



Isomonodromic deformations through differential Galois theory

Juan Sebastián Díaz Arboleda

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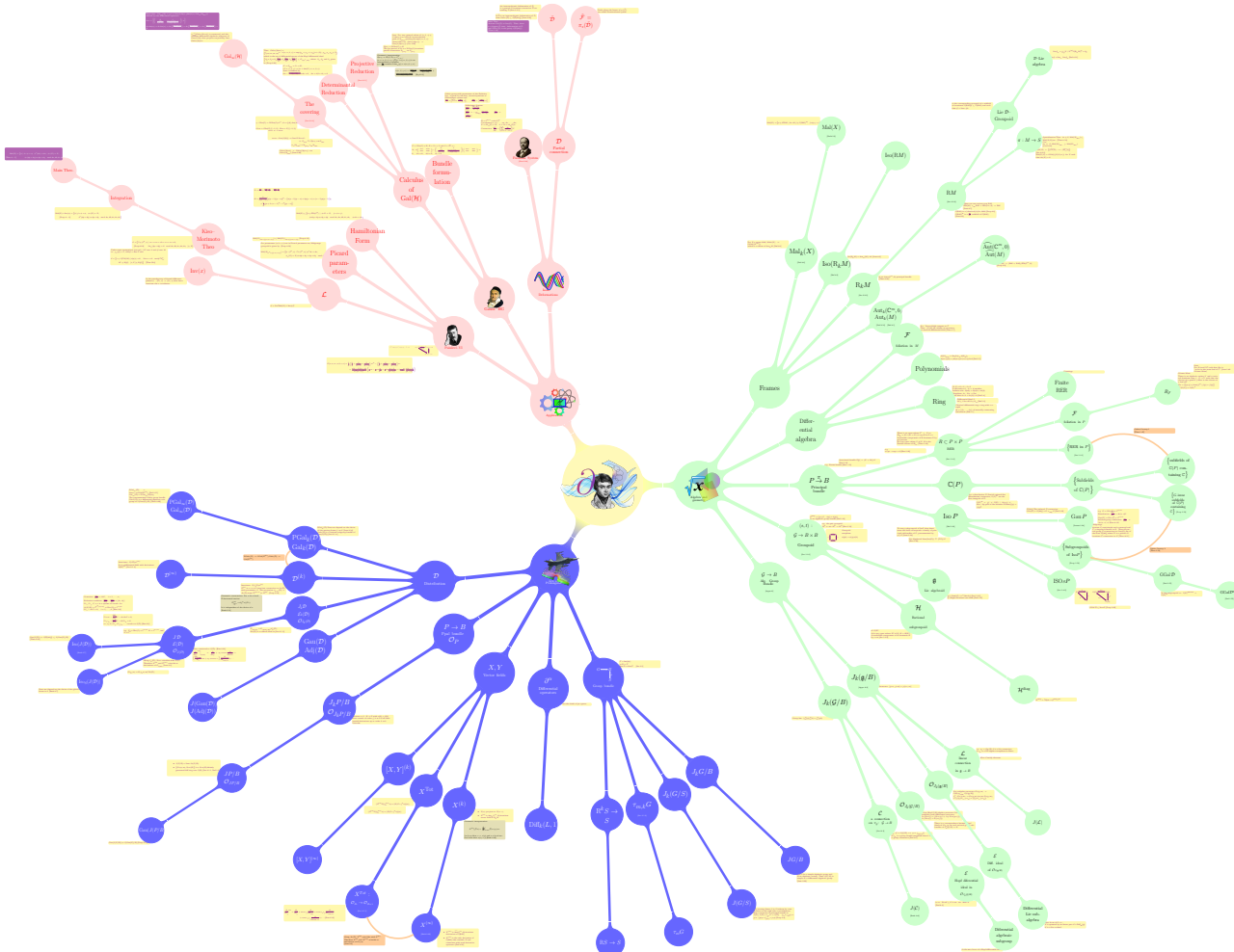
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Par

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Isomonodromic deformations through differential Galois theory

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Isomonodromic deformations through differential Galois theory

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Universidad Nacional de Colombia
Facultad de Ciencias
Medellín, Colombia
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To my family, master David and all those I met
through this journey

Wanting to understand language is like a
person made of salt wanting to explore the
undersea depths.
—ZEN SAYING

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Abstract

The text begins with a brief description of differential Galois theory from a geometrical perspective. Then, parameterized Galois theory is developed by means of prolongation of partial connections to the jet bundles. The relation between the parameterized differential Galois groups and isomonodromic deformations is unfolded as an application of Kiso-Cassidy theorem. It follows the computation of the parameterized Galois groups of the general fuchsian equation and Gauss hypergeometric equation. Finally, some non-linear applications are developed. By means of a non-linear analog, Kiso-Morimoto theorem, the Malgrange groupoid of Painlevé VI equation with variable parameters is calculated.

Résumé

Le texte commence par une brève description de théorie différentielle de Galois dans une perspective géométrique. Ensuite, la théorie paramétrée de Galois est développée au moyen d'une prolongation des connexions partielles avec les fibrés de jets. La relation entre les groupes de Galois différentiels à paramètres et les déformations isomonodromiques est développée comme une application du théorème de Kiso-Cassidy. Il s'ensuit le calcul des groupes de Galois à paramètres de l'équation générale fuchsienne et de l'équation hypergéométrique de Gauss. Enfin, certaines applications non linéaires sont développées. Au moyen d'un théorème de Kiso-Morimoto, un analogue non linéaire, on calcule le groupoïde de Malgrange de l'équation de Painlevé VI à paramètres variables.

Resumen

El texto comienza con una breve descripción de la teoría de Galois diferencial desde una perspectiva geométrica. Luego la teoría de Galois con parámetros se presenta mediante las prolongaciones de conexiones parciales en los fibrados de jets. La relación entre el grupo de Galois con parámetros y las deformaciones isomonodrómicas se desarrolla como una aplicación del teorema de Kiso-Cassidy. Se calculan los grupos de Galois con parámetros de la ecuación fuchsiana general y de la ecuación hiper-geométrica de Gauss. Finalmente se desarrollan algunas aplicaciones no lineales. Mediante un análogo no lineal, a saber el teorema de Kiso-Morimoto, se calcula el grupoide de Malgrange de la ecuación de Painlevé VI con parámetros variables.

Keywords: Algebraic groups, Jet bundles, Parameterized differential Galois theory, Gauss hypergeometric equation , Painlevé VI equation

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Résumé

Contexte historique

Au début du XIX ème siècle, Emile Picard et Ernest Vessiot ont développé une théorie de Galois pour les équations différentielles linéaires, [Pic87], [Ves04b] and [Ves04a]. Ces travaux ont été étendus par Ellis Kolchin a un contexte multivarié et permettent de considérer, pour toute solution fondamentale Φ d'une connection intégrable :

$$\frac{\partial U}{\partial x_i} = A_i(x)U, \quad A_i(x) \in \mathfrak{gl}_n(\mathbb{C}(x)), \quad \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = [A_i, A_j]$$

Un groupe de Galois définit par

$$\text{Gal}_\Phi = \left\{ \sigma \in \text{GL}_n(\mathbb{C}) \mid \forall P \in \mathbb{C}(u_{11}, \dots, u_{nn}) \ P(\Phi) \in \mathbb{C}(x) \Rightarrow P(\Phi) = P(\Phi\sigma) \right\}$$

Michael Singer et Phyllis Cassidy dans [CS05] et Peter Landesman dans [Lan08] ont développé une théorie de Galois pour les équations différentielles linéaires contenant des paramètres:

$$\frac{\partial U}{\partial x} = A(x, \lambda_1, \dots, \lambda_m)U, \quad A(x) \in \mathfrak{gl}_n(\mathbb{C}(x, \lambda_1, \dots, \lambda_m)).$$

Dans leurs travaux, le groupe de Galois est un groupe de matrices dont les coefficients sont des fonctions des paramètres, λ , agissant sur une solution fondamentale Φ par multiplication à droite et préservant toutes les relations différentielles entre les coefficients de Φ lorsque l'on s'autorise à dériver ces coefficients par rapport aux paramètres. Le groupe obtenu n'est plus un groupe algébrique de matrices mais un groupe différentiel, c'est-à-dire décrit par des équations aux dérivées partielles. Cette théorie a été appliquée avec succès aux problèmes d'hypertranscendences. Elle a aussi permis d'interpréter la notion de déformation isomonodromique d'une équation différentielle fuchsienne en terme de structure du groupe de Galois de la famille d'équations différentielles.

Parallèlement, Bernard Malgrange dans [Mal01] et Hiroshi Umemura dans [Ume96] ont développé une théorie de Galois pour les équations différentielles non-linéaires. Même si elles sont équivalentes, ces deux théories paraissent très différentes. H. Umemura étudie les extensions de corps différentiels et l'objet analogue au groupe de Galois et un foncteur de Lie-Ritt. B. Malgrange étudie les feuilletages holomorphes singuliers, l'analogue du groupe

de Galois est un \mathcal{D} -groupe de Lie.

Contenu de la thèse

Le but principal de ce travail est de clarifier les relations entre les déformations isomonodromique et les différentes théories de Galois différentielles, pour les équations différentielles linéaires ou non linéaires. Nous appliquons le formalisme développé pour déterminer le groupe de Galois ou le groupe de Galois de familles remarquables équations différentielles

Dans leurs travaux sur la théorie de Galois à paramètres, [CS05] p.128 section 5, P. Cassidy et M. Singer donnent une relation entre le groupe de Galois paramétré d'une famille fuchsienne et son caractère isomonodromique.

Nous proposons une version entièrement géométrique de la théorie de Galois à paramètres basée sur les prolongations de connections partielles \mathcal{D} sur une fibré principal $P \rightarrow B$ au fibré des jets $J(P/B)$. Ce point de vue géométrique nous permet de clarifier les relations entre le groupe de Galois et les déformations isomonodromiques. Un point clef est le théorème de Kiso-Cassidy (Theorem A.9). Notre méthode permet de calculer simplement le groupe de Galois de la famille générale des systèmes fuchsien de trace nulle avec n singularités ainsi que de la famille hypergéométrique de Gauss.

Lorsque l'on s'intéresse aux équations non linéaires, il n'existe pas de théorie de Galois à paramètres proprement dit. Plus précisément, la théorie développée par B. Malgrange contient déjà les informations.

Dans sa thèse Damien Davy [Dav16] a fait le chemin inverse. Il a défini le groupe de Galois d'une famille d'équations différentielles non linéaires en préservant le statut spécial des paramètres. Il a ainsi pu obtenir un théorème très général de spécialisation des paramètres et retrouver les résultats de Serge Cantat et Frank Loray, [CL09], sur le groupe de Galois de la sixième famille d'équations de Painlevé. Ces résultats ne déterminent que partiellement le groupe de Galois de la sixième équation de Painlevé.

En utilisant le théorème de Kiso-Morimoto (Theorem B.4), nous montrons que le groupe de Galois de la sixième équation de Painlevé est déterminé par les chemins isomonodromiques de l'espace des paramètres. Une conséquence des travaux de classification des solutions algébriques de ces équations est l'absence de tels chemins.

Présentation des résultats

Ce travail commence par une présentation en termes géométriques de la théorie de Galois différentielle. Nous travaillons sur un fibré principal $P \rightarrow B$ où la base est munie d'un feuilletage \mathcal{F} donnant les directions des variables indépendantes, l'équation est alors une connexion \mathcal{D} au-dessus de \mathcal{F} intégrable et G -invariante.

Un fibré principal vient toujours avec un groupe de Galois $\text{Iso}(P)$, le groupe de Galois des isomorphismes de P . Il existe une correspondance naturelle entre les sous-groupes de $\text{Iso}(P)$ qui

dominant la base et certains sous-fibrés de P .

La nature des objets étudiés nous impose de travailler avec des groupoides rationnels : ce sont des sous-ensemble Zariski fermés d'un groupoïde qui, en restriction à un ouvert de Zariski de la base ont une structure de sous-groupoides. Nous avons alors une correspondance de Galois:

$$\left\{ \begin{array}{c} \text{Sous-groupoides} \\ \text{de Iso}(P) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{sous-corps } G\text{-invariant de } \mathbb{C}(P) \\ \text{contenant } \mathbb{C} \end{array} \right\}$$

Le groupoïde de Galois est le sous-groupoïde correspondant au corps des intégrales premières rationnelles de \mathcal{D} . Le groupe de Galois classique est obtenu en prenant le stabilisateur d'un point dans le groupoïde de Galois. Le groupe de Galois intrinsèque ([Kat82]) est le fibré en groupes sur B obtenu en prenant la diagonale du groupoïde de Galois.

Le chapitre 2 présente la théorie des fibrés de jets à travers la théorie de Weil des points infinitésimalement proches. Nous suivons les constructions très générales faites dans [MnRM00], ce qui nous amène à présenter le mécanisme des prolongements d'un champs de vecteurs aux espaces de jets.

Le chapitre 3 est dédié à l'exposition de la théorie de Galois différentielle linéaire avec paramètres. P. Cassidy and M. Singer dans [CS05] ont établi une relation entre les familles isomonodromiques et la structure du groupe de Galois à paramètres. Afin de décrire cette relation d'un point de vu géométrique, nous interprétons les déformations isomonodromiques en terme de feuilletage: une extension d'une \mathcal{F} -connection \mathcal{D} est une $\tilde{\mathcal{F}}$ -connection $\tilde{\mathcal{D}}$ de reng plus grand telle que $\mathcal{D} \subset \tilde{\mathcal{D}}$ et $\mathcal{F} \subset \tilde{\mathcal{F}}$.

En utilisant les prolongations $\mathcal{D}^{(k)}$ et $\mathcal{D}^{(\infty)}$ de la connection partielle aux fibrés de jets, nous pouvons définir le groupoïde de Galois à paramètres comme le groupoïde des isomorphismes de $J(\mathcal{D})$ qui fixe les invariants différentiels rationnels de \mathcal{D} . Le fibré en groupes de Galois est la diagonale de ce groupoïde. Il peut être vu comme sous-fibré rationnel en groupe $\text{Gal}_{\infty}(\mathcal{D}) \subset J(\text{Gau}(P)/B)$. Le fibré en groupe $J(\text{Gau}(P)/B)$ est naturellement muni d'une structure différentielle.

Théorème 3.18 *Le fibré en groupe des automorphismes de jauge $J(\mathcal{D})$ peut s'identifier au sous-groupe algébrique différentiel des symétries de \mathcal{D} , définie par la connexion de groupe $\mathcal{D}[\text{Adj}]$.*

$$\text{Gau}(J(\mathcal{D})) \simeq J(\mathcal{D}[\text{Adj}]) \subset J(\text{Gau}(P)/B).$$

Théorème 3.20 *Le fibré en groupes de Galois à paramètres $\text{Gal}_{\infty}(\mathcal{D})$ est un sous-groupe algébrique différentiel de $J(\text{Gau}(P)/B)$.*

Sous l'hypothèse de simplicité du groupe structural du fibré principal, les équations des déformations isomonodromiques sont les équations du groupoïde de Galois :

Théorème 3.23 Soient $P \rightarrow B$ un fibré principal affine avec un groupe de structure G simple, $\rho: B \rightarrow S$ une application dominante, $\mathcal{F} = \ker(d\rho)$, et \mathcal{D} une connexion principal avec paramètres dans S . Supposons que le groupe de Galois \mathcal{D} est $\text{Gau}(P)$ alors il existe une déformation isomonodromique maximale $\tilde{\mathcal{D}}$ de \mathcal{D} et le groupe de Galois à paramètres, $\text{Gal}_\infty(\mathcal{D})$, est le groupe $J(\tilde{\mathcal{D}}[\text{Adj}])$ des symétries de jauge de $\tilde{\mathcal{D}}$.

Dans l'exemple 3.24 nous considérons le fibré $P \rightarrow B$ où $B = \mathbb{C}^{k+1} \times \mathfrak{sl}_n(\mathbb{C})^k$ avec comme coordonnées $x, a_1, \dots, a_k, A_1, \dots, A_k$; $S = \mathbb{C}^k \times \mathfrak{sl}_n(\mathbb{C})^k$ et $P = \text{SL}_n(\mathbb{C}) \times B$. Nous notons ρ la projection $B \rightarrow S$. Le système fuchsien général avec k singularités et sans trace est la $\text{SL}_n(\mathbb{C})$ -connection donnée par

$$\frac{\partial U}{\partial x} = \left(\sum_{i=1}^n \frac{A_i}{x - a_i} \right) U.$$

Ces déformations isomonodromiques sont bien connues et données par le système de Schlesinger :

$$\begin{aligned} \frac{\partial A_i}{\partial a_j} &= \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \\ \frac{\partial A_i}{\partial a_i} &= - \sum_{i \neq j} \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \\ \frac{\partial U}{\partial a_i} &= \frac{A_i}{x - a_i} U. \end{aligned}$$

Les deux premières équations définissent le feuilletage $\rho_*(\tilde{\mathcal{F}})$ des chemins isomonodromiques dans S . La troisième équation définit la déformation isomonodromique $\tilde{\mathcal{D}}$ en tant que $\tilde{\mathcal{F}}$ -connexion. Le groupe de Galois à paramètre de l'équation fuchsienne générale est un sous-groupe algébrique différentiel de $J(\text{SL}_n(\mathbb{C})/B)$ donné par les deux premières équations et

$$\frac{\partial \sigma}{\partial x} = \left[\sum_{i=1}^n \frac{A_i}{x - a_i}, \sigma \right], \quad \frac{\partial \sigma}{\partial a_i} = \left[\frac{A_i}{x - a_i}, \sigma \right].$$

Nous étudions l'équation hypergéométrique de Gauss

$$x(1-x) \frac{d^2 u}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\} \frac{du}{dx} - \alpha\beta u = 0$$

Cette équation s'interprète comme une connexion linéaire avec paramètres. Le fibré est $P = \text{GL}_2(\mathbb{C}) \times B \rightarrow B$ où $B = \mathbb{C}_x \times S$ et $S = \mathbb{C}_{\alpha, \beta, \gamma}^3$. Par une série d'arguments dont l'étude de la réduction projective à $\text{PGL}_2(\mathbb{C}) \times B$, l'absence de chemin isomonodromique

pour l'équation projectivée et un analogue différentiel d'un théorème de Kolchin, nous obtenons:

Théorème 3.31 *Le groupe de Galois à paramètre de l'équation hypergéométrique de Gauss est donné par :*

$$\mathrm{Gal}_\infty(\mathcal{H}) = \{j\sigma \in \mathrm{Gau}(J(\mathcal{H})) \mid j\det(\sigma) \in \mathrm{Gal}_\infty(\mathcal{H}_{\det})\}$$

qui est défini par les équations différentielles,

$$\partial_x \sigma = \left[\begin{pmatrix} 0 & 1 \\ \frac{\alpha\beta}{x(1-x)} & \frac{(\alpha+\beta+1)x-\gamma}{x(1-x)} \end{pmatrix}, \sigma \right],$$

$$\partial_x \det(\sigma) = \partial_c \det(\sigma) = \partial_a \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_b \det(\sigma)}{\det(\sigma)} \right) = 0.$$

où σ est une matrice 2×2 inversible dépendante de x, α, β, γ et les opérateurs différentiels $\partial_a, \partial_b, \partial_c$ sont donnés par la proposition 3.28.

Le quatrième chapitre est dédié à la théorie de Galois–Malgrange des équations différentielles non linéaires. Le groupoïde de Malgrange d'un champ de vecteurs est introduit suivant les notations et outils mis en place au chapitre 2.

Un repère d'ordre k sur une variété M est le jet d'ordre k de germe de biholomorphisme $\phi : (\mathbb{C}^m, 0) \rightarrow M$. Le fibré des repères d'ordre k $R_k M \rightarrow M$ est un fibré principal avec groupe de structure $\mathrm{Aut}_k(\mathbb{C}^m, 0)$.

Les prolongations $X^{(k)}$ de X sur $R_k M$ est $\mathrm{Aut}_k(\mathbb{C}^m, 0)$ -invariant. La distribution $\langle X^{(k)} \rangle$ est une $\langle X \rangle$ -connexion partielle. Nous pouvons utiliser la théorie de Galois telle qu'exposée dans le chapitre 3. Le *Groupoïde de Malgrange d'ordre k* $\mathrm{Mal}_k(X)$ d'un champ de vecteurs X est le groupoïde de Galois de sa prolongation d'ordre k $\mathrm{GGal}(\langle X^{(k)} \rangle)$. La limite projective ($k \rightarrow \infty$) donne le \mathcal{D} -groupoïde $\mathrm{Mal}(X)$ sur M .

Le groupoïde de Malgrange a une \mathcal{D} -algebra de Lie. Un théorème de Kiso and Morimoto sur la structure des \mathcal{D} -algèbres de Lie, qui peut être vu comme un analogue non-linéaire du théorème de Kiso-Cassidy theorem, peut être utilisé de manière analogue à notre traitement de l'hypergéométrie de Gauss.

Nous appliquons cette stratégie à l'équation de Painlevé VI :

$$u'' = F(x, u, u', a, b, c, e);$$

où $F \in \mathbb{C}(x, u, v, a, b, c, e)$ est

$$F(x, u, v, a, b, c, e) = \frac{1}{2} \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x} \right) v^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x} \right) v \\ + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \left(\frac{1}{2}c^2 - \frac{1}{2}a^2 \frac{x}{u^2} + \frac{1}{2}b^2 \frac{x-1}{(u-1)^2} + \left(\frac{1-e^2}{2}\right) \frac{x(x-1)}{(u-x)^2} \right).$$

qui peut être vu comme un champ de vecteurs hamiltonien non autonome X sur \mathbb{C}^7 . Le champ X donne une connexion $\langle X \rangle$ sur le fibré trivial $M \rightarrow B$ où $M = \mathbb{C}^7$, $B = \mathbb{C}_{x,a,b,c,e}^5$ et $S = \mathbb{C}_{a,b,c,e}^4$. Nous obtenons :

Théorème 4.26 *Le groupoïde de Malgrange de la sixième équation de Painlevé VI est donné par*

$$\text{Mal}(X) = \left\{ \phi \mid \bar{\pi} \circ \phi = \phi; \quad \phi^*(dx) = dx; \quad \phi_*(X) = X; \right. \\ \left. \phi^*(dp \wedge dq) \equiv dp \wedge dq \mod da, db, dc, de, dx \right\}$$

Cette thèse se finit avec deux annexes contenant les preuves des résultats centraux utilisés dans le texte: La théorie des groupes algébriques différentiels suivant Malgrange [Mal10a], et le théorème de Kiso-Morimoto suivant et complétant [Kis79].

Introduction

Historical context

Geometrical and algebraic methods play an important role in the theory of ordinary differential equations. In particular, Picard-Vessiot theory deals with integrable systems of linear differential equations.

$$\frac{\partial U}{\partial x_i} = A_i(x)U, \quad A_i(x) \in \mathfrak{gl}_n(\mathbb{C}(x)), \quad \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = [A_i, A_j] \quad (0-1)$$

Its main object, the Galois group, is composed of transformations of the solutions of a system that preserve the algebraic relations existing between them. If Φ is a fundamental matrix of solutions of (0-1) then,

$$\mathrm{Gal}_\Phi = \left\{ \sigma \in \mathrm{GL}_n(\mathbb{C}) \mid \forall P \in \mathbb{C}(u_{11}, \dots, u_{nn}), P(\Phi) \in \mathbb{C}(x) \Rightarrow P(\Phi) = P(\Phi\sigma) \right\}$$

This theory was developed for the first time by Émile Picard [Pic87] and Ernest Vessiot [Ves04b], [Ves04a] at the end of XIXth century, inspired by Galois theory of algebraic equations, and was formalized and generalized to equations with several independent variables by Kolchin [Kol73] in the mid of XXth century, in terms of differential field extensions. When the coefficients of such differential equations are rational functions on a variety B , then the differential equation (0-1) is seen as a G -invariant connection \mathcal{D} in a principal bundle $\pi: P \rightarrow B$. There are some advantages in considering not the Galois group but the Galois groupoid $\mathrm{GGal}(\mathcal{D})$ which is the rational subgroupoid of $\mathrm{Iso}(P)$ preserving the field $\mathbb{C}(P)^\mathcal{D}$ of rational first integrals of \mathcal{D} . This can be seen as a particular case of a vast generalization of Picard-Vessiot theory proposed by Y. André [And01].

An application of Picard–Vessiot theory is the classification of linear differential equations by whether its solutions admit expressions by means of quadratures or not. A classical example is given by the Gauss hypergeometric equation, where the works of Schwarz [Sch73] and Kimura [Kim69] give a complete answer to this question.

An analytic invariant of a linear differential equation is its monodromy representation. This measures the analytic prolongation of solutions along closed paths in \mathbb{CP}_1 avoiding singularities. It is natural to consider families of linear differential equations depending

rationally on some parameters.

$$\frac{\partial U}{\partial x} = A(x, \lambda_1, \dots, \lambda_m)U, \quad A(x) \in \mathfrak{gl}_n(\mathbb{C}(x, \lambda_1, \dots, \lambda_m)). \quad (0-2)$$

Equations of the form (0-2) can be seen geometrically as partial connections $\pi: (P, \mathcal{D}) \rightarrow (B, \mathcal{F})$ where \mathcal{F} is the foliation given by the parameters. The geometrical differential Galois theory applies as it is and there is a Galois groupoid $\mathrm{GGal}(\mathcal{D})$.

Singer–Cassidy–Landesman [CS05], [Lan08] offer a theory for differential equations with parameters(0-2). In their work, Galois group preserves not only algebraic relations but differential relations appearing when derivation with respect to the parameters take place. These Galois groups turn out to be differential algebraic groups. For some work and calculus on differential galois group with parameters see [GGO13, Dre14, MS14, MOS15, Arr16, LSN16, HMO17]. In Picard–Vessiot theory, the Galois group of an equation is an algebraic group of matrices. It was Kolchin who set the foundations of the subject differential algebraic geometry with the study of differential algebraic groups.

In parallel, Malgrange [Mal01] and Umemura [Ume96] continued a project begun by J. Drach [Dra98] to extend Picard Vessiot theory to non linear differential equations. Both paths lead to equivalent theories. In Malgrange line, the object corresponding to a non linear equation (a vector field X in a variety M) is a \mathcal{D} -groupoid $\mathrm{Mal}(X) \subset \widehat{\mathrm{Aut}}(M)$ that is, in general terms, a groupoid of jets of diffeomorphisms defined by a system of PDE's.

Isomonodromic deformations are curves in the parameter space that preserve the monodromy representation. Monodromy deformations can be seen geometrically as unfolding of the underlying foliation Isomonodromic deformations play also an important role in the theory of non-linear differential equations. When one studies isomonodromic deformations of a fuchsian equation with four singularities there appear the so-called Schlesinger systems. An important reduction of the Schlesinger system corresponding to an equation with four singularities is Painlevé VI. There is an analogy between Painlevé VI and Gauss hypergeometric equation; it can be seen in [IKSY13]. The other Painlevé equations (PI – PV) can be seen as some limit cases of the PVI, as the classical special linear differential equations appear as degenerations of the Hypergeometric equation. There is huge literature about the non-linear monodromy, symmetries and closed form solutions of the PVI equation. From the works of Iwasaki [Iwa08], and Lisovyy and Thykyy [LT14] we have a precise description of the non-linear monodromy and a complete classification of the algebraic solutions of PVI.

Casale, Loray and Cantat [CL09] have calculated Malgrange groupoid for Painlevé VI equation, for all values of the parameters. There is also a recent specialization theorem [Dav16] that relates the Galois group obtained for specific values of the parameters with some partial Malgrange groupoid (where the derivative with respect to the parameters is skipped in the computation of the groupoid).

General purpose of this work

The main purpose on this work is to unfold the relation between isomonodromic deformations and differential Galois theory (linear and non-linear). As application we will be able to compute the differential Galois group and groupoid of some remarkable differential equations.

In the paper on parameterized Galois theory, [CS05] p.128 section 5 Cassidy and Singer find a relationship between parameterized Galois theory and isomonodromic families of linear differential equations.

Here we propose a completely geometric version of parameterized Galois theory, based on the prolongation of the partial connection \mathcal{D} in $P \rightarrow B$ to the jet bundle $J(P/B)$. Such a geometrical perspective allow us to clarify the relationship between Galois group and isomonodromic deformations. A key point in this link is Kiso–Cassidy (Theorem A.9) theorem. By our method it is possible to calculate the Galois group of a general fuchsian equation and of Gauss hypergeometric equation.

In the case of non linear Galois–Malgrange theory, there does not exists a parameterized version. More precisely, the non-linear Galois–Malgrange theory in its usual presentation already incorporates the parameters. In [Dav16] some part of the Malgrange groupoid (called partial Malgrange groupoid) of Painlevé VI equation was calculated. However, a question remains open: Which is its Galois groupoid when the parameters are considered as additional variables? This groupoid should preserve differential relations satisfied by the solutions of Painlevé VI with respect to the parameters.

By using Kiso–Morimoto theorem (Theorem B.4) and the lack of non-linear isomonodromy for P-VI (a consequence of the classification of its algebraic solutions) we are able to compute the total Malgrange groupoid of Painlevé VI including the parameters.

Brief summary of the text and main results

We begin by a presentation of linear differential Galois theory in geometrical terms. To do this it is proper to speak about principal bundles, $P \rightarrow B$ where the base comes with a foliation \mathcal{F} in the direction of independant variable, and the equation is a connection \mathcal{D} above \mathcal{F} that is flat and G -invariant.

Each principal bundle gives rise to the groupoid $\text{Iso}(P)$ of gauge isomorphisms, and there is a correspondence between subgroupoids of $\text{Iso}(P)$ that dominate the base and certain families of subbundles of P . In classical terms, the Galois groupoid is defined to be the invariance groupoid of the first rational integrals of \mathcal{D} .

Because of the nature of objects to appear, we introduce the concept of rational groupoids: these are Zariski-closed subsets of a groupoid that are groupoids when restricted to an open set in the base. This setting leads naturally to a Galois correspondence

$$\left\{ \begin{array}{c} \text{Rational subgroupoids} \\ \text{of } \text{Iso}(P) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} G\text{-invariant subfields of } \mathbb{C}(P) \\ \text{containing } \mathbb{C} \end{array} \right\}$$

Classical Galois group is obtained as the stabilizer of the Galois groupoid at a point. Intrinsic Galois group is then seen [Kat82] as a bundle in groups over the base B and is obtained as the diagonal part of the groupoid.

In order to handle a parameterized theory, we use the concept of G -invariant connection with parameters in S , that is a G -invariant connection \mathcal{D} in P in the direction of \mathcal{F} .

Chapter 2 presents jet bundles and the theory of Weil infinitesimally near points. There appear frame bundles and prolongation mechanisms for vector fields. We follow the general construction as done in [MnRM00]. However, we consider the ring of functions in Weil and jet bundles as non-classical differential fields by means of the total derivative operator.

Chapter 3 is devoted to linear differential Galois theory with parameters. Cassidy and Singer in [CS05] established a relationship between isomonodromic families and parameterized Galois theory. To see this relation through a geometrical perspective, we interpret isomonodromic deformations as extensions of vector field distributions. An extension of a \mathcal{F} -connection \mathcal{D} is a $\tilde{\mathcal{F}}$ -connection $\tilde{\mathcal{D}}$ of bigger rank with $\mathcal{D} \subset \tilde{\mathcal{D}}$ and $\mathcal{F} \subset \tilde{\mathcal{F}}$. For more on isomonodromic deformations see [GO14].

We define the prolongations $\mathcal{D}^{(k)}$ and $\mathcal{D}^{(\infty)}$ of a partial connection in a principal bundle to the jet spaces. This allows us to introduce the Galois groupoid with parameters as the groupoid of gauge isomorphisms of $J(\mathcal{D})$ that fixes rational differential invariants of \mathcal{D} . Then the Galois group bundle is the diagonal part of the groupoid. Is it going to be seen as a rational group sub-bundle $\text{Gal}_\infty(\mathcal{D}) \subset J(\text{Gau}(P)/B)$. This group bundle $J(\text{Gau}(P)/B)$ has a differential structure given by the total derivatives and a group sub-bundle is a *differential algebraic subgroup* if it is defined by a differential ideal of its ring of regular functions on the generic point of B . In this setting we prove:

Theorem 3.18 *The group bundle of gauge automorphisms $J(\mathcal{D})$ is identified with the differential algebraic subgroup of gauge symmetries of \mathcal{D} , defined by the group connection $\mathcal{D}[\text{Adj}]$.*

$$\text{Gau}(J(\mathcal{D})) \simeq J(\mathcal{D}[\text{Adj}]) \subset J(\text{Gau}(P)/B).$$

Theorem 3.20 *The parameterized Galois group bundle $\text{Gal}_\infty(\mathcal{D})$ is a differential algebraic subgroup of $J(\text{Gau}(P)/B)$.*

Here $J(\mathcal{D}[\text{Adj}]) \subset J(\text{Gau}(P)/B)$ is the differential algebraic group of Gauge symmetries of the connection \mathcal{D} . Under the hypothesis of simple structural group, isomonodromic deformations provide the equations for the Galois group with parameters. Here the main result appears:

Theorem 3.23 *Let $P \rightarrow B$ be an affine principal bundle with simple group G , $\rho: B \rightarrow S$ a dominant map, $\mathcal{F} = \ker(d\rho)$, and \mathcal{D} a principal G -invariant connection with parameters in S . Let us assume that the Galois group of \mathcal{D}*

is $\text{Gau}(P)$. Then, there is a biggest isomonodromic deformation $\tilde{\mathcal{D}}$ of \mathcal{D} and the Galois group with parameters $\text{Gal}_\infty(\mathcal{D})$ is the group $J(\tilde{\mathcal{D}}[\text{Adj}])$ of gauge symmetries of $\tilde{\mathcal{D}}$.

In **Example 3.24** we consider the bundle $P \rightarrow B$ where $B = \mathbb{C}^{k+1} \times \mathfrak{sl}_n(\mathbb{C})^k$ with coordinates $x, a_1, \dots, a_k, A_1, \dots, A_k$; $S = \mathbb{C}^k \times \mathfrak{sl}_n(\mathbb{C})^k$ and $P = \text{SL}_n(\mathbb{C}) \times B$. We also consider the projection $\rho: B \rightarrow S$. In this setting we study the general fuchsian system with k singularities and trace free matrices, with $\text{SL}_n(\mathbb{C})$ -connection given by

$$\frac{\partial U}{\partial x} = \left(\sum_{i=1}^n \frac{A_i}{x - a_i} \right) U. \quad (0-3)$$

Its isomonodromic deformations are well known and given by the Schlesinger system:

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad (0-4)$$

$$\frac{\partial A_i}{\partial a_i} = - \sum_{i \neq j} \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad (0-5)$$

$$\frac{\partial U}{\partial a_i} = \frac{A_i}{x - a_i} U. \quad (0-6)$$

Equations (0-4), (0-5) define the foliation $\rho_*(\tilde{\mathcal{F}})$ of isomonodromic paths in S , and equations (3-5) - (0-6) define the isomonodromic deformation $\tilde{\mathcal{D}}$ as a partial $\tilde{\mathcal{F}}$ -connection. The Galois group with parameters of 3-5 is the differential algebraic subgroup of $J(\text{SL}_n(\mathbb{C})/B)$ given by equations (0-4), (0-5), and

$$\frac{\partial \sigma}{\partial x} = \left[\sum_{i=1}^n \frac{A_i}{x - a_i}, \sigma \right], \quad \frac{\partial \sigma}{\partial a_i} = \left[\frac{A_i}{x - a_i}, \sigma \right]. \quad (0-7)$$

We also study Gauss hypergeometric equation

$$x(1-x) \frac{d^2 u}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\} \frac{du}{dx} - \alpha\beta u = 0 \quad (0-8)$$

It is possible to interpret Equation (0-8) as a linear connection with parameters, in the bundle $P = \text{GL}_2(\mathbb{C}) \times B \rightarrow B$, where $B = \mathbb{C}_x \times S$ and $S = \mathbb{C}_{\alpha, \beta, \gamma}^3$. Then by a series of arguments, that include reducing the equation to $\text{PGL}_2(\mathbb{C}) \times B$, the absence of isomonodromic deformations for the reduced equation and also the projection given by the determinant, we arrive at the following description of its the Galois group:

Theorem 3.31 *The differential Galois group with parameters of Gauss' hypergeometric equation is given by*

$$\mathrm{Gal}_\infty(\mathcal{H}) = \{j\sigma \in \mathrm{Gau}(J(\mathcal{H})) \mid j\det(\sigma) \in \mathrm{Gal}_\infty(\mathcal{H}_{\det})\}$$

which is defined by the differential equations,

$$\partial_x \sigma = \left[\begin{pmatrix} 0 & 1 \\ \frac{\alpha\beta}{x(1-x)} & \frac{(\alpha+\beta+1)x-\gamma}{x(1-x)} \end{pmatrix}, \sigma \right],$$

$$\partial_x \det(\sigma) = \partial_c \det(\sigma) = \partial_a \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_b \det(\sigma)}{\det(\sigma)} \right) = 0.$$

where σ is an invertible 2×2 matrix depending on x, α, β, γ and the differential operators $\partial_a, \partial_b, \partial_c$ are those given in Proposition 3.28.

Chapter 4 is devoted to non-linear Galois–Malgrange theory. In order to introduce the Malgrange groupoid of a vector field we use the theory of frames as exposed in chapter 2. A k -frame in an variety M is the jet of order k of germ of biholomorphism $\phi : (\mathbb{C}^m, 0) \rightarrow M$. The bundle of k -frames $R_k M \rightarrow M$ is a principal bundle with structure group $\mathrm{Aut}_k(\mathbb{C}^m, 0)$.

Our framework to treat with non-linear vector field X is to consider its linearization to each finite order. The prolongation $X^{(k)}$ of X to $R_k M$ is $\mathrm{Aut}_k(\mathbb{C}^m, 0)$ -invariant. Then the distribution $\langle X^{(k)} \rangle$ is a partial $\langle X \rangle$ -connection. We can apply differential Galois theory as exposed in Chapter 3. The *order k Malgrange groupoid* $\mathrm{Mal}_k(X)$ of a vector field X is the Galois groupoid of its k -th prolongation $\mathrm{GGal}(\langle X^{(k)} \rangle)$. By taking the projective limit ($k \rightarrow \infty$) we obtain a \mathcal{D} -groupoid $\mathrm{Mal}(X)$ in M .

The Malgrange groupoid has a \mathcal{D} -Lie algebra. There is a result from Kiso and Morimoto about the structure of \mathcal{D} -Lie algebras that can be seen a non-linear analog of Kiso-Cassidy theorem (the foliation \mathcal{G} appearing in Theorem B.4 can be seen as a non-linear isomonodromic deformation). We then apply this result to Painlevé VI equation:

$$u'' = F(x, u, u', a, b, c, e); \tag{0-9}$$

where $F \in \mathbb{C}(x, u, v, a, b, c, e)$ is

$$\begin{aligned} F(x, u, v, a, b, c, e) = & \frac{1}{2} \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x} \right) v^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x} \right) v \\ & + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \left(\frac{1}{2}c^2 - \frac{1}{2}a^2 \frac{x}{u^2} + \frac{1}{2}b^2 \frac{x-1}{(u-1)^2} + \left(\frac{1-e^2}{2} \right) \frac{x(x-1)}{(u-x)^2} \right). \end{aligned}$$

Which we see in its hamiltonian form as a vector field X in \mathbb{C}^7 . By means of Kiso–Morimoto theorem and interpreting X as a connection $\langle X \rangle$ in the trivial bundle $M \rightarrow B$ where $M = \mathbb{C}^7$, $B = \mathbb{C}_{x,a,b,c,e}^5$ and $S = \mathbb{C}_{a,b,c,e}^4$ we are able to compute the Malgrange groupoid of PVI:

Theorem 4.26 *The Malgrange groupoid of Painlevé VI equation is given by*

$$\text{Mal}(X) = \left\{ \phi \mid \bar{\pi} \circ \phi = \phi; \quad \phi^*(dx) = dx; \quad \phi_*(X) = X; \right. \\ \left. \phi^*(dp \wedge dq) \equiv dp \wedge dq \mod da, db, dc, de, dx \right\}$$

Finally Appendix A. and B. are endowed to the theory of differential algebraic groups, following Malgrange [Mal10a], and Kiso-Morimoto theorem, following [Kis79].

Introducción

Contexto histórico

Los métodos geométricos y algebraicos juegan un papel importante en la teoría de las ecuaciones diferenciales ordinarias. En particular, la teoría de Picard–Vessiot se ocupa de los sistemas integrables de ecuaciones diferenciales lineales:

$$\frac{\partial U}{\partial x} = A(x)U, \quad A(x) \in \mathfrak{gl}_n(\mathbb{C}(x)), \quad \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = [A_i, A_j] \quad (0-10)$$

Su objeto principal, el grupo de Galois, se compone de transformaciones de las soluciones de un sistema, que preservan las relaciones algebraicas que existan entre dichas soluciones. Si Φ es una matriz fundamental de soluciones de (0-10) entonces,

$$\text{Gal}_\Phi = \left\{ \sigma \in \text{GL}_n(\mathbb{C}) \mid \forall P(U) \in \mathbb{C}(u_{11}, \dots, u_{nn}) \ P(\Phi) \in \mathbb{C}(x) \rightarrow P(\Phi) = P(\Phi\sigma) \right\}$$

Esta teoría fue desarrollada en primera instancia por Émile Picard [Pic87] y Ernest Vessiot [Ves04b], [Ves04a] a finales del siglo XIX, inspirados en la teoría de Galois de las ecuaciones algebraicas, y fue formalizada y generalizada a ecuaciones con varias variables independientes por Kolchin [Kol73] a mediados del siglo XX, en términos de extensiones diferenciales de cuerpos. Cuando los coeficientes de dichas ecuaciones son funciones racionales en una variedad B , entonces la ecuación diferencial (0-10) puede verse como una conexión G -invariante \mathcal{D} en un fibrado principal $\pi: P \rightarrow B$. Existen ciertas ventajas al considerar no solo el grupo de Galois sino el grupoide de Galois $\text{GGal}(\mathcal{D})$ que es el subgrupoide racional de $\text{Iso}(P)$ que preserva el campo $\mathbb{C}(P)^\mathcal{D}$ de integrales primeras racionales de \mathcal{D} . Esto puede verse como un caso particular de una vasta generalización de la teoría de Picard–Vessiot propuesta por Y. André [And01].

Una aplicación de la teoría de Picard–Vessiot es la clasificación de ecuaciones diferenciales lineales bajo el criterio de si sus soluciones pueden ser expresadas mediante cuadraturas o no. Un ejemplo clásico está dado por la ecuación hipergeométrica de Gauss, para la cual los trabajos de Schwartz [Sch73] y Kimura [Kim69] dan una respuesta completa a dicha pregunta.

Un invariante analítico de una ecuación diferencial lineal es su representación de monodromía. Ésta mide la prolongación analítica de soluciones sobre caminos cerrados en \mathbb{CP}_1 que evitan las singularidades. Es natural considerar familias de ecuaciones diferenciales lineales que depende racionalmente de algunos parámetros:

$$\frac{\partial U}{\partial x} = A(x, \lambda_1, \dots, \lambda_m)U, \quad A(x) \in \mathfrak{gl}_n(\mathbb{C}(x, \lambda_1, \dots, \lambda_m)). \quad (0-11)$$

Las ecuaciones de la forma (0-11) pueden verse geoméricamente como conexiones parciales $\pi: (P, \mathcal{D}) \rightarrow (B, \mathcal{F})$ donde \mathcal{F} es una foliación determinada por los parámetros. La teoría de Galois diferencial aplica como tal y existe un grupoide de Galois $\text{GGal}(\mathcal{D})$.

Singer–Cassidy–Landesman [CS05], [Lan08] ofrecen una teoría para las ecuaciones diferenciales con parámetros (0-11). En dicho trabajo, El grupo de Galois preserva no solo las relaciones algebraicas sino también las relaciones diferenciales que aparecen cuando se deriva respecto a los parámetros. Estos grupos de Galois resultan ser grupos algebraicos diferenciales. Algunos cálculos y desarrollos de la teoría de Galois con parámetros puede verse en [GGO13, Dre14, MS14, MOS15, Arr16, LSN16, HMO17]. En la teoría de Picard–Vessiot, los grupos de Galois de una ecuación son grupos algebraicos de matrices. Fué Kolchin quien estableció los fundamentos de la geometría algebraica diferencial con el estudio de los grupos diferenciales algebraicos.

En paralelo, Malgrange [Mal01] y Umemura [Ume96] retoman un proyecto comenzado por J. Drach [Dra98] para extender la teoría de Picard–Vessiot a ecuaciones diferenciales no lineales. Ambos caminos llevan a teorías equivalentes. En la línea de Malgrange, el objeto correspondiente a una ecuación no lineal (un campo vectorial X en una variedad M) es un \mathcal{D} -grupoide $\text{Mal}(X) \subset \widehat{\text{Aut}}(M)$ que es, en términos generales, un grupoide de jets de difeomorfismos definido por un sistema de EDP's.

Las deformaciones isomonodrómicas son curvas en el espacio de parámetros que preservan la representación de monodromía. Dichas deformaciones pueden verse geoméricamente como desarrollos de la foliación subyacente. Las deformaciones isomonodrómicas juegan un papel importante en la teoría de las ecuaciones diferenciales no lineales. Cuando se estudian las deformaciones isomonodrómicas de una ecuación fuchsiana con cuatro singularidades aparecen los llamados sistemas de Schlesinger. Una reducción importante del sistema de Schlesinger correspondiente a una ecuación con cuatro singularidades es Painlevé VI. Hay una analogía entre Painlevé VI y la ecuación hipergeométrica de Gauss; esta puede verse en [IKSY13]. Las otras ecuaciones de Painlevé (PI–PV) pueden verse como ciertos casos límite de PVI, así como las ecuaciones diferenciales lineales clásicas especiales aparecen como degeneraciones de la ecuación hipergeométrica. Hay una literatura extensa sobre la monodromía no lineal, las simetrías y las soluciones en forma cerrada de la ecuación PVI. A partir de los trabajos de Iwasaki [Iwa08], y Lisovyy y Thykyy [LT14] tenemos una descripción precisa de la monodromía no lineal y una clasificación completa de las soluciones algebraicas de PVI.

Casale, Loray y Cantat [CL09] han calculado el grupoide de Malgrange de la ecuación de Painlevé VI, para todos los valores de los parámetros. Hay también un teorema reciente sobre especialización [Dav16] que relaciona el grupo de Galois que se obtiene para valores específicos de los parámetros con el grupoide de Malgrange parcial, en donde la derivada respecto a los parámetros se evita en el cómputo del grupoide.

Propósito general de este trabajo

El propósito principal de este trabajo es desarrollar la relación entre deformaciones isomonodrómicas y la teoría de Galois diferencial (lineal y no lineal). Como aplicación seremos capaces de computar el grupo de Galois diferencial y el grupoide de ciertas ecuaciones diferenciales importantes.

En su artículo sobre teoría de Galois parametrizada, [CS05] p.128 sección 5, Cassidy y Singer encuentran una relación entre la teoría de Galois con parámetros y las familias isomonodrómicas de ecuaciones diferenciales lineales.

Nosotros proponemos una versión completamente geométrica de la teoría de Galois con parámetros de una conexión parcial \mathcal{D} en $P \rightarrow B$ al fibrado de jets $J(P/B)$. Dicha perspectiva geométrica nos permite clarificar la relación entre el grupo de Galois y las deformaciones isomonodrómicas. Un punto clave en este vínculo es el teorema de Kiso–Cassidy (Theorem A.9). Mediante nuestro método es posible calcular el grupo de Galois de una ecuación fuchsiana general y de la ecuación hipergeométrica de Gauss.

En el caso de la teoría de Galois no lineal de Malgrange, no existe una versión con parámetros. Mas precisamente, la teoría de Galois no lineal de Malgrange en su presentación usual incorpora los parámetros. En [Dav16] se calcula cierta parte del grupoide de Malgrange (llamado grupoide parcial de Malgrange) de la ecuación de Painlevé VI. Sin embargo, una pregunta queda abierta: ¿cuál es su grupoide de Galois cuando los parámetros son considerados como variables adicionales? Este grupoide debería preservar las relaciones diferenciales que satisfacen las soluciones de la ecuación de Painlevé VI respecto a los parámetros.

Usando el teorema de Kiso–Morimoto (Teorema B.4) y la carencia de isomonodromía no lineal de PVI (que es una consecuencia de la clasificación de sus soluciones algebraicas) somos capaces de calcular el grupoide de Malgrange total de la ecuación de Painlevé VI incluyendo los parámetros.

Breve sumario del texto y los resultados principales

Comenzamos con una presentación de la teoría de Galois diferencial en términos geométricos. Para hacer esto es apropiado hablar de fibrados principales $P \rightarrow B$ donde la base viene con una foliación \mathcal{F} en la dirección de las variables independientes y una conexión \mathcal{D} sobre \mathcal{F} que es plana y G -invariante.

Cada fibrado principal da lugar a un grupoide $\text{Iso}(P)$ de isomorfismos gauge, y existe una correspondencia entre subgrupos de $\text{Iso}(P)$ que dominan la base y ciertas familias de subfibrados de P . En términos clásicos, el grupoide de Galois se define como el grupoide de invarianza de las integrales primeras de \mathcal{D} .

Debido a la naturaleza de los objetos que aparecen, introducimos el concepto de grupoide racional: son los subconjuntos Zariski cerrados de un grupoide que son grupoides cuando se restringen a un conjunto abierto en la base. Este escenario conduce naturalmente a la

correspondencia de Galois:

$$\left\{ \begin{array}{c} \text{Subgrupoides Racionales} \\ \text{de Iso}(P) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Subcampos } G\text{-invariantes de } \mathbb{C}(P) \\ \text{que contienen a } \mathbb{C} \end{array} \right\}$$

El grupo de Galois clásico se obtiene como el estabilizador del grupoide de Galois en un punto. El grupo de Galois intrínseco es visto [Kat82] como un fibrado en grupos sobre la base B y se obtiene como la parte diagonal del grupoide.

Con el objetivo de manejar la teoría con parámetros, usamos el concepto de conexión G -invariante con parámetros en S , esto es una conexión G -invariante \mathcal{D} en P en la dirección de \mathcal{F} .

El capítulo 2 presenta los fibrados de jets y la teoría de puntos infinitesimalmente próximos de Weil. Allí aparecen los fibrados de marcos y el mecanismo de prolongación para los campos vectoriales. Seguimos la construcción general presentada en [MnRM00]. Sin embargo, consideramos el anillo de funciones en los fibrados de Weil y de jets como un campo diferencial no clásico mediante el operador de derivada total.

El capítulo 3 se dedica a la teoría de Galois diferencial lineal con parámetros. Cassidy y Singer en [CS05] establecen una relación entre familias isomonodrómicas y la teoría de Galois con parámetros. Para ver esta relación a través de una perspectiva geométrica, interpretamos las deformaciones isomonodrómicas como extensiones de distribuciones de campos vectoriales. Una extensión de una \mathcal{F} -conexión \mathcal{D} es una $\tilde{\mathcal{F}}$ -conexión $\tilde{\mathcal{D}}$ de rango mayor tal que $\mathcal{D} \subset \tilde{\mathcal{D}}$ y $\mathcal{F} \subset \tilde{\mathcal{F}}$. Para otra presentación sobre las deformaciones isomonodrómicas referise a [GO14].

Definimos las prolongaciones $\mathcal{D}^{(k)}$ y $\mathcal{D}^{(\infty)}$ de una conexión parcial en un fibrado principal a los espacios de jets. Esto permite introducir el grupoide de Galois con parámetros como el grupoide de isomorfismos gauge de $J(\mathcal{D})$ que deja fijos los invariantes diferenciales racionales de \mathcal{D} . Entonces el fibrado en grupos de Galois es la parte diagonal de grupoide. Este será visto como un subfibrado en grupos racional $\text{Gal}_{\infty}(\mathcal{D}) \subset J(\text{Gau}(P)/B)$. Este fibrado en grupos $J(\text{Gau}(P)/B)$ tiene estructura diferencial dada por las derivadas totales y un subfibrado en grupos es *subgrupo algebraico diferencial* si está definido por un ideal diferencial del anillo de funciones regulares en el punto genérico de B . En este contexto probamos:

Teorema 3.18 *El grupo de los automorfismos gauge $J(\mathcal{D})$ se identifica con un subgrupo diferencial algebraico de simetrías gauge de \mathcal{D} , definido por la conexión en grupos $\mathcal{D}[\text{Adj}]$.*

$$\text{Gau}(J(\mathcal{D})) \simeq J(\mathcal{D}[\text{Adj}]) \subset J(\text{Gau}(P)/B).$$

Teorema 3.20 *El fibrado en grupos de Galois con parámetros $\text{Gal}_{\infty}(\mathcal{D})$ es un subgrupo algebraico diferencial de $J(\text{Gau}(P)/B)$.*

Aquí $J(\mathcal{D}[\text{Adj}]) \subset J(\text{Gau}(P)/B)$ es el grupo algebraico diferencial de simetrías gauge de la conexión \mathcal{D} . Bajo la hipótesis de grupo estructural simple, las deformaciones isomonodrómicas proveen las ecuaciones para el grupo de Galois con parámetros. Aquí aparece el resultado principal:

Teorema 3.23 *Sea $P \rightarrow B$ Un fibrado principal con grupo simple G , $\rho: B \rightarrow S$ un mapa dominante, $\mathcal{F} = \ker(d\rho)$, y \mathcal{D} una conexión G -invariante con parámetros en S . Asuma que el grupo de Galois \mathcal{D} es $\text{Gau}(P)$. Entonces hay una deformación isomonodrómica maximal $\tilde{\mathcal{D}}$ de \mathcal{D} y el grupo de Galois con parámetros $\text{Gal}_\infty(\mathcal{D})$ es el grupo $J(\tilde{\mathcal{D}}[\text{Adj}])$ de simetrías gauge de $\tilde{\mathcal{D}}$.*

En el **Ejemplo 3.24** consideramos el fibrado $P \rightarrow B$ donde $B = \mathbb{C}^{k+1} \times \mathfrak{sl}_n(\mathbb{C})^k$ con coordenadas $x, a_1, \dots, a_k, A_1, \dots, A_k$; $S = \mathbb{C}^k \times \mathfrak{sl}_n(\mathbb{C})^k$ y $P = \text{SL}_n(\mathbb{C}) \times B$. Además consideramos la proyección $\rho: B \rightarrow S$. En este escenario estudiamos el sistema fuchsiano general con k singularidades y matrices sin traza, donde tenemos una $\text{SL}_n(\mathbb{C})$ -conexión dada por

$$\frac{\partial U}{\partial x} = \left(\sum_{i=1}^n \frac{A_i}{x - a_i} \right) U. \quad (0-12)$$

Sus deformaciones isomonodrómicas son bien conocidas y están dadas por el sistema de Schlesinger:

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad (0-13)$$

$$\frac{\partial A_i}{\partial a_i} = - \sum_{j \neq i} \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad (0-14)$$

$$\frac{\partial U}{\partial a_i} = \frac{A_i}{x - a_i} U. \quad (0-15)$$

Las ecuaciones (0-13), (0-14) definen la foliación $\rho_*(\tilde{\mathcal{F}})$ de caminos isomonodrólicos en S , y las ecuaciones (0-12) - (0-15) definen la deformación isomonodrómica $\tilde{\mathcal{D}}$ como una $\tilde{\mathcal{F}}$ -conexión parcial. El grupo de Galois con parámetros de (0-12) es el subgrupo algebraico diferencial $J(\text{SL}_n(\mathbb{C})/B)$ dado por las ecuaciones (0-13), (0-14), y

$$\frac{\partial \sigma}{\partial x} = \left[\sum_{i=1}^n \frac{A_i}{x - a_i}, \sigma \right], \quad \frac{\partial \sigma}{\partial a_i} = \left[\frac{A_i}{x - a_i}, \sigma \right]. \quad (0-16)$$

También se estudia la ecuación hipergeométrica de Gauss

$$x(1-x)\frac{d^2u}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\}\frac{du}{dx} - \alpha\beta u = 0 \quad (0-17)$$

Es posible interpretar la ecuación (0-17) como una conexión lineal con parámetros, en el fibrado $P = \mathrm{GL}_2(\mathbb{C}) \times B \rightarrow B$, donde $B = \mathbb{C}_x \times S$ y $S = \mathbb{C}_{\alpha, \beta, \gamma}^3$. Entonces, por una serie de argumentos, que incluyen la reducción de la ecuación a $\mathrm{PGL}_2(\mathbb{C}) \times B$, la carencia de deformaciones isomonodrómicas para la ecuación reducida y además la proyección obtenida mediante el determinante, se llega a la siguiente descripción de su grupo de Galois:

Teorema 3.31 *El grupo diferencial de Galois con parámetros de la ecuación hipergeométrica de Gauss está dado por*

$$\mathrm{Gal}_\infty(\mathcal{H}) = \{j\sigma \in \mathrm{Gau}(J(\mathcal{H})) \mid j\det(\sigma) \in \mathrm{Gal}_\infty(\mathcal{H}_{\det})\}$$

el cual está definido por las ecuaciones diferenciales

$$\partial_x \sigma = \left[\begin{pmatrix} 0 & 1 \\ \frac{\alpha\beta}{x(1-x)} & \frac{(\alpha+\beta+1)x-\gamma}{x(1-x)} \end{pmatrix}, \sigma \right],$$

$$\partial_x \det(\sigma) = \partial_c \det(\sigma) = \partial_a \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_b \det(\sigma)}{\det(\sigma)} \right) = 0.$$

donde σ es una matriz invertible 2×2 que depende de x, α, β, γ y los operadores diferenciales $\partial_a, \partial_b, \partial_c$ son los que aparecen en la Proposición 3.28.

El capítulo 4 se dedica a la teoría no lineal de Galois–Malgrange. Para introducir el grupoide de Malgrange de un campo vectorial usamos la teoría de los marcos como se expone en el capítulo 2. Un k -marco en una variedad M está dado por el jet de orden k de un germen de biholomorfismo $\phi : (\mathbb{C}^m, 0) \rightarrow M$. El fibrado de k -marcos $R_k M \rightarrow M$ es un fibrado principal con grupo estructural $\mathrm{Aut}_k(\mathbb{C}^m, 0)$.

Nuestra estrategia para tratar con los campos vectoriales no lineales X es considerar su linealización a cada orden finito. La prolongación $X^{(k)}$ de X a $R_k M$ es $\mathrm{Aut}_k(\mathbb{C}^m, 0)$ -invariante. Entonces la distribución $\langle X^{(k)} \rangle$ es una conexión $\langle X \rangle$ -parcial. Podemos aplicar la teoría de Galois diferencial como se expone en el Capítulo 3. El grupoide de Malgrange de orden k $\mathrm{Mal}_k(X)$ de un campo vectorial X es el grupoide de Galois de su k -ésima prolongación $\mathrm{GGal}(\langle X^{(k)} \rangle)$. Tomando el límite proyectivo ($k \rightarrow \infty$) obtenemos un \mathcal{D} -grupoide $\mathrm{Mal}(X)$ en M .

El grupoide de Malgrange tiene una \mathcal{D} -álgebra de Lie. Hay un resultado de Kiso y Morimoto sobre la estructura de las \mathcal{D} -álgebras de Lie que puede ser visto como un análogo no lineal del teorema de Kiso-Cassidy (la foliación \mathcal{G} que aparece en el Teorema B.4 puede

verse como una deformación isomonodrómica no lineal). Luego aplicamos este resultado a la sexta ecuación de Painlevé:

$$u'' = F(x, u, u', a, b, c, e); \quad u' = v \quad (0-18)$$

donde $F \in \mathbb{C}(x, u, v, a, b, c, e)$ es

$$\begin{aligned} F(x, u, u', a, b, c, e) = & \frac{1}{2} \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x} \right) u'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x} \right) u' \\ & + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \left(\frac{1}{2}c^2 - \frac{1}{2}a^2 \frac{x}{u^2} + \frac{1}{2}b^2 \frac{x-1}{(u-1)^2} + \left(\frac{1-e^2}{2} \right) \frac{x(x-1)}{(u-x)^2} \right). \end{aligned}$$

La cual es considerada en su forma hamiltoniana como campo vectorial X en \mathbb{C}^7 . Mediante el teorema de Kiso–Morimoto e interpretando X como una conexión $\langle X \rangle$ en el fibrado trivial $M \rightarrow B$ donde $M = \mathbb{C}^7$, $B = \mathbb{C}_{x,a,b,c,e}^5$ y $S = \mathbb{C}_{a,b,c,e}^4$ se hace posible computar el grupoide de Malgrange de PVI:

Teorema 4.26 *El grupoide de Malgrange de la ecuación Painlevé VI está dado por*

$$\begin{aligned} \text{Mal}(X) = & \left\{ \phi \mid \bar{\pi} \circ \phi = \phi; \quad \phi^*(dx) = dx; \quad \phi_*(X) = X; \right. \\ & \left. \phi^*(dp \wedge dq) \equiv dp \wedge dq \mod da, db, dc, de, dx \right\} \end{aligned}$$

Finalmente los apéndices A. y B. consisten de la teoría de los grupos algebraicos diferenciales, siguiendo a Malgrange [Mal10a], y el teorema de Kiso–Morimoto, siguiendo a [Kis79].

1. Some algebra and geometry

1.1. Differential Algebra

Some concepts from differential algebra are presented here with a geometrical flavour. Results to be used later are stated and some proofs are given.

Definition 1.1 A *derivation* in a ring A is an additive map $\partial : A \rightarrow A$ that satisfies Leibniz rule: $\partial(ab) = \partial(a)b + a\partial(b)$. Derivations of A form an A -module: $\text{Der } A$. The commutator $[\partial, \delta] = \partial \circ \delta - \delta \circ \partial$ of two derivations is itself a derivation.

A *differential structure* in a ring A is a A -module \mathfrak{X}_A of derivations of A closed under the commutator bracket. A *differential ring* consist of a pair (A, \mathfrak{X}_A) where A is a ring and \mathfrak{X}_A is a differential structure in A . We will say interchangeably that (A, \mathfrak{X}_A) is a differential ring or that A is a \mathfrak{X}_A -ring.

Example 1.2 The ring of rational functions $\mathbb{C}(x_1, \dots, x_n)$ on an open subset of \mathbb{C}^n , with module of derivations generated by $\partial_1, \dots, \partial_n$ where $\partial_i x_j = \delta_j^i$ and $\partial_i c = 0$ for all $c \in \mathbb{C}$ is a differential ring.

A *classical differential ring* (A, Δ) , or a Δ -ring A , is a ring with a n -tuple $\Delta = \{\delta_1, \dots, \delta_n\}$ of mutually commuting derivations. It is clear that a classical differential ring is a differential ring, just take as the module of derivations the A -module spanned by Δ .

Definition 1.3 An ideal I in a differential ring A with derivations \mathfrak{X}_A is called differential ideal (or \mathfrak{X}_A -ideal) if it is closed under all the derivations : $\partial I \subset I$ for all $\partial \in \mathfrak{X}_A$. An element $c \in A$ is said to be a constant if $\partial c = 0$ for every $\partial \in \mathfrak{X}_A$.

In the classical setting, the condition $\mathfrak{X}_A I \subset I$ becomes $\partial_i I \subset I$ for all $i = 1, \dots, n$, and the property of being a constant is expressed as $\partial_i c = 0$ for all $i = 1, \dots, n$. The set of all \mathfrak{X}_A -constants of A is a subring of A denoted by C_A or $A^{\mathfrak{X}_A}$ when it is necessary to point out the differential structure.

Definition 1.4 Let A and B be two differential rings with modules of derivations \mathfrak{X}_A and \mathfrak{X}_B . A ring morphism $\theta : A \rightarrow B$ is differential if there is a map $\theta_* : \mathfrak{X}_A \rightarrow \mathfrak{X}_B$ such that $\theta \circ \partial = \theta_*(\partial) \circ \theta$ for any $\partial \in \mathfrak{X}_A$ and the image $\theta_*(\mathfrak{X}_A)$ generates \mathfrak{X}_B as a B -module. This condition assures that the image of any constant is a constant.

In classical terms, a morphism of classical differential rings is a ring morphism that commutes with the derivations. If $\Delta_A = \{\delta_1, \dots, \delta_n\}$ is a system of commuting generators of \mathfrak{X}_A , and if $\Delta_B = \{\theta_*(\delta_1), \dots, \theta_*(\delta_n)\}$, then $\theta: (A, \Delta_A) \rightarrow (B, \Delta_B)$ is a morphism of classical differential rings. Thus, the notion of morphism of differential rings extends that of morphism of classical differential ring.

Definition 1.5 The algebra of linear differential operators is the non-commutative algebra $A[\mathfrak{X}_A]$, spanned by \mathfrak{X}_A and A inside $\text{End}_{C_A} A$, with composition denoted by \circ . The *ring of differential polynomials in the unknowns* x_1, \dots, x_n is defined in the following way. Let $L = \bigoplus_{i=1}^n Ax_i$ be the free A -module spanned by the variables. Then, the ring of differential polynomials spanned by L is:

$$A\{L\}_{\mathfrak{X}_A} = S^\bullet(L \otimes_A A[\mathfrak{X}_A])$$

where its differential structure is given by the derivations $\partial \in \mathfrak{X}_A$ acting, for $a \in A$ as follows:

$$\partial(ax_i \otimes \theta) = (\partial a)x_i \otimes \theta + ax_i \otimes \partial \circ \theta$$

In the classical context the ring of differential polynomials is given by the next construction: See [VdPS12], Appendix D. Begin with a basis of derivations $\Delta = \{\partial_1, \dots, \partial_r\}$ and variables x_1, \dots, x_n . Take Θ to be the free commutative semigroup generated by the elements of Δ . Let $\{\theta x_i\}_{\theta \in \Theta, i \in \{1, \dots, n\}}$ be a set of indeterminates. Then define $A\{x_1, \dots, x_n\}_\Delta$ to be $A[\theta x_i]_{\theta \in \Theta, i \in \{1, \dots, n\}}$. Derivations in Δ give to $A\{x_1, \dots, x_n\}_\Delta$ a structure of differential ring by setting $\partial_i(\theta x_j) = \partial_i \theta x_j$.

The given definition of differential polynomials extends the classical one. In what follows it is useful to use multi-index notation. For α a multi-index write $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_r^{\alpha_r}$.

Proposition 1.6 Let (A, \mathfrak{X}_A) be a differential ring and $\Delta = \{\partial_1, \dots, \partial_r\}$ a system of commuting generators of \mathfrak{X}_A . Let us assume that $A[\mathfrak{X}_A]$ is a free A -module spanned by the powers ∂^α . Then,

$$A\{L\}_{\mathfrak{X}_A} = A\{x_1, \dots, x_n\}_\Delta;$$

that is, $A\{L\}$ is the ring of polynomials in the variables θx_i where θ varies in the free commutative semigroup Θ spanned by Δ .

PROOF. As $A[\mathfrak{X}_A]$ is spanned by $\{\partial^\alpha\}$, write $A[\mathfrak{X}_A] = \bigoplus_\alpha A\partial^\alpha$. Then

$$(\bigoplus Ax_i) \otimes A[\mathfrak{X}_A] = (\bigoplus Ax_i) \otimes \bigoplus_\alpha A\partial^\alpha = \bigoplus_{\alpha, i} A(x_i \otimes \partial^\alpha)$$

A useful alternative notation is $x_i \otimes \partial^\alpha := \partial^\alpha x_i = x_{i:\alpha}$. Finally, the derivation given by

$\partial(ax_i \otimes \theta) = (\partial a)x_i \otimes \theta + ax_i \otimes \partial \circ \theta$ behaves well with tensor relations:

$$\partial(a_i x_i \otimes \theta) = \partial(a_i)x_i \otimes \theta + a_i x_i \otimes \partial \circ \theta$$

$$\partial(x_i \otimes a_i \theta) = x_i \otimes (\partial a_i)\theta + x_i \otimes a_i \partial \circ \theta$$

□

1.1.1. Differential field associated to a foliation

All previous notation applies to the particular case in which the ring happens to be a field. Thus, we shall speak of differential fields, Δ -fields, \mathfrak{X}_K -fields, and so on. The following result is about the differential algebraic structure of singular foliations, the reader may check p.33 for the terminology on foliations.

Theorem 1.7 *Let M be an irreducible smooth algebraic variety over \mathbb{C} , \mathcal{F} a singular foliation on M of rank r and $\mathfrak{X}_{\mathcal{F}}$ the $\mathbb{C}(M)$ -module of derivations tangent to \mathcal{F} . Then there exists a basis \mathcal{B} of derivations of $\mathfrak{X}_{\mathcal{F}}$ that commute among each other, and so $(\mathbb{C}(M), \mathcal{B})$ admits an structure of classical differential field with $|\Delta| = \text{rank}(\mathcal{F})$.*

PROOF. This is a restatement of Frobenius theorem. Take x_1, \dots, x_n a transcendence basis for $\mathbb{C}(M)$. Write ∂_i for $\partial/\partial x_i$. The collection $\Delta = \{\partial_i\}$ is basis of $\text{Der}(\mathbb{C}(M))$.

Take any basis \tilde{X} for $\mathfrak{X}_{\mathcal{F}}$. As $\mathfrak{X}_{\mathcal{F}}$ is a vector subspace of \mathfrak{X}_M it is possible to express the basis \tilde{X} in terms of the basis Δ , so there is a matrix \tilde{A} such that

$$\begin{pmatrix} \tilde{X}_1 \\ \vdots \\ \tilde{X}_r \end{pmatrix} = \tilde{A} \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}$$

Matrix \tilde{A} have rank r and it is possible to assume that the first r columns are linearly independent. Call B the sub-matrix of \tilde{A} that consist of those columns. Then consider the following vector fields

$$\begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = B^{-1} \tilde{A} \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \quad A := B^{-1} \tilde{A} = \begin{pmatrix} 1 & & a_{1r+1} & \cdots & a_{1n} \\ & \ddots & \vdots & & \\ & & 1 & a_{rr+1} & \cdots & a_{rn} \end{pmatrix}$$

A direct calculation shows that for $1 \leq k \leq r$, $[X_i, X_j]x_k = 0$ and for $r+1 \leq k \leq n$ it is given by

$$[X_i, X_j]x_k = \partial_i a_{jk} - \partial_j a_{ik} + \sum_{t=r+1}^n a_{it} \partial_t a_{jk} - \sum_{s=r+1}^n a_{js} \partial_s a_{ik}$$

Calling these coefficients $C_{ij}^k := [X_i, X_j]x_k$ write $[X_i, X_j] = \sum_{s=r+1}^n C_{ij}^k \partial_s$.

As \mathcal{F} is involutive, it is possible to write $[X_i, X_j] = \sum g_{ij}^k X_k$. Expanding this expression yields

$$\begin{aligned} [X_i, X_j] &= \sum g_{ij}^k X_k \\ &= \sum_{k=0} g_{ij}^k \partial_k + \sum_{k=0}^r \sum_{l=r+1}^n g_{ij}^k a_{kl} \partial_l \end{aligned}$$

It follows that $g_{ij}^k = 0$ for $k = 1, \dots, r$, and this way the fields X_i commute. \square

1.1.2. Differentially finitely generated differential field extensions

Let (K, Δ) be a characteristic 0 Δ -field, and $K \subset L$ a Δ -field extension. Given a set S of elements of L we denote by $K\langle S \rangle$ to the Δ -field spanned K and the elements of S . A classical differential field extension $K \subset L$ with derivations set Δ is said to be Δ -finitely generated if there are elements $\{a_1, \dots, a_m\}$ such that $L = K\langle a_1, \dots, a_m \rangle_\Delta$. The following result, due to E. Kolchin has deep implications.

Theorem 1.8 *Given a classical differential field extension $K \subset L$ of characteristic 0, Δ -finitely generated over K then any intermediate Δ -field extension $K \subset L' \subset L$ is also Δ -finitely generated.*

PROOF. See [Kol73] p.112, Proposition 14. \square

It is clear that it applies to the non-classical case whenever it can be reduced to the classical, as in Theorem 1.7.

1.2. Principal bundles and groupoids

A principal bundle $\pi : P \rightarrow B$ with structure group G , consist of the following data: Two algebraic varieties P, B and an algebraic group G such that:

1. G acts on P on the right.
2. The map π is smooth and onto.
3. The map $m : P \times G \rightarrow P \times_B P$ such that $(x, g) \mapsto (x, x.g)$ is an isomorphism.

Example 1.9 Let G be the Lie group $\mathrm{GL}_n(\mathbb{C})$, the set of all linear automorphisms $\mathbb{C}^n \rightarrow \mathbb{C}^n$. Let M be a smooth variety of dimension m . For a point $x \in M$ define the set of 1-frames over x as the collection of all linear isomorphisms $\mathbb{C}^m \rightarrow T_x M$. An element of g the group $G = \mathrm{GL}_n(\mathbb{C})$ acts on a 1-frame F by composition from the right or source *i.e.* $Fg = F \circ g$. The collection of all 1-frames is called the bundle of 1-frames, denoted as $R_1 M$. As it is shown

in Subsection 2.2.2 it admit an smooth algebraic variety structure such that $R_1M \rightarrow M$ is a principal bundle. \square

If U is an open subset of B then $\pi^{-1}(U) \rightarrow U$ is also a principal bundle with structural group G . The difference with the classical definition of smooth principal bundles in differential geometry is that those are locally trivial, but in the algebraic context such requirement might not be satisfied, and instead a different notion is needed: local isotriviality. The bundle π is said to be *locally isotrivial* if for all $b \in B$ there exist a Zariski open set $U \subset B$ containing b and a non ramified covering $V \rightarrow U$ such that $P \times_U V \rightarrow V$ is isomorphic to the trivial bundle $G \times V \rightarrow V$.

Proposition 1.10 *Let $\pi : P \rightarrow B$ be a morphism of varieties. Assume that an algebraic group G acts on P in a way that π is G -invariant. Then $P \rightarrow B$ is a principal bundle if and only if it is locally isotrivial.*

PROOF. See [Ser58]. \square

An important consequence of the previous proposition is the following:

Corollary 1.11 *Let $P \rightarrow B$ be a principal bundle with structural group G . Then there exist an affine open set $B_0 \subset B$ such that $\pi^{-1}B_0$ is affine.*

PROOF. See [Dav16] p.18 \square

Let $\pi : P \rightarrow B$ be a principal bundle with structure group G , let M be a variety on which G acts on the left: $\rho : G \times M \rightarrow M$. The group G acts on $P \times M$ on the right *diagonally* by setting

$$(p, m).g = (pg, g^{-1}m)$$

Definition 1.12 The *associated bundle* of P with fiber M over B is the quotient space $P[\rho] = (P \times M)/G$, under the diagonal action. The projection $\pi_B : P[\rho] \rightarrow B$ is given by $\pi_B(pg, g^{-1}m) = \pi(p)$. When the fiber M happens to be a vector space, then $P[\rho]$ is called the *associated vector bundle* of P . When there are no doubts about the action, it is customary to write $P[M]$ instead of $P[\rho]$.

Example 1.13 The tangent bundle TM can be identified to $R_1M[\mathbb{C}^m]$. This identification is attained as follows: given a frame $F \in R_1M$ and a vector $v \in \mathbb{C}^m$ we can get a tangent vector at the point in which the frame sits, $F(v)$. To see this assignation is well defined, observe the pair (F, v) under diagonal action of $GL_m(\mathbb{C})$ goes to the pair $(F \circ A, A^{-1}v)$ for $A \in GL_m(\mathbb{C})$. This last pair produces the same tangent vector because $F \circ A \circ A^{-1} \circ v = F(v)$. In this way we obtain a bundle isomorphism $R_1M[\mathbb{C}^m] \rightarrow TM$.

Example 1.14 Let $\text{Gr}(d, m)$ denote the set of all d -dimensional linear subspaces of the vector space \mathbb{C}^m . The *Grassmann bundle* of d -dimensional linear subspaces is the associated bundle $\text{Gr}(d, TM) := R_1 M[\text{Gr}(d, m)]$. Observe that sections of the Grassmann bundle coincide with smooth distributions on the tangent space.

Definition 1.15 Let $P \rightarrow G$ be a G -principal bundle, and $H \subset G$ a subgroup. Let $P' \subset P$ be an H -invariant algebraic subvariety. We say that P' is a reduction of P from G to H if:

1. $P' \rightarrow B$ is surjective.
2. $P' \times H \rightarrow P' \times_B P'$ is an isomorphism.

Note that $P' \rightarrow B$ is a H -principal bundle.

It is possible to talk about reduction or extension of the structure group in terms of associated bundles. Let $H \subset G$ be an algebraic subgroup, and suppose $P \rightarrow B$ is an H -principal bundle. Take $M = G$ in the associated bundle construction. Observe that H acts naturally on the left of M , therefore $P[G] = P \times G/H$ is an H -bundle. The group G acts on the right on itself, thus it acts also on $P \times G$ by composition on the second factor. This action commutes with the diagonal action of H and induces an action of G on $P[G]$, turning it into a G -bundle.

Definition 1.16 Let $P \rightarrow B$ be a H -principal bundle with H a subgroup of G . We say the G -bundle $P[G]$ is an extension of P with group H to group G .

Note that $P \subset P[G]$ is a reduction of $P[G]$ from the group G to the subgroup H . This mechanism works the other way. If $H' \subset H$ is a subgroup, and $P' \subset P$ is a reduction of P to H' then

$$P'[H] \rightarrow P, \quad (p, h) \rightarrow ph$$

is an isomorphism of H -principal bundles.

Example 1.17 Another particular case of associated bundle is where M is an homogeneous space G/H . In such case we have a projection,

$$P \rightarrow P[G/H],$$

where the pre-images are the H -orbits, therefore $P[G/H] \simeq P/H$. In particular, if $H \triangleleft G$ then P/H is a G/H -principal bundle.

Observe that an associated bundle needs not to be principal.

1.2.1. Groupoids

A groupoid consist of two varieties \mathcal{G} and B called the groupoid and the base respectively, equipped with the following morphisms:

- Source $s : \mathcal{G} \rightarrow B$ and target $t : \mathcal{G} \rightarrow B$.
- Identity $e : B \rightarrow \mathcal{G}$
- Partial composition $m : \mathcal{G} \times_{sMt} \mathcal{G} \rightarrow \mathcal{G}$ where the product is fibered over s and t .
- Inverse $\text{inv} : \mathcal{G} \rightarrow \mathcal{G}$

Partial composition $m(g, h)$ is usually written simply as gh . These maps are required to satisfy, for $f, g, h \in \mathcal{G}$ with $s(g) = t(h)$ and $s(f) = t(g)$, the identities listed below:

- $s(gh) = s(h), \quad t(gh) = t(g)$
- $f(gh) = (fg)h$
- $e(t(g))g = g = ge(s(g))$
- $s(\text{inv}(g)) = t(g), \quad t(\text{inv}(g)) = s(g),$
- $\text{inv}(g)g = e(s(g)), \quad g\text{inv}(g) = e(t(g))$

Example 1.18 Given a variety M , consider the product $M^2 = M \times M$, equipped with the following morphisms $pr_1, pr_2 : M^2 \rightarrow M$ of projection in the first and second component respectively. Identity map is given by the diagonal inclusion $M \rightarrow M^2$. Partial composition

$$M^2 \times_{sMt} M^2 \rightarrow M^2$$

is determined by the rule $(y, z) \times (x, y) \mapsto (x, z)$ and inversion $\text{inv} : M^2 \rightarrow M^2$ is such that $(x, y) \mapsto (y, x)$. With these maps M^2 is referred to as the *pair groupoid* with base M , see [MM03] p.113. Key to note in this example, the subgroupoids of M^2 are the equivalence relations.

Definition 1.19 A *morphism of groupoids* \mathcal{G} and \mathcal{G}' with base B and B' respectively is a pair of maps $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ and $\phi_0 : B \rightarrow B'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\phi} & \mathcal{G}' \\ s/t \downarrow & & \downarrow s/t \\ B & \xrightarrow{\phi_0} & B' \end{array}$$

and such that $\phi(gh) = \phi(g)\phi(h)$. Note that commutation of diagram guarantees this last formula makes sense.

Definition 1.20 Let \mathcal{G} be a groupoid with base B . A subgroupoid of \mathcal{G} is a groupoid \mathcal{H} with base B' together with injective immersions $i : \mathcal{H} \rightarrow \mathcal{G}$ and $i_0 : B' \rightarrow B$ such that (i, i_0) is a morphism of groupoids. See [Mac05] p.13.

Definition 1.21 The *diagonal part* of \mathcal{G} is the restriction of the groupoid \mathcal{G} to the diagonal of B :

$$\mathcal{G}^{\text{diag}} = \{g \in \mathcal{G} : s(g) = t(g)\}.$$

It is an algebraic group bundle $s : \mathcal{G}^{\text{diag}} \rightarrow B$.

1.2.2. Groupoid of gauge isomorphisms

Definition 1.22 Let $P \rightarrow B$ be a principal bundle with structural group G . Denote by $\text{Iso}P$ the set of all equivariant bijections between one fiber of P to another (possibly the same). In other words, for every pair $(b, b') \in B^2$ all equivariant bijections $P_b \rightarrow P_{b'}$.

The set $\text{Iso}P$ has the structure of a groupoid with the following mappings

- Source s and target t which assign to an equivariant map the point of the base over which the domain fiber sits, and the base point that the codomain projects to, respectively.
- The identity map assigns to every base point the corresponding identity transformation of the fiber.
- Partial composition is allowed whenever the domain of one map coincides with the codomain of the other.
- The inversion map takes a bijection $P_b \rightarrow P_{b'}$ and gives its inverse $P_{b'} \rightarrow P_b$.

$\text{Iso}P$ is an algebraic groupoid acting on B . This groupoid can be described as an associated bundle,

$$\text{Iso}(P) \simeq (P \times P)/G = P[P].$$

The class $[p, q]$ of a pair (p, q) is identified with the unique G -equivariant map $\phi_{p,q} : P_{\pi(p)} \rightarrow P_{\pi(q)}$ satisfying $\phi_{p,q}(p) = q$. It is straightforward to see that $\phi_{p,q}(pg) = qg$ is a well defined equivariant mapping.

Definition 1.23 By $\text{Gau}(P)$ we mean the group bundle of *gauge automorphisms* of P ; the diagonal part of $\text{Iso}(P)$. Another common notation for $\text{Gau}P$ is $\text{Iso}_{s=t}P$.

$\text{Gau}P$ is an algebraic group bundle over B . Its elements are G -equivariant maps from a fiber of P to itself. The group bundle $\text{Gau}(P)$ can also be constructed as the associated bundle, the balanced construction $P[\text{Adj}] = P \times_{\text{Adj}} G$.

Lemma 1.24 *Let P be a principal bundle with structural group G . The groupoid $\text{Iso}P$ can be given a variety structure in a way that the map $\Phi : P^2 \rightarrow \text{Iso}P$ such that $\Phi(x, y) = \phi_{xy}$ is an algebraic morphism of groupoids. The map Φ descends to the quotient by diagonal action, and is going to be denoted also by Φ . It gives a bijection between equivalence relations in P^2 that are invariant under the diagonal action of G and the set of subgroupoids of $\text{Iso}P$.*

PROOF. A detailed one appears in [Dav16] p. 22. \square

Lemma 1.25 *Let $P \rightarrow B$ be a principal bundle with structure group G .*

- *The evaluation map $\text{Iso}P \times_{sB} P \rightarrow P$ given by $(\phi_{xy}, z) \mapsto \phi_{xy}(z)$ is algebraic for a any chosen point $x \in P$.*
- *The variety $\text{Gau}P$ is an algebraic group bundle over B , each fiber being isomorphic to the group G ; such isomorphisms depend on the choice of a point in the fiber.*
- *The bundle of groups $\text{Gau}P = (\text{Iso}P)^{\text{diag}}$ acts on $\text{Iso}P$ over B by composition from the source. The groupoid $\text{Iso}P$ is a bundle of principal bundles with projection over B given by the target. This means each fiber is a principal bundle.*

Each of the fibers of $\text{Iso}P$ is isomorphic to P , again the morphism depending on the choice of a point in P .

- *To every subgroupoid of $\text{Iso}P$ that dominates the base corresponds a family of principal subbundles of P , parameterized by $p \in P$, and every pair in a family are related through translation by an element of the group. This correspondence depends on the choice of a point in P .*

Every pair of subbundles in a family have structure groups that are conjugated.

PROOF. We prove the last item. Denote by $\text{Subgrd}(\text{Iso}(P))$ the set of subgroupoids of $\text{Iso}(P)$ that dominate the base B , and $\text{Subbnd}(P)$ the collection of principal subbundles of π . Lets say that $L, L' \in \text{Subbnd}(P)$ are related if there exists $g \in G$ such that $L'g = L$. It is an equivalence relation, call it \sim . For any choice of $x \in P$, define a map $\Theta_x : \text{Subgrd}(\text{Iso}(P)) \rightarrow \text{Subbnd}(P)/\sim$ by $\mathcal{G} \mapsto [\mathcal{G}.x]$ Where $[\mathcal{G}.x]$ denotes the class of

$$\mathcal{G}.x = \{\phi(x) \mid \phi \in \mathcal{G}, s(\phi) = \pi(x)\}$$

Define also the map $\Upsilon : \text{Subbnd}(P)/\sim \rightarrow \text{Subgrd}(\text{Iso}(P))$ by $[L] \mapsto \text{Iso}(L)$.

First we observe that given $x, y \in P$ there exists $g \in G$ such that $L = \Theta_x(\mathcal{G}) = \Theta_y(\mathcal{G})g = L'g$. Indeed, assume we have $\pi(x) = b$ and $\pi(y) = b'$. As \mathcal{G} dominates the base, there exist $\sigma_{bb'} \in \mathcal{G}$; Define $z = \sigma(x)$. We have $\mathcal{G}x = \mathcal{G}\sigma(x) = \mathcal{G}z$. As z and y are in the same fiber there is $g \in G$ such that $z = yg$. Then

$$L'g = (\mathcal{G}y)g = \mathcal{G}(yg) = \mathcal{G}z = \mathcal{G}x = L.$$

Second, let's see that $\text{Iso}(\mathcal{G}.x) = \mathcal{G}$ no matter what the choice of $x \in P$ be. The first inclusion $\text{Iso}(\mathcal{G}.x) \subset \mathcal{G}$: For p, q in $\mathcal{G}.x$, let's see f_{pq} is in \mathcal{G} . By definition there exist f and f' both in \mathcal{G} such that $f(x) = p$ and $f'(x) = q$. Then $f' \circ f^{-1} = f_{pq}$ is in \mathcal{G} because f' and f^{-1} are.

The remaining inclusion: For $f_{wz} \in \mathcal{G}$ we want to find $f_{w'z'}$ that coincides with f_{wz} and such that $f_{w'z'} \in \text{Iso}(\mathcal{G}.x)$. Let $b = \pi(x)$ and $c = s(f)$. It suffices to find h such that $s(h) = b$, $t(h) = c$. Such h exists by the hypothesis that \mathcal{G} dominates B . Define $w' = h(x)$ and $z' = f(w')$. Observe that there exists $g \in G$ such that $w' = wg$. Then $f(w') = f(wg) = f(w)g = zg$ and $z' = f \circ h(x)$ then $z \in \mathcal{G}.x$, because $f, h \in \mathcal{G}$. We conclude that $f_{wz} = f_{w'z'} \in \text{Iso}(\mathcal{G}.x)$. Observe that the previous arguments works for any $x \in P$ so $\text{Iso}(\mathcal{G}.x) = \mathcal{G}$ for all $x \in P$.

Finally we see structure groups are conjugated: Let L and Lg be subbundles, with structure groups H and H' respectively. From $LH \subset L$ it follows $Lgg^{-1}Hg \subset Lg$. As H' is the structural group of Lg it follows that $gHg^{-1} \subset H'$.

Also we have for $L' = Lg$ that $L'g^{-1}gH'g^{-1} \subset L'g^{-1}$ and from here, $LgH'g^{-1} \subset L$. But as H is the structural group of L then it follows that $gH'g^{-1} \subset H$, and $H \subset g^{-1}H'g$. This proves $H' = g^{-1}Hg$. \square

Example 1.26 Let $\pi : P = \mathbb{C}^n \times \text{GL}_n(\mathbb{C}) \rightarrow \mathbb{C}^n$ be the frame bundle on \mathbb{C}^n . An equivariant map between two fibers of P is determined by two points in the base and an invertible matrix. This way $\text{Iso}P = \mathbb{C}^n \times \mathbb{C}^n \times \text{GL}_n(\mathbb{C})$ and $(\text{Iso}P)^{\text{diag}} = \mathbb{C}^n \times \text{GL}_n(\mathbb{C})$. There are two possibilities for $\mathbb{C}^n \times \text{GL}_n(\mathbb{C})$ to act on $\text{Iso}P$, by pre- or post- composition:

$$\begin{aligned} \left(\mathbb{C}^n \times \mathbb{C}^n \times \text{GL}_n(\mathbb{C}) \right) \times_{s, \pi} \left(\mathbb{C}^n \times \text{GL}_n(\mathbb{C}) \right) &\longrightarrow \mathbb{C}^n \times \mathbb{C}^n \times \text{GL}_n(\mathbb{C}) \\ (a, b, C) \times (a, A) &\longmapsto (a, b, CA) \end{aligned}$$

$$\begin{aligned} \left(\mathbb{C}^n \times \text{GL}_n(\mathbb{C}) \right) \times_{\pi, t} \left(\mathbb{C}^n \times \mathbb{C}^n \times \text{GL}_n(\mathbb{C}) \right) &\longrightarrow \mathbb{C}^n \times \mathbb{C}^n \times \text{GL}_n(\mathbb{C}) \\ (a, A) \times (b, a, C) &\longmapsto (b, a, AC) \end{aligned}$$

Observe, for example, in the second case, the domains and codomains of $C : P_b \rightarrow P_a$, and $A : P_a \rightarrow P_a$.

1.2.3. Lie algebroid of a Lie groupoid

Let \mathcal{G} be a Lie groupoid with base B . There is an action of \mathcal{G} on \mathcal{G} itself as follows: For $h \in \mathcal{G}$, we have the right translation by h , $R_h : s^{-1}(t(h)) \rightarrow s^{-1}(s(h))$, given by composition $R_h g = gh$. Observe it is defined only for those elements g such that $s(g) = t(h)$. Then we obtain an action of \mathcal{G} on $T\mathcal{G}$: for $\xi \in T_g \mathcal{G}$, define $\xi h = dR_h(\xi)$. This action restricts naturally to $\Gamma(\ker ds)$, which is a Lie subalgebra of $\Gamma(T\mathcal{G})$.

Definition 1.27 A vector field $X \in \Gamma(\ker ds)$ is said to be \mathcal{G} -right invariant if $X_{gh} = X_g h$ whenever the action is defined.

Let $\mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ be the Lie subalgebra of $\Gamma(\ker ds)$ consisting of right invariant vector fields. Any $X \in \mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ is projectable along t to B . The crucial observation is that $X \in \mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ is determined by its restriction to the set of units $e(B)$, because $X_g = X_{e(t(g))}g$. Let $\mathfrak{g} = e^*(\ker ds)$. We have $\Gamma(\mathfrak{g}) \cong \mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ as vector spaces. Each $X \in \Gamma(\mathfrak{g})$ can be extended uniquely to a \mathcal{G} -invariant field \tilde{X} and each $Y \in \mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ restricts uniquely to a section of \mathfrak{g} over B .

Then as $\mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ is a Lie algebra, there is a structure of Lie algebra on $\Gamma(\mathfrak{g})$ such that the isomorphism $\Gamma(\mathfrak{g}) \cong \mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ is also a Lie algebra isomorphism.

Definition 1.28 The *Lie algebroid* of the groupoid \mathcal{G} is $\mathfrak{g} = \text{Lie}_R(\mathcal{G}) = e^*(\ker ds)$. It comes with an *anchor map* $\text{an} : \mathfrak{g} \rightarrow TB$ that is the restriction of $dt : T\mathcal{G} \rightarrow TB$ to $e^*(\ker ds)$ seen as a subbundle of $\ker ds$.

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \ker ds \\ \downarrow & & \downarrow \\ B & \xrightarrow{e} & \mathcal{G} \end{array}$$

If we write $\mathcal{G}_{b\bullet} = s^{-1}(b)$ then for a point $b \in B$ we have $\mathfrak{g}_b = T_{e(b)}\mathcal{G}_{b\bullet}$. The subindex R in $\text{Lie}_R(\mathcal{G})$ appears because we have used the right invariant fields to define the algebroid. Analogous constructions can be made using left invariant vector fields.

The relationship between the structure of Lie algebra and module of $\Gamma(\mathfrak{g})$ is expressed in the following:

$$[X, fY] = (f \circ t)[X, Y] + \text{an}(X)(f)Y$$

where $\text{an} : \Gamma(\mathfrak{g}) \rightarrow \Gamma(TB) = \mathfrak{X}(B)$ such that $\text{an}(X) = dt(\tilde{X})$.

Example 1.29 Let $\pi : P \rightarrow B$ be a principal bundle with structure group G . We are going to describe the Lie algebroid of $\text{Iso}(P)$. Let $p, q \in P$ be points such that $\pi(p) = \pi(q) = b$ and $p = qg$ for $g \in G$. Observe that $\text{Iso}(P)_{b\bullet} = (P_b \times P)/G$

We have a sequence

$$0 \longrightarrow P_b \xrightarrow{\Delta} P_b \times P \longrightarrow (P_b \times P)/G \longrightarrow 0$$

Where Δ is the diagonal inclusion $\Delta(q) = (q, q)$. The last sequence induces the following:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_q(P_b) & \longrightarrow & T_{(q,q)}(P_b \times P) & \longrightarrow & \text{Lie}_R(\text{Iso}(P))_b \longrightarrow 0 \\ & & \uparrow dR_g & & \uparrow & & \parallel \\ 0 & \longrightarrow & T_p(P_b) & \longrightarrow & T_{(p,p)}(P_b \times P) & \longrightarrow & \text{Lie}_R(\text{Iso}(P))_b \longrightarrow 0 \end{array}$$

We can decompose as follows

$$T_{(p,p)}(P_b \times P) = T_{(p,p)}(\Delta P_b) \oplus T_{(p,p)}(\{p\} \times P)$$

$$T_{(q,q)}(P_b \times P) = T_{(q,q)}(\Delta P_b) \oplus T_{(q,q)}(\{q\} \times P)$$

There is an isomorphism $T_{(p,p)}(\{p\} \times P) \cong T_p P$, and similarly for q . From this we get $\text{Lie}_R(\text{Iso}(P))_b \cong T_p P \cong T_q P$. We conclude that $\text{Lie}_R(\text{Iso}(P)) \cong (TP)/G$. Note that a G -invariant field \tilde{X} on P induces a section X of $(TP)/G$ over B :

$$\begin{array}{ccc} TP & \xrightleftharpoons{\tilde{X}} & P \\ \downarrow & & \downarrow \\ (TP)/G & \xrightarrow{\quad} & B \\ & \xleftarrow{X} & \end{array}$$

Finally $d\pi : TP \rightarrow TB$ induces the anchor map $\text{an} : (TP)/G \rightarrow TB$.

1.2.4. Distributions

Let us recall that a d -rank distribution \mathcal{D} on M is a smoothly varying family of d -dimensional linear subspaces of the tangent space of a variety, therefore $\mathcal{D} \subset TM$ is a linear subbundle of TM over M .

We say that \mathcal{D} is a d -rank rational distribution if:

- (a) \mathcal{D} is a subvariety of TM .
- (b) There is a dense open subset $M' \subset M$ such that $\mathcal{D}|_{M'}$ is Zariski dense in \mathcal{D} and a d -rank distribution on M' .

Distributions can be described locally by d linearly independent vector fields on M . These span, at each $x \in M$ a d -dimensional linear subspace $\mathcal{D}_x \subset T_x M$. Rational distributions are spanned by rational vector fields, and therefore there is a bijective correspondence between d -rank rational distribution in M and d -dimensional $\mathbb{C}(M)$ -vector subspaces of $\mathfrak{X}_{\text{rat}}(M)$. To the distribution \mathcal{D} it corresponds the $\mathbb{C}(M)$ -vector space $\mathfrak{X}_{\mathcal{D}}$ of rational sections of \mathcal{D} .

A rational function f is said to be a *rational first integral* of \mathcal{D} if $Xf = 0$ for any section of \mathcal{D} . The field $\mathbb{C}(M)^{\mathcal{D}}$ of rational first integrals of \mathcal{D} is a relatively algebraically closed subfield of $\mathbb{C}(M)$. It is the field of constants of the differential field $(\mathbb{C}(M), \mathfrak{X}_{\mathcal{D}})$.

Foliations

Let \mathcal{D} be a d -rank distribution in M . An analytic submanifold $S \subset M$ is said to be an *integral submanifold* of M if for all $x \in S$ it happens that $T_x S = \mathcal{D}_x$. Let us recall that

a d -rank distribution is *involutive* if its sheaf of sections is closed with respect to the Lie bracket.

Frobenius theorem, [Ste99] p.132, says that a distribution is involutive if and only if its integral submanifolds form a foliation of M . On the other hand, the tangent spaces to the leaves of a foliation define an involutive distribution. Therefore, by a *foliation* in M we understand an analytic involutive distribution. Its integral manifolds are called *leaves* of the foliation.

Rational involutive distributions in M are called *singular foliations* in M . A singular foliation \mathcal{F} in M gives a foliation in a maximal open subset $M' \subset M$. Its complement $\text{Sing}\mathcal{F}$ is known to be of codimension ≥ 2 , see for example [CCD13] p.40. We say that \mathcal{F} is a *regular foliation* if $\text{Sing}\mathcal{F} = \emptyset$.

There is a natural one to one correspondence between d -rank singular foliations and Lie subalgebras of $\mathfrak{X}_{\text{rat}}(M)$ of dimension d over $\mathbb{C}(M)$. To each foliation \mathcal{F} corresponds its sheaf of rational sections $\mathfrak{X}_{\mathcal{F}}$ which is the Lie algebra of rational vector fields in M tangent to the leaves of \mathcal{F} .

Invariant and associated distributions

Let $P \rightarrow B$ be a G -principal bundle and let \mathcal{D} be a rational distribution in P . We say that \mathcal{D} is *G -invariant* if for each $g \in G$ we have $R_{g*}(\mathcal{D}) = \mathcal{D}$. The distribution \mathcal{D} is said to *dominate the base* if for $x \in P$ generic it happens that $d_x\pi(\mathcal{D}_x) = T_{\pi(x)}B$ where $d_x\pi : T_xP \rightarrow T_{\pi(x)}B$.

Let \mathcal{D} be a G -invariant distribution in P . Let $\rho : G \times M \rightarrow M$ be an action of G , so that we have an associated bundle $P[\rho]$ (or $P[M]$ whenever the action is chosen canonically). Consider the distribution $0_M \in TM$ that assigns the zero vector space to every point of M . As a subset of TM it is $\{(p, 0_{T_pM}) \mid p \in M\}$. Consider the following commutative diagram,

$$\begin{array}{ccc} P \times M & \xleftarrow{\quad} & T(P \times M) \\ \downarrow \pi & & \downarrow d\pi \\ P[\rho] = (P \times M)/G & \xleftarrow{\quad} & T(P \times M/G) \end{array}$$

Definition 1.30 For \mathcal{D} a G -invariant distribution, the *induced distribution* on $P[\rho]$ is given by $\mathcal{D}[\rho] = d\pi(\mathcal{D} \times 0_M)$. If the action ρ is canonically chosen and it does not lead to confusion we write $\mathcal{D}[M]$ instead of $\mathcal{D}[\rho]$.

Example 1.31 Let us consider the canonical action of G in P by the left side given by $(g, p) = pg^{-1}$. Then, $P[P] = \text{Iso } P$. The associated distribution $\mathcal{D}[P]$ is denoted by \mathcal{D}^s . However there is some ambiguity in the definition. If we consider the first factor of $P \times P$ as a left G -space and the second factor as a G -principal bundle we obtain $d\pi(0_P \times \mathcal{D})$ that we denote \mathcal{D}^t . Note that $\text{inv}_*(\mathcal{D}^s) = \mathcal{D}^t$.

Definition 1.32 $\mathcal{D} \boxtimes \mathcal{D}$ is the distribution $\mathcal{D}^s + \mathcal{D}^t$.

Remark 1.33 The distribution $pr_1^{-1}(\mathcal{D}) \cap pr_2^{-1}(\mathcal{D})$ on $P \times P$ is invariant under the diagonal action of G . The image on the quotient $\text{Iso}(P)$ is $\mathcal{D} \boxtimes \mathcal{D}$.

Remark 1.34 Let us recall that $\text{Gau } P$ can be viewed as the associated bundle $P[\text{Adj}]$. We have indeed $\mathcal{D} \boxtimes \mathcal{D}|_{\text{Gau}(P)} = \mathcal{D}[\text{Adj}]$.

When P is $B \times \text{GL}_n$ over a curve B with connection given by solutions of $\frac{\partial Y}{\partial x} - A(x)Y$ then $\mathcal{D} \boxtimes \mathcal{D}$ on $\text{Iso}(P)$ is given by $\frac{\partial U}{\partial s} - A(s)U$ and $\frac{\partial U}{\partial t} + UA(t)$ and $\mathcal{D}[\text{Adj}]$ is given by $\frac{\partial U}{\partial x} - [A(x), U]$.

Partial connections

Definition 1.35 A rational partial flat Ehresmann connection in the direction of \mathcal{F} in P is a π -projectable, singular foliation \mathcal{D} in P of the same rank that \mathcal{F} such that $\pi_*(\mathcal{D}) = \mathcal{F}$.

From now on we will write \mathcal{F} -connection, to refer to partial flat Ehresmann connections.

A local analytic section u of P is \mathcal{D} -horizontal if $\mathcal{D}_{u(x)} \subset d_x u(T_x B)$ for each x in the domain of u . This means that each restriction of u to a leaf \mathcal{L} of \mathcal{F} is $\mathcal{D}|_{\pi^{-1}(\mathcal{L})}$ -horizontal, and vice versa. A vector field in P tangent to \mathcal{D} is also called \mathcal{D} -horizontal. Given a local analytic vector field X tangent to \mathcal{F} in B , there is a unique \mathcal{D} -horizontal vector field \tilde{X} in P such that $\pi_*(\tilde{X}) = X$. This \tilde{X} is called the \mathcal{D} -horizontal lift of X . It is clear that if X is a rational vector field then \tilde{X} is also a rational vector field, moreover, if X is regular in $B' \subset B$ and \mathcal{D} is regular in $\pi^{-1}(B')$ then \tilde{X} is regular in $\pi^{-1}(B')$. The \mathcal{D} -horizontal lifting is compatible with the Lie bracket $\widetilde{[X, Y]} = [\tilde{X}, \tilde{Y}]$.

1.3. Galois Correspondence

1.3.1. Rational equivalence relations

Let P be a complex irreducible algebraic affine smooth variety. We say that a Zariski-closed subset $R \subset P \times P$ is a *rational equivalence relation* if:

1. there is an open subset P' such that $R|_{P'} := R \cap (P' \times P')$ is a equivalence relation in P' ; and it satisfies any of the following equivalent conditions:
2. for any open subset U of P , R is the Zariski closure of $R|_U$;
3. irreducible components of R dominate P by projections.

Remark 1.36 Meromorphic equivalence relations, and generic quotients by them, have been already studied in [Gra86]. Generic quotients in a broader sense have been studied in SGA3

[DG70]. Galois correspondences shown here (Propositions 1.47, 1.53) can be seen as a direct consequence of results shown in SGA3 (Théorème 8.1).

Remark 1.37 Let us recall that the diagonal $\Delta_P \subset P \times P$ is included in any rational equivalence relation, and it dominates P by projections. Therefore, if $R \subset P \times P$ is an irreducible Zariski-closed subset then it is a rational equivalence relation if and only if it satisfies condition (1).

The following result is a particular case of [Mal01] Théorème 1.1.

Theorem 1.38 *Let R be a rational equivalence relation in P . There is an open subset $P' \subset P$ such that $R|_{P'}$ is a smooth equivalence relation in P' .*

Example 1.39 Let P be \mathbb{C}^2 with coordinates x, y ; we consider coordinates x_1, y_1, x_2, y_2 in $P \times P$. Then the closed subset R defined by the equation

$$x_1 y_2 - x_2 y_1 = 0$$

is rational equivalence relation in P .

Rational equivalence relations are reflexive and symmetric. They are not in general transitive (as relations), but generically transitive, *i.e.* they become transitive after restriction to a dense open subset. For $p \in P$ let us denote $[p]_R$ its *class*, the set of elements that are related by R to p , *i.e.* $\pi_1(\pi_2^{-1}(\{p\}))$. It is a Zariski closed subset of P . For generic $p \in P$ we have $\dim([p]_R) = \dim(R) - \dim(P)$. Thus, we say that a rational equivalence relation is *finite* if its dimension equals the dimension of P ; it implies that generic equivalence classes are finite.

Irreducible rational equivalence relations

Let us proceed to the description of all irreducible rational equivalence relations in P . Let us consider \mathcal{F} a singular foliation in P , and let $P' \subset P$ be the complement of $\text{sing}(\mathcal{F})$. We say that \mathcal{F} is a *foliation with algebraic leaves* if for all $p \in P'$ the leaf of \mathcal{F} that passes through p is open in its Zariski closure. A result of Gómez-Mont (Theorem 3 in [GM89]) ensures that there is an algebraic variety V and a rational dominant map $f: P \dashrightarrow V$ such that the closure of a generic f -fibre is the closure of a leaf of \mathcal{F} .

The foliation \mathcal{F} induces an rational equivalence relation in P ,

$$R_{\mathcal{F}} = \overline{\{(p, q) \in \text{dom}(f)^2 \mid f(p) = f(q)\}}.$$

where $\text{dom}(f)$ denotes the domain of definition of f . We say that the rational equivalence relation $R_{\mathcal{F}}$ is the relation “to be on the same leaf of \mathcal{F} ”. For generic $p \in P$ the class $[p]_{R_{\mathcal{F}}}$ is the Zariski closure of the leaf passing through p .

Lemma 1.40 *Let R be a rational irreducible equivalence relation in P . Then, there is a singular foliation \mathcal{F} with algebraic leaves in P such that, R is the relation $R_{\mathcal{F}}$ “to be on the same leaf of \mathcal{F} .”*

PROOF. Let us consider R an irreducible rational equivalence relation in P . For each $p \in P$ we define $\mathcal{F}_p = T_p([p]_R)$. This \mathcal{F} is a singular foliation of rank $\dim(R) - \dim(P)$. For generic $p \in P$ the leaf of \mathcal{F} that passes through p is the regular part of the irreducible component of $[p]_R$ that contains p . Then \mathcal{F} is a foliation with algebraic leaves. We consider now an algebraic variety V of dimension $\text{codim}(R)$ and a rational dominant map $f: P \dashrightarrow V$ such that for generic p in P the leaf that passes through p is Zariski dense in $\overline{f^{-1}\{f(p)\}}$. By definition $R_{\mathcal{F}}$ is a rational equivalence relation of the same dimension of R , and contained in R . From the irreducibility of R we conclude $R = R_{\mathcal{F}}$. \square

Remark 1.41 In the open subset in which the foliation \mathcal{F} is regular, leaves of \mathcal{F} are irreducible. A generic equivalence class is the Zariski closure of a generic leaf of \mathcal{F} thus it is irreducible.

In the previous discussion, the algebraic variety V is defined up to birational equivalence. The field $f^*(\mathbb{C}(V)) = \mathbb{C}(P)^{\mathcal{F}}$ of rational first integrals of \mathcal{F} allows us to recover the rational equivalence relation.

$$R_{\mathcal{F}} = \overline{\{(p, q) \in P \times P \mid \forall h \in \mathbb{C}(P)^{\mathcal{F}} \text{ if } p, q \text{ are in the domain of } h \text{ then } h(p) = h(q)\}}$$

Let us recall that a subfield \mathbb{K} of $\mathbb{C}(P)$ is *relatively algebraically closed* if all elements of $\mathbb{C}(P)$ that are algebraic over \mathbb{K} are in \mathbb{K} . It is also well known that any function which is algebraic over $\mathbb{C}(P)^{\mathcal{F}}$ is also a first integral of \mathcal{F} . Thus, $\mathbb{C}(P)^{\mathcal{F}}$ is relatively algebraically closed in $\mathbb{C}(P)$. Given a rational equivalence relation R , we can always consider its field of first integrals,

$$\mathbb{C}(P)^R = \{f \in \mathbb{C}(P) \mid \forall (p, q) \in R \text{ if } p, q \text{ are in the domain of } f \text{ then } f(p) = f(q)\}.$$

Reciprocally, from any relatively algebraically closed subfield $\mathbb{K} \subset \mathbb{C}(P)$ containing \mathbb{C} we can recover a unique irreducible rational equivalence relation R having $\mathbb{C}(P)^R = \mathbb{K}$. All this can be summarized in the following lemma.

Lemma 1.42 *The following maps are bijective:*

- (a) *The assignation $\mathcal{F} \rightsquigarrow R_{\mathcal{F}}$ that sends any singular foliation \mathcal{F} with algebraic leaves to the irreducible rational equivalence relation “to be on the same leaf of \mathcal{F} ”.*
- (b) *The assignation $R \rightsquigarrow \mathbb{C}(P)^R$ that sends any irreducible rational equivalence relation to its field of rational invariants.*

- (c) The assignation that sends any relatively algebraically closed subfield $\mathbb{K} \subset \mathbb{C}(P)$ to the foliation $\mathcal{F} = \ker(d\mathbb{K})$.

Finite rational equivalence relations

In this case $\dim R = \dim P$ and the projection $\pi_1: R \rightarrow P$ is a dominant map. Thus, there is an open subset P' such that $\pi_1: R|_{P'} \rightarrow P'$ is a covering map. Note that, in order to construct this open set P' we may remove not only the ramification but its orbit by the equivalence relation. Let us distinguish between two cases. First, let us assume $\pi_1|_{P'}$ is a trivial covering. We have the following result.

Lemma 1.43 *Let us assume that there is an open subset $P' \subset P$ such that $\pi_1: R|_{P'} \rightarrow P'$ is a trivial covering of order d . Then, there is a free action of a finite group of order d , $G \subset \text{Aut}(P')$ such that:*

$$R = \overline{\{(p, g(p)) \mid p \in P', g \in G\}}.$$

PROOF. Let $R|_{P'} = R'_0 \cup R'_1 \cup \dots \cup R'_{d-1}$ be the decomposition of R in irreducible components. For each $i = 0, \dots, d-1$ we define $g_i = \pi_2 \circ \pi_1|_{R'_i}^{-1}$. By the reflexive and transitive properties of R we have that $\{g_0, \dots, g_{d-1}\}$ is a group of regular automorphisms of P' . By definition of the g_i , a pair (p, q) is in $R_i|_{P'}$ if and only if $q = g_i(p)$. Thus,

$$R|_{P'} = \{(p, g_i(p)) \mid p \in P', i = 0, 1, \dots, d-1\}.$$

□

The quotient by free actions of finite groups is well known (see, for instance [DG70]) and then there is a rational quotient map $f: P \dashrightarrow P'/G$ of degree d . Note that the finite group G acts on $\mathbb{C}(P)$, and that this action fixes $f^*(\mathbb{C}(P/G))$ which is identified with $\mathbb{C}(P)^G$. We have the following:

Lemma 1.44 *The assignation $R \rightsquigarrow \mathbb{C}(P)^R$ establishes a bijective correspondence between finite rational equivalence equations that are trivial coverings of open subsets of P , and subfields \mathbb{K} of $\mathbb{C}(P)$ such that $\mathbb{K} \subset \mathbb{C}(P)$ is an algebraic Galois extension.*

PROOF. We need only to check that starting with a Galois subfield $\mathbb{K} \subset \mathbb{C}(P)$ we recover a trivial covering. Let G be the group of automorphisms of $\mathbb{C}(P)$ fixing \mathbb{K} . They are birational automorphisms of P . We may find an open subset P' invariant by those automorphisms. It is clear that this allows us to recover a trivial rational equivalence relation in P . □

Finally let us consider the second case in which $\pi_1: R|_{P'} \rightarrow P'$ is a covering, but not trivial. In such case there is finite covering $\rho: P'' \rightarrow P'$ that trivializes π_1 (and π_2). We define $R'' \subset P'' \times P''$ the pullback of the equivalence relation $R|_{P'}$ to P'' . Now, R'' is a trivial cover of P'' by π_1 or π_2 , and thus there is a group of automorphisms of P'' inducing R'' . We have proven the following.

Lemma 1.45 *Let R be a finite rational equivalence relation in P . There is an open subset P' , a finite covering $\rho: P'' \rightarrow P'$ and a finite group G of regular automorphisms of P'' such that:*

$$R = \overline{\{(\rho(p), \rho(g(p))) \mid p \in P'' \text{ } g \in G\}}.$$

In this case, we have that the generic quotient P/R coincides with the quotient P''/G which is, up to birational equivalence, determined by its field of rational functions. We also have:

Lemma 1.46 *The assignation $R \rightsquigarrow \mathbb{C}(P)^R$ establishes a bijective correspondence between finite rational equivalence relations in P , and subfields \mathbb{K} of $\mathbb{C}(P)$ such that $\mathbb{K} \subset \mathbb{C}(P)$ is an algebraic extension.*

General rational equivalence relations

Finally, given a non-irreducible rational equivalence relation R , we may proceed to construct the quotient in two steps. First, we consider R^0 the irreducible component containing the diagonal Δ , see [Gra86] p.133. In this case, the quotient is given by Lemma 1.40. Then, we project the rational equivalence relation to the quotient, obtaining a finite rational equivalence relation. Then we apply Lemma 1.46. Finally we obtain that rational equivalence relations are determined by their fields of invariants, and vice versa. Summarizing:

Proposition 1.47 (Galois corresp. for rational equivalence relations) *The map,*

$$\begin{aligned} \{\text{rational equivalence relations in } P\} &\rightarrow \{\text{subfields of } \mathbb{C}(P) \text{ containing } \mathbb{C}\} \\ R &\rightarrow \mathbb{C}(P)^R \end{aligned}$$

is a bijective correspondence, and anti-isomorphism of lattices.

Remark 1.48 In the correspondence given by Proposition 1.47:

1. irreducible rational equivalence relations are given by rational foliations with algebraic leaves and correspond to relatively algebraically closed subfields of $\mathbb{C}(P)$;
2. finite trivial rational equivalence relations correspond to actions of finite groups on open subsets of P .

1.3.2. Rational groupoids

Let us fix B a complex irreducible smooth affine algebraic variety. By an *algebraic groupoid* acting on B we mean a smooth algebraic complex Lie groupoid $(\alpha, \beta): \mathcal{G} \rightarrow B \times B$.

Definition 1.49 A Zariski-closed subset $\mathcal{H} \subset \mathcal{G}$ is called a *rational subgroupoid* if it satisfies:

1. there is an open subset B' of B such that $\mathcal{H}|_{B'}$ is a Lie subgroupoid of $\mathcal{G}|_{B'}$;
- and it satisfies any of the following equivalent conditions:
2. for any open subset B' of B , $\mathcal{H} = \overline{\mathcal{H}|_{B'}}$;
 3. irreducible components of \mathcal{H} dominate B by the source (or target) map.

Remark 1.50 If \mathcal{H} is irreducible, it automatically satisfies (3); it contains the identity.

It is clear that the notion of rational subgroupoid generalizes that of rational equivalence relations, the latter being the rational subgroupoids of $B \times B$.

Let $\pi_{\mathcal{G}}: \mathcal{G} \rightarrow B$ be an *algebraic group bundle*. The fibers of $\pi_{\mathcal{G}}$ are complex algebraic groups where the group operation and inversion are defined as morphisms over B . There is a natural dictionary between group bundles over open subsets of B and algebraic groups defined over $\mathbb{C}(B)$. Thus, sections of \mathcal{G} define an algebraic group $\tilde{\mathcal{G}}$. Algebraic $\mathbb{C}(B)$ -subgroups of $\tilde{\mathcal{G}}$ are in bijective correspondence with *rational group subbundles* of \mathcal{G} . That is, Zariski closed subsets $\mathcal{H} \subset \mathcal{G}$ such that:

1. there is an open subset B' such that $\pi_{\mathcal{G}}: \mathcal{H}|_{B'} \rightarrow B'$ is an algebraic group bundle;
2. irreducible components of \mathcal{H} dominate B by $\pi_{\mathcal{G}}$.

Given a rational subgroupoid $\mathcal{H} \subset \mathcal{G}$, the intersection $\mathcal{H} \cap \mathcal{G}^{\text{diag}}$ is not a rational group bundle over B , it may contain some irreducible components that do not dominate the basis.

Example 1.51 Let us consider $B = \mathbb{C}^2$, and $\mathcal{G} = \mathbb{C}^* \times B \times B$ where the composition law is given by $(\mu, x', y', x'', y'') \cdot (\lambda, x, y, x', y') = (\lambda\mu, x, y, x'', y'')$. The equation,

$$\lambda = \frac{x' y}{y' x}$$

defines an algebraic subgroupoid outside of the curve $xy = 0$, and thus,

$$\mathcal{H} = \{(\lambda, x, y, x', y') : \lambda xy' - x'y = 0\}$$

is a rational subgroupoid of \mathcal{G} , which is in fact, irreducible. However the intersection with the diagonal group bundle,

$$\mathcal{H} \cap \mathcal{G}^{\text{diag}} = \{(\lambda, x, y) \mid (\lambda - 1)xy = 0\}$$

has three irreducible components. Two of them, of equations $x = 0$ and $y = 0$ do not dominate B ; the other one, of equation $\lambda = 1$ defines a rational group subbundle.

Hence, in order to define the diagonal part of a rational groupoid we have to restrict our attention to a suitable affine open subset. It is clear that the following definition does not depend on the choice of B' .

Definition 1.52 Let \mathcal{H} be a rational subgroupoid of \mathcal{G} . Let us consider an affine subset $B' \subset B$ such that $\mathcal{H}|_{B'}$ is an algebraic groupoid acting on B' . We define the diagonal part of \mathcal{H} ,

$$\mathcal{H}^{\text{diag}} = \overline{\mathcal{H}|_{B'}} \cap \mathcal{G}^{\text{diag}}.$$

1.3.3. Rational groupoids of gauge isomorphisms

Let G be a linear algebraic group, $\pi: P \rightarrow B$ a principal connected fiber bundle with structure group G . From now on, *we assume that B is affine and P is affine over B* ; many of our results can be stated in a more general setting.

By a *rational reduction* of $\pi: P \rightarrow B$ we mean a Zariski closed subset $L \subset P$ such that:

1. there is an open subset $B' \subset B$ such that $L|_{B'} \rightarrow B'$ is a reduction of the bundle $P|_{B'} \rightarrow B'$;
2. irreducible components of L dominate B by projection.

As before, if L is irreducible then condition (1) implies condition (2).

Given a rational subgroupoid $\mathcal{G} \subset \text{Iso}(P)$, there is a corresponding field of rational invariants,

$$\mathbb{C}(P)^{\mathcal{G}} = \left\{ f \mid f(p) = f(g(p)) \text{ for all } p \in P \text{ and } g \in \mathcal{G} \right. \\ \left. \text{whenever } p \in \text{dom}(g) \text{ and } p, g(p) \text{ are in } \text{dom}(f) \right\}.$$

We would like to characterize rational subgroupoids of $\text{Iso}(P)$ in terms of their invariants. We say that a subfield $\mathbb{K} \subset \mathbb{C}(P)$ is G -invariant if right translations in P map \mathbb{K} into itself.

Proposition 1.53 (Galois correspondence) *The map,*

$$\begin{aligned} \{\text{Rational subgroupoids of } \text{Iso}(P)\} &\rightarrow \{G\text{-invariant subfields of } \mathbb{C}(P) \text{ containing } \mathbb{C}\} \\ \mathcal{G} &\rightarrow \mathbb{C}(P)^{\mathcal{G}} \end{aligned}$$

is a bijective correspondence, and anti-isomorphism of lattices.

PROOF. The quotient map $\pi_{P \times P}: P \times P \rightarrow \text{Iso}(P)$, $(p, q) \mapsto \phi_{p,q}$ establishes a bijective correspondence between the set of subgroupoids of $\text{Iso}(P)$ and the set of G -invariant equivalence relations in P . This induces a bijective correspondence between the set of rational subgroupoids of $\text{Iso}(P)$ and G -invariant rational equivalence relations in P . Moreover, we have $\mathbb{C}(P)^{\mathcal{G}} = \mathbb{C}(P)^{\pi_{P \times P}^{-1}(\mathcal{G})}$.

The field of invariants of a G -invariant rational equivalence relation is G -invariant. Reciprocally, the rational equivalence relation corresponding to a G -invariant field, is G -invariant. Thus, by Proposition 1.47 we finish the proof. \square

Remark 1.54 Let us note that irreducible rational subgroupoids of $\text{Iso}(P)$ correspond to G -invariant relatively algebraically closed subfields of $\mathbb{C}(P)$ containing \mathbb{C} . Hence, they correspond to G -invariant singular foliations in P with algebraic leaves.

1.4. Differential Galois theory

1.4.1. Galois groupoid of a partial connection

Here we give a geometric presentation based on rational subgroupoids of the classical Galois theory for G -invariant connections. The ideas of this section can be found in [Car08], see also [Mal10b] for a non linear version. Let us consider G and $\pi: P \rightarrow B$ as in section 1.3.3. Additionally, we also consider a singular foliation \mathcal{F} in B . Note that, replacing B by a suitable affine subset, we may assume that any G -invariant rational \mathcal{F} -connection \mathcal{D} is regular.

Let \mathcal{D} be a G -invariant \mathcal{F} -connection on P . We consider the field $\mathbb{C}(P)^{\mathcal{D}}$ of rational first integrals of the foliation. This field is, by definition, G -invariant. Thus, by Proposition 1.53 it corresponds to an irreducible rational subgroupoid of $\text{Iso}(P)$. The following definition is equivalent to that given in [Dav16].

Definition 1.55 The *Galois groupoid* of \mathcal{D} is the rational subgroupoid $\text{GGal}(\mathcal{D})$ of $\text{Iso}(P)$ such that $\mathbb{C}(P)^{\text{GGal}(\mathcal{D})} = \mathbb{C}(P)^{\mathcal{D}}$. The *Galois group bundle* is the diagonal part of the Galois groupoid, $\text{Gal}(\mathcal{D}) = \text{GGal}(\mathcal{D})^{\text{diag}}$. It is a rational group subbundle of $\text{Gau}(P)$.

By definition, it is clear that $\text{GGal}(\mathcal{D})$ is irreducible in the Zariski topology. The Galois group bundle may not be irreducible.

Remark 1.56 The classical case of Picard-Vessiot theory corresponds to the case in which the foliation \mathcal{F} has no rational first integrals, *i.e.*, $\mathbb{C}(B)^{\mathcal{F}} = \mathbb{C}$. This setting has been exposed in [BSC17]. The fiber of the Galois group bundle is isomorphic to the classical Galois group. The intrinsic Galois group is the algebraic $\mathbb{C}(B)$ -group whose elements are the sections of the Galois group bundle. It has been defined algebraically, with a different approach, by Katz in [Kat82] (see also book review [Ber96] of [Mag94]).

Remark 1.57 The Galois group bundle admits also a structure of differential algebraic group. The induced associated connection $\mathcal{D}[\text{Adj}]$ is a partial group connection (see Appendix A) in $\text{Gau}(P)$. It restricts to the group subbundle $\text{Gal}(\mathcal{D})$. Thus, the pair $(\text{Gal}(\mathcal{D}), \mathcal{D}[\text{Adj}])$ can be seen as a differential algebraic group of order 1. This differential algebraic group structure was already pointed out by Pillay in [Pil04].

Remark 1.58 As the group bundle $\text{Gau}(P) \rightarrow B$ admits two equivalent descriptions, so the associated connection does. We may consider the distribution $\mathcal{D} \times \mathcal{D}$ in $P \times P$. This distribution is G -invariant under the diagonal action, thus it projects onto a distribution

$\mathcal{D} \boxtimes \mathcal{D}$ in $\text{Iso}(P)$. The intersection of this distribution with the tangent bundle $T(\text{Gau}(P))$ gives us the associated connection $\mathcal{D}[\text{Adj}]$.

Remark 1.59 The Galois groupoid can be also constructed geometrically as the smallest rational subgroupoid of $\text{Iso}(P)$ which is tangent to the distribution $\mathcal{D} \boxtimes \mathcal{D}$, see [Dav16] or [Car08].

Remark 1.60 By construction, the action $\text{Gau}(P) \times_B P \rightarrow P$ maps the distribution $\mathcal{D}[\text{Adj}] \times_B \mathcal{D}$ onto \mathcal{D} . Therefore, $\mathcal{D}[\text{Adj}]$ is the *equation of gauge symmetries of \mathcal{D}* . A gauge transformation is $\mathcal{D}[\text{Adj}]$ -horizontal if and only if it has the property of transforming \mathcal{D} -horizontal sections into \mathcal{D} -horizontal sections.

1.4.2. Connections with parameters

Let us also consider a submersion $\rho: B \rightarrow S$, so that we have commutative diagrams as follows,

$$\begin{array}{ccc} P & \xrightarrow{\pi} & B \\ & \searrow \bar{\pi} & \downarrow \rho \\ & & S \end{array} \quad \begin{array}{ccc} \mathbb{C}(P) & \xleftarrow{\pi^*} & \mathbb{C}(B) \\ & \nwarrow \bar{\pi}^* & \uparrow \rho^* \\ & & \mathbb{C}(S) \end{array}$$

we assume that the fibers of $\bar{\pi}$ are irreducible. Let \mathcal{F} be the foliation with algebraic leaves defined by $\ker(d\rho)$. By a *G-invariant connection in P with parameters in S* we mean a G -invariant \mathcal{F} -connection \mathcal{D} in P [Mal10a, Mal10b].

For each $s \in S$ we denote $B_s = \rho^{-1}(s)$, $P_s = \bar{\pi}^{-1}(s)$, and $\pi_s = \pi|_{P_s}$. Thus, $\pi_s: P_s \rightarrow B_s$ is a principal fiber bundle with structure group G . The foliation \mathcal{D} is tangent to P_s its restriction \mathcal{D}_s is a G -invariant connection in P_s .

We identify $\mathbb{C}(S)$ with its image in $\mathbb{C}(B)$ by ρ^* and $\mathbb{C}(B)$ with its image in $\mathbb{C}(P)$ by π^* so that we have a sequence of extensions, $\mathbb{C}(S) \subset \mathbb{C}(B) \subset \mathbb{C}(P)$. Note that $\mathbb{C}(B)$ is the fixed field $\mathbb{C}(P)^G$ of rational invariants by the action of G and $\mathbb{C}(S)$ is the field $\mathbb{C}(B)^{\mathcal{F}}$ of rational first integrals of \mathcal{F} .

By $\text{Iso}_S(P)$ we mean the subgroupoid of $\text{Iso}(P)$ consisting on fiber isomorphisms that respect the projection ρ .

Remark 1.61 In terms of Proposition 1.53 $\text{Iso}_S(P)$ corresponds to the subfield $\mathbb{C}(S) \subset \mathbb{C}(P)$ and to the equivalence relation $P \times_S P$. Thus, we have a bijective correspondence between irreducible rational subgroupoids of $\text{Iso}_S(P)$ and G -invariant subfields of $\mathbb{C}(P)$ containing $\mathbb{C}(S)$.

Proposition 1.62 *Let \mathcal{D} be a G -invariant connection with parameters in S . We have $\text{GGal}(\mathcal{D}) \subset \text{Iso}_S(P)$.*

PROOF. Taking into account $\mathbb{C}(S) \subset \mathbb{C}(P)^{\mathcal{D}}$ we conclude using Remark 1.61. \square

2. Jets and infinitesimally near points

2.1. Contact of order k

Let us discuss the notion of contact order of sections of a bundle $\pi: P \rightarrow B$. Local analytic sections have contact of order k at a point of B in its domain of definition if their Taylor expansions in any system of adapted coordinates coincide up to terms of degree k . This notion is, in fact, coordinate independent. It can be seen geometrically or algebraically in several intrinsic ways. Namely.

1. Sections α, β are said to have 0-order contact at $b \in B$ if $\alpha(b) = \beta(b)$. Any section $\sigma: B \rightarrow P$ induces a section $d\sigma: TB \rightarrow TP$. Let us write $T^k P$ for $T(\cdots T(P))$.

$$\begin{array}{ccccccc}
 P & & TP & & T(TP) & \cdots & T^k P \\
 \pi \downarrow \nearrow \sigma & & d\pi \downarrow \nearrow d\sigma & & \downarrow d(d\pi) & & \downarrow d^k \pi \\
 B & & TB & & T(TB) & \cdots & T^k B
 \end{array}$$

Sections α, β have contact of order $\geq k$ at b if $d\alpha$ and $d\beta$ have the same contact of order $k-1$ at b .

2. In terms of function rings and ideals, two sections $P \xrightleftharpoons[\pi]{\alpha, \beta} B$ with $\alpha(b) = \beta(b)$ induce maps

$$\mathcal{O}_{P, \alpha(b)} \xrightarrow{\alpha^*, \beta^*} \mathcal{O}_{B, b}$$

In these terms, contact order $\geq k$ means that $\alpha^* - \beta^*$ take values in \mathfrak{m}_b^{k+1} .

3. Using differential operators in B , sections α, β have contact of order $\geq k$ at b if for every function f in P and θ a differential operator in B of order $\leq k$ it happens that $(\theta(\alpha^* f - \beta^* f))(b) = 0$.
4. Using adapted coordinates, contact of order $\geq k$ at $b \in B$ is expressed as follows. Introduce the following relation on the set of sections of $P \rightarrow B$: Two sections $\sigma, \beta: B \rightarrow P$ have the same 1-jet at $b \in B$ if all their first order partial derivatives coincide at b . Sections $\sigma, \beta: B \rightarrow P$ with $\sigma(b) = \beta(b)$ will have contact of order $\leq k$ at b if all their partial derivatives up to order k at b coincide.

All these definitions are equivalent. The contact relation grasps the idea of two sections having the same Taylor series development up to order k .

Definition 2.1 A k -jet at a point $p \in B$ is an equivalence class by the relation “contact of order $\geq k$ at p ” on the space of local analytic sections of π .

The bundle of order k -jets $J_k(P/B)$ is defined as $\cup_{p \in B} (J_k(P/M))_p$ where $(J_k(P/B))_p$ is the set of k -jets at p of local analytic sections of π .

For a local analytic section α defined at p we write $j_p^k \alpha$ for the k -jet of α at p . There are projections between jet bundles of different order:

$$B \xleftarrow{\pi} P \xleftarrow{\pi_1} J_1(P/B) \xleftarrow{\quad} \cdots \xleftarrow{\pi_k} J_k(P/B) \xleftarrow{\quad} \cdots$$

The projection $\pi_k : J_k(P/B) \rightarrow J_{k-1}(P/B)$ is simply given by $\pi_k(j_p^k \sigma) = j_p^{k-1} \sigma$. They form a projective system and therefore we may consider the limit,

$$J(P/B) = \lim_k J_k(P/B).$$

Remark 2.2 Note that the local analytic maps of from B to M are the sections of the trivial bundle $B \times M \rightarrow B$. Therefore we may speak of k -jets of local analytic maps. In this sense,

$$J_k(B \rightarrow M) = J_k(B \times M/B).$$

In particular, if B and M have the same dimension we can speak of jets of local analytic biholomorphisms $J_k^*(B \rightarrow M)$.

In what follows we will explore the differential algebro-geometric structure of the jet bundles.

2.2. Infinitesimally near points

Definition 2.3 Let M be a smooth irreducible algebraic manifold of dimension n . A k -frame in M is a k -jet of a germ of biholomorphism from a neighborhood of zero in \mathbb{C}^n to a neighborhood of a point $p \in M$. Define $R_k M = J_k^*((\mathbb{C}^n, 0) \rightarrow M)$ to be set of k -frames in M . The star denotes the invertible jets.

There are natural projections $\pi_{k,r} : R_k M \rightarrow R_r M$ for $k \geq r \geq 0$. The point $\pi_{k,0}(j_0^k \psi) = \psi(0)$ is the base point of $j_0^k \psi$. There is a projective limit $RM \rightarrow M$, namely, the set of formal frames in M . In what follows we will describe the structure of RM , for a smooth irreducible affine algebraic variety M , as a affine pro-algebraic bundle over M . See Malgrange [Mal10b] or Davy [Dav16] p.44.

2.2.1. Weil bundle of infinitesimally near points

In order to see the pro-algebraic structure of \mathbf{RM} it is useful to see it as a subset of a more general construction, the Weil bundle of infinitely near points.

Weil algebras

A Weil algebra is a local, finite dimensional, commutative \mathbb{C} -algebra. For any Weil algebra there is a canonical decomposition $A = \mathbb{C} \oplus \mathfrak{m}_A$ where \mathfrak{m}_A denote the maximal ideal of A . The projection onto the first component $\omega_A : \mathbb{C} \oplus \mathfrak{m}_A \rightarrow \mathbb{C}$ is called the *rational point* of A .

Let $\mathbb{C}_{(m,k)} = \mathbb{C}[\varepsilon]/(\varepsilon)^{k+1}$ be the algebra of polynomials in m variables $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, truncated up to order k . It is straightforward to see that $\mathbb{C}_{(m,k)}$ are Weil algebras. These are the only ones we need to develop the theory that meets our purposes.

There is a natural projection from $\mathbb{C}_{(m,k)}$ to $\mathbb{C}_{(m,k-1)}$ given by the quotient map $\mathbb{C}_{(m,k)} \rightarrow \mathbb{C}_{(m,k)}/\mathfrak{m}_{(m,k)}^k = \mathbb{C}_{(m,k-1)}$. This implies we have a projective system

$$\cdots \longrightarrow \mathbb{C}_{(m,k)} \longrightarrow \mathbb{C}_{(m,k-1)} \longrightarrow \cdots \longrightarrow \mathbb{C}$$

The projective limit $\varprojlim_k \mathbb{C}_{(m,k)}$ is denoted $\mathbb{C}_{(m,\infty)}$. It is the ring $\mathbb{C}[[\varepsilon]]$ of formal power series in m variables. It is not a Weil algebra, but it is a local \mathbb{C} -algebra and it is endowed with the (ε) -adic topology.

Infinitesimally near points

Let M be an algebraic variety and A a Weil algebra. A near point of M of type A , or an A -point of M is a \mathbb{C} -scheme morphism $\mathrm{Spec}(A) \rightarrow M$. By composition with the dual of the rational point, it is clear that a near A -point has a base point in M . An A -point with $A = \mathbb{C}_{(m,k)}$ is called a (m,k) -near point or a generalized vector of order k and width m . The set of (m,k) -near points in M is denoted by $\tau_{m,k}M$.

Note that any analytic germ of map $\varphi : (\mathbb{C}^m, 0) \rightarrow M$ by truncation to order k induces a (m,k) -near point. This (m,k) -point depends on and characterizes the k -jet on φ at $0 \in \mathbb{C}^m$. Therefore we have a canonical identification $\tau_{m,k}M \simeq J_k((\mathbb{C}^m, 0) \rightarrow M)$.

From now on, let M be an affine algebraic variety. Thus, an A -point of M is given by a morphism of \mathbb{C} -algebras $\mathbb{C}[M] \rightarrow A$. By abuse of notation we will write

$$M(A) = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[M], A), \quad \tau_{m,k}M = M(\mathbb{C}_{(m,k)}).$$

For a fixed m , the canonical truncation morphisms $\mathbb{C}_{m,k} \rightarrow \mathbb{C}_{m,r}$ with $r \leq k$ induce a system of compatible projections,

$$\cdots \longrightarrow \tau_{m,k}M \longrightarrow \tau_{m,k-1}M \longrightarrow \cdots \longrightarrow M$$

and thus there is a projective limit,

$$\hat{\tau}_m M = \varprojlim_k \tau_{m,k} M \simeq \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[M], \mathbb{C}[[\varepsilon_1, \dots, \varepsilon_m]])$$

of formal generalized vectors of with m , that we identify with \mathbb{C} -algebra morphisms from $\mathbb{C}[M]$ to $\mathbb{C}_{(m,\infty)}$. The set $\hat{\tau}_m M$ is naturally identified with the set $J((\mathbb{C}^m, 0) \rightarrow M)$ of formal maps from $(\mathbb{C}^m, 0)$ to M .

In [KSM99] chapter 35 and [MnRM00] it is shown that for a smooth variety M the space of (real) near points $M(A)$ is smooth bundle over M . These results can be easily translated to the algebraic case. Since it fits our purposes, and it has a simpler presentation, we will restrict ourselves to the particular case in which M is a smooth affine algebraic variety and A is $\mathbb{C}_{(m,k)}$. Moreover, we will see that $\hat{\tau}_m M$ is a pro-algebraic affine variety endowed with a canonical differential structure.

Prolongation of functions

A natural basis of $\mathbb{C}_{(m,k)}$ is the set of monomials $\{\varepsilon^\alpha / \alpha!\}_{0 \leq |\alpha| \leq k}$. Let us denote by $\{\omega_\alpha\}_{0 \leq |\alpha| \leq k}$ the dual basis. Here α is a multi-index. The rational point of $\mathbb{C}_{(m,k)}$ is ω_0 .

Any regular $f \in \mathbb{C}[M]$ induces a function, $f^{(m,k)}: \tau_{m,k} M \rightarrow \mathbb{C}_{(m,k)}$ given by $f^{(m,k)}(p^{(m,k)}) = p^{(m,k)}(f)$. We will give to $\tau_{m,k} M$ the algebraic variety structure that makes the functions $f^{(m,k)}$ regular. We define $f_{:\alpha}$ the α -th total derivative of f to be the composition $\omega_\alpha \circ f^{(m,k)}$. The total derivatives of f up to order k are also called the *complex components* of the prolongation $f^{(m,k)}$. We have:

$$f^{(m,k)} = \sum_{0 \leq |\alpha| \leq k} f_{:\alpha} \frac{\varepsilon^\alpha}{\alpha!}.$$

This notation is justified by the following fact. Let $\varphi: (\mathbb{C}^m, 0) \rightarrow M$ be a germ of analytic map. Its k -jet $j_0^k \varphi$ is a (m, k) -near point in M with base point $\varphi(0)$. For any regular function f in M we have,

$$f_{:\alpha}(j_0^k \varphi) = \left. \frac{\partial^{|\alpha|}}{\partial \varepsilon^\alpha} \right|_{\varepsilon=0} (f \circ \varphi).$$

Affine variety structure of $\tau_{(m,r)} M$

Let $\mathbb{C}[\tau_{m,k} M]$ be $\mathbb{C}[f_{:\alpha}]_{f \in \mathbb{C}[M], 0 \leq |\alpha| \leq k}$, the \mathbb{C} -algebra of complex functions in $\tau_{m,k} M$ spanned by the α -th total derivatives of regular functions in M with $0 \leq |\alpha| \leq k$.

Definition 2.4 Given an ideal $J \subset \mathbb{C}[M]$ we call it *prolongation* to $\mathbb{C}[\tau_{m,k} M]$ to the ideal spanned by the real components of the prolongations of elements of J .

$$J^{(m,k)} = (\{P_{:\alpha} \mid P \in J, 0 \leq |\alpha| \leq k\})$$

Let us see that $\tau_{m,k}M$ is an affine variety whose ring of regular functions is $\mathbb{C}[\tau_{m,k}M]$. First, let us check that the above statement is true in the case of the affine space \mathbb{C}^n .

Proposition 2.5 $\tau_{m,k}\mathbb{C}^n = (\mathbb{C}_{(m,k)})^n$ and $\mathbb{C}[\tau_{m,k}\mathbb{C}^n] = \mathbb{C}[x_{i:\alpha}]_{1 \leq i \leq n, 0 \leq |\alpha| \leq k}$

PROOF. Let $\mathbb{C}[x_1, \dots, x_n]$ be the ring of regular functions in \mathbb{C}^n . A (m,k) -point is a \mathbb{C} -algebra morphism $p^{(m,k)}$ from $\mathbb{C}[x_1, \dots, x_n]$ to $\mathbb{C}_{(m,k)}$ and it is determined by the images $x_1(p^{(m,k)}), \dots, x_n(p^{(m,k)})$ that are arbitrary elements of $\mathbb{C}_{(m,k)}$. The set,

$$(\mathbb{C}_{(m,k)})^n = \left\{ \left(\sum_{0 \leq |\alpha| \leq k} x_{1:\alpha} \frac{\varepsilon^\alpha}{\alpha!}, \dots, \sum_{0 \leq |\alpha| \leq k} x_{n:\alpha} \frac{\varepsilon^\alpha}{\alpha!} \right) : x_{i:\alpha} \in \mathbb{C} \right\}$$

is an affine space whose coordinates are the functions $x_{i:\alpha}$. \square

Let $\Phi: M \rightarrow N$ be a regular map between affine algebraic varieties. It induces a map $\Phi^{(m,k)}: \tau_{m,k}M \rightarrow \tau_{m,k}N$ by setting $\Phi^{(m,k)}(j_0^k \varphi) = j_0^k(\Phi \circ \varphi)$. This map is compatible with the prolongation of functions, so that, we have that for $f \in \mathbb{C}[N]$, $f^{(m,k)} \circ \Phi^{(m,k)} = (f \circ \Phi)^{(m,k)}$; and therefore with the total derivatives, $f_{:\alpha}(\Phi^{(m,k)}(j_0^k \varphi)) = f_{:\alpha}(j_0^k(\Phi \circ \varphi))$.

Proposition 2.6 Let $M \subset \mathbb{C}^n$ be an smooth affine subvariety of \mathbb{C}^n , and let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be the ideal of M . Then, the induced inclusion $\tau_{m,k}M \subset (\mathbb{C}_{(m,k)})^n$ identifies $\tau_{m,k}M$ with the subvariety of $(\mathbb{C}_{(m,k)})^n$ of ideal

$$I^{(m,k)} = (\{P_{:\alpha} \mid P \in I, 0 \leq |\alpha| \leq k\}) \subset \mathbb{C}[x_{i:\alpha}]_{1 \leq i \leq n, 0 \leq |\alpha| \leq k}.$$

PROOF. Let $p^{(m,k)}: \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}_{(m,k)}$ be a (m,k) -near point in \mathbb{C}^n . The condition to be an (m,k) -near point in M is that, as a \mathbb{C} -algebra morphism, it factorizes through $\mathbb{C}[M]$, i.e., $I \subset \ker(p^{(m,k)})$. That is, that for each $P \in I$ we have $P^{(m,k)}(p^{(m,k)}) = 0$. By taking complex components of $P^{(m,k)}$ we obtain the equations of $\tau_{m,k}M$ inside $\mathbb{C}_{(m,k)}^n$. \square

Therefore, if $\Phi: M \rightarrow N$ is a regular map between affine varieties, then $\Phi^{(m,k)}: \tau_{m,k}M \rightarrow \tau_{m,k}N$ is a regular map. In particular, if $U \subset M$ is an affine open subset, then $\tau_{m,k}U$ is an affine open subset of $\tau_{m,k}M$.

Affine $\frac{\partial}{\partial \varepsilon}$ -pro-variety structure of $\hat{\tau}_m M$

By definition, $\hat{\tau}_m M$ is a pro-algebraic affine variety, and its ring of regular functions is:

$$\mathbb{C}[\hat{\tau}_m M] = \bigcup_k \mathbb{C}[\tau_{m,k}M] = \mathbb{C}[f_{:\alpha}]_{f \in \mathbb{C}[M], \alpha \in \mathbb{Z}_{\geq 0}^m}$$

This ring is naturally graded by the *order*. By setting,

$$\frac{\partial}{\partial \varepsilon_i} f_{:\alpha} = f_{:\alpha + \epsilon_i}, \quad \frac{\partial}{\partial \varepsilon_i} : \mathbb{C}[\hat{\tau}_m M] \rightarrow \mathbb{C}[\hat{\tau}_m M]$$

we obtain regular derivations of degree +1 of $\mathbb{C}[\hat{\tau}_m M]$. We consider in $\mathbb{C}[\hat{\tau}_m M]$ the classical differential ring structure spanned by the m derivations $\frac{\partial}{\partial \varepsilon_i}$. In order to distinguish this differential structure from other differential structures in the same ring we call it the $\frac{\partial}{\partial \varepsilon}$ -ring structure of $\mathbb{C}[\hat{\tau}_m M]$.

Proposition 2.7 *Let $I \subset \mathbb{C}[M]$ be a radical ideal. Then, the radical $\frac{\partial}{\partial \varepsilon}$ -ideal $\{I\} \subset \mathbb{C}[\hat{\tau}_m M]$ spanned by I is,*

$$\{I\} = \bigcup_{r=0}^{\infty} I^{(m,r)}$$

PROOF. It is clear that $\bigcup_{r=0}^{\infty} I^{(m,r)}$ is the $\frac{\partial}{\partial \varepsilon}$ -ideal spanned by I . It suffices to check that, if I is radical, then $I^{(m,r)}$ is also radical. \square

Proposition 2.8 *Let M be a smooth affine subvariety of \mathbb{C}^n with ideal $I \subset \mathbb{C}[M]$. Then $\hat{\tau}_m M$ is a smooth pro-affine subvariety of $(\mathbb{C}_{(m,\infty)})^n$ and its ideal is $\{I\}$.*

Proposition 2.9 *Let $\Phi: M \rightarrow N$ be a regular map between smooth affine varieties, and let $\hat{\Phi}^{(m)}: \hat{\tau}_m M \rightarrow \hat{\tau}_m N$ be the induced map.*

- (a) $(\hat{\Phi}^{(m)})^*: \mathbb{C}[\hat{\tau}_m N] \rightarrow \mathbb{C}[\hat{\tau}_m M]$ is a $\frac{\partial}{\partial \varepsilon}$ -ring morphism.
- (b) $(\hat{\Phi}^{(m)})^*$ is the only extension of $\Phi^*: \mathbb{C}[N] \rightarrow \mathbb{C}[M]$ compatible with the $\frac{\partial}{\partial \varepsilon}$ -ring structures.

Prolongation of vector fields to Weil near point bundles

Proposition 2.10 *Let $p^{(m,k)}$ be in $\tau_{m,k} M$. The space $T_{p^{(m,k)}} \tau_{m,k} M$ is identified with the set of derivations $\text{Der}_{p^{(m,k)}}(\mathbb{C}[M], \mathbb{C}_{(m,k)})$.*

PROOF. A tangent vector at $p^{(m,k)}$ is seen as an equivalence class of curves that passes through that point. Let $p_t^{(m,k)}$ be a smooth curve such that $p_t^{(m,k)}|_{t=0} = p^{(m,k)}$. Developing around zero we get

$$p_t^{(m,k)}(\varepsilon) = \gamma = \gamma_0 + \gamma_1 \varepsilon + o(\varepsilon^2)$$

Where

$$\gamma_0 = p^{(m,k)}; \quad \gamma_1 = \left. \frac{d}{dt} \right|_{t=0} p_t^{(m,k)}$$

As $p_t^{(m,k)}$ is a morphism for every t we have $\gamma(fg) = \gamma_0(fg) + \gamma_1(fg)\varepsilon$ and

$$\begin{aligned} \gamma(f)\gamma(g) &= (\gamma_0(f) + \gamma_1(f)\varepsilon)(\gamma_0(g) + \gamma_1(g)\varepsilon) \\ &= \gamma_0(f)\gamma_0(g) + (\gamma_1(f)\gamma_0(g) + \gamma_0(f)\gamma_1(g))\varepsilon + o(\varepsilon^2) \end{aligned}$$

From this we see $\gamma_1(fg) = (\gamma_1(f)\gamma_0(g) + \gamma_0(f)\gamma_1(g))$ so γ_1 is a derivation over γ_0 , this meaning that γ_0 gives $\mathbb{C}_{(m,k)}$ the structure of $\mathbb{C}[M]$ -module.

If $\{\partial_i\}$ is a basis for $\text{Der } \mathbb{C}[M]$ and $\{a_\alpha\}$ is a basis for $\mathbb{C}_{(m,k)}$ then $\{a_\alpha \partial_i\}$ is a basis for $\text{Der}(\mathbb{C}[M], \mathbb{C}_{(m,k)})$. This way the dimensions of $T_{p^{(m,k)}} \tau_{m,k} M$ and $\text{Der}(\mathbb{C}[M], \mathbb{C}_{(m,k)})$ coincide so they are isomorphic. \square

Let X be a regular vector field in M . We see X as a derivation of $\mathbb{C}[M]$. For each $p^{(m,k)} \in \tau_{m,k} M$ we have that the composition $p^{(m,k)} \circ X$ is a derivation of $\mathbb{C}[M]$ with values in $\mathbb{C}_{(m,k)}$, thus an element of $T_{p^{(m,k)}}(\tau_{m,k} M)$.

Definition 2.11 For a regular vector field X in M its *prolongation* $X^{(m,k)}$ to $\tau_{m,k} M$ is the regular vector field in $\tau_{m,k} M$ whose value at each $p^{(m,k)} \in \tau_{m,k} M$ is $p^{(m,k)} \circ X$.

$$X_{p^{(m,k)}}^{(m,k)} = p^{m,k} \circ X.$$

The fields $X^{(m,k)}$ can be described more explicitly as follows: if $X = \sum f_i \partial_i$ then the coefficient of $X^{(m,k)}$ in ∂_s is given at a point $p^{(m,k)}$ by

$$X_{p^{(m,k)}}^{(m,k)}(x_s) = p^{(m,k)} \circ X(x_s) = p^{(m,k)}(f_s) = f_s^{(m,k)}(p^{(m,k)})$$

so that $X^{(m,k)} = \sum f_s^{(m,k)} \partial_s$.

This prolongation can be seen geometrically as follows. Let us call $\{\sigma_t\}$ the flow of X . Each σ_t defines a local diffeomorphism $\sigma_t^{(m,k)}$ of $\tau_{m,k} M$. Then, $X^{(m,k)}$ is the vector field in $\tau_{m,k} M$ whose flow is $\{\sigma_t^{(m,k)}\}$.

Let $\phi : (\mathbb{C}^m, 0) \rightarrow M$ be a germ of analytic map. The flow σ_t prolongs to $\tau_{m,k} M$:

$$\begin{array}{ccc} M & \longleftarrow & \tau_{m,k} M \\ \sigma_t \downarrow & & \downarrow \sigma_t^{(m,k)} \\ M & \longleftarrow & \tau_{m,k} M \end{array}$$

By the formula $\sigma_t^{(m,k)}(j_0^k \phi) = j_0^k(\sigma_t \circ \phi)$ we obtain a geometric expression for the prolongation $X^{(m,k)}$:

$$X^{(m,k)}(j_0^k \phi) = \left. \frac{d}{dt} \right|_{t=0} \sigma_t^{(m,k)}(j_0^k \phi) = \left. \frac{d}{dt} \right|_{t=0} j_0^k(\sigma_t \circ \phi)$$

The vector field $X^{(m,k)}$ projects naturally to $X^{(m,k-1)}$ and there is a corresponding projective system:

$$\cdots \longrightarrow \text{Der}(\mathbb{C}[M], \mathbb{C}_{(m,k)}) \longrightarrow \text{Der}(\mathbb{C}[M], \mathbb{C}_{(m,k-1)}) \longrightarrow \cdots$$

The limit $X^{(m,\infty)}$ is an element in $\text{Der}(\mathbb{C}[M], \mathbb{C}[[\varepsilon]])$. We have that $X^{(m,\infty)}$ is a derivation in $\mathbb{C}[\hat{\tau}_m M]$.

Theorem 2.12 *The prolongation $X^{(m,\infty)}$ is the unique derivation of $\mathbb{C}[\hat{\tau}_m M]$ that commutes with all total derivative operators and such that its restriction to $\mathbb{C}[M]$ coincides with X .*

PROOF. Let us see that $X^{(m,\infty)}$ commutes with total derivatives:

$$\begin{aligned} \frac{\partial}{\partial \varepsilon_i} \circ \hat{X} F &= \frac{\partial}{\partial \varepsilon_i} \left(f_{j:\alpha} \frac{\partial}{\partial x_{j:\alpha}} F \right) \\ &= f_{j:\alpha+\epsilon_i} \frac{\partial}{\partial x_{j:\alpha}} F + f_{j:\alpha} \frac{\partial}{\partial \varepsilon_i} \frac{\partial}{\partial x_{j:\alpha}} F \\ &= f_{j:\alpha+\epsilon_i} \frac{\partial}{\partial x_{j:\alpha}} F + f_{j:\alpha} x_{k:\beta+\epsilon_i} \frac{\partial}{\partial x_{k:\beta}} \frac{\partial}{\partial x_{j:\alpha}} F \end{aligned}$$

and on the other side

$$\begin{aligned} \hat{X} \left(\frac{\partial}{\partial \varepsilon_i} F \right) &= \hat{X} \left(x_{k:\beta+\epsilon_i} \frac{\partial}{\partial x_{k:\beta}} F \right) \\ &= f_{j:\alpha} \left(\frac{\partial}{\partial x_{j:\alpha}} x_{k:\beta+\epsilon_i} \frac{\partial}{\partial x_{k:\beta}} F + x_{k:\beta+\epsilon_i} \frac{\partial}{\partial x_{j:\alpha}} \frac{\partial}{\partial x_{k:\beta}} F \right) \end{aligned}$$

Observe that $\frac{\partial}{\partial x_{j:\alpha}} x_{k:\beta+\epsilon_i}$ is 1 for $j = k$ and $\alpha = \beta + 1$ and zero otherwise. Then by an appropriate change of summation multi-index we get the desired result. \square

Finally let \hat{Y} be another vector field that coincides with X in $\mathbb{C}[M]$ and commutes with total derivatives. Take $F \in \mathbb{C}[\hat{\tau}_m M]$, then F is a function in some $\mathbb{C}[\tau_{m,k} M]$. For $F = \left(\frac{\partial}{\partial \varepsilon} \right)^\alpha f$ for $f \in \mathbb{C}[M]$ then

$$\hat{Y} F = \hat{Y} \left(\frac{\partial}{\partial \varepsilon} \right)^\alpha f = \left(\frac{\partial}{\partial \varepsilon} \right)^\alpha Y f = \left(\frac{\partial}{\partial \varepsilon} \right)^\alpha X f = \hat{X} \left(\frac{\partial}{\partial \varepsilon} \right)^\alpha f = \hat{X} F$$

As the total derivatives of functions in $\mathbb{C}[M]$ span the ring $\mathbb{C}[\hat{\tau}_m M]$ the proof is complete. \square

Action of $\text{Aut}_k(\mathbb{C}^m, 0)$

Definition 2.13 Let $\text{Aut}(\mathbb{C}^m, 0)$ be the group of invertible analytic maps from a neighborhood of zero into itself. The truncation of $\text{Aut}(\mathbb{C}^m, 0)$ to order k is the group of invertible jets of order k of biholomorphisms from $(\mathbb{C}^n, 0)$ into itself and it is denoted by $\text{Aut}_k(\mathbb{C}^m, 0)$.

In what follows let us discuss the canonical algebraic group isomorphism $\text{Aut}_k(\mathbb{C}^m, 0) \cong \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}_{(m,k)})$. A diffeomorphism $g : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^m, 0)$ induces an automorphism of algebras $g^* : \mathbb{C}[[\varepsilon]] \rightarrow \mathbb{C}[[\varepsilon]]$ where $g^*(a(\varepsilon))$ is the Taylor expansion of $a(g(\varepsilon))$ w.r.t. ε . Let Tr^k be the truncation morphism $\mathbb{C}[[\varepsilon]] \rightarrow \mathbb{C}[[\varepsilon]]/(\varepsilon)^{k+1} = \mathbb{C}_{(m,k)}$. Since g^* respects the ideal

(ε) it defines a quotient isomorphism $g_{(k)}^*$.

$$\begin{array}{ccc} \mathbb{C}[[x]] & \xrightarrow{g^*} & \mathbb{C}[[x]] \\ \text{Tr}^k \downarrow & & \downarrow \text{Tr}^k \\ \mathbb{C}_{(m,k)} & \xrightarrow{g_{(k)}^*} & \mathbb{C}_{(m,k)} \end{array}$$

The canonical isomorphism maps $j_0^k g$ to $(g_{(k)}^*)^{-1}$.

We have the projective system of groups,

$$\cdots \rightarrow \text{Aut}_{k+1}(\mathbb{C}^m, 0) \rightarrow \text{Aut}_k(\mathbb{C}^m, 0) \rightarrow \cdots \rightarrow \{e\}$$

It makes sense to construct the projective limit $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ which is the group of formal jets of biholomorphisms of $(\mathbb{C}^m, 0)$. On the other hand, the projective limit of the system,

$$\cdots \rightarrow \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}_{(m,k+1)}) \rightarrow \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}_{(m,k)}) \rightarrow \cdots \rightarrow \{e\}$$

is the group of (ε) -continuous automorphisms of $\mathbb{C}_{(m,\infty)}$. Let us recall that a (ε) -continuous automorphisms is given by a formal series substitution,

$$(\varepsilon_1, \dots, \varepsilon_m) \mapsto (\hat{y}_1(\varepsilon), \dots, \hat{y}_m(\varepsilon))$$

where $\hat{y}_1(\varepsilon), \dots, \hat{y}_m(\varepsilon)$ is a system of generators of the maximal (ε) . Therefore, $\widehat{\text{Aut}}(\mathbb{C}^m, 0) \cong \text{Aut}_{(\varepsilon)\text{-cont}}(\mathbb{C}_{(m,\infty)})$.

Remark 2.14 There are automorphisms of $\mathbb{C}_{(m,\infty)}$ that are not (ε) -continuous. For instance, the series ε and e^ε are algebraically independent. Therefore, it should exist an automorphism of $\mathbb{C}_{(1,\infty)}$ that maps $\varepsilon \mapsto \varepsilon$, $e^\varepsilon \mapsto 2e^\varepsilon$. The sequence of polynomials $1, 1 + \varepsilon, 1 + \varepsilon + \varepsilon^2/2! \dots$ tends to e^ε but this limit does not commute with the automorphism, that therefore can not be continuous.

Two actions We begin by describing the action of $\text{Aut}_k(\mathbb{C}^m, 0)$ on the right over $\tau_{m,k}M$. When we see $\tau_{m,k}M$ as points representing the jets of mappings $F : (\mathbb{C}^m, 0) \rightarrow M$ then the action is given by composition on the right: $F \cdot g = F \circ g$

$$(\mathbb{C}^m, 0) \xrightarrow{g} (\mathbb{C}^m, 0) \xrightarrow{F} M$$

On the other hand $\text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}_{(m,k)})$ acts on the left over $\tau_{m,k}M$. If we look at $\tau_{m,k}M$ as a set of algebra morphisms, then an element $p^{(m,k)} : \mathbb{C}[M] \rightarrow \mathbb{C}_{(m,k)}$ can be composed on the

left with an algebra morphism $g_{(k)}^* : \mathbb{C}_{(m,k)} \rightarrow \mathbb{C}_{(m,k)}$:

$$\mathbb{C}[M] \xrightarrow{p^{(m,k)}} \mathbb{C}_{(m,k)} \xrightarrow{g_{(k)}^*} \mathbb{C}_{(m,k)}$$

giving rise the the left action $g_{(k)}^* \cdot p^{(m,k)} = g_{(k)}^* \circ p^{(m,k)}$.

Relationship between these two actions. As described earlier, the morphism g induces a morphism of algebras $g_{(k)}^*$. We note that $g_{(k)}^*$ and $j_0^k g$ determine each other and the map $j_0^k g \mapsto (g_{(k)}^*)^{-1}$ is a group isomorphism¹. These two actions by left and right sides are equivalent in the following sense:

$$p^{(m,k)} \cdot (j_0^k g) = g_{(k)}^{*-1} \circ p^{(m,k)}.$$

Limit of the action of $\text{Aut}_k(\mathbb{C}^m, 0)$ For each k there is an action of $\text{Aut}_k(\mathbb{C}^m, 0)$ on $\tau_{m,k}M$. These actions induce an action on the projective limit: If $\hat{f} = \{f_k\}$ is in $\hat{\tau}_m$ and $g = \{g_k\}$ a group element in $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ then $\hat{g} \cdot \hat{f} = \{g_k \circ f_k\}$ defines an action. The action of $\text{Aut}(\mathbb{C}^m, 0)$ respects prolongation. This can be seen by following the diagram:

$$\begin{array}{ccccc} \mathbb{C}[M] & \xrightarrow{p^{(m,k)}} & \mathbb{C}_{(m,k)} & \xrightarrow{g_k} & \mathbb{C}_{(m,k)} \\ \parallel & & \uparrow \pi & & \uparrow \pi \\ \mathbb{C}[M] & \xrightarrow{p^{(m,k+1)}} & \mathbb{C}_{(m,k+1)} & \xrightarrow{g_{k+1}} & \mathbb{C}_{(m,k+1)} \end{array}$$

Each square is by definition commutative. Then $\pi \circ g_{k+1} \circ p^{(m,k+1)} = g_k \circ p^{(m,k)}$ and the action behaves well with prolongations.

Proposition 2.15 *Regular map of varieties induces equivariant maps in Weil bundles.*

PROOF. Let $\Phi : M \rightarrow N$ be a regular map. The prolongation $\Phi^{(m,k)}$ is such that, for $h \in \mathbb{C}[N]$, we have $\Phi^{(m,k)}(p^{(m,k)})(h) = p^{(m,k)}(h \circ \Phi)$. Let σ be a \mathbb{C} -algebra automorphism of $\mathbb{C}_{(m,k)}$. To see that $\Phi^{(m,k)}$ is an equivariant map let us write:

$$\Phi^{(m,k)}(\sigma \circ p^{(m,k)})(h) = (\sigma \circ p^{(m,k)})(h \circ \Phi) = \sigma \circ (p^{(m,k)}(h \circ \Phi)) = \sigma \circ \Phi^{(m,k)}(p^{(m,k)})(h)$$

□

Lemma 2.16 *Let $g \in \text{Aut}_k(\mathbb{C}^m, 0)$ and $q^{(m,k)} \cdot g = p^{(m,k)}$ and R_g be the right translation by g in $R_k M$. Let g^{*-1} the induced automorphism of $\mathbb{C}_{(m,k)}$ given by the canonical isomorphism. Then the differential satisfies $d_{q^{(m,k)}}(X_{q^{(m,k)}})R_g = g^{*-1} \circ X_{q^{(m,k)}}$ for any $X_{q^{(m,k)}}$ that we see as a derivation of $\mathbb{C}[M]$ with values on $\mathbb{C}_{(m,k)}$.*

¹Observe that if we omit the inverse then we get an anti morphism.

PROOF. Let γ be a germ of path such that $\gamma(0) = p^{(m,k)}$ and $\frac{d}{dt}\big|_{t=0} \gamma = X_{p^{(m,k)}}$. Develop γ as $\gamma(t) = p^{(m,k)} + tX_{p^{(m,k)}} + o(t^2)$, then

$$\gamma(t) \cdot g = g^{*-1} \left(p^{(m,k)} + tX_{p^{(m,k)}} + o(t^2) \right)$$

We get:

$$\begin{aligned} d_{p^{(m,k)}} R_g(X_{p^{(m,k)}}) &= \frac{d}{dt} \bigg|_{t=0} \gamma(t)g \\ &= \frac{d}{dt} \bigg|_{t=0} (p^{(m,k)}g + t(g^{*-1} \circ X_{p^{(m,k)}}) + o(t^2)) \\ &= g^{*-1} \circ X_{p^{(m,k)}} \end{aligned}$$

□

Proposition 2.17 *The prolongation $X^{(m,k)}$ of a vector field X is $\text{Aut}_k(\mathbb{C}^m, 0)$ -invariant for every k .*

PROOF. Let $g \in \text{Aut}_k(\mathbb{C}^m, 0)$. We want to prove $R_{g*}X^{(m,k)} = X^{(m,k)}$.

$$\begin{array}{ccc} \mathbb{C}[M] & \xrightarrow{X} & \mathbb{C}[M] \\ & \searrow & \downarrow q^{(m,k)} \quad \swarrow p^{(m,k)} \\ & & \mathbb{C}_{(m,k)} \xleftarrow{g^*} \mathbb{C}_{(m,k)} \end{array}$$

From the previous diagram we get

$$\begin{aligned} R_{g*}(X^{(m,k)})(p^{(m,k)}) &= dR_g(X^{(m,k)}(q^{(m,k)})) \\ &= g^{*-1} \circ X_{q^{(m,k)}} = g^{*-1} \circ q^{(m,k)} \circ X = p^{(m,k)} \circ X = X^{(m,k)}(p^{(m,k)}) \end{aligned}$$

□

It follows from Proposition 2.17 that $X^{(m,\infty)}$ is $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -invariant in $\hat{\tau}_m M$.

Example 2.18 There is an identification $\text{GL}_n(\mathbb{C}) \cong \text{Aut}_1(\mathbb{C}^n, 0) \cong \text{Aut}_{\mathbb{C}} T_0 \mathbb{C}^n$.

Example 2.19 Lets describe in some detail $\Gamma = \widehat{\text{Aut}}(\mathbb{C}^n, 0)$ for one variable, *i.e.* $n = 1$. Let $\Gamma_k = \text{Aut}(\mathbb{C}[[\varepsilon]]/(\varepsilon)^{k+1})$. As vector space $\mathbb{C}[[\varepsilon]]/(\varepsilon)^{k+1}$ is generated by $1, \varepsilon, \dots, \varepsilon^k$. For an automorphism g of $\mathbb{C}[[\varepsilon]]/(\varepsilon)^{k+1}$ such that $\varepsilon \mapsto y$ we have

$$\begin{aligned} \varepsilon &\mapsto y = y'\varepsilon + y''/2!\varepsilon^2 + \dots \\ \varepsilon^2 &\mapsto y^2 = (y')^2\varepsilon^2 + y'y''\varepsilon^3 + (2/3!y'y^{(3)} + (y'')^2/2!2!)\varepsilon^4 + \dots \\ \varepsilon^3 &\mapsto y^3 = (y')^3\varepsilon^3 + \dots \end{aligned}$$

From these we get a matrix:

$$\begin{pmatrix} y' & y''/2! & y^{(3)}/3! & \cdots \\ 0 & (y')^2 & 2\frac{y'y^{(3)}}{3!} + \frac{y''^2}{2!2!} & \\ 0 & 0 & (y')^3 & \\ \vdots & & & \end{pmatrix}$$

□

Example 2.20 Let $\Gamma = \widehat{\text{Aut}}(\mathbb{C}, 0)$, the group in one variable say ε . Let $V = J((\mathbb{C}, 0) \rightarrow \mathbb{C})$ the set of all jets. Consider the action $V \times \Gamma \rightarrow V$ such that $(\phi, g) \mapsto \phi \circ g$. Observe that: If $g \in \Gamma$ and $\phi \in V$ then by developing in the basis $\{\varepsilon^k/k!\}$:

$$\phi = \phi_0 + \phi_1\varepsilon + \phi_2\frac{\varepsilon^2}{2!} + \cdots$$

$$g = g_1\varepsilon + g_2\frac{\varepsilon^2}{2!} + g_3\frac{\varepsilon^3}{3!} + \cdots$$

and developing $\phi \circ g$ we have

$$\phi \circ g = \phi_0 + \phi_1g_1\varepsilon + (\phi_1g_2 + \phi_2g_1^2)\frac{\varepsilon^2}{2!} + (\phi_1g_3 + 3\phi_2g_1g_2 + \phi_3g_1^3)\frac{\varepsilon^3}{3!} + \cdots$$

In matrix form we have

$$\phi \circ g = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ g_1 & 0 & 0 & \\ g_2 & g_1^2 & 0 & \\ g_3 & 3g_1g_2 & g_1^3 & \\ \vdots & & & \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix}$$

2.2.2. Frame bundle

A k -frame in M is the jet of order k of an invertible map $\phi : (\mathbb{C}^m, 0) \rightarrow M$ where m is the dimension of M . We write $R_k M$ for the set of all k -frames in M . The set of frames is an affine Zariski open subset of the generalized tangent $\tau_{m,k} M$.

Lemma 2.21 *The open subset $\det x_i \neq 0$ is independent of the basis of regular functions chosen on U .*

Given a system x_1, \dots, x_m of regular functions whose differentials are linearly independent in an open subset $U \subset M$, the equation of $R_k U$ inside $\tau_{m,k} U$ are given by the non-vanishing of the Jacobian,

$$R_k U = \{p^{m,k} \in \tau_{m,k} U \mid \det(x_{i:\epsilon_j}) \neq 0\}.$$

this way $R_k M$ becomes itself an affine smooth algebraic variety. In the same way, the set of formal frames RM is an affine open subset of $\hat{\tau}_m M$.

Moreover, $\mathbb{C}[RM]$ is a $\frac{\partial}{\partial \varepsilon}$ -differential ring, endowed with the total derivative operators. The ring of regular functions of $\mathbb{C}[RM]$ is spanned by total derivative of regular functions in M and the inverse of the jacobian in any system of coordinates. Let x_1, \dots, x_m be a coordinate system on M . The ring of regular functions for $R_k M$ is $\mathbb{C}[x_{i:\alpha}, 1/\det x_{i:\epsilon_j}]$.

Principal structure

Let us note that RM is an $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -invariant set for the natural action of $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ in $\hat{\tau}_m M$. Moreover:

Theorem 2.22 *For each k , $R_k M \rightarrow M$ is a left $\text{Aut}_k(\mathbb{C}^m, 0)$ -principal bundle.*

PROOF. Lets see that the map

$$\Psi : \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}_{(m,r)}) \times R_k M \longrightarrow R_k M \times_M R_k M$$

such that $(g, r) \mapsto (gr, r)$, is an isomorphism. Let $(p^{(m,k)}, q^{(m,k)}) \in R_k M \times_M R_k M$. As \mathfrak{m}_p is zero under $p^{(m,k)}$ there is an isomorphism $\tilde{p}^{(m,k)} : \mathbb{C}[M]/\mathfrak{m}_p^{k+1} \rightarrow \mathbb{C}_{(m,k)}$ and the same for $q^{(m,k)}$, there is an isomorphism $\tilde{q}^{(m,k)} : \mathbb{C}[M]/\mathfrak{m}_p^{k+1} \rightarrow \mathbb{C}_{(m,k)}$.

Then $\tilde{q}^{(m,k)} \circ \tilde{p}^{(m,k)-1}$ is an algebra automorphism of $\mathbb{C}_{(m,k)}$ and the map $R_k M \times_M R_k M \rightarrow R_k M \times \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}_{(m,r)})$ given by $(p^{(m,k)}, q^{(m,k)}) \mapsto (\tilde{q}^{(m,k)} \circ \tilde{p}^{(m,k)-1}, p^{(m,k)})$ provides an inverse for Ψ . \square

As a consequence of the previous theorem the limit RM is an $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -principal bundle.

Prolongation of vector fields

Definition 2.23 For a given regular vector field X in M let us denote $X^{(k)}$ to the restriction of $X^{(m,k)}$ (see p.50, Definition 2.11) to $R_k M$, and $X^{(\infty)}$ to the restriction of $X^{(m,\infty)}$ to RM .

The main properties of the prolongation of vector fields have been stated in Theorem 2.12 and Proposition 2.17. Summarizing, if X is a regular vector field in M :

1. $X^{(k)}$ projects to $X^{(k-1)}$.
2. $X^{(\infty)}$ is the only derivation of $\mathbb{C}[RM]$ that extends X and commutes with total derivative operators.
3. $X^{(k)}$ is $\text{Aut}_k(\mathbb{C}^m, 0)$ -invariant vector field in $R_k M$.
4. $X^{(\infty)}$ is $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -invariant derivation of $\mathbb{C}[RM]$.

2.3. Jet bundles

Let $\pi: P \rightarrow B$ be a smooth affine algebraic bundle with irreducible affine B . Here we explore the differential geometric algebraic structure of $J(P/B)$, as a pro-algebraic variety with a differential structure on its ring of regular functions.

2.3.1. Algebraic and differential structure

Jets of functions

Let us consider the case $P = B \times \mathbb{C}$. In such case $J(P/B) = J(B \rightarrow \mathbb{C})$, and we speak of jets of analytic functions. From now on by $J_k B$ we mean $J_k(B \rightarrow \mathbb{C})$. For a given $p \in B$ let us consider $\widehat{\mathbb{C}[B]}_p$ the \mathfrak{m}_p -adic completion of $\mathbb{C}[B]$. It is a \mathbb{C} -algebra isomorphic to $\mathbb{C}_{(m,\infty)}$ with $m = \dim B$.

Let $p \in B$; the fiber $(JB)_p$ is naturally identified with $\widehat{\mathbb{C}[B]}_p$, we have $j_p f = j_p g$ if and only if both local analytic functions have the same development in $\widehat{\mathbb{C}[B]}_p$. For each order k the fiber $(J_k B)_p$ is naturally identified with $\widehat{\mathbb{C}[B]}_p / \mathfrak{m}_p^{k+1}$. This \mathbb{C} -algebra is isomorphic to $\mathbb{C}_{(m,k)}$.

We consider $(\mathbb{C}[B], \mathfrak{X}_B)$ as a differential ring whose differential structure is given by the whole module of derivations $\mathfrak{X}_B = \text{Der}(\mathbb{C}[B])$. The ring of differential operators

$$\mathcal{D}_B = \mathbb{C}[B][\mathfrak{X}_B],$$

is graded and we have,

$$\mathcal{D}_B^0 \subset \mathcal{D}_B^1 \subset \mathcal{D}_B^2 \subset \cdots \subset \mathcal{D}_B,$$

where \mathcal{D}_B^k is the left $\mathbb{C}[B]$ -module of differential operators of order k . It is not difficult to see that \mathcal{D}_B^k is an space of fiber-wise \mathbb{C} -linear functions θ in $J_k(B \rightarrow \mathbb{C})$ by taking,

$$\theta(j_p f) = (\theta f)(p).$$

More over, this gives an identification of $\mathcal{D}_B^k|_{(J_k B)}$ and $(J_k B)^*$. Therefore we can give to $J_k B \rightarrow B$ the structure of an algebraic vector bundle, affine over B , such that $\Gamma(J_k B) = (\mathcal{D}_B^k)^*$. In order to construct the ring of regular functions we have to consider the symmetric algebra over the linear functions. Therefore, we take $\mathbb{C}[J_k B]$ as the symmetric algebra $S_{\mathcal{D}_B^k}^\bullet(\mathcal{D}_B^k)$. In the limit, we take $JB \rightarrow B$ as a pro-algebraic vector bundle, affine over B , with regular functions $\mathbb{C}[JB] = S_{\mathbb{C}[B]}^\bullet(\mathcal{D}_B)$.

Remark 2.24 If we compare the previous with the construction done in Chapter 1, it follows that $\mathbb{C}[JB] = \mathbb{C}[B]\{u\}_{\mathfrak{X}_B}$, the ring of differential polynomials in a variable u with coefficients in $\mathbb{C}[B]$. This identification is obtained by seeing the variable u as the coordinate in the fibre \mathbb{C} for the projection $\mathbb{C} \times B \rightarrow B$ and the operator ∂^α becomes identified with

the differential function given by $\partial^\alpha u = u_{:\alpha}$.

Remark 2.25 Differential operators of order $\leq k$ can be seen as sections of the dual bundle $(J_k B)^* \rightarrow B$.

Definition 2.26 A vector field $X \in \mathfrak{X}_B$ acts on \mathcal{D}_B by composition on the left: $X : \theta \mapsto X \circ \theta$. By means of Leibniz rule this action extends to $\mathbb{C}[JB]$. In this way we obtain a derivation

$$X^{\text{tot}} : \mathbb{C}[JB] \rightarrow \mathbb{C}[JB]$$

that coincides with X in $\mathbb{C}[B] \subset \mathbb{C}[JB]$. This derivation is called the *total derivative* in the direction of X .

As X^{tot} sends $\mathbb{C}[J_k B]$ into $\mathbb{C}[J_{k+1} B]$, its degree is $+1$. If x_1, \dots, x_n are regular functions whose differentials are linearly independent at any point of B (something we may find replacing B by a suitable Zariski open) then the partial derivatives with respect to the variables span \mathfrak{X}_B . We have a system of linear adapted coordinates where $u_{:\alpha}(j_p f) = \frac{\partial^\alpha f}{\partial x^\alpha}(p)$ for which,

$$\frac{\partial}{\partial x_i}^{\text{tot}} = \frac{\partial}{\partial x_i} + \sum_{0 \leq |\alpha| < \infty} u_{:\alpha + \epsilon_i} \frac{\partial}{\partial u_{:\alpha}}$$

The map,

$$\text{Der}(\mathbb{C}[B]) \rightarrow \text{Der}(\mathbb{C}[JB]), \quad X \mapsto X^{\text{tot}}$$

is an injective Lie algebra, and $\mathbb{C}[B]$ -module morphism, therefore $(\mathbb{C}[B], \mathfrak{X}_B) \subset (\mathbb{C}[JB], \mathfrak{X}_B)$ is an extension of \mathfrak{X}_B -differential rings.

As we are interested in objects that can be defined generically *i.e.* in some Zariski-open subset of B , it is meaningful to allow division by non-vanishing functions on B . Let us define

$$\mathcal{O}_{JB} := \mathbb{C}(B) \otimes_{\mathbb{C}[B]} \mathbb{C}[JB]$$

the ring of *differential functions regular along a generic open set in B* . The module of total derivatives becomes bigger, because for every $X \in \mathfrak{X}_B^{\text{rat}}$ it is possible to calculate X^{tot} and $(\mathcal{O}_{JB}, \mathfrak{X}_B^{\text{rat}})$ is a $\mathfrak{X}_B^{\text{rat}}$ -differential ring extension of $(\mathbb{C}(B), \mathfrak{X}_B^{\text{rat}})$.

2.3.2. Jets of sections of vector bundles

Let $E \rightarrow B$ be a vector bundle and assume that E is affine over B . Then there are enough regular sections in $\Gamma(E)$. Consider

$$\text{Diff}(\Gamma(E), \mathbb{C}[B]) = \mathcal{D}_B \otimes_{\mathbb{C}[M]} \Gamma(E)^*$$

Elements in $\text{Diff}(\Gamma(E), \mathbb{C}[B])$ are differential operators $\Gamma(E) \rightarrow \mathbb{C}[B]$ where $(\theta \otimes \omega)(s) = \theta(\omega(s))$ for $s \in \Gamma(E)$. As before, we consider $\text{Diff}(\Gamma(E), \mathbb{C}[B])$ filtered by the order and

$\text{Diff}^k(\Gamma(E), \mathbb{C}[B])$ the $\mathbb{C}[B]$ -module of differential operators of order $\leq k$. For every $p \in B$, the set $(J_k(E/B))_p$ is a \mathbb{C} -vector space with $j_p^k \lambda s = \lambda j_p^k s$ and $j_p^k s + j_p^k t = j_p^k (s+t)$. Differential operators $\psi \in \text{Diff}^k(\Gamma(E), \mathbb{C}[E])$ are linear along the fibers of $J_k(E/B) \rightarrow B$ with $\psi(j_p^k s) = \psi(s)(p)$. Again $J_k(E/B) \rightarrow B$ is a vector bundle and $\Gamma(J_k(E/B)) = \text{Diff}^k(\Gamma(E), \mathbb{C}[E])^*$. The ring of regular functions on $J_k(E/B)$ is $\mathbb{C}[J_k(E/B)] = S_{\mathbb{C}[B]}^\bullet(\text{Diff}^k(\Gamma(E), \mathbb{C}[B]))$.

Example 2.27 Let $E = \mathbb{C}^n \times \mathbb{C}^m \rightarrow \mathbb{C}^m = B$ be the trivial bundle with coordinates x_1, \dots, x_m in the base and y_1, \dots, y_n in the fibers. Let us denote by $y_{j:\alpha}$ the differential operator $\partial_x^\alpha \otimes y_j$ seen as a regular function on $J(E/B)$. Then, jet bundles $J_k(E/B)$ are coordinated by $x_i, y_{j:\alpha}$ for α multi-index subject to the restriction $|\alpha| \leq k$.

We have a chain of inclusions,

$$\mathbb{C}[B] \subseteq \mathbb{C}[J_1 B] \subseteq \dots$$

As before we may consider the limit as the ring of regular functions in the pro-algebraic bundle $J(E/B) \rightarrow B$,

$$\mathbb{C}[J(E/B)] = S_{\mathbb{C}[B]}^\bullet(\text{Diff}(\Gamma(E), \mathbb{C}[B])).$$

We may also define total derivative operators. Let X be a regular vector field in B . First, for $F \in \text{Diff}(\Gamma(E), \mathbb{C}[E]) \subset \mathbb{C}[J(E/B)]$ we define,

$$X^{\text{tot}} F = X \circ F.$$

Then, X^{tot} extends from $\text{Diff}(\Gamma E, \mathbb{C}[E])$ to $\mathbb{C}[J(E/B)]$ by means of the Leibniz rule. As before,

$$\mathfrak{X}_B \rightarrow \text{Der} J(E/B), \quad X \mapsto X^{\text{tot}}$$

is a Lie algebra and $\mathbb{C}[B]$ -module morphism and we have that $\mathbb{C}[J(E/B)]$ is a \mathfrak{X}_B -differential ring extension of $\mathbb{C}[B]$. As before, we define $\mathcal{O}_{J(E/B)} := \mathbb{C}(B) \otimes_{\mathbb{C}} \mathbb{C}[J(E/B)]$ the *ring of differential functions in a generic open set of B* and $(\mathcal{O}_{J(E/B)}, \mathfrak{X}_B^{\text{rat}})$ is a differential ring extension of $(\mathbb{C}(B), \mathfrak{X}_B^{\text{rat}})$.

2.3.3. Jets of sections of a bundle

Let $\pi : P \rightarrow B$ be a smooth bundle. Assume B is affine and irreducible and P is affine over B . Then There are u_1, \dots, u_n regular functions in P such that

$$\mathbb{C}[P] = \mathbb{C}[B][u_1, \dots, u_n]/I$$

where I is the ideal of dependence relations between the u_i 's. Therefore P can be seen as a subbundle of a trivial bundle

$$\begin{array}{ccc} P & \xrightarrow{i} & E = B \times \mathbb{C}_{u_1, \dots, u_n}^n \\ & \searrow & \downarrow \\ & & B \end{array}$$

where $p \mapsto (\pi(p), u_1(p), \dots, u_n(p))$.

By derivation of i we obtain that $J(P/B) \hookrightarrow (JB) \times \mathbb{C}^n/B$ as the zero locus of the \mathfrak{X}_B -ideal $\{I\}$ generated by I in $\mathbb{C}[E/B]$. Each $J_k(P/B)$ is identified with the zero locus of $\{I\} \cap \mathbb{C}[J_k(E/B)]$ in $J_k(E/B)$.

Thus, we consider the quotient $\mathbb{C}[J(E/B)]/\{I\}$ as the ring of regular functions $\mathbb{C}[J(P/B)]$. As $\{I\}$ is a \mathfrak{X}_B -ideal, total derivatives descend to the quotient and for each $X \in \text{Der}(\mathbb{C}[B])$ we have $X^{\text{tot}} : \mathbb{C}[J(P/B)] \rightarrow \mathbb{C}[J(P/B)]$. As before $(\mathbb{C}[J(P/B)], \mathfrak{X}_B)$ is seen as a \mathfrak{X}_B -ring extension of $(\mathbb{C}[B], \mathfrak{X}_B)$. We also consider the *ring of functions on $J(P/B)$ that are regular along a generic open subset of B* ,

$$\mathcal{O}_{J(P/B)} = \mathbb{C}(B) \otimes_{\mathbb{C}[B]} \mathbb{C}[J(P/B)],$$

which we see as a $\mathfrak{X}_B^{\text{rat}}$ -ring extension of $(\mathbb{C}(B), \mathfrak{X}_B^{\text{rat}})$.

2.3.4. Prolongation of vector fields

There is a mechanism of prolongation of vector fields in P to vector fields in $J(P/B)$, similar to the one described in the case of frames. The following result can be verified in coordinates, see for example, [KSM99].

Theorem 2.28 *Let X be a rational vector field in P . Then for every k there is a rational vector field $X^{(k)}$ in $J_k(P/B)$ such that*

1. $X^{(k+1)}$ is projectable over $X^{(k)}$ by $\pi_{k+1,k} : J_{k+1}(P/B) \rightarrow J_k(P/B)$, and $X^{(0)} = X$.
2. Prolongation commutes with total derivatives: For all $Y \in \mathfrak{X}_B^{\text{rat}}$ we have $X^{(k+1)} \circ Y^{\text{tot}} = Y^{\text{tot}} \circ X^{(k)}$.

PROOF. The result is known for the smooth and analytic cases. Here, the reader only needs to check that if X is rational then its prolongation $X^{(k)}$ is also rational. \square

We can put together the successive prolongations of X , to define $X^{(\infty)}$, a derivation of $\mathcal{O}_{J(P/B)}$,

$$X^{(\infty)} F = X^{(k)} F$$

for $F \in \mathcal{O}_{J_k(P/B)} \subset \mathcal{O}_{J(P/B)}$.

Remark 2.29 Prolongation of vector fields is compatible with the Lie bracket $[X, Y]^{(k)} = [X^{(k)}, Y^{(k)}]$, for $k = 0, \dots, \infty$.

Remark 2.30 The prolongation of vector fields to $J(P/B)$ has the following geometrical interpretation. Let us assume that X has a real flow $\{\sigma_t\}_{t \in \mathbb{R}}$ in a neighborhood of $p \in P$, let $x = \pi(p)$ and u be a local section such that $u(x) = p$. Let Γ_u be the graph of u . For small t we have that $\sigma_t(\Gamma_u)$ is, around x , also the graph of a local section that we name u_t . Then,

$$X^{(k)}(j_x^k u) = \left. \frac{d}{dt} \right|_{t=0} j_{\pi(\sigma_t(p))}^k u_t.$$

Thus, the flow of $X^{(k)}$ is the projection in the jet space of the flow naturally induced by Y in the space of pointed germs of submanifolds of P .

3. Linear differential Galois theory with parameters

3.1. Galois groupoid with parameters

Let us consider, as in Section 1.4.2 an affine principal bundle $\pi: P \rightarrow B$ with structure group bundle $G \rightarrow S$, a submersion $\rho: B \rightarrow S$, a foliation $\mathcal{F} = \ker(d\rho)$, and \mathcal{D} a G -invariant connection with parameters in S meaning that $\dim \mathcal{D} = \dim \mathcal{F}$ and $\pi_* \mathcal{D} = \mathcal{F}$. Some of our arguments will require to restrict these objects to some affine subset of B . Accordingly, without loss of generality, *we assume that \mathcal{F} and \mathcal{D} are regular.*

3.1.1. Prolongation of a principal bundle

The concepts of jet bundles, prolongation and total derivatives were introduced in Sections 2.1 and 2.3.2; here we begin with a brief recall of those. Two germs of analytic sections of P at $x \in B$ have the same jet of order k if they are tangent up to the order k . This means, that in any analytic system of coordinates, they have the same power series development up to degree k . The space $J_k(P/B)$ of sections is a smooth affine algebraic variety and then $J(P/B) = \lim J_k(P/B)$ is a smooth pro-algebraic affine variety. The jet of order k at $x \in B$ of the section u is denoted by $j_x^k u$, and the section of $J_k(P/B)$ that assigns to each x in the domain of u the jet $j_x^k u$ is denoted by $j^k u$ (we omit the index $k = \infty$ for jets of infinite order).

By $\mathcal{O}_{J_k(P/B)}$ we understand the ring of regular functions in $J_k(P/B)$ with rational coefficients in B , that is, $\mathcal{O}_{J_k(P/B)} = \mathbb{C}(B) \otimes_{\mathbb{C}[B]} \mathbb{C}[J_k(P/B)]$. Elements of $\mathcal{O}_{J_k(P/B)}$ are regular *differential functions of order k with rational coefficients* in B , or equivalently, regular differential functions in $J_k(P|_{B'}/B')$ for some open subset $B' \subset B$. We have a chain of ring extensions:

$$\mathbb{C}(B) \subset \mathcal{O}_{P/B} \subset \mathcal{O}_{J_1(P/B)} \subset \dots \subset \mathcal{O}_{J_k(P/B)} \subset \dots \subset \mathcal{O}_{J(P/B)}.$$

Differential structure of $\mathcal{O}_{J(P/B)}$

There exists a total derivative prolongation: if X is a rational vector field in B with domain of regularity $B' \subset B$ then its *total derivative operator* X^{tot} is a derivation of the ring of

differential functions, defined by the rule,

$$(X^{\text{tot}} F)(j_x^{k+1} u) = (X(F \circ j_x^k u))(x),$$

for any differential function F of order k ; X^{tot} is a derivation of $\mathcal{O}_{J(P/B)}$. Note that the total derivative increases the order of differential functions, thus X^{tot} is not a rational vector field in $J_k(P/B)$ for k finite. We consider the differential ring $(\mathcal{O}_{J(P/B)}, \mathfrak{X}_{\text{rat}}(B))$. This is a $\mathfrak{X}_{\text{rat}}(B)$ -finitely generated differential ring over $\mathbb{C}(B)$. See Definition 1.1, on p.22.

Remark 3.1 Here we work with a broader definition of differential field than usual (cf. [Kol73]). We consider a differential field to be a field endowed with an arbitrary Lie algebra of derivations. However, it is always possible to choose a finite commuting basis of $\mathfrak{X}_{\text{rat}}(B)$ in order to recover the usual definition, see Proposition 1.6 on p.23. For instance, let us take a transcendence basis of $\mathbb{C}(B)$ over \mathbb{C} . Then, the partial derivatives with respect to the basis elements are commuting rational vector fields spanning $\mathfrak{X}_{\text{rat}}(B)$. Their corresponding total derivative operators are a basis of derivations for $\mathcal{O}_{J(P/B)}$.

Prolongation of rational vector fields in P

Given a local analytic vector field Y in P , its k -th prolongation $Y^{(k)}$ to $J_k(P/B)$ can be seen geometrically in the following way. Let us assume that Y has a real flow $\{\sigma_t\}_{t \in \mathbb{R}}$ in a neighborhood of $p \in P$, let $x = \pi(p)$ and u be a local section such that $u(x) = p$. Let Γ_u be the graph of u . For small t we have that $\sigma_t(\Gamma_u)$ is, around x , also the graph of a local section that we name u_t . Then,

$$Y^{(k)}(j_x^k u) = \left. \frac{d}{dt} \right|_{t=0} j_{\pi(\sigma_t(p))}^k u_t.$$

Thus, the flow of $Y^{(k)}$ is the projection in the jet space of the flow naturally induced by Y in the space of pointed germs of submanifolds of P . It is easy to check that prolongation of vector fields is compatible with the Lie bracket $[X, Y]^{(k)} = [X^{(k)}, Y^{(k)}]$. The prolongation $Y^{(k)}$ of a regular vector field Y is also regular. Thus, if Y is a rational vector field in P whose domain of definition contains an open subset of the form $\pi^{-1}(B')$ with B' open in B , then $Y^{(k)}$ is a derivation of $\mathcal{O}_{J_k(P/B)}$. This prolongation is possible up to arbitrary order, so that we obtain a derivation $Y^{(\infty)}$ of $\mathcal{O}_{J(P/B)}$ that respects the order of differential functions.

Homogeneous structure of the fibers.

In general, if $\mathcal{G} \rightarrow B$ is a group bundle, the group composition lifts to the jets of sections of \mathcal{G} ,

$$J_k(\mathcal{G}/B) \times_B J_k(\mathcal{G}/B) \rightarrow J_k(\mathcal{G}/B), \quad (j_x^k g)(j_x^k h) = j_x^k(gh).$$

Thus, for all k , $J_k(\mathcal{G}/B) \rightarrow B$ is an algebraic group bundle and $J(\mathcal{G}/B) \rightarrow B$ is a pro-algebraic group bundle.

Let us consider the trivial group bundle $G \times B \rightarrow B$; $J_k(G/B)$ will stand for $J_k(G \times B/B)$. There is a natural fibered action of this group bundle on $J_k(P/B)$,

$$J_k(P/B) \times_B J_k(G/B), \quad (j_x^k u, j_x^k g) \mapsto j_x^k(u \cdot g)$$

so that, for all $x \in B$, $J_k(P/B)_x$ becomes a principal homogeneous space for the action of the algebraic group $J_k(G/B)_x$. Those actions are compatible with the truncations of order; hence, taking the limit to infinite order we have that $J(P/B)_x$ is a principal homogeneous space for the action of the pro-algebraic group $J(G/B)_x$.

3.1.2. Prolongation of a principal connection

The G -invariant integrable distribution \mathcal{D} defines closed subsets $J_k(\mathcal{D}) \subset J_k(P/B)$. They consist of k -jets of \mathcal{D} -horizontal sections. This closed subset $J_k(\mathcal{D})$ is the algebraic differential equation of order k associated to \mathcal{D} . The chain of maps,

$$J(\mathcal{D}) = \lim_{\leftarrow} J_k(\mathcal{D}) \rightarrow \dots \rightarrow J_k(\mathcal{D}) \rightarrow \dots \rightarrow J_2(\mathcal{D}) \rightarrow J_1(\mathcal{D}) \rightarrow P,$$

are surjective.

$J_k(\mathcal{D})$ is defined by a radical ideal $\mathcal{E}_k(\mathcal{D})$ of $\mathcal{O}_{J_k(P/B)}$. Indeed, from the Frobenius integrability condition (flatness of \mathcal{D}) we obtain that $\mathcal{E}(\mathcal{D})$ is the radical $\mathfrak{X}_{\text{rat}}(B)$ -differential ideal of $\mathcal{O}_{J(P/B)}$ spanned by $\mathcal{E}_1(\mathcal{D})$. For each k , $\mathcal{E}_k(\mathcal{D})$ is the intersection of $\mathcal{E}(\mathcal{D})$ with the ring $\mathcal{O}_{J_k(P/B)}$. The ring of *differential functions* on horizontal sections is the $\mathfrak{X}_{\text{rat}}(B)$ -differential ring quotient $\mathcal{O}_{J(\mathcal{D})} = \mathcal{O}_{J(P/B)}/\mathcal{E}(\mathcal{D})$.

The composition with ρ induces a map $\rho^*: J_k(G/S) \rightarrow J_k(G/B)$. Therefore we also have an action defined over S ,

$$J_k(P/B) \times_S J_k(G/S), \quad (j_x^k f, j_{\rho(x)}^k g) \mapsto j_x^k(f \cdot (g \circ \rho))$$

Proposition 3.2 *The action of $J_k(G/S)_{\rho(x)}$ on $J_k(\mathcal{D})_x$ is free and transitive for any $x \in B$.*

PROOF. Let $j_x^k u$ and $j_x^k v$ be in $J_k(\mathcal{D})_x$. Representatives u and v can be taken as \mathcal{D} -horizontal sections defined in a neighborhood of x . There is a unique section g defined in this neighborhood G such that $u = v \cdot g$. Taking into account that u and v are \mathcal{D} -horizontal, we have that g is constant along the fibers of ρ , so it can be thought as a function from a neighborhood of $\rho(x)$ in S to G . Finally we have, $j_x^k u = (j_x^k v) \cdot (j_{\rho(x)}^k g)$. \square

Remark 3.3 It is important to note that the action is compatible with the truncation maps $J_{k+1}(G/S) \rightarrow J_k(G/S)$ and $J_{k+1}(\mathcal{D}) \rightarrow J_k(\mathcal{D})$. Hence, we have that $J(G/S)_{\rho(x)}$ acts freely and transitively in $J(\mathcal{D})_x$.

Trivialization of $J(G/S)$

By Proposition 3.2 we have that $J(\mathcal{D}) \rightarrow B$ is a bundle by principal homogeneous spaces. However, the structure group varies with the parameter in S . In order to have a principal bundle structure we should trivialize the group bundle $J(G/S)$.

Let m be dimension of S and $\tau_{m,k}G$ be the generalized tangent bundle of G of rank m and order k , *i.e.*, the space of jets of order k of local analytic maps from $(\mathbb{C}^m, 0)$ to G . It is an algebraic group, with the composition law, $(j_0^k g)(j_0^k h) = j_0^k(gh)$. The group $\text{Aut}_k(\mathbb{C}^m, 0)$ of jets of automorphisms act on $\tau_{m,k}G$ by group automorphisms on the right side by composition.

For this fixed m we may take the projective limit in k and thus we obtain the pro-algebraic group $\hat{\tau}_m G$ of formal maps from $(\mathbb{C}^m, 0)$ to G . In $\hat{\tau}_m G$ the group of formal automorphisms $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ acts by group automorphisms on the right side by composition.

By a *frame* of order k at $s \in S$ we mean a k -jet at $0 \in \mathbb{C}^m$ of a locally invertible map from $(\mathbb{C}^m, 0)$ to (S, s) . The frame bundle $R^k S \rightarrow S$ is a principal bundle, modeled over the group $\text{Aut}_k(\mathbb{C}^m, 0)$. Those bundles form a projective system and the projective limit $RS \rightarrow S$ is the frame bundle. It is a pro-algebraic principal bundle modeled over the group $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ of formal analytic automorphisms. A section of the frame bundle is called a *moving frame*.

Proposition 3.4 Any moving frame $\hat{\varphi}$ in S induces by composition at the right side a trivialization,

$$\begin{array}{ccc} J(G/S) & \xrightarrow{\hat{\varphi}_*} & \hat{\tau}_m G \times S \\ & \searrow & \swarrow \\ & S & \end{array}$$

and thus a structure of principal bundle in $J(\mathcal{D})$,

$$J(\mathcal{D}) \times \hat{\tau}_m G \rightarrow J(\mathcal{D}), \quad (j_x u, j_0 g) \mapsto j_x u \cdot (j_0 g \circ \hat{\varphi}_{\rho(x)}^{-1} \circ j_x \rho)$$

PROOF. Let $\hat{\varphi}$ be a moving frame in S . Then, for $s \in S$ we have that $\hat{\varphi}_s$ is a formal isomorphism between $(\mathbb{C}^m, 0)$ and (S, s) . Thus, it induces a pro-algebraic group isomorphism from $J(G/S)_s$ to $\hat{\tau}_m G$ that sends $j_s g$ to $j_0(g \circ \hat{\varphi}_s)$. In this way we obtain the trivialization of the statement. Finally, by Proposition 3.2 and Remark 3.3 we have that the induced action of $\hat{\tau}_m G$ in $J(\mathcal{D})$ gives the structure of a principal bundle. \square

Remark 3.5 Let $\hat{\varphi}$ and $\hat{\psi}$ be two different moving frames in S . Then, there is a regular map $\hat{\gamma}$ from S to $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ such that $\hat{\varphi}(s) = \hat{\psi}(s)\hat{\gamma}(s)$. We have a commutative diagram:

$$\begin{array}{ccc} J(G/S) & \xrightarrow{\hat{\varphi}_*} & \hat{\tau}_m G \times S \\ & \searrow \hat{\psi}_* & \downarrow \hat{\gamma} \times \text{Id}_S \\ & & \hat{\tau}_m G \times S \end{array} \quad \begin{array}{c} (j_0 g, s) \\ \downarrow \\ (j_0 g \circ \hat{\gamma}(s), s) \end{array}$$

The group $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ acts on $\hat{\tau}_m G$ by group automorphisms. Thus, the two $\hat{\tau}_m G$ -principal bundle structures induced in $J(\mathcal{D})$ by $\hat{\varphi}$ and $\hat{\psi}$ are related by the fibered automorphism $\hat{\gamma}$ of $\hat{\tau}_m G \times S \rightarrow S$.

Remark 3.6 As before, our constructions are compatible with the truncation to any finite order. The moving frame φ induces frames of any order, and thus, also induces compatible principal bundle structures in $J_k(\mathcal{D})$ modeled over $\tau_{m,k} G$ in the sense that the following diagram,

$$\begin{array}{ccc} J(\mathcal{D}) \times \hat{\tau}_m G & \longrightarrow & J(\mathcal{D}) \\ \downarrow & & \downarrow \\ J_k(\mathcal{D}) \times \tau_{m,k} G & \longrightarrow & J_k(\mathcal{D}) \end{array}$$

is commutative.

Remark 3.7 For $k > \ell$ there are natural surjective morphisms $\text{Iso}(J_k(\mathcal{D})) \rightarrow \text{Iso}(J_\ell(\mathcal{D}))$. The projective limit of this system yields $\text{Iso}(J(\mathcal{D}))$ which is a pro-algebraic Lie groupoid over B . The construction of $\text{Iso}(J_k(\mathcal{D}))$ may depend on the global frame $\hat{\varphi}$, a map between two fibers $J(\mathcal{D})_x$ and $J(\mathcal{D})_y$ may or not be $\hat{\tau}_m G$ -equivariant for the structure induced by some global frame. However, according to Remark 3.5, if $\rho(x) = \rho(y)$, then a map between the fibers $J(\mathcal{D})_x$ and $J(\mathcal{D})_y$ is $\hat{\tau}_m G$ -equivariant for some induced structure if and only if it is $\hat{\tau}_m G$ -equivariant for any induced structure. Accordingly, the pro-algebraic Lie groupoid $\text{Iso}_S(J(\mathcal{D}))$ does not depend on the choice of the global frame in S .

Example 3.8 Our main examples are linear differential equations with parameters. Here $P = \text{GL}_k(\mathbb{C}) \times \mathbb{C}^{n+m}$, $B = \mathbb{C}^{n+m}$, and $S = \mathbb{C}^m$. We consider a linear differential system with parameters

$$\frac{\partial U}{\partial x_i} = A_i U, \quad i = 1, \dots, n, \quad (3-1)$$

where A_i are matrices with coefficients in $\mathbb{C}(x_1, \dots, x_n, s_1, \dots, s_m)$. The Frobenius integrability condition imposes here the compatibility equations:

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = [A_i, A_j].$$

Equation (3-1) can be seen itself as a sub-bundle of $J_1(P/B)$. In this case, $J_1(P/B) = \mathbb{C}^{k^2(n+m)} \times \mathrm{GL}_k(\mathbb{C}) \times \mathbb{C}^{n+m}$, and $U_{x_i}, U_{s_j}, U, x, s$ is a system of coordinates. Equations

$$U_{x_i} - A_i U = 0, \quad i = 1, \dots, n, \quad (3-2)$$

define $J_1(\mathcal{D})$ as a subset of $J_1(P/B)$. A basis of the space of total derivative operators is given by $\partial_{x_i}^{\mathrm{tot}}, \partial_{s_j}^{\mathrm{tot}}$. We differentiate equations (3-2) and obtain

$$U_{x_i x_k} - \frac{\partial A_i}{\partial x_k} U - A_i A_k U = 0, \quad (3-3)$$

$$U_{x_i s_k} - \frac{\partial A_i}{\partial s_k} U - A_i U_{s_k} = 0. \quad (3-4)$$

Equations (3-2), (3-3), and (3-4) define $J_2(\mathcal{D})$, and so on. Note that $x_i, s_j, U, U_{s_j}, U_{s_j s_k}, \dots$ are coordinates on $J(\mathcal{D})$.

Prolongation of \mathcal{D} to $J(\mathcal{D})$.

Here we show how to lift the G -invariant connection \mathcal{D} to a projective system of $\tau_{m,k}G$ -invariant connections $\mathcal{D}^{(k)}$ in $J_k(\mathcal{D})$, and thus, a $\hat{\tau}_m G$ -invariant connection in $J(\mathcal{D})$.

Lemma 3.9 *Let X be a rational vector field tangent to \mathcal{F} . Along $J_k(\mathcal{D})$, the total derivative X^{tot} coincides with $\tilde{X}^{(k)}$, the jet prolongation of the \mathcal{D} -horizontal lift of X . Therefore X^{tot} and $\tilde{X}^{(\infty)}$ coincide as derivations of $\mathcal{O}_{J(\mathcal{D})}$.*

PROOF. Let X be a rational field tangent to \mathcal{F} . First, we have that the ideal of $J(\mathcal{D})$ is a differential ideal, so that, X^{tot} defines a derivation of $\mathcal{O}_{J(\mathcal{D})}$.

Let us fix $j_x^k u \in J_k(\mathcal{D})$. Let us assume that u is a \mathcal{D} -horizontal local section. The vector field \tilde{X} is tangent to the graph of u . Thus, the flow of \tilde{X} leaves the section u invariant. Let us consider $\{\sigma_t\}_{t \in \mathbb{R}}$ the local flow of X around x . Then, we have,

$$\tilde{X}^{(\infty)}(j_x u) = \left. \frac{d}{dt} \right|_{t=0} j_{\sigma_t(x)} u$$

Thus, for any differential function F of arbitrary order, we have,

$$(X^{\mathrm{tot}} F)(j_x u) = X_x(F \circ j u) = \left. \frac{d}{dt} \right|_{t=0} F(j_{\sigma_t(x)} u) = (\tilde{X}^{(\infty)} F)(j_x u).$$

We have that X^{tot} coincides with $\tilde{X}^{(\infty)}$ along $J(\mathcal{D})$, and therefore they preserve the order of differential functions. \square

Example 3.10 Let us compute $\partial_{x_j}^{\mathrm{tot}}$ restricted to $J(\mathcal{D})$ in Example 3.8. Let u_{ij} be the (i, j) -entry of coordinate matrix U , and $a_{i,jk}$ be the (j, k) -entry of matrix A_i . The usual expression

for the total derivative is:

$$\frac{\partial}{\partial x_j}^{\text{tot}} = \frac{\partial}{\partial x_j} + u_{mk;x_j} \frac{\partial}{\partial u_{mk}} + u_{mk;x_\ell x_j} \frac{\partial}{\partial u_{mk;x_\ell}} + u_{mk;s_\ell x_j} \frac{\partial}{\partial u_{mk;s_\ell}} + \dots$$

We have that $x_j, s_\ell, u_{mk}, u_{mk;s_\ell}, \dots$ is a system of coordinates on $J(\mathcal{D})$. We obtain:

$$\left. \frac{\partial}{\partial x_j}^{\text{tot}} \right|_{J(\mathcal{D})} = \frac{\partial}{\partial x_j} + a_{j,m\ell} u_{\ell k} \frac{\partial}{\partial u_{mk}} + \left(\frac{\partial a_{j,m\ell}}{\partial s_r} u_{\ell k} + a_{j,m\ell} u_{\ell k;s_r} \right) \frac{\partial}{\partial u_{mk;s_r}} + \dots$$

Here we see that the total derivative of a function on $J_k(\mathcal{D})$ is also defined on $J_k(\mathcal{D})$.

Definition 3.11 For each $k = 1, \dots, \infty$ we call the k -th prolongation of \mathcal{D} the distribution $\mathcal{D}^{(k)}$ spanned by the vector fields of the form $X^{\text{tot}}|_{J_k(\mathcal{D})}$, where X is tangent to \mathcal{F} .

Remark 3.12 The above definition ensures $\mathcal{D}^{(k)}$ is a regular connection on $J_k(\mathcal{D})$. Another way of defining $\mathcal{D}^{(k)}$ in geometrical terms is the following. Let u be a local \mathcal{D} -horizontal section defined around $x \in B$,

$$\mathcal{D}_{j_x^k u}^{(k)} = d(j^k u)(\mathcal{F}_x).$$

The space $\mathcal{D}_{j_x^k u}^{(k)} \subset T_{j_x^k u}(J_k(\mathcal{D}))$ does not depend on the choice of u . It is confined into the fiber of $\rho(x)$. Any other \mathcal{D} -horizontal section passing through the same point $u(x)$ shall coincide with u along the fiber of $\rho(x)$.

Proposition 3.13 *The prolongation $\mathcal{D}^{(k)}$ is a $\tau_{m,k}G$ -invariant connection on $J_k(\mathcal{D})$ with parameters S . The projection $J_{k+1}(\mathcal{D}) \rightarrow J_k(\mathcal{D})$ maps $\mathcal{D}^{(k+1)}$ to $\mathcal{D}^{(k)}$.*

PROOF. Let $\{X_i\}$ be a commuting basis of the $\mathbb{C}(B)$ -space of rational vector fields tangent to \mathcal{F} . Their values span \mathcal{F} on the generic point of B . In virtue of Lemma 3.9 we have that $\mathcal{D}^{(k)}$ is the distribution of vector fields spanned by $\{\tilde{X}_i^{(k)}|_{J_k(\mathcal{D})}\}$. From the compatibility of the Lie bracket and the lift of vector fields we obtain that $\mathcal{D}^{(k)}$ is a Frobenius integrable distribution in $J(\mathcal{D})$. Also, from the properties of the lift we obtain that $\mathcal{D}^{(k+1)}$ projects onto $\mathcal{D}^{(k)}$. The invariance of $\mathcal{D}^{(k)}$ with respect to $\tau_{m,k}G$ follows automatically from remark 3.12. \square

The projective limit $\mathcal{D}^{(\infty)} = \lim_{\leftarrow} \mathcal{D}^{(k)}$ is called the *jet prolongation with respect to parameters* of \mathcal{D} . In virtue of Proposition 3.13 $\mathcal{D}^{(\infty)}$ is a $\hat{\tau}_m G$ invariant \mathcal{F} -connection in $J(\mathcal{D})$. Note that the construction of $\mathcal{D}^{(\infty)}$ is independent of the moving frame $\hat{\varphi}$ in S that induces the $\hat{\tau}_m G$ -principal bundle structure in $J(\mathcal{D})$. Different moving frames give different actions that are not necessarily equivalent, but at least they are equivalent along the fibers of ρ .

3.1.3. The parameterized Galois groupoid

The field of *rational differential invariants* of order $\leq k$ of \mathcal{D} is the field $\mathbb{C}(J_k(\mathcal{D}))^{\mathcal{D}^{(k)}}$ of rational first integrals of $\mathcal{D}^{(k)}$. We have a tower of field extensions,

$$\mathbb{C}(S) \subset \mathbb{C}(P)^{\mathcal{D}} \subset \dots \subset \mathbb{C}(J_k(\mathcal{D}))^{\mathcal{D}^{(k)}} \subset \mathbb{C}(J_{k+1}(\mathcal{D}))^{\mathcal{D}^{(k+1)}} \subset \dots$$

and the union of all its members is the field of rational differential invariants of \mathcal{D} . If X is a rational vector field in B and f a rational invariant of \mathcal{D} of order $\leq k$ then $X^{\text{tot}}f$ is a rational invariant of \mathcal{D} of order $\leq k+1$. Thus, we have that $\mathbb{C}(J(\mathcal{D}))^{\mathcal{D}^{(\infty)}}$ is a differential field with derivations $\mathfrak{X}(B)^{\text{tot}}$. Moreover, by the finiteness of differential subfields (for instance, by Theorem 14 in [Kol73], p. 112) we have that there is some ℓ such that the field of differential invariants is spanned by $\mathbb{C}(J_\ell(\mathcal{D}))^{\mathcal{D}^{(\ell)}}$ and total differentiation.

Now let us translate, by means of the Galois correspondence 1.61, this tower of fields into a system of groupoids, and the field of rational differential invariants into a pro-algebraic groupoid.

In order to have a principal structure in $J(\mathcal{D})$ we need to take a moving frame on S . It is well known that there always exists a rational moving frame on S . We may, for instance, take the frame induced by a transcendence basis of $\mathbb{C}(S)$. Thus, we fix a global frame $\hat{\varphi}$ in S , that we assume to be regular, after restriction to a suitable affine open subset.

Lemma 3.14 *The Galois groupoid $\text{GGal}(\mathcal{D}^{(k)})$ does not depend on the choice of the moving frame $\hat{\varphi}$ on S .*

PROOF. Let $\hat{\varphi}$ and $\hat{\psi}$ two different moving frames on S , whose truncations to order k induce respective $\tau_{m,k}G$ -structures in $J_k(\mathcal{D}) \rightarrow B$. Thus, there are two different groupoids of gauge isomorphisms $\text{Iso}^{\hat{\varphi}}(J_k(\mathcal{D}))$ and $\text{Iso}^{\hat{\psi}}(J_k(\mathcal{D}))$. However, by Remark 3.7 we have $\text{Iso}_S(J_k(\mathcal{D})) \subset \text{Iso}^{\hat{\varphi}}(J_k(\mathcal{D})) \cap \text{Iso}^{\hat{\psi}}(J_k(\mathcal{D}))$. We also have $\mathbb{C}(S) \subset \mathbb{C}(J_k(\mathcal{D}))^{\mathcal{D}^{(k)}}$. By Galois correspondence 1.61 we have that the Galois groupoid of $\mathcal{D}^{(k)}$ is inside $\text{Iso}_S(J_k(\mathcal{D}))$, hence independent of the choice of the global frame. \square

Definition 3.15 Let us consider \mathcal{D} a principal G -invariant connection on P with parameters in S .

1. The Galois groupoid (and group bundle) of order k with parameters of \mathcal{D} is the Galois groupoid (and group bundle) of gauge isomorphisms (automorphisms) of $J_k(\mathcal{D})$ that fixes rational differential invariants of \mathcal{D} of order $\leq k$.

$$\text{PGal}_k(\mathcal{D}) = \text{GGal}(\mathcal{D}^{(k)}), \quad \text{Gal}_k(\mathcal{D}) = \text{Gal}(\mathcal{D}^{(k)}).$$

2. The Galois groupoid with parameters of \mathcal{D} is the groupoid of gauge isomorphisms of

$J(\mathcal{D})$ that fixes rational differential invariants of \mathcal{D} .

$$\mathrm{PGal}_\infty(\mathcal{D}) = \lim_{\infty \leftarrow k} \mathrm{GGal}(\mathcal{D}^{(k)}).$$

3. The Galois group with parameters of \mathcal{D} is the diagonal part of the Galois groupoid with parameters of \mathcal{D} ,

$$\mathrm{Gal}_\infty(\mathcal{D}) = \mathrm{PGal}_\infty(\mathcal{D})^{\mathrm{diag}}.$$

3.1.4. Differential structure of the $\mathrm{Gau}(J(\mathcal{D}))$

The Galois group with parameters, as in Definition 3.15, is a rational subgroup bundle of $\mathrm{Gau}(J(\mathcal{D}))$. In what follows we present its natural differential group structure.

The fibers of $J(P/B) \rightarrow B$ are principal homogeneous spaces for the fibers of the group bundle

$J(G/B) \rightarrow B$ (see 3.1.1). We can consider the bundle $\mathrm{Gau}(J(P/B)) \rightarrow B$ of gauge automorphisms as the quotient $(J(P/B) \times_B J(P/B))/J(G/B)$. This quotient exists as a pro-algebraic variety (at least on the generic point of B): it can be constructed, for instance, trivializing the group bundle $J(G/B) \rightarrow B$ by means of a moving frame in B .

Lemma 3.16 *Let $E \rightarrow B$ be a fiber bundle and G an algebraic group acting on E and preserving the projection to B . Assume E/G exists as smooth algebraic variety. Then there is a canonical isomorphism $J(E/B)/J(G/B) \simeq J((E/G)/B)$ of bundles over B .*

PROOF. Let f be the projection $E \rightarrow E/G$ and jf the induced map $J(E/B) \rightarrow J((E/G)/B)$. Because $jf(j\sigma) = j(f(\sigma))$, this map induces the wanted isomorphism. \square

Proposition 3.17 *For the principal bundle $\pi: P \rightarrow B$ there is a canonical isomorphism $\mathrm{Gau}(J(P/B)) \simeq J(\mathrm{Gau}(P)/B)$ of bundles over B .*

PROOF. It suffices to consider the case $E = P \times_B P$ in Lemma 3.16. \square

From this, we know that $\mathrm{Gau}(J(P/B))$ has a structure of differential algebraic group. Let us consider now the bundle of jets of \mathcal{D} -horizontal sections $J(\mathcal{D}) \subset J(P/B)$. It is a *differential subvariety* of $J(P/B)$ in the sense that it is the zero locus of the differential ideal $\mathcal{E}(\mathcal{D}) \subset \mathcal{O}_{J(P/B)}$ (see Subsection 3.1.2).

Theorem 3.18 *The group bundle of gauge automorphisms $J(\mathcal{D})$ is identified with the differential algebraic subgroup of gauge symmetries of \mathcal{D} , defined by the group connection $\mathcal{D}[\mathrm{Adj}]$.*

$$\mathrm{Gau}(J(\mathcal{D})) \simeq J(\mathcal{D}[\mathrm{Adj}]) \subset J(\mathrm{Gau}(P)/B).$$

PROOF. The differential subvariety $J(\mathcal{D}) \subset J(P/B)$ is principal for the action of the subgroup bundle $J(G/S) \times_S B \subset J(G/B)$ (note that two \mathcal{D} -horizontal sections are related by

a right translation which is constant along the fibers of ρ , this being a differential algebraic subgroup of $J(G/B)$.

By means of the general construction of $\text{Gau}(P) = (P \times_B P)/G$ and Proposition 3.17 we have a commutative diagram

$$\begin{array}{ccc} J(\mathcal{D}) \times_B J(\mathcal{D}) & \longrightarrow & J(P/B) \times J(P/B) \\ \downarrow & & \downarrow \\ \text{Gau}(J(\mathcal{D})) & \xrightarrow{i} & J(\text{Gau}(P)/B) \end{array}$$

It is clear that if a pair $(j_x u, j_x v)$ of jets of \mathcal{D} -horizontal sections are related by an element of $J(G/B)_x$, then this element is constant along the fibers of ρ . Therefore, the map i is injective. Finally, an element $j_x \sigma \in J(\text{Gau}(P)/B)$ is in the image of i if and only if it transform \mathcal{D} -horizontal sections into \mathcal{D} -horizontal sections, if and only if it is a jet of a gauge symmetry of \mathcal{D} , if and only if it is $\mathcal{D}[\text{Adj}]$ -horizontal. \square

Remark 3.19 The action of $\text{Gau}(J(\mathcal{D}))$ on $J(\mathcal{D})$ is, by definition, compatible with the differential structure, in the sense that the co-action:

$$\alpha^* : \mathcal{O}_{J(\mathcal{D})} \rightarrow \mathcal{O}_{\text{Gau}(J(\mathcal{D}))} \otimes_{\mathbb{C}(B)} \mathcal{O}_{J(\mathcal{D})}$$

is compatible with the total derivative operators.

Theorem 3.20 *The parameterized Galois group bundle $\text{Gal}_\infty(\mathcal{D})$ is a differential algebraic subgroup of $J(\text{Gau}(P)/B)$.*

PROOF. Let us compute the equations of $\text{Gal}_\infty(\mathcal{D})$ inside $\text{Gau}(J(\mathcal{D}))$ in order to verify that it is the zero locus of a differential Hopf ideal in $\mathcal{O}_{\text{Gau}(J(\mathcal{D}))}$. For the concept of Hopf differential ideal see Definition A.3. From the group bundle structure we already know that the ideal of $\text{Gal}_\infty(\mathcal{D})$ is a Hopf ideal; it suffices to check that it is a differential ideal.

By definition, $\text{Gal}_\infty(\mathcal{D})$ is the group of gauge automorphisms commuting with rational differential invariants $\mathbb{C}(J(\mathcal{D}))^{\mathcal{D}^{(\infty)}}$. This field is differentially finitely generated over $\mathbb{C}(S)$, let us take a system of generators.

$$\mathbb{C}(J(\mathcal{D}))^{\mathcal{D}^{(\infty)}} = \mathbb{C}(S) \langle h_1, \dots, h_k \rangle$$

There is some N big enough such that the differential functions h_i belong to $\mathbb{C}(J_N(\mathcal{D}))$. By construction (and restricting our considerations to a suitable open subset of B) $J_N(\mathcal{D})$ is an affine algebraic variety, so that we have,

$$h_i = \frac{P_i}{Q_i}$$

with P_i, Q_i in $\mathcal{O}_{J(\mathcal{D})}$. Now, we define

$$F_i = \alpha^*(P_i)(1 \otimes Q_i) - \alpha^*(Q_i)(1 \otimes P_i)$$

which are elements of $\mathcal{O}_{\text{Gau}(J(\mathcal{D}))} \otimes_{\mathbb{C}(B)} \mathcal{O}_{J(\mathcal{D})}$. We consider \mathcal{I} the radical differential ideal $\{F_1, \dots, F_k\}$. From Remark 3.19 we have that the zero locus of \mathcal{I} is $\text{Gal}_\infty(\mathcal{D}) \times_B J(\mathcal{D})$. Therefore, the ideal of the Galois group bundle is the differential-Hopf ideal $i^*(\mathcal{I})$ for the canonical inclusion $i^*: \mathcal{O}_{\text{Gau}(J(\mathcal{D}))} \rightarrow \mathcal{O}_{\text{Gau}(J(\mathcal{D}))} \otimes_{\mathbb{C}(B)} \mathcal{O}_{J(\mathcal{D})}$. \square

3.2. Examples

3.2.1. Isomonodromic deformations

An *isomonodromic deformation* of the connection \mathcal{D} with parameters in S is a partial G -invariant connection $\tilde{\mathcal{D}}$ extending \mathcal{D} . If we take $\tilde{\mathcal{F}} = \pi_*(\tilde{\mathcal{D}})$, then $\tilde{\mathcal{D}}$ is a partial $\tilde{\mathcal{F}}$ -connection. Paths along the leaves of $\rho_*(\tilde{\mathcal{F}})$ are called *isomonodromic paths*. The main application is that the monodromy representation of the connection with parameters is constant (up to conjugation) along isomonodromic paths.

Remark 3.21 In the analytic context, the restriction of \mathcal{D} to a leaf of $\tilde{\mathcal{F}}$ is an isomonodromic family. Under some assumptions we have that \mathcal{D} is an isomonodromic family if and only if it admits an isomonodromic deformation which is a TB -connection (see [CS05] §5 and references therein).

An important point about isomonodromic deformations is that they give us first order equations for the Galois group with parameters. The following lemma is a generalization (in our geometrical setting) of Proposition 5.4 in [CS05].

Lemma 3.22 *If $\tilde{\mathcal{D}}$ is an isomonodromic deformation of \mathcal{D} , then $\text{Gal}_\infty(\mathcal{D}) \subset J(\tilde{\mathcal{D}}[\text{Adj}])$.*

PROOF. It suffices to verify that $\text{Gal}_k(\mathcal{D}) \subset J_k(\tilde{\mathcal{D}}[\text{Adj}])$ for any finite k . Any $\tilde{\mathcal{D}}$ -horizontal section is, in particular, \mathcal{D} -horizontal. It follows that $J_k(\tilde{\mathcal{D}}) \subset J_k(\mathcal{D})$.

Let us consider a moving frame in S , so that we may make use of the $\tau_{m,k}G$ -structure in $J(\mathcal{D})$. From the construction of the prolongation $\mathcal{D}^{(k)}$ we have that for any $j_x^k u \in J_k(\tilde{\mathcal{D}})$ there is an inclusion of vector spaces $\mathcal{D}_{j_x^k u}^{(k)} \subset \tilde{\mathcal{D}}_{j_x^k u}^{(k)}$. Therefore $J_k(\tilde{\mathcal{D}})$ is tangent to the distribution $\mathcal{D}^{(k)}$. It is in fact a reduction of structure group to $\tau_{m-d,k}G$, where d is the codimension of \mathcal{D} in $\tilde{\mathcal{D}}$.

From there, we have that $\text{Iso}(J_k(\tilde{\mathcal{D}}))$ is a subgroupoid of $\text{Iso}(J_k(\mathcal{D}))$ and it is tangent to the distribution $\mathcal{D}^{(k)} \boxtimes \mathcal{D}^{(k)}$, as defined in Remark 1.58. From the topological construction of the Galois groupoid (see Remark 1.59) we have $\text{GGal}(\mathcal{D}^{(k)}) \subset \text{Iso}(J_k(\tilde{\mathcal{D}}))$. Now, by taking the diagonal parts we have $\text{Gal}_k(\mathcal{D}) \subset J_k(\tilde{\mathcal{D}}[\text{Adj}])$. \square

Theorems A.9 and A.11 describe all *Zariski dense* differential subgroups H of $J(\text{Gau}(P)/B)$ for simple algebraic groups G . There exists a unique partial G -invariant connection $\tilde{\mathcal{D}}$ such that $H = J(\tilde{\mathcal{D}}[\text{Adj}])$. Note that $\text{Gal}_\infty(\mathcal{D})$ is Zariski dense in $J(\text{Gau}(P)/B)$ if and only if $\text{Gal}(\mathcal{D}) = \text{Gau}(P)$. We can state our main result.

Theorem 3.23 *Let $P \rightarrow B$ be an affine principal bundle with simple group G , $\rho: B \rightarrow S$ a dominant map, $\mathcal{F} = \ker(d\rho)$, and \mathcal{D} a principal G -invariant connection with parameters in S . Let us assume that the Galois group of \mathcal{D} is $\text{Gau}(P)$. Then, there is a biggest isomonodromic deformation $\tilde{\mathcal{D}}$ of \mathcal{D} and the Galois group with parameters $\text{Gal}_\infty(\mathcal{D})$ is the group $J(\tilde{\mathcal{D}}[\text{Adj}])$ of gauge symmetries of $\tilde{\mathcal{D}}$.*

PROOF. From Lemma 3.22 and Theorem A.9 it is clear that the Galois group with parameters is the group of gauge symmetries of a certain isomonodromic deformation $\tilde{\mathcal{D}}$ of \mathcal{D} . It suffices to check that any other isomonodromic deformation is extended by $\tilde{\mathcal{D}}$. Let \mathcal{D}_1 be an isomonodromic deformation of \mathcal{D} . By Lemma 3.22 we have $\text{Gal}_\infty(\mathcal{D}) = J(\tilde{\mathcal{D}}[\text{Adj}]) \subset J(\mathcal{D}_1[\text{Adj}])$. From there, and Theorem A.11 we have $\mathcal{D}_1 \subset \tilde{\mathcal{D}}$. \square

Example 3.24 Let us consider $B = \mathbb{C}^{k+1} \times \mathfrak{sl}_n(\mathbb{C})^k$ with coordinates $x, a_1, \dots, a_k, A_1, \dots, A_k$; $S = \mathbb{C}^k \times \mathfrak{sl}_n(\mathbb{C})^k$ with the obvious projection (forgetting the coordinate x), and $P = \text{SL}_n(\mathbb{C}) \times B$. The general Fuchsian system with k singularities and trace free matrices,

$$\frac{\partial U}{\partial x} = \left(\sum_{i=1}^n \frac{A_i}{x - a_i} \right) U \quad (3-5)$$

is a $\text{SL}_n(\mathbb{C})$ -invariant connection \mathcal{D} on P with parameters in S . The general isomonodromic deformation (see, for instance, [IKSY13] §3.5) admitted by (3-5) is the well-known Schlesinger system. It consists in the system of equations formed by (3-5) together with:

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad (3-6)$$

$$\frac{\partial A_i}{\partial a_i} = - \sum_{j \neq i} \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad (3-7)$$

$$\frac{\partial U}{\partial a_i} = \frac{A_i}{x - a_i} U. \quad (3-8)$$

Equations (3-6), (3-7) define the foliation $\rho_*(\tilde{\mathcal{F}})$ of isomonodromic paths in S , and equations (3-5) - (3-8) define the isomonodromic deformation $\tilde{\mathcal{D}}$ as a partial $\tilde{\mathcal{F}}$ -connection. Thus, the Galois group with parameters of 3-5 is the differential algebraic subgroup of $J(\text{SL}_n(\mathbb{C})/B)$

given by equations (3-6), (3-7), and

$$\frac{\partial \sigma}{\partial x} = \left[\sum_{i=1}^n \frac{A_i}{x - a_i}, \sigma \right], \quad (3-9)$$

$$\frac{\partial \sigma}{\partial a_i} = \left[\frac{A_i}{x - a_i}, \sigma \right]. \quad (3-10)$$

3.3. Gauss hypergeometric equation

Let us consider Gauss hypergeometric equation:

$$x(1-x) \frac{d^2 u}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\} \frac{du}{dx} - \alpha\beta u = 0 \quad (3-11)$$

We see equation (3-11) as a linear connection with parameters in the following way. We set $P = \mathrm{GL}_2(\mathbb{C}) \times B$, $B = \mathbb{C}_x \times S$ and $S = \mathbb{C}_{\alpha, \beta, \gamma}^3$ with the obvious projections. Then, the hypergeometric equation is the $\mathrm{GL}_2(\mathbb{C})$ -invariant partial connection \mathcal{H} in P defined by the differential system:

$$d \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \frac{\alpha\beta}{x(1-x)} & \frac{(\alpha+\beta+1)x-\gamma}{x(1-x)} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = 0, \quad d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0. \quad (3-12)$$

3.3.1. Projective reduction

Let us set $P_{\mathrm{proj}} = \mathrm{PGL}_2(\mathbb{C}) \times B$. There is a canonical quotient map $P \rightarrow P_{\mathrm{proj}}$ which is a morphism of principal bundles. The projection of \mathcal{H} is a $\mathrm{PGL}_2(\mathbb{C})$ -invariant partial connection $\mathcal{H}_{\mathrm{proj}}$ on P_{proj} . This connection has a nice geometric interpretation. Let us consider $J_2^*(\mathbb{C}^1, \mathbb{P}_1)$ the space of 2-jets of local diffeomorphisms from \mathbb{C}^1 to \mathbb{P}_1 . Let us choose an affine embedding φ of \mathbb{C}^1 in \mathbb{P}_1 , for instance $\varphi(\varepsilon) = [\varepsilon : 1]$. Let us consider the action of $\mathrm{PGL}_2(\mathbb{C})$ in \mathbb{P}_1 on the right side, by setting:

$$[t_0 : t_1] \star \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [at_0 + ct_1 : bt_0 + dt_1].$$

We get an isomorphism of $\mathrm{PGL}_2(\mathbb{C})$ bundles,

$$P_{\mathrm{proj}} \rightarrow J_2^*(\mathbb{C}, \mathbb{P}_1) \times \mathbb{C}_{\alpha, \beta, \gamma}^3, \quad (\sigma, x, \alpha, \beta, \gamma) \mapsto (j_x^2(\varphi \star \sigma), \alpha, \beta, \gamma).$$

Through such isomorphism, the reduced connection is seen as a third order differential equation in the projective space, the well known Schwartzian equation: if $\{u_1, u_2\}$ is a local

basis of solutions of (3-11) then $y = \frac{u_1}{u_2}$ satisfies,

$$\text{Schw}_x(y) = \nu(\alpha, \beta, \gamma; x), \quad (3-13)$$

with

$$\nu(\alpha, \beta, \gamma; x) = \frac{\gamma(2-\gamma)}{4x^2} + \frac{1-(\gamma-\alpha-\beta)^2}{4(1-x^2)} + \frac{\gamma(1-\gamma+\alpha+\beta)-2\alpha\beta}{2x(1-x)}.$$

Lemma 3.25 *For very general values of $(\alpha, \beta, \gamma) \in S$ there is no smooth isomonodromic path of $\mathcal{H}_{\text{proj}}$ passing through (α, β, γ) .*

PROOF. Here we apply the description of the monodromy of (3-11) in terms of the parameters, as explained in [IKSY13]. The eigenvalues of the monodromy matrices at 0, 1, ∞ are respectively

$$\{1, e^{-2\gamma\pi i}\}, \quad \{1, e^{2(\gamma-\alpha-\beta)\pi i}\}, \quad \{e^{2\alpha\pi i}, e^{2\beta\pi i}\}.$$

The transcendental functions,

$$f = e^{2\gamma\pi i} + e^{-2\gamma\pi i}, \quad g = e^{2(\gamma-\alpha-\beta)\pi i} + e^{2(\alpha+\beta-\gamma)\pi i}, \quad h = e^{2(\alpha-\beta)\pi i} + e^{2(\beta-\alpha)\pi i},$$

are, by construction, invariants of the monodromy of $\mathcal{H}_{\text{proj}}$. Thus, they are constant along any isomonodromic path. A simple calculus shows that they are functionally independent on the very general point of S . It yields the result. \square

Corollary 3.26 *The differential Galois group with parameters of $\mathcal{H}_{\text{proj}}$ consists in all the gauge symmetries of $\mathcal{H}_{\text{proj}}$, $\text{Gal}_{\infty}(\mathcal{H}_{\text{proj}}) = \text{Gau}(J(\mathcal{H}_{\text{proj}}))$.*

PROOF. It is a direct consequence of Lemma 3.25 and Theorem 3.23. \square

3.3.2. Differential equation for the determinant

Another reduction of (3-11) is given by the quotient map $P \rightarrow P_{\text{det}} = \mathbb{C}^* \times B$, $(U, x, \alpha, \beta, \gamma) \mapsto (w = \det(U), x, \alpha, \beta, \gamma)$. The projection is a morphism of principal bundles and it sends \mathcal{H} to a partial connection \mathcal{H}_{det} in P_{det} defined by the differential system:

$$dw - \frac{(\alpha + \beta + 1)x - \gamma}{x(1-x)} w dx = 0, \quad d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0. \quad (3-14)$$

It is convenient to consider an affine change of variables in the parameter space, by setting:

$$\begin{aligned} a &= -\gamma \\ b &= \alpha + \beta - \gamma + 1 \\ c &= \alpha \end{aligned}$$

The equation for horizontal sections of \mathcal{H}_{\det} is now written as:

$$\frac{dw}{dx} = \left(\frac{a}{x} - \frac{b}{1-x} \right) w. \quad (3-15)$$

And its general solution is:

$$w = f(a, b, c) x^a (1-x)^b.$$

Let us compute the Galois group with parameters of such an equation. Note that

$$\mathbb{C}(J(\mathcal{H}_{\det})) = \mathbb{C}(x, a, b, c, w, w_a, w_b, w_c, w_{aa}, \dots) = \mathbb{C}(x, a, b, c) \langle w \rangle.$$

Parameter c does not appear in the equation. This means that in $\mathbb{C}(J(\mathcal{H}_{\det}))$, the differential function w_c is an invariant. Thus,

$$\mathbb{C} \langle w_c \rangle \subset \mathbb{C}(J(\mathcal{H}_{\det}))^{\mathcal{H}_{\det}^\infty}.$$

For the rest of the computation let us ignore the derivatives with respect to the parameter c and let us consider the equation as dependent only of parameters a and b . This is equivalent to projecting the equation onto a quotient basis B' . Let us consider the equation for the second prolongation $\mathcal{H}_{\det}^{(2)}$. It is:

$$\begin{aligned} w_x &= \left(\frac{a}{x} - \frac{b}{1-x} \right) w \\ w_{ax} &= \frac{1}{x} w + \left(\frac{a}{x} - \frac{b}{1-x} \right) w_a \\ w_{bx} &= \frac{1}{1-x} w + \left(\frac{a}{x} - \frac{b}{1-x} \right) w_b \\ w_{aax} &= \frac{2}{x} w_a + \left(\frac{a}{x} - \frac{b}{1-x} \right) w_{aa} \\ w_{abx} &= \frac{1}{1-x} w_a + \frac{1}{x} w_b + \left(\frac{a}{x} - \frac{b}{1-x} \right) w_{ab} \\ w_{bbx} &= \frac{2}{1-x} w_b + \left(\frac{a}{x} - \frac{b}{1-x} \right) w_{bb} \end{aligned}$$

At this level, there appear three rational first integrals of $\mathcal{H}_{\det}^{(2)}$, namely:

$$R = \frac{w_{aa}}{w} - \frac{w_a^2}{w^2}, \quad S = \frac{w_{ab}}{w} - \frac{w_a w_b}{w^2}, \quad T = \frac{w_{bb}}{w} - \frac{w_b^2}{w^2}.$$

Thus, $\mathbb{C}(J(\mathcal{H}_{\det}))^{\mathcal{H}_{\det}^\infty}$ contains the \mathbb{C}^* -invariant $\mathfrak{X}(B)^{\text{tot}}$ -differential field spanned by w_c, R, S and T . Let us compute the differential B -group associated to this field.

The gauge group $J(\text{Gau}(P/B))$ is in this case isomorphic to $J(B, \mathbb{C}^*)$. On the other

hand, we have that the adjoint equation in dimension 1 vanish, so we also have $\sigma_x = 0$. From the rational invariants we have that the equations for the Galois group $\text{Gal}_\infty(\mathcal{H}_{\det})$ should include:

$$\sigma_x = 0, \quad \sigma_c = 0, \quad \sigma_{aa} = \frac{\sigma_a^2}{\sigma}, \quad \sigma_{ab} = \frac{\sigma_a \sigma_b}{\sigma}, \quad \sigma_{bb} = \frac{\sigma_b^2}{\sigma}.$$

The three last equations are equivalent to say that σ has constant logarithmic derivative with respect to a and b . So, these equations define a differential algebraic group, which is in fact a group bundle with fibers of dimension 3 over B .

$$\mathcal{G} = \{j_p \sigma : \sigma_x = \sigma_c = \partial_a(\sigma_a/\sigma) = \partial_b(\sigma_a/\sigma) = \partial_b(\sigma_b/\sigma) = 0\}.$$

Let us observe that,

$$\mathcal{G} \simeq \dots \simeq \mathcal{G}_k \simeq \dots \simeq \mathcal{G}_3 \simeq \mathcal{G}_2 \simeq \mathcal{G}_1 \rightarrow B.$$

Let us see that the Galois group is exactly this group.

Lemma 3.27 *Let us fix a and b complex numbers \mathbb{Q} -linearly independent. Let us consider the differential system with coefficients in $\mathbb{C}(x)$:*

$$\frac{d}{dx} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{a}{x} - \frac{b}{1-x} & 0 & 0 \\ \frac{1}{x} & \frac{a}{x} - \frac{b}{1-x} & 0 \\ \frac{-1}{1-x} & 0 & \frac{a}{x} - \frac{b}{1-x} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}$$

Its Galois group is a connected group of dimension 3.

PROOF. A fundamental matrix of solutions:

$$\begin{pmatrix} x^a(1-x)^b & 0 & 0 \\ \log(x)x^a(1-x)^b & x^a(1-x)^b & 0 \\ \log(1-x)x^a(1-x)^b & 0 & x^a(1-x)^b \end{pmatrix}.$$

We may compute its monodromy group. For rationally independent a and b its Zariski closure is a connected group of dimension 3. \square

We have that $\text{Gal}_1(\mathcal{H}_{\det}^{(1)})_{(x,a,b,c)}$ must contain the monodromy at (x, a, b, c) of the connection P_{\det} . By Lemma 3.27 we have that the Zariski closure of the monodromy at (x, a, b, c) contains $\mathcal{G}_{1,(x,a,b,c)}$. Therefore we have:

$$\text{Gal}_1(\mathcal{H}_{\det}^{(1)})_{(x,a,b,c)} \subset \mathcal{G}_{1,(x,a,b,c)} \subset \text{Gal}_1(\mathcal{H}_{\det}^{(1)})_{(x,a,b,c)}.$$

Thus, we have the equality. The equations of the Galois group can be easily integrated, so we obtain a description of the functions whose jets are elements of $\text{Gal}_\infty(\mathcal{H}_{\det})$.

Proposition 3.28 *The differential Galois group with parameters of \mathcal{H}_{\det} is*

$$\mathrm{Gal}_{\infty}(\mathcal{H}_{\det}) = \left\{ j_{(x_0, \alpha_0, \beta_0, \gamma_0)} \sigma : \sigma(x, \alpha, \beta, \gamma) = \exp(\mu_0 + \mu_1 \gamma + \mu_2(\alpha + \beta)), \mu_0, \mu_1, \mu_2 \in \mathbb{C} \right\},$$

which is the set of differential zeroes of the Hopf differential ideal,

$$\left\{ \partial_x \sigma, \partial_c \sigma, \partial_a \frac{\partial_a \sigma}{\sigma}, \partial_a \frac{\partial_b \sigma}{\sigma}, \partial_b \frac{\partial_b \sigma}{\sigma} \right\} \subset \mathcal{O}_{J(\mathbb{C}^*/B)},$$

where:

$$\partial_a = -\partial_{\beta} - \partial_{\gamma},$$

$$\partial_b = \partial_{\beta},$$

$$\partial_c = \partial_{\alpha} - \partial_{\beta}.$$

3.3.3. Computation of the Galois group

We have a 2-covering group morphism

$$\varphi: \mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{PGL}_2(\mathbb{C}) \times \mathbb{C}^*, \quad A \mapsto ([A], \det(A)).$$

Its kernel is $\{-1, 1\}$. Let us set $G_{\mathrm{red}} = \mathrm{GL}_2(\mathbb{C})/\{-1, 1\}$, $P_{\mathrm{red}} = P/\{-1, 1\}$. So P_{red} is G_{red} -principal bundle. The hypergeometric distribution \mathcal{H} induces $\mathcal{H}_{\mathrm{red}}$ in P_{red} .

We have an isomorphism

$$\psi: P_{\mathrm{red}} \xrightarrow{\sim} P_{\mathrm{proj}} \times_B P_{\mathrm{det}},$$

that yields a decoupling $\psi_*(\mathcal{H}_{\mathrm{red}}) = \mathcal{H}_{\mathrm{proj}} \times_B \mathcal{H}_{\mathrm{det}}$. From the commutation of direct products and jets, we have that $\mathrm{Gal}_{\infty}(\mathcal{H}_{\mathrm{red}})$ is a differential algebraic subgroup of $\mathrm{Gal}_{\infty}(\mathcal{H}_{\mathrm{proj}}) \times_B \mathrm{Gal}_{\infty}(\mathcal{H}_{\mathrm{det}})$ that projects surjectively onto both components. We will make use of the following.

Lemma 3.29 *Let G be a simple algebraic group and B an algebraic variety. $J(G/B)$ is simple as a differential algebraic group.*

PROOF. Let $H \triangleleft J(G/B)$ be a differential algebraic subgroup. Then H_0 is rational group subbundle of $G \times B \rightarrow B$, therefore either $H_0 = G \times B$ or $H_0 = \{\mathrm{Id}\} \times B$. In the latter case H is the identity subgroup bundle $J(G/B)$. In the former case, by Theorem A.9 we have $H = J(\mathcal{C})$ for a singular foliation \mathcal{F} on B and a group \mathcal{F} -connection \mathcal{C} .

Let us recall the classical decomposition of the tangent group $TG \simeq \mathrm{Lie}(G) \ltimes G$. Let us fix x a regular point of \mathcal{F} . The 1-jet at x of a function from B to G can be identified with its derivative which is a linear map from $T_x B$ to a fiber of $TG \rightarrow G$. Therefore we have

semidirect product decomposition,

$$J_1(G/B)_x \simeq \text{Lin}_{\mathbb{C}}(T_x B, \text{Lie}(G)) \ltimes G.$$

On the other hand, let us consider \mathcal{L} the leaf of \mathcal{F} that passes through x . The same argument applies, so we have

$$J_1(G/\mathcal{L})_x \simeq \text{Lin}_{\mathbb{C}}(\mathcal{F}_x, \text{Lie}(G)) \ltimes G.$$

The value of the group \mathcal{F} -connection \mathcal{C} at x can be seen as a section \mathcal{C}_x from G to $J_1(G/\mathcal{L})_x$ that assigns to any initial condition g in G the only \mathcal{C} -horizontal leaf that passes through (x, g) . In the following diagram, where π_x is the restriction to the leaf \mathcal{L} , we have $H_{1,x} = \pi_x^{-1}(\mathcal{C}_x(G))$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Lin}_{\mathbb{C}}(T_x B, \text{Lie}(G)) & \longrightarrow & J_1(G/B)_x & \longrightarrow & G \longrightarrow \{\text{Id}\} \\ & & \downarrow & & \downarrow \pi_x & & \downarrow \\ 0 & \longrightarrow & \text{Lin}_{\mathbb{C}}(\mathcal{F}_x, \text{Lie}(G)) & \longrightarrow & J_1(G/\mathcal{L})_x & \longrightarrow & G \longrightarrow \{\text{Id}\} \\ & & & & \swarrow \mathcal{C}_x & & \end{array}$$

If $H_{1,x} \triangleleft J_1(G/B)_x$ then $\mathcal{C}_x(G) \triangleleft J_1(G/\mathcal{L})_x$. However, this would produce an isomorphic direct product decomposition $J_1(G/\mathcal{L})_x \simeq \text{Lin}(\mathcal{F}_x, \text{Lie}(G)) \times G$, which does not exist as the adjoint representation is not trivial. Hence, $H_{1,x}$ is not a normal subgroup, except in the case in which \mathcal{F} is the foliation by points. \square

Lemma 3.30 $\text{Gal}_{\infty}(\mathcal{H}_{\text{red}}) = \text{Gal}_{\infty}(\mathcal{H}_{\text{proj}}) \times_B \text{Gal}_{\infty}(\mathcal{H}_{\text{det}}).$

PROOF. Here we follow a reasoning of Kolchin [Kol68]. Consider the projections

$$\pi_{\text{proj}}: \text{Gal}_{\infty}(\mathcal{H}_{\text{red}}) \rightarrow \text{Gal}_{\infty}(\mathcal{H}_{\text{proj}}) \quad \text{and} \quad \pi_{\text{proj}}: \text{Gal}_{\infty}(\mathcal{H}_{\text{red}}) \rightarrow \text{Gal}_{\infty}(\mathcal{H}_{\text{det}}).$$

The kernel of π_{red} is a differential algebraic group of the form,

$$\ker(\pi_{\text{proj}}) = \text{Id} \times_B K^{\text{det}}$$

where K^{det} is a differential algebraic subgroup of $\text{Gal}_{\infty}(\mathcal{H}_{\text{det}})$. Moreover, for generic $p \in B$ and any order k we have $K_{k,p}^{\text{det}}$ is a normal algebraic subgroup of $\text{Gal}_k(\mathcal{H}_{\text{det}})_p$. For the same reason we have that the kernel of π_{proj} is a differential algebraic group of the form,

$$\ker(\pi_{\text{proj}}) = K^{\text{proj}} \times_B \text{Id}$$

where K^{proj} is a differential algebraic subgroup of $\text{Gal}_{\infty}(\mathcal{H}_{\text{det}})$. Moreover, for generic $p \in B$ and any order k we have $K_{k,p}^{\text{proj}}$ is a normal algebraic subgroup of $\text{Gal}_k(\mathcal{H}_{\text{proj}})_p$. The subgroup

$\text{Gal}_k(\mathcal{H})_p$ gives the graph of an algebraic group isomorphism

$$\text{Gal}_k(\mathcal{H}_{\text{proj}})_p / K_{k,p}^{\text{proj}} \simeq \text{Gal}_k(\mathcal{H}_{\text{det}})_p / K_{k,p}^{\text{det}}.$$

Therefore, the quotient $\text{Gal}_k(\mathcal{H}_{\text{proj}})_p / K_{k,p}^{\text{proj}}$ is for any k an abelian group of dimension smaller or equal than three. By Lemma 3.29 the only possibility is $\text{Gal}_k(\mathcal{H}_{\text{proj}})_p = K_{k,p}^{\text{proj}}$. This is for generic p and all k , therefore $\text{Gal}_{\infty}(\mathcal{H}_{\text{proj}}) = K^{\text{proj}}$ and it follows the required statement. \square

The quotient map $\pi_{\text{red}}: P \rightarrow P_{\text{red}}$ induces a 2-covering group bundle morphism

$$\pi_{\text{red}*}: \text{Gau}(J(\mathcal{H})) \rightarrow \text{Gau}(J(\mathcal{H}_{\text{red}}))$$

whose restriction to the Galois group bundles gives a surjective map $\pi_{\text{red}}: \text{Gal}_{\infty}(\mathcal{H}) \rightarrow \text{Gal}_{\infty}(\mathcal{H}_{\text{red}})$. This morphism may be either an isomorphism or a 2-covering. It turns out that $\pi_{\text{red}}^{-1}(\text{Gal}_{\infty}(\mathcal{H}_{\text{red}}))$ is connected, and the smallest differential algebraic subgroup of $\text{Gau}(J(\mathcal{H}))$ that projects surjectively onto $\text{Gal}_{\infty}(\mathcal{H})_{\text{red}}$. Therefore,

$$\text{Gal}_{\infty}(\mathcal{H}) = \pi_{\text{red}*}^{-1}(\text{Gal}_{\infty}(\mathcal{H}_{\text{red}})).$$

Theorem 3.31 *The differential Galois group with parameters of Gauss' hypergeometric equation is given by*

$$\text{Gal}_{\infty}(\mathcal{H}) = \{j\sigma \in \text{Gau}(J(\mathcal{H})) \mid j\det(\sigma) \in \text{Gal}_{\infty}(\mathcal{H}_{\text{det}})\} \quad (3-16)$$

which is defined by the differential equations

$$\partial_x \sigma = \left[\begin{pmatrix} 0 & 1 \\ \frac{\alpha\beta}{x(1-x)} & \frac{(\alpha+\beta+1)x-\gamma}{x(1-x)} \end{pmatrix}, \sigma \right],$$

$$\partial_x \det(\sigma) = \partial_c \det(\sigma) = \partial_a \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_a \det(\sigma)}{\det(\sigma)} \right) = \partial_b \left(\frac{\partial_b \det(\sigma)}{\det(\sigma)} \right) = 0,$$

where σ is an invertible 2×2 matrix depending on x, α, β, γ and the differential operators $\partial_a, \partial_b, \partial_c$ are those given in Proposition 3.28.

4. Non linear Galois theory

4.1. Gauge groupoid of isomorphisms

We introduce the following notation: Let $\mathbb{C}[M]_p$ denote the set of germs functions in M at p . Write $(M, p) = \text{spec } \mathcal{O}_{M,p}$ and $(M, p)_k = \text{spec } \mathbb{C}[M]_p / \mathfrak{m}_p^{k+1}$ where \mathfrak{m}_p is the maximal ideal of functions vanishing at p . In the limit we get $(\widehat{M}, p) = \text{spf } \widehat{\mathbb{C}[M]_p}$, its completion in the \mathfrak{m}_p -adic topology. The purpose of this notation is the following. A k -jet of a local holomorphic map sending p to q is interpreted as a morphism $(M, p)_k \rightarrow (M, q)_k$. In the same way, a jet of infinite order is seen as a morphism $(\widehat{M}, p) \rightarrow (\widehat{M}, q)$. This interpretation is compatible with the composition, and therefore it allows us to illustrate some situations of jet composition with commutative diagrams and without the necessity of taking representatives where it is no need of it.

Definition 4.1 Let $\text{Aut}_k(M)$ be the groupoid of k -jets of local invertible biholomorphisms of M .

We have that its limit, $\widehat{\text{Aut}}(M)$, is a pro-algebraic groupoid. This space comes with source and target maps: An element ϕ of $\widehat{\text{Aut}}(M)$ is a formal map $\phi : (\widehat{M}, p) \rightarrow (\widehat{M}, q)$. Then source and target are given by $s(\phi) = p$ and $t(\phi) = q$. Partial composition of two compatible jets ϕ_1 and ϕ_2 is given by $\phi_1 \circ \phi_2$. Finally inversion map is $i(\phi) = \phi^{-1}$.

From Theorem 2.22 in Chapter 2, p.44, it follows that $R_k M$ is an $\text{Aut}_k(\mathbb{C}^m, 0)$ -principal bundle. Therefore $RM \rightarrow M$ is a pro-algebraic principal $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -bundle. They have their corresponding groupoid of isomorphisms:

Definition 4.2 Let $\text{Iso}(R_k M)$ be the set of $\text{Aut}_k(\mathbb{C}^m, 0)$ -equivariant bijections between one fiber of $R_k M$ and another. The limit $\lim_{\leftarrow} \text{Iso}(R_k M) = \text{Iso}(RM)$ is the set of $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -equivariant bijections between one fiber of RM and another.

Lemma 4.3 For each k , $\text{Iso}(R_k M) = \text{Aut}_k(M)$.

Observe that $\text{Aut}_k(M)$ is an algebraic Lie groupoid. There is an additional equivalent definition for $\text{Iso}(RM)$: it is the quotient of RM by the diagonal action of $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$:

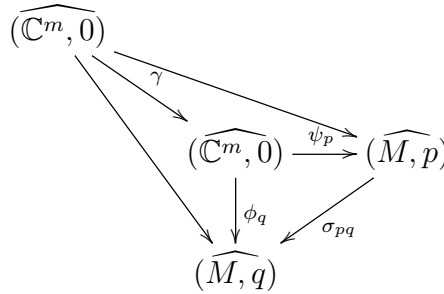
$$\text{Iso}(RM) = (RM \times RM) / \widehat{\text{Aut}}(\mathbb{C}^m, 0)$$

We have mappings

- $\Phi : RM \times RM \rightarrow \text{Iso}(RM)$ such that $j\theta \times j\beta \mapsto \phi_{j\theta, j\beta}$.
- $\Psi : RM \times RM \rightarrow \widehat{\text{Aut}}M$ such that $j\theta \times j\beta \mapsto j(\beta \circ \theta^{-1})$.

Proposition 4.4 $\widehat{\text{Aut}}(M) = (RM \times RM) / \widehat{\text{Aut}}(\mathbb{C}^m, 0)$.

PROOF. The map Ψ descends into the quotient and is well defined there: Let ψ_p be a frame at $p \in M$, ϕ_q a frame at $q \in M$ and $\sigma_{pq} : (\widehat{M}, p) \rightarrow (\widehat{M}, q)$. Let $\gamma \in \widehat{\text{Aut}}(\mathbb{C}^m, 0)$ be a group element. Then $\Psi(\psi_p \circ \gamma, \phi_q \circ \gamma) = \phi_q \circ \gamma \circ (\psi_p \circ \gamma)^{-1} = \phi_q \circ \psi_p^{-1} = \Psi(\psi_p, \phi_q)$. The map Ψ is readily seen to be a bijection and groupoid morphism so the desired result follows. □



Lemma 4.5 *There are two actions over RM ,*

$$\widehat{\text{Aut}}(M) \times_M RM \times \widehat{\text{Aut}}(\mathbb{C}^m, 0) \rightarrow RM$$

The first action $\widehat{\text{Aut}}(M) \times_M RM$ is the action of the isomorphism groupoid. The second is the action $RM \times \widehat{\text{Aut}}(\mathbb{C}^m, 0)$ of the structural group.

PROOF. The action of $RM \times \widehat{\text{Aut}}(\mathbb{C}^m, 0)$ was described in Theorem 2.22. The action of the groupoid is given by the action of $\text{Iso } RM$ on RM . □

It follows that $\text{Iso}(RM)$ coincides with $\widehat{\text{Aut}}(M)$.

4.2. Malgrange Pseudogroup

4.2.1. Malgrange Groupoid of a vector field

Let X be a rational vector field on M which, by definition, is regular in some open subset M' of M . We are interested in applying differential Galois theory, as developed in Chapter 1 to some rational connections on M that are regular along M' . Thus, without loss of generality let us assume that $M = M'$.

By Theorem 2.22 the bundle $R_k M \rightarrow M$ is a principal $\text{Aut}_k(\mathbb{C}^m, 0)$ -bundle. By Proposition 2.17 the prolongation $X^{(k)}$ (see p.56, Definition 2.23) is $\text{Aut}_k(\mathbb{C}^m, 0)$ -invariant. The

distribution $\langle X^{(k)} \rangle$, spanned by $X^{(k)}$, projects onto $\langle X \rangle$. Therefore $\langle X^{(k)} \rangle$ is a $\text{Aut}_k(\mathbb{C}^m, 0)$ -invariant $\langle X \rangle$ -connection. There is a differential Galois groupoid associated to it by means of Definition 1.55.

Definition 4.6 The *order k Malgrange groupoid* of a vector field X is $\text{Mal}_k(X) := \text{GGal}(\langle X^{(k)} \rangle)$ which is a subset of $\text{Aut}_k M$.

Definition 4.7 A *order k differential invariant* of X is a rational first integral of $X^{(k)}$. Order k rational invariants of X form a field, $\mathbb{C}(\text{R}_k M)^X \subset \mathbb{C}(\text{R}_k M)$.

An analysis of Definition 4.6 and Galois correspondence 1.53 tell us that $\text{Mal}_k(X)$ is the rational subgroupoid of $\text{Aut}_k M$ corresponding to the field of order k rational differential invariants $\mathbb{C}(\text{R}_k M)^X$. Note that, by definition of differential invariants $\mathbb{C}(\text{R}_k M)^X = \mathbb{C}(\text{R}_k M)^{\langle X^{(k)} \rangle}$. There is a chain of field inclusions and groupoid projections:

$$\begin{aligned} \cdots \subset \mathbb{C}(\text{R}_k M)^X \subset \mathbb{C}(\text{R}_{k+1} M)^X \subset \cdots \\ \cdots \longleftarrow \text{Mal}_k(X) \longleftarrow \text{Mal}_{k+1}(X) \longleftarrow \cdots \end{aligned}$$

Let us see that this chain is compatible with the differential structure given in RM by the total derivatives with respect to the infinitesimal parameters ε . Let us recall that we use $\partial/\partial\varepsilon$ as an abbreviation of $\{\partial/\partial\varepsilon_1, \dots, \partial/\partial\varepsilon_m\}$.

Proposition 4.8 $\mathbb{C}(RM)$ is a (classical) $\partial/\partial\varepsilon$ -field.

PROOF. We know from chapter 2 that $\mathbb{C}[\hat{\tau}_m M]$ is a classical differential ring with derivations given by the total derivatives $\partial/\partial\varepsilon$. As $RM \subset \hat{\tau}_m M$, we have $\mathbb{C}[RM]$ inherits the $\partial/\partial\varepsilon$ -ring structure. A result from differential algebra, for example [CH11] p.121, tell us that $\mathbb{C}(RM)$ can be given a $\partial/\partial\varepsilon$ -ring structure as desired. \square

Let us define the *field of rational differential invariants* of X as the union of the fields of rational differential invariants on any order, $\mathbb{C}(RM)^X = \bigcup_k \mathbb{C}(\text{R}_k M)^X$. Note that, since the restriction of $X^{(\infty)}$ to any field $\mathbb{C}(\text{R}_k M)$ yields $X^{(k)}$ we have $\mathbb{C}(RM)^X = \mathbb{C}(RM)^{\langle X^{(\infty)} \rangle}$.

Lemma 4.9 $\mathbb{C}(RM)^X$ is a $\frac{\partial}{\partial\varepsilon}$ -subfield.

PROOF. If f is an invariant of order k , then $\frac{\partial}{\partial\varepsilon_i} f$ is an invariant of order $k+1$. This follows because prolongation commutes with total derivatives. \square

Corollary 4.10 $\mathbb{C}(RM)^X$ is a $\frac{\partial}{\partial\varepsilon}$ -finitely generated.

PROOF. It comes automatically from Lemma 4.9 and Theorem 1.8. \square

Definition 4.11 Let \mathbb{K} be a $\frac{\partial}{\partial \varepsilon}$ -subfield of $\mathbb{C}(RM)$. We consider its truncations, $\mathbb{K}_j = \mathbb{K} \cap \mathbb{C}(R_j M)$. For each \mathbb{K}_j defines the rational grupoid $\mathcal{G}_j \subset \text{Aut}_k(M)$ of invariance of \mathbb{K}_j . The \mathcal{D} -groupoid defined by \mathbb{K} is $\mathcal{G} = \lim_{\leftarrow} \mathcal{G}_k$.

Theorem 1.8 guarantees that \mathbb{K} is differentially spanned by one of the truncations \mathbb{K}_i . Therefore \mathcal{G} is determined by \mathcal{G}_k for some finite k , and \mathcal{G} is rational subgroupoid of $\widehat{\text{Aut}}(M)$.

Definition 4.12 The Malgrange groupoid of X is $\widehat{\text{Mal}}(X)$ the rational \mathcal{D} -groupoid corresponding to the field of rational differential invariants of X .

As before we have a sequence,

$$\widehat{\text{Mal}}(X) \dots \longrightarrow \text{Mal}_{k+1}(X) \longrightarrow \text{Mal}_k(X) \longrightarrow \dots \longrightarrow \text{Mal}_0(X) \subset M \times M,$$

and $\widehat{\text{Mal}}(X)$ is identified with the limit $\varprojlim \text{Mal}_k(X) \subset \widehat{\text{Aut}}(M)$. It is determined by $\text{Mal}_k(X)$ for some finite k .

4.2.2. \mathcal{D} -Lie algebra of a \mathcal{D} -groupoid

Let \mathcal{G} be \mathcal{D} -groupoid defined as the corresponding groupoid of a subfield of invariants $\mathbb{C}(RM)^{\mathcal{G}} \subset \mathbb{C}(RM)$ and such that $\mathcal{G} = \lim_{\leftarrow} \mathcal{G}_k$.

Definition 4.13 The \mathcal{D} -Lie algebra of \mathcal{G}_k is given by

$$\text{Lie } \mathcal{G}_k = \{j_p X \mid X^{(k)} \mathbb{C}(R_k M)^{\mathcal{G}} = 0\}$$

and $\text{Lie } \mathcal{G}$, the \mathcal{D} -Lie algebra of \mathcal{G} , is the limit $\lim_{\leftarrow} \text{Lie } \mathcal{G}_k$.

For more details on Malgrange groupoid, see [Mal01, Mal10b, Dav16].

4.2.3. Specialization theorem

Let $\pi : M \rightarrow S$ be a surjective smooth map of smooth varieties with smooth connected fibers. For each $s \in S$ we consider M_s the π -fibre of $\{s\}$. Let X be a rational vector field tangent to π i.e. such that $d\pi(X) = 0$. We assume that X restricts to a rational vector field $X|_{M_s}$ at each fiber M_s . Davy in [Dav16] p.63 introduces the set of partial frames over the fibers of π ,

$$R(M/S) = \left\{ (\widehat{\mathbb{C}^m}, 0) \longrightarrow (\widehat{M_s}, p) \right\}.$$

The set $R(M/S)$ is an $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -principal bundle. Observe that for $s \in S$ we have $R(M/S)|_{M_s} = R(M_s)$. The vector field X can be prolonged to a vector field $(X/S)^{(\infty)}$ such that $(X/S)^{(\infty)}|_{R(M_s)} =$

$(X|_{M_s})^{(\infty)}$. We have that $\langle (X/S)^{(\infty)} \rangle$ is a partial $\widehat{\text{Aut}}(\mathbb{C}^m, 0)$ -invariant $\langle X \rangle$ -connection. Partial Malgrange groupoid is defined as,

$$\text{Mal}(X/S) = \text{GGal}(\langle (X/S)^{\infty} \rangle).$$

For each $s \in S$ we have that $\widehat{\text{Aut}}(M_s)$ is a Zarisky closed subset of $\text{Iso}(\text{R}(M/S))$. Thus, we can speak of the restriction of the partial Galois groupoid to a fibre,

$$\text{Mal}(X/S)|_{M_s} = \text{Mal}(X/S) \cap \widehat{\text{Aut}}(M_s).$$

Note that this restriction is \mathcal{D} -groupoid in M for generic values of s . The following result is a direct consequence of the specialization Theorem ([Dav16], Th. 3.6.1).

Theorem 4.14 *For all $s \in S$, $\text{Mal}(X|_{M_s}) \subset \text{Mal}(X/S)|_{M_s}$.*

4.2.4. Projection theorem

Let $P \rightarrow M$ be a G -bundle, \mathcal{F} a foliation in M and \mathcal{D} a G -invariant \mathcal{F} -connection. Assume we have a subgroup K of G and write $\overline{G} = G/K$. The quotient $q : P \rightarrow \overline{P} = P/K$ is a map over M . It follows that \overline{P} is a \overline{G} -principal bundle. The actions commute: $q(xg) = q(x)\phi(g)$. The induced distribution $\overline{\mathcal{D}} = q_*\mathcal{D}$ is \overline{G} -principal distribution. There is also a projection of groupoids

$$\begin{array}{ccc} \text{Iso } P & \longrightarrow & \text{Iso } \overline{P} \\ & \searrow s,t & \downarrow s,t \\ & & M \end{array}$$

where the class $[p, q]$ is sent to $[\overline{p}, \overline{q}]$.

Theorem 4.15 *The map $\text{GGal}(\mathcal{D}) \longrightarrow \text{GGal}(\overline{\mathcal{D}})$ is surjective.*

PROOF. See [Cas09]. □

Let G be the subgroup of $\text{Aut}(\mathbb{C}^m, 0)$ of maps that leave $\mathbb{C}^m \subset \mathbb{C}^n$ invariant, where we identify \mathbb{C}^m inside \mathbb{C}^n as $\mathbb{C}^m = \{\varepsilon_{m+1} = \dots = \varepsilon_n = 0\}$. Explicitly we have

$$G = \left\{ \phi : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^m, 0) \mid \frac{\phi_i}{\partial \varepsilon_j} = 0 \text{ for } i = 1, \dots, m; \quad j = m+1, \dots, n \right\}$$

Let us consider $\text{R}_k^\pi M$ the set of k -jets of biholomorphisms $(\mathbb{C}^m, 0) \rightarrow (M, p)$ that send the subspace \mathbb{C}^m to the fiber $M_{\pi(p)}$. If a frame is in $\text{R}_k^\pi M$ then it can be restricted to $(\mathbb{C}^m, 0)$ obtaining a frame of the fiber. By taking projective limit we obtain $\text{R}^\pi M \subset \text{RM}$, a reduction of RM to the subgroup G . The gauge groupoid $\text{Iso}(\text{R}^\pi M)$ is identified with $\widehat{\text{Aut}}(M)^\pi$ the subgroupoid of $\widehat{\text{Aut}}(M)$ of formal maps respecting the projection π . As $d\pi(X) = 0$ we have

$X^{(\infty)}$ is tangent to $R^\pi M$ and $\text{Mal}(X) \subset \widehat{\text{Aut}}(M)$. G acts on $R^\pi M$. There is a natural exact sequence,

$$0 \rightarrow K \rightarrow G \rightarrow \text{Aut}(\mathbb{C}^m, 0) \rightarrow 0$$

given by the restriction to \mathbb{C}^m . Here, K is the subgroup of formal maps in \mathbb{C}^n inducing the identity in \mathbb{C}^m . A frame in $R^\pi M$ can be restricted to \mathbb{C}^m , and therefore we obtain a frame on a fiber of π . Thus, we have a projection,

$$R^\pi M \longrightarrow R^\pi(M)/K \simeq R(M/S), \quad \phi \mapsto \phi|_{\mathbb{C}^m}.$$

It is straightforward that $X^{(\infty)}$ is projectable and projects onto $(X/S)^\infty$. By Theorem 4.15 we get a surjective map,

$$\text{Mal}(X) \longrightarrow \text{Mal}(M/S).$$

We define the restriction of the Malgrange groupoid of X to a fiber as $\text{Mal}(X)|_{M_s} = \text{Mal}(X/S)|_{M_s}$. From Theorem 4.14 we get:

Corollary 4.16 *For all $s \in S$, $\text{Mal}(X)|_{M_s} \supset \text{Mal}(X|_{M_s})$.*

4.3. Malgrange pseudogroup of Painlevé VI equation

4.3.1. Hamiltonian form for Painlevé VI

Painlevé VI equation is a second order differential equation for function u of x a complex variable of the form (see, for example, [IKSY13] p.119, and [O⁺80])

$$u'' = F(x, u, u', a, b, c, e); \quad u' = v \tag{4-1}$$

where $F \in \mathbb{C}(x, u, v, a, b, c, e)$ is

$$\begin{aligned} F(x, u, u', a, b, c, e) = & \frac{1}{2} \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x} \right) u'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x} \right) u' \\ & + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \left(\frac{1}{2}c^2 - \frac{1}{2}a^2 \frac{x}{u^2} + \frac{1}{2}b^2 \frac{x-1}{(u-1)^2} + \left(\frac{1-\epsilon^2}{2} \right) \frac{x(x-1)}{(u-x)^2} \right). \end{aligned}$$

The variables a, b, c, e are the parameters of Painlevé VI equation. However, our interest is to consider the role of the parameters in non-linear differential Galois theory. Therefore, we see Painlevé VI equation as the following rational vector field in \mathbb{C}^7 ,

$$Y = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + F(x, u, u', a, b, c, e) \frac{\partial}{\partial y'}. \tag{P_{VI}}$$

The trajectories of this vector field, parameterized by x are $(x, u(x), u'(x), a, b, c, e)$ for u a solution of P_{VI} with fixed parameters.

Equation Painlevé VI (4-1) admits the following equivalent Hamiltonian form (see [IKSY13] p.140) with hamiltonian function $H \in \mathbb{C}(x, p, q, a, b, c, e)$ such that

$$H = \frac{1}{x(x-1)} \left[p(p-1)(p-x)q^2 - \left(a(p-1)(p-x) + bp(p-x) + (e-1)p(p-1) \right) q \right. \\ \left. + \frac{1}{4} \left((a+b+e-1)^2 - c^2 \right) (p-x) \right].$$

and hamiltonian vector field

$$X = \frac{\partial}{\partial x} + \frac{\partial H}{\partial q} \frac{\partial}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial}{\partial q}. \quad (HP_{VI})$$

The map $\phi : \mathbb{C}^7 \rightarrow \mathbb{C}^7$ such that

$$(x, p, q, a, b, c, e) \mapsto \left(x, p, \partial H / \partial q, \frac{1}{2}c^2, \frac{1}{2}a^2, \frac{1}{2}b^2, \frac{1}{2}e^2 \right)$$

sends the vector field X onto the vector field Y , giving the equivalence between systems (4-1) and (HP_{VI}) . Our purpose is to compute the Malgrange groupoid of vector field (HP_{VI}) .

4.3.2. Malgrange groupoid for P_{VI}

Painlevé VI associated vector field HP_{VI} gives raise to a connection $\langle X \rangle$ in the trivial bundle

$$\begin{array}{ccc} M = \mathbb{C}^7 & \xrightarrow{\pi} & \mathbb{C}_{x,a,b,c,e}^5 = B \\ & \searrow \bar{\rho} & \downarrow \rho \\ & & \mathbb{C}_{a,b,c,e}^4 = S \end{array}$$

From definition 4.12 the Malgrange groupoid of a vector field X is given by:

$$\text{Mal}(X) = \left\{ \phi \in \widehat{\text{Aut}} M \mid \text{for all } f \in \mathbb{C}(\text{RM})^X, \ f \circ \phi_{p,q} = f \right\}$$

where $\phi : (\widehat{M}, p) \rightarrow (\widehat{M}, q)$ is an invertible formal map, and $\phi_{p,q} : R_p M \rightarrow R_q M$ is its corresponding $\widehat{\text{Aut}}(\mathbb{C}^7, 0)$ -equivariant map.

Remark 4.17 From the known invariants of X we can obtain some information of its Malgrange groupoid. First, the conserved quantities a, b, c, e are by themselves rational differential invariants of order 0. Therefore $\mathbb{C}\langle a, b, c, e \rangle_{\frac{\partial}{\partial \varepsilon}} \subset \mathbb{C}(\text{RM})^X$. The \mathcal{D} -groupoid corresponding to such field is the groupoid of formal maps respecting the projection $\bar{\rho}$. Therefore,

$$\text{Mal}(X) \subset \left\{ \phi \in \widehat{\text{Aut}}(\mathbb{C}^7) \mid \bar{\rho} \circ \phi = \bar{\rho} \right\}$$

Also, from geometric invariants we can obtain information. There is an intrinsic connection between geometric structures and \mathcal{D} -groupoids. As it is shown ([Cas04] Theorem 1.3.2 p. 10) any transitive \mathcal{D} -groupoid is the groupoid of invariance of a geometric structure. Therefore, any geometric structure invariant by X , gives us a restriction on its Malgrange groupoid.

- We know $Xx = 1$ and therefore $\text{Lie}_X(dx) = 0$. Therefore, 1-form dx can be seen as a geometric structure invariant by X , so that

$$\text{Mal}(X) \subset \left\{ \phi \in \widehat{\text{Aut}}(\mathbb{C}^7) \mid \phi^*(dx) = dx \right\}$$

How can we obtain differential invariants of dx . Note that, in any given frame φ , the pullback $\varphi^*(dx)(0)$ is a co-vector in \mathbb{C}^m . The coordinates of $\varphi^*(dx)(0)$ are rational functions on \mathbb{R}_1M .

- The coordinates Y_i of a rational vector field Y can also be seen as rational functions in \mathbb{R}_1M . If $\text{Lie}_X Y = 0$ then we have that $X^{(1)}Y_i = 0$, and therefore the coordinates of Y are the rational differential invariants associated to a symmetry Y of X . In particular X is a symmetry of itself and therefore,

$$\text{Mal}(X) \subset \left\{ \phi \in \widehat{\text{Aut}}(\mathbb{C}^7) \mid \phi_*(X) = X \right\}.$$

- $\text{Lie}_X(dq \wedge dp) = 0 \pmod{\pi^* \Omega_{\mathbb{C}^5}^1}$. This means that the rank 2 bundle $\ker(d(\pi \circ \rho))$ is endowed with an X -invariant volume form, this can also be seen as a geometric structure in \mathbb{C}^7 and yields the following restriction,

$$\text{Mal}(X) \subset \left\{ \phi \in \widehat{\text{Aut}}(\mathbb{C}^7) \mid \phi^*(dq \wedge dp) \equiv dq \wedge dp \pmod{dx, da, db, dc, de} \right\}.$$

Summarizing Remark 4.17, we have the following restrictions of the Malgrange groupoid of X ,

$$\begin{aligned} \text{Mal}(X) \subset \left\{ \phi \in \widehat{\text{Aut}}(\mathbb{C}^7) \mid \phi_* X = X; \quad \bar{\rho} \circ \phi = \bar{\rho}; \right. \\ \left. \phi^* dq \wedge dp \equiv dq \wedge dp \pmod{dx, da, db, dc, de}; \quad \phi^* dx = dx \right\} \end{aligned} \quad (4-2)$$

The *Picard parameters* set is known in the literature to be the following subset of \mathbb{C}^4 :

$$\left\{ (a, b, c, e) \in \left(\frac{1}{2} + \mathbb{Z}^4\right) \right\} \cup \left\{ (a, b, c, e) \in \mathbb{Z}^4 \mid a + b + c + e \text{ is even} \right\}$$

Malgrange groupoid of Painlevé VI equation has been found for fixed values of the parameters by Casale and Cantat–Loray. In what follows it is necessary to distinguish from Picard parameters and the other parameters. In particular, from Cantat–Loray theorem in [CL09] p.2972 Theorem 6.1 we get the following:

Proposition 4.18 *For parameters (a, b, c, e) not in Picard parameter set, Malgrange groupoid is given by:*

$$\text{Mal}(X|_{\mathbb{C}^3 \times \{(a,b,c,e)\}}) = \left\{ \phi : (\mathbb{C}^3, *) \xrightarrow{\sim} (\mathbb{C}^3, \star) \mid \phi^* dx = dx; \phi_*(X) = X; \right. \\ \left. \phi^* dp \wedge dq = dp \wedge dq \mod dx \right\}$$

In this formula a, b, c, e are fixed parameters. The asterisks and the star stands for any point in \mathbb{C}^3 .

Davy in [Dav16] proves a specialization theorem, that gives the relationship between the Malgrange groupoid of a vector field tangent to the fibers of $\rho : M \rightarrow S$ and the Malgrange groupoid of the restricted field $X|_{M_q}$ for a fixed parameter q . As a direct consequence of [Dav16] Proposition 3.6.1 p.70 we have the following:

Proposition 4.19 *There is an inclusion $\text{Mal}(X|_{\mathbb{C}^3 \times \{(a,b,c,e)\}}) \subset \text{Mal}(X)|_{\mathbb{C}^3 \times \{(a,b,c,e)\}}$.*

Transversal part of $\text{Mal}(X)$

Let us introduce $\text{Inv}(x)$, the pseudogroup of formal diffeomorphisms $\phi : (\mathbb{C}^7, *) \rightarrow (\mathbb{C}^7, \star)$ that leave invariant the x coordinate, *i.e.* such that $x \circ \phi = x$. We want to prove the equality in Equation (4-2). To do this we look at $\text{Mal}(X) \cap \text{Inv}(x)$. We have that

$$\text{Mal}(X) \cap \text{Inv}(x) \subset \left\{ \phi : (\mathbb{C}^7, *) \rightarrow (\mathbb{C}^7, \star) \mid \phi^* X = X; \quad \pi \circ \phi = \pi; \right. \\ \left. \phi^*(dq \wedge dp) = dq \wedge dp \mod \pi^* \Omega_{\mathbb{C}^5}^1 \right\} \quad (4-3)$$

The plan of the remaining proof is to obtain equality in Equation (4-3). To see this we restrict our attention to the \mathcal{D} -Lie algebra $\mathcal{L} = \text{Lie}(\text{Mal}(X) \cap \text{Inv}(x))$. This Lie algebra is seen to be of the kind described in Kiso–morimoto Theorem B.4 so we apply it. By these means we get the desired equality but for $\text{Lie}(\text{Mal}(X) \cap \text{Inv}(x))$ instead of $\text{Mal}(X) \cap \text{Inv}(x)$. The final steps will be to “get rid” of the terms Lie and $\text{Inv}(x)$.

Proposition 4.20 *The \mathcal{D} -Lie algebra \mathcal{L} and $\pi : \mathbb{C}^7 \rightarrow \mathbb{C}^4$ are under the hypothesis of Theorem B.4, namely: first, that \mathcal{L} is tangent to ρ and second, that there exists a form $\omega \in \Omega^m M$ such that for all $s \in S$, $\omega|_{M_s}$ is a non-identically zero m -form on M_s satisfying $\mathcal{L}|_{M_s} = \{v \text{ over } M_s \text{ s.t. } \text{Lie}_v \omega|_{M_s} = 0\}$.*

PROOF. First, \mathcal{L} is tangent to the fibers of $\pi : \mathbb{C}^7 \rightarrow \mathbb{C}^5$. Second, it preserves the volume form $dq \wedge dp$ on the fibers. Finally, by Proposition 4.18 and Corollary 4.16 the restriction of all the vector fields to the fibers coincide with the set of all vector fields that preserve volume. \square

Therefore, from Kiso–Morimoto Theorem B.4, we have that:

- (a) There exists a foliation \mathcal{F} over \mathbb{C}^5 ,
- (b) there exists \mathcal{G} over \mathbb{C}^7 such that $\pi(\mathcal{G}) = \mathcal{F}$, $\dim \mathcal{G} = \dim \mathcal{F}$ and
- (c) \mathcal{L} is the \mathcal{D} -Lie algebra of vector fields tangent to the fibers of π , preserving the volume form $dp \wedge dq$, and preserving the \mathcal{F} -connection \mathcal{G} .

$$\mathcal{L} = \left\{ v \in J(TM/M) \mid d\pi(v) = 0; \text{ Lie}_v \omega = 0 \pmod{\pi^* \Omega_S^1}; \right. \\ \left. \forall Y \in \mathfrak{X}(\mathcal{G}) \quad [v, Y] \in \mathfrak{X}(\mathcal{G}) \right\}$$

In particular, the vector field X is tangent to \mathcal{G} . Therefore $\pi_*(X) = \frac{\partial}{\partial x}$ is tangent to \mathcal{F} . Thus, the foliation \mathcal{F} is ρ -projectable. Let us denote by $\overline{\mathcal{F}}$ its projection onto \mathbb{C}^4 .

Affine Weyl group

Let us note that if ϕ is a birational automorphism of M and X is a rational vector field on M , then the pullback by the prolongation, $\phi^{(\infty)}: \text{RM} \rightarrow \text{RM}$ sends rational differential invariants of $\phi_*(X)$ to rational differential invariants of X . It follows clearly that ϕ induces an isomorphism between $\text{Mal}(X)$ and $\text{Mal}(\phi_*(X))$. In particular, if ϕ is a *discrete birational symmetry* of X , that is, a birational automorphism of M such that $\phi_*(X) = X$, then ϕ leaves $\text{Mal}(X)$ invariant. That is,

$$j_p \sigma \in \text{Mal}(X) \iff j_{\phi(p)}(\phi \circ \sigma \circ \phi^{-1}) \in \text{Mal}(X).$$

Lemma 4.21 *Foliation $\overline{\mathcal{F}}$ in \mathbb{C}^4 is regular.*

PROOF. Equation (HP_{VI}) admits a discrete group \widetilde{W} of symmetries, known as Backlund transformations, isomorphic to the extended affine Weyl group $D_4^{(1)}$. All elements of this group are birational transformations of \mathbb{C}^7 . In particular this group contains a subgroup G_4 isomorphic to \mathbb{Z}^4 with the following characteristics:

- (a) The action of G_4 in \mathbb{C}^7 is projectable by $\bar{\rho}$ to the action of translations with integer displacements in \mathbb{C}^4 .
- (b) The function x is invariant by the action.

This group of translations is listed in [NY02] p.6.

The action of G_4 preserves $\text{Mal}(X)$. The function x is an invariant for this action, then $\text{Inv}(x)$ is also preserved. It follows that G_4 leaves \mathcal{L} invariant, and its defining equations, in particular $\phi_*(\mathcal{G}) = \mathcal{G}$. Denote by $\bar{\phi}$ the projection of ϕ to \mathbb{C}^4 , observe $\bar{\phi}$ is a translation.

Then $\bar{\phi}$ sends $\bar{\mathcal{F}}$ into $\bar{\mathcal{F}}$, i.e. $\phi_*(\bar{\mathcal{F}}) = \bar{\mathcal{F}}$. As the set of singularities of $\bar{\mathcal{F}}$ is a proper Zariski closed of \mathbb{C}^4 , invariant under \mathbb{Z}^4 , it must be empty. \square

A result of Iwasaki, Theorem 1.3 in [Iwa08] implies the following:

Lemma 4.22 *The solutions to P_{VI} with finite monodromy are algebraic.*

Lemma 4.23 *$\bar{\mathcal{F}}$ is a foliation by points.*

PROOF. In [LT14] a list of all possible algebraic solutions of Painlevé VI equation is presented. These appear at special values of the parameters. The solution numbered as 45 in [LT14] p.52 is algebraic with 72 branches, and happens at the parameter $\theta = (1/12, 1/12, 1/12, 11/12)$.

If the dimension of $\bar{\mathcal{F}}$ were greater than 0, we could find a path on $\bar{\mathcal{F}}$ along which we could prolong the given solution to a solution at another parameter, with also 72 branches. By Iwasaki, this solution is algebraic. But we know the solution at θ is unique, and all other algebraic solutions have less than 72 branches. We conclude that the foliation cannot have dimension greater than 1 so it must be a foliation by points. \square

Therefore $\mathcal{F} = \langle \frac{\partial}{\partial x} \rangle$ and $\mathcal{G} = \langle X \rangle$. Note that, for a vector field v tangent to $\bar{\rho}$, as X is transversal to $\bar{\rho}$, it is equivalent to say that $\text{Lie}_v \mathcal{G} \subset \mathcal{G}$ or $[v, X] = 0$. This completes the proof of the following result.

Proposition 4.24

$$\mathcal{L} = \left\{ \vec{v} \in (\mathbb{C}^7, *) \mid vx = va = vb = vc = ve = 0; \right. \\ \left. \text{Lie}_v(dp \wedge dq) = 0 \mod da, db, dc, de, dx; [v, X] = 0 \right\}$$

By integration of the \mathcal{D} -Lie algebra \mathcal{L} we obtain:

Proposition 4.25

$$\text{Mal}(X) \cap \text{Inv}(x) = \left\{ \phi \mid \bar{\rho} \circ \phi = \phi; \phi_*(X) = X; \right. \\ \left. \phi^*(dp \wedge dq) \equiv dp \wedge dq \mod da, db, dc, de, dx \right\} \quad (4-4)$$

PROOF. The Lie algebra \mathcal{L} is determined by \mathcal{L}_1 , its first order part. Let us call \mathcal{G} the right hand side of (4-4). \mathcal{G} is determined by \mathcal{G}_1 , its first order part, and also $\text{Mal}(X)$ is determined by $\text{Mal}_1(X)$. It can be proved that \mathcal{G}_1 is connected with respect to source and target. Then \mathcal{G}_1 is the least Lie subgroupoid of $\text{Aut}_1(\mathbb{C}^7)$ such that $\text{Lie}(\mathcal{G}_1) = \mathcal{L}_1$. By (4-3) we know that $\text{Mal}_1(X) \subset \mathcal{G}_1$. As $\text{Lie}(\text{Mal}(X)) = \mathcal{L}_1$ then $\text{Mal}_1(X) = \mathcal{G}_1$. We conclude that $\text{Mal}(X) = \mathcal{G}$. \square

Theorem 4.26 *The Malgrange groupoid of Painlevé VI equation is given by*

$$\text{Mal}(X) = \left\{ \phi \mid \bar{\pi} \circ \phi = \phi; \quad \phi^*(dx) = dx; \quad \phi_*(X) = X; \right. \\ \left. \phi^*(dp \wedge dq) \equiv dp \wedge dq \pmod{da, db, dc, de, dx} \right\} \quad (4-5)$$

PROOF. We already have pointed in (4-2) that $\text{Mal}(X)$ is contained in the right side set of Equation (4-5). Let us prove the remaining inclusion. Fix an integer k large enough and let $\psi: (\mathbb{C}^7, z_0) \rightarrow (\mathbb{C}^7, z_1)$ be a map that satisfy equations in the right hand side of (4-5). We are going to show that $j_{z_0}^k \psi$ is in $\text{Mal}_k(X)$. Let $\varepsilon = x(z_1) - x(z_0)$. Without loss of generality, assume that $\exp(-\varepsilon X)$ is defined in a neighborhood such that if $z_2 = \exp(-\varepsilon X)(z_1)$ then all the points z_0, z_1 and z_2 are inside an open set where $\text{Mal}(X)$ is effectively a groupoid. Let the map $\phi: (\mathbb{C}^7, z_0) \rightarrow (\mathbb{C}^7, z_2)$ be defined by $\phi = \exp(-\varepsilon X) \circ \psi$. Observe that ϕ respects $\bar{\rho}$, $dp \wedge dq|_{\mathbb{C}_{p,q}^2}$ and the field X . By Proposition 4.25 $j_{z_0}^k \phi$ is in $\text{Mal}_k(X)$. Then $j_{z_0}^k \psi = j_{z_2}^k \exp(\varepsilon X) \circ j_{z_0}^k \phi$ is in $\text{Mal}_k(X)$. \square

Corollary 4.27 *If $y(x, a, b, c, e)$ is a parameters dependant solution of the sixth Painlevé equation then its annihilator in $\mathcal{O}_{J(\mathbb{C}^5, \mathbb{C})}$ is the ∂ -ideal generated by the sixth Painlevé equation.*

Using the Hamiltonian formulation, we want to prove that the Zariski closure of a parameters dependant solution $p(x, a, b, c, e); q(x, a, b, c, e)$ in $J(\mathbb{C}^5, \mathbb{C}^2)$ is V : the subvariety defined by the differential ideal generated by $\partial_x p - \frac{\partial H}{\partial q}$ and $\partial_x q + \frac{\partial H}{\partial p}$.

To prove this one studies the action of $\text{Mal}(X)$ on ∂_x^{tot} . This action is well defined because the elements of $\text{Mal}(X)$ are fiber preserving transformations of $\mathbb{C}^5 \times \mathbb{C}^2 \rightarrow \mathbb{C}^5$. Let $W \subset V$ be the Zariski closure of a parameter dependant solution.

Lemma 4.28 *$\text{Mal}(X)$ preserves W*

Remark. The restriction of ∂_x^{tot} on $J_0(V) = \mathbb{C}^7$ is the vector field X . As V or W are define by ∂_x^{tot} -ideals, the prolongation of X on $J(\mathbb{C}^5, \mathbb{C}^2)$ is tangent to V and to W . The stabilizer of W is a \mathcal{D} -groupoid containing the flows of X : it must contain $\text{Mal}(X)$.

Lemma 4.29 *$\text{Mal}(X)$ acts simply transitively on V over the parameter space.*

This means that fibers $\text{Mal}(X)_{a,b,c,e}$ act simply transitively on fibers $V_{a,b,c,e}$.

A. Differential algebraic groups

Our treatment of differential algebraic groups is the geometrical theory proposed by Malgrange in [Mal10a], chapter 7. This formulation is slightly different from the better known algebraic formulation in [Kol85]. Here we recall and comment the most important results, including Malgrange's version of Kiso-Cassidy theorem.

A.1. Differential algebraic groups

Let $\pi_{\mathcal{G}}: \mathcal{G} \rightarrow B$ a group bundle. As before, we assume \mathcal{G} to be affine over B . The jet bundle $J_k(\mathcal{G}/B) \rightarrow B$ of k -jets of sections of $\pi_{\mathcal{G}}$ is an algebraic group bundle with the group operation $(j_x^k g)(j_x^k h) = j_x^k(gh)$. We consider $\mathcal{O}_{J_k(\mathcal{G}/B)}$ the ring of differential functions of order $\leq k$ with rational coefficients in B . We may interpret it as the ring of regular functions in $J_k(\mathcal{G}/B)$ seen as an algebraic $\mathbb{C}(B)$ -group. The group structure induces, by duality, a Hopf $\mathbb{C}(B)$ -algebra structure on $\mathcal{O}_{J_k(\mathcal{G}/B)}$.

Remark A.1 Note that, by Cartier theorem (for instance, in [DG70]), $\mathbb{C}(B)$ -algebraic groups are smooth (and therefore reduced), so that, Hopf ideals of $\mathcal{O}_{J_k(\mathcal{G}/B)}$ are radical.

Remark A.2 There is a natural bijective correspondence between Hopf ideals of $\mathcal{O}_{J_k(\mathcal{G}/B)}$ and rational group subbundles of $J_k(\mathcal{G}/B) \rightarrow B$. Given a Hopf ideal \mathcal{E} , the set of zeros of $\mathcal{E} \cap \mathbb{C}[J_k(\mathcal{G}/B)]$ is rational group subbundle. By abuse of notation we will refer to it as the set of zeros of \mathcal{E} .

The group multiplication is compatible with the order truncation, therefore we may take the projective limit, so that the jet bundle $J(\mathcal{G}/B)$ is a pro-algebraic group bundle and its ring of coordinates (on the generic point of B) $\mathcal{O}_{J(\mathcal{G}/B)} = \mathbb{C}(B) \otimes_{\mathbb{C}[B]} \mathbb{C}[J(\mathcal{G}/B)]$ a Hopf $\mathbb{C}(B)$ -algebra.

Indeed, $\mathcal{O}_{J(\mathcal{G}/B)}$ is endowed with a differential ring structure, with the total derivative operators $\mathfrak{X}(B)^{\text{tot}}$. The Hopf and the differential structure are compatible in the following sense:

$$\mu^*(X^{\text{tot}} f) = (X^{\text{tot}} \otimes 1 + 1 \otimes X^{\text{tot}}) \mu^*(f), \quad e^*(X^{\text{tot}} f) = X(e^*(f)),$$

where μ^* and e^* stand for the coproduct and coidentity operators. Hence, we may say that $\mathcal{O}_{J(\mathcal{G}/B)}$ is a Hopf-differential algebra.

Definition A.3

1. A *Hopf-differential* ideal of $\mathcal{O}_{J(\mathcal{G}/B)}$ is a Hopf ideal \mathcal{E} of $\mathcal{O}_{J(\mathcal{G}/B)}$ such that $X^{\text{tot}}\mathcal{E} \subset \mathcal{E}$ for every rational vector field X on B .
2. A *differential algebraic* subgroup $H \subset J(\mathcal{G}/B)$ is a Zariski closed subset consisting of the zero locus of some Hopf-differential ideal \mathcal{E} of $\mathcal{O}_{J(\mathcal{G}/B)}$

As differential-Hopf ideals are radical (Remark A.1) we have that the natural correspondence between differential-Hopf ideals of $\mathcal{O}_{J(\mathcal{G}/B)}$ and differential algebraic subgroups of $J(\mathcal{G}/B)$ is, in fact, bijective.

Example A.4 Let $\mathcal{G} = \text{GL}_n(\mathbb{C}) \times B$. A differential algebraic subgroup $H \subset J(\mathcal{G}/B)$ is a differential algebraic group of $\mathcal{G}(\mathcal{U})$ defined over $\mathbb{C}(B)$ in the sense of Kolchin (as in [Kol85]), where \mathcal{U} a universal differential field extension of $\mathbb{C}(B)$.

Given a Hopf-differential ideal \mathcal{E} , its truncation $\mathcal{E}_k = \mathcal{E} \cap \mathcal{O}_{J_k(\mathcal{G}/B)}$ is Hopf-ideal of $\mathcal{O}_{J_k(\mathcal{G}/B)}$. Thus, a differential algebraic subgroup of $J(\mathcal{G}/B)$ is given by a chain of Hopf ideals

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_k \subset \dots \subset \mathcal{E}$$

and thus, a system of rational group subbundles of the $J_k(\mathcal{G}/B)$,

$$H \rightarrow \dots \rightarrow H_k \rightarrow \dots \rightarrow H_1 \rightarrow H_0 \rightarrow B.$$

The differential ring $\mathcal{O}_{J(\mathcal{G}/B)}$ is, by construction, the quotient of a ring of differential polynomials with coefficients in $\mathbb{C}(B)$. Thus, it has the *Ritt property* (see, for instance, [Kol73]): radical differential ideals are finitely generated as radical differential ideals. Thus, there is a smallest order k such that \mathcal{E} coincides with $\{\mathcal{E}_k\}$, the radical differential ideal spanned by \mathcal{E}_k . We have that H is completely determined by H_k and we say that H is a differential algebraic subgroup of $J(\mathcal{G}/B)$ of *order* k .

A *section* of H is a section g of $\pi_{\mathcal{G}}: \mathcal{G} \rightarrow B$ such that $j_x g \in H$ for any x in the domain of definition of g . We can deal with rational, local analytic, convergent, or formal sections of H . We can also speak of sections of H with coefficients in any differential field extension of $\mathbb{C}(B)$.

Example A.5 The points of H in the sense of Kolchin (as in Example A.4) are sections of H with coefficients in arbitrary differential field extensions of $\mathbb{C}(B)$. In that context, it is useful to fix a differentially closed field extension of $\mathbb{C}(B)$ so that H is determined by its sections.

A.2. Differential Lie algebras

Let us consider $\pi_{\mathcal{G}}: \mathcal{G} \rightarrow B$ as before; let us define $\mathfrak{g} = e^*(\ker(d\pi_{\mathcal{G}}))$, therefore $\pi_{\mathfrak{g}}: \mathfrak{g} \rightarrow B$ is the Lie algebra bundle associated to \mathcal{G} .

In an analogous way the jet bundles of finite order $J_k(\mathfrak{g}/B) \rightarrow B$ are Lie algebra bundles and the limit $J(\mathfrak{g}/B) \rightarrow B$ is a pro-algebraic linear bundle of Lie algebras; the Lie bracket is defined as $[j_x v_1, j_x v_2] = j_x[v_1, v_2]$ for jets of local analytic sections.

We consider its ring of coordinates on the generic point of B , $\mathcal{O}_{J(\mathfrak{g}/B)} = \mathbb{C}(B) \otimes_{\mathbb{C}[B]} \mathbb{C}[J(\mathfrak{g}/B)]$. It has a Lie coalgebra structure with the induced comultiplication,

$$\beta^*: \mathcal{O}_{J(\mathfrak{g}/B)} \rightarrow \mathcal{O}_{J(\mathfrak{g}/B)} \otimes_{\mathbb{C}(B)} \mathcal{O}_{J(\mathfrak{g}/B)}, \quad \beta^*(f)(j_x v_1, j_x v_2) = f([j_x v_1, j_x v_2]).$$

Moreover, $\mathcal{O}_{J(\mathfrak{g}/B)}$ is a differential ring with the differential structure induced by total derivative operators $\mathfrak{X}(B)^{\text{tot}}$. Indeed, the linear bundle structure of $J(\mathfrak{g}/B)$ is reflected in the ring $\mathcal{O}_{J(\mathfrak{g}/B)}$: linear differential functions are identified with linear differential operators from the space of sections of \mathfrak{g} on $\mathbb{C}(B)$. We have a sub- $\mathfrak{X}(B)$ -module $\text{Diff}_B(\mathfrak{g}) \subset \mathcal{O}_{J(\mathfrak{g}/B)}$ that spans the ring of differential functions (as in [Mal10a] section 4).

Definition A.6 A differential Lie subalgebra $\mathfrak{h} \subset J(\mathfrak{g}/B)$ is a Zariski closed subset consisting of the zeroes of a differential ideal \mathcal{E} of $\mathcal{O}_{J(\mathfrak{g}/B)}$ satisfying:

1. \mathcal{E} is spanned by its linear part $\mathcal{E} \cap \text{Diff}_B(\mathfrak{g})$.
2. \mathcal{E} is a Lie co-ideal.

Note that a differential Lie subalgebra is also a differential subgroup with respect to the additive structure of \mathfrak{g} . There is an smallest k such that $\{\mathcal{E}_k\} = \mathcal{E}$, the *order* of \mathfrak{h} ; and \mathfrak{h} is determined by its truncation \mathfrak{h}_k .

A differential algebraic group H has an associated differential Lie algebra \mathfrak{h} of the same order. For each ℓ , the Lie algebra bundle $\mathfrak{h}_\ell \rightarrow B$ is (on the generic point of B) the Lie algebra of the group bundle $H_\ell \rightarrow B$. Equations of \mathfrak{h} can be found by linearization of the equations of H along the identity (see [Mal10a], section 4.5).

A.3. Group connections

Let us consider a regular connection \mathcal{C} on $\pi_{\mathcal{G}}: \mathcal{G} \rightarrow B$, that is, a regular foliation in \mathcal{G} transversal to the fibers of $\pi_{\mathcal{G}}$. For each $g \in \mathcal{G}$ the leaf of \mathcal{C} that passes through g is the graph of a local analytic section \tilde{g} of $\pi_{\mathcal{G}}$. The connection \mathcal{C} defines a regular map,

$$\gamma_{\mathcal{C}}: \mathcal{G} \rightarrow J(\mathcal{G}/B), \quad g \mapsto j_{\pi_{\mathcal{G}}(g)} \tilde{g},$$

that is, a section of $J(\mathcal{G}/B) \rightarrow \mathcal{G}$ and it identifies \mathcal{G} with the set of jets of \mathcal{C} -horizontal sections $\Gamma(\mathcal{C})$.

Definition A.7 We say that \mathcal{C} is a *group connection* if $\gamma_{\mathcal{C}}$ is a group bundle morphism.

In general, a *rational group connection* is a regular group connection defined on some dense Zariski open subset of B . Given a rational group connection \mathcal{C} , the variety $J(\mathcal{C})$ of \mathcal{C} -horizontal leaves is an order 1 differential algebraic subgroup of $J(\mathcal{G}/B)$ isomorphic (as group bundle, on the generic point of B) to \mathcal{G} .

Remark A.8 Differential algebraic groups of the form $J(\mathcal{C})$, for a group connection \mathcal{C} , correspond to *constant* differential algebraic groups in the sense of Kolchin [Kol85]. The group connection is the differential equation of the conjugation of the differential algebraic group with an algebraic group over the constants.

Analogously, a regular *linear connection* \mathcal{L} in $\mathfrak{g} \rightarrow B$ induces a section $\gamma_{\mathcal{L}}$ of $J(\mathfrak{g}/B) \rightarrow \mathfrak{g}$ which is linear in fibers. We say that \mathcal{L} is a *Lie connection* if $\gamma_{\mathcal{L}}$ is a Lie algebra morphism in fibers. A rational Lie connection is a regular Lie connection in some Zariski open subset of B . The variety of horizontal sections $J(\mathcal{L})$ of a rational Lie connection is a differential Lie subalgebra of $J(\mathfrak{g}/B)$ of order 1.

It is clear that a group connection \mathcal{C} can be linearized along the identity, producing Lie connection \mathcal{C}' with the same domain of regularity. In this case, $J(\mathcal{C}')$ is the Lie algebra of $J(\mathcal{C})$.

We may also define partial group and Lie connections. Let us fix \mathcal{F} a singular foliation in B . Let us recall that a \mathcal{F} -connection in \mathcal{G} (resp. \mathfrak{g}) is just an foliation \mathcal{C} of the same rank of \mathcal{F} and that projects onto \mathcal{F} . A local section is \mathcal{C} -horizontal if its derivative maps \mathcal{F} to \mathcal{C} . The jets of local sections of \mathcal{C} define a closed subset $J(\mathcal{C})$ in $J(\mathcal{G}/B)$ (resp. in $J(\mathfrak{g}/B)$).

$$J(\mathcal{C}) = \overline{\{j_x s \mid x \in B, s \text{ local } \mathcal{C} - \text{horizontal section}\}}.$$

A *partial group \mathcal{F} -connection* is a partial Ehresmann \mathcal{F} -connection \mathcal{C} whose set of horizontal sections $J(\mathcal{C}) \subset J(\mathcal{G}/B)$ is a differential algebraic group. A *partial Lie \mathcal{F} -connection* is a \mathcal{F} -connection whose set of horizontal sections is a differential Lie subalgebra. As before, those objects are of order 1, and the linearization of a group \mathcal{F} -connection is a Lie \mathcal{F} -connection.

We say that a differential subgroup $H \subset J(\mathcal{G}/B)$ is *Zariski dense* in \mathcal{G} if $H_0 = \mathcal{G}$, see [Mal10a] p.308. In the algebraic setting, the classification of Zariski dense subgroups in simple groups is due to Cassidy [Cas89]. We follow the exposition of Malgrange, that finds the key argument of the proof in the a result of Kiso [Kis79].

Theorem A.9 (Kiso-Cassidy, Th. 8.1 in [Mal10a]) *Let $\mathfrak{g} \rightarrow B$ be a Lie algebra bundle with simple fibers. Let $\mathfrak{h} \subset J(\mathfrak{g}/B)$ be a differential subalgebra such that $\mathfrak{h}_0 = \mathfrak{g}$. Then there is a singular foliation \mathcal{F} in B and a partial Lie \mathcal{F} -connection \mathcal{L} in \mathfrak{g} such that $\mathfrak{h} = J(\mathcal{L})$.*

Corollary A.10 *Let $\mathcal{G} \rightarrow B$ be a group bundle with simple fibers. Let $H \subset J(\mathcal{G}/B)$ be a differential algebraic subgroup such that $H_0 = \mathcal{G}$. Then there is a singular foliation \mathcal{F} in B and a partial group \mathcal{F} -connection \mathcal{C} in \mathcal{G} such that $H = J(\mathcal{C})$.*

A.4. The adjoint connection

Let us consider now $\pi: P \rightarrow B$ a principal bundle with structural group G . Let \mathcal{F} be a singular foliation on B and \mathcal{D} be a partial G -invariant \mathcal{F} -connection on P . Let F be an affine G -space with an action $\alpha: G \rightarrow \text{Aut}(F)$. The associated bundle $P[\alpha]$ is defined as the balanced construction $P \times_{\alpha} F$; it is a bundle $P[\alpha] \rightarrow B$ with fibers F . The G -invariant foliation $\mathcal{D} \times \{0\}$ in $P \times F$ projects onto a foliation on $P[\alpha]$ that we call $\alpha(\mathcal{D})$; it is the induced partial Ehresmann \mathcal{F} -connection on $P[\alpha]$.

Let $\text{Gau}(P) \rightarrow B$ be the group bundle of gauge automorphisms of P . Let us recall that $\text{Gau}(P)$ can be constructed as the associated bundle $P[\text{Adj}] = (P \times_{\text{Adj}} G)$ for the adjoint action of G on itself. The induced \mathcal{F} -connection $\mathcal{D}[\text{Adj}]$ turns out to be a group \mathcal{F} -connection in $\text{Gau}(P)$.

Theorem A.11 (Th. 7.6 in [Mal10a]) *Assume that G is semi-simple and connected, and let \mathcal{F} be a singular foliation in B . Then all partial group \mathcal{F} -connections in $\text{Gau}(P)$ are of the form $\mathcal{D}[\text{Adj}]$ where \mathcal{D} is a partial G -invariant \mathcal{F} -connection in P .*

Remark A.12 Note that, since the adjoint action of semi-simple G in itself is faithful, the G -invariant partial connection \mathcal{D} inducing $\mathcal{D}[\text{Adj}]$ is unique.

Example A.13 Let us consider compatible linear equations in $P = \text{GL}_n(\mathbb{C}) \times \mathbb{C}^{n+m}$:

$$\frac{\partial U}{\partial x_i} = A_i(x, s)U.$$

Then $\text{Gau}(P) = \text{GL}_n(\mathbb{C}) \times \mathbb{C}^{n+m}$ and the induced group connection is

$$\frac{\partial \sigma}{\partial x_i} = [A_i(x, s), \sigma].$$

The horizontal sections $\sigma(x, s)$ are gauge symmetries of the equation.

Theorems A.9 and A.11 give us a complete description of *Zariski dense* differential algebraic subgroups of $J(\text{Gau}(P)/B)$ when G is simple and connected. Such subgroups are of the form $J(\mathcal{D}[\text{Adj}])$ for a partial G -invariant connections \mathcal{D} in P .

B. Kiso–Morimoto theorem

B.1. Recall about sections on jet bundle

Let M, S be smooth irreducible algebraic affine varieties and $\rho : M \rightarrow S$ be a bundle. We set m to be the dimension of fibers and n the dimension of S .

Definition B.1 Let $\mathcal{D}_M \subset \text{End}_{\mathbb{C}} \mathbb{C}[M]$ be the ring of operators generated by

- $\mathbb{C}[M]$ acting by multiplication
- The set of derivations $\text{Der}(\mathbb{C}[M])$

It is a non commutative ring. One has two different $\mathbb{C}[M]$ –algebra structures on \mathcal{D}_M :

1. Left $\mathbb{C}[M]$ –algebra structure given by $\mathbb{C}[M] \otimes_{\mathbb{C}} \mathcal{D}_M \rightarrow \mathcal{D}_M$ such that $f \otimes P \mapsto fP$ where $(fP)(g) = fP(g)$.
2. Right $\mathbb{C}[M]$ –structure $\mathcal{D}_M \otimes \mathbb{C}[M] \rightarrow \mathcal{D}_M$ given by $P \otimes f \mapsto P \circ f$ where $(P \circ f)(g) = P(fg)$. This action will be denoted by $\circ \mathbb{C}[M]$.

Let $V \rightarrow M$ be a vector bundle. Its sheaf of linear functions $\Gamma(V^*)$ is a $\mathbb{C}[M]$ –module. Then the jet bundle of V over M is the vector bundle whose sheaf of linear functions is given by $\mathcal{D}_M \otimes_{\circ \mathbb{C}[M]} \Gamma(V^*)$. It is a left \mathcal{D}_M –module. Note that we are using here the right $\mathbb{C}[M]$ –algebra structure for \mathcal{D}_M . That is:

$$\Gamma(J(V/M)) = (\mathcal{D}_M \otimes_{\circ \mathbb{C}[M]} \Gamma(V^*))^*.$$

Definition B.2 A \mathcal{D} –linear subspace of $J(V/M)$ is the linear sub-bundle $\mathcal{L} \subset J(V/M)$ annihilated by a \mathcal{D}_M –submodule $N \subset \mathcal{D}_M \otimes_{\circ \mathbb{C}[M]} \Gamma(V^*)$.

B.2. \mathcal{D} -Lie algebras

Now we are looking specifically at $V = TM$. The space $J(TM/M)$ has a Lie bracket written in coordinates as

$$\begin{aligned} \left[j^k \sum a_i(x) \frac{\partial}{\partial x_i}, j^k \sum b_i(x) \frac{\partial}{\partial x_i} \right] &= j^{k-1} \left[\sum a_i(x) \frac{\partial}{\partial x_i}, \sum b_i(x) \frac{\partial}{\partial x_i} \right] \\ &= \sum_{\substack{|\alpha+\beta| \leq k-1 \\ 1 \leq i, j \leq n}} \left(a_{i:\alpha} b_{j:\beta+\epsilon_i} - b_{i:\beta} a_{j:\alpha+\epsilon_i} \right) \frac{x^{\alpha+\beta}}{\alpha! \beta!} \frac{\partial}{\partial x_i} \end{aligned}$$

Definition B.3 A \mathcal{D}_M -Lie subalgebra is a \mathcal{D}_M -subspace stable by the Lie bracket.

Let us discuss some hypothesis on a \mathcal{D} -Lie algebra \mathcal{L} used in the theorem we want to prove.

Hyp1 \mathcal{L} is tangent to ρ , meaning that $\mathcal{L} \subset \ker d\rho$, or in other terms $\rho^* \Omega_S^1 \subset N$

Hyp2 There exists a form $\omega \in \Omega^m M$ such that for all $s \in S$, $\omega|_{M_s}$ is a non-identically zero m -form on M_s satisfying

$$\mathcal{L}|_{M_s} = \{v \text{ over } M_s \text{ s.t. } \text{Lie}_v \omega|_{M_s} = 0\}. \quad (\text{B-1})$$

Let $\mathcal{D}_{M/S}$ be the set of differential operators that are tangent to the fibers, and $\Omega_{M/S}^1$ the restriction of the 1-forms to the fibers, *i.e.* the sections of the dual of $\ker d\rho$. The equations of N restricted to the fibers are given by the image of N in $\mathcal{D}_{M/S} \otimes \Omega_{M/S}^1$. To explain this let us note that we have the following morphisms:

- $\mathcal{D}_M \leftrightarrow \mathcal{D}_{M/S}$ because $\text{End}_{\mathbb{C}(S)} \mathbb{C}[M] \subset \text{End}_{\mathbb{C}} \mathbb{C}[M]$,
- the map $\Omega_M^1 \rightarrow \Omega_{M/S}^1$ is given by $l \mapsto l|_{\ker d\rho}$ whose kernel is $\rho^* \Omega_S^1$.

From hypothesis 1: $\rho^* \Omega_S^1 \subset N$. N is given by its image in $\mathcal{D}_M \otimes \Omega_{M/S}^1$, meaning the following: By denoting $\text{rest} : \mathcal{D}_M \otimes \Omega_M^1 \rightarrow \mathcal{D}_M \otimes \Omega_{M/S}^1$

$$0 \longrightarrow \mathcal{D}_M \otimes \Omega_S^1 \longrightarrow \mathcal{D}_M \otimes \Omega_M^1 \longrightarrow \mathcal{D}_M \otimes \Omega_{M/S}^1 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{D}_M \otimes \Omega_S^1 \longrightarrow N \xrightarrow{\text{rest}} \text{rest}(N) \longrightarrow 0$$

as $\text{rest}^{-1}(\text{rest}(N)) = N + \rho^* \Omega_S^2$ and $\rho^* \Omega_S^1 \subset N$, it follows that $\text{rest}^{-1}(\text{rest}(N)) = N$. Now $\text{rest}(N) \cap \mathcal{D}_{M/S} \otimes \Omega_{M/S}^1 = \overline{N}$ are the equations of the restriction of \mathcal{L} to the fibers.

The conditions given in (B-1) Hyp2 gives order 1 linear differential equation for \mathcal{L} in \overline{N} . In local coordinates $x_1, \dots, x_m, s_1, \dots, s_n$ $\omega = f(x, s)dx_1 \wedge \dots \wedge dx_m$ and $v = \sum a_i(x, s)\frac{\partial}{\partial x_i}$. Using Cartan's formula for $\text{Lie}_v\omega$ we obtain:

$$\begin{aligned} \iota_v d\omega + d\iota_v \omega &= \iota_v \left(\sum (\partial f / \partial s_i) ds_i \wedge dx_1 \wedge \dots \wedge dx_m \right) \\ &\quad + d \left(\sum f(x, s) a_i(x, s) dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m \right) \\ &= 0 + \left(\sum_{i=1}^m \frac{\partial(f a_i)}{\partial x_i} \right) dx_1 \wedge \dots \wedge dx_m \mod \rho^* \Omega_S^1 \end{aligned}$$

so we get $\sum_i^m \frac{\partial(f a_i)}{\partial x_i} = 0$. Thus the form ω induces an operator of order one $\text{div}_\omega \in \mathcal{D}_{M/S} \otimes \Omega_{M/S}^1$. By hypothesis 2 we write $\overline{N} = \mathcal{D}_{M/S} \text{div}_\omega$ and it follows $\mathcal{D}_M \overline{N} = \mathcal{D}_M \text{div}_\omega$.

Theorem B.4 (Kiso–Morimoto) *Let $\rho : M \rightarrow S$ and \mathcal{L} as above. Assume that the hypothesis Hyp1 and Hyp2 hold. Then there exists a foliation \mathcal{F} over S and a foliation \mathcal{G} over M such that $\rho_*(\mathcal{G}) = \mathcal{F}$, $\dim \mathcal{G} = \dim \mathcal{F}$ and*

$$\mathcal{L} = \left\{ v \in J(TM/M) \mid d\rho(v) = 0, \quad \text{Lie}_v \omega = 0 \mod \rho^* \Omega_S^1, \right. \\ \left. \forall Y \in \mathfrak{X}(\mathcal{G}) \quad [v, Y] \in \mathfrak{X}(\mathcal{G}) \right\}$$

Remark B.5 If $X \in \mathfrak{X}(\mathcal{F})$ and $Y \in \mathfrak{X}(\mathcal{G})$ with $\rho_* Y = X$ and $v \in \mathcal{L}$ then we have $\rho_* v = 0$ therefore $\rho_*[v, Y] = [\rho_* v, \rho_* Y] = 0$. Also $[v, Y] \in \mathfrak{X}(M)$ implies $[v, Y] = 0$.

Before developing the proof of Kiso–Morimoto theorem let us analyze how the conclusion of the theorem give us relations between the distribution \mathcal{G} and the linear isotropy of \mathcal{L} .

Definition B.6 For $p \in M$ let $\mathcal{L}_p^{\geq k} = \mathcal{L}_p \cap \mathfrak{m}_p^k J(TM/M)_p$ the set of all vector fields that vanish up to order k at p . Let $\mathfrak{g}^k = \mathcal{L}^{\geq k+1} / \mathcal{L}^{\geq k+2}$.

This \mathfrak{g}^k set is, up to restriction to an open Zariski subset, a subbundle of $S^{k+1} T^* M \otimes TM/S$ described by the principal symbol of the equations of order $k+1$ of \mathcal{L} . The linear isotropy of \mathcal{L} is $\mathfrak{g}^0 \subset T^* M \otimes TM/S \subset \text{End}(TM)$.

\mathfrak{g}^0 is a Lie algebra bundle.

Lemma B.7 *If \mathcal{L} is described by the expression given in Kiso–Morimoto theorem B.4 then $\mathcal{G} \subset \ker \mathfrak{g}^0$.*

PROOF. Let $Y \in \mathfrak{X}(\mathcal{G})$ be a vector field projectable on S and $v \in \mathcal{L}$ that vanishes at $p \in M$ with linear part v^1 . Then writing $Y = Y_i \frac{\partial}{\partial x_i} + o(x)$ and $v = v_j^1 x_i \frac{\partial}{\partial x_j} + o(x^2)$

$$0 = [Y, v] = Y \circ v - v \circ Y = Y_i v_j^1 \frac{\partial}{\partial x_j} + o(x) - o(x).$$

By evaluating at p we get $0 = v_j^i Y_i$ and this implies $Y(p) \in \ker v(p)$ for $v(p) \in \text{End}(TM, TM/S)$. \square

B.3. Proof of Kiso–Morimoto theorem

The proof is done in 3 steps.

1. Structure of \mathfrak{g}^0 We have a projection $T^*M \otimes TM/S \rightarrow T^*M/S \otimes TM/S$ with kernel \mathfrak{b} .

Lemma B.8 *The image of \mathfrak{g}^0 denoted by \mathfrak{h} is $\mathfrak{sl}(TM/S)$ the lie algebra of trace free endomorphisms of TM/S .*

PROOF. Let's see that $\pi(\mathfrak{g}_p^0) \subset \mathfrak{sl}(T_p M/S)$. Let $\phi \in \mathfrak{g}_p^0$. We can write $\phi = j_p^1 X$ for X a vector field in \mathcal{L} , with a zero at p and tangent to M_s . The restriction $X|_{M_s}$ preserves the volume $\omega|_{M_s}$. We have $\pi(\phi) = j_p^1 X|_{TM/S} = j_p^1 (X|_{M_s})$ and by Hyp.2 these have zero trace.

To see $\mathfrak{sl}(T_p M/S) \subset \pi(\mathfrak{g}_p^0)$, fix $\bar{\phi} \in \mathcal{G}|_{T_p M_s}$. As $\mathcal{L}|_{M_s} = \text{Sym}(\omega|_{M_s})$ there exists a field \bar{X} defined around p in M_s such that $j_p^1 \bar{X} = \bar{\phi}$. \square

Lemma B.9 *If $b_1, b_2 \in \mathfrak{b}$ then we have $b_1 \circ b_2 = 0$*

PROOF. We have $\ker b_1 \supset TM/S$ and $\text{Im } b_2 \supset T(M/S)$. \square

Define $\mathfrak{a} = \mathfrak{b}|_{\mathfrak{g}^0}$. we have an exact sequence:

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g}^0 \longrightarrow \mathfrak{h} \longrightarrow 0$$

with $\mathfrak{a} \subset \mathfrak{b}$ commutative.

Lemma B.10 *\mathfrak{h} acts on \mathfrak{b} and preserves \mathfrak{a} . Moreover $\mathfrak{b} \cong \rho^* T^* S \otimes TM/S$ and the action of \mathfrak{h} on \mathfrak{b} by the Lie bracket is the action of \mathfrak{h} on the second factor.*

PROOF. For $h \in \mathfrak{h}$ and $a \in \mathfrak{a}$, we have that the bracket $[h, a] = h \circ a - a \circ h = h \circ a$. Also $\mathfrak{a} \subseteq \mathfrak{b}$ is stable by \mathfrak{h} because when seen as maps, $h : TM/S \rightarrow TM/S$ and $a : TM \rightarrow TM/S$ are such that their composition $h \circ a$ is a map $TM \rightarrow TM/S$. \square

Observe that \mathfrak{h} is the set of endomorphisms of TM/S that preserve the volume ω . This follows from $\mathcal{L}_{M_s} = \text{Sym}(\omega)$, the symmetries of ω . Observe also that $\mathfrak{h} = \mathfrak{sl}(TM/S)$.

Lemma B.11 *There exists $A \subset \rho^* T^* S$ a $\mathbb{C}[M]$ –submodule such that $\mathfrak{a} = A \otimes TM/S$.*

PROOF Lets see the case \mathfrak{a} irreducible. Let $u \in \rho^*TS$ and $i(u) : \mathfrak{a} \rightarrow TM/S$ be the contraction, $i(u)(\omega \otimes X) = \omega(u)X$. Observe that $i(u)\mathfrak{a}$ is irreducible. Then there exists $u_1 \in \rho^*TS$ such that $I(u_1)\mathfrak{a} = TM/S$. If u_2 satisfies also that $i(u_2)\mathfrak{a} = TM/S$, we have $i(u_1) \circ i(u_2)^{-1}$ commutes with the action of \mathfrak{h} and by Schur's lemma there exists $\lambda \in \mathbb{C}$ such that $i(u_2) = \lambda i(u_1)$. It follows that $N(\mathfrak{a}) = \{u \in \rho^*TS \mid i(u)\mathfrak{a} = 0\}$ have codimension 1 and we can write $\mathfrak{a} = u_0 \otimes TM/S$, for u_0 in $N(\mathfrak{a})$. See [Kis79] Proposition 4.1 p.122. \square

2. Construction of the foliations. We define $\mathcal{A} = \ker \mathfrak{a}$, $\mathcal{G} = \ker \mathfrak{g}^0$. Easily it follows $\mathcal{G} \subset \mathcal{A}$

Lemma B.12 $\mathcal{G} \cap TM/S = \{0\}$.

PROOF. Take $v \in \mathcal{G} \cap TM/S$. Recall that $\mathfrak{a} \subset \rho^*T^*S \otimes TM/S$. For $h \in \mathfrak{h}$ chose $a \in \mathfrak{a}$ such that $h + a \in \mathfrak{g}^0$. Then $h(v) + a(v) \in \mathfrak{g}^0(\mathcal{G}) = 0$. We have $a(v) = 0$ and $h(v) = 0$. \square

Proposition B.13 *The distribution \mathcal{G} is integrable.*

PROOF.

Fact 1. If $X \in \ker \mathfrak{g}^0$ we have that for all $l \in T^*M/S$ it happens $X \otimes l \in \text{Ann } \mathfrak{g}^0$. This means that for every $l \in T^*M/S$ and all $g \in \mathfrak{g}^0$ $(X \otimes l)(g) = l(g(X)) = 0$. Observe that $\text{Ann } \mathfrak{g}^0 \subset TM \otimes T^*M/S$.

Fact 2. By definition of \mathfrak{g}^0 it follows that $X \otimes l \in \text{Ann } \mathfrak{g}^0$ if and only if there is an equation of the form $X \otimes l + \dots$ in N .

From the two previous facts we have that if l_1, \dots, l_m is a basis for T^*M/S and $X \in \ker \mathfrak{g}^0$ then exists A_i^j with coefficients in $\mathbb{C}[M]$ such that $X \otimes l_i + A_i^j \otimes l_j \in N \subset \mathcal{D}_M \otimes T^*M/S$ is an equation in N .

Now take $X, Y \in \ker \mathfrak{g}^0$. Then the following equation belongs to N :

$$\begin{aligned} & Y(X \otimes l_i + A_i^j \otimes l_j) - X(Y \otimes l_i - B_i^j \otimes l_j) \\ & \quad - A_i^j(Y \otimes l_j + B_j^k \otimes l_k) + B_i^j(X \otimes l_j + A_j^k \otimes l_k) \\ &= Y \circ X \otimes l_i + Y(A_i^j) \otimes l_j + A_i^j Y \otimes l_j - X \circ Y \otimes l_i - Y(B_i^j) \otimes l_j - B_i^j X \otimes l_j \\ & \quad - A_i^j(Y \otimes l_j + B_j^k \otimes l_k) + B_i^j(X \otimes l_j + A_j^k \otimes l_k) \\ &= [X, Y] \otimes l_i - A_i^j B_j^k \otimes l_k - B_i^j A_j^k \otimes l_k + (Y(A_i^j) - Y(B_i^j)) \otimes l_j \end{aligned}$$

In this last equation the terms $-A_i^j B_j^k \otimes l_k - B_i^j A_j^k \otimes l_k$ and $(Y(A_i^j) - Y(B_i^j)) \otimes l_j$ are of order zero. Then we obtain that $[X, Y]$ is in the annihilator of \mathfrak{g}^0 and so $[X, Y] \in \ker \mathfrak{g}^0$. \square

Lemma B.14 \mathcal{G} is ρ -projectable.

PROOF. Let X be a vector field inside \mathcal{L} that do not vanish at p and is tangent to the fibers of ρ . Let σ_t be the flow of X at time t .

As X is in \mathcal{L} , the flow of X transform \mathcal{L} into \mathcal{L} . Then $d\sigma : T_p M \rightarrow T_{\sigma(p)} M$ conjugates \mathcal{G}_p^0 with $\mathcal{G}_{\sigma(p)}^0$. We get an isomorphism $\mathcal{G}_p \cong \mathcal{G}_{\sigma(p)}$. As $\rho \circ \sigma^{-1} = \rho$ it follows $d_p \rho(\mathcal{G}_p) = d_{\sigma(p)} \rho(\mathcal{G}_{\sigma(p)})$.

Then $d_p \rho(\mathcal{G}_p) = d_q \rho(\mathcal{G}_q)$ whenever p and q can be joined by an integral curve in \mathcal{L} . As the action of \mathcal{L} is transitive in the fibers of ρ the lemma follows. \square

Lemma B.15 *We have the decomposition $\mathcal{A} = \ker \mathfrak{a} = TM/S \oplus \mathcal{G}$*

Consequences: $\ker \mathfrak{a}$ is projectable on S and $\rho_* \ker \mathfrak{a} = \mathcal{F} = \rho_* \mathcal{G}$ thus \mathcal{G} is a connection along \mathcal{F} .

PROOF OF THE LEMMA.

Consider the exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g}^0 \longrightarrow \mathfrak{h} \longrightarrow 0$$

and take restriction to \mathcal{A} .

$$0 \longrightarrow \mathfrak{g}^0|_{\mathcal{A}} \longrightarrow \mathfrak{h} \longrightarrow 0$$

observe that $\mathfrak{g}^0|_{\mathcal{A}} \subset A^* \otimes TM/S \rightarrow T^*M/S \otimes TM/S$ given by $TM/S \subset \mathcal{A}$ i.e. $\mathcal{A}^* \rightarrow T^*M/S$.

Then $\mathfrak{g}^0|_{\mathcal{A}} \subset \text{End}(A, A)$ is a Lie algebra isomorphic to \mathfrak{h} , so it is a simple Lie algebra.

We have the following facts:

1. $\mathfrak{g}^0|_{\mathcal{A}}$ preserves TM/S .
2. Because $\mathfrak{g}^0|_{\mathcal{A}}$ is simple, it is completely reducible.

There exist $K \subset A$ such that $A = K \oplus TM/S$ and $\mathfrak{g}^0|_{\mathcal{A}}(K) \subset K$. Then $\mathfrak{g}^0|_{\mathcal{A}}(K) \subset K \cap TM/S = 0$ so $K \subset \mathfrak{g}$. Finally $A \subset \mathcal{G} \oplus TM/S$.

Here we have used a theorem that says that simple Lie algebras are completely reducible. \square

Let

$$\overline{\mathcal{L}} = \left\{ v \in J((TM)/M) \mid v \in \ker(\rho^* \Omega_S^1), \quad (\iota_v d\omega + d\iota_v \omega) \equiv 0 \pmod{\rho^* \Omega_S^1}, \quad \text{Lie}_v \mathcal{G} \subset \mathcal{G} \right\}$$

Observe that the condition $\text{Lie}_v \mathcal{G} \subset \mathcal{G}$ is a first order PDE on v .

Lemma B.16 *We have the inclusion $\mathcal{L} \subset \overline{\mathcal{L}}$.*

PROOF. By Hyp.1 \mathcal{L} preserves the fibers of ρ . By Hyp.2 \mathcal{L} preserves volume in the transversal. As seen in the proof of B.14 the flow of vector fields transforms \mathcal{G} into itself. This is to say that Lie derivative of fields in \mathcal{G} in the direction of fields in \mathcal{L} gives fields in \mathcal{G} .

\mathcal{L} preserves \mathcal{L} then \mathcal{L} preserves \mathfrak{g}^0 and then \mathcal{L} preserves $\ker \mathfrak{g}^0$.

□

3. Proof of the equality $\mathcal{L} = \overline{\mathcal{L}}$. Let us consider around a point $p \in M$ a system of adapted coordinates $s_1, \dots, s_q, t_1, \dots, t_r, x_1, \dots, x_m$, vanishing at p with the following properties. Roughly speaking t_i is going to be s_{q+i} in the previous notation. This is chosen to simplify the writing.

- (a) $s_1, \dots, s_q, t_1, \dots, t_r$ form system of coordinates in S around $\rho(p)$, that is, as local functions in M they are constant along the fibers of ρ .
- (b) The foliation \mathcal{F} in S rectified by the coordinates t, s , that is:

$$\mathcal{F} = \{ds_1 = \dots = ds_q = 0\} = \left\langle \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r} \right\rangle.$$

- (c) The functions x_i are first integrals of \mathcal{G} , thus:

$$\mathcal{G} = \left\langle \frac{\partial}{\partial s_{q+1}}, \dots, \frac{\partial}{\partial s_n} \right\rangle.$$

- (d) The volume form ω is written in canonical form with respect to the x_i functions,

$$\omega = dx_1 \wedge \dots \wedge dx_m.$$

Let X be an analytic vector field in $\overline{\mathcal{L}}$ defined around p . Since X preserves the projection ρ its local expression in such system of adapted coordinates has the form

$$X = \sum_i f_i(x, s, t) \frac{\partial}{\partial x_i}.$$

X preserves \mathcal{G} therefore for $j = 1, \dots, r$ we have that $\text{Lie}_X \frac{\partial}{\partial t_j}$ is a vector field tangent to G .

$$\text{Lie}_X \frac{\partial}{\partial t_j} = -\left[\frac{\partial}{\partial t_j}, X \right] = -\sum_{i=1}^m \frac{\partial f_i}{\partial t_j} \frac{\partial}{\partial x_i}$$

Therefore, $\frac{\partial f_i}{\partial t_j} = 0$ for all $j = 1, \dots, r$. We obtain that the expression of X has the form,

$$X = \sum_{i=1}^m f_i(x, s) \frac{\partial}{\partial x_i}. \tag{B-2}$$

We also know that X preserves ω . Finally we have that X is in $\overline{\mathcal{L}}$ if and only if its local expression is of the form (B-2), satisfying

$$\operatorname{div}_\omega X = \sum_{i=1}^m \frac{\partial f_i}{\partial x_i} = 0.$$

From now restrict our analysis to the point p , therefore by \mathfrak{g}^k we mean the space of homogeneous parts of degree $(k+1)$ of vector fields in \mathcal{L} having a zero of order $(k+2)$ at p (Definition B.6). By definition, it is identified with a vector subspace of $S^k(T_p^*M) \otimes T_pM$. Analogously we have $\overline{\mathfrak{g}}^k$ the space of homogeneous parts of degree $(k+1)$ of vector fields in $\overline{\mathcal{L}}$ having a zero of order $(k+2)$ at p .

The Lie homogeneous part of a Lie bracket depends only of the homogeneous part of the factors, therefore the Lie bracket is defined as a graded operation,

$$[\cdot, \cdot]: \overline{\mathfrak{g}}^k \times \overline{\mathfrak{g}}^l \rightarrow \overline{\mathfrak{g}}^{k+l}.$$

Thus $\overline{\mathfrak{g}} = \bigoplus_{k=0}^{\infty} \overline{\mathfrak{g}}^k$ can be seen as a graded Lie algebra, *the symbol* of $\overline{\mathcal{L}}$.

Remark B.17 It is clear that $\mathfrak{g}^k \subset \overline{\mathfrak{g}}^k$ for all degree k (and all point p of M). Moreover, we obtain the equality if and only if $\mathcal{L} = \overline{\mathcal{L}}$. Therefore, *the proof of Kiso–Morimoto theorem is reduced to the proof the equality of such finite dimensional spaces.*

Let $\mathbb{C}[x, s]_{k+1}$ the space of homogeneous polynomials of degree $(k+1)$ in the variables $x_1, \dots, x_m, s_1, \dots, s_q$. From the local expression of X in $\overline{\mathcal{L}}$ around p we have that the vector space $\overline{\mathfrak{g}}^k$ is identified with a subspace of $\bigoplus_{i=1}^m \mathbb{C}[x, s]_{k+1} \frac{\partial}{\partial x_i}$.

Let $P_{(k+1)}(x, s)$ be an homogeneous polynomial of degree $(k+1)$. It can be decomposed in a unique way as a sum,

$$P_{k+1}(x, s) = P_{k+1}^{(0)}(x) + P_{k+1}^{(1)}(x, s) + \dots + P_{k+1}^{(k)}(x, s) + P_{k+1}^{(k+1)}(s),$$

where $P_{k+1}^{(j)}$ is in $(\mathbb{C}[x]_{k+1-j})[s]_j$, this is, it is an homogeneous polynomial of degree j in the variables s_1, \dots, s_q whose coefficients are homogeneous polynomials in x_1, \dots, x_m of degree $(k+1-j)$.

Given $X \in \overline{\mathfrak{g}}^k$ we can define the valuation $\operatorname{val}_s(X)$ as the minimum grade in s appearing in X . It allows us to define a filtration in $\overline{\mathfrak{g}}^k$.

Definition B.18 Let $(\overline{\mathfrak{g}}^k)^{\geq l}$ be the set of elements of $\overline{\mathfrak{g}}^k$ with valuations in s bigger than l ,

$$(\overline{\mathfrak{g}}^k)^{\geq l} = \{X \in \overline{\mathfrak{g}}^k \mid \operatorname{val}_s(X) \geq l\}.$$

so that we have a filtration,

$$(\overline{\mathfrak{g}}^k)^{\geq k+2} = \{0\} \subset (\overline{\mathfrak{g}}^k)^{\geq k+1} \subset \dots \subset (\overline{\mathfrak{g}}^k)^{\geq 0} = \overline{\mathfrak{g}}^k.$$

And let $\bar{\mathfrak{g}}^{k,l}$ be the consecutive quotient $(\bar{\mathfrak{g}}^k)^{\geq l}/(\bar{\mathfrak{g}}^k)^{\geq l+1}$.

By restriction of this filtration to the subspace \mathfrak{g}^k we define the consecutive quotients $\mathfrak{g}^{k,l}$. By construction $\mathfrak{g}^{k,l} \subseteq \bar{\mathfrak{g}}^{k,l}$. Since we have a finite filtration, we have that $\mathfrak{g}^k = \bar{\mathfrak{g}}^k$ if and only if $\mathfrak{g}^{k,l} = \bar{\mathfrak{g}}^{k,l}$ for all $l = 0, \dots, k+1$. Thus, *the proof of Kiso–Morimoto theorem is reduced to the comparison of the spaces $\mathfrak{g}^{k,l}$ to $\bar{\mathfrak{g}}^{k,l}$.*

From the fundamental hypothesis on \mathcal{L} we have the following.

Lemma B.19 *We have $\mathfrak{g}^{0,1} = \bar{\mathfrak{g}}^{0,1} = \mathfrak{a}$.*

PROOF For $v^1 \in \mathfrak{g}^{0,1} \subset \mathfrak{g}^0$ then its restriction to $s = 0$ is 0. This means that $v^1 \in \mathfrak{a}$. \square

Lemma B.20 *(Kiso hyp 2) $\mathfrak{g}^{k,0} = \bar{\mathfrak{g}}^{k,0} = \mathfrak{h}^k$.*

Next step is to propagate the equalities by means of the Lie bracket.

Lemma B.21 *The Lie bracket in $\bar{\mathfrak{g}}$ is compatible with the filtration of the spaces $\bar{\mathfrak{g}}^k$, and therefore it induces a Lie bracket between the intermediate quotients,*

$$[\ , \]: \bar{\mathfrak{g}}^{k,l} \times \bar{\mathfrak{g}}^{k',l'} \rightarrow \bar{\mathfrak{g}}^{k+k',l+l'}.$$

PROOF. Lets take $X \in \bar{\mathfrak{g}}^{k,l}$ and $Y \in \bar{\mathfrak{g}}^{k',l'}$. We write here ∂_i for $\partial/\partial x_i$. We have the Taylor development $X = \sum P_i \partial_i + R_i \partial_i$ where P_i is an homogeneous polynomial in x and s , with grades $k+1-l$ and l respectively. The R_i have degree $l+1$ in s and $k+2$ in x . Similarly write $Y = \sum P'_i \partial_i + R'_i \partial_i$ where P'_i is an homogeneous polynomial in x and s , with grades $k'+1-l'$ and l' respectively. The R'_i have degree $l'+1$ in s and $k'+2$ in x . The bracket can be expressed as

$$[X, Y] = \sum_{i,j} (P_i \partial_i q_j - Q_i \partial_i P_j) \partial_i + (R_i \partial_i Q_j - R'_i \partial_i P_j + (P_i + R_i) \partial_i R'_j - (Q_i + R'_i) \partial_i R_j) \partial_i$$

Observe that:

$$P_i \partial_i q_j - Q_i \partial_i P_j \quad \text{have degree } \geq l+l' \text{ in } s \text{ and } k+k'+1 \text{ in } x,$$

$$R_i \partial_i Q_j - R'_i \partial_i R_j \quad \text{have degree } \geq l+l'+1 \text{ in } s,$$

$$(P_i + R_i) \partial_i R'_j - (Q_i + R'_i) \partial_i R_j \quad \text{have degree } \geq l+l'+1 \text{ in } s.$$

It follows that the bracket is an element in $\bar{\mathfrak{g}}^{k+k',l+l'}$. \square

because $Y(Q) = \partial/\partial y_1(P) = 0$. As the elements of the form QPX form a basis for $\bar{\mathfrak{g}}^{k,l+1}$ we get the lemma. \square

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List of symbols

Symbol	Term
$ \alpha $	Multi-index absolute value
$\widehat{\text{Aut}}(\mathbb{C}^m, 0)$	Limit of $\text{Aut}_k(\mathbb{C}^m, 0)$
$\text{Aut}_k(\mathbb{C}^m, 0)$	Truncation to order k of $\text{Aut}(\mathbb{C}^m, 0)$
$\text{Aut}_k(M)$	k -jets of local invertible biholomorphisms of M
$\text{Aut}(\mathbb{C}^m, 0)$	Invertible analytic maps from a neighborhood of zero into itself
$\mathbb{C}_{(m,k)}$	Polynomial algebra in m variables and height k
$\mathbb{C}(J_k(\mathcal{D}))^{\mathcal{D}^{(k)}}$	Field of rational first integrals of $\mathcal{D}^{(k)}$
$\mathbb{C}(M)^{\mathcal{D}}$	Field of rational first integrals
$\mathbb{C}[[\varepsilon]]$	Ring of formal power series in m variables
$\widehat{\mathbb{C}[B]}_p$	\mathfrak{m}_p -adic completion of $\mathbb{C}[B]$
$\mathcal{D}, \mathcal{D}^{(k)}$	Distribution and its k th order prolongation
$\mathcal{D}_{M/S}$	Differential operators that are tangent to the fibers
\mathcal{D}_B	Ring of differential operators $\mathbb{C}[B][\mathfrak{X}_B]$
\mathcal{D}_M	Ring of operators on M
$D^\alpha x_i$	Derivative of x_i
$\mathcal{D}^{(\infty)}$	Limit prolongation of \mathcal{D}
$\mathcal{D}^s \boxtimes \mathcal{D}^t$	Sum of \mathcal{D}^s and \mathcal{D}^t
Δ_P	Diagonal in $P \times P$
$\text{Der} A$	Module of derivations on A
$\text{Diff}(\Gamma(E), \mathbb{C}[B])$	Differential operators $\Gamma(E) \rightarrow \mathbb{C}[B]$
div_ω	Divergence of ω
ϵ_i	Multi-index with 1 in the i -th position
\mathcal{F}	Foliation

Symbol	Term
$f^{(m,k)}$	Prolongation of f to $\tau_{m,k}M$
$\mathcal{G} \times_{s_M^t} \mathcal{G}$	Fibered product over source and target
$\mathrm{Gal}_\infty(\mathcal{D})$	Diagonal part of $\mathrm{PGal}_\infty(\mathcal{D})$
$\mathrm{Gal}(\mathcal{D})$	Diagonal part of $\mathrm{GGal}(\mathcal{D})$
$\mathrm{Gau} P$	Diagonal part of $\mathrm{Iso} P$
$\mathrm{GGal}(\mathcal{D})$	Subgroupoid corresponding to $\mathbb{C}(P)^{\mathcal{D}}$
$I^{(m,k)}$	Prolongation of the ideal I
$\mathrm{Iso}(\mathrm{RM})$	$\widehat{\mathrm{Aut}}(\mathbb{C}^m, 0)$ -equivariant bijections between fibers of RM
$\mathrm{Iso}(P)_{b\bullet}$	Equivariant bijections with source b
$\mathrm{Iso}_S(P)$	Fiber isomorphisms respecting the projection ρ
$\mathrm{Iso} P$	Equivariant bijections between fibers of P
$J_k(P/B)$	Order k Jet bundle of sections of $P \rightarrow B$
$J_k(\mathcal{D})$	k -jets of \mathcal{D} -horizontal sections
$J(P/B)$	Infinite order Jet bundle
$\mathrm{Lie}_v \omega$	Lie derivative of ω along v
$\mathrm{Lie} \mathcal{G}_k$	\mathcal{D} -Lie algebra of \mathcal{G}_k
$M(A)$	Points of type A in M
$\mathrm{Mal}_k(X)$	Order k Malgrange groupoid
$\mathrm{Mal}(X)$	Limit of $\mathrm{Mal}_k(X)$
\mathcal{O}_{JB}	Differential functions regular along a generic open set in B
\mathcal{O}_P	Ring of regular functions on P
$\Omega_{M/S}^1$	Restriction of the 1-forms to the fibers
Ω_S^1	1-forms on S
$P[\rho]$	Associated bundle
$P \times_S P$	Fibered product over S
$\mathrm{PGal}_\infty(\mathcal{D})$	Limit of $\mathrm{PGal}_k(\mathcal{D})$
$\mathrm{PGal}_k(\mathcal{D})$	Galois groupoid of order k with parameters of \mathcal{D}
$\mathrm{R}_k M \rightarrow M$	Bundle of k -frames on M

Symbol	Term
S^\bullet	Algebra of symmetric tensors
$\tau_{m,k}M$	Generalized tangent bundle of type m, k
$x_{i:\alpha}$	i : α -coordinate in the jet space
\mathfrak{X}_A	Submodule of $\text{Der}A$ stable by the Lie bracket
\tilde{X}	\mathcal{D} -horizontal lifting of X
$X^{(m,\infty)}$	Prolongation of X to $\mathbb{C}[\hat{\tau}_m M]$
$X^{(m,k)}$	Prolongation of the vector field X
X^{tot}	Total derivative in the direction of X
$\mathfrak{X}_{\mathcal{F}}$	$\mathbb{C}(M)$ -module of derivations tangent to \mathcal{F}

Abbreviations

Abbreviation	Term
HG	Gauss' Hypergeometric Equation
PHG	Projection of Gauss Hypergeometric equation
P_{VI}	Painlevé VI equation
HP_{VI}	Hamiltonian vector field of Painlevé VI equation
Diag	Diagonal
Tot	Total
Proj	Projective
Red	Reduced

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Titre : Déformations isomonodromiques à travers de l'théorie de Galois différentielle

Mots clés: Théorie paramétrée de Galois, déformations isomonodromiques, équation hypergéométrique de Gauss, équation de Painlevé VI

Résumé : Le texte commence par une brève description de théorie différentielle de Galois dans une perspective géométrique. Ensuite, la théorie paramétrée de Galois est développée au moyen d'une prolongation des connexions partielles avec les fibrés de jets. La relation entre les groupes de Galois différentiels à paramètres et les déformations isomonodromiques est développée comme une application du théorème de Kiso-Cassidy.

Il s'ensuit le calcul des groupes de Galois à paramètres de l'équation générale fuchsienne et de l'équation hypergéométrique de Gauss. Enfin, certaines applications non linéaires sont développées. Au moyen d'un théorème de Kiso-Morimoto, un analogue non linéaire, on calcule le groupoïde de Malgrange de l'équation de Painlevé VI à paramètres variables.

Title : Isomonodromic deformations through differential Galois theory

Keywords: Parameterized Galois theory, isomonodromic deformations, Gauss hypergeometric equation, Painlevé VI equation

Abstract: The text begins with a brief description of differential Galois theory from a geometrical perspective. Then, parameterized Galois theory is developed by means of prolongation of partial connections to the jet bundles. The relation between the parameterized differential Galois groups and isomonodromic deformations is unfold as an application of Kiso-Cassidy theorem.

It follows the computation of the parameterized Galois groups of the general fuchsian equation and Gauss hypergeometric equation. Finally, some non-linear applications are developed. By means of a non-linear analog, Kiso-Morimoto theorem, the Malgrange groupoid of Painlevé VI equation with variable parameters is calculated.