_i be a preimage of Q'_i in G'. In order to establish the existence of Γ and Γ', we shall use Lemma 8.33. To that end, it is enough to verify that the following three conditions are satisfied (up to renumbering the G_i):

- $p = p'$,
- φ maps each G_i isomorphically to a conjugate of G'_i,
- and φ' maps each G'_i isomorphically to a conjugate of G_i.

Since every finite subgroup of G' is contained in a conjugate of some G'_i, there exists a map $\sigma : [\![1, p]\!] \to [\![1, p']\!]$ such that $\varphi(G_i)$ is contained in a conjugate of $G'_{\sigma(i)}$, for every $1 \leq i \leq p$. Likewise, there exists a map $\sigma' : [\![1, p']\!] \to [\![1, p]\!]$ such that $\varphi'(G'_i)$ is contained in a conjugate of $G_{\sigma'(i)}$, for every $1 \leq i \leq p'$.

Since (φ, φ') is a strongly $(\geq m_{G,G'})$-special pair and $|G_i| \geq |C| \geq m_{G,G'}$ for every $1 \leq i \leq p$, the subgroup $\varphi' \circ \varphi(G_i)$ of G is conjugate to G_i. Hence, $\sigma' \circ \sigma$ is the identity of $[\![1, p]\!]$. Likewise, $\sigma \circ \sigma'$ is the identity of $[\![1, p']\!]$. It follows that $p = p'$ and that $\sigma' = \sigma^{-1}$, which concludes the proof. □

Note that the previous proposition holds in particular if G and G' are both finite-by-free. We need to prove that this result remains true if only one of these two groups is assumed to be finite-by-free.

COROLLARY 8.38. *Let G and G' be two virtually free groups. Suppose that G is finite-by-free (possibly finite or finite-by-$\mathbb{Z}$). Suppose that there exists a strongly $(\geq m_{G,G'})$-special pair of homomorphisms $(\varphi : G \to G', \varphi' : G' \to G)$. Then there exists a $(\geq m_{G,G'})$-expansion $(\Gamma, \Gamma', \psi, \psi')$ of $(G, G', \varphi, \varphi')$ such that $\psi : \Gamma \to \Gamma'$ and $\psi' : \Gamma' \to \Gamma$ are bijective.*

PROOF. Since G is finite-by-free, there is a unique finite normal subgroup $C \subset G$ such that $G/C \simeq F_n$, with $n \geq 0$. Let T' be a reduced Stallings tree of G'. First, note that all vertex groups of T' have order equal to $|C|$. Indeed, if v is a vertex of T', then $\varphi'(G'_v)$ is a subgroup of C. Hence $\varphi \circ \varphi(G'_v)$ is contained in $\varphi(C)$, which is of order $|C|$. The vertex group G'_v being finite maximal (since T' is reduced), and $\varphi \circ \varphi(G'_v)$ being conjugate to G'_v, we have $|G'_v| = |C|$.

In order to prove that G' is finite-by-free, it suffices to show that all edge groups of T' have order equal to $|C|$, as all vertex groups of T' have order equal to $|C|$. Assume towards a contradiction that there is an edge $e = [v, w]$ of T' such that $|G'_e| < |C|$. Since G'_v and G'_w have order $|C|$, we have $\varphi'(G'_v) = \varphi'(G'_w) = C$. It follows that G'_v and G'_w are conjugate in G' since φ', being $(\geq m_{G,G'})$-strongly special, maps non-conjugate finite subgroups of G' of order $\geq m_{G,G'}$ to non-conjugate finite subgroups of G. Let t be an element of G' such that $G'_w = tG'_v t^{-1}$, let $E_1 := G'_e$ and $E_2 := tE_1t^{-1}$.

Let S be the tree obtained from T' by collapsing all edges of T' that are not in the G'-orbit of e. The segment $[w, tv]$ fixed by $G'_w = tG'_v t^{-1}$ is collapsed to a point in S. Hence t has a translation length equal to 1 in S, i.e. t is a stable letter in the splitting of G as an HNN extension whose S is the Bass-Serre tree. Let x be the image of v (or w) in S and let $H = G'_x$ be its stabilizer. Note that $E_1 \subset G'_w \subset H$ and $E_2 \subset G'_w \subset H$. We have $G' = \langle H, t \mid txt^{-1} = \alpha(x), \forall x \in E_1 \rangle = H_{\alpha : E_1 \to E_2}$ where α denotes an isomorphism from

E_1 to E_2. Observe that x is the unique vertex of S fixed by G'_w, because $|G'_w| > |E_1|$. Therefore $N_{G'}(G'_w)$ fixes x, i.e. $N_{G'}(G'_w) \subset H$.

Now, let us observe that the homomorphism $\varphi \circ \varphi'$ coincides on the finite subgroup G'_w with an inner automorphism $\mathrm{ad}(g')$, for a certain $g' \in G'$. Up to replacing φ by $\mathrm{ad}(g'^{-1}) \circ \varphi$, one can assume without loss of generality that $\varphi \circ \varphi'$ coincides with the identity on G'_w. In particular, $\varphi \circ \varphi'$ coincides with the identity on E_1 and E_2. Let $z := \varphi \circ \varphi'(t)$. We have $zE_1z^{-1} = E_2$ and $tE_1t^{-1} = E_2$. Therefore $z^{-1}t$ normalizes E_1. In addition, z normalizes G'_w; indeed, $G'_w = tG'_v t^{-1}$ and $G'_w = \varphi \circ \varphi'(G'_w) = \varphi \circ \varphi'(G'_v)$, thus $G'_w = \varphi \circ \varphi'(t) G'_w \varphi' \circ \varphi'(t)^{-1}$. Hence, z belongs to H since $N_{G'}(G'_w) \subset H$. Thus, up to replacing t by $z^{-1}t$ in the previous splitting of G' as an HNN extension, we get a splitting of G' of the form $G' = \langle H, t \mid txt^{-1} = \alpha(x), \forall x \in E_1 \rangle$ where α denotes an automorphism of E_1. This shows that $N_{G'}(E_1)$ is infinite and that $E_{G'}(N_{G'}(E_1)) = E_1$. By applying the strongly special homomorphism φ' to this equality, we get $E_G(N_G(\varphi'(E_1))) = \varphi'(E_1)$. This is a contradiction, because $|\varphi'(E_1)| < |C|$ and C is normal in G.

As a conclusion, all edge groups of T' have order $|C|$. As mentionned above, this shows that the group G' is finite-by-free. Let $C' \subset G'$ be the unique finite normal subgroup such that G'/C' is free. Note that $|C| = |C'| = m_{G,G'}$ and $G' = N_{G'}(C')$. Since φ' is strongly $(\geq m_{G,G'})$-special, $\varphi'(G')$ is non-elementary as soon as G' is non-elementary, and $\varphi'(G')$ is infinite as soon as G' is infinite. By symmetry, $\varphi(G)$ is non-elementary as soon as G is non-elementary, and $\varphi(G)$ is infinite as soon as G' is infinite. Consequently, G and G' are simultaneously finite, virtually $\mathbb{Z}$ or non-elementary. We treat the three cases separately.

First case. Suppose that G and G' are finite. Since the homomorphisms φ and φ' are injective on finite groups, they are bijective, and one can take $\Gamma = G$ and $\Gamma' = G'$.

Second case. Suppose that G and G' are virtually $\mathbb{Z}$. Note that G' can be written as $G' = G *_G G'$, where the embedding of G into G is the identity, and the embedding of G into G' is the nice embedding $\varphi : G \hookrightarrow G'$. Moreover, $\varphi' : G' \hookrightarrow G$ is a nice embedding. Hence G' is a legal small extension of G. One can take $\Gamma = \Gamma' = G'$.

Third case. Suppose that G and G' are non-elementary. Then the existence of Γ and Γ' is an immediate consequence of Proposition 8.37 above. ☐

8.3. A property of the tree of cylinders. Let G be a virtually free group. Let $m \geq 1$ be an integer, and let T be a m-splitting of G. Recall that the tree of cylinders T_c (see [**GL11**] and Section 2.7) is the bipartite tree whose set of vertices $V(T_c)$ is the disjoint union of the following two sets:

- the set of vertices x of T which belong to at least two cylinders, denoted by $V_0(T_c)$;
- the set of cylinders of T, denoted by $V_1(T_c)$.

There is an edge $\varepsilon = (x, Y)$ between $x \in V_0(T_c)$ and $Y \in V_1(T_c)$ in T_c if and only if $x \in Y$. If $Y = \mathrm{Fix}(G_e)$ is the cylinder associated with an edge $e \in T$, then the stabilizer G_Y of Y is $N_G(G_e)$.

LEMMA 8.39. *Let G be a virtually free group. Let $m \geq 1$ be an integer, and let T be a m-splitting of G. Let T_c denote the tree of cylinders of T. Let φ be an endomorphism of G. Suppose that φ maps every vertex group of T_c isomorphically to a conjugate of itself, and every finite subgroup of G isomorphically to a conjugate of itself. Then φ is an automorphism.*

PROOF. If T_c is a point (i.e. if there is a unique cylinder in T), then G is a vertex group of T_c and φ is an automorphism. From now on, we suppose that T_c is not a point.

As a first step, we build a ϕ-equivariant map $f : T \to T$. Let $v_1, \ldots, v_n$ be some representatives of the orbits of vertices of T. For every $1 \leq i \leq n$, there exists an element $g_i \in G$ such that $\phi(G_{v_i}) = G_{v_i}^{g_i}$. We let $f(v_i) = g_i \cdot v_i$. Then we extend f linearly on edges of T.

We claim that the map f induces a ϕ-equivariant map $f_c : T_c \to T_c$. Indeed, for each cylinder $Y = \mathrm{Fix}(C) \subset T$, the image $f(Y)$ is contained in $\mathrm{Fix}(\varphi(C))$ of T, which is a cylinder (not a point) since $\varphi(C)$ is conjugate to C. If $v \in T$ belongs to two cylinders, so does $f(v)$. This allows us to define f_c on vertices of T_c, by sending $v \in V_0(T_c)$ to $f(v) \in V_0(T_c)$ and $Y \in V_1(T_c)$ to $f(Y) \in V_1(T_c)$. If (v, Y) is an edge of T_c, then $f_c(v)$ and $f_c(Y)$ are adjacent in T_c.

We shall prove that f_c does not fold any pair of edges and, therefore, that f_c is injective. Assume towards a contradiction that there exist a vertex v of T_c, and two distinct vertices w and w' adjacent to v such that $f_c(w) = f_c(w')$.

First, assume that v is not a cylinder. Since T_c is bipartite, w and w' are two cylinders, associated with two edges e and e' of T. Since $f_c(w) = f_c(w')$, we have $\phi(G_\varepsilon) = \phi(G_{\varepsilon'})$ by definition of f_c. But ϕ is injective on G_v by hypothesis, and $G_\varepsilon, G_{\varepsilon'}$ are two distinct subgroups of G_v (by definition of a cylinder). This is a contradiction.

Now, assume that $v = Y_\varepsilon$ is a cylinder. Since $f_c(w) = f_c(w')$, there exists an element $g \in G$ such that $w' = g \cdot w$. As a consequence $\phi(g)$ belongs to $\phi(G_w)$, so one can assume that $\phi(g) = 1$ up to multiplying g by an element of G_w. In particular, it follows that g does not belong to $N_G(G_\varepsilon) = G_v$, since ϕ is injective in restriction to $N_G(G_\varepsilon) = G_v$. Then observe that $G_\varepsilon \subset G_w$, $G_\varepsilon \subset G_{w'}$ and $gG_\varepsilon g^{-1} \subset gG_w g^{-1} = G_{w'}$. We have $G_\varepsilon \neq gG_\varepsilon g^{-1}$ since g does not lie in $N_G(G_\varepsilon)$, but $\phi(G_\varepsilon) = \phi(gG_\varepsilon g^{-1})$ since $\phi(g) = 1$. This contradicts the injectivity of ϕ on $G_{w'}$.

Hence, f_c is injective. It follows that ϕ is injective. Indeed, let g be an element of G such that $\phi(g) = 1$. Then $f_c(g \cdot v) = f_c(v)$ for each vertex v of T_c, so $g \cdot v = v$ for each vertex v of T_c. But ϕ is injective on vertex groups of T_c, so $g = 1$.

It remains to prove the surjectivity of ϕ. We begin by proving the surjectivity of f_c. It suffices to prove the local surjectivity. Let v be a vertex of T and e an edge adjacent to v. Up to conjugacy, we can assume that $f_c(v) = v$ and $f_c(e) = e$. We thus have $f_c(G_v \cdot e) = \phi(G_v) \cdot f_c(e) = G_v \cdot f_c(e) = G_v \cdot e$. Therefore, all the translates of e by an element of G_v are in the image of f_c, which proves the surjectivity of f_c. It remains to prove the surjectivity of ϕ. Let $g \in G$ and let w be a vertex. There are two vertices v and v' such that $f_c(v) = w$ and $f_c(v') = gw$. Hence there exists $h \in G$ such that $v' = hv$, so $f_c(v') = f_c(hv) = \phi(h)w$, i.e. $gw = \phi(h)w$, so $g^{-1}\phi(h)$ belongs to $G_w = \phi(G_v)$, hence $g = \phi(h)g'$ with $g' \in G_v$. As a consequence, ϕ is surjective. □

8.4. A key proposition.

Definition 8.40. Let G and G' be two virtually free groups, and let $\varphi, \psi : G \to G'$ and $\varphi', \psi' : G' \to G$ be homomorphisms. We say that (ψ, ψ') is a *power* of (φ, φ') if there exist two integers $n, n' \geq 0$ such that $\psi_{|G} = \varphi \circ (\varphi' \circ \varphi)^n$ and $\psi'_{|G'} = \varphi' \circ (\varphi \circ \varphi')^{n'}$.

Definition 8.41. Let G and G' be two virtually free groups, and let $\varphi, \psi : G \to G'$ and $\varphi', \psi' : G' \to G$ be homomorphisms. The notation $(\psi, \psi') \sim (\varphi, \varphi')$ means that $\psi \sim \varphi$ and $\psi' \sim \varphi'$.

Proposition 8.42. *Let G and G' be two virtually free groups. Let m denote the integer $m_{G,G'}$. Let $\varphi : G \to G'$ and $\varphi' : G' \to G$ be two homomorphisms. If $(G, G', \varphi, \varphi')$ has property $\mathcal{P}_m$, then there exists a pair $(\psi : G \to G', \psi' : G' \to G)$ which is equivalent (in the sense of $\sim$) to a power of (φ, φ') and a m-expansion $(\widehat{G}, \widehat{G}', \widehat{\varphi}, \widehat{\varphi}')$ of (G, G', ψ, ψ') such that $\widehat{\varphi}$ and $\widehat{\varphi}'$ are isomorphisms.*

Proof. Let T and T' be reduced m-JSJ splittings of G and G'. First, let us observe that φ maps every m-factor of G isomorphically to a m-factor of G' and that φ' maps every m-factor of G' isomorphically to a m-factor of G (see Remark 8.30).

If G or G' is finite-by-free, the result is an immediate consequence of Corollary 8.38. From now on, we suppose that G and G' are not finite-by-free. We decompose the proof of the lemma into three steps.

Step 1. We claim that there exists a m-expansion $(G_1, G_1', \varphi_1, \varphi_1')$ of $(G, G', \varphi, \varphi')$ such that φ_1 and φ_1' maps edge groups of m-JSJ splittings T_1 and T_1' of G_1 and G_1' to edge groups of T_1' and T_1 respectively.[1]

Proof of Step 1. If $\varphi(C)$ is an edge group of T' for every edge group C of T, and if $\varphi'(C')$ is an edge group of T for every edge group C' of T', then one can take $G_1 = G$, $\varphi_1 = \varphi$, $G_1' = G'$, and $\varphi_1' = \varphi'$. Now, let us suppose that there is an edge e of T such that $\varphi(G_e)$ is not the stabilizer of an edge of T'. Let $C := G_e$. We shall prove that the group $\widehat{G}' = \langle G', t \mid \mathrm{ad}(t)_{|\varphi(C)} = \mathrm{id}_{\varphi(C)}\rangle$ is a legal m-extension of G'. Note that $\varphi(C)$ is an edge group in any m-JSJ tree $\widehat{T}'$ of $\widehat{G}$. In addition, one easily sees that the homomorphism $\widehat{\varphi}' : \widehat{G}' \to G$ defined by $\widehat{\varphi}'(t) = 1$ and $\widehat{\varphi}'(g') = \varphi(g')$ for every $g' \in G'$ satisfies the following three conditions:

- the pair $(\varphi, \widehat{\varphi}')$ is strongly $(\geq m)$-special;
- $\widehat{\varphi}'_{|G'} \sim \varphi'$;
- $\widehat{\varphi}'$ maps every m-factor of $\widehat{G}'$ isomorphically to a m-factor of G.

Hence, one can define the group G_1', and symmetrically the group G_1, by iterating the construction described above finitely many times, since T and T' have only finitely many orbits of edges.

It remains to prove that the group $\widehat{G}' = \langle G', t \mid \mathrm{ad}(t)_{|\varphi(C)} = \mathrm{id}_{\varphi(C)}\rangle$ is a legal m-extension of G', under the hypothesis that $\varphi(C)$ is not contained in the stabilizer of an edge of T'.

Since the group $\varphi(C)$ is not contained in the stabilizer of an edge of T', it fixes a unique vertex v' of T'. There exists an edge $e = [v, w]$ of T such that $G_e = C$, $\varphi(G_v) \subset G'_{v'}$ and $\varphi(G_w) \subset G'_{v'}$. Moreover, recall that φ maps G_v and G_w isomorphically to m-factors of G'. Therefore, the following equalities hold: $\varphi(G_v) = \varphi(G_w) = G'_{v'}$ and $\varphi' \circ \varphi(G_v) = \varphi' \circ \varphi(G_w)$. Since $\varphi' \circ \varphi$ maps every m-factor of G to a conjugate of itself, there exists an element $g \in G$ such that $G_w = gG_vg^{-1}$. Thus we have $\varphi(G_v) = \varphi(gG_vg^{-1}) = \varphi(g)\varphi(G_v)\varphi(g)^{-1}$. This shows that $\varphi(g)$ belongs to $N_{G'}(\varphi(G_v))$. Since G is not finite-by-free, Lemma 8.8 asserts that G_v has order $> m$. This implies that $N_{G'}(\varphi(G_v)) = \varphi(G_v)$. Hence $\varphi(g)$ belongs to $\varphi(G_v)$. There is an element $h \in G_v$ such that $\varphi(g) = \varphi(h)$. Let $k = gh^{-1}$, so that $\varphi(k) = 1$ and $w = gv = kv$. Note that $k \neq 1$, since $w = kv \neq v$. Now, note that $\varphi(C) = \varphi(kCk^{-1})$. Since C and kCk^{-1} are contained in G_w, and since φ is injective on G_w, this proves that $C = kCk^{-1}$, i.e. that k belongs to the normalizer $N_G(C)$ of C in G. In particular, $N_G(C)$ must be non-elementary, otherwise φ would be injective on $N_G(C)$, contradicting the fact that $\varphi(k) = 1$. In order to prove that the group $\widehat{G}'$ defined above is a legal large m-extension of G', we need to prove that $\varphi(C) = E_{G'}(N_{G'}(\varphi(C)))$. First, let us prove that $C = E_G(N_G(C))$. Note that the inclusion $C \subset E_G(N_G(C))$ always holds, because $E_G(N_G(C))$ is the unique maximal finite subgroup of C normalized by $N_G(C)$. Assume towards a contradicton that the inclusion $C \subset E_G(N_G(C))$ is strict. Then $E_G(N_G(C))$ has order $> m$, so it fixes a unique vertex $x \in T$, because edge groups of T have order equal to m. This implies that x is fixed by $N_G(C)$. But $N_G(C)$ is not elliptic in T since the element $k \in N_G(C)$ acts hyperbolically on T. This is a contradiction. We have proved that $E_G(N_G(C)) = C$. It follows that $E_{G'}(N_{G'}(\varphi(C))) = \varphi(C)$, because φ is strongly $(\geq m)$-special and C has order m. Hence, the group $G_1' = \langle G', t \mid \mathrm{ad}(t)_{|\varphi(C)} = \mathrm{id}_{\varphi(C)}\rangle$ is a legal large m-extension of G, which concludes the proof of the first step.

Now, up to replacing $(G, G', \varphi, \varphi')$ with $(G_1, G_1', \varphi_1, \varphi_1')$, one can suppose that $(G, G', \varphi, \varphi')$ has the property of Step 1.

Step 2. We claim that there exists a power (ρ, ρ') of (φ, φ') and a m-expansion $(G_2, G_2', \varphi_2, \varphi_2')$ of (G, G', ρ, ρ') such that the following two conditions hold:

[1] Note that this condition is not automatically satisfied, as shown by the following example: take G that does not split non-trivially as a free product, and $G' = G * F_n$.

- for every edge group C of T_2, the group $\varphi_2(C)$ is an edge group of T_2' and φ_2 maps $N_{G_2}(C)$ isomorphically to $N_{G_2'}(\varphi_2(C))$;
- for every edge group C' of T_2', the group φ_2' is an edge group of T_2 and φ_2' maps $N_{G_2'}(C')$ isomorphically to $N_{G_2}(\varphi_2'(C'))$.

Proof of Step 2. Let T be a m-JSJ splitting of G, and let T' be a m-JSJ splitting of G'. Let $C_1, \ldots, C_q$ be a set of representatives of the conjugacy classes of edge groups of T. Let $C_1', \ldots, C_{q'}'$ be a set of representatives of the conjugacy classes of edge groups of T'. Thanks to the first step, we know that $q = q'$ and, up to renumbering the edges of T, one can assume that $\varphi(C_i) = g_i' C_i' g_i'^{-1}$ for a certain element $g_i' \in G_1'$ and that $\varphi'(C_i') = g_i C_i g_i^{-1}$ for a certain element $g_i \in G$. For every $1 \leq i \leq q$, let $N_i = N_G(C_i)$ and $N_i' = N_{G'}(C_i')$. Let T_c and T_c' be the trees of cylinders of T and T'. Recall that $N_1, \ldots, N_q$ are the new vertex groups of T_c and that $N_1', \ldots, N_q'$ are the new vertex groups of T_c'.

Let $i \in [\![1, q]\!]$. Let $\varphi_i = \mathrm{ad}(g_i'^{-1}) \circ \varphi_{|N_i}$ and $\varphi_i' = \mathrm{ad}(g_i^{-1}) \circ \varphi'_{|N_i'}$. Let Y_i be the cylinder of C_i in T. Recall that Y_i is connected (see Section 2.7). Moreover, note that N_i acts cocompactly on Y_i, because two edges of Y_i are in the same G-orbit if and only if they are in the same N_i-orbit. As a consequence, the action of N_i on Y_i gives a decomposition of N_i as a graph of groups, all of whose edges groups are equal to C_i. Let Y_i' be the cylinder of C_i' in T_1'.

We claim that, for every $1 \leq i \leq q$, there exists a power (ρ_i, ρ_i') of (φ_i, φ_i') and a m-expansion $(\widehat{N}_i, \widehat{N}_i', \widehat{\varphi}_i, \widehat{\varphi}_i')$ of $(N_i, N_i', \rho_i, \rho_i')$ such that $\widehat{\varphi}_i : \widehat{N}_i \to \widehat{N}_i'$ and $\widehat{\varphi}_i' : \widehat{N}_i' \to \widehat{N}_i$ are bijective and coincide with ρ_i and ρ_i', up to conjugacy, on the edge groups adjacent to the vertices fixed by N_i and N_i' in T_c and T_c' (this condition on the edge groups will be necessary in order to apply Lemma 2.27).

The homomorphisms φ and φ' being strongly $(\geq m)$-special, and the edge groups of T and T' having order equal to m, the groups N_i and N_i' are simultaneously finite, virtually cyclic infinite, or non-elementary. We treat separately the three cases.

First case. If N_i and N_i' are finite, $\varphi_{|N_i} : N_i \to N_i'$ and $\varphi_{|N_i'} : N_i' \to N_i$ are injective. Thus, they are bijective. There is nothing to be done: we simply take $\widehat{N}_i = N_i$, $\widehat{N}_i' = N_i'$, $\widehat{\varphi}_i = \varphi_{|N_i}$ and $\widehat{\varphi}_i' = \varphi'_{|N_i'}$.

Second case. Suppose that N_i and N_i' are virtually cyclic infinite. Let us observe that N_i is elliptic in T if and only if N_i' is elliptic in T. Indeed, if N_i fixes a vertex $v \in T$, then $\varphi(N_i)$ fixes the vertex v' fixed by $\varphi(G_v)$ in T'. Then, note that $\varphi(N_i)$ has finite index in N_i', because $\varphi(N_i)$ is infinite and N_i' is virtually cyclic. It follows that N_i' fixes v'.

First subcase. If N_i and N_i' are elliptic in T and T', then φ and φ' induce isomorphisms between N_i and N_i', because φ and φ' induce isomorphisms between vertex groups of T and T'. There is nothing to be done: we simply take $\widehat{N}_i = N_i$, $\widehat{N}_i' = N_i'$, $\widehat{\varphi}_i = \varphi_{|N_i}$ and $\widehat{\varphi}_i' = \varphi'_{|N_i'}$.

Second subcase. If N_i and N_i' are not elliptic in T and T', we take $\widehat{N}_i = \widehat{N}_i' = N_i'$, and we take for $\widehat{\varphi}_i$ and $\widehat{\varphi}_i'$ the identity of N_i'. The group $\widehat{N}_i$ is a legal small extension of N_i, with nice embeddings $\varphi_i : N_i \hookrightarrow \widehat{N}_i$ and $\varphi_i' : \widehat{N}_i \hookrightarrow N_i$, and the group $\widehat{N}_i'$ is a legal small extension of N_i', with nice embeddings $\iota = \varphi_i \circ \varphi_i' : \widehat{N}_i' \hookrightarrow N_i'$ and $\iota' = \varphi_i \circ \varphi_i' : N_i' \hookrightarrow \widehat{N}_i'$.

$$\begin{tikzcd}
\widehat{N}_i = N_i' \arrow[r, "\widehat{\varphi}_i = \mathrm{id}"] \arrow[d, "\varphi_i'"] & \widehat{N}_i' = N_i' \arrow[d, "\iota"] \\
N_i \arrow[r, "\varphi_i"] & N_i'
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{N}_i = N_i' & \widehat{N}_i' = N_i' \arrow[l, "\widehat{\varphi}_i' = \mathrm{id}"'] \\
N_i \arrow[u, "\varphi_i"] & N_i' \arrow[l, "\varphi_i'"] \arrow[u, "\iota'"']
\end{tikzcd}$$

Note that the edge groups adjacent to the vertices fixed by N_i and N_i' in T_c and T_c' are finite. Indeed, let e be an edge of T_c adjacent to the vertex fixed by N_i in T_c. Observe that G_e is elliptic in T, by definition of the tree of cylinders T_c. As a consequence, if $G_e \subset N_i$ were infinite, then N_i would be ellitptic in T, contradicting our hypothesis. As a conclusion, $\widehat{\varphi}_i$ and $\widehat{\varphi}'_i$ coincide with φ_i and φ'_i, up to conjugacy, on the edge groups adjacent to the vertices fixed by N_i and N'_i in T_c and T'_c

Third case. Last, suppose that N_i and N'_i are non-elementary. Recall that the inclusion $C_i \subset E_G(N_i)$ always holds because $E_G(N_i)$ is the unique maximal finite subgroup of G normalized by N_i. Likewise, C'_i is contained in $E_{G'}(N'_i)$. We distinguish two cases.

First subcase. If $E_i := E_G(N_i)$ contains C_i strictly, it has order $> m$. Therefore, it fixes a unique vertex $v_i \in T$, and N_i fixes v_i as well. By definition of a strongly $(\geq m)$-special homomorphism, $\varphi(E_i) = E'_i := E_{G'}(N_{G'}(\varphi(C_i)))$ and $\varphi(C_i)$ is conjugate to C'_i. Consequently, N'_i fixes the unique vertex v'_i of T' fixed by $\varphi(E_i)$. Since φ induces an isomorphism from $(G)_{v_i}$ to $(G')_{v'_i}$, it induces an isomorphism from N_i to N'_i. There is nothing to be done: we simply take $\widehat{N}_i = N_i$, $\widehat{N}'_i = N'_i$, $\widehat{\varphi}_i = \varphi_{|N_i}$ and $\widehat{\varphi}'_i = \varphi'_{|N'_i}$.

Second subcase. If $E_G(N_i) = C_i$, then $E_{G'}(N'_i) = C'_i$ since φ is strongly $(\geq m)$-special. We will use Proposition 8.33 in order to establish the existence of a power (ρ_i, ρ'_i) of (φ_i, φ'_i) and of a m-expansion $(\widehat{N}_i, \widehat{N}'_i, \widehat{\varphi}_i, \widehat{\varphi}'_i)$ of $(N_i, N'_i, \rho_i, \rho'_i)$ satisfying the properties announced above. Before using Proposition 8.33, we will prove that the cylinders Y_i and Y'_i of C_i and C'_i, endowed with actions of N_i and N'_i respectively, satisfy the following conditions:

(1) Y_i and Y'_i are m-splittings of N_i and N'_i with the same number of orbit of vertices;
(2) φ_i and φ'_i induce bijections between the conjugacy classes of vertices of Y_i and Y'_i, and induce isomorphisms between the vertex groups.

First, note that each vertex group of the N_i-tree Y_i is of the form $N_i \cap G_v$ for some $v \in Y_i$. By hypothesis, $\varphi' \circ \varphi(G_v) = gG_vg^{-1}$ for some $g \in G$. Let $\psi' = \mathrm{ad}(g^{-1}) \circ \varphi'$ and $\psi = \varphi$, so that $\psi' \circ \psi(G_v) = G_v$. Since $\psi' \circ \psi$ maps non-conjugate finite subgroups to non-conjugate finite subgroups, and since G_v has only finitely many conjugacy classes of finite subgroups, there exists an integer $n \geq 1$ such that $(\psi' \circ \psi)^n(C_i) = g_vC_ig_v^{-1}$ with $g_v \in G_v$. Let $\rho' = \mathrm{ad}(g_v) \circ (\psi' \circ \psi)^{n-1} \circ \psi'$ and $\rho = \psi$, so that $\rho' \circ \rho(G_v) = G_v$ and $\rho' \circ \rho(C_i) = C_i$. As a consequence, $\rho' \circ \rho(N_i) \subset N_i$.

Let $v_1, \ldots, v_r$ be some representatives of the G-orbits of vertices of Y_i. We can define iteratively two homomorphisms $\rho : G \to G'$ and $\rho' : G' \to G$ such that, for every $1 \leq j \leq r$, there exists an element $g_j \in G$ such that $\mathrm{ad}(g_j) \circ \rho' \circ \rho(G_{v_j}) = G_{v_j}$ and $\mathrm{ad}(g_j) \circ \rho' \circ \rho(C_i) = C_i$. Up to replacing ρ' by $\mathrm{ad}(g_j^{-1}) \circ \rho'$, we can suppose without loss of generality that $g_1 = 1$. Hence $\rho' \circ \rho(C_i) = C_i$ and $\mathrm{ad}(g_j) \circ \rho' \circ \rho(C_i) = C_i$ for every $j \geq 2$. This shows that g_j normalizes C_i, i.e. that g_j belongs to N_i.

We have proved that the homomorphism $\rho' \circ \rho$ maps every vertex group $N_i \cap G_v$ of N_i isomorphically to a N_i-conjugate of itself. Moreover, one can suppose that $\rho \circ \rho'$ maps every vertex group of N'_i isomorphically to a N'_i-conjugate of itself (we repeat the same operation described above with N'_i instead of N_i, and this does not affect the property satisfied by $\rho' \circ \rho$). Note that the pair (ρ, ρ') is a power of (φ, φ').

Now, the existence of a m-expansion $(\widehat{N}_i, \widehat{N}'_i, \widehat{\varphi}_i, \widehat{\varphi}'_i)$ of $(N_i, N'_i, \rho_i, \rho'_i)$ such that $\widehat{\varphi}_i : \widehat{N}_i \to \widehat{N}'_i$ and $\widehat{\varphi}'_i : \widehat{N}'_i \to \widehat{N}_i$ are bijective and coincide with ρ_i and ρ'_i, up to conjugacy, on the edge groups adjacent to the vertices fixed by N_i and N'_i in T_c and T'_c follows from Lemma 8.33. This concludes the proof of the second subcase.

The group G_2 obtained from G by replacing each vertex group N_i of T_c by $\widehat{N}_i$ is a multiple legal m-extension of G. Similarly, the group G'_2 obtained from G' by replacing each vertex group N'_i of T_c by $\widehat{N}'_i$ is a multiple legal m-extension of G'.

Last, since the morphisms $\widehat{\varphi}_i : \widehat{N}_i \to \widehat{N}'_i$ and $\widehat{\varphi}'_i : \widehat{N}'_i \to \widehat{N}_i$ coincide with φ_i and φ'_i, up to conjugacy, on the vertex groups of Y_i and Y'_i, and since every edge group of T_c and T'_c is contained in a vertex group of Y_i or Y'_i, Lemma 2.27 guarantees the existence of the homomorphisms $\varphi_2 : G_2 \to G'_2$ and $\varphi'_2 : G'_2 \to G_2$ announced above.

Step 3. We now prove that the homomorphisms $\varphi_2 : G_2 \to G_2'$ and $\varphi_2' : G_2' \to G_2$ constructed previously are bijective. To that end, we use Lemma 8.39. Let T_2 and T_2' denote two recuced m-JSJ splittings of G_2 and G_2', and let $T_{2,c}$ and $T'_{c2,c}$ be their trees of cylinders. Recall that $V(T_{2,c}) = V_0(T_{2,c}) \sqcup V_1(T_{2,c})$, where $V_0(T_{2,c})$ denotes the set of vertices of T belonging to at least two distinct cylinders, and $V_1(T_{2,c})$ denotes the set of cylinders of T. Observe that the following facts hold.

- φ_2 maps every vertex group of $V_1(T_{2,c})$ isomorphically to a vertex group of $V_1(T'_{2,c})$, and φ_2' maps every vertex group of $V_1(T'_{2,c})$ isomorphically to a vertex group of $V_1(T_{2,c})$.
- φ_2 maps every vertex group of $V_0(T_{2,c})$ isomorphically to a vertex group of $V_0(T'_{2,c})$, and φ_2' maps every vertex group of $V_0(T'_{2,c})$ isomorphically to a vertex group of $V_0(T_{2,c})$.

The first point is a consequence of Step 2. The second point follows from the fact that $v \in T$ belongs to two cylinders if and only if there exists two distinct edge groups $C_1 \subset (G_2)_v$ and $C_2 \subset (G_2)_v$. By Step 1, $\varphi_2(C_1)$ and $\varphi_2(C_2)$ are edge groups of T', and they are distinct because φ_1 is injective on vertex groups of T. Consequently, $\varphi_2((G_2)_v)$ is a vertex group of T'. Last, Lemma 8.39 ensures that $\varphi_2' \circ \varphi_2$ is an automorphism of G_2. Thus, φ_2 and φ_2' are two isomorphisms. This concludes the proof. □

8.5. Proof of (4) ⇒ (5).

8.5.1. *A particular case.* We first prove the implication (4) ⇒ (5) of Theorem 1.23 in a particular case.

PROPOSITION 8.43. *Let G and G' be two virtually free groups. Let T and T' be two reduced JSJ-splittings of G and G' over finite groups. Suppose that all edges groups of T and T' have the same order, say m. If there exist two homomorphisms $\varphi : G \to G'$ and $\varphi' : G' \to G$ such that the pair (φ, φ') is strongly special, then there exist two multiple legal extensions Γ and Γ' of G and G' such that $\Gamma \simeq \Gamma'$.*

PROOF. This particular case is an immediate consequence of Proposition 8.42. □

8.5.2. *The general case.* We shall now prove the implication (4) ⇒ (5) of Theorem 1.23 in the general case, that is the following result.

PROPOSITION 8.44. *Let G and G' be two virtually free groups. Suppose that there exists a strongly special pair of homomorphisms $(\varphi : G \to G', \varphi' : G' \to G)$. Then there exist two multiple legal extensions Γ and Γ' of G and G' respectively, such that $\Gamma \simeq \Gamma'$.*

PROOF. We define the complexity of the pair (G, G'), denoted by $c(G, G')$, as the sum of the number of edges in reduced Stallings splittings of G and G'. We will prove Proposition 8.44 by induction on the complexity $c(G, G')$. In fact, we will prove a slightly stronger result (see the induction hypothesis below). If G is infinite, recall that we denote by $m(G)$ the smallest order of an edge group in a reduced Stallings splitting of G. If G is finite, we set $m(G) = |G|$. We denote by $m_{G,G'}$ the integer $\min(m(G), m(G'))$.

Induction hypothesis $H(n)$. For every pair of virtually free groups (G, G') such that $c(G, G') \leq n$, if there exist two homomorphisms $\varphi : G \to G'$ and $\varphi' : G' \to G$ such that the pair (φ, φ') is strongly $(\geq m_{G,G'})$-special, then there exists a power (ϕ, ϕ') of (φ, φ') and a $(\geq m_{G,G'})$-expansion $(\Gamma, \Gamma', \psi, \psi')$ of (G, G', ϕ, ϕ') such that $\psi : \Gamma \to \Gamma'$ and $\psi' : \Gamma' \to \Gamma$ are two isomorphisms.

The base case $H(0)$ is obvious: if reduced Stallings splittings of G and G' have 0 edge, then G and G' are finite, and one can take $\Gamma = G$ and $\Gamma' = G'$.

We now prove the induction step. Let $n \geq 0$ be an integer. Suppose that $H(n)$ holds, and let us prove that $H(n+1)$ holds. To that end, let us consider two virtually free groups G and G' such that $c(G,G')=n+1$, and suppose that there exists a strongly $(\geq m_{G,G'})$-special pair of homomorphisms $(\varphi : G\to G', \varphi' : G'\to G)$.

Let us fix two reduced $m_{G,G'}$-JSJ decompositions of G and G', and let T and T' denote their respective Bass-Serre trees. Note that T or T' can be trivial, but not both at the same time. Let $G_1\ldots,G_p$ and $G'_1,\ldots,G'_{p'}$ denote the $m_{G,G'}$-factors of G and G', well-defined up to conjugacy. In other words, $G_1\ldots,G_p$ and $G'_1,\ldots,G'_{p'}$ are some representatives of the conjugacy classes of vertex groups of T and T' respectively. These groups are $m_{G,G'}$-rigid by definition.

We now prove that there exists a $(\geq m_{G,G'})$-expansion $(\widehat{G},\widehat{G}',\widehat{\varphi},\widehat{\varphi}')$ of (G,G',φ,φ') with property $\mathcal{P}_{m_{G,G'}}$, which means that the pair $(\widehat{\varphi},\widehat{\varphi}')$ is strongly $(\geq m)$-special, and that $\widehat{\varphi}'\circ\widehat{\varphi}$ and $\widehat{\varphi}\circ\widehat{\varphi}'$ map every $m_{G,G'}$-factor of G and G' isomorphically to a conjugate of itself.

Note that $m(G)=m(\widehat{G})$, because $\widehat{G}$ is a $(\geq m_{G,G'})$-legal extension of G, with $m_{G,G'}\geq m(G)$. Likewise, $m(G')=m(\widehat{G}')$. Therefore, $m_{G,G'}=m_{\widehat{G},\widehat{G}'}$.

If G or G' is finite-by-free, the result is a consequence of Proposition 8.38. From now on, we suppose that G and G' are not finite-by-free. In particular, according to Lemma 8.8, for every vertex v of T or T' and for every edge e incident to v, the edge group G_e is strictly contained in G_v.

Claim: the integers p and p' are equal. Moreover, $\varphi(G_i)$ is contained in $g'_iG'_ig'^{-1}_i$ for some $g'_i\in G'$, and $\varphi'(G'_i)$ is contained in $g_iG_ig_i^{-1}$ for some $g_i\in G$, up to renumbering the subgroups G_i.

Let us prove this claim. First, we prove that $\varphi(G_i)$ fixes a unique vertex of T', for every $1\leq i\leq p$. Note that there exists a vertex $v_i\in T$ such that $G_i=G_{v_i}$. Let T_i be a reduced Stallings splitting of G_i.

As a first step, we will prove that each vertex group $(G_i)_w$ of T_i has order $>m_{G,G'}$. If T_i is not reduced to a point, each vertex group $(G_i)_w$ of T_i contains an edge group, and edge groups of T_i have order $>m_{G,G'}$ since G_i is $m_{G,G'}$-rigid. If T_i is reduced to a point, the group $G_i=(G_w)_i$ is finite. By Lemma 8.8, we have $|G_i|>m_{G,G'}$, because G is not finite-by-free by assumption.

In the previous paragraph, we have proved that each vertex group $(G_i)_w$ of T_i has order $>m_{G,G'}$. Since φ is injective on finite subgroups of G, the finite group $\varphi((G_i)_w)$ has order $>m_{G,G'}$ as well. But edge groups of T' have order exactly $m_{G,G'}$, so $\varphi((G_i)_w)$ fixes a unique vertex v' of T'. We shall prove that $\varphi(G_i)$ fixes this vertex v'. Let us consider a vertex w_2 adjacent to w in T_i. The same argument shows that $\varphi((G_i)_{w_2})$ fixes a unique vertex v'_2 of T'. But G_i is $m_{G,G'}$-rigid, as a $m_{G,G'}$-factor of G, so $(G_i)_w\cap(G_i)_{w_2}$ has order $>m_{G,G'}$. As a consequence, $v'_2=v'$. It follows from the connectedness of T_i that, for every vertex x of T_i, the group $\varphi((G_i)_x)$ fixes v' and only v'. Now, let g be an element of G_i. For any vertex x of T_i, $\varphi((G_i)_x)$ and $\varphi((G_i)_{gx})$ fixes v' and only v', so g fixes v'. Hence, $\varphi(G_i)$ fixes v' and only v'.

Symmetrically, $\varphi'(G_{v'})$ fixes a unique vertex v of T. Since $\varphi'\circ\varphi$ is a conjugacy on finite subgroups, v is a translate of v_i (because G_i has a finite subgroup of order $>m_{G,G'}$, see above), i.e. $\varphi'\circ\varphi(G_i)$ is contained in a conjugate of G_i. Hence, φ and φ' induce two inverses bijections of the conjugacy classes of $m_{G,G'}$-factors of G and G'. Now, up to renumbering the $m_{G,G'}$-factors, one can assume that $\varphi(G_i)$ is contained in a conjugate $g'_iG'_ig'^{-1}_i$ of G'_i, with $g'_i\in G$, and that $\varphi'(G'_i)$ is contained in a conjugate $g_iG_ig_i^{-1}$ of G_i, with $g_i\in G$. This concludes the proof of the claim.

We aim to apply the induction hypothesis $H(n)$ to the pair (G_i,G'_i), for every $1\leq i\leq p$. First, let us observe that the complexity $c(G_i,G'_i)$ is less than n. Indeed, at least one of the trees T or T' is not reduced to a point, say T, and one gets a Stallings splitting of G by replacing the vertex of T fixed by G_i by the splitting T_i of G_i; moreover, the resulting

Stallings tree is reduced because T_i is reduced and the vertex groups of T_i have order $> m_{G,G'}$, whereas edge groups of T are of order $m_{G,G'}$, by Lemma 8.8.

In order to apply the induction hypothesis $H(n)$ to (G_i, G_i'), we need a strongly $(\geq m_{G_i,G_i'})$-special pair of homomorphisms $(\varphi_i : G_i \to G_i', \varphi_i' : G_i' \to G_i)$. We define $\varphi_i = \mathrm{ad}(g_i'^{-1}) \circ \varphi_{|G_i} : G_i \to G_i'$ and $\varphi_i' = \mathrm{ad}(g_i^{-1}) \circ \varphi'_{|G_i'} : G_i' \to G_i$. By Remark 8.26, the pair (φ_i, φ_i') is strongly $(> m_{G,G'})$-special. But $m_{G_i,G_i'} = \min(m(G_i), m(G_i')) > m_{G,G'}$. Thus, (φ_i, φ_i') is strongly $(\geq m_{G_i,G_i'})$-special.

Now, for every $1 \leq i \leq p$, the induction hypothesis $H(n)$ applied to the vertex groups G_i and G_i' together with the pair of homomorphisms (φ_i, φ_i') endows us a $(> m_{G,G'})$-expansion $(\widehat{G}_i, \widehat{G}_i', \widehat{\varphi}_i, \widehat{\varphi}_i')$ of $(G_i, G_i', \varphi_i, \varphi_i')$ such that $\widehat{\varphi}_i : \widehat{G}_i \to \widehat{G}_i'$ and $\widehat{\varphi}_i' : \widehat{G}_i' \to \widehat{G}_i$ are isomorphisms.

One defines $\widehat{G}$ from G by replacing every vertex group G_i by $\widehat{G}_i \supset G_i$ in the $m_{G,G'}$-JSJ splitting T of G, and one defines $\widehat{G}'$ symmetrically. Thanks to Corollary 8.28, the groups $\widehat{G}$ and $\widehat{G}'$ are multiple legal $(> m_{G,G'})$-extensions of G and G'. In particular, they are multiple legal $(\geq m_{G,G'})$-extensions of G and G'.

We define below a strongly $(\geq m_{G,G'})$-special homomorphism $\widehat{\varphi} : \widehat{G} \to \widehat{G}'$ that coincides up to conjugacy with $\widehat{\varphi}_i$ on each subgroup G_i (in particular, $\widehat{\varphi}_{|G} \sim \varphi$). Thanks to Lemma 8.20, in order to prove that this morphism $\widehat{\varphi}$ is strongly $(\geq m_{G,G'})$-special, it suffices to prove that the fourth condition of Definition 8.9 holds, namely: for every finite subgroup A of $\widehat{G}$ of order $\geq m_{G,G'}$, if $N_{\widehat{G}}(A)$ is virtually $\mathbb{Z}$ maximal, then $N_{\widehat{G}'}(\widehat{\varphi}(A))$ is virtually $\mathbb{Z}$ maximal as well, and the restriction of $\widehat{\varphi}$ to $N_{\widehat{G}}(A)$ is nice.

Construction of $\widehat{\varphi}$. We proceed by induction on the number of edges of the m-JSJ decomposition T/G of G. It is enough to construct $\widehat{\varphi}$ in the case where T/G has only one edge.

First case. Suppose that $G = G_1 *_C G_2$. If $N_G(C)$ is virtually $\mathbb{Z}$, then there exists two finite subgroups $C_i \subset G_i$ such that $[C_i : C] = 2$ and $N_G(C) = \langle C_1, C_2 \rangle = C_1 *_C C_2$. If $N_G(C)$ is not virtually $\mathbb{Z}$, let $C_1 := C$ and $C_2 := C$. Since C_1 and C_2 are finite, there exist two elements $g_1', g_2' \in G'$ such that

$$(\widehat{\varphi}_1)_{|C_1} = \mathrm{ad}(g_1') \circ \varphi_{|C_1} \quad \text{and} \quad (\widehat{\varphi}_2)_{|C_2} = \mathrm{ad}(g_2') \circ \varphi_{|C_2}.$$

The homomorphisms $\mathrm{ad}(g_1'^{-1}) \circ \widehat{\varphi}_1$ and $\mathrm{ad}(g_2'^{-1}) \circ \widehat{\varphi}_2$ coincide on $C \subset C_1, C_2$. Hence, one can define $\widehat{\varphi} : \widehat{G} \to \widehat{G}'$ by

$$\widehat{\varphi}_{|\widehat{G}_1} = \mathrm{ad}(g_1'^{-1}) \circ \widehat{\varphi}_1 \quad \text{and} \quad \widehat{\varphi}_{|\widehat{G}_2} = \mathrm{ad}(g_2'^{-1}) \circ \widehat{\varphi}_2.$$

Note that $\widehat{\varphi}$ coincides with φ on C_1 and on C_2. By Lemma 8.20, in order to prove that $\widehat{\varphi}$ is strongly $(\geq m_{G,G'})$-special, we only need to prove that $\widehat{\varphi}$ satisfies the fourth condition of Definition 8.9. Let A be a finite subgroup of $\widehat{G}$ of order $\geq m$ such that $N_{\widehat{G}}(A)$ is virtually $\mathbb{Z}$ maximal. Since every finite subgroup of $\widehat{G}$ is conjugate to a finite subgroup of G, one can suppose without loss of generality that $A \subset G$. As a finite group, A is elliptic in the m-JSJ tree T of G, i.e. A is contained in at least one conjugate of $\widehat{G}_1$ or $\widehat{G}_2$. There are two possibilities.

First possibility. The group A may be contained in only one conjugate of $\widehat{G}_1$ or $\widehat{G}_2$, which is the case for instance if $|A| > m$. Then $N_{\widehat{G}'}(\widehat{\varphi}(A))$ is virtually $\mathbb{Z}$ maximal and the restriction of $\widehat{\varphi}$ to $N_{\widehat{G}}(A)$ is nice, because $\widehat{\varphi}$ induces an isomorphism from $\widehat{G}_i$ to its image (for $i \in \{1, 2\}$).

Second possibiliy. The group A may be contained in at least two distinct conjugates of $\widehat{G}_1$ or $\widehat{G}_2$. Then A is contained in an edge group of the m-JSJ splitting T of G. Since A has order m, one can suppose without loss of generality that $A = C$. The normalizer $N_{\widehat{G}}(A)$ is finite-by-D_∞, equal to $C_1 *_C C_2$. By construction, $\widehat{\varphi}$ coincides with φ on C_1 and C_2. Hence, $\widehat{\varphi}$ coincides with φ on $N_{\widehat{G}}(A) = C_1 *_C C_2$. Since φ is strongly $(\geq m)$-special, $N_{G'}(\varphi(A))$ is virtually $\mathbb{Z}$ maximal, and the restriction of φ to $N_G(A)$ is nice. Let us observe that $\varphi(A)$ is an edge group of the m-JSJ splitting T' of G'. This implies that $N_{\widehat{G}'}(\varphi(A)) = N_{G'}(\varphi(A))$. As a conclusion, $N_{\widehat{G}'}(\varphi(A))$ is virtually $\mathbb{Z}$ maximal and the restriction of $\widehat{\varphi}$ to $N_{\widehat{G}}(A)$ is nice.

Second case. Suppose that

$$G = G_1 *_C = \langle G_1, t \mid tct^{-1} = \alpha(c),\ \forall c \in C\rangle.$$

If $N_G(C)$ is virtually $\mathbb{Z}$ with finite center (i.e. finite-by-D_∞), then C and tCt^{-1} are non-conjugate in G_1 and C has index 2 in $C_1 := N_{G_1}(C)$ and $C_2 := N_{tG_1t^{-1}}(C)$. If $N_G(C)$ is not virtually $\mathbb{Z}$ with finite center, let $C_1 := C$ and $C_2 := tCt^{-1}$. Since C_1 and C_2 are finite, there exist two elements $g_1', g_2' \in G'$ such that such that

$$(\widehat{\varphi}_1)_{|C_1} = \mathrm{ad}(g_1') \circ \varphi_{|C_1} \quad \text{and} \quad (\widehat{\varphi}_1)_{|C_2} = \mathrm{ad}(g_2') \circ \varphi_{|C_2}.$$

One can define $\widehat{\varphi} : \widehat{G} \to \widehat{G}'$ by

$$\widehat{\varphi}_{|\widehat{G}_1} = \widehat{\varphi}_1 \quad \text{and} \quad \widehat{\varphi}(t) = g_2'\varphi(t){g_1'}^{-1}.$$

We need to prove that $\widehat{\varphi}$ satisfies the fourth condition of Definition 8.9. Let A be a finite subgroup of $\widehat{G}$ of order $\geq m$ such that $N_{\widehat{G}}(A)$ is virtually $\mathbb{Z}$ maximal. One can suppose without loss of generality that A is contained in G. As a finite group, A is elliptic in the m-JSJ tree T of G, i.e. A is contained in at least one conjugate of $\widehat{G}_1$.

First possibility. The group A may be contained in only one conjugate of $\widehat{G}_1$, which is the case for instance if $|A| > m$. Then $N_{\widehat{G}'}(\widehat{\varphi}(A))$ is virtually $\mathbb{Z}$ maximal, and the restriction of $\widehat{\varphi}$ to $N_{\widehat{G}}(A)$ is nice, because $\widehat{\varphi}$ induces an isomorphism from $\widehat{G}_1$ to its image.

Second possibility. The group A may be contained in at least two distinct conjugates of $\widehat{G}_1$. Then A is contained in an edge group of T. Since A has order m, one can suppose without loss of generality that $A = C$. There are two subcases.

First subcase. The groups C and $tCt^{-1} = \alpha(C)$ are conjugate in G_1. Up to replacing t with gt for some $g \in G_1$, one can suppose without loss of generality that $tCt^{-1} = C$, i.e. $t \in N_G(C)$. Thus we have $C_1 = C_2 = C$ and one can suppose that $g_1' = g_2' = 1$. Hence, $\widehat{\varphi}$ coincides with φ on C and on t. This implies that $\widehat{\varphi}$ coincides with φ on $N_G(C) = \langle C, t\rangle$. Since φ is strongly ($\geq m$)-special, $N_{G'}(\varphi(A))$ is virtually $\mathbb{Z}$ maximal, and the restriction of φ to $N_G(A)$ is nice. Let us observe that $\varphi(A)$ is an edge group of the m-JSJ splitting T' of G' (see the first step of Proposition 8.42). This implies that $N_{\widehat{G}'}(\varphi(A)) = N_{G'}(\varphi(A))$. As a conclusion, $N_{\widehat{G}'}(\varphi(A))$ is virtually $\mathbb{Z}$ maximal and the restriction of $\widehat{\varphi}$ to $N_{\widehat{G}}(A)$ is nice.

Second subcase. The groups C and $tCt^{-1} = \alpha(C)$ are not conjugate in G_1. Then $N_G(C)$ is C-by-D_∞ and we conclude as in the first case.

Then, note that the morphism $\widehat{\varphi}$ constructed above is strongly ($\geq m_{G,G'}$)-special, thanks to Lemma 8.20. Likewise, there exists a strongly ($\geq m_{G,G'}$)-special homomorphism $\widehat{\varphi}' : \widehat{G}' \to \widehat{G}$ such that, for every G_i', the restriction $\widehat{\varphi}'_{|\widehat{G}_i'}$ coincides with $\widehat{\varphi}_i'$ up to conjugacy.

Let $m = m_{G,G'}$. Let us observe that the tuple $(\widehat{G}, \widehat{G}', \widehat{\varphi}, \widehat{\varphi}')$ is a ($\geq m$)-expansion of $(G, G', \varphi, \varphi')$ and has property $\mathcal{P}_m$. In addition, recall that $m_{\widehat{G},\widehat{G}'} = m$, since $\widehat{G}$ and $\widehat{G}'$ are ($\geq m$)-legal extensions of G and G'. Now, the key proposition 8.42 claims that there exists a pair $(\psi : G \to G', \psi' : G' \to G)$ which is equivalent (in the sense of $\sim$) to a power of $(\widehat{\varphi}, \widehat{\varphi}')$ and a ($\geq m$)-expansion $(\Gamma, \Gamma', \psi, \psi')$ of $(\widehat{G}, \widehat{G}', \widehat{\varphi}, \widehat{\varphi}')$ such that ψ and ψ' are isomorphisms. By transitivity of the relation "to be a ($\geq m$)-expansion of", the tuple $(\Gamma, \Gamma', \psi, \psi')$ is a ($\geq m$)-expansion of $(G, G', \varphi, \varphi')$. This concludes the proof. $\square$

Bibliography

[Ago13] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.

[ALM17] N. Avni, A. Lubotzky, and C. Meiri. First order rigidity of non-uniform higher rank arithmetic lattices. *ArXiv e-prints*, September 2017.

[And18a] S. André. Hyperbolicity and cubulability are preserved under elementary equivalence. arXiv:1801.09411, 2018.

[And18b] S. André. Virtually free groups are almost homogeneous. arXiv:1810.11200, 2018.

[AS12] J. Aramayona and J. Souto. Homomorphisms between mapping class groups. *Geom. Topol.*, 16(4):2285–2341, 2012.

[Bau62] G. Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.

[Bau67] B. Baumslag. Residually free groups. *Proc. London Math. Soc. (3)*, 17:402–418, 1967.

[BdlHV08] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[Bes88] M. Bestvina. Degenerations of the hyperbolic space. *Duke Math. J.*, 56(1):143–161, 1988.

[BF92] M. Bestvina and M. Feighn. A combination theorem for negatively curved groups. *J. Differential Geom.*, 35(1):85–101, 1992.

[BF95] M. Bestvina and M. Feighn. Stable actions of groups on real trees. *Invent. Math.*, 121(2):287–321, 1995.

[BH05] M. Bridson and J. Howie. Conjugacy of finite subsets in hyperbolic groups. *Internat. J. Algebra Comput.*, 15(4):725–756, 2005.

[BMR99] G. Baumslag, A. Myasnikov, and V. Remeslennikov. Algebraic geometry over groups. I. Algebraic sets and ideal theory. *J. Algebra*, 219(1):16–79, 1999.

[Bou16] É. Bouscaren. Introduction à la théorie des modèles. *Gaz. Math.*, (149):18–32, 2016.

[Bow98] B. Bowditch. Cut points and canonical splittings of hyperbolic groups. *Acta Math.*, 180(2):145–186, 1998.

[Bow06] B. Bowditch. Intersection numbers and the hyperbolicity of the curve complex. *J. Reine Angew. Math.*, 598:105–129, 2006.

[Bra99] N. Brady. Branched coverings of cubical complexes and subgroups of hyperbolic groups. *J. London Math. Soc. (2)*, 60(2):461–480, 1999.

[Bra00] N. Brady. Finite subgroups of hyperbolic groups. *Internat. J. Algebra Comput.*, 10(4):399–405, 2000.

[Bri63] J. Britton. The word problem. *Ann. of Math. (2)*, 77:16–32, 1963.

[BV00] M. Bridson and K. Vogtmann. Automorphisms of automorphism groups of free groups. *J. Algebra*, 229(2):785–792, 2000.

[BV06] M. Bridson and K. Vogtmann. Automorphism groups of free groups, surface groups and free abelian groups. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 301–316. Amer. Math. Soc., Providence, RI, 2006.

[CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary.

[CG05] C. Champetier and V. Guirardel. Limit groups as limits of free groups. *Israel J. Math.*, 146:1–75, 2005.

[Cha12] V. Chaynikov. *Properties of hyperbolic groups: Free normal subgroups, quasiconvex subgroups and actions of maximal growth.* ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Vanderbilt University.

[Cou] TITLE = Théorie de la petite simplification: une approche géométrique [d'après F. Dahmani, V. Guirardel, D. Osin et S. Cantat, S. Lamy] JOURNAL = Astérisque FJOURNAL = Astérisque NUMBER = 380, Séminaire Bourbaki. Vol. 2014/2015 YEAR = 2016 PAGES = Exp. No. 1089, 1–33 ISSN = 0303-1179 ISBN = 978-2-85629-836-7 MRCLASS = 20F67 (20F06 20F65 57M07 57M50) MRNUMBER = 3522169 MRREVIEWER = Matthew Stover Coulon, R.

[Cou13] R. Coulon. Small cancellation theory and burnside problem. Internat. J. Algebra Comput. 24 (2014), no. 3, 251-345, 2013.

[DBP19] A. Dente-Byron and C. Perin. Homogeneity of torsion-free hyperbolic groups. arXiv:1907.03264, 2019.

[Del95] T. Delzant. L'image d'un groupe dans un groupe hyperbolique. *Comment. Math. Helv.*, 70(2):267–284, 1995.

[DG11] F. Dahmani and V. Guirardel. The isomorphism problem for all hyperbolic groups. *Geom. Funct. Anal.*, 21(2):223–300, 2011.

[DGO17] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.*, 245(1156):v+152, 2017.

[DGP11] F. Dahmani, V. Guirardel, and P. Przytycki. Random groups do not split. *Math. Ann.*, 349(3):657–673, 2011.

[DK18] C. Druţu and M. Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018. With an appendix by B. Nica.

[Dun93] M. Dunwoody. An inaccessible group. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 75–78. Cambridge Univ. Press, Cambridge, 1993.

[GH17a] D. Groves and M. Hull. Homomorphisms to acylindrically hyperbolic groups I: Equationally noetherian groups and families. *ArXiv e-prints*, April 2017.

[GH17b] V. Guirardel and C. Horbez. Algebraic laminations for free products and arational trees. arXiv:1709.05664, 2017.

[GL11] V. Guirardel and G. Levitt. Trees of cylinders and canonical splittings. *Geom. Topol.*, 15(2):977–1012, 2011.

[GL15] V. Guirardel and G. Levitt. Splittings and automorphisms of relatively hyperbolic groups. *Groups Geom. Dyn.*, 9(2):599–663, 2015.

[GL17] V. Guirardel and G. Levitt. JSJ decompositions of groups. *Astérisque*, (395):vii+165, 2017.

[GL19] V. Guirardel and G. Levitt. Random subgroups, automorphisms, splittings. arXiv:1906.09654, 2019.

[GLP94] D. Gaboriau, G. Levitt, and F. Paulin. Pseudogroups of isometries of $\mathbf{R}$ and Rips' theorem on free actions on $\mathbf{R}$-trees. *Israel J. Math.*, 87(1-3):403–428, 1994.

[Gre60] M. Greendlinger. Dehn's algorithm for the word problem. *Comm. Pure Appl. Math.*, 13:67–83, 1960.

[Gui01] V. Guirardel. Bounding the complexity of small actions on $\mathbb{R}$-trees. 2001.

[Gui04] V. Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. *Geom. Topol.*, 8:1427–1470, 2004.

[Gui08] V. Guirardel. Actions of finitely generated groups on $\mathbb{R}$-trees. *Ann. Inst. Fourier (Grenoble)*, 58(1):159–211, 2008.

[Hag08] F. Haglund. Finite index subgroups of graph products. *Geom. Dedicata*, 135:167–209, 2008.

[Hei18] S. Heil. Test sequences and formal solutions over hyperbolic groups. arXiv:1811.06430, 2018.

[Hor17] C. Horbez. The boundary of the outer space of a free product. *Israel J. Math.*, 221(1):179–234, 2017.

[HW15] T. Hsu and D. Wise. Cubulating malnormal amalgams. *Invent. Math.*, 199(2):293–331, 2015.

[Joh79] K. Johannson. *Homotopy equivalences of* 3*-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[JS79] W. Jaco and P. Shalen. Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.*, 21(220):viii+192, 1979.

[Kap05] M. Kapovich. Representations of polygons of finite groups. *Geom. Topol.*, 9:1915–1951, 2005.

[Khr05] D. Khramtsov. Endomorphisms of automorphism groups of free groups. *Algebra Logika*, 44(2):211–237, 256–257, 2005.

[KM98] O. Kharlampovich and A. Myasnikov. Irreducible affine varieties over a free group. I. Irreducibility of quadratic equations and Nullstellensatz. *J. Algebra*, 200(2):472–516, 1998.

[KM05] O. Kharlampovich and A. Myasnikov. Implicit function theorem over free groups. *J. Algebra*, 290(1):1–203, 2005.

[KM06] O. Kharlampovich and A. Myasnikov. Elementary theory of free non-abelian groups. *J. Algebra*, 302(2):451–552, 2006.

[KM13] O. Kharlampovich and A. Myasnikov. Definable sets in a hyperbolic group. *Internat. J. Algebra Comput.*, 23(1):91–110, 2013.

[KPS73] A. Karrass, A. Pietrowski, and D. Solitar. Finite and infinite cyclic extensions of free groups. *J. Austral. Math. Soc.*, 16:458–466, 1973. Collection of articles dedicated to the memory of Hanna Neumann, IV.

[Las98] D. Lascar. Perspective historique sur les rapports entre la théorie des modèles et l'algèbre. Un point de vue tendancieux. *Rev. Histoire Math.*, 4(2):237–260, 1998.

[Las13] C. Lasserre. Polycyclic-by-finite groups and first-order sentences. *J. Algebra*, 396:18–38, 2013.

[Lev05] G. Levitt. Automorphisms of hyperbolic groups and graphs of groups. *Geom. Dedicata*, 114:49–70, 2005.

[Lin83] P. Linnell. On accessibility of groups. *J. Pure Appl. Algebra*, 30(1):39–46, 1983.

[LPS13] L. Louder, C. Perin, and R. Sklinos. Hyperbolic towers and independent generic sets in the theory of free groups. *Notre Dame J. Form. Log.*, 54(3-4):521–539, 2013.

[Lyn66] R. Lyndon. On Dehn's algorithm. *Math. Ann.*, 166:208–228, 1966.

[Mak82] G. Makanin. Equations in a free group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(6):1199–1273, 1344, 1982.

[Mak84] G. Makanin. Decidability of the universal and positive theories of a free group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(4):735–749, 1984.

[Mar02] D. Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.

[Mer66] J. Merzljakov. Positive formulae on free groups. *Algebra i Logika Sem.*, 5(4):25–42, 1966.

[MM99] H. Masur and Y. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[Moi13] C. Moioli. Graphes de groupes et groupes co-hopfiens. *PhD thesis*, 2013.

[MS84] J. Morgan and P. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.

[MS17] J. Maher and A. Sisto. Random subgroups of acylindrically hyperbolic groups and hyperbolic embeddings. arXiv:1701.00253, 2017.

[MW05] J. McCammond and D. Wise. Coherence, local quasiconvexity, and the perimeter of 2-complexes. *Geom. Funct. Anal.*, 15(4):859–927, 2005.

[MW08] J. McCammond and D. Wise. Locally quasiconvex small-cancellation groups. *Trans. Amer. Math. Soc.*, 360(1):237–271, 2008.

[Nie03] A. Nies. Aspects of free groups. *J. Algebra*, 263(1):119–125, 2003.

[Oge83] F. Oger. Cancellation and elementary equivalence of groups. *J. Pure Appl. Algebra*, 30(3):293–299, 1983.

[Oge91] F. Oger. Cancellation and elementary equivalence of finitely generated finite-by-nilpotent groups. *J. London Math. Soc. (2)*, 44(1):173–183, 1991.

[Oge06] F. Oger. Quasi-finitely axiomatizable groups and groups which are prime models. *J. Group Theory*, 9(1):107–116, 2006.

[OH07] A. Ould Houcine. Limit groups of equationally Noetherian groups. In *Geometric group theory*, Trends Math., pages 103–119. Birkhäuser, Basel, 2007.

[OH11] A. Ould Houcine. Homogeneity and prime models in torsion-free hyperbolic groups. *Confluentes Math.*, 3(1):121–155, 2011.

[Os93] A. Ol′ shanskiĭ. On residualing homomorphisms and G-subgroups of hyperbolic groups. *Internat. J. Algebra Comput.*, 3(4):365–409, 1993.

[Osi16] D. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.

[OW11] Y. Ollivier and D. Wise. Cubulating random groups at density less than 1/6. *Trans. Amer. Math. Soc.*, 363(9):4701–4733, 2011.

[Pau88] F. Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. *Invent. Math.*, 94(1):53–80, 1988.

[Pau91] F. Paulin. Outer automorphisms of hyperbolic groups and small actions on $\mathbf{R}$-trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 331–343. Springer, New York, 1991.

[Pau97] F. Paulin. Sur les automorphismes extérieurs des groupes hyperboliques. *Ann. Sci. École Norm. Sup. (4)*, 30(2):147–167, 1997.

[Pau04] F. Paulin. Sur la théorie élémentaire des groupes libres (d'après Sela). *Astérisque*, (294):ix, 363–402, 2004.

[Per08] C. Perin. *Elementary embeddings in torsion-free hyperbolic groups*. Theses, Université de Caen, October 2008. Thèse rédigée en anglais, avec une introduction détaillée en français.

[Per11] C. Perin. Elementary embeddings in torsion-free hyperbolic groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(4):631–681, 2011.

[Per13] C. Perin. Erratum to: "elementary embeddings in torsion-free hyperbolic groups". *Annales scientifiques de l'École Normale Supérieure*, 46(5):851–856, 2013.

[Pil09] A. Pillay. On genericity and weight in the free group. *Proc. Amer. Math. Soc.*, 137(11):3911–3917, 2009.

[PPST12] C. Perin, A. Pillay, R. Sklinos, and K. Tent. On groups and fields interpretable in torsion-free hyperbolic groups. arXiv:1210.5757, 2012.

[PS12] C. Perin and R. Sklinos. Homogeneity in the free group. *Duke Math. J.*, 161(13):2635–2668, 2012.

[Raz84] A. Razborov. Systems of equations in a free group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(4):779–832, 1984.

[Rem89] V. Remeslennikov. $\exists$-free groups. *Sibirsk. Mat. Zh.*, 30(6):193–197, 1989.

[Rip82] E. Rips. Subgroups of small cancellation groups. *Bull. London Math. Soc.*, 14(1):45–47, 1982.

[RS94] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. *Geom. Funct. Anal.*, 4(3):337–371, 1994.

[RS97] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)*, 146(1):53–109, 1997.

[RSS86] P. Rogers, H. Smith, and D. Solitar. Tarski's problem for solvable groups. *Proc. Amer. Math. Soc.*, 96(4):668–672, 1986.

[RW14] C. Reinfeldt and R. Weidmann. Makanin-razborov diagrams for hyperbolic groups. 2014.

[Sac73a] G. Sacerdote. Almost all free products of groups have the same positive theory. *J. Algebra*, 27:475–485, 1973.

[Sac73b] G. Sacerdote. Elementary properties of free groups. *Trans. Amer. Math. Soc.*, 178:127–138, 1973.

[Sag14] M. Sageev. CAT(0) cube complexes and groups. In *Geometric group theory*, volume 21 of *IAS/Park City Math. Ser.*, pages 7–54. Amer. Math. Soc., Providence, RI, 2014.

[Sch68] P. Schupp. On Dehn's algorithm and the conjugacy problem. *Math. Ann.*, 178:119–130, 1968.

[Sel97] Z. Sela. Acylindrical accessibility for groups. *Invent. Math.*, 129(3):527–565, 1997.

[Sel01] Z. Sela. Diophantine geometry over groups. I. Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.*, (93):31–105, 2001.

[Sel04] Z. Sela. Diophantine geometry over groups. IV. An iterative procedure for validation of a sentence. *Israel J. Math.*, 143:1–130, 2004.

[Sel05] Z. Sela. Diophantine geometry over groups. V_1. Quantifier elimination. I. *Israel J. Math.*, 150:1–197, 2005.

[Sel06a] Z. Sela. Diophantine geometry over groups. V_2. Quantifier elimination. II. *Geom. Funct. Anal.*, 16(3):537–706, 2006.

[Sel06b] Z. Sela. Diophantine geometry over groups. VI. The elementary theory of a free group. *Geom. Funct. Anal.*, 16(3):707–730, 2006.

[Sel09] Z. Sela. Diophantine geometry over groups. VII. The elementary theory of a hyperbolic group. *Proc. Lond. Math. Soc. (3)*, 99(1):217–273, 2009.

[Sho91] H. Short. Quasiconvexity and a theorem of Howson's. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 168–176. World Sci. Publ., River Edge, NJ, 1991.

[Sta83] J. Stallings. Topology of finite graphs. *Invent. Math.*, 71(3):551–565, 1983.

[SW79] P. Scott and T. Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge-New York, 1979.

[Syk05] M. Sykiotis. On subgroups of finite complexity in groups acting on trees. *J. Pure Appl. Algebra*, 200(1-2):1–23, 2005.

[Szm55] W. Szmielew. Elementary properties of Abelian groups. *Fund. Math.*, 41:203–271, 1955.

[Tar49] V. Tartakovskiĭ. Solution of the word problem for groups with a k-reduced basis for $k > 6$. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 13:483–494, 1949.

[TV58] A. Tarski and R. Vaught. Arithmetical extensions of relational systems. *Compositio Math*, 13:81–102, 1958.

[Vau55] R. Vaught. On the arithmetical equivalence of free algebras. *Bulletin of the American Mathematical Society*, 1955.

[Wis12] D. Wise. The structure of groups with a quasiconvex hierarchy. 2012.