Problémes bien-posés et étude qualitative pour des équations cinétiques et des équations dissipatives.

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Problèmes bien posés et étude qualitative pour des équations cinétiques et des équations dissipatives
Well-posedness and qualitative study for some kinetic equations and some dissipative equations

Présentée et soutenue par
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Résumé

Dans cette thèse, nous étudions certaines équations différentielles partielles avec mécanisme dissipatif, telles que l'équation de Boltzmann, l'équation de Landau et certaines systèmes hyperboliques symétriques avec type de dissipation. L'existence globale de solutions ou les taux de dégradation optimaux des solutions pour ces systèmes sont envisagées dans les espaces de Sobolev ou de Besov. Les propriétés de lissage des solutions sont également étudiées. Dans cette thèse, nous prouvons principalement les quatre suivants résultats, voir les chapitres 3-6 pour plus de détails.

Pour le premier résultat, nous étudions le problème de Cauchy pour le non linéaire inhomogène équation de Landau avec des molécules Maxwelliennes ($\gamma = 0$). Voir des résultats connus pour l'équation de Boltzmann et l'équation de Landau, leur existence globale de solutions est principalement prouvée dans certains espaces de Sobolev (pondérés) et nécessite un indice de régularité élevé, voir Guo [62], une série d'œuvres d'Alexander Morimoto-Ukai-Xu-Yang [5, 6, 7, 9] et des références à ce sujet. Récemment, Duan-Liu-Xu [52] et Morimoto-Sakamoto [145] ont obtenu les résultats de l'existence globale de solutions à l'équation de Boltzmann dans l'espace critique de Besov. Motivés par leurs œuvres, nous établissons l'existence globale de la solution dans des espaces de Besov spatialement critiques dans le cadre de perturbation. Précisément, si le datum initial est une petite perturbation de la distribution d'équilibre dans l'espace Chemin-Lerner $\widetilde{L}^2_v(B^{3/2}_{2,1})$, alors le problème de Cauchy de Landau admet qu'une solution globale appartient à $\widetilde{L}^\infty_t\widetilde{L}^2_v(B^{3/2}_{2,1})$. Notre résultat améliore le résultat dans [62] et étend le résultat d'existence globale de l'équation de Boltzmann dans [52, 145] à l'équation de Landau.

Deuxiémement, nous considérons le problème de Cauchy pour l'équation de Kac non-coupée spatialement inhomogène. Lerner-Morimoto-Pravda-Starov-Xu a considéré l'équation de Kac non-coupée spatialement inhomogéne dans les espaces de Sobolev et a montré que le problème de Cauchy pour la fluctuation autour de la distribution maxwellienne admise $S^{1+\frac{1}{2s}}_{1+\frac{1}{2s}}$ Propriétés de régularité Gelfand-Shilov par rapport à la variable de vélocité et propriétés de régularisation $G^{1+\frac{1}{2s}}$ Gevrey à la variable de position. Et les auteurs ont supposé qu'il restait encore à déterminer si les indices de régularité $1 + \frac{1}{2s}$ étaient nets ou non. Dans cette thèse, si la donnée initiale appartient à l'espace de Besov spatialement critique, nous pouvons prouver que l'équation de Kac inhomogène est bien posée dans un cadre de perturbation. De plus, il est montré que la solution bénéficie des propriétés de régularisation de Gelfand-Shilov en ce qui concerne la variable de vitesse et des propriétés de régularisation de Gevrey en ce qui concerne la variable de position. Dans notre thèse, l'indice de régularité de Gelfand-Shilov est amélioré pour être optimal. Et ce résultat est le premier qui présente un effet de lissage pour l'équation cinétique dans les espaces de Besov.

A propos du troisième résultat, nous considérons les équations de Navier-Stokes-Maxwell compressibles apparaissant dans la physique des plasmas, qui est un exemple concret de systèmes composites hyperboliques-paraboliques à dissipation non symétrique. On observe que le

problème de Cauchy pour les équations de Navier-Stokes-Maxwell admet le mécanisme dissipatif de type perte de régularité. Par conséquent, une régularité plus élevée est généralement nécessaire pour obtenir le taux de dégradation optimal de $L^1(\mathbb{R}^3)$-$L^2(\mathbb{R}^3)$ type, en comparaison avec cela pour l'existence globale dans le temps de solutions lisses. Pour les équations de Navier-Stokes-Maxwell, nous avons prouvé l'existence globale de solutions dans les espaces de Sobolev et obtenu la régularité minimale de désintégration des solutions globales lisses avec une aide de $L^p(\mathbb{R}^d)$-$L^q(\mathbb{R}^d)$-$L^r(\mathbb{R}^d)$ estimations.

Enfin, nous sommes consacrés à étudier le système de Timoshenko-Fourier en thermoélasticité, qui décrit la dynamique de la propagation des ondes élastiques et de la dissipation thermique dans un faisceau vibrant. Dans des travaux antérieurs, Kawashima [92] et Umeda-Kawashima-Shizuta [171] ont donné une théorie complète sur l'existence globale et le comportement à grande échelle de solutions lisses à une classe assez générale de systèmes composites hyperboliques-paraboliques symétriques, cependant, jusqu'à présent, il n'y a pas de résultats généraux en cas de dissipation non symétrique. C'est notre motivation pour étudier le système Timoshenko-Fourier, car il admet une dissipation non symétrique. En outre, il est observé que le mécanisme dissipatif de type perte de régularité se produira si les vitesses des vagues sont différentes. Merci à L^p-L^q-L^r estimations, nous obtenons le taux de dégradation optimal de L^1-L^2 type pour les solutions globales lisses avec l'indice de régularité de décroissance minimal dans les espaces Sobolev.

Dans le chapitre 1, il présente une revue générale de la littérature et une brève vue des principaux résultats obtenus dans cette thèse où ces systèmes sont principalement étudiés dans des espaces de régularité critiques. Le chapitre 2 fournit des notations, des définitions d'espaces fonctionnels et des outils d'analyse fondamentale qui seront utilisés dans les chapitres suivants. Dans le chapitre 7, il présente un résumé de cette thèse et offre de nouvelles perspectives pour les questions de recherche. Le chapitre 8 montre une liste des papiers d'auteurs.

Mots-clés: Équation de Landau inhomogène, Équation de Kac inhomogène, Équations de Navier-Stokes-Maxwell, Système de Timoshenko-Fourier, Espace critique de Besov, Régularité de Gelfand-Shilov, Régularité de Gevrey, L^p-L^q-L^r estimations, Régularité minimale de decadency, Régularité-perte, Décomposition spectrale, Solution globale.

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Abstract

In this thesis, we study some kinetic equations and some partial differential equations with dissipative mechanism, such as Boltzmann equation, Landau equation and some non-symmetric hyperbolic systems with dissipation type. Global existence of solutions or optimal decay rates of solutions for these systems are considered in Sobolev spaces or Besov spaces. Also the smoothing properties of solutions are studied. In this thesis, we mainly prove the following four results, see Chapters 3-6 for more details.

For the first result, we investigate the Cauchy problem for the inhomogeneous nonlinear Landau equation with Maxwellian molecules ($\gamma = 0$). See from some known results for Boltzmann equation and Landau equation, their global existence of solutions are mainly proved in some (weighted) Sobolev spaces and require a high regularity index, see Guo [62], a series works of Alexandre-Morimoto-Ukai-Xu-Yang [5, 6, 7, 9] and references therein. Recently, Duan-Liu-Xu [52] and Morimoto-Sakamoto [145] obtained the global existence results of solutions to the Boltzmann equation in critical Besov spaces. Motivated by their works, we establish the global existence of solutions for Landau equation in spatially critical Besov spaces in perturbation framework. Precisely, if the initial datum is a small perturbation of the equilibrium distribution in the Chemin-Lerner space $\widetilde{L}^2_v(B^{3/2}_{2,1})$, then the Cauchy problem of Landau equation admits a global solution belongs to $\widetilde{L}^\infty_t\widetilde{L}^2_v(B^{3/2}_{2,1})$. Our results improve the result in [62] and extend the global existence result for Boltzmann equation in [52, 145] to Landau equation.

Secondly, we consider the Cauchy problem for the spatially inhomogeneous non-cutoff Kac equation. Lerner-Morimoto-Pravda-Starov-Xu [117] considered the spatially inhomogeneous non-cutoff Kac equation in Sobolev spaces and showed that the Cauchy problem for the fluctuation around the Maxwellian distribution admitted $S^{1+\frac{1}{2s}}_{1+\frac{1}{2s}}$ Gelfand-Shilov regularity properties with respect to the velocity variable and $G^{1+\frac{1}{2s}}$ Gevrey regularizing properties with respect to the position variable. And the authors conjectured that it remained still open to determine whether the regularity indices $1+\frac{1}{2s}$ is sharp or not. In this thesis, if the initial datum belongs to the spatially critical Besov space $\widetilde{L}^2_v(B^{1/2}_{2,1})$, we prove the well-posedness to the inhomogeneous Kac equation under a perturbation framework. Furthermore, it is shown that the weak solution enjoys $S^{\frac{3s+1}{2s(s+1)}}_{\frac{3s+1}{2s(s+1)}}$ Gelfand-Shilov regularizing properties with respect to the velocity variable and $G^{1+\frac{1}{2s}}$ Gevrey regularizing properties with respect to the position variable. In our results, the Gelfand-Shilov regularity index is improved to be optimal. And this result is the first one that exhibits smoothing effect for the kinetic equation in Besov spaces.

About the third result, we consider compressible Navier-Stokes-Maxwell equations arising in plasmas physics, which is a concrete example of hyperbolic-parabolic composite systems with non-symmetric dissipation. It is observed that the Cauchy problem for Navier-Stokes-Maxwell equations admits the dissipative mechanism of regularity-loss type. Consequently, extra higher regularity is usually needed to obtain the optimal decay rate of $L^1(\mathbb{R}^3)$-$L^2(\mathbb{R}^3)$

type, in comparison with that for the global-in-time existence of smooth solutions. For the Navier-Stokes-Maxwell equations, we proved the global existence of solutions in Sobolev spaces and obtain the minimal decay regularity of global smooth solutions with aid of $L^p(\mathbb{R}^d)$-$L^q(\mathbb{R}^d)$-$L^r(\mathbb{R}^d)$ estimates.

Finally, we are devoted to investigate the Timoshenko-Fourier system in thermoelasticity, which describes the dynamics of elastic wave propagation and thermal dissipation in a vibrating beam. In earlier works, Kawashima [92] and Umeda-Kawashima-Shizuta [171] gave a complete theory on the global existence and large-time behavior of smooth solutions to a rather general class of symmetric hyperbolic-parabolic composite systems, however, so far there are no general results in the case of non-symmetric dissipation. This is our motivation to study the Timoshenko-Fourier system since it admits a non-symmetric dissipation. Furthermore, it is observed that the dissipative mechanism of regularity-loss type will occur if wave speeds are different. Thanks to L^p-L^q-L^r estimates, we get the optimal decay rate of L^1-L^2 type for global smooth solutions with the minimal decay regularity index in Sobolev spaces.

In Chapter 1, it presents a general literature review and a short view of the main results obtained in this thesis where these systems are mainly studied in critical regularity spaces. Chapter 2 provides some notations, definitions of functional spaces and fundamental analysis tools which will be used in the subsequent chapters. In Chapter 7, it shows a summary of this thesis and puts forward further prospects for research issues. Chapter 8 shows a list of author's papers.

Keywords: Inhomogeneous Landau equation, Inhomogeneous Kac equation, Navier-Stokes-Maxwell equations, Timoshenko-Fourier system, Critical Besov space, Gelfand-Shilov regularity, Gevrey regularity, L^p-L^q-L^r estimates, Minimal decay regularity, Regularity-loss, Spectral decomposition, Global solution.

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Contents

Chapter 1

Introduction

In this introduction, we will give some specific mathematical equations and its basic properties, mainly see [14, 24, 68, 69, 105, 114, 137, 165, 166, 173, 176] and the references therein. Then, we show our main results in this thesis. Finally, we recall some known results to the Cauchy problem for these equations and give a comparison with our main results.

1.1 Mathematical systems

1.1.1 Landau equation

In gas and fluid dynamics, there are many famous equations of motion, which have been derived by focusing the attention on different aspects of gases and fluids in different physical scales. Most of them are classical, dating back to the 19th century or earlier. The Landau equation, which was proposed by Landau in 1936 [105], is a fundamental model in kinetic theory that describes the evolution of the density of particles in a plasma. In this thesis, we consider the Cauchy problem to the spatially inhomogeneous Landau equation in $\mathbb{R}^3$, which is given by

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q_L(F, F), \\ F|_{t=0} = F_0, \end{cases} \tag{1.1}$$

where $F = F(t, x, v) \geq 0$ is the density of particles on position $x \in \mathbb{R}^3$ and with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The collision operator Q_L is a bilinear operator acting only on the velocity variable v and reads as

$$Q_L(F, G)(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v - v_*)[F_* \nabla_v G - \nabla_v F_* G] dv_* \right),$$

where we used the usual shorthand $F_* = F(t, x, v_*), \nabla_v G = \nabla_v G(t, x, v), \nabla_v F_* = \nabla_v F(t, x, v_*)$ and $G = G(t, x, v)$. The matrix-valued function $a(v) = (a_{i,j}(v))_{1 \leq i,j \leq 3}$ is non-negative, symmetric and depends on the interaction between particles, which is usually assumed by

$$a(v) = (|v|^2 I - v \otimes v)|v|^\gamma = |v|^{\gamma+2} \mathbb{P}_{v^\perp}, \quad -3 < \gamma \leq 1,$$

where $I = I_{3\times 3}$ is the unit matrix on $\mathbb{R}^3$, $v \otimes v = (v_i v_j)_{1 \leq i,j \leq 3}$ and $\mathbb{P}_{v^\perp}$ is the orthogonal projection onto $v^\perp$. Here, according to the value of γ, we have the following classification:

- $0 < \gamma < 1$, one calls hard potentials;

- $\gamma = 0$, it is Maxwellian molecules ($a(v) = |v|^2 \mathbb{P}_{v^\perp}$);
- $-3 < \gamma < 0$, one calls soft potentials;
- $\gamma = -3$, it is Coulombian potential ($a(v) = |v|^{-1} \mathbb{P}_{v^\perp}$).

■ **Entropy principle**

The solutions of the Landau equation have some elementary properties. And these properties correspond to basic conservation laws (mass, momentum, kinetic energy) and therefore have a simple physical meaning.

The entropy H can be defined as

$$H = \int_{\mathbb{R}^d \times \mathbb{R}^d} F \log(1/F) dx dv.$$

We recall the mathematical expression of the entropy principle for Landau equation, see Villani [173],

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} F \log(1/F) dx dv - \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(v - v_*) F F_* \\ \times \left(\frac{\partial}{\partial v_i} (\log F) - \frac{\partial}{\partial v_{*,i}} (\log F_*) \right) \left(\frac{\partial}{\partial v_j} (\log F) - \frac{\partial}{\partial v_{*,j}} (\log F_*) \right) dx dv dv_* = 0.$$

Define $F(t, x, v) = e^{-\phi(t,x,v)}$, then the entropy

$$H = \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi e^{-\phi} dx dv$$

is increasing with time t (Landau version of Boltzmann's famous H-theorem), due to that the right-hand side is nonnegative.

In this thesis, we study the Cauchy problem (1.1) with Maxwellian molecules, since the Landau operator enjoys very nice spectral property in that case. Here, we are concerned with the Landau equation around the absolute Gaussian distribution in $\mathbb{R}^3$:

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

With the perturbation $F(t, x, v) = \mu(v) + \sqrt{\mu}(v) g(t, x, v)$, the Cauchy problem (1.1) can be rewritten as

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L} g = \mathbf{L}(g, g), \\ g|_{t=0} = g_0, \end{cases} \tag{1.2}$$

with $F_0 = \mu(v) + \sqrt{\mu}(v) g_0$, where

$$\mathcal{L}(g) = \mathcal{L}_1(g) + \mathcal{L}_2(g) \tag{1.3}$$

is a linear operator and non-negative (see [114]) with

$$\mathcal{L}_1(g) = -\mu^{-\frac{1}{2}} Q_L(\mu, \sqrt{\mu} g), \quad \mathcal{L}_2(g) = -\mu^{-\frac{1}{2}} Q_L(\sqrt{\mu} g, \mu)$$

and

$$\mathbf{L}(g,g)=\mu^{-\frac{1}{2}}Q_L(\sqrt{\mu}g,\sqrt{\mu}g). \tag{1.4}$$

The null space of operator $\mathcal{L}$ is given by

$$\mathcal{N}=\operatorname{span}\left\{\sqrt{\mu},\, v_1\sqrt{\mu},\, v_2\sqrt{\mu},\, v_3\sqrt{\mu},\, |v|^2\sqrt{\mu}\right\}.$$

$\mathbf{P}$ denotes the orthogonal projector onto the null space $\mathcal{N}$.

Denote the Harmonic Oscillator

$$\mathcal{H}\triangleq -\Delta_v+\frac{|v|^2}{4}$$

and the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$

$$\Delta_{\mathbb{S}^2}\triangleq\frac{1}{2}\sum_{\substack{1\le j,k\le 3\\ j\neq k}}(v_j\partial_k-v_k\partial_j)^2.$$

Let $\{\varphi_{n,l,m};n,l\in\mathbb{N},|m|\le l\}$ be the orthonormal basis of $L^2(\mathbb{R}^3)$ defined in [65, 114, 122], composed by the eigenfunction of the Harmonic Oscillator and Laplace-Beltrami operator. We have

$$\mathcal{H}(\varphi_{n,l,m})=(2n+l+\frac{3}{2})\,\varphi_{n,l,m},\quad -\Delta_{\mathbb{S}^2}(\varphi_{n,l,m})=l(l+1)\varphi_{n,l,m}.$$

Then, we have the spectral decomposition of the linear Landau operator (see [16, 114])

$$\mathcal{L}(\varphi_{n,l,m})=\lambda_{n,l}\,\varphi_{n,l,m},\quad n,\,l\in\mathbb{N},\ -l\le m\le l$$

with $\lambda_{0,0}=\lambda_{0,1}=\lambda_{1,0}=0$, $\lambda_{0,2}=12$ and for $2n+l>2$,

$$\lambda_{n,l}=2(2n+l)+l(l+1).$$

Using this spectral decomposition, we have

$$\mathcal{L}=(2\mathcal{H}-3-\Delta_{\mathbb{S}^2})-[2\mathcal{H}-3-\Delta_{\mathbb{S}^2}]\mathbb{P}_1+[-\Delta_{\mathbb{S}^2}-(2\mathcal{H}-3)]\mathbb{P}_2,$$

where $\mathbb{P}_k(k=1,2)$ is the orthogonal projection onto the Hermite basis.

1.1.2 Kac equation

Kac equation is a simplified model of Boltzmann equation but still keeping some of the main features of Boltzmann's. And the Kac operator is a one-dimensional collision model for the radically symmetric Boltzmann operator. In this thesis, we also consider the spatially inhomogeneous Kac's model of the non-cutoff Boltzmann equation. Hence, firstly we introduce the Boltzmann equation and its basic property briefly.

■ **Boltzmann equation**

Ludwig Boltzmann derived the Boltzmann equation in 1872, which is the basic kinetic equation in statistical mechanics and describes the dynamics of particles of dilute gas, see [14]. Since the advent of Boltzmann equation, it has been received much attention from mathematicians and physicists. The Boltzmann equation describes the behavior of a dilute gas when the only interactions taken into account are binary collisions [24]. It reads as the equation

$$\begin{cases} \partial_t G + v \cdot \nabla_x G = Q_B(G, G), \\ G|_{t=0} = G_0, \end{cases} \tag{1.5}$$

where the unknown $G = G(t, x, v) \geq 0$ is a density distribution function in phase space with position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$ at time $t \geq 0$, and G_0 is an initial data. The Boltzmann equation is one of the fundamental equations of mathematical physics, in particular, a cornerstone of statistical physics.

The Boltzmann collision operator, $Q_B(\cdot, \cdot)$ is a bilinear operator which is defined by

$$Q_B(F, G) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*|^\gamma B_0(\theta)(F'_* G' - F_* G) dv_* d\omega$$

with $d \geq 2$, where

$$F'_* = F(t, x, v'_*), \quad G' = G(t, x, v'), \quad F_* = F(t, x, v_*), \quad G = G(t, x, v).$$

In this expression, v, v_* and v', v'_* are the velocities in $\mathbb{R}^d$ of a pair of particles before and after the collision. v' and v'_* are shown through the formulas

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma,$$

where $\sigma \in \mathbb{S}^{d-1}$ belongs to the unit sphere. Those relations correspond physically to elastic collisions with the conservations of momentum and kinetic energy in the binary collisions

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$.

For monatomic gas, the cross section $B(v - v_*, \sigma)$ is a non-negative function which only depends on the relative velocity $|v - v_*|$ and on the deviation angle θ defined through the scalar product in $\mathbb{R}^d$

$$\cos\theta = z \cdot \sigma, \qquad z = \frac{v - v_*}{|v - v_*|}.$$

Without loss of generality, we may assume that $B(v - v_*, \sigma)$ is supported on the set where $z \cdot \sigma \geq 0$, i.e. where $0 \leq \theta \leq \frac{\pi}{2}$. Otherwise and since the term F', F'_* appearing in the Boltzmann operator $Q_B(F, F)$ is invariant under the mapping $\sigma \to -\sigma$, we can reduce to this situation with the customary symmetrization

$$\tilde{B}(v - v_*, \sigma) = [B(v - v_*, \sigma) + B(v - v_*, -\sigma)] \mathbb{1}_{\{\frac{v - v_*}{|v - v_*|} \cdot \sigma \geq 0\}},$$

where $\mathbb{1}_E$ is the characteristic function of the set E. For more details about collision kernel, see Villani[176] and references therein. More specifically, we consider cross sections of the type

$$B(v-v_*,\sigma)=\Phi(|v-v_*|)b(\cos\theta),\quad \cos\theta=\frac{v-v_*}{|v-v_*|}\cdot\sigma,\ 0\le\theta\le\frac{\pi}{2},$$

where $\Phi(|v-v_*|)=|v-v_*|^\gamma,\ \gamma>-3$ is a kinetic factor and it is related to the deviation angle with a singularity

$$(\sin\theta)^{d-2}b(\cos\theta)\underset{\theta\to 0_+}{\approx}\theta^{-1-2s},\quad 0<s<1\ \text{ and }\ 0\le\theta\le\frac{\pi}{2}. \tag{1.6}$$

This singularity is not integrable

$$\int_0^{\frac{\pi}{2}}(\sin\theta)^{d-2}b(\cos\theta)d\theta=+\infty.$$

Here, this non-integrability property plays an important effect when considering the qualitative behaviour of solutions to the Boltzmann equation. Actually, Desvillettes [35] firstly observed that grazing collisions that account for the non-integrability of the angular factor near $\theta=0$ to induce smoothing effects for the solutions of the non-cutoff Kac equation, or more generally for the solutions of the non-cutoff Boltzmann equation. On the other hand, when the cross section is assumed to be integrable, or by using a cutoff function to remove the singularity (Grad's angular cutoff assumption), then these solutions are at most as regular as the initial data, see Wennberg [178]. For some details on the physics background and the derivation of the Boltzmann equation, we refer the reader to the extensive expositions Cercignani [24] and Villani [176].

Boltzmann equation and Landau equation both provide a mathematical model for the description of the evolution of a large number of particles interacting through "collisions". Similar to the Landau equation, one can consider the fluctuation of density distribution in $\mathbb{R}^3$

$$G(t,x,v)=M(v)+\sqrt{M}(v)g(t,x,v)$$

near the absolute Maxwellian distribution

$$M(v)=(2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}}.$$

With the perturbation and since $Q_B(M,M)=0$ by the conservation of the kinetic energy, the Cauchy problem (1.5) can be rewritten as

$$\begin{cases}\partial_t g+v\cdot\nabla_x g+\bar{\mathcal{L}}g=\Gamma(g,g),\\ g|_{t=0}=g_0,\end{cases} \tag{1.7}$$

with $G_0=M(v)+\sqrt{M}(v)g_0$, where

$$\bar{\mathcal{L}}(g)=-\mu^{-\frac{1}{2}}(Q_B(M,\sqrt{M}g)+Q_B(\sqrt{M}g,M)),\quad \Gamma(g,g)=M^{-\frac{1}{2}}Q_B(\sqrt{M}g,\sqrt{M}g).$$

The operator $\bar{\mathcal{L}}$ is linear, non-negative (see Cercignani [24] and Lerner-Morimoto-Pravda-Starov-Xu [114, 115, 116]) and its null space of operator is given by

$$\mathcal{N} = \text{span}\left\{\sqrt{M},\, v_1\sqrt{M},\, v_2\sqrt{M},\, v_3\sqrt{M},\, |v|^2\sqrt{M}\right\}.$$

■ **Kac equation**

After a brief introduction to the Boltzmann equation, we introduce the spatially inhomogeneous non-cutoff Kac's equation that we study in this thesis, precisely,

$$\begin{cases} \partial_t f + v\partial_x f = K(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{1.8}$$

where $f = f(t,x,v) \geq 0$ is the density distribution function depending on the position $x \in \mathbb{R}$, the velocity $v \in \mathbb{R}$ and the time $t \geq 0$. The Kac collision operator is a one-dimensional collision model for the radially symmetric Boltzmann operator defined as

$$K(f,g)(v) = \int_{|\theta|\leq\frac{\pi}{4}} \beta(\theta)\left(\int_{\mathbb{R}} \left(f'_* g' - f_* g\right) dv_*\right) d\theta$$

with the standard shorthand $f'_* = f(t,x,v'_*), f' = f(t,x,v'), f_* = f(t,x,v_*), f = f(t,x,v)$, where the relations between pre and post collisional velocities given by

$$v' + iv'_* = e^{i\theta}(v + iv_*),\ i.e.,\quad v' = v\cos\theta - v_*\sin\theta,\quad v'_* = v\sin\theta + v_*\cos\theta,\ v, v_* \in \mathbb{R},$$

follow from the conversation of the kinetic energy in the binary collision

$$v^2 + v_*^2 = v'^2 + v'^2_*.$$

Here, the cross section is assumed to be an even non-negative function satisfying

$$\beta \geq 0, \quad \beta \in L^1_{loc}(0,1), \quad \beta(-\theta) = \beta(\theta)$$

with a non-integrable singularity for grazing collisions

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta) d\theta = +\infty.$$

This non-integrability plays an important role when considering the qualitative behaviour of solutions to the Kac equation and this feature is essential for the smoothing effect.

In this thesis, we consider a cross section with a non-integrable singularity of the type

$$\beta(\theta) \underset{\theta\to 0}{\approx} |\theta|^{-1-2s} \tag{1.9}$$

for some given parameter $0 < s < 1$. For more details on the physics background, we can refer to [24, 176] and the references therein. Under the assumption, the Kac collision operator

can be defined as a finite part integral. For details about this as a finite part integral, we refer the reader to Lerner-Morimoto-Pravda-Starov-Xu [115, 117]. Desvillettes [35] pointed out that non-integrability feature (1.9) accounts for the diffusive properties of the non-cutoff Kac equation. For more details on the physics background, we can refer to Cercignani [24], Villani [176] and the references therein. In fact, we take the following choice for the cross section

$$\beta(\theta) = \frac{|\cos\frac{\theta}{2}|}{|\sin\frac{\theta}{2}|^{1+2s}}, \quad |\theta| \leq \frac{\pi}{4}, \tag{1.10}$$

in part because of the usage of those results in [115] directly.

Here, we consider the Kac equation (1.8) around the normalized Maxwellian distribution

$$M(v) = (2\pi)^{-\frac{1}{2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}.$$

In a close to equilibrium framework, consider the fluctuation of density distribution function

$$f(t,x,v) = M(v) + \sqrt{M}(v) g(t,x,v).$$

Since $K(M,M) = 0$ by conservation of the kinetic energy, the Cauchy problem (1.8) reads as

$$\begin{cases} \partial_t g + v\partial_x g + \mathcal{K} g = \Gamma(g,g), \ \ t > 0, \, v \in \mathbb{R}, \\ g|_{t=0} = g_0, \end{cases} \tag{1.11}$$

with $f_0 = M(v) + \sqrt{M}(v) g_0$, where

$$\mathcal{K}(g) = \mathcal{K}_1(g) + \mathcal{K}_2(g) \tag{1.12}$$

with

$$\mathcal{K}_1(g) = -M^{-1/2} K(M, M^{1/2} g), \quad \mathcal{K}_2(g) = -M^{-1/2} K(M^{1/2} g, M)$$

and

$$\Gamma(g,g) = M^{-1/2} K(M^{1/2} g, M^{1/2} g). \tag{1.13}$$

The linearized Kac operator $\mathcal{K}$ is a non-negative unbounded operator on $L^2(\mathbb{R}_v)$ (see [115]) with a kernel given by

$$\text{Ker } \mathcal{K} = \text{Span}\{e_0, e_2\},$$

where the Hermite basis $(e_n)_{n\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R})$, which is recalled in Section 4.2.1.

1.1.3 Navier-Stokes-Maxwell equations

In this thesis, we study the motion of one fluid described by the compressible Navier-Stokes equations coupled with the Maxwell equations through the Lorentz force, that is, Navier-Stokes-Maxwell (N-S-M) equations. In the isentropic case, it read as

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) \\ \qquad = -n(E + u \times B) - \lambda nu + \mu \Delta u + (\mu + \mu')\nabla \mathrm{div} u, \\ \partial_t E - \nabla \times B = nu, \\ \partial_t B + \nabla \times E = 0 \end{cases} \tag{1.14}$$

with constraints

$$\nabla \cdot E = n_\infty(x) - n, \quad \nabla \cdot B = 0. \tag{1.15}$$

Here the mass density $n \in \mathbb{R}^+$, the velocity field $u \in \mathbb{R}^3$, the electric field $E \in \mathbb{R}^3$ and the magnetic induction $B \in \mathbb{R}^3$ are unknown functions for $(t, x) \in [0, +\infty) \times \mathbb{R}^3$, respectively. The pressure $p(n)$ is a given smooth function of n satisfying $p'(n) > 0$ for $n > 0$. We denote by μ and μ' the two Lamé coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $2\mu + \mu' > 0$, and $\lambda > 0$ is the damping constant. $n_\infty(x)$ stands for the density of positively charged background ions, which is assumed to be a positive constant for simplicity. Notice that the system (1.14) with (1.15) admits a constant equilibrium state of the form

$$(n, u, E, B) = (n_\infty, 0, 0, B_\infty),$$

where $B_\infty \in \mathbb{R}^3$ is an arbitrarily fixed constant vector. In this thesis, we concerned with the Cauchy problem to (1.14)-(1.15), so the initial data are supplemented by

$$(n, u, E, B)|_{t=0} = (n_0, u_0, E_0, B_0)(x), \quad x \in \mathbb{R}^3. \tag{1.16}$$

Also, it is simple to see that the constraint (1.15) holds true for any $t > 0$, if it holds initially. Namely, we only assume that

$$\nabla \cdot E_0 = n_\infty(x) - n_0, \quad \nabla \cdot B_0 = 0, \quad x \in \mathbb{R}^3. \tag{1.17}$$

We rewrite (1.14) as the linearized perturbation form around the equilibrium state $w_\infty := (n_\infty, 0, 0, B_\infty)$. Without loss of generality, we set the physical parameters to be $\lambda = 1 = \mu$ and $\mu + \mu' = 0$. By taking change of variables

$$\rho = n - n_\infty, \ \upsilon = nu/n_\infty, \ E = E, \ h = B - B_\infty,$$

it is not difficulty to verify

$$\begin{cases} \partial_t \rho + n_\infty \mathrm{div} \upsilon = 0, \\ \partial_t \upsilon + a_\infty \nabla \rho + \upsilon \times B_\infty + E + \upsilon - \Delta \upsilon / n_\infty = (\mathrm{div} q_2 + r_2)/n_\infty + \Delta s_2, \\ \partial_t E - \nabla \times h - n_\infty \upsilon = 0, \\ \partial_t h + \nabla \times E = 0, \end{cases} \tag{1.18}$$

where $a_\infty = p'(n_\infty)/n_\infty$,

$$\begin{aligned} q_2 &= -n_\infty^2 \upsilon \otimes \upsilon/n - [p(n) - p(n_\infty) - p'(n_\infty)\rho]I, \\ r_2 &= -\rho E - n_\infty \upsilon \times h, \\ s_2 &= (1/(\rho + n_\infty) - 1/n_\infty)\upsilon := \tilde{h}(\rho)\upsilon. \end{aligned}$$

For simplicity, we set $z := (\rho, \upsilon, E, h)^\top$. The corresponding initial data are given by

$$z|_{t=0} = (\rho_0, \upsilon_0, E_0, h_0)^\top(x)$$

with $\rho_0 = n_0 - n_\infty$, $\upsilon_0 = n_0 u_0/n_\infty$ and $h_0 = B_0 - B_\infty$. System (1.18) is also rewritten in vector form as

$$A_\infty^0 z_t + \sum_{j=1}^3 A_\infty^j z_{x_j} + L_\infty z = \sum_{j,k=1}^3 B_\infty^{jk} z_{x_j x_k} + \sum_{j=1}^3 Q_{x_j} + R + \Delta S, \tag{1.19}$$

where $Q(z) = (0, q_2^j/n_\infty, 0, 0)^\top$, $R(z) = (0, r_2/n_\infty, 0, 0)^\top$ and $S(z) = (0, s_2, 0, 0)^\top$. The coefficient matrices $A_\infty^0 = A^0(w_\infty), A_\infty^j = A^j(w_\infty), B_\infty^{jk} = B^{jk}(w_\infty)$ and $L_\infty = L(w_\infty)$ are explicitly given by

$$A_\infty^0 = \begin{pmatrix} a_\infty & 0 & 0 & 0 \\ 0 & n_\infty I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad L_\infty = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & n_\infty(I - \Omega_{B_\infty}) & n_\infty I & 0 \\ 0 & -n_\infty I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sum_{j=1}^3 A_\infty^j \xi_j = \begin{pmatrix} 0 & p'(n_\infty)\xi & 0 & 0 \\ p'(n_\infty)\xi^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Omega_\xi \\ 0 & 0 & \Omega_\xi & 0 \end{pmatrix}$$

and

$$\sum_{j,k=1}^3 B_\infty^{jk} \xi_j \xi_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |\xi|^2 I/n_\infty & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, I is the identity matrix of third order, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, and Ω_ξ is the skew-symmetric matrix defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix},$$

which implies that $\Omega_\xi E^\top = (\xi \times E)^\top$ (as a column vector in $\mathbb{R}^3$) for $E = (E_1, E_2, E_3) \in \mathbb{R}^3$. In (1.19), matrices $A_\infty^j (j = 0, 1, 2, 3)$ and $B_\infty^{j,k} (j, k = 1, 2, 3)$ are real symmetric, and L_∞ and $\sum_{j,k=1}^3 B_\infty^{jk} \xi_j \xi_k$ are degenerate and nonnegative definite; however, L_∞ *is not real symmetric*. Therefore, the dissipation forces (1.19) to go beyond the general class of equations of symmetric hyperbolic-parabolic type as in [171], which is our motivation to study the N-S-M equations (1.14).

1.1.4 Timoshenko-Fourier system

It is well known that thermoelastic systems are thermoelastic equations, which are mathematical models established by the distribution of temperature and the deformation of thermoelastic bodies. Thermoelasticity is called thermoelastic theory and is a branch of solid mechanics. The research problem is mainly the deformation and stress generated by the non-uniform temperature field of the object due to heat in the elastic range. Based on the important physical background of thermoelasticity, thermoelastic equations have attracted the attention of many physicists and mathematicians and have yielded rich research results. In this thesis, we investigate the Timoshenko-Fourier system in thermoelasticity, which describes the dynamics of elastic wave propagation and thermal dissipation in a vibrating beam. Precisely,

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - [\sigma(\psi_x)]_x - (\varphi_x - \psi) + \gamma\psi_t + \beta\theta_x = 0, \\ \theta_t - \kappa\theta_{xx} + \beta\psi_{tx} = 0, \end{cases} \tag{1.20}$$

where $t \in (0, +\infty)$ is the time variable and $x \in \mathbb{R}$ is the space variable. The function φ denotes the transversal displacement of the beam from an equilibrium state, and ψ is the rotation angle of the filament of the beam, θ is the temperature difference of the solid elastic material. γ, β and κ are physically positive constants depending on the material elastic and thermal properties. The smooth function $\sigma(\eta)$ satisfies $\sigma'(\eta) > 0$ for any $\eta > 0$. In this chapter, we focus on the Cauchy problem to (1.20), hence initial conditions are prescribed as

$$(\varphi, \varphi_t, \psi, \psi_t, \theta)|_{t=0} = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)(x), \quad x \in \mathbb{R}. \tag{1.21}$$

The linearized version of (1.20) reads correspondingly

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \gamma\psi_t + \beta\theta_x = 0, \\ \theta_t - \kappa\theta_{xx} + \beta\psi_{tx} = 0, \end{cases} \tag{1.22}$$

where $a > 0$ denotes the sound speed defined by $a^2 = \sigma'(0)$. The case $a = 1$ corresponds to the Timoshenko-Fourier system with equal wave speeds, whereas the case $a \neq 1$ implies that the wave speeds of the first two equations are different.

In this thesis, we are interested in the Timoshenko-Fourier system (1.20), where the heat conduction is described by the classical Fourier law. In the whole space, by virtue of the change of variable in [85]

$$v = \varphi_x - \psi,\ u = \varphi_t,\ z = a\psi_x,\ y = \psi_t.$$

System (1.22) is rewritten as the first-order system of hyperbolic-parabolic type

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - a y_x = 0, \\ y_t - a z_x - v + \gamma y + \beta \theta_x = 0, \\ \theta_t - \kappa \theta_{xx} + \beta y_x = 0. \end{cases} \tag{1.23}$$

For simplicity, we set $U := (v, u, z, y, \theta)^\top$($\top$ transpose). The corresponding initial data are as follows

$$U|_{t=0} = (v_0, u_0, z_0, y_0, \theta_0)^\top(x) \tag{1.24}$$

with

$$v_0 = \varphi_{0,x} - \psi_0,\ u_0 = \varphi_1,\ z_0 = a\psi_{0,x},\ y_0 = \psi_1.$$

Furthermore, (1.23)-(1.24) are equivalent to the following vector form

$$\begin{cases} U_t + AU_x + LU = BU_{xx}, \\ U(x,0) = U_0(x), \end{cases} \tag{1.25}$$

where the coefficient matrices A, L and B are explicitly given by

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & -a & 0 & \beta \\ 0 & 0 & 0 & \beta & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa \end{pmatrix}.$$

Observe that matrices A is real symmetric, L is non-negative definite but non-symmetric and B is real symmetric.

It is convenient to rewrite (1.20)-(1.21) as the following Cauchy problem for the hyperbolic-parabolic system

$$\begin{cases} U_t + AU_x + LU = BU_{xx} + G(U)_x, \\ U(x,0) = U_0(x), \end{cases} \tag{1.26}$$

where $G(U) = (0,0,0,g(z),0)^\top(x)$ with $g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0)z/a := O(z^2)$ near $z = 0$.

To show the pointwise estimate of solutions to (1.25) in the Fourier space, taking the Fourier transform of (1.25) gives

$$\begin{cases} \hat{U}_t + (i\xi A + L + \xi^2 B)\hat{U} = 0, \\ \hat{U}(\xi,0) = \hat{U}_0(\xi). \end{cases} \tag{1.27}$$

The solution to (1.27) is given by $\hat{U}(\xi,t) = e^{t\hat{\Phi}(i\xi)}\hat{U}_0(\xi)$, where

$$\hat{\Phi}(\zeta) = -(L + \zeta A - \zeta^2 B), \quad \zeta \in \mathbb{C}. \tag{1.28}$$

The corresponding eigenvalue problem along with (1.27) is as follows:

$$\lambda\phi + (i\xi A + L + \xi^2 B)\phi = 0, \tag{1.29}$$

where we denote by $\lambda = \lambda(i\xi) \in \mathbb{C}$ the eigenvalue value. Said Houari-Kasimov [160] showed that the dissipative structure of (1.25) is characterized by

$$\operatorname{Re}\lambda(i\xi) \le \begin{cases} -c\eta_1(\xi), & \text{for} \quad a = 1; \\ -c\eta_2(\xi), & \text{for} \quad a \neq 1 \end{cases}$$

for some constant $c > 0$, where the dissipative rates $\eta_1(\xi) = \frac{\xi^2}{1+\xi^2}$ and $\eta_2(\xi) = \frac{\xi^2}{(1+\xi^2)^2}$, respectively. Furthermore, it follows that the same decay properties (1.40)-(1.41) are available for the Cauchy problem (1.23)-(1.24). In addition, they strengthened those decay properties with an additional rate of $t^{-\gamma/2}$ ($\gamma \in [0,1]$), by restricting the initial data in the integral space $L^{1,\gamma}(\mathbb{R})$. Such sharp decay rates were shown in [156] for the semilinear Timoshenko system. Regarding the case of $\gamma = 0$ in (1.23), it was firstly investigated by Said-Houari and Kasimov in [161], however, their dissipative rates are not optimal. Recently, Mori-Kawashima [134] have improved them and got

$$\operatorname{Re}\lambda(i\xi) \le \begin{cases} -c\widetilde{\eta}_1(\xi), & \text{for} \quad a = 1; \\ -c\widetilde{\eta}_2(\xi), & \text{for} \quad a \neq 1 \end{cases}$$

with $\widetilde{\eta}_1(\xi) = \frac{\xi^4}{(1+\xi^2)^2}$ and $\widetilde{\eta}_2(\xi) = \frac{\xi^4}{(1+\xi^2)^3}$, from which we see that the weaker dissipative mechanism occurs in the absence of damping $y = \psi_t$.

So far there are few stability results for the nonlinear Timoshenko-Fourier system (1.20)-(1.21). As a first step, we focus on the cases $\gamma > 0$ and $a \neq 1$.

1.2 Main results

In this thesis, we mainly study well-posedness and qualitative study to the above four mathematical systems, that is, Landau equation (1.1), Kac equation (1.8), Navier-Stokes-Maxwell equations (1.14) with (1.15) and Timoshenko-Fourier system (1.20). Our main results in this thesis are shown as follows.

1.2.1 Global existence for the inhomogeneous Landau equation

The Landau equation is not only interesting in itself, but also because we can hope to have through it a better understanding of the Boltzmann equation in the case when grazing collisions are not neglected. On the other hand, the Bony's para-product decomposition have been widely used in the study of fluid dynamics, see for example [26, 27, 32, 40, 43] and references therein, however, there are few results available that one applies the Besov space theory to the global existence of kinetic equations. The recent works [52, 145] are devoted to the Boltzmann equation. And there are few results concerning the global existence for the Landau equation in spatially critical Besov spaces. So it is very interesting to work a result for (1.1), since the collision operator between the Boltzmann equation and Landau equation are fundamentally different. As the first step, by using the spectral analysis on the nonlinear Landau operator, we investigate the Cauchy problem (1.2) with Maxwellian molecules ($\gamma = 0$) in this thesis and main result is shown as follows.

★ **Main Theorem A**

There exists a constant $\varepsilon_0 > 0$ such that if $g_0 \in \widetilde{L}^2_v(B^{3/2}_{2,1})$ and

$$\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \leq \varepsilon_0,$$

then the Cauchy problem (1.2) *admits a global solution satisfying*

$$g \in \widetilde{L}^\infty_t \widetilde{L}^2_v(B^{3/2}_{2,1}) \quad \textit{and} \quad \mathcal{L}^{\frac{1}{2}} g \in \widetilde{L}^2_t \widetilde{L}^2_v(B^{3/2}_{2,1}).$$

Moreover, if $F_0(x,v) = \mu(v) + \sqrt{\mu}(v) g_0(x,v) \geq 0$, then $F(t,x,v) = \mu(v) + \sqrt{\mu}(v) g(t,x,v) \geq 0$.

♦ We state some comments.

(1) Above norms of the Chemin-Lerner space will be rigorously defined in Chapter 2. The Chemin-Lerner space without involving the microscopic velocity was initiated by [30] to establish the global existence of solutions to the incompressible Navier-Stokes equations. This is the first global existence result of solutions to the inhomogeneous Landau equation in Besov space.

(2) Observe that the regularity index $s = 3/2$ that the Besov space is subjected to $B^{3/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, but the Sobolev space $H^{3/2}(\mathbb{R}^3)$ is not embedded into $L^\infty(\mathbb{R}^3)$, it thus is critical for the algebra with respect to the spatial variable.

(3) We remark that the Chemin-Lerner space $\widetilde{L}^{\infty}_t\widetilde{L}^2_v(B^{3/2}_{2,1})$ enjoys stronger topology than the usual mixed space $L^{\infty}_t L^2_v(B^{3/2}_{2,1})$, which allows to get a good control for nonlinear collision terms.

1.2.2 Smoothing effect for the inhomogeneous Kac equation

In this thesis, we consider the Gelfand-Shilov and Gevrey smoothing effect of solutions for the inhomogeneous non-cutoff Kac equation. Firstly, the definitions of Gelfand-Shilov space and Gevrey class is shown as follows.

Definition 1.1. *Let the two constants $\mu, \nu > 0$ satisfy $\mu + \nu \geq 1$. The Gevrey class $G^{\mu}(\mathbb{R}^d)$ is defined by*

$$G^{\mu} = \Big\{ f \in C^{\infty}(\mathbb{R}^d), \exists A, C \geq 0, \quad \forall \alpha \in \mathbb{N}^d, \quad \sup_{v\in\mathbb{R}^d} |\partial_v^{\alpha} f(v)| \leq C A^{|\alpha|}(\alpha!)^{\mu} \Big\}.$$

The Gelfand-Shilov spaces $S^{\mu}_{\nu}(\mathbb{R}^d)$ are defined by

$$\begin{aligned} S^{\mu}_{\nu}(\mathbb{R}^d) &= \Big\{ f \in C^{\infty}(\mathbb{R}^d), \exists A, C \geq 0, \quad \forall \alpha, \beta \in \mathbb{N}^d, \quad |\partial_v^{\alpha} f(v)| \leq C A^{|\alpha|}(\alpha!)^{\mu} e^{-\frac{1}{C}|v|^{1/\nu}} \Big\} \\ &= \Big\{ f \in C^{\infty}(\mathbb{R}^d), \exists A, C \geq 0, \quad \forall \alpha, \beta \in \mathbb{N}^d, \quad \sup_{v\in\mathbb{R}^d} |v^{\beta}\partial_v^{\alpha} f(v)| \leq C A^{|\alpha|+|\beta|}(\alpha!)^{\mu}(\beta!)^{\nu} \Big\} \end{aligned}$$

with $v \in \mathbb{R}^d$. These Gelfand-Shilov spaces $S^{\mu}_{\nu}(\mathbb{R}^d)$ may also be characterized as the spaces of Schwarz functions belonging to the Gevrey space $G^{\nu}(\mathbb{R}^d)$, whose Fourier transforms belong to the Gevrey space $G^{\mu}(\mathbb{R}^d)$. That is,

$$S^{\mu}_{\nu}(\mathbb{R}^d) = \Big\{ f \in \mathcal{S}(\mathbb{R}^d), \exists C \geq 0, \varepsilon > 0, \quad |f(v)| \leq C e^{-\varepsilon|v|^{1/\nu}}, \quad |\widehat{f}(\xi)| \leq C e^{-\varepsilon|\xi|^{1/\mu}} \Big\}$$

with $v \in \mathbb{R}^d, \xi \in \mathbb{R}^d$.

In particular, we notice that Hermite functions belong to the symmetric Gelfand-Shilov spaces $S^{1/2}_{1/2}(\mathbb{R})$. The symmetric Gelfand-Shilov spaces $S^{\mu}_{\mu}(\mathbb{R})$ can be given as:

Remark 1.1. The symmetric Gelfand-Shilov spaces $S^{\mu}_{\mu}(\mathbb{R})$ with $\mu \geq 1/2$ can be characterized through the decomposition into the Hermite basis $(e_n)_{n\geq 0}$ see e.g. [167] (Proposition 1.2)

$$\begin{aligned} f \in S^{\mu}_{\mu}(\mathbb{R}) &\Leftrightarrow f \in L^2(\mathbb{R}),\ \exists t_0 > 0,\ \|((f, e_n)_{L^2} \exp(t_0 n^{\frac{1}{2\mu}}))_{n\geq 0}\|_{l^2(\mathbb{N})} < +\infty \\ &\Leftrightarrow f \in L^2(\mathbb{R}),\ \exists t_0 > 0,\ \|e^{t_0\mathcal{H}^{1/2\mu}} f\|_{L^2} < +\infty, \end{aligned}$$

where $\mathcal{H} = -\Delta_v + \frac{v^2}{4}$ is the Harmonic Oscillator and $(e_n)_{n\geq 0}$ is Hermite basis given in Section 4.2.1.

For extensive expositions of the Gelfand-Shilov regularity theory, we refer the reader to the works [69, 68, 150, 167] and the references therein.

Recently, Lerner-Morimoto-Pravda-Starov-Xu [117] considered the spatially inhomogeneous non-cutoff Kac equation in the Sobolev space and showed that the Cauchy problem for the fluctuation around the Maxwellian distribution admitted $S^{1+\frac{1}{2s}}_{1+\frac{1}{2s}}$ Gelfand-Shilov regularity properties with respect to the velocity variable and $G^{1+\frac{1}{2s}}$ Gevrey regularizing properties with respect to the position variable. In [117], the authors conjectured that it remained still open to determine whether the regularity indices $1+\frac{1}{2s}$ is sharp or not. On the other hand, Duan-Liu-Xu [52] and Morimoto-Sakamoto [145] studied the Cauchy problem for the Boltzmann equation with the initial datum belonging to critical Besov space. Motivated by those works, we intend to study the inhomogeneous non-cutoff Kac equation in critical Besov space and then improve the Gelfand-Shilov regularizing properties with respect to the velocity variable.

Our main results are stated as follows.

★ Main Theorem B

Let $0 < T < +\infty$. We suppose that the collision cross section satisfies (1.10) *with $0 < s < 1$. There exists a constant $\varepsilon_0 > 0$ such that for all $g_0 \in \widetilde{L}^2_v(B^{1/2}_{2,1})$ satisfying*

$$\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \varepsilon_0,$$

then the Cauchy problem (1.11) *admits a unique weak solution $g \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$ satisfying*

$$\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \|\mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq c_0 e^T \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}$$

for some constant $c_0 > 1$. Furthermore, this solution is smooth for all positive time $0 < t \leq T$, which satisfies the following Gelfand-Shilov and Gevrey type estimates: For $\delta > 0$, there exists $C > 1$ such that for all $0 < t \leq T$ and for all $k, l, q \geq 0$,

$$\|v^k \partial_v^l \partial_x^q g(t)\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{C^{k+l+q+1}}{t^{\frac{3s+1}{2s(s+1)}(k+l+2)+\frac{2s+1}{2s}q+\delta}} (k!)^{\frac{3s+1}{2s(s+1)}} (l!)^{\frac{3s+1}{2s(s+1)}} (q!)^{\frac{2s+1}{2s}} \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

♦ Our result deserves some comments in contrast to the result of [117].

(1) We show the well-posedness of Cauchy problem with the initial datum belonging to the spatially critical Besov space $\widetilde{L}^2_v(B^{1/2}_{2,1})$, rather than in the Sobolev space $L^2_v(H^1_x)$.

(2) For the regularizing effect, our result indicates that

$$\forall t > 0,\ \forall x \in \mathbb{R},\quad g(t,x,\cdot) \in S^{\frac{3s+1}{2s(s+1)}}_{\frac{3s+1}{2s(s+1)}}(\mathbb{R});\quad \forall t > 0,\ \forall v \in \mathbb{R},\quad g(t,\cdot,v) \in G^{1+\frac{1}{2s}}(\mathbb{R}).$$

Actually, the Gelfand-Shilov index for the velocity variable is sharp for $0 < s < 1$, if noticing that

$$\frac{3s+1}{2s(s+1)} = \frac{2s+1}{2s}\frac{3s+1}{(2s+1)(s+1)} < 1 + \frac{1}{2s}.$$

(3) If s is close to 1, the solution is almost analytic in the velocity variable, since

$$\frac{3s+1}{2s(s+1)} \to 1.$$

Therefore, our Gelfand-Shilov index for the velocity variable should be optimal.

1.2.3 Optimal decay rate for Navier-Stokes-Maxwell equations

We study the global existence and decay rate of solutions for Navier-Stokes-Maxwell equations (1.14)-(1.15) in $\mathbb{R}^3$. To state our main results, set $w = (n, u, E, B)^\top$($\top$ transpose) and $w_0 = (n_0, u_0, E_0, B_0)^\top$, which are column vectors in $\mathbb{R}^{10}$. We define the following energy norm $N_0(t)$ and the corresponding dissipation norm $D_0(t)$:

$$\begin{aligned} N_0(t) &:= \sup_{0\le\tau\le t} \|(w-w_\infty)(\tau)\|_{H^s}, \\ D_0(t)^2 &:= \int_0^t (\|(n-n_\infty)(\tau)\|_{H^s}^2 + \|u(\tau)\|_{H^{s+1}}^2 + \|E(\tau)\|_{H^{s-1}}^2 + \|\nabla B(\tau)\|_{H^{s-2}}^2)d\tau. \end{aligned}$$

Since the dissipative mechanism of regularity loss, our aim is to seek the minimal decay regularity of solutions to the N-S-M equations (1.14)-(1.15). Firstly, we recall a notion of minimal decay regularity which is formulated by Xu-Mori-Kawashima [182] for generally dissipative systems of regularity-loss.

Definition 1.2. *If the optimal decay rate of $L^1(\mathbb{R}^d)$-$L^2(\mathbb{R}^d)$ type is achieved under the lowest regularity assumption, then the lowest index is called the minimal decay regularity for dissipative systems of regularity-loss, which is labeled as s_D.*

We show our main results as follows.

★ Main Theorem C

(1) The global-in-time existence of smooth solutions

Let $s \ge 3$ and suppose that the initial data satisfy $w_0 - w_\infty \in H^s$ and (1.17). Put $I_0 := \|w_0 - w_\infty\|_{H^s}$. Then there exists a constant $\varepsilon_0 > 0$ such that if $I_0 \le \varepsilon_0$, the Cauchy problem (1.14) and (1.16) has a unique global solution $w(t,x)$ with

$$\begin{aligned} &w - w_\infty \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1}), \\ &n \in L^2([0,\infty); H^s), u \in L^2([0,\infty); H^{s+1}), \\ &E \in L^2([0,\infty); H^{s-1}), \nabla B \in L^2([0,\infty); H^{s-2}). \end{aligned}$$

The solution satisfies the uniform energy estimate

$$N_0(t)^2 + D_0(t)^2 \lesssim I_0^2$$

for $t \geq 0$. Moreover, the solution $w(t,x)$ converges to the constant state w_∞ uniformly in $x \in \mathbb{R}^3$ as $t \to \infty$. More precisely, we have

$$\|(n - n_\infty, u, E)(t)\|_{W^{s-2,\infty}} \to 0 \quad as \quad t \to \infty,$$
$$\|(B - B_\infty)(t)\|_{W^{s-4,\infty}} \to 0 \quad as \quad t \to \infty,$$

where the asymptotic behavior for the magnetic field B holds true only by assuming the additional regularity $s \geq 4$.

(2) The optimal decay rates of smooth solutions

Let $s \geq 3$. Assume that the initial data satisfy $w_0 - w_\infty \in H^s \cap L^1$ and (1.17). Put $I_1 := \|w_0 - w_\infty\|_{H^s \cap L^1}$. Then there exists a constant $\varepsilon_1 > 0$ such that if $I_1 \leq \varepsilon_1$, the above global solution admits the decay estimate

$$\|w - w_\infty\|_{L^2} \lesssim I_1(1+t)^{-3/4}.$$

♦ We give some comments about the above results.

(1) Let us mention that the above result is of the regularity-loss type because we have 1-regularity loss for (E,B) in the dissipation part $D_0(t)$.

(2) From the result of decay estimate, we see that the $L^1(\mathbb{R}^3)$-$L^2(\mathbb{R}^3)$ decay rate of solutions is available with the critical regularity $s = 3$; that is, we arrive at $s_D \leq 3$ in the sense of Definition 1.2, and the extra regularity is not necessary, which improves previous works great, for example, [50, 169].

(3) Our decay rate coincides with those results for compressible N-S equations by Matsumura-Nishida [138] and Ponce [152].

1.2.4 Optimal decay rate for Timoshenko-Fourier system

To state our main results, define $N_0(t)$ and $D_0(t)$ by the following energy norm and the corresponding dissipation, respectively:

$$N_0(t) := \sup_{0\leq\tau\leq t} \|U(\tau)\|_{H^s(\mathbb{R})},$$
$$D_0(t)^2 := \int_0^t (\|v(\tau)\|^2_{H^{s-1}(\mathbb{R})} + \|\partial_x u(\tau)\|^2_{H^{s-2}(\mathbb{R})} + \|\partial_x z(\tau)\|^2_{H^{s-1}(\mathbb{R})} + \|y(\tau)\|^2_{H^s(\mathbb{R})} + \|\partial_x \theta(\tau)\|^2_{H^s(\mathbb{R})})d\tau.$$

Our main results are shown as follows.

★ Main Theorem D

(1) The global-in-time existence of smooth solutions

Let $s \geq 2$ and suppose that the initial data satisfy $U_0 \in H^s(\mathbb{R})$. Put $I_0 := \|U_0\|_{H^s(\mathbb{R})}$. Then

there exists a constant $\varepsilon_0 > 0$ such that if $I_0 \le \varepsilon_0$, the Cauchy problem (1.26) has a unique global solution $U(t,x)$ with

$$U \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})).$$

Moreover, the uniform energy inequality

$$N_0(t)^2 + D_0(t)^2 \lesssim I_0^2$$

holds for $t \ge 0$.
(2) The optimal decay rates of smooth solutions
Let $m = 0, 1$ and $s \ge \ell_0 + m + 1$ with $\ell_0 = m + 1$. Assume that the initial data satisfy $U_0 \in H^s \cap L^1(\Omega)$. The addition initial condition $\bar{z}_0 = 0$ is assumed in the periodic space $\Omega = \mathbb{T}$. Put $I_1 := \|U_0\|_{H^s \cap L^1(\Omega)}$. Then there exists a constant $\varepsilon_1 > 0$ such that if $I_1 \le \varepsilon_1$, the solution of (1.26) admits the decay estimate

$$\|\partial_x^m U\|_{L^2(\Omega)} \lesssim I_1 (1+t)^{-1/4 - m/2}$$

in case of $\Omega = \mathbb{R}, m = 0$ as well as in case of $\Omega = \mathbb{T}$ and $m = 0, 1$.

♦ Our result deserves the following comments.

(1) We mention that the energy inequality is of the regularity-loss type because there is 1-regularity loss for (v, u) in the dissipation part $D_0(t)$.

(2) The above result also holds in the periodic domain $\mathbb{T}$. It is not difficult to see that $\bar{z}(t) = 0$ for all $t > 0$, if $\bar{z}_0 = 0$ is additionally assumed, where $\bar{f} := \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} f(x)dx$.

(2) Due to the weaker dissipative mechanism of regularity-loss type, it is more interesting to seek the possibly lower regularity. By virtue of L^p-L^q-L^r inequalities which is given by Xu-Mori-Kawashima in [182], we could achieve the optimal algebraic rate of L^1-L^2 type under the critical regularity $s_c = 2$, that is, the minimal decay regularity $s_D \le 2$ follows for the Timoshenko-Fourier system.

1.3 Comparison with known results

In this section, we recall some known results to the above mathematical systems. Also, we state our motivations and innovation of results for these mathematical problems.

1.3.1 Well-posedness for Landau equation

Landau equation is a fundamental equation to describe collisions among charged particles interacting with their Coulombic force. The Landau equation is not only interesting in itself, but also because we can hope to have through it a better understanding of the Boltzmann

equation in the case when grazing collisions are not neglected. There are lots of known results concerning the well-posedness to the Landau equation and large-time behavior of its solutions.

■ **In spatially homogeneous case**

In 1989, DiPerna-Lions [51] proved the global existence of a so-called "re-normalized solution" of Boltzmann equation with angular cut-off and arbitrary initial data. Then, in 1994, Lions [107] proved similar results on the compactness properties of the Landau equation. Villani [173] used the above these two results to construct global weak solutions for the Coulombic interaction with $\gamma = -3$, up to some defect measures. Subsequently, in [175], he extended the result to the Landau equation without the presence of spatial dependence. In the Maxwellian molecules case $\gamma = 0$, Villani proved an exponential in time convergence to equilibrium. For the hard potential $\gamma \in (0, 1]$, Desvillettes-Villani [57, 58] investigated the existence, uniqueness and smoothness of classical solutions. They proved a functional inequality for entropy-entropy dissipation that is not linear, from which the polynomial in time convergence of solutions towards equilibrium was also shown.

Recently, Carrapatoso [23] proved the optimal exponential decay to equilibrium with the decay rate given by the spectral gap of the associated linearized operator, by using the method developed by Gualdani-Mischler-Mouhot [67]. Morimoto-Xu [148] proved the ultra-analytic effect for the Cauchy problem of homogeneous nonlinear Landau equation in the case of $\gamma = 0$. Li-Xu [122] studied the homogeneous nonlinear Landau operator by introducing the spectral analysis and proved the existence of weak solution for the Cauchy problem, provided that the initial datum belonging to Shubin space of negative index which contains the probability measures.

■ **In spatially inhomogeneous case**

Firstly, Guo [62] proved the global-in-time existence of classical solutions to the Landau equation in a period box by employing an energy method.

■ **Guo's main result in [62] for Landau equation in Sobolev space**

Let $\gamma \geq -3$ and $N \geq 8$. Assume $F_0(x, v) = \mu + \sqrt{\mu} g_0(x, v) \geq 0$. There is a constant $\varepsilon_0 > 0$ such that if

$$\sum_{|\alpha|+|\beta|\leq N} \|\partial_x^\alpha \partial_v^\beta g_0\|^2_{|\beta|} < \varepsilon_0, \quad \text{if } \gamma + 2 < 0;$$
$$\sum_{|\alpha|+|\beta|\leq N} \|\partial_x^\alpha \partial_v^\beta g_0\|^2 < \varepsilon_0, \quad \text{if } \gamma + 2 \geq 0,$$

then there exists a unique global solution $g(t, x, v)$ to the Landau equation (1.2) *with $F(t, x, v) = \mu + \sqrt{\mu} g(t, x, v) \geq 0$.*

Later, Hsiao-Yu [82] extended Guo's results [62] to the whole space with the same regularity requirement $N \geq 8$. Baranger-Mouhot [22] studied the explicit spectral gap estimates to the linearized Landau operator with hard potentials. Mouhot [131] established the coercivity estimates for a general class of interactions including hard potentials and soft potentials. There

are many other works on the Landau equation; see [22, 23, 28, 78, 80, 114, 121, 122, 146, 159, 174].

To the best of our knowledge, there are various studies concerning the well-posedness of solutions to the Boltzmann, see for example, [5, 6, 7, 9, 16, 63, 70, 82, 123, 124] and references therein. For the inhomogeneous Boltzmann equation, Alexandre-Morimoto-Ukai-Xu-Yang made a continuous work by introducing a new non-isotropic norm which captures the main feature of the singularity in the cross-section. This new norm is in fact the counterpart of the coercive norm which was introduced by Guo [62] as an essential step for Landau equation. In their series of works, they established the global existence of solutions in some (weighted) Sobolev space for the inhomogeneous Boltzmann equation near a global equilibrium and proved the regularity effects of solutions with respect to all variables, see [5, 6, 7, 9]. Precisely,

■ **Alexandre-Morimoto-Ukai-Xu-Yang's main results for Boltzmann equation without angular cut-off in some (weighted) Sobolev space**

• *Global existence result in [6] for case of Maxwellian molecules* $\gamma = 0$

Let $N \geq 3$ *and* $\ell \geq 3$. *Assume that the cross-section satisfies* (1.6) *with* $0 < s < 1/2$ *and* $g_0 \in H^N_\ell(\mathbb{R}^6)$ *satisfies*

$$G_0(x,v) = M(v) + \sqrt{M}(v) g_0(x,v) \geq 0.$$

Then there exists a constant $\varepsilon_0 > 0$ *such that if* $\|g_0\|_{H^N_\ell(\mathbb{R}^6)} \leq \varepsilon_0$, *the Cauchy problem* (1.7) *admits a unique global solution*

$$g \in L^\infty([0,\infty); H^N_\ell(\mathbb{R}^6)).$$

Moreover, $G(t,x,v) = M(v) + \sqrt{M}(v) g(t,x,v) \geq 0$ *and* $g \in C^\infty([0,\infty) \times \mathbb{R}^6)$.

• *Global existence result in [7] for case of hard potential* $\gamma + 2s > 0$

Let $N \geq 6$ *and* $\ell \geq 3/2 + 2s + \gamma$. *Assume that the cross-section satisfies* (1.6) *with* $0 < s < 1, \gamma + 2s > 0$ *and* $g_0 \in H^N_\ell(\mathbb{R}^6)$. *There exists a constant* $\varepsilon_0 > 0$ *such that if* $\|g_0\|_{H^N_\ell(\mathbb{R}^6)} \leq \varepsilon_0$, *then the Cauchy problem* (1.7) *admits a global solution*

$$g \in L^\infty([0,\infty); H^N_\ell(\mathbb{R}^6)).$$

• *Global existence result in [9] for case of soft potential* $\gamma + 2s \leq 0$

Let $N \geq 4$ *and* $\gamma > \max\{-3, -2s - 3/2\}$. *Assume that the cross-section satisfies* (1.6) *with* $0 < s < 1, \gamma + 2s \leq 0$ *and* $g_0 \in H^N(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))$. *There exists a small constant* $\varepsilon_0 > 0$ *such that if* $\|g_0\|_{H^N(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))} \leq \varepsilon_0$, *then the Cauchy problem* (1.7) *admits a global solution*

$$g \in L^\infty([0,\infty); H^N(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))).$$

Moreover, let $\ell \geq N$, *the Cauchy problem* (1.7) *admits a global solution*

$$g \in L^\infty([0,\infty); \widetilde{\mathcal{H}}^N_\ell(\mathbb{R}^6)).$$

Through the above results, we mention that the global existence of the Boltzmann equation and Landau equation are all proved in some (weighted) Sobolev space and require a high regularity index. On one hand, there are many results for fluid dynamics equations and kinetic equations by using the fundamental (or weighted) energy methods. On the other hand, the Bony's para-product decomposition have been widely used in the study of fluid dynamics, see for example [26, 27, 32, 40, 43] and references therein, however, there are few results available that one applies the Besov space theory to the global existence of kinetic equations. Very recently, Duan-Liu-Xu [52] first introduced the Chemin-Lerner type spaces (defined in Chapter 2) involving the microscopic velocity and established the global existence of strong solutions near Maxwellian for the cut-off Boltzmann equation with the hard potential ($\gamma > 0$). Subsequently, Morimoto-Sakamoto [145] extended Duan-Liu-Xu's result to the non-cutoff Boltzmann equation by using the triple norm that was introduced by Alexandre-Morimoto-Ukai-Xu-Yang [6, 9]. Their results are shown as follows.

■ **Global existence result for Boltzmann equation in critical Besov space**

• *Duan-Liu-Xu's main result in [52] for case of angular cut-off*

Assumed $0 \le \gamma \le 1$ *and* $0 \le B_0(\theta) \le C|\cos\theta|$. *There exists a constant* $\varepsilon_0 > 0$ *such that if* $\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \le \varepsilon_0$, *then the Cauchy problem* (1.7) *admits a unique global strong solution*

$$g(t,x,v) \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1}) \quad in \ \ \mathbb{R}^3$$

for $T > 0$. *Moreover, if* $G_0(x,v) = M(v) + \sqrt{M}(v)g_0(x,v) \ge 0$, *then* $G(t,x,v) = M(v) + \sqrt{M}(v)g(t,x,v) \ge 0$.

• *Morimoto-Sakamoto's main results in [145] for case of non-cutoff*

Assume $0 < s < 2$ *and* $\gamma > \max\{-3, -s - 3/2\}$. *There exists a constant* $\varepsilon_0 > 0$ *such that if* $\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \le \varepsilon_0$, *then the Cauchy problem* (1.7) *admits a unique global solution*

$$g(t,x,v) \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1}) \quad in \ \ \mathbb{R}^3$$

for $T > 0$.

Based on the above result, one may carry out the same proof for the angular cutoff case which formally corresponds to the limiting situation $\gamma > -3/2$ as $s \to 0$. Consequently, the existing approaches as in [52, 145] cannot be directly applied to treat the case of $-3 < \gamma \le -3/2$. Recently, Duan-Sakamoto [56] considered the Boltzmann equation near global Maxwellians in the d-dimensional whole space and obtained a unique global-in-time mild solution in a Chemin-Lerner type space with respect to the phase variable (x, v). Both hard and soft potentials with angular cutoff are considered. The new function space for global well-posedness is introduced to essentially treat the case of soft potentials, and the key point is that the velocity variable is taken in the weighted supremum norm, and the space variable is in the s-order Besov space with $s \ge d/2$ including the spatially critical regularity.

As pointed out in [52], the most interesting value of the index s under consideration is $d/2$. One may not expect to take this value if one seeks a solution in the usual Sobolev space $H^{d/2}$, since $H^{d/2}$ is not embedded into L^∞ while the Besov space $B^{d/2}_{2,1}$ is. In such sense, the regularity index $s = d/2$ is said to be spatially critical. There are few results concerning the global existence for the Landau equation in spatially critical Besov spaces. We refer readers to [12, 164] for applications of the Besov space to the kinetic theory from different perspectives. In this thesis, we investigate the Cauchy problem (1.2) with Maxwellian molecules ($\gamma = 0$) and obtain the global existence of solutions in spatially Besov space with critical regularity index (**Main Theorem A**). Our results improve the results in [62] and extend the global existence result for Boltzmann equation in [52, 145] to Landau equation.

1.3.2 Gelfand-Shilov and Gevrey smoothing effect for Kac equation

■ **Smoothing effect for homogeneous Boltzmann and Landau equation**

It is known that Boltzmann equation without angular cutoff can enjoy smoothing effects for solutions of the associated Cauchy problems. There are many results of C^∞ smoothing properties for the spatially homogeneous Boltzmann equation without angular cut-off, see [1, 13, 59, 65, 175] and references therein. These studies demonstrate that the singularity of the collision cross-section improves the regularity on weak solutions for the Cauchy problem. One can also obtain the C^∞ regularity of weak solutions for the spatially homogeneous Boltzmann operator without cut-off. The non-integrability of the cross section is essential for the smoothing effect, see for example [35].

In the local setting, the Gevrey regularity has been first proved in [168] for the initial data that has the same Gevrey regularity. A more general result on the Gevrey regularity is obtained in [147] for the spatially homogeneous linear Boltzmann equation with any initial Cauchy data. Hence, one sees a similar smoothness effect for the homogeneous Boltzmann equations as in the case of the heat equation. Recently, Barbaroux-Hundertmark-Ried-Vugalter [21] prove any weak solution of the fully non-linear homogenous Boltzmann equation for Maxwellian molecules belongs to the Gevrey class at positive time, and the Gevrey index therein is optimal. Recently, Li-Xu [121, 122] studied the nonlinear spatially homogeneous Boltzmann equation and Landau equation with the initial datum belonging to Shubin space, respectively, also obtained the corresponding Gelfand-Shilov smoothing effect by using the spectral decomposition.

■ **Smoothing effect for inhomogeneous Boltzmann and Landau equation**

For the spatially inhomogeneous Landau equation, Chen-Desvillettes-He [28] obtained the C^∞ smoothing effect of solutions under a mild regularity assumption on the initial data. Recently, Henderson-Snelson [81] and Snelson [159] showed C^∞ smoothing of bounded weak solutions where the pointwise Gaussian upper bound played a key role. Imbert-Mouhot-Silvestre [89] establish upper bounds for Boltzmann equation. For the study of the regularizing effect for the inhomogeneous Boltzmann equation, one of the main difficulties comes from the coupling

of the transport operator with the collision operator, which is similar to the Landau equation studied in [28]. Alexandre-Morimoto-Ukai-Xu-Yang [8] highlighted the importance of regularization effects for Boltzmann equation (see also [5, 11, 39, 55]). They studied C^∞ smoothing properties of the spatially inhomogeneous non-cutoff Boltzmann equation in [4, 5, 6, 8]. Also, in [3] they investigated a kinetic equation with the diffusion term as a non-linear function of the velocity variable and obtained a C^∞ regularity result for the linear spatially inhomogeneous Boltzmann equation without cut-off by using uncertainty principle and microlocal analysis.

Alexandre-Hérau-Li [2] obtained the global sharp estimate for the linearized Boltzmann operator rather than the model operators, using additionally symbolic calculus for the collisional cross-section; see [78, 80, 112] for the earlier works on the hypoelliptic properties of other related models. It is mentioned that the above works about hypoellipticity don't involve the initial data as a matter of fact the time variable is supposed to vary in the whole space. Hence, Fourier analysis can be applied when deriving the subelliptic estimates in time variable. Recently, Chen-Hu-Li-Zhan [29] considered Gevrey regularization effect for the spatially inhomogeneous Boltzmann equation without angular cutoff and obtained any solution with mild regularity will become smooth in Gevrey class at positive time, with Gevrey index depending on the angular singularity. Their proof depends on the symbolic calculus for the collision operator and the subelliptic estimates for the Cauchy problem of linearized Boltzmann equation, here the initial data is involved in the analysis.

■ **Regularity properties and analysis of results for Kac's equation**

In [114], Lerner-Morimoto-Pravda-Starov-Xu studied the linearized Landau and Boltzmann equation and proved that the linearized non-cutoff Boltzmann operator with Maxwellian is exactly equal to a fractional power of the linearized Landau operator which is the sum of the Harmonic Oscillator and the spherical Laplacian. In addition, Lekrine-Xu [120] investigated the Gevrey regularizing effect of the Cauchy problem for non-cutoff homogeneous Kac equation, a one-dimensional Boltzmann model. Later, Lerner-Morimoto-Pravda-Starov-Xu [115] considered the linearized non-cutoff Kac collision operator around the Maxwellian distribution and found that it behaved like a fractional power of the Harmonic Oscillator and was diagonal in the hermite basis. Moreover, it was shown in [116] by Lerner-Morimoto-Pravda-Starov-Xu that the Cauchy problem to the homogeneous non-cutoff Kac equation

$$\begin{cases} \partial_t g + \mathcal{K} g = \Gamma(g, g), \\ g|_{t=0} = g_0 \in L^2(\mathbb{R}_v) \end{cases}$$

enjoys the following Gefand-Shilov regularizing properties

$$\forall t > 0, \quad g(t) \in S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}).$$

The analysis of the Gevrey regularizing properties of spatially inhomogeneous kinetic equations with respect to both position and velocity variables is more complicated. Up to now, there

are few results expect for a very simplified model of the linearized inhomogeneous non-cutoff Boltzmann equation, say the generalized Kolomogorov equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + (-\Delta_v)^s g = 0, \\ g|_{t=0} = g_0 \in L^2(\mathbb{R}^{2d}_{x,v}) \end{cases} \tag{1.30}$$

with $0 < s < 1$. Morimoto-Xu [148] showed that the solution to (1.30) satisfied

$$\exists c > 0, \forall t > 0, \quad e^{c(t^{2s+1}(-\Delta_x)^s + t(-\Delta_v)^s)} g(t) \in L^2(\mathbb{R}^{2d}_{x,v}),$$

which implies that the generalized Kolomogorov equation enjoys a $G^{\frac{1}{2s}}(\mathbb{R}^{2d}_{x,v})$ Gevrey smoothing effect with respect to both position and velocity variables. The phenomenon of hypoellipticity arises from non-commutation and non-trivial interactions between the transport part $v \cdot \nabla_x$ and the diffusion part $(-\Delta_v)^s$ in this evolution equation, see [31, 113] for the further improvement on the exponent of subelliptic estimate. The operator $(-\Delta_v)^s$ is just a local model of Boltzmann equation, inspirited by the diffusion property in v velocity obtained by Alexandre-Desvillettes-Villani-Wennberg in [1]. On the other hand, for the Cauchy problem of the linear model of spatially inhomogeneous Landau equation,

$$\begin{cases} \partial_t g + v \cdot \nabla_x g = \nabla_v \left(\bar{a}(\mu) \cdot \nabla_v g - \bar{b}(\mu) g \right), \\ g|_{t=0} = g_0, \end{cases} \tag{1.31}$$

with

$$\begin{aligned} \bar{a}_{ij}(\mu) &= \delta_{ij}(|v|^2 + 1) - v_i v_j, \\ \bar{b}_j(\mu) &= -v_j, \quad i, j = 1, \cdots, d, \end{aligned}$$

they showed in [148] that the solution to (1.31) enjoyed a $G^1(\mathbb{R}^{2d}_{x,v})$ Gevrey smoothing effect with respect to both position and velocity variables with the estimate

$$\exists c > 0, \forall t > 0, \quad e^{c(t^2(-\Delta_x)^{\frac{1}{2}} + t(-\Delta_v)^{\frac{1}{2}})} g(t) \in L^2(\mathbb{R}^{2d}_{x,v}),$$

which coincides with the fact that the Landau equation can be regarded as the limit $s = 1$ of the Boltzmann equation. There are many other regularity studies for non-cutoff Kac equation; see [18, 35, 36, 38, 66] and references therein.

Recently, Lerner-Morimoto-Pravda-Starov-Xu [117] considered the spatially inhomogeneous non-cutoff Kac equation in Sobolev space and showed that the Cauchy problem for the fluctuation around the Maxwellian distribution admitted $S^{1+\frac{1}{2s}}_{1+\frac{1}{2s}}$ Gelfand-Shilov regularity properties with respect to the velocity variable and $G^{1+\frac{1}{2s}}$ Gevrey regularizing properties with respect to the position variable. Precisely,

■ **Lerner-Morimoto-Pravda-Starov-Xu's main results in [117]**

Let $0 < T < +\infty$. We assume that the collision cross section satisfies (1.10) with $0 < s < 1$. Then there exist some positive constants $\varepsilon_0 > 0, c_0 > 0$ such that for all $g_0 \in H^{1,0}(\mathbb{R}^2_{x,v})$ satisfying

$$\|g_0\|_{(1,0)} \leq \varepsilon_0,$$

the Cauchy problem associated to the spatially inhomogeneous Kac equation (1.11) admits a weak solution $g \in L^\infty([0,T]; H^{(1,0)}(\mathbb{R}^2_{x,v}))$ satisfying

$$\|g\|_{L^\infty([0,T];H^{(1,0)}(\mathbb{R}^2_{x,v}))} + \|\mathcal{H}^{\frac{s}{2}} g\|_{L^2([0,T];H^{(1,0)}(\mathbb{R}^2_{x,v}))} \leq c_0 e^T \|g_0\|_{(1,0)}$$

with $\mathcal{H} = -\Delta_v + \frac{v^2}{4}$. Furthermore, this solution is smooth for all time $0 < t \leq T$, and satisfies the Gelfand-Shilov and Gevrey type estimates:

1) $\exists C > 1,\ \forall 0 < t \leq T,\ \forall k \geq 0,$

$$\left\| \left(\mathcal{H}^{\frac{1}{2}} + \langle D_x \rangle \right)^k g(t) \right\|_{(1,0)} \leq \frac{C^{k+1}}{t^{\frac{2s+1}{2s}k}} (k!)^{\frac{2s+1}{2s}} \|g_0\|_{(1,0)},$$

2) $\forall \delta > 0,\ \exists C > 1,\ \forall 0 < t \leq T,\ \forall k, l, q \geq 0,$

$$\|v^k \partial_v^l \partial_x^q g(t)\|_{L^\infty(\mathbb{R}^2_{x,v})} \leq \frac{C^{k+l+q+1}}{t^{\frac{2s+1}{2s}(k+l+q+3)+\delta}} (k!)^{\frac{2s+1}{2s}} (l!)^{\frac{2s+1}{2s}} (q!)^{\frac{2s+1}{2s}} \|g_0\|_{(1,0)}.$$

The above results in [117], the authors conjectured that it remained still open to determine whether the regularity indices $1+\frac{1}{2s}$ is sharp or not. Duan-Liu-Xu [52] and Morimoto-Sakamoto [145] both studied the Cauchy problem for the Boltzmann equation with the initial datum belonging to critical Besov space. In this thesis, we intend to study the inhomogeneous non-cutoff Kac equation in critical Besov space and then improve the Gelfand-Shilov regularizing properties with respect to the velocity variable (**Main Theorem B**). Comparing with the result of [117], the well-posedness to the Kac equation is proved in critical Besov space $\widetilde{L}^2_v(B^{1/2}_{2,1})$ rather than in the Sobolev space $L^2_v(H^1_x)$, and our Gelfand-Shilov index for the velocity variable should be optimal.

1.3.3 Large time behavior for Navier-Stokes-Maxwell equations

Navier-Stokes-Maxwell equations (1.14)-(1.15) appear in the modelling of magnetized plasmas under the frequency-collision conditions. If there is no Lorentz force coupled to self-consistent electromagnetic fields, system (1.14) will reduce to the usual compressible Navier-Stokes (N-S) equations. There are many results for the global existence and large time behavior of classical solution for the N-S equations.

■ **Global existence of solutions for the N-S equations**

For the following Navier-Stokes equations in the context of gasdynamics,

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) = \mu \Delta u, \end{cases} \tag{1.32}$$

where the electromagnetic field is absent and thus the Maxwell system is decoupled from the fluid dynamic equations. The N-S equations is hyperbolic-parabolic type. It is essential to understand the interaction between the hyperbolic conservative part and the parabolic diffusive part. As in [92, 162, 177], this kind of interaction induced some weak smoothness and dissipation of the hyperbolic component of the coupled system. Kanel [90] firstly observed this property in the proof of global existence of solutions to (1.32) in $\mathbb{R}$. Later, in $\mathbb{R}^3$, Matsumura-Nishida [137, 138, 139] showed the global well-posedness of the near constant equilibrium solutions to the Cauchy problem or the initial boundary value problem by using the classical energy method. Since then, many works are devoted to consider the perturbation theory of the Navier-Stokes system, see [27, 32, 40, 64, 88, 132, 152].

■ **Decay estimates of solutions for the N-S equations**

To the best of our knowledge, the study of the dissipative property of the equations (1.32) is not only important for the global existence of solutions under small perturbations but also useful for obtaining the rates of convergence of solutions trending towards the equilibrium. The decay rate of solutions to the Cauchy problem of N-S equations has been extensively investigated since Matsumura-Nishida achieved the first global existence of small solutions in [137]. For the initial perturbation small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, Matsumura-Nishida [138] showed

$$\|(n - n_\infty, u)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}}.$$

Subsequently, Ponce [152] obtained the optimal L^p-decay rates

$$\|\nabla^k (n - n_\infty, u)\|_{L^p} \lesssim (1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \quad 2 \le p \le \infty, \quad 0 \le k \le 2.$$

For the optimal time decay of solutions, Kagei-Kobayashi [97, 98] proved the large time behavior of solutions over the half space in $\mathbb{R}^3$. Kobayashi-Shibata [102] and later Kobayashi [95] gave an analysis of the Green's function and large time behavior of solutions for the compressible N-S equations in an exterior domain of $\mathbb{R}^3$. By introducing the Sobolev space of negative order, Guo-Wang [71] developed new energy approaches with regularity interpolation technique and proved the optimal time-decay rates.

Let us emphasize that all the aforementioned works concern solutions with high Sobolev regularity. Danchin [40] obtained the global existence for the isentropic compressible Navier-Stokes equations in the L^2 critical regularity framework. Also, for the global existence results in the general L^p critical regularity framework, see Charve-Danchin [27], Chen-Miao-Zhang [32] and Haspot [73]. Okati [151] performed low- and high-frequency decompositions and showed the L^2-decay rate in the framework of spatially Besov spaces. In the survey paper [42], Danchin proposed another description of the time decay which allows to handle dimension $d \ge 2$ in the L^2 critical framework. Later, Danchin-Xu [43] developed the method of [42] and established the optimal decay results in the general L^p critical Besov space in any dimension $d \ge 2$.

■ **Decay estimates of solutions for the general hyperbolic-parabolic systems**

Kawashima [92, 94] and Shizuta-Kawashima [162] established a general approach for obtaining the optimal time decay of solutions in L^p space with $p \geq 2$. In [92, 94], Kawashima applied the Fourier analysis to the linearized homogeneous system and constructed some compensation function to capture the dissipation of the hyperbolic component in the solution and then obtained an estimate on the Fourier transform $\hat{U}$ of the solution U as

$$|\hat{U}(t,\xi)| \lesssim e^{-\frac{c|\xi|^2}{1+|\xi|^2}}|\hat{U}_0| \tag{1.33}$$

for any t and ξ. From (1.33), it is well known the solution over the high frequency domain decays exponentially while over the low frequency domain it decays polynomially with the rate of the heat kernel. Besides, in [162], Shizuta-Kawashima proposed the Shizuta-Kawashima condition in order to assure the time decay of solutions to linear systems of equations of the hyperbolic-parabolic type. This condition played an important role when considering the stability and large time behavior of the solution to the nonlinear system; see Hanouzet-Natalini [79], Yong [184], Bianchini-Hanouzet-Natalini [20] and references therein. For the time decay in L^p space with $1 \leq p \leq 2$, Zeng [186] provided a complete analysis of the Green matrix with some sharp, pointwise bounds for the one dimensional viscous heat-conductive fluid system. As an extension of [186], see Liu-Zeng [126] in the case of zero heat conductivity and Liu-Zeng [125, 127] for the general hyperbolic-parabolic systems of conservation laws in $\mathbb{R}$. Besides, with aid of detailed analysis on the Green function, Hoff-Zumbrun [83, 84] relaxed the value of p as $1 \leq p \leq \infty$ and established the same optimal L^p-decay rates of solutions towards diffusion waves.

■ **A coupled system of the hyperbolic-parabolic type**

For the following compressible Navier-Stokes-Poisson system

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) = -nE + \mu \Delta u, \\ E = \nabla \Phi, \ \nabla \cdot E = n_\infty - n. \end{cases}$$

Compared with the Navier-Stokes-Maxwell system, the magnetic field B is omitted and the electric field E is generated by a consistent potential function Φ therefore satisfying the Poisson equation

$$\Delta \Phi = n_\infty - n.$$

Note that the compressible Navier-Stokes-Poisson system is a coupled system of the hyperbolic-parabolic-elliptic type and it has some special dissipative and time decay properties. Li-Matsumura-Zhang [118] made a delicate analysis of the linearized Navier-Stokes-Poisson system and investigated the spectrum of the linear semigroup in terms of the decomposition of wave modes at lower frequency and higher frequency respectively. For the sufficiently small initial perturbation in $H^s \cap L^1$ with s properly large, they showed that the solutions in $\mathbb{R}^3$ holds the

following optimal decay rates

$$\|n-n_\infty\|_{L^2}\lesssim(1+t)^{-\frac{3}{4}},\qquad \|u\|_{L^2}\lesssim(1+t)^{-\frac{1}{4}},$$
$$\|E\|_{L^2}\lesssim(1+t)^{-\frac{1}{4}},$$

for $t\geq 0$. Later, Wang-Wu [179] obtained the pointwise estimates of solutions to the Navier-Stokes-Poisson system through the analysis of the Green's function; see Hao-Li [76], Hsiao-Li [77], Zhang-Li-Zhu [187], Zhang-Tan [188] and the references therein for more results.

For the magnetohydrodynamic system in [17, 34], in the isentropic case, it is written as

$$\begin{cases}\partial_t n+\nabla\cdot(nu)=0,\\ \partial_t(nu)+\nabla\cdot(nu\otimes u)+\nabla p(n)=(\nabla\times B)\times B+\mu\Delta u,\\ \partial_t B+\nabla\times(\nabla\times B-u\times B)=0,\quad \nabla\cdot B=0.\end{cases}$$

The above system simulates the interaction between the viscous fluid and the variable magnetic field. Kawashima-Okada [101] and Kawashima [93] proved the global existence of smooth solutions near constant equilibrium states in $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Later, Umeda-Kawashima-Shizuta [171] considered a rather general class of symmetric hyperbolic-parabolic systems:

$$A^0 z_t+\sum_{j=1}^n A^j z_{x_j}+Lz=\sum_{j,k=1}^n B^{jk}z_{x_jx_k},\tag{1.34}$$

where $z=z(t,x)$ is an $\mathbb{R}^m$-valued function, and $A^j(j=0,1,2\cdots,n)$, L and $B^{jk}(j,k=1,2,\cdots,n)$ are real constant matrices of order m. With certain dissipative assumptions (say, Shizuta-Kawashima condition), they showed the same L^2-decay rate as that for linearized N-S equations. As a matter of fact, the general decay framework could be well applied to not only N-S equations but also lots of other equations of fluid dynamics, see, e.g., [171]. Chen-Tan [33] and Zhang-Zhao [189] showed the time decay rates for the nonlinear system magnetohydrodynamic equation.

For the fluid and kinetic systems in the presence of the electromagnetic field, firstly, about the following Euler-Maxwell system

$$\begin{cases}\partial_t n+\nabla\cdot(nu)=0,\\ \partial_t(nu)+\nabla\cdot(nu\otimes u)+\nabla p(n)=-n(E+u\times B)-nu,\\ \partial_t E-\nabla\times B=nu,\\ \partial_t B+\nabla\times E=0\end{cases}\tag{1.35}$$

with constraints

$$\nabla\cdot E=n_\infty-n,\quad \nabla\cdot B=0.\tag{1.36}$$

Duan [48], Ueda-Wang-Kawashima [172] and Ueda-Kawashima [169] proved some global existence and decay estimates of solutions as follows.

■ **Global existence and decay estimates for Euler-Maxwell system**

• *Duan's results in [48]*

Let $s \geq 4$ and (1.36) holds. Then there is a constant $\varepsilon_0 > 0$ such that if

$$\|(n_0 - 1, u_0, E_0, B_0)\|_{H^s} \leq \varepsilon_0,$$

the Cauchy problem (1.35)-(1.36) with relaxation admits a unique global solution (n, u, E, B) with

$$(n, u, E, B) \in C([0, \infty); H^s(\mathbb{R}^3)) \cap Lip([0, \infty); H^{s-1}(\mathbb{R}^3)).$$

• *Ueda-Wang-Kawashima's results in [172]*

Let $s \geq 3$ and suppose that the initial data satisfy $(n_0 - n_\infty, u_0, E_0, B_0 - B_\infty) \in H^s$. Then there exists a constant $\varepsilon_0 > 0$ such that if

$$\|(n_0 - n_\infty, u_0, E_0, B_0 - B_\infty)\|_{H^s} \leq \varepsilon_0,$$

the Cauchy problem (1.35)-(1.36) has a unique global solution $(n(t,x), u(t,x), E(t,x), B(t,x))$ with

$$(n, u, E, B) \in C([0, \infty); H^s(\mathbb{R}^3)) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}^3)).$$

Moreover, the solution (n, u, E, B) converges to the constant state $(n_\infty, 0, 0, B_\infty)$ uniformly in $x \in \mathbb{R}^3$ as $t \to \infty$. More precisely,

$$\begin{aligned} &\|(n - n_\infty, u, E)\|_{W^{s-2,\infty}} \to 0 \qquad as \qquad t \to \infty, \\ &\|B - B_\infty\|_{W^{s-4,\infty}} \to 0 \qquad as \qquad t \to \infty, \end{aligned}$$

where the asymptotic convergence of B holds true only by assuming the additional regularity $s \geq 4$.

• *Ueda-Kawashima's results in [169]*

Let $s \geq 6$ and assume that the initial data satisfy $(n_0 - n_\infty, u_0, E_0, B_0 - B_\infty) \in H^s \cap L^1$. Put $I_0 = \|(n_0 - n_\infty, u_0, E_0, B_0 - B_\infty)\|_{H^s \cap L^1}$. Then there exists a constant $\varepsilon_0 > 0$ such that if $I_0 \leq \varepsilon_0$, the global solution constructed in [172] satisfies the time weighted energy estimate and admits the following decay estimates

$$\begin{aligned} &\|\partial_x^k (n - n_\infty, u, E, B - B_\infty)(t)\|_{H^{s-2k-3}} \lesssim I_0 (1+t)^{-3/4-k/2}, \\ &\|\partial_x^k (n - n_\infty, u, E)\|_{H^{s-2k-5}} \lesssim I_0 (1+t)^{-5/4-k/2}, \\ &\|\partial_x^k (n - n_\infty)\|_{H^{s-2k-7}} \lesssim I_0 (1+t)^{-7/4-k/2}, \end{aligned}$$

where taking $0 \leq s \leq [\frac{s-1}{2}] - 1$, $0 \leq s \leq [\frac{s-1}{2}] - 2$ and $0 \leq s \leq [\frac{s-1}{2}] - 3$ with $s \geq 7$ in the three inequalities, respectively.

Duan-Strain [54] showed the optimal convergence rates of solutions near equilibriums to the Vlasov-Maxwell-Boltzmann system (V-M-B) in the two-species case. In the one-species case,

Duan [49] proved the global existence of classical solutions to the V-M-B system under small perturbation. Masmoudi [132] proved the global existence and exponential convergence rate of regular solutions to the Maxwell-Navier-Stokes system over two dimensional bounded domain. Later, Duan [50] proved the global existence and decay rates of solutions in the whole space as follows.

■ **Duan's results in [50] for the compressible N-S-M equations**

Let $s \geq 4$. Set $\rho = n - n_\infty, v = \frac{n_\infty}{\sqrt{p'(n_\infty)}}, \bar{E} = \frac{n_\infty}{p'(n_\infty)}^{1/2} E, \bar{B} = \frac{n_\infty}{p'(n_\infty)}^{1/2} B$. Assume that initial data $(\rho_0, v_0, \bar{E}_0, \bar{B}_0) \in H^s$ and satisfies the norm $\|(\rho_0, v_0, \bar{E}_0, \bar{B}_0)\|_{H^s}$ is sufficiently small. Then the Cauchy problem of the N-S-M equations (that is, the system without the damping term nu in (1.14)*) admits a unique global solution $(\rho, v, \bar{E}, \bar{B})$ with*

$$
\begin{aligned}
&(\rho, v, \bar{E}, \bar{B}) \in C([0,\infty); H^s(\mathbb{R}^3));\\
&\rho \in L^2([0,\infty); H^s(\mathbb{R}^3)), \quad \nabla v \in L^2([0,\infty); H^s(\mathbb{R}^3));\\
&\nabla \bar{E} \in L^2([0,\infty); H^{s-2}(\mathbb{R}^3)), \quad \nabla^2 \bar{B} \in L^2([0,\infty); H^{s-3}(\mathbb{R}^3)).
\end{aligned}
$$

Moreover, if $\|(\rho_0, v_0, \bar{E}_0, \bar{B}_0)\|_{H^{s+2}\cap L^1}$ is sufficiently small, then the obtained solution $(\rho, v, \bar{E}, \bar{B})$ satisfies

$$
\begin{aligned}
&\|\rho\|_{L^2} \lesssim (1+t)^{-1}, \quad \|v\|_{L^2} \lesssim (1+t)^{-\frac{5}{8}},\\
&\|\bar{E}\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}} \ln(3+t), \quad \|\bar{B}\|_{L^2} \lesssim (1+t)^{-\frac{3}{8}}
\end{aligned}
$$

for any $t \geq 0$.

Also, Duan [50] obtained pointwise estimate of the linearized N-S-M system by using the Green's function and found that it admits the same time decay property with the one-species linearized V-M-B system.

From the dissipative structure, the above some systems admit the regularity-loss type, such as the compressible Euler-Maxwell system in [48, 169, 172] and V-M-B system in [54]; see other many dissipative systems, quasilinear hyperbolic systems of viscoelasticity in [53], hyperbolic-elliptic systems of radiating gas in [75, 99], dissipative Timoshenko system in [85, 87, 111, 141], and a plate equation with rotational inertia effect in [44, 108, 109, 163]. Back to the results in [171], the dissipation matrices L and $B^{jk}(j,k = 1,\cdots,n)$ are both assumed to be real symmetric. In this setting, the typical feature of the time-decay property of solutions is that the low-frequency part of solutions decays polynomially at the rate of the heat kernel, while the high-frequency part has a better exponential decay. However, so far there are few general results in case that L and B^{jk} are not necessary symmetric, which is one of our motivation to study the N-S-M equations (1.14).

On the other hand, since for the dissipative systems with the regularity-loss type, the extra regularity will be posted than that for global smooth solutions, if the optimal time-decay rate $L^1(\mathbb{R}^d)$-$L^2(\mathbb{R}^d)$ is expected; see the results in [20, 49, 50, 87, 103, 169, 184].

Thanks to the following key $L^p(\mathbb{R}^d)$-$L^q(\mathbb{R}^d)$-$L^r(\mathbb{R}^d)$ estimates, which was recently established by Xu-Mori-Kawashima [182] for a class of dissipative systems of regularity-loss type:

■ $L^p(\mathbb{R}^d)$-$L^q(\mathbb{R}^d)$-$L^r(\mathbb{R}^d)$ **estimates in [182]**

Let $\eta(\xi)$ be a positive, continuous and real-valued function in $\mathbb{R}^d$ satisfying

$$\eta(\xi) \sim \begin{cases} |\xi|^{\sigma_1}, & |\xi| \to 0; \\ |\xi|^{-\sigma_2}, & |\xi| \to \infty \end{cases}$$

for $\sigma_1, \sigma_2 > 0$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$, it holds that

$$\begin{aligned} & \|\mathcal{F}^{-1}[|\xi|^k e^{-\eta(\xi)t}|\hat{\phi}(\xi)|]\|_{L^p} \\ \lesssim\ & \underbrace{(1+t)^{-\gamma_{\sigma_1}(q,p)-\frac{k-j}{\sigma_1}}\|\partial_x^j \phi\|_{L^q}}_{Low-frequency\ Estimate} + \underbrace{(1+t)^{-\frac{\ell}{\sigma_2}+\gamma_{\sigma_2}(r,p)}\|\partial_x^{k+\ell}\phi\|_{L^r}}_{High-frequency\ Estimate} \end{aligned} \tag{1.37}$$

for $\ell > d(\frac{1}{r}-\frac{1}{p})$ [1], $1 \le q, r \le 2 \le p \le \infty$ and $0 \le j \le k$, where $\gamma_\sigma(q,p) := \frac{n}{\sigma}(\frac{1}{q}-\frac{1}{p})(\sigma > 0)$. In [182], Xu-Mori-Kawashima applied the above Lemma to Euler-Maxwell system and obtained the optimal decay rate $t^{-3/4}$ as $t \to \infty$, provided that the initial data are in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Thus, our another motivation to study the for the N-S-M system (1.14) of regularity-loss type is to obtain the optimal decay estimates with the minimal decay regularity. In this thesis, we obtain the global smooth solution to the N-S-M system (1.14) in $\mathbb{R}^3$ and optimal decay estimate with the minimal regularity index $s = 3$ in Sobolev space, which improved the result of Duan [50].

1.3.4 Decay estimates of solutions for Timoshenko-Fourier system

In 1920s, Timoshenko [165, 166] derived the original model which consists of a coupled system of two wave equations (*i.e.*, without thermal effect in (1.20)). The Timoshenko system consists of two coupled wave equations with non-symmetric relaxation, and describes vibrations of the beam with shear deformation and rotational inertia effect. Normally, if the relaxation is not symmetric, the dissipation is produced through the complicated interaction of the components of the system, and the dissipative structure is of regularity-loss type.

■ **Global existence and Decay property for Timoshenko systems**

The subject of stability of Timoshenko systems has received much attention in the last years, and quite a number of results concerning uniform and asymptotic decay of solutions have been established, see for instance [133, 136, 141, 142, 144] and references therein. In a bounded domain, Timoshenko system is exponentially stable if the damping term φ_t is also present on the left-hand side of the first equation, see, e.g., [153]. Soufyane [158] showed that Timoshenko system could not be exponentially stable by considering only the damping term of the form ψ_t, unless for the case of $a = 1$ (equal wave speeds). A similar result was obtained by

[1] We would like to remark that $\ell \ge 0$ in the case of $p = r = 2$.

Rivera-Racke [141] with an alternative proof. In the whole space, Ide-Haramoto-Kawashima [85] performed the linear analysis for the Timoshenko system, and observed that the linearized solution $\widetilde{U}$ decayed at different rates in cases of $a = 1$ and $a \neq 1$. Precisely, one has

$$\|\partial_x^k \widetilde{U}(t)\|_{L^2(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}} \|\widetilde{U}_0\|_{L^1(\mathbb{R})} + e^{-ct} \|\partial_x^k \widetilde{U}_0\|_{L^2(\mathbb{R})} \tag{1.38}$$

for $a = 1$ and

$$\|\partial_x^k \widetilde{U}(t)\|_{L^2(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}} \|\widetilde{U}_0\|_{L^1(\mathbb{R})} + (1+t)^{-\frac{l}{2}} \|\partial_x^{k+l} \widetilde{U}_0\|_{L^2(\mathbb{R})} \tag{1.39}$$

for $a \neq 1$, where k and l are non-negative integers. Clearly, it verifies that the decay property (1.39) is of regularity-loss type in comparison with (1.38), since $(1+t)^{-l/2}$ is created by assuming the additional ℓ-th order regularity. Consequently, extra initial regularity is usually posed for the nonlinear Timoshenko system if the optimal decay rate could be expected, see for example [87]. Ide-Kawashima proved the global existence of smooth solutions and achieved the decay estimates of L^1-L^2 type in $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with suitably larger s ($s \geq 7$). Very recently, Xu-Mori-Kawashima [183] got the similar stability results as the initial data in $B^{3/2}_{2,1}(\mathbb{R}) \cap \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$, that is, the regularity assumption was reduced heavily. Additionally, the interested reader is also referred to [136, 142] for the Timoshenko systems with frictional damping, to [110, 111, 133, 135, 144] for Timoshenko systems with memory effect and to [143, 155] for Timoshenko system with Cattaneo's law.

■ **Some results of solutions for Timoshenko system with Cattaneo's law or Fourier's law**

Racke-Said Houari [155] considered the following Timoshenko-Cattaneo system in $\mathbb{R}$

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - [\sigma(\psi_x)]_x - (\varphi_x - \psi) + \lambda\psi_t + \beta\theta_x = 0, \\ \theta_t \tilde{q}_x + \beta\psi_{tx} = 0, \\ \tau_0 \tilde{q}_t + \tilde{q} + \kappa\theta_x = 0 \end{cases}$$

for a small parameter $\tau_0 \in (0, 1]$. They proved the global existence and the decay estimate of solutions provided with the small initial data in $H^8 \cap L^1$. Here, the high regularity index is to control the weak dissipation. Recently, Mori-Racke [143] improved the results of [155]. They obtained the global existence in H^2 by energy methods without any negative weights and proved the optimal decay estimate in $H^2 \cap L^1$ by using the L^p-L^q-L^r estimates. The original Timoshenko system consists of the first two equations with $\lambda = \beta = 0$ and $\theta = \tilde{q} = 0$ which was founded by Timoshenko [165, 166] to describe the vibration of the alleged Timoshenko beams. Also, the last two equations with $\beta = 0$ represent the heat conduction described by Cattaneo's law, which is the first-order approximation of Fourier's law ($\tilde{q} + \kappa\theta_x = 0$) with a time-delay effect $\tilde{q}(t + \tau_0) + \kappa\theta_x = 0$. Formally letting $\tau_0 \to 0$, it has $\tilde{q} = -\kappa\theta_x$, which gives the Timoshenko-Fourier system (1.20) with parabolic heat conduction.

To the best of our knowledge, few stability efforts can be available for Timoshenko-Fourier system (1.20). In a bounded domain, Rivera-Racke [140] proved several exponential decay results for the linearized version (1.22) of Timoshenko-Fourier system (1.20) and found a non-exponential stability for the case of different wave speeds. Mori-Kawashima [134] studied similar linearized Timoshenko-Fourier system, that is, removing the term $\gamma\psi_t$ in the second equation of (1.22), they obtained the optimal decay estimates by using the pointwise estimates of solutions in the Fourier space and a refinement of the energy method. Namely,

■ **Mori-Kawashima's optimal decay estimates for Timoshenko-Fourier system in [134]**

The solution U of the linearized Timoshenko-Fourier system(removing the term $\gamma\psi_t$ in the second equation of (1.22)) satisfies the following decay estimates for $t \geq 0$,

$$\|\partial_x^k U(t)\|_{L^2(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^p(\mathbb{R})} + e^{-ct}\|\partial_x^k U_0\|_{L^2(\mathbb{R})} \tag{1.40}$$

for $a = 1$ and

$$\|\partial_x^k U(t)\|_{L^2(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^p(\mathbb{R})} + (1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} U_0\|_{L^2(\mathbb{R})} \tag{1.41}$$

for $a \neq 1$, where $1 \leq p \leq 2$, k and l are non-negative integers.

Also, Mori-Kawashima [134] proved that its dissipative structure is more weaker that that of Timoshenko system. There is no global existence results and decay estimates for the nonlinear system of Timoshenko-Fourier system studied in [134]. Since Timoshenko-Fourier system is the nonsymmetric hyperbolic-parabolic system, the general theory of symmetric hyperbolic-parabolic systems established by Umeda-Kawashima-Shizuta [171] can not be applied to the Timoshenko-Fourier system with regularity-loss type, which is one of our motivations to study the Timoshenko-Fourier system. On the other hand, since the weak dissipative structure, seeking the minimal decay regularity in Sobolev space corresponding to the optimal decay rate is our another aim of considering the Timoshenko-Fourier system.

In this thesis, we study the nonlinear Timoshenko-Fourier system (1.20) and obtain the global existence of smooth solutions in H^2. Also we show the optimal decay rate of L^1-L^2 type provided that the initial data are in $H^s \cap L^1$ with $s \geq 2$. Here, we also obtain the decay estimate of first derivative order of solutions provided that the initial data are in $H^s \cap L^1$ with $s \geq 3$ in the periodic space.

Chapter 2

Preliminaries

Contents

In this chapter, we recall some definitions of functional spaces and its fundamental analytical tools, see [19, 30, 37, 41, 46, 61, 74, 75, 86, 96, 100, 99, 130, 128, 149, 154, 185] for more details.

2.1 Notations

- Let X be an interval in $[0,\infty)$ and Y be a Banach space over $\mathbb{R}(\mathbb{R}^3)$. Then, for a nonnegative integer k, $C^k(X;Y)$ denotes the space of k-times continuously differential functions on X with values in Y.

- $\mathcal{S}$ stands for the Schwartz space of smooth functions over $\mathbb{R}^d$ whose derivatives of all order decay at infinity. The space $\mathcal{S}$ is endowed with the topology generated by the following family of semi-norms:

$$\|u\|_{M,\mathcal{S}} := \sup_{\substack{x\in\mathbb{R}^d \\ |\alpha|\leq M}} (1+|x|)^M |\partial^\alpha u(x)| \quad \text{for all } u\in\mathcal{S} \quad \text{and} \quad M\in\mathbb{N}.$$

- The set $\mathcal{S}'(\mathbb{R}^d)$ of temperate distributions is the dual set of $\mathcal{S}(\mathbb{R}^d)$ for the usual pairing.

- For any $u \in \mathcal{S}$, the Fourier transform of u denoted by $\hat{u}$ or $\mathcal{F}[u]$ is defined by

$$\forall \xi \in \mathbb{R}^d, \hat{u}(\xi) = \mathcal{F}[u](\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} u(x) dx.$$

  The Fourier transform maps $\mathcal{S}$ into and onto itself, and the inverse Fourier transform is given by the formula

$$\mathcal{F}^{-1}u(x) = \bar{\mathcal{F}}u(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} u(\xi) d\xi.$$

- The Fourier transform is extended by duality to the whole $\mathcal{S}'$ by setting

$$\langle \hat{u}, \phi \rangle := \langle u, \hat{\phi} \rangle_{\mathcal{S}',\mathcal{S}}$$

  whenever $u \in \mathcal{S}'$ and $\phi \in \mathcal{S}$.

- Derivatives: for all multi-index $\alpha \in \mathbb{R}^d$, we have

$$\mathcal{F}[\partial_x^\alpha u] = (i\xi)^\alpha \mathcal{F}[u] \quad \text{and} \quad \mathcal{F}[x^\alpha u] = (-i)^{|\alpha|} \partial_\xi^\alpha \mathcal{F}[u].$$

- Algebraic properties: for $(u, v) \in \mathcal{S} \times \mathcal{S}'$, we have $u \star v \in \mathcal{S}'$ and

$$\mathcal{F}[u \star v] = \mathcal{F}[u]\mathcal{F}[v].$$

  The above formula also holds true for couples of distributions with compact supports.

- Multipliers: if A is a smooth function with polynomial growth at infinity and $u \in \mathcal{S}'(\mathbb{R}^d)$, then we set $A(D)u := \mathcal{F}^{-1}(A\mathcal{F}[u])$.

- $\langle \cdot, \cdot \rangle$ denotes the standard inner product in the complex vector space $\mathbb{C}^d$ $(d \geq 1)$.

- The open ball (resp. closed) with radius R centered at $x_0 \in \mathbb{R}^d$ is denoted by $\mathbf{B}(x_0, R)$(resp.$\overline{\mathbf{B}}(x_0, R)$).

- The shell $\{\xi \in \mathbb{R}^d | R_1 \leq |\xi| \leq R_2\}$ is denoted by $\mathbf{C}(0, R_1, R_2)$.

- The notation $A \lesssim B$ means $A \leq CB$ for some "irrelevant" constant C. $A \approx B$ means $A \lesssim B$ and $B \lesssim A$ simultaneously.

- For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{N}^d$, we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} \quad \text{and} \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

- For $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \mathbb{N}^d, \beta \leq \alpha$ stands for $\beta_j \leq \alpha_j$ for $j = 1, 2, \cdots, d$, and $\beta < \alpha$ stands for $\beta < \alpha$ and $\beta \neq \alpha$. The Leibniz formulas:

$$\partial^\alpha(uv) = u\partial^\alpha v + \sum_{\beta<\alpha} C_\alpha^\beta \partial^{\alpha-\beta} u \partial^\beta v, \quad \forall \alpha \in \mathbb{N}^d,$$

  where C_α^β for $\beta < \alpha$ are constants.

- On $\mathbb{R}^3$, the differential operators $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\Delta = \sum_{j=1}^{3} \partial_{x_j}^2$ denote the standard gradient and Laplacian with respect to $x = (x_1, x_2, x_3)$, respectively. Moreover, the differential operators $\mathrm{div}(\nabla\cdot)$ and $\mathrm{rot}(\nabla\times)$ are defined by

$$\nabla \cdot u = \sum_{j=1}^{3} \partial_{x_j} u_j, \qquad \nabla \times u = (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)$$

for vector function $u = (u_1, u_2, u_3)$.

2.2 Littlewood-Paley theory

One of the most important roles of Littlewood-Paley theory is to localize the frequency space. Fourier transform transforms differential operations in physical space into algebraic operations in frequency space. Littlewood-Paley decomposition writes the temperate distribution form as the countable sums of smooth functions that are almost orthogonal in the sense of frequency space. The advantage of this localization method is that for the distributions whose Fourier transforms are supported in a ball or an annulus, it can make full use of Bernstein estimates to realize the algebraication of the derivation or differential operation. Littlewood-Paley theory is fundamental tools for researching nonlinear evolution equation. The Littlewood-Paley theory, precisely, dyadic partition of unity and Bony's decomposition are introduced as follows. The reader is also referred to [19] for more details.

2.2.1 Dyadic partition of unity

Now, let us introduce a dyadic partition of unity in $\mathbb{R}^d$.

Proposition 2.1. *([19]) Let (φ, χ) be a couple of smooth functions valued in the closed interval $[0,1]$ such that φ is supported in the shell $\mathbf{C}(0, \frac{3}{4}, \frac{8}{3}) = \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\}$ and χ is supported in the ball $\mathbf{B}(0, \frac{4}{3}) = \left\{\xi \in \mathbb{R}^d : |\xi| \le \frac{4}{3}\right\}$. In terms of the two functions, it holds that*

$$\chi(\xi) + \sum_{q \ge 0} \varphi\left(2^{-q}\xi\right) = 1, \qquad \forall \xi \in \mathbb{R}^d, \tag{2.1}$$

$$\sum_{q \in \mathbb{Z}} \varphi\left(2^{-q}\xi\right) = 1, \qquad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \tag{2.2}$$

$$\mathrm{Supp}\ \varphi(2^{-p}\cdot) \cap \mathrm{Supp}\ \varphi(2^{-q}\cdot) = \emptyset \qquad \textit{if} \quad |p-q| \ge 2, \tag{2.3}$$

$$\mathrm{Supp}\ \chi \cap \mathrm{Supp}\ \varphi(2^{-q}\cdot) = \emptyset \qquad \textit{if} \quad q \ge 1. \tag{2.4}$$

If $\widetilde{\mathbf{C}} := \mathbf{B}(0, \frac{2}{3}) + \mathbf{C}$*, then* $\widetilde{\mathbf{C}}$ *is a ring and it holds that*

$$2^p\widetilde{\mathbf{C}} \cap 2^q\mathbf{C} = \emptyset, \qquad \text{if} \quad |p-q| \geq 5, \tag{2.5}$$

$$\frac{1}{3} \leq \chi(\xi)^2 + \sum_{q\geq 0} \varphi^2\left(2^{-q}\xi\right) \leq 1, \qquad \forall \xi \in \mathbb{R}^d, \tag{2.6}$$

$$\frac{1}{2} \leq \sum_{q\in\mathbb{Z}} \varphi^2\left(2^{-q}\xi\right) \leq 1, \qquad \forall \xi \in \mathbb{R}^d \setminus \{0\}. \tag{2.7}$$

Proof. Let us take $\alpha \in (1, \frac{4}{3})$ and denote $\mathbf{C}' = \left\{\xi \in \mathbb{R}^d : \alpha^{-1} \leq |\xi| \leq 2\alpha\right\}$. We choose a smooth function θ valued in the closed interval $[0,1]$ such that θ is supported in $\mathbf{C}$ with value 1 in the neighbourhood of $\mathbf{C}'$. Firstly, we prove (2.3). If we suppose that $2^p\mathbf{C} \cap 2^q\mathbf{C} \neq \emptyset$ and that $p \geq q$. It turns out that $2^p \times \frac{3}{4} \leq 2^q \times \frac{8}{3}$ which implies that $p - q \leq 1$. That is,

$$|p-q| \geq 2 \Rightarrow 2^p\mathbf{C} \cap 2^q\mathbf{C} = \emptyset, \tag{2.8}$$

which gives (2.3). We set

$$S(\xi) = \sum_{q\in\mathbb{Z}} \theta(2^{-q}\xi).$$

This sum is locally finite on the space $\mathbb{R}^d \setminus \{0\}$ due to (2.8). Hence, the function S is smooth on this space. As α is greater than 1, one has

$$\bigcup_{q\in\mathbb{Z}} 2^q\mathbf{C}' = \mathbb{R}^d \setminus \{0\}.$$

Since the function θ is nonnegative and has value 1 near $\mathbf{C}'$, it comes from the above property that the above function is positive. Then we put

$$\varphi(\xi) = \frac{\theta(\xi)}{S(\xi)}.$$

Now let us check that φ fits. It is obviously that φ is smooth and Supp $\varphi \subset \mathbf{C}$. Seeing that the support of θ is included in $\mathbf{C}$, if $|\xi \geq \frac{4}{3}$, we have

$$\sum_{q\in\mathbb{Z}} \varphi\left(2^{-q}\xi\right) = \sum_{q\in\mathbb{Z}} \frac{\theta(2^{-q}\xi)}{S(2^{-q}\xi)} = \frac{\sum_{q\in\mathbb{Z}} \theta(2^{-q}\xi)}{\sum_{q\in\mathbb{Z}} \theta(2^{-q}\xi)} = 1, \tag{2.9}$$

where we have used the fact $S(2^{-q}\xi) = \sum_{q\in\mathbb{Z}} \theta(2^{-(p+q)}\xi) = \sum_{q_1\in\mathbb{Z}} \theta(2^{-q_1}\xi) = S(\xi)$. Since (2.8), the function $1 - \sum_{q\geq 0} \varphi\left(2^{-q}\xi\right)$ is smooth. Set

$$\chi(\xi) = 1 - \sum_{q\geq 0} \varphi\left(2^{-q}\xi\right)$$

and Supp $\chi = \mathbf{B}(0, \frac{4}{3})$ by using (2.8) again. (2.4) is a obvious consequence of (2.8) and (2.9). Hence, (2.1)-(2.4) is proved. It is clear that the ring $\widetilde{\mathbf{C}}$ is the ring of center 0, of small radius $\frac{1}{12}$ and of big radius $\frac{10}{3}$. Then it turns out that

$$2^p\widetilde{\mathbf{C}} \cap 2^q\mathbf{C} \neq \emptyset \Rightarrow (\frac{3}{4} \times 2^q \leq 2^p \times \frac{10}{3} \text{ or } \frac{1}{12} \times 2^p \leq 2^q \times \frac{8}{3}),$$

which implies (2.5). As χ and φ have their values in $[0,1]$, it is obvious that

$$\chi^2(\xi)+\sum_{q\geq 0}\varphi^2\left(2^{-q}\xi\right)\leq 1.$$

Let us give the lower bound for the sum of squares. Since

$$1=\Big(\chi(\xi)+\sum_{k\in\mathbb{N}}\sum_{q=2k}\varphi\left(2^{-q}\xi\right)+\sum_{k\in\mathbb{N}}\sum_{q=2k+1}\varphi\left(2^{-q}\xi\right)\Big)^2,$$

then it comes that

$$1\leq 3\Big(\chi^2(\xi)+\sum_{k\in\mathbb{N}}\sum_{q=2k}\varphi^2\left(2^{-q}\xi\right)+\sum_{k\in\mathbb{N}}\sum_{q=2k+1}\varphi^2\left(2^{-q}\xi\right)\Big).$$

With (2.3), we have

$$1=\Big(\sum_{q\in\mathbb{Z}}\varphi\left(2^{-q}\xi\right)\Big)^2=\sum_{q\in\mathbb{Z}}\varphi^2\left(2^{-q}\xi\right)+2\sum_{q\in\mathbb{Z}}\varphi\left(2^{-q}\xi\right)\varphi\left(2^{-(q+1)}\xi\right)\leq 2\sum_{q\in\mathbb{Z}}\varphi^2\left(2^{-q}\xi\right).$$

Hence, the proof of Proposition 2.1 is completed. □

More generally, the Fourier transform of a tempered distribution $u\in\mathcal{S}'(\mathbb{R}^d_x)$ is defined by the dual argument in the standard way. Then the nonhomogeneous dyadic blocks of $u=u(x)\in\mathcal{S}'(\mathbb{R}^d_x)$ are defined by

$$\begin{aligned}
&\Delta_q u\triangleq 0 \ \text{ if } \ q\leq -2;\\
&\Delta_{-1}u\triangleq\chi(D)u=\Psi\star u=\int_{\mathbb{R}^d}\Psi(y)u(x-y)dy \ \text{ with } \ \Psi\triangleq\mathcal{F}^{-1}\chi;\\
&\Delta_q u\triangleq\varphi(2^{-q}D)u=2^{qd}\Phi(2^q\cdot)\star u=\Phi_q\star u\\
&\qquad=2^{qd}\int_{\mathbb{R}^d}\Phi(2^qy)u(x-y)dy \ \text{ with } \ \Phi\triangleq\mathcal{F}^{-1}\varphi,\quad \Phi_q(x)=2^{qd}\Phi(2^qx),\quad q\geq 0,
\end{aligned}\tag{2.10}$$

where $\star$ is the convolution with respect to the variable x and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

The Littlewood-Paley decomposition of a general tempered distribution $u\in\mathcal{S}'(\mathbb{R}^d_x)$ reads as the following lemma.

Lemma 2.1. *For $u\in\mathcal{S}'(\mathbb{R}^d_x)$, we have $u=\sum_{q\geq -1}\Delta_q u$, in sense of $\mathcal{S}'(\mathbb{R}^d_x)$.*

Proof. For any $\phi\in\mathcal{S}(\mathbb{R}^d_x)$, one has $\hat{\phi}(\xi)=(\chi(\xi)+\sum_{q\geq 0}\varphi\left(2^{-q}\xi\right))\hat{\phi}(\xi)$ in the sense of $\mathcal{S}(\mathbb{R}^d_x)$,

then for any $u \in \mathcal{S}'(\mathbb{R}^d_x)$, it follows from the Parseval formula that

$$\begin{aligned}(u,\phi) &= (2\pi)^{-d}(\hat{u},\hat{\phi}) = (2\pi)^{-d}\Big(\hat{u}(\xi), \Big(\chi(\xi) + \sum_{q\geq 0}\varphi\left(2^{-q}\xi\right)\Big)\hat{\phi}(\xi)\Big)\\ &= (2\pi)^{-d}\Big(\Big(\chi(\xi)\hat{u}(\xi), \hat{\phi}(\xi)\Big) + \sum_{q\geq 0}\Big(\varphi(2^{-q}\xi)\hat{u}(\xi), \hat{\phi}(\xi)\Big)\Big)\\ &= (2\pi)^{-d}\Big(\Big(\chi(\xi) + \sum_{q\geq 0}\varphi\left(2^{-q}\xi\right)\Big)\hat{u}(\xi), \hat{\phi}(\xi)\Big)\\ &= \Big(\sum_{q\geq -1}\Delta_q u, \phi\Big).\end{aligned}$$

This proves the Lemma 2.1. □

It is also convenient to introduce the following low-frequency cut-off:

$$S_q u = \sum_{p\leq q-1}\Delta_p u,$$

where $S_0 u = \Delta_{-1}u$ for $q = 0$, and $S_q u = 0$ in the case of $q \leq -1$. Because $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ for all $\xi \in \mathbb{R}^d$, it holds that

$$S_q u = \chi(2^{-q}D)u = 2^{qd}\int_{\mathbb{R}^d}\Psi(2^q y)u(x-y)dy \quad \text{for all} \quad q \in \mathbb{N}. \tag{2.11}$$

Let $\mathcal{P}$ be the class of all polynomials on $\mathbb{R}^d_x$ and denote by $\mathcal{S}'(\mathbb{R}^d_x)/\mathcal{P}$ the tempered distributions modulo polynomials. Therefore, the homogeneous dyadic blocks $\dot{\Delta}_q$ are defined as follows:

$$\dot{\Delta}_q u \triangleq \varphi(2^{-q}D)u = 2^{qd}\Phi(2^q\cdot)\star u = \Phi_q \star u = 2^{qd}\int_{\mathbb{R}^d}\Phi(2^q y)u(x-y)dy$$

for any $u \in \mathcal{S}'(\mathbb{R}^d_x)/\mathcal{P}$ and $q \in \mathbb{Z}$, where $\Phi_q(x) = 2^{qd}\Phi(2^q x)$. And for any $u \in \mathcal{S}'(\mathbb{R}^d_x)/\mathcal{P}$, we have

$$u = \sum_{q\in\mathbb{Z}}\dot{\Delta}_q u.$$

2.2.2 Bony's decomposition

When dealing with nonlinear problems, one often has to study the functional properties of products of two temperate distributions u and v. Characterizing distributions such that the product uv makes sense is an intricate question which is intimately related to the notion of wavefront (see e.g [10] for an elementary introduction).

In this section, we shall see that very simple arguments based on the use of Littlewood-Paley decomposition yield sufficient conditions for uv to be defined, and continuity results for the map $(u,v) \mapsto uv$.

For two temperate distributions u and v, we have the following formal decomposition:

$$uv = \sum_{p,p'} \Delta_{p'} u \Delta_p v.$$

The fundamental idea of paradifferential calculus is to split uv into three parts, both of them being always defined. The first part, denoted by $\mathbf{T}_u v$ corresponds to the low frequencies of u multiplied by the high frequencies of v. The second term, $\mathbf{T}_v u$ is the symmetric counterpart of $\mathbf{T}_u v$. The third and last term (the remainder term) corresponds to the dyadic blocks of u and v with comparable frequencies.

This split device goes back to the pioneering work by Bony in [15]. In the following, we adopt the following definition for paraproduct and remainder:

Definition 2.1. *Let u and v be two temperate distributions. We denote*

$$\mathbf{T}_u v = \sum_{p' \leq p-2} \Delta_{p'} u \Delta_p v = \sum_p S_{p-1} u \Delta_p v$$

and

$$\mathbf{R}(u,v) = \sum_{|p-p'|\leq 1} \Delta_{p'} u \Delta_p v,$$

where the above operators $\mathbf{T}$ and $\mathbf{R}$ are called "paraproduct" and "remainder", respectively.

At least formally, we have the following Bony's decomposition:

$$uv = \mathbf{T}_u v + \mathbf{T}_v u + \mathbf{R}(u,v).$$

Of course, it may happen that the product uv is not defined. However, the reader may retain the following principles:

• The paraproduct of two temperate distributions u and v is always defined. This is due to the fact that the general term of the paraproduct is spectrally localized in dyadic shells. Besides, the regularity of $\mathbf{T}_u v$ is mainly determined by the regularity of v. In particular, $\mathbf{T}_u v$ cannot be more regular than v.

• The remainder may not be defined. Roughly, it is defined as soon as u and v belong to functional spaces whose sum of regularity index is positive. In that case, the regularity exponent of $\mathbf{R}(u,v)$ is the sum of the regularity exponents of u and v.

2.3 Functional spaces

In this section, we introduce some function spaces.

2.3.1 Functional spaces with one variable

Firstly, we give the definition of Lebesgue space for all L^p integrable functions.

Definition 2.2. *$L^p(\mathbb{R}^d)(1 \le p < \infty)$ denotes the space of measurable functions whose p-th powers are integrable on $\mathbb{R}^d$, with the norm*

$$\|u\|_{L^p} = \Big(\int_{\mathbb{R}^d} |u(x)|^p dx \Big)^{1/p}.$$

$L^\infty(\mathbb{R}^d)$ denotes the space of bounded measurable functions on $\mathbb{R}^d$, with the norm

$$\|u\|_{L^\infty} = \operatorname{ess.\,sup}_{x \in \mathbb{R}^d} |u(x)|.$$

Next, let us recall how nonhomogeneous Sobolev spaces H^s are defined.

Definition 2.3. *For $s \in \mathbb{R}$. A tempered distribution u belongs to $H^s(\mathbb{R}^d)$ if $\hat{u} \in L^2_{loc}(\mathbb{R}^d)$ and*

$$\|u\|_{H^s} \triangleq \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty.$$

It is classical that H^s endowed with the norm $\|\cdot\|_{H^s}$ is a Banach space and note that $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

Definition 2.4. *For a nonnegative integer s, the s-th order Sobolev space $W^{s,p}(\mathbb{R}^d)$ is defined as*

$$W^{s,p}(\mathbb{R}^d) = \Big\{ u \in L^p(\mathbb{R}^d) | \partial^\alpha u \in L^p(\mathbb{R}^d), |\alpha| \le s, \quad with$$
$$\|u\|_{W^{s,p}} = \Big(\sum_{|\alpha|=0}^{s} \|\partial^\alpha u\|^p_{L^p} \Big)^{1/p} < \infty \Big\}.$$

When $s = 0, W^{0,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$; when $p = 2$, $W^{s,2}(\mathbb{R}^d)$ is denoted as $H^s(\mathbb{R}^d)$, the norm of $u \in H^s(\mathbb{R}^d)$ is also defined as

$$\|u\|_{H^s} \triangleq \Big(\sum_{|\alpha|=0}^{s} \|\partial^\alpha u\|^2_{L^2} \Big)^{1/2}.$$

In addition, we also define some weighted functional spaces $L^p_\ell(\mathbb{R}^d_x), H^s_\ell(\mathbb{R}^d_x)$. For $p \ge 1, \ell \in \mathbb{R}$, one defines

$$\|u\|_{L^p_\ell} = \left(\int_{\mathbb{R}^d} |\langle x \rangle^\ell u(x)|^p dx \right)^{1/p}$$

and for $s \in \mathbb{R}$

$$\|u\|_{H^s_\ell} = \left(\int_{\mathbb{R}^d} |\langle D_x \rangle^s (\langle x \rangle^\ell u(x))|^2 dx \right)^{1/2},$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $\langle D_x \rangle = (1 - \Delta_x)^{1/2}$.

Then, let us focus on Hölder spaces.

Definition 2.5. *Let $r \in (0,1)$. We denote by C^r the set of bounded functions $u : \mathbb{R}^d \to \mathbb{R}$ such that there exists $C \geq 0$ with*

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,\ |u(x) - u(y)| \leq C|x-y|^r.$$

More generally, if $r > 0$ is not an integer, we denote by C^r the set of $[r]$ times differentiable functions u such that $\partial^\alpha u \in C^{r-[r]}$ for all $|\alpha| \leq r$.

The characterizations of Sobolev and Hölder spaces naturally lead to the following definition of Besov spaces.

Definition 2.6. *Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. For $1 \leq r < \infty$, the nonhomogeneous Besov space $B^s_{p,r}$ is defined by*

$$B^s_{p,r} := \Big\{ u \in \mathcal{S}'(\mathbb{R}^d_x) :\ u = \sum_{q \geq -1} \Delta_q u \quad in \quad \mathcal{S}', \quad with$$
$$\|u\|_{B^s_{p,r}} \triangleq \left(\sum_{q \geq -1} \left(2^{qs} \|\Delta_q u\|_{L^p_x}\right)^r \right)^{1/r} < \infty \Big\},$$

where in the case $r = \infty$ we set

$$\|u\|_{B^s_{p,\infty}} = \sup_{q \geq -1} 2^{qs} \|\Delta_q u\|_{L^p_x} < \infty.$$

Definition 2.7. *Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. For $1 \leq r < \infty$, the homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by*

$$\dot{B}^s_{p,r} := \Big\{ u \in \mathcal{S}'/\mathcal{P} :\ u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \quad in \quad \mathcal{S}'/\mathcal{P}, \quad with$$
$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \left(\sum_{q \in \mathbb{Z}} \left(2^{qs} \|\dot{\Delta}_q u\|_{L^p_x}\right)^r \right)^{1/r} < \infty \Big\},$$

where in the case $r = \infty$ we set

$$\|u\|_{\dot{B}^s_{p,\infty}} = \sup_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p_x} < \infty.$$

2.3.2 Functional spaces with three variables

For the kinetic equations, such as Boltzmann equation, Landau equation, there is three variable of function, that is, time t, position x and velocity v. Here, we define functional space with three variables t, x, v. For the distribution $u = u(t,x,v)$, we define the Banach space valued function space.

$$L_T^{p_1}L_v^{p_2}L_x^{p_3} \triangleq L^{p_1}([0,T];L^{p_2}(\mathbb{R}_v^d;L^{p_3}(\mathbb{R}_x^d)))$$

for $0<T\leq\infty, 1\leq p_1,p_2,p_3\leq\infty$, where the norm is given by

$$\|u\|_{L_T^{p_1}L_v^{p_2}L_x^{p_3}}=\left(\int_0^T\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|u(t,x,v)|^{p_3}\,dx\right)^{p_2/p_3}dv\right)^{p_1/p_2}dt\right)^{1/p_1}$$

with the usual convention if $p_1,p_2,p_3=\infty$. We also define the following velocity weighted norm

$$\|u\|_{L_T^{p_1}L_{v,w}^{p_2}L_x^{p_3}}=\left(\int_0^T\left(\int_{\mathbb{R}^d}w(v)^{p_2}\left(\int_{\mathbb{R}^d}|u(t,x,v)|^{p_3}\,dx\right)^{p_2/p_3}dv\right)^{p_1/p_2}dt\right)^{1/p_1}$$

We present the definition of the Chemin-Lerner type space, which were initialed in [30].

Definition 2.8. *Let $s\in\mathbb{R}$ and $1\leq\varrho_1,\varrho_2,p,r\leq\infty$. For $0<T\leq\infty$, the space $\widetilde{L}_T^{\varrho_1}\widetilde{L}_v^{\varrho_2}(B_{p,r}^s)$ is defined by*

$$\widetilde{L}_T^{\varrho_1}\widetilde{L}_v^{\varrho_2}(B_{p,r}^s)=\left\{u(t,x,v)\in\mathcal{S}':\|u\|_{\widetilde{L}_T^{\varrho_1}\widetilde{L}_v^{\varrho_2}(B_{p,r}^s)}<\infty\right\},$$

where

$$\|u\|_{\widetilde{L}_T^{\varrho_1}\widetilde{L}_v^{\varrho_2}(B_{p,r}^s)}=\left(\sum_{q\geq-1}\left(2^{qs}\|\Delta_q u\|_{L_T^{\varrho_1}L_v^{\varrho_2}L_x^p}\right)^r\right)^{1/r}$$

with the usual convention for $\varrho_1,\varrho_2,p,r=\infty$. Similarly, one also denotes

$$\|u\|_{\widetilde{L}_T^{\varrho_1}\widetilde{L}_v^{\varrho_2}(\dot{B}_{p,r}^s)}=\left(\sum_{q\in\mathbb{Z}}\left(2^{qs}\|\dot{\Delta}_q u\|_{L_T^{\varrho_1}L_v^{\varrho_2}L_x^p}\right)^r\right)^{1/r}$$

with the usual convention for $\varrho_1,\varrho_2,p,r=\infty$.

2.4 Fundamental inequalities

2.4.1 A few properties of Besov spaces

Lemma 2.2. *([19, 41]) Let $k\in\mathbb{N}$ and (R_1,R_2) satisfy $0<R_1<R_2$. There exists a constant $C>0$ depending only on R_1,R_2,d such that for all $1\leq a\leq b\leq\infty$ and $u\in L^a$, we have*

$$\text{Supp }\hat{u}\subset\mathbf{B}(0,R_1\lambda)\ \Rightarrow\ \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b}\leq C^{k+1}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},\tag{2.12}$$

$$\text{Supp }\hat{u}\subset\mathbf{C}(0,R_1\lambda,R_2\lambda)\ \Rightarrow\ C^{-k-1}\lambda^k\|u\|_{L^a}\leq\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a}\leq C^{k+1}\lambda^k\|u\|_{L^a}.\tag{2.13}$$

Proof. Arguing by rescaling with a dilation of size λ, one can assume with no loss of generality that $\lambda = 1$. Now, fix a smooth function ϕ compactly supported and such that $\phi \equiv 1$ in a neighborhood of the ball $\mathbf{B}(0, R_1)$. We notice that $\hat{u} = \phi\hat{u}$. Hence, denoting $g := \mathcal{F}^{-1}\phi$, we get for all multi-index α

$$\partial^\alpha u(x) = \int_{\mathbb{R}^d} \partial^\alpha g(y) u(x-y)dy = \partial^\alpha g \star u.$$

Taking advantage of Young inequality, we thus get

$$\|\partial^\alpha g \star u\|_{L^b} \leq \|\partial^\alpha g\|_{L^c}\|u\|_{L^a} \quad \text{with} \quad \frac{1}{c} = 1 + \frac{1}{b} - \frac{1}{a}.$$

Since for $1 \leq c \leq \infty$,

$$\begin{aligned}
\|\partial^\alpha g\|_{L^c} &\leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \\
&\leq C\|(1+|x|^2)^d \partial^\alpha g\|_{L^\infty} \\
&\leq C\|(Id-\Delta)^d((i\xi)^\alpha \phi)\|_{L^1} \\
&\leq C^{k+1},
\end{aligned}$$

the proof of the first inequality (2.12) is completed. To prove the second inequality, we first notice that the inequality on the right is a particular case of the first inequality. Next, we introduce a smooth function $\widetilde{\varphi}$ with compact support in $\mathbb{R}^d \setminus \{0\}$ and such that $\widetilde{\varphi} \equiv 1$ in a neighborhood of the shell $\mathbf{C}(0, R_1, R_2)$. Using the following algebraic identity

$$|\xi|^{2k} = (\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2)^k = \sum_{1 \leq j_1, \cdots, j_k \leq d} \xi_{j_1}^2 \cdots \xi_{j_k}^2 = \sum_{|\alpha|=k} (i\xi)^\alpha (-i\xi)^\alpha,$$

we have

$$\sum_{|\alpha|=k} \frac{(i\xi)^\alpha(-i\xi)^\alpha}{|\xi|^{2k}} = 1.$$

Giving the definition $g_\alpha := \mathcal{F}^{-1}((-i\xi)^\alpha |\xi|^{-2k}\widetilde{\varphi}(\xi))$, we write as $\hat{u} = \widetilde{\varphi}\hat{u}$ so that

$$\hat{u} = \sum_{|\alpha|=k} \frac{(-i\xi)^\alpha}{|\xi|^{2k}} \widetilde{\varphi}(\xi)(i\xi)^\alpha \hat{u}(\xi) = \sum_{|\alpha|=k} \hat{g}_\alpha(\xi)(i\xi)^\alpha \hat{u}(\xi),$$

which implies that

$$u = \sum_{|\alpha|=k} g_\alpha \star \partial^\alpha u.$$

Making use of Young inequality, one can now conclude to the left inequality in (2.13). Hence, It ends the proof of Lemma 2.2. □

As a direct corollary of the above inequality, one has

Remark 2.1. Let $s \in \mathbb{R}$ and $1 \le p, r \le \infty$. It holds that

$$\frac{1}{C}\|u\|_{\dot{B}^{s+|\alpha|}_{p,r}} \le \|\partial^\alpha u\|_{\dot{B}^{s}_{p,r}} \le C\|u\|_{\dot{B}^{s+|\alpha|}_{p,r}}, \qquad \alpha \in \mathbb{R},$$
$$\|\partial^\alpha u\|_{B^{s}_{p,r}} \le C\|u\|_{B^{s+|\alpha|}_{p,r}}, \qquad \alpha \ge 0,$$

for some constant $C > 0$.

The Littlewood-Paley decomposition is "almost" orthogonal.

Lemma 2.3. *([19, 41]) For any $u \in \mathcal{S}'(\mathbb{R}^d)$ and $v \in \mathcal{S}'(\mathbb{R}^d)$, the following properties hold:*

$$\Delta_p\Delta_q u \equiv 0, \qquad \text{if} \quad |p-q| \ge 2, \tag{2.14}$$
$$\Delta_q(S_{p-1}u\Delta_p v) \equiv 0, \qquad \text{if} \quad |p-q| \ge 5. \tag{2.15}$$

Proof. Proving (2.14) is equivalent to prove

$$|p-q| \ge 2 \Rightarrow \text{Supp } \varphi(2^{-p}\cdot) \cap \text{Supp } \varphi(2^{-q}\cdot) = \emptyset.$$

Then, (2.14) is given with (2.3) in Proposition 2.1. The proof of (2.15) is equivalent to prove

$$\text{Supp } \varphi(2^{-q}\cdot) \cap (\text{Supp } \varphi(2^{-p}\cdot) + \text{Supp } \chi(2^{-(p-1)}\cdot)) = \emptyset, \quad \text{if} \quad |p-q| \ge 5,$$

that is, to prove

$$2^p\widetilde{\mathbf{C}} \cap 2^q\mathbf{C} = \emptyset, \quad \text{if} \quad |p-q| \ge 5,$$

where $\widetilde{\mathbf{C}} := \mathbf{B}(0, \frac{2}{3}) + \mathbf{C}$. With the aid of (2.5) in Proposition 2.1, (2.15) is obtained. Therefore, the proof of Lemma 2.3 is completed. □

Additionally, it is crucial that we have

Lemma 2.4. *Let $1 \le p \le \infty$ and $u \in L^p_x$, then there exists a constant $C > 0$ independent of p, q and u such that*

$$\|\Delta_q u\|_{L^p_x} \le C\|u\|_{L^p_x}, \qquad \|S_q u\|_{L^p_x} \le C\|u\|_{L^p_x}.$$

Proof. With the form for $\Delta_q u$ in (2.10), $S_q u$ in (2.11), and using the Young inequality, we can prove the Lemma 2.4. □

Based on the Lemma 2.4, by the direct calculation we obtain

Lemma 2.5. *Let $s \in \mathbb{R}$ and $1 \le \varrho, p, r \le \infty$. It holds that*

$$\|\Delta_q u\|_{\widetilde{L}^\varrho_v(B^s_{p,r})} \lesssim \|u\|_{\widetilde{L}^\varrho_v(B^s_{p,r})}, \qquad \|S_q u\|_{\widetilde{L}^\varrho_v(B^s_{p,r})} \lesssim \|u\|_{\widetilde{L}^\varrho_v(B^s_{p,r})}.$$

Lemma 2.6. *(Topological properties, see [41]) Let $s \in \mathbb{R}$ and $1 \le p, r \le \infty$. $B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. Let us first prove that $B^s_{p,r}$ is continuously embedded in $\mathcal{S}'$. By definition, $B^s_{p,r}$ is a subspace of $\mathcal{S}'$. To do this, for any test function ϕ in $\mathcal{S}$, we have

$$\langle \Delta_q u, \phi\rangle = (-1)^k 2^{-qk} \sum_{|\alpha|=k} \langle \Delta_q u, 2^{qd} g_\alpha(-2^q\cdot) \star \partial^\alpha \phi\rangle.$$

We see that for all $k \in \mathbb{N}$, there exists an integer M_k and a constant $C_k > 0$ such that

$$|\langle \Delta_q u, \phi\rangle| \le C_k 2^{-q}(2^{q(1-k)}\|\Delta_q u\|_{L^\infty})\|\phi\|_{M_k,\mathcal{S}}.$$

With aid of Bernstein lemma, we have $\|\Delta_q u\|_{L^\infty} \le C 2^{q\frac{d}{p}}\|\Delta_q u\|_{L^p}$. Hence, if k has been chosen so large as to satisfy $s - \frac{d}{p} \ge 1 - k$, we get

$$|\langle \Delta_q u, \phi\rangle| \le C_k 2^{-q}\|u\|_{B^s_{p,r}}\|\phi\|_{M_k,\mathcal{S}}.$$

Sum up the above inequality over q, we can obtain that there exist an integer M and a constant $C > 0$ for any ϕ in $\mathcal{S}$

$$|\langle u, \phi\rangle| \le C\|u\|_{B^s_{p,r}}\|\phi\|_{M,\mathcal{S}}. \tag{2.16}$$

Next, inequality (2.16) implies that for any test function ϕ in $\mathcal{S}$, we have

$$|\langle u^{(k)} - u^{(l)}, \phi\rangle| \lesssim \|u^{(k)} - u^{(l)}\|_{B^s_{p,r}}\|\phi\|_{M,\mathcal{S}}, \quad \forall k, l \in \mathbb{N}.$$

Hence, sequence $(\langle u^{(n)}, \phi\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Thus the formula

$$\langle u, \phi\rangle := \lim_{n\to\infty}\langle u^{(n)}, \phi\rangle$$

defines a temperate distribution. By definition of the norm of $B^s_{p,r}$, sequence $(\Delta_q u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in L^p for any q. Thus an element u_q of L^p exists such that $(\Delta_q u^{(n)})_{n\in\mathbb{N}}$ converges to u_q in L^p, that is, $\lim_{n\to\infty}\|\Delta_q u^{(n)} - u_q\|_{L^p} = 0$. On the other hand, as the sequence $(u^{(n)})_{n\in\mathbb{N}}$ converges to u in $\mathcal{S}'$, we actually have $\Delta_q = u_q$.

Fix a $Q \in \mathbb{N}$ and a positive constant ε. Since for all $q \ge -1$, $\Delta_q u^{(n)}$ tends to $\Delta_q u$ in L^p, for all $n \in \mathbb{N}$ we have

$$\Big(\sum_{q\le Q}(2^{qs}\|\Delta_q(u^{(n)} - u)\|_{L^p})^r\Big)^{1/r} = \lim_{m\to\infty}\Big(\sum_{q\le Q}(2^{qs}\|\Delta_q(u^{(n)} - u^{(m)})\|_{L^p})^r\Big)^{1/r}.$$

Because the argument of the limit in the right-hand side is bounded by $\|u^{(n)} - u^{(m)}\|_{B^s_{p,r}}$ and $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $B^s_{p,r}$, one can now conclude that there exists a n_0 (independent of Q) such that for all $n \ge n_0$, we have

$$\Big(\sum_{q\le Q}(2^{qs}\|\Delta_q(u^{(n)} - u)\|_{L^p})^r\Big)^{1/r} \le \varepsilon.$$

Letting Q go to infinity insures that $(u^{(n)})_{n\in\mathbb{N}}$ tends to u in $B^s_{p,r}$. This proves the Lemma 2.6.

□

Lemma 2.7. *(Density, see [41]) Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The space $\mathcal{C}_c^\infty$ of smooth functions with compact support is dense in $B^s_{p,r}$ if and only if p and r are finite.*

Proof. We consider firstly the case $r < \infty$. Let $u \in B^s_{p,r}$ and $\varepsilon > 0$. Since $r < \infty$, there exists an integer q such that

$$\|u - S_q u\|_{B^s_{p,r}} < \frac{\varepsilon}{2}. \tag{2.17}$$

Now let $\phi \in \mathcal{C}_c^\infty$. For any $q' \in \mathbb{N}$, using Bernstein lemma 2.2 gives

$$\begin{aligned} 2^{q's}\|\Delta_{q'}(\phi S_q u - S_q u)\|_{L^p} &\leq 2^{-q'}2^{q'([s]+2)}\|\Delta_{q'}(\phi S_q u - S_q u)\|_{L^p} \\ &\leq C_s 2^{-q'} \sup_{|\alpha|=[s]+2} \|\partial^\alpha(\phi S_q u - S_q u)\|_{L^p}. \end{aligned}$$

From the above inequality, we obtain

$$\|\phi S_q u - S_q u\|_{B^s_{p,r}} \leq C_s\Big(\|(1-\phi)S_q u\|_{L^p} + \sup_{|\alpha|=[s]+2} \|\partial^\alpha((1-\phi)S_q u)\|_{L^p}\Big). \tag{2.18}$$

Let us consider a sequence $(\phi_n)_{n\in\mathbb{N}}$ such that all the derivatives of ϕ_n of order less than or equal to $[s]+2$ are uniformly bounded with respect to n and such that $\phi_n \equiv 1$ in a neighborhood of the ball $\mathbf{B}(0,n)$. If p is finite, combining Leibniz formula and Lebesgue theorem, we discover that

$$\lim_{n\to\infty} \|(1-\phi_n)S_q u\|_{L^p} + \sup_{|\alpha|=[s]+2} \|\partial^\alpha((1-\phi_n)S_q u)\|_{L^p} = 0.$$

Thus a function $\phi \in \mathcal{C}_c^\infty$ exists such that

$$C_s\Big(\|(1-\phi)S_q u\|_{L^p} + \sup_{|\alpha|=[s]+2} \|\partial^\alpha((1-\phi)S_q u)\|_{L^p}\Big) < \frac{\varepsilon}{2}.$$

Combining (2.17) and (2.18), we end up with

$$\|\phi S_q u - u\|_{B^s_{p,r}} < \varepsilon.$$

As $S_q u$ is a smooth function, this completes the proof in the case $p, r < \infty$.

Now, it is obvious that the set $\mathcal{C}_b^\infty$ of smooth functions with bounded derivatives at all orders is embedded in any space $B^s_{\infty,r}$. Therefore $\mathcal{C}_c^\infty$ cannot be a dense subset of $B^s_{\infty,r}$. Finally, the closure of $\mathcal{C}_c^\infty$ for the Besov norm $B^s_{p,\infty}$ is the space of temperate distributions u such that

$$\lim_{q\to\infty} 2^{qs}\|\Delta_q u\|_{L^p} = 0,$$

which is a strict subspace of $B^s_{p,\infty}$. This ends the proof of Lemma 2.7. □

We have the following embedding properties in Besov spaces.

Lemma 2.8. *(Embedding properties for Besov spaces on $\mathbb{R}^d$, see [41, 181]) Let $s \in \mathbb{R}$ and $1 \le p, r \le \infty$. Then*

(a) If $s > 0$, then $B^s_{p,r} = L^p \cap \dot{B}^s_{p,r}$.

(b) If $\widetilde{s} < s$ or $\widetilde{s} = s$ and $1 \le r \le \tilde{r} \le \infty$, then $B^s_{p,r} \hookrightarrow B^{\widetilde{s}}_{p,\tilde{r}}$.

(c) If $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$ and $B^s_{p_1,r_1} \hookrightarrow B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.

(d) It holds that $B^0_{\infty,1} \hookrightarrow \mathcal{C} \cap L^\infty$. If $p < \infty$, then $B^{d/p}_{p,1} \hookrightarrow \mathcal{C}_0$, where $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

Proof. Let us first prove (a). We prove mainly (a) for $r < \infty$ without loss of generality. If $u \in L^p$, we have

$$\|\Delta_{-1}u\|_{L^p} = \|\Psi \star u\|_{L^p} \le C\|u\|_{L^p}.$$

On the other hand, if $u \in \dot{B}^s_{p,r}$, we get

$$\Big(\sum_{q\in\mathbb{Z}} \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)^r\Big)^{1/r} = \Big(\sum_{q\in\mathbb{Z}} (2^{qs}\|\Phi_q \star u\|_{L^p})^r\Big)^{1/r} < +\infty.$$

Hence, we also have

$$\Big(\sum_{q\ge 0} \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)^r\Big)^{1/r} = \Big(\sum_{q\ge 0} (2^{qs}\|\Phi_q \star u\|_{L^p})^r\Big)^{1/r} < +\infty.$$

Thus, we obtain

$$\|u\|_{B^s_{p,r}} = \Big(\sum_{q\ge -1} \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)^r\Big)^{1/r} < +\infty.$$

That is, $u \in B^s_{p,r}$. On the other hand, it holds that the embedding $B^s_{p,r} \hookrightarrow L^p$ with $s > 0$. This gives (a).

Considering that $\ell^r(\mathbb{Z}) \subset \ell^{\tilde{r}}(\mathbb{Z})$ for $r \le \tilde{r}$. And the first embedding in (b) is straightforward. This shows (b). In order to prove the embedding in (c), we apply Bernstein lemma 2.2 and get

$$\|S_0 u\|_{L^{p_2}} \lesssim \|S_0 u\|_{L^{p_1}} \quad \text{and} \quad \|\Delta_q u\|_{L^{p_2}} \lesssim 2^{qd(\frac{1}{p_1}-\frac{1}{p_2})}\|\Delta_q u\|_{L^{p_1}} \quad \text{if} \quad q \in \mathbb{N}.$$

Multiplying the above inequality by 2^{qs} and summing up the resultant over $q \ge -1$, we are led to the embedding result in (c).

For proving that $B^{d/p}_{p,1} \hookrightarrow \mathcal{C}_0$, we use again Bernstein lemma 2.2 and get that

$$\|\Delta_q u\|_{L^\infty} \le 2^{q\frac{d}{p}}\|\Delta_q u\|_{L^p}.$$

This insures that the series $\sum_{q\ge -1}\Delta_q u$ of continuous bounded functions converges uniformly on $\mathbb{R}^d$. Hence u is a bounded continuous function. Besides, it is obvious that the embedding is continuous. If p is finite, one can use in addition that $\mathcal{C}^\infty_c$ is dense in $B^{d/p}_{p,1}$ and conclude that u decays at infinity. □

Also, it follows from [52] that

Lemma 2.9. *Let $s \in \mathbb{R}$ and $1 \le \varrho_1, \varrho_2, p, r \le \infty$.*

(1) If $r \le \min\{\varrho_1, \varrho_2\}$, then it holds that

$$\|u\|_{L_T^{\varrho_1} L_v^{\varrho_2}(B^s_{p,r})} \le \|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B^s_{p,r})}. \tag{2.19}$$

(2) If $r \ge \max\{\varrho_1, \varrho_2\}$, then it holds that

$$\|u\|_{L_T^{\varrho_1} L_v^{\varrho_2}(B^s_{p,r})} \ge \|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B^s_{p,r})}. \tag{2.20}$$

Proof. We only prove (2.19) in terms of $1 \le r, \varrho_1, \varrho_2 < +\infty$, the other cases and (2.20) can be proved similarly. Since $\varrho_2/r \ge 1$ and $\varrho_1/r \ge 1$, by applying the Generalized Minkowski's inequality twice, one can see that

$$\begin{aligned}
\|u\|_{L_T^{\varrho_1} L_v^{\varrho_2}(B^s_{p,r})} &= \left(\int_0^T \left(\int_{\mathbb{R}^d} \Big(\sum_{q \ge -1} 2^{qsr} \|\Delta_q u\|_{L_x^p}^r \Big)^{\varrho_2/r} dv \right)^{\varrho_1/\varrho_2} dt \right)^{1/\varrho_1} \\
&= \left(\int_0^T \left(\int_{\mathbb{R}^d} \Big(\sum_{q \ge -1} 2^{qsr} \|\Delta_q u\|_{L_x^p}^r \Big)^{\varrho_2/r} dv \right)^{\frac{r}{\varrho_2} \cdot \frac{\varrho_1}{r}} dt \right)^{1/\varrho_1} \\
&\le \left(\int_0^T \left(\sum_{q \ge -1} 2^{qsr} \Big(\int_{\mathbb{R}^d} \|\Delta_q u\|_{L_x^p}^{\varrho_2} dv \Big)^{r/\varrho_2} \right)^{\frac{\varrho_1}{r}} dt \right)^{1/\varrho_1} \\
&= \left(\int_0^T \left(\sum_{q \ge -1} 2^{qsr} \Big(\int_{\mathbb{R}^d} \|\Delta_q u\|_{L_x^p}^{\varrho_2} dv \Big)^{r/\varrho_2} \right)^{\frac{\varrho_1}{r}} dt \right)^{\frac{r}{\varrho_1} \cdot \frac{1}{r}} \\
&\le \left(\sum_{q \ge -1} 2^{qsr} \left(\int_0^T \Big(\int_{\mathbb{R}^d} \|\Delta_q u\|_{L_x^p}^{\varrho_2} dv \Big)^{\varrho_1/\varrho_2} dt \right)^{r/\varrho_1} \right)^{1/r} \\
&= \|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B^s_{p,r})},
\end{aligned}$$

which gives (2.19). □

As in [180], we also obtain the relation between homogeneous Chemin-Lerner spaces and nonhomogeneous Chemin-Lerner spaces as the following proposition.

Proposition 2.2. *Let $s \in \mathbb{R}$ and $1 \le \varrho_1, \varrho_2, p, r \le \infty$.*

(1) It holds that

$$L_T^{\varrho_1} L_v^{\varrho_2} L_x^p \cap \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(\dot{B}^s_{p,r}) \subset \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B^s_{p,r}). \tag{2.21}$$

(2) Furthermore, if $s > 0$ and $r \le \min\{\varrho_1, \varrho_2\}$, then

$$L_T^{\varrho_1} L_v^{\varrho_2} L_x^p \cap \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(\dot{B}^s_{p,r}) = \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B^s_{p,r}) \tag{2.22}$$

for any $T > 0$.

Proof. The proof is the similar with that in [180] but with some changes. Here, we give the detailed proof as follows. We prove mainly this proposition for $r < \infty$ without loss of generality. For $s \in \mathbb{R}$, if $u \in L_T^{\varrho_1} L_v^{\varrho_2} L_x^p$, we have

$$\|\Delta_{-1} u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} = \|\Psi \star u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \leq C \|u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p}.$$

On the other hand, if $u \in \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(\dot{B}_{p,r}^s)$, we get

$$\Big(\sum_{q \in \mathbb{Z}} \Big(2^{qs} \|\dot{\Delta}_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} = \Big(\sum_{q \in \mathbb{Z}} \Big(2^{qs} \|\Phi_q \star u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} < +\infty.$$

Hence, we also have

$$\Big(\sum_{q \geq 0} \Big(2^{qs} \|\Delta_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} = \Big(\sum_{q \geq 0} \Big(2^{qs} \|\Phi_q \star u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} < +\infty.$$

Thus, we obtain

$$\|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B_{p,r}^s)} = \Big(\sum_{q \geq -1} \Big(2^{qs} \|\Delta_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} < +\infty.$$

That is, $u \in \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B_{p,r}^s)$. The formula (2.21) is obtained.

For (2.22), for $s > 0$ and $r \leq \min\{\varrho_1, \varrho_2\}$ and with Lemma 2.8, we have the embedding as

$$\widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B_{p,r}^s) \hookrightarrow L_T^{\varrho_1} L_v^{\varrho_2}(B_{p,r}^s) \hookrightarrow L_T^{\varrho_1} L_v^{\varrho_2}(B_{p,1}^0) \hookrightarrow L_T^{\varrho_1} L_v^{\varrho_2} L_x^p,$$

so if $u \in \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B_{p,r}^s)$ then $u \in L_T^{\varrho_1} L_v^{\varrho_2} L_x^p$.

For $q < 0$, we get

$$\Big\| \dot{\Delta}_q u \Big\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} = \begin{cases} \|\Phi_q \star \Psi \star u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \leq C \|\Delta_{-1} u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p}, & q < -1, \\ \|\dot{\Delta}_{-1} u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \leq C \|u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p}, & q = -1. \end{cases}$$

Therefore, for $s > 0$ and $r \leq \min\{\varrho_1, \varrho_2\}$, we obtain

$$\begin{aligned}
\|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(\dot{B}_{p,r}^s)} &= \Big(\sum_{q \in \mathbb{Z}} \Big(2^{qs} \|\dot{\Delta}_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} \\
&\leq \Big(\sum_{q < 0} \Big(2^{qs} \|\dot{\Delta}_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} + \Big(\sum_{q \geq 0} \Big(2^{qs} \|\dot{\Delta}_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} \\
&\leq C \Big(\sum_{q < -1} \Big(2^{qs} \|\dot{\Delta}_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} + C 2^{-s} \|\dot{\Delta}_{-1} u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} + \Big(\sum_{q \geq 0} \Big(2^{qs} \|\dot{\Delta}_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} \\
&\leq C \|\Delta_{-1} u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} + C \|u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} + \Big(\sum_{q \geq 0} \Big(2^{qs} \|\Delta_q u\|_{L_T^{\varrho_1} L_v^{\varrho_2} L_x^p} \Big)^r \Big)^{1/r} \\
&\leq C \|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B_{p,r}^s)}.
\end{aligned}$$

Hence, if $u \in \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B_{p,r}^s)$ then $u \in \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(\dot{B}_{p,r}^s)$. That is, if $u \in \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(B_{p,r}^s)$ then $u \in L_T^{\varrho_1} L_v^{\varrho_2} L_x^p \cap \widetilde{L}_T^{\varrho_1} \widetilde{L}_v^{\varrho_2}(\dot{B}_{p,r}^s)$. Together with (2.21), we obtain the desired (2.22) directly. The proof of Proposition 2.2 is completed. $\square$

2.4.2 Analysis tools of Sobolev spaces

Lemma 2.10. *(Young's inequality for products, see [185]) Assume* $1 < p, q < \infty$ *with* $\frac{1}{p}+\frac{1}{q} = 1$. *For all* $a, b, \varepsilon > 0$. *Then the following inequalities hold*

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{2.23}$$

$$a \cdot b \leq \frac{\varepsilon}{p} a^p + \frac{1}{q\varepsilon^{\frac{1}{p-1}}} b^q. \tag{2.24}$$

Proof. Obviously, it is true if $a = 0$ or $b = 0$. Assume $a, b > 0$. Set $\theta = \frac{1}{p}$ and $1 - \theta = \frac{1}{q}$. Since the logarithm function is concave,

$$\ln(\theta a^p + (1-\theta)b^q) \geq \theta \ln(a^p) + (1-\theta)\ln(b^q) = \ln(a) + \ln(b) = \ln(ab),$$

where the equality holds if and only if $a^p = b^q$. (2.23) follows by exponentiating. For $\varepsilon > 0$, substituting $a' = a\varepsilon^{1/p}, b' = b\varepsilon^{-1/p}$ into (2.23) gives the Young's inequality with ε in (2.24). □

Lemma 2.11. *(Hölder's inequality, see [96]) Assume* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $x, y \in \mathbb{C}^d(\mathbb{R}^d)$, *it holds that*

$$\sum_{i=1}^{d} |x_i||y_i| \leq \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{d} |y_i|^q\right)^{1/q}.$$

Proof. If $x = 0$ or $y = 0$, it is certainly true. Assume $x \neq 0, y \neq 0$. Set $X = (\sum_{i=1}^{d} |x_i|^p)^{1/p}$ and $Y = (\sum_{i=1}^{d} |y_i|^q)^{1/q}$. It follows from Young's inequality in Lemma 2.10 that

$$\begin{aligned}\sum_{i=1}^{d} \frac{|x_i|}{X}\frac{|y_i|}{Y} &\leq \sum_{i=1}^{d} \left(\frac{1}{p}(\frac{|x_i|}{X})^p + \frac{1}{q}(\frac{|y_i|}{Y})^q\right) \\ &= \frac{1}{p}\frac{1}{X^p}\sum_{i=1}^{d} |x_i|^p + \frac{1}{q}\frac{1}{Y^q}\sum_{i=1}^{d} |y_i|^q = \frac{1}{p} + \frac{1}{q} = 1,\end{aligned}$$

which implies

$$\sum_{i=1}^{d} |x_i||y_i| \leq XY = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{d} |y_i|^q\right)^{1/q}.$$

This ends the proof. □

Lemma 2.12. *(Hölder's inequality, see [19]) Let* (X, μ) *be a measure space and* (p, q, r) *in* $[1, \infty]^3$ *be*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

If (f, g) *belongs to* $L^p(X, \mu) \times L^q(X, \mu)$, *then* fg *belongs to* $L^r(X, \mu)$ *and*

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q},$$

in particular, in case of $r = 1, p = q = 2$, *the above inequality is called Schwarz inequality.*

Proof. The cases where $p = 1$ or $p = \infty$ being trivial, we assume from now on that p is a real number greater than 1. The concavity of the logarithm function entails that for any positive real numbers a and b and any θ in $[0, 1]$,

$$\theta \log(a) + (1 - \theta) \log(b) \leq \log(\theta a + (1 - \theta)b),$$

which obviously implies that

$$a^\theta b^{1-\theta} \leq \theta a + (1 - \theta)b.$$

Hence, assuming that $\|f\|_{L^p} = \|g\|_{L^q} = 1$, we can write

$$\begin{aligned} \int_X |fg|^r d\mu &= \int_X (|f|^p)^{\frac{r}{p}} (|g|^q)^{\frac{r}{q}} d\mu \\ &\leq \frac{r}{p} \int_X |f|^p d\mu + \frac{r}{q} \int_X |g|^q d\mu \\ &\leq \frac{r}{p} + \frac{r}{q} = 1. \end{aligned}$$

The Lemma is thus proved. □

The following lemma is an easy extension of Hölder inequality.

Lemma 2.13. *([46]) Suppose that $f_i : X \to \mathbb{C}$ are measurable functions for $i = 1, \cdots, n$ and $p_1, \cdots, p_n$ and r are positive constants such that $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}$, then it holds that*

$$\Big\| \prod_{j=1}^{n} f_j \Big\|_{L^r} \leq \prod_{j=1}^{n} \|f_j\|_{L^{p_j}}.$$

Proof. Firstly, we prove this inequality with $n = 2$, then for any $p \in [1, \infty]$,

$$\|fg\|_{L^r}^r = \int f^r g^r d\mu \leq \|f^r\|_{L^p} \|g^r\|_{L^{p^*}},$$

where $p^* = \frac{p}{p-1}$ is the conjugate exponent. Let $p_1 = pr$ and $p_2 = p^* r$ so that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r}$ are desired. Then the above inequality becomes

$$\|fg\|_{L^r} \leq \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

The general case is now proved by induction. Indeed,

$$\Big\| \prod_{j=1}^{n+1} f_j \Big\|_{L^r} = \Big\| \prod_{j=1}^{n} f_j \cdot f_{n+1} \Big\|_{L^r} \leq \Big\| \prod_{j=1}^{n} f_j \Big\|_{L^q} \|f_{n+1}\|_{L^{p_{n+1}}},$$

where $\frac{1}{q} + \frac{1}{p_{n+1}} = \frac{1}{r}$. Since $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{q}$, we use the induction hypothesis to conclude

$$\Big\| \prod_{j=1}^{n} f_j \Big\|_{L^q} \leq \prod_{j=1}^{n} \|f_j\|_{L^{p_j}},$$

which combined with the previous displayed equation proved the generalized form of Hölder's inequality. □

Lemma 2.14. *(Young's convolution inequality, see [46]) Let $p, q, r \in [1, \infty]$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{2.25}$$

If $f \in L^p$ and $g \in L^q$ then $|f| \star |g|(x) < \infty$ for $m - a.e.x$ and

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}. \tag{2.26}$$

In particular L^1 is closed under convolution. (The space $(L^1, \star)$ is an example of a "Banach algebra" without unit.)

Proof. Before the proof, let us understand (2.25) by the following scaling argument. For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$, then after a few simple change of variables we find

$$\|f_\lambda\|_{L^p} = \lambda^{-1/p} \|f\|_{L^p} \quad \text{and} \quad (f \star g)_\lambda = \lambda f_\lambda \star g_\lambda.$$

Therefore if (2.26) holds for some $p, q, r \in [1, \infty]$, we would also have

$$\|f \star g\|_{L^r} = \lambda^{1/r} \|(f \star g)_\lambda\|_{L^r} \leq \lambda^{1/r} \lambda \|f_\lambda\|_{L^p} \|g_\lambda\|_{L^q} = \lambda^{1+1/r-1/p-1/q} \|f\|_{L^p} \|g\|_{L^q}$$

for all $\lambda > 0$. This is only possible if (2.25) holds.

Then we begin the proof. Let $\alpha, \beta \in [0, 1]$ and $p_1, p_2 \in [0, \infty]$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = 1$. Then by Hölder's inequality,

$$\begin{aligned}
|f \star g(x)| &= \left| \int f(x-y) g(y) dy \right| \leq \int |f(x-y)|^{(1-\alpha)} |g(y)|^{(1-\beta)} |f(x-y)|^{\alpha} |g(y)|^{\beta} dy \\
&\leq \left(\int |f(x-y)|^{(1-\alpha)r} |g(y)|^{(1-\beta)r} dy \right)^{1/r} \left(\int |f(x-y)|^{\alpha p_1} dy \right)^{1/p_1} \left(\int |g(y)|^{\beta p_2} dy \right)^{1/p_2} \\
&= \left(\int |f(x-y)|^{(1-\alpha)r} |g(y)|^{(1-\beta)r} dy \right)^{1/r} \|f\|_{L^{\alpha p_1}}^{\alpha} \|g\|_{L^{\beta p_2}}^{\beta}.
\end{aligned}$$

Taking the r^{th} power of the above equation and integrating on x, we obtain

$$\begin{aligned}
\|f \star g\|_{L^r}^r &\leq \int \left(\int |f(x-y)|^{(1-\alpha)r} |g(y)|^{(1-\beta)r} dy \right) dx \cdot \|f\|_{L^{\alpha p_1}}^{\alpha} \|g\|_{L^{\beta p_2}}^{\beta} \\
&\leq \|f\|_{L^{(1-\alpha)r}}^{(1-\alpha)r} \|g\|_{L^{(1-\beta)r}}^{(1-\beta)r} \|f\|_{L^{\alpha p_1}}^{\alpha r} \|g\|_{L^{\beta p_2}}^{\beta r}.
\end{aligned} \tag{2.27}$$

Let us now suppose, $(1-\alpha)r = \alpha p_1$ and $(1-\beta)r = \beta p_2$, so (2.27) becomes

$$\|f \star g\|_{L^r}^r \leq \|f\|_{L^{\alpha p_1}}^r \|g\|_{L^{\beta p_2}}^r,$$

which is the desired (2.26) with

$$p := (1-\alpha)r = \alpha p_1 \quad \text{and} \quad q := (1-\beta)r = \beta p_2. \tag{2.28}$$

Thus, to finish the proof, it suffices to show p and q are arbitrary indices in $[1,\infty]$ satisfying $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$.

If α,β,p_1,p_2 satisfy the above relation (2.28), then

$$\alpha=\frac{r}{r+p_1},\qquad \beta=\frac{r}{r+p_2}$$

and

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{p_1}\frac{r+p_1}{r}+\frac{1}{p_2}\frac{r+p_2}{r}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{2}{r}=1+\frac{1}{r}.$$

Conversely, if p,q,r satisfy (2.25), then let α and β satisfy $p=(1-\alpha)r$ and $q=(1-\beta)r$, i.e.

$$\alpha:=\frac{r-p}{r}=1-\frac{p}{r}\le 1\ \text{ and }\ \beta=\frac{r-q}{r}=1-\frac{q}{r}\le 1.$$

Due to (2.25), it holds that $\alpha=p(1-\frac{1}{q})\ge 0$ and $\beta=q(1-\frac{1}{p})\ge 0$, so that $\alpha,\beta\in[0,1]$. We define $p_1:=p/\alpha$ and $p_2:=q/\beta$, then

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{r}=\beta\frac{1}{q}+\alpha\frac{1}{p}+\frac{1}{r}=\frac{1}{q}-\frac{1}{r}+\frac{1}{p}-\frac{1}{r}+\frac{1}{r}=1.$$

Hence, the proof of Lemma 2.14 is completed. □

Lemma 2.15. *(Minkowski's inequality, see [19]) Let (X,μ_1) and (Y,μ_2) be two measure spaces and f is a nonnegative measurable function over $X\times Y$. For $1\le p\le q\le\infty$, we have*

$$\left\|\|f(\cdot,y)\|_{L^p(X,\mu_1)}\right\|_{L^q(Y,\mu_2)}\le\left\|\|f(x,\cdot)\|_{L^q(Y,\mu_2)}\right\|_{L^p(X,\mu_1)}.$$

Proof. The result is obvious if $q=\infty$. If q is finite, then, using Fubini's theorem and $r:=(q/p)'$, we have

$$\begin{aligned}\left\|\|f(\cdot,y)\|_{L^p(X,\mu_1)}\right\|_{L^q(Y,\mu_2)}&=\left(\int_Y\left(\int_X f^p(x,y)d\mu_1(x)\right)^{\frac{q}{p}}d\mu_2(y)\right)^{\frac{1}{q}}\\&=\left(\sup_{\substack{\|g\|_{L^r(Y,\mu_2)}=1\\ g\ge 0}}\int_{X\times Y}f^p(x,y)g(y)d\mu_1(x)d\mu_2(y)\right)^{\frac{1}{p}}\\&\le\left(\int_X\left(\sup_{\substack{\|g\|_{L^r(Y,\mu_2)}=1\\ g\ge 0}}\int_Y f^p(x,y)g(y)d\mu_2(y)\right)d\mu_1(x)\right)^{\frac{1}{p}}.\end{aligned}$$

Using Hölder's inequality we may then infer that

$$\left\|\|f(\cdot,y)\|_{L^p(X,\mu_1)}\right\|_{L^q(Y,\mu_2)}\le\left(\int_X\left(\int_Y f^q(x,y)d\mu_2(y)\right)^{\frac{p}{q}}d\mu_1(x)\right)^{\frac{1}{p}},$$

which is the desired inequality. □

Lemma 2.16. *(Gagliardo-Nirenberg interpolation inequality, see [61, 149]) Let $1 \leq q, r \leq \infty$, and let $k > 0$ be an integer. Assume that $u \in L^q(\mathbb{R}^d)$ and $D^k u \in L^r(\mathbb{R}^d)$, then for the derivatives $D^j u, 0 \leq j \leq k$, the following inequalities hold*

$$\|D^j u\|_{L^p} \leq C \|D^k u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \tag{2.29}$$

where $\frac{1}{p} = \frac{j}{d} + \alpha(\frac{1}{r} - \frac{k}{d}) + (1-\alpha)\frac{1}{q}$ for all α in the interval $\frac{j}{k} \leq \alpha \leq 1$ and the constant C is a positive constant depending only on d, k, j, q, r, α. There exist the following exceptional cases:

(1) If $j = 0, rk < d$ and $q = \infty$, then we make the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$ or $u \in L^{q'}(\mathbb{R}^d)$ for some finite $q' > 0$.

(2) If $1 < r < \infty$ and $k - j - d/r$ is a non-negative integer, then (2.29) holds only for a satisfying $j/k \leq \alpha < 1$.

Proof. We recall the proof of this Lemma that is given by Nirenberg in [149]. Here, we state the main steps. The proof of the this lemma is elementary and contains in particular an elementary proof for the Sobolev case $\alpha = 1$. In order to prove (2.29) for any given j, on has only to prove it for the extreme values of $\alpha, j/k$ and unity. For in general there is a simple interpolation lemma, that is, if $-\infty < \lambda_1 \leq \lambda \leq \lambda_2 < \infty$, then with some simple calculations, it holds that

$$\|u\|_{L^{\frac{1}{\lambda}}} \leq c \|u\|_{L^{\frac{1}{\lambda_1}}}^{\frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1}} \|u\|_{L^{\frac{1}{\lambda_2}}}^{\frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}},$$

where c is independent of u. For $\lambda_1 > 0$ it is merely the usual interpolation inequality for L^p norms. Now we turn the main proof of this lemma. Firstly consider the Sobolev case, $\alpha = 1$. It suffices to consider the case $j = 0, k = 1$, from which the general result may then be derived. If $r > d$, (2.29) asserts that u satisfies a certain Hölder condition, and an elementary proof due to Morrey has long been known. We shall sketch it here for functions defined in a general domain $\mathcal{D}$.

Definition: A domain $\mathcal{D}$ is said to have the strong cone property if there exists positive constants C, λ and a closed solid right spherical cone V of fixed opening and height such that any points P, Q in $\overline{\mathcal{D}}$ (the closure of $\mathcal{D}$) with $|P - Q| \leq C$ are vertices of cones V_P, V_Q lying in $\overline{\mathcal{D}}$ which are congruent to V and have the following property: the volume of the intersection of the sets: V_P, V_Q and the two spheres with contents P, Q and radius $|P - Q|$, is not less than $\lambda |P - Q|^d$.

Now it turns to prove the assertion.

If u has first derivatives in $L^r, r > d$, in a domain $\mathcal{D}$ having the strong cone property, then for points P, Q in $\mathcal{D}$ with $|P - Q| \leq C$, we have

$$\frac{|u(P) - u(Q)|}{|P - Q|^{1 - \frac{d}{r}}} \leq c \|Du\|_{L^r}, \tag{2.30}$$

where c is some constant depending only on C, λ, V, d and r. With (2.30), the estimate for $[u]_{1-\frac{d}{r}}$, depending on the domain. Now, we prove (2.30). Set $s = |P - Q|$ and let $S_P (S_Q)$ be

the intersection of $V_P(V_Q)$ with the sphere about $P(Q)$ radius s. Set $S_P \cap S_Q = S$. If R is a point in S, integrating with respect to R over S, we have

$$\begin{aligned}\text{Volume } of\ S \cdot |u(P) - u(Q)| \le &\int_S |u(P) - u(R)| dR \\ &+ \int_S |u(R) - u(Q)| dR \triangleq I_1 + I_2.\end{aligned}$$

Because of the strong cone property, the left hand of the above inequality is not less than

$$\lambda s^d |u(P) - u(Q)|.$$

To estimate I_1, introducing polar coordinates ϱ, η about P, where η is a unit vector, we find easily that I_1 is bounded by

$$\begin{aligned}\int_{S_P} \varrho^{d-1} d\omega_\eta d\varrho \int_0^\varrho \Big|\frac{\partial u}{\partial \varrho}\Big| d\varrho &\le c_0 s^d \int_{S_P} \Big|\frac{\partial u}{\partial \varrho}\Big| \frac{dx}{\varrho^{d-1}} \\ &\le c_0 s^d \Big(\int_{S_P} \Big|\frac{\partial u}{\partial \varrho}\Big|^r dx \Big)^{\frac{1}{r}} \Big(\int_{S_P} \varrho^{(1-d)\frac{r}{r-1}} dx \Big)^{\frac{r-1}{r}} \\ &\le c_0 s^{d+1-\frac{d}{r}} \Big(\int_{S_P} \Big|\frac{\partial u}{\partial \varrho}\Big|^r dx \Big)^{\frac{1}{r}},\end{aligned}$$

where $d\omega$ is the element of area on the unit sphere, dx is the elementary of volume and the Hölder's inequality is used in the last inequality. Similarly, the term I_2 can be bounded by a similar estimate with that for I_1. This gives (2.30).

In the following, we return to functions defined in the full d- space.

Suppose $r < d$. We shall prove a strong formulation of (2.29), that is

$$\|u\|_{L^{\frac{dr}{d-r}}} \le \frac{r}{2} \frac{d-1}{d-r} \prod_i \|\frac{\partial u}{\partial x_i}\|_{L^r}^{\frac{1}{d}}. \tag{2.31}$$

For $1 < r < d$, (2.31) follows from the special case $r = 1$, as one readily verifies, by simply applying the inequality for $r = 1$ to the function $V = |u|^{\frac{d-1}{d-r}r}$ and using Hölder's inequality in a sultable way. Thus, it suffices to prove (2.31) for the case $r = 1$ as the following

$$\|u\|_{L^{\frac{d}{d-1}}} \le \frac{1}{2} \prod_i \|\frac{\partial u}{\partial x_i}\|_{L^1}^{\frac{1}{d}}. \tag{2.32}$$

We prove (2.32) for $d = 3$. One holds easily that

$$|u(x)| \le \frac{1}{2} \int_i \Big|\frac{\partial u}{\partial x_i}\Big| dx_i, \qquad i = 1, 2, 3,$$

where $\int_i$ denotes integration along the full line through x parallel to x_i axis. Thus, we have

$$|2u(x)|^{\frac{3}{2}} \le \Big(\int_1 \Big|\frac{\partial u}{\partial x_1}\Big| dx_1 \Big)^{\frac{1}{2}} \Big(\int_2 \Big|\frac{\partial u}{\partial x_2}\Big| dx_2 \Big)^{\frac{1}{2}} \Big(\int_3 \Big|\frac{\partial u}{\partial x_3}\Big| dx_3 \Big)^{\frac{1}{2}}.$$

Integrating with respect to x_1 then x_2 and then x_3, with the aid of Schwarz's inequality we have

$$\int_1 |2u(x)|^{\frac{3}{2}} dx_1 \le \Big(\int_1 \Big| \frac{\partial u}{\partial x_1} \Big| dx_1 \Big)^{\frac{1}{2}} \Big(\int_1 \int_2 \Big| \frac{\partial u}{\partial x_2} \Big| dx_2 dx_1 \Big)^{\frac{1}{2}} \Big(\int_1 \int_3 \Big| \frac{\partial u}{\partial x_3} \Big| dx_3 dx_1 \Big)^{\frac{1}{2}},$$
$$\int_2 \int_1 |2u(x)|^{\frac{3}{2}} dx_1 dx_2 \le \Big(\int_2 \int_1 \Big| \frac{\partial u}{\partial x_1} \Big| dx_1 dx_2 \Big)^{\frac{1}{2}} \Big(\int_1 \int_2 \Big| \frac{\partial u}{\partial x_2} \Big| dx_2 dx_1 \Big)^{\frac{1}{2}} \Big(\int \int \int \Big| \frac{\partial u}{\partial x_3} \Big| dx \Big)^{\frac{1}{2}},$$
$$\int \int \int |2u(x)|^{\frac{3}{2}} dx \le \Big(\int \int \int \Big| \frac{\partial u}{\partial x_1} \Big| dx \Big)^{\frac{1}{2}} \Big(\int \int \int \Big| \frac{\partial u}{\partial x_2} \Big| d \Big)^{\frac{1}{2}} \Big(\int \int \int \Big| \frac{\partial u}{\partial x_3} \Big| dx \Big)^{\frac{1}{2}},$$

which is the desired (2.32) for $d = 3$. For general d the inequality is proved in the same way with the aid of Hölder's inequality. It is worth pointing out here that the combinatorial argument of [119] provide a nice shortcut for the proof of the isoperimetric inequality, which is essentially (2.31).

Finally, for $j = 0, k = 1$, that $r = d$, this is the exceptional case. We claim that

$$\|u\|_{L^p} \le c_1 \|Du\|_{L^d}^{\frac{1-q}{p}} \|u\|_{L^p}^{\frac{q}{p}}, \quad 0 < q \le p < \infty,$$

where c_1 is a constant depending on d, p and q. It suffices to show this for large p and this is easily done by applying (2.32) to the function $v = |u|^{p(1-1/d)}$, and using Hölder's inequality in a judicious manner.

Now let us consider the other extreme case $\alpha = j/k$. It suffices to consider the case $j = 1, k = 2$, the general case may then be proved by induction on k. We claim that the following inequality holds

$$\|Du\|_{L^p} \le \tilde{c} \|D^2 u\|_{L^r}^{\frac{1}{2}} \|u\|_{L^q}^{\frac{1}{2}} \quad \text{for} \quad \frac{2}{p} = \frac{1}{r} + \frac{1}{q}, \ 1 \le q, r \le \infty, \tag{2.33}$$

with $\tilde{c}$ an absolute constant. Inequality (2.33) follows from the corresponding inequality in one dimension

$$\int |\partial_x u|^p dx \le \tilde{c}^p \Big(\int |\partial_x^2 u|^r dx \Big)^{\frac{p}{2r}} \Big(\int |u|^q dx \Big)^{\frac{p}{2q}}, \quad \frac{2}{p} = \frac{1}{r} + \frac{1}{q}, \tag{2.34}$$

which holds for the full, or half-infinite line (with $\tilde{c}$ an absolute constant) by integrating with respect to the other variables and applying Hölder's inequality.

The proof of (2.34) is slightly tricky and is based on the following inequality: On an interval λ, whose length we also denote by λ, we have

$$\int_\lambda |\partial_x u|^p dx \le \bar{c}^p \lambda^{1+p-\frac{p}{r}} \Big(\int_\lambda |\partial_x^2 u|^r dx \Big)^{\frac{p}{r}} + \bar{c}^p \lambda^{-(1+p-\frac{p}{r})} \Big(\int_\lambda u^q dx \Big)^{\frac{p}{q}} \tag{2.35}$$

with $\bar{c}$ an absolute constant. we shall prove that for any interval $L : 0 \le x \le L$ the following inequality holds

$$\int_0^L |\partial_x u|^p dx \le 2\bar{c}^p \Big(\int_0^\infty |\partial_x^2 u|^r dx \Big)^{\frac{p}{2r}} \Big(\int_0^\infty |u|^q dx \Big)^{\frac{p}{2q}}. \tag{2.36}$$

(2.34) follows easily from (2.36). Now, we prove (2.36). Suppose that $\|\partial_x^2 u\|_{L^r} = 1$. we shall cover the interval L by a finite constant of successive intervals $\lambda_1, \lambda_2, \cdots$, each one having as initial point the end point of the preceding. For m a fixed positive integer, choose first the interval $\lambda : 0 \le x \le \frac{L}{m}$, and consider (2.35) for this interval. If the first term on the right of (2.35) is greater than the second term, set $\lambda_1 = \lambda$, then we have

$$\int_{\lambda_1} |\partial_x u|^p dx \le 2\bar{c}^p \Big(\frac{L}{m}\Big)^{1+p-\frac{p}{r}}$$

since $\|\partial_x^2 u\|_{L^r} = 1$. However, if the second term of (2.35) is the greater extend interval λ (keeping its left endpoint fixed) until the two terms of the right of (2.35) become equal. Since $1 + p - \frac{p}{r} > 0$, equality of these two terms must occur for a finite value of λ. Let λ_1 be the resulting interval. Then we have

$$\int_{\lambda_1} |\partial_x u|^p dx \le 2\bar{c}^p \Big(\int_{\lambda_1} |\partial_x^2 u|^r dx\Big)^{\frac{p}{2r}} \Big(\int_{\lambda_1} |u|^q dx\Big)^{\frac{p}{2q}}.$$

Starting at the end point of λ_1 repeat this process, keeping m fixed, choosing $\lambda_2, \lambda_3, \cdots$, until L is covered. There are clearly at most m such intervals λ_j. If we sum these estimates for $\int_{\lambda_i} |\partial_x u|^p dx$, with the aid of Hölder's inequality (recall that $\frac{p}{2r} + \frac{p}{2q} = 1$), we have

$$\int_0^L |\partial_x u|^p dx \le 2\bar{c}^p \Big(\frac{L}{m}\Big)^{1+p-\frac{p}{r}} \cdot m + 2\bar{c}^p \Big(\int_0^\infty |\partial_x^2 u|^r dx\Big)^{\frac{p}{2r}} \Big(\int_0^\infty |u|^q dx\Big)^{\frac{p}{2q}}.$$

By letting $m \to \infty$, the first term on the right of the preceding tends to 0 since $r > 1$. Then (2.36) is obtained and the proof of (2.33) is given. □

Lemma 2.17. *(Moser-type calculus inequalities, see [100, 130]) Let $s \ge 1$ be an integer. Suppose $u \in H^s(\mathbb{R}^d), \nabla u \in L^\infty(\mathbb{R}^d)$ and $v \in H^{s-1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then for all multi-index α with $1 \le |\alpha| \le s$. One has $\partial^\alpha(uv) \in L^2(\mathbb{R}^d), \partial^\alpha(uv) - u\partial^\alpha v \in L^2(\mathbb{R}^d)$ and*

$$\|\partial^\alpha(uv)\|_{L^2} \le C_s(\|u\|_{L^\infty}\|D^{|\alpha|}v\|_{L^2} + \|v\|_{L^\infty}\|D^{|\alpha|}u\|_{L^2}), \tag{2.37}$$

$$\|\partial^\alpha(uv) - u\partial^\alpha v\|_{L^2} \le C_s(\|\nabla u\|_{L^\infty}\|D^{|\alpha|-1}v\|_{L^2} + \|v\|_{L^\infty}\|D^{|\alpha|}u\|_{L^2}), \tag{2.38}$$

where $\|D^{s'}u\| = \sum_{|\alpha|=s'} \|\partial^\alpha u\|$. In particular, if $s > \frac{d}{2} + 1$, then the embedding $H^{s-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ is continuous and we have

$$\|uv\|_{s-1} \le C_s\|u\|_{s-1}\|v\|_{s-1}, \quad \forall u, v \in H^{s-1}(\mathbb{R}^d) \tag{2.39}$$

and for all $|\alpha| \le s$,

$$\|\partial^\alpha(uv) - u\partial^\alpha v\|_{L^2} \le C_s\|u\|_s\|v\|_{s-1}, \quad \forall u, v \in H^s(\mathbb{R}^d). \tag{2.40}$$

Proof. As in Moser [128], We shall be using the following version of the well-known Gagliardo-Nirenberg calculus inequality:

$$\|\partial^i u\|_{L^{2r/i}} \le C_r\|u\|_{L^\infty}^{1-i/r}\|\partial^r u\|_{L^2}^{i/r}, \tag{2.41}$$

where $r \geq i \geq 0$. Applying a Hölder inequality and then using the above inequality (2.41), for $\delta \geq \max(|\beta|, |\gamma|)$ we can obtain

$$\begin{aligned}
\|\partial^\alpha (uv)\|_{L^2} &\leq C_\alpha \sum_{\beta+\gamma=\alpha} \|\partial^\beta u \partial^\gamma v\|_{L^2} \\
&\leq C_\alpha \sum_{\beta+\gamma=\alpha} \|\partial^\beta u\|_{L^{2\delta/|\beta|}} \|\partial^\gamma v\|_{L^{2\delta/|\gamma|}} \\
&\leq C_\alpha \sum_{\beta+\gamma=\alpha} C_\beta C_\gamma \|u\|_{L^\infty}^{1-|\beta|/\delta} \|\partial^\delta u\|_{L^2}^{|\beta|/\delta} \|v\|_{L^\infty}^{1-|\gamma|/\delta} \|\partial^\delta v\|_{L^2}^{|\gamma|/\delta} \\
&\leq C_\alpha \sum_{\beta+\gamma=\alpha} (\|u\|_{L^\infty} \|\partial^\delta v\|_{L^2})^{|\gamma|/\delta} (\|v\|_{L^\infty} \|\partial^\delta u\|_{L^2})^{|\beta|/\delta} \\
&\leq C_s (\|u\|_{L^\infty} \|D^{|\alpha|} v\|_{L^2} + \|v\|_{L^\infty} \|D^{|\alpha|} u\|_{L^2}),
\end{aligned} \tag{2.42}$$

which is (2.37). To prove (2.38), we make the following modifications:

$$\begin{aligned}
\|\partial^\alpha (uv) - u\partial^\alpha v\|_{L^2} &\leq C_\alpha \sum_{\beta+\gamma=\alpha, \beta\neq 0} \|\partial^\beta u \partial^\gamma v\|_{L^2} \\
&= C_\alpha \sum_{|\beta'|+|\gamma|\leq s-1} \|\partial^{\beta'} (\partial u) \partial^\gamma v\|_{L^2},
\end{aligned}$$

then replacing f by ∂f and N by $N-1$ in (2.42), we obtain (2.38). (2.39) and (2.40) are given by the embedding $H^{s-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ with $s > \frac{d}{2} + 1$. Hence, the proof of Lemma 2.17 is completed. □

Lemma 2.18. *(see [74, 75]) Let $1 \leq p, q, r \leq \infty$, and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then we have*

$$\|\partial_x^k (uv)\|_{L^p} \lesssim \|u\|_{L^q} \|\partial_x^k v\|_{L^r} + \|v\|_{L^q} \|\partial_x^k u\|_{L^r} \quad for \quad k \geq 0, \tag{2.43}$$

$$\|[\partial_x^k, u]\partial_x v\|_{L^p} \lesssim \|\partial_x u\|_{L^q} \|\partial_x^k v\|_{L^r} + \|\partial_x v\|_{L^q} \|\partial_x^k u\|_{L^r} \quad for \quad k \geq 1, \tag{2.44}$$

where $[\ ,\ ]$ stands for the commutator which is defined by $[A, B] := AB - BA$.

Proof. We state the detailed proof given in [74, 75]. Firstly, we show the following estimate: Let k_1 and k_2 be non-negative integers and put $k = k_1 + k_2$. Let $1 \leq p, q, r \leq \infty$ and $1/p = 1/q + 1/r$. Then we have

$$\|\partial_x^{k_1} u_1 \partial_x^{k_2} u_2\|_{L^p} \lesssim \|u_1\|_{L^q} \|\partial_x^k u_2\|_{L^r} + \|u_2\|_{L^q} \|\partial_x^k u_1\|_{L^r}. \tag{2.45}$$

To prove this, we put $\theta_j = k_j/k, j = 1, 2$ and define p_j by

$$\frac{1}{p_j} = \frac{1-\theta_j}{q} + \frac{\theta_j}{r},$$

where $j = 1, 2$. Since $\theta_1 + \theta_2 = 1$, we have $1/p = 1/p_1 + 1/p_2$. Therefore, applying the Hölder inequality and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\partial_x^{k_1} u_1 \partial_x^{k_2} u_2\|_{L^p} &\le \|\partial_x^{k_1} u_1\|_{L^{p_1}} \|\partial_x^{k_2} u_2\|_{L^{p_2}} \\ &\lesssim (\|u_1\|_{L^q}^{1-\theta_1} \|\partial_x^k u_1\|_{L^r}^{\theta_1})(\|u_2\|_{L^q}^{1-\theta_2} \|\partial_x^k u_2\|_{L^r}^{\theta_2}) \\ &\lesssim (\|u_1\|_{L^q} \|\partial_x^k u_2\|_{L^r})^{\theta_2} (\|u_2\|_{L^q} \|\partial_x^k u_1\|_{L^r})^{\theta_1} \\ &\lesssim \|u_1\|_{L^q} \|\partial_x^k u_2\|_{L^r} + \|u_2\|_{L^q} \|\partial_x^k u_1\|_{L^r},\end{aligned}$$

where we have used the Young's inequality based on the relation $\theta_1 + \theta_2 = 1$. This proves (2.45). Since $\partial_x^k(uv)$ consists of terms of the form $\partial_x^{k_1} u \partial_x^{k_2} v$ with $k_1 + k_2 = k$, the estimate (2.43) directly follows from (2.45).

Next, we show (2.44) for $k \ge 1$. We see that $[\partial_x^k, u]\partial_x v = \partial_x^k(u\partial_x v) - u\partial_x^k(\partial_x v)$ consists of the form $\partial_x^{k_1} u \partial_x^{k_2}(\partial_x v)$ with $k_1 + k_2 = k$, where $k_1 \ge 1, k_2 \ge 0$. These terms are written as $\partial_x^{l_1}(\partial_x u)\partial_x^{k_2}(\partial_x v)$ with $l_1 + k_2 = k - 1$, where $l_1 \ge 0, k_2 \ge 0$. Therefore, applying (2.45), we have

$$\begin{aligned}\|\partial_x^{l_1}(\partial_x u)\partial_x^{k_2}(\partial_x v)\|_{L^p} &\lesssim \|\partial_x u\|_{L^q} \|\partial_x^{k-1}\partial_x v\|_{L^r} + \|\partial_x v\|_{L^q} \|\partial_x^{k-1}\partial_x u\|_{L^r} \\ &\lesssim \|\partial_x u\|_{L^q} \|\partial_x^k v\|_{L^r} + \|\partial_x v\|_{L^q} \|\partial_x^k u\|_{L^r}.\end{aligned}$$

This shows (2.44). Hence, the proof of Lemma 2.18 is completed. □

In [74, 86], the authors showed a variant of the estimate (2.45) as the the following.

Lemma 2.19. *([74, 86]) Assume that $n \ge 2$ is an integer, $k_1, k_2, \cdots, k_n$ are non-negative integers, $1 \le p, q, r \le \infty$ and $1/p = 1/q + 1/r$. Put $k = k_1 + k_2 + \cdots + k_n$. Then there exists a positive constant $C = C(n, p, q, r, k)$ such that the inequality*

$$\Big\| \prod_{j=1}^n (\partial_x^{k_j} u_j) \Big\|_{L^p} \le C \|u\|_{L^\infty}^{n-2} \|u\|_{L^q} \|\partial_x^k u\|_{L^r} \tag{2.46}$$

holds for any $u = (u_1, u_2, \cdots, u_n)$.

Proof. Define p_j by

$$\frac{1}{p_j} = (1 - \frac{k_j}{k})\frac{1}{(n-1)q} + \frac{k_j}{k}\frac{1}{r}$$

for $j = 1, 2, \cdots, n$. Then it holds that $p \le p_j \le \infty$ for $j = 1, 2, \cdots, n$ and $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_n$. Therefore, by Hölder's inequality we obtain

$$\Big\| \prod_{j=1}^n (\partial_x^{k_j} u_j) \Big\|_{L^p} \le \prod_{j=1}^n \|\partial_x^{k_j} u_j\|_{L^{p_j}}. \tag{2.47}$$

Here, Gagliardo-Nirenberg's interpolation inequalities imply that

$$\begin{aligned}\|\partial_x^{k_j} u_j\|_{L^{p_j}} &\le C\|u_j\|_{L^{(n-1)q}}^{1-\frac{k_j}{k}}\|\partial_x^k u_j\|_{L^r}^{\frac{k_j}{k}}\\ &\le C(\|u_j\|_{L^\infty}^{1-\frac{1}{n-1}}\|u_j\|_{L^q}^{\frac{1}{n-1}})^{1-\frac{k_j}{k}}\|\partial_x^k u_j\|_{L^r}^{\frac{k_j}{k}}\\ &\le C\|u_j\|_{L^\infty}^{(1-\frac{1}{n-1})(1-\frac{k_j}{k})}\|u_j\|_{L^q}^{\frac{1}{n-1}(1-\frac{k_j}{k})}\|\partial_x^k u_j\|_{L^r}^{\frac{k_j}{k}}.\end{aligned}$$

Substituting the above inequality into (2.47) yields the desired inequality. □

Lemma 2.20. *([99]) Let $1 \le p, q, r \le \infty$ with $1/p = 1/q + 1/r$, and let k be a non-negative integer. Suppose that $f(u)$ is a smooth function of u satisfying $f(u) = O(1)$. Then we have*

$$\|\partial_x^k(f(u)u)\|_{L^p} \le C(1+\|u\|_{L^\infty})^k\|\partial_x^k u\|_{L^p}, \tag{2.48}$$

$$\|\partial_x^k(f(u)u^N)\|_{L^p} \le C(1+\|u\|_{L^\infty})^k\|u\|_{L^\infty}^{N-2}\|u\|_{L^q}\|\partial_x^k u\|_{L^r}, \tag{2.49}$$

where N is an integer with $N \ge 2$. Here $C = C(\|u\|_{L^\infty})$ is a quantity depending on $\|u\|_{L^\infty}$ and is regarded as an increasing function of $\|u\|_{L^\infty}$.

Proof. The proof is from [99]. Firstly, we prove (2.48). The estimate (2.48) is trivial for $k = 0$. When $k \ge 1$, we have

$$\partial_x^k(f(u)u) = f(u)\partial_x^k u + [\partial_x^k, f(u)]u. \tag{2.50}$$

Obviously, we have $\|f(u)\partial_x^k u\|_{L^p} \le C\|\partial_x^k u\|_{L^p}$ for some $C = C(\|u\|_{L^\infty})$. On the other hand, $[\partial_x^k, f(u)]u$ consists of the terms of the form

$$g(u)\prod_{j=1}^{J}\partial_x^{k_j}u,$$

where $2 \le J \le k+1, k = k_1 + \cdots + k_J$, and $g(u) = O(1)$. By using (2.47) with $q = \infty$ and $r = p$, we can estimate this term as

$$\Big\|g(u)\prod_{j=1}^{J}\partial_x^{k_j}u\Big\|_{L^p} \le C\|u\|_{L^\infty}^{J-1}\|\partial_x^k u\|_{L^p},$$

where $C = C(\|u\|_{L^\infty})$. Therefore, we obtain

$$\begin{aligned}\|\partial_x^k(f(u)u)\|_{L^p} &\le C\|\partial_x^k u\|_{L^p} + C\sum_{J=2}^{k+1}\|u\|_{L^\infty}^{J-1}\|\partial_x^k u\|_{L^p}\\ &\le C(1+\|u\|_{L^\infty})^k\|\partial_x^k u\|_{L^p}.\end{aligned}$$

Next we prove (2.49) for $N = 2$. By applying (2.43) and (2.48), we get

$$\begin{aligned}\|\partial_x^k(f(u)u^2)\|_{L^p} &\le C(\|\partial_x^k(f(u)u)\|_{L^r}\|u\|_{L^q} + \|f(u)u\|_{L^q}\|\partial_x^k u\|_{L^r})\\ &\le C((1+\|u\|_{L^\infty})^k\|\partial_x^k u\|_{L^r}\|u\|_{L^q} + \|u\|_{L^q}\|\partial_x^k u\|_{L^r})\\ &\le C(1+\|u\|_{L^\infty})^k\|u\|_{L^q}\|\partial_x^k u\|_{L^r},\end{aligned}$$

which prove (2.49) for $N = 2$. The general case can be proved by the induction argument on N so that we omit the details. Hence, the proof of Lemma 2.20 is completed. □

Lemma 2.21. *(Gronwall's inequality, see [46]) Let $x(t), f(t)$ and $g(t)$ be real continuous functions and $g(t) \geq 0$ for $t \in [a, b]$. Suppose that for $t \in [a, b]$ it holds that the following inequality*

$$x(t) \leq f(t) + \int_a^t g(s)x(s)ds.$$

Then, for $t \in [a, b]$ it holds that

$$x(t) \leq f(t) + \int_a^t f(s)g(s) \exp\left[\int_s^t g(\tau)d\tau\right] ds.$$

Proof. Consider the function $y(t) := \int_a^t g(\tau)x(\tau)d\tau, t \in [a, b]$. Then we have $y(a) = 0$ and

$$\begin{aligned} y'(t) = g(t)x(t) &\leq g(t)f(t) + g(t)\int_a^t g(s)x(s)ds \\ &= f(t)g(t) + g(t)y(t), \quad t \in [a, b]. \end{aligned}$$

Multiply the above equality with $\exp(-\int_a^t g(s)ds) > 0$, we obtain

$$\frac{d}{dt}\left(y(t)\exp\left(-\int_a^t g(s)ds\right)\right) \leq f(t)g(t)\exp\left(-\int_a^t g(s)ds\right).$$

Integrating the above inequality over $[a, t]$, one has

$$y(t)\exp\left(-\int_a^t g(s)ds\right) \leq \int_a^t f(\tau)g(\tau)\exp\left(-\int_a^\tau g(s)ds\right)d\tau.$$

Then we have

$$y(t) \leq \int_a^t f(\tau)g(\tau)\exp\left(\int_\tau^t g(s)ds\right)d\tau, \quad t \in [a, b].$$

Thus, the proof of Lemma 2.21 is completed. □

Lemma 2.22. *(see [129, 154, 157]) Let $a > 0$ and $b > 0$ be constants. Then, it holds that*

$$\begin{aligned} &\int_0^t (1 + t - s)^{-a}(1 + s)^{-b}ds \lesssim (1 + t)^{-\min(a,b)}, \quad \text{if } \max(a, b) > 1, \\ &\int_0^t (1 + t - s)^{-a}(1 + s)^{-b}ds \lesssim (1 + t)^{-\min(a,b)}\ln(2 + t), \quad \text{if } \max(a, b) = 1, \\ &\int_0^t (1 + t - s)^{-a}(1 + s)^{-b}ds \lesssim (1 + t)^{1-a-b}, \quad \text{if } \max(a, b) < 1. \end{aligned}$$

Proof. We write

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds = \Big(\int_0^{t/2} + \int_{t/2}^t\Big)(\cdots)ds := J_1 + J_2.$$

We estimate J_1, J_2 in case of $\max(a,b) > 1$ and $\max(a,b) \le 1$. If $\max(a,b) > 1$ and assume $a < b$ with loss of generality, we have

$$\begin{aligned} J_1 + J_2 &\lesssim (1+t)^{-a}\int_0^{t/2}(1+s)^{-b}ds + \Big(1+\frac{t}{2}\Big)^{-b}\int_{t/2}^t (1+t-s)^{-a}ds \\ &\lesssim (1+t)^{-a}\left(1-\Big(1+\frac{t}{2}\Big)^{-b+1}\right) + \Big(1+\frac{t}{2}\Big)^{-b}\left(\Big(1+\frac{t}{2}\Big)^{-a+1}-1\right) \\ &\lesssim (1+t)^{-a}, \end{aligned}$$

which implies the first inequality. On one hand, if $\max(a,b) \le 1$ and assume $a < b$, for J_1 we get

$$\begin{aligned} J_1 &\lesssim (1+t)^{-a}\int_0^{t/2}(1+s)^{-b}ds \\ &\lesssim \begin{cases} (1+t)^{-a}\ln\big(1+\frac{t}{2}\big) & \text{if } \max(a,b)=1; \\ (1+t)^{-a}\Big(1+\big(1+\frac{t}{2}\big)^{-b+1}\Big) & \text{if } \max(a,b)<1 \end{cases} \\ &\lesssim \begin{cases} (1+t)^{-a}\ln(2+t) & \text{if } \max(a,b)=1; \\ (1+t)^{1-a-b} & \text{if } \max(a,b)<1. \end{cases} \end{aligned}$$

On the other hand, for J_2 we obtain

$$\begin{aligned} J_2 &\lesssim \Big(1+\frac{t}{2}\Big)^{-b}\int_{t/2}^t (1+t-s)^{-a}ds \\ &\lesssim \Big(1+\frac{t}{2}\Big)^{-b}\left(\Big(1+\frac{t}{2}\Big)^{-a+1}-1\right) \\ &\lesssim \begin{cases} (1+t)^{-a} & \text{if } \max(a,b)=1; \\ (1+t)^{1-a-b} & \text{if } \max(a,b)<1. \end{cases} \end{aligned}$$

Combining the above inequalities gives the last two inequalities in Lemma 2.22. Hence, the proof of Lemma 2.22 is completed. □

Chapter 3

Well-posedness of Cauchy problem for Landau equation

Contents

In this chapter, we are devoted to study global existence of solutions to the inhomogeneous nonlinear Landau equation with Maxwellian molecules motivated by the results of [52, 122, 145]. Here, we consider this Cauchy problem in Besov space if the initial datum is a small perturbation of the equilibrium distribution in the Chemin-Lerner space $\widetilde{L}^2_v(B^{3/2}_{2,1})$.

3.1 Description of problem

In this chapter, we study the Cauchy problem (1.1) with Maxwellian molecules, since the Landau operator enjoys very nice spectral property in that case. Here, we are concerned with the Landau equation around the absolute Gaussian distribution in $\mathbb{R}^3$:

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

With the perturbation $F(t,x,v) = \mu(v) + \sqrt{\mu}(v)g(t,x,v)$, the Cauchy problem (1.1) can be rewritten as

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}g = \mathbf{L}(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{3.1}$$

with $F_0 = \mu(v) + \sqrt{\mu}(v)g_0$, where $\mathcal{L}(g), \mathbf{L}(g,g)$ are given by (1.3)-(1.4).

There are many results about the well-posedness to the Boltzmann and Landau equation, see for example, [57, 58, 62, 82, 146, 174, 175] and references therein. However, there are few results concerning the global existence for the Landau equation in spatially critical Besov spaces. Recently, Duan-Liu-Xu [52] first introduced the Chemin-Lerner type spaces involving the microscopic velocity and established the global existence of strong solutions near Maxwellian for the cut-off Boltzmann equation in the critical Besov space $\widetilde{L}^{\infty}_T \widetilde{L}^2_v(B^{3/2}_{2,1})$. Morimoto and Sakamoto [145] studied global solutions of the Cauchy problem to the non-cutoff Boltzmann equation in a critical Chemin-Lerner space by using the triple norm that was introduced by Alexandre-Morimoto-Ukai-Xu-Yang [6, 9].

Motivated by those works, the main goal of this chapter is to consider the global existence of solutions to (3.1). We investigate the Cauchy problem (3.1) with Maxwellian molecules ($\gamma = 0$) in this thesis. Our main result is shown as the following theorem (also, **Main Theorem A** in Chapter 1).

Theorem 3.1. *There exists a constant $\varepsilon_0 > 0$ such that if $g_0 \in \widetilde{L}^2_v(B^{3/2}_{2,1})$ and*

$$\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \leq \varepsilon_0,$$

then the Cauchy problem (3.1) *admits a global solution satisfying*

$$g \in \widetilde{L}^{\infty}_t \widetilde{L}^2_v(B^{3/2}_{2,1}) \quad \text{and} \quad \mathcal{L}^{\frac{1}{2}} g \in \widetilde{L}^2_t \widetilde{L}^2_v(B^{3/2}_{2,1}).$$

Moreover, if $F_0(x,v) = \mu(v) + \sqrt{\mu}(v)g_0(x,v) \geq 0$, then $F(t,x,v) = \mu(v) + \sqrt{\mu}(v)g(t,x,v) \geq 0$.

■ **The schema of proof of Theorem 3.1**

The proof of Theorem 3.1 is elaborated and its schema is stated as followings.

• Spectral analysis of Landau collision operator (Section 3.2)

Firstly, we recall the spectral analysis properties and some key estimates of Landau collision operator, which will be used in subsequent sections.

• Nonlinear estimates of Landau collision operator (Section 3.3)

Nonlinear estimates of Landau collision operator play a key role to our proof. Here, the methods of estimating collision operator in [52, 145] will be ineffective because the collision operator between the Boltzmann equation and Landau equation are fundamentally different. Recently, Lerner-Morimoto-Pravda-Starov-Xu [114] showed that the linearized non-cutoff Boltzmann operator with Maxwellian molecules is exactly equal to a fractional power of the linearized Landau operator which is the sum of the Harmonic Oscillator and the spherical Laplacian. Li-Xu [122] proved some estimates of nonlinear Landau term by using spectral analysis and proved the existence of weak solution for the Cauchy problem with initial datum belonging to Shubin space of negative index which includes the probability measures. With the aid of these spectral analysis results, we establish some new trilinear estimates of the Landau collision operator.

• Commutator estimates (Section 3.4)

To improve the regularity of weak solution, we proved some commutator estimates for the Landau collision operator, which will be used to prove the local existence of solutions.

• The local-in-time existence (Section 3.5)

Firstly, we are devoted to the local existence to following linearized Landau equation in spatially critical Besov space $\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})$

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = \mathbf{L}(f,g) - \mathcal{L}_2 f, \\ g(t,x,v)|_{t=0} = g_0(x,v). \end{cases} \tag{3.2}$$

Step 1) The existence of weak solution

Due to the fact that the dual space of $\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})$ is complicated. Usually, the dual space of $L^{\infty}(\mu)$ appears as the finitely additive finite (signed) measures which are absolutely continuous with respect to μ, equipped with the total variation norm. To do this, we firstly try to find a weak solution g to the linearized equation (3.2) in the wider space $L^{\infty}([0,T];L^{2}(\mathbb{R}^{6}_{x,v}))$ by using the duality argument and the Hahn-Banach extension theorem in Theorem 3.13.

Step 2) Mollifier of weak solution

Since for $g \in L^{\infty}([0,T];L^{2}(\mathbb{R}^{6}_{x,v}))$, for the terms in (3.2) we only have

$$v \cdot \nabla_x g \in H^{-1}_{x}L^{2}_{v,-1}, \quad \mathcal{L}_1 g \in L^{2}_{x}H^{-2}_{v,-2}, \quad \mathbf{L}(f,g) \in L^{2}_{x}H^{-2}_{v,-2}, \quad \mathcal{L}_2 f \in B^{3/2}_{x,2,1}H^{-2}_{v,-2}.$$

So, we need to mollifier the function g and take its as right test function. Here, taking different $1 > \delta, \delta' > 0$, we use a weighted function $W_{\delta'}(v) = \langle \delta' v \rangle^{-2}$ and mollifiers $M^{\delta}(D_v), S_{\delta}(D_x)$ defined in Section 3.4. So that we can take

$$W_{\delta'}S_{\delta}(M^{\delta})^{2}S_{\delta}W_{\delta'}g \in H^{+\infty}_{x}H^{4}_{v,4},$$

as test function to the equation (3.2). The estimates of commutators with the mollifier operators $W_{\delta'}, S_{\delta}$ and M^{δ} are complicated and shown in Section 3.4. We also need to mollifier the function

f as following $f_N = \sum_{p\geq -1}^{N-1} \Delta_p f$ for $N \in \mathbb{N}$, then for f_N we get

$$f_N \in \widetilde{L}_T^{\infty}\widetilde{L}_v^2(H_x^{+\infty}) \text{ and } \|f_N\|_{\widetilde{L}_T^{\infty}\widetilde{L}_v^2(B_{2,1}^{3/2})} \leq \|f\|_{\widetilde{L}_T^{\infty}\widetilde{L}_v^2(B_{2,1}^{3/2})}.$$

For the sequence $\{f_N, N \in \mathbb{N}\}$, we consider a sequence of weak solution $\{g_N, N \in \mathbb{N}\} \subset L^{\infty}([0,T]; L^2(\mathbb{R}^6_{x,v}))$ to the following Cauchy problem

$$\begin{cases} \partial_t g_N + v \cdot \nabla_x g_N + \mathcal{L}_1 g_N = \mathbf{L}(f_N, g_N) - \mathcal{L}_2 f_N, \\ g_N(t,x,v)|_{t=0} = g_0(x,v). \end{cases} \tag{3.3}$$

Step 3) Regularity of weak solution in velocity variable

For the weak solution g_N, based on these commutator estimates given in Section 3.4, we prove Proposition 3.14

$$\|g_N\|_{L_T^{\infty}L_v^2L_x^2} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{L_T^2L_v^2L_x^2} \leq C\|g_0\|_{L_v^2L_x^2} + C\sqrt{T}\|\mathbb{S}_2 f\|_{L_T^{\infty}L_v^2L_x^2}$$

under the assumption that $\|f\|_{\widetilde{L}_T^{\infty}\widetilde{L}_v^2(B_{2,1}^{3/2})}$ is sufficiently small. Remark that we can't do the iteration by using above estimates, since there is no regularity in position variable x for the weak solution g_N while f satisfies the condition $f \in \widetilde{L}_T^{\infty}\widetilde{L}_v^2(B_{2,1}^{3/2})$ and its norm is small.

Step 4) Regularity of weak solution in position variable

In this step, we mainly prove the regularity of weak solution g_N in position variable x for Besov norm, we obtain Proposition 3.15 that

$$\|g_N\|_{\widetilde{L}_T^{\infty}\widetilde{L}_v^2(B_{2,1}^{3/2})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{\widetilde{L}_T^2\widetilde{L}_v^2(B_{2,1}^{3/2})} < +\infty.$$

But here, we can only get a upper bound dependent of N. So that this step is a technical step, but very important to give a rigorous proof.

Step 5) Energy estimates in Besov space

We prove the following energy estimate for the weak solution g_N in Proposition 3.16,

$$\|g_N\|_{\widetilde{L}_T^{\infty}\widetilde{L}_v^2(B_{2,1}^{3/2})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{\widetilde{L}_T^2\widetilde{L}_v^2(B_{2,1}^{3/2})} \leq C\|g_0\|_{\widetilde{L}_v^2(B_{2,1}^{3/2})} + C\sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}_T^2\widetilde{L}_v^2(B_{2,1}^{3/2})}.$$

The convergence of this sequence is then standard. The local existence to the linearized Landau equation is proved in Theorem 3.6 and shown as

$$g(t,v,x) \in \widetilde{L}_T^{\infty}\widetilde{L}_v^2(B_{2,1}^{3/2}) \quad \text{and} \quad \widetilde{\mathcal{L}}^{\frac{1}{2}} g(t,v,x) \in \widetilde{L}_T^2\widetilde{L}_v^2(B_{2,1}^{3/2}).$$

Step 6) The local existence to the nonlinear Landau equation

Finally based on the Theorem 3.6, we employ the Picard's iteration scheme to prove the local existence of solution to (3.1) in Proposition 3.17.

• The global-in-time solution (Section 3.6)

A priori estimates is crucial to develop the local existence of solution to global existence. To obtain that, the following steps is necessary.

Step 1) Estimate on the macroscopic dissipation

We cannot deduce the dissipative estimate for the macroscopic part $\mathbf{P}g$ directly. To overcome the difficulty, as in [52, 145], we shall perform the macro-micro decomposition and deduce a fluid dynamics system of macroscopic projection of g. Consequently, by using the standard energy method, we can obtain the estimate on the macroscopic dissipation in Proposition 3.18.

Step 2) A priori estimate

With the aid of macro-micro decomposition, we can obtain another form of estimate on the nonlinear term $\mathbf{L}(\mathbf{g},\mathbf{g})$ in Proposition 3.19. With Propositions 3.18-3.19, a global priori estimate is given in Proposition 3.20. Combining the local existence result in Proposition 3.17 and a priori estimate in Proposition 3.20, we prove the global-in-time existence in Theorem 3.1 by the standard continuity argument.

3.2 Spectral analysis of Landau collision operator

We recall some spectral properties of Landau operator briefly, see [16, 114, 121, 122] for more details. Firstly, one has an explicit expression for the linearized Landau operator with Maxwellian molecules.

Lemma 3.1. *([114]) The linearized Landau operator with Maxwellian molecules can be written as*

$$\mathcal{L}g = -\mu^{-\frac{1}{2}}\Big(Q_L(\mu,\sqrt{\mu}g) + Q_L(\sqrt{\mu}g,\mu)\Big) := \mathcal{L}_1 g + \mathcal{L}_2 g,$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are equal to

$$\mathcal{L}_1 = (d-1)(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2}) - \Delta_{\mathbb{S}^{d-1}},$$
$$\mathcal{L}_2 = [\Delta_{\mathbb{S}^{d-1}} - (d-1)(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2})]\mathbb{P}_1 + [-\Delta_{\mathbb{S}^{d-1}} - (d-1)(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2})]\mathbb{P}_2.$$

Here and below, $\Delta_{\mathbb{S}^{d-1}}$ stands for the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{d-1}$ and $\mathbb{P}_k(k=1,2)$ is the orthogonal projection onto the Hermite basis.

There is the algebra property of nonlinear Landau operators on the basis $\{\varphi_{n,l,m}\}$ (see [121, 122] and referencein), for $n,l \in \mathbb{N}, m \in \mathbb{Z}, |m| \le l$, let's denote

$$\varphi_{n,l,m}(v) = \left(\frac{n!}{\sqrt{2}\,\Gamma(n+l+3/2)}\right)^{1/2}\left(\frac{|v|}{\sqrt{2}}\right)^l e^{-\frac{|v|^2}{4}} L_n^{(l+1/2)}\left(\frac{|v|^2}{2}\right) Y_l^m\left(\frac{v}{|v|}\right),$$

where $\Gamma(\,\cdot\,)$ is the standard Gamma function, and – $L_n^{(\alpha)}$ is the Laguerre polynomial of order α and degree n,

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n}(-1)^{n-r}\frac{\Gamma(\alpha+n+1)}{r!(n-r)!\Gamma(\alpha+n-r+1)}x^{n-r};$$

– $Y_l^m(\sigma)$ is the orthonormal basis of spherical harmonics

$$Y_l^m(\sigma)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)e^{im\phi},\ \ |m|\le l.$$

The notations $\sigma=(\cos\theta,\sin\theta\cos\phi,\sin\theta\sin\phi)$, where the angle θ is the zenith angle and belongs to $(0,\pi)$, the angle ϕ is the azimuth angle and belongs to $(-\pi,\pi)$; $N_{l,m}$ are the normalisation factor and $P_l^{|m|}$ is the Legendre functions of the first kind of order l and degree $|m|$

$$P_l^{|m|}(x)=(1-x^2)^{\frac{|m|}{2}}\frac{d^{|m|}}{dx}\left(\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l\right).$$

From [114, 116], we see that those spherical harmonics are equivalent to the real spherical harmonics $\widetilde{Y_l^m}(\sigma)$ for $l\ge 0$ and $-l\le m\le l$, which are defined by $\widetilde{Y_0^0}(\sigma)=\frac{1}{\sqrt{4\pi}}$ and for any $l\ge 1$,

$$\widetilde{Y_l^m}(\sigma)=\begin{cases}\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta), & m=0;\\ \sqrt{\frac{2l+1}{2\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)\cos m\phi, & m=1,2,\cdots,l;\\ \sqrt{\frac{2l+1}{2\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)\sin m\phi, & m=-1,-2,\cdots,-l.\end{cases}$$

Therefore, we deduce that $\{\varphi_{n,l,m}\}\subset\mathcal{S}(\mathbb{R}^3)$ and

$$\begin{aligned}&\varphi_{0,0,0}(v)=\sqrt{\mu}, && \varphi_{0,1,0}(v)=v_1\sqrt{\mu},\\ &\varphi_{0,1,1}(v)=\frac{v_2+iv_3}{\sqrt{2}}\sqrt{\mu}, && \varphi_{0,1,-1}(v)=\frac{v_2-iv_3}{\sqrt{2}}\sqrt{\mu},\\ &\varphi_{1,0,0}(v)=\sqrt{\frac{2}{3}}\left(\frac{3}{2}-\frac{|v|^2}{2}\right)\sqrt{\mu}.\end{aligned}$$

In addition, the explicit form of the eigenfunctions $\{\varphi_{0,2,m_2},|m_2|\le 2\}$ satisfies that

$$\begin{aligned}&\varphi_{0,2,0}(v)=\sqrt{\tfrac{1}{3}}\left(\tfrac{3}{2}v_1^2-\tfrac{1}{2}|v|^2\right)\sqrt{\mu}, && \varphi_{0,2,1}(v)=\tfrac{v_1v_2+iv_1v_3}{\sqrt{2}}\sqrt{\mu},\\ &\varphi_{0,2,-1}(v)=\tfrac{v_1v_2-iv_1v_3}{\sqrt{2}}\sqrt{\mu}, && \varphi_{0,2,2}(v)=\left(\tfrac{v_2^2-v_3^2}{2\sqrt{2}}+i\tfrac{v_2v_3}{\sqrt{2}}\right)\sqrt{\mu},\\ &\varphi_{0,2,-2}(v)=\left(\tfrac{v_2^2-v_3^2}{2\sqrt{2}}-i\tfrac{v_2v_3}{\sqrt{2}}\right)\sqrt{\mu}\end{aligned}$$

and the eigenfunctions $\{\varphi_{1,1,m_1},|m_1|\le 1\}$ are given by

$$\begin{aligned}&\varphi_{1,1,0}(v)=\tfrac{1}{\sqrt{10}}\left(5-|v|^2\right)v_1\sqrt{\mu}, && \varphi_{1,1,1}(v)=\tfrac{1}{\sqrt{10}}\left(5-|v|^2\right)\tfrac{v_2+iv_3}{\sqrt{2}}\sqrt{\mu},\\ &\varphi_{1,1,-1}(v)=\tfrac{1}{\sqrt{10}}\left(5-|v|^2\right)\tfrac{v_2-iv_3}{\sqrt{2}}\sqrt{\mu}.\end{aligned}$$

In the following, we recall the algebraic property of nonlinear Landau operator that is given in [122].

Proposition 3.1. *([122]) For $n, l \in \mathbb{N}$, $|m| \le l$, we have*

$$
\begin{aligned}
&(i)\quad \mathbf{L}(\varphi_{0,0,0}, \varphi_{n,l,m}) = -\left(2(2n+l)+l(l+1)\right)\varphi_{n,l,m};\\
&(ii)\quad \mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m})\\
&\qquad = A^{-}_{n,l,m,m_1}\varphi_{n+1,l-1,m_1+m} + A^{+}_{n,l,m,m_1}\varphi_{n,l+1,m_1+m},\quad \forall\, |m_1| \le 1;\\
&(iii)\quad \mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m}) = \frac{4\sqrt{3(n+1)(2n+2l+3)}}{3}\varphi_{n+1,l,m};\\
&(iv)\quad \mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m}) = A^{1}_{n,l,m,m_2}\varphi_{n+2,l-2,m+m_2}\\
&\qquad + A^{2}_{n,l,m,m_2}\varphi_{n+1,l,m+m_2} + A^{3}_{n,l,m,m_2}\varphi_{n,l+2,m+m_2},\quad \forall\, |m_2| \le 2;\\
&(v)\quad \mathbf{L}(\varphi_{\tilde{n},\tilde{l},\tilde{m}}, \varphi_{n,l,m}) = 0,\quad \forall\, 2\tilde{n}+\tilde{l} > 2,\quad |\tilde{m}| \le \tilde{l}.
\end{aligned}
$$

where the coefficients are defined as follows

$$
\begin{aligned}
A^{-}_{n,l,m,m_1} &= 4\sqrt{\frac{\pi}{3}}(l-1)\sqrt{2(n+1)}\int_{\mathbb{S}^2} Y_1^{m_1}(\omega)Y_l^{m}(\omega)Y_{l-1}^{-m_1-m}(\omega)d\omega;\\
A^{+}_{n,l,m,m_1} &= 4\sqrt{\frac{\pi}{3}}(l+2)\sqrt{2n+2l+3}\int_{\mathbb{S}^2} Y_1^{m_1}(\omega)Y_l^{m}(\omega)Y_{l+1}^{-m_1-m}(\omega)d\omega;
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
A^{1}_{n,l,m,m_2} &= -4\sqrt{\frac{\pi}{15}}\sqrt{4(n+2)(n+1)}\int_{\mathbb{S}^2} Y_2^{m_2}(\omega)Y_l^{m}(\omega)Y_{l-2}^{-m_2-m}(\omega)d\omega;\\
A^{2}_{n,l,m,m_2} &= 4\sqrt{\frac{\pi}{15}}\sqrt{2(n+1)(2n+2l+3)}\\
&\qquad\times\int_{\mathbb{S}^2} Y_2^{m_2}(\omega)Y_l^{m}(\omega)Y_{l}^{-m_2-m}(\omega)d\omega;\\
A^{3}_{n,l,m,m_2} &= -4\sqrt{\frac{\pi}{15}}\sqrt{(2n+2l+5)(2n+2l+3)}\\
&\qquad\times\int_{\mathbb{S}^2} Y_2^{m_2}(\omega)Y_l^{m}(\omega)Y_{l+2}^{-m_2-m}(\omega)d\omega.
\end{aligned}
$$

It follows from Proposition 3.1 that, for suitable function f, g,

$$
\begin{aligned}
&\Big(\mathbf{L}(f,g), \varphi_{n,l,m}\Big)_{L^2}\\
&= -\left(2(2n+l)+l(l+1)\right) f_{0,0,0}(t)g_{n,l,m}(t)\\
&\quad + \sum_{\substack{|m^*|\le l+1, |m_1|\le 1\\ m_1+m^*=m}} A^{-}_{n-1,l+1,m^*,m_1} f_{0,1,m_1}(t)g_{n-1,l+1,m^*}(t)\\
&\quad + \sum_{\substack{|m^*|\le l-1, |m_1|\le 1\\ m_1+m^*=m}} A^{+}_{n,l-1,m^*,m_1} f_{0,1,m_1}(t)g_{n,l-1,m^*}(t)\\
&\quad + \frac{4\sqrt{3n(2n+2l+1)}}{3} f_{1,0,0}(t)g_{n-1,l,m}(t)
\end{aligned}
$$

$$+\sum_{\substack{|m^*|\le l+2,|m_2|\le 2\\ m^*+m_2=m}} A^1_{n-2,l+2,m^*,m_2} f_{0,2,m_2}(t) g_{n-2,l+2,m^*}(t)$$
$$+\sum_{\substack{|m^*|\le l,|m_2|\le 2\\ m^*+m_2=m}} A^2_{n-1,l,m^*,m_2} f_{0,2,m_2}(t) g_{n-1,l,m^*}(t)$$
$$+\sum_{\substack{|m^*|\le l-2,|m_2|\le 2\\ m^*+m_2=m}} A^3_{n,l-2,m^*,m_2} f_{0,2,m_2}(t) g_{n,l-2,m^*}(t)$$

with the conventions

$$g_{n,l,m} \equiv 0, \qquad \text{if} \quad n<0 \quad \text{or} \quad l<0$$

and

$$(\mathcal{L}(g), \varphi_{n,l,m})_{L^2} = \lambda_{n,l}\, g_{n,l,m}(t)$$

with $g_{n,l,m}(t) = \langle g,\, \varphi_{n,l,m}\rangle$, $n, l \in \mathbb{N}$, $m \in \mathbb{Z}$, $|m| \le l$.

Furthermore, those coefficients A^1, A^2 and A^3 satisfy the the following estimates.

Proposition 3.2. *([122]) It holds that*
i) For $n, l \in \mathbb{N}$, $n \ge 2$,

$$\max_{|m^*|\le l} \sum_{\substack{|m|\le l+2,|m_2|\le 2\\ m+m_2=m^*}} \left|A^1_{n-2,l+2,m,m_2}\right|^2 \le \frac{16n(n-1)}{3}; \tag{3.5}$$

ii) For $n, l \in \mathbb{N}$, $n \ge 1$,

$$A^2_{n-1,0,0,0} = 0;$$
$$\max_{|m^*|\le l} \sum_{\substack{|m|\le l,|m_2|\le 2\\ m+m_2=m^*}} \left|A^2_{n-1,l,m,m_2}\right|^2 \le \frac{4n(2n+2l+1)}{3}, \quad \forall\, l \ge 1; \tag{3.6}$$

iii) For $n, l \in \mathbb{N}$, $l \ge 2$,

$$\max_{|m^*|\le l} \sum_{\substack{|m|\le l-2,|m_2|\le 2\\ m+m_2=m^*}} \left|A^3_{n,l-2,m,m_2}\right|^2 \le \frac{(2n+2l+1)(2n+2l-1)}{2}. \tag{3.7}$$

We prove the following estimates for the coefficients A^- and A^+ .

Proposition 3.3. *For the coefficients of the Proposition 3.1 defined in* (3.4)*, we have*
i) For $n, l \in \mathbb{N}$, $n \ge 1$,

$$\max_{|m^*|\le l} \sum_{\substack{|m|\le l+1,|m_1|\le 1\\ m+m_1=m^*}} \left|A^-_{n-1,l+1,m,m_1}\right|^2 \le 4nl(l+1). \tag{3.8}$$

ii) For $n, l \in \mathbb{N}$, $l \ge 1$,

$$\max_{|m^*|\le l} \sum_{\substack{|m|\le l-1,|m_1|\le 1\\ m+m_1=m^*}} \left|A^+_{n,l-1,m,m_1}\right|^2 \le 2(2n+2l+1)(l+1)^2. \tag{3.9}$$

Proof. It follows from (3.4) that

$$A^-_{n-1,l+1,m,m_1} = 4l\sqrt{\frac{2n\pi}{3}}\Big(\int_{\mathbb{S}^2} Y_1^{m_1}(\omega)Y_{l+1}^m(\omega)Y_l^{-m_1-m}(\omega)d\omega\Big);$$
$$A^+_{n,l-1,m,m_1} = 4(l+1)\sqrt{\frac{(2n+2l+1)\pi}{3}}\Big(\int_{\mathbb{S}^2} Y_1^{m_1}(\omega)Y_{l-1}^m(\omega)Y_l^{-m_1-m}(\omega)d\omega\Big).$$

Then

$$\begin{aligned}
&\sum_{\substack{|m|\le l+1,|m_1|\le 1\\ m+m_1=m^*}} \Big|A^-_{n-1,l+1,m,m_1}\Big|^2\\
=&\frac{32nl^2\pi}{3}\sum_{\substack{|m|\le l+1,|m_1|\le 1\\ m+m_1=m^*}}\left|\int_{\mathbb{S}^2} Y_1^{m_1}(\omega)Y_{l+1}^m(\omega)Y_l^{-m_1-m}(\omega)d\omega\right|^2\\
=&\frac{32nl^2\pi}{3}\sum_{|m|\le l+1}\sum_{|m_1|\le 1}\left|\int_{\mathbb{S}^2} Y_1^{m_1}(\omega)Y_{l+1}^m(\omega)Y_l^{-m^*}(\omega)d\omega\right|^2\\
=&\frac{32nl^2\pi}{3}\sum_{|m|\le l+1}\sum_{|m_1|\le 1}\\
&\times\int_{\mathbb{S}^2_\omega}\int_{\mathbb{S}^2_\sigma} Y_1^{m_1}(\omega)Y_1^{-m_1}(\sigma)Y_{l+1}^m(\omega)Y_{l+1}^{-m}(\sigma)Y_l^{-m^*}(\omega)Y_l^{m^*}(\sigma)d\omega d\sigma.
\end{aligned}$$

We recall again that, for $\sigma,\kappa\in\mathbb{S}^2$,

$$P_k(\sigma\cdot\kappa) = \frac{4\pi}{2k+1}\sum_{|m|\le k} Y_k^m(\sigma)Y_k^{-m}(\kappa),\quad \forall\, k\in\mathbb{N}.$$

Then

$$\begin{aligned}
&\sum_{\substack{|m|\le l+1,|m_1|\le 1\\ m+m_1=m^*}} \Big|A^-_{n-1,l+1,m,m_1}\Big|^2\\
&= \frac{32nl^2\pi}{3}\frac{3}{4\pi}\frac{2l+3}{4\pi}\int_{\mathbb{S}^2_\omega}\int_{\mathbb{S}^2_\sigma} P_1(\omega\cdot\sigma)P_{l+1}(\omega\cdot\sigma)Y_l^{-m^*}(\omega)Y_l^{m^*}(\sigma)d\omega d\sigma.
\end{aligned}$$

By using the fact that,

$$P_1(\omega\cdot\sigma)P_{l+1}(\omega\cdot\sigma) = \frac{l+2}{2l+3}P_{l+2}(\omega\cdot\sigma) + \frac{l+1}{2l+3}P_l(\omega\cdot\sigma)$$

and the orthogonal of $\{Y_l^{-m^*}(\omega), l\in\mathbb{N}, |m^*|\le l\}$ on $\mathbb{S}^2$, one can verify that

$$\begin{aligned}
&\sum_{\substack{|m|\le l+1,|m_1|\le 1\\ m+m_1=m^*}} \Big|A^-_{n-1,l+1,m,m_1}\Big|^2\\
&= \frac{32nl^2\pi}{3}\frac{3}{4\pi}\frac{2l+3}{4\pi}\frac{4\pi}{2l+1}\frac{l+1}{2l+3} \le 4nl(l+1).
\end{aligned}$$

Hence, we arrive at (3.8). On the other hand, we have

$$\begin{aligned}
&\sum_{\substack{|m|\le l-1,|m_1|\le 1\\ m+m_1=m^*}}\left|A^+_{n,l-1,m,m_1}\right|^2\\
=&\frac{16(2n+2l+1)(l+1)^2\pi}{3}\sum_{\substack{|m|\le l-1,|m_1|\le 1\\ m+m_1=m^*}}\left|\int_{\mathbb{S}^2}Y_1^{m_1}(\omega)Y_{l-1}^{m}(\omega)Y_l^{-m_1-m}(\omega)d\omega\right|^2\\
=&\frac{16(2n+2l+1)(l+1)^2\pi}{3}\sum_{|m|\le l-1}\sum_{|m_1|\le 1}\left|\int_{\mathbb{S}^2}Y_1^{m_1}(\omega)Y_{l-1}^{m}(\omega)Y_l^{-m^*}(\omega)d\omega\right|^2\\
=&\frac{16(2n+2l+1)(l+1)^2\pi}{3}\sum_{|m|\le l-1}\sum_{|m_1|\le 1}\\
&\times\int_{\mathbb{S}^2_\omega}\int_{\mathbb{S}^2_\sigma}Y_1^{m_1}(\omega)Y_1^{-m_1}(\sigma)Y_{l-1}^{m}(\omega)Y_{l-1}^{-m}(\sigma)Y_l^{-m^*}(\omega)Y_l^{m^*}(\sigma)d\omega d\sigma\\
=&\frac{16(2n+2l+1)(l+1)^2\pi}{3}\frac{3}{4\pi}\frac{2l-1}{4\pi}\\
&\times\int_{\mathbb{S}^2_\omega}\int_{\mathbb{S}^2_\sigma}P_1(\omega\cdot\sigma)P_{l-1}(\omega\cdot\sigma)Y_l^{-m^*}(\omega)Y_l^{m^*}(\sigma)d\omega.
\end{aligned}$$

By employing the fact for $l\ge 1$ that,

$$P_1(\omega\cdot\sigma)P_{l-1}(\omega\cdot\sigma)=\frac{l}{2l-1}P_l(\omega\cdot\sigma)+\cdots$$

and the orthogonal of the $\{Y_l^{-m^*}(\omega), l\in\mathbb{N}, |m^*|\le l\}$ on $\mathbb{S}^2$, one can verify that

$$\begin{aligned}
\sum_{\substack{|m|\le l-1,|m_1|\le 1\\ m+m_1=m^*}}\left|A^+_{n,l-1,m,m_1}\right|^2=&\frac{16(2n+2l+1)(l+1)^2\pi}{3}\frac{3}{4\pi}\frac{2l-1}{4\pi}\frac{4\pi}{2l+1}\frac{l}{2l-1}\\
\le&2(2n+2l+1)(l+1)^2.
\end{aligned}$$

This leads to the inequality (3.9). □

3.3 Nonlinear estimates of Landau collision operator

In this section, we establish nonlinear estimates of Landau collision operator.

3.3.1 Trilinear estimates

The orthogonal projectors $\{\mathbb{S}_N, N\in\mathbb{N}\}$ and $\{\tilde{\mathbb{S}}_N, N\in\mathbb{N}\}$ are defined as follows, for $f\in\mathcal{S}'(\mathbb{R}^3_v)$,

$$\mathbb{S}_N f=\sum_{0\le 2n+l\le N}\sum_{|m|\le l}f_{n,l,m}\,\varphi_{n,l,m}\in\mathcal{S}(\mathbb{R}^3_v)$$

and

$$\tilde{\mathbb{S}}_N f = \sum_{\substack{2\le 2n+l\le N\\ n+l\ge 2}} \sum_{|m|\le l} f_{n,l,m}\,\varphi_{n,l,m} \in \mathcal{S}(\mathbb{R}^3_v),$$

where $f_{n,l,m} = \langle f, \varphi_{n,l,m}\rangle$ and then we have

$$\mathbf{P}f = (\mathbb{S}_N - \tilde{\mathbb{S}}_N)f = f_{0,0,0}\,\varphi_{0,0,0} + f_{1,0,0}\,\varphi_{1,0,0} + \sum_{|m|\le 1} f_{0,1,m}\,\varphi_{0,1,m}.$$

Firstly, we bound the nonlinear term $(\mathbf{L}(f,g),\, h)_{L^2_{x,v}}$ as the following theorem.

Theorem 3.2. *Let $f, g, h \in \mathcal{S}(\mathbb{R}^6_{x,v})$. It holds that*

$$\left|(\mathbf{L}(f,g),h)_{L^2_{x,v}}\right| \lesssim \min\left\{\|\mathbb{S}_2 f\|_{L^2_v(L^\infty_x)}\,\|\widetilde{\mathcal{L}}^{\frac12} g\|_{L^2_{x,v}}, \|\mathbb{S}_2 f\|_{L^2_{x,v}}\,\|\widetilde{\mathcal{L}}^{\frac12} g\|_{L^2_v(L^\infty_x)}\right\}\|\widetilde{\mathcal{L}}^{\frac12} h\|_{L^2_{x,v}}, \quad (3.10)$$

where the self-adjoint operator $\widetilde{\mathcal{L}}$ is defined by

$$\widetilde{\mathcal{L}} \triangleq 2\mathcal{H} - \Delta_{\mathbb{S}^2}.$$

Proof. For any temperate functions $f, g, h \in \mathcal{S}(\mathbb{R}^6_{x,v})$, one has the decomposition

$$z = \sum_{2n+l\ge 0}\sum_{|m|\le l} z_{n,l,m}\varphi_{n,l,m}, \qquad z_{n,l,m} \triangleq \langle z, \varphi_{n,l,m}\rangle$$

with $z = f, g, h$. We can deduce from Proposition 3.1 that

$$\begin{aligned}
\mathbf{L}(f,g) = &-\sum_{2n+l\ge 0}\sum_{|m|\le l}(2(2n+l)+l(l+1))\,f_{0,0,0}g_{n,l,m}\varphi_{n,l,m}\\
&+\sum_{\substack{2n+l\ge 0\\ l\ge 1}}\sum_{|m|\le l}\sum_{|m_1|\le 1} A^-_{n,l,m,m_1} f_{0,1,m_1} g_{n,l,m}\varphi_{n+1,l-1,m_1+m}\\
&+\sum_{2n+l\ge 0}\sum_{|m|\le l}\sum_{|m_1|\le 1} A^+_{n,l,m,m_1} f_{0,1,m_1} g_{n,l,m}\varphi_{n,l+1,m_1+m}\\
&+\sum_{2n+l\ge 0}\sum_{|m|\le l}\frac{4\sqrt{3(n+1)(2n+2l+3)}}{3} f_{1,0,0}g_{n,l,m}\varphi_{n+1,l,m}\\
&+\sum_{\substack{2n+l\ge 0\\ l\ge 2}}\sum_{|m|\le l}\sum_{|m_2|\le 2} A^1_{n,l,m,m_2} f_{0,2,m_2} g_{n,l,m}\varphi_{n+2,l-2,m+m_2}\\
&+\sum_{2n+l\ge 0}\sum_{|m|\le l}\sum_{|m_2|\le 2} A^2_{n,l,m,m_2} f_{0,2,m_2} g_{n,l,m}\varphi_{n+1,l,m+m_2}\\
&+\sum_{2n+l\ge 0}\sum_{|m|\le l}\sum_{|m_2|\le 2} A^3_{n,l,m,m_2} f_{0,2,m_2} g_{n,l,m}\varphi_{n,l+2,m+m_2}.
\end{aligned}$$

Then it follows from the above equality that

$$\left|(\mathbf{L}(f,g),h)_{L^2_{x,v}}\right| \le \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \mathbf{J}_4 + \mathbf{J}_5 + \mathbf{J}_6 + \mathbf{J}_7$$

with

$$\mathbf{J}_1 = \left| \sum_{2n+l\geq 0} \sum_{|m|\leq l} \left(2(2n+l)+l(l+1)\right) \left(f_{0,0,0} g_{n,l,m}, h_{n,l,m}\right)_{L^2_x} \right|,$$

$$\mathbf{J}_2 = \left| \sum_{\substack{2n+l\geq 0 \\ n\geq 1}} \sum_{\substack{|m|\leq l+1, |m_1|\leq 1 \\ |m+m_1|\leq l}} A^-_{n-1,l+1,m,m_1} (f_{0,1,m_1} g_{n-1,l+1,m}, h_{n,l,m_1+m})_{L^2_x} \right|,$$

$$\mathbf{J}_3 = \left| \sum_{\substack{2n+l\geq 0 \\ l\geq 1}} \sum_{\substack{|m|\leq l-1, |m_1|\leq 1 \\ |m+m_1|\leq l}} A^+_{n,l-1,m,m_1} (f_{0,1,m_1} g_{n,l-1,m}, h_{n,l,m_1+m})_{L^2_x} \right|,$$

$$\mathbf{J}_4 = \left| \sum_{\substack{2n+l\geq 0 \\ n\geq 1}} \sum_{|m|\leq l} \frac{4\sqrt{3n(2n+2l+1)}}{3} (f_{1,0,0} g_{n-1,l,m}, h_{n,l,m})_{L^2_x} \right|,$$

$$\mathbf{J}_5 = \left| \sum_{\substack{2n+l\geq 0 \\ n\geq 2}} \sum_{\substack{|m|\leq l+2, |m_2|\leq 2 \\ |m+m_2|\leq l}} A^1_{n-2,l+2,m,m_2} (f_{0,2,m_2} g_{n-2,l+2,m}, h_{n,l,m+m_2})_{L^2_x} \right|,$$

$$\mathbf{J}_6 = \left| \sum_{\substack{2n+l\geq 0 \\ n\geq 1, l\geq 1}} \sum_{\substack{|m|\leq l, |m_2|\leq 2 \\ |m+m_2|\leq l}} A^2_{n-1,l,m,m_2} (f_{0,2,m_2} g_{n-1,l,m}, h_{n,l,m+m_2})_{L^2_x} \right|,$$

$$\mathbf{J}_7 = \left| \sum_{\substack{2n+l\geq 0 \\ l\geq 2}} \sum_{\substack{|m|\leq l-2, |m_2|\leq 2 \\ |m+m_2|\leq l}} A^3_{n,l-2,m,m_2} (f_{0,2,m_2} g_{n,l-2,m}, h_{n,l,m+m_2})_{L^2_x} \right|.$$

For $\mathbf{J}_1$, by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\mathbf{J}_1 &\lesssim \sum_{2n+l\geq 0} \sum_{|m|\leq l} \left(2(2n+l)+l(l+1)\right) \|f_{0,0,0} g_{n,l,m}\|_{L^2_x} \|h_{n,l,m}\|_{L^2_x} \\
&\lesssim \sum_{2n+l\geq 0} \sum_{|m|\leq l} \left(2(2n+l)+l(l+1)\right) \|f_{0,0,0}\|_{L^2_x} \|g_{n,l,m}\|_{L^\infty_x} \|h_{n,l,m}\|_{L^2_x} \\
&\lesssim \|f_{0,0,0}\|_{L^2_x} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|_{L^2_{x,v}}.
\end{aligned}$$

Similarly, we bound the term $\mathbf{J}_4$ that

$$\mathbf{J}_4 \lesssim \|f_{1,0,0}\|_{L^2_x} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|_{L^2_{x,v}}.$$

Next, we turn to estimate $\mathbf{J}_2$. Precisely,

$$\begin{aligned}
\mathbf{J}_2 &\lesssim \sum_{\substack{2n+l\geq 0\\ n\geq 1}} \sum_{\substack{|m|\leq l+1,|m_1|\leq 1\\ |m+m_1|\leq l}} \|f_{0,1,m_1}\|_{L^2_x}\|g_{n-1,l+1,m}\|_{L^\infty_x}\|A^-_{n-1,l+1,m,m_1}h_{n,l,m_1+m}\|_{L^2_x}\\
&\lesssim \sum_{\substack{2n+l\geq 0\\ n\geq 1}} \Big(\sum_{|m_1|\leq 1}\|f_{0,1,m_1}\|^2_{L^2_x}\Big)^{1/2}\\
&\qquad\times\left(\sum_{\substack{|m|\leq l+1,|m_1|\leq 1\\ |m+m_1|\leq l}}\|g_{n-1,l+1,m}\|^2_{L^\infty_x}\left\|A^-_{n-1,l+1,m,m_1}h_{n,l,m_1+m}\right\|^2_{L^2_x}\right)^{1/2}\\
&\lesssim \sum_{\substack{2n+l\geq 0\\ n\geq 1}} \Big(\sum_{|m_1|\leq 1}\|f_{0,1,m_1}\|^2_{L^2_x}\Big)^{1/2}\Big(\sum_{|m|\leq l+1}\|g_{n-1,l+1,m}\|^2_{L^\infty_x}\Big)^{1/2}\\
&\qquad\times\left(\sum_{\substack{|m_1|\leq 1,|m|\leq l+1\\ |m+m_1|\leq l}}\left\|A^-_{n-1,l+1,m,m_1}h_{n,l,m_1+m}\right\|^2_{L^2_x}\right)^{1/2}.
\end{aligned}$$

By exchanging the order in the last summation, we deduce from (3.8) of Proposition 3.3 that

$$\begin{aligned}
&\sum_{\substack{|m_1|\leq 1,|m|\leq l+1\\ |m+m_1|\leq l}}\left\|A^-_{n-1,l+1,m,m_1}h_{n,l,m_1+m}\right\|^2_{L^2_x}\\
&=\sum_{|m^*|\leq l}\left(\sum_{\substack{|m_1|\leq 1,|m|\leq l+1\\ m_1+m=m^*}}|A^-_{n-1,l+1,m,m_1}|^2\right)\|h_{n,l,m^*}\|^2_{L^2_x}\\
&\leq\left(\max_{|m^*|\leq l}\sum_{\substack{|m_1|\leq 1,|m|\leq l+1\\ m_1+m=m^*}}|A^-_{n-1,l+1,m,m_1}|^2\right)\sum_{|m^*|\leq l}\|h_{n,l,m^*}\|^2_{L^2_x}\\
&\leq 4(n+1)l(l+1)\sum_{|m^*|\leq l}\|h_{n,l,m^*}\|^2_{L^2_x}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\mathbf{J}_2 &\lesssim \sum_{\substack{2n+l\geq 0\\ n\geq 1}} \Big(\sum_{|m_1|\leq 1}\|f_{0,1,m_1}\|^2_{L^2_x}\Big)^{1/2}\sqrt{4(n+1)l(l+1)}\\
&\qquad\times\Big(\sum_{|m|\leq l+1}\|g_{n-1,l+1,m}\|^2_{L^\infty_x}\Big)^{1/2}\Big(\sum_{|m^*|\leq l}\|h_{n,l,m^*}\|^2_{L^2_x}\Big)^{1/2}\\
&\lesssim \Big(\sum_{|m_1|\leq 1}\|f_{0,1,m_1}\|^2_{L^2_x}\Big)^{1/2}\left\|\widetilde{\mathcal{L}}^{\frac12}g\right\|_{L^2_v(L^\infty_x)}\left\|\widetilde{\mathcal{L}}^{\frac12}h\right\|_{L^2_{x,v}}.
\end{aligned}$$

Those terms $\mathbf{J}_3,\ \mathbf{J}_5,\ \mathbf{J}_6$ and $\mathbf{J}_7$ may be treated along the same line as $\mathbf{J}_2$ with aid of (3.5), (3.6), (3.7) of Proposition 3.2 and (3.9) of Proposition 3.3. Therefore, we can deduce that

$$\mathbf{J}_3 \lesssim \Big(\sum_{|m_1|\le 1} \|f_{0,1,m_1}\|_{L^2_x}^2 \Big)^{1/2} \Big\| \widetilde{\mathcal{L}}^{\frac{1}{2}} g \Big\|_{L^2_v(L^\infty_x)} \Big\| \widetilde{\mathcal{L}}^{\frac{1}{2}} h \Big\|_{L^2_{x,v}}$$

and

$$\mathbf{J}_5,\ \mathbf{J}_6,\ \mathbf{J}_7 \lesssim \Big(\sum_{|m_2|\le 2} \|f_{0,2,m_2}\|_{L^2_x}^2 \Big)^{1/2} \Big\| \widetilde{\mathcal{L}}^{\frac{1}{2}} g \Big\|_{L^2_v(L^\infty_x)} \Big\| \widetilde{\mathcal{L}}^{\frac{1}{2}} h \Big\|_{L^2_{x,v}}.$$

Together with those estimates on $\mathbf{J}_1$-$\mathbf{J}_7$, we get

$$\Big| (\mathbf{L}(f,g),h)_{L^2_{x,v}} \Big| \lesssim \|\mathbb{S}_2 f\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|_{L^2_{x,v}}. \tag{3.11}$$

Similarly, when estimating the terms $\mathbf{J}_1$-$\mathbf{J}_7$, taking L^∞ norm on the position variable x for f and taking L^2 norm on the position variable x for g, we obtain

$$\Big| (\mathbf{L}(f,g),h)_{L^2_{x,v}} \Big| \lesssim \|\mathbb{S}_2 f\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|_{L^2_{x,v}}. \tag{3.12}$$

Combining (3.11)-(3.12) gives the desired inequality (3.10). Hence, the proof of Theorem 3.2 is finished. □

Furthermore, optimal information will be obtained if splitting the functions into frequency packets of comparable sizes. Indeed, one has

Theorem 3.3. *Let $f, g, h \in \mathcal{S}(\mathbb{R}^6_{x,v})$. It holds that*

$$\begin{aligned} &\Big| (\Delta_j \mathbf{L}(f,g), \Delta_j h)_{L^2_{x,v}} \Big| \\ &\lesssim \sum_{|p-j|\le 4} \|\Delta_p \mathbb{S}_2 f\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} \Delta_j h\|_{L^2_{x,v}} \\ &\quad + \sum_{p\ge j-4} \|\mathbb{S}_2 f\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} \Delta_p g\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} \Delta_j h\|_{L^2_{x,v}} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} |(\Delta_j \mathbf{L}(f,g), \Delta_j e^{2c_1 t\mathcal{H}} h)_{L^2_{x,v}}| &\lesssim e^{-\frac{3}{2}c_1 t} \\ &\times \Big[\sum_{|p-j|\le 4} \|\Delta_p \mathbb{S}_2 e^{c_1 t\mathcal{H}} f\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} e^{c_1 t\mathcal{H}} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} e^{c_1 t\mathcal{H}} \Delta_j h\|_{L^2_{x,v}} \\ &+ \sum_{p\ge j-4} \|\mathbb{S}_2 e^{c_1 t\mathcal{H}} f\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} e^{c_1 t\mathcal{H}} \Delta_p g\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} e^{c_1 t\mathcal{H}} \Delta_j h\|_{L^2_{x,v}} \Big] \end{aligned} \tag{3.14}$$

for any $j \ge -1$.

Proof. For $f, g, h \in \mathcal{S}(\mathbb{R}^6_{x,v})$, it follows from Proposition 3.1 and the orthogonal property of $\{\varphi_{n,l,m};\, n, l \in \mathbb{N}, |m| \le l\}$, that

$$\left|(\Delta_j \mathbf{L}(f,g), \Delta_j h)_{L^2_{x,v}}\right| \le \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5 + \mathbf{I}_6 + \mathbf{I}_7$$

with

$$\mathbf{I}_1 = \left| \sum_{2n+l\ge 0} \sum_{|m|\le l} \left(2(2n+l) + l(l+1)\right) \left(\Delta_j \left(f_{0,0,0} g_{n,l,m}\right), \Delta_j h_{n,l,m}\right)_{L^2_x} \right|,$$

$$\mathbf{I}_2 = \left| \sum_{\substack{2n+l\ge 0\\ n\ge 1}} \sum_{\substack{|m|\le l+1, |m_1|\le 1\\ |m+m_1|\le l}} A^-_{n-1,l+1,m,m_1} \left(\Delta_j \left(f_{0,1,m_1} g_{n-1,l+1,m}\right), \Delta_j h_{n,l,m+m_1}\right)_{L^2_x} \right|,$$

$$\mathbf{I}_3 = \left| \sum_{\substack{2n+l\ge 0\\ l\ge 1}} \sum_{\substack{|m|\le l-1, |m_1|\le 1\\ |m+m_1|\le l}} A^+_{n,l-1,m,m_1} \left(\Delta_j \left(f_{0,1,m_1} g_{n,l-1,m}\right), \Delta_j h_{n,l,m+m_1}\right)_{L^2_x} \right|,$$

$$\mathbf{I}_4 = \left| \sum_{\substack{2n+l\ge 0\\ n\ge 1}} \sum_{|m|\le l} \frac{4\sqrt{3n(2n+2l+1)}}{3} \left(\Delta_j \left(f_{1,0,0} g_{n-1,l,m}\right), \Delta_j h_{n,l,m}\right)_{L^2_x} \right|,$$

$$\mathbf{I}_5 = \left| \sum_{\substack{2n+l\ge 0\\ n\ge 2}} \sum_{\substack{|m|\le l+2, |m_2|\le 2\\ |m+m_2|\le l}} A^1_{n-2,l+2,m,m_2} \left(\Delta_j \left(f_{0,2,m_2} g_{n-2,l+2,m}\right), \Delta_j h_{n,l,m+m_2}\right)_{L^2_x} \right|,$$

$$\mathbf{I}_6 = \left| \sum_{\substack{2n+l\ge 0\\ n\ge 1, l\ge 1}} \sum_{\substack{|m|\le l, |m_2|\le 2\\ |m+m_2|\le l}} A^2_{n-1,l,m,m_2} \left(\Delta_j \left(f_{0,2,m_2} g_{n-1,l,m}\right), \Delta_j h_{n,l,m+m_2}\right)_{L^2_x} \right|,$$

$$\mathbf{I}_7 = \left| \sum_{\substack{2n+l\ge 0\\ l\ge 2}} \sum_{\substack{|m|\le l-2, |m_2|\le 2\\ |m+m_2|\le l}} A^3_{n,l-2,m,m_2} \left(\Delta_j \left(f_{0,2,m_2} g_{n,l-2,m}\right), \Delta_j h_{n,l,m+m_2}\right)_{L^2_x} \right|.$$

To estimate $\mathbf{I}_1$, Bony's decomposition comes into play in our context. The product of u and v can be decomposed into

$$uv = \mathbf{T}_u v + \mathbf{T}_v u + \mathbf{R}(u,v)$$

with

$$\mathbf{T}_u v = \sum_p S_{p-1} u \Delta_p v, \quad \mathbf{R}(u,v) = \sum_{|p-p'|\le 1} \Delta_{p'} u \Delta_p v, \quad \text{for } u, v \in \mathcal{S}'(\mathbb{R}^3).$$

The above operators $\mathbf{T}$ and $\mathbf{R}$ are called "paraproduct" and "remainder", respectively. Moreover, it follows from Lemma 2.3 that

$$\begin{aligned}
\Delta_j \mathbf{T}_u v &= \sum_p \Delta_j (S_{p-1} u \Delta_p v) = \sum_{|p-j|\le 4} \Delta_j (S_{p-1} u \Delta_p v),\\
\Delta_j \mathbf{T}_v u &= \sum_p \Delta_j (S_{p-1} v \Delta_p u) = \sum_{|p-j|\le 4} \Delta_j (S_{p-1} v \Delta_p u),\\
\Delta_j \mathbf{R}(u,v) &= \sum_{|p-p'|\le 1} \Delta_j (\Delta_{p'} u \Delta_p v) = \sum_{\max(p,p')\ge j-2} \sum_{|p-p'|\le 1} \Delta_j (\Delta_{p'} u \Delta_p v).
\end{aligned}$$

Consequently, we are led to the inequality

$$\begin{aligned}
\mathbf{I}_1 &\le \sum_{2n+l\ge 0} \sum_{|m|\le l} (2(2n+l)+l(l+1)) \left| \left(\Delta_j \left(f_{0,0,0} g_{n,l,m}\right), \Delta_j h_{n,l,m}\right)_{L^2(\mathbb{R}^3_x)} \right|\\
&\le \sum_{|p-j|\le 4} \sum_{2n+l\ge 0} \sum_{|m|\le l} (2(2n+l)+l(l+1))\\
&\quad \times \|\Delta_j (S_{p-1} f_{0,0,0} \Delta_p g_{n,l,m} + S_{p-1} g_{n,l,m} \Delta_p f_{0,0,0})\|_{L^2_x} \|\Delta_j h_{n,l,m}\|_{L^2_x}\\
&\quad + \sum_{\max(p,p')\ge j-2} \sum_{|p-p'|\le 1} \sum_{2n+l\ge 0} \sum_{|m|\le l} (2(2n+l)+l(l+1))\\
&\quad \times \|\Delta_j (\Delta_{p'} f_{0,0,0} \Delta_p g_{n,l,m})\|_{L^2_x} \|\Delta_j h_{n,l,m}\|_{L^2_x}.
\end{aligned}$$

Furthermore, with aid of Lemma 2.4, we arrive at

$$\begin{aligned}
\mathbf{I}_1 &\lesssim \sum_{|p-j|\le 4} \sum_{2n+l\ge 0} \sum_{|m|\le l} (2(2n+l)+l(l+1))\\
&\quad \times \left(\|f_{0,0,0}\|_{L^\infty_x} \|\Delta_p g_{n,l,m}\|_{L^2_x} + \|g_{n,l,m}\|_{L^\infty_x} \|\Delta_p f_{0,0,0}\|_{L^2_x}\right) \|\Delta_j h_{n,l,m}\|_{L^2_x}\\
&\quad + \sum_{p\ge j-3} \sum_{2n+l\ge 0} \sum_{|m|\le l} (2(2n+l)+l(l+1)) \|f_{0,0,0}\|_{L^\infty_x} \|\Delta_p g_{n,l,m}\|_{L^2_x} \|\Delta_j h_{n,l,m}\|_{L^2_x}.
\end{aligned}$$

The Cauchy-Schwarz inequality enables us to get

$$\begin{aligned}
\mathbf{I}_1 &\le \sum_{|p-j|\le 4} \|f_{0,0,0}\|_{L^\infty_x} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_p g\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_j h\|_{L^2_{x,v}}\\
&\quad + \sum_{|p-j|\le 4} \|\Delta_p f_{0,0,0}\|_{L^2_x} \|\widetilde{\mathcal{L}}^{\frac12} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_j h\|_{L^2_{x,v}}\\
&\quad + \sum_{p\ge j-3} \|f_{0,0,0}\|_{L^\infty_x} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_p g\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_j h\|_{L^2_{x,v}}\\
&\lesssim \sum_{|p-j|\le 4} \|\Delta_p f_{0,0,0}\|_{L^2_x} \|\widetilde{\mathcal{L}}^{\frac12} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_j h\|_{L^2_{x,v}}\\
&\quad + \sum_{p\ge j-4} \|f_{0,0,0}\|_{L^\infty_x} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_p g\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_j h\|_{L^2_{x,v}}.
\end{aligned}$$

Similar to the proof of the term $\mathbf{I}_1$, one can verify that

$$\begin{aligned}\mathbf{I}_4 \lesssim & \sum_{|p-j|\le 4} \|\Delta_p f_{1,0,0}\|_{L^2_x} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2_v(L^\infty_x)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} \Delta_j h\|_{L^2_{x,v}} \\ & + \sum_{p\ge j-4} \|f_{1,0,0}\|_{L^\infty_x} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} \Delta_p g\|_{L^2_{x,v}} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} \Delta_j h\|_{L^2_{x,v}}.\end{aligned}$$

We now estimate the remaining terms $\mathbf{I}_2,\ \mathbf{I}_3,\ \mathbf{I}_5,\ \mathbf{I}_6,\ \mathbf{I}_7$. The process of the proofs with respect to these five terms are almost the same, so we can take $\mathbf{I}_5$ as the example.

$$\begin{aligned}\mathbf{I}_5 \le & \sum_{|p-j|\le 4} \sum_{\substack{2n+l\ge 0\\ n\ge 2}} \Big| \sum_{\substack{|m|\le l+2, |m_2|\le 2\\ |m+m_2|\le l}} A^1_{n-2,l+2,m,m_2} \left(\Delta_j(S_{p-1} f_{0,2,m_2} \Delta_p g_{n-2,l+2,m}), \Delta_j h_{n,l,m+m_2}\right)_{L^2_x} \Big| \\ & + \sum_{|p-j|\le 4} \sum_{\substack{2n+l\ge 0\\ n\ge 2}} \Big| \sum_{\substack{|m|\le l+2, |m_2|\le 2\\ |m+m_2|\le l}} A^1_{n-2,l+2,m,m_2} (\Delta_j(S_{p-1} g_{n-2,l+2,m} \Delta_p f_{0,2,m_2}), \Delta_j h_{n,l,m+m_2})_{L^2_x} \Big| \\ & + \sum_{\max(p,p')\ge j-2} \sum_{|p-p'|\le 1} \sum_{\substack{2n+l\ge 0\\ n\ge 2}} \Big| \sum_{\substack{|m|\le l+2, |m_2|\le 2\\ |m+m_2|\le l}} A^1_{n-2,l+2,m,m_2} \\ & \qquad \times (\Delta_j(\Delta_{p'} f_{0,2,m_2} \Delta_p g_{n-2,l+2,m}), \Delta_j h_{n,l,m+m_2})_{L^2_x} \Big| \\ & \triangleq \mathbf{I}_{51} + \mathbf{I}_{52} + \mathbf{I}_{53}.\end{aligned}$$

Regarding the term $\mathbf{I}_{51}$, we use Cauchy-Schwarz inequality again and get

$$\begin{aligned} & \left| \sum_{\substack{|m|\le l+2, |m_2|\le 2\\ |m+m_2|\le l}} A^1_{n-2,l+2,m,m_2} (\Delta_j(S_{p-1} f_{0,2,m_2} \Delta_p g_{n-2,l+2,m}), \Delta_j h_{n,l,m+m_2})_{L^2_x} \right| \\ & \le \sum_{\substack{|m|\le l+2, |m_2|\le 2\\ |m+m_2|\le l}} \|f_{0,2,m_2}\|_{L^\infty_x} \|\Delta_p g_{n-2,l+2,m}\|_{L^2_x} \|A^1_{n-2,l+2,m,m_2} \Delta_j h_{n,l,m+m_2}\|_{L^2_x} \\ & \le \Big(\sum_{|m_2|\le 2} \|f_{0,2,m_2}\|^2_{L^\infty_x} \Big)^{1/2} \Big(\sum_{|m|\le l+2} \|\Delta_p g_{n-2,l+2,m}\|^2_{L^2_x} \Big)^{1/2} \\ & \quad \times \left(\sum_{\substack{|m_2|\le 2, |m|\le l+2\\ |m+m_2|\le l}} \left\| A^1_{n-2,l+2,m,m_2} \Delta_j h_{n,l,m+m_2} \right\|^2_{L^2_x} \right)^{1/2}, \end{aligned}$$

where the last summation can be estimates by (3.5) of Proposition 3.2,

$$\begin{aligned}
&\sum_{\substack{|m_2|\le 2,|m|\le l+2\\ |m+m_2|\le l}} \left\|A^1_{n-2,l+2,m,m_2}\Delta_j h_{n,l,m+m_2}\right\|^2_{L^2_x}\\
&= \sum_{|m^*|\le l}\left(\sum_{\substack{|m_2|\le 2,|m|\le l+2\\ m+m_2=m^*}} \left|A^1_{n-2,l+2,m,m_2}\right|^2\right)\|\Delta_j h_{n,l,m^*}\|^2_{L^2_x}\\
&\le \left(\max_{|m^*|\le l}\sum_{\substack{|m_2|\le 2,|m|\le l+2\\ m+m_2=m^*}} \left|A^1_{n-2,l+2,m,m_2}\right|^2\right)\sum_{|m^*|\le l}\|\Delta_j h_{n,l,m^*}\|^2_{L^2_x}\\
&\le \frac{16n(n-1)}{3}\sum_{|m^*|\le l}\|\Delta_j h_{n,l,m^*}\|^2_{L^2_x}.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\mathbf{I}_{51} &\le \sum_{|p-j|\le 4}\Big(\sum_{|m_2|\le 2}\|f_{0,2,m_2}\|^2_{L^\infty_x}\Big)^{1/2}\sum_{\substack{2n+l\ge 0\\ n\ge 2}}\sqrt{\frac{16n(n-1)}{3}}\\
&\quad\times\Big(\sum_{|m|\le l+1}\|\Delta_p g_{n-2,l+2,m}\|^2_{L^2_x}\Big)^{1/2}\Big(\sum_{|m^*|\le l}\|\Delta_j h_{n,l,m^*}\|^2_{L^2_x}\Big)^{1/2}\\
&\le \sum_{|p-j|\le 4}\Big(\sum_{|m_2|\le 2}\|f_{0,2,m_2}\|^2_{L^\infty_x}\Big)^{1/2}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_p g\right\|_{L^2_{x,v}}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\right\|_{L^2_{x,v}}.
\end{aligned}$$

Bounding $\mathbf{I}_{52}$ and $\mathbf{I}_{53}$ essentially follows from the same procedure as $\mathbf{I}_{51}$, so we get

$$\mathbf{I}_{52} \le \sum_{|p-j|\le 4}\Big(\sum_{|m_2|\le 2}\|\Delta_p f_{0,2,m_2}\|^2_{L^2_x}\Big)^{1/2}\left\|\widetilde{\mathcal{L}}^{\frac12} g\right\|_{L^2_v(L^\infty_x)}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\right\|_{L^2_{x,v}}$$

and

$$\mathbf{I}_{53} \le \sum_{p\ge j-4}\Big(\sum_{|m_2|\le 2}\|f_{0,2,m_2}\|^2_{L^\infty_x}\Big)^{1/2}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_p g\right\|_{L^2_{x,v}}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\right\|_{L^2_{x,v}}.$$

Therefore, by combining those estimates, we conclude that

$$\begin{aligned}
\mathbf{I}_5 &\lesssim \sum_{|p-j|\le 4}\Big(\sum_{|m_2|\le 2}\|\Delta_p f_{0,2,m_2}\|^2_{L^2_x}\Big)^{1/2}\left\|\widetilde{\mathcal{L}}^{\frac12} g\right\|_{L^2_v(L^\infty_x)}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\right\|_{L^2_{x,v}}\\
&\quad+\sum_{p\ge j-4}\Big(\sum_{|m_2|\le 2}\|f_{0,2,m_2}\|^2_{L^\infty_x}\Big)^{1/2}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_p g\right\|_{L^2_{x,v}}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\right\|_{L^2_{x,v}}.
\end{aligned}$$

Similarly, we can get the following estimates for $\mathbf{I}_2,\ \mathbf{I}_3,\ \mathbf{I}_6,\ \mathbf{I}_7$:

$$\begin{aligned}\mathbf{I}_2,\ \mathbf{I}_3 \lesssim & \sum_{|p-j|\le 4}\Big(\sum_{|m_1|\le 1}\|\Delta_p f_{0,1,m_1}\|_{L^2_x}^2\Big)^{1/2}\Big\|\widetilde{\mathcal{L}}^{\frac12}g\Big\|_{L^2_v(L^\infty_x)}\Big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\Big\|_{L^2_{x,v}}\\ &+\sum_{p\ge j-4}\Big(\sum_{|m_1|\le 1}\|f_{0,1,m_1}\|_{L^\infty_x}^2\Big)^{1/2}\Big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_p g\Big\|_{L^2_{x,v}}\Big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\Big\|_{L^2_{x,v}}\end{aligned}$$

and

$$\begin{aligned}\mathbf{I}_6,\ \mathbf{I}_7 \lesssim & \sum_{|p-j|\le 4}\Big(\sum_{|m_2|\le 2}\|\Delta_p f_{0,2,m_2}\|_{L^2_x}^2\Big)^{1/2}\Big\|\widetilde{\mathcal{L}}^{\frac12}g\Big\|_{L^2_v(L^\infty_x)}\Big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\Big\|_{L^2_{x,v}}\\ &+\sum_{p\ge j-4}\Big(\sum_{|m_2|\le 2}\|f_{0,2,m_2}\|_{L^\infty_x}^2\Big)^{1/2}\Big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_p g\Big\|_{L^2_{x,v}}\Big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j h\Big\|_{L^2_{x,v}}.\end{aligned}$$

Putting above estimate of $\mathbf{I}_1$-$\mathbf{I}_7$ together, we eventually conclude that (3.13).

Similarly, we obtain the second inequality (3.14) of Theorem 3.3 by using the following equality

$$\begin{aligned}e^{c_1t(2n+l+\frac32)} = & e^{\frac32c_1t}e^{c_1t(2n+l+\frac32)}e^{-\frac32c_1t}\\ &+e^{\frac52c_1t}e^{c_1t(2(n-1)+(l+1)+\frac32)}e^{-\frac32c_1t}\\ &+e^{\frac52c_1t}e^{c_1t(2n+(l-1)+\frac32)}e^{-\frac32c_1t}\\ &+e^{\frac72c_1t}e^{c_1t(2(n-1)+l+\frac32)}e^{-\frac32c_1t}\\ &+e^{\frac72c_1t}e^{c_1t(2(n-2)+(l+2)+\frac32)}e^{-\frac32c_1t}\\ &+e^{\frac72c_1t}e^{c_1t(2n+(l-2)+\frac32)}e^{-\frac32c_1t}.\end{aligned}$$

This ends the proof of Theorem 3.3. □

With Theorem 3.3, we establish crucial estimates for Landau collision operator in the framework of Besov space, which are used to achieve the global-in-time existence.

Theorem 3.4. *Assume $s>0, 0<t,T\le+\infty$. Let $f=f(t,x,v)$, $g=g(t,x,v)$ and $h=h(t,x,v)$ be three suitably functions, then it holds that*

$$\begin{aligned}&\sum_{j\ge -1}2^{js}\left[\int_0^T\Big|(\Delta_j\mathbf{L}(f,g),\Delta_j h)_{L^2_{x,v}}\Big|\,dt\right]^{1/2}\\ &\lesssim\left\{\|\mathbb{S}_2 f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g\|^{1/2}_{L^2_TL^2_v(L^\infty_x)}+\|\mathbb{S}_2 f\|^{1/2}_{L^\infty_TL^2_v(L^\infty_x)}\|\widetilde{\mathcal{L}}^{\frac12}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\right\}\|\widetilde{\mathcal{L}}^{\frac12}h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}\sum_{j\geq -1}2^{js}\bigg[\int_0^T|\big(\Delta_j\mathbf{L}(f,g),\Delta_j e^{2c_1t\mathcal{H}}h\big)_{L^2_{x,v}}|dt\bigg]^{\frac12}&\lesssim e^{-\frac32c_1t}\\ \times\Big[\big\|\mathbb{S}_2e^{c_1t\mathcal{H}}f\big\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac12}e^{c_1t\mathcal{H}}g\big\|^{\frac12}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}e^{c_1t\mathcal{H}}h\big\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}&\\ +\big\|\mathbb{S}_2e^{c_1t\mathcal{H}}f\big\|_{L^\infty_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}e^{c_1t\mathcal{H}}g\big\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac12}e^{c_1t\mathcal{H}}h\big\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\Big].&\end{aligned}\tag{3.16}$$

Proof. It follows from Theorem 3.3 and Cauchy-Schwarz inequality that

$$\begin{aligned}&\sum_{j\geq -1}2^{js}\bigg[\int_0^T\Big|(\Delta_j\mathbf{L}(f,g),\Delta_jh)_{L^2_{x,v}}\Big|\,dt\bigg]^{1/2}\\ &\lesssim\sum_{j\geq -1}2^{js}\bigg[\sum_{|p-j|\leq4}\int_0^T\|\Delta_p\mathbb{S}_2f\|_{L^2_{x,v}}\big\|\widetilde{\mathcal{L}}^{\frac12}g\big\|_{L^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_jh\big\|_{L^2_{x,v}}dt\bigg]^{1/2}\\ &\quad+\sum_{j\geq -1}2^{js}\bigg[\sum_{p\geq j-4}\int_0^T\|\mathbb{S}_2f\|_{L^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_pg\big\|_{L^2_{x,v}}\big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_jh\big\|_{L^2_{x,v}}dt\bigg]^{1/2}\\ &\lesssim\sum_{j\geq -1}2^{js}\Big(\sum_{|p-j|\leq4}\|\Delta_p\mathbb{S}_2f\|_{L^\infty_TL^2_{x,v}}\Big)^{1/2}\big\|\widetilde{\mathcal{L}}^{\frac12}g\big\|^{1/2}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_jh\big\|^{1/2}_{L^2_TL^2_{x,v}}\\ &\quad+\sum_{j\geq -1}2^{js}\|\mathbb{S}_2f\|^{1/2}_{L^\infty_TL^2_v(L^\infty_x)}\Big(\sum_{p\geq j-4}\big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_pg\big\|_{L^2_TL^2_{x,v}}\Big)^{1/2}\big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_jh\big\|^{1/2}_{L^2_TL^2_{x,v}}.\end{aligned}$$

By changing the order of the summation, we have

$$\begin{aligned}&\sum_{j\geq -1}2^{js}\bigg[\int_0^T\Big|(\Delta_j\mathbf{L}(f,g),\Delta_jh)_{L^2_{x,v}}\Big|\,dt\bigg]^{1/2}\\ &\lesssim\|\mathbb{S}_2f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac12}g\big\|^{1/2}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}h\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\Big(\sum_{j\geq -1}\sum_{|p-j|\leq4}2^{(j-p)s}c(p)\Big)^{1/2}\\ &\quad+\|\mathbb{S}_2f\|^{1/2}_{L^\infty_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}h\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\Big(\sum_{j\geq -1}\sum_{p\geq j-4}2^{js}\big\|\widetilde{\mathcal{L}}^{\frac12}\Delta_pg\big\|_{L^2_TL^2_{x,v}}\Big)^{1/2},\end{aligned}$$

where

$$c(p)=\frac{2^{ps}\|\Delta_p\mathbb{S}_2f\|_{L^\infty_TL^2_{x,v}}}{\|\mathbb{S}_2f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}}$$

fulfills $\|c(p)\|_{\ell^1}\leq1$. Hence, by Fubini's theorem and Young's inequality, we get

$$\begin{aligned}\sum_{j\geq -1}\sum_{|p-j|\leq4}2^{(j-p)s}c(p)&=\sum_{j\geq -1}[(\mathbf{1}_{|p|\leq4}2^{ps})*c(p)](j)\\ &\leq\|\mathbf{1}_{|p|\leq4}2^{ps}\|_{\ell^1}\|c(p)\|_{\ell^1}<+\infty.\end{aligned}$$

We exchange the order in the summation that

$$\sum_{j\geq -1}\sum_{p\geq j-4}2^{js}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g\big\|_{L^2_TL^2_{x,v}}=\sum_{p\geq -1}2^{ps}\Big(\sum_{-1\leq j\leq p+4}2^{(j-p)s}\Big)\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g\big\|_{L^2_TL^2_{x,v}}.$$

It is obviously that, for $s>0$,

$$\sum_{-1\leq j\leq p+4}2^{(j-p)s}=2^{-(p+1)s}+2^{-ps}+\cdots+2^{4s}=\frac{2^{4s}(1-2^{-ps})}{1-2^{-s}}<+\infty,$$

then it follows that

$$\sum_{j\geq -1}2^{js}\sum_{p\geq j-4}\Big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g\Big\|_{L^2_TL^2_{x,v}}\lesssim\sum_{p\geq -1}2^{ps}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g\big\|_{L^2_TL^2_{x,v}}=\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g\big\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}.$$

Consequently, we conclude that

$$\begin{aligned}&\sum_{j\geq -1}2^{js}\left[\int_0^T\Big|(\Delta_j\mathbf{L}(f,g),\Delta_j h)_{L^2_{x,v}}\Big|\,dt\right]^{1/2}\\&\lesssim\left\{\|\mathbb{S}_2f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g\|^{1/2}_{L^2_TL^2_v(L^\infty_x)}+\|\mathbb{S}_2f\|^{1/2}_{L^\infty_TL^2_v(L^\infty_x)}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\right\}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})},\end{aligned}$$

which is just (3.15). Similarly, we obtain the inequality (3.16) with the aid of (3.14). The proof of Theorem 3.4 is completed. □

Remark 3.1. In fact, from the proof of the above Theorem 3.4, we also have

$$\begin{aligned}&\sum_{j\geq -1}2^{js}\left[\int_0^T|(\Delta_j\mathbf{L}(f,g),\Delta_j h)|dt\right]^{\frac{1}{2}}\\&\lesssim\|\mathbb{S}_2f\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g\big\|^{\frac{1}{2}}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}h\big\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}\\&\quad+\|\mathbb{S}_2f\|^{\frac{1}{2}}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g\big\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}h\big\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^s_{2,1})}.\end{aligned}$$

3.3.2 Coercivity of linear Landau operator

In order to obtain energy estimates, the coercivity of linear operator which indicates the microscopic dissipation plays a key role. The lower estimate of $(\mathcal{L}_1g,g)$ and the upper estimate of $(\mathcal{L}_2f,g)$ are shown in the following theorem.

Theorem 3.5. *For the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$, it holds that*

$$\begin{aligned}&(\mathcal{L}_1g,g)_{L^2_v}\geq\frac{1}{2}(\mathcal{L}g,g)_{L^2_v}=\frac{1}{2}\|\mathcal{L}^{\frac{1}{2}}g\|^2_{L^2_v}\geq\frac{1}{2}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g\|^2_{L^2_v}-C\|g\|^2_{L^2_v},\\&\Big|(\mathcal{L}_2f,g)_{L^2_v}\Big|\lesssim\|\mathbb{S}_2f\|_{L^2_v}\|\mathbb{S}_2g\|_{L^2_v}\end{aligned}\tag{3.17}$$

for some positive constant C.

Proof. By Proposition 3.1, we have

$$\begin{aligned}
\mathcal{L}_1 g &= -\mathbf{L}\left(\varphi_{0,0,0}, g\right) \\
&= -\sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} g_{n,l,m}\mathbf{L}\left(\varphi_{0,0,0}, \varphi_{n,l,m}\right) \\
&= \sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} \left(2(2n+l)+l(l+1)\right)\varphi_{n,l,m} g_{n,l,m}, \\
\mathcal{L}_2 g &= -\mathbf{L}\left(g, \varphi_{0,0,0}\right) \\
&= -\sum_{2\tilde{n}+\tilde{l}\le 2} g_{\tilde{n},\tilde{l},\tilde{m}}\mathbf{L}\left(\varphi_{\tilde{n},\tilde{l},\tilde{m}}, \varphi_{0,0,0}\right) \\
&= -\Big\{\sum_{|m_1|\le 1} A^{+}_{0,0,0,m_1} g_{0,1,m_1}\varphi_{0,1,m_1} + 4 g_{1,0,0}\varphi_{1,0,0} \\
&\quad + \sum_{|m_2|\le 2}\left(A^{2}_{0,0,0,m_2} g_{0,2,m_2}\varphi_{0,2,m_2} + A^{3}_{0,0,0,m_2} g_{0,2,m_2}\varphi_{0,2,m_2}\right)\Big\} \\
&= -4\sum_{|m_1|\le 1} g_{0,1,m_1}\varphi_{0,1,m_1} - 4 g_{1,0,0}\varphi_{1,0,0} + 2\sum_{|m_2|\le 2} g_{0,2,m_2}\varphi_{0,2,m_2}.
\end{aligned}$$

Furthermore, we see that

$$\begin{aligned}
\left(\mathcal{L}_1 g, g\right)_{L^2_v} &= \sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{|m|\le l}\left(2(2n+l)+l(l+1)\right)|g_{n,l,m}|^2 \\
&= 4\sum_{|m|\le 1}|g_{0,1,m}|^2 + 4|g_{1,0,0}|^2 + 10\sum_{|m|\le 2}|g_{0,2,m}|^2 \\
&\quad + \sum_{2n+l>2}\sum_{|m|\le l}\left(2(2n+l)+l(l+1)|g_{n,l,m}|^2\right., \\
\left(\mathcal{L}_2 f, g\right)_{L^2_v} &= -4\sum_{|m|\le 1} f_{0,1,m} g_{0,1,m} - 4 f_{1,0,0} g_{1,0,0} + 2\sum_{|m|\le 2} f_{0,2,m} g_{0,2,m}\,, \\
\left(\mathcal{L}_2 g, g\right)_{L^2_v} &= -4\sum_{|m|\le 1}|g_{0,1,m}|^2 - 4|g_{1,0,0}|^2 + 2\sum_{|m|\le 2}|g_{0,2,m}|^2\,.
\end{aligned}$$

Thus, keeping in mind that $\left(\mathcal{L}(g), g\right)_{L^2_v} = \|\mathcal{L}^{\frac{1}{2}} g\|^2_{L^2_v}$, we can obtain (3.17) with aid of Young's inequality. The proof of Theorem 3.5 is completed. □

Since the operator Δ_j acts on the position variable x only, we have the direct consequence of Theorem 3.5.

Corollary 3.1. *For the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$, it holds that*

$$\begin{aligned}
\left(\Delta_j\mathcal{L}_1 g, \Delta_j g\right)_{L^2_{x,v}} &= \frac{1}{2}\left(\Delta_j\mathcal{L} g, \Delta_j g\right)_{L^2_{x,v}} \ge \frac{1}{2}\|\mathcal{L}^{\frac{1}{2}}\Delta_j g\|^2_{L^2_{x,v}}, \\
\left(\Delta_j\mathcal{L}_2 f, \Delta_j g\right)_{L^2_{x,v}} &\lesssim \|\mathbb{S}_2\Delta_j f\|_{L^2_{x,v}}\|\mathbb{S}_2\Delta_j g\|_{L^2_{x,v}}
\end{aligned}$$

for each $j \geq -1$. Furthermore, we have

$$\sum_{j\geq -1} 2^{js}\left(\int_0^T (\Delta_j \mathcal{L} g, \Delta_j g)_{L^2_{x,v}}\, dt\right)^{1/2} = \frac{1}{2}\big\|\mathcal{L}^{\frac{1}{2}} g\big\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^s_{2,1})}$$

for $s > 0$ and $0 < T \leq \infty$.

3.3.3 Macro projections of nonlinear operator

For the nonlinear Landau operator $\mathbf{L}(g, g)$, we have the following macro projections.

Proposition 3.4. *Let $\varphi_{n,l,m}$ be the set of eigenfunctions. For $g, h \in \mathcal{S}(\mathbb{R}^3)$, we have*

$$(\mathbf{L}(g,g), \mathbf{P}h)_{L^2(\mathbb{R}^3)} = 0; \tag{3.18}$$

$$\begin{aligned}(\mathbf{L}(g,g), \varphi_{0,2,m})_{L^2(\mathbb{R}^3)} &= -12 g_{0,0,0} g_{0,2,m} \\ &+ \sum_{\substack{|m_1|\leq 1, |m'|\leq 1 \\ m_1+m'=m}} 4\sqrt{15\pi}\Big(\int_{\mathbb{S}^2} Y_1^{m_1} Y_1^{m'} Y_2^{-m}\Big) g_{0,1,m_1} g_{0,1,m'}, \quad \textit{for } |m| \leq 2;\end{aligned} \tag{3.19}$$

$$\begin{aligned}(\mathbf{L}(g,g), \varphi_{1,1,m})_{L^2(\mathbb{R}^3)} &= -8 g_{0,0,0} g_{1,1,m} + \frac{8\sqrt{15}}{3} g_{1,0,0} g_{0,1,m} \\ &+ \sum_{\substack{|m_1|\leq 1, |m'|\leq 2 \\ m_1+m'=m}} \frac{8\sqrt{6\pi}}{3}\Big(\int_{\mathbb{S}^2} Y_1^{m_1} Y_2^{m'} Y_1^{-m}\Big) g_{0,1,m_1} g_{0,2,m'}, \quad \textit{for } |m| \leq 1.\end{aligned} \tag{3.20}$$

Proof. For temperate functions $f, g \in \mathcal{S}(\mathbb{R}^3)$, it follows from Proposition 3.1 that

$$\begin{aligned}(\mathbf{L}(f,g), \mathbf{P}h)_{L^2(\mathbb{R}^3)} =& -4 f_{0,0,0} g_{1,0,0} h_{1,0,0} - 4 \sum_{|m|\leq 1} f_{0,0,0} g_{0,1,m} h_{0,1,m} \\ &+ \sum_{\substack{|m|\leq 1, |m_1|\leq 1 \\ m+m_1=0}} A^-_{0,1,m,m_1} f_{0,1,m_1} g_{0,1,m} h_{1,0,0} \\ &+ \sum_{|m_1|\leq 1} A^+_{0,0,0,m_1} f_{0,1,m_1} g_{0,0,0} h_{0,1,m_1} + 4 f_{1,0,0} g_{0,0,0} h_{1,0,0}.\end{aligned}$$

We need to compute $A^+_{0,0,0,m_1}, A^-_{0,1,m,m_1}$ for $|m| \leq 1$ and $|m_1| \leq 1$. The direct computation shows that

$$A^+_{0,0,0,m_1} = 4, \quad A^-_{0,1,m,m_1} \equiv 0.$$

Therefore, if we take the case of $f = g$, then we obtain

$$\begin{aligned}(\mathbf{L}(g,g), \mathbf{P}h)_{L^2(\mathbb{R}^3)} =& -4 g_{0,0,0} g_{1,0,0} h_{1,0,0} - 4 \sum_{|m|\leq 1} g_{0,0,0} g_{0,1,m} h_{0,1,m} \\ &+ \sum_{|m_1|\leq 1} 4 g_{0,1,m_1} g_{0,0,0} h_{0,1,m_1} + 4 g_{1,0,0} g_{0,0,0} h_{1,0,0} = 0,\end{aligned}$$

which is just (3.18).

Now we prove the equality (3.19). For $|m| \le 2$,

$$\begin{aligned}&(\mathbf{L}(g,g),\varphi_{0,2,m})_{L^2(\mathbb{R}^3)}\\&= -10g_{0,0,0}g_{0,2,m} + \sum_{\substack{|m_1|\le 1,|m'|\le 1\\ m_1+m'=m}} A^+_{0,1,m',m_1} g_{0,1,m_1}g_{0,1,m'} + A^3_{0,0,0,m}g_{0,2,m}g_{0,0,0}.\end{aligned}$$

It is not difficult to find that

$$A^+_{0,1,m',m_1} = 4\sqrt{15\pi}\Big(\int_{\mathbb{S}^2} Y_1^{m_1}Y_1^{m}Y_2^{-m'-m_1}\Big), \quad A^3_{0,0,0,m} = -2.$$

Substituting into the above formula, we get

$$\begin{aligned}&(\mathbf{L}(g,g),\varphi_{0,2,m})_{L^2(\mathbb{R}^3)}\\&= -12g_{0,0,0}g_{0,2,m} + \sum_{\substack{|m_1|\le 1,|m'|\le 1\\ m_1+m'=m}} 4\sqrt{15\pi}\Big(\int_{\mathbb{S}^2} Y_1^{m_1}Y_1^{m}Y_2^{-m}\Big) g_{0,1,m_1}g_{0,1,m'}.\end{aligned}$$

On the other hand, by a direct computation, we obtain

$$\begin{aligned}(\mathbf{L}(g,g),\varphi_{1,1,m})_{L^2(\mathbb{R}^3)} &= -8g_{0,0,0}g_{1,1,m} + \sum_{\substack{|m'|\le 2,|m_1|\le 1\\ m'+m_1=m}} A^-_{0,2,m',m_1} g_{0,1,m_1}g_{0,2,m'}\\ &\quad + A^+_{1,0,0,m}g_{0,1,m}g_{1,0,0} + \frac{4\sqrt{15}}{3}g_{1,0,0}g_{0,1,m}\\ &\quad + \sum_{\substack{|m_2|\le 2,|m'|\le 1\\ m_2+m'=m}} A^2_{0,1,m',m_2}g_{0,2,m_2}g_{0,1,m'},\end{aligned}$$

where

$$\begin{aligned}A^+_{1,0,0,m_1} &= \frac{4\sqrt{15}}{3};\\ A^-_{0,2,m',m_1} &= \frac{4\sqrt{6\pi}}{3}\Big(\int_{\mathbb{S}^2} Y_1^{m_1}Y_2^{m'}Y_1^{-m'-m_1}\Big);\\ A^2_{0,1,m',m_2} &= \frac{4\sqrt{6\pi}}{3}\Big(\int_{\mathbb{S}^2} Y_2^{m_2}Y_1^{m'}Y_1^{-m'-m_2}\Big).\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}(\mathbf{L}(g,g),\varphi_{1,1,m})_{L^2(\mathbb{R}^3)} &= -8g_{0,0,0}g_{1,1,m} + \frac{8\sqrt{15}}{3}g_{1,0,0}g_{0,1,m}\\ &\quad + \sum_{\substack{|m_1|\le 1,|m'|\le 2\\ m'+m_1=m}} \frac{8\sqrt{6\pi}}{3}\Big(\int_{\mathbb{S}^2} Y_1^{m_1}Y_2^{m'}Y_1^{-m}\Big) g_{0,1,m_1}g_{0,2,m'}.\end{aligned}$$

Hence, the proof of Proposition 3.4 is completed. □

Based on Proposition 3.4, we have the following estimate in spatially Besov spaces.

Proposition 3.5. *Let $s > 0$ and $\phi(v)$ be the finite combination of the eigenfunctions*

$$\{\varphi_{0,2,m_2}(v), \varphi_{1,1,m_1}(v)\}_{|m_1|\le 1, |m_2|\le 2}\,.$$

Then it holds that

$$\begin{aligned}&\sum_{j\ge -1} 2^{js}\left(\int_0^T \left\|(\Delta_j \mathbf{L}(g,g), \phi(v))_{L^2_v}\right\|^2_{L^2_x} dt\right)^{1/2}\\ &\lesssim \|\mathbb{S}_3 g\|_{L^2_T L^2_v(L^\infty_x)} \|\mathbf{P}g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^s_{2,1})} + \|\mathbf{P}g\|_{L^2_T L^2_v(L^\infty_x)} \|\mathbb{S}_3 g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^s_{2,1})}\end{aligned} \tag{3.21}$$

for any $T > 0$.

Proof. We need to prove (3.21) for all $\{\varphi_{0,2,m_2}(v), \varphi_{1,1,m_1}(v)\}$ with $|m_1| \le 1, |m_2| \le 2$ one by one. For $|m| \le 2$, it follows from (3.19) that

$$\begin{aligned}&\left\|(\Delta_j \mathbf{L}(g,g), \varphi_{0,2,m})_{L^2_v(\mathbb{R}^3)}\right\|^2_{L^2_x}\\ &\lesssim \|\Delta_j(g_{0,0,0} g_{0,2,m})\|^2_{L^2_x}\\ &\quad + \Big\| \sum_{\substack{|m_1|\le 1, |m'|\le 1\\ m_1+m'=m}} \Big(\int_{\mathbb{S}^2} Y_1^{m_1} Y_1^{m'} Y_2^{-m}\Big) \Delta_j(g_{0,1,m_1} g_{0,1,m'})\Big\|^2_{L^2_x}\\ &\lesssim \|\Delta_j(g_{0,0,0} g_{0,2,m})\|^2_{L^2_x} + \sum_{|m_1|\le 1}\sum_{|m'|\le 1} \left\|\Delta_j(g_{0,1,m_1} g_{0,1,m'})\right\|^2_{L^2_x}\,.\end{aligned} \tag{3.22}$$

By using Theorem 2.4, the right side of (3.22) can be estimated as

$$\begin{aligned}&\|\Delta_j(g_{0,0,0} g_{0,2,m})\|^2_{L^2_x}\\ &\le \sum_{|p-j|\le 4} \|\Delta_j(S_{p-1} g_{0,0,0} \Delta_p g_{0,2,m} + \Delta_p g_{0,0,0} S_{p-1} g_{0,2,m})\|^2_{L^2_x}\\ &\quad + \sum_{\max(p,p')\ge j-2} \sum_{|p-p'|\le 1} \left\|\Delta_j(\Delta_{p'} g_{0,0,0} \Delta_p g_{0,2,m})\right\|^2_{L^2_x}\\ &\le \sum_{|p-j|\le 4} \Big(\|g_{0,0,0}\|^2_{L^\infty_x} \|\Delta_p g_{0,2,m}\|^2_{L^2_x} + \|g_{0,2,m}\|^2_{L^\infty_x} \|\Delta_p g_{0,0,0}\|^2_{L^2_x}\Big)\\ &\quad + \sum_{p\ge j-3} \|g_{0,0,0}\|^2_{L^\infty_x} \|\Delta_p g_{0,2,m}\|^2_{L^2_x}\\ &\lesssim \sum_{|p-j|\le 4} \|g_{0,2,m}\|^2_{L^\infty_x} \|\Delta_p g_{0,0,0}\|^2_{L^2_x} + \sum_{p\ge j-4} \|g_{0,0,0}\|^2_{L^\infty_x} \|\Delta_p g_{0,2,m}\|^2_{L^2_x}\end{aligned}$$

and

$$\begin{aligned}&\left\|\Delta_j(g_{0,1,m_1} g_{0,1,m'})\right\|^2_{L^2_x}\\ &\lesssim \sum_{|p-j|\le 4} \|g_{0,1,m_1}\|^2_{L^\infty_x} \left\|\Delta_p g_{0,1,m'}\right\|^2_{L^2_x} + \sum_{p\ge j-4} \|\Delta_p g_{0,1,m_1}\|^2_{L^2_x} \left\|g_{0,1,m'}\right\|^2_{L^\infty_x}\,.\end{aligned}$$

Substituting these two estimates into (3.22) implies that

$$\begin{aligned}
&\left\|\left(\Delta_j\mathbf{L}(g,g),\varphi_{0,2,m}\right)_{L^2_v(\mathbb{R}^3)}\right\|^2_{L^2_x}\\
&\lesssim \sum_{|p-j|\le 4}\|\mathbb{S}_2 g\|^2_{L^2_vL^\infty_x}\|\Delta_p\mathbf{P}g\|^2_{L^2_vL^2_x}+\sum_{p\ge j-4}\|\mathbf{P}g\|^2_{L^2_vL^\infty_x}\|\Delta_p\mathbb{S}_2 g\|^2_{L^2_vL^2_x}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{j\ge -1}2^{js}\left(\int_0^T\left\|\left(\Delta_j\mathbf{L}(g,g),\varphi_{0,2,m}(v)\right)_{L^2_v}\right\|^2_{L^2_x}dt\right)^{1/2}\\
&\lesssim \sum_{j\ge -1}\sum_{|p-j|\le 4}2^{js}\left(\int_0^T\|\mathbb{S}_2 g\|^2_{L^2_vL^\infty_x}\|\Delta_p\mathbf{P}g\|^2_{L^2_vL^2_x}dt\right)^{1/2}\\
&\quad+\sum_{j\ge -1}\sum_{p\ge j-4}2^{js}\left(\int_0^T\|\mathbf{P}g\|^2_{L^2_vL^\infty_x}\|\Delta_p\mathbb{S}_2 g\|^2_{L^2_vL^2_x}dt\right)^{1/2}\\
&\lesssim \|\mathbb{S}_2 g\|_{L^2_TL^2_v(L^\infty_x)}\|\mathbf{P}g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}+\|\mathbf{P}g\|_{L^2_TL^2_v(L^\infty_x)}\|\mathbb{S}_2 g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}.
\end{aligned}$$

Similarly, it follows from (3.20) in Proposition 3.4 that

$$\begin{aligned}
&\sum_{j\ge -1}2^{js}\left(\int_0^T\left\|\left(\Delta_j\mathbf{L}(g,g),\varphi_{1,1,m}(v)\right)_{L^2_v}\right\|^2_{L^2_x}dt\right)^{1/2}\\
&\lesssim \|\mathbb{S}_3 g\|_{L^2_TL^2_v(L^\infty_x)}\|\mathbf{P}g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}+\|\mathbf{P}g\|_{L^2_TL^2_v(L^\infty_x)}\|\mathbb{S}_3 g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}
\end{aligned}$$

for all $|m|\le 1$. It ends the proof of Proposition 3.5. □

3.4 Commutators estimates

To improve the regularity of weak solution, we need delicate commutator estimates involving the nonlinear term $\mathbf{L}(f,g)$ and cut-off functions with respect to variables x and v. Notice that

$$\begin{aligned}
\mathbf{L}(f,g)=&\sum_{i,j=1}^3\partial_i\int_{\mathbb{R}^3}a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)\partial_j g(v)dv_*\\
&-\frac{1}{2}\sum_{i,j=1}^3\int_{\mathbb{R}^3}a^{ij}(v-v_*)v_{*i}\mu^{1/2}(v_*)f(v_*)\partial_j g(v)dv_*\\
&-\sum_{i,j=1}^3\partial_i\int_{\mathbb{R}^3}a^{ij}(v-v_*)\mu^{1/2}(v_*)(\partial_j f)(v_*)g(v)dv_*\\
&+\frac{1}{2}\sum_{i,j=1}^3\int_{\mathbb{R}^3}a^{ij}(v-v_*)v_{*i}\mu^{1/2}(v_*)(\partial_j f)(v_*)g(v)dv_*
\end{aligned}\tag{3.23}$$

with $a^{ij}(v)=\delta_{ij}|v|^2-v_iv_j$ (δ_{ij} is Kronecker's delta), where we used the fact that

$$\sum_i a^{ij}(v-v_*)(v_i-v_{*i})=\sum_j a^{ij}(v-v_*)(v_j-v_{*j})=0.$$

In addition, we have

$$\begin{aligned}(\mathcal{L}g,g)_{L^2_v}=\|\mathcal{L}^{\frac12}g\|^2_{L^2_v}=&2\|\nabla_v(\mathbf{I}-\mathbf{P})g\|^2_{L^2_v}+\frac12\|v(\mathbf{I}-\mathbf{P})g\|^2_{L^2_v}-3\|(\mathbf{I}-\mathbf{P})g\|^2_{L^2_v}\\&+\frac12\sum_{\substack{1\le j,k\le 3\\ j\neq k}}\|(v_j\partial_k-v_k\partial_j)(\mathbf{I}-\mathbf{P})g\|^2_{L^2_v},\end{aligned}$$

which leads to

$$\|\nabla_v(\mathbf{I}-\mathbf{P})g\|^2_{L^2_v}+\|(\mathbf{I}-\mathbf{P})g\|^2_{L^2_1}\lesssim\|\mathcal{L}^{\frac12}g\|^2_{L^2_v}\lesssim\|(\mathbf{I}-\mathbf{P})g\|^2_{H^1_1}\tag{3.24}$$

and

$$\|\widetilde{\mathcal{L}}^{\frac12}g\|_{L^2_v}\le\|\mathcal{L}^{\frac12}g\|_{L^2_v}+\|\mathcal{L}^{\frac12}\mathbf{P}g\|_{L^2_v}\le\|\mathcal{L}^{\frac12}g\|_{L^2_v}+C\|g\|_{L^2_v}.$$

3.4.1 Commutators with moments

Let

$$W_{\delta'}(v)=\langle\delta'v\rangle^{-2}=\frac{1}{1+|\delta'v|^2}$$

for $0<\delta'<1$. Here and below, we agree with the norm

$$\|f\|_{L^2_1}=\left(\int_{\mathbb{R}^3}|f(v)|^2(1+|v|)^2dv\right)^{\frac12}.$$

Proposition 3.6. *For $0<\delta'<1$ and $f,g,h\in\mathcal{S}(\mathbb{R}^3)$, one has*

$$\begin{aligned}&\left|(W_{\delta'}\mathbf{L}(f,g)-\mathbf{L}(f,W_{\delta'}g),h)_{L^2_v}\right|\\&\lesssim\|f\|_{L^2}\left(\|W_{\delta'}g\|_{L^2_1}\|h\|_{L^2}+\|\nabla_vW_{\delta'}g\|_{L^2}\|h\|_{L^2_1}+\|W_{\delta'}g\|_{L^2_1}\|\nabla_vh\|_{L^2}\right).\end{aligned}\tag{3.25}$$

Proof. It follows from (3.23) that

$$\begin{aligned}&W_{\delta'}\mathbf{L}(f,g)-\mathbf{L}(f,W_{\delta'}g)\\&=\sum_{i,j=1}^3W_{\delta'}\partial_i\int_{\mathbb{R}^3}a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)\left(2\delta'^2v_jW_{\delta'}g+W_{\delta'}^{-1}\partial_j(W_{\delta'}g(v))\right)dv_*\\&\quad-\frac12\sum_{i,j=1}^3W_{\delta'}\int_{\mathbb{R}^3}a^{ij}(v-v_*)v_{i*}\mu^{1/2}(v_*)f(v_*)\left(2\delta'^2v_jW_{\delta'}g+W_{\delta'}^{-1}\partial_j\left(W_{\delta'}g(v)\right)\right)dv_*\\&\quad+\sum_{i,j=1}^3\partial_i\left(W_{\delta'}\right)\int_{\mathbb{R}^3}a^{ij}(v-v_*)\mu^{1/2}(v_*)(\partial_jf)(v_*)g(v)dv_*\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i,j=1}^{3} \partial_i \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)\partial_j(W_{\delta'}g(v))dv_* \\
&+ \frac{1}{2}\sum_{i,j=1}^{3} \int_{\mathbb{R}^3} a^{ij}(v-v_*)v_{i*}\mu^{1/2}(v_*)f(v_*)\partial_j(W_{\delta'}g(v))dv_*,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&W_{\delta'}\mathbf{L}(f,g) - \mathbf{L}(f,W_{\delta'}g) \\
=&\sum_{i,j=1}^{3} W_{\delta'}\partial_i \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)2\delta'^2 v_j W_{\delta'}g(v)dv_* \\
&- \sum_{i,j=1}^{3} \partial_i\left(W_{\delta'}\right) \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)W_{\delta'}^{-1}\partial_j(W_{\delta'}g(v))dv_* \\
&- \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} a^{ij}(v-v_*)v_{i*}\mu^{1/2}(v_*)f(v_*)\delta'^2 v_j W_{\delta'}^2 g(v)dv_* \\
&+ \sum_{i,j=1}^{3} \partial_i\left(W_{\delta'}\right) \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)(\partial_j f)(v_*)g(v)dv_*.
\end{aligned}
$$

Therefore, we arrive at

$$
\begin{aligned}
&W_{\delta'}\mathbf{L}(f,g) - \mathbf{L}(f,W_{\delta'}g) \\
=&\sum_{i,j=1}^{3} \partial_i \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)2\delta'^2 v_j W_{\delta'}^2 g(v)dv_* \\
&- \sum_{i,j=1}^{3} \partial_i\left(W_{\delta'}\right) \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)2\delta'^2 v_j W_{\delta'}g(v)dv_* \\
&- \sum_{i,j=1}^{3} \partial_i\left(W_{\delta'}\right) \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)W_{\delta'}^{-1}\partial_j(W_{\delta'}g(v))dv_* \\
&- \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} a^{ij}(v-v_*)v_{i*}\mu^{1/2}(v_*)f(v_*)\delta'^2 v_j W_{\delta'}^2 g(v)dv_* \\
&+ \sum_{i,j=1}^{3} \partial_i\left(W_{\delta'}\right) \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)(\partial_j f)(v_*)g(v)dv_* \\
=&\sum_{i,j=1}^{3} \partial_i \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)2\delta'^2 v_j W_{\delta'}^2 g(v)dv_* \\
&+ \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)4\delta'^4 v_i v_j W_{\delta'}^3 g(v)dv_*
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)2\delta'^2v_iW_{\delta'}\partial_j(W_{\delta'}g(v))dv_*\\
&-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}a^{ij}(v-v_*)v_{i*}\mu^{1/2}(v_*)f(v_*)2\delta'^2v_jW_{\delta'}^2g(v)dv_*\\
&-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}a^{ij}(v-v_*)\mu^{1/2}(v_*)(\partial_jf)(v_*)2\delta'^2v_iW_{\delta'}^2g(v)dv_*.\\
\triangleq&A_1+A_2+A_3+A_4+A_5,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1=&-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)2\delta'^2v_jW_{\delta'}^2g(v)dv_*\right)\partial_ih(v)dv;\\
A_2=&\sum_{i,j=1}^{3}\int_{\mathbb{R}^6}a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)4\delta'^4v_iv_jW_{\delta'}^3g(v)h(v)dvdv_*;\\
A_3=&\sum_{i,j=1}^{3}\int_{\mathbb{R}^6}a^{ij}(v-v_*)\mu^{1/2}(v_*)f(v_*)2\delta'^2v_iW_{\delta'}\partial_j(W_{\delta'}g(v))h(v)dvdv_*;\\
A_4=&-\sum_{i,j=1}^{3}\int_{\mathbb{R}^6}a^{ij}(v-v_*)v_{*i}\mu^{1/2}(v_*)f(v_*)2\delta'^2v_jW_{\delta'}^2gh(v)dvdv_*;\\
A_5=&-\sum_{i,j=1}^{3}\int_{\mathbb{R}^6}a^{ij}(v-v_*)\mu^{1/2}(v_*)(\partial_jf)(v_*)2\delta'^2v_iW_{\delta'}^2g(v)h(v)dvdv_*.
\end{aligned}
$$

In the following, $A_1,\cdots,A_5$ can be estimated one by one. For A_1, by using integration by parts and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|A_1|\lesssim&\sum_{i,j=1}^{3}\Big|\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}a^{ij}(v-v_*)^2\mu(v_*)dv_*\right)^{1/2}\left(\int_{\mathbb{R}^3}|f(v_*)|^2\,dv_*\right)^{1/2}\\
&\qquad\times\delta'^2|v|\left|W_{\delta'}^2g(v)\partial_ih(v)\right|dv\Big|\\
\lesssim&\sum_{i=1}^{3}\left|\int_{\mathbb{R}^3}\|f\|_{L_v^2}(1+|v|)^2\delta'^2|v|W_{\delta'}\left|W_{\delta'}g(v)\partial_ih(v)\right|dv\right|\\
\lesssim&\|f\|_{L_v^2}\left(\int_{\mathbb{R}^3}(1+|v|)^2\left|W_{\delta'}g\right|^2dv\right)^{1/2}\left(\int_{\mathbb{R}^3}|\nabla_vh|^2\,dv\right)^{1/2}\\
\lesssim&\|f\|_{L_v^2}\left\|W_{\delta'}g\right\|_{L_1^2}\left\|\nabla_vh\right\|_{L_v^2},
\end{aligned}
\tag{3.26}
$$

where we used the fact

$$
(1+|v|)\delta'^2|v|W_{\delta'}=\frac{\delta'^2|v|}{1+|\delta'v|^2}+\frac{\delta'^2|v|^2}{1+|\delta'v|^2}\le\delta'+1\le2.
$$

Next, we calculate A_2-A_5 and obtain

$$
\begin{aligned}
|A_2| &\lesssim \sum_{i,j=1}^{3} \Big| \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} a^{ij}(v-v_*)^2 \mu(v_*) dv_* \right)^{1/2} \left(\int_{\mathbb{R}^3} |f(v_*)|^2 dv_* \right)^{1/2} \\
&\qquad \times \delta'^4 |v|^2 |W^3_{\delta'} g(v) h(v)| dv \Big| \\
&\lesssim \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1+|v|)^2 \delta'^4 |v|^2 W^2_{\delta'} \left| W_{\delta'} g(v) h(v) \right| dv \\
&\lesssim \|f\|_{L^2_v} \left(\int_{\mathbb{R}^3} |W_{\delta'} g|^2 \, dv \right)^{1/2} \left(\int_{\mathbb{R}^3} |h|^2 \, dv \right)^{1/2} \\
&\lesssim \|f\|_{L^2_v} \|W_{\delta'} g\|_{L^2} \|h\|_{L^2_v} ; \\
|A_3| &\lesssim \sum_{i,j=1}^{3} \Big| \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} a^{ij}(v-v_*)^2 \mu(v_*) dv_* \right)^{1/2} \left(\int_{\mathbb{R}^3} |f(v_*)|^2 \, dv_* \right)^{1/2} \\
&\qquad \times \delta'^2 |v| W_{\delta'} |\partial_j \left(W_{\delta'} g(v) \right) h(v)| dv \Big| \\
&\lesssim \sum_{j=1}^{3} \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1+|v|)^2 \delta'^2 |v| W_{\delta'} \left| \partial_j \left(W_{\delta'} g(v) \right) h(v) \right| dv \\
&\lesssim \|f\|_{L^2_v} \|\nabla_v W_{\delta'} g\|_{L^2_v} \|h\|_{L^2_1} ; \\
|A_4| &\lesssim \sum_{i,j=1}^{3} \Big| \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} a^{ij}(v-v_*)^2 v^2_{*i} \mu(v_*) dv_* \right)^{1/2} \left(\int_{\mathbb{R}^3} |f(v_*)|^2 \, dv_* \right)^{1/2} \\
&\qquad \times \delta'^2 |v| |W^2_{\delta'} g(v) h(v)| dv \Big| \\
&\lesssim \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1+|v|)^2 \delta'^2 |v| W_{\delta'} \left| W_{\delta'} g(v) h(v) \right| dv \\
&\lesssim \|f\|_{L^2_v} \|W_{\delta'} g\|_{L^2_1} \|h\|_{L^2_v} ; \\
|A_5| &\lesssim \sum_{i,j=1}^{3} \Big| \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |\partial_{j*} (a^{ij}(v-v_*) \mu(v_*))|^2 dv_* \right)^{1/2} \left(\int_{\mathbb{R}^3} |f(v_*)|^2 \, dv_* \right)^{1/2} \\
&\qquad \times \delta'^2 |v| \left| W^2_{\delta'} g(v) h(v) \right| dv \Big| \\
&\lesssim \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1+|v|)^2 \delta'^2 |v| W_{\delta'} \left| W_{\delta'} g(v) h(v) \right| dv \\
&\lesssim \|f\|_{L^2_v} \|W_{\delta'} g\|_{L^2_1} \|h\|_{L^2_v} .
\end{aligned}
\tag{3.27}
$$

Hence, (3.25) follows from (3.26)-(3.27) directly. □

3.4.2 Commutators with a mollifier in the x variable

Inspired by [5, 145], we can obtain the commutator of the collision operator with a mollifier in the x variable.

Proposition 3.7. *Let $S \in C_0^\infty(\mathbb{R})$ satisfying $0 \le S \le 1$ and*

$$S(\tau) = 1, \quad |\tau| \le 1; \quad S(\tau) = 0, \quad |\tau| \ge 2.$$

Let $S_\delta(D_x) = S(\delta D_x)$ for $\delta > 0$. Then for $f, g, h \in \mathcal{S}(\mathbb{R}^6_{x,v})$, we have

$$\begin{aligned} &\left| \int_0^T (S_\delta(D_x)\mathbf{L}(f,g) - \mathbf{L}(f, S_\delta(D_x)g), h)_{L^2_{x,v}} \, dt \right| \\ &\lesssim \delta \|\nabla_x \mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x; L^2_v)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})}. \end{aligned} \tag{3.28}$$

Proof. As in [5, 145], let us introduce

$$K_\delta(z) = -\frac{z}{\delta^4} \mathcal{F}^{-1}(S)(\frac{z}{\delta}).$$

Then, we have

$$\begin{aligned} &\left| \int_0^T (S_\delta(D_x)\mathbf{L}(f,g) - \mathbf{L}(f, S_\delta(D_x)g), h)_{L^2(\mathbb{R}^6_{x,v})} \, dt \right| \\ &= \Big| \int_0^1 \Big(\int_{\mathbb{R}^3_x \times \mathbb{R}^3_y} K_\delta(x-y) \\ &\qquad \times \int_0^T (\mathbf{L}(\nabla_x f(t, x+\tau(y-x), v), \delta g(t,y,v)), h(t,x,v))_{L^2(\mathbb{R}^3_v)} \, dt dx dy \Big) d\tau \Big| \\ &\lesssim \delta \, \|\nabla_x \mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x; L^2_v)} \int_0^T \int_{\mathbb{R}^3_x} \left(|K_\delta| * \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2_v} \right)(x) \|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|_{L^2_v} dx dt \\ &\lesssim \delta \|\nabla_x \mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x; L^2_v)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})}, \end{aligned}$$

which is just (3.28) and where we used Theorem 3.2 and $\|K_\delta\|_{L^1_x} = \|K_1\|_{L^1_x}$. □

3.4.3 Commutators with a mollifier in the v variable

Let $M^\delta(D_v) = \frac{1}{1-\delta\Delta_v}$ for $0 < \delta \le 1$, which is a pseudo-differential operator of symbol $M^\delta(\xi) = \frac{1}{1+\delta|\xi|^2}$.

Proposition 3.8. *Let $0 < \delta \le 1$, and $f, g, h \in \mathcal{S}(\mathbb{R}^3)$. It holds that*

$$\left| \left(M^\delta \mathbf{L}(f,g) - \mathbf{L}(f, M^\delta g), h \right)_{L^2_v} \right| \lesssim \|f\|_{L^2_v} \left(\|M^\delta g\|_{L^2_1} \|\nabla_v h\|_{L^2_v} + \|M^\delta g\|_{L^2_v} \|h\|_{L^2_1} \right). \tag{3.29}$$

Proof. By using the nonlinear term $\mathbf{L}(f,g)$ of (3.23) and integration by parts, we have

$$\begin{aligned}
&\Big(M^\delta \mathbf{L}(f,g) - \mathbf{L}(f, M^\delta g), h\Big)_{L^2_v} \\
&= -\sum_{i,j=1}^3 \int_{\mathbb{R}^6} \Big[M^\delta, a^{ij}(v-v_*)\Big] \partial_j g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) dv_* dv \\
&\quad - \frac{1}{2}\sum_{i,j=1}^3 \int_{\mathbb{R}^6} \Big[M^\delta, a^{ij}(v-v_*)\Big] \partial_j g(v) \mu^{1/2}(v_*) f(v_*) h(v) dv_* dv \\
&\quad + \sum_{i,j=1}^3 \int_{\mathbb{R}^6} \Big[M^\delta, a^{ij}(v-v_*)\Big] g(v) \mu^{1/2}(v_*) (\partial_j f)(v_*) \partial_i h(v) dv_* dv \\
&\quad + \frac{1}{2}\sum_{i,j=1}^3 \int_{\mathbb{R}^6} \Big[M^\delta, a^{ij}(v-v_*)\Big] g(v) \mu^{1/2}(v_*) (\partial_j f)(v_*) h(v) dv_* dv \\
&\triangleq B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

For B_1, note that $(M^\delta)^{-1} = 1 - \delta\Delta_v$, we get

$$\begin{aligned}
B_1 &= -\sum_{i,j=1}^3 \int_{\mathbb{R}^6} M^\delta \Big[a^{ij}(v-v_*), (M^\delta)^{-1}\Big] M^\delta \partial_j g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) dv_* dv \\
&= \sum_{i,j=1}^3 \int_{\mathbb{R}^6} M^\delta \left[1 - \delta\Delta_v, a^{ij}(v-v_*)\right] \partial_j M^\delta g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) dv_* dv.
\end{aligned}$$

Owing to the fact

$$\left[1 - \delta\Delta_v, a^{ij}(v-v_*)\right] = -\delta\left(2\sum_{l=1}^3 \partial_l a^{ij}(v-v_*)\partial_l + \sum_{l=1}^3 \partial_l^2 a^{ij}(v-v_*)\right),$$

furthermore, one can deduce that

$$\begin{aligned}
B_1 &= -2\delta \sum_{i,j=1}^3 \sum_{l=1}^3 \int_{\mathbb{R}^6} M^\delta \partial_l a^{ij}(v-v_*) \partial_l \partial_j M^\delta g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) dv_* dv \\
&\quad - \delta \sum_{i,j=1}^3 \sum_{l=1}^3 \int_{\mathbb{R}^6} M^\delta \partial_l^2 a^{ij}(v-v_*) \partial_j M^\delta g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) dv_* dv,
\end{aligned}$$

where we agree with $\partial_l = \partial_{v_l}$. The direct calculation enables us to get

$$\begin{aligned}
M^\delta \partial_l a^{ij}(v-v_*) &= \partial_l a^{ij}(v-v_*) M^\delta + [M^\delta, \partial_l a^{ij}(v-v_*)] \\
&= \partial_l a^{ij}(v-v_*) M^\delta - M^\delta \left[1 - \delta\Delta_v, \partial_l a^{ij}(v-v_*)\right] M^\delta \\
&= \partial_l a^{ij}(v-v_*) M^\delta + 2\delta M^\delta \sum_{k=1}^3 \partial_k \partial_l a^{ij}(v-v_*) \partial_k M^\delta.
\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}
&\partial_l a^{ij}(v-v_*) = 2\delta_{ij}(v_l - v_{l*}) - (\delta_{li}(v_j - v_{j*}) + \delta_{lj}(v_i - v_{i*}))\,;\\
&\partial_l^2 a^{ij}(v-v_*) = 2\delta_{ij} - 2\delta_{li}\delta_{lj}, \quad l = 1,2,3;\\
&\partial_l\partial_k a^{ij}(v-v_*) = -\delta_{ki}\delta_{lj}, \quad k \neq l,
\end{aligned}$$

where the last two derivations are constants. Consequently, we are led to

$$\begin{aligned}
B_1 = &-2\delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l a^{ij}(v-v_*)\partial_l\partial_j(M^\delta)^2 g(v)\mu^{1/2}(v_*)f(v_*)\partial_i h(v)dv_*dv\\
&- 4\delta^2 \sum_{i,j=1}^{3}\sum_{l=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^6} \partial_k\partial_l a^{ij}(v-v_*)\partial_l\partial_j\partial_k(M^\delta)^3 g(v)\mu^{1/2}(v_*)f(v_*)\partial_i h(v)dv_*dv\\
&- \delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l^2 a^{ij}(v-v_*)\partial_j(M^\delta)^2 g(v)\mu^{1/2}(v_*)f(v_*)\partial_i h(v)dv_*dv.
\end{aligned}$$

So, by Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned}
|B_1| &\lesssim \delta\|f\|_{L^2}\sum_{i,j,l=1}^{3}\int_{\mathbb{R}^3}(1+|v|)|\partial_l\partial_j(M^\delta)^2 g(v)||\partial_i h(v)|dv\\
&\quad + \delta^2\|f\|_{L^2}\sum_{i,j,l,k=1}^{3}\int_{\mathbb{R}^3}|\partial_l\partial_j\partial_k(M^\delta)^3 g(v)||\partial_i h(v)|dv\\
&\quad + \delta\|f\|_{L^2}\sum_{i,j,l=1}^{3}\int_{\mathbb{R}^3}|\partial_j(M^\delta)^2 g(v)||\partial_i h(v)|dv\\
&\lesssim \|f\|_{L^2}\left\|M^\delta g\right\|_{L^2_1}\|\nabla_v h\|_{L^2_v},
\end{aligned}$$

where the following estimates are used:

$$\begin{aligned}
&\delta\|(1+|v|)\partial_l\partial_j(M^\delta)u\|_{L^2(\mathbb{R}^3_v)} \le \|u\|_{L^2_1}\,;\\
&\delta^2\|\partial_l\partial_j\partial_k(M^\delta)^2 u\| \le \|u\|_{L^2} \le \|u\|_{L^2_1}\,;\\
&\delta\|\partial_j(M^\delta)u\| \le \|u\|_{L^2} \le \|u\|_{L^2_1}
\end{aligned}$$

for $u \in L^2_1(\mathbb{R}^3)$.

By employing the similar calculations as B_1, we have

$$\begin{aligned}
B_2 = &-\delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l a^{ij}(v-v_*)\partial_l\partial_j(M^\delta)^2 g(v)\mu^{1/2}(v_*)f(v_*)h(v)dv_*dv\\
&- 2\delta^2 \sum_{i,j=1}^{3}\sum_{l=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^6} \partial_k\partial_l a^{ij}(v-v_*)\partial_l\partial_j\partial_k(M^\delta)^3 g(v)\mu^{1/2}(v_*)f(v_*)h(v)dv_*dv\\
&- \frac{1}{2}\delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l^2 a^{ij}(v-v_*)\partial_j(M^\delta)^2 g(v)\mu^{1/2}(v_*)f(v_*)h(v)dv_*dv,
\end{aligned}$$

which gives

$$|B_2| \lesssim \|f\|_{L^2}\|M^\delta g\|_{L^2}\|h\|_{L^2_1}.$$

Similarly, for B_3, B_4, we obtain

$$\begin{aligned}
B_3 = {} & 2\delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l a^{ij}(v-v_*)\partial_l (M^\delta)^2 g(v)\mu^{1/2}(v_*)\partial_j f(v_*)\partial_i h(v)dv_*dv \\
& + 4\delta^2 \sum_{i,j=1}^{3}\sum_{l=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^6} \partial_k\partial_l a^{ij}(v-v_*)\partial_l\partial_k (M^\delta)^3 g(v)\mu^{1/2}(v_*)\partial_j f(v_*)\partial_i h(v)dv_*dv \\
& + \delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l^2 a^{ij}(v-v_*)(M^\delta)^2 g(v)\mu^{1/2}(v_*)\partial_j f(v_*)\partial_i h(v)dv_*dv
\end{aligned}$$

and

$$\begin{aligned}
B_4 = {} & \delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l a^{ij}(v-v_*)\partial_l (M^\delta)^2 g(v)\mu^{1/2}(v_*)\partial_j f(v_*)h(v)dv_*dv \\
& + 2\delta^2 \sum_{i,j=1}^{3}\sum_{l=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^6} \partial_k\partial_l a^{ij}(v-v_*)\partial_l\partial_k (M^\delta)^3 g(v)\mu^{1/2}(v_*)\partial_j f(v_*)h(v)dv_*dv \\
& + \frac{1}{2}\delta \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\mathbb{R}^6} \partial_l^2 a^{ij}(v-v_*)(M^\delta)^2 g(v)\mu^{1/2}(v_*)\partial_j f(v_*)h(v)dv_*dv.
\end{aligned}$$

Integration by parts with respect to the variable v_*, we arrive at

$$\begin{aligned}
|B_3| &\lesssim \|f\|_{L^2_v}\|M^\delta g\|_{L^2_1}\|\nabla_v h\|_{L^2_v}, \\
|B_4| &\lesssim \|f\|_{L^2_v}\|M^\delta g\|_{L^2_v}\|h\|_{L^2_1}.
\end{aligned}$$

Putting the above estimates for B_1-B_4 together, we achieve (3.29) eventually. □

Based on Propositions 3.6-3.8, we obtain commutator estimates involving the nonlinear term $\mathbf{L}$ and various cut-off functions.

Proposition 3.9. *For the nonlinear term, for any $f, g \in \mathcal{S}(\mathbb{R}^6_{x,v})$, it holds that*

$$\begin{aligned}
& \int_0^T \left| \left(M^\delta S_\delta W_{\delta'}\mathbf{L}(f,g) - \mathbf{L}(f, M^\delta S_\delta W_{\delta'} g),\ M^\delta S_\delta W_{\delta'} g \right)_{x,v} \right| dt \\
& \lesssim \|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} \left(\|W_{\delta'} g\|^2_{L^2([0,T]\times\mathbb{R}^3_x;L^2_1(\mathbb{R}^3_v))} + \|\nabla_v W_{\delta'} g\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})} \right) \\
& \quad + \left(\|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} + \epsilon \right) \|||M^\delta S_\delta W_{\delta'} g|||^2_{L^2([0,T]\times\mathbb{R}^3_x)} \\
& \quad + \delta C_\epsilon \|\nabla_x \mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} W_{\delta'} g\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}, \\
& \int_0^T \left| \left(\mathbf{L}(f, M^\delta S_\delta W_{\delta'} g), M^\delta S_\delta W_{\delta'} g \right)_{x,v} \right| dt \\
& \quad \lesssim \|\mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} M^\delta S_\delta W_{\delta'} g\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}
\end{aligned}$$

for $0 < \delta, \delta' < 1$ and $\epsilon > 0$, where $W_{\delta'}$, S_δ, M^δ are defined by Propositions 3.6-3.8, respectively.

Proof. Note that

$$\begin{aligned}
&\Big(M^\delta S_\delta W_{\delta'}\mathbf{L}(f,g) - \mathbf{L}(f, M^\delta S_\delta W_{\delta'}g),\ M^\delta S_\delta W_{\delta'}g\Big)_{x,v}\\
&= \Big(W_{\delta'}\mathbf{L}(f,g) - \mathbf{L}(f, W_{\delta'}g), S_\delta (M^\delta)^2 S_\delta W_{\delta'}g\Big)_{x,v}\\
&\quad + \Big(S_\delta\mathbf{L}(f, W_{\delta'}g) - \mathbf{L}(f, S_\delta W_{\delta'}g), (M^\delta)^2 S_\delta W_{\delta'}g\Big)_{x,v}\\
&\quad + \Big(M^\delta\mathbf{L}(f, S_\delta W_{\delta'}g) - \mathbf{L}(f, M^\delta S_\delta W_{\delta'}g), M^\delta S_\delta W_{\delta'}g\Big)_{x,v}\\
&\quad \triangleq J_1 + J_2 + J_3.
\end{aligned}$$

For J_1, it follows from Proposition 3.6 that

$$\begin{aligned}
\int_0^T J_1 dt &\lesssim \|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}\\
&\qquad \times \Big(\|W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_1(\mathbb{R}^3_v))}\|M^\delta S_\delta W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\\
&\qquad + \|\nabla_v W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\|M^\delta S_\delta W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_1(\mathbb{R}^3_v))}\\
&\qquad + \|W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_1(\mathbb{R}^3_v))}\|\nabla_v M^\delta S_\delta W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\Big)\\
&\lesssim \|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}|||M^\delta S_\delta W_{\delta'}g|||^2_{L^2([0,T]\times\mathbb{R}^3_x)}\\
&\qquad \times \Big(\|W_{\delta'}g\|^2_{L^2([0,T]\times\mathbb{R}^3_x;L^2_1(\mathbb{R}^3_v))} + \|\nabla_v W_{\delta'}g\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\Big).
\end{aligned}$$

For J_2, thanks to Proposition 3.7 and Young's inequality, we get

$$\begin{aligned}
\int_0^T J_2 dt &\lesssim \delta\|\nabla_x\mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\\
&\qquad \times \|\widetilde{\mathcal{L}}^{\frac{1}{2}}M^\delta S_\delta W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\\
&\lesssim \delta C_\epsilon\|\nabla_x\mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}W_{\delta'}g\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\\
&\qquad + \epsilon|||M^\delta S_\delta W_{\delta'}g|||^2_{L^2([0,T]\times\mathbb{R}^3_x)}
\end{aligned}$$

for $\epsilon > 0$, where $C_\epsilon > 0$ is some constant depending only on ϵ. For J_3, by Proposition 3.8, we obtain

$$\begin{aligned}
\int_0^T J_3 dt &\lesssim \|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}\|M^\delta S_\delta W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_1(\mathbb{R}^3_v))}\\
&\qquad \times \Big(\|\nabla_v M^\delta S_\delta W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_v)} + \|M^\delta S_\delta W_{\delta'}g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_v)}\Big)\\
&\lesssim \|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}|||M^\delta S_\delta W_{\delta'}g|||^2_{L^2([0,T]\times\mathbb{R}^3_x)}.
\end{aligned}$$

The first inequality in Proposition 3.9 follows from above estimates for J_1, J_2 and J_3. In addition, we can get the second inequality according to Theorem 3.2. $\square$

3.4.4 Commutators for linear operator

Moreover, we also need some commutators with the linear operator $\mathcal{L}_1$.

Proposition 3.10. *For $0 < \delta' < 1$, it holds that*

$$\left|(W_{\delta'}\mathcal{L}_1 g - \mathcal{L}_1 W_{\delta'} g, h)_{L^2_v}\right| \lesssim \|W_{\delta'} g\|_{L^2_v} \left(\|h\|_{L^2_v} + \|\nabla_v h\|_{L^2_v}\right). \tag{3.30}$$

Proof. By using Lemma 3.1, we have

$$\begin{aligned}
&(W_{\delta'}\mathcal{L}_1 g - \mathcal{L}_1 W_{\delta'} g, h)_{L^2_v} \\
=&\left(2\left[\Delta_v, W_{\delta'}\right] g, h\right)_{L^2_v} + \left(\left[\Delta_{\mathbb{S}^2}, W_{\delta'}\right] g, h\right)_{L^2_v} \\
\triangleq& D_1 + D_2.
\end{aligned}$$

For the term D_1, integration by parts allows to get

$$\begin{aligned}
D_1 &= 2\sum_{l=1}^{3}\int_{\mathbb{R}^3}\left(2(\partial_l W_{\delta'})\partial_l g + (\partial_l^2 W_{\delta'})g\right) h dv \\
&= -4\sum_{l=1}^{3}\int_{\mathbb{R}^3}\left(\partial_l W_{\delta'}\right)\left(\partial_l h\right) g dv - 2\sum_{l=1}^{3}\int_{\mathbb{R}^3}(\partial_l^2 W_{\delta'}) g h dv.
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
|D_1| &\le 4\Big|\sum_{l=1}^{3}\int_{\mathbb{R}^3}(\partial_l W_{\delta'}) g \partial_l h dv\Big| + 2\Big|\sum_{l=1}^{3}\int_{\mathbb{R}^3}(\partial_l^2 W_{\delta'}) g h dv\Big| \\
&\lesssim \|W_{\delta'} g\|_{L^2_v}\|\nabla_v h\|_{L^2_v} + \|W_{\delta'} g\|_{L^2_v}\|h\|_{L^2_v}.
\end{aligned}$$

On the other hand, it follows from the direct calculation that

$$\begin{aligned}
D_2 =& \frac{1}{2}\sum_{\substack{1\le j,k\le 3\\ j\neq k}}\int_{\mathbb{R}^3}\left(\left(v_j\partial_k - v_k\partial_j\right)^2\left(W_{\delta'} g\right) - W_{\delta'}\left(v_j\partial_k - v_k\partial_j\right)^2 g\right) h dv \\
=& \frac{1}{2}\sum_{\substack{1\le j,k\le 3\\ j\neq k}}\int_{\mathbb{R}^3}\left(\left[\left(v_j\partial_k - v_k\partial_j\right)^2 W_{\delta'}\right] g + 2\left[\left(v_j\partial_k - v_k\partial_j\right) W_{\delta'}\right]\left(v_j\partial_k - v_k\partial_j\right) g\right) h dv \\
=& 0,
\end{aligned}$$

where we used the relation

$$(v_j\partial_k - v_k\partial_j) W_{\delta'} = -2\delta'^2 v_j v_k W_{\delta'}^2 + 2\delta'^2 v_k v_j W_{\delta'}^2 = 0.$$

We obtain (3.30) directly. Therefore, the proof of Proposition 3.10 is completed. □

Proposition 3.11. *For $0 < \delta \leq 1$, it holds that*

$$\left|\left(M^\delta \mathcal{L}_1 g - \mathcal{L}_1 M^\delta g, h\right)_{L^2_v}\right| \lesssim \|M^\delta g\|_{L^2_v}\|h\|_{L^2_1}. \tag{3.31}$$

Proof. From Lemma 3.1, we obtain

$$\begin{aligned}\left(M^\delta \mathcal{L}_1 g - \mathcal{L}_1 M^\delta g, h\right)_{L^2_v} =& \frac{1}{2}\left(\left[M^\delta, v^2\right] g, h\right)_{L^2_v} - \left(\left[M^\delta, \Delta_{\mathbb{S}^2}\right] g, h\right)_{L^2_v} \\ \triangleq& E_1 + E_2.\end{aligned}$$

For E_1, by the direct calculation, we arrive at

$$\begin{aligned}E_1 =& -\frac{1}{2}\int_{\mathbb{R}^3} M^\delta \left[(1-\delta\Delta), v^2\right] M^\delta g h dv \\ =& \delta \int_{\mathbb{R}^3} \sum_{l=1}^3 v_l \partial_l (M^\delta)^2 g h dv + 2\delta^2 \int_{\mathbb{R}^3} \sum_{l,k=1}^3 \delta_{k,l}\partial_l\partial_k (M^\delta)^3 g h dv \\ &+ \frac{1}{2}\delta \int_{\mathbb{R}^3} \sum_{l=1}^3 (M^\delta)^2 g h dv,\end{aligned}$$

which indicates that

$$|E_1| \lesssim \|M^\delta g\|_{L^2_v}\|h\|_{L^2_1}.$$

Note that

$$(v_j\partial_k - v_k\partial_j)^2 = v_j^2\partial_k^2 - 2v_kv_j\partial_k\partial_j + v_k^2\partial_j^2 - v_k\partial_k - v_j\partial_j,$$

then for $1 \leq k \neq j \leq 3$, we get

$$\begin{aligned}&\Delta\left(v_j\partial_k - v_k\partial_j\right)^2 u - \left(v_j\partial_k - v_k\partial_j\right)^2 \Delta u \\ =& \sum_{l=1}^3 \partial_l^2\left[\left(v_j^2\partial_k^2 - 2v_kv_j\partial_k\partial_j + v_k^2\partial_j^2 - v_k\partial_k - v_j\partial_j\right) u\right] \\ &- \sum_{l=1}^3 \left(v_j^2\partial_k^2 - 2v_kv_j\partial_k\partial_j + v_k^2\partial_j^2 - v_k\partial_k - v_j\partial_j\right)\partial_l^2 u \\ =& 2\sum_{l=1}^3 (\delta_{lj}\partial_k^2 + \delta_{lk}\partial_j^2 - \delta_{lk}\partial_k\partial_l - \delta_{lj}\partial_j\partial_l) u \\ &+ 4\left(\delta_{lj}v_j\partial_k^2 - \delta_{lj}v_k\partial_k\partial_j\right)\partial_l u + \left(\delta_{lk}v_k\partial_j^2 - \delta_{lk}v_j\partial_k\partial_j\right)\partial_l u \\ =& 2\left(\partial_k^2 + \partial_j^2 - \partial_j^2 - \partial_k^2\right) u\end{aligned}$$

which shows that

$$E_2 = -\sum_{\substack{1\leq j,k\leq 3\\ j\neq k}} \left(M^\delta\left[(1-\delta\Delta), \left(v_j\partial_k - v_k\partial_j\right)^2\right] M^\delta g, h\right)_{L^2_{x,v}} = 0.$$

The inequality (3.31) is followed directly. Hence, the proof of Proposition 3.11 is completed. □

As a consequence of Propositions 3.10-3.11, we get the commutator estimate.

Proposition 3.12. *Let $g \in \mathcal{S}(\mathbb{R}^6_{x,v})$. There exist some positive constants $C_\epsilon > 0$ independent of $T > 0$ such that*

$$\begin{aligned}
&\int_0^T \Big| \Big(\mathcal{L}_1 g, W_{\delta'} S_\delta (M^\delta)^2 S_\delta W_{\delta'} g\Big)_{x,v} - \Big(\mathcal{L}_1 M^\delta S_\delta W_{\delta'} g, M^\delta S_\delta W_{\delta'} g\Big)_{x,v} \Big| dt \\
&\le \epsilon |||| M^\delta S_\delta W_{\delta'} g |||\!|^2_{L^2([0,T]\times\mathbb{R}^3_x)} + C_\epsilon T \|S_\delta W_{\delta'} g\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} \\
&\quad + C_\epsilon T \|M^\delta S_\delta W_{\delta'} g\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))},
\end{aligned}$$

where $\epsilon > 0$ is sufficiently small.

Proof. Obviously, we see that

$$\begin{aligned}
&\Big(\mathcal{L}_1 g, W_{\delta'} S_\delta (M^\delta)^2 S_\delta W_{\delta'} g\Big)_{x,v} - \Big(\mathcal{L}_1 M^\delta S_\delta W_{\delta'} g, M^\delta S_\delta W_{\delta'} g\Big)_{x,v} \\
&= \Big(S_\delta W_{\delta'} \mathcal{L}_1 g - \mathcal{L}_1 S_\delta W_{\delta'} g, (M^\delta)^2 S_\delta W_{\delta'} g\Big)_{x,v} \\
&\quad + \Big(M^\delta \mathcal{L}_1 S_\delta W_{\delta'} g - \mathcal{L}_1 M^\delta S_\delta W_{\delta'} g, M^\delta S_\delta W_{\delta'} g\Big)_{x,v} \\
&\triangleq F_1 + F_2.
\end{aligned}$$

For F_1, it follows from the Proposition 3.10 that

$$\begin{aligned}
\int_0^T F_1 dt &\lesssim \sqrt{T} \|S_\delta W_{\delta'} g\|_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} \|M^\delta S_\delta W_{\delta'} g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_v(\mathbb{R}^3_v))} \\
&\quad + \sqrt{T} \|S_\delta W_{\delta'} g\|_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} \|\nabla_v M^\delta S_\delta W_{\delta'} g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_v(\mathbb{R}^3_v))} \\
&\le \frac{\epsilon}{2} |||| M^\delta S_\delta W_{\delta'} g |||\!|^2_{L^2([0,T]\times\mathbb{R}^3_x)} + C_\epsilon T \|S_\delta W_{\delta'} g\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))},
\end{aligned}$$

where we used (3.24). For F_2, thanks to Proposition 3.11, we obtain

$$\begin{aligned}
\int_0^T F_2 dt &\lesssim \sqrt{T} \|M^\delta S_\delta W_{\delta'} g\|_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} \|M^\delta S_\delta W_{\delta'} g\|_{L^2([0,T]\times\mathbb{R}^3_x;L^2_1(\mathbb{R}^3_v))} \\
&\le \frac{\epsilon}{2} |||| M^\delta S_\delta W_{\delta'} g |||\!|^2_{L^2([0,T]\times\mathbb{R}^3_x)} + C_\epsilon T \|M^\delta S_\delta W_{\delta'} g\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}.
\end{aligned}$$

Therefore, we finish the proof of Proposition 3.12. □

3.5 The local-in-time existence

In this section, we establish the local-in-time existence of solution to the Cauchy problem (3.1) in critical Besov space $\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})$. Due to the fact that the dual space of $\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})$ is

complicated. In fact, usually, the dual space of $L^\infty(\mu)$ appears as the finitely additive finite (signed) measures which are absolutely continuous with respect to μ, equipped with the total variation norm. Consequently, the proof of this local existence is a little bit complicated. For clarity, we divided it into several parts.

3.5.1 The local existence of weak solution

Firstly, we are devoted to establish the local existence of weak solution to following linearized Landau equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = \mathbf{L}(f,g) - \mathcal{L}_2 f, \\ g(t,x,v)|_{t=0} = g_0(x,v). \end{cases} \tag{3.32}$$

By using the duality argument and Hahn-Banach extension theorem, we obtain

Proposition 3.13. *There exist $\epsilon_1 > 0$ and $T_0 > 0$ such that for all $0 < T \le T_0$, $f \in L^\infty([0,T]\times\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))$, $g_0 \in L^2(\mathbb{R}^6_{x,v})$ satisfying*

$$\|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} \le \epsilon_1,$$

then the Cauchy problem (3.32) admits a weak solution

$$g \in L^\infty([0,T]; L^2(\mathbb{R}^6_{x,v})).$$

Proof. The strategy of that proof was originated from [9], and well developed by [145]. We consider the joint operator

$$\mathcal{G} = -\partial_t + (v \cdot \nabla_x + \mathcal{L}_1 - \mathbf{L}(f,\cdot))^*,$$

where $(\cdot)^*$ is taken with respect to the scalar product in $L^2(\mathbb{R}^6_{x,v})$. For all $h \in C^\infty([0,T], \mathcal{S}(\mathbb{R}^6_{x,v}))$ with $h(T) = 0$, we have

$$\begin{aligned} &\mathrm{Re}\,(\mathcal{G}h(t), h(t))_{x,v} \\ &= -\frac{1}{2}\frac{d}{dt}(\|h(t)\|^2_{L^2_{x,v}}) + \mathrm{Re}(v\cdot\nabla_x h, h)_{x,v} + \mathrm{Re}\,(\mathcal{L}_1 h, h)_{x,v} - \mathrm{Re}\,(\mathbf{L}(f,h), h)_{x,v} \\ &\ge -\frac{1}{2}\frac{d}{dt}(\|h(t)\|^2_{L^2_{x,v}}) + \frac{1}{C}\|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|^2_{L^2_{x,v}} - C\|h\|^2_{L^2_{x,v}} - C\,\|\mathbb{S}_2 f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}\,\|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|^2_{L^2_{x,v}} \end{aligned}$$

for $0 \le t \le T$, where we used the fact that $\mathrm{Re}(v\cdot\nabla_x h, h) = 0$.

Since $\|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}$ is sufficiently small, we obtain

$$-\frac{d}{dt}(e^{2Ct}\|h(t)\|^2_{L^2_{x,v}}) + \frac{1}{C}e^{2Ct}\|\widetilde{\mathcal{L}}^{\frac{1}{2}} h\|^2_{L^2_{x,v}} \le 2e^{2Ct}\|h\|_{L^2_{x,v}}\,\|\mathcal{G}h\|_{L^2_{x,v}}.$$

Furthermore, we arrive at

$$\begin{aligned}\|h(t)\|^2_{L^2_{x,v}} &+ \frac{1}{C}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}h\|^2_{L^2([t,T],L^2_{x,v})}\\ &\leq 2\int_t^T e^{2C(\tau-t)}\|h(\tau)\|_{L^2_{x,v}}\|\mathcal{G}h(\tau)\|_{L^2_{x,v}}d\tau\\ &\leq 2e^{2CT}\|h\|_{L^\infty([0,T],L^2_{x,v})}\|\mathcal{G}h\|_{L^1([0,T],L^2_{x,v})},\end{aligned}$$

which leads to

$$\|h\|_{L^\infty([0,T],L^2_{x,v})} \leq 2e^{2CT}\|\mathcal{G}h\|_{L^1([0,T],L^2_{x,v})}. \tag{3.33}$$

In the following, we consider the vector subspace

$$\begin{aligned}\mathbb{U} = \big\{u = \mathcal{G}h : h \in C^\infty([0,T],\mathcal{S}(\mathbb{R}^6_{x,v})), h(T) = 0\big\}\\ \subset L^1([0,T],L^2(\mathbb{R}^6_{x,v})).\end{aligned}$$

Indeed, the above inclusion is true due to similar calculations of Theorem 3.2. For $g \in L^2_{x,v}$, we get

$$|(\mathbf{L}(f,\cdot)^*h,g)_{L^2_{x,v}}| = |(h,\mathbf{L}(f,g))_{L^2_{x,v}}| \lesssim \|\mathbb{S}_2 f\|_{L^\infty_x(L^2_v)}\|g\|_{L^2_{x,v}}\|\widetilde{\mathcal{L}}h\|_{L^2_{x,v}},$$

which implies that

$$\|\mathbf{L}(f,\cdot)^*h\|_{L^2_{x,v}} \lesssim \|\mathbb{S}_2 f\|_{L^\infty_x(L^2_v)}\|\widetilde{\mathcal{L}}h\|_{L^2_{x,v}}$$

for any $t \in [0,T]$.

For $g_0 \in L^2(\mathbb{R}^6_{x,v})$, we define the linear functional as follows

$$\begin{aligned}\mathcal{Q}:\ &\mathbb{U} \to \mathbb{C}\\ &u = \mathcal{G}h \mapsto (g_0,h(0))_{L^2_{x,v}} - (\mathcal{L}_2 f,h)_{L^2([0,T],L^2_{x,v})},\end{aligned}$$

where $h \in C^\infty([0,T],\mathcal{S}(\mathbb{R}^6_{x,v}))$ with $h(T) = 0$. It follows from (3.33) that the operator $\mathcal{G}$ is injective. The linear functional $\mathcal{Q}$ is hence well-defined. We obtain

$$\begin{aligned}|\mathcal{Q}(u)| &\leq \|g_0\|_{L^2_{x,v}}\|h(0)\|_{L^2_{x,v}} + C_T\|\mathbb{S}_2 f\|_{L^\infty([0,T],L^2_{x,v})}\|\widetilde{\mathcal{L}}h\|_{L^2_{x,v}}\\ &\leq \widetilde{C}_T\left(\|g_0\|_{L^2_{x,v}} + \|\mathbb{S}_2 f\|_{L^\infty([0,T],L^2_{x,v})}\right)\|\mathcal{G}h\|_{L^2_{x,v}}\\ &= \widetilde{C}_T\left(\|g_0\|_{L^2_{x,v}} + \|\mathbb{S}_2 f\|_{L^\infty([0,T],L^2_{x,v})}\right)\|u\|_{L^2_{x,v}}.\end{aligned}$$

Hence, $\mathcal{Q}$ is a continuous linear form on $(\mathbb{U},\|\cdot\|_{L^1([0,T],L^2_{x,v})})$. By using the Hahn-Banach theorem, $\mathcal{Q}$ can be extended as a continuous linear form on $L^1([0,T];L^2(\mathbb{R}^6_{x,v}))$. It follows that there exists $g \in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$ satisfying

$$\|g\|_{L^\infty([0,T],L^2_{x,v})} \leq \widetilde{C}_T\left(\|g_0\|_{L^2_{x,v}} + \|\mathbb{S}_2 f\|_{L^\infty([0,T],L^2_{x,v})}\right),$$

such that

$$\forall u \in L^1([0,T];L^2(\mathbb{R}^6_{x,v})), \qquad \mathcal{Q}(u)=\int_0^T (g(t),u(t))_{L^2_{x,v}}\,dt.$$

It implies that for all $h\in C_0^\infty((-\infty,T),\mathcal{S}(\mathbb{R}^6_{x,v}))$,

$$\begin{aligned}\mathcal{Q}(\mathcal{G}h)&=\int_0^T (g(t),\mathcal{G}h(t))_{L^2_{x,v}}\,dt\\&=(g_0,h(0))_{L^2_{x,v}}-\int_0^T(\mathcal{L}_2 f(t),h(t))_{L^2_{x,v}}\,dt.\end{aligned}$$

Therefore, $g\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$ is a weak solution of the Cauchy problem (3.32). The proof of Proposition 3.13 is completed. □

In next steps, we need to improve the regularity of weak solution $g\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$ in both velocity and position variables.

3.5.2 Regularity of weak solution in velocity variable

To do this, we smooth out the function f. Set $f_N=S_Nf=\sum_{p\geq -1}^{N-1}\Delta_p f$ for $N\in\mathbb{N}$. Then, we have $f_N\in\widetilde{L}^\infty_T\widetilde{L}^2_v(H^\infty_x)$ and the following property.

Lemma 3.2. *For any $f\in\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})$, it holds that*

(i) $\{f_N\}$ is a Cauchy sequence in $\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})$;

(ii) For $0<s\leq 3/2$, f_N satisfies $\|f_N\|_{L^\infty_T L^2_v L^2_x}\leq C\|f\|_{L^\infty_T L^2_v L^2_x}$ and

$$\|f_N\|_{L^\infty_T L^2_v L^2_x}\leq C_1\|f_N\|_{L^\infty_T L^2_v(B^s_{2,1})}\leq C_2\|f_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})}\leq C_3\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^s_{2,1})},\tag{3.34}$$

where $C_1,C_2,C_3>0$ are some constants independent of N.

Proof. Firstly, for any $M',M''\in\mathbb{N}(M'>M'')$ big enough, then we have

$$\begin{aligned}\|f_{M'}-f_{M''}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}&=\sum_{q\geq -1}2^{3q/2}\|\Delta_q(f_{M'}-f_{M''})\|_{L^\infty_T L^2_v L^2_x}\\&=\sum_{q\geq -1}2^{3q/2}\|\sum_{p=M''-1}^{M'-1}\Delta_q\Delta_p f\|_{L^\infty_T L^2_v L^2_x}\\&\leq\sum_{q=p-1}^{q=p+1}\sum_{p=M''-1}^{M'-1}2^{3q/2}\|\Delta_q\Delta_p f\|_{L^\infty_T L^2_v L^2_x}\\&\leq 3\times 2^{3/2}\sum_{p\geq M''-1}2^{3p/2}\|\Delta_p f\|_{L^\infty_T L^2_v L^2_x},\end{aligned}$$

where we used $\Delta_q\Delta_p f=0$ if $|p-q|\geq 2$. Since $f\in\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})$, by the definition of the norm $\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})$, that is, $\{2^{3p/2}\|\Delta_p f\|_{L^\infty_T L^2_v L^2_x}\}\in\ell^1$, this deduce that $\{f_N,N\in\mathbb{N}\}$ is a Cauchy sequence in $\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})$.

It follows from Lemma 2.4 that $\|f_N\|_{L^\infty_T L^2_v L^2_x} \le C\|f\|_{L^\infty_T L^2_v L^2_x}$. For the left hand side of (3.34), it can be obtained from Lemma 2.8. For the right hand side, we have

$$\begin{aligned}\|f_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^s_{2,1})} &= \sum_{q\ge -1} 2^{qs}\|\Delta_q f_N\|_{L^\infty_T L^2_v L^2_x}\\ &\le \sum_{q\ge -1}\sum_{p\ge -1}^{N-1} 2^{qs}\|\Delta_p f\|_{L^\infty_T L^2_v L^2_x}\\ &\le \sum_{q\ge -1}\sum_{|p-q|\le 1} 2^{(q-p)s}c(p)\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^s_{2,1})}\\ &\le C_3\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^s_{2,1})},\end{aligned}$$

where $c(p) = 2^{ps}\|\Delta_p f\|_{L^\infty_T L^2_v L^2_x}/\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^s_{2,1})}$ and we used the estimate

$$\begin{aligned}\sum_{q\ge -1}\sum_{|p-q|\le 1} 2^{(q-p)s}c(p) &\le \sum_{q\ge -1}[(\mathbf{1}_{|p|\le 1}2^{ps}) * c(p)]\\ &\le \|\mathbf{1}_{|p|\le 1}2^{ps}\|_{\ell^1}\|c(p)\|_{\ell^1} < +\infty.\end{aligned}$$

Therefore, the proof of Lemma 3.2 is completed. □

Then, according to the commutator estimate in Section 3.4, we have the following estimate for the weak solution.

Proposition 3.14. *Set $f_N = S_N f$ for each $N \in \mathbb{N}$. There exist $\epsilon_1 > 0$ and $T_0 > 0$ such that for all $0 < T \le T_0, g_0 \in L^2(\mathbb{R}^6_{x,v}), f \in L^\infty([0,T]\times\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))$ satisfying*

$$\|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} \le \epsilon_1,$$

then the Cauchy problem

$$\begin{cases}\partial_t g_N + v\cdot\nabla_x g_N + \mathcal{L}_1 g_N = \mathbf{L}(f_N, g_N) - \mathcal{L}_2 f_N,\\ g_N(t,x,v)|_{t=0} = g_0(x,v),\end{cases} \tag{3.35}$$

admits a weak solution $g_N(t,x,v) \in L^\infty([0,T]; L^2(\mathbb{R}^6_{x,v}))$ satisfying

$$\|g_N\|_{L^\infty_T L^2_v L^2_x} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{L^2_T L^2_v L^2_x} \le C\|g_0\|_{L^2_v L^2_x} + C\sqrt{T}\|\mathbb{S}_2 f\|_{L^\infty_T L^2_v L^2_x}. \tag{3.36}$$

Proof. It follows from Proposition 3.13 that the Cauchy problem (3.35) admits a weak solution $g_N(t,x,v) \in L^\infty([0,T]; L^2(\mathbb{R}^6_{x,v}))$. In what follows, we show (3.36) under the assumption that $\|f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}$ is sufficiently small (independent of N) and $\|\nabla_x f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} < +\infty$.

Let $0 < \delta, \delta' < 1$. We use a weighted function $W_{\delta'}(v) = \langle \delta' v\rangle^{-2}$ and mollifiers $M^\delta(D_v), S_\delta(D_x)$ defined as in Section 3.4. Taking the inner products of (3.35) with $W_{\delta'} S_\delta (M^\delta)^2 S_\delta W_{\delta'} g_N \in$

$H_x^{+\infty}H_{v,4}^4$ and integrating the resulting equation with respect to the time $t \in [0,T]$ and $(x,v) \in \mathbb{R}^6$. We obtain

$$\begin{aligned}&\|M^\delta S_\delta W_{\delta'} g_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} + \int_0^T \Big(v\cdot\nabla_x g_N + \mathcal{L}_1 g_N, W_{\delta'}S_\delta(M^\delta)^2 S_\delta W_{\delta'} g_N\Big)\,dt \\ &= \int_0^T \Big(\mathbf{L}(f_N,g_N) - \mathcal{L}_2 f_N, W_{\delta'}S_\delta(M^\delta)^2 S_\delta W_{\delta'} g_N\Big)\,dt + \|M^\delta S_\delta W_{\delta'} g_0\|^2_{L^2(\mathbb{R}^6_{x,v})}.\end{aligned} \tag{3.37}$$

To the term $v\cdot\nabla_x g_N$, we bound it as

$$\begin{aligned}&\Big(v\cdot\nabla_x g_N, W_{\delta'}S_\delta(M^\delta)^2 S_\delta W_{\delta'} g_N\Big)_{x,v} \\ &= \Big(\Big[M^\delta, v\Big]\cdot\nabla_x S_\delta W_{\delta'} g_N, M^\delta S_\delta W_{\delta'} g_N\Big)_{x,v} \le 2\|M^\delta S_\delta W_{\delta'} g_N\|^2_{L^2_{x,v}},\end{aligned}$$

where we have used $|(\nabla_\xi M^\delta(\xi))\cdot(\eta)S_\delta(\eta)| \le M^\delta(\xi)|\delta\eta|S(\delta\eta) \le 2M^\delta(\xi)S_\delta(\eta)$. It follows from Proposition 3.9 that

$$\begin{aligned}&\int_0^T \Big(\mathbf{L}(f_N,g_N), W_{\delta'}S_\delta(M^\delta)^2 S_\delta W_{\delta'} g_N\Big)_{x,v}\,dt \\ &\lesssim \big(\|f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} + \epsilon\big)\,|||\!|M^\delta S_\delta W_{\delta'} g_N|\!|||^2_{L^2([0,T]\times\mathbb{R}^3_x)} \\ &\quad + \|f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}|||\!|W_{\delta'} g_N|\!|||^2_{L^2([0,T]\times\mathbb{R}^3_x)} \\ &\quad + \delta\|\nabla_x\mathbb{S}_2 f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}|||\!|W_{\delta'} g_N|\!|||^2_{L^2([0,T]\times\mathbb{R}^3_x)}.\end{aligned}$$

On the other hand, regarding linear terms $\mathcal{L}_1, \mathcal{L}_2$, we deduce from Proposition 3.12 and Theorem 3.5 that

$$\begin{aligned}&\int_0^T \Big(\mathcal{L}_1 g_N, W_{\delta'}S_\delta(M^\delta)^2 S_\delta W_{\delta'} g_N\Big)_{x,v}\,dt \\ &\ge \left(\frac{1}{2}-\epsilon\right)|||\!|M^\delta S_\delta W_{\delta'} g_N|\!|||^2_{L^2([0,T]\times\mathbb{R}^3_x)} - C_\epsilon T\|S_\delta W_{\delta'} g_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \Big(\mathcal{L}_2 f_N, W_{\delta'}S_\delta(M^\delta)^2 S_\delta W_{\delta'} g_N\Big)\,dt \\ &\le CT\|\mathbb{S}_2 f_N\|_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}\|M^\delta S_\delta W_{\delta'} g_N\|_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))},\end{aligned} \tag{3.38}$$

where $\epsilon > 0$ is sufficiently small.

Combining (3.37)-(3.38), it is shown that

$$\begin{aligned}&\|M^\delta S_\delta W_{\delta'} g_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} M^\delta S_\delta W_{\delta'} g_N\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})} \\ &\le C\|M^\delta S_\delta W_{\delta'} g_0\|^2_{L^2(\mathbb{R}^6_{x,v})} + CT\|S_\delta W_{\delta'} g_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} \\ &+ CT\|M^\delta S_\delta W_{\delta'} g_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} + CT\|\mathbb{S}_2 f_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))} \\ &+ C(\|f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)} + \delta\|\nabla_x\mathbb{S}_2 f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)})|||\!|W_{\delta'} g_N|\!|||^2_{L^2([0,T]\times\mathbb{R}^3_x)},\end{aligned}$$

since $\|f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}$ is small enough. Taking T sufficiently small (for example, $CT<\frac{1}{4}$) and passing to the limit $\delta\to 0$, we get

$$\begin{aligned}
&\|W_{\delta'}g_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}+\|\widetilde{\mathcal{L}}^{\frac12}W_{\delta'}g_N\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\\
&\le C\|g_0\|^2_{L^2(\mathbb{R}^6_{x,v})}+CT\|\mathbb{S}_2f_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}\\
&\quad+C\|f_N\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}|||W_{\delta'}g_N|||^2_{L^2([0,T]\times\mathbb{R}^3_x)}\\
&\le C\|g_0\|^2_{L^2(\mathbb{R}^6_{x,v})}+CT\|\mathbb{S}_2f\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}\\
&\quad+C\|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}\|\widetilde{\mathcal{L}}^{\frac12}W_{\delta'}g_N\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})},
\end{aligned}$$

where we used the fact $\|\widetilde{\mathcal{L}}^{\frac12}g\|_{L^2_v}\sim|||g|||$ and Proposition 3.2. Since $\|f\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2_v)}$ and T are both small, we obtain

$$\begin{aligned}
&\|W_{\delta'}g_N\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}+\|\widetilde{\mathcal{L}}^{\frac12}W_{\delta'}g_N\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\\
&\le C\|g_0\|^2_{L^2(\mathbb{R}^6_{x,v})}+CT\|\mathbb{S}_2f\|^2_{L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))}<+\infty,
\end{aligned}$$

where we used $\|W_{\delta'}g\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}\lesssim\|g\|^2_{L^2([0,T]\times\mathbb{R}^6_{x,v})}$ for $0<\delta'<1$. Now letting $\delta'\to 0$ and taking square root to the resulting inequality give the desired (3.36) for a weak solution $g_N\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$. Hence, the proof of Proposition 3.14 is finished. □

3.5.3 Regularity of weak solution in position variable

In the following, we need to obtain the regularity of g_N with respect to the position variable x.

Lemma 3.3. *Let $0<s\le 3/2$ and $0<T\le\infty$. Set $f_N=S_Nf$ for each $N\in\mathbb{N}$. If g_N satisfies*

$$g_N\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v})),\qquad \widetilde{\mathcal{L}}^{\frac12}g_N\in L^2([0,T]\times\mathbb{R}^6_{x,v}),$$

then there exists a constant $C>0$ (independent of N) such that

$$\begin{aligned}
&\sum_{p\ge-1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\int_0^T\left|(\Delta_p\mathbf{L}(f_N,g_N),\Delta_pg_N)_{x,v}\right|dt\right)^{1/2}\\
&\quad\le C\|\mathbb{S}_2f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{s,\kappa}_{2,1})}\\
&\qquad+C_N\|\mathbb{S}_2f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{L^2_TL^2_vL^2_x}
\end{aligned}\tag{3.39}$$

for any $\kappa>0$, where

$$\|g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{s,\kappa}_{2,1})}\triangleq\sum_{p\ge-1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\|\Delta_pg_N\|_{L^2_TL^2_vL^2_x}$$

and $C_N>0$ is a constant depending only on N.

Proof. Due to $\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N \in L^2([0,T]\times\mathbb{R}^6_{x,v})$, we obtain

$$\begin{aligned}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\big\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{s,\kappa}_{2,1})} &= \sum_{p\geq -1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\big\|\Delta_p\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\big\|_{L^2_TL^2_vL^2_x}\\ &\leq C_\kappa\sum_{p\geq -1}2^{-ps}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\big\|_{L^2_TL^2_vL^2_x}\leq C_\kappa\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\big\|_{L^2_TL^2_vL^2_x}<+\infty,\end{aligned}$$

where $C_\kappa>0$ is a constant depending only on $\kappa>0$.

With the Bony's decomposition, we divide the inner product into three terms:

$$\begin{aligned}(\Delta_p\mathbf{L}(f_N,g_N),\Delta_p g_N) &= \left(\Delta_p\left(\sum_j\mathbf{L}(S_{j-1}f_N,\Delta_j g_N)\right),\Delta_p g_N\right)\\ &\quad+\left(\Delta_p\left(\sum_j\mathbf{L}(\Delta_j f_N,S_{j-1}g_N)\right),\Delta_p g_N\right)\\ &\quad+\left(\Delta_p\left(\sum_j\sum_{|j-j'|\leq 1}\mathbf{L}(\Delta_{j'}f_N,\Delta_j g_N)\right),\Delta_p g_N\right)\\ &\triangleq\ H_1+H_2+H_3.\end{aligned}$$

For the term H_1, noticing that

$$\Delta_p\sum_j(S_{j-1}f_N,\Delta_j g_N)=\Delta_p\sum_{|j-p|\leq 4}(S_{j-1}f_N,\Delta_j g_N),$$

and using the Theorem 3.3 we have

$$|H_1|\leq\sum_{|j-p|\leq 4}\|\mathbb{S}_2S_{j-1}f_N\|_{L^2_vL^\infty_x}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_j g_N\big\|_{L^2_{x,v}}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g_N\big\|_{L^2_{x,v}}.$$

Hence, it follows that

$$\begin{aligned}&\sum_{p\geq -1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\int_0^T|H_1|\,dt\right)^{1/2}\\ &\lesssim\sum_{p\geq -1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\sum_{|j-p|\leq 4}\int_0^T\|\mathbb{S}_2S_{j-1}f_N\|_{L^2_vL^\infty_x}\left\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_j g_N\right\|_{L^2_{x,v}}\left\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g_N\right\|_{L^2_{x,v}}dt\right)^{1/2}\\ &\lesssim\|\mathbb{S}_2f_N\|^{1/2}_{L^\infty_TL^2_v(L^\infty_x)}\left(\sum_{p\geq -1}\sum_{|j-p|\leq 4}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\int_0^T\left\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_j g_N\right\|^2_{L^2_{x,v}}dt\right)^{1/2}\right)^{1/2}\\ &\quad\times\left(\sum_{p\geq -1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\int_0^T\left\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g_N\right\|^2_{L^2_{x,v}}dt\right)^{1/2}\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \|\widetilde{\mathcal{L}}^{\frac12} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{s,\kappa}_{2,1})} \\
&\qquad \times \left(\sum_{p\geq -1} \sum_{|j-p|\leq 4} \frac{1+\kappa 2^{2js}}{1+\kappa 2^{2ps}} 2^{(p-j)s} c(j) \right)^{1/2} \\
&\lesssim \|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \|\widetilde{\mathcal{L}}^{\frac12} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{s,\kappa}_{2,1})},
\end{aligned}
$$

where we used Lemmas 2.4, 2.8 and 2.9 and the following estimate

$$
\begin{aligned}
&\sum_{p\geq -1} \sum_{|j-p|\leq 4} \frac{1+\kappa 2^{2js}}{1+\kappa 2^{2ps}} 2^{(p-j)s} c(j) \\
&\leq C \sum_{p\geq -1} \sum_{|j-p|\leq 4} 2^{(p-j)s} c(j) \leq C \sum_{p\geq -1} [(\mathbf{1}_{|j|\leq 4} 2^{js}) * c(j)](p) \\
&\leq C \|\mathbf{1}_{|j|\leq 4} 2^{js}\|_{\ell^1} \|c(j)\|_{\ell^1} < +\infty
\end{aligned}
$$

with

$$
c(j) := \frac{\frac{2^{js}}{1+\kappa 2^{2js}} \|\widetilde{\mathcal{L}}^{\frac12} \Delta_j g_N\|_{L^2_T L^2_{x,v}}}{\|\widetilde{\mathcal{L}}^{\frac12} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{s,\kappa}_{2,1})}} \quad \text{satisfying} \quad \|c(j)\|_{\ell^1} \leq 1.
$$

For H_2, we get

$$
\begin{aligned}
&\sum_{p\geq -1} \frac{2^{ps}}{1+\kappa 2^{2ps}} \left(\int_0^T |H_2|\, dt \right)^{1/2} \\
&\lesssim \sum_{p\geq -1} \frac{2^{ps}}{1+\kappa 2^{2ps}} \left(\int_0^T \sum_{\substack{|j-p|\leq 4 \\ j\leq N}} \|\mathbb{S}_2 \Delta_j f_N\|_{L^2_v L^\infty_x} \left\| \widetilde{\mathcal{L}}^{\frac12} S_{j-1} g_N \right\|_{L^2_{x,v}} \left\| \widetilde{\mathcal{L}}^{\frac12} \Delta_p g_N \right\|_{L^2_{x,v}} dt \right)^{1/2} \\
&\lesssim N \sum_{p\geq -1}^{N+4} \frac{2^{ps}}{1+\kappa 2^{2ps}} \left(\|\mathbb{S}_2 f_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \left\| \widetilde{\mathcal{L}}^{\frac12} g_N \right\|_{L^2_T L^2_{x,v}} \left\| \widetilde{\mathcal{L}}^{\frac12} \Delta_p g_N \right\|_{L^2_T L^2_{x,v}} \right)^{1/2} \\
&\leq C_N \|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \|\widetilde{\mathcal{L}}^{\frac12} g_N\|_{L^2_T L^2_v L^2_x},
\end{aligned}
$$

where $C_N = CN2^{Ns}$. Owing to

$$
\begin{aligned}
\Delta_p \left(\sum_j \sum_{|j-j'|\leq 1} (\Delta_{j'} f_N \Delta_j g_N) \right) &= \Delta_p \left(\sum_{\max(j,j')\geq p-2} \sum_{|j-j'|\leq 1} (\Delta_{j'} f_N \Delta_j g_N) \right) \\
&= 0, \qquad \text{if } \ p \geq N+3.
\end{aligned}
$$

Finally, H_3 can be estimated as follows

$$\begin{aligned}
&\sum_{p\geq -1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\int_0^T |H_3|\,dt\right)^{1/2}\\
&=\sum_{p\geq -1}^{N+2}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\int_0^T \left|(\Delta_p\mathbf{L}^3(f_N,g_N),\Delta_p g_N)_{x,v}\right|dt\right)^{1/2}\\
&\lesssim \sum_{p\geq -1}^{N+2}\sum_{j\leq N+1}\frac{2^{ps}}{1+\kappa 2^{2ps}}\left(\int_0^T \left\|\mathbb{S}_2\Delta_{j'}f_N\right\|_{L^2_vL^\infty_x}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_j g_N\right\|_{L^2_{x,v}}\left\|\widetilde{\mathcal{L}}^{\frac12}\Delta_p g_N\right\|_{L^2_{x,v}}dt\right)^{1/2}\\
&\lesssim C_N\|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{L^2_TL^2_vL^2_x}.
\end{aligned}$$

Together with above three inequalities, we achieve (3.39). This ends the proof of Lemma 3.3. □

Based on Proposition 3.14 and Lemma 3.3, we can obtain the regularity of the weak solution g_N to the Cauchy problem (3.35) and get the corresponding energy estimate dependent of N.

Proposition 3.15. *There exist $\epsilon_2>0$ and $T_0>0$ such that for all $0<T\leq T_0, f\in \widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1}), g_0\in\widetilde{L}^2_v(B^{3/2}_{2,1})$ satisfying*

$$\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\leq\epsilon_2,$$

then the Cauchy problem (3.35) *admits a weak solution $g_N\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$ satisfying*

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\leq C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+C_N\|\mathbb{S}_2 f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{L^2_TL^2_vL^2_x},
\end{aligned}\tag{3.40}$$

where $C_N>0$ is a constant depending on N and $f_N=S_Nf$.

Proof. We consider a weak solution $g_N\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$ to the above Cauchy problem (3.35). To do this, applying $\Delta_p(p\geq -1)$ to (3.35) and taking the inner product with $\Delta_p g_N$ over $\mathbb{R}^3_x\times\mathbb{R}^3_v$ give

$$\begin{aligned}
&\frac{d}{dt}\|\Delta_p g_N\|^2_{L^2_{x,v}}+\|\widetilde{\mathcal{L}}^{\frac12}\Delta_p g_N\|^2_{L^2_{x,v}}\\
&\leq 2\left(\Delta_p\mathbf{L}(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}+C(\|\Delta_p\mathbb{S}_2 f_N\|^2_{L^2_{x,v}}+\|\Delta_p g_N\|^2_{L^2_{x,v}}),
\end{aligned}$$

where we used Corollary 3.1. Integrating the above inequality with respect to the time variable over $[0,t]$ with $0\leq t\leq T$ and taking the square root of both sides of the resulting inequality.

Multiplying the resulting inequality by $\frac{2^{3p/2}}{1+\kappa 2^{3p}}$ gives

$$\begin{aligned}
&\frac{2^{3p/2}}{1+\kappa 2^{3p}}\|\Delta_p g_N\|_{L^2_v L^2_x}+\frac{2^{3p/2}}{1+\kappa 2^{3p}}\left(\int_0^t\left\|\widetilde{\mathcal{L}}^{\frac12}g_N\right\|^2_{L^2_vL^2_x}dt\right)^{1/2}\\
&\le \frac{2^{3p/2}}{1+\kappa 2^{3p}}\|\Delta_p g_0\|_{L^2_vL^2_x}+\frac{2^{3p/2}}{1+\kappa 2^{3p}}\left(\int_0^t\left|(\Delta_p\mathbf{L}(f_N,g_N),\Delta_p g_N)_{L^2_{x,v}}\right|dt\right)^{1/2}\\
&\quad+C\sqrt{T}\frac{2^{3p/2}}{1+\kappa 2^{3p}}\left(\|\mathbb{S}_2\Delta_p f_N\|_{L^\infty_T L^2_vL^2_x}+\|\Delta_p g_N\|_{L^\infty_T L^2_vL^2_x}\right)\\
&\le C2^{3p/2}\|\Delta_p g_0\|_{L^2_vL^2_x}+\frac{2^{3p/2}}{1+\kappa 2^{3p}}\left(\int_0^t\left|(\Delta_p\mathbf{L}(f_N,g_N),\Delta_p g_N)_{L^2_{x,v}}\right|dt\right)^{1/2}\\
&\quad+C\sqrt{T}2^{3p/2}\|\mathbb{S}_2\Delta_p f_N\|_{L^\infty_T L^2_vL^2_x}+C\sqrt{T}\frac{2^{3p/2}}{1+\kappa 2^{3p}}\|\Delta_p g_N\|_{L^\infty_T L^2_vL^2_x}.
\end{aligned}$$

Taking supremum over $0\le t\le T$ on the left side and summing up over $p\ge -1$, we obtain

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})}\\
&\le C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})}\\
&\quad+C_N\|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{L^2_TL^2_vL^2_x}+C\sqrt{T}\|\mathbb{S}_2 f_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\quad+C\sqrt{T}\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})}\\
&\le C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})}\\
&\quad+C_N\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{L^2_TL^2_vL^2_x}+C\sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\quad+C\sqrt{T}\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})},
\end{aligned}$$

where we used the Proposition 3.2 and Lemma 3.3 because the weak solution g_N satisfies the Proposition 3.14. Then, for the small constant $T>0$ and the small norm $\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}$ (for example, taking $C\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}, C\sqrt{T}<\frac14$), we obtain

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2,\kappa}_{2,1})}\\
&\le C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+C_N\|\mathbb{S}_2 f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{L^2_TL^2_vL^2_x}.
\end{aligned}$$

Letting $\kappa\to 0$, we deduce that

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\le C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+C_N\|\mathbb{S}_2 f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g_N\|_{L^2_TL^2_vL^2_x},
\end{aligned}$$

which ends the proof of Proposition 3.15. □

3.5.4 Energy estimates in Besov space

It follows from (3.40) in Proposition 3.15 that

$$\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}<+\infty.$$

Then applying (3.15) in Theorem 3.4 to f_N, g_N and using the Lemma 2.9, we get the following inequality

$$\begin{aligned}&\sum_{p\geq -1}2^{\frac{3}{2}p}\left(\int_0^T\left|(\Delta_p\mathbf{L}(f_N,g_N),\Delta_p g_N)_{x,v}\right|dt\right)^{1/2}\\ \leq& C\|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\end{aligned}\tag{3.41}$$

for some constant $C>0$ independent of N.

With the aid of (3.41), one can obtain the further energy estimate, which is independent of N for the weak solution g_N.

Proposition 3.16. *There exist $\epsilon_2>0$ and $T_0>0$ such that for all $0<T\leq T_0, f\in\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1}), g_0\in\widetilde{L}^2_v(B^{3/2}_{2,1})$ satisfying*

$$\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\leq\epsilon_2,$$

then the Cauchy problem (3.35) *admits a weak solution $g_N\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$ satisfying*

$$\begin{aligned}&\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\ &\qquad\leq C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})},\end{aligned}\tag{3.42}$$

where $C>0$ is a constant independent of N.

Proof. We consider the weak solution $g_N\in L^\infty([0,T];L^2(\mathbb{R}^6_{x,v}))$ to (3.35). Applying $\Delta_p(p\geq -1)$ to (3.35) and taking the inner product with $\Delta_p g_N$ over $\mathbb{R}^3_x\times\mathbb{R}^3_v$ give

$$\begin{aligned}&\frac{d}{dt}\|\Delta_p g_N\|^2_{L^2_{x,v}}+\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g_N\|^2_{L^2_{x,v}}\\ &\leq 2\left(\Delta_p\mathbf{L}(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}+C(\|\Delta_p\mathbb{S}_2 f_N\|^2_{L^2_{x,v}}+\|\Delta_p g_N\|^2_{L^2_{x,v}}),\end{aligned}$$

where we used corollary 3.1. Integrating the above inequality with respect to the time variable over $[0,t]$ with $0\leq t\leq T$ and taking the square root of both sides of the resulting inequality. Multiplying the resulting inequality by $2^{\frac{3}{2}p}$ gives

$$\begin{aligned}&2^{\frac{3}{2}p}\|\Delta_p g_N\|_{L^2_vL^2_x}+2^{\frac{3}{2}p}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p g_N\|_{L^2_vL^2_x}\\ &\leq 2^{\frac{3}{2}p}\|\Delta_p g_0\|_{L^2_vL^2_x}+2^{\frac{3}{2}p}\left(\int_0^t\left|(\Delta_p\mathbf{L}(f_N,g_N),\Delta_p g_N)_{L^2_{x,v}}\right|dt\right)^{1/2}\\ &\quad+C\sqrt{T}2^{\frac{3}{2}p}\left(\|\mathbb{S}_2 f_N\|_{L^\infty_TL^2_vL^2_x}+\|\Delta_p g_N\|_{L^\infty_TL^2_vL^2_x}\right).\end{aligned}$$

Take supremum over $0 \le t \le T$ on the left side and sum up over $p \ge -1$, we obtain

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \\
&\le C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + C\|\mathbb{S}_2 f_N\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \\
&\quad + C\sqrt{T}\left(\|\mathbb{S}_2 f_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|g_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})}\right) \\
&\le C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + C\|f\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \\
&\quad + C\sqrt{T}\left(\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|g_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})}\right),
\end{aligned}$$

where we used the Proposition 3.2 and (3.41) because the weak solution g_N satisfies the Proposition 3.15. Then, for a small $T > 0$ and small $\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})}$ (taking $C\sqrt{T}, C\|f\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} < \frac{1}{4}$), we get

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \\
&\qquad \le C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + C\sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})},
\end{aligned}$$

which is the desired (3.42). Hence, the proof of Proposition 3.16 is finished. □

3.5.5 The local existence to linearized Landau equation

We obtain the following local existence to the linearized Landau equation by using Proposition 3.16.

Theorem 3.6. *There exist $C_0 > 1, \epsilon_0 > 0$ and $T_0 > 0$ such that for all $0 < T \le T_0, f \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1}), g_0 \in \widetilde{L}^2_v(B^{3/2}_{2,1})$ satisfying*

$$\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \le \epsilon_0,$$

then the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = \mathbf{L}(f, g) - \mathcal{L}_2 f, \\ g(t, x, v)|_{t=0} = g_0(x, v), \end{cases} \tag{3.43}$$

admits a weak solution $g \in L^\infty([0, T]; L^2(\mathbb{R}^6_{x,v}))$ satisfying

$$\begin{aligned}
&\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \\
&\qquad \le C_0\left(\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + \sqrt{T}\|\mathbb{S}_2 f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})}\right).
\end{aligned} \tag{3.44}$$

Proof. As a first step, let us show that

$$g(t,x,v)\in \widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1}) \quad \text{and} \quad \widetilde{\mathcal{L}}^{\frac{1}{2}}g(t,x,v)\in \widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1}).$$

Based on Proposition 3.16, it suffices to prove the convergence of the sequences

$$\{g_N, N\in\mathbb{N}\}\subset \widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1}) \quad \text{and} \quad \{\widetilde{\mathcal{L}}^{\frac{1}{2}}g_N, N\in\mathbb{N}\}\subset \widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1}).$$

Set $w_{M,M'}=g_M-g_{M'}, M,M'\in\mathbb{N}$. Then it follows that (3.43) that

$$\begin{aligned}&\partial_t w_{M,M'}+v\cdot\nabla_x w_{M,M'}+\mathcal{L}_1 w_{M,M'}\\&=\mathbf{L}(f_{M'},w_{M,M'})+\mathbf{L}(f_M-f_{M'},g_N)-\mathcal{L}_2(f_M-f_{M'}).\end{aligned}\tag{3.45}$$

Applying $\Delta_p(p\geq -1)$ to (3.45) and taking the inner product with $2^{3p}\Delta_p w_{M,M'}$ over $\mathbb{R}^3_x\times\mathbb{R}^3_v$, we get

$$\begin{aligned}&\frac{d}{dt}2^{3p}\|\Delta_p w_{M,M'}\|^2_{L^2_{x,v}}+2^{3p}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}\Delta_p w_{M,M'}\|^2_{L^2_{x,v}}\\&\leq C2^{3p}\|\Delta_p w_{M,M'}\|^2_{L^2_{x,v}}+2^{3p+1}\left(\Delta_p\mathbf{L}(f_{M'},w_{M,M'}),\Delta_p w_{M,M'}\right)_{L^2_{x,v}}\\&\quad+2^{3p+1}\left(\Delta_p\mathbf{L}(f_M-f_{M'},g_M),\Delta_p w_{M,M'}\right)_{L^2_{x,v}}\\&\quad+C2^{3p}\|\Delta_p\mathbb{S}_2(f_M-f_{M'})\|_{L^2_{x,v}}\|\Delta_p w_{M,M'}\|_{L^2_{x,v}}.\end{aligned}\tag{3.46}$$

Integrating (3.46) with respect to the time variable over $[0,t]$ with $0\leq t\leq T$, taking the square root of both sides of the resulting inequality and summing up over $p\geq -1$. Following from Lemma 3.4, we obtain

$$\begin{aligned}&\|w_{M,M'}\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac{1}{2}}w_{M,M'}\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\\&\leq C\sqrt{T}\|w_{M,M'}\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}+C\|\mathbb{S}_2 f_{M'}\|^{1/2}_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}w_{M,M'}\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\\&\quad+C\|\mathbb{S}_2(f_M-f_{M'})\|^{1/2}_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}g_M\|^{1/2}_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}w_{M,M'}\|^{1/2}_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\\&\quad+C\sqrt{T}\|\mathbb{S}_2(f_M-f_{M'})\|^{1/2}_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\|w_{M,M'}\|^{1/2}_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}.\end{aligned}\tag{3.47}$$

It follows from Proposition 3.16 and Young's inequality that

$$\begin{aligned}&\|w_{M,M'}\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac{1}{2}}w_{M,M'}\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\\&\leq C\sqrt{T}\|w_{M,M'}\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}+C\|f\|^{1/2}_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac{1}{2}}w_{M,M'}\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\\&\quad+C\left(\|g_0\|_{\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}+\sqrt{T}\|f\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\right)\|\widetilde{\mathcal{L}}^{\frac{1}{2}}w_{M,M'}\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}\\&\quad+\sqrt{T}\|\mathbb{S}_2(f_M-f_{M'})\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{3/2}_{2,1})}.\end{aligned}$$

Thanks to the smallness of $\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}$, $\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}$ and T, we can choose

$$C\left(\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+\sqrt{T}\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\right)\le\frac{1}{8}.$$

Then it is shown that

$$\begin{aligned}&\|w_{M,M'}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}w_{M,M'}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\&\quad\le\tilde\lambda\|\mathbb{S}_2(f_M-f_{M'})\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\end{aligned}$$

for $0<\tilde\lambda<1$. As $\{f_N\}$ is a Cauchy sequence in $\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})$, we deduce that $\{g_N\}$ is also true and satisfies

$$\{g_N,N\in\mathbb{N}\}\subset\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})\quad\text{and}\quad\{\widetilde{\mathcal{L}}^{\frac12}g_N,N\in\mathbb{N}\}\subset\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1}).$$

Set $g=\lim_{N\to\infty}g_N$. Therefore we obtain

$$g(t,v,x)\in\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})\quad\text{and}\quad\widetilde{\mathcal{L}}^{\frac12}g(t,v,x)\in\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1}).$$

Next, we prove (3.44). Applying $\Delta_p(p\ge-1)$ to (3.43), taking the inner product with $2^{3p}\Delta_pg$ over $\mathbb{R}^3_x\times\mathbb{R}^3_v$ and using Corollary 3.1, we get

$$\begin{aligned}&\frac{d}{dt}2^{3p}\|\Delta_pg\|^2_{L^2_{x,v}}+2^{3p}\|\widetilde{\mathcal{L}}^{\frac12}\Delta_pg\|^2_{L^2_x}\\&\le2^{3p+1}\left(\Delta_p\mathbf{L}(f,g),\Delta_pg\right)_{L^2_{x,v}}+2^{3p}C(\|\Delta_p\mathbb{S}_2f\|^2_{L^2_{x,v}}+\|\Delta_pg\|^2_{L^2_{x,v}}).\end{aligned}\tag{3.48}$$

Integrating (3.48) with respect to the time variable over $[0,t]$ with $0\le t\le T$, taking the square root of both sides of the resulting inequality and summing up over $p\ge-1$. It follows from Lemma 3.4 that

$$\begin{aligned}\|g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}&\le\tilde C\|\mathbb{S}_2f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\&+C\left(\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+\sqrt{T}\|\mathbb{S}_2f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\sqrt{T}\|g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\right).\end{aligned}$$

Since $\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}$ and T are small (taking $\tilde C\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})},C\sqrt{T}<\frac14$), we get the desired inequality (3.44). □

3.5.6 The local existence to nonlinear Landau equation

The local-in-time existence of solutions to the Cauchy problem (3.1) is shown as follows.

Proposition 3.17. *For a sufficiently small constant $\varepsilon_0 > 0$, there exists $T > 0$ such that if*

$$\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \leq \varepsilon_0,$$

then the Cauchy problem (3.1) *admits a local solution $g(t,x,v) \in (0,T)\times\mathbb{R}^3\times\mathbb{R}^3$ satisfying*

$$g \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1}) \quad \textit{and} \quad \widetilde{\mathcal{L}}^{\frac{1}{2}} g \in \widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1}).$$

Proof. Firstly, we construct the following sequence of iterating approximate solutions:

$$\begin{cases} \partial_t g^{n+1} + v\cdot\nabla_x g^{n+1} + \mathcal{L}_1 g^{n+1} = \mathbf{L}(g^n, g^{n+1}) - \mathcal{L}_2 g^n, \ \ t>0,\, v\in\mathbb{R}^3, \\ g^{n+1}(t,x,v)|_{t=0} = g_0(x,v), \end{cases} \tag{3.49}$$

starting from $g^0(t,x,v) \equiv g_0(x,v)$. Taking $g = g^{n+1}, f = g^n$ and $T = \min\{T_0, 1/(4C_0^2)\}$ in Theorem 3.6 gives

$$\|g^n\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g^n\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \leq \varepsilon,$$

where $\varepsilon_0 > 0$ is chosen such that $2C_0\varepsilon_0 \leq \varepsilon$. Secondly, it suffices to prove the convergence of the sequence $\{g^n\}$ in the space Y, which is defined by

$$Y = \{g | g \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1}), \quad \widetilde{\mathcal{L}}^{\frac{1}{2}} g \in \widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})\}.$$

Set $w_n = g^{n+1} - g^n$. It follows from (3.49) that

$$\partial_t w^n + v\cdot\nabla_x w^n + \mathcal{L}_1 w^n = \mathbf{L}(g^n, w^n) + \mathbf{L}(w^{n-1}, g^n) - \mathcal{L}_2 w^{n-1}$$

with $w^n|_{t=0} = 0$. The following calculation is similar to the energy estimate leading to (3.47). Here, for completeness, we also give the basic estimates. Apply $\Delta_p (p \geq -1)$ to the above equation and take the inner product with $2^{3p}\Delta_p w^n$ over $\mathbb{R}^3_x \times \mathbb{R}^3_v$, we obtain

$$\begin{aligned} &\frac{d}{dt} 2^{3p} \|\Delta_p w^n\|^2_{L^2_{x,v}} + 2^{3p} \|\widetilde{\mathcal{L}}^{\frac{1}{2}} \Delta_p w^n\|^2_{L^2_{x,v}} \\ &\leq C 2^{3p} \|\Delta_p w^n\|^2_{L^2_{x,v}} + 2^{3p+1} \left(\Delta_p \mathbf{L}(g^n, w^n), \Delta_p w^n\right)_{L^2_{x,v}} \\ &\quad + 2^{3p+1} \left(\Delta_p \mathbf{L}(w^{n-1}, g^n), \Delta_p w^n\right)_{L^2_{x,v}} \\ &\quad + C 2^{3p} \|\Delta_p \mathbb{S}_2 w^{n-1}\|_{L^2_{x,v}} \|\Delta_p w^n\|_{L^2_{x,v}}. \end{aligned}$$

Then, Integrating the resultant with respect to the time variable over $[0,t]$ with $0 \leq t \leq T$, taking the square root of both sides of the resulting inequality and summing up over $p \geq -1$.

It follows from Lemma 3.4 that

$$\begin{aligned}
&\|w^n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}w^n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\le C\sqrt{T}\|w^n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\|\mathbb{S}_2g^n\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}w^n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\quad+C\|\mathbb{S}_2w^{n-1}\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}g^n\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\widetilde{\mathcal{L}}^{\frac12}w^n\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\quad+C\sqrt{T}\|\mathbb{S}_2w^{n-1}\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|w^n\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\le C\sqrt{T}\|w^n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+C\sqrt{\varepsilon}\|\widetilde{\mathcal{L}}^{\frac12}w^n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\quad+C(\sqrt{T}+\sqrt{\varepsilon})\|\mathbb{S}_2w^{n-1}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}.
\end{aligned}$$

Furthermore, if ε and T are sufficiently small (for example, taking $C\sqrt{\varepsilon}, C\sqrt{T}<\frac18$), then

$$\begin{aligned}
\|w^n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\widetilde{\mathcal{L}}^{\frac12}w^n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}&\le\lambda\|\mathbb{S}_2w^{n-1}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\
&\le\lambda^{n-1}\|\mathbb{S}_2w^1\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}
\end{aligned}$$

for some $0<\lambda<1$. Clearly, we see that $\{g^n\}$ is a Cauchy sequence in Y, so there is some limit function $g\in Y$ such that $g^n\to g$ as $n\to\infty$. The standard procedure enables us to know that g is the desired solution to the Cauchy problem (3.1) satisfying

$$g(t,v,x)\in\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})\quad\text{and}\quad\widetilde{\mathcal{L}}^{\frac12}g(t,v,x)\in\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1}).$$

The proof of Proposition 3.17 is finished. □

3.6 The global-in-time solution

Now we prove that we can extend the above local-in-time solution to a global-in-time solution, which heavily depends on the key a priori estimate. For this end, we define the energy functional

$$\mathcal{E}_T(g)=\|g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}$$

and the dissipation functional

$$\mathcal{D}_T(g)=\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}+\|\mathcal{L}^{\frac12}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})},$$

respectively. The following sections is needed to prove a priori estimate.

3.6.1 Estimate on the macroscopic dissipation

In this part, we bound the macroscopic dissipation arising from Landau collision operator. We by $\mathbf{P}$ denote the projection operator on $\ker\mathcal{N}$, which is given by

$$\mathbf{P}g = \big(a(t,x) + v\cdot\, b(t,x) + (|v|^2-3)c(t,x)\big)\sqrt{\mu}. \tag{3.50}$$

In terms of the macro-micro decomposition, the distribution function $g(t,v,x)$ can be decomposed as

$$g = \mathbf{P}g + (\mathbf{I}-\mathbf{P})g.$$

Precisely, the macroscopic dissipation of g is included in the following proposition.

Proposition 3.18. *It holds that*

$$\begin{aligned}\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})} \lesssim{}& \|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + \mathcal{E}_T(g)\\ &+ \|\mathcal{L}^{\frac{1}{2}}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})} + \mathcal{E}_T(g)\mathcal{D}_T(g)\end{aligned} \tag{3.51}$$

for any $T>0$.

Proof. We take the velocity moments

$$\sqrt{\mu},\ v_i\sqrt{\mu},\ \frac{1}{6}\left(|v|^2-3\right)\sqrt{\mu},\ (v_iv_j-\delta_{ij})\sqrt{\mu},\ \frac{1}{\sqrt{10}}\left(|v|^2-5\right)v_i\sqrt{\mu}$$

with $i,j=1,2,3$ for the Landau equation (3.1). Define

$$\phi_{i,j} = (v_iv_j-\delta_{ij})\sqrt{\mu},\quad \phi_{1,1,i} = \frac{1}{\sqrt{10}}\left(|v|^2-5\right)v_i\sqrt{\mu}$$

and

$$A_{i,j}(g) = (g,\phi_{i,j}),\quad B_i(g) = (g,\phi_{1,1,i}).$$

Noticing that

$$\phi_{i,j}\in \operatorname{span}\{\varphi_{0,0,0},\varphi_{1,0,0},\varphi_{0,2,0},\varphi_{0,2,1},\varphi_{0,2,-1},\varphi_{0,2,2},\varphi_{0,2,-2}\}$$

and

$$\phi_{1,1,1} = -\varphi_{1,1,0};\quad \phi_{1,1,2} = -\frac{\varphi_{1,1,1}+\varphi_{1,1,-1}}{\sqrt{2}};\quad \phi_{1,1,3} = -\frac{\varphi_{1,1,1}-\varphi_{1,1,-1}}{\sqrt{2}i},$$

we deduce that

$$\mathcal{L}\phi_{i,j} = 12\widetilde{\mathbb{S}}_2\phi_{i,j}\quad\text{with}\quad |C_{i,j}|\le 2;\quad \mathcal{L}\phi_{1,1,i} = 8\phi_{1,1,i},\quad i,j=1,2,3.$$

With the aid of the orthogonal of $\{\varphi_{n,l,m}\}$, we infer that (a,b,c) which is the coefficient of the

macroscopic component $\mathbf{P}g$ in (3.50) satisfies the fluid-type system

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b_i + \partial_{x_i}(a+2c) + \sum\limits_{j=1}^{3} \partial_{x_j} A_{i,j}((\mathbf{I}-\mathbf{P})g) = 0, \\ \partial_t c + \frac{1}{3}\nabla_x \cdot b + \frac{\sqrt{10}}{6}\sum\limits_{i=1}^{3} \partial_{x_i} B_i((\mathbf{I}-\mathbf{P})g) = 0, \\ \partial_t \left(A_{i,j}((\mathbf{I}-\mathbf{P})g) + 2c\delta_{i,j}\right) + \partial_{x_i} b_j + \partial_{x_j} b_i + 12\left(\widetilde{\mathbb{S}}_2 g, \phi_{i,j}\right) \\ \qquad\qquad = A_{i,j}\left(\mathbf{L}(g,g) - v\cdot\nabla_x(\mathbf{I}-\mathbf{P})g\right), \\ \partial_t B_i((\mathbf{I}-\mathbf{P})g) + \sqrt{10}\partial_{x_i} c + 8B_i(g) = B_i\left(\mathbf{L}(g,g) - v\cdot\ \nabla_x(\mathbf{I}-\mathbf{P})g\right), \end{cases} \tag{3.52}$$

where $i,j = 1,2,3$. Applying the cut-off operator Δ_p with $p \geq -1$ to the system (3.52) implies that

$$\begin{cases} \partial_t \Delta_p a + \nabla_x \cdot \Delta_p b = 0, \\ \partial_t \Delta_p b_i + \partial_{x_i}(\Delta_p a + 2\Delta_p c) + \sum\limits_{j=1}^{3} \partial_{x_j} A_{i,j}(\Delta_p(\mathbf{I}-\mathbf{P})g) = 0, \\ \partial_t \Delta_p c + \frac{1}{3}\nabla_x \cdot \Delta_p b + \frac{\sqrt{10}}{6}\sum\limits_{j=1}^{3} \partial_{x_i} B_i(\Delta_p(\mathbf{I}-\mathbf{P})g) = 0, \\ \partial_t \left(A_{i,j}(\Delta_p(\mathbf{I}-\mathbf{P})g) + 2\Delta_p c\delta_{i,j}\right) + \partial_{x_i}\Delta_p b_j + \partial_{x_j}\Delta_p b_i + 12\left(\widetilde{\mathbb{S}}_2\Delta_p g, \phi_{i,j}\right) \\ \qquad\qquad = A_{i,j}\left(\Delta_p\mathbf{L}(g,g) - v\cdot\nabla_x\Delta_p(\mathbf{I}-\mathbf{P})g\right), \\ \partial_t B_i(\Delta_p(\mathbf{I}-\mathbf{P})g) + \sqrt{10}\partial_{x_i}\Delta_p c + 8B_i(\Delta_p g) \\ \qquad\qquad = B_i\left(\Delta_p\mathbf{L}(g,g) - v\cdot\nabla_x\Delta_p(\mathbf{I}-\mathbf{P})g\right). \end{cases} \tag{3.53}$$

As in [47, 52], we denote the temporal interactive functionals as follows:

$$\begin{aligned} \mathcal{E}_0(t) &= \sum_{i=1}^{3} \left(\partial_{x_i}\Delta_p c, B_i(\Delta_p(\mathbf{I}-\mathbf{P})g)\right)_{L^2_x}, \\ \mathcal{E}_1(t) &= \sum_{i,j=1}^{3} \left(\partial_{x_i}\Delta_p b_j + \partial_{x_j}\Delta_p b_i, A_{i,j}(\Delta_p(\mathbf{I}-\mathbf{P})g) + 2\Delta_p c\delta_{i,j}\right)_{L^2_x}, \\ \mathcal{E}_2(t) &= \sum_{i=1}^{3} \left(\partial_{x_i}\Delta_p a, \Delta_p b_i\right)_{L^2_x}. \end{aligned}$$

Set $\mathcal{E}_p^{int}(g(t)) = \mathcal{E}_0(t) + \kappa_1\mathcal{E}_1(t) + \kappa_2\mathcal{E}_2(t)$, where $0 < \kappa_2 < \kappa_1 \ll 1$ are some constants (to be confirmed below).

By multiplying the fifth equality of (3.53) by $\partial_{x_i}\Delta_p c$ and summing up on i with $1 \leq i \leq 3$,

one can get

$$\frac{d\mathcal{E}_0(t)}{dt} - \sum_{i=1}^{3} \left(\partial_{x_i}\Delta_p\partial_t c, B_i(\Delta_p(\mathbf{I}-\mathbf{P})g)\right)_{L^2_x} + \sqrt{10}\sum_{i=1}^{3}\|\partial_{x_i}\Delta_p c\|^2_{L^2_x} + 8\sum_{i=1}^{3}\left(\partial_{x_i}\Delta_p c, B_i(\Delta_p g)\right)_{L^2_x} = \sum_{i=1}^{3}\left(\partial_{x_i}\Delta_p c, B_i\left(\Delta_p\mathbf{L}(g,g) - v\cdot\nabla_x\Delta_p(\mathbf{I}-\mathbf{P})g\right)\right)_{L^2_x}.$$

Using the third equality of (3.53) and Young's inequality in Lemma 2.10, we arrive at

$$\begin{aligned}
&\sum_{i=1}^{3}\left|\left(\partial_{x_i}\Delta_p\partial_t c, B_i(\Delta_p(\mathbf{I}-\mathbf{P})g)\right)_{L^2_x}\right| \\
&= \sum_{i=1}^{3}\left|\left(\frac{1}{3}\nabla_x\cdot\,\Delta_p b + \frac{\sqrt{10}}{6}\sum_{j=1}^{3}\partial_{x_i}B_i(\Delta_p(\mathbf{I}-\mathbf{P})g), \partial_{x_i}B_i(\Delta_p(\mathbf{I}-\mathbf{P})g)\right)_{L^2_x}\right| \\
&\le \epsilon_{01}\|\nabla_x\Delta_p b\|^2_{L^2_x} + \frac{C}{\epsilon_{01}}\|\nabla_x(\mathbf{I}-\mathbf{P})\Delta_p g\|^2_{L^2_v L^2_x}
\end{aligned}$$

for $\epsilon_{01}>0$, where $C_{\epsilon_{01}}$ is a constant depending on ϵ_{01}. Furthermore, by using Young's equality again, we are led to

$$\begin{aligned}
&\frac{d\mathcal{E}_0(t)}{dt} + \lambda_1\|\nabla_x\Delta_p c\|^2_{L^2_x} \\
&\le \epsilon_{01}\|\nabla_x\Delta_p b\|^2_{L^2_x} + \sum_{i=1}^{3} C_{\epsilon_{02}}\|B_i(\Delta_p\mathbf{L}(g,g))\|^2_{L^2_x} \\
&\quad + C_{\epsilon_{01},\epsilon_{02}}\Big(\|\nabla_x(\mathbf{I}-\mathbf{P})\Delta_p g\|^2_{L^2_v L^2_x} + \|(\mathbf{I}-\mathbf{P})\Delta_p g\|^2_{L^2_v L^2_x}\Big)
\end{aligned}$$

for $\lambda_1>0$ and $\epsilon_{01},\epsilon_{02}>0$, where $C_{\epsilon_{02}}$ and $C_{\epsilon_{01},\epsilon_{02}}$ are some constants depending on $\epsilon_{01},\epsilon_{02}$.

Multiplying the fourth equality of (3.53) by $\partial_{x_i}\Delta_p b_j + \partial_{x_j}\Delta_p b_i$ and summing up $1\le i,j\le 3$, we obtain

$$\begin{aligned}
&\frac{d\mathcal{E}_1(t)}{dt} - \sum_{i,j=1}^{3}\left(\partial_{x_i}\Delta_p\partial_t b_j + \partial_{x_j}\Delta_p\partial_t b_i, A_{ij}\left(\Delta_p(\mathbf{I}-\mathbf{P})g\right) + 2\Delta_p c\delta_{i,j}\right)_{L^2_x} \\
&+ 2\left(\|\nabla_x\Delta_p b\|^2_{L^2_x} + \|\nabla_x\cdot\Delta_p b\|^2_{L^2_x}\right) + \sum_{i,j=1}^{3} 12\left(\partial_{x_i}\Delta_p b_j + \partial_{x_j}\Delta_p b_i, \left(\widetilde{\mathbb{S}}_2\Delta_p g, \phi_{i,j}\right)\right)_{L^2_x} \\
&= \sum_{i,j=1}^{3}\left(\partial_{x_i}\Delta_p b_j + \partial_{x_j}\Delta_p b_i, A_{i,j}\left(\Delta_p\mathbf{L}(g,g) - v\cdot\nabla_x\Delta_p(\mathbf{I}-\mathbf{P})g\right)\right)_{L^2_x}.
\end{aligned}$$

Then, substitute the second equality of (3.53) to eliminate $\Delta_p\partial_t b$ and get

$$\begin{aligned}
&\sum_{i,j=1}^{3}\left|(\partial_{x_i}\Delta_p\partial_t b_j, A_{ij}(\Delta_p(\mathbf{I}-\mathbf{P})g)+2\Delta_p c\delta_{i,j})_{L^2_x}\right| \\
&=\sum_{i,j=1}^{3}\Big|\Big(\partial_{x_i}(\Delta_p a+2\Delta_p c)+\sum_{j=1}^{3}\partial_{x_j}A_{i,j}(\Delta_p(\mathbf{I}-\mathbf{P})g), \\
&\qquad\qquad \partial_{x_i}A_{ij}(\Delta_p(\mathbf{I}-\mathbf{P})g)+2\partial_{x_i}\Delta_p c\delta_{i,j}\Big)_{L^2_x}\Big| \\
&\le \epsilon_{11}\left\|\nabla_x\Delta_p a\right\|^2_{L^2_x}+C_{\epsilon_{11}}\left\|\nabla_x\Delta_p c\right\|^2_{L^2_x}+C_{\epsilon_{11}}\sum_{i,j=1}^{3}\left\|\nabla_x A_{ij}(\Delta_p(\mathbf{I}-\mathbf{P})g)\right\|^2_{L^2_x}
\end{aligned}$$

for $\epsilon_{11}>0$, where $C_{\epsilon_{11}}$ is a constant depending on ϵ_{11}. Consequently, there exist some constant $\lambda_2>0$ such that the following inequality holds

$$\begin{aligned}
&\frac{d\mathcal{E}_1(t)}{dt}+\lambda_2\left\|\nabla_x\Delta_p b\right\|^2_{L^2_x} \\
&\le \epsilon_{11}\left\|\nabla_x\Delta_p a\right\|^2_{L^2_x}+C_{\epsilon_{11}}\left\|\nabla_x\Delta_p c\right\|^2_{L^2_x}+C_{\epsilon_{11},\epsilon_{12}}\Big(\left\|\nabla_x\Delta_p(\mathbf{I}-\mathbf{P})g\right\|^2_{L^2_vL^2_x} \\
&\quad+\left\|\Delta_p(\mathbf{I}-\mathbf{P})g\right\|^2_{L^2_vL^2_x}+\sum_{i,j=1}^{3}\left\|A_{ij}(\Delta_p\mathbf{L}(g,g))\right\|^2_{L^2_x}\Big),
\end{aligned}$$

where $C_{\epsilon_{11}}$ and $C_{\epsilon_{11},\epsilon_{12}}$ are positive constants depending on $\epsilon_{11},\epsilon_{12}>0$. Multiplying the second equality by $\partial_{x_i}\Delta_p a$ and summing up $1\le i\le 3$,

$$\begin{aligned}
\frac{d\mathcal{E}_2(t)}{dt}-\sum_{i=1}^{3}(\Delta_p b_i,\partial_{x_i}\Delta_p\partial_t a)_{L^2_x}+\left\|\nabla_x\Delta_p a\right\|^2_{L^2_x}&=-2c\sum_{i=1}^{3}(\partial_{x_i}\Delta_p c,\partial_{x_i}\Delta_p a)_{L^2_x} \\
&\quad-\sum_{i,j=1}^{3}(\partial_{x_j}A_{i,j}(\Delta_p(\mathbf{I}-\mathbf{P})g),\partial_{x_i}\Delta_p a)_{L^2_x}.
\end{aligned}$$

Eliminate $\Delta_p\partial_t b$ by the first equality of (3.53), we get

$$\frac{d\mathcal{E}_2(t)}{dt}+\lambda_3\left\|\nabla_x\Delta_p a\right\|^2_{L^2_x}\le C_{\epsilon_{21}}\left\|\nabla_x\Delta_p(b,c)\right\|^2_{L^2_x}+C_{\epsilon_{21}}\left\|\nabla_x(\mathbf{I}-\mathbf{P})\Delta_p g\right\|^2_{L^2_vL^2_x}$$

for $\lambda_3>0$, where $C_{\epsilon_{21}}$ is a positive constant depending on ϵ_{21}.

Put above energy estimates together and choose $\epsilon_{01},\epsilon_{11},\kappa_1,\kappa_2$ small enough. Consequently, there exists a positive constant $\lambda>0$ such that

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_p^{int}(g(t))+\lambda\left\|\nabla_x\Delta_p(a,b,c)\right\|^2_{L^2_x} \\
&\lesssim \left\|\nabla_x(\mathbf{I}-\mathbf{P})\Delta_p g\right\|^2_{L^2_vL^2_x}+\left\|\Delta_p(\mathbf{I}-\mathbf{P})g\right\|^2_{L^2_vL^2_x}+\left\|\Delta_p g\right\|^2_{L^2_vL^2_x} \\
&\quad+\sum_{i,j=1}^{3}\left\|A_{ij}(\Delta_p\mathbf{L}(g,g))\right\|^2_{L^2_x}+\sum_{i=1}^{3}\left\|B_i(\Delta_p\mathbf{L}(g,g))\right\|^2_{L^2_x}.
\end{aligned}$$

Integrating the above inequality with respect to t over $[0,T]$ and taking square roots on both sides give

$$\begin{aligned}
&\bigg(\int_0^T \|\nabla_x \Delta_p(a,b,c)\|_{L_x^2}^2\,dt\bigg)^{1/2}\\
&\lesssim \sqrt{|\mathcal{E}_p^{int}(g(T))|}+\sqrt{|\mathcal{E}_p^{int}(g(0))|}+\|\nabla_x(\mathbf{I}-\mathbf{P})\Delta_p g\|_{L_T^2L_v^2L_x^2}+\|(\mathbf{I}-\mathbf{P})\Delta_p g\|_{L_T^2L_v^2L_x^2}\\
&\quad+\sum_{i,j=1}^3\|A_{ij}(\Delta_p\mathbf{L}(g,g))\|_{L_T^2L_x^2}+\sum_{i=1}^3\|B_i(\Delta_p\mathbf{L}(g,g))\|_{L_T^2L_x^2}.
\end{aligned}$$

Multiplying the above inequality by $2^{\frac{p}{2}}$ and then taking the summation over $p\geq -1$ to get

$$\begin{aligned}
&\|\nabla_x(a,b,c)\|_{\widetilde{L}_T^2(B_{2,1}^{1/2})}\\
&\lesssim \sum_{p\geq -1}2^{\frac{p}{2}}\sqrt{|\mathcal{E}_p^{int}(g(T))|}+\sum_{p\geq -1}2^{\frac{p}{2}}\sqrt{|\mathcal{E}_p^{int}(g(0))|}+\|(\mathbf{I}-\mathbf{P})g\|_{\widetilde{L}_T^2\widetilde{L}_v^2(B_{2,1}^{3/2})}\\
&\quad+\sum_{p\geq -1}\sum_{i,j=1}^3 2^{\frac{p}{2}}\|A_{ij}(\Delta_p\mathbf{L}(g,g))\|_{L_T^2L_x^2}+\sum_{p\geq -1}\sum_{i=1}^3 2^{\frac{p}{2}}\|B_i(\Delta_p\mathbf{L}(g,g))\|_{L_T^2L_x^2}.
\end{aligned}\tag{3.54}$$

Clearly, it is not difficult to check that

$$\begin{aligned}
\sum_{p\geq -1}2^{\frac{p}{2}}\sqrt{|\mathcal{E}_p^{int}(g(t))|}&\lesssim \|\nabla_x(a,b,c)\|_{B_{2,1}^{1/2}}+\|g\|_{\widetilde{L}_v^2(B_{2,1}^{1/2})}\\
&\lesssim \|(a,b,c)\|_{\widetilde{L}_T^\infty(B_{2,1}^{3/2})}+\|g\|_{\widetilde{L}_T^\infty\widetilde{L}_v^2(B_{2,1}^{3/2})},
\end{aligned}$$

which implies that

$$\sum_{p\geq -1}2^{\frac{p}{2}}\sqrt{|\mathcal{E}_p^{int}(g(T))|}\lesssim \mathcal{E}_T(g),\quad \sum_{p\geq -1}2^{\frac{p}{2}}\sqrt{|\mathcal{E}_p^{int}(g(0))|}\lesssim \|g_0\|_{\widetilde{L}_v^2(B_{2,1}^{3/2})}.\tag{3.55}$$

By using Proposition 3.5 (taking $s=\frac{1}{2}$) and the Sobolev embedding $B_{2,1}^{3/2}(\mathbb{R}_x^3)\hookrightarrow \dot{B}_{2,1}^{3/2}(\mathbb{R}_x^3)\hookrightarrow L^\infty(\mathbb{R}_x^3)$, we obtain

$$\begin{aligned}
&\sum_{p\geq -1}\sum_{i,j=1}^3 2^{\frac{p}{2}}\|A_{ij}(\Delta_p\mathbf{L}(g,g))\|_{L_T^2L_x^2}+\sum_{p\geq -1}\sum_{i=1}^3 2^{\frac{p}{2}}\|B_i(\Delta_p\mathbf{L}(g,g))\|_{L_T^2L_x^2}\\
&\lesssim \|\mathbb{S}_3 g\|_{L_T^2L_v^2L_x^\infty}\|\mathbf{P}g\|_{\widetilde{L}_T^\infty\widetilde{L}_v^2(B_{2,1}^{1/2})}+\|\mathbf{P}g\|_{L_T^2L_v^2L_x^\infty}\|\mathbb{S}_3 g\|_{\widetilde{L}_T^\infty\widetilde{L}_v^2(B_{2,1}^{1/2})}\lesssim \mathcal{E}_T(g)\mathcal{D}_T(g),
\end{aligned}\tag{3.56}$$

where we have used the following estimates:

$$\begin{aligned}
\|\mathbf{P}g\|_{L_T^2L_v^2L_x^\infty}&\lesssim \|\mathbf{P}g\|_{\widetilde{L}_T^2\widetilde{L}_v^2(\dot{B}_{2,1}^{3/2})}\lesssim \|\nabla_x(a,b,c)\|_{\widetilde{L}_T^2(B_{2,1}^{1/2}(\mathbb{R}_x^3))}\lesssim \mathcal{D}_T(g),\\
\|\mathbb{S}_3 g\|_{L_T^2L_v^2L_x^\infty}&\leq \|\mathbf{P}g\|_{L_T^2L_v^2L_x^\infty}+\big\|\widetilde{\mathbb{S}}_3 g\big\|_{L_T^2L_v^2L_x^\infty}\\
&\lesssim \|\nabla_x(a,b,c)\|_{\widetilde{L}_T^2(B_{2,1}^{1/2}(\mathbb{R}_x^3))}+\big\|\widetilde{\mathbb{S}}_3 g\big\|_{\widetilde{L}_T^2\widetilde{L}_v^2(B_{2,1}^{3/2})}\lesssim \mathcal{D}_T(g).
\end{aligned}$$

By inserting (3.55)-(3.56) into (3.54), we conclude that

$$\begin{aligned}&\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}\\&\lesssim \|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})}+\|\mathcal{L}^{\frac12}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}+\mathcal{E}_T(g)+\mathcal{E}_T(g)\mathcal{D}_T(g),\end{aligned}$$

which is just (3.51). □

3.6.2 Estimate on the nonlinear term $\mathbf{L}(g,g)$

In this section, we estimate the nonlinear term $\mathbf{L}(g,g)$. It follows from (3.18) in Proposition 3.4 that

$$(\mathbf{L}(g,g),\mathbf{P}g)_{L^2(\mathbb{R}^3)}=0,\quad \forall\, g\in\mathcal{S}(\mathbb{R}^3),$$

so it suffices to bound

$$(\mathbf{L}(g,g),(\mathbf{I}-\mathbf{P})g)_{L^2(\mathbb{R}^3)}.$$

Proposition 3.19. *Let $g=g(t,x,v)$ be suitably smooth function. It holds that*

$$\sum_{p\geq -1}2^{\frac32 p}\left(\int_0^T|(\Delta_p\mathbf{L}(g,g),\Delta_p(\mathbf{I}-\mathbf{P})g)_{x,v}|\,dt\right)^{1/2}\lesssim\sqrt{\mathcal{E}_T(g)}\mathcal{D}_T(g)$$

for $0<T\leq+\infty$.

Proof. With aid of the macro-micro decomposition, we split $\mathbf{L}(g,g)$ into four terms:

$$\mathbf{L}(g,g)=\mathbf{L}(\mathbf{P}g,\mathbf{P}g)+\mathbf{L}(\mathbf{P}g,(\mathbf{I}-\mathbf{P})g)+\mathbf{L}((\mathbf{I}-\mathbf{P})g,\mathbf{P}g)+\mathbf{L}((\mathbf{I}-\mathbf{P})g,(\mathbf{I}-\mathbf{P})g).$$

Then it follows from Theorem 3.4 (taking $s=\frac32$) that

$$\begin{aligned}&\sum_{p\geq -1}2^{\frac32 p}\left(\int_0^T|(\Delta_p\mathbf{L}(\mathbf{P}g,\mathbf{P}g),\Delta_p(\mathbf{I}-\mathbf{P})g)_{x,v}|\,dt\right)^{1/2}\\&\lesssim\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac12}\mathbf{P}g\big\|^{1/2}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\&\quad+\|\mathbf{P}g\|^{1/2}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}\mathbf{P}g\big\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac12}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\&\lesssim\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\mathbf{P}g\|^{1/2}_{L^2_TL^2_v(L^\infty_x)}\big\|\widetilde{\mathcal{L}}^{\frac12}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}.\end{aligned}$$

Note that $\dot{B}^{3/2}_{2,1}(\mathbb{R}^3_x)\hookrightarrow L^\infty(\mathbb{R}^3_x)$, Proposition 2.2 and Lemma 2.9, we arrive at

$$\begin{aligned}&\sum_{p\geq -1}2^{\frac32 p}\left(\int_0^T|(\Delta_p\mathbf{L}(\mathbf{P}g,\mathbf{P}g),\Delta_p(\mathbf{I}-\mathbf{P})g)_{x,v}|\,dt\right)^{1/2}\\&\lesssim\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(\dot{B}^{3/2}_{2,1})}\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac12}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})},\end{aligned}$$

where

$$\begin{aligned}\|\mathbf{P}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(\dot{B}^{3/2}_{2,1})} &\lesssim \|(a,b,c)\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} \lesssim \|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}\\ &\lesssim \|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \lesssim \sqrt{\mathcal{D}_T(g)}.\end{aligned}$$

Hence, we conclude that

$$\sum_{p\geq -1} 2^{\frac{3}{2}p}\left(\int_0^T |(\Delta_p\mathbf{L}(\mathbf{P}g,\mathbf{P}g),\Delta_p(\mathbf{I}-\mathbf{P})g)_{x,v}|\,dt\right)^{1/2} \lesssim \sqrt{\mathcal{E}_T(g)}\mathcal{D}_T(g).$$

In fact, other collision terms can be estimated at a similar way. Precisely,

$$\begin{aligned}&\sum_{p\geq -1} 2^{\frac{3}{2}p}\left(\int_0^T |(\Delta_p\mathbf{L}((\mathbf{I}-\mathbf{P})g,\mathbf{P}g),\Delta_p(\mathbf{I}-\mathbf{P})g)_{x,v}|\,dt\right)^{1/2}\\ &\lesssim \Big\{ \|(\mathbf{I}-\mathbf{P})g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(\dot{B}^{3/2}_{2,1})} + \|(\mathbf{I}-\mathbf{P})g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\Big\}\\ &\quad\times \big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\ &\lesssim \sqrt{\mathcal{E}_T(g)}\mathcal{D}_T(g),\\ &\sum_{p\geq -1} 2^{\frac{3}{2}p}\left(\int_0^T |(\Delta_p\mathbf{L}(\mathbf{P}g,(\mathbf{I}-\mathbf{P})g),\Delta_p(\mathbf{I}-\mathbf{P})g)_{x,v}|\,dt\right)^{1/2}\\ &\lesssim \|\mathbf{P}g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\ &\lesssim \sqrt{\mathcal{E}_T(g)}\mathcal{D}_T(g)\end{aligned}$$

and

$$\begin{aligned}&\sum_{p\geq -1} 2^{\frac{3}{2}p}\left(\int_0^T |(\Delta_p\mathbf{L}((\mathbf{I}-\mathbf{P})g,(\mathbf{I}-\mathbf{P})g),\Delta_p(\mathbf{I}-\mathbf{P})g)_{x,v}|\,dt\right)^{1/2}\\ &\lesssim \|(\mathbf{I}-\mathbf{P})g\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\big\|\widetilde{\mathcal{L}}^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\big\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{3/2}_{2,1})}\\ &\lesssim \sqrt{\mathcal{E}_T(g)}\mathcal{D}_T(g).\end{aligned}$$

Therefore, combine those estimates to finish the proof of Proposition 3.19. □

3.6.3 The global existence to nonlinear Landau equation

Having Propositions 3.18-3.19, we can establish the following priori estimate.

Proposition 3.20. *Let $g \in Y$ be the solution to the Cauchy problem* (3.1)*. It holds that*

$$\mathcal{E}_T(g)+\mathcal{D}_T(g) \leq C\,\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + C\left(\mathcal{E}_T(g)+\sqrt{\mathcal{E}_T(g)}\right)\mathcal{D}_T(g) \tag{3.57}$$

for any $T>0$, where $C>0$ is some constant independent of T.

Proof. Applying $\Delta_p (p \geq -1)$ to (3.1) and taking the inner product with $\Delta_p g$ over $\mathbb{R}^3_x \times \mathbb{R}^3_v$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta_p g\|^2_{L^2_{x,v}} + \|\mathcal{L}^{\frac{1}{2}}\Delta_p g\|^2_{L^2_{x,v}} \leq (\Delta_p \mathbf{L}(g,g), \Delta_p(\mathbf{I}-\mathbf{P})g)_{L^2_{x,v}},$$

where we used $(v\cdot\Delta_p\nabla_x g, \Delta_p g)_{L^2_x} = 0$. Integrate the above inequality with respect to the time variable over $[0,t]$ with $0 \leq t \leq T$ and then take the square root of both sides of the resulting inequality. Consequently, we obtain

$$\begin{aligned}
&2^{\frac{3}{2}p}\|\Delta_p g\|_{L^2_v L^2_x} + 2^{\frac{3}{2}p}\|\mathcal{L}^{\frac{1}{2}}\Delta_p g\|_{L^2_v L^2_x} \\
&\leq 2^{\frac{3}{2}p}\|\Delta_p g_0\|_{L^2_v L^2_x} + 2^{\frac{3}{2}p}\left(\int_0^t \left|(\Delta_p \mathbf{L}(g,g), \Delta_p g)_{L^2_{x,v}}\right| dt\right)^{1/2},
\end{aligned}$$

which implies that (by using Proposition 3.19)

$$\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\mathcal{L}^{\frac{1}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \leq C\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + \sqrt{\mathcal{E}_T(g)}\mathcal{D}_T(g). \tag{3.58}$$

It follows from Proposition 3.18 that (performing the calculation $\alpha \times (3.51) + (3.58)$ in fact) that

$$\begin{aligned}
(1-\alpha)\mathcal{E}_T(g) + (1-\alpha)\|\mathcal{L}^{\frac{1}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \alpha\,\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
\lesssim \|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + (\sqrt{\mathcal{E}_T(g)} + \mathcal{E}_T(g))\mathcal{D}_T(g),
\end{aligned}$$

which leads to (3.57) directly if taking $\alpha > 0$ sufficiently small. □

Proof of Theorem 3.1.

Firstly, we prove the non-negativity of the solution to (1.1). For the solution of the Cauchy problem (3.1) that obtained in Proposition 3.17, which is the limit of the sequence of (3.49), coming back to the original Landau equation, it is also the limit of a sequence constructed successively by the following linear Cauchy problem

$$\begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} = Q_L(f^n, f^{n+1}), \\ f^{n+1}|_{t=0} = f_0 = \mu + \sqrt{\mu} g_0 \geq 0. \end{cases}$$

Then, the non-negativity of solutions to the Cauchy problem (1.1) can be proved by the same methods as Section 5 in [8]. Based on the global a priori estimate (3.57) and the local existence result (Proposition 3.17), Theorem 3.1 is followed by the standard continuity argument. The detailed calculation is similar to that of Section 8 in [52]. For completeness, we give the detailed process.

From the local existence of solutions in Proposition 3.17, that is, for a sufficiently small $m_0 > 0$, there exists $T^* = T^*(m_0) > 0$ such that if

$$\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \leq m_0,$$

then the Cauchy problem (3.1) admits a local solution $g(t,x,v) \in (0,T^*) \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying

$$\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})} \leq 2m_0, \quad T \in [0,T^*). \tag{3.59}$$

It is also needed to prove that $T \mapsto \|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})}$ is continuous in $[0,T^*)$. The proof is similar to that of Section 7 (Page 27) in [52]. We also state the detailed proof. Firstly, prove that

$$t \mapsto \sum_{p\geq -1} 2^{\frac{3}{2}p} \|\Delta_p g(t)\|_{L^2_v L^2_x} \tag{3.60}$$

is continuous in $[0,T^*)$. In fact, take $0 \leq t_1 \leq t_2 < T^*$. Letting $f = g$ in (3.48), integrating the resultant inequality with respect to the time variable over $[t_1,t_2]$ with $0 \leq t \leq T$, taking the square root of both sides of the resulting inequality and summing up over $p \geq -1$, we obtain

$$\left| \sum_{p\geq -1} 2^{\frac{3}{2}p} \left(\|\Delta_p g(t_2)\|_{L^2_v L^2_x} - \|\Delta_p g(t_1)\|_{L^2_v L^2_x} \right) \right| \lesssim (1+\sqrt{m_0}) \sum_{p\geq -1} 2^{\frac{3}{2}p} \left(\int_{t_1}^{t_2} \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2},$$

where Lemma 3.4 is used. Hence, it is sufficient to prove

$$\lim_{t_2 \to t_1} \sum_{p\geq -1} 2^{\frac{3}{2}p} \left(\int_{t_1}^{t_2} \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2} = 0. \tag{3.61}$$

Due to that $\sum_{p\geq -1} 2^{\frac{3}{2}p} \left(\int_0^T \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2}$ is finite for a fixed time T with $\max\{t_1,t_2\} < T < T^*$, there exists an integer N such that

$$\sum_{p\geq N+1} 2^{\frac{3}{2}p} \left(\int_{t_1}^{t_2} \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2} \leq \sum_{p\geq N+1} 2^{\frac{3}{2}p} \left(\int_0^{T} \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2} < \frac{\varepsilon}{2}$$

for $\varepsilon > 0$. Directly, it holds that

$$\lim_{t_2 \to t_1} \sum_{-1\leq p\leq N} 2^{\frac{3}{2}p} \left(\int_{t_1}^{t_2} \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2} = 0$$

which implies that there is $\delta > 0$ such that

$$\sum_{-1\leq p\leq N} 2^{\frac{3}{2}p} \left(\int_{t_1}^{t_2} \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2} < \frac{\varepsilon}{2},$$

for $|t_2 - t_1| < \delta$. Thus, $|t_2 - t_1| < \delta$, one has

$$\sum_{p\geq -1} 2^{\frac{3}{2}p} \left(\int_{t_1}^{t_2} \|\Delta_p g(t)\|^2_{L^2_v L^2_x} dt \right)^{1/2} = \left(\sum_{-1\leq p\leq N} + \sum_{p\geq N+1} \right) \cdots < \varepsilon.$$

The limit formula (3.61) is proved and thus $t \mapsto \sum_{p\geq -1} 2^{\frac{3}{2}p}\|\Delta_p g(t)\|_{L^2_v L^2_x}$ is continuous in $[0, T^*)$. Particularly, for $t \in [0, T^*)$ the norm $\sum_{p\geq -1} 2^{\frac{3}{2}p}\|\Delta_p g(t)\|_{L^2_v L^2_x}$ in (3.61) is well defined. Then, we show that $T \mapsto \|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})}$ is continuous in $[0, T^*)$. As a matter of fact, take T_1, T_2 with $0 \leq T_1 \leq T_2 < T^*$. Observed that $\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{3/2}_{2,1})}$ is non-decreasing in T. After, we calculate that

$$
\begin{aligned}
0 \leq & \left(\|g\|_{\widetilde{L}^\infty_{T_2} \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{\widetilde{L}^2_{T_2} \widetilde{L}^2_v(B^{3/2}_{2,1})}\right) - \left(\|g\|_{\widetilde{L}^\infty_{T_1} \widetilde{L}^2_v(B^{3/2}_{2,1})} + \|\widetilde{\mathcal{L}}^{\frac{1}{2}} g\|_{\widetilde{L}^2_{T_1} \widetilde{L}^2_v(B^{3/2}_{2,1})}\right) \\
\leq & \sum_{p\geq -1} 2^{\frac{3}{2}p} \left(\sup_{T_1 \leq t \leq T_2} \|\Delta_p g(t)\|_{L^2_v L^2_x} - \|\Delta_p g(T_1)\|_{L^2_v L^2_x}\right) \\
& + \sum_{p\geq -1} 2^{\frac{3}{2}p} \left(\int_{T_1}^{T_2} \|\Delta_p g(t)\|_{L^2_v L^2_x}\right)^{1/2} \triangleq A + B.
\end{aligned}
$$

It follows from (3.61) that $B \to 0$ as $T_2 \to T_1$. Since $t \mapsto \|\Delta_p g(t)\|_{L^2_v L^2_x}$ is continuous and $\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{3/2}_{2,1})}$ is finite for some fixed time T with $T_2 < T < T^*$, we can obtain that $A \to 0$ as $T_2 \to T_1$ with the completely same way as proof of (3.61). Also, one can obtain $T \mapsto \mathcal{E}_T(g) + \mathcal{D}_T(g)$ is continuous with the same proof.

Now we begin to prove the global existence of solutions to (3.1). Let us redefine the constant C on the right of (3.57) to be $C_1 \geq 1$, and choose $m_1 > 0$ such that

$$
C_1(m_1 + \sqrt{m_1}) \leq \frac{1}{2}.
$$

Set $m = \min\{m_0, m_1\}$. Choose g_0 such that

$$
\|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \leq \frac{M}{4C_1} \leq \frac{M_0}{2}.
$$

Define

$$
\widetilde{T} = \sup\{T : \mathcal{E}_T(g) + \mathcal{D}_T(g) \leq m\}.
$$

Then, $\widetilde{T} > 0$ holds true since the solution g exists locally in time from (3.59) and $T \mapsto \mathcal{E}_T(g) + \mathcal{D}_T(g)$ is continuous. Furthermore, we deduce from (3.57) that for $0 \leq T \leq \widetilde{T}$,

$$
\mathcal{E}_T(g) + \mathcal{D}_T(g) \leq C_1 \|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} + C_1 (m_1 + \sqrt{m_1}) (\mathcal{E}_T(g) + \mathcal{D}_T(g))
$$

That is, $0 \leq T \leq \widetilde{T}$ one can obtain

$$
\mathcal{E}_T(g) + \mathcal{D}_T(g) \leq 2C_1 \|g_0\|_{\widetilde{L}^2_v(B^{3/2}_{2,1})} \leq \frac{m}{2} < m
$$

which implies $\widetilde{T} = \infty$. Hence, the proof of Theorem 3.1 is completed.

Chapter 4

Sharp Gelfand-Shilov and Gevrey smoothing effect for the inhomogeneous non-cutoff Kac equation

Contents

In this chapter, we investigate the Cauchy problem for the spatially inhomogeneous non-cutoff Kac equation. If the initial datum belongs to the spatially critical Besov space, we prove the well-posedness for the inhomogeneous Kac equation under a perturbation framework. Furthermore, it is shown that the solution enjoys Gelfand-Shilov regularizing properties with

respect to the velocity variable and Gevrey regularizing properties with respect to the position variable. In comparison with the recent result of [117], the Gelfand-Shilov regularity index is improved to be optimal.

4.1 Description of problem

In this chapter, we consider the Kac equation (1.8) around the normalized Maxwellian distribution

$$M(v) = (2\pi)^{-\frac{1}{2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}.$$

In a close to equilibrium framework, considering the fluctuation of density distribution function

$$f(t,x,v) = M(v) + \sqrt{M}(v) g(t,x,v).$$

Since $K(M,M) = 0$ by conservation of the kinetic energy, the Cauchy problem (1.8) reads as

$$\begin{cases} \partial_t g + v\partial_x g + \mathcal{K} g = \Gamma(g,g), \ \ t > 0,\ v \in \mathbb{R}, \\ g|_{t=0} = g_0 \end{cases} \tag{4.1}$$

with $f_0 = M(v) + \sqrt{M}(v) g_0$, where $\mathcal{K}(g), \Gamma(g,g)$ are given by (1.12)-(1.13).

Our main results are stated as follows (also, **Main Theorem B** in Chapter 1).

Theorem 4.1. *Let $0 < T < +\infty$. We suppose that the collision cross section satisfies (4.7) with $0 < s < 1$. There exists a constant $\varepsilon_0 > 0$ such that for all $g_0 \in \widetilde{L}^2_v(B^{1/2}_{2,1})$ satisfying*

$$\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \varepsilon_0,$$

then the Cauchy problem (4.1) admits a unique weak solution $g \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$ satisfying

$$\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \|\mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq c_0 e^T \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}$$

for some constant $c_0 > 1$. Furthermore, this solution is smooth for all positive time $0 < t \leq T$, which satisfies the following Gelfand-Shilov and Gevrey type estimates:

1) $\exists C > 1,\ \forall 0 < t \leq T,\ \forall k \geq 0,$

$$\left\| \left(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x \rangle^{\frac{1+3s}{1+2s}} \right)^k g(t) \right\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{C^{k+1}}{t^{\frac{3s+1}{2s}k}} (k!)^{\frac{3s+1}{2s}} \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})},$$

2) $\forall \delta > 0,\ \exists C > 1,\ \forall 0 < t \leq T,\ \forall k,l,q \geq 0,$

$$\|v^k \partial_v^l \partial_x^q g(t)\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{C^{k+l+q+1}}{t^{\frac{3s+1}{2s(s+1)}(k+l+2)+\frac{2s+1}{2s}q+\delta}} (k!)^{\frac{3s+1}{2s(s+1)}} (l!)^{\frac{3s+1}{2s(s+1)}} (q!)^{\frac{2s+1}{2s}} \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

Our result deserves some comments in contrast to the result of [117].

Remark 4.1. (1) We show the well-posedness of Cauchy problem with the initial datum belonging to the spatially critical Besov space $\widetilde{L}^2_v(B^{1/2}_{2,1})$, rather than in the Sobolev space $L^2_v(H^1_x)$.

(2) For the regularizing effect, our result indicates that

$$\forall t>0,\ \forall x\in\mathbb{R},\quad g(t,x,\cdot)\in S^{\frac{3s+1}{2s(s+1)}}_{\frac{3s+1}{2s(s+1)}}(\mathbb{R});\quad \forall t>0,\ \forall v\in\mathbb{R},\quad g(t,\cdot,v)\in G^{1+\frac{1}{2s}}(\mathbb{R}).$$

Actually, the Gelfand-Shilov index for the velocity variable is sharp for $0<s<1$, if noticing that

$$\frac{3s+1}{2s(s+1)}=\frac{2s+1}{2s}\frac{3s+1}{(2s+1)(s+1)}<1+\frac{1}{2s}.$$

(3) If s is close to 1, the solution is almost analytic in the velocity variable, since

$$\frac{3s+1}{2s(s+1)}\to 1.$$

Therefore, our Gelfand-Shilov index for the velocity variable should be optimal.

■ **The schema of proof of Theorem 4.1**

The proof of Theorem 4.1 is a little complex since that we need to prove the local weak solution in critical Besov space $\widetilde{L}^2_v(B^{1/2}_{2,1})$ but also prove the Gelfand-Shilov index with respect to the velocity variable v and Gevrey regularizing effect for position variable x. The summary of the proof is shown as follows.

• Analysis of Kac collision operator (Section 4.3)

The estimates of Kac collision operator make an important effect on our proof. We prove some estimates of Kac collision operator along the Hermite basis, of which the estimate of the nonlinear collision operator given in Lemma 4.13 is new and critical to obtain the more sharp Gelfand-Shilov regularity index. Here, by taking a new mollifier different from that in [117], we obtain some key estimates.

• The local existence of weak solution (Section 4.4)

The proof of the local existence is similar to that for Landau equation in Chapter 3. However, the difference is that we take a new mollifier with respect to the position variable and velocity variable. See Section 4.4 for more details.

• Gelfand-Shilov and Gevrey regularizing effects (Section 4.5)

To show Gelfand-Shilov and Gevrey smoothing effects, we firstly prove a priori estimate with exponential weights in Proposition 4.6. To do this, some critical estimates are needed, hypoelliptic estimate of the linear inhomogeneous Kac operator in Lemma 4.17 and Lemmas 4.18-4.20. With a key estimate on the Hermite functions in Lemma 4.9 and Proposition 4.6, we prove the Gelfand-Shilov regularizing effect for velocity variable v and Gevrey regularizing effect for position variable x in Theorem 4.1.

4.2 Analysis tools

4.2.1 Hermite functions

The standard Hermite functions $(\phi_n)_{n\in\mathbb{N}}$ are defined for $v \in \mathbb{R}$,

$$\phi_n(v) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}} e^{\frac{v^2}{2}} \frac{d^n}{dv^n}(e^{-\frac{v^2}{2}}) = -\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}(v - \frac{d}{dv})^n(e^{-\frac{v^2}{2}}) = \frac{a_+^n \phi_0}{\sqrt{n!}},$$

where a_+ is the creation operator

$$a_+ = \frac{1}{\sqrt{2}}\Big(v - \frac{d}{dv}\Big).$$

The family $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$. We set for $n \in \mathbb{N}, v \in \mathbb{R}$,

$$e_n(v) = 2^{-1/4}\phi_n(2^{-1/2}v), \quad e_n = \frac{1}{\sqrt{n!}}(\frac{v}{2} - \frac{d}{dv})^n e_0.$$

The family $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$ composed by the eigenfunctions of the Harmonic Oscillator

$$\mathcal{H} \triangleq -\Delta_v + \frac{v^2}{4} = \sum_{n\geq 0}(n + \frac{1}{2})\mathbb{P}_n, \quad 1 = \sum_{n\geq 0}\mathbb{P}_n,$$

where $\mathbb{P}_n$ stands for the orthogonal projection

$$\mathbb{P}_n f = (f, e_n)_{L^2(\mathbb{R}_v)} e_n.$$

It satisfies the identities

$$A_+ e_n = \sqrt{n+1}e_{n+1}, \quad A_- e_n = \sqrt{n}e_{n-1}, \tag{4.2}$$

where

$$A_\pm = \frac{v}{2} \mp \frac{d}{dv}. \tag{4.3}$$

Furthermore, the fractional Harmonic Oscillator

$$\mathcal{H}^s = \sum_{k\geq 0}(k + \frac{1}{2})^s \mathbb{P}_k$$

can be defined by the functional calculus.

4.2.2 The Kac collision operator

The linearized Kac operator $\mathcal{K}$ is a non-negative unbounded operator on $L^2(\mathbb{R}_v)$ (see [115]) with a kernel given by

$$\text{Ker } \mathcal{K} = \text{Span}\{e_0, e_2\}.$$

where the Hermite basis $(e_n)_{n\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R})$, which is recalled in Section 4.2.1. The linearized Kac operator is diagonal in the Hermite basis

$$\mathcal{K} = \sum_{k\geq 1} \lambda_k \mathbb{P}_k \tag{4.4}$$

with a spectrum only composed by the non-negative eigenvalues

$$\lambda_{2k+1} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1-(\cos\theta)^{2k+1})d\theta \geq 0, \quad k \geq 0,$$
$$\lambda_{2k} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1-(\cos\theta)^{2k}-(\sin\theta)^{2k})d\theta \geq 0, \quad k \geq 1$$

satisfying the asymptotic estimates

$$\lambda_k \approx k^s, \tag{4.5}$$

when $k \to +\infty$. We notice that

$$0 = \lambda_2 \leq \lambda_{2k} < \lambda_{2l}, \quad 0 < \lambda_1 < \lambda_{2k+1} < \lambda_{2l+1},$$

when $1 \leq k < l$, and that λ_1 is the lowest positive eigenvalue. The linearized Kac operator enjoys the coercive estimates

$$\exists C > 0, \forall f \in \mathcal{S}(\mathbb{R}_v), \quad \frac{1}{C}\|\mathcal{H}^{\frac{s}{2}} f\|_{L^2}^2 \leq (\mathcal{K}f, f)_{L^2} + \|f\|_{L^2}^2 \leq C\|\mathcal{H}^{\frac{s}{2}} f\|_{L^2}^2. \tag{4.6}$$

Now we take the following choice for the cross section

$$\beta(\theta) = \frac{|\cos\frac{\theta}{2}|}{|\sin\frac{\theta}{2}|^{1+2s}}, \quad |\theta| \leq \frac{\pi}{4}, \tag{4.7}$$

in part because of the usage of those results in [115] directly. In that case, the eigenvalues satisfy the asymptotic equivalence

$$\lambda_k \underset{k\to+\infty}{\sim} \frac{2^{1+s}}{s}\mathbf{\Gamma}(1-s)k^s, \tag{4.8}$$

where $\mathbf{\Gamma}$ denotes the Gamma function. Moreover, the linearized Kac operator

$$\mathcal{K}u = l^w(v, D_v)u = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(v-y)\eta} l(\frac{v+y}{2}, \eta)u(y)dyd\eta$$

is a pseudo-differential operator whose Weyl symbol belongs to $\mathbf{S}^s(\mathbb{R}^2)$, where for $m \in \mathbb{R}$, the symbol class $\mathbf{S}^m(\mathbb{R}^2)$ is defined as the set of smooth functions $a : \mathbb{R}^2 \to \mathbb{C}$ satisfying

$$\forall(\alpha,\beta) \in \mathbb{N}^2,\ \exists C_{\alpha,\beta} > 0,\ \forall(v,\eta) \in \mathbb{R}^2,\ \ |\partial_v^\alpha \partial_\eta^\beta a(v,\eta)| \leq C_{\alpha,\beta}\langle(v,\eta)\rangle^{2m-|\alpha|-|\beta|}$$

with $\langle(v,\eta)\rangle = \sqrt{1+|v|^2+|\eta|^2}$. More specifically, the Weyl symbol $l(v,\eta)$ admit the following asymptotic expansion

$$\begin{aligned}\forall N \geq 1,\ \ l(v,\eta) \equiv \frac{2^{1+s}}{s}\mathbf{\Gamma}(1-s)(1+\eta^2+\frac{v^2}{4})^s - \frac{2^{1+s}(2+\sqrt{2})^s}{s}\\ + \sum_{k=1}^{N} c_k(1+\eta^2+\frac{v^2}{4})^{s-k} \bmod \mathbf{S}^{s-N-1}(\mathbb{R}^2),\end{aligned}$$

where $(c_k)_{k\geq 1}$ is a sequence of real numbers.

Next, we recall the following lemma given by Lerner-Morimoto-Pravda-Starov-Xu in [115], which gives the details about the Kac collision operator can be defined as a finite part integral under the assumption (1.9).

For φ a function defined on $\mathbb{R}$, we denote its even part by

$$\breve{\varphi}(\theta) = \frac{1}{2}(\varphi(\theta)+\varphi(-\theta)).$$

The lemma in [115] (Lemma A.1) is shows as:

Lemma 4.1. *Let $\nu \in L^1_{\mathrm{loc}}(\mathbb{R}^*)$ be an even function such that $\theta^2\nu(\theta) \in L^1(\mathbb{R})$. Then, the mapping*

$$\varphi \in C_c^2(\mathbb{R}) \mapsto \lim_{\varepsilon\to 0_+}\int_{|\theta|\geq\varepsilon}\nu(\theta)(\varphi(\theta)-\varphi(0))d\theta = \int_0^1\int_{\mathbb{R}}(1-t)\theta^2\nu(\theta)\varphi''(t\theta)d\theta dt,$$

defines a distribution of order 2 denoted $\mathrm{fp}(\nu)$. *The linear form* $\mathrm{fp}(\nu)$ *can be extended to $C^{1,1}$ functions (C^1 functions whose second derivative is L^∞). For $\varphi \in C^{1,1}$ satisfying $\varphi(0) = 0$, the function $\nu\breve{\varphi}$ belongs to $L^1(\mathbb{R})$ and*

$$\langle \mathrm{fp}(\nu), \varphi\rangle = \int \nu(\theta)\breve{\varphi}(\theta)d\theta.$$

Let $f, g \in \mathcal{S}(\mathbb{R})$ be Schwartz functions. We define

$$F_{f,g}(\underbrace{v,v_*}_{w}) = f(v)g(v_*),\ \ \varphi_{f,g}(\theta,v) = \int_{\mathbb{R}}(F_{f,g}(R_\theta w) - F_{f,g}(w))dv_*,$$

where R_θ stands for the rotation of angle θ in $\mathbb{R}^2$,

$$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \exp(\theta J),\ \ J = R_{\frac{\pi}{2}}.$$

We have

$$F_{f,g}(R_\theta w) - F_{f,g}(w) = f(v\cos\theta - v_*\sin\theta)g(v\sin\theta + v_*\cos\theta) - f(v)g(v_*),$$

so that by using the notation $f'_* = f(v'_*), f' = f(v'), f_* = f(v_*), f = f(v)$ with

$$v' = v\cos\theta - v_*\sin\theta, \quad v'_* = v\sin\theta + v_*\cos\theta, \ v, v_* \in \mathbb{R},$$

we may write

$$\varphi_{f,g}(\theta, v) = \int_{\mathbb{R}} (g'_* f' - g_* f) dv_*.$$

Furthermore, we easily check that its even part as a function of the variable θ is given by

$$\breve{\varphi}_{f,g}(\theta, v) = \int_{\mathbb{R}} ((\breve{g})'_* f' - g_* f) dv_* = \int_{\mathbb{R}} ((\breve{g})'_* f' - (\breve{g})_* f) dv_*.$$

Notice that for each $\theta \in \mathbb{R}$, the mapping

$$(f, g) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \mapsto \varphi_{f,g}(\theta, \cdot) \in \mathcal{S}(\mathbb{R})$$

is continuous uniformly with respect to θ. In fact, the function $F_{f,g}$ belongs to $\mathcal{S}(\mathbb{R}^2)$. By denoting Π_1 the projection onto the first variable, this implies that the function

$$v^l \partial_v^k \varphi_{f,g}(\theta, v) = \int \Pi_1(w)^l \partial_v^k \Phi_{f,g}(\theta, w) dv_*$$

is bounded since

$$\Phi_{f,g}(\theta, w) = F_{f,g}(R_\theta w) - F_{f,g}(w) \in \mathcal{S}(\mathbb{R}^2).$$

As a result, the function $v \mapsto \varphi_{f,g}(\theta, v)$ belongs to $\mathcal{S}(\mathbb{R})$ uniformly with respect to θ. Moreover, the second derivative with respect to θ of the function $\Phi_{f,g}$,

$$F''_{f,g}(e^{\theta J} w)(e^{\theta J} Jw, e^{\theta J} Jw) - F'_{f,g}(e^{\theta J} w) e^{\theta J} w$$

belongs to $\mathcal{S}(\mathbb{R}^2)$ uniformly with respect to θ. This implies that the second derivative with respect to θ of the function $\varphi_{f,g}$ is in $\mathcal{S}(\mathbb{R})$ uniformly with respect to θ. We define the non-cutoff Kac operator as

$$K(g, f)(v) = \langle \mathrm{fp}(\mathbb{I}_{(-\frac{\pi}{4}, \frac{\pi}{4})}\beta), \varphi_{f,g}(\cdot, v)\rangle,$$

when β is a function satisfying (1.9). Since $\Phi_{f,g}(0, v) \equiv 0$, Lemma 4.1 allows to replace the finite part by the absolutely converging integral

$$K(g, f)(v) = \int_{|\theta| \le \frac{\pi}{4}} \beta(\theta) \Big(\int_{\mathbb{R}} ((\breve{g})'_* f' - (\breve{g})_* f) dv_* \Big) d\theta = K(\breve{g}, f)(v).$$

It was established in [115] (Lemma A.2) that $K(g, f) \in \mathcal{S}(\mathbb{R})$, when $g, f \in \mathcal{S}(\mathbb{R})$. We also recall the Bobylev formula [16] providing an explicit formula for the Fourier transform of the Kac operator

$$\widehat{K(g, f)}(\xi) = \int_{|\theta| \le \frac{\pi}{4}} \beta(\theta) \Big[\widehat{\breve{g}}(\xi \sin\theta) \widehat{f}(\xi\cos\theta) - \widehat{g}(0)\widehat{f}(\xi) \Big] d\theta,$$

when $f, g \in \mathcal{S}(\mathbb{R})$. The proof of this formula may be found in [115] (Lemma A.4).

4.2.3 Linear inhomogeneous Kac operator

We recall some spectral analysis for the linear inhomogeneous Kac operator that are given in [115, 117]. Consider the operator acting in the velocity variable

$$P = iv\xi + a_0^w(v, D_v) \tag{4.9}$$

with parameter $\xi \in \mathbb{R}$, where the operator $A = a_0^w(v, D_v)$ stands for the pseudo-differential operator

$$a_0^w(v, D_v)u = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(v-w)\eta} a_0(\frac{v+w}{2}, \eta)u(w)dwd\eta$$

defined by the Weyl quantization of the symbol

$$a_0(v, \eta) = c_0(1 + \eta^2 + \frac{v^2}{4})^s$$

with some constants $c_0 > 0, 0 < s < 1$. This operator corresponds to the principle part of the linear inhomogeneous Kac operator

$$v\partial_x + \mathcal{K}$$

on the Fourier side in the position variable.

Let ψ be a $C_0^\infty(\mathbb{R}, [0, 1])$ function satisfying

$$\psi = 1 \ \text{ on } \ [-1, 1], \ \ \text{supp } \psi \subset [-2, 2].$$

We define the real-valued symbol

$$m = -\frac{\xi\eta}{\lambda^{\frac{2s+2}{2s+1}}}\psi(\frac{\eta^2 + v^2}{\lambda^{\frac{2}{2s+1}}}) \tag{4.10}$$

with

$$\lambda = (1 + v^2 + \eta^2 + \xi^2)^{\frac{1}{2}}.$$

It holds that the following equivalence of norms

$$\forall r \in \mathbb{R}, \exists C_r > 0, \frac{1}{C_r}\|\mathcal{H}^r u\|_{L^2} \leq \Big\|\mathrm{Op}^w\Big(\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^r\Big)u\Big\|_{L^2} \leq C_r\|\mathcal{H}^r u\|_{L^2}, \tag{4.11}$$

where $\mathcal{H} = -\Delta_v + \frac{v^2}{4}$ stands for the Harmonic Oscillator.

To obtain the hypoelliptic estimate of the inhomogeneous Kac operator, we recall some notations and metrics on the phase space, see [72, 106, 117]. We consider the following metrics on the phase space $\mathbb{R}^2_{v,\eta}$

$$\Gamma_0 = \frac{dv^2 + d\eta^2}{\langle(v, \eta)\rangle^2}, \qquad \Gamma_0 = \frac{dv^2 + d\eta^2}{M(v, \eta, \xi)}$$

with

$$\langle (v,\eta)\rangle^2 = 1+v^2+\eta^2,$$
$$N(v,\eta,\xi) = 1+v^2+\eta^2+\lambda^{\frac{2}{2s+1}} = 1+v^2+\eta^2+(1+v^2+\eta^2+\xi^2)^{\frac{1}{2s+1}}.$$

Notice that the second metric depends on the parameter $\xi \in \mathbb{R}$. For a positive function $\mu \geq 1$, we define the space $S(\mu,\Gamma_0)$ as the set of functions possibly depending on the parameter ξ satisfying

$$\forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0, \forall (v,\eta,\xi) \in \mathbb{R}^3, \quad |\partial^\alpha_{v,\eta} a(v,\eta,\xi)| \leq C_\alpha \mu(v,\eta,\xi)\langle (v,\eta)\rangle^{-|\alpha|},$$

whereas the space $S(\mu,\Gamma_1)$ corresponds to $C^\infty(\mathbb{R}^2_{v,\eta},\mathbb{C})$ functions depending on the parameter ξ satisfying

$$\forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0, \forall (v,\eta,\xi) \in \mathbb{R}^3, \quad |\partial^\alpha_{v,\eta} a(v,\eta,\xi)| \leq C_\alpha \mu(v,\eta,\xi) N(v,\eta,\xi)^{-\frac{|\alpha|}{2}}.$$

The metrics Γ_0 and Γ_1 are admissible (slowly varying, satisfying the uncertainty principle and temperate), see Section A.4 in [117]. In [117], the authors verified some properties for the weight $\mu \geq 1$ with respect to the metric Γ_j, namely the slowly varying property of μ with respect to Γ_j, for the function space $S(\mu,\Gamma_j)$ to enjoy nice symbolic calculus properties. Also, they studied in the symbol classes

$$S(\langle (v,\eta)\rangle^n,\Gamma_0), \quad S(N^n,\Gamma_1)$$

with $n \in \mathbb{R}$, which enjoy nice symbolic calculus since the function $\langle (v,\eta)\rangle^n$ is a Γ_0-slowly varying weight and that the function N is a Γ_1-slowly varying weight uniformly with respect to the parameter $\xi \in \mathbb{R}$, see Section A.4 in [117]. The gain functions in the symbolic calculus associated to these two symbol classes $S(\langle (v,\eta)\rangle^n,\Gamma_0)$ and $S(N^n,\Gamma_1)$ are respectively given by

$$\Lambda_{\Gamma_0} = \langle (v,\eta)\rangle^2, \quad \Lambda_{\Gamma_1} = N(x,\eta,\xi).$$

Since $\langle (v,\eta)\rangle^2 \leq N(x,\eta,\xi)$, it holds that $S(n,\Gamma_1) \subset S(n,\Gamma_0)$.

In [117], the authors show the following symbolic estimates.

Lemma 4.2. *For $n \in \mathbb{R}$, the following symbols belongs to their respective function spaces*

$$1)\langle \xi\rangle^n \in S(\lambda^n,\Gamma_1) \qquad 2)\lambda^n \in S(\lambda^n,\Gamma_1) \qquad 3)\psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big) \in S(1,\Gamma_1)$$
$$4)m \in S(1,\Gamma_1) \qquad 5)\frac{\xi^2}{\lambda^{\frac{2+2s}{2s+1}}}\psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big) \in S(N,\Gamma_1)$$
$$6)\Big(1-\psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big)\Big)\Big(1+\eta^2+\frac{v^2}{4}\Big)^s \in S(N,\Gamma_1),$$

uniformly with respect to the parameter $\xi \in \mathbb{R}$. It holds that

$$\{v\xi, m\} = \frac{\xi^2}{\lambda^{\frac{2+2s}{2s+1}}}\psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big) + r$$

with a remainder belongs to the symbol class $S(\langle(v,\eta)\rangle^{2s}, \Gamma_0)$ uniformly with respect to the parameter $\xi \in \mathbb{R}$.

4.2.4 Fundamental inequalities

We recall some estimates for the Kac collision operator along with the Hermite basis, see [117] for details.

Lemma 4.3. *Let P be the operator defined in (4.9) and $M = m^w$ the self-adjoint operator defined by the Weyl quantization of the symbol (4.10). Then, the operator M is uniformly bounded on $L^2(\mathbb{R}_v)$ with respect to the parameter $\xi \in \mathbb{R}$, and there exist some positive constants $0 < \varepsilon_0 \leq 1, c_1, c_2 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, u \in \mathcal{S}(\mathbb{R}_v), \xi \in \mathbb{R}$,*

$$\mathrm{Re}(Pu, (1-\varepsilon M)u) \geq c_1\|\mathcal{H}^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}_v)} + c_1\varepsilon\langle\xi\rangle^{\frac{2s}{2s+1}}\|u\|^2_{L^2(\mathbb{R}_v)} - c_2\|u\|^2_{L^2(\mathbb{R}_v)},$$

where $\mathcal{H} = -\Delta_v + \frac{v^2}{4}$ stands for the Harmonic Oscillator.

Lemma 4.4. *Let $(e_n)_{n\geq 0}$ be the Hermite basis of $L^2(\mathbb{R})$ describes in Section 4.2.1. We have*

$$\Gamma(e_k, e_l) = \alpha_{k,l}e_{k+l}, \quad k, l \geq 0$$

with

$$\alpha_{2n,m} = \sqrt{C^{2n}_{2n+m}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2n}(\cos\theta)^m d\theta, \quad n \geq 1,\ m \geq 0,$$
$$\alpha_{0,m} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)((\cos\theta)^m - 1)d\theta, \quad m \geq 1; \quad \alpha_{0,0} = \alpha_{2n+1,m} = 0, \quad n, m \geq 0,$$

where $C^k_n = \frac{n!}{k!(n-k)!}$ stands for the binomial coefficients.

Lemma 4.5. *We assume that the cross section satisfies (4.7) with $0 < s < 1$. Then, there exists a positive constant $C > 0$ such that for all $n \geq 1, m \geq 0$,*

$$0 \leq \alpha_{2n,m} = \sqrt{C^{2n}_{2n+m}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2n}(\cos\theta)^m d\theta \leq \frac{C}{n^{\frac{3}{4}}}\tilde{\mu}_{n,m},$$

where $\tilde{\mu}_{n,m} = (1+\frac{m}{n})^s(1+\frac{n}{m+1})^{\frac{1}{4}}$.

Lemma 4.6. *Let $r > 1/2$. There exists a positive constant $C_r > 0$ such that for all $f, g \in \mathcal{S}(\mathbb{R}_x), t \geq 0, 0 < \kappa \leq 1, m, n \geq 0$,*

$$\|\mathcal{M}_{\kappa,m+n}(t)([(\mathcal{M}_{\kappa,m}(t))^{-1}f][(\mathcal{M}_{\kappa,n}(t))^{-1}g])\|_{H^r} \leq C_r\|f\|_{H^r}\|g\|_{H^r}$$

with Fourier multiplier

$$\mathcal{M}_{\kappa,n}(t) = \frac{\exp\Big(t\Big(\sqrt{n+\frac{1}{2}} + \langle D_x\rangle\Big)^{\frac{2s}{2s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\sqrt{n+\frac{1}{2}} + \langle D_x\rangle\Big)^{\frac{2s}{2s+1}}\Big)},$$

where $\|\cdot\|_{H^r}$ stands for the Sobolev norm $H^r(\mathbb{R}_x)$.

Lemma 4.7. *Let $r > 1/2$. Then, there exists a positive constant $C_r > 0$ such that for all $f, g, h \in \mathcal{S}(\mathbb{R}^2_{x,v}), t \geq 0, 0 \leq c \leq 1, 0 \leq \kappa \leq 1, j_1, j_2 \geq 0$ with $j_1 + j_2 \leq 1$,*

$$\begin{aligned}
&|(\Gamma(f,g),h)_{L^2(\mathbb{R}^2_{x,v})}| \leq C_r\|f\|_{(r,0)}\|\mathcal{H}^{\frac{s}{2}}g\|_{L^2(\mathbb{R}^2_{x,v})}\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2(\mathbb{R}^2_{x,v})},\\
&|(\Gamma(\langle D_x\rangle f,g),h)_{L^2(\mathbb{R}^2_{x,v})}| \leq C_r\|f\|_{(1,0)}\|\mathcal{H}^{\frac{s}{2}}g\|_{(r,0)}\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2(\mathbb{R}^2_{x,v})},\\
&\Big|\Big((1+c\sqrt{\mathcal{H}}+c\langle D_x\rangle)^{-1}M_\kappa(t)\\
&\times\Gamma\big((M_\kappa(t))^{-1}(1+c\sqrt{\mathcal{H}})^{j_1}f,(M_\kappa(t))^{-1}(1+c\sqrt{\mathcal{H}})^{j_2}g\big),h\Big)_{(r,0)}\Big|\\
&\leq C_r\|f\|_{(r,0)}\|\mathcal{H}^{\frac{s}{2}}g\|_{(r,0)}\|\mathcal{H}^{\frac{s}{2}}h\|_{(r,0)}
\end{aligned}$$

with

$$\begin{aligned}
M_\kappa(t) &= \frac{\exp\Big(t\Big(\sqrt{\mathcal{H}} + \langle D_x\rangle\Big)^{\frac{2s}{2s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\sqrt{\mathcal{H}} + \langle D_x\rangle\Big)^{\frac{2s}{2s+1}}\Big)} = \sum_{n=0}^{+\infty}\mathcal{M}_{\kappa,n}\mathbb{P}_n\\
\mathcal{M}_{\kappa,n} &= \frac{\exp\Big(t\Big(\sqrt{n+\frac{1}{2}} + \langle D_x\rangle\Big)^{\frac{2s}{2s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\sqrt{n+\frac{1}{2}} + \langle D_x\rangle\Big)^{\frac{2s}{2s+1}}\Big)},
\end{aligned}$$

where $\mathbb{P}_n$ denotes the orthogonal projections onto the Hermite basis described in Section 4.2.1. In particular, we also have for all $f, g, h \in \mathcal{S}(\mathbb{R}^2_{x,v})$,

$$|(\Gamma(f,g),h)_{(r,0)}| \leq C_r\|f\|_{(r,0)}\|\mathcal{H}^{\frac{s}{2}}g\|_{(r,0)}\|\mathcal{H}^{\frac{s}{2}}h\|_{(r,0)}.$$

Lemma 4.8. *Let $r > 1/2$. Then, there exists a positive constant $c_r > 0$ such that for all $f, g \in \mathcal{S}(\mathbb{R}^2_{x,v})$,*

$$\|\mathcal{H}^{-s}\Gamma(f,g)\|_{(r,0)} \leq c_r\|f\|_{(r,0)}\|g\|_{(r,0)}.$$

In [117], the authors showed a key estimate on the Hermite functions.

Lemma 4.9. *It holds that*

$$1)\ \forall n,k,l\geq 0,\quad \|v^k\partial_v^l e_n\|_{L^2(\mathbb{R})}\leq 2^k\sqrt{\frac{(k+l+n)!}{n!}};$$

$$2)\ \forall r\geq\frac{1}{2},\forall\varepsilon>0,\forall n,k,l\geq 0,$$

$$\|v^k\partial_v^l e_n\|_{L^2(\mathbb{R})}\leq\sqrt{2}((1-\delta_{n,0})\exp(\varepsilon r n^{\frac{1}{2r}})+\delta_{n,0})\Big(\frac{2^{\frac{3}{2}+r}e^r}{\inf(\varepsilon^r,1)}\Big)^{k+l}(k!)^r(l!)^r,$$

where $\delta_{n,0}$ stands for the Kronecker delta i.e., $\delta_{n,0}=1$ if $n=0,\delta_{n,0}=0$ if $n\neq 0$.

4.3 Analysis of Kac collision operator

In this section, we present the trilinear estimates of the Kac collision operator which will be used in the subsequent analysis.

4.3.1 Trilinear estimates I

Due to the coercivity of the linearized Kac collision operator $\mathcal{K}$, we have the following result.

Lemma 4.10. *For the linear term $\mathcal{K}$, there exists a constant $C>0$ such that for the suitable function f,g,*

$$\frac{1}{C}\|\Delta_p\mathcal{H}^{\frac{s}{2}}f\|^2_{L^2(\mathbb{R}_v)}\leq(\Delta_p\mathcal{K}f,\Delta_p f)_{L^2(\mathbb{R}_v)}+\|\Delta_p f\|^2_{L^2(\mathbb{R}_v)}\leq C\|\Delta_p\mathcal{H}^{\frac{s}{2}}f\|^2_{L^2(\mathbb{R}_v)}$$

for each $p\geq -1$. Moreover, for $\sigma>0$ and $T>0$, it holds that

$$\sum_{p\geq -1}2^{p\sigma}\left(\int_0^T(\Delta_p\mathcal{K}g,\Delta_p g)_{L^2(\mathbb{R}^2_{x,v})}\,dt\right)^{1/2}\geq\frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^\sigma_{2,1})}-\|g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^\sigma_{2,1})}.$$

Proof. Observe that the inner product $(\cdot,\cdot)_{L^2(\mathbb{R}_v)}$ is with respect to v, Δ_p acts on x and the linearized non-cutoff Kac operator $\mathcal{K}$ is independent of x. Thus, the first inequality can be obtained by using the spectral estimate for $\mathcal{K}$ in Section 4.2.3. The second inequality just follows from the first inequality and the definition of Chemin-Lerner spaces. □

For the nonlinear term $\Gamma(f,g)$, in [117] the authors showed some trilinear estimates in Sobolev space. Here, we establish the trilinear estimates with minor changes, which will be used in the proof of local existence of solutions to (4.1).

Lemma 4.11. *Let $f,g,h\in\mathcal{S}(\mathbb{R}^2_{x,v})$. Then there exists a constant $C_0>0$ such that for all $0\leq\delta\leq 1, j_1,j_2\geq 0$ with $j_1+j_2\leq 1$, it holds that*

$$\begin{aligned}
&\Big|\Big((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma((1+\delta\sqrt{\mathcal{H}})^{j_1}f,(1+\delta\sqrt{\mathcal{H}})^{j_2}g),h\Big)_{L^2_{x,v}}\Big|\\
&\qquad\leq C_0\,\|f\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2_{x,v}},\\
&\Big|\Big((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma(f,\delta\langle D_x\rangle(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}g),h\Big)_{L^2_{x,v}}\Big|\\
&\qquad\leq C_0\|f\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2_{x,v}}.
\end{aligned}\tag{4.12}$$

Proof. For $f, g, h \in \mathcal{S}(\mathbb{R}^2_{x,v})$, we decompose these functions into the Hermite basis in the velocity variable

$$f = \sum_{n=0}^{+\infty} f_n(x) e_n(v), \quad f_n = \langle f(x,\cdot), e_n \rangle_{L^2(\mathbb{R}_v)}$$

and similar decomposition for g, h. Notice that

$$\|f\|_{L^2_v L^2_x} = \Big(\sum_{n=0}^{+\infty} \|f_n\|^2_{L^2(\mathbb{R}_x)} \Big)^{\frac{1}{2}}, \quad \|\mathcal{H}^m f\|_{L^2_v L^2_x} = \Big(\sum_{n=0}^{+\infty} \Big(n+\frac{1}{2}\Big)^{2m} \|f_n\|^2_{L^2(\mathbb{R}_x)} \Big)^{\frac{1}{2}} \tag{4.13}$$

with $m \in \mathbb{R}$. Following from Lemma 4.4 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\Big| \Big((1+\delta\sqrt{\mathcal{H}} + \delta\langle D_x\rangle)^{-1} \Gamma((1+\delta\sqrt{\mathcal{H}})^{j_1} f, \Big(1+\delta\sqrt{\mathcal{H}}\Big)^{j_2} g), h \Big)_{L^2_{x,v}} \Big| \\
&= \Big| \sum_{n=0}^{+\infty} \sum_{\substack{k+l=n \\ k,l\geq 0}} \alpha_{k,l} \Big(\Big(1+\delta\sqrt{n+\frac{1}{2}} + \delta\langle D_x\rangle\Big)^{-1} \Big(1+\delta\sqrt{k+\frac{1}{2}}\Big)^{j_1} \\
&\qquad \times \Big(1+\delta\sqrt{l+\frac{1}{2}}\Big)^{j_2} f_k g_l, h_n \Big)_{L^2_x} \Big| \\
&\leq \sum_{n=0}^{+\infty} |\alpha_{0,n}| \|f_0\|_{L^\infty_x} \|g_n\|_{L^2_x} \|h_n\|_{L^2_x} + \sum_{n=0}^{+\infty} \sum_{\substack{2k+l=n \\ k\geq 1, l\geq 0}} |\alpha_{2k,l}| \|f_{2k}\|_{L^\infty_x} \|g_l\|_{L^2_x} \|h_n\|_{L^2_x},
\end{aligned}$$

where we used that

$$\Big\| \Big(1+\delta\sqrt{n+\frac{1}{2}} + \delta\langle D_x\rangle\Big)^{-1} \Big(1+\delta\sqrt{k+\frac{1}{2}}\Big)^{j_1} \Big(1+\delta\sqrt{l+\frac{1}{2}}\Big)^{j_2} \Big\|_{\mathcal{L}(L^2(\mathbb{R}_x))} \leq 1,$$

since $k+l=n$ and $j_1, j_2 \geq 0$ with $j_1 + j_2 \leq 1$. Under the assumption (4.7), we use the formula (A.17) in [115] (Section A.4.2), that is

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1-(\cos\theta)^n) d\theta \sim \frac{2^{1+s}}{s} \mathbf{\Gamma}(1-s) n^s,$$

when $n \to +\infty$, where $\mathbf{\Gamma}$ denotes the Gamma function. It follows from Lemma 4.5 and the above formula that for $f, g, h \in \mathcal{S}(\mathbb{R}^2_{x,v})$,

$$\begin{aligned}
&\Big| \Big((1+\delta\sqrt{\mathcal{H}} + \delta\langle D_x\rangle)^{-1} \Gamma((1+\delta\sqrt{\mathcal{H}})^{j_1} f, \Big(1+\delta\sqrt{\mathcal{H}}\Big)^{j_2} g), h \Big)_{L^2_{x,v}} \Big| \\
&\lesssim \|f_0\|_{L^\infty_x} \sum_{n=0}^{+\infty} (n+\frac{1}{2})^s \|g_n\|_{L^2_x} \|h_n\|_{L^2_x} + \sum_{n=0}^{+\infty} \|h_n\|_{L^2_x} \Big(\sum_{\substack{2k+l=n \\ k\geq 1, l\geq 0}} \frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}} \|f_{2k}\|_{L^\infty_x} \|g_l\|_{L^2_x} \Big) \\
&\triangleq I_1 + I_2.
\end{aligned}$$

By using (4.13), we have

$$\begin{aligned} I_1 &\le \|f_0\|_{L^\infty_x}\Big(\sum_{n=0}^{+\infty}(n+\frac{1}{2})^s\|g_n\|^2_{L^2_x}\Big)^{\frac{1}{2}}\Big(\sum_{n=0}^{+\infty}(n+\frac{1}{2})^s\|h_n\|^2_{L^2_x}\Big)^{\frac{1}{2}} \\ &\le \|f_0\|_{L^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2_{x,v}}. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} I_2 &= \sum_{k\ge1,l\ge0}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}}\|f_{2k}\|_{L^\infty_x}\|g_l\|_{L^2_x}\|h_{2k+l}\|_{L^2_x} \\ &= \sum_{l=0}^{+\infty}(l+\frac{1}{2})^{\frac{s}{2}}\|g_l\|_{L^2_x}\Big(\sum_{k=1}^{+\infty}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}(l+\frac{1}{2})^{\frac{s}{2}}}\|f_{2k}\|_{L^\infty_x}\|h_{2k+l}\|_{L^2_x}\Big) \\ &\le \|f\|_{L^2_v(L^\infty_x)}\|\mathcal{H}^{\frac{s}{2}}g\|_{L^2_{x,v}}\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}}. \end{aligned}$$

Here, we may calculate that

$$\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}} = \Big[\sum_{n=0}^{+\infty}\|h_n\|^2_{L^2_x}\Big(\sum_{\substack{2k+l=n\\k\ge1,l\ge0}}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\Big)\Big]^{\frac{1}{2}}.$$

Since

$$\tilde{\mu}_{k,l}\lesssim k^{\frac{1}{4}}\text{ when }k\ge l,k\ge1,l\ge0;\quad \tilde{\mu}_{k,l}\lesssim(l+\frac{1}{2})^s\text{ when }1\le k\le l,$$

it follows from Lemma 4.5 that

$$\sum_{\substack{2k+l=n\\k\ge1,l\ge0}}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\lesssim\sum_{\substack{2k+l=n\\k\ge1,l\ge0\\k\ge l}}\frac{k^{\frac{1}{2}}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}+\sum_{\substack{2k+l=n\\k\le1,l\ge0\\k\le l}}\frac{(l+\frac{1}{2})^s}{k^{\frac{3}{2}}}\lesssim(n+\frac{1}{2})^s. \tag{4.14}$$

Thus, we are led to

$$\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}}\lesssim\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2_{x,v}}.$$

We can conclude that there a positive constant $C_0>0$ such that

$$\begin{aligned} &\Big|\Big((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma((1+\delta\sqrt{\mathcal{H}})^{j_1}f,(1+\delta\sqrt{\mathcal{H}})^{j_2}g),h\Big)_{L^2_{x,v}}\Big| \\ &\quad\le C_0\,\|f\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2_{x,v}}, \end{aligned}$$

which leads to the first inequality in (4.12). Similarly, we have

$$\begin{aligned}
&\Big|\Big((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma(f,\delta\langle D_x\rangle(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}g),h\Big)_{x,v}\Big| \\
&=\Big|\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}\alpha_{k,l}\int_{\mathbb{R}}\frac{1}{(1+\delta\sqrt{n+\frac{1}{2}}+\delta\langle D_x\rangle)}(f_k\frac{\delta\langle D_x\rangle}{(1+\delta\sqrt{l+\frac{1}{2}}+\delta\langle D_x\rangle)}g_l)\overline{h_n(x)}dx\Big| \\
&\leq\sum_{n=0}^{+\infty}|\alpha_{0,n}|\|f_0\|_{L^\infty_x}\|g_n\|_{L^2_x}\|h_{\delta,n}\|_{L^2_x}+\sum_{n=0}^{+\infty}\sum_{\substack{2k+l=n\\k\geq 1,l\geq 0}}|\alpha_{2k,l}|\|f_{2k}\|_{L^\infty_x}\|g_l\|_{L^2_x}\|h_n\|_{L^2_x},
\end{aligned}$$

where we used

$$\begin{aligned}
&\Big\|\delta\langle D_x\rangle\Big(1+\delta\sqrt{n+\frac{1}{2}}+\delta\langle D_x\rangle\Big)^{-1}\Big\|_{\mathcal{L}(L^2(\mathbb{R}_x))}\leq 1,\\
&\Big\|\Big(1+\delta\sqrt{n+\frac{1}{2}}+\delta\langle D_x\rangle\Big)^{-1}\Big\|_{\mathcal{L}(L^2(\mathbb{R}_x))}\leq 1.
\end{aligned}$$

Hence, by proceeding the similar procedure, we can obtain the second inequality in (4.12). □

Putting $\delta=0$ in Lemma 4.11, which coincides with Lemma 3.5 of [116].

Remark 4.2. Let $f,g,h\in\mathcal{S}(\mathbb{R}^2_{x,v})$, then there exists a constant $C_0>0$ such that

$$\Big|(\Gamma(f,g),h)_{L^2_{x,v}}\Big|\leq C_0\,\|f\|_{L^2_vL^\infty_x}\,\|\mathcal{H}^{\frac{s}{2}}g\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}h\|_{L^2_{x,v}}.$$

By the similar proof as in [117], we also have

Lemma 4.12. *Let $f,g,h\in\mathcal{S}(\mathbb{R}^2_{x,v})$. Then there exists a constant $C_0>0$ such that*

$$\big\|\mathcal{H}^{-s}\Gamma(f,g)\big\|_{L^2_{x,v}}\leq C_0\,\|f\|_{L^2_vL^\infty_x}\,\|g\|_{L^2_{x,v}}.$$

We prove the following result in order to estimate the nonlinear collision operator in the framework of Besov spaces.

Lemma 4.13. *There exists a constant $C_1>0$ such that for all $f,g\in\mathcal{S}(\mathbb{R}_x),t\geq 0,0<\kappa\leq 1,m,n\geq 0$,*

$$\|\mathcal{G}_{\kappa,m+n}(t)([(\mathcal{G}_{\kappa,m}(t))^{-1}f][(\mathcal{G}_{\kappa,n}(t))^{-1}g])\|_{L^2_x}\leq C_1\|f\|_{B^{1/2}_{2,1}}\|g\|_{L^2_x}\tag{4.15}$$

with the Fourier multiplier

$$\mathcal{G}_{\kappa,n}(t)=\frac{\exp\Big(t\Big((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}.\tag{4.16}$$

Proof. Notice that the operator $\mathcal{G}_{\kappa,n}(t)$ is a bounded isomorphism of $L^2(\mathbb{R}_x)$ such that

$$(\mathcal{G}_{\kappa,n}(t))^{-1}=\frac{1+\kappa\exp\Big(t\Big((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{\exp\Big(t(n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}.$$

Set

$$h=\mathcal{G}_{\kappa,m+n}(t)([(\mathcal{G}_{\kappa,m}(t))^{-1}f][(\mathcal{G}_{\kappa,n}(t))^{-1}g]),$$

then we have

$$\begin{aligned}\hat{h}(\xi)&=\frac{\exp\Big(t\Big(\big(m+n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\big(m+n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}\mathcal{F}(([[(\mathcal{G}_{\kappa,m}(t))^{-1}f][(\mathcal{G}_{\kappa,n}(t))^{-1}g]))\\&=\frac{1}{2\pi}\frac{\exp\Big(t\Big(\big(m+n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\big(m+n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}\mathcal{F}((\mathcal{G}_{\kappa,m}(t))^{-1}f)*\mathcal{F}((\mathcal{G}_{\kappa,n}(t))^{-1}g),\end{aligned}\tag{4.17}$$

where $\mathcal{F}$ denotes the Fourier transform. Consider the increasing function

$$Z(x)=\frac{e^x}{1+\kappa e^x},$$

we can calculate and obtain

$$\forall x,y\geq 0,\quad \frac{Z(x+y)}{Z(x)Z(y)}=\kappa+\frac{1-\kappa}{1+\kappa e^{x+y}}+\frac{\kappa(e^x+e^y)}{1+\kappa e^{x+y}}\leq 1+\frac{1}{e^x}+\frac{1}{e^y}\leq 3,$$

which implies that the function $Z(x)$ satisfies the inequality

$$\forall x,y\geq 0,\quad Z(x+y)\leq 3Z(x)Z(y).\tag{4.18}$$

Since for all $m,n\geq 0,\xi,\eta\in\mathbb{R}$,

$$\begin{aligned}&\Big(\Big(m+n+\frac{1}{2}\Big)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\\&\leq\Big(\Big(m+\frac{1}{2}\Big)^{\frac{s+1}{2}}+\langle\eta\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}+\Big(\Big(n+\frac{1}{2}\Big)^{\frac{s+1}{2}}+\langle\xi-\eta\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}},\end{aligned}$$

by using the inequality (4.18), we obtain

$$\begin{aligned}
&\frac{\exp\Big(t\Big(\big(m+n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\big(m+n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}\\
&\le\frac{3\exp\Big(t\Big(\big(m+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\eta\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\big(m+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\eta\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}\\
&\quad\times\frac{\exp\Big(t\Big(\big(n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi-\eta\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\big(n+\frac{1}{2}\big)^{\frac{s+1}{2}}+\langle\xi-\eta\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}.
\end{aligned}\tag{4.19}$$

Then it follows from (4.17) and (4.19) that

$$\begin{aligned}
\|h\|_{L^2_x}&\le\frac{3}{(2\pi)^{\frac{3}{2}}}\||\hat f|*|\hat g|\|_{L^2_x}=\frac{3}{(2\pi)^{\frac{3}{2}}}\|\mathcal{F}\mathcal{F}^{-1}(|\hat f|*|\hat g|)\|_{L^2_x}\\
&=\frac{3}{2\pi}\|\mathcal{F}^{-1}(|\hat f|*|\hat g|)\|_{L^2_x}=3\|\mathcal{F}^{-1}(|\hat f|)\mathcal{F}^{-1}(|\hat g|)\|_{L^2_x}\\
&\le 3\|\mathcal{F}^{-1}(|\hat f|)\|_{L^\infty_x}\|\mathcal{F}^{-1}(|\hat g|)\|_{L^2_x}\le C_1\|f\|_{B^{1/2}_{2,1}}\|g\|_{L^2_x},
\end{aligned}$$

which leads to the desired (4.15). Here we used $\|\mathcal{F}^{-1}(|\hat u|)\|_{L^\infty_x}\le C_d\|u\|_{B^{d/2}_{2,1}}$ on $\mathbb{R}^d(d\ge 1)$ for $u\in\mathcal{S}(\mathbb{R}^d)$. Indeed,

$$\begin{aligned}
\|\mathcal{F}^{-1}(|\hat u|)\|_{L^\infty_x}&\le\|\hat u\|_{L^1_x}\le\sum_{p\ge -1}\|\widehat{\Delta_p u}\|_{L^1_x}\\
&=\int_{|\xi|\le\frac{4}{3}}|\widehat{\Delta_{-1}u}|d\xi+\sum_{p\ge 0}\int_{\frac{3}{4}2^p\le|\xi|\le\frac{8}{3}2^p}|\widehat{\Delta_p u}|d\xi\\
&\le\sum_{p\ge -1}C_d2^{\frac{d}{2}p}\|\Delta_p u\|_{L^2_x}=C_d\|u\|_{B^{d/2}_{2,1}},
\end{aligned}$$

where $C_d>0$ is a positive constant depending on the dimension d. Hence, the proof of Lemma 4.13 is finished. □

4.3.2 Trilinear estimates II

Now, we establish the key trilinear estimates for nonlinear term $\Gamma(f,g)$.

Lemma 4.14. *Let $f,g,h\in\mathcal{S}(\mathbb{R}^2_{x,v})$. Then it holds that for all $t\ge 0$ and $0<\kappa\le 1$, $p,j,j'\ge$*

-1,

$$\begin{aligned}
&\Big|\big(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h\big)_{L^2_{x,v}}\Big|\\
&\lesssim \sum_{|j-p|\le 4}\|S_{j-1}f\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_j g\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}\\
&\quad+\sum_{|j-p|\le 4}\|\Delta_j f\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}S_{j-1}g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}\\
&\quad+\sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\|\Delta_j f\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}\Delta_{j'} g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}},
\end{aligned}\tag{4.20}$$

with

$$G_\kappa(t)=\frac{\exp\Big(t\Big(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}.$$

Proof. Firstly, we have

$$\begin{aligned}
\Delta_p(uv)=&\sum_{|j-p|\le 4}\Delta_p(S_{j-1}u\Delta_j v)+\sum_{|j-p|\le 4}\Delta_p(\Delta_j uS_{j-1}v)\\
&+\sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\Delta_p(\Delta_j u\Delta_{j'}v).
\end{aligned}\tag{4.21}$$

In fact, recalling Bony's decomposition, one can write $\Delta_p(uv)$ as follows

$$\Delta_p(uv)=\Delta_p\left(\mathbf{T}_u v+\mathbf{T}_v u+\mathbf{R}(u,v)\right),$$

where $\mathbf{T}$ and $\mathbf{R}$ are called as "paraproduct" and "remainder". They are defined formally by

$$\mathbf{T}_u v=\sum_j S_{j-1}u\Delta_j v,\quad \mathbf{R}(u,v)=\sum_j\sum_{|j-j'|\le 1}\Delta_j u\Delta_{j'}v,\quad \text{for } u,v\in\mathcal{S}'(\mathbb{R}).$$

Notice that

$$\begin{aligned}
\Delta_p\mathbf{T}_u v&=\sum_j\Delta_p(S_{j-1}u\Delta_j v)=\sum_{|j-p|\le 4}\Delta_p(S_{j-1}u\Delta_j v),\\
\Delta_p\mathbf{T}_v u&=\sum_j\Delta_p(\Delta_j uS_{j-1}v)=\sum_{|j-p|\le 4}\Delta_p(\Delta_j uS_{j-1}v),\\
\Delta_p\mathbf{R}(u,v)&=\sum_j\sum_{|j-j'|\le 1}\Delta_p(\Delta_j u\Delta_{j'}v)=\sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\Delta_p(\Delta_j u\Delta_{j'}v),
\end{aligned}$$

which implies the equality in (5.34). With the eigenfunctions of the Harmonic Oscillator, one has

$$G_\kappa(t)=\frac{\exp\Big(t\Big(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}{1+\kappa\exp\Big(t\Big(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)}=\sum_{n=0}^{+\infty}\mathcal{G}_{\kappa,n}\mathbb{P}_n,$$

where $\mathcal{G}_{\kappa,n}$ is given by (4.16) and $\mathbb{P}_n$ denotes the orthogonal projections onto the Hermite basis described in Section 4.2.1. For $f, g \in \mathcal{S}(\mathbb{R}^2_{x,v})$, following from Lemma 4.4 and then using (5.34) with $u = (\mathcal{G}_{\kappa,k}(t))^{-1} f_k, v = (\mathcal{G}_{\kappa,l}(t))^{-1} g_l$, we obtain

$$
\begin{aligned}
&\left(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h\right)_{L^2_{x,v}}\\
&=\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}\alpha_{k,l}\left(\mathcal{G}_{\kappa,n}(t)\Delta_p\left([(\mathcal{G}_{\kappa,k}(t))^{-1}f_k][(\mathcal{G}_{\kappa,l}(t))^{-1}g_l]\right),\Delta_p h_n\right)_{L^2_x}\\
&=\sum_{|j-p|\leq 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}\alpha_{k,l}\left(\mathcal{G}_{\kappa,n}(t)\Delta_p\left(S_{j-1}[(\mathcal{G}_{\kappa,k}(t))^{-1}f_k]\Delta_j[(\mathcal{G}_{\kappa,l}(t))^{-1}g_l]\right),\Delta_p h_n\right)_{L^2_x}\\
&+\sum_{|j-p|\leq 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}\alpha_{k,l}\left(\mathcal{G}_{\kappa,n}(t)\Delta_p\left(\Delta_j[(\mathcal{G}_{\kappa,k}(t))^{-1}f_k]S_{j-1}[(\mathcal{G}_{\kappa,l}(t))^{-1}g_l]\right),\Delta_p h_n\right)_{L^2_x}\\
&+\sum_{\max(j,j')\geq p-2}\sum_{|j-j'|\leq 1}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}\alpha_{k,l}\left(\mathcal{G}_{\kappa,n}(t)\Delta_p\left(\Delta_j[(\mathcal{G}_{\kappa,k}(t))^{-1}f_k]\Delta_{j'}[(\mathcal{G}_{\kappa,l}(t))^{-1}g_l]\right),\Delta_p h_n\right)_{L^2_x}\\
&\triangleq A_1+A_2+A_3.
\end{aligned}
$$

For A_1, since $[S_{j-1},(\mathcal{G}_{\kappa,k}(t))^{-1}]=0$ and $[\Delta_j,(\mathcal{G}_{\kappa,l}(t))^{-1}]=0$ with $j\geq 1, 0<\kappa\leq 1, k\geq 0$, we obtain

$$
\begin{aligned}
&|A_1|\\
&\leq\sum_{|j-p|\leq 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}|\alpha_{k,l}|\|\mathcal{G}_{\kappa,n}(t)\left(S_{j-1}[(\mathcal{G}_{\kappa,k}(t))^{-1}f_k]\Delta_j[(\mathcal{G}_{\kappa,l}(t))^{-1}g_l]\right)\|_{L^2_x}\|\Delta_p^2 h_n\|_{L^2_x}\\
&=\sum_{|j-p|\leq 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}|\alpha_{k,l}|\|\mathcal{G}_{\kappa,n}(t)\left((\mathcal{G}_{\kappa,k}(t))^{-1}[S_{j-1}f_k](\mathcal{G}_{\kappa,l}(t))^{-1}[\Delta_j g_l]\right)\|_{L^2_x}\|\Delta_p^2 h_n\|_{L^2_x}\\
&\leq\sum_{|j-p|\leq 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\geq 0}}|\alpha_{k,l}|\|S_{j-1}f_k\|_{B^{1/2}_{2,1}}\|\Delta_j g_l\|_{L^2_x}\|\Delta_p h_n\|_{L^2_x}\\
&\leq\sum_{|j-p|\leq 4}\sum_{n=0}^{+\infty}|\alpha_{0,n}|\|S_{j-1}f_0\|_{B^{1/2}_{2,1}}\|\Delta_j g_n\|_{L^2_x}\|\Delta_p h_n\|_{L^2_x}\\
&\quad+\sum_{|j-p|\leq 4}\sum_{n=0}^{+\infty}\sum_{\substack{2k+l=n\\k\geq 1,l\geq 0}}|\alpha_{2k,l}|\|S_{j-1}f_{2k}\|_{B^{1/2}_{2,1}}\|\Delta_j g_l\|_{L^2_x}\|\Delta_p h_n\|_{L^2_x},
\end{aligned}
$$

where we used Lemma 4.13 and Lemma 4.4 in last three and two line, respectively. Bounding

A_2, A_3 are similar, we have

$$
\begin{aligned}
&|A_2| \\
&\le \sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\ge 0}} |\alpha_{k,l}|\|\mathcal{G}_{\kappa,n}(t)\left(\Delta_j[(\mathcal{G}_{\kappa,k}(t))^{-1}f_k]S_{j-1}[(\mathcal{G}_{\kappa,l}(t))^{-1}g_l]\right)\|_{L^2_x}\|\Delta_p^2 h_n\|_{L^2_x} \\
&= \sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\ge 0}} |\alpha_{k,l}|\|\mathcal{G}_{\kappa,n}(t)\left((\mathcal{G}_{\kappa,k}(t))^{-1}[\Delta_j f_k](\mathcal{G}_{\kappa,l}(t))^{-1}[S_{j-1}g_l]\right)\|_{L^2_x}\|\Delta_p^2 h_n\|_{L^2_x} \\
&\le \sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\ge 0}} |\alpha_{k,l}|\|\Delta_j f_k\|_{L^2_x}\|S_{j-1}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x} \\
&\le \sum_{|j-p|\le 4}\sum_{n=0}^{+\infty} |\alpha_{0,n}|\|\Delta_j f_0\|_{L^2_x}\|S_{j-1}g_n\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x} \\
&\quad + \sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\sum_{\substack{2k+l=n\\k\ge 1,l\ge 0}} |\alpha_{2k,l}|\|\Delta_j f_{2k}\|_{L^2_x}\|S_{j-1}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x},
\end{aligned}
$$

$$
\begin{aligned}
&|A_3| \\
&\le \sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\ge 0}} |\alpha_{k,l}|\|\mathcal{G}_{\kappa,n}(t)\left(\Delta_j[(\mathcal{G}_{\kappa,k}(t))^{-1}f_k]\Delta_{j'}[(\mathcal{G}_{\kappa,l}(t))^{-1}g_l]\right)\|_{L^2_x}\|\Delta_p^2 h_n\|_{L^2_x} \\
&= \sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\ge 0}} |\alpha_{k,l}|\|\mathcal{G}_{\kappa,n}(t)\left((\mathcal{G}_{\kappa,k}(t))^{-1}[\Delta_j f_k](\mathcal{G}_{\kappa,l}(t))^{-1}[\Delta_{j'}g_l]\right)\|_{L^2_x}\|\Delta_p^2 h_n\|_{L^2_x} \\
&\le \sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty}\sum_{\substack{k+l=n\\k,l\ge 0}} |\alpha_{k,l}|\|\Delta_j f_k\|_{L^2_x}\|\Delta_{j'}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x} \\
&\le \sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty} |\alpha_{0,n}|\|\Delta_j f_0\|_{L^2_x}\|\Delta_{j'}g_n\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x} \\
&\quad + \sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty}\sum_{\substack{2k+l=n\\k\ge 1,l\ge 0}} |\alpha_{2k,l}|\|\Delta_j f_{2k}\|_{L^2_x}\|\Delta_{j'}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x}.
\end{aligned}
$$

Combining the three estimates for A_1, A_2, A_3 implies that

$$
\begin{aligned}
&\left|\left(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h\right)_{L^2_{x,v}}\right| \\
&\le \sum_{|j-p|\le 4}\sum_{n=0}^{+\infty} |\alpha_{0,n}|\|S_{j-1}f_0\|_{B^{1/2}_{2,1}}\|\Delta_j g_n\|_{L^2_x}\|\Delta_p h_n\|_{L^2_x}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}|\alpha_{0,n}|\|\Delta_j f_0\|_{L^2_x}\|S_{j-1}g_n\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x}\\
&+\sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty}|\alpha_{0,n}|\|\Delta_j f_0\|_{L^2_x}\|\Delta_{j'}g_n\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x}\\
&+\sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\sum_{\substack{2k+l=n\\k\ge 1,l\ge 0}}|\alpha_{2k,l}|\|S_{j-1}f_{2k}\|_{B^{1/2}_{2,1}}\|\Delta_j g_l\|_{L^2_x}\|\Delta_p h_n\|_{L^2_x}\\
&+\sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\sum_{\substack{2k+l=n\\k\ge 1,l\ge 0}}|\alpha_{2k,l}|\|\Delta_j f_{2k}\|_{L^2_x}\|S_{j-1}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x}\\
&+\sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty}\sum_{\substack{2k+l=n\\k\ge 1,l\ge 0}}|\alpha_{2k,l}|\|\Delta_j f_{2k}\|_{L^2_x}\|\Delta_{j'}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x}\\
&\triangleq J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}
$$

By using the formula (4.8) and (4.13), for J_1, J_2 and J_3, we arrive at

$$
\begin{aligned}
&J_1+J_2+J_3\\
&\le\sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\big(n+\frac{1}{2}\big)^s\|S_{j-1}f_0\|_{B^{1/2}_{2,1}}\|\Delta_j g_n\|_{L^2_x}\|\Delta_p h_n\|_{L^2_x}\\
&\quad+\sum_{|j-p|\le 4}\sum_{n=0}^{+\infty}\big(n+\frac{1}{2}\big)^s\|\Delta_j f_0\|_{L^2_x}\|S_{j-1}g_n\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x}\\
&\quad+\sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\sum_{n=0}^{+\infty}\big(n+\frac{1}{2}\big)^s\|\Delta_j f_0\|_{L^2_x}\|\Delta_{j'}g_n\|_{B^{1/2}_{2,1}}\|\Delta_p h_n\|_{L^2_x}\\
&\le\sum_{|j-p|\le 4}\|S_{j-1}f_0\|_{B^{1/2}_{2,1}}\|\mathcal{H}^{\frac{s}{2}}\Delta_j g\|_{L^2_v L^2_x}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}\\
&\quad+\sum_{|j-p|\le 4}\|\Delta_j f_0\|_{L^2_x}\|\mathcal{H}^{\frac{s}{2}}S_{j-1}g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}\\
&\quad+\sum_{\max(j,j')\ge p-2}\sum_{|j-j'|\le 1}\|\Delta_j f_0\|_{L^2_x}\|\mathcal{H}^{\frac{s}{2}}\Delta_{j'}g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}.
\end{aligned}
$$

On the other hand, for J_4, J_5 and J_6, by using the Lemma 4.5 once again, we obtain

$$
\begin{aligned}
&J_4+J_5+J_6\\
&\le\sum_{|j-p|\le 4}\sum_{k\ge 1,l\ge 0}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}}\|S_{j-1}f_{2k}\|_{B^{1/2}_{2,1}}\|\Delta_j g_l\|_{L^2_x}\|\Delta_p h_{2k+l}\|_{L^2_x}\\
&\quad+\sum_{|j-p|\le 4}\sum_{k\ge 1,l\ge 0}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}}\|\Delta_j f_{2k}\|_{L^2_x}\|S_{j-1}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_{2k+l}\|_{L^2_x}
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
&+\sum_{\max(j,j')\geq p-2}\sum_{|j-j'|\leq 1}\sum_{k\geq 1,l\geq 0}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}}\|\Delta_j f_{2k}\|_{L^2_x}\|\Delta_{j'}g_l\|_{B^{1/2}_{2,1}}\|\Delta_p h_{2k+l}\|_{L^2_x}\\
&\leq \sum_{|j-p|\leq 4}\sum_{l=0}\Big((l+\frac{1}{2})^{\frac{s}{2}}\|\Delta_j g_l\|_{L^2_x}\Big)\Big(\sum_{k=1}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}(l+\frac{1}{2})^{\frac{s}{2}}}\|S_{j-1}f_{2k}\|_{B^{1/2}_{2,1}}\|\Delta_p h_{2k+l}\|_{L^2_x}\Big)\\
&\quad+\sum_{|j-p|\leq 4}\sum_{l=0}\Big((l+\frac{1}{2})^{\frac{s}{2}}\|S_{j-1}g_l\|_{B^{1/2}_{2,1}}\Big)\Big(\sum_{k=1}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}(l+\frac{1}{2})^{\frac{s}{2}}}\|\Delta_j f_{2k}\|_{L^2_x}\|\Delta_p h_{2k+l}\|_{L^2_x}\Big)\\
&\quad+\sum_{\max(j,j')\geq p-2}\sum_{|j-j'|\leq 1}\sum_{l=0}\Big((l+\frac{1}{2})^{\frac{s}{2}}\|\Delta_{j'} g_l\|_{B^{1/2}_{2,1}}\Big)\Big(\sum_{k=1}\frac{\tilde{\mu}_{k,l}}{k^{\frac{3}{4}}(l+\frac{1}{2})^{\frac{s}{2}}}\|\Delta_j f_{2k}\|_{L^2_x}\|\Delta_p h_{2k+l}\|_{L^2_x}\Big)\\
&\leq \sum_{|j-p|\leq 4}\|S_{j-1}f\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_j g\|_{L^2_{x,v}}\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|\Delta_p h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}}\\
&\quad+\sum_{|j-p|\leq 4}\|\Delta_j f\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}S_{j-1}g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|\Delta_p h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}}\\
&\quad+\sum_{\max(j,j')\geq p-2}\sum_{|j-j'|\leq 1}\|\Delta_j f\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}\Delta_{j'}g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|\Delta_p h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Here, we may calculate that

$$
\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|\Delta_p h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}}=\Big[\sum_{n=0}^{+\infty}\|\Delta_p h_n\|^2_{L^2_x}\Big(\sum_{\substack{2k+l=n\\ k\geq 1,l\geq 0}}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\Big)\Big]^{\frac{1}{2}}.
$$

It follows from (4.14) again that

$$
\Big(\sum_{l=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{\tilde{\mu}^2_{k,l}}{k^{\frac{3}{2}}(l+\frac{1}{2})^s}\|\Delta_p h_{2k+l}\|^2_{L^2_x}\Big)^{\frac{1}{2}}\lesssim\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}. \tag{4.23}
$$

Therefore, we conclude from the above estimates of J_1-J_6 and (4.23) that

$$
\begin{aligned}
&\Big|(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h)_{L^2_{x,v}}\Big|\\
&\lesssim\sum_{|j-p|\leq 4}\|S_{j-1}f\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_j g\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}\\
&\quad+\sum_{|j-p|\leq 4}\|\Delta_j f\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}S_{j-1}g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}\\
&\quad+\sum_{\max(j,j')\geq p-2}\sum_{|j-j'|\leq 1}\|\Delta_j f\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}\Delta_{j'}g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}},
\end{aligned}
$$

which is (4.20) exactly. The proof of Lemma 4.14 is completed. □

Apply Lemma 2.5 to the above Lemma 4.14, and then combining the last two terms on the right hand side of the resultant inequality, we obtain the following remark which will be used mainly in the subsequent calculation.

Remark 4.3. Let $f, g, h \in \mathcal{S}(\mathbb{R}^2_{x,v})$. Then it holds that for all $t \geq 0$ and $0 < \kappa \leq 1, p, j \geq -1$,

$$\begin{aligned}
&\left|\left(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h\right)_{L^2_{x,v}}\right| \\
&\lesssim \sum_{|j-p|\leq 4} \|f\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}}\Delta_j g\|_{L^2_{x,v}} \|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}} \\
&\quad + \sum_{j\geq p-4} \|\Delta_j f\|_{L^2_{x,v}} \|\mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}},
\end{aligned}$$

where $G_\kappa(t)$ is given in Lemma 4.14.

Take $\kappa = t = 0$ in Lemma 4.14 and use the similar calculation, we have the following trilinear estimate without exponential weights.

Remark 4.4. Let $f, g, h \in \mathcal{S}(\mathbb{R}^2_{x,v})$. Then it holds that for $p, j \geq -1$,

$$\begin{aligned}
&\left|(\Delta_p\Gamma(f,g),\Delta_p h)_{L^2_{x,v}}\right| \\
&\lesssim \sum_{|j-p|\leq 4} \|f\|_{L^2_v L^\infty_x} \|\mathcal{H}^{\frac{s}{2}}\Delta_j g\|_{L^2_{x,v}} \|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}} \\
&\quad + \sum_{j\geq p-4} \|\Delta_j f\|_{L^2_{x,v}} \|\mathcal{H}^{\frac{s}{2}} g\|_{L^2_v L^\infty_x} \|\mathcal{H}^{\frac{s}{2}}\Delta_p h\|_{L^2_{x,v}}.
\end{aligned}$$

Having Remark 4.3, we can obtain the following result.

Lemma 4.15. *Assume $\sigma > 0, T > 0$ and $0 < \kappa \leq 1$. Let $f = f(t,x,v)$, $g = g(t,x,v)$ and $h = h(t,x,v)$ be three suitably functions such that all norms on the right of the following inequalities are well defined. Then there exists a constant $C_1 > 0$ such that*

$$\begin{aligned}
&\sum_{p\geq -1} 2^{p\sigma}\left[\int_0^T \left|\left(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h\right)_{L^2_{x,v}}\right| dt\right]^{1/2} \\
&\leq C_1 \|f\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} g\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{\sigma}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} h\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{\sigma}_{2,1})} \\
&\quad + C_1 \|f\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{\sigma}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} g\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} h\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{\sigma}_{2,1})}.
\end{aligned}$$

Proof. Based on Remark 4.3, it follows from Cauchy-Schwarz inequality that

$$\begin{aligned}
&\sum_{p\geq -1} 2^{p\sigma}\left[\int_0^T \left|\left(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h\right)_{L^2_{x,v}}\right| dt\right]^{1/2} \\
&\lesssim \sum_{p\geq -1} 2^{p\sigma}\left[\sum_{|j-p|\leq 4}\int_0^T \|f\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \left\|\mathcal{H}^{\frac{s}{2}}\Delta_j g\right\|_{L^2_{x,v}} \left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\right\|_{L^2_{x,v}} dt\right]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{p\geq -1}2^{p\sigma}\left[\sum_{j\geq p-4}\int_0^T\|\Delta_j f\|_{L^2_{x,v}}\left\|\mathcal{H}^{\frac{s}{2}}g\right\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\right\|_{L^2_{x,v}}dt\right]^{1/2}\\
&\lesssim \|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\sum_{p\geq -1}2^{p\sigma}\Big(\sum_{|j-p|\leq 4}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_j g\right\|_{L^2_TL^2_{x,v}}\Big)^{1/2}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\right\|^{1/2}_{L^2_TL^2_{x,v}}\\
&\quad+\left\|\mathcal{H}^{\frac{s}{2}}g\right\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\sum_{p\geq -1}2^{p\sigma}\Big(\sum_{j\geq p-4}\|\Delta_j f\|_{L^\infty_TL^2_{x,v}}\Big)^{1/2}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h\right\|^{1/2}_{L^2_TL^2_{x,v}}\\
&\lesssim \|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}\left(\sum_{p\geq -1}\sum_{|j-p|\leq 4}2^{(p-j)\sigma}c(j)\right)^{1/2}\\
&\quad+\|\mathcal{H}^{\frac{s}{2}}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}\left(\sum_{p\geq -1}\sum_{j\geq p-4}2^{p\sigma}\|\Delta_j f\|_{L^\infty_TL^2_{x,v}}\right)^{1/2},
\end{aligned}
$$

where $c(j)=2^{j\sigma}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_j g\right\|_{L^2_TL^2_{x,v}}\Big/\left\|\mathcal{H}^{\frac{s}{2}}g\right\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}$ and $\|c(j)\|_{\ell^1}\leq 1$. Hence, with Fubini's theorem and Young's inequality, we have

$$
\begin{aligned}
\sum_{p\geq -1}\sum_{|j-p|\leq 4}2^{(p-j)\sigma}c(j)&=\sum_{p\geq -1}[(\mathbf{1}_{|j|\leq 4}2^{j\sigma})*c(j)](p)\\
&\leq \|\mathbf{1}_{|j|\leq 4}2^{j\sigma}\|_{\ell^1}\|c(j)\|_{\ell^1}<+\infty.
\end{aligned}
$$

Since

$$
\sum_{-1\leq p\leq j+4}2^{(p-j)\sigma}=2^{-(j+1)\sigma}+2^{-j\sigma}+\cdots+2^{4\sigma}=\frac{2^{4\sigma}(1-2^{-j\sigma})}{1-2^{-\sigma}}<+\infty,
$$

it follows that

$$
\begin{aligned}
\sum_{p\geq -1}\sum_{j\geq p-4}2^{p\sigma}\|\Delta_j f\|_{L^\infty_TL^2_{x,v}}&=\sum_{j\geq -1}2^{j\sigma}\Big(\sum_{-1\leq p\leq j+4}2^{(p-j)\sigma}\Big)\|\Delta_j f\|_{L^\infty_TL^2_{x,v}}\\
&\lesssim\sum_{j\geq -1}2^{j\sigma}\|\Delta_j f\|_{L^\infty_TL^2_{x,v}}=\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}.
\end{aligned}
$$

Consequently, we conclude that

$$
\begin{aligned}
&\sum_{p\geq -1}2^{p\sigma}\left[\int_0^T\left|\big(G_\kappa(t)\Delta_p\Gamma((G_\kappa(t))^{-1}f,(G_\kappa(t))^{-1}g),\Delta_p h\big)\right|dt\right]^{1/2}\\
&\leq C_1\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}\\
&\quad+C_1\|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma}_{2,1})}.
\end{aligned}
$$

Hence, the proof of Lemma 4.15 is complete. □

Similarly, it follows from Remark 4.4, Lemma 2.8 and Lemma 2.9 that

Corollary 4.1. *Set $T>0$. Let $f=f(t,x,v)$, $g=g(t,x,v)$ and $h=h(t,x,v)$ be three suitably functions. Then it holds that*

$$\sum_{p\geq -1}2^{\frac{p}{2}}\left(\int_0^T\left|(\Delta_p\Gamma(f,g),\Delta_p g)_{L^2_{x,v}}\right|dt\right)^{1/2}\lesssim \|f\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}h\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

4.4 The local existence of weak solution

This section is devoted to proving the local existence of weak solution to the Cauchy problem (4.1). To do this, we firstly consider the local existence to the following linearized Kac equation

$$\begin{cases}\partial_t g+v\partial_x g+\mathcal{K}g=\Gamma(f,g),\\ g(t,x,v)|_{t=0}=g_0(x,v).\end{cases}\tag{4.24}$$

4.4.1 Schema of the proof for the linearized Kac equation

Due to that there is a problem that the dual space of $\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})$ is complicated. Therefore, the proof of local existence of weak solution for the linearized Kac equation is a little complicated, we present a schema of its proof as the following five steps.

Step I) The existence of weak solution

Firstly, by using the duality argument and the Hahn-Banach extension theorem, we find a weak solution g to a linearized Kac equation in a wider space $L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$.

Step II) Mollifier of weak solution

Since for $g\in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$, for the terms in (4.24) we only have

$$v\partial_x g\in H^{-1}_x L^2_{v,-1},\quad \mathcal{H}^{-s}\mathcal{K}g\in L^2_vL^2_x,\quad \mathcal{H}^{-s}\Gamma(f,g)\in L^2_vL^2_x.$$

So, we need to mollifier the function g and take its as right test function. Here, taking different $1>\delta>0$, we use mollifiers $(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}$. So that we can take

$$(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-2}g\in H^{+\infty}_xH^2_{v,2},$$

as test function to the equation (4.24). We also need to mollifier the function f as following $f_N=\sum_{p\geq -1}^{N-1}\Delta_p f$ for $N\in\mathbb{N}$, then for f_N we get

$$f_N\in\widetilde{L}^\infty_T\widetilde{L}^2_v(H^{+\infty}_x)\ \text{ and }\ \|f_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\leq\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

For the sequence $\{f_N,N\in\mathbb{N}\}$, we consider a sequence of weak solution $\{g_N,N\in\mathbb{N}\}\subset L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ to the following Cauchy problem

$$\begin{cases}\partial_t g_N+v\partial_x g_N+\mathcal{K}g_N=\Gamma(f_N,g_N),\\ g_N(t,x,v)|_{t=0}=g_0(x,v).\end{cases}\tag{4.25}$$

Step III) Regularity of weak solution in velocity variable

For the weak solution g_N, based on these commutator estimates, we prove

$$\|g_N\|_{L^\infty_T L^2_v L^2_x} + \|\mathcal{H}^{\frac{s}{2}} g_N\|_{L^2_T L^2_v L^2_x} \leq C\|g_0\|_{L^2_v L^2_x},$$

under the assumption that $\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})}$ is sufficiently small. Remark that we can't do the iteration by using above estimates, since there is no regularity in position variable x for the weak solution g_N while f satisfies the condition $f \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$ and its norm is small.

Step IV) Regularity of weak solution in position variable

In this step, we mainly prove the regularity of weak solution g_N in position variable x for Besov norm, we obtain

$$\|g_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \|\mathcal{H}^{\frac{s}{2}} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} < +\infty.$$

But here, we can only get a upper bound dependent of N. So that this step is a technical step, but very important to give a rigorous proof.

Step V) Energy estimates in Besov space

We prove finally the following energy estimate for the weak solution g_N,

$$\|g_N\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \|\mathcal{H}^{\frac{s}{2}} g_N\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq C\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

The convergence of this sequence is then standard, that is, $g = \lim_{N\to\infty} g_N$ and

$$g(t,x,v) \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1}) \quad \text{and} \quad \mathcal{H}^{\frac{s}{2}} g(t,x,v) \in \widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1}).$$

4.4.2 The local weak solution of linearized Kac equation

In the first step, we give the existence of local weak solution with the rough initial datum as the following proposition.

Proposition 4.1. *There exists a constant $\varepsilon_0 > 0$ such that for all $T > 0, g_0 \in L^2(\mathbb{R}^2_{x,v}), f \in L^\infty([0,T] \times \mathbb{R}_x; L^2(\mathbb{R}_v))$ satisfying*

$$\|f\|_{L^\infty([0,T]\times\mathbb{R}_x; L^2_v)} \leq \varepsilon_0,$$

then the Cauchy problem (4.24) *admits a weak solution*

$$g \in L^\infty([0,T]; L^2(\mathbb{R}^2_{x,v})).$$

Proof. The proof is similar as that in [117]. We also consider the joint operator

$$\mathcal{Q} = -\partial_t + (v\partial_x + \mathcal{K} - \Gamma(f,\cdot))^*,$$

where $(\cdot)^*$ is taken with respect to the scalar product in $L^2(\mathbb{R}^2_{x,v})$. For all $h \in C^\infty([0,T], \mathcal{S}(\mathbb{R}^2_{x,v}))$ with $h(T) = 0$, it follows from (4.6) and Remark 4.2 that

$$\begin{aligned}
&\mathrm{Re}\,(h(t), \mathcal{Q}h(t))_{x,v} \\
&= -\frac{1}{2}\frac{d}{dt}\left(\|h\|^2_{L^2_{x,v}}\right) + \mathrm{Re}(v\partial_x h, h)_{x,v} + \mathrm{Re}\,(\mathcal{K}h, h)_{x,v} - \mathrm{Re}\,(\Gamma(f,h), h)_{x,v} \\
&\geq -\frac{1}{2}\frac{d}{dt}\left(\|h\|^2_{L^2_{x,v}}\right) + \frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}h\|^2_{L^2_{x,v}} - \|h\|^2_{L^2_{x,v}} \\
&\quad - C_0\,\|f\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)}\,\|\mathcal{H}^{\frac{s}{2}}h\|^2_{L^2_{x,v}}
\end{aligned}$$

for $0 \leq t \leq T$, since $\mathcal{K}$ is a self-adjoint operator and the fact $\mathrm{Re}(v\partial_x h, h) = 0$.

Since $\|f\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)}$ is sufficiently small, taking

$$\|f\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)} \leq \frac{1}{4CC_0}, \quad T > 0$$

and following from the Cauchy-Schwarz inequality, we obtain

$$-\frac{d}{dt}\Big(\|h\|^2_{L^2_{x,v}}\Big) + \frac{3}{2C}\|\mathcal{H}^{\frac{s}{2}}h\|^2_{L^2_{x,v}} \leq 2\|h\|_{L^2_{x,v}}\,\|\mathcal{Q}h\|_{L^2_{x,v}} + 2\|h\|^2_{L^2_{x,v}},$$

that is

$$-\frac{d}{dt}\Big(e^{2t}\|h\|^2_{L^2_{x,v}}\Big) + \frac{3}{2C}e^{2t}\|\mathcal{H}^{\frac{s}{2}}h\|^2_{L^2_{x,v}} \leq 2e^{2t}\|h\|_{L^2_{x,v}}\,\|\mathcal{Q}h\|_{L^2_{x,v}}.$$

Since $h(T) = 0$, we have

$$\begin{aligned}
&\|h\|^2_{L^2_{x,v}} + \frac{3}{2C}\|\mathcal{H}^{\frac{s}{2}}h\|^2_{L^2([t,T],L^2_{x,v})} \\
&\quad \leq \|h\|^2_{L^2_{x,v}} + \frac{3}{2C}\int_t^T e^{2(\tau-t)}\|\mathcal{H}^{\frac{s}{2}}h(\tau)\|^2_{L^2([t,T],L^2_{x,v})}d\tau \\
&\quad \leq 2\int_t^T e^{2(\tau-t)}\|h(\tau)\|_{L^2_{x,v}}\|\mathcal{Q}h(\tau)\|_{L^2_{x,v}}d\tau \\
&\quad \leq 2e^{2T}\|h\|_{L^\infty([0,T],L^2_{x,v})}\|\mathcal{Q}h\|_{L^1([0,T],L^2_{x,v})},
\end{aligned}$$

which leads

$$\|h\|_{L^\infty([0,T],L^2_{x,v})} \leq 2e^{2T}\|\mathcal{Q}h\|_{L^1([0,T],L^2_{x,v})}. \tag{4.26}$$

In the following, we consider the vector subspace

$$\begin{aligned}
\mathbb{U} = \big\{u = \mathcal{Q}h : h \in C^\infty([0,T], \mathcal{S}(\mathbb{R}^2_{x,v})), h(T) = 0\big\} \\
\subset L^1([0,T], L^2(\mathbb{R}^2_{x,v})).
\end{aligned}$$

Mainly, the above inclusion is right due to similar calculations in Lemma 4.11. For $g \in L^2_{x,v}$, we have

$$\begin{aligned}
|(\Gamma(f,\cdot)^*h, g)_{L^2_{x,v}}| &= |(h, \Gamma(f,g))_{L^2_{x,v}}| = |(\mathcal{H}^s h, \mathcal{H}^{-s}\Gamma(f,g))_{L^2_{x,v}}| \\
&\leq \|\mathcal{H}^{-s}\Gamma(f,g)\|_{L^2_{x,v}}\|\mathcal{H}^s h\|_{L^2_{x,v}} \leq \tilde{c}_0\|f\|_{L^2_v L^\infty_x}\|g\|_{L^2_{x,v}}\|\mathcal{H}^s h\|_{L^2_{x,v}}.
\end{aligned}$$

Thus, for any $t \in [0,T]$ we have

$$\|\Gamma(f,\cdot)^* h\|_{L^2_{x,v}} \leq \tilde{c}_0 \|f\|_{L^2_v L^\infty_x} \|\mathcal{H}^s h\|_{L^2_{x,v}}.$$

Since $g_0 \in L^2(\mathbb{R}^2_{x,v})$, we define the linear functional

$$\begin{aligned} &\mathcal{G} : \mathbb{U} \to \mathbb{C} \\ &u = \mathcal{Q}h \mapsto (g_0, h(0))_{L^2_{x,v}}, \end{aligned}$$

where $h \in C^\infty([0,T], \mathcal{S}(\mathbb{R}^2_{x,v}))$ with $h(T) = 0$. From (4.26), the operator $\mathcal{Q}$ is injective. Therefore the linear functional $\mathcal{G}$ is well-defined. We can obtain

$$\begin{aligned} |\mathcal{G}(u)| &\leq \|g_0\|_{L^2_{x,v}} \|h(0)\|_{L^2_{x,v}} \leq \|g_0\|_{L^2_{x,v}} \|h\|_{L^\infty([0,T];L^2_{x,v})} \\ &\leq 2e^{2T} \|g_0\|_{L^2_{x,v}} \|\mathcal{Q}h\|_{L^1([0,T],L^2_{x,v})} = 2e^{2T} \|g_0\|_{L^2_{x,v}} \|u\|_{L^1([0,T];L^2_{x,v})}. \end{aligned}$$

Hence, $\mathcal{G}$ is a continuous linear form on $(\mathbb{U}, \|\cdot\|_{L^1([0,T],L^2_{x,v})})$. By using the Hahn-Banach theorem, $\mathcal{G}$ can be extended as a continuous linear form on $L^1([0,T];L^2(\mathbb{R}^2_{x,v}))$ with a norm smaller than $2e^{2T}\|g_0\|_{L^2_{x,v}}$. It follows that there exists $g \in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ satisfying

$$\|g\|_{L^\infty([0,T];L^2_{x,v})} \leq 2e^{2T} \|g_0\|_{L^2_{x,v}},$$

such that

$$\forall u \in L^1([0,T];L^2(\mathbb{R}^2_{x,v})), \qquad \mathcal{G}(u) = \int_0^T (g(t), u(t))_{L^2_{x,v}} \, dt.$$

It implies that for all $h \in C^\infty_0((-\infty,T), \mathcal{S}(\mathbb{R}^2_{x,v}))$,

$$\mathcal{G}(\mathcal{Q}h) = \int_0^T (g(t), \mathcal{Q}h(t))_{L^2_{x,v}} \, dt = (g_0, h(0))_{L^2_{x,v}}.$$

Therefore, $g \in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ is a weak solution of the Cauchy problem (4.24). The proof of Proposition 4.1 is completed. □

Next, we turn to prove the regularity of the weak solution with respect to x and v, which is shown by the following two subsections.

4.4.3 Regularity of weak solution in velocity variable

To obtain the above solution g belongs to the critical Besov space $\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$, we need to mollifier the weak solution $g \in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ in velocity and position variables. To do this, we mollifier the function f, that is, setting $f_N = \sum_{p\geq -1}^{N-1} \Delta_p f$ for $N \in \mathbb{N}$, then we have $f_N \in \widetilde{L}^\infty_T \widetilde{L}^2_v(H^{+\infty}_x)$. For each $f_N (N \in \mathbb{N})$, we consider a weak solution $g_N \in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ to the following Cauchy problem

$$\begin{cases} \partial_t g_N + v\partial_x g_N + \mathcal{K} g_N = \Gamma(f_N, g_N), \\ g_N(t,x,v)|_{t=0} = g_0(x,v). \end{cases} \tag{4.27}$$

Some simple calculations enable us to obtain the following proposition for f_N.

Proposition 4.2. *If $f \in \widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1})$. For $N \in \mathbb{N}$, put $f_N = S_N f = \sum_{p\geq -1}^{N-1}\Delta_p f$. Then we get*

i) If $f_N \in \widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1})$, then $\{f_N\}$ is a Cauchy sequence in $\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1})$.

ii) For $0<\sigma\leq 1/2$, f_N satisfies $\|f_N\|_{L^{\infty}_{T}L^{2}_{v}L^{2}_{x}} \leq C\|f\|_{L^{\infty}_{T}L^{2}_{v}L^{2}_{x}}$ and

$$\|f_N\|_{L^{\infty}_{T}L^{2}_{v}L^{2}_{x}} \leq C_2\|f_N\|_{L^{\infty}_{T}L^{2}_{v}(B^{\sigma}_{2,1})} \leq C_3\|f_N\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{\sigma}_{2,1})} \leq C_4\|f\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{\sigma}_{2,1})},$$

where $C_2, C_3, C_4 > 0$ are constants independent of N.

Then, we can establish the following proposition for the weak solution g_N.

Proposition 4.3. *For $N \in \mathbb{N}$, put $f_N = \sum_{p\geq -1}^{N-1}\Delta_p f$. There exists a constant $\varepsilon_0 > 0$ such that for all $T>0, g_0 \in L^2(\mathbb{R}^2_{x,v}), f \in L^{\infty}([0,T]\times\mathbb{R}_x; L^2(\mathbb{R}_v))$ satisfying*

$$\|f\|_{L^{\infty}([0,T]\times\mathbb{R}_x;L^2_v)} \leq \varepsilon_0,$$

then the Cauchy problem (4.27) admits a weak solution $g_N(t,x,v) \in L^{\infty}([0,T]; L^2(\mathbb{R}^2_{x,v}))$ such that

$$\|g_N\|_{L^{\infty}_{T}L^{2}_{v}L^{2}_{x}} + \|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^{2}_{T}L^{2}_{v}L^{2}_{x}} \leq c_0 e^{3T}\|g_0\|_{L^{2}_{v}L^{2}_{x}}, \tag{4.28}$$

for some constant $c_0 > 1$.

Proof. By applying Proposition 4.1, we see that the Cauchy problem (4.27) admits a weak solution $g_N(t,x,v) \in L^{\infty}([0,T]; L^2(\mathbb{R}^2_{x,v}))$. It only need to show (4.28) for a weak solution $g_N \in L^{\infty}([0,T]; L^2(\mathbb{R}^2_{x,v}))$ under the assumption that $\|f_N\|_{L^{\infty}([0,T]\times\mathbb{R}_x;L^2_v)}$ is sufficiently small, independent of N.

It follows from (4.4), (4.5) and Lemma 4.12 that

$$\mathcal{H}^{-s}\mathcal{K}g \in L^{\infty}([0,T]; L^2(\mathbb{R}^2_{x,v})), \quad \mathcal{H}^{-s}\Gamma(f,g) \in L^{\infty}([0,T]; L^2(\mathbb{R}^2_{x,v})),$$

for $f \in L^{\infty}([0,T]\times\mathbb{R}_x; L^2(\mathbb{R}_v)), g \in L^{\infty}([0,T]; L^2(\mathbb{R}^2_{x,v}))$. Define

$$g_\delta = (1+\delta\sqrt{\mathcal{H}} + \delta\langle D_x\rangle)^{-1}g_N, \quad 0<\delta\leq 1. \tag{4.29}$$

Notice that

$$(1+\delta\sqrt{\mathcal{H}} + \delta\langle D_x\rangle)g_\delta \in L^{\infty}([0,T]; L^2(\mathbb{R}^2_{x,v})) \subset L^2([0,T]; L^2(\mathbb{R}^2_{x,v})).$$

According to Theorem 3 of [60], we deduce that the mapping

$$t \mapsto \|g_\delta(t)\|^2_{L^2_{x,v}},$$

is absolutely continuous with

$$\frac{d}{dt}\Big(\|g_\delta\|^2_{L^2_{x,v}}\Big) = 2\mathrm{Re}(\partial_t g_\delta, g_\delta)_{L^2_{x,v}}. \tag{4.30}$$

Taking the inner product of (4.27) with $(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-2}g$ and integrating the resulting inequality with respect to $(x,v)\in\mathbb{R}^2$. It follows from (4.29) and (4.30) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|g_\delta\|^2_{L^2_{x,v}}\right)+\mathrm{Re}\,(\mathcal{K}g_\delta,g_\delta)_{L^2_{x,v}}+\mathrm{Re}\,(v\partial_x g_\delta,g_\delta)_{L^2_{x,v}}\\
&\quad+\mathrm{Re}([(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1},v](1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)\partial_x g_\delta,g_\delta)_{L^2_{x,v}}\\
&=\mathrm{Re}((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma(f_N,(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)g_\delta),g_\delta)_{L^2_{x,v}},
\end{aligned}$$

since $[(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1},\mathcal{K}]=0$. Due to the coercivity estimate of the linearized Kac collision operator $\mathcal{K}$, we obtain, for all $0\le t\le T$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|g_\delta\|^2_{L^2_{x,v}})+\frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}g_\delta\|^2_{L^2_{x,v}}-\|g_\delta\|^2_{L^2_{x,v}}\\
&\le|([(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1},v](1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)\partial_x g_\delta,g_\delta)_{L^2_{x,v}}|\\
&\quad+|((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma(f_N,(1+\delta\sqrt{\mathcal{H}})g_\delta),g_\delta)_{L^2_{x,v}}|\\
&\quad+|((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma(f_N,\delta\langle D_x\rangle g_\delta),g_\delta)_{L^2_{x,v}}|,
\end{aligned}\tag{4.31}$$

since $\mathcal{K}$ is a selfadjoint operator and $\mathrm{Re}(v\partial_x g_\delta,g_\delta)_{L^2_{x,v}}=0$. Furthermore, it follows from Lemma 4.11 with $j_1=0,j_2=1$ that for all $0\le\delta\le1$,

$$\begin{aligned}
&\left|\left(\left(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle\right)^{-1}\Gamma\left(f_N,(1+\delta\sqrt{\mathcal{H}})g_\delta\right),g_\delta\right)_{L^2_{x,v}}\right|\\
&\qquad\le C_0\|f_N\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g_\delta\|^2_{L^2_{x,v}}.
\end{aligned}\tag{4.32}$$

Similarly, we obtain

$$\begin{aligned}
&\left|\left((1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}\Gamma(f_N,\delta\langle D_x\rangle g_\delta),g_\delta\right)_{L^2_{x,v}}\right|\\
&=\left|\left(\Gamma(f_N,\delta\langle D_x\rangle(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}g_N),M_\delta g_\delta\right)_{L^2_{x,v}}\right|\\
&\le C_0\|f_N\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_{x,v}}\|\mathcal{H}^{\frac{s}{2}}g_\delta\|_{L^2_{x,v}}.
\end{aligned}\tag{4.33}$$

Thanks to the commutator estimate (4.10) of [117], we have

$$\left\|\left[(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1},v\right]\left(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle\right)\partial_x f\right\|^2_{L^2_{x,v}}\le4\|f\|^2_{L^2_{x,v}},$$

which leads to

$$\left|\left(\left[\left(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle\right)^{-1},v\right]\left(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle\right)\partial_x g_\delta,g_\delta\right)_{L^2_{x,v}}\right|\le2\|g_\delta\|^2_{L^2_{x,v}}.\tag{4.34}$$

Consequently, we can deduce from (4.31), (4.32), (4.33) and (4.34) that for all $0\le t\le T,0<\delta\le1$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|g_\delta\|^2_{L^2_{x,v}}\right)+\frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}g_\delta\|^2_{L^2_{x,v}}\\
&\le3\|g_\delta\|^2_{L^2_{x,v}}+C_0\|f_N\|_{L^2_vL^\infty_x}(\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_{x,v}}+\|\mathcal{H}^{\frac{s}{2}}g_\delta\|_{L^2_{x,v}})\|\mathcal{H}^{\frac{s}{2}}g_\delta\|_{L^2_{x,v}}\\
&\le3\|g_\delta\|^2_{L^2_{x,v}}+2C_0\|f_N\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g_\delta\|^2_{L^2_{x,v}}+C_0\|f_N\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g_N\|^2_{L^2_{x,v}}.
\end{aligned}$$

Furthermore, if taking

$$\|f\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)} \le \frac{1}{4CC_0}, \quad T>0,$$

then we obtain

$$\frac{d}{dt}\Big(\|g_\delta\|^2_{L^2_{x,v}}\Big)+\frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}g_\delta\|^2_{L^2_{x,v}} \le 6\|g_\delta\|^2_{L^2_{x,v}}+2C_0\|f_N\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g_N\|^2_{L^2_{x,v}},$$

which leads to

$$\begin{aligned}
&\|g_\delta\|^2_{L^2_{x,v}}+\frac{1}{C}\int_0^t e^{6(t-\tau)}\|\mathcal{H}^{\frac{s}{2}}g_\delta(\tau)\|^2_{L^2_{x,v}}d\tau\\
&\le e^{6t}\|(1+\delta\sqrt{\mathcal{H}}+\delta\langle D_x\rangle)^{-1}g_0\|^2_{L^2_{x,v}}\\
&\quad+2C_0\int_0^t e^{6(t-\tau)}\|f_N(\tau)\|_{L^2_vL^\infty_x}\|\mathcal{H}^{\frac{s}{2}}g_N(\tau)\|^2_{L^2_{x,v}}d\tau\\
&\le e^{6t}\|g_0\|^2_{L^2_{x,v}}+2C_0\|f_N(\tau)\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)}\int_0^t e^{6(t-\tau)}\|\mathcal{H}^{\frac{s}{2}}g_N(\tau)\|^2_{L^2_{x,v}}d\tau
\end{aligned}$$

for $0\le t\le T$ and $0<\delta\le 1$. Consequently, we obtain

$$\begin{aligned}
&\|g_\delta\|^2_{L^\infty_TL^2_{x,v}}+\frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}g_\delta(\tau)\|^2_{L^2([0,T]\times\mathbb{R}^2_{x,v})}\\
&\le e^{6T}\|g_0\|^2_{L^2_{x,v}}+2C_0e^{6T}\|f_N(\tau)\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)}\|\mathcal{H}^{\frac{s}{2}}g_N(\tau)\|^2_{L^2([0,T]\times\mathbb{R}^2_{x,v})}.
\end{aligned}$$

On the other hand, noticing that

$$\begin{aligned}
&\|g_\delta\|^2_{L^2_{x,v}}=\frac{1}{2\pi}\sum_{n=0}^{+\infty}\int_{\mathbb{R}}\Big(1+\delta\sqrt{n+\frac{1}{2}}+\delta\langle\xi\rangle\Big)^{-2}|\mathcal{F}_x\bar{g}_N(t,\xi)|^2d\xi,\\
&\|\mathcal{H}^{\frac{s}{2}}g_\delta(\tau)\|^2_{L^2([0,T]\times\mathbb{R}^2_{x,v})}\\
&=\frac{1}{2\pi}\int_0^T\sum_{n=0}^{+\infty}(n+\frac{1}{2})^s\int_{\mathbb{R}}\Big(1+\delta\sqrt{n+\frac{1}{2}}+\delta\langle\xi\rangle\Big)^{-2}|\mathcal{F}_x\bar{g}_N(t,\xi)|^2d\xi dt,
\end{aligned}$$

with $\bar{g}_N=(g_N(t,x,\cdot),e_n)_{L^2(\mathbb{R}_v)}$, where $\mathcal{F}_x$ denotes the partial Fourier transform in the position variable, it follows from the monotone convergence theorem (passing to the limit $\delta\to 0_+$) that

$$\begin{aligned}
&\|g_N\|^2_{L^\infty_TL^2_{x,v}}+\frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}g_N(\tau)\|^2_{L^2([0,T]\times\mathbb{R}^2_{x,v})}\\
&\le e^{6T}\|g_0\|^2_{L^2_{x,v}}+2C_0e^{6T}\|f_N(\tau)\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)}\|\mathcal{H}^{\frac{s}{2}}g_N(\tau)\|^2_{L^2([0,T]\times\mathbb{R}^2_{x,v})}.
\end{aligned}$$

Thanks to the smallness of $\|f_N(\tau)\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)}$ (taking $\|f_N(\tau)\|_{L^\infty([0,T]\times\mathbb{R}_x;L^2_v)}\le\frac{1}{4CC_0e^{6T}}$), we arrive at

$$\|g_N\|^2_{L^\infty_TL^2_{x,v}}+\|\mathcal{H}^{\frac{s}{2}}g_N(\tau)\|^2_{L^2([0,T]\times\mathbb{R}^2_{x,v})}\le 2(C+1)e^{6T}\|g_0\|^2_{L^2_{x,v}}.$$

Hence, the proof of Proposition 4.3 is finished. □

Remark 4.5. Owing to the embedding $\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1}) \hookrightarrow L^{\infty}([0,T]\times\mathbb{R}_x; L^2(\mathbb{R}_v))$, we deduce that the norm $\|f\|_{L^{\infty}([0,T]\times\mathbb{R}_x;L^2_v)}$ is small, since $\|f\|_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1})}$ is sufficiently small. However, there is no regularity available in position variable x for the weak solution g_N according to Proposition 4.3. In that case, we cannot attain desired solutions to (4.24) which belongs to the critical Besov space $\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1})$.

4.4.4 Regularity of weak solution in position variable

In what follow, we establish the regularity of g_N with respect to x. Firstly, for the nonlinear term $\Gamma(f_N, g_N)$, we obtain the following lemma if g_N belongs to the suitable space.

Lemma 4.16. *Let $0<\sigma\le 1/2$ and $0<T<+\infty$. For $N\in\mathbb{N}$, setting $f_N=\sum_{p\ge -1}^{N-1}\Delta_p f$. If g_N satisfies*

$$g_N\in L^{\infty}([0,T];L^2(\mathbb{R}^2_{x,v})),\qquad \mathcal{H}^{\frac{s}{2}}g_N\in L^2([0,T]\times\mathbb{R}^2_{x,v}),$$

then there exists a $\tilde{C}>0$ independent of N such that for any $\kappa>0$,

$$\begin{aligned}
&\sum_{p\ge -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\int_0^T\left|\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{x,v}\right|dt\right)^{1/2}\\
&\quad\le \widetilde{C}\|f_N\|^{1/2}_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{\sigma,\kappa}_{2,1})}\\
&\qquad + C_N\|f_N\|^{1/2}_{\widetilde{L}^{\infty}_{T}\widetilde{L}^{2}_{v}(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_TL^2_vL^2_x},
\end{aligned}\tag{4.35}$$

where $C_N>0$ is a constant depending only on N and

$$\|g_N\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{\sigma,\kappa}_{2,1})}=\sum_{p\ge -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\|\Delta_p g_N\|_{L^2_TL^2_vL^2_x}.$$

Proof. For $\sigma,\kappa>0$, we have

$$\begin{aligned}
\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{\sigma,\kappa}_{2,1})}&=\sum_{p\ge -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\|\Delta_p\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_TL^2_vL^2_x}\\
&\le C_\kappa\sum_{p\ge -1}2^{-p\sigma}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_TL^2_vL^2_x}\le C_\kappa\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_TL^2_vL^2_x},
\end{aligned}$$

where $C_\kappa>0$ is a constant depending only on κ. Hence, one has $\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^{2}_{T}\widetilde{L}^{2}_{v}(B^{\sigma,\kappa}_{2,1})}<+\infty$ due to $\mathcal{H}^{\frac{s}{2}}g_N\in L^2([0,T]\times\mathbb{R}^2_{x,v})$.

By using Bony's decomposition, we divide the inner product into three parts:

$$\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)=\left(\Delta_p(\Gamma^1(f_N,g_N)+\Gamma^2(f_N,g_N)+\Gamma^3(f_N,g_N)),\Delta_p g_N\right),$$

where $\Gamma^1(f_N,g_N)\triangleq\sum_j\Gamma(S_{j-1}f_N,\Delta_j g_N)$, $\Gamma^2(f_N,g_N)\triangleq\sum_j\Gamma(\Delta_j f_N,S_{j-1}g_N)$ and $\Gamma^3(f_N,g_N)\triangleq\sum_j\sum_{|j-j'|\le 1}\Gamma(\Delta_{j'}f_N,\Delta_j g_N)$. For $\Gamma^1(f_N,g_N)$, note that

$$\Delta_p\sum_j(S_{j-1}f_N\Delta_j g_N)=\Delta_p\sum_{|j-p|\le 4}(S_{j-1}f_N\Delta_j g_N).$$

It follows from Remark 4.4 that

$$\begin{aligned}
&\sum_{p\geq -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\int_0^T\left|\left(\Delta_p\Gamma^1(f_N,g_N),\Delta_p g_N\right)_{x,v}\right|dt\right)^{1/2}\\
&\lesssim \sum_{p\geq -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\sum_{|j-p|\leq 4}\int_0^T\|S_{j-1}f_N\|_{L^2_vL^\infty_x}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_j g_N\right\|_{L^2_{x,v}}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\right\|_{L^2_{x,v}}dt\right)^{1/2}\\
&\lesssim \|f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma,\kappa}_{2,1})}\left(\sum_{p\geq -1}\sum_{|j-p|\leq 4}\frac{1+\kappa 2^{2j\sigma}}{1+\kappa 2^{2p\sigma}}2^{(p-j)s}c(j)\right)^{1/2}\\
&\lesssim \|f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma,\kappa}_{2,1})},
\end{aligned}$$

where we used Lemmas 2.4, 2.8 and 2.9 in the third line and the following sequence $\{c(j)\}$

$$\begin{aligned}
c(j)&:=\frac{\frac{2^{j\sigma}}{1+\kappa 2^{2j\sigma}}\|\mathcal{H}^{\frac{s}{2}}\Delta_j g_N\|_{L^2_TL^2_{x,v}}}{\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{\sigma,\kappa}_{2,1})}}\\
&\sum_{p\geq -1}\sum_{|j-p|\leq 4}\frac{1+\kappa 2^{2j\sigma}}{1+\kappa 2^{2p\sigma}}2^{(p-j)\sigma}c(j)\\
&\leq C\sum_{p\geq -1}\sum_{|j-p|\leq 4}2^{(p-j)\sigma}c(j)\leq C\sum_{p\geq -1}[(\mathbf{1}_{|j|\leq 4}2^{j\sigma})*c(j)](p)\\
&\leq C\|\mathbf{1}_{|j|\leq 4}2^{j\sigma}\|_{\ell^1}\|c(j)\|_{\ell^1}<+\infty,
\end{aligned}$$

satisfying $\|c(j)\|_{\ell^1}\leq 1$.

For $\Gamma^2(f_N,g_N)$, similarly, we get

$$\begin{aligned}
&\sum_{p\geq -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\int_0^T\left|\left(\Delta_p\Gamma^2(f_N,g_N),\Delta_p g_N\right)_{x,v}\right|dt\right)^{1/2}\\
&\lesssim \sum_{p\geq -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\int_0^T\sum_{\substack{|j-p|\leq 4\\ j\leq N}}\|\Delta_j f_N\|_{L^2_vL^\infty_x}\left\|\mathcal{H}^{\frac{s}{2}}S_{j-1}g_N\right\|_{L^2_{x,v}}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\right\|_{L^2_{x,v}}dt\right)^{1/2}\\
&\lesssim N\sum_{p\geq -1}^{N+4}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\|f_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\left\|\mathcal{H}^{\frac{s}{2}}g_N\right\|_{L^2_TL^2_{x,v}}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\right\|_{L^2_TL^2_{x,v}}\right)^{1/2}\\
&\leq C_N\|f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_TL^2_vL^2_x},
\end{aligned}$$

where $C_N=CN2^{N\sigma}$ with $N\in\mathbb{N}$. Owing to

$$\begin{aligned}
\Delta_p\left(\sum_j\sum_{|j-j'|\leq 1}(\Delta_{j'}f_N\Delta_j g_N)\right)&=\Delta_p\left(\sum_{\max(j,j')\geq p-2}\sum_{|j-j'|\leq 1}(\Delta_{j'}f_N\Delta_j g_N)\right)\\
&=0,\quad\text{if}\quad p\geq N+3,
\end{aligned}$$

then $\Gamma^3(f_N, g_N)$ can be estimated as follows:

$$\begin{aligned}
&\sum_{p\geq -1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\int_0^T\left|\left(\Delta_p\Gamma^3(f_N,g_N),\Delta_p g_N\right)_{x,v}\right|dt\right)^{1/2}\\
&=\sum_{p\geq -1}^{N+2}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\int_0^T\left|\left(\Delta_p\Gamma^3(f_N,g_N),\Delta_p g_N\right)_{x,v}\right|dt\right)^{1/2}\\
&\lesssim\sum_{p\geq -1}^{N+2}\sum_{j\leq N+1}\frac{2^{p\sigma}}{1+\kappa 2^{2p\sigma}}\left(\int_0^T\|\Delta_{j'}f_N\|_{L^2_vL^\infty_x}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_j g_N\right\|_{L^2_{x,v}}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\right\|_{L^2_{x,v}}dt\right)^{1/2}\\
&\lesssim C_N\|f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_TL^2_vL^2_x}.
\end{aligned}$$

Together the above three inequalities, we can get (4.35). □

Based on Proposition 4.3 and Lemma 4.16, we obtain the regularity of the weak solution g_N to (4.27).

Proposition 4.4. *There exists a constant $\varepsilon_0>0$ such that for all $T>0, g_0\in\widetilde{L}^2_v(B^{1/2}_{2,1}), f\in\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})$ fulfilling*

$$\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\leq\varepsilon_0,$$

then (4.27) *admits a weak solution $g_N\in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ satisfying*

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\frac{1}{\sqrt{2C}}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\leq e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{2}C_Ne^T\|f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{L^2_TL^2_vL^2_x},
\end{aligned}\tag{4.36}$$

where $C_N>0$ is some constant depending on N.

Proof. Applying $\Delta_p(p\geq -1)$ to (4.27), and then taking the inner product with $\Delta_p g_N$ over $\mathbb{R}_x\times\mathbb{R}_v$ gives

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta_p g_N\|^2_{L^2_{x,v}}\right)+\frac{1}{C}\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\|^2_{L^2_{x,v}}\leq\|\Delta_p g_N\|^2_{L^2_{x,v}}+\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}},$$

where we used the coercivity estimate of $\mathcal{K}$. It follows that

$$\frac{d}{dt}\left(e^{-2t}\|\Delta_p g_N\|^2_{L^2_{x,v}}\right)+\frac{2}{C}e^{-2t}\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\|^2_{L^2_{x,v}}\leq 2e^{-2t}|\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}|$$

for $0\leq t\leq T$.

Integrating the above inequality with respect to the time variable over $[0,t]$ with $0\leq t\leq T$ and taking the square root of both sides of the resulting inequality, we get

$$\begin{aligned}
&\|\Delta_p g_N\|_{L^2_{x,v}}+\sqrt{\frac{2}{C}}\Big(\int_0^t e^{2(t-\tau)}\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N(\tau)\|^2_{L^2_{x,v}}d\tau\Big)^{1/2}\\
&\leq e^t\|\Delta_p g_0\|_{L^2_vL^2_x}+\sqrt{2}\Big(\int_0^t e^{2(t-\tau)}|\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}|d\tau\Big)^{1/2},
\end{aligned}$$

then, taking supremum over $0 \le t \le T$ on the left side and multiplying the resulting inequality by $\frac{2^{p/2}}{1+\kappa 2^p}$, we obtain

$$\begin{aligned}
&\frac{2^{p/2}}{1+\kappa 2^p}\|\Delta_p g_N\|_{L_v^2 L_x^2} + \sqrt{\frac{2}{C}}\frac{2^{p/2}}{1+\kappa 2^p}\left(\int_0^t \left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\right\|_{L_v^2 L_x^2}^2 d\tau\right)^{1/2} \\
&\le e^T \frac{2^{p/2}}{1+\kappa 2^p}\|\Delta_p g_0\|_{L_v^2 L_x^2} + \sqrt{2}e^T \frac{2^{p/2}}{1+\kappa 2^p}\left(\int_0^T \left|(\Delta_p \Gamma(f_N, g_N), \Delta_p g_N)_{L^2_{x,v}}\right| dt\right)^{1/2}.
\end{aligned}$$

Further taking the summation over $p \ge -1$, the above inequality implies

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2,\kappa})} + \sqrt{\frac{2}{C}}\|\mathcal{H}^{\frac{s}{2}} g_N\|_{\widetilde{L}_T^2 \widetilde{L}_v^2(B_{2,1}^{1/2,\kappa})} \\
&\le e^T \|g_0\|_{\widetilde{L}_v^2(B_{2,1}^{1/2})} + \sqrt{2}\widetilde{C}e^T \|f_N\|^{1/2}_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} \|\mathcal{H}^{\frac{s}{2}} g_N\|_{\widetilde{L}_T^2 \widetilde{L}_v^2(B_{2,1}^{1/2,\kappa})} \\
&\quad + \sqrt{2}C_N e^T \|f_N\|^{1/2}_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} \|\mathcal{H}^{\frac{s}{2}} g_N\|_{L_T^2 L_v^2 L_x^2},
\end{aligned}$$

where we used the Proposition 4.2 and Lemma 4.16. Then, by taking $\|f\|_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} \le \frac{1}{4e^{2T}C\widetilde{C}^2}$ and letting $\kappa \to 0$, we obtain

$$\begin{aligned}
&\|g_N\|_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} + \frac{1}{\sqrt{2C}}\|\mathcal{H}^{\frac{s}{2}} g_N\|_{\widetilde{L}_T^2 \widetilde{L}_v^2(B_{2,1}^{1/2})} \\
&\le e^T \|g_0\|_{\widetilde{L}_v^2(B_{2,1}^{1/2})} + \sqrt{2}C_N e^T \|f_N\|^{1/2}_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} \|\mathcal{H}^{\frac{s}{2}} g_N\|_{L_T^2 L_v^2 L_x^2},
\end{aligned}$$

which ends the proof of Proposition 4.4. □

4.4.5 Energy estimates in Besov space

It follows from (4.36) in Proposition 4.4 that

$$\|g_N\|_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} + \|\mathcal{H}^{\frac{s}{2}} g_N\|_{\widetilde{L}_T^2 \widetilde{L}_v^2(B_{2,1}^{1/2})} < +\infty.$$

Then applying the Corollary 4.1 to f_N and g_N, we get the following inequality

$$\sum_{p\ge -1} 2^{\frac{p}{2}} \left(\int_0^T \left|(\Delta_p \Gamma(f_N, g_N), \Delta_p g_N)_{x,v}\right| dt\right)^{1/2} \le C_1 \|f_N\|^{1/2}_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} \|\mathcal{H}^{\frac{s}{2}} g_N\|_{\widetilde{L}_T^2 \widetilde{L}_v^2(B_{2,1}^{1/2})} \quad (4.37)$$

for some constant $C_1 > 0$ independent of N.

With aid of (4.37), one can obtain the further energy estimate, which is independent of N for the weak solution g_N.

Proposition 4.5. *There exists a constant $\varepsilon_0 > 0$ such that for $T > 0, g_0 \in \widetilde{L}_v^2(B_{2,1}^{1/2}), f \in \widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})$ fulfilling*

$$\|f\|_{\widetilde{L}_T^\infty \widetilde{L}_v^2(B_{2,1}^{1/2})} \le \varepsilon_0,$$

then (4.27) *admits a weak solution* $g_N \in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ *satisfying*

$$\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\frac{1}{\sqrt{2C}}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}, \tag{4.38}$$

where $C>0$ *is some constant independent of* N.

Proof. Applying $2^p\Delta_p(p\ge -1)$ to (4.27) and taking the inner product with $\Delta_p g_N$ over $\mathbb{R}_x\times\mathbb{R}_v$ give

$$\begin{aligned}&\frac12\frac{d}{dt}\left(2^p\|\Delta_p g_N\|^2_{L^2_{x,v}}\right)+\frac{1}{C}2^p\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\|^2_{L^2_{x,v}}\\&\qquad\le 2^p\|\Delta_p g_N\|^2_{L^2_{x,v}}+2^p\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}.\end{aligned}$$

It follows that

$$\frac{d}{dt}\left(e^{-2t}2^p\|\Delta_p g_N\|^2_{L^2_{x,v}}\right)+\frac{2}{C}2^pe^{-2t}\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N\|^2_{L^2_{x,v}}\le 2e^{-2t}2^p|\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}|$$

for all $0\le t\le T$.

Integrating the above inequality with respect to the time variable over $[0,t]$ with $0\le t\le T$ and taking the square root, we obtain

$$\begin{aligned}&2^{\frac p2}\|\Delta_p g_N\|_{L^2_{x,v}}+\sqrt{\frac{2}{C}}2^{\frac p2}\Big(\int_0^t e^{2(t-\tau)}\|\mathcal{H}^{\frac{s}{2}}\Delta_p g_N(\tau)\|^2_{L^2_{x,v}}d\tau\Big)^{1/2}\\&\qquad\le e^t2^{\frac p2}\|\Delta_p g_0\|_{L^2_vL^2_x}+\sqrt{2}2^{\frac p2}\Big(\int_0^t e^{2(t-\tau)}|\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}|d\tau\Big)^{1/2}.\end{aligned}$$

Taking supremum over $0\le t\le T$ on the left side and summing up over $p\ge -1$, we get

$$\begin{aligned}&\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{\frac{2}{C}}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\&\le e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{2}e^T\sum_{p\ge -1}2^{\frac p2}\left(\int_0^T\Big|\left(\Delta_p\Gamma(f_N,g_N),\Delta_p g_N\right)_{L^2_{x,v}}\Big|dt\right)^{1/2}\\&\le e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{2}e^TC_1\|f_N\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})},\end{aligned}$$

where we used Proposition 4.2 and (4.37). It follows from the smallness of $\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}$ (taking $\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le\frac{1}{4e^{2T}CC_1^2}$) that

$$\|g_N\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\frac{1}{\sqrt{2C}}\|\mathcal{H}^{\frac{s}{2}}g_N\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})},$$

which indicates the desired inequality (4.38). The proof of Proposition 4.5 is completed. □

4.4.6 The local existence for linearized Kac equation

Based on the above propositions, we can obtain the local existence of solutions for linearized Kac equation.

Theorem 4.2 (**Local existence for linearized equation).** *There exists a constant $\varepsilon_0 > 0$ such that for all $T > 0, g_0 \in \widetilde{L}^2_v(B^{1/2}_{2,1}), f \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$, satisfying*

$$\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \varepsilon_0,$$

then the Cauchy problem (4.24) *admits a weak solution $g \in L^\infty([0,T]; L^2(\mathbb{R}^2_{x,v}))$ satisfying*

$$\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \|\mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq c_0 e^T \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \tag{4.39}$$

for some constant $c_0 > 1$.

Proof. A) The first step

$$\textbf{Prove} \quad g(t,x,v) \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1}) \quad \textbf{and} \quad \mathcal{H}^{\frac{s}{2}} g(t,x,v) \in \widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1}).$$

It suffices to show that the sequence $\{g_N, N \in \mathbb{N}\}$ is Cauchy in the space

$$X = \{g \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1}) | \mathcal{H}^{\frac{s}{2}} g \in \widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})\}.$$

Set $w_{M,M'} = g_M - g_{M'}$ for $M, M' \in \mathbb{N}$. Then it follows that (4.27) that

$$\partial_t w_{M,M'} + v\partial_x w_{M,M'} + \mathcal{K} w_{M,M'} = \Gamma(f_{M'}, w_{M,M'}) + \Gamma(f_M - f_{M'}, g_M).$$

Applying $\Delta_p (p \geq -1)$ to the above equality and taking the inner product with $2^p \Delta_p w_{M,M'}$ over $\mathbb{R}_x \times \mathbb{R}_v$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(2^p \|\Delta_p w_{M,M'}\|^2_{L^2_{x,v}}\right) + \frac{2}{C} 2^p \|\mathcal{H}^{\frac{s}{2}} \Delta_p w_{M,M'}\|^2_{L^2_{x,v}} \\
&\leq 2^{p+1} \|\Delta_p w_{M,M'}\|^2_{L^2_{x,v}} + 2^{p+1} \left(\Delta_p \Gamma(f_{M'}, w_{M,M'}), \Delta_p w_{M,M'}\right)_{L^2_{x,v}} \\
&\quad + 2^{p+1} \left(\Delta_p \Gamma(f_M - f_{M'}, g_M), \Delta_p w_{M,M'}\right)_{L^2_{x,v}},
\end{aligned}$$

then it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(2^p e^{-2t} \|\Delta_p w_{M,M'}\|^2_{L^2_{x,v}}\right) + \frac{2}{C} e^{-2t} 2^p \|\mathcal{H}^{\frac{s}{2}} \Delta_p w_{M,M'}\|^2_{L^2_{x,v}} \\
&\leq 2^{p+1} e^{-2t} \left|\left(\Delta_p \Gamma(f_{M'}, w_{M,M'}), \Delta_p w_{M,M'}\right)_{L^2_{x,v}}\right| \\
&\quad + 2^{p+1} e^{-2t} \left|\left(\Delta_p \Gamma(f_M - f_{M'}, g_M), \Delta_p w_{M,M'}\right)_{L^2_{x,v}}\right|.
\end{aligned}$$

Integrating the above inequality with respect to the time variable over $[0,t]$ with $0 \le t \le T$ and taking the square root of both sides of the resulting inequality, we are led to

$$\begin{aligned}
&2^{\frac{p}{2}}\|\Delta_p w_{M,M'}\|_{L^2_{x,v}} + \sqrt{\frac{2}{C}}2^{\frac{p}{2}}\Big(\int_0^t e^{2(t-\tau)}\|\mathcal{H}^{\frac{s}{2}}\Delta_p w_{M,M'}(\tau)\|^2_{L^2_{x,v}}d\tau\Big)^{1/2}\\
&\le \sqrt{2}2^{\frac{p}{2}}\Big(\int_0^t e^{2(t-\tau)}\Big|\big(\Delta_p\Gamma(f_{M'},w_{M,M'}),\Delta_p w_{M,M'}\big)_{L^2_{x,v}}\Big|d\tau\Big)^{1/2}\\
&+\sqrt{2}2^{\frac{p}{2}}\Big(\int_0^t e^{2(t-\tau)}\Big|\big(\Delta_p\Gamma(f_M-f_{M'},g_M),\Delta_p w_{M,M'}\big)_{L^2_{x,v}}\Big|d\tau\Big)^{1/2}.
\end{aligned}$$

Taking the supremum over $0 \le t \le T$ on the left side and summing up over $p \ge -1$. It follows from the condition for small $\|f\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}$ and Young's inequality that

$$\begin{aligned}
&\big\|w_{M,M'}\big\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{\frac{2}{C}}\Big\|\mathcal{H}^{\frac{s}{2}}w_{M,M'}\Big\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\le \sqrt{2}e^T\sum_{p\ge -1}2^{\frac{p}{2}}\Big(\int_0^T\Big|\big(\Delta_p\Gamma(f_{M'},w_{M,M'}),\Delta_p w_{M,M'}\big)_{L^2_{x,v}}\Big|dt\Big)^{1/2}\\
&\quad+\sqrt{2}e^T\sum_{p\ge -1}2^{\frac{p}{2}}\Big(\int_0^T\Big|\big(\Delta_p\Gamma(f_M-f_{M'},g_M),\Delta_p w_{M,M'}\big)_{L^2_{x,v}}\Big|dt\Big)^{1/2}\\
&\le \sqrt{2}C_1e^T\|f_{M'}\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}w_{M,M'}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\quad+\sqrt{2}C_1e^T\|(f_M-f_{M'})\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_M\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}w_{M,M'}\|^{1/2}_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\le \sqrt{2C}C_1^2e^{2T}\|(f_M-f_{M'})\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}g_M\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\quad+\frac{3}{4}\sqrt{\frac{2}{C}}\|\mathcal{H}^{\frac{s}{2}}w_{M,M'}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}.
\end{aligned}$$

The smallness of $\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}$ (taking $\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \le \frac{\tilde{\lambda}}{2CC_1^2e^{3T}}$) and Proposition 4.5 enable us to obtain

$$\begin{aligned}
&\|w_{M,M'}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} + \frac{1}{\sqrt{8C}}\|\mathcal{H}^{\frac{s}{2}}w_{M,M'}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\le 2CC_1^2e^{3T}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\|(f_M-f_{M'})\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\le \tilde{\lambda}\|(f_M-f_{M'})\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}
\end{aligned}$$

for $0<\tilde{\lambda}<1$. It follows from Proposition 4.2 that $\{f_N\}$ is a Cauchy sequence in $\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})$ which implies that $\{g_N\}$ is a Cauchy sequence in X. Letting $g=\lim_{N\to\infty}g_N$, we can get the desired result.

B) The proof of energy estimate (4.39)

To prove (4.39), applying $\Delta_p (p \geq -1)$ to (4.24), taking the inner product with $2^p \Delta_p g$ over $\mathbb{R}_x \times \mathbb{R}_v$ and using (4.6), we get

$$\frac{1}{2}\frac{d}{dt}\left(2^p \|\Delta_p g\|^2_{L^2_{x,v}}\right) + \frac{1}{C}2^p \left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g\right\|^2_{L^2_{x,v}} \leq 2^p \|\Delta_p g\|^2_{L^2_{x,v}} + 2^p \left(\Delta_p \Gamma(f,g), \Delta_p g\right)_{L^2_{x,v}}.$$

It follows that for all $0 \leq t \leq T$,

$$\frac{d}{dt}\left(e^{-2t}2^p \|\Delta_p g\|^2_{L^2_{x,v}}\right) + \frac{2}{C}2^p e^{-2t} \left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g\right\|^2_{L^2_{x,v}} \leq 2e^{-2t}2^p |\left(\Delta_p \Gamma(f,g), \Delta_p g\right)_{L^2_{x,v}}|.$$

Integrating the above inequality with respect to the time variable over $[0,t]$ with $0 \leq t \leq T$ and taking the square root, we can obtain

$$\begin{aligned}
&2^{\frac{p}{2}} \|\Delta_p g\|_{L^2_{x,v}} + \sqrt{\frac{2}{C}}2^{\frac{p}{2}}\Big(\int_0^t e^{2(t-\tau)} \left\|\mathcal{H}^{\frac{s}{2}}\Delta_p g(\tau)\right\|^2_{L^2_{x,v}} d\tau\Big)^{1/2} \\
&\quad \leq e^t 2^{\frac{p}{2}} \|\Delta_p g_0\|_{L^2_v L^2_x} + \sqrt{2}2^{\frac{p}{2}}\Big(\int_0^t e^{2(t-\tau)} \left|\left(\Delta_p \Gamma(f,g), \Delta_p g\right)_{L^2_{x,v}}\right| d\tau\Big)^{1/2}.
\end{aligned}$$

Taking supremum over $0 \leq t \leq T$ on the left side and summing up over $p \geq -1$, we get

$$\begin{aligned}
&\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{\frac{2}{C}} \left\|\mathcal{H}^{\frac{s}{2}} g\right\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq e^T \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{2}e^T \sum_{p\geq -1} 2^{\frac{p}{2}} \left(\int_0^T \left|\left(\Delta_p \Gamma(f,g), \Delta_p g\right)_{L^2_{x,v}}\right| dt\right)^{1/2} \\
&\leq e^T \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{2}e^T C_1 \|f\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})},
\end{aligned}$$

where we used Proposition 4.2 and Corollary 4.1. Then, with the small norm $\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})}$ (taking $\|f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{1}{4e^{2T}CC_1^2}$), we get

$$\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \frac{1}{\sqrt{2C}} \|\mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq e^T \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})},$$

which is the desired (4.39). Due to the steps **A)** and **B)**, this ends the proof of Theorem 4.2. □

4.4.7 The local existence for nonlinear Kac equation

Based on Theorem 4.2, We prove the local-in-time existence of weak solution to (4.1).

Theorem 4.3 (**Local existence).** *Let $0 < T < +\infty$. We assume that the collision cross section satisfies* (4.7) *with $0 < s < 1$. There exists a constant $\varepsilon_0 > 0$ such that for all $g_0 \in \widetilde{L}^2_v(B^{1/2}_{2,1})$ fulfilling*

$$\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \varepsilon_0,$$

then (4.1) *admits a unique weak solution* $g \in L^\infty([0,T];L^2(\mathbb{R}^2_{x,v}))$ *satisfying*

$$\|g\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\|\mathcal{H}^{\frac{s}{2}}g\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le c_0e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}$$

for some constant $c_0>1$.

Proof. Let $0<\lambda<1, T>0$ and $g_0\in\widetilde{L}^2_v(B^{1/2}_{2,1})$ be an initial fluctuation satisfying

$$\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\le\tilde{\varepsilon}_0,\ \text{with}\ \ 0<\tilde{\varepsilon}_0=\inf\Big(\frac{\varepsilon_0}{c_0e^T},\frac{1}{4c_0CC_1^2e^{3T}},\frac{\lambda}{\sqrt{2C}c_0C_1^2e^{3T}}\Big)\le\varepsilon_0, \tag{4.40}$$

where c_0,ε_0,C_1,C are the constants defined in (4.6), Theorem 3.4 and Theorem 4.3. We define

$$\tilde{g}_0=\exp\left(-\delta t\left(\sqrt{\mathcal{H}}+\langle D_x\rangle\right)^{\frac{2s}{2s+1}}\right)g_0,\quad 0\le t\le T$$

with $0\le\delta\le1$. Notice that

$$\|\tilde{g}_0\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\le\varepsilon_0.$$

By using the Theorem 4.2, we can construct the local solution and prove the local existence of solutions to the nonlinear Kac equation. Iteration for the local existence of solution to nonlinear equation (4.1), we consider the following sequence of iterating approximate solutions:

$$\begin{cases}\partial_t\tilde{g}_{n+1}+v\partial_x\tilde{g}_{n+1}+\mathcal{K}\tilde{g}_{n+1}=\Gamma(\tilde{g}_n,\tilde{g}_{n+1}),\ \ t>0,\ v,x\in\mathbb{R},\\ \tilde{g}_{n+1}(t,x,v)|_{t=0}=g_0(x,v).\end{cases} \tag{4.41}$$

Taking $g=\tilde{g}_{n+1}, f=\tilde{g}_n$ in Theorem 4.2 gives

$$\|\tilde{g}_n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\|\mathcal{H}^{\frac{s}{2}}\tilde{g}_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le c_0e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\le\varepsilon_0 \tag{4.42}$$

for all $n\ge1$. Indeed, if we assume that for some $n\ge0$,

$$\|\tilde{g}_n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\|\mathcal{H}^{\frac{s}{2}}\tilde{g}_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le\varepsilon_0,$$

then it follows from (4.40) and Theorem 4.3 that there exists a solution

$$\tilde{g}_{n+1}\in\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})$$

to the Cauchy problem (4.41) satisfying

$$\|\tilde{g}_{n+1}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\|\mathcal{H}^{\frac{s}{2}}\tilde{g}_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\le c_0e^T\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\le\varepsilon_0.$$

Then it remains to prove the convergence of the sequence

$$\{\tilde{g}_n,n\in\mathbb{N}\}\subset\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1}),\qquad\left\{\mathcal{H}^{\frac{s}{2}}\tilde{g}_n,n\in\mathbb{N}\right\}\subset\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1}).$$

Set $w_n = \tilde{g}_{n+1} - \tilde{g}_n$ and (4.41), we obtain

$$\partial_t w_n + v\partial x w_n + \mathcal{K} w_n = \Gamma(\tilde{g}_n, w_n) + \Gamma(w_{n-1}, \tilde{g}_n)$$

with $w_n|_{t=0} = 0$. The following calculation is similar to that of estimate (4.39) but with little changes. To be completed, we give the detailed process. Applying $\Delta_p (p \geq -1)$ to the above equality and taking the inner product with $2^p \Delta_p w_n$ over $\mathbb{R}_x \times \mathbb{R}_v$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(2^p \|\Delta_p w_n\|^2_{L^2_{x,v}}\right) + \frac{2}{C} 2^p \|\mathcal{H}^{\frac{s}{2}} \Delta_p w_n\|^2_{L^2_{x,v}} \\
&\leq 2^{p+1} \|\Delta_p w_n\|^2_{L^2_{x,v}} + 2^{p+1} \left(\Delta_p \Gamma(\tilde{g}_n, w_n), \Delta_p w_n\right)_{L^2_{x,v}} \\
&\quad + 2^{p+1} \left(\Delta_p \Gamma(w_{n-1}, \tilde{g}_n), \Delta_p w_n\right)_{L^2_{x,v}}.
\end{aligned}$$

Taking the supremum over $0 \leq t \leq T$ on the left side and summing up over $p \geq -1$. We deduce from (4.37) that

$$\begin{aligned}
&\|w_n\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{\frac{2}{C}} \left\|\mathcal{H}^{\frac{s}{2}} w_n\right\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \sqrt{2} e^T \sum_{p\geq -1} 2^{\frac{p}{2}} \Big(\int_0^T \left| \left(\Delta_p \Gamma\left(\tilde{g}_n, w_n\right), \Delta_p w_n\right)_{L^2_{x,v}} \right| dt \Big)^{1/2} \\
&\quad + \sqrt{2} e^T \sum_{p\geq -1} 2^{\frac{p}{2}} \Big(\int_0^T \left| \left(\Delta_p \Gamma\left(w_{n-1}, \tilde{g}_n\right), \Delta_p w_n\right)_{L^2_{x,v}} \right| dt \Big)^{1/2} \\
&\leq \sqrt{2} C_1 e^T \|\tilde{g}_n\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} w_n\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\quad + \sqrt{2} C_1 e^T \|w_{n-1}\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} \tilde{g}_n\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} w_n\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \sqrt{2c_0} C_1 e^{\frac{3}{2}T} \|g_0\|^{1/2}_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} w_n\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\quad + \sqrt{2C} C_1^2 e^{2T} \|w_{n-1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} \tilde{g}_n\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\quad + \frac{1}{\sqrt{8C}} \|\mathcal{H}^{\frac{s}{2}} w_n\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})}.
\end{aligned} \tag{4.43}$$

It follows from (4.40) that

$$\begin{aligned}
&\|w_n\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \frac{1}{\sqrt{8C}} \left\|\mathcal{H}^{\frac{s}{2}} w_n\right\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \sqrt{2C} C_1^2 e^{2T} \|\mathcal{H}^{\frac{s}{2}} \tilde{g}_n\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|w_{n-1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \sqrt{2C} c_0 C_1^2 e^{3T} \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \|w_{n-1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \lambda \|w_{n-1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})}
\end{aligned}$$

for some constant $0 < \lambda < 1$. Then for all $n \geq 1$ we have

$$\begin{aligned}\|w_n\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} &\leq \lambda \|w_{n-1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\ &\leq \lambda^n \|w_0\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})}, \\ \|\mathcal{H}^{\frac{s}{2}} w_n\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} &\leq \sqrt{8C}\lambda \|w_{n-1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\ &\leq \sqrt{8C}\lambda^n \|w_0\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})}.\end{aligned}$$

It concludes that $\{\tilde{g}^n\}$ is a Cauchy sequence and the convergence of the sequences

$$\begin{aligned} g &= \lim_{n\to+\infty} \tilde{g}_n \quad \text{in} \quad \widetilde{L}^\infty_T([0,T];\widetilde{L}^2_v(B^{1/2}_{2,1})), \\ G &= \lim_{n\to+\infty} \mathcal{H}^{\frac{s}{2}} \tilde{g}_n \quad \text{in} \quad \widetilde{L}^2_T([0,T];\widetilde{L}^2_v(B^{1/2}_{2,1})). \end{aligned} \tag{4.44}$$

Let $H = \mathbf{1}_{\mathbb{R}_+}$ be the Heaviside function. The convergence

$$\begin{aligned} H(t)g &= \lim_{n\to+\infty} H(t)\tilde{g}_n \quad \text{in} \quad \widetilde{L}^\infty_T((-\infty,T];\widetilde{L}^2_v(B^{1/2}_{2,1})), \\ H(t)G &= \lim_{n\to+\infty} H(t)\mathcal{H}^{\frac{s}{2}}\tilde{g}_n \quad \text{in} \quad \widetilde{L}^2_T((-\infty,T];\widetilde{L}^2_v(B^{1/2}_{2,1})) \end{aligned}$$

imply

$$H(t)g = \lim_{n\to+\infty} H(t)\tilde{g}_n, \quad H(t)G = \lim_{n\to+\infty} H(t)\mathcal{H}^{\frac{s}{2}}\tilde{g}_n \quad \text{in } \mathcal{D}'((-\infty,T),\mathcal{S}'(\mathbb{R}^2_{x,v})). \tag{4.45}$$

We obtain for all $\varphi \in \mathfrak{F} = C^\infty_0((-\infty,T),\mathcal{S}(\mathbb{R}^2_{x,v}))$

$$\begin{aligned} \langle H(t)G,\varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}} &= \lim_{n\to+\infty}\langle H(t)\mathcal{H}^{\frac{s}{2}}\tilde{g}_n,\varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}} = \lim_{n\to+\infty}\langle H(t)\tilde{g}_n,\mathcal{H}^{\frac{s}{2}}\varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}} \\ &= \langle H(t)g,\mathcal{H}^{\frac{s}{2}}\varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}} = \langle H(t)\mathcal{H}^{\frac{s}{2}}g,\varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}}, \end{aligned}$$

where $\mathfrak{F}^*$ stands for the anti-dual (anti-linear forms) of $\mathfrak{F}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{F}^*,\mathfrak{F}}$ denotes the duality bracket. Thus, it follows that

$$G = \mathcal{H}^{\frac{s}{2}} g. \tag{4.46}$$

Therefore, the limit function g is a desired solution to the Cauchy problem (4.1) and g satisfies

$$g(t,v,x) \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1}) \quad \text{and} \quad \mathcal{H}^{\frac{s}{2}} g(t,v,x) \in \widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1}).$$

Passing to the limit when $n \to \infty$ in the estimate (4.42) and following from (4.44), (4.45) and (4.46), we are led to

$$\|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \|\mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq c_0 e^T \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

We deduce from (4.45) that

$$\partial_t(H(t)g) = \lim_{n\to+\infty} \partial_t(H(t)\tilde{g}_n), \quad H(t)v\partial_x g = \lim_{n\to+\infty} H(t)v\partial_x\tilde{g}_n \tag{4.47}$$

with limits in $\mathfrak{F}$. Following from (4.4), (4.5) and (4.45), we get

$$H(t)\mathcal{K}g = \lim_{n\to+\infty} H(t)\mathcal{K}\tilde{g}_n \tag{4.48}$$

with limit in $\mathfrak{F}$. On the other hand, for all $\varphi \in \mathfrak{F}$ we obtain

$$\begin{aligned} &|\langle H(t)\Gamma(\tilde{g}_n, \tilde{g}_{n+1}), \varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}} - \langle H(t)\Gamma(g,g), \varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}}| \\ &\leq |\langle H(t)\Gamma(\tilde{g}_n - g, \tilde{g}_{n+1}), \varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}}| + |\langle H(t)\Gamma(g, \tilde{g}_{n+1} - g), \varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}}|. \end{aligned} \tag{4.49}$$

We deduce from Remark 4.1 that

$$\begin{aligned} &\sum_{p\geq -1} 2^{\frac{p}{2}} |\langle H(t)\Delta_p\Gamma(\tilde{g}_n - g, \tilde{g}_{n+1}), \Delta_p\varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}}|^{1/2} \\ &\leq \sum_{p\geq -1} 2^{\frac{p}{2}} \Big(\int_{\mathbb{R}} H(t) \Big| (\Delta_p\Gamma(\tilde{g}_n - g, \tilde{g}_{n+1}), \Delta_p\varphi)_{L^2(\mathbb{R}^2_{x,v})} \Big| \, dt \Big)^{1/2} \\ &\leq C_1 \|\tilde{g}_n - g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} \tilde{g}_{n+1}\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} \varphi\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\ &\leq \varepsilon_0 C_1 \|\mathcal{H}^{\frac{s}{2}} \varphi\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\tilde{g}_n - g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})}. \end{aligned} \tag{4.50}$$

We deduce from (4.44) and (4.50) that

$$\lim_{n\to+\infty} \langle H(t)\Gamma(\tilde{g}_n - g, \tilde{g}_{n+1}), \varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}} = 0. \tag{4.51}$$

Similarly, we deduce from Remark 4.1 that

$$\begin{aligned} &\sum_{p\geq -1} 2^{\frac{p}{2}} |\langle H(t)\Delta_p\Gamma(g, \tilde{g}_{n+1} - g), \Delta_p\varphi\rangle_{\mathfrak{F}^*,\mathfrak{F}}|^{1/2} \\ &\leq \sum_{p\geq -1} 2^{\frac{p}{2}} \Big(\int_{\mathbb{R}} H(t) \Big| (\Delta_p\Gamma(g, \tilde{g}_{n+1} - g), \Delta_p\varphi)_{L^2(\mathbb{R}^2_{x,v})} \Big| \, dt \Big)^{1/2} \\ &\leq C_1 \|g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} \tilde{g}_{n+1} - \mathcal{H}^{\frac{s}{2}} g\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} \varphi\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})}. \end{aligned} \tag{4.52}$$

It follows from (4.44) and (4.52) that

$$\lim_{n\to+\infty} H(t)\Gamma(g, \tilde{g}_{n+1} - g) = 0 \tag{4.53}$$

with limit in $\mathfrak{F}$. We deduce from (4.49), (4.51) and (4.53) that

$$\lim_{n\to+\infty} H(t)\Gamma(\tilde{g}_n, \tilde{g}_{n+1}) = H(t)\Gamma(g,g) \tag{4.54}$$

with limit in $\mathfrak{F}$. It follows from (4.41), (4.47), (4.48) and (4.54) that

$$\partial_t(H(t)g) + v\partial_x H(t)g + \mathcal{K}H(t)g = H(t)\Gamma(g,g) + \delta_0(t) \otimes g_0,$$

namely,

$$\begin{cases} \partial_t g + v\partial_x g + \mathcal{K} g = \Gamma(f,g), \\ g(t,x,v)|_{t=0} = g_0(x,v). \end{cases}$$

Next, we prove the uniqueness of solutions to the Cauchy problem (4.1) when the initial fluctuation is sufficiently small as in (4.40). Let $g_1, g_2 \in \widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$ be two solutions of the Cauchy problem (4.1) satisfying

$$\|g_j\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{1}{4C_1^2 C^2 e^{2T}} \quad \text{and} \quad \mathcal{H}^{\frac{s}{2}} g_j(t,v,x) \in \widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1}), \quad \text{j=1,2}$$

where C, C_1 are the constants defined in (4.6) and Theorem 3.4. Set $h = g_1 - g_2$. This function satisfies the Cauchy problem

$$\begin{cases} \partial_t h + v\partial_x h + \mathcal{K} h = \Gamma(g_1, f) + \Gamma(f, g_2), \\ h(t,x,v)|_{t=0} = 0. \end{cases}$$

Then, with the similar computation as that of estimate (4.39), precisely, applying $\Delta_p (p \geq -1)$ to the above equality and taking the inner product with $2^p \Delta_p h$ over $\mathbb{R}_x \times \mathbb{R}_v$, we get

$$\begin{aligned} &\frac{d}{dt}\left(2^p \|\Delta_p h\|^2_{L^2_{x,v}}\right) + \frac{2}{C} 2^p \|\mathcal{H}^{\frac{s}{2}} \Delta_p h\|^2_{L^2_{x,v}} \\ &\leq 2^{p+1} \|\Delta_p h\|^2_{L^2_{x,v}} + 2^{p+1} \left(\Delta_p \Gamma(g_1, f), \Delta_p h\right)_{L^2_{x,v}} \\ &\quad + 2^{p+1} \left(\Delta_p \Gamma(f, g_2), \Delta_p h\right)_{L^2_{x,v}}. \end{aligned}$$

Taking the supremum over $0 \leq t \leq T$ on the left side and summing up over $p \geq -1$. We deduce from (4.37) that

$$\begin{aligned} &\|h\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{\frac{2}{C}} \|h\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\ &\leq \sqrt{2} e^T \sum_{p \geq -1} 2^{\frac{p}{2}} \Big(\int_0^T \Big| \left(\Delta_p \Gamma\left(g_2, h\right), \Delta_p h\right)_{L^2_{x,v}} \Big| dt \Big)^{1/2} \\ &+ \sqrt{2} e^T \sum_{p \geq -1} 2^{\frac{p}{2}} \Big(\int_0^T \Big| \left(\Delta_p \Gamma\left(h, g_1\right), \Delta_p h\right)_{L^2_{x,v}} \Big| dt \Big)^{1/2} \\ &\leq \sqrt{2} C_1 e^T \|g_2\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} h\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\ &\quad + \sqrt{2} C_1 e^T \|h\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} g_1\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} h\|^{1/2}_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\ &\leq \frac{2}{\sqrt{2C}} \|h\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} + 2\sqrt{C} C_1 e^T \|h\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|\mathcal{H}^{\frac{s}{2}} g_1\|_{\widetilde{L}^2_T \widetilde{L}^2_v(B^{1/2}_{2,1})}, \end{aligned} \tag{4.55}$$

which implies $h = 0$ with the aid of (4.40). Hence, we finish the proof of local existence in Theorem 4.3. $\square$

4.5 Gelfand-Shilov and Gevrey regularizing effect

In this section, we prove that the Cauchy problem (4.1) enjoys the Gelfand-Shilov regularizing properties with respect to the velocity variable v and Gevrey regularizing properties with respect to the position variable x.

4.5.1 A priori estimate with exponential weights

Firstly, it is shown that the sequence of approximate solutions $(\tilde{g}_n)_{n\geq 0}$ (which defined by (4.41)) satisfies a priori estimate with exponential weights for sufficiently small initial data.

Proposition 4.6. *Let $T > 0$. There exist some positive constants $C, \varepsilon_1 > 0, 0 < c_0 \leq 1$ such that for all initial data $\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \varepsilon_1$, the sequence of approximate solutions $(\tilde{g}_n)_{n\geq 0}$ satisfies*

$$\begin{aligned}&\|G_\kappa(ct)\tilde{g}_n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} + \|\mathcal{H}^{\frac{s}{2}}G_\kappa(ct)\tilde{g}_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\ &+ \|\langle D_x\rangle^{\frac{s}{2s+1}}G_\kappa(ct)\tilde{g}_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq Ce^{CT}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\end{aligned} \tag{4.56}$$

for $0 < \kappa \leq 1, 0 < c \leq c_0, n \geq 1$, where

$$G_\kappa(t) = \frac{\exp(t(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}{1+\kappa\exp(t(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}.$$

To prove Proposition 4.6, we need some lemmas. Firstly, we prove the hypoelliptic estimate for the principal part of the linear inhomogeneous Kac operator. Since the operator Δ_p acts on the position variable x only, we obtain the following estimate based on Lemma 4.3.

Lemma 4.17. *Let P be the operator defined in (4.9) and $M = m^w$ the self-adjoint operator defined by the Weyl quantization of the symbol (4.10). Then, the operator M is uniformly bounded on $L^2(\mathbb{R}_v)$ with respect to the parameter $\xi \in \mathbb{R}$, and there exist some positive constants $0 < \varepsilon_0 \leq 1, c_3, c_4 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, u \in \mathcal{S}(\mathbb{R}_v), \xi \in \mathbb{R}, p \geq -1$,*

$$\begin{aligned}&\mathrm{Re}(P\Delta_p u, (1-\varepsilon M)\Delta_p u)_{L^2(\mathbb{R}^2_v)}\\ &\geq c_3\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2(\mathbb{R}^2_v)} + c_3\langle\xi\rangle^{\frac{2s}{2s+1}}\|\Delta_p u\|^2_{L^2(\mathbb{R}^2_v)} - c_4\|\Delta_p u\|^2_{L^2(\mathbb{R}^2_v)},\end{aligned}$$

where $\mathcal{H} = -\Delta_v + \frac{v^2}{4}$ stands for the Harmonic Oscillator.

Proof. The proof is similar with that in [117]. Following from (4.9), for $0 < \varepsilon \leq 1$ we have

$$\begin{aligned}\mathrm{Re}(P\Delta_p u, (1-\varepsilon M)\Delta_p u)_{L^2(\mathbb{R}^2_v)} &= \mathrm{Re}(a^w_0(v, D_v)\Delta_p u, \Delta_p u)_{L^2(\mathbb{R}^2_v)}\\ &- \varepsilon\mathrm{Re}(iv\xi\Delta_p u, M\Delta_p u)_{L^2(\mathbb{R}^2_v)} - \varepsilon\mathrm{Re}(a^w_0(v, D_v)\Delta_p u, M\Delta_p u)_{L^2(\mathbb{R}^2_v)}.\end{aligned} \tag{4.57}$$

Since a_0 is real-valued, we have $\mathrm{Re}(a_0^w(v,D_v)\Delta_p u,\Delta_p u)_{L^2(\mathbb{R}^2_v)} = (a_0^w(v,D_v)\Delta_p u,\Delta_p u)_{L^2(\mathbb{R}^2_v)}$. And in the class $S(\langle(v,\eta)\rangle^s,\Gamma_0)$, it holds that

$$a_0 - c_0\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\sharp^w\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}} \in S(\langle(v,\eta)\rangle^{2s-2},\Gamma_0)\subset S(1,\Gamma_0) \tag{4.58}$$

for $0<s<1$. Following from (4.11), there exist some positive constants $\tilde{c}_0, C>0$ such that for all $u\in\mathcal{S}(\mathbb{R}_v)$,

$$\begin{aligned}(a_0^w(v,D_v)\Delta_p u,\Delta_p u)_{L^2(\mathbb{R}^2_v)} &\geq c_0\Big\|\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big)\Delta_p u\Big\|^2_{L^2} - C\|\Delta_p u\|^2_{L^2}\\ &\geq \tilde{c}_0\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2} - C\|\Delta_p u\|^2_{L^2}.\end{aligned} \tag{4.59}$$

For the third term in (4.57), since that the operator $M=m^w$ is uniformly bounded on $L^2(\mathbb{R}_v)$ with respect to the parameter $\xi\in\mathbb{R}$ and (4.11), (4.58), we have

$$\begin{aligned}&\mathrm{Re}(a_0^w(v,D_v)\Delta_p u, M\Delta_p u)_{L^2(\mathbb{R}^2_v)}\\ &= c_0\mathrm{Re}\Big(\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big)\Delta_p u, \mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big)M\Delta_p u\Big)_{L^2(\mathbb{R}^2_v)} + \mathcal{O}(\|\Delta_p u\|^2_{L^2})\\ &= R_1+R_2+\mathcal{O}(\|\Delta_p u\|^2_{L^2}),\end{aligned} \tag{4.60}$$

with

$$\begin{aligned}R_1 &= c_0\mathrm{Re}\Big(\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big)\Delta_p u, M\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big)\Delta_p u\Big)_{L^2(\mathbb{R}^2_v)}\\ &= \mathcal{O}(\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2}),\\ R_2 &= c_0\mathrm{Re}\Big(\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big)\Delta_p u, \Big[\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big), m^w\Big]\Delta_p u\Big)_{L^2(\mathbb{R}^2_v)}.\end{aligned} \tag{4.61}$$

On the other hand, since $m\in S(1,\Gamma_0)$ respecting to $\xi\in\mathbb{R}$ and using some symbolic calculus, it holds that

$$\Big[\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\frac{s}{2}}\Big), m^w\Big]\in \mathrm{Op}^w(S(\langle(v,\eta)\rangle^{s-2},\Gamma_0))\subset \mathrm{Op}^w(S(1,\Gamma_0)), \tag{4.62}$$

since $0<s<1$. Combining (4.11), (4.60), (4.61) and (4.62) gives that

$$\mathrm{Re}(a_0^w(v,D_v)\Delta_p u, M\Delta_p u)_{L^2(\mathbb{R}^2_v)} = \mathcal{O}(\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2}) \tag{4.63}$$

uniformly with respect to the parameter $\xi\in\mathbb{R}$. Finally, since the operator M and $iv\xi$ are respectively formally self-adjoint and skew-selfadjoint on L^2, we obtain

$$-\mathrm{Re}(iv\xi\Delta_p u, M\Delta_p u)_{L^2} = \frac{1}{2}([iv\xi, M]\Delta_p u,\Delta_p u)_{L^2}. \tag{4.64}$$

By using some symbolic calculus, it holds that $\frac{1}{2}\{v\xi, m\}$ is the Weyl symbol of the commutator $\frac{1}{2}[v\xi, M]$. With Lemma 4.2, the symbol r is a first order symbol for the symbolic calculus associated to the class $S(\langle(v,\eta)\rangle^2, \Gamma_0)$ and r satisfies

$$|r| \lesssim \Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s.$$

Then, in the class $S(\langle(v,\eta)\rangle^2, \Gamma_0)$ we can deduce from the Gårding inequality (see e.g. [106], Theorem 2.5.4) that

$$\big|(r^w \Delta_p u, \Delta_p u)_{L^2}\big| \leq \Big(\mathrm{Op}^w\Big(\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s\Big)\Delta_p u, \Delta_p u\Big) + \mathcal{O}\left(\|\Delta_p u\|^2_{L^2}\right) \tag{4.65}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}$. Therefore, it follows from (4.11) and (4.65) that

$$|(r^w \Delta_p u, \Delta_p u)_{L^2}| \leq \Big\|\mathrm{Op}^w\Big(\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^{\frac{s}{2}}\Big)\Delta_p u\Big\|^2_{L^2} + \|\Delta_p u\|^2_{L^2} = \mathcal{O}\left(\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2}\right) \tag{4.66}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}$. Put

$$\Psi = \frac{\xi^2}{2\lambda^{\frac{2+2s}{2s+1}}}\psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big), \tag{4.67}$$

we deduce from (4.64), (4.65) and (4.66) that there exists a constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v), \xi \in \mathbb{R}$,

$$-\mathrm{Re}(iv\xi\Delta_p u, M\Delta_p u)_{L^2} \geq (\Psi^w \Delta_p u, \Delta_p u)_{L^2} - c\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2}. \tag{4.68}$$

It follows from (4.57), (4.59), (4.63) and (4.68) that

$$\begin{aligned}&\mathrm{Re}(P\Delta_p u, (1-\varepsilon M)\Delta_p u)_{L^2(\mathbb{R}^2_v)}\\ &\geq \tilde{c}_0\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2} + \varepsilon(\Psi^w\Delta_p u, \Delta_p u)_{L^2} - C\|\Delta_p u\|^2_{L^2} - \varepsilon\mathcal{O}(\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2})\end{aligned}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}$. Choosing a value of the parameter $0 < \varepsilon_0 < 1$ and a new constant $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v), 0 < \varepsilon \leq \varepsilon_0, \xi \in \mathbb{R}$,

$$\begin{aligned}&\mathrm{Re}(P\Delta_p u, (1-\varepsilon M)\Delta_p u)_{L^2(\mathbb{R}^2_v)}\\ &\geq \frac{\tilde{c}_0}{2}\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2} + \varepsilon(\Psi^w\Delta_p u, \Delta_p u)_{L^2} - C\|\Delta_p u\|^2_{L^2}.\end{aligned} \tag{4.69}$$

To estimate the term $(\Psi^w\Delta_p u, \Delta_p u)_{L^2}$, according to the support of the function

$$\psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big),$$

we consider the calculation separately on the two regions of the phase space

$$v^2 + \eta^2 \lesssim \lambda^{\frac{2}{2s+1}}, \qquad v^2 + \eta^2 \gtrsim \lambda^{\frac{2}{2s+1}}.$$

One can find a constant $c_1 > 0$ such that for all $(v, \eta, \xi) \in \mathbb{R}^3$,

$$\Psi + \Phi \geq c_1 \lambda^{\frac{2s}{2s+1}} \geq c_1 \langle \xi \rangle^{\frac{2s}{2s+1}}, \tag{4.70}$$

where

$$\Phi = \Big(1 - \psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big)\Big)\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s.$$

Here, it follows from Lemma 4.2 that Ψ, Φ and $\langle \xi \rangle^{\frac{2s}{2s+1}}$ are first order symbols in the class $S(M, \Gamma_1)$ uniformly with respect to the parameter $\xi \in \mathbb{R}$. We can deduce from (4.67), (4.70) and the Gårding inequality that there exists a constant $c_2 > 0$ such that for $u \in \mathcal{S}(\mathbb{R}_v), \xi \in \mathbb{R}$,

$$\begin{aligned} &(\Psi^w \Delta_p u, \Delta_p u)_{L^2} + \Big(\mathrm{Op}^w\Big(\Big(1 - \psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big)\Big)\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s\Big)\Delta_p u, \Delta_p u\Big)_{L^2} \\ &\quad \geq c_1 \langle \xi \rangle^{\frac{2s}{2s+1}} \|\Delta_p u\|_{L^2}^2 - c_2 \|\Delta_p u\|_{L^2}^2. \end{aligned} \tag{4.71}$$

Additionally, since $0 < s < 1$, Lemma 4.2 and $S(n, \Gamma_1) \subset S(n, \Gamma_0)$, it holds that

$$\begin{aligned} &1 - \psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big) \in S(1, \Gamma_1) \subset S(1, \Gamma_0) \\ \text{and} \qquad &\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s \in S(\langle (v,\eta) \rangle^{2s}, \Gamma_0) \subset S(\langle (v,\eta) \rangle^2, \Gamma_0) \end{aligned}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}$. Thus, since

$$\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s \geq \Big(1 - \psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big)\Big)\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s$$

and using the Gårding inequality, there exists a constant $\tilde{c}_3 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v), \xi \in \mathbb{R}$,

$$\begin{aligned} &\Big(\mathrm{Op}^w\Big(\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s\Big)\Delta_p u, \Delta_p u\Big)_{L^2} \\ &\quad \geq \Big(\mathrm{Op}^w\Big(\Big(1 - \psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big)\Big)\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s\Big)\Delta_p u, \Delta_p u\Big)_{L^2} - \tilde{c}_3 \|\Delta_p u\|_{L^2}^2. \end{aligned}$$

Thus, there exists a positive constant $\tilde{c}_4 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v), \xi \in \mathbb{R}$,

$$\Big(\mathrm{Op}^w\Big(\Big(1 - \psi\Big(\frac{v^2+\eta^2}{\lambda^{\frac{2}{2s+1}}}\Big)\Big)\Big(1 + \eta^2 + \frac{v^2}{4}\Big)^s\Big)\Delta_p u, \Delta_p u\Big)_{L^2} \leq \tilde{c}_4 \|\mathcal{H}^{\frac{s}{2}} \Delta_p u\|_{L^2}^2. \tag{4.72}$$

It follow from (4.69), (4.71) and (4.72) that for all $u \in \mathcal{S}(\mathbb{R}_v), 0 < \varepsilon \leq \varepsilon_0, \xi \in \mathbb{R}$,

$$\begin{aligned} &\mathrm{Re}(P\Delta_p u, (1 - \varepsilon M)\Delta_p u)_{L^2(\mathbb{R}^2_v)} \\ &\geq (\frac{\tilde{c}_0}{2} - \tilde{c}_4 \varepsilon) \|\mathcal{H}^{\frac{s}{2}} \Delta_p u\|_{L^2}^2 + c_1 \varepsilon \langle \xi \rangle^{\frac{2s}{2s+1}} \|\Delta_p u\|_{L^2}^2 - (C + c_2 \varepsilon) \|\Delta_p u\|_{L^2}^2. \end{aligned}$$

There exist some positive constants $c_3, c_4 > 0$ with a new parameter $0 < \varepsilon_0 \le 1$ such that for all $u \in \mathcal{S}(\mathbb{R}_v), 0 < \varepsilon \le \varepsilon_0, \xi \in \mathbb{R}$,

$$\begin{aligned}&\mathrm{Re}(P\Delta_p u, (1-\varepsilon M)\Delta_p u)_{L^2(\mathbb{R}^2_v)}\\ &\ge c_3\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2(\mathbb{R}^2_v)} + c_3\langle\xi\rangle^{\frac{2s}{2s+1}}\|\Delta_p u\|^2_{L^2(\mathbb{R}^2_v)} - c_4\|\Delta_p u\|^2_{L^2(\mathbb{R}^2_v)}.\end{aligned}$$

The proof of Lemma 4.17 is completed. □

Lemma 4.18. *There exists a constant $c_5 > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^2_{x,v})$,*

$$\|(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{s}{3s+1}} f\|_{L^2_{x,v}} \le c_5\|\mathcal{H}^{\frac{(1+s)s}{2(3s+1)}} f\|_{L^2_{x,v}} + c_5\|\langle D_x\rangle^{\frac{s}{2s+1}} f\|_{L^2_{x,v}}.$$

Proof. We decompose f into the Hermite basis in the velocity variable

$$f(x,v) = \sum_{n=0}^{+\infty} f_n(x)e_n(v), \quad \text{with} \quad f_n(x) = (f(x,\cdot), e_n)_{L^2(\mathbb{R}_v)}. \tag{4.73}$$

Since

$$\forall 0 < \alpha < 1, \quad \forall a, b \ge 0, \quad (a+b)^\alpha \le a^\alpha + b^\alpha,$$

one can verify that

$$\begin{aligned}\|(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{s}{3s+1}} f\|_{L^2_{x,v}} &= \Big(\frac{1}{2\pi}\sum_{n=0}^{+\infty}\int_{\mathbb{R}}\Big((n+\frac{1}{2})^{\frac{s+1}{2}} + \langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}|\widehat{f_n}(\xi)|^2 d\xi\Big)^{1/2}\\ &\le \Big(\frac{1}{2\pi}\sum_{n=0}^{+\infty}\int_{\mathbb{R}}\Big[(n+\frac{1}{2})^{\frac{s(1+s)}{3s+1}} + \langle\xi\rangle^{\frac{2s}{2s+1}}\Big]|\widehat{f_n}(\xi)|^2 d\xi\Big)^{1/2}\\ &= \Big(\|\mathcal{H}^{\frac{s(s+1)}{2(3s+1)}} f\|^2_{L^2_{x,v}} + \|\langle D_x\rangle^{\frac{s}{2s+1}} f\|^2_{L^2_{x,v}}\Big)^{1/2}.\end{aligned}$$

□

Remark 4.6. Since the indices

$$\frac{(1+s)s}{2(3s+1)} < \frac{s}{2},$$

we always use the following result

$$\|(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{s}{3s+1}} f\|_{L^2_{x,v}} \le c_5\|\mathcal{H}^{\frac{s}{2}} f\|_{L^2_{x,v}} + c_5\|\langle D_x\rangle^{\frac{s}{2s+1}} f\|_{L^2_{x,v}}.$$

Lemma 4.19. *For all $0 \le \alpha \le 1$, there exists a constant $\tilde{c}_\alpha > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^2_{x,v})$,*

$$\|\mathcal{H}^\alpha Q\Delta_p f\|_{L^2_{x,v}} \le \tilde{c}_\alpha\|\mathcal{H}^\alpha\Delta_p f\|_{L^2_{x,v}}.$$

Proof. We can deduce from (4.11) that

$$\begin{aligned}\|\mathcal{H}^{\alpha}Q\Delta_p f\|_{L^2_{x,v}} &\lesssim \Big\|\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\alpha}\Big)Q\Delta_p f\Big\|_{L^2_{x,v}}\\ &\lesssim \Big\|Q\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\alpha}\Big)\Delta_p f\Big\|_{L^2_{x,v}}\\ &\quad+\Big\|\Big[\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\alpha}\Big),Q\Big]\Delta_p f\Big\|_{L^2_{x,v}},\end{aligned}\tag{4.74}$$

since $[Q,\Delta_p]=0$. Due to the fact that the multiplier Q is a bounded operator on $L^2(\mathbb{R}^2_{x,v})$ and (4.11), we can obtain

$$\begin{aligned}\Big\|Q\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\alpha}\Big)\Delta_p f\Big\|_{L^2_{x,v}} &\lesssim \Big\|\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\alpha}\Big)\Delta_p f\Big\|_{L^2_{x,v}}\\ &\lesssim \|\mathcal{H}^{\alpha}\Delta_p f\|_{L^2_{x,v}}.\end{aligned}\tag{4.75}$$

On the other hand, we deduce from (3.7), (3.8), (4.84) and Lemma 4.2 that

$$\Big[\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\alpha}\Big),Q\Big]\in\mathrm{Op}^w\Big(S(\langle(v,\eta)\rangle^{2\alpha-2},\Gamma_0)\Big)\subset\mathrm{Op}^w(S(1,\Gamma_0))$$

uniformly with respect to the parameter $\xi\in\mathbb{R}$ because $0\le\alpha\le 1$. It implies that

$$\Big\|\Big[\mathrm{Op}^w\Big(\Big(1+\eta^2+\frac{v^2}{4}\Big)^{\alpha}\Big),Q\Big]\Delta_p f\Big\|_{L^2_{x,v}}\lesssim\|\Delta_p f\|_{L^2_{x,v}}.\tag{4.76}$$

Combining (4.74), (4.75) and (4.76) gives

$$\|\mathcal{H}^{\alpha}Q\Delta_p f\|_{L^2_{x,v}}\le\tilde{c}_{\alpha}\|\mathcal{H}^{\alpha}\Delta_p f\|_{L^2_{x,v}}.$$

This gives the proof of Lemma 4.19. □

Lemma 4.20. *There exists a constant $\tilde{c}_1>0$ such that for all $f\in\mathcal{S}(\mathbb{R}^2_{x,v})$, $0\le c\le 1, 0<\kappa\le 1, t\ge 0$,*

$$\Big\|\mathcal{H}^{-\frac{s}{2}}\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\Delta_p\partial_x f\Big\|_{L^2_{x,v}}\le\tilde{c}_1 cte^{\tilde{c}_1 ct}\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p f\|_{L^2_{x,v}}.$$

Proof. For all $f\in\mathcal{S}(\mathbb{R}^2_{x,v})$, with the aid of the decomposition (4.73), using the identities (4.2)-(4.3) satisfied by the creation and annihilation operators

$$\begin{aligned}A_+e_n&=(\frac{v}{2}-\partial_v)e_n=\sqrt{n+1}e_{n+1};\\ A_-e_n&=(\frac{v}{2}+\partial_v)e_n=\sqrt{n}e_{n-1},\end{aligned}$$

we have immediately

$$\begin{aligned}ve_n&=A_+e_n+A_-e_n=\sqrt{n+1}e_{n+1}+\sqrt{n}e_{n-1},\\ \mathcal{H}&=\frac{1}{2}\left(A_+A_-+A_-A_+\right)e_n=\left(n+\frac{1}{2}\right)e_n.\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{\exp(ct(\mathcal{H}^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}{1+\kappa\exp(ct(\mathcal{H}^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}v\frac{1+\kappa\exp(ct(\mathcal{H}^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}{\exp(ct(\mathcal{H}^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}e_n\\
&=\frac{\exp(ct((n+\frac{3}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}{1+\kappa\exp(ct((n+\frac{3}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}\\
&\quad\times\frac{1+\kappa\exp(ct((n+\frac{1}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}{\exp(ct((n+\frac{1}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}\sqrt{n+1}e_{n+1}\\
&+\frac{\exp(ct((n-\frac{1}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}{1+\kappa\exp(ct((n-\frac{1}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}\\
&\quad\times\frac{1+\kappa\exp(ct((n+\frac{1}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}{\exp(ct((n+\frac{1}{2})^{\frac{1+s}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}\sqrt{n}e_{n-1}.
\end{aligned}$$

One can verify that

$$\begin{aligned}
&\mathcal{F}_x\left(\mathcal{H}^{-\frac{s}{2}}\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\Delta_p\partial_x f\right)\\
&=\sum_{n=0}^{+\infty}i\xi\widehat{\Delta_p f_n}(\xi)\sqrt{n+1}(n+\frac{3}{2})^{-\frac{s}{2}}A^+_{n,\langle\xi\rangle}e_{n+1}\\
&\quad+\sum_{n=1}^{+\infty}i\xi\widehat{\Delta_p f_n}(\xi)\sqrt{n}(n-\frac{1}{2})^{-\frac{s}{2}}A^-_{n,\langle\xi\rangle}e_{n-1},
\end{aligned}$$

where $\mathcal{F}_x$ stands for the partial Fourier transform with respect to the position variable x and

$$\begin{aligned}
A^+_{n,\langle\xi\rangle}&=\frac{\exp\left(ct\left((n+\frac{3}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}-ct\left((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\right)-1}{\left(1+\kappa\exp\left(ct\left((n+\frac{3}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\right)\right)};\\
A^-_{n,\langle\xi\rangle}&=\frac{\exp\left(ct\left((n-\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}-ct\left((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\right)-1}{\left(1+\kappa\exp\left(ct\left((n-\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\right)\right)}.
\end{aligned}\tag{4.77}$$

Then by the Plancherel theorem and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\sqrt{2\pi}\left\|\mathcal{H}^{-\frac{s}{2}}\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\Delta_p\partial_x f\right\|_{L^2_{x,v}}\\
&=\left\|\mathcal{F}_x\left(\mathcal{H}^{-\frac{s}{2}}\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\Delta_p\partial_x f\right)\right\|_{L^2_{\xi,v}}\\
&\lesssim\left(\sum_{n=0}^{+\infty}\|\xi\widehat{\Delta_p f_n}(\xi)A^+_{n,\langle\xi\rangle}\|^2_{L^2_\xi}\langle n\rangle^{1-s}\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{+\infty}\|\xi\widehat{\Delta_p f_n}(\xi)A^-_{n,\langle\xi\rangle}\|^2_{L^2_\xi}\langle n\rangle^{1-s}\right)^{\frac{1}{2}}
\end{aligned}\tag{4.78}$$

with $A^+_{n,\langle\xi\rangle}, A^-_{n,\langle\xi\rangle}$ defined in (4.77). Now we come to estimate $|A^+_{n,\langle\xi\rangle}|$ and $|A^-_{n,\langle\xi\rangle}|$. It follows from the mean value theorem that

$$\begin{aligned}
&\left((n+\frac{3}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}-\left((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\\
&=\frac{cts(1+s)}{3s+1}\left((n+\frac{1}{2}+\theta)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{-\frac{1+s}{3s+1}}(n+\frac{1}{2}+\theta)^{\frac{s-1}{2}}\quad(0<\theta<1),
\end{aligned}$$

which leads to

$$\begin{aligned}
0\le A^+_{n,\langle\xi\rangle}&\le\exp\left(\frac{2cts(1+s)}{3s+1}\right)\frac{cts(1+s)}{3s+1}\left((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{-\frac{1+s}{3s+1}}(n+\frac{1}{2})^{\frac{s-1}{2}}\\
&\le\exp\left(\frac{2cts(1+s)}{3s+1}\right)\frac{2cts(1+s)}{3s+1}\langle\xi\rangle^{-\frac{1+s}{2s+1}}\langle n\rangle^{\frac{s-1}{2}}.
\end{aligned}$$

This shows that, for all $n\ge 0$,

$$|A^+_{n,\langle\xi\rangle}|\le\exp\left(\frac{2cts(1+s)}{3s+1}\right)\frac{2cts(1+s)}{3s+1}\langle\xi\rangle^{-\frac{1+s}{2s+1}}\langle n\rangle^{\frac{s-1}{2}}. \tag{4.79}$$

On the other hand, we use the mean value theorem again,

$$\begin{aligned}
0\ge A^-_{n,\langle\xi\rangle}&\ge\exp\left(ct\left(\left(n-\frac{1}{2}\right)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}-ct\left(\left(n+\frac{1}{2}\right)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\right)-1\\
&=\exp\left(-\frac{cts(1+s)}{3s+1}\left(\left(n-\frac{1}{2}+\theta\right)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{-\frac{1+s}{3s+1}}\left(n-\frac{1}{2}+\theta\right)^{\frac{s-1}{2}}\right)-1\quad(0<\theta<1)\\
&=\exp\left(-\frac{c\tau ts(1+s)}{3s+1}\left(\left(n-\frac{1}{2}+\theta\right)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{-\frac{1+s}{3s+1}}\left(n-\frac{1}{2}+\theta\right)^{\frac{s-1}{2}}\right)\\
&\quad\times\frac{cts(1+s)}{3s+1}\left(\left(n-\frac{1}{2}+\theta\right)^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\right)^{-\frac{1+s}{3s+1}}\left(n-\frac{1}{2}+\theta\right)^{\frac{s-1}{2}}\quad(0<\theta<1).
\end{aligned}$$

Then for all $n\ge 1$, we have

$$|A^-_{n,\langle\xi\rangle}|\le\frac{2cts(1+s)}{3s+1}\langle\xi\rangle^{-\frac{1+s}{2s+1}}\langle n\rangle^{\frac{s-1}{2}}. \tag{4.80}$$

Substituting the results (4.79) and (4.80) into (4.78), we conclude that

$$\begin{aligned}
&\left\|\mathcal{H}^{-\frac{s}{2}}\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\Delta_p\partial_x f\right\|_{L^2_{x,v}}\\
&\le\exp\left(\frac{2cts(1+s)}{3s+1}\right)\frac{2cts(1+s)}{3s+1}\frac{1}{\sqrt{2\pi}}\\
&\quad\times\left(\left(\sum_{n=0}^{+\infty}\|\langle\xi\rangle^{\frac{s}{2s+1}}\widehat{\Delta_p f_n}(\xi)\|^2_{L^2_\xi}\right)^{\frac12}+\left(\sum_{n=1}^{+\infty}\|\langle\xi\rangle^{\frac{s}{2s+1}}\widehat{\Delta_p f_n}(\xi)\|^2_{L^2_\xi}\right)^{\frac12}\right)\\
&\le\tilde{c}_1cte^{\tilde{c}_1ct}\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p f\|_{L^2_{x,v}}.
\end{aligned}$$

The proof Lemma 4.20 is completed. □

The proof of Proposition 4.6

Proof.

Let $0 \leq c \leq 1$ and $0 < \kappa \leq 1$. Define

$$h_{n,c,\kappa} = G_\kappa(ct)\tilde{g}_n, \quad n \geq 0. \tag{4.81}$$

The function $h_{n,c,\kappa}$ depends on the parameters $0 \leq c \leq 1$ and $0 < \kappa \leq 1$. Here, we write h_n for $h_{n,c,\kappa}$ for simplicity. Notice that

$$h_0(t) = (1 + \kappa \exp(t(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}))^{-1} g_0, \quad 0 \leq t \leq T$$

satisfies

$$\|h_0\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}. \tag{4.82}$$

By using (4.81) that

$$\tilde{g}_n = (G_\kappa(ct))^{-1} h_n = (\kappa + \exp(-ct(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}))h_n,$$

then the equation

$$\partial_t \tilde{g}_{n+1} + v\partial_x \tilde{g}_{n+1} + \mathcal{K}\tilde{g}_{n+1} = \Gamma(\tilde{g}_n, \tilde{g}_{n+1})$$

can be rewritten as

$$\begin{aligned}
&(G_\kappa(ct))^{-1}\partial_t h_{n+1} + v\partial_x (G_\kappa(ct))^{-1} h_{n+1} + (G_\kappa(ct))^{-1}\mathcal{K}h_{n+1} \\
&\quad - c(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}} \exp(-ct(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}) h_{n+1} \\
&= \Gamma((G_\kappa(ct))^{-1} h_n, (G_\kappa(ct))^{-1} h_{n+1}).
\end{aligned}$$

Due to (4.4), the linearized Kac operator $\mathcal{K} = f(\mathcal{H})$ is a function of the Harmonic Oscillator acting on the velocity variable v, which can commute with the exponential weight $(G_\kappa(ct))^{-1}$. Applying $\Delta_p (p \geq -1)$ to the resulting equality, we have

$$\begin{aligned}
&\partial_t \Delta_p h_{n+1} + G_\kappa(ct) v (G_\kappa(ct))^{-1} \partial_x \Delta_p h_{n+1} + \mathcal{K}\Delta_p h_{n+1} \\
&\quad - \frac{c(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}}{1 + \kappa \exp(ct(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})} \Delta_p h_{n+1} \\
&= G_\kappa(ct)\Delta_p \Gamma((G_\kappa(ct))^{-1} h_n, (G_\kappa(ct))^{-1} h_{n+1}).
\end{aligned} \tag{4.83}$$

According to Lemma 4.3 and (4.10) in Section 4.2, we choose the positive parameter $0 < \varepsilon \leq \varepsilon_0$ in order to ensure that the multiplier

$$Q = Q(v, D_v, D_x) = 1 - \varepsilon m^w(v, D_v, D_x) \tag{4.84}$$

is a positive bounded isomorphism on $L^2(\mathbb{R}^2_{x,v})$.

By integrating with respect to the $\xi-$variable and using the multiplier $Q\Delta_p h_{n+1}$ in $L^2(\mathbb{R}^2_{x,v})$, we deduce from (4.83) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Q^{1/2}\Delta_p h_{n+1}\|^2_{L^2_{x,v}} + \mathrm{Re}(\mathcal{K}\Delta_p h_{n+1}, Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}\\
&\quad + \mathrm{Re}(G_\kappa(ct)v(G_\kappa(ct))^{-1}\partial_x\Delta_p h_{n+1}, Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}\\
&\quad - \mathrm{Re}\Big(\frac{c(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}}{\exp(-ct(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}})}\Delta_p h_{n+1}, Q\Delta_p h_{n+1}\Big)_{L^2(\mathbb{R}^2_{x,v})}\\
&= \mathrm{Re}(G_\kappa(ct)\Delta_p\Gamma((G_\kappa(ct))^{-1}h_n, (G_\kappa(ct))^{-1}h_{n+1}), Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})},
\end{aligned} \tag{4.85}$$

which leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Q^{1/2}\Delta_p h_{n+1}\|^2_{L^2_{x,v}} + \mathrm{Re}((v\partial_x + \mathcal{K})\Delta_p h_{n+1}, Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}\\
&\quad + \mathrm{Re}([G_\kappa(ct)v(G_\kappa(ct))^{-1} - v]\partial_x\Delta_p h_{n+1}, Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}\\
&\le c\|(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{s}{3s+1}}\Delta_p h_{n+1}\|_{L^2_{x,v}}\|(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{s}{3s+1}}Q\Delta_p h_{n+1}\|_{L^2_{x,v}}\\
&\quad + |(G_\kappa(ct)\Delta_p\Gamma((G_\kappa(ct))^{-1}h_n, (G_\kappa(ct))^{-1}h_{n+1}), Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}|.
\end{aligned} \tag{4.86}$$

It follows from Lemma 4.17 that

$$\begin{aligned}
&\mathrm{Re}((v\partial_x + \mathcal{K})\Delta_p u, Q\Delta_p u)_{L^2(\mathbb{R}^2_{x,v})}\\
&\ge c_3\|\mathcal{H}^{\frac{s}{2}}\Delta_p u\|^2_{L^2(\mathbb{R}^2_{x,v})} + c_3\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p u\|^2_{L^2(\mathbb{R}^2_{x,v})} - c_4\|\Delta_p u\|^2_{L^2(\mathbb{R}^2_{x,v})}
\end{aligned} \tag{4.87}$$

for some constants $c_3, c_4 > 0$. We deduce from (4.86) and (4.87) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Q^{1/2}\Delta_p h_{n+1}\|^2_{L^2_{x,v}} + c_3\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}\|^2_{L^2(\mathbb{R}^2_{x,v})} + c_3\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}\|^2_{L^2(\mathbb{R}^2_{x,v})}\\
&\le c\|(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{s}{3s+1}}\Delta_p h_{n+1}\|_{L^2_{x,v}}\|(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{s}{3s+1}}Q\Delta_p h_{n+1}\|_{L^2_{x,v}}\\
&\quad + |([G_\kappa(ct)v(G_\kappa(ct))^{-1} - v]\partial_x\Delta_p h_{n+1}, Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}|\\
&\quad + |(G_\kappa(ct)\Delta_p\Gamma((G_\kappa(ct))^{-1}h_n, (G_\kappa(ct))^{-1}h_{n+1}), Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}|\\
&\quad + c_4\|\Delta_p h_{n+1}\|^2_{L^2(\mathbb{R}^2_{x,v})}.
\end{aligned} \tag{4.88}$$

By using Lemma 4.18 for f being replaced by $\Delta_p h_{n+1}, Q\Delta_p h_{n+1}$ and Lemma 4.19, we have

$$\begin{aligned}
&\Big\|\Big(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{s}{3s+1}}\Delta_p h_{n+1}\Big\|_{L^2_{x,v}}\Big\|\Big(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{s}{3s+1}}Q\Delta_p h_{n+1}\Big\|_{L^2_{x,v}}\\
&\lesssim \Big(\Big\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}\Big\|_{L^2_{x,v}} + \Big\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}\Big\|_{L^2_{x,v}}\Big)\\
&\qquad\times\Big(\Big\|\mathcal{H}^{\frac{s}{2}}Q\Delta_p h_{n+1}\Big\|_{L^2_{x,v}} + \Big\|\langle D_x\rangle^{\frac{s}{2s+1}}Q\Delta_p h_{n+1}\Big\|_{L^2_{x,v}}\Big)\\
&\lesssim \Big\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}\Big\|^2_{L^2_{x,v}} + \Big\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}\Big\|^2_{L^2_{x,v}}.
\end{aligned}$$

Based on Lemma 4.19 and Lemma 4.20, we obtain

$$\begin{aligned}
&\left|\left(\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\partial_x\Delta_p h_{n+1}, Q\Delta_p h_{n+1}\right)_{L^2(\mathbb{R}^2_{x,v})}\right| \\
&= \left|\left(\mathcal{H}^{-\frac{s}{2}}\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\partial_x\Delta_p h_{n+1}, \mathcal{H}^{\frac{s}{2}}Q\Delta_p h_{n+1}\right)_{L^2(\mathbb{R}^2_{x,v})}\right| \\
&\le \left\|\mathcal{H}^{-\frac{s}{2}}\left[G_\kappa(ct)v(G_\kappa(ct))^{-1}-v\right]\partial_x\Delta_p h_{n+1}\right\|_{L^2_{x,v}}\left\|\mathcal{H}^{\frac{s}{2}}Q\Delta_p h_{n+1}\right\|_{L^2_{x,v}} \\
&\le c_8 cte^{c_8 ct}\left\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}\right\|_{L^2_{x,v}}\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}\right\|_{L^2_{x,v}},
\end{aligned}$$

since Q is commuting with any function of the operator D_x. Then, it follows from (4.88) that there exists some positive constants $0<c_0\le 1, c_9>0$ such that for $0\le c\le c_0, 0<\kappa\le 1, 0\le t\le T$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|Q^{1/2}\Delta_p h_{n+1}\right\|^2_{L^2_{x,v}} + \left(c_9 - c_8 cTe^{c_8 T}\right)\Big(\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}\right\|^2_{L^2(\mathbb{R}^2_{x,v})} \\
&+ \left\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}\right\|^2_{L^2(\mathbb{R}^2_{x,v})}\Big) \le c_4\left\|\Delta_p h_{n+1}\right\|^2_{L^2(\mathbb{R}^2_{x,v})} \\
&\quad + \left|\left(G_\kappa(ct)\Delta_p\Gamma((G_\kappa(ct))^{-1}h_n, (G_\kappa(ct))^{-1}h_{n+1}), Q\Delta_p h_{n+1}\right)_{L^2(\mathbb{R}^2_{x,v})}\right|.
\end{aligned}$$

Here, the constant $0<c_0\le 1$ is chosen sufficiently small so that

$$c_8 cTe^{c_8 T}\le\frac{c_9}{2},$$

then we obtain that for all $0\le c\le c_0, 0<\kappa\le 1, 0\le t\le T$,

$$\begin{aligned}
&\frac{d}{dt}\left\|Q^{1/2}\Delta_p h_{n+1}\right\|^2_{L^2_{x,v}} + c_9\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}\right\|^2_{L^2_{x,v}} + c_9\left\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}\right\|^2_{L^2_{x,v}} \\
&\le 2c_{10}\left\|Q^{1/2}\Delta_p h_{n+1}\right\|^2_{L^2_{x,v}} + 2\left|\left(G_\kappa(ct)\Delta_p\Gamma((G_\kappa(ct))^{-1}h_n, (G_\kappa(ct))^{-1}h_{n+1}), Q\Delta_p h_{n+1}\right)_{L^2_{x,v}}\right|
\end{aligned}$$

with $c_{10}=c_4\|(Q^{1/2})^{-1}\|_{\mathcal{L}(L^2)}>0$. Following from (4.41), (4.81) and (4.82), for all $0\le c\le c_0, 0<\kappa\le 1, 0\le t\le T$ we are led to

$$\begin{aligned}
&\left\|Q^{1/2}\Delta_p h_{n+1}\right\|^2_{L^2_{x,v}} \\
&+ c_9\int_0^t e^{2c_{10}(t-\tau)}\left(\left\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}(\tau)\right\|^2_{L^2_{x,v}} + \left\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}(\tau)\right\|^2_{L^2_{x,v}}\right)d\tau \\
&\le e^{2c_{10}t}\left\|Q^{1/2}\Delta_p g_0\right\|^2_{L^2_{x,v}} + 2\int_0^t e^{2c_{10}(t-\tau)} \\
&\times\left|\left(G_\kappa(c\tau)\Delta_p\Gamma((G_\kappa(c\tau))^{-1}h_n(\tau), (G_\kappa(c\tau))^{-1}h_{n+1}(\tau)), Q\Delta_p h_{n+1}(\tau)\right)_{L^2(\mathbb{R}^2_{x,v})}\right|d\tau \\
&\le e^{2c_{10}t}\left\|Q^{1/2}\right\|^2_{\mathcal{L}(L^2)}\left\|\Delta_p g_0\right\|^2_{L^2_{x,v}} \\
&+ 2\int_0^t e^{2c_{10}(t-\tau)} \\
&\times\left|\left(G_\kappa(c\tau)\Delta_p\Gamma((G_\kappa(c\tau))^{-1}h_n(\tau), (G_\kappa(c\tau))^{-1}h_{n+1}(\tau)), Q\Delta_p h_{n+1}(\tau)\right)_{L^2(\mathbb{R}^2_{x,v})}\right|d\tau.
\end{aligned}$$

Taking the square root of the above inequality and taking the supremum over $0 \leq t \leq T$ give

$$\begin{aligned}
&\big\|Q^{1/2}\Delta_p h_{n+1}\big\|_{L^\infty_T L^2_{x,v}} + \sqrt{c_9}\,\Big\|\mathcal{H}^{\frac{s}{2}}\Delta_p h_{n+1}\Big\|_{L^2_T L^2_{x,v}} + \sqrt{c_9}\,\Big\|\langle D_x\rangle^{\frac{s}{2s+1}}\Delta_p h_{n+1}\Big\|_{L^2_T L^2_{x,v}}\\
&\leq e^{c_{10}T}\|Q^{1/2}\|_{\mathcal{L}(L^2)}\,\|\Delta_p g_0\|_{L^2_{x,v}}\\
&+ \sqrt{2}e^{c_{10}T}\left(\int_0^T \Big|\big(G_\kappa(ct)\Delta_p\Gamma((G_\kappa(ct))^{-1}h_n,(G_\kappa(ct))^{-1}h_{n+1}),Q\Delta_p h_{n+1}\big)_{L^2_{x,v}}\Big|\,dt\right)^{1/2}.
\end{aligned}$$

Multiply the above inequality by $2^{\frac{p}{2}}$ and take the summation over $p \geq -1$, we have

$$\begin{aligned}
&\|Q^{1/2}h_{n+1}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{c_9}\|\mathcal{H}^{\frac{s}{2}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{c_9}\|\langle D_x\rangle^{\frac{s}{2s+1}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\leq e^{c_{10}T}\|Q^{1/2}\|_{\mathcal{L}(L^2)}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{2}e^{c_{10}T}\sum_{p\geq -1}2^{\frac{p}{2}}\\
&\times\left(\int_0^T\Big|\Big(G_\kappa(ct)\Delta_p\Gamma((G_\kappa(ct))^{-1}h_n,(G_\kappa(ct))^{-1}h_{n+1}),Q\Delta_p h_{n+1})_{L^2(\mathbb{R}^2_{x,v})}\Big)\Big|\,dt\right)^{1/2}.
\end{aligned}$$

It follows from Lemma 4.15 that

$$\begin{aligned}
&\|h_{n+1}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{c_9}\|\mathcal{H}^{\frac{s}{2}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})} + \sqrt{c_9}\|\langle D_x\rangle^{\frac{s}{2s+1}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\leq e^{c_{10}T}\|Q^{1/2}\|_{\mathcal{L}(L^2)}\|(Q^{1/2})^{-1}\|_{\mathcal{L}(L^2)}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\quad + \sqrt{2}e^{c_{10}T}C_1\|h_n\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})} \qquad (4.89)\\
&\leq e^{c_{10}T}\|Q^{1/2}\|_{\mathcal{L}(L^2)}\|(Q^{1/2})^{-1}\|_{\mathcal{L}(L^2)}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\quad + e^{c_{10}T}c_{11}\|h_n\|^{1/2}_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\|\mathcal{H}^{\frac{s}{2}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}.
\end{aligned}$$

Next, we use the mathematical induction argument to show that

$$\|h_n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{c_9}{2c_{11}^2e^{2c_{10}T}}$$

for $n \geq 0$. In the case of $n = 0$, owing to the assumption

$$\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \varepsilon_1, \text{ with } 0 < \varepsilon_1 = \inf(\tilde{\varepsilon}_0, \frac{c_9}{2c_{11}^2e^{2c_{10}T}}, \frac{c_9}{2c_{11}^2e^{3c_{10}T}\|Q^{1/2}\|_{\mathcal{L}(L^2)}}) \leq \tilde{\varepsilon}_0, \qquad (4.90)$$

where the positive parameter $\tilde{\varepsilon}_0 > 0$ is defined in (4.40), we deduce from (4.82) that

$$\|h_0\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{c_9}{2c_{11}^2e^{2c_{10}T}}.$$

In the case of $n \geq 1$, if we assume that

$$\|h_n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{c_9}{2c_{11}^2e^{2c_{10}T}},$$

then it follows from (4.89) that

$$\begin{aligned}&\|h_{n+1}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{\frac{c_9}{2}}\|\mathcal{H}^{\frac{s}{2}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{c_9}\|\langle D_x\rangle^{\frac{s}{2s+1}}h_{n+1}\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\&\leq e^{c_{10}T}\|Q^{1/2}\|_{\mathcal{L}(L^2)}\|(Q^{1/2})^{-1}\|_{\mathcal{L}(L^2)}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.\end{aligned}\tag{4.91}$$

Together with (4.90) and (4.91), we deduce that

$$\|h_{n+1}\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\leq\frac{c_9}{2c_{11}^2e^{2c_{10}T}}.$$

Hence, it follows from (4.91) that for $0\leq c\leq c_0, 0<\kappa\leq 1, n\geq 1$,

$$\begin{aligned}&\|h_n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{\frac{c_9}{2}}\|\mathcal{H}^{\frac{s}{2}}h_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\sqrt{c_9}\|\langle D_x\rangle^{\frac{s}{2s+1}}h_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\&\leq e^{c_{10}T}\|Q^{1/2}\|_{\mathcal{L}(L^2)}\|(Q^{1/2})^{-1}\|_{\mathcal{L}(L^2)}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.\end{aligned}$$

This ends the proof of Proposition 4.6. □

Based on Proposition 4.6, by passing the limit when $\kappa\to 0_+$ in the estimate (4.56), it follows from the monotone convergence theorem that the following lemma:

Lemma 4.21. *Let $T>0$. Then, there exist some constants $C,\varepsilon_1>0, 0<c_0\leq 1$ such that for all initial data $\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\leq\varepsilon_1$, the sequence of approximate solutions $(\tilde{g}_n)_{n\geq 0}$ satisfies*

$$\begin{aligned}&\|G_0(ct)\tilde{g}_n\|_{\widetilde{L}^\infty_T\widetilde{L}^2_v(B^{1/2}_{2,1})}+\|\mathcal{H}^{\frac{s}{2}}G_0(ct)\tilde{g}_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\\&\quad+\|\langle D_x\rangle^{\frac{s}{2s+1}}G_0(ct)\tilde{g}_n\|_{\widetilde{L}^2_T\widetilde{L}^2_v(B^{1/2}_{2,1})}\leq Ce^{CT}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\end{aligned}$$

for all $0<c\leq c_0, n\geq 1$, where

$$G_0(t)=\exp(t(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}).$$

4.5.2 Gelfand-Shilov and Gevrey regularities

It follows from the Cauchy-Schwarz inequality that for all $0<c\leq c_0, 0<\kappa\leq 1$,

$$\begin{aligned}\|G_\kappa(ct)\Delta_p f\|^2_{L^2_{x,v}}&=\left(\frac{\exp(2ct(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{1+3s}{1+2s}})^{\frac{2s}{1+3s}})}{(1+\kappa\exp(t(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{1+3s}{1+2s}})^{\frac{2s}{1+3s}}))^2}\Delta_p f,\Delta_p f\right)_{L^2(\mathbb{R}^2_{x,v})}\\&\leq\|G_0(2ct)\Delta_p f\|_{L^2_{x,v}}\|\Delta_p f\|_{L^2_{x,v}}.\end{aligned}$$

By passing to the limit $\kappa\to 0_+$ in the above inequality, it follows from the monotone convergence theorem that for all $0<c\leq c_0$,

$$\|G_0(ct)\Delta_p f\|^2_{L^2_{x,v}}\leq\|G_0(2ct)\Delta_p f\|_{L^2_{x,v}}\|\Delta_p f\|_{L^2_{x,v}}.$$

It implies that for all $0 < c \leq c_0$,

$$\|G_\kappa(ct)\Delta_p f\|_{L^\infty_T L^2_{x,v}} \leq \|G_0(2ct)\Delta_p f\|^{1/2}_{L^\infty_T L^2_{x,v}} \|\Delta_p f\|^{1/2}_{L^\infty_T L^2_{x,v}}.$$

We can obtain

$$\begin{aligned}
\|G_0(ct)f\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} &= \sum_{p\geq -1} 2^{\frac{p}{2}} \|G_\kappa(ct)\Delta_p f\|_{L^\infty_T L^2_{x,v}} \\
&\leq \Big(\sum_{p\geq -1} 2^{\frac{p}{2}} \|G_0(2ct)\Delta_p f\|_{L^\infty_T L^2_{x,v}}\Big)^{1/2} \Big(\sum_{p\geq -1} 2^{\frac{p}{2}} \|\Delta_p f\|_{L^\infty_T L^2_{x,v}}\Big)^{1/2} \\
&\leq \|G_0(2ct)f\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \|f\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})}.
\end{aligned} \tag{4.92}$$

For the solutions $(\tilde{g}_n)_{n\geq 0}$ defined in (4.41), by using Lemma 4.21 and (4.92), we can obtain that for $0 \leq c \leq \frac{c_0}{2}$,

$$\begin{aligned}
&\|G_0(ct)\tilde{g}_{n+p} - G_0(ct)\tilde{g}_n\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq 2\sqrt{C} e^{\frac{CT}{2}} \|g_0\|^{1/2}_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \|\tilde{g}_{n+p} - \tilde{g}_n\|^{1/2}_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})},
\end{aligned}$$

which implies that $(G_0(ct)\tilde{g}_n)_{n\geq 1}$ is a Cauchy sequence in $\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$. Let h be the limit of the Cauchy sequence $(G_0(\frac{c_0}{2}t)\tilde{g}_n)_{n\geq 1}$ in the space $\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$. Notice that

$$\begin{aligned}
&\Big\|\tilde{g}_n - \Big(G_0\Big(\frac{c_0}{2}t\Big)\Big)^{-1} h\Big\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&= \Big\|(G_0(\frac{c_0}{2}t))^{-1}\Big(G_0\Big(\frac{c_0}{2}t\Big)\tilde{g}_n - h\Big)\Big\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \Big\|G_0\Big(\frac{c_0}{2}t\Big)\tilde{g}_n - h\Big\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})},
\end{aligned}$$

then following from the convergence of the sequences $\{\tilde{g}_n\}$ in $\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})$ and the uniqueness of the solution to the Cauchy problem (4.1), we have

$$g = \Big(G_0\Big(\frac{c_0}{2}t\Big)\Big)^{-1} h = \exp\left(-\frac{c_0}{2}t\left(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\right) h.$$

On the other hand, we can deduce from Lemma 4.21 that

$$\Big\|G_0(\frac{c_0}{2}t)\tilde{g}_n\Big\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \leq Ce^{CT}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}. \tag{4.93}$$

Passing to the limit in the above estimate (4.93) when $n \to +\infty$, we obtain

$$\begin{aligned}
\|h\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} &= \|\exp\big(\frac{c_0 t}{2}\big(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\big)^{\frac{2s}{3s+1}}\big) g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq Ce^{CT}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.
\end{aligned} \tag{4.94}$$

By using the following elementary inequality

$$\forall x, c > 0, \quad x^k \exp\left(-\frac{3s+1}{s} c x^{\frac{s}{3s+1}}\right) \leq \frac{(k!)^{\frac{3s+1}{s}}}{c^{\frac{3s+1}{s}k}},$$

we can deduce that for $f \in \mathcal{S}(\mathbb{R}^2_{x,v})$ and all $k \geq 0$,

$$\begin{aligned}
&\left\| \left(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\right)^k \exp\left(-\frac{c_0 t}{2}\left(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\right)^{\frac{2s}{3s+1}}\right) \Delta_p f \right\|_{L^2_{x,v}} \\
&= \left(\frac{1}{2\pi}\sum_{n=0}^{+\infty} \int_{\mathbb{R}} \left((n+\frac{1}{2})^{\frac{s+1}{2}} + \langle \xi\rangle^{\frac{1+3s}{1+2s}}\right)^{2k} \exp\Big(-c_0 t\Big((n+\frac{1}{2})^{\frac{s+1}{2}} + \langle \xi\rangle^{\frac{1+3s}{1+2s}}\Big)^{\frac{2s}{1+3s}}\Big) \left|\widehat{\Delta_p f_n}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\leq \Big(\frac{3s+1}{s c_0}\Big)^{\frac{3s+1}{2s}k} \frac{(k!)^{\frac{3s+1}{2s}}}{t^{\frac{3s+1}{2s}k}} \|\Delta_p f\|_{L^2_{x,v}}.
\end{aligned} \tag{4.95}$$

Then it follows from (4.94) and (4.95) that the solution to the Cauchy problem (4.1) satisfies for all $0 < t \leq T, k \geq 0$,

$$\begin{aligned}
&\| \left(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\right)^k g\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \Big(\frac{3s+1}{s c_0}\Big)^{\frac{3s+1}{2s}k} \frac{(k!)^{\frac{3s+1}{2s}}}{t^{\frac{3s+1}{2s}k}} \| \exp\big(\frac{c_0 t}{2}\big(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\big)^{\frac{2s}{3s+1}}\big) g\|_{\widetilde{L}^\infty_T \widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&\leq \Big(\frac{3s+1}{s c_0}\Big)^{\frac{3s+1}{2s}k} \frac{(k!)^{\frac{3s+1}{2s}}}{t^{\frac{3s+1}{2s}k}} C e^{CT} \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.
\end{aligned} \tag{4.96}$$

By (4.96), we obtain that there exists a positive constant $C > 1$ such that $\forall 0 < t \leq T, k \geq 0$,

$$\left\| \left(\mathcal{H}^{\frac{s+1}{2}} + \langle D_x\rangle^{\frac{3s+1}{2s+1}}\right)^k g(t) \right\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \leq \frac{C^{k+1}}{t^{\frac{3s+1}{2s}k}} (k!)^{\frac{3s+1}{2s}} \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

This proves the Gelfand-Shilov property in Theorem 4.1.

On the other hand, we have for $p \geq -1, q \geq 0$,

$$\partial_x^q \Delta_p g(x,v) = \sum_{n=0}^{+\infty} \partial_x^q \Delta_p g_n(x) e_n(x), \quad \text{with} \quad g_n(x) = (g(x,\cdot), e_n)_{L^2(\mathbb{R}_v)} \tag{4.97}$$

and

$$\partial_x^q \Delta_p g_n(x) = (\partial_x^q \Delta_p g(x,\cdot), e_n)_{L^2(\mathbb{R}_v)}. \tag{4.98}$$

We deduce from (4.97)-(4.98) and Lemma 4.9 with $r = \frac{3s+1}{2s(s+1)}$ that there exist some con-

stants $C_1, C_2 > 0$ such that for all $k, l, q \geq 0, \varepsilon > 0$,

$$\begin{aligned}
\|v^k\partial_v^l\partial_x^q g(t)\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} &= \sum_{p\geq -1} 2^{\frac{p}{2}}\|v^k\partial_v^l\partial_x^q\Delta_p g(t)\|_{L^2_{x,v}} \\
&\leq \sum_{p\geq -1}\sum_{n=0}^{+\infty} 2^{\frac{p}{2}}\|\partial_x^q\Delta_p g_n(t)\|_{L^2_x}\|v^k\partial_v^l e_n\|_{L^2_v} \\
&\leq C_1\Big(\frac{C_2}{\inf(\varepsilon^{\frac{3s+1}{2s(s+1)}},1)}\Big)^{k+l}(k!)^{\frac{3s+1}{2s(s+1)}}(l!)^{\frac{3s+1}{2s(s+1)}}\sum_{n=0}^{+\infty}\sum_{p\geq -1} 2^{\frac{p}{2}}\|\partial_x^q\Delta_p g_n(t)\|_{L^2_x} \\
&\quad\times\Big((1-\delta_{n,0})\exp\Big(\varepsilon\frac{3s+1}{2s(s+1)}n^{\frac{s(s+1)}{3s+1}}\Big)+\delta_{n,0}\Big),
\end{aligned} \tag{4.99}$$

where $\delta_{n,0}$ stands for the Kronecker delta, i.e., $\delta_{n,0} = 1$ if $n = 0, \delta_{n,0} = 0$ if $n \neq 0$. It follows from (4.94) that for all $0 \leq t \leq T$,

$$\begin{aligned}
&\Big\|\exp\Big(\frac{c_0 t}{2}(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}\Big)g\Big\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \\
&= \sum_{p\geq -1} 2^{\frac{p}{2}}\Big\|\exp\Big(\frac{c_0 t}{2}(\mathcal{H}^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{1+3s}{1+2s}})^{\frac{2s}{1+3s}}\Big)\Delta_p g(t)\Big\|_{L^2_{x,v}} \\
&= \sum_{p\geq -1} 2^{\frac{p}{2}}\Big(\sum_{n=0}^{+\infty}\Big\|\exp\Big(\frac{c_0 t}{2}((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}\Big)\Delta_p g_n(t)\Big\|^2_{L^2_x}\Big)^{1/2} \\
&\leq Ce^{CT}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})},
\end{aligned}$$

which implies that for all $0 \leq t \leq T$,

$$\sum_{p\geq -1} 2^{\frac{p}{2}}\sup_{n\geq 0}\Big\|\exp\Big(\frac{c_0 t}{2}((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}\Big)\Delta_p g_n(t)\Big\|_{L^2_x} \leq Ce^{CT}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}. \tag{4.100}$$

Then we obtain that for all $n, q \geq 0, p \geq -1$,

$$\begin{aligned}
\|\partial_x^q\Delta_p g_n(t)\|_{L^2_x} &= \Big(\frac{1}{2\pi}\int_{\mathbb{R}}|\xi|^{2q}|\widehat{\Delta_p g_n}(\xi)|^2 d\xi\Big)^{1/2} \\
&= \Big(\frac{1}{2\pi}\int_{\mathbb{R}}|\xi|^{2q}\exp\Big(-c_0 t((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}})^{\frac{2s}{3s+1}}\Big) \\
&\quad\times\Big|\exp\Big(\frac{c_0 t}{2}\Big((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)\widehat{\Delta_p g_n}(\xi)\Big|^2 d\xi\Big)^{1/2} \\
&\leq \exp\Big(-\frac{c_0 t}{2}\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big)\Big(\frac{1}{2\pi}\int_{\mathbb{R}}\langle\xi\rangle^{2q}\exp\Big(-c_0 t\langle\xi\rangle^{\frac{2s}{2s+1}}\Big) \\
&\quad\times\Big|\exp\Big(\frac{c_0 t}{2}\Big((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle\xi\rangle^{\frac{1+3s}{1+2s}}\Big)^{\frac{2s}{3s+1}}\Big)\widehat{\Delta_p g_n}(\xi)\Big|^2 d\xi\Big)^{1/2} \\
&\leq \Big(\frac{2s+1}{sc_0 t}\Big)^{\frac{2s+1}{2s}q}(q!)^{\frac{2s+1}{2s}}\exp\Big(-\frac{c_0 t}{2}\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big) \\
&\quad\times\Big\|\exp\Big(\frac{c_0 t}{2}\Big((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{3s+1}}\Big)\Delta_p g_n(t)\Big\|_{L^2_x},
\end{aligned} \tag{4.101}$$

where we used the following inequality that, $\forall 0 < t \leq T, q \geq 0, \xi \in \mathbb{R}$,

$$\langle\xi\rangle^{2q} e^{-c_0 t\langle\xi\rangle^{\frac{2s}{2s+1}}} \leq \Big(\frac{2s+1}{sc_0t}\Big)^{\frac{2s+1}{s}q}(q!)^{\frac{2s+1}{s}}.$$

Then it follows from (4.99), (4.100) and (4.101) that for all $0 \leq t \leq T, k, l, q \geq 0$,

$$\begin{aligned}
&\|v^k\partial_v^l\partial_x^q g(t)\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\leq C_1\Big(\frac{C_2}{\inf(\varepsilon^{\frac{3s+1}{2s(s+1)}},1)}\Big)^{k+l}\Big(\frac{2s+1}{sc_0t}\Big)^{\frac{2s+1}{2s}q}(k!)^{\frac{3s+1}{2s(s+1)}}(l!)^{\frac{3s+1}{2s(s+1)}}(q!)^{\frac{2s+1}{2s}}\\
&\quad\times\sum_{n=0}^{+\infty}\sum_{p\geq -1}2^{\frac{p}{2}}\Big\|\exp\Big(\frac{c_0t}{2}\Big((n+\frac{1}{2})^{\frac{s+1}{2}}+\langle D_x\rangle^{\frac{3s+1}{2s+1}}\Big)^{\frac{2s}{2s+1}}\Big)\Delta_p g_n(t)\Big\|_{L^2_x}\\
&\quad\times\Big((1-\delta_{n,0})\exp\Big(\varepsilon\frac{3s+1}{2s(s+1)}n^{\frac{s(s+1)}{3s+1}}-\frac{c_0}{4}t\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big)\\
&\qquad+\delta_{n,0}\exp\Big(-\frac{c_0}{4}t\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big)\Big)\\
&\leq Ce^{CT}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}C_1\Big(\frac{C_2}{\inf(\varepsilon^{\frac{3s+1}{2s(s+1)}},1)}\Big)^{k+l}\Big(\frac{2s+1}{sc_0t}\Big)^{\frac{2s+1}{2s}q}(k!)^{\frac{3s+1}{2s(s+1)}}(l!)^{\frac{3s+1}{2s(s+1)}}(q!)^{\frac{2s+1}{2s}}\\
&\quad\times\sum_{n=0}^{+\infty}\Big((1-\delta_{n,0})\exp\Big(\varepsilon\frac{3s+1}{2s(s+1)}n^{\frac{s(s+1)}{3s+1}}-\frac{c_0}{4}t\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big)\\
&\qquad+\delta_{n,0}\exp\Big(-\frac{c_0}{4}t\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big)\Big).
\end{aligned}$$

If we choose

$$\varepsilon = \frac{s(s+1)c_0t}{12s+4} > 0,$$

then there exist some constants $C_3, C_4 > 0$ such that for all $0 \leq t \leq T, k, l, q \geq 0$,

$$\begin{aligned}
&\|v^k\partial_v^l\partial_x^q g(t)\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}\\
&\leq C_3C_4^{k+l+q}\frac{F(t)}{t^{\frac{3s+1}{s(s+1)}(k+l)+\frac{2s+1}{2s}q}}(k!)^{\frac{3s+1}{2s(s+1)}}(l!)^{\frac{3s+1}{2s(s+1)}}(q!)^{\frac{2s+1}{2s}}\|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})},
\end{aligned}\tag{4.102}$$

where

$$F(x) = \sum_{n=0}^{+\infty}\exp\Big(-\frac{c_0x}{8}\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big),\quad x>0.$$

For any $x > 0$, we obtain that

$$\begin{aligned} F(x) &= x^{-(\frac{3s+1}{s(s+1)}+\alpha)} \sum_{n=0}^{+\infty} \Big(\frac{c_0 x}{8}\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big)^{\frac{3s+1}{s(s+1)}+\alpha} \exp\Big(-\frac{c_0 x}{8}\Big(n+\frac{1}{2}\Big)^{\frac{s(s+1)}{3s+1}}\Big) \\ &\quad \times \frac{1}{(\frac{c_0}{8})^{\frac{3s+1}{s(s+1)}+\alpha}(n+\frac{1}{2})^{1+\frac{\alpha s(s+1)}{3s+1}}} \\ &\lesssim x^{-(\frac{3s+1}{s(s+1)}+\alpha)} \|z^{\frac{3s+1}{s(s+1)}+\alpha} e^{-z}\|_{L^\infty([0,+\infty))} \sum_{n=0}^{+\infty} \frac{1}{(\frac{c_0}{8})^{\frac{3s+1}{s(s+1)}+\alpha}(n+\frac{1}{2})^{1+\frac{\alpha s(s+1)}{3s+1}}} \\ &\lesssim x^{-(\frac{3s+1}{s(s+1)}+\alpha)} \end{aligned}$$

with a positive parameter $\alpha > 0$. We deduce from (4.102) that for $\forall \alpha > 0$, there exist some constants $C_5, C_6 > 0$ such that for all $0 \le t \le T, k, l, q \ge 0$,

$$\|v^k \partial_v^l \partial_x^q g(t)\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})} \le \frac{C_5 C_6^{k+l+q}}{t^{\frac{3s+1}{2s(s+1)}(k+l+2)+\frac{2s+1}{2s}q+\alpha}} (k!)^{\frac{3s+1}{2s(s+1)}} (l!)^{\frac{3s+1}{2s(s+1)}} (q!)^{\frac{2s+1}{2s}} \|g_0\|_{\widetilde{L}^2_v(B^{1/2}_{2,1})}.$$

This proves the Gevrey smoothing property in Theorem 4.1.

Chapter 5

Global smooth flows for compressible Navier-Stokes-Maxwell equations

Contents

In this chapter, we investigate compressible Navier-Stokes-Maxwell (N-S-M) equations arising in plasmas physics, which is a concrete example of hyperbolic-parabolic composite systems with non-symmetric dissipation. It is observed that the Cauchy problem for N-S-M equations admits the dissipative mechanism of regularity-loss type. Consequently, extra higher regularity is usually needed to obtain the optimal decay rate of $L^1(\mathbb{R}^3)$-$L^2(\mathbb{R}^3)$ type, in comparison with that for the global-in-time existence of smooth solutions. In this chapter, we obtain the minimal decay regularity of global smooth solutions to N-S-M equations, with aid of $L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$-$L^r(\mathbb{R}^n)$ estimates.

5.1 Description of problem

Plasma dynamics is a field of studying flow problems of electrically conducting fluids, for instance, ionized gases, interacting with their own self-consistent electromagnetic field. In the macroscopic continuous level, the transport process is typically governed by the relaxed Euler-Maxwell equations; see for example [25]. In this chapter, we consider electrically conducting fluids in the viscous case, which takes the following form of Navier-Stokes-Maxwell (N-S-M)

equations:

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) \\ \qquad = -n(E + u \times B) - \lambda nu + \mu \Delta u + (\mu + \mu')\nabla \mathrm{div} u, \\ \partial_t E - \nabla \times B = nu, \\ \partial_t B + \nabla \times E = 0, \end{cases} \tag{5.1}$$

with constraints

$$\nabla \cdot E = n_\infty(x) - n, \quad \nabla \cdot B = 0. \tag{5.2}$$

Notice that the system (5.1) with (5.2) admits a constant equilibrium state of the form

$$(n, u, E, B) = (n_\infty, 0, 0, B_\infty), \tag{5.3}$$

where $B_\infty \in \mathbb{R}^3$ is an arbitrarily fixed constant vector. In this chapter, we concerned with the Cauchy problem to (5.1)-(5.2), so the initial data are supplemented by

$$(n, u, E, B)|_{t=0} = (n_0, u_0, E_0, B_0)(x), \quad x \in \mathbb{R}^3. \tag{5.4}$$

Also, it is simple to see that the constraint (5.2) holds true for any $t > 0$, if it holds initially. Namely, we only assume that

$$\nabla \cdot E_0 = n_\infty(x) - n_0, \quad \nabla \cdot B_0 = 0, \quad x \in \mathbb{R}^3. \tag{5.5}$$

We rewrite (5.1) as the linearized perturbation form around the equilibrium state $w_\infty := (n_\infty, 0, 0, B_\infty)$. Without loss of generality, we set the physical parameters to be $\lambda = 1 = \mu$ and $\mu + \mu' = 0$. By taking change of variables

$$\rho = n - n_\infty, \ \upsilon = nu/n_\infty, \ E = E, \ h = B - B_\infty.$$

For simplicity, we set $z := (\rho, \upsilon, E, h)^\top$. System (5.1) is also rewritten in vector form as

$$A^0_\infty z_t + \sum_{j=1}^{3} A^j_\infty z_{x_j} + L_\infty z = \sum_{j,k=1}^{3} B^{jk}_\infty z_{x_j x_k} + \sum_{j=1}^{3} Q_{x_j} + R + \Delta S, \tag{5.6}$$

where $Q(z), R(z), S(z)$ and the coefficient matrices $A^0_\infty, A^j_\infty, B^{jk}_\infty, L_\infty$ are given in Chapter 1.1.3. The corresponding initial data are given by

$$z|_{t=0} = (\rho_0, \upsilon_0, E_0, h_0)^\top(x) \tag{5.7}$$

with $\rho_0 = n_0 - n_\infty$, $\upsilon_0 = n_0 u_0/n_\infty$ and $h_0 = B_0 - B_\infty$. In (5.6), matrices $A^j_\infty (j = 0, 1, 2, 3)$ and $B^{j,k}_\infty (j, k = 1, 2, 3)$ are real symmetric, and L_∞ and $\sum_{j,k=1}^{3} B^{jk}_\infty \xi_j \xi_k$ are degenerate and nonnegative definite; however, L_∞ *is not real symmetric*. Therefore, the dissipation forces

(5.6) to go beyond the general class of equations of symmetric hyperbolic-parabolic type as in [171], which is our motivation to study the N-S-M equations (5.1).

Indeed, the non-symmetric dissipation affected (5.6) such that the weak dissipative mechanism of regularity-loss is present. Let $z_{\mathcal{L}}$ be linearized solutions to (5.6)-(5.7) without nonlinear source (Q, R, S). As shown in Section 5.3, the Fourier image of $z_{\mathcal{L}}$ satisfies the following pointwise estimate

$$|\widehat{z_{\mathcal{L}}}(t,\xi)| \lesssim e^{-c_0\eta(\xi)t}|\hat{z}_0(\xi)|$$

for any $t \geq 0$ and $\xi \in \mathbb{R}^3$, where the dissipative rate $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$ and $c_0 > 0$ is a constant. Consequently, the decay property of $z_{\mathcal{L}}$ satisfies that

$$\|z_{\mathcal{L}}\|_{L^2} \lesssim (1+t)^{-3/4}\|z_0\|_{L^1} + (1+t)^{-\ell/2}\|\partial_x^\ell z_0\|_{L^2}, \tag{5.8}$$

where ℓ is a nonnegative integer. Clearly, the decay (5.8) is of regularity-loss type at the high-frequency part, since $(1+t)^{-\ell/2}$ is created by assuming the additional ℓ-th order regularity on the initial data, which is different from that of symmetric hyperbolic-parabolic system (1.34) studied as in [171]. More important, it follows from (5.8) that the extra regularity will be posted than that for global smooth solutions, if the optimal time-decay rate $(1+t)^{-3/4}$ is expected. To the best of our knowledge, similar phenomena also appear in the study of other many dissipative systems, such as quasilinear hyperbolic systems of viscoelasticity in [53], hyperbolic-elliptic systems of radiating gas in [75], dissipative Timoshenko system in [87, 111], compressible Euler-Maxwell system in [48, 169], Vlasov-Maxwell-Boltzmann system in [54], and a plate equation with rotational inertia effect in [163].

Our main goal is to prove the global existence and establish the optimal decay estimates of solutions with the minimal regularity assumption. Now, we begin to state main results in this chapter (that is, **Main Theorem C** in Chapter 1). For this purpose, set $w = (n, u, E, B)^\top$($\top$ transpose) and $w_0 = (n_0, u_0, E_0, B_0)^\top$, which are column vectors in $\mathbb{R}^{10}$. We define the following energy norm $N_0(t)$ and the corresponding dissipation norm $D_0(t)$:

$$N_0(t) := \sup_{0\leq\tau\leq t}\|(w-w_\infty)(\tau)\|_{H^s},$$
$$D_0(t)^2 := \int_0^t (\|(n-n_\infty)(\tau)\|_{H^s}^2 + \|u(\tau)\|_{H^{s+1}}^2 + \|E(\tau)\|_{H^{s-1}}^2 + \|\nabla B(\tau)\|_{H^{s-2}}^2)d\tau.$$

Theorem 5.1. *(***Global-in-time existence***) Let $s \geq 3$ and suppose that the initial data satisfy $w_0 - w_\infty \in H^s$ and (5.5). Put $I_0 := \|w_0 - w_\infty\|_{H^s}$. Then there exists a constant $\varepsilon_0 > 0$ such that if $I_0 \leq \varepsilon_0$, the Cauchy problem (5.1) and (5.4) has a unique global solution $w(t,x)$ with*

$$w - w_\infty \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1});$$
$$n \in L^2([0,\infty); H^s), \quad u \in L^2([0,\infty); H^{s+1});$$
$$E \in L^2([0,\infty); H^{s-1}), \quad \nabla B \in L^2([0,\infty); H^{s-2}).$$

The solution satisfies the uniform energy estimate

$$N_0(t)^2 + D_0(t)^2 \lesssim I_0^2 \tag{5.9}$$

for $t \geq 0$. Moreover, the solution $w(t,x)$ converges to the constant state w_∞ uniformly in $x \in \mathbb{R}^3$ as $t \to \infty$. More precisely, we have

$$\|(n - n_\infty, u, E)(t)\|_{W^{s-2,\infty}} \to 0 \quad as \quad t \to \infty \tag{5.10}$$

and

$$\|(B - B_\infty)(t)\|_{W^{s-4,\infty}} \to 0 \quad as \quad t \to \infty, \tag{5.11}$$

where the asymptotic behavior for the magnetic field B holds true only by assuming the additional regularity $s \geq 4$.

Remark 5.1. Let us mention that (5.9) is of the regularity-loss type because we have 1-regularity loss for (E, B) in the dissipation part $D_0(t)$.

Theorem 5.2. *(***Optimal decay estimate***) Let $s \geq 3$. Assume that the initial data satisfy $w_0 - w_\infty \in H^s \cap L^1$ and (5.5). Put $I_1 := \|w_0 - w_\infty\|_{H^s \cap L^1}$. Then there exists a constant $\varepsilon_1 > 0$ such that if $I_1 \leq \varepsilon_1$, the global solution constructed in Theorem 5.1 admits the decay estimate*

$$\|w - w_\infty\|_{L^2} \lesssim I_1 (1+t)^{-3/4}.$$

Remark 5.2. From Theorem 5.2, we see that the $L^1(\mathbb{R}^3)$-$L^2(\mathbb{R}^3)$ decay rate of solutions is available with the critical regularity $s = 3$; that is, we arrive at $s_D \leq 3$ in the sense of Definition 1.2, and the extra regularity is not necessary, which improves previous works great, for example, [50, 169]. Our decay rate coincides with those results for compressible N-S equations by Matsumura-Nishida [138] and Ponce [152].

Finally, we would like to mention that the same terminology of "Navier-Stokes-Maxwell equations" is also used by Masmoudi [132], Ibrahim-Keraani [88] and Germain-Ibrahim-Masmoudi [64], where they investigated global well-posedness for the forced (Lorentz force) *incompressible Navier-Stokes equations* coupled with Maxwell equations.

■ *The schema of proof of main theorem*

Our aim is to seek minimal decay regularity for smooth solutions to the Cauchy problem of (5.1)-(5.2). As a first step, we shall construct a priori energy estimates and establish the global-in-time existence of smooth solutions near the constant equilibrium $w_\infty = (n_\infty, 0, 0, B_\infty)$.

• The proof of global smooth solution (Section 5.2)

The proof of the global smooth solution in Theorem 5.1 is based on the energy methods.

Step 1) The mathematical entropy

Due to the non-symmetric dissipation, the general theory in [92] cannot be applied directly. Fortunately, compared to the Euler-Maxwell system as in [172], the viscous term $(\Delta u + \nabla \mathrm{div} u)$ in (5.1) does not influence the mathematical entropy from the point of view of energy approaches (see Section 5.2.1).

Step 2) A priori energy estimates

With the similar calculations as that in [172], and also dealing with the new content arising from the viscosity, for instance, see (5.24) and (5.30), we obtain global a priori estimates in Proposition 5.1. It is observed that the dissipative structure of regularity-loss appears not only in the time-decay estimate, but also in the dissipation part of nonlinear energy estimates. Based on the Proposition 5.1, global-in-time existence of smooth solutions is follows from the standard continuation argument. And the large-time behavior of solutions is given with the aid of Gagliardo-Nirenberg's inequality.

• The proof of optimal decay estimates (Section 5.3)

Extra higher regularity is usually posted in order to overcome the difficulty arising from the regularity-loss mechanism. It seems impossible to obtain the optimal decay rate with the lower regularity, even if a combination of elaborate spectral analysis and Duhamel's principle is used; see for example, [50, 169] (the regularity was assumed to be $s \geq 6$). Actually, Duan [50] first investigated compressible N-S-M equations, where the dissipative rate is subjected to the weaker form $\tilde{\eta}(\xi) = |\xi|^4/(1+|\xi|^2)^3$ due to the absence of damping. To do this, we emloy some different techniques.

Step 1) Energy methods in Fourier spaces

We perform the energy method in Fourier spaces and establish the "square formula of Duhamel principle" for (5.6)-(5.7), where the dissipative rate satisfies $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$, see Proposition 5.2. We would like to mention that similar ideas have been used by Kawashima [91] for the Boltzman equation, and then well developed in [104] for hyperbolic systems of balance laws.

Step 2) Optimal decay rates

We are led to the optimal decay rates of solution for (5.1) and (5.4) with the aid of $L^p(\mathbb{R}^d)$-$L^q(\mathbb{R}^d)$-$L^r(\mathbb{R}^d)$ estimates in Lemma 5.1, which was recently established for a class of dissipative systems of regularity-loss type, see Proposition 5.3.

5.2 Global existence

In this section, we prove crucial a priori estimates, which are used to establish the global-in-time existence of smooth solutions to (5.1) and (5.4). To do this, we first review the basic property of (5.1) from the point of view of energy approaches.

5.2.1 The mathematical entropy

We remark that our system (5.1) admits the following energy balance law:

$$\begin{aligned}&\frac{d}{dt}\Big\{n\mathcal{E}+\frac{1}{2}(|E|^2+|B|^2)\Big\}\\&+\operatorname{div}\Big\{nu\mathcal{E}+p(n)u+E\times B-\nabla\Big(\frac{|u|^2}{2}\Big)\Big\}+n|u|^2+|\nabla u|^2=0,\end{aligned}\tag{5.12}$$

where the function $\mathcal{E}=\mathcal{E}(n,u)$ is defined by

$$\mathcal{E}(n,u):=\frac{1}{2}|u|^2+\Phi(n),\quad \Phi(n):=\int_a^n\frac{p(\eta)}{\eta^2}d\eta$$

for an arbitrary constant a.

Note that we do not use (5.2) in deriving (5.12). It is easy to see that the total energy $\mathcal{H}:=n\mathcal{E}+(|E|^2+|B|^2)/2$ is a strictly convex function of the conserved quantities $\widetilde{w}:=(n,nu,E,B)$; for a similar convexity property of the total energy, we refer to [92, 93]. This total energy can be regarded as a mathematical entropy defined in [103] for symmetric hyperbolic systems with dissipation (cf.[79, 184]). Also, the potential energy $\Phi(n)$ can be regarded as a strictly convex function of $\upsilon:=1/n$, so that $\mathcal{E}$ is also a strictly convex function of (υ,u). Based on the convexity of the total energy $\mathcal{H}=\mathcal{H}(\widetilde{w})$, we introduce the associated energy form $\widetilde{\mathcal{H}}$ by

$$\widetilde{\mathcal{H}}:=\mathcal{H}(\widetilde{w})-\mathcal{H}(\widetilde{w}_\infty)-D_{\widetilde{w}}\mathcal{H}(\widetilde{w}_\infty)(\widetilde{w}-\widetilde{w}_\infty),$$

where $\widetilde{w}_\infty=(n_\infty,0,0,B_\infty)$ is the constant state of $\widetilde{w}:=(n,nu,E,B)$ that corresponds to (5.1), and $D_{\widetilde{w}}\mathcal{H}(\widetilde{w})$ denotes the Fréchet derivative of $\mathcal{H}(\widetilde{w})$ with respect to $\widetilde{w}$. Similarly, by direct computations, we find that

$$\widetilde{\mathcal{H}}:=n\widetilde{\mathcal{E}}+\frac{1}{2}(|E|^2+|B-B_\infty|^2),$$

where

$$\widetilde{\mathcal{E}}(n,u):=\frac{1}{2}|u|^2+\widetilde{\Phi}(n),\quad \widetilde{\Phi}(n):=\int_a^n\frac{p(\eta)-p(n_\infty)}{\eta^2}d\eta$$

for the constant a.

It seems here not so difficult to give a complete argument for the following entropy equality

$$\begin{aligned}&\frac{d}{dt}\widetilde{\mathcal{H}}+\operatorname{div}\Big\{nu\widetilde{\mathcal{E}}+(p(n)-p(n_\infty))u+E\times(B-B_\infty)-\nabla\Big(\frac{|u|^2}{2}\Big)\Big\}\\&\quad+n|u|^2+|\nabla u|^2=0.\end{aligned}\tag{5.13}$$

As above, it is observed that N-S-M equations (5.1) have the same mathematical entropy in comparison with the Euler-Maxwell system as in [172], which allows to construct global a priori estimates of smooth solutions in the same spirt as [172]. Actually, we may enhance the regularity of velocity field u in the dissipation norm $D_0(t)$ due to the viscous term Δu.

5.2.2 The proof of global existence

This section is devoted to the global existence of smooth solutions in Theorem 5.1. To show that the solutions of (5.1) and (5.4) are globally defined, we need further a priori estimate. To do this, for any time $T > 0$, and for any solution $w - w_\infty \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})(s \geq 3)$, we define the following quantities:

$$M(t) := \sup_{0\leq\tau\leq t} \|(w - w_\infty)(\tau)\|_{W^{1,\infty}}, \quad I(t)^2 = \int_0^t \|(n - n_\infty, u)(\tau)\|_{L^\infty}^2 d\tau$$

for $t \in [0,T]$, which are used to the subsequent analysis. For clarity, we give a priori estimate by the following proposition.

Proposition 5.1. *Suppose that the initial data satisfy $w_0 - w_\infty \in H^s$ with $s \geq 3$ and (5.5). Let $w(t,x)$ be a solution to (5.1) and (5.4) satisfying $w - w_\infty \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})$ for any $T > 0$. There is a constant $\varepsilon_2 > 0$ independent of T such that if $N_0(T) \leq \varepsilon_2$, then a priori estimate*

$$N_0(t)^2 + D_0(t)^2 \lesssim \|w_0 - w_\infty\|_{H^s}^2 \tag{5.14}$$

holds for $t \in [0,T]$.

Proof. The proof is separated into four steps for clarity.

Step 1. The $L_T^\infty(L^2)$ estimate of $(w - w_\infty)$ and the $L_T^2(H^1)$ one of u

In view of the strictly convexity of the total energy $\mathcal{H}$, it follows that $\widetilde{\mathcal{H}}$ is equivalent to the quadratic function $|w - w_\infty|^2$, since $|w - w_\infty| \lesssim N_0(t)$ is suitablely small. Hence, integrating (5.13) over $[0,t] \times \mathbb{R}^3$ yields

$$\|(w - w_\infty)(t)\|_{L^2}^2 + \int_0^t \|u(\tau)\|_{H^1}^2 d\tau \lesssim \|w_0 - w_\infty\|_{L^2}^2 \tag{5.15}$$

for $t \in [0,T]$.

Step 2. The $L_T^\infty(H^{s-1})$ estimate of ∇w and the $L_T^2(H^s)$ one of ∇u

Actually, we derive the following energy inequality for derivatives of solutions:

$$\begin{aligned} &\|\nabla w(t)\|_{H^{s-1}}^2 + \int_0^t \|\nabla u(\tau)\|_{H^s}^2 d\tau \\ &\lesssim \|\partial_x w_0\|_{H^{s-1}}^2 + (M(t) + N_0(t)) D_0(t)^2 + N_0(t) D_0(t) I(t) \end{aligned} \tag{5.16}$$

for $t \in [0,T]$.

To do this, we first rewrite system (5.1) as

$$\begin{cases} \partial_t n + u \cdot \nabla n + n \mathrm{div} u = 0, \\ \partial_t u + (u \cdot \nabla) u + \frac{p'(n)}{n} \nabla n + E + u \times B_\infty + u = -u \times (B - B_\infty) + \frac{\Delta u}{n}, \\ \partial_t E - \nabla \times B - n_\infty u = (n - n_\infty) u, \\ \partial_t B + \nabla \times E = 0. \end{cases} \tag{5.17}$$

Applying the differential operator $\partial_x^l (1 \le l \le s)$ to (5.17), we get

$$\begin{cases} n_\infty \frac{p'(n)}{n^2}(\partial_x^l n_t + u\cdot\nabla\partial_x^l n) + n_\infty a(n)\mathrm{div}\partial_x^l u = n_\infty \frac{p'(n)}{n^2} f_1^l, \\ n_\infty(\partial_x^l u_t + (u\cdot\nabla)\partial_x^l u) + n_\infty a(n)\nabla\partial_x^l n + n_\infty(\partial_x^l E + \partial_x^l u \times B_\infty) \\ \quad + n_\infty \partial_x^l u - \partial_x^l \Delta u = n_\infty f_2^l - n_\infty \partial_x^l (u\times(B-B_\infty)) + n_\infty \partial_x^l(\tilde{h}(n)\Delta u), \\ \partial_x^l E_t - \nabla\times\partial_x^l B - n_\infty \partial_x^l u = \partial_x^l((n-n_\infty)u), \\ \partial_x^l B_t + \nabla\times\partial_x^l E = 0, \end{cases} \tag{5.18}$$

where $a(n) = \frac{p'(n)}{n}$, $\tilde{h}(n) = (\frac{1}{n} - \frac{1}{n_\infty})\Delta u$,

$$f_1^l = -[\partial_x^l, u]\cdot\nabla n - [\partial_x^l, n]\mathrm{div}u \quad \text{and} \quad f_2^l = -\Big[\partial_x^l, a(n)\Big]\nabla n - [\partial_x^l, u]\cdot\nabla u.$$

Taking the inner product of (5.18) with $\partial_x^l n, \partial_x^l u, \partial_x^l E$ and $\partial_x^l B$, respectively. Then, adding the resultant equations implies that

$$\partial_t \mathcal{H}^l + n_\infty |\partial_x^l u|^2 + |\partial_x^l \nabla u|^2 + \mathrm{div}\mathcal{F}^l = \mathcal{R}^l + \mathcal{S}^l + \mathcal{U}^l, \tag{5.19}$$

where

$$\begin{aligned} \mathcal{H}^l &= \frac{1}{2} n_\infty \{\frac{p'(n)}{n^2}(\partial_x^l n)^2 + |\partial_x^l u|^2\} + \frac{1}{2}(|\partial_x^l E|^2 + |\partial_x^l B|^2), \\ \mathcal{F}^l &= \frac{1}{2} n_\infty u\big\{\frac{p'(n)}{n^2}(\partial_x^l n)^2 + |\partial_x^l u|^2\big\} + n_\infty a(n)\partial_x^l n\ \partial_x^l u \\ &\quad + \partial_x^l E \times \partial_x^l B - \nabla(\frac{|\partial_x^l u|^2}{2}), \\ \mathcal{R}^l &= \frac{1}{2} n_\infty \{\partial_t \frac{p'(n)}{n^2} + \mathrm{div}(\frac{p'(n)}{n^2}u)\}(\partial_x^l n)^2 + \frac{1}{2} n_\infty \mathrm{div}u |\partial_x^l u|^2 \\ &\quad + n_\infty \nabla a(n)\cdot\partial_x^l n \partial_x^l u + n_\infty \frac{p'(n)}{n^2}\partial_x^l n f_1^l + n_\infty \partial_x^l u \cdot f_2^l, \\ \mathcal{S}^l &= \partial_x^l E\cdot\partial_x^l((n-n_\infty)u) - n_\infty \partial_x^l u\cdot\partial_x^l(u\times(B-B_\infty)), \\ \mathcal{U}^l &= n_\infty \partial_x^l u\cdot\partial_x^l(\tilde{h}(n)\Delta u). \end{aligned}$$

Integrating (5.19) over $x\in\mathbb{R}^3$ gives

$$\frac{d}{dt}\int_{\mathbb{R}^3} \mathcal{H}^l dx + n_\infty \|\partial_x^l u\|_{L^2}^2 + \|\partial_x^l \nabla u\|_{L^2}^2 \le R^l + S^l + U^l, \tag{5.20}$$

where we set $R^l := \int_{\mathbb{R}^3} |\mathcal{R}^l| dx, S^l := \int_{\mathbb{R}^3} |\mathcal{S}^l| dx$ and $U^l := \int_{\mathbb{R}^3} \mathcal{U}^l dx$. Notice that $\mathcal{H}^l$ is equivalent to the quadratic function $|\partial_x^l w|^2$, since $|w - w_\infty| \lesssim N_0(t)$ is sufficiently small.

Hence, by integrating (5.20) with respect to $t > 0$ and summing the resulting inequality for l with $1 \le l \le s$ together, we are led to

$$\begin{aligned} &\|\partial_x w(t)\|_{H^{s-1}}^2 + \int_0^t \|\nabla u(\tau)\|_{H^s}^2 d\tau \\ &\quad \lesssim \|\partial_x w_0\|_{H^{s-1}}^2 + \sum_{l=1}^s \int_0^t (R^l + S^l + U^l)(\tau) d\tau. \end{aligned} \tag{5.21}$$

The next step is to evaluate these integrals related to R^l, S^l and U^l. The estimates for terms R^l, S^l is the same as that in [172], that is

$$\begin{aligned} R^l &\lesssim (\|\partial_t n\|_{L^\infty} + \|\partial_x(n,u)\|_{L^\infty})\|\partial_x^l(n,u)\|_{L^2}^2 + \|\partial_x^l(n,u)\|_{L^2}\|(f_1^l, f_2^l)\|_{L^2} \\ &\lesssim \|\partial_x(n,u)\|_{L^\infty}\|\partial_x^l(n,u)\|_{L^2}^2, \end{aligned}$$

where we used $|\partial_t n| \lesssim |\partial_x(n,u)|$. And we used $\|(f_1^l, f_2^l)\|_{L^2} \lesssim \|\partial_x(n,u)\|_{L^\infty}\|\partial_x^l(n,u)\|_{L^2}$, which comes from the estimates for the commutator in Lemma 2.18. We have

$$\int_0^t R^l(\tau)d\tau \lesssim \sup_{0\le\tau\le t}\|\partial_x(n,u)(\tau)\|_{L^\infty}\int_0^t \|\partial_x^l(n,u)\|_{L^2}^2 d\tau \lesssim M(t)D_0(t)^2 \tag{5.22}$$

for $1 \le l \le s$. Similarly, for S^l it holds that

$$\begin{aligned} S^l &\lesssim \|\partial_x^l E\|_{L^2}\|\partial_x((n-n_\infty)u)\|_{L^2} + \|\partial_x^l u\|_{L^2}\|\partial_x^l(u\times(B-B_\infty))\|_{L^2} \\ &\lesssim \|B-B_\infty\|_{L^\infty}\|\partial_x^l u\|_{L^2}^2 + \|(n-n_\infty,u)\|_{L^\infty}\|\partial_x^l(n,u)\|_{L^2}\|\partial_x^l(E,B)\|_{L^2}, \end{aligned}$$

where Lemma 2.18 is used. As a result, we get

$$\begin{aligned} \int_0^t S^l(\tau)d\tau &\lesssim \sup_{0\le\tau\le t}\|B-B_\infty(\tau)\|_{L^\infty}\int_0^t \|\partial_x^l u(\tau)\|_{L^2}^2 d\tau \\ &\quad + \sup_{0\le\tau\le t}\|\partial_x^l(E,B)(\tau)\|_{L^2}\int_0^t \|(n-n_\infty,u)(\tau)\|_{L^\infty}\|\partial_x^l(n,u)(\tau)\|_{L^2}d\tau \\ &\lesssim M(t)D_0(t)^2 + N_0(t)D_0(t)I(t) \end{aligned} \tag{5.23}$$

for $1 \le l \le s$. For the nonlinear term U^l, we write $U^l = U_1^l + U_2^l$ with

$$U_1^l := \int_{\mathbb{R}^3} n_\infty \partial_x^l u \cdot \{\partial_x^l(\tilde{h}(n)\Delta u) - \tilde{h}(n)\partial_x^l \Delta u\}dx$$

and

$$U_2^l := \int_{\mathbb{R}^3} n_\infty \partial_x^l u \cdot \tilde{h}(n)\partial_x^l \Delta u dx.$$

By using the Cauchy-Schwarz inequality, Lemma 2.17 and Lemma 2.20, we arrive at

$$\begin{aligned} U_1^l &\lesssim \|\partial_x^l u\|_{L^2}\|\partial_x^l(\tilde{h}(n)\Delta u) - \tilde{h}(n)\partial_x^l\Delta u\|_{L^2} \\ &\lesssim \|\partial_x^l u\|_{L^2}(\|\nabla\tilde{h}(n)\|_{L^\infty}\|\partial_x^{l-1}\Delta u\|_{L^2} + \|\partial_x^l\tilde{h}(n)\|_{L^2}\|\Delta u\|_{L^\infty}) \\ &\lesssim \|n\|_{H^s}\|u\|_{H^s}\|\nabla u\|_{H^s} \\ &\lesssim \|n\|_{H^s}(\|u\|_{H^s}^2 + \|\nabla u\|_{H^s}^2), \end{aligned}$$

where we used the embedding inequality $\|\Delta u\|_{L^\infty} \lesssim \|\nabla u\|_{H^s} (s \ge 3)$ in second line.

On the other hand, using integration by parts implies that for U_2^l,

$$\begin{aligned} U_2^l &\lesssim \|\tilde{h}(n)\|_{L^\infty}\|\partial_x^l\nabla u\|_{L^2}^2 + \|\nabla\tilde{h}(n)\|_{L^\infty}\|\partial_x^l u\|_{L^2}\|\partial_x^l\nabla u\|_{L^2} \\ &\lesssim \|(n,\nabla n)\|_{L^\infty}\|\partial_x^l\nabla u\|_{L^2}(\|\partial_x^l u\|_{L^2} + \|\partial_x^l\nabla u\|_{L^2}) \\ &\lesssim \|n\|_{H^s}(\|u\|_{H^s}^2 + \|\nabla u\|_{H^s}^2). \end{aligned}$$

Hence, we obtain

$$\begin{aligned}\int_0^t U^l(\tau)d\tau &\lesssim \int_0^t \|n(\tau)\|_{H^s}(\|u(\tau)\|_{H^s}^2+\|\nabla u(\tau)\|_{H^s}^2)d\tau\\ &\lesssim \sup_{0\le\tau\le t}\|n(\tau)\|_{H^s}\int_0^t(\|u(\tau)\|_{H^s}^2+\|\nabla u(\tau)\|_{H^s}^2)d\tau\\ &\lesssim N_0(t)D_0(t)^2\end{aligned} \tag{5.24}$$

for $1\le l\le s$. Substituting (5.22), (5.23) and (5.24) into (5.21), we can get the desired inequality (5.16).

Finally, together with (5.15) and (5.16), we conclude that

$$\begin{aligned}&\|(w-w_\infty)(t)\|_{H^s}^2+\int_0^t\|u(\tau)\|_{H^{s+1}}^2d\tau\\ &\lesssim \|w_0-w_\infty\|_{H^s}^2+(M(t)+N_0(t))D_0(t)^2+N_0(t)D_0(t)I(t).\end{aligned} \tag{5.25}$$

Step 3. The $L_T^2(H^s)$ estimate of $n-n_\infty$ and the $L_T^2(H^{s-1})$ one of E

Under conditions in Proposition 5.1, we can derive the following third energy estimate of solutions for $t\in[0,T]$:

$$\begin{aligned}&\int_0^t(\|(n-n_\infty)(\tau)\|_{H^s}^2+\|E(\tau)\|_{H^{s-1}}^2)d\tau\\ &\le C\|w_0-w_\infty\|_{H^s}^2+\epsilon\int_0^t\|\nabla B(\tau)\|_{H^{s-2}}^2d\tau+C_\epsilon\int_0^t\|u(\tau)\|_{H^{s+1}}^2d\tau\\ &\quad+C(M(t)+N_0(t))D_0(t)^2\end{aligned} \tag{5.26}$$

for any $\epsilon>0$, where C_ϵ is a positive constant depending on ϵ.

Set $\rho=n-n_\infty, h=B-B_\infty$. Then (5.17) can be rewritten as

$$\begin{cases}\partial_t\rho+n_\infty\mathrm{div}u=g_1,\\ \partial_t u+a_\infty\nabla\rho+E+u\times B_\infty+u=g_2\triangleq g_{21}+g_{22},\\ \partial_t E-\nabla\times B-n_\infty u=g_3,\\ \partial_t B+\nabla\times E=0,\end{cases} \tag{5.27}$$

where

$$\begin{aligned}g_1&=-u\cdot\nabla\rho+\rho\mathrm{div}u,\\ g_{21}&=-\big\{(u\cdot\nabla)u+\big(a(\rho)-a_\infty\big)\nabla\rho+u\times h\big\},\\ g_{22}&=\tilde h(\rho)\Delta u+\frac{1}{n_\infty}\Delta u,\\ g_3&=\rho u\end{aligned}$$

with $a(\rho)=\frac{p'(\rho+n_\infty)}{\rho+n_\infty}$, $a_\infty=\frac{p'(n_\infty)}{n_\infty}$ and $\tilde h(\rho)=\frac{1}{\rho+n_\infty}-\frac{1}{n_\infty}$.

Applying the differential operator $\partial_x^l (l \geq 0)$ to (5.27) and taking the inner product of the first, second and the third equations of the resulting equations with $-a_\infty \mathrm{div}\partial_x^l u, a_\infty \nabla \partial_x^l \rho + \partial_x^l E$ and $\partial_x^l u$, respectively, we obtain

$$\partial_t \mathcal{H}_1^l + \mathcal{D}_1^l + \mathrm{div}\mathcal{F}_1^l = \mathcal{M}_1^l + \mathcal{G}_1^l,$$

where $l \geq 0$, and $\mathcal{H}_1^l, \mathcal{D}_1^l, \mathcal{F}_1^l, \mathcal{M}_1^l$ and $\mathcal{G}_1^l$ are correspondingly defined as

$$\begin{aligned}
\mathcal{H}_1^l &:= -a_\infty \partial_x^l \rho \mathrm{div}\partial_x^l u + \partial_x^l u \cdot \partial_x^l E, \\
\mathcal{D}_1^l &:= 2a_\infty (\partial_x^l \rho)^2 + a_\infty^2 |\nabla \partial_x^l \rho|^2 + |\partial_x^l E|^2, \\
\mathcal{F}_1^l &:= a_\infty \partial_x^l \rho (\partial_x^l u_t + 2\partial_x^l E) + \partial_x^l u \times \partial_x^l h, \\
\mathcal{M}_1^l &:= \nabla \times \partial_x^l u \cdot \partial_x^l h - (a_\infty \nabla \partial_x^l \rho + \partial_x^l E) \cdot (\partial_x^l u \times B_\infty + \partial_x^l u) \\
&\quad + n_\infty |\partial_x^l u|^2 + a_\infty n_\infty (\mathrm{div}\partial_x^l u)^2, \\
\mathcal{G}_1^l &:= -a_\infty \partial_x^l g_1 \mathrm{div}\partial_x^l u + (a_\infty \nabla \partial_x^l \rho + \partial_x^l E) \cdot \partial_x^l g_2 + \partial_x^l u \cdot \partial_x^l g_3.
\end{aligned}$$

Furthermore, there exists a constant $c_0 > 0$ such that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^3} \mathcal{H}_1^l dx + c_0 (\|\partial_x^l \rho\|_{H^1}^2 + \|\partial_x^l E\|_{L^2}^2) \\
&\qquad \leq \epsilon \|\partial_x^l h\|_{L^2}^2 + C_\epsilon \|\partial_x^l u\|_{H^1}^2 + G_1^l
\end{aligned}$$

for $0 \leq l \leq s-1$ and $\epsilon > 0$, where C_ϵ is a positive constant depending only on ϵ and $G_1^l := \int_{\mathbb{R}^3} |\mathcal{G}_1^l| dx$.

Notice that the fact $|\int_{\mathbb{R}^3} \mathcal{H}_1^l dx| \leq \varepsilon \|\partial_x^l (\rho, E)\|_{L^2}^2 + C_\varepsilon \|\partial_x^l u\|_{H^1}^2$ for $0 \leq l \leq s-1$ and $\varepsilon > 0$, we can arrive at

$$\begin{aligned}
&\int_0^t (\|\rho(\tau)\|_{H^s}^2 + \|E(\tau)\|_{H^{s-1}}^2) d\tau \\
&\leq C \|w_0 - w_\infty\|_{H^s}^2 + \epsilon \int_0^t \|\nabla h(\tau)\|_{H^{s-2}}^2 d\tau + C_\epsilon \int_0^t \|u(\tau)\|_{H^s}^2 d\tau \\
&\quad + C \sum_{l=0}^{s-1} \int_0^t G_1^l(\tau) d\tau.
\end{aligned} \tag{5.28}$$

It follows from [172] that

$$\begin{aligned}
\|\partial_x^l g_1\|_{L^2} &\lesssim \|(\rho, u)\|_{L^\infty} \|\partial_x^{l+1} (\rho, u)\|_{L^2}, \\
\|\partial_x^l g_{21}\|_{L^2} &\lesssim \|(\rho, u, h)\|_{L^\infty} \|\partial_x^l (\rho, u)\|_{H^1} + \|u\|_{L^\infty} \|\partial_x^l h\|_{L^2}, \\
\|\partial_x^l g_3\|_{L^2} &\lesssim \|(\rho, u)\|_{L^\infty} \|\partial_x^l (\rho, u)\|_{L^2}
\end{aligned} \tag{5.29}$$

for $0 \leq l \leq s-1$, where the term $\|u\|_{L^\infty} \|\partial_x^l h\|_{L^2}$ in second line can be omitted if $l = 0$. On the other hand, by using Lemma 2.18 and Lemma 2.20, we obtain

$$\begin{aligned}
\|\partial_x^l g_{22}\|_{L^2} &\lesssim (\|\tilde{h}(\rho)\|_{L^\infty} \|\partial_x^{l+1} \nabla u\|_{L^2} + \|\partial_x^l \tilde{h}(\rho)\|_{L^2} \|\Delta u\|_{L^\infty}) + \|\partial_x^{l+1} \nabla u\|_{L^2} \\
&\lesssim \|\rho\|_{H^s} \|\nabla u\|_{H^s} + \|\partial_x^{l+1} \nabla u\|_{L^2}.
\end{aligned} \tag{5.30}$$

Due to (5.29) and (5.30), we are led to

$$\begin{aligned} G_1^l &\lesssim \|\partial_x^{l+1}u\|_{L^2}\|\partial_x^l g_1\|_{L^2} + (\|\partial_x^{l+1}n\|_{L^2} + \|\partial_x^l E\|_{L^2})\|\partial_x^l g_2\|_{L^2} + \|\partial_x^l u\|_{L^2}\|\partial_x^l g_3\|_{L^2} \\ &\lesssim \|w - w_\infty\|_{L^\infty}(\|\partial_x^l(\rho, u, \nabla u)\|_{H^1}^2 + \|\partial_x^l(E, h)\|_{L^2}^2) + \|\rho\|_{H^s}\|\nabla u\|_{H^s}. \end{aligned}$$

Consequently, we get following inequality for G_1^l:

$$\int_0^t G_1^l(\tau)d\tau \lesssim \int_0^t \|\nabla u(\tau)\|_{H^s}^2 d\tau + (M(t) + N_0(t))D_0(t)^2 \tag{5.31}$$

for $0 \le l \le s-1$ and $t \in [0, T]$. Finally, substituting (5.31) into (5.28) leads to (5.26).

Step 4. The $L_T^2(H^{s-2})$ estimate of ∇B

We have the following dissipation rate from of ∇B inequality

$$\begin{aligned} \int_0^t \|\nabla B(\tau)\|_{H^{s-2}}^2 d\tau \lesssim \|w_0 - w_\infty\|_{H^{s-1}}^2 + \int_0^t \|E(\tau)\|_{H^{s-1}}^2 d\tau \\ + (M(t) + N_0(t))D_0(t)^2 + N_0(t)D_0(t)I(t) \end{aligned} \tag{5.32}$$

for $t \in [0, T]$.

The proof of (5.32) is the same as that in [172], here, for completeness we give the detailed calculations. Taking the inner product of the third and fourth equations of (5.18) with $-\nabla \times \partial_x^l B$ and $-\nabla \times \partial_x^l E$, respectively, and adding the resultant two equations, we obtain

$$-\partial_t(\partial_x^l E \cdot \nabla \times \partial_x^l B) + |\nabla \times \partial_x^l B|^2 - \operatorname{div}(\partial_x^l E \times \partial_x^l \partial_t B) = \mathcal{M}_2^l + \mathcal{G}_2^l, \tag{5.33}$$

where

$$\mathcal{M}_2^l := -n_\infty \partial_x^l u \cdot \nabla \times \partial_x^l B + |\nabla \times \partial_x^l E|^2, \qquad \mathcal{G}_2^l := -\partial_x^l((n - n_\infty)u) \cdot \nabla \times \partial_x^l B.$$

Integrating (5.33) over $\mathbb{R}^3$ and using Cauchy-Schwarz inequality give

$$-\frac{d}{dt}\int_{\mathbb{R}^3} \partial_x^l E \cdot \nabla \times \partial_x^l B dx + c_1\|\nabla \times \partial_x^l B\|_{L^2}^2 \le \|\nabla \times \partial_x^l E\|_{L^2}^2 + C\|\partial_x^l u\|_{L^2}^2 + G_2^l, \tag{5.34}$$

where $G_2^l := \int_{\mathbb{R}^3} |\mathcal{G}_2^l| dx$. Integrate (5.34) with respect to $t > 0$ and sum the resultant inequality for l with $0 \le l \le s-2$, we have

$$\begin{aligned} &\int_0^t \|\nabla B(\tau)\|_{H^{s-2}}^2 d\tau \\ &\le C(\|w_0 - w_\infty\|_{H^{s-1}}^2 + \|(w - w_\infty)(t)\|_{H^{s-1}}^2) \\ &\quad + C\left(\int_0^t \|\partial_x E(\tau)\|_{H^{s-2}}^2 d\tau + \int_0^t \|u(\tau)\|_{H^{s-2}}^2 d\tau + \sum_{l=0}^{s-2}\int_0^t G_2^l(\tau)d\tau\right), \end{aligned} \tag{5.35}$$

where we used the equality $\|\partial_x^l B\|_{L^2}^2 = \|\mathrm{div} B\|_{L^2}^2 + \|\nabla \times B\|_{L^2}^2$ due to (5.2). It also holds that

$$\begin{aligned} G_2^l &\le \|\partial_x^l((n-n_\infty)u)\|_{L^2}\|\partial_x^{l+1}B\|_{L^2} \\ &\lesssim \|(\rho,u)\|_{L^\infty}\|\partial_x^l(\rho,u)\|_{L^2}\|\partial_x^{l+1}B\|_{L^2}, \\ \int_0^t G_2^l(\tau)d\tau &\lesssim \sup_{0\le\tau\le t}\|(\rho,u)\|_{L^\infty}\int_0^t \|\partial_x^l(\rho,u)(\tau)\|_{L^2}\|\partial_x^{l+1}B(\tau)\|_{L^2}d\tau \\ &\lesssim M(t)D_0(t)^2 \end{aligned}$$

for $0 \le l \le s-2$. By substituting the above inequality for G_2^l into (5.35) and with the aid of (5.25), we obtain the desired (5.32).

Next, we make the suitable linear combination for (5.25), (5.26) and (5.32). Firstly, substituting (5.32) into (5.26) and taking $\epsilon > 0$ suitably small give

$$\begin{aligned} &\int_0^t (\|(n-n_\infty)(\tau)\|_{H^s}^2 + \|E(\tau)\|_{H^{s-1}}^2)d\tau \\ &\le C(\|w_0 - w_\infty\|_{H^s}^2 + (M(t)+N_0(t))D_0(t)^2 + N_0(t)D_0(t)I(t)). \end{aligned} \tag{5.36}$$

Then, substitute (5.36) into (5.32) to show

$$\begin{aligned} &\int_0^t \|\nabla B(\tau)\|_{H^{s-2}}^2 d\tau \\ &\le C(\|w_0 - w_\infty\|_{H^s}^2 + (M(t)+N_0(t))D_0(t)^2 + N_0(t)D_0(t)I(t)). \end{aligned} \tag{5.37}$$

Together with (5.25), (5.36) and (5.37), we have

$$\begin{aligned} &\|(w-w_\infty)(t)\|_{H^s}^2 + \int_0^t (\|\rho(\tau)\|_{H^s}^2 + \|u(\tau)\|_{H^{s+1}}^2 + \|E(\tau)\|_{H^{s-1}}^2 + \|\nabla B(\tau)\|_{H^{s-2}}^2)d\tau \\ &\le C(\|w_0 - w_\infty\|_{H^s}^2 + (M(t)+N_0(t))D_0(t)^2 + N_0(t)D_0(t)I(t)). \end{aligned}$$

That is

$$N_0(t)^2 + D_0(t)^2 \le C(\|w_0 - w_\infty\|_{H^s}^2 + (M(t)+N_0(t))D_0(t)^2 + N_0(t)D_0(t)I(t)). \tag{5.38}$$

Note that $M(t) \lesssim N_0(t)$ for $s \ge 3$ and $I(t) \lesssim D_0(t)$ for $s \ge 2$, (5.38) is reduced to

$$N_0(t)^2 + D_0(t)^2 \le C\|w_0 - w_\infty\|_{H^s}^2 + \tilde{C}N_0(t)D_0(t)^2$$

which gives a priori estimate (5.14) by using some smallness condition for $M(t)$ and $N_0(t)$. The proof of Proposition 5.1 is completed. □

The proof of Theorem 5.1.

Based on Proposition 5.1, we can obtain the global-in-time existence of smooth solutions to (5.1) and (5.4) by the standard continuation argument. Additionally, the large-time behavior of solutions stated in (5.10) and (5.11) is a consequence of the usage of Gagliardo-Nirenberg's

inequality and its proof is similar to that in [172]. We show the the detailed proof as follows. Firstly, we obtain the following uniform estimates for the time derivative $\partial_t w$

$$\|\partial_t w(t)\|^2_{H^{s-1}} + \int_0^t (\|\partial_t(n,u)(\tau)\|^2_{H^{s-1}} + \|\partial_t(E,B)(\tau)\|^2_{H^{s-2}})d\tau \lesssim \|w_0 - w_\infty\|^2_{H^s}. \tag{5.39}$$

To do this, by using (5.17) and the uniform estimate $\|w - w_\infty\|_{H^s} \le C$ in (5.14), we have

$$\begin{aligned}
&\|\partial_t w\|_{H^{s-1}} \lesssim \|w - w_\infty\|_{H^s},\\
&\|\partial_t(n,u)\|_{H^{s-1}} \lesssim \|\partial_x(n,u)\|_{H^{s-1}} + \|(u,E)\|_{H^{s-1}},\\
&\|\partial_t(E,B)\|_{H^{s-2}} \lesssim \|\partial_x(E,B)\|_{H^{s-2}} + \|u\|_{H^{s-1}},
\end{aligned}$$

which gives the desired (5.39) by using (5.14) again. Next, by using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_0^\infty \left|\frac{d}{dt}\|(n-n_\infty)(t)\|^2_{H^{s-1}}\right| dt \le 2\int_0^\infty \|(n-n_\infty)(t)\|_{H^{s-1}}\|\partial_t n(t)\|_{H^{s-1}}dt\\
&\le \int_0^\infty \left(\|(n-n_\infty)(t)\|^2_{H^{s-1}} + \|\partial_t n(t)\|^2_{H^{s-1}}\right) dt \lesssim \|w - w_\infty\|^2_{H^s},
\end{aligned}$$

where (5.14) and (5.39) are used. Combine the above inequality with (5.14) imply that $\|(n - n_\infty)(t)\|^2_{H^{s-1}}$ and $\frac{d}{dt}\|(n-n_\infty)(t)\|^2_{H^{s-1}}$ are integrable in $t \in (0,\infty)$, that is $\|(n-n_\infty)(t)\|^2_{H^{s-1}} \in W^{1,1}(0,\infty)$. It implies that $\|(n-n_\infty)(t)\|_{H^{s-1}} \to 0$ as $t \to \infty$. We deduce from the Gagliardo-Nirenberg's inequality $\|f\|_{L^\infty} \lesssim \|f\|^{1/4}_{L^2}\|\partial_x^2 f\|^{3/4}_{L^2}$ that

$$\|(n-n_\infty)(t)\|_{W^{s-2,\infty}} \lesssim \|(n-n_\infty)(t)\|^{1/4}_{H^{s-2}}\|\partial_x^2 n(t)\|^{3/4}_{H^{s-2}} \to 0 \ \text{ as } \ t \to \infty,$$

where we used the fact that $\|(n-n_\infty)(t)\|_{H^s}$ is uniformly bounded by (5.14). Similarly, we also obtain $\|u(t)\|_{W^{s-2,\infty}} \to 0$ as $t \to \infty$.

For the term E, it follows from (5.14) and (5.39) that $\|E(t)\|^2_{H^{s-2}} \in W^{1,1}(0,\infty)$ and hence $\|E(t)\|_{H^{s-2}} \to 0$ as $t \to \infty$. Then with the same way as for $n - n_\infty$, we have $\|E(t)\|_{W^{s-2,\infty}} \to 0$ as $t \to \infty$.

Finally, we deduce from (5.14) and (5.39) for B that $\|\partial_x B(t)\|^2_{H^{s-3}} \in W^{1,1}(0,\infty)$ and hence we have $\|\partial_x B(t)\|_{H^{s-3}} \to 0$ as $t \to \infty$. It follows from the above Gagliardo-Nirenberg's inequality that

$$\begin{aligned}
\|(B-B_\infty)(t)\|_{W^{s-4,\infty}} &\lesssim \|(B-B_\infty)(t)\|^{1/4}_{H^{s-4}}\|\partial_x^2 B(t)\|^{3/4}_{H^{s-4}}\\
&\lesssim \|(B-B_\infty)(t)\|^{1/4}_{H^{s-4}}\|\partial_x B(t)\|^{3/4}_{H^{s-3}} \to 0 \ \text{ as } \ t \to \infty,
\end{aligned}$$

provided that $s \ge 4$. The proof of the asymptotic stability result in (5.10) and (5.11) is completed. This completes the proof of Theorem 5.1.

5.3 The proof of optimal decay estimates

In the following, the analysis focuses on (5.6)-(5.7). Actually, we can achieve a similar global existence for the solution z. For simplification, allow us to abuse the notations $N_0(t)$ and $D_0(t)$ a little, which means that the corresponding functional norms with respect to z is still labeled as $N_0(t)$ and $D_0(t)$. Hence it follows from Theorem 5.1 that they may be bounded by $\|z_0\|_{H^s}(s \geq 3)$. The proof of Theorem 5.2 is divided into two parts, in order to show optimal decay rates of L^1-L^2 type for (5.1) and (5.4) with the minimal regularity assumption. Let us mention that the approach of this chapter is totally different from the elaborate spectral analysis and Duhamel's principle as in [50]. To overcome the difficulty arising from the weak dissipative mechanism of regularity-loss, as a first step, we establish the energy inequality for (5.6)-(5.7) in Fourier spaces, which is something like "square formula of Duhamel principle". Then, by using the L^p-L^q-L^r time-decay estimates in Lemma 5.1, we can obtain the optimal decay estimates in Theorem 5.2.

5.3.1 Energy methods in Fourier spaces

In this section, our task is to establish the "square formula of Duhamel principle" for (5.6)-(5.7) in Fourier spaces. Precisely,

Proposition 5.2. *Let $z = (\rho, v, E, h)^\top$ be the global smooth solutions to (5.6)-(5.7). Then the Fourier image of solutions satisfies the following pointwise estimate*

$$\begin{aligned} |\hat{z}(t,\xi)|^2 \lesssim\ & e^{-c_0\eta(\xi)t}|\hat{z}_0(\xi)|^2 \\ & + \int_0^t e^{-c_0\eta(\xi)(t-\tau)}(|\xi|^2|(\hat{Q},\hat{S})(\tau,\xi)|^2 + |\hat{R}(\tau,\xi)|^2)d\tau \end{aligned} \tag{5.40}$$

for any $t \geq 0$ and $\xi \in \mathbb{R}^3$, where the dissipative rate $\eta(\xi) := |\xi|^2/(1+|\xi|^2)^2$ and $c_0 > 0$ is a constant.

Proof. Applying the Fourier transform to (5.6) gives

$$\begin{cases} \partial_t\hat{\rho} + n_\infty i|\xi|\hat{v}\cdot\omega = 0, \\ \partial_t\hat{v} + a_\infty i|\xi|\hat{\rho}\omega + \hat{E} + \hat{v} + \hat{v}\times B_\infty + |\xi|^2\hat{v}/n_\infty \\ \qquad\qquad = (i|\xi|\hat{q}_2\cdot\omega + \hat{r}_2)/n_\infty - |\xi|^2\hat{s}_2, \\ \partial_t\hat{E} + i|\xi|\hat{h}\times\omega - n_\infty\hat{v} = 0, \\ \partial_t\hat{h} - i|\xi|\hat{E}\times\omega = 0, \end{cases} \tag{5.41}$$

for $\omega = \xi/|\xi| \in \mathbb{S}^2$, where "$i$" is the imaginary unit such that $i^2 = -1$. Also, it follows from (5.2) that

$$i|\xi|\hat{E}\cdot\omega = -\hat{\rho}, \quad i|\xi|\hat{h}\cdot\omega = 0. \tag{5.42}$$

Step 1. Estimate for dissipative term of $\hat{v}$

By taking the inner product of (5.41) in $\mathbb{C}^3$ with $a_\infty\hat{\rho},\ n_\infty\hat{v}, \hat{E}$ and $\hat{h}$, respectively, we arrive at

$$\begin{cases} \frac{1}{2}a_\infty\frac{d}{dt}|\hat{\rho}|^2 + a_\infty n_\infty\langle i|\xi|\hat{v}\cdot\omega,\hat{\rho}\rangle = 0, \\ \frac{1}{2}n_\infty\frac{d}{dt}|\hat{v}|^2 + a_\infty n_\infty\langle i|\xi|\hat{\rho}\omega,\hat{v}\rangle + \langle\hat{E}, n_\infty\hat{v}\rangle + \langle\hat{v}\times B_\infty + \hat{v}, n_\infty\hat{v}\rangle \\ \qquad\qquad +|\xi|^2|\hat{v}|^2 = \langle(i|\xi|\hat{q}_2\cdot\omega + \hat{r}_2 - n_\infty|\xi|^2\hat{s}_2),\hat{v}\rangle, \\ \frac{1}{2}\frac{d}{dt}|\hat{E}|^2 + \langle i|\xi|\hat{h}\times\omega,\hat{E}\rangle - \langle n_\infty\hat{v},\hat{E}\rangle = 0, \\ \frac{1}{2}\frac{d}{dt}|\hat{h}|^2 - \langle i|\xi|\hat{E}\times\omega,\hat{h}\rangle = 0. \end{cases} \tag{5.43}$$

Adding the equations of (5.43) together, and then taking the real part to get

$$\frac{d}{dt}\mathcal{E}_0 + 2(n_\infty + |\xi|^2)|\hat{v}|^2 = 2\mathrm{Re}\langle i|\xi|\hat{q}_2\cdot\omega + \hat{r}_2 - n_\infty|\xi|^2\hat{s}_2,\hat{v}\rangle,$$

where $\mathcal{E}_0 := a_\infty|\hat{\rho}|^2 + n_\infty|\hat{v}|^2 + |\hat{E}|^2 + |\hat{h}|^2 \approx |\hat{z}|^2$. It follows from Young's inequality that

$$\frac{d}{dt}\mathcal{E}_0 + c_1(1+|\xi|^2)|\hat{v}|^2 \lesssim (|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2 + |\xi|^2|\hat{S}|^2) \tag{5.44}$$

for $c_1 > 0$.

Step 2. Estimate for dissipative term of $\hat{\rho}$

By taking the inner product of the second equation of (5.41) with $a_\infty i|\xi|\hat{\rho}\omega$, we get

$$\begin{aligned} &\{\langle a_\infty i|\xi|\hat{\rho}_t\omega,\hat{v}\rangle + \langle\hat{v}_t, a_\infty i|\xi|\hat{\rho}\omega\rangle\} + |a_\infty i|\xi|\hat{\rho}\omega|^2 - n_\infty a_\infty|\xi|^2|\hat{v}\cdot\omega|^2 \\ &\quad + \langle\hat{v}\times B_\infty + \hat{E} + \hat{v}, a_\infty i|\xi|\hat{\rho}\omega\rangle + \langle|\xi|^2/n_\infty\hat{v}, a_\infty i|\xi|\hat{\rho}\omega\rangle \\ &\qquad = \langle(i|\xi|\hat{q}_2\cdot\omega + \hat{r}_2)/n_\infty - |\xi|^2\hat{s}_2, a_\infty i|\xi|\hat{\rho}\omega\rangle. \end{aligned} \tag{5.45}$$

By taking the real part on both sides of (5.45), then using the Young's inequality and the first equation in (5.42), we can obtain

$$\frac{d}{dt}\Big(\frac{a_\infty|\xi|}{1+|\xi|^2}\mathcal{E}_1\Big) + c_2|\hat{\rho}|^2 \lesssim (1+|\xi|^2)|\hat{v}|^2 + (|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2 + |\xi|^2|\hat{S}|^2) \tag{5.46}$$

for $c_2 > 0$, where $\mathcal{E}_1 := \mathrm{Re}\langle i\hat{\rho}\omega,\hat{v}\rangle$.

Step 3. Estimate for dissipative term of $\hat{E}$

By taking the inner product of the second and third equations of (5.41) with $\hat{E}$ and $\hat{v}$, respectively, we have

$$\begin{aligned} &\langle\hat{v}_t,\hat{E}\rangle + \langle\hat{E}_t,\hat{v}\rangle + |\hat{E}|^2 + a_\infty|\xi|^2|\hat{E}\cdot\omega|^2 \\ &= n_\infty|\hat{v}|^2 - \langle\hat{v}\times B_\infty + \hat{v},\hat{E}\rangle - \langle i|\xi|\hat{h}\times\omega,\hat{v}\rangle \\ &- \langle|\xi|^2/n_\infty\hat{v},\hat{E}\rangle + \langle(i|\xi|\hat{q}_2\cdot\omega + \hat{r}_2)/n_\infty - |\xi|^2\hat{s}_2,\hat{E}\rangle. \end{aligned} \tag{5.47}$$

By taking the real part of (5.47), and then using the Young's inequality, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{\mathcal{E}_2}{1+|\xi|^2}\Big) + \frac{c_3}{1+|\xi|^2}|\hat{E}|^2 \\ &\le \epsilon\frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{h}|^2 + C_\epsilon(1+|\xi|^2)|\hat{v}|^2 + C(|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2 + |\xi|^2|\hat{S}|^2) \end{aligned} \tag{5.48}$$

for any $\epsilon > 0$, where $\mathcal{E}_2 := \mathrm{Re}\langle \hat{v}, \hat{E}\rangle$, and c_3, C_ϵ (depending on ϵ) are some positive constants.

Step 4. Estimate for dissipative term of $\hat{h}$

Performing the inner product of the third and fourth equations of (5.41) with $i|\xi|\hat{h}\times\omega$ and $i|\xi|\hat{E}\times\omega$, then taking the real part gives

$$\frac{d}{dt}(|\xi|\mathcal{E}_3) + |\xi|^2|\hat{h}\times\omega|^2 = |\xi|^2|\hat{E}\times\omega|^2 + \mathrm{Re}\langle n_\infty \hat{v}, i|\xi|\hat{h}\times\omega\rangle, \tag{5.49}$$

where $\mathcal{E}_3 := \mathrm{Re}\langle \hat{E}, i\hat{h}\times\omega\rangle$. Due to the second equation in (5.42), we have $|\hat{h}\times\omega| \approx |\hat{h}|$. Furthermore, dividing (5.49) by $(1+|\xi|^2)^2$, with the aid of Young's inequality, we reach

$$\frac{d}{dt}\Big\{\frac{|\xi|}{(1+|\xi|^2)^2}\mathcal{E}_3\Big\} + c_4\frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{h}|^2 \lesssim \frac{1}{1+|\xi|^2}|\hat{E}|^2 + |\hat{v}|^2 \tag{5.50}$$

for $c_4 > 0$.

Step 5. Combining the above analysis

Together with energy inequalities (5.44), (5.46), (5.48) and (5.50), the final step is to make the suitable linear combination for them, see [169] for similar details. That is, the Navier-Stokes-Maxwell system admits Lyapunov function

$$\mathcal{E}[\hat{z}] := \mathcal{E}_0 + \frac{\alpha_1}{1+|\xi|^2}\Big\{a_\infty|\xi|\mathcal{E}_1 + \mathcal{E}_2 + \frac{\alpha_2|\xi|}{1+|\xi|^2}\mathcal{E}_3\Big\}$$

such that the following differential inequality holds

$$\frac{d}{dt}\mathcal{E}[\hat{z}] + c_0\mathcal{D}[\hat{z}] \lesssim (|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2 + |\xi|^2|\hat{S}|^2), \tag{5.51}$$

where

$$\mathcal{D}[\hat{z}] = |\hat{\rho}|^2 + (1+|\xi|^2)|\hat{v}|^2 + \frac{1}{1+|\xi|^2}|\hat{E}|^2 + \frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{h}|^2$$

and $\alpha_1, \alpha_2 > 0$ are suitable small constants which ensure that $\mathcal{E}[\hat{z}] \approx |\hat{z}|^2$. It follows from (5.51) that

$$\frac{d}{dt}\mathcal{E}[\hat{z}] + c_0\eta(\xi)\mathcal{E}[\hat{z}] \lesssim (|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2 + |\xi|^2|\hat{S}|^2),$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$. Finally, the inequality (5.40) is followed from the standard Gronwall's inequality. This ends the proof of Proposition 5.2. □

5.3.2 Optimal decay rates

In this section, we prove the optimal decay estimates of smooth solutions to (5.6)-(5.7) in Theorem 5.2. To this end, we recall L^p-L^q-L^r estimates, which has been developed in [182] recently.

Lemma 5.1. *([182]) Let $\eta(\xi)$ be a positive, continuous and real-valued function in $\mathbb{R}^d$ satisfying*

$$\eta(\xi)\sim\begin{cases}|\xi|^{\sigma_1}, & |\xi|\to 0;\\ |\xi|^{-\sigma_2}, & |\xi|\to\infty;\end{cases}$$

for $\sigma_1,\sigma_2>0$. For $\phi\in\mathcal{S}(\mathbb{R}^d)$, it holds that

$$\begin{aligned}&\|\mathcal{F}^{-1}[|\xi|^k e^{-\eta(\xi)t}|\hat{\phi}(\xi)|]\|_{L^p}\\ &\lesssim \underbrace{(1+t)^{-\gamma_{\sigma_1}(q,p)-\frac{k-j}{\sigma_1}}\|\partial_x^j\phi\|_{L^q}}_{Low-frequency\ \ Estimate}+\underbrace{(1+t)^{-\frac{\ell}{\sigma_2}+\gamma_{\sigma_2}(r,p)}\|\partial_x^{k+\ell}\phi\|_{L^r}}_{High-frequency\ \ Estimate}\end{aligned}\tag{5.52}$$

for $\ell>d(\frac{1}{r}-\frac{1}{p})$ [1], $1\le q,r\le 2\le p\le\infty$ and $0\le j\le k$, where $\gamma_\sigma(q,p):=\frac{d}{\sigma}(\frac{1}{q}-\frac{1}{p})(\sigma>0)$.

Obviously, we see that the major contribution of (5.52) with respect to (5.8) is that the integrability information is captured in the high-frequency estimate, which leads to achieve the minimal decay regularity $s=3$ for (5.6)-(5.7). We define time-weighted energy functionals:

$$\begin{aligned}N(t)&=\sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{4}}\|z(\tau)\|_{L^2},\\ D(t)^2&=\int_0^t\Big(\|\rho(\tau)\|_{H^s}^2+\|\upsilon(\tau)\|_{H^{s+1}}^2+\|E(\tau)\|_{H^{s-1}}^2+\|\nabla h(\tau)\|_{H^{s-2}}^2\Big)d\tau\end{aligned}$$

for $s\ge 3$. In what follows, let us show a nonlinear energy inequality in terms of $N(t)$ and $D(t)$, which is included in the following

Proposition 5.3. *Let $z=(\rho,\upsilon,E,h)^\top$ be the global smooth solutions of (5.6)-(5.7). Additionally, if $z_0\in L^1$, then we have*

$$N(t)\lesssim\|z_0\|_{H^s\cap L^1}+N(t)D(t)+N(t)^2\tag{5.53}$$

for $s\ge 3$.

Proof. It follows from (5.40) that

$$\begin{aligned}&\int_{\mathbb{R}^3}|\hat{z}(\xi)|^2d\xi\\ &\lesssim\int_{\mathbb{R}^3}e^{-c\eta(\xi)t}|\hat{z}_0(\xi)|^2d\xi+\int_{\mathbb{R}^3}\int_0^t e^{-c\eta(\xi)(t-\tau)}\Big\{|\xi|^2\Big(|\hat{Q}(\tau,\xi)|^2\\ &\quad+|\hat{S}(\tau,\xi)|^2\Big)+|\hat{R}(\tau,\xi)|^2\Big\}d\tau d\xi\\ &\triangleq J_1+J_2+J_3+J_4.\end{aligned}\tag{5.54}$$

For J_1, by taking $\sigma_1=\sigma_2=2,p=2,k=0,j=0,q=1$ and $r=2,\ell\ge 2$ in Lemma 5.1, we arrive at

$$J_1\lesssim(1+t)^{-\frac{3}{2}}\|z_0\|_{L^1}^2+(1+t)^{-\ell}\|\partial_x^\ell z_0\|_{L^2}^2.\tag{5.55}$$

[1] We would like to remark that $\ell\ge 0$ in the case of $p=r=2$.

Next, we begin to evaluate nonlinear terms. For J_2, it is written as the sum of low-frequency and high-frequency parts

$$J_2 := J_{2L} + J_{2H}.$$

For J_{2L}, by taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 1, j = 0, q = 1$ in Lemma 5.1, we have

$$\begin{aligned} J_{2L} &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|Q(\tau)\|_{L^1}^2 d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|z(\tau)\|_{L^2}^4 d\tau \\ &\lesssim N(t)^4 \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-3} d\tau \\ &\lesssim (1+t)^{-\frac{5}{2}} N(t)^4, \end{aligned} \tag{5.56}$$

where we have used the fact $Q(z) = O(|(\rho, \upsilon)|^2)$. For simplicity, we set $z^\perp := (\rho, \upsilon)$.

For J_{2H}, more elaborate estimates are proceeded. For this purpose, we write

$$J_{2H} = \Big(\int_0^{t/2} + \int_{t/2}^t \Big)(\cdots) d\tau := J_{2H1} + J_{2H2}.$$

Taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 1, r = 2, \ell \geq 2$ in Lemma 5.1 gives

$$\begin{aligned} J_{2H1} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} \|\partial_x^{\ell+1} Q(\tau)\|_{L^2}^2 d\tau \\ &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} \|z^\perp\|_{L^\infty}^2 \|\partial_x^{\ell+1} z^\perp\|_{L^2}^2 d\tau \\ &\lesssim \sup_{0 \leq \tau \leq t/2} \Big\{ (1+t-\tau)^{-\ell} \|z\|_{L^\infty}^2 \Big\} \int_0^{t/2} \|\partial_x^{\ell+1} z^\perp\|_{L^2}^2 d\tau \\ &\lesssim (1+t)^{-\ell} N_0(t)^2 D(t)^2 \\ &\lesssim (1+t)^{-\ell} \|z_0\|_{H^s}^2 \end{aligned} \tag{5.57}$$

for $s \geq 3$, where we have used the fact $Q(z) = O(|z^\perp|^2)$ and Lemma 2.18. On the other hand, by taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 1, r = 1, \ell = 2$ in Lemma 5.1, we get

$$\begin{aligned} J_{2H2} &\lesssim \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^3 Q(\tau)\|_{L^1}^2 d\tau \\ &\lesssim \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|z^\perp\|_{L^2}^2 \|\partial_x^3 z^\perp\|_{L^2}^2 d\tau \\ &\lesssim N(t)^2 \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{3}{2}} \|\partial_x^3 z^\perp\|_{L^2}^2 d\tau \\ &\lesssim N(t)^2 \sup_{t/2 \leq \tau \leq t} \Big\{ (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{3}{2}} \Big\} \int_0^t \|\partial_x^3 z^\perp\|_{L^2}^2 d\tau \\ &\lesssim (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2. \end{aligned} \tag{5.58}$$

For J_3, we write

$$J_3 := J_{3L} + J_{3H}.$$

Recalling $S(z) = O(|\tilde{h}(\rho)v|)$, where $\tilde{h}(\rho)$ is a smooth function satisfying $\tilde{h}(0) = 0$. By taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 1, j = 0, q = 1$ in Lemma 5.1, we obtain

$$\begin{aligned} J_{3L} &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|S(\tau)\|_{L^1}^2 d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|\tilde{h}(\rho)(\tau)\|_{L^2}^2 \|v(\tau)\|_{L^2}^2 d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|\rho(\tau)\|_{L^2}^2 \|v(\tau)\|_{L^2}^2 d\tau \\ &\lesssim N(t)^4 \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-3} d\tau \\ &\lesssim (1+t)^{-\frac{5}{2}} N(t)^4. \end{aligned} \tag{5.59}$$

Similarly, we split the high-frequency part J_{3H} as follows:

$$J_{3H} = \Big(\int_0^{t/2} + \int_{t/2}^t \Big)(\cdots) d\tau := J_{3H1} + J_{3H2}.$$

Taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 1, r = 2, \ell \geq 2$ in Lemma 5.1 gives

$$\begin{aligned} J_{3H1} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} \|\partial_x^{\ell+1} S(\tau)\|_{L^2}^2 d\tau \\ &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} \|\partial_x^{\ell+1} (\tilde{h}(\rho)v)\|_{L^2}^2 d\tau \\ &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} (\|\tilde{h}(\rho)\|_{L^\infty}^2 \|\partial_x^{\ell+1} v\|_{L^2}^2 + \|v\|_{L^\infty}^2 \|\partial_x^{\ell+1} \tilde{h}(\rho)\|_{L^2}^2) d\tau \\ &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} (\|\rho\|_{L^\infty}^2 \|\partial_x^{\ell+1} v\|_{L^2}^2 + \|v\|_{L^\infty}^2 \|\partial_x^{\ell+1} \rho\|_{L^2}^2) d\tau \\ &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} \|z^\perp\|_{L^\infty}^2 \|\partial_x^{\ell+1} z^\perp\|_{L^2}^2 d\tau \\ &\lesssim \sup_{0 \leq \tau \leq t/2} \Big\{ (1+t-\tau)^{-\ell} \|z\|_{L^\infty}^2 \Big\} \int_0^{t/2} \|\partial_x^{\ell+1} z^\perp\|_{L^2}^2 d\tau \\ &\lesssim (1+t)^{-\ell} N_0(t)^2 D(t)^2 \\ &\lesssim (1+t)^{-\ell} \|z_0\|_{H^s}^2 \end{aligned} \tag{5.60}$$

for $s \geq 3$, where Lemma 2.18 and Lemma 2.20 were used in the third and fourth lines, respectively. On the other hand, by taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 1, r = 1, \ell = 2$ in Lemma 5.1,

we get

$$\begin{aligned}
J_{3H2} &\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^3 S(\tau)\|_{L^1}^2 d\tau.\\
&\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}(\|\tilde{h}(\rho)\|_{L^2}^2\|\partial_x^3 v\|_{L^2}^2+\|v\|_{L^2}^2\|\partial_x^3\tilde{h}(\rho)\|_{L^2}^2)d\tau\\
&\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}(\|\rho\|_{L^2}^2\|\partial_x^3 v\|_{L^2}^2+\|v\|_{L^2}^2\|\partial_x^3\rho\|_{L^2}^2)d\tau\\
&\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\|z^{\perp}\|_{L^2}^2\|\partial_x^3 z^{\perp}\|_{L^2}^2 d\tau\\
&\lesssim N(t)^2\int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}\|\partial_x^3 z^{\perp}\|_{L^2}^2 d\tau\\
&\lesssim N(t)^2\sup_{t/2\le\tau\le t}\Big\{(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}\Big\}\int_0^t\|\partial_x^3 z^{\perp}\|_{L^2}^2 d\tau\\
&\lesssim (1+t)^{-\frac{3}{2}}N(t)^2 D(t)^2.
\end{aligned}\tag{5.61}$$

For J_4, we write

$$J_4 := J_{4L}+J_{4H}.$$

Note that $R(z)=O(\rho|E|+|v||h|)$, by taking $\sigma_1=\sigma_2=2, p=2$ and $k=0, j=0, q=1$ in Lemma 5.1, we obtain

$$\begin{aligned}
J_{4L} &\lesssim \int_0^t(1+t-\tau)^{-\frac{3}{2}}\|R(\tau)\|_{L^1}^2 d\tau\\
&\lesssim \int_0^t(1+t-\tau)^{-\frac{3}{2}}\|z(\tau)\|_{L^2}^4 d\tau\\
&\lesssim N(t)^4\int_0^t(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-3}d\tau\\
&\lesssim (1+t)^{-\frac{3}{2}}N(t)^4.
\end{aligned}\tag{5.62}$$

Similarly, we separate the high-frequency part J_{4H} as follows

$$J_{4H}=\Big(\int_0^{t/2}+\int_{t/2}^{t}\Big)(\cdots)d\tau := J_{4H1}+J_{4H2}.$$

Taking $\sigma_1=\sigma_2=2, p=2$ and $k=0, r=2, \ell\ge 2$ in Lemma 5.1 leads to

$$\begin{aligned}
J_{4H1} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\ell}\|\partial_x^\ell R(\tau)\|_{L^2}^2 d\tau\\
&\lesssim \int_0^{t/2}(1+t-\tau)^{-\ell}\|z\|_{L^\infty}^2\|\partial_x^\ell z\|_{L^2}^2 d\tau\\
&\lesssim \sup_{0\le\tau\le t/2}\Big\{(1+t-\tau)^{-\ell}\|z\|_{L^\infty}^2\Big\}\int_0^{t/2}\|\partial_x^\ell z\|_{L^2}^2 d\tau\\
&\lesssim (1+t)^{-\ell}N_0(t)^2 D(t)^2\\
&\lesssim (1+t)^{-\ell}\|z_0\|_{H^s}^2,
\end{aligned}\tag{5.63}$$

where we have used Lemma 2.18. On the other hand, by taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 0, r = 1, \ell = 2$ in Lemma 5.1, we arrive at

$$\begin{aligned} J_{4H2} &\lesssim \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^2 R(\tau)\|_{L^1}^2 d\tau \\ &\lesssim \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} \|z\|_{L^2}^2 \|\partial_x^2 z\|_{L^2}^2 d\tau \\ &\lesssim N(t)^2 \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{3}{2}} \|\partial_x^2 z\|_{L^2}^2 d\tau \\ &\lesssim N(t)^2 \sup_{t/2\le\tau\le t} \left\{ (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{3}{2}} \right\} \int_0^t \|\partial_x^2 z\|_{L^2}^2 d\tau \\ &\lesssim (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2. \end{aligned} \tag{5.64}$$

Therefore, together with inequalities (5.54)-(5.64), it follows from Plancherel's theorem that

$$\begin{aligned} \|z\|_{L^2}^2 \lesssim\ & (1+t)^{-\frac{3}{2}} \|z_0\|_{H^s\cap L^1}^2 + (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2 \\ & + (1+t)^{-\frac{3}{2}} N(t)^4, \end{aligned}$$

which leads to (5.53) exactly. □

The proof of Theorem 5.2. The dissipation norm $D(t) \lesssim \|z_0\|_{H^s} \lesssim \|z_0\|_{H^s\cap L^1}$, according to the similar energy inequality for the solution z as (5.9) in Theorem 5.1. Thus, if $\|z_0\|_{H^s\cap L^1}$ is sufficient small, then it holds that

$$N(t) \lesssim \|z_0\|_{H^s\cap L^1} + N(t)^2$$

for $s \ge 3$, which implies that $N(t) \lesssim \|z_0\|_{H^s\cap L^1}$, provided that $\|z_0\|_{H^s\cap L^1}$ is sufficient small. Consequently, the decay estimates in Theorem 5.2 are followed.

Chapter 6

Global existence and large-time behavior of the Timoshenko-Fourier system

Contents

This chapter is devoted to investigate the Timoshenko-Fourier system in thermoelasticity, which describes the dynamics of elastic wave propagation and thermal dissipation in a vibrating beam. Indeed, the Timoshenko-Fourier system admits a *non-symmetric dissipation*, furthermore, it is observed that the dissipative mechanism of regularity-loss type will occur if wave speeds are different, which makes those classical energy methods in Ide-Kawashima [87] and Racke-Said Houari [155] cannot work. Consequently, we obtain the optimal decay rate of L^1-L^2 type for global smooth solutions by using L^p-L^q-L^r estimates.

6.1 Description of problem

In this chapter, we consider the following coupled set of two wave equations of Timoshenko theory of a vibrating beam [165, 166] with additional effect of heat conduction according to the Fourier law,

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - [\sigma(\psi_x)]_x - (\varphi_x - \psi) + \gamma\psi_t + \beta\theta_x = 0, \\ \theta_t - \kappa\theta_{xx} + \beta\psi_{tx} = 0, \end{cases} \tag{6.1}$$

where initial conditions are prescribed as

$$(\varphi, \varphi_t, \psi, \psi_t, \theta)|_{t=0} = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)(x), \quad x \in \mathbb{R}. \tag{6.2}$$

The linearized version of (6.1) reads correspondingly

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \gamma\psi_t + \beta\theta_x = 0, \\ \theta_t - \kappa\theta_{xx} + \beta\psi_{tx} = 0, \end{cases} \tag{6.3}$$

where $a > 0$ denotes the sound speed defined by $a^2 = \sigma'(0)$. The case $a = 1$ corresponds to the Timoshenko-Fourier system with equal wave speeds, whereas the case $a \neq 1$ implies that the wave speeds of the first two equations are different.

We are interested in the Timoshenko-Fourier system (6.1), where the heat conduction is described by the classical Fourier law. To the best of our knowledge, few stability efforts can be available for (6.1). In a bounded domain, Rivera-Racke [140] proved several exponential decay results for (6.3) and found a non-exponential stability for the case of different wave speeds. In the whole space, by virtue of the change of variable in [85]

$$v = \varphi_x - \psi,\ u = \varphi_t,\ z = a\psi_x,\ y = \psi_t.$$

For simplicity, we set $U := (v, u, z, y, \theta)^\top$($\top$ transpose). System (6.3) is rewritten as the first-order system of hyperbolic-parabolic type

$$\begin{cases} U_t + AU_x + LU = BU_{xx}, \\ U(x,0) = U_0(x), \end{cases} \tag{6.4}$$

where the coefficient matrices A, L and B are explicitly given in Chapter 1.1.4. The corresponding initial data are as follows

$$U|_{t=0} = (v_0, u_0, z_0, y_0, \theta_0)^\top(x),$$

with

$$v_0 = \varphi_{0,x} - \psi_0,\ u_0 = \varphi_1,\ z_0 = a\psi_{0,x},\ y_0 = \psi_1.$$

It is convenient to rewrite (6.1)-(6.2) as the following Cauchy problem for the hyperbolic-parabolic system

$$\begin{cases} U_t + AU_x + LU = BU_{xx} + G(U)_x, \\ U(x,0) = U_0(x), \end{cases} \tag{6.5}$$

where $G(U) = (0, 0, 0, g(z), 0)^\top(x)$ with $g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0)z/a := O(z^2)$ near $z = 0$.

As yet there are few stability results for the nonlinear Timoshenko-Fourier system (6.1)-(6.2). As a first step, we focus on the cases $\gamma > 0$ and $a \neq 1$. Now we show our main results, to do this, define $N_0(t)$ and $D_0(t)$ by the following energy norm and the corresponding dissipation,

respectively:

$$N_0(t) := \sup_{0\le\tau\le t} \|U(\tau)\|_{H^s(\mathbb{R})},$$
$$D_0(t)^2 := \int_0^t \Big(\|v(\tau)\|^2_{H^{s-1}(\mathbb{R})} + \|\partial_x u(\tau)\|^2_{H^{s-2}(\mathbb{R})} + \|\partial_x z(\tau)\|^2_{H^{s-1}(\mathbb{R})} + \|y(\tau)\|^2_{H^s(\mathbb{R})} + \|\partial_x \theta(\tau)\|^2_{H^s(\mathbb{R})} \Big) d\tau.$$

Then we state the global-in-time existence of smooth solutions to (6.5) in the small amplitude regime.

Theorem 6.1. *Let $s \ge 2$ and suppose that the initial data satisfy $U_0 \in H^s(\mathbb{R})$. Put $I_0 := \|U_0\|_{H^s(\mathbb{R})}$. Then there exists a constant $\varepsilon_0 > 0$ such that if $I_0 \le \varepsilon_0$, the Cauchy problem (6.5) has a unique global solution $U(t,x)$ with*

$$U \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})).$$

Moreover, the uniform energy inequality

$$N_0(t)^2 + D_0(t)^2 \lesssim I_0^2 \tag{6.6}$$

holds for $t \ge 0$.

Remark 6.1. We mention that the energy inequality (6.6) is of the regularity-loss type because there is 1-regularity loss for (v,u) in the dissipation part $D_0(t)$. In addition, Theorem 6.1 also holds in the periodic domain $\mathbb{T}$. It is not difficult to see that $\bar{z}(t) = 0$ for all $t > 0$, if $\bar{z}_0 = 0$ is additionally assumed, where $\bar{f} := \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} f(x)dx$.

Furthermore, based on Theorem 6.1, the optimal decay rates for solution to (6.5) are available.

Theorem 6.2. *Let $m = 0, 1$ and $s \ge \ell_0 + m + 1$ with $\ell_0 = m + 1$. Assume that the initial data satisfy $U_0 \in H^s(\Omega) \cap L^1(\Omega)$. The addition initial condition $\bar{z}_0 = 0$ is assumed in the periodic space $\Omega = \mathbb{T}$. Put $I_1 := \|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}$. Then there exists a constant $\varepsilon_1 > 0$ such that if $I_1 \le \varepsilon_1$, the solution of (6.5) admits the decay estimate*

$$\|\partial_x^m U\|_{L^2(\Omega)} \lesssim I_1(1+t)^{-1/4-m/2},$$

in case of $\Omega = \mathbb{R}, m = 0$ as well as in case of $\Omega = \mathbb{T}$ and $m = 0, 1$.

Remark 6.2. Due to the weaker dissipative mechanism of regularity-loss type, it is more interesting to seek the possibly lower regularity. By virtue of L^p-L^q-L^r inequalities in Lemma 5.1, we could achieve the optimal algebraic rate of L^1-L^2 type under the critical regularity $s_c = 2$, that is, the minimal decay regularity $s_D \le 2$ follows for the Timoshenko-Fourier system.

■ *The schema of proof of main theorem*

• The proof of global smooth solution (Section 6.2)

By a careful observation, we think the viscosity term in (6.5) does not influence the mathematical entropy, which inspires us to obtain global a priori estimates in Proposition 6.1 from the point of view of hyperbolic energy approaches. The global existence in Theorem 6.1 is obtained by using the standard boot-strap argument.

• The proof of optimal decay estimates (Section 6.3)

System (6.4) admits the decay property (1.41), which is of regularity-loss type at the high frequency. Consequently, it seems impossible to obtain the optimal decay rate with the relatively lower regularity. To overcome it, we skip the traditional Duhamel's principle and employ a different technique.

Step 1) Asymptotic expansion of eigenvalues

Through the asymptotic expansions (6.32) and (6.33), we obtain a result that the analysis of eigenvalues in [134] is improved due to the existence of the damping term $\gamma\psi_t$. Also, the decay behavior of eigenvalues also confirms the optimality of those linear results in [160].

Step 2) Energy methods in Fourier spaces

We establish the pointwise energy estimate in Fourier spaces for (6.5), which is something like "square formula of Duhamel principle", see Proposition 6.2. From this proposition, we obtain that the dissipative rate is regularity-loss type satisfying $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$.

Step 3) Optimal decay rates

To get the optimal decay rate of L^1-L^2 type, we employ the L^p-L^q-L^r estimates. In particular, the high-frequency estimate is divided into two parts, and on each part, one can proceed different spatial integrals (for example, $r = 1$ or $r = 2$), which leads to desired decay estimates effectively. Additionally, it follows from (6.5) that $z(t,x)$ is a conservative quantity for all time $t \geq 0$. Therefore, by Poincaré inequality, one further achieves the optimal estimate for the derivative of solutions in the periodic domain. See the detailed proof in Proposition 6.3.

6.2 The proof of global existence

In this section, the central task is to construct a priori estimate according to the dissipative mechanism produced by the Timoshenko-Fourier system, which ensures that the solution of (6.5) is globally defined. Define the quantity

$$M_0(t) := \sup_{0\leq\tau\leq t} \|U(\tau)\|_{W^{1,\infty}(\mathbb{R})}. \tag{6.7}$$

For clarity, a priori estimate are shown by the following.

Proposition 6.1. *Let $U(t,x)$ be a solution to (6.5) satisfying*

$$U \in C([0,T);H^s(\mathbb{R})) \cap C^1([0,T);H^{s-1}(\mathbb{R}))$$

for any $T > 0$ and $s \geq 2$. There are a constant $\varepsilon_2 > 0$ independent of T such that if $N_0(T) \leq \varepsilon_2$, then a priori estimate

$$N_0(t)^2 + D_0(t)^2 \lesssim \|U_0\|^2_{H^s(\mathbb{R})} \tag{6.8}$$

holds for $t \in [0, T]$.

Proof. Let us begin with the following equations

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - a y_x = 0, \\ y_t - [\sigma(z/a)]_x - v + \gamma y + \beta \theta_x = 0, \\ \theta_t - \kappa \theta_{xx} + \beta y_x = 0. \end{cases} \tag{6.9}$$

Indeed, the proof is separated into several steps.

Step 1. The $L^\infty_T(L^2)$ estimate of U and the L^2_T one of (y, θ_x)

By multiplying (6.9) by $v, u, (\sigma(z/a) - \sigma(0))/a, y$ and θ, respectively, and adding the resulting equalities, we arrive at

$$\partial_t \mathcal{H}^0 - \partial_x \mathcal{F}^0 + \gamma y^2 + \kappa \theta_x^2 = 0, \tag{6.10}$$

where

$$\begin{aligned} \mathcal{H}^0 &= \frac{1}{2}(v^2 + u^2 + F(z) + y^2 + \theta^2), \\ \mathcal{F}^0 &= vu + (\sigma(z/a) - \sigma(0))y + \beta \theta y + \kappa \theta_x \theta, \\ F(z) &= 2 \int_0^{z/a} (\sigma(\eta) - \sigma(0)) d\eta. \end{aligned}$$

Due to $\sigma'(\eta) > 0$ and (6.7), we see that $F(z) \approx |z|^2$. Hence, integrating (6.10) over $[0, t] \times \mathbb{R}$ yields

$$\|U(t)\|^2_{L^2} + \int_0^t 2(\gamma \|y(\tau)\|^2_{L^2} + \kappa \|\theta_x(\tau)\|^2_{L^2}) d\tau \lesssim \|U_0\|^2_{L^2} \tag{6.11}$$

for $t \in [0, T]$.

Step 2. The $L^\infty_T(H^{s-1})$ estimate of $\partial_x U$ and the $L^2_T(H^{s-1})$ one of $\partial_x(u, \theta_x)$

Actually, the energy inequality for derivatives of solutions is stated as follows

$$\begin{aligned} &\|\partial_x U(t)\|^2_{H^{s-1}} + \int_0^t (\|\partial_x y(\tau)\|^2_{H^{s-1}} + \|\partial_x \theta_x(\tau)\|^2_{H^{s-1}}) d\tau \\ &\quad \lesssim \|\partial_x U_0\|^2_{H^{s-1}} + M_0(t) D_0(t)^2 \end{aligned} \tag{6.12}$$

for $t \in [0, T]$.

To do this, by applying the differential operator $\partial_x^l (1 \le l \le s)$ to (6.9), we obtain

$$\begin{cases} \partial_x^l v_t - \partial_x^{l+1} u + \partial_x^l y = 0, \\ \partial_x^l u_t - \partial_x^{l+1} v = 0, \\ \partial_x^l z_t - a\partial_x^{l+1} y = 0, \\ \partial_x^l y_t - \sigma'(z/a)\partial_x^{l+1} z/a - \partial_x^l v + \gamma \partial_x^l y + \beta \partial_x^{l+1}\theta = [\partial_x^l, \sigma'(z/a)](z/a)_x, \\ \partial_x^l \theta_t - \kappa \partial_x^{l+2}\theta + \beta \partial_x^{l+1} y = 0. \end{cases} \tag{6.13}$$

Multiplying (6.13) by $\partial_x^l v, \partial_x^l u, (1/a^2)\sigma'(z/a)\partial_x^l z, \partial_x^l y$ and $\partial_x^l \theta$, respectively, then adding the resultant equations implies that

$$\partial_t \mathcal{H}^l + \partial_x \mathcal{F}^l + \gamma(\partial_x^l y)^2 + \kappa(\partial_x^l \theta_x)^2 = \mathcal{R}^l, \tag{6.14}$$

where

$$\begin{aligned} \mathcal{H}^l &:= \frac{1}{2}\{(\partial_x^l v)^2 + (\partial_x^l u)^2 + \sigma'(z/a)(\partial_x^l z/a)^2 + (\partial_x^l y)^2 + (\partial_x^l \theta)^2\}, \\ \mathcal{F}^l &:= \partial_x^l v \partial_x^l u + \partial_x^l y \partial_x^l \theta + \sigma'(z/a)(\partial_x^l z/a)\partial_x^l y + \kappa \partial_x^{l+1}\theta \partial_x^l \theta, \\ \mathcal{R}^l &:= \frac{1}{2}\sigma'(z/a)_t(\partial_x^l z/a)^2 - \sigma'(z/a)_x(\partial_x^l z/a)\partial_x^l y + \partial_x^l y[\partial_x^l, \sigma'(z/a)](z/a)_x, \end{aligned}$$

where we used $z_t = a y_x$ for the third equation of $\mathcal{F}^l$. Integrating (6.14) over $x \in \mathbb{R}$ gives

$$\frac{d}{dt}\int_{\mathbb{R}} \mathcal{H}^l dx + \gamma \|\partial_x^l y\|_{L^2}^2 + \kappa \|\partial_x^l \theta_x\|_{L^2}^2 \le R^l, \tag{6.15}$$

where we set $R^l := \int_{\mathbb{R}} |\mathcal{R}^l| dx$. For R^l, it follows from Lemmas 2.17-2.18 that

$$R^l \lesssim \|\partial_x(z,y)\|_{L^\infty} \|\partial_x^l(z,y)\|_{L^2}^2, \quad \int_0^t R^l(\tau)d\tau \lesssim M_0(t) D_0(t)^2. \tag{6.16}$$

Here, $\int_{\mathbb{R}} \mathcal{H}^l dx$ is equivalent to the quadratic norm $\|\partial_x^l U\|_{L^2}^2$ due to (6.7). Integrating (6.15) with respect to $t > 0$ and summing the resulting inequality for l with $1 \le l \le s$ together. Furthermore, (6.12) is followed by (6.16). Finally, together with (6.11) and (6.12), we can conclude that

$$\|U(t)\|_{H^s}^2 + \int_0^t (\|y(\tau)\|_{H^s}^2 + \|\partial_x \theta(\tau)\|_{H^s}^2) d\tau \lesssim \|U_0\|_{H^s}^2 + M_0(t) D_0(t)^2 \tag{6.17}$$

for $t \in [0, T]$.

Step 3. The $L^2_T(H^{s-2})$ estimate of $\partial_x u$ and the $L^2_T(H^{s-1})$ one of $\partial_x z$

Applying ∂_x^l to the system (6.5) gives

$$\begin{cases} \partial_x^l v_t - \partial_x^{l+1} u + \partial_x^l y = 0, \\ \partial_x^l u_t - \partial_x^{l+1} v = 0, \\ \partial_x^l z_t - a\partial_x^{l+1} y = 0, \\ \partial_x^l y_t - a\partial_x^{l+1} z - \partial_x^l v + \gamma \partial_x^l y + \beta \partial_x^{l+1}\theta = \partial_x^{l+1} g(z), \\ \partial_x^l \theta_t - \kappa \partial_x^{l+2}\theta + \beta \partial_x^{l+1} y = 0, \end{cases} \tag{6.18}$$

where $g(z) := \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$. By multiplying the first and second equations of (6.18) with $-\partial_x^{l+1}u$ and $\partial_x^{l+1}v$, respectively, and adding the resultant equations together, we arrive at

$$-\partial_t(\partial_x^l v\partial_x^{l+1}u) + \partial_x(\partial_x^l v\partial_x^l u_t) + (\partial_x^{l+1}u)^2 - (\partial_x^{l+1}v)^2 = \partial_x^{l+1}u\partial_x^l y. \tag{6.19}$$

Hence, integrate (6.19) over $[0,t]\times\mathbb{R}$ and sum the resulting inequality for l with $0 \le l \le s-2$ give that

$$\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \lesssim \int_0^t (\|\partial_x v(\tau)\|_{H^{s-2}}^2 + \|y(\tau)\|_{H^{s-2}}^2)d\tau + \|U_0\|_{H^s}^2 + N_0(t)^2. \tag{6.20}$$

Similarly, multiplying the third and forth equations of (6.18) with $\partial_x^{l+1}y$ and $-\partial_x^{l+1}z$, respectively, and adding the resultant equations. We are led to

$$-\partial_t(\partial_x^{l+1}z\partial_x^l y) + \partial_x(\partial_x^l y\partial_x^l z_t) + a\{(\partial_x^{l+1}z)^2 - (\partial_x^{l+1}y)^2\} = \mathcal{R}_1^l, \tag{6.21}$$

where $\mathcal{R}_1^l := \partial_x^{l+1}z(\gamma\partial_x^l y + \beta\partial_x^{l+1}\theta - \partial_x^l v) - \partial_x^{l+1}z\partial_x^{l+1}g(z)$. Hence, integrating (6.21) with respect to x implies that

$$a\|\partial_x^{l+1}z\|_{L^2}^2 \le a\|\partial_x^{l+1}y\|_{L^2}^2 + \frac{d}{dt}\int_{\mathbb{R}} |\partial_x^{l+1}z||\partial_x^l y|dx + \int_{\mathbb{R}} |\mathcal{R}_1^l|dx. \tag{6.22}$$

By employing Young's inequality and Lemma 2.18, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |\partial_x^{l+1}z||\partial_x^l y|dx \le \epsilon\|\partial_x^{l+1}z\|_{L^2}^2 + C_\epsilon\|\partial_x^l y\|_{L^2}^2,\\
&\int_{\mathbb{R}} |\mathcal{R}_1^l|dx \le \epsilon\|\partial_x^{l+1}z\|_{L^2}^2 + C_\epsilon\|\partial_x^l(v,y,\theta_x)\|_{L^2}^2 + C\|z\|_{L^\infty}\|\partial_x^{l+1}z\|_{L^2}^2
\end{aligned} \tag{6.23}$$

for any $\epsilon > 0$ and $0 \le l \le s-1$, where C_ϵ (depending on ϵ) is a positive constant. Furthermore, with aid of (6.23), integrating (6.22) with respect to t and summing the resulting inequality for l with $0 \le l \le s-1$ together. We are lead to the following inequality

$$\begin{aligned}
\int_0^t \|\partial_x z(\tau)\|_{H^{s-1}}^2 d\tau \le C_\epsilon \int_0^t (\|v(\tau)\|_{H^{s-1}}^2 + \|(y,\partial_x\theta)(\tau)\|_{H^s}^2)d\tau\\
+ C(\|U_0\|_{H^s}^2 + N_0(t)^2 + M_0(t)D_0(t)^2).
\end{aligned} \tag{6.24}$$

Therefore, together with (6.20) and (6.24), we can derive the following energy estimate of solutions

$$\begin{aligned}
&\int_0^t \left(\|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2\right) d\tau\\
&\lesssim \|U_0\|_{H^s}^2 + N_0(t)^2 + M_0(t)D_0(t)^2 + \int_0^t \left(\|v(\tau)\|_{H^{s-1}}^2 + \|(y,\partial_x\theta)(\tau)\|_{H^s}^2\right) d\tau
\end{aligned} \tag{6.25}$$

for $t \in [0,T]$.

Step 4. The $L^2_T(H^{s-1})$ estimate of v

Multiplying the first and forth equations of (6.18) with $-\partial_x^l y$ and $-\partial_x^l v$, respectively. Also, multiplying the second and third equations of (6.18) with $-a\partial_x^l z$ and $-a\partial_x^l u$, respectively. Then adding all resultant equations gives

$$-\partial_t \mathcal{H}_2^l + \partial_x \mathcal{F}_2^l + (\partial_x^l v)^2 - (\partial_x^l y)^2 = \mathcal{R}_2^l, \tag{6.26}$$

where

$$\begin{aligned}
\mathcal{H}_2^l &:= \partial_x^l v \partial_x^l y + a\partial_x^l u \partial_x^l z,\\
\mathcal{F}_2^l &:= a\partial_x^l v \partial_x^l z + a^2 \partial_x^l u \partial_x^l y,\\
\mathcal{R}_2^l &:= \gamma \partial_x^l v \partial_x^l y + \beta \partial_x^l v \partial_x^{l+1}\theta + (a^2-1)\partial_x^{l+1} u \partial_x^l y - \partial_x^l v \partial_x^{l+1} g(z).
\end{aligned}$$

Integrating (6.26) with respect to x leads to

$$\|\partial_x^l v\|_{L^2}^2 \le \|\partial_x^l y\|_{L^2}^2 + \frac{d}{dt}\int_{\mathbb{R}} |\mathcal{H}_2^l| dx + \int_{\mathbb{R}} \mathcal{R}_2^l dx. \tag{6.27}$$

By using Young's inequality and Lemma 2.18, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{H}_2^l| dx \lesssim \|\partial_x^l (v,u,y,z)\|_{L^2}^2,\\
&\int_{\mathbb{R}} \mathcal{R}_2^l dx \le \epsilon \left(\|\partial_x^l v\|_{L^2}^2 + \|\partial_x u\|_{H^{s-2}}^2 \right) + C_\epsilon \left(\|\partial_x^l y\|_{H^1}^2 + \|\partial_x^{l+1}\theta\|_{L^2}^2 \right)\\
&\qquad\qquad + C\|z\|_{L^\infty} \|\partial_x^l (v, z_x)\|_{L^2}^2,
\end{aligned}$$

for any $\epsilon > 0$, where C, C_ϵ (depending on ϵ) are some positive constants. Here, we would like to mention that the integration by parts was used, which takes care of the regularity for $\partial_x u$. Thus, integrating (6.27) with respect to t and summing the resulting inequality for l with $0 \le l \le s-1$ together, we get the energy estimates of v as

$$\int_0^t \|v(\tau)\|_{H^{s-1}}^2 d\tau \lesssim \int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|(y, \partial_x \theta)(\tau)\|_{H^s}^2 d\tau + \|U_0\|_{H^s}^2 + M_0(t) D_0(t)^2 \tag{6.28}$$

for $t \in [0, T]$.

Finally, it suffices to make the suitable linear combination between (6.17), (6.25) and (6.28), which leads to (6.8) eventually. Hence, the proof of Proposition 6.1 is completed. □

The local existence of smooth solutions to (6.5) follows directly based on the general theory framework (see for example [92]). By using the standard boot-strap argument, see for example [138] (Theorem 7.1, page 100), we can finish the proof of Theorem 6.1.

Additionally, note that $z_t = a y_x$ in (6.9), we have the fact $\frac{d}{dt}\int_{\mathbb{T}} z dx = 0$ in periodic case, which leads to $\int_{\mathbb{T}} z dx = \int_{\mathbb{T}} z_0 dx$. That is, if $\bar{z}_0 = 0$, one has $\bar{z} = 0$ for all $t > 0$. Thus, the latter part of Remark 6.1 is also obtained.

6.3 The proof of optimal decay estimates

In the following, we focus on proving Theorem 6.2. To show our energy methods are optimal in Fourier spaces, we first investigate the asymptotic expansion of eigenvalues of (1.27) at $|\xi| \to 0$ and $|\xi| \to \infty$. Then we establish the energy inequality for (6.5), which is something like "square formula of Duhamel principle". Finally, by using L^p-L^q-L^r decay estimates (see Lemma 5.1), we deduce those decay estimates in Theorem 6.2.

6.3.1 Asymptotic expansion of eigenvalues

We denote by $\lambda_j(\zeta), j = 1, 2, 3, 4, 5$, the eigenvalues of the matrix $\hat{\Phi}(\zeta)$ in (1.28), which are solutions to the characteristic equation

$$\begin{aligned} &\det(\lambda I - \hat{\Phi}(\zeta)) := \lambda^5 + (\gamma - \kappa\zeta^2)\lambda^4 + \{1 - (1 + a^2 + b^2 + \kappa\gamma)\zeta^2\}\lambda^3 \\ &- \{\gamma + \kappa - \kappa(a^2 + 1)\zeta^2\}\zeta^2\lambda^2 + (a^2 + b^2 + \kappa\gamma)\zeta^4\lambda - a^2\zeta^6 = 0. \end{aligned} \tag{6.29}$$

Case 1: When $|\zeta| \to 0$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) := \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \cdots. \tag{6.30}$$

Substituting $\lambda = \lambda_j(\zeta)$ in (6.30) into the characteristic equation (6.29) and calculating the coefficients $\lambda_j^{(k)}, k = 0, 1, 2, \cdots$ successively, we obtain

$$\begin{aligned} &\lambda_j^{(0)} = \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \alpha_j, \quad \lambda_j^{(3)} = 0, \text{ for } \quad j = 1, 2, 3; \\ &\lambda_j^{(0)} = \beta_j, \quad \lambda_j^{(1)} = 0, \quad \text{for } \ j = 4, 5, \end{aligned}$$

where $\beta_j = -\frac{1}{2}(\gamma \pm \sqrt{\gamma^2 - 4})$ and α_j are the solutions of the algebraic equation

$$Y^3 - (\kappa + \gamma)Y^2 + (a^2 + b^2 + \kappa\gamma)Y - a^2\kappa = 0. \tag{6.31}$$

By the simple analysis of (6.31), we find that $\mathrm{Re}\alpha_j > 0$. Also, $\mathrm{Re}\beta_j < 0$. Consequently, for $|\xi| \to 0$, we have

$$\mathrm{Re}\,\lambda_j(i\xi) = \begin{cases} -(\mathrm{Re}\alpha_j)\xi^2 + O(|\xi|^4), & \text{for} \quad j = 1, 2, 3; \\ \mathrm{Re}\beta_j + O(|\xi|^2), & \text{for} \quad j = 4, 5. \end{cases} \tag{6.32}$$

Case 2: To obtain the asymptotic expansion of the eigenvalues $\lambda_j(\zeta)$ for $|\zeta| \to \infty$, we define the matrix $\hat{\Psi}(\zeta^{-1}) = B - \zeta^{-1}A - \zeta^{-2}L$. Thus, we have the relation $\hat{\Phi}(\zeta) = \zeta^2\hat{\Psi}(\zeta^{-1})$. Let $\mu_j(\zeta^{-1})$ be the eigenvalues of the matrix $\hat{\Psi}(\zeta^{-1})$, which are the solutions to the characteristic equation

$$\begin{aligned} &\det(\mu I - \hat{\Psi}(\zeta^{-1})) := \mu^5 - (\kappa - \gamma\zeta^{-2})\mu^4 \\ &- \{(1 + a^2 + b^2 + \kappa\gamma) - \zeta^{-2}\}\zeta^{-2}\mu^3 + \{\kappa(1 + a^2) - (\kappa + \gamma)\zeta^{-2}\}\zeta^{-2}\mu^2 \\ &+ (a^2 + b^2 + \kappa\gamma)\zeta^{-4}\mu - a^2\kappa\zeta^{-4} = 0. \end{aligned}$$

As $|\zeta| \to \infty$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) = \mu_j^{(2)}\zeta^2 + \mu_j^{(1)}\zeta + \mu_j^{(0)} + \mu_j^{(-1)}\zeta^{-1} + \mu_j^{(-2)}\zeta^{-2} + \cdots.$$

We have the relation $\lambda_j(\zeta) = \zeta^2\mu_j(\zeta^{-1})$. By direct computations, the coefficients $\mu_j^{(k)}, k = 2, 1, 0, -1, -2, \cdots$ are shown as follows:

$$\begin{aligned}
&\mu_j^{(2)} = 0,\ \mu_j^{(1)} = \pm 1,\ \mu_j^{(0)} = 0,\ \mu_j^{(-1)} = \pm\frac{1}{2P},\\
&\mu_j^{(-2)} = \frac{b^2+\kappa\gamma}{2\kappa P^2} \quad \text{for}\ \ j = 1, 2;\\
&\mu_j^{(2)} = 0,\ \mu_j^{(1)} = \pm a,\ \mu_j^{(0)} = -\frac{b^2+\kappa\gamma}{2\kappa} \quad \text{for}\ \ j = 3, 4;\\
&\mu_j^{(2)} = \kappa,\ \mu_j^{(1)} = 0 \quad \text{for}\ \ j = 5,
\end{aligned}$$

where $P = a^2 - 1 (a \neq 1)$. Consequently, for $|\xi| \to \infty$, we have

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} -\frac{b^2+\kappa\gamma}{2\kappa P^2} + O(|\xi|^{-3}), & \text{for} \quad j = 1, 2;\\ -\frac{b^2+\kappa\gamma}{2\kappa} + O(|\xi|^{-1}), & \text{for} \quad j = 3, 4;\\ -\kappa\xi^2 + O(1) & \text{for} \quad j = 5. \end{cases} \tag{6.33}$$

From the expansion (6.33) for $|\xi| \to \infty$, we obtain that one eigenvalue satisfies $\mathrm{Re}\lambda_j(i\xi) \sim -c\xi^2$, two eigenvalues are of the standard type and satisfy $\mathrm{Re}\lambda_j(i\xi) \sim -c$, and the last two are not standard and satisfy $\mathrm{Re}\lambda_j(i\xi) \sim -c\xi^{-2}$.

Here, we would like to mention that the asymptotic expansions (6.32) and (6.33) improved that analysis in [134] due to the existence of the damping term $\gamma\psi_t$. Precisely, the dissipative rate $\widetilde{\eta}_2(\xi)$ becomes into $\eta_2(\xi)$. See [170] for the notion of dissipative rates. At the same time, the decay behavior of eigenvalues also confirms the optimality of those linear results in [160]. Due to the dissipative mechanism of regularity-loss, extra higher regularity is usually needed, see for example [87, 155]. Therefore, it is interesting to seek the possibly lower regularity for the optimal decay rate of $L^1(\mathbb{R})$-$L^2(\mathbb{R})$ type.

6.3.2 Energy methods in Fourier spaces

Based on the asymptotic analysis, the next task is to establish the "square formula of Duhamel principle" for (6.1)-(6.2) in Fourier spaces, which leads to the optimal decay rates.

Proposition 6.2. *Let $U = (v, u, z, y, \theta)^\top$ be the global smooth solution to (6.1)-(6.2). Then the Fourier image of solution satisfies the following pointwise estimate*

$$\begin{aligned}
|\hat{U}(t,\xi)|^2 \lesssim\ & e^{-c_0\eta(\xi)t}|\hat{U}_0(\xi)|^2\\
&+ \int_0^t e^{-c_0\eta(\xi)(t-\tau)}\xi^2|\hat{G}(\tau,\xi)|^2 d\tau
\end{aligned} \tag{6.34}$$

for any $t \geq 0$ and $\xi \in \mathbb{R}$, where the dissipative rate $\eta(\xi) := \xi^2/(1+\xi^2)^2$ and $c_0 > 0$ is a constant.

Proof. Applying the Fourier transform to (6.5) gives

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{6a}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{6b}$$

$$\hat{z}_t - ai\xi\hat{y} = 0, \tag{6c}$$

$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + \gamma\hat{y} + \beta i\xi\hat{\theta} = i\xi\hat{g}, \tag{6d}$$

$$\hat{\theta}_t + \kappa\xi^2\hat{\theta} + \beta i\xi\hat{y} = 0. \tag{6e}$$

Step 1. Estimate for dissipative term of $(\hat{y}, \hat{\theta})$

Multiplying $(6a)$-$(6e)$ with $\bar{\hat{v}}, \bar{\hat{u}}, \bar{\hat{z}}, \bar{\hat{y}}$ and $\bar{\hat{\theta}}$, respectively. Then adding the resultant equations and taking the real part, we obtain

$$\frac{d}{dt}\mathcal{E}_0 + 2(\gamma|\hat{y}|^2 + \kappa\xi^2|\hat{\theta}|^2) = 2\mathrm{Re}(i\xi\hat{g}\bar{\hat{y}}),$$

where $\mathcal{E}_0 := |\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2 = |\hat{U}|^2$. It follows from Young's inequality in Lemma 2.10 that

$$\frac{d}{dt}\mathcal{E}_0 + c_1|\hat{y}|^2 + c_2\xi^2|\hat{\theta}|^2 \lesssim \xi^2|\hat{G}|^2 \tag{6.35}$$

for $c_1, c_2 > 0$.

Step 2. Estimate for dissipative term of $\hat{v}$

To create the dissipative term $|\hat{v}|^2$, we compute as $(6a)\times(-\bar{\hat{y}}) + (6d)\times(-\bar{\hat{v}}) + (6b)\times(-a\bar{\hat{z}}) + (6c)\times(-a\bar{\hat{u}})$ and take the real part. We obtain

$$\frac{d}{dt}\mathcal{E}_1 + |\hat{v}|^2 - |\hat{y}|^2 = \gamma\mathrm{Re}(\bar{\hat{v}}\hat{y}) + (1-a^2)\xi\mathrm{Re}(i\bar{\hat{u}}\hat{y}) + \xi\mathrm{Re}\{i\bar{\hat{v}}(\beta\hat{\theta} - \hat{g})\}, \tag{6.36}$$

where $\mathcal{E}_1 := -\mathrm{Re}(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}})$. Compute (6.36) $\times(1+\xi^2)$ and use the Young's inequality. This gives

$$\frac{d}{dt}(1+\xi^2)\mathcal{E}_1 + c_3(1+\xi^2)|\hat{v}|^2 \le \epsilon\xi^2|\hat{u}|^2 + C_\epsilon(1+\xi^2)^2|\hat{y}|^2 + C\xi^2(1+\xi^2)(|\hat{\theta}|^2 + |\hat{G}|^2) \tag{6.37}$$

for any $\epsilon \in (0,1)$,where c_3, C_ϵ (depending on ϵ) are some positive constants.

Step 3. Estimate for dissipative term of $(\hat{u}, \hat{z})$

For the dissipative term $|\hat{u}|^2$, we compute as $(6a)\times(i\xi\bar{\hat{u}}) + (6b)\times(-i\xi\bar{\hat{v}})$ and take the real part. This gives

$$\xi\frac{d}{dt}\mathcal{E}_2 + \xi^2(|\hat{u}|^2 - |\hat{v}|^2) = -\xi\mathrm{Re}(i\bar{\hat{u}}\hat{y}), \tag{6.38}$$

where $\mathcal{E}_2 := \mathrm{Re}(i\hat{v}\bar{\hat{u}})$. Here we use the fact that $\mathrm{Re}(i\hat{v}_t\bar{\hat{u}} - i\bar{\hat{v}}\hat{u}_t) = \{\mathrm{Re}(i\hat{v}\bar{\hat{u}})\}_t$.

To get the dissipative term for $\hat{z}$, we compute as $(6c)\times(-i\xi\bar{\hat{y}}) + (6d)\times(i\xi\bar{\hat{z}})$ and take the real part. Then we obtain

$$\xi\frac{d}{dt}\mathcal{E}_3 + a\xi^2(|\hat{z}|^2 - |\hat{y}|^2) = \xi^2\mathrm{Re}\{\bar{\hat{z}}(\beta\hat{\theta} - \hat{g})\} + \xi\mathrm{Re}\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\}, \tag{6.39}$$

where $\mathcal{E}_3 := \mathrm{Re}(i\hat{y}\bar{\hat{z}})$.

We combine (6.38) and (6.39) such that (6.38)+(6.39)$\times(1+\xi^2)$ by using Young's inequality. Then we arrive at

$$\begin{aligned}&\frac{d}{dt}\xi\{\mathcal{E}_2+(1+\xi^2)\mathcal{E}_3\}+c_4\xi^2|\hat{u}|^2+c_5\xi^2(1+\xi^2)|\hat{z}|^2\\&\le C_\epsilon(1+\xi^2)|\hat{v}|^2+C(1+\xi^2)^2|\hat{y}|^2+C\xi^2(1+\xi^2)(|\hat{\theta}|^2+|\hat{G}|^2)\end{aligned}\tag{6.40}$$

for any $\epsilon\in(0,1)$, where c_4, c_5 and C_ϵ (depending on ϵ) are some positive constants.

Step 4. Combining the above analysis

We combine (6.35), (6.37) and (6.40) such that (6.35) + $\{(6.37)+(6.40)\times\alpha_1\}\times\frac{\alpha_2}{(1+\xi^2)^2}$. Thus we obtain the following Lyapunov function to system (6.5)

$$\mathcal{E}[\hat{U}] := \mathcal{E}_0+\frac{\alpha_2}{1+\xi^2}\Big\{\mathcal{E}_1+\frac{\alpha_1\xi}{1+\xi^2}\mathcal{E}_2+\alpha_1\xi\mathcal{E}_3\Big\}$$

such that the following differential inequality holds

$$\frac{d}{dt}\mathcal{E}[\hat{U}]+c_0\mathcal{D}[\hat{U}]\lesssim\xi^2|\hat{G}|^2,\tag{6.41}$$

where

$$\mathcal{D}[\hat{U}]=\frac{1}{1+\xi^2}|\hat{v}|^2+\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2+\frac{\xi^2}{1+\xi^2}|\hat{z}|^2+|\hat{y}|^2+\xi^2|\hat{\theta}|^2$$

and $\alpha_1,\alpha_2>0$ are suitable small constants which ensure that $\mathcal{E}[\hat{U}]\approx|\hat{U}|^2$. Also, we mention that the asymptotic expansions (6.32) and (6.33) indicate the energy inequality (6.41) is optimal. Furthermore, it follows from (6.41) that

$$\frac{d}{dt}\mathcal{E}[\hat{U}]+c_0\eta(\xi)\mathcal{E}[\hat{U}]\lesssim\xi^2|\hat{G}|^2,$$

where $\eta(\xi)=\xi^2/(1+\xi^2)^2$. Finally, the inequality (6.34) is followed from the standard Gronwall's inequality. This ends the proof of Proposition 6.2. □

6.3.3 Optimal decay rates

With above preparations, the final section is devoted to the optimal decay estimates of smooth solutions to (6.5) with the regularity $s_c=2$, i.e., $s_D\le 2$ follows.

Define new time-weighted energy functionals:

$$N(t)\triangleq\begin{cases}\sup\limits_{0\le\tau\le t}(1+\tau)^{\frac{1}{4}}\|U(\tau)\|_{L^2(\mathbb{R})},\\ \sup\limits_{m=0,1}\sup\limits_{0\le\tau\le t}(1+\tau)^{\frac{1}{4}+\frac{m}{2}}\|\partial_x^m U(\tau)\|_{L^2(\mathbb{T})}.\end{cases}\tag{6.42}$$

In what follows, we shall get a nonlinear energy inequality in terms of $N(t)$ and $D_0(t)$,which shows the optimal decay rate of solutions to (6.5).

Proposition 6.3. *Let $U=(v,u,z,y,\theta)^\top$ be the global smooth solutions of (6.5). Additionally, if $U_0\in L^1(\Omega)$ with $\Omega=\mathbb{R}$ for $m=0$, and $\Omega=\mathbb{T}$ for $m=0,1$, then it holds that*

$$N(t)\lesssim \|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}+N(t)D_0(t)+N(t)^2 \tag{6.43}$$

for $s\geq \ell_0+m+1$ with $\ell_0=m+1$ $(m=0,1)$.

Proof. For $m=0,1$, it follows from (6.34) that

$$\begin{aligned}&\int_\Omega \xi^{2m}|\hat{U}(\xi)|^2d\xi\\ &\lesssim \int_\Omega \xi^{2m}e^{-c\eta(\xi)t}|\hat{U}_0(\xi)|^2d\xi+\int_\Omega\int_0^t e^{-c\eta(\xi)(t-\tau)}\xi^{2m+2}|\hat{G}(\tau,\xi)|^2d\tau d\xi\\ &\triangleq J_1+J_2.\end{aligned} \tag{6.44}$$

For J_1, by taking $\sigma_1=\sigma_2=2,p=2,k=m,j=0,q=1$ and $r=2,\ell\geq\ell_0$ with $\ell_0=m+1$ in Lemma 5.1, we arrive at

$$J_1\lesssim (1+t)^{-\frac{1}{2}-m}\|U_0\|^2_{L^1(\Omega)}+(1+t)^{-\ell}\|\partial_x^{m+\ell}U_0\|^2_{L^2(\Omega)}. \tag{6.45}$$

Next, we begin to bound nonlinear terms. For J_2, it is written as the sum of low-frequency and high-frequency parts

$$J_2:=J_{2L}+J_{2H}.$$

To get the precise estimate of J_{2L}, we compute J_{2L} with $m=0$ and $m=1$. In the case for $m=0$, by taking $\sigma_1=\sigma_2=2,p=2$ and $k=1,j=0,q=1$ in Lemma 5.1, we have

$$\begin{aligned}J_{2L}&\lesssim\int_0^t(1+t-\tau)^{-\frac{3}{2}}\|G(\tau)\|^2_{L^1}d\tau\\ &\lesssim\int_0^t(1+t-\tau)^{-\frac{3}{2}}\|U(\tau)\|^4_{L^2}d\tau\\ &\lesssim N(t)^4\int_0^t(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-1}d\tau\\ &\lesssim (1+t)^{-1}N(t)^4.\end{aligned} \tag{6.46}$$

When $m=1$, by taking $\sigma_1=\sigma_2=2,p=2$ and $k=2,j=1,q=1$ in Lemma 5.1, we obtain

$$\begin{aligned}J_{2L}&\lesssim\int_0^t(1+t-\tau)^{-\frac{3}{2}}\|\partial_xG(\tau)\|^2_{L^1(\mathbb{T})}d\tau\\ &\lesssim\int_0^t(1+t-\tau)^{-\frac{3}{2}}\|U(\tau)\|^2_{L^2(\mathbb{T})}\|\partial_xU(\tau)\|^2_{L^2(\mathbb{T})}d\tau\\ &\lesssim N(t)^4\int_0^t(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-2}d\tau\\ &\lesssim (1+t)^{-\frac{3}{2}}N(t)^4,\end{aligned} \tag{6.47}$$

where we have used Lemma 2.18 and the fact $G(U) = O(z^2)$. Thus, combining (6.46) and (6.47), we are led to

$$J_{2L} \lesssim (1+t)^{-\frac{1}{2}-m} N(t)^4, \quad m = 0, 1. \tag{6.48}$$

For J_{2H}, more elaborate estimates are proceeded. For this purpose, we write

$$J_{2H} = \Big(\int_0^{t/2} + \int_{t/2}^{t} \Big)(\cdots) d\tau := J_{2H1} + J_{2H2}.$$

Taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = m+1, r = 2, \ell \geq \ell_0$ with $\ell_0 = m+1$ in Lemma 5.1 gives

$$\begin{aligned} J_{2H1} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} \|\partial_x^{\ell+m+1} G(\tau)\|_{L^2(\Omega)}^2 d\tau \\ &\lesssim \int_0^{t/2} (1+t-\tau)^{-\ell} \|z\|_{L^\infty(\Omega)}^2 \|\partial_x^{\ell+m+1} z\|_{L^2(\Omega)}^2 d\tau \\ &\lesssim \sup_{0\leq\tau\leq t/2} \Big\{ (1+t-\tau)^{-\ell} \|z\|_{L^\infty(\Omega)}^2 \Big\} \int_0^{t/2} \|\partial_x^{\ell+m+1} z\|_{L^2(\Omega)}^2 d\tau \\ &\lesssim (1+t)^{-\ell} N_0(t)^2 D_0(t)^2 \\ &\lesssim (1+t)^{-\ell} \|U_0\|_{H^s(\Omega)}^2 \end{aligned} \tag{6.49}$$

for $s \geq \ell_0 + m + 1$ with $\ell_0 = m+1$ $(m = 0, 1)$, where we have used Lemma 2.18 and the fact $G(U) = O(z^2)$.

On the other hand, we compute J_{2H2} with $m = 0$ and $m = 1$ differently. For $m = 0$, by taking $\sigma_1 = \sigma_2 = 2, p = 2$ and $k = 1, r = 1, \ell = 1$ in Lemma 5.1, we get

$$\begin{aligned} J_{2H2} &\lesssim \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^2 G(\tau)\|_{L^1}^2 d\tau \\ &\lesssim \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} \|z\|_{L^2}^2 \|\partial_x^2 z\|_{L^2}^2 d\tau \\ &\lesssim N(t)^2 \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{1}{2}} \|\partial_x^2 z\|_{L^2}^2 d\tau \\ &\lesssim N(t)^2 \sup_{t/2\leq\tau\leq t} \Big\{ (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{1}{2}} \Big\} \int_0^{t} \|\partial_x^2 z\|_{L^2}^2 d\tau \\ &\lesssim (1+t)^{-\frac{1}{2}} N(t)^2 D_0(t)^2. \end{aligned} \tag{6.50}$$

For $m=1$, by taking $\sigma_1=\sigma_2=2, p=2$ and $k=2, r=1, \ell=1$ in Lemma 5.1, we get

$$\begin{aligned} J_{2H2} &\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^3 G(\tau)\|_{L^1(\mathbb{T})}^2 d\tau \\ &\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\|z\|_{L^2(\mathbb{T})}^2\|\partial_x^3 z\|_{L^2(\mathbb{T})}^2 d\tau \\ &\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x z\|_{L^2(\mathbb{T})}^2\|\partial_x^3 z\|_{L^2(\mathbb{T})}^2 d\tau \\ &\lesssim N(t)^2 \sup_{t/2\le\tau\le t}\left\{(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}\right\}\int_0^t \|\partial_x^3 z\|_{L^2}^2 d\tau \\ &\lesssim (1+t)^{-\frac{3}{2}}N(t)^2 D_0(t)^2, \end{aligned} \tag{6.51}$$

where we used Remark 6.1 and $\|z\|_{L^2(\mathbb{T})}=\|z-\bar{z}\|_{L^2(\mathbb{T})}\le\|\nabla z\|_{L^2(\mathbb{T})}$ led by Poincaré inequality. Thus, combining (6.50) and (6.51), we arrive at

$$J_{2H2}\lesssim (1+t)^{-\frac{1}{2}-m}N(t)^2 D_0(t)^2, \quad m=0,1. \tag{6.52}$$

Therefore, together with inequalities (6.44)-(6.52), it follows from Plancherel's theorem that

$$\begin{aligned} \|\partial_x^m U\|_{L^2(\Omega)}^2 \lesssim{}& (1+t)^{-\frac{1}{2}-m}\|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}^2 + (1+t)^{-\frac{1}{2}-m}N(t)^2 D_0(t)^2 \\ &+ (1+t)^{-\frac{1}{2}-m}N(t)^4, \end{aligned}$$

which leads to (6.43) exactly. □

The proof of Theorem 6.2. Due to the similar energy inequality for the solution U as (6.6) in Theorem 6.1, we have $D_0(t)\lesssim\|U_0\|_{H^s(\Omega)}\lesssim\|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}$. Thus, if $\|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}$ is sufficient small, then it holds that

$$N(t)\lesssim\|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}+N(t)^2$$

for $s\ge\ell_0+m+1$ with $\ell_0=m+1$ $(m=0,1)$, which implies that $N(t)\lesssim\|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}$, provided that $\|U_0\|_{H^s(\Omega)\cap L^1(\Omega)}$ is sufficient small. Consequently, the decay inequality in Theorem 6.2 is followed.

Chapter 7

Summary and further prospects

7.1 Summary

With the fast expansion of breadth and depth of natural phenomena, the study of partial differential equations arising in physics, mechanics and other various branches of natural sciences has become more and more important, which not only provides us the effective solution methods, but also rigorously characterizes some physical effects, for example, vacuum, convention, compression, dissipation, dispersion, relaxation and chemical reaction. In recent years, some dissipative equations, such as Euler-Maxwell system, Navier-Stokes equation and Timoshenko system etc; kinetic equations, such as Boltzmann equation, Landau equation and its coupling equations, etc; These systems received extensive attention, many experts and scholars conducted in-depth research on the structure of these equations, and carried out detailed analysis on the well-posedness, smoothness and decay estimate of solutions to these systems. Many important and meaningful results are shown.

In this thesis, we investigate well-posedness and qualitative study for some kinetic equations and some dissipative equations. In this thesis, we study two dissipative equations Navier-Stokes-Maxwell system and Timoshenko-Fourier system, and both systems have non-symmetric dissipation and they admit a weakly dissipative mechanism of regularity loss type, to overcome this difficulty, by using some new methods and detailed analysis, we obtain the global existence of solutions and optimal decay estimate with minimal decay regularity for both systems in Sobolev spaces, this gives a special reference example for solving the opening problem left in [171]. In addition, we also studied the kinetic equations, Landau equation and Boltzmann equation in one-dimensional space, namely the Kac equation. Here, we study the inhomogeneous nonlinear problems of these systems and give a rigorous analysis of collision operator and some key trilinear estimates, hypoelliptic estimates by using analytical tools for spectral analysis, Bony's decomposition and pseudo-differential operator.

As we all know, one of the most important roles of Littlewood-Paley theory is to localize the frequency space. Fourier transform transforms differential operations in physical space into algebraic operations in frequency space. Littlewood-Paley decomposition writes the temperate distribution form as the countable sums of smooth functions that are almost orthogonal in the sense of frequency space. Bony's decomposition was introduced by Bony in the study of the singular propagation of the solution of hyperbolic equations. It is the brilliant achievement of

Littlewood-Paley theory and has been widely used in the study of partial differential equations. In recent years, the application of Bony's decomposition in fluid mechanics equations has been done by many scholars. However, there are not many research results for the dynamic equations. In this thesis, we study the dynamic equations in the critical Besov space: Landau equation and Kac equation. In addition to using Bony's decomposition, we apply theoretical and analytical tools such as spectral decomposition, iterative sequence and pseudo-differential operator. The well-posedness for the systems are proved in the Besov space. In addition, for Kac equation, we also obtain the Gelfand-Shilov regularity effect on the velocity variable and the Gevrey regularity effect on the spatial variable. And compared to the Gelfand-Shilov regularity index in [117], we improve the index to the optimal one.

7.2 Further prospects

Dissipative partial differential equations and kinetic equations have been the focus of many experts and professors. They have been extensively studied in Sobolev space and have been shown many well-established systematic results. However, the well-posedness and decay property for partial differential equation with non-symmetric dissipation, the well-posedness or smoothness effect for kinetic equation in critical Besov space, the research results about these are relatively few. These systems of this thesis are studied under relatively ideal or certain circumstances. Here, for these systems and their results, it is possible to further deepen the research:

(1) The non-symmetric dissipative systems studied in this thesis: Navier-Stokes-Maxwell system and Timoshenko-Fourier system, both are all studied under the condition that the damping term affects the dissipative structure, and the optimal decay estimate is obtained with the minimal decay regularity. We can guess: if there is no damping term effect, can we get similar results under a weaker dissipation mechanism? Further research is needed.

(2) In critical Besov space, we obtain the well-posedness to the Landau equation with Maxwellian molecules. However, for the case of soft potential ($-3 < \gamma < 0$) and hard potential ($0 < \gamma < 1$), the study of well-posedness to the Landau equation requires further in-depth analysis and research.

(3) In addition, for the Landau equation, the smoothing effect of solutions has not yet been studied. This is a question that is meaningful and worthy of further study.

(4) The uniqueness of solutions for the above systems are obtained when the small perturbation of the initial value is bounded, that is, the solution is bounded. For any solution, if there is no bounded condition, whether the solution is unique is the further work we will consider and study.

Chapter 8

List of author's papers

1. Jiang Xu and Hongmei Cao, Global smooth flows for compressible Navier-Stokes-Maxwell equations, *Z. Angew. Math. Phys.*, 67(4), 2016.
2. Hongmei Cao and Jiang Xu, Global existence and large-time behavior of the Timoshenko-Fourier system in thermoelasticity, *Nonlinear Analysis: Real World Applications*, 39:246–260, 2018.
3. Hongmei Cao, Haoguang Li, Chaojiang Xu and Jiang Xu, Well-posedness of Cauchy problem for Landau equation in critical Besov space, *Kinet. Relat. Models*, 12(4):829–884, 2019.
4. Hongmei Cao, Haoguang Li, Chaojiang Xu and Jiang Xu, The Cauchy problem for the inhomogeneous non-cutoff Kac equation in critical Besov space, arXiv:1902.06699.

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