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## THÈSE

présentée par

**Bianca BARUCCHIERI**

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**DOCTEUR**

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École Doctorale de Mathématiques et d'Informatique

Spécialité Mathématiques Pures

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# AFFINE HERMITE-LORENTZ MANIFOLDS

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dirigée par Vincent KOZIARZ et Pierre MOUNOUD

soutenue le 26 septembre 2019

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# AFFINE HERMITE-LORENTZ MANIFOLDS

*Abstract.* — In this work we deal with crystallographic groups, i.e. the subgroups of the group of affine transformations that act properly discontinuously and cocompactly on affine space. In other words they are the fundamental groups of compact and complete affine manifolds. In this thesis we classify such groups with the additional hypothesis that the linear part preserves a Hermitian form of signature  $(n, 1)$ . Grunewald and Margulis proved that such crystallographic groups are virtually solvable (the Auslander conjecture states that this is always true). Our classification is for  $n \leq 3$ . It corresponds to a classification, up to finite covering, and for complex dimension at most 4, of flat compact complete Hermite-Lorentz manifolds. This is inspired by the works done by Bieberbach, then Fried, and finally Grunewald and Margulis who classified crystallographic groups whose linear part preserves a positive definite or Lorentzian quadratic form. Making this classification we had to classify a family of 8-dimensional nilpotent Lie algebras. We then extended this classification to all the 8-dimensional 3-step nilpotent Lie algebras having the free 2-step nilpotent Lie algebra on 3 generators as quotient. This result can be seen as a step in the direction of a general classification of nilpotent Lie algebras of dimension 8. We then wondered which of these Lie algebras admit flat pseudo-Riemannian metrics and gave a partial answer to this question.

**Key words:** Affine manifolds, crystallographic groups, Hermite-Lorentz manifolds, nilpotent Lie algebras.

# VARIÉTÉS AFFINES HERMITE-LORENTZ

*Résumé.* — Dans ce travail nous nous intéressons aux groupes cristallographiques, i.e. aux sous-groupes du groupe des transformations affines qui agissent proprement discontinûment et de façon cocompacte sur l'espace affine. Ce sont les groupes fondamentaux des variétés affines compactes et complètes. Nous classifions les groupes cristallographiques dont la partie linéaire préserve une forme hermitienne de signature  $(n, 1)$ . Grunewald et Margulis ont prouvé que ces groupes cristallographiques sont virtuellement résolubles (la conjecture d'Auslander affirme que c'est toujours le cas). Notre classification est effectuée pour  $n \leq 3$ . Elle correspond à la classification, à revêtement fini près, des variétés Hermite-Lorentz plates, compactes et complètes en dimension complexe inférieure ou égale à 4. Ce travail est inspiré par ceux menés par Bieberbach, puis Fried, et enfin Grunewald et Margulis sur les groupes cristallographiques dont la partie linéaire préserve une forme quadratique définie positive ou lorentzienne. En effectuant cette classification, nous avons été amené à étudier certaines familles d'algèbres de Lie nilpotentes de dimension 8. Nous avons ensuite étendu cette classification à celle de toutes les algèbres de Lie 3-nilpotentes de dimension 8 ayant l'algèbre de Lie libre 3-nilpotente à 3 générateurs pour quotient. Ce résultat peut être vu comme un pas dans la direction d'une classification des algèbres de Lie nilpotentes de dimension 8. Ensuite nous nous sommes demandé lesquelles de ces algèbres admettent une métrique pseudo-riemannienne plate et nous avons donné une réponse partielle.

**Mots-clés:** Variétés affines, groupes cristallographiques, variétés Hermite-Lorentz, algèbres de Lie nilpotentes.

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# Introduction

For a vector space  $V$ , subgroups  $\Gamma$  of the affine group  $\text{Aff}(V) = \text{GL}(V) \ltimes V$  that act properly discontinuously and cocompactly on  $V$  are called crystallographic groups and have a long history. The first results date back to Bieberbach around the years 1911-12, when he studied the case  $\Gamma \leq \text{O}(n) \ltimes \mathbb{R}^n$ . His results show that such groups are, up to finite index, abelian groups made of translations. Later the Lorentzian case,  $\Gamma \leq \text{O}(n, 1) \ltimes \mathbb{R}^{n+1}$ , was studied. For this case the first results were obtained by Auslander and Markus in dimension 3, [9], and, more generally, Fried and Goldman studied all 3-dimensional crystallographic groups in [26]. Still for the Lorentzian case, in the years 1987-88 Fried investigated the dimension 4, [25], and afterwards results for all dimensions were given by Grunewald and Margulis, [31]. In this thesis we are interested in the Hermite-Lorentz case, namely  $\Gamma \leq \text{U}(n, 1) \ltimes \mathbb{C}^{n+1}$ . Following the strategy used by Grunewald and Margulis in [31] we give a classification, up to finite index, of these crystallographic groups for  $n \leq 3$ . These groups are the fundamental groups of flat compact complete Hermite-Lorentz manifolds and they determine the manifold. Indeed, after a classical result of Mostow [38], if two such groups are isomorphic the corresponding manifolds are diffeomorphic. Since we are interested in classifying the manifolds up to diffeomorphism we will only be concerned with isomorphism classes of these groups and not with the different ways one can realise them as subgroups of  $\mathcal{H}(n, 1) = \text{U}(n, 1) \ltimes \mathbb{C}^{n+1}$ . Therefore, according to Mostow's Theorem, the classification, up to finite index, of crystallographic subgroups  $\Gamma \leq \text{U}(n, 1) \ltimes \mathbb{C}^{n+1}$  given in this thesis corresponds to a classification, up to finite covering, and for complex dimension at most 4, of flat compact complete Hermite-Lorentz manifolds. Indeed, let  $\mathfrak{a}(\mathbb{C}^{n+1})$  be the affine space associated to the complex vector space  $\mathbb{C}^{n+1}$  endowed with a Hermitian form of signature  $(n, 1)$  and  $\text{U}(n, 1)$  be the subgroup of the general linear group preserving this Hermitian form. Flat compact complete Hermite-Lorentz manifolds can then be presented as the quotient  $\Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$  where the group  $\Gamma$  is a subgroup of  $\text{U}(n, 1) \ltimes \mathbb{C}^{n+1}$  acting properly discontinuously and cocompactly on  $\mathfrak{a}(\mathbb{C}^{n+1})$ . These manifolds can be thought as analogue of both Hermitian manifolds (definite positive) in complex geometry and Lorentzian manifolds in real differential geometry, see [4]. Let us notice that Scheuneman in [43] already studied the case of compact complete affine complex surfaces.

About crystallographic groups there is a long-standing conjecture due to Auslander that says that such groups are virtually solvable. Otherwise said, from the manifold point of view, the fundamental group of every compact complete flat affine manifold is virtually solvable. In some special cases the Auslander conjecture is known to be true. One such case is the Riemannian case as we have

seen in Bieberbach's result. Another such case is the Lorentzian case, which was proved in [29] by Goldman and Kamishima, see also [15]. Finally, after a result of Grunewald and Margulis, [31], also the case that interests us, namely the Hermite-Lorentz case  $\Gamma \leq \mathrm{U}(n, 1) \ltimes \mathbb{C}^{n+1}$ , is known to satisfy the Auslander conjecture. For virtually solvable crystallographic groups Fried and Goldman gave the following generalisation of Bieberbach's result, see Theorem 1.2.37.

**Theorem** ([26, Corollary 1.5.]). *Let  $V$  be a finite dimensional real vector space and  $\Gamma \leq \mathrm{Aff}(V)$  be a virtually solvable group acting properly discontinuously and cocompactly on  $\mathfrak{a}(V)$ . Then there exists a closed solvable subgroup  $H \leq \mathrm{Aff}(V)$  such that*

1.  $H$  acts simply transitively on  $\mathfrak{a}(V)$ ;
2.  $\Gamma \cap H$  has finite index in  $\Gamma$ ;
3.  $\Gamma \cap H$  is a lattice in  $H$ .

Indeed, when the linear part of  $\Gamma$  preserves a positive definite bilinear form, Bieberbach's results tell us that the simply transitive group of the theorem is the group of translations. The simply transitive groups appearing in this theorem are sometimes called connected crystallographic hulls. This theorem will be our starting point for the investigation of the Hermite-Lorentz manifolds  $\Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$ . More precisely, in order to study crystallographic subgroups  $\Gamma \leq \mathrm{U}(n, 1) \ltimes \mathbb{C}^{n+1}$  we will study lattices in simply transitive affine groups  $H \leq \mathrm{U}(n, 1) \ltimes \mathbb{C}^{n+1}$ . Hence, even though the Hermite-Lorentz manifolds are complex manifolds, we see them as real objects with an integrable complex structure. More precisely we will look at this manifolds, up to finite index, as  $\Gamma \backslash H$  where  $H$  is a simply transitive subgroup of  $\mathrm{U}(n, 1) \ltimes \mathbb{C}^{n+1}$  and  $\Gamma$  is a lattice in it. The Lie groups  $H$  that act simply transitively on  $\mathbb{C}^{n+1}$  are real Lie groups which inherit an integrable left invariant complex structure from  $\mathbb{C}^{n+1}$ . The integrability condition, see Definition 3.1.15, is more general than the condition for a Lie group to be a complex Lie group. Indeed the Lie groups we are looking at are not complex Lie groups.

This thesis is organised as follows.

In Chapter 1 we recall some known facts about crystallographic groups and affine manifolds. We will recall the results of Bieberbach for Riemannian crystallographic groups and of Fried, Grunewald and Margulis for Lorentzian crystallographic groups.

Chapter 2 is the heart of the results of this thesis and is organised in the following way. In Section 2.1 we will study unipotent groups  $H$  acting simply transitively on  $\mathfrak{a}(\mathbb{C}^{n+1})$ . Let us remember the following definition.

**Definition.** *A subgroup  $U$  of the affine group  $\mathrm{Aff}(V)$  is unipotent if its linear part consists of endomorphisms whose eigenvalues are all 1.*

Notice that a unipotent group is, in particular, a nilpotent group. The converse in general is not true. But every simply transitive nilpotent affine group of motion that appears as a connected

crystallographic hull is unipotent, see Theorem 1.2.13. Thus in Section 2.1 we give a presentation,  $u(\gamma_2, \gamma_3, b_2, b_3)$ , valid in any dimension, of unipotent simply transitive groups of Hermite-Lorentz affine motion, and give their classification up to conjugacy.

In Section 2.2 we reduce, as for the Lorentzian case, the study of crystallographic groups to the study of lattices in simply transitive unipotent Lie groups. Let us define some terminology.

**Definition.** *A group  $\Gamma$  is said to be abelian-by-cyclic if it has a normal abelian subgroup  $\Gamma_1$  such that  $\Gamma/\Gamma_1$  is cyclic.*

We then prove the following.

**Theorem.** *Let  $\Gamma$  be a subgroup of  $\mathcal{H}(n, 1)$  that acts properly discontinuously and cocompactly on  $\mathfrak{a}(V)$ . Then  $\Gamma$  is either virtually nilpotent or virtually abelian-by-cyclic.*

According to this theorem, the unipotent hypothesis we make in Section 2.1 only leaves out the easy abelian-by-cyclic case. We end this section with the classification of the latter case.

In Section 2.3 we start the classification, up to isomorphism, of those Lie groups found in Section 2.1. We complete their classification for the dimensions 2 and 3. In dimension 4 we give the classification for some particular cases, which we call degenerate cases.

In Section 2.4 we finish the classification of the unipotent simply transitive groups in dimension 4. For this more general case we are left with a classification problem of 8-dimensional nilpotent Lie algebras defined by three parameters. Since there are no complete classifications of nilpotent Lie algebras in dimension larger than 7 we introduce an ad-hoc method to study our particular family of Lie algebras. Using the fact that these Lie algebras are Carnot, see Subsection 2.4.2, we can identify their isomorphism classes with the orbits of the  $SL(3, \mathbb{R})$ -action on the Grassmannian of 2-dimensional subspaces of  $\mathfrak{sl}(3, \mathbb{R})$  induced by the adjoint action. Hence studying the orbits for this action is the same as studying the isomorphism classes for the family of Lie algebras. In contrast with the Lorentzian case where, for every fixed dimension, there are finitely many isomorphism classes of unipotent subgroups of  $O(n, 1) \ltimes \mathbb{R}^{n+1}$  acting simply transitively on  $\mathfrak{a}(\mathbb{R}^{n+1})$  we find that these Lie algebras constitute an infinite family of pairwise non-isomorphic Lie algebras of Lie groups acting simply transitively on  $\mathfrak{a}(\mathbb{C}^{3+1})$ . In order to classify unipotent simply transitive subgroups of  $U(3, 1) \ltimes \mathbb{C}^{3+1}$  up to isomorphism it is sufficient to classify their Lie algebras up to Lie algebra isomorphism. Thus in Section 2.4 we complete the proof of the following.

**Theorem.** *The list given in Appendix A with  $\mathbb{K} = \mathbb{R}$  is a complete non-redundant list of isomorphism classes of unipotent simply transitive subgroups of  $U(3, 1) \ltimes \mathbb{C}^{3+1}$ .*

In Section 2.5 we study the Hermite-Lorentz crystallographic groups in the nilpotent case. Indeed we have reduced it to the study of lattices in simply transitive unipotent groups. Since in Sections 2.3 and 2.4 we have given the complete list of those simply transitive groups we just need to understand which of them admit lattices and what are their abstract commensurability classes. Let us give the following definition.

**Definition.** Two groups  $\Gamma_1$  and  $\Gamma_2$  are said to be abstractly commensurable if there exist two subgroups  $\Delta_1 \leq \Gamma_1$  and  $\Delta_2 \leq \Gamma_2$  of finite index such that  $\Delta_1$  is isomorphic to  $\Delta_2$ .

Lattices in nilpotent Lie groups are described by Malcev's theorems, see Section 2.5 and, for more details, see [19] or [39, Section II]. These theorems say that a nilpotent Lie group  $G$  admits a lattice if and only if its Lie algebra  $\mathfrak{g}$  admits a  $\mathbb{Q}$ -form, that is a rational subalgebra  $\mathfrak{g}_0$  such that  $\mathfrak{g}_0 \otimes \mathbb{R}$  is isomorphic to  $\mathfrak{g}$ . Furthermore abstract commensurability classes of lattices in  $G$  are in correspondence with  $\mathbb{Q}$ -isomorphism classes of  $\mathbb{Q}$ -forms of  $\mathfrak{g}$ . To be more precise let us remember that for a simply connected nilpotent Lie group the exponential map from its Lie algebra is a diffeomorphism and we will call  $\log$  its inverse. Then from a lattice  $\Gamma$  in  $G$ ,  $\text{span}_{\mathbb{Q}}\{\log \Gamma\}$  provides a  $\mathbb{Q}$ -form of  $\mathfrak{g}$ . On the other hand let  $\mathfrak{g}$  be a Lie algebra with a basis with respect to which the structure constants are rational and let  $\mathfrak{g}_0$  be the  $\mathbb{Q}$ -span of this basis. If  $\mathcal{L}$  is any lattice of  $\mathfrak{g}$  contained in  $\mathfrak{g}_0$  the group generated by  $\exp \mathcal{L}$  is a lattice in  $G$ . The main result of this section is then the following.

**Theorem.** The list given in Appendix A with  $\mathbb{K} = \mathbb{Q}$  is a complete list without repetition of abstract commensurability classes of nilpotent crystallographic subgroups of  $U(3, 1) \ltimes \mathbb{C}^{3+1}$ .

Finally in Section 2.6 we give some topological considerations about the manifolds  $\Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$  that are virtually nilpotent, namely that they are finitely covered by torus bundles over tori.

In Chapter 3 we adopt the point of view of metric structures on Lie groups. Indeed Lie groups acting simply transitively on  $\mathfrak{a}(\mathbb{C}^{n+1})$  can be seen as real Lie groups that are endowed with a left invariant flat Hermite-Lorentz metric. There is a vast literature that treats similar questions with the point of view of metric Lie algebras. The first structure one can put on a Lie algebra is an affine structure. By this we mean a representation of a Lie algebra  $\mathfrak{g}$  of dimension  $n$  in the Lie algebra of the affine group of  $\mathbb{R}^n$  with the property that the span of all the translations is the whole  $\mathbb{R}^n$ . Two affine structures are isomorphic if they are isomorphic as representations. A work of Scheuneman, [42] see also Proposition 3.1.10, proves that one can always put an affine structure on a 3-step nilpotent Lie algebra by an explicit construction. Remember that the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are 3-step nilpotent, then we have the following, see also Proposition 3.2.22.

**Proposition.** The affine structure on  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  coming from Proposition 2.1.5 is in general not isomorphic to the one defined by Scheuneman.

In Section 3.1 we recall some general facts and notations. In particular we recall the construction results for Lorentzian and pseudo-Riemannian nilpotent Lie algebras of signature  $(n-2, 2)$  given in [5] and [12]. They construct such Lie algebras using a process called double extension, see Definition 3.1.24. But in these articles they are not interested in the classification problem. The Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ , that appear in Proposition 2.1.5 are a particular case of pseudo-Riemannian nilpotent Lie algebras of signature  $(n-2, 2)$ . A posteriori we can show, Proposition 3.1.28, how they can be seen as a double extension as in [12]. Furthermore we give conditions, Proposition 3.1.30, for

a flat metric Lie algebra of signature  $(n - 2, 2)$  constructed with a double extension process to be Hermite-Lorentz.

In the Section 3.2 we concentrate our study to a family of 8-dimensional nilpotent Lie algebras,  $\mathfrak{g}(Q)$ , defined in Definition 2.4.18, which contains, up to isomorphism,  $u(\gamma_2, \gamma_3, b_2, b_3)$  in the non-degenerate case in dimension 4. We first study their isomorphism classes. After showing the first steps we did in this direction using techniques coming from geometric invariant theory we prove the following with a more ad-hoc method, see Theorem 3.2.14.

**Theorem.** *Each Lie algebra  $\mathfrak{g}(Q)$  belongs to one of the non-isomorphic families represented in Appendix B.*

Secondly we are interested in studying complex and metric structures on  $\mathfrak{g}(Q)$ . Indeed, starting from the complex and metric structure that the Lie algebras  $u(\gamma_2, \gamma_3, b_2, b_3)$  naturally have, we can deform them to get some complex and metric structures on the Lie algebras  $\mathfrak{g}(Q)$ . Inside the family  $\mathfrak{g}(Q)$  there are two special families,  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$ , see Definitions 3.2.3 and 3.2.6. The union of these two families is an open dense subset of the space of isomorphism classes of the Lie algebras  $\mathfrak{g}(Q)$ . We have the following result, see Proposition 3.2.24 and Proposition 3.2.31.

**Proposition.** *We can put a flat pseudo-Riemannian metric of signature  $(6, 2)$  and one of signature  $(4, 4)$  on both  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$ .*



# Introduction en français

Soit  $V$  un espace vectoriel de dimension finie et  $\mathfrak{a}(V)$  l'espace affine associé à  $V$ . On appelle groupe cristallographique un sous-groupe  $\Gamma$  du groupe des transformations affines de  $V$ , noté  $\text{Aff}(V)$ , qui agit de façon proprement discontinue et cocompacte sur  $\mathfrak{a}(V)$ . Les premiers résultats sur les groupes cristallographiques sont dus à Bieberbach dans les années 1911-12, qui montre que, lorsque la partie linéaire de  $\Gamma$  préserve une forme quadratique définie positive,  $\Gamma$  est, à indice fini près, un groupe abélien composé de translation. Par la suite, ses travaux ont été généralisés au cas où la partie linéaire de  $\Gamma$  préserve une forme lorentzienne, i.e. de signature  $(n, 1)$  avec  $\dim(V) = n + 1$ . D'abord Auslander et Markus ont étudié ce cas pour  $V$  de dimension 3, [9], et, de façon plus générale, Fried et Goldman ont étudié tous les groupes cristallographiques de dimension 3 dans [26]. Pour le cas lorentzien, dans les années 1987-88 Fried a étudié le cas de la dimension 4, [25], et ensuite Grunewald et Margulis ont donné des résultats pour  $V$  de dimension quelconque, [31]. Dans cette thèse on s'intéresse à la classification des groupes cristallographiques dont la partie linéaire préserve une forme hermitienne de signature  $(n, 1)$ , appelée ici forme Hermite-Lorentz, c'est-à-dire lorsque  $\Gamma$  est un sous-groupe de  $\mathcal{H}(n, 1) = \text{U}(n, 1) \ltimes \mathbb{C}^{n+1}$ . En suivant la stratégie utilisée par Grunewald et Margulis dans [31] on classe ces groupes cristallographiques pour  $n \leq 3$ . Ces groupes sont les groupes fondamentaux des variétés Hermite-Lorentz plates, compactes et complètes et ils déterminent la variété. Plus précisément, d'après un résultat de Mostow [38], si deux groupes de ce type sont isomorphes alors les variétés associées sont difféomorphes. Étant donné que notre intérêt est de classifier les variétés à difféomorphisme près, on sera concerné seulement par les classes d'isomorphismes de ces groupes et non par les différentes façons de les réaliser comme sous-groupes de  $\mathcal{H}(n, 1)$ . Par conséquent, d'après le théorème de Mostow, la classification faite dans cette thèse, à indice fini près, des groupes cristallographiques  $\Gamma \leq \mathcal{H}(n, 1)$ , correspond à la classification, à revêtement fini près, des variétés Hermite-Lorentz plates, compactes et complètes en dimension complexe inférieure ou égale à 4. En effet, une variété Hermite-Lorentz plate, compacte et complète peut être présentée comme le quotient  $\Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$  où  $\Gamma$  est un sous-groupe cristallographique de  $\mathcal{H}(n, 1)$ . Ces variétés peuvent être vues comme étant l'analogue à la fois des variétés hermitiennes (définies positives) en géométrie complexe et des variétés lorentziennes en géométrie différentielle réelle, voir pour référence [4]. On peut remarquer que Scheuneman avait déjà étudié les surfaces affines complexes compactes dans [43].

En ce qui concerne les groupes cristallographiques il existe une conjecture, attribuée à Auslander, qui affirme que tout groupe cristallographique est, à indice fini près, résoluble. Cette conjecture a été

prouvée dans certains cas. Par exemple, lorsque le groupe préserve une forme euclidienne, voir le résultat de Bieberbach, ou une forme lorentzienne, ce qui a été prouvé par Goldman et Kamishima, [29], voir aussi [15]. Enfin, d'après Grunewald et Margulis, [31], on sait que pour  $\Gamma \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$ , soit le cadre de cette thèse, la conjecture d'Auslander est vraie. Pour les groupes cristallographiques virtuellement résolubles Fried et Goldman ont donné la généralisation suivante du théorème de Bieberbach, voir aussi théorème 1.2.37.

**Théorème** ([26, Corollaire 1.5.]). *Soit  $V$  un espace vectoriel réel de dimension finie et  $\Gamma \leq \text{Aff}(V)$  un groupe virtuellement résoluble qui agit proprement discontinûment et de façon cocompacte sur  $\mathfrak{a}(V)$ . Alors il existe un sous-groupe fermé résoluble  $H \leq \text{Aff}(V)$  tel que*

1.  $H$  agit simplement transitivement sur  $\mathfrak{a}(V)$ ;
2.  $\Gamma \cap H$  est d'indice fini dans  $\Gamma$ ;
3.  $\Gamma \cap H$  est un réseau dans  $H$ .

Les groupes simplement transitifs qui apparaissent dans ce théorème sont appelés enveloppes cristallographiques connexes. Ce théorème sera notre point de départ pour la classification des variétés Hermite-Lorentz  $\Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$ . De façon plus exacte, pour étudier les groupes cristallographiques  $\Gamma \leq \mathcal{H}(n, 1)$  on étudiera les réseaux dans les groupes affines simplement transitifs  $H \leq \mathcal{H}(n, 1)$ . Donc, même si les variétés Hermite-Lorentz sont des variétés complexes, on les regarde comme étant des objets réels munis d'une structure complexe intégrable. Plus précisément, on regarde ces variétés, à indice fini près, comme étant de la forme  $\Gamma \backslash H$  où  $H$  est un sous-groupe simplement transitif de  $\mathcal{H}(n, 1)$  et  $\Gamma$  est un réseau dans  $H$ . Les groupes  $H$  sont des groupes de Lie réels qui héritent d'une structure complexe intégrable invariante à gauche provenant de  $\mathbb{C}^{n+1}$ . La condition d'intégrabilité, voir définition 3.1.15, est plus générale que la condition pour qu'un groupe de Lie soit complexe. En effet, les groupes de Lie qu'on étudiera ne sont pas des groupes de Lie complexes.

Cette thèse est organisée de la façon suivante. Dans le chapitre 1 on rappelle des faits connus sur les groupes cristallographiques et les variétés affines. On rappelle les résultats de Bieberbach pour les groupes cristallographiques riemanniens et ceux de Fried, Grunewald et Margulis sur les groupes cristallographiques lorentziens.

Le chapitre 2 est le cœur de cette thèse et contient les résultats principaux. Il est organisé de la façon suivante. Dans la section 2.1 on étudie les groupes unipotents  $H$  qui agissent simplement transitivement sur  $\mathfrak{a}(\mathbb{C}^{n+1})$ . On rappelle la définition suivante.

**Définition.** *Un sous-groupe du groupe des transformations affines  $\text{Aff}(V)$  est unipotent si sa partie linéaire est uniquement composée d'endomorphismes dont la seule valeur propre est 1.*

Remarquons qu'un groupe unipotent est, en particulier, nilpotent. L'inverse est, en général, faux. Cependant, toute enveloppe cristallographique connexe nilpotente est unipotente, voir théorème 1.2.13. Dans la section 2.1 on donne une présentation,  $u(\gamma_2, \gamma_3, b_2, b_3)$ , valable en toute dimension, des

groupes unipotents simplement transitifs des transformations affines Hermite-Lorentz et on montre une proposition sur leurs classes de conjugaison.

Dans la section 2.2 on réduit, comme dans le cas lorentzien, l'étude des groupes cristallographiques, à l'étude des réseaux dans les groupes unipotents simplement transitifs. On y montre le théorème suivant.

**Théorème.** *Soit  $\Gamma$  un sous-groupe de  $\mathcal{H}(n, 1)$  qui agit proprement discontinûment et de façon cocompacte sur  $\mathfrak{a}(V)$ . Alors  $\Gamma$  est soit virtuellement nilpotent soit contient un sous-groupe d'indice fini qui est l'extension d'un groupe cyclique par un groupe abélien.*

On termine cette section par la classification des groupes cristallographiques qui sont une extension d'un groupe cyclique par un groupe abélien.

Dans la section 2.3 on commence la classification, à isomorphisme près, des groupes de Lie trouvés dans la section 2.1. On les classe en dimension 2 et 3. En dimension 4 on donne leur classification dans certains cas qu'on appelle cas dégénérés.

Dans la section 2.4 on termine la classification des groupes unipotents simplement transitifs en dimension 4. Pour cela on doit classifier des algèbres de Lie nilpotentes de dimension 8 définies par 3 paramètres. Vu qu'il n'y a pas de classification complète des algèbres de Lie nilpotentes en dimension plus grande que 7 on introduit une méthode ad-hoc afin d'étudier notre famille particulière. En utilisant le fait que ces algèbres sont de Carnot, voir la sous-section 2.4.2, on peut identifier leurs classes d'isomorphisme avec les orbites de l'action de  $SL(3, \mathbb{R})$ , induite par l'action adjointe, sur la grassmannienne des sous-espaces de dimension 2 de  $\mathfrak{sl}(3, \mathbb{R})$ . Par conséquent, étudier les orbites de cette action revient à étudier les classes d'isomorphisme de la famille d'algèbres de Lie à laquelle on s'intéresse. On montre que cette famille constitue une famille infinie d'algèbres de Lie deux à deux non isomorphes dont les groupes de Lie associés agissent simplement transitivement sur  $\mathfrak{a}(\mathbb{C}^{3+1})$ , à l'opposé du cas lorentzien où, pour chaque dimension fixée, il n'existe qu'un nombre fini de classes d'isomorphisme de sous-groupes unipotents de  $O(n, 1) \ltimes \mathbb{R}^{n+1}$  qui agissent simplement transitivement sur  $\mathfrak{a}(\mathbb{R}^{n+1})$ . Si on veut classifier les sous-groupes unipotents simplement transitifs de  $\mathcal{H}(3, 1)$  il suffit de classifier leurs algèbres de Lie. Dans cette section on complète donc la preuve du théorème suivant.

**Théorème.** *La liste donnée dans l'appendice A avec  $\mathbb{K} = \mathbb{R}$  est une liste complète et non redondante des classes d'isomorphisme des sous-groupes unipotents simplement transitifs de  $\mathcal{H}(3, 1)$ .*

Dans la section 2.5 on étudie les groupes cristallographiques Hermite-Lorentz dans le cas nilpotent. En effet, on a réduit leur étude à celle des réseaux dans les groupes unipotents simplement transitifs. Il suffit donc de déterminer quels groupes unipotents de la liste donnée dans les sections 2.3 et 2.4 admettent un réseau et classifier ces réseaux. On donne la définition suivante.

**Définition.** *Deux groupes  $\Gamma_1$  et  $\Gamma_2$  sont abstraitement commensurables s'il existe deux sous-groupes  $\Delta_1 \leq \Gamma_1$  et  $\Delta_2 \leq \Gamma_2$  d'indice fini tels que  $\Delta_1$  soit isomorphe à  $\Delta_2$ .*

La classification des groupes cristallographiques sera faite par rapport à cette relation d'équivalence que l'on vient de définir. La théorie des réseaux dans des groupes de Lie nilpotents est décrite par les théorèmes de Malcev, voir section 2.5 et, pour plus des détails, voir [19] ou [39, section II]. Ces théorèmes montrent qu'un groupe de Lie nilpotent  $G$  contient un réseau si et seulement si son algèbre de Lie,  $\mathfrak{g}$ , admet une  $\mathbb{Q}$ -forme, c'est-à-dire une sous-algèbre rationnelle  $\mathfrak{g}_0$  telle que  $\mathfrak{g}_0 \otimes \mathbb{R}$  est isomorphe à  $\mathfrak{g}$ . De plus, les classes de commensurabilité abstraite des réseaux de  $G$  correspondent aux classes d'isomorphisme sur  $\mathbb{Q}$  des  $\mathbb{Q}$ -formes de  $\mathfrak{g}$ . On rappelle que pour un groupe de Lie simplement connexe et nilpotent l'application exponentielle depuis son algèbre de Lie est un difféomorphisme, on appelle  $\log$  son inverse. Alors, si  $\Gamma$  est un réseau dans  $G$ ,  $\text{span}_{\mathbb{Q}}\{\log \Gamma\}$  donne une  $\mathbb{Q}$ -forme de  $\mathfrak{g}$ . Réciproquement, soit  $\mathfrak{g}$  une algèbre de Lie avec une  $\mathbb{Q}$ -forme  $\mathfrak{g}_0$ . Si  $\mathcal{L}$  est un réseau de  $\mathfrak{g}$  contenu dans  $\mathfrak{g}_0$ , le groupe engendré par  $\exp \mathcal{L}$  est un réseau dans  $G$ . Le résultat principal de cette section est le suivant.

**Théorème.** *La liste donnée dans l'appendice A avec  $\mathbb{K} = \mathbb{Q}$  est une liste complète, sans répétition des classes de commensurabilité abstraite des sous-groupes cristallographiques nilpotents de  $U(3, 1) \times \mathbb{C}^{3+1}$ .*

Enfin, dans la section 2.6 on fait quelques remarques de nature topologique sur les variétés  $\Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$  qui sont virtuellement nilpotentes, i.e. elles admettent un revêtement fini composé de fibrés en tores sur des tores.

Dans le chapitre 3 on s'intéresse aux structures invariantes à gauche sur les groupes de Lie. En fait, les groupes de Lie qui agissent simplement transitivement sur  $\mathfrak{a}(\mathbb{C}^{n+1})$  peuvent être vus comme des groupes de Lie avec une métrique plate Hermite-Lorentz invariante à gauche. Il y a de nombreux travaux sur ce sujet. La première structure qu'on peut mettre sur une algèbre de Lie est une structure affine. Par cela on entend une représentation d'une algèbre de Lie  $\mathfrak{g}$  de dimension  $n$  dans l'algèbre de Lie du groupe affine de  $\mathbb{R}^n$  tel que le sous-espace engendré par les translations soit tout  $\mathbb{R}^n$ . Deux structures affines sont isomorphes si elles sont isomorphes comme représentations. Dans un travail de Scheuneman, [42] et voir aussi proposition 3.1.10, il est prouvé par une construction explicite qu'il existe toujours une structure affine sur une algèbre de Lie 3-nilpotente. Les algèbres de Lie  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  étant 3-nilpotentes on peut comparer la structure affine de Scheuneman et celle induite par la métrique Hermite-Lorentz. On obtient :

**Proposition.** *La structure affine sur  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  définie dans la Proposition 2.1.5 n'est, en général, pas isomorphe à celle définie par Scheuneman.*

Dans la section 3.1 on rappellera des faits généraux sur les algèbres métriques. En particulier on rappellera la construction donnée dans [5] et [12] appelée "double extension" pour les algèbres de Lie lorentziennes et pseudo-riemanniennes de signature  $(n-2, 2)$ . Dans ces papiers ces algèbres de Lie sont construites avec un processus appelé double extension, voir définition 3.1.24. Cependant, ces travaux ne proposent pas de classification de ces algèbres. Les algèbres de Lie  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ , qui apparaissent dans la proposition 2.1.5, sont des cas particuliers d'algèbres de Lie nilpotentes pseudo-riemanniennes de signature  $(n-2, 2)$ . On montre a posteriori, voir proposition 3.1.28, comment

écrire ces algèbres comme des doubles extensions. De plus, on donne des conditions, proposition 3.1.30, pour qu'une algèbre de Lie plate de signature  $(n - 2, 2)$  construite avec un processus de double extension soit Hermite-Lorentz.

Dans la section 3.2 on se concentre sur l'étude d'une famille d'algèbres de Lie nilpotentes,  $\mathfrak{g}(Q)$ , définie dans la définition 2.4.18, qui contient, à isomorphisme près, les algèbres  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  non dégénérées de dimension 4. On étudie d'abord leurs classes d'isomorphismes. Après avoir décrit les premiers pas que l'on a effectués dans cette étude en utilisant des techniques provenant de la théorie des invariants géométriques, on prouvera le théorème suivant en utilisant une méthode ad-hoc.

**Théorème.** *Chaque algèbre de Lie  $\mathfrak{g}(Q)$  appartient à une des familles non isomorphes représentées dans l'appendice B.*

Dans une seconde partie on s'intéresse aux structures complexes et métriques dont on peut munir  $\mathfrak{g}(Q)$ . On veut déformer les structures complexes et métriques apparaissant naturellement sur  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \cap \mathfrak{g}(Q)$  pour obtenir des structures complexes et métriques sur  $\mathfrak{g}(Q)$ . Dans la famille  $\mathfrak{g}(Q)$  il y a deux familles,  $\mathfrak{g}(a, b, c, d)$  et  $\mathfrak{g}'(a, b, c, d)$ , voir définitions 3.2.3 et 3.2.6. L'union de ces deux familles définit un ouvert dense de l'espace des classes d'isomorphismes des algèbres  $\mathfrak{g}(Q)$ . On a les résultats suivants, voir proposition 3.2.24 et proposition 3.2.31.

**Proposition.** *On peut définir une métrique pseudo-riemannienne plate de signature  $(6, 2)$  et une de signature  $(4, 4)$  à la fois sur  $\mathfrak{g}(a, b, c, d)$  et  $\mathfrak{g}'(a, b, c, d)$ .*



# Chapter 1

## Preliminaries

The beginning of the story of crystallographic groups is often associated with Hilbert's 18th problem concerning the classification of groups of Euclidean isometries with compact fundamental domain. In this chapter we will say a few words about Bieberbach's answer to Hilbert's question. Afterwards we will put the study and classification of crystallographic groups into perspective and link them to the world of affine manifolds. Some material of this section has been inspired by [1]. Finally we will say some words about the answer to the classification problem in the Lorentzian case.

Let  $V$  be a vector space of dimension  $n$ , denote by  $\mathfrak{a}(V)$  the affine space associated to  $V$  and by  $\text{Aff}(V)$  the group of affine transformations of  $\mathfrak{a}(V)$ . Remember that the affine group can be seen as a linear group as follows

$$\text{Aff}(V) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in \text{GL}(V), v \in V \right\}.$$

Denote furthermore by  $L : \text{Aff}(V) \rightarrow \text{GL}(V)$  the homomorphism that associates to each affine transformation its linear part. Its kernel consists of translations and let us denote it by

$$T = \left\{ \begin{pmatrix} \text{Id} & v \\ 0 & 1 \end{pmatrix} \mid v \in V \right\}.$$

For a subgroup  $G \leq \text{Aff}(V)$ , let us denote by  $T_G = G \cap T$  the subgroup of translations in  $G$ . Denote furthermore by  $t : \text{Aff}(V) \rightarrow V$  the map that associates to each element its translation part. Notice that the exact sequence

$$1 \longrightarrow T \longrightarrow \text{Aff}(V) \xrightarrow{L} \text{GL}(V) \longrightarrow 1$$

splits and we have  $\text{Aff}(V) \cong \text{GL}(V) \ltimes V$ .

## 1.1 Classical case: Bieberbach's theorems

Endow  $V$  with a positive definite bilinear form  $\langle \cdot, \cdot \rangle$  and let  $O(n)$  be the subgroup of linear transformations of  $V$  preserving  $\langle \cdot, \cdot \rangle$ . Denote by  $\mathcal{E}(n)$  the group of affine transformation whose linear part is in  $O(n)$ .

**Definition 1.1.1.** A subgroup  $\Gamma$  of  $\mathcal{E}(n)$  is called an *Euclidean crystallographic group* if it is discrete and cocompact, i.e. the quotient  $\Gamma \backslash \mathfrak{a}(V)$  is compact.

**Theorem 1.1.2** (Bieberbach). *Every Euclidean crystallographic group has a finite index subgroup that is abelian made of translations.*

*Proof.* We will write a geometrical proof coming from [13]. Let  $|\cdot|$  be the norm associated to  $\langle \cdot, \cdot \rangle$  and let us define for an element  $A \in O(n)$  the following quantity called the rotational part of  $A$

$$m(A) = \max \left\{ \frac{|Ax - x|}{|x|} \mid x \in V \setminus \{0\} \right\}$$

and

$$E_A = \{x \in V \mid |Ax - x| = m(A)|x|\}.$$

We can consider the orthogonal decomposition  $V = E_A \oplus E_A^\perp$  and write  $x = x^E + x^\perp$ . Define furthermore  $m^\perp(A) = \max \left\{ \frac{|Ax^\perp - x^\perp|}{|x^\perp|} \mid x^\perp \in E_A^\perp \setminus \{0\} \right\}$  if  $E_A \neq \{0\}$  and 0 otherwise. It follows that

$$m^\perp(A) < m(A) \quad \text{if} \quad A \neq \text{Id}$$

and

$$|Ax^E - x^E| = m(A)|x^E|, \quad |Ax^\perp - x^\perp| \leq m^\perp(A)|x^\perp|.$$

Notice that, for  $[A, B]$  the commutator of  $A$  and  $B$ , the following also holds

$$m([A, B]) \leq 2m(A)m(B).$$

Let now  $\Gamma$  be an Euclidean crystallographic group.

**Claim 1.1.2.1.** For all  $u \in V$  unit vector and for all  $\varepsilon, \delta > 0$  there exists  $\beta \in \Gamma$  such that  $t(\beta) \neq 0$ , the angle between  $u$  and  $t(\beta)$  is  $\leq \delta$  and  $m(L(\beta)) \leq \varepsilon$ .

*Proof.* The cocompactness of the action implies that there exists some constant  $d$  such that for all  $u \in V$  unit vector there exists an element  $\beta_k \in \Gamma$  such that  $|t(\beta_k) - ku| \leq d$  for every  $k \in \mathbb{N}$ . The sequence  $\{\beta_k\}$  is such that  $|t(\beta_k)| \rightarrow \infty$  and the angle between  $u$  and  $t(\beta_k)$  tends to 0 as  $k \rightarrow \infty$ . Since  $O(n)$  is compact we can find a subsequence such that  $L(\beta_k)$  converges. Hence for all  $\varepsilon, \delta > 0$  there exists  $i < j$  such that  $m(L(\beta_j\beta_i^{-1})) \leq \varepsilon$ , the angle between  $u$  and  $t(\beta_j)$  is  $\leq \frac{\delta}{2}$  and  $|t(\beta_i)| \leq \frac{\delta}{4}|t(\beta_j)|$ . Then the affine transformation  $\beta = \beta_j\beta_i^{-1}$  is such that  $t(\beta) \neq 0$ , the angle between  $u$  and  $t(\beta)$  is  $\leq \delta$  and  $m(L(\beta)) \leq \varepsilon$ .  $\square$

**Claim 1.1.2.2.** If  $\gamma \in \Gamma$  satisfies  $m(L(\gamma)) \leq \frac{1}{2}$  then  $\gamma$  is a translation.

*Proof.* Assume by contradiction that there exists  $\gamma \in \Gamma$  such that  $m(L(\gamma)) \leq \frac{1}{2}$  and  $\gamma$  is not a translation. Then  $m(L(\gamma)) > 0$ . Since the group is discrete there are only finitely many elements in  $\Gamma$  with translational part of modulus less than a certain constant. Hence we might take the  $\gamma \in \Gamma$  such that  $0 < m(L(\gamma)) \leq \frac{1}{2}$  and  $|t(\gamma)|$  is a minimum. Now Claim 1.1.2.1 applied to any unit vector in  $E_{L(\gamma)}$  implies that there exist elements  $\beta \in \Gamma$  such that

$$t(\beta) \neq 0, \quad |t(\beta)^\perp| \leq |t(\beta)^E| \quad \text{and} \quad m(L(\beta)) \leq \frac{1}{8}(m(L(\gamma)) - m^\perp(L(\gamma))). \quad (1.1)$$

Again among these  $\beta$ 's let us take the one for which  $|t(\beta)|$  is a minimum. Now let us consider the commutator  $\tilde{\beta} = [\gamma, \beta]$ . One can check that  $\tilde{\beta}$  satisfies

$$m(L(\tilde{\beta})) \leq 2m(L(\gamma))m(L(\beta)) \leq m(L(\beta))$$

and

$$t(\tilde{\beta}) = (L(\gamma) - \text{Id})t(\beta)^E + (L(\gamma) - \text{Id})t(\beta)^\perp + r$$

with

$$r = (\text{Id} - L(\tilde{\beta}))t(\beta) + L(\gamma)(\text{Id} - L(\beta))L(\gamma)^{-1}t(\gamma).$$

If  $\beta$  is a translation then  $L(\beta) = \text{Id} = L(\tilde{\beta})$  hence  $r = 0$ . Otherwise, from our assumption on  $\gamma$ ,  $|t(\gamma)| \leq |t(\beta)|$  and therefore  $|r| \leq (m(L(\tilde{\beta})) + m(L(\beta))|t(\beta)| \leq 2m(L(\beta))|t(\beta)| \leq 4m(L(\beta))|t(\beta)^E|$ . Hence in either case we have

$$|r| \leq \frac{1}{2}(m(L(\gamma)) - m^\perp(L(\gamma))|t(\beta)^E|$$

but then we obtain

$$|t(\tilde{\beta})^\perp| \leq \frac{1}{2}(m(L(\gamma)) + m^\perp(L(\gamma))|t(\beta)^E| \leq |t(\tilde{\beta})^E|.$$

Hence  $\tilde{\beta}$  also satisfies (1.1) but on the other hand we have  $|t(\tilde{\beta})| \leq m(L(\gamma))|t(\beta)| + |r| < |t(\beta)|$  hence a contradiction.  $\square$

From Claim 1.1.2.1 we can find elements in  $\Gamma$  with  $n$  linearly independent translation parts whose rotation part is smaller than  $\frac{1}{2}$  and from Claim 1.1.2.2 we know that they are translations. Hence there are  $n$  independent translations in  $\Gamma$ . Finally the subgroup of  $\Gamma$  generated by these translation has finite index since  $O(n)$  is compact.  $\square$

There are other two important results due to Bieberbach.

**Theorem 1.1.3.** *The following holds.*

- *Every isomorphism of two Euclidean crystallographic groups is induced by an affine transformation of  $\mathbb{R}^n$ .*
- *There are finitely many isomorphism classes of Euclidean crystallographic groups in each dimension.*

*Remark 1.1.4.* Let us notice that none of these results is true in the general context of affine crystallographic groups. See Remark 1.2.48 and Remark 1.2.52 for counterexamples.

## 1.2 Affine manifolds and crystallographic groups

Let us now introduce the language of  $(G, X)$ -structures for convenience.

**Definition 1.2.1.** Let  $G$  be a Lie group acting smoothly, transitively and analytically on a manifold  $X$  and let  $M$  be a manifold of the same dimension as  $X$ . A  $(G, X)$ -structure on  $M$  is a maximal atlas of coordinate charts  $\phi_i : U_i \rightarrow X$  such that the restriction of a coordinate transformation  $\phi_j \circ \phi_i^{-1}$  to a connected component of  $U_j \cap U_i$  is an element of  $G$ .

**Definition 1.2.2.** A differentiable map  $f : M \rightarrow N$  between  $(G, X)$ -manifolds is called a  $(G, X)$ -map if in the local charts it is given by the action of an element of  $G$ .

In the theory of  $(G, X)$ -structures we have the following fundamental theorem.

**Theorem 1.2.3** ([24]). *Let  $M$  be a  $(G, X)$ -manifold, let  $\tilde{M}$  be its universal cover endowed with the  $(G, X)$ -structure such that the covering map becomes a  $(G, X)$ -map, then there exists a  $(G, X)$ -morphism*

$$D : \tilde{M} \rightarrow X$$

*and a group homomorphism*

$$\rho : \pi_1(M) \rightarrow G$$

*such that*

$$D \circ \gamma = \rho(\gamma) \circ D \quad \forall \gamma \in \pi_1(M).$$

*Such a pair  $(D, \rho)$  is uniquely determined up to the action of  $G$ : any other pair has the form  $(g \circ D, g \circ \rho \circ g^{-1})$  for some  $g \in G$ .*

**Definition 1.2.4.** The map  $D : \tilde{M} \rightarrow X$  arising from this theorem is called the *developing map* and the morphism  $\rho : \pi_1(M) \rightarrow G$  is called the *holonomy morphism* of the  $(G, X)$ -structure on  $M$ . Such a pair  $(D, \rho)$  completely determines the  $(G, X)$ -structure on  $M$ .

*Remark 1.2.5.* Notice that the developing map is actually a local diffeomorphism.

**Definition 1.2.6.** In this context an *affine manifold* is a manifold with an  $(\text{Aff}(V), \mathfrak{a}(V))$ -structure. A *flat Riemannian manifold* is just an  $(\text{O}(n) \times \mathbb{R}^n, \mathbb{R}^n)$ -manifold. A *flat Lorentzian manifold* is a  $(\text{O}(n, 1) \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ -manifold and finally a *flat Hermite-Lorentz manifold* is a manifold with a  $(\text{U}(n, 1) \times \mathbb{C}^{n+1}, \mathbb{C}^{n+1})$ -structure.

**Definition 1.2.7.** An affine manifold  $M$  is said to be *complete* if the developing map is a diffeomorphism. In this case the holonomy morphism,  $\rho : \pi_1(M) \rightarrow \text{Aff}(V)$ , becomes injective and the developing map induces a diffeomorphism  $M \cong \rho(\pi_1(M)) \backslash \mathfrak{a}(V)$ .

*Remark 1.2.8.* If  $M$  is an affine manifold then it inherits the flat affine connection from  $\mathfrak{a}(V)$ . Then,  $M$  is complete in the above sense if and only if  $M$  is geodesically complete in the sense of the affine flat connection on  $M$ , see [8].

*Remark 1.2.9.* For flat Riemannian manifolds, compactness implies completeness. This is also true for flat Lorentzian manifolds and was proved by Carrière in [14]. Under the additional assumptions that  $D$  is a covering map onto its image and that  $\pi_1(M)/\ker \rho$  acts properly discontinuously and freely on  $D(\tilde{M})$ , it was generalised also to Hermite-Lorentz manifolds by Tholozan in [46]. The general statement is still a conjecture.

**Conjecture 1.2.10** (Markus). A compact affine manifold is complete if and only if its linear holonomy has values in  $\text{SL}(V)$ .

### 1.2.1 Affine manifolds with nilpotent holonomy

**Definition 1.2.11.** Let  $M$  be an affine manifold. We will say that  $M$  has nilpotent holonomy if  $\rho(\pi_1(M))$  is nilpotent.

**Definition 1.2.12.** Recall that an endomorphism  $A$  of a vector space  $V$  is called *unipotent* if all its eigenvalues are 1 or, equivalently, if  $(A - \text{Id})^n = 0$  for some  $n$ . A subgroup  $G \leq \text{GL}(V)$  is called *unipotent* if all its elements are unipotent.

**Theorem 1.2.13** ([27]). *The linear holonomy,  $L \circ \rho$ , of a compact affine manifold with nilpotent holonomy is unipotent if and only if the manifold is complete.*

*Remark 1.2.14.* This proves Markus's conjecture in the case of affine manifolds with nilpotent holonomy.

The main examples of compact complete affine manifolds with nilpotent holonomy are quotients of nilpotent Lie groups with an affine structure by a lattice.

**Definition 1.2.15.** A subgroup  $\Gamma$  of a Lie group  $G$  is called a *lattice* if it is discrete and the quotient  $\Gamma \backslash G$  is compact.

*Remark 1.2.16.* Notice that this definition of lattice is what is sometimes called a *uniform lattice* in the literature.

For lattices in nilpotent Lie groups the theory is illustrated by Malcev's theorems. Let us recall them briefly for convenience.

**Theorem 1.2.17** ([39, Section II, Theorem 2.11.]). *Let  $U_1$  and  $U_2$  be two nilpotent simply connected Lie groups and let  $\Gamma$  be a lattice inside  $U_1$ . Then any homomorphism  $\rho : \Gamma \rightarrow U_2$  can be uniquely extended to a continuous homomorphism  $\tilde{\rho} : U_1 \rightarrow U_2$ .*

**Theorem 1.2.18** ([39, Section II, Theorem 2.12.]). *Let  $U$  be a nilpotent simply connected Lie group and  $\mathfrak{u}$  its Lie algebra. Then  $U$  admits a lattice if and only if  $\mathfrak{u}$  admits a basis with respect to which the structure constants are rational.*

*Remark 1.2.19.* For  $\mathfrak{u}$  to have a basis with respect to which the structure constants are rational is equivalent to  $\mathfrak{u}$  admitting a  $\mathbb{Q}$ -form, i.e. a rational Lie subalgebra  $\mathfrak{u}_{\mathbb{Q}}$  such that  $\mathfrak{u}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathfrak{u}$ . To be more precise about the statement of Theorem 1.2.18 let us remember that for a simply connected nilpotent Lie group the exponential map from its Lie algebra is a diffeomorphism and we will call  $\log$  its inverse. Then from a lattice  $\Gamma$  in  $U$ ,  $\text{span}_{\mathbb{Q}}\{\log \Gamma\}$  provides a  $\mathbb{Q}$ -form of  $\mathfrak{u}$ . On the other hand let  $\mathfrak{u}$  be a Lie algebra with a basis with respect to which the structure constants are rational and let  $\mathfrak{u}_0$  be the  $\mathbb{Q}$ -span of this basis. If  $\mathcal{L}$  is any lattice of  $\mathfrak{u}$  contained in  $\mathfrak{u}_0$  the subgroup generated by  $\exp \mathcal{L}$  is a lattice in  $U$ .

**Theorem 1.2.20** ([39, Section II, Theorem 2.18.]). *A group  $\Gamma$  is isomorphic to a lattice inside a simply connected nilpotent Lie group  $U$  if and only if  $\Gamma$  is finitely generated, torsion-free and nilpotent.*

*Remark 1.2.21.* The group  $U$  of Theorem 1.2.20 is unique and it is called the real Malcev completion of  $\Gamma$ .

**Definition 1.2.22.** Two groups  $\Gamma_1$  and  $\Gamma_2$  are said to be *abstractly commensurable* if there exist two subgroups  $\Delta_1 \leq \Gamma_1$  and  $\Delta_2 \leq \Gamma_2$  of finite index such that  $\Delta_1$  is isomorphic to  $\Delta_2$ . Two subgroups  $\Gamma_1$  and  $\Gamma_2$  of a group  $G$  are said to be *commensurable* if  $\Gamma_1 \cap \Gamma_2$  has finite index in both  $\Gamma_1$  and  $\Gamma_2$ .

**Theorem 1.2.23** ([19],[31]). *Let  $\Gamma_1$  and  $\Gamma_2$  be lattices in a nilpotent Lie group  $G$ . Then  $\Gamma_1$  and  $\Gamma_2$  induce isomorphic rational structures on  $\mathfrak{g}$  if and only if they are abstractly commensurable.*

## 1.2.2 Crystallographic groups

Let us now introduce the general definition of a crystallographic group.

**Definition 1.2.24.** The action of a group  $\Gamma$  on a locally compact topological space  $X$  is said to be *properly discontinuous* if and only if for every compact subset  $K \subseteq X$  the set  $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$  is finite. The action is *cocompact* if the quotient  $\Gamma \backslash X$  is compact.

**Definition 1.2.25.** A subgroup  $\Gamma \leq \text{Aff}(V)$  is called a *crystallographic group* if it acts properly discontinuously and cocompactly on  $\mathfrak{a}(V)$ .

*Remark 1.2.26.* A subgroup  $\Gamma$  of  $\text{Aff}(V)$  which acts properly discontinuously on  $\mathfrak{a}(V)$  is discrete in  $\text{Aff}(V)$ . The converse is not true, for a counterexample consider the action on  $\mathbb{R}^2$  of the group generated by  $g : (x, y) \rightarrow (\lambda x, \lambda^{-1}y)$ , with  $0 < \lambda < 1$ . This group is discrete in  $\text{Aff}(\mathbb{R}^2)$  but it does not act properly discontinuously. However, in the case of Euclidean crystallographic groups the action is properly discontinuous if and only if the group is discrete. The reason is that the whole group  $\mathcal{E}(n)$  acts properly on  $\mathfrak{a}(V)$ , i.e. the set  $\{g \in \mathcal{E}(n) \mid gK \cap K \neq \emptyset\}$  is compact for every compact subset  $K$  of  $\mathfrak{a}(V)$ .

*Remark 1.2.27.* Notice that if  $\Gamma$  acts properly discontinuously then its linear part,  $L(\Gamma)$ , need not be discrete. For counterexamples see [26].

**Definition 1.2.28.** We will say that a group has *virtually* a property  $\mathcal{P}$  if it contains a finite index subgroup which has the property  $\mathcal{P}$ .

*Remark 1.2.29.* One can see that crystallographic groups are finitely generated. From a theorem of Selberg every finitely generated linear group contains a torsion-free subgroup of finite index. Hence every crystallographic group is virtually torsion-free. Furthermore since the action is properly discontinuous then the stabilizer of a point is finite. Therefore, the action of a crystallographic group is virtually free.

**Lemma 1.2.30.** *If  $\gamma \in \text{Aff}(V)$  has no fixed point then 1 is an eigenvalue of  $L(\gamma)$ .*

*Proof.* Write  $\gamma(x) = Ax + t$ . Then  $\gamma$  has a fixed point if and only if there exists  $x \in \mathfrak{a}(V)$  such that  $Ax + t = x$ . The last condition holds if and only if  $t \in \text{Im}(A - \text{Id})$ . Hence if  $\gamma$  has no fixed point then  $t \notin \text{Im}(A - \text{Id})$  hence 1 is an eigenvalue of  $A$ .  $\square$

Let us see the connection between crystallographic groups and affine manifolds. On the one hand if  $\Gamma$  is a crystallographic group then, up to finite index, we can suppose that  $\Gamma$  is torsion-free and hence that  $\Gamma \backslash \mathfrak{a}(V)$  is a compact complete affine manifold. On the other hand if  $M$  is a compact complete affine manifold then, if we consider its holonomy morphism  $\rho : \pi_1(M) \rightarrow \text{Aff}(V)$ , the group  $\Gamma = \rho(\pi_1(M))$  is crystallographic. We therefore have the following bijection

$$\left\{ \begin{array}{c} \text{Compact complete} \\ \text{affine manifolds} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Torsion-free} \\ \text{crystallographic groups} \end{array} \right\}.$$

As a corollary to Bieberbach's result we have the following proposition about flat compact Riemannian manifolds.

**Corollary 1.2.31.** *Every flat compact Riemannian manifold is finitely covered by a flat torus  $\mathbb{T}^n$ .*

After Bieberbach's results an effort was made to study more generally crystallographic groups. Auslander conjectured the following.

**Conjecture 1.2.32 (Auslander).** Every crystallographic group is virtually solvable.

Sometimes Auslander's conjecture is equivalently formulated using the term polycyclic instead of solvable. Let us recall the definition.

**Definition 1.2.33.** A group  $\Gamma$  is polycyclic if it contains a sequence of normal subgroups

$$\Gamma = \Gamma_0 \triangleright \Gamma_1 \triangleright \dots \triangleright \Gamma_t = \{\text{Id}\}$$

such that  $\Gamma_i/\Gamma_{i+1}$  is cyclic for  $i = 0, \dots, t-1$ .

*Remark 1.2.34.* A polycyclic group is solvable. The converse is not true in general. But every discrete solvable subgroup of  $\text{GL}(\mathbb{R}^n)$  is polycyclic, [39, Proposition 3.10]. Hence we will use the word polycyclic or solvable interchangeably.

In some special cases the Auslander conjecture is known to be true. One such case is the Riemannian case as we have seen in Bieberbach's result, see Theorem 1.1.2. Another such case is the Lorentzian case, which was proved in [29] by Goldman and Kamishima, see also [15]. The conjecture is also known to be true when the linear part of  $\Gamma$  preserves a quadratic form of signature  $(n-2, 2)$ , see [2]. Furthermore it has been proved in dimension 3 in [26] and up to dimension 6 in [3]. Finally, Grunewald and Margulis [31] proved it for all groups whose linear part is contained in a group  $G$ , where  $G \leq \text{GL}(V)$  is a closed subgroup which is reductive of real rank less or equal than 1 and which has finitely many connected components. We will give a sketch of the proof of this fact in the particular case we are interested in, see [31, Theorem 1.3.] for all the details.

**Theorem 1.2.35** ([31]). *Every Hermite-Lorentz crystallographic group is virtually solvable.*

*Proof.* Let  $(V, h)$  be a vector space endowed with a Hermite-Lorentz metric on which  $\Gamma$ , a Hermite-Lorentz crystallographic group, acts. Since the result is known to be true in full generality when the dimension is  $\leq 6$  we will proceed by induction on the dimension of  $V$ .

Assume first that  $T_\Gamma$ , the subgroup of translations in  $\Gamma$ , is not trivial. Then we can consider the subspace  $V_0 = \left\{ v \in V \mid \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in T_\Gamma \right\}$ . Consider the natural morphism

$$\rho_{V_0} : L^{-1}(\text{Stab}_{\text{GL}(V)}(V_0)) \rightarrow \text{Aff}(V/V_0).$$

It can be proved that  $\rho_{V_0}(\Gamma)$  acts properly discontinuously and cocompactly on  $\mathfrak{a}(V/V_0)$ . Since of course  $\dim(V/V_0) < \dim V$  we can apply the inductive hypothesis after having studied the subgroup  $\text{Stab}_{\text{GL}(V)}(V_0) \leq \text{GL}(V/V_0)$ . If the restriction of  $h$  to  $V_0$  is non-degenerate then  $\text{Stab}_{\text{GL}(V)}(V_0)$  is contained in the unitary group of the metric induced on  $V/V_0$  that can be either Hermite-Lorentz or Hermitian. In the first case we can apply the inductive hypothesis and in the second Bieberbach's result imply that  $\rho_{V_0}(\Gamma)$  is virtually abelian, hence virtually solvable. If instead the restriction of  $h$  to  $V_0$  is degenerate, then  $L(\Gamma)$  fixes  $\ker(h|_{V_0})$ , that is a point in the boundary at infinity of  $\mathbb{H}_{\mathbb{C}}^n$ , the complex hyperbolic space. From [28] the stabiliser of a point at infinity is isomorphic to  $\mathbb{R}^* \text{U}(n-1) \times H^{2n-1}$ , with  $H^{2n-1}$  the Heisenberg group, and this group is amenable. Hence  $L(\Gamma)$  is contained in an amenable

group and  $\Gamma \cong L(\Gamma) \times T_\Gamma$  is contained in a group extension of an amenable group by a group that is amenable as well. Being amenability closed under group extension  $\Gamma$  is contained in an amenable group. Then a discrete subgroup of an amenable group is virtually polycyclic, see [37, Lemma 2.2.], and the result follows.

Assume now that  $T_\Gamma$  is trivial. Let us distinguish two cases. Assume first that  $L(\Gamma)$  is discrete in  $U(n, 1)$ . We will apply a cohomological argument to find a contradiction. Up to replacing  $\Gamma$  by a finite index subgroup we can assume that  $\Gamma$  is torsion-free and that  $\Gamma \backslash \mathfrak{a}(V)$  is a compact manifold of real dimension  $2n + 2$ . On the other hand since  $L(\Gamma) \leq U(n, 1)$  is discrete and torsion-free then  $L(\Gamma)$  acts on the homogeneous space associated to  $U(n, 1)$ ,  $\mathbb{H}_\mathbb{C}^n$ , freely and properly discontinuously. Furthermore since  $T_\Gamma$  is trivial we have  $\Gamma \cong L(\Gamma)$ . But then  $\Gamma$  is also the fundamental group of  $\Gamma \backslash \mathbb{H}_\mathbb{C}^n$ . This implies that the cohomological dimension of  $\Gamma$  would be at most  $2n$  which is a contradiction. Assume now that  $L(\Gamma)$  is not discrete in  $U(n, 1)$ . From [15, Theorem 1.2.1] we know that  $L(\Gamma)_{nd} = L(\Gamma) \cap \overline{L(\Gamma)}^0$  is unipotent. After noticing that the stabiliser of a point at infinity of  $\mathbb{H}_\mathbb{C}^n$  is amenable we can adapt [15, Lemma 3.2.2] to our situation and conclude that the normaliser of a unipotent group inside  $U(n, 1)$  is amenable. Since  $L(\Gamma)$  normalises  $L(\Gamma)_{nd}$  we can deduce that  $L(\Gamma)$  is contained in an amenable group. Hence  $\Gamma$  is contained in an amenable group. Since a discrete subgroup of an amenable group is virtually solvable,  $\Gamma$  would be virtually solvable.  $\square$

**Example 1.2.36.** There are examples of solvable, non-nilpotent, crystallographic groups. Indeed let us consider  $\{e_1, \dots, e_n\}$ , the standard basis for  $\mathbb{R}^n$ , then let  $\Gamma$  be the group generated by the  $n - 1$  translations  $e_1, \dots, e_{n-1}$  and the map

$$x \mapsto Ax + e_n \text{ with } A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B \in \text{GL}(\mathbb{Z}^{n-1}).$$

Then  $\Gamma$  is a crystallographic group isomorphic to  $\mathbb{Z} \times_B \mathbb{Z}^{n-1}$  and it is nilpotent if and only if  $B$  is unipotent.

The reason why virtual solvability is nice is the following generalisation of Bieberbach's result given by Fried and Goldman.

**Theorem 1.2.37** ([26, Corollary 1.5.]). *Let  $V$  be a finite-dimensional real vector space and  $\Gamma \leq \text{Aff}(V)$  be a virtually solvable group acting properly discontinuously and cocompactly on  $\mathfrak{a}(V)$ . Then there exists a closed solvable subgroup  $H \leq \text{Aff}(V)$  such that*

1.  $H$  acts simply transitively on  $\mathfrak{a}(V)$ ;
2.  $\Gamma \cap H$  has finite index in  $\Gamma$ ;
3.  $\Gamma \cap H$  is a lattice in  $H$ .

*Proof.* Let us give an idea of the proof in the simpler case of nilpotent crystallographic groups and let the reader see the proof in [26] for details. Since  $\Gamma$  is finitely generated we can find a finite index

subgroup that is torsion-free by a theorem of Selberg. Hence, up to replacing  $\Gamma$  with a finite index subgroup, we can assume that  $\Gamma$  is a finitely generated, torsion-free and nilpotent group. Then from Theorem 1.2.20 there exists a simply connected nilpotent Lie group  $G$  such that  $\Gamma$  is a lattice in  $G$ . From Theorem 1.2.13 since  $\Gamma$  is nilpotent, its linear action is unipotent, i.e.  $\Gamma \hookrightarrow U \subseteq \text{Aff}(V)$ , where  $U$  is a unipotent subgroup. Then from Theorem 1.2.17 we can extend the inclusion  $\Gamma \hookrightarrow \text{Aff}(V)$  to a morphism  $G \rightarrow \text{Aff}(V)$ . Let  $H$  be the image of  $G$  under this morphism. Since for all  $h \in H$  the linear part of  $\log h$  is a nilpotent transformation we have that the exponential map  $\exp : \mathfrak{h} \rightarrow H$  is a diffeomorphism hence  $H$  is contractible. Let us define the map  $\tilde{f} : H \rightarrow V$  by  $\tilde{f}(h) = h(0)$  and let  $f$  be the induced map  $f : \Gamma \backslash H \rightarrow \Gamma \backslash V$ . Let us notice that both  $\Gamma \backslash H$  and  $\Gamma \backslash V$  are spaces of type  $K(\Gamma, 1)$ , i.e. spaces whose universal cover is contractible and whose fundamental group is isomorphic to  $\Gamma$ . Let us show that  $f$  is an homotopy equivalence. From [32, Proposition 1.B.9] we know that all we need to prove is that the map induced by  $f$  on fundamental groups,  $f_*$ , is an isomorphism. Now let us fix as basepoint,  $x_0$ , in  $\Gamma \backslash H$  the image of  $1 \in H$  by the covering projection. Elements of  $\pi_1(\Gamma \backslash H, x_0)$  are in one to one correspondence with homotopy classes of paths  $\tilde{\gamma} : [0, 1] \rightarrow H$  such that  $\tilde{\gamma}(0) = 1$  and  $\tilde{\gamma}(1) = g \in \Gamma$ . If we fix  $g \in \Gamma$  and let  $\tilde{\gamma}_g$  be such a path, since  $\tilde{f}$  is continuous,  $\tilde{f} \circ \tilde{\gamma}_g$  is a path in  $V$  such that  $\tilde{f}(\tilde{\gamma}_g(0)) = 0$  and  $\tilde{f}(\tilde{\gamma}_g(1)) = g(0)$ . But this homotopy class of path is precisely the one that corresponds to  $g$  as element of the fundamental group of  $\Gamma \backslash V$ . Since  $f_*$  sends, via the correspondence,  $\tilde{\gamma}$  to  $\tilde{f} \circ \tilde{\gamma}$ , we see that  $f_*$  is an isomorphism. Since  $\Gamma \backslash H$  and  $\Gamma \backslash V$  are compact manifolds, they have the same dimension and the map  $f$  is surjective. This implies that  $\tilde{f}$  is surjective as well and, since  $\dim(H) = \dim(V)$ ,  $\tilde{f}$  is a covering map. Since the group  $H$  is simply connected, the map  $\tilde{f}$  is a homeomorphism and hence  $H$  acts simply transitively on  $\mathfrak{a}(V)$ .  $\square$

**Definition 1.2.38.** A *solvmanifold (nilmanifold)* is a manifold of the form  $G/\Gamma$  with  $G$  a solvable (nilpotent) Lie group and  $\Gamma$  a lattice in  $G$ .

**Corollary 1.2.39.** *Every compact complete affine manifold with virtually solvable (nilpotent) fundamental group is finitely covered by a solvmanifold (nilmanifold).*

In light of Theorem 1.2.37 a strategy of classification of virtually solvable crystallographic groups is to first classify all the possible simply transitive groups  $H$  and then the possible lattices inside the  $H$ . Let us then state the following fact about simply transitive actions.

**Proposition 1.2.40** ([7, Theorem I.1]). *A Lie group acting simply transitively on an affine space is solvable.*

On the other hand let us cite the following.

**Proposition 1.2.41** ([11, Théorème 1.]). *There exists a nilpotent compact manifold that has no complete affine structure.*

*Remark 1.2.42.* The above proposition implies that there are examples of simply connected nilpotent Lie groups which do not admit any simply transitive affine action.

In the more general effort of understanding groups acting properly discontinuously without the cocompact condition Milnor proved the following theorem.

**Theorem 1.2.43** ([37]). *If a group is torsion-free and is virtually polycyclic, then it is isomorphic to the fundamental group of some (possibly non-compact) complete affine manifold.*

Therefore he asked whether the converse holds, formulating a generalisation of Auslander's conjecture to complete manifolds. But this last conjecture was proven to be false by Margulis. Indeed Margulis proved that there exists free non-abelian subgroup of  $\text{Isom}(\mathbb{R}^{2+1}) \cong \text{O}(2, 1) \ltimes \mathbb{R}^{2+1}$  which acts properly discontinuously on  $\mathfrak{a}(\mathbb{R}^{2+1})$ . To put this into perspective let us recall the following.

**Theorem 1.2.44** (Tits alternative). *Let  $\Gamma \leq \text{GL}(\mathbb{C}^n)$ . Then either  $\Gamma$  is virtually solvable or it contains a free non-abelian group.*

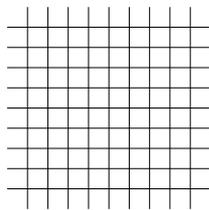
In order to have some examples in mind let us see what happens for crystallographic groups in dimension 2. This was studied by Kuiper.

**Theorem 1.2.45** ([36]). *Let  $\Gamma \in \text{Aff}(\mathbb{R}^2)$  be a crystallographic group. Then  $\Gamma$  is virtually isomorphic to  $\mathbb{Z}^2$ . More precisely  $\Gamma$  contains a finite index subgroup which, up to conjugation, is a lattice in either the group of translations  $T$  or the group*

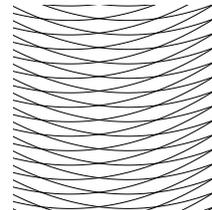
$$U = \left\{ \left( \begin{array}{ccc} 1 & s & \frac{s^2}{2} + t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{array} \right) \mid s, t \in \mathbb{R} \right\}.$$

**Corollary 1.2.46.** *Up to diffeomorphism, the only complete affine structures on the torus  $\mathbb{T}^2$ , other than the standard Riemannian one  $\mathbb{Z}^2 \backslash \mathbb{R}^2$ , are  $\Gamma \backslash U$  where  $\Gamma$  is a lattice in the unipotent group described above.*

**Example 1.2.47.** Here are two examples of 2-dimensional crystallographic groups. The first one,  $\Gamma_1 = \{(x, y) \mapsto (x + p, y + q) \mid (p, q) \in \mathbb{Z}^2\}$ , is the standard lattice in the group of translations  $T \cong \mathbb{R}^2$ . The second,  $\Gamma_2 = \{(x, y) \mapsto (x + qy + \frac{q^2}{2} + p, y + q) \mid (p, q) \in \mathbb{Z}^2\}$ , is a lattice in  $U$ .



(a) Tessellation associated with  $\Gamma_1$ .



(b) Tessellation associated with  $\Gamma_2$ .

**Remark 1.2.48.** Consider the two crystallographic groups of Example 1.2.47. They are isomorphic since both are isomorphic to  $\mathbb{Z}^2$ . However these two groups are not affinely conjugate. This shows

that the first part of Theorem 1.1.3 does not generalise to affine crystallographic groups. Let us say en passant that there exists a polynomial automorphism of  $\mathbb{R}^2$  that conjugates  $\Gamma_1$  and  $\Gamma_2$ . This is a general phenomenon for virtually solvable crystallographic groups, see [26, Theorem 1.20].

Let us also say some words about crystallographic groups in dimension 3. They were studied by Fried and Goldman.

**Theorem 1.2.49** ([26]). *Every crystallographic subgroup of  $\text{Aff}(\mathbb{R}^3)$  is virtually isomorphic to  $\mathbb{Z} \rtimes_A \mathbb{Z}^2$  with  $A \in \text{SL}(\mathbb{Z}^2)$  with positive eigenvalues.*

*Remark 1.2.50.* A way to embed  $\mathbb{Z} \rtimes_A \mathbb{Z}^2$  inside  $\text{Aff}(\mathbb{R}^3)$  is as follows

$$\left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & A^t & v \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{Z}, v \in \mathbb{Z}^2 \right\}.$$

*Remark 1.2.51.* The group  $\mathbb{Z} \rtimes_A \mathbb{Z}^2$  of Theorem 1.2.49 is a cocompact lattice in  $G = \mathbb{R} \rtimes_A \mathbb{R}^2$ . Furthermore if  $A$  is hyperbolic, i.e.  $\text{tr}(A) > 2$  then  $G$  is solvable and not nilpotent; if  $A = \text{Id}$  then  $G$  is abelian; otherwise  $G$  is nilpotent isomorphic to the three-dimensional Heisenberg group. The groups  $G = \mathbb{R} \rtimes_A \mathbb{R}^2$  fall into three isomorphism classes. Representatives of these classes are, in terms of their Lie algebra,  $\mathbb{R}^3$ ,  $L_{3,2}$  and  $S$ . Here  $\mathbb{R}^3$  is the three-dimensional abelian Lie algebra,  $L_{3,2}$  is the Heisenberg Lie algebra:  $[x, y] = z$  and  $S$  is the solvable Lie algebra  $[x, z] = x$ ,  $[y, z] = -y$ , both given in terms of a basis  $(x, y, z)$ .

*Remark 1.2.52.* Theorem 1.2.49 shows that already in dimension 3 there are infinitely many isomorphism classes of affine crystallographic groups. Indeed  $\mathbb{Z} \rtimes_A \mathbb{Z}^2$  and  $\mathbb{Z} \rtimes_B \mathbb{Z}^2$  are isomorphic if and only if  $A$  is  $\text{GL}(2, \mathbb{Z})$ -conjugate to either  $B$  or  $B^{-1}$ . Hence this illustrates that also the second part of Theorem 1.1.3 is not true without the Euclidean assumption. Notice that in dimension 2 we still have finitely many isomorphism classes. This holds for the same reason as for Euclidean crystallographic groups. The point is that in both cases every crystallographic group  $\Gamma$  is a finite extension of  $\mathbb{Z}^n$  without non-trivial finite normal subgroups. There are finitely many of those. Indeed, these groups  $\Gamma$  sit in an exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow \theta \longrightarrow 1$$

with  $\theta$  a finite group. Isomorphism classes of groups sitting in such exact sequences are classified by  $H^2(\theta, \mathbb{Z}^n)$ , the second cohomology group of  $\theta$  with coefficients in  $\mathbb{Z}^n$ , that is finite since  $\theta$  is finite. Finally one should show that there are finitely many isomorphism classes of modules over finite groups which are free of rank  $n$  as  $\mathbb{Z}$ -modules, see [16, Theorem 7.1.1].

### 1.3 Flat compact Lorentzian manifolds

Let  $V$  be a real vector space of dimension  $n + 1$  and endow it with a symmetric, non-degenerate, bilinear form  $\langle \cdot, \cdot \rangle$  of signature  $(n, 1)$ , called a Lorentzian bilinear form. Let  $q$  be the quadratic form associated to  $\langle \cdot, \cdot \rangle$ . Let furthermore  $O(n, 1)$  be the group of linear transformations of  $V$  that preserve  $\langle \cdot, \cdot \rangle$  and  $\mathcal{E}(n, 1)$  the group of affine transformations of  $\mathfrak{a}(V)$  whose linear part is in  $O(n, 1)$ . We will explicitly recall the results on Lorentzian crystallographic groups obtained by Fried in dimension 4 and summarize those in higher dimensions. The techniques we will use in Chapter 2 are mostly inspired by those used to prove the following results. Hence, since we will write the details later, we will not say much about the proofs in this section. Let us start with the dimension 4 where we see already all the interesting features.

**Theorem 1.3.1** ([25]). *If  $H$  is a subgroup of  $\mathcal{E}(3, 1)$  acting simply transitively on  $\mathfrak{a}(\mathbb{R}^{3+1})$  then, up to conjugacy in  $\mathcal{E}(3, 1)$ ,  $H$  is one of  $T, U_\rho, \mathcal{A}$  or  $\mathcal{B}$  with  $\rho = \frac{\beta^2}{\varepsilon} \in [0, \infty]$  and  $\beta, \varepsilon \geq 0$ ,  $(\beta, \varepsilon) \neq (0, 0)$ . A typical element of each group is as follows, where  $(w, x, y, z) \in \mathbb{R}^4$  and  $q(w, x, y, z) = 2wz + x^2 + y^2$*

$T$  : translations,

$$U_\rho : \begin{pmatrix} 1 & -\beta y & -\varepsilon z & -\frac{1}{2}(\beta^2 y^2 + \varepsilon^2 z^2) & w - \frac{1}{2}(\beta xy + \varepsilon xz) - \frac{1}{2}(\beta^2 y^2 z + \varepsilon^2 z^3) \\ 0 & 1 & 0 & \beta y & x + \frac{1}{2}\beta yz \\ 0 & 0 & 1 & \varepsilon z & y + \frac{1}{2}\varepsilon z^2 \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{A} : \begin{pmatrix} e^y & 0 & 0 & 0 & w \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & y \\ 0 & 0 & 0 & e^{-y} & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathcal{B} : \begin{pmatrix} 1 & 0 & 0 & 0 & w \\ 0 & \cos z & \sin z & 0 & x \\ 0 & -\sin z & \cos z & 0 & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

*Remark 1.3.2.* The groups  $U_\rho$  are nilpotent Lie groups and, up to isomorphism, they fall into three classes represented by the following Lie algebras:  $\mathbb{R}^4, L_{3,2} \oplus \mathbb{R}$  and  $L_{4,3}$ . Here  $\mathbb{R}^4$  is the abelian Lie algebra of dimension 4,  $L_{3,2}$  is the Heisenberg Lie algebra and  $L_{4,3}$  is defined by the following Lie brackets in the basis  $(\tau, e_1, e_2, \xi)$ :  $[e_2, \xi] = e_1, [e_1, e_2] = \tau$ . Notice that these are all the 4-dimensional nilpotent Lie algebras up to isomorphism.

**Theorem 1.3.3** ([25]). *Let  $M$  be a flat compact Lorentzian manifold of dimension 4 with fundamental group  $\Gamma$ . Then there is a uniquely determined simply transitive subgroup  $H$  of  $\mathcal{E}(3,1)$  such that  $\Gamma \cap H = \pi$  has finite index in  $\Gamma$ . The group  $\pi$  is a semidirect product  $\mathbb{Z} \rtimes_A \mathbb{Z}^3$  with  $A \in \text{SL}(\mathbb{Z}^3)$  and eigenvalues  $1, \lambda, \lambda^{-1}$  with  $\lambda \in \mathbb{R}$ . More precisely  $H$  is conjugate to*

- $\mathcal{A}$  if  $\lambda \neq 1$ ,
- $U_\rho$  with  $\rho > 0$  if  $A$  is unipotent of index 3,
- $U_0$  if  $A$  is unipotent of index 2,
- $T$  if  $A = \text{Id}$ .

**Corollary 1.3.4.** *Every compact flat Lorentzian manifold is finitely covered by a  $\mathbb{T}^3$ -bundle over the circle.*

In general we have the following results.

**Definition 1.3.5.** A group  $\Gamma$  is said to be *abelian-by-cyclic* if it has a normal abelian subgroup  $\Gamma_1$  such that  $\Gamma/\Gamma_1$  is cyclic.

**Theorem 1.3.6** ([31]). *Let  $\Gamma$  be a Lorentzian crystallographic group. Then, either  $\Gamma$  is virtually nilpotent, or it is virtually abelian-by-cyclic.*

Each of the possibilities can be further specified as follows.

**Definition 1.3.7.** Let  $L$  be a field,  $k, m \geq 0$  be integers. Let  $E = L^k, F = L^k, G = L^m$  be vector spaces and let  $e_i, f_i, g_i$  be the respective bases. Let  $\hat{\cdot} : F \rightarrow E$  be the isomorphism defined by  $f_i \mapsto e_i$ . Put  $W = E \oplus F \oplus G$  and let  $\varphi : W \rightarrow L$  be a linear form. Let furthermore  $S \in \text{Sym}(k, L)$  be a  $k \times k$  non-degenerate symmetric matrix. Define the Lie algebra  $\mathfrak{g}_L(2k + m + 2, k, \varphi, S)$  as the vector space  $L \oplus W \oplus L$  with Lie brackets

$$[(r, u, s), (r', u', s')] = (\hat{f}'^\top S e - \hat{f}^\top S e' + s' \varphi(u) - s \varphi(u'), s' \hat{f} - s \hat{f}', 0)$$

where  $u = e + f + g \in W$ .

**Definition 1.3.8.** For  $n \geq 1$  and  $0 \leq k \leq \frac{n-1}{2}$  let  $\mathcal{L}_{\mathbb{R}}^i(n+1, k)$  with  $i = 1, 2$  be the Lie group whose Lie algebra is  $\mathfrak{g}_{\mathbb{R}}(n+1, k, \varphi_i, \text{Id}_k)$ , where  $\varphi_1 = 0$  and, if  $k \neq \frac{n-1}{2}$ ,  $\varphi_2 = g_1^*$ . Let  $\Gamma_i(n+1, k, S)$  be the subgroup of  $\mathcal{L}_{\mathbb{R}}^i(n+1, k)$  that is defined by the  $\mathbb{Q}$ -form  $\mathfrak{g}_{\mathbb{Q}}(n+1, k, \varphi_i, S)$  for any rational symmetric matrix  $S$ .

**Proposition 1.3.9** ([31]). *Every unipotent subgroup of  $\mathcal{E}(n, 1)$  acting simply transitively on  $\mathfrak{a}(\mathbb{R}^{n+1})$  is isomorphic to exactly one of  $\mathcal{L}_{\mathbb{R}}^i(n+1, k)$  with  $0 \leq k \leq \frac{n-1}{2}$ .*

*Remark 1.3.10.* More explicitly, in dimension 5, the flat Lorentzian nilpotent Lie algebras are, up to isomorphism, the following defined in the basis  $(\tau, e_1, f_1, e_3, \xi)$ :

- $\mathbb{R}^5$
- $L_{3,2} \oplus \mathbb{R}^2: [\xi, e_1] = -\tau,$
- $L_{4,3} \oplus \mathbb{R}: [\xi, f_1] = -e_1, [e_1, f_1] = \tau,$
- $L_{5,5}: [\xi, e_3] = -\tau, [\xi, f_1] = -e_1$  and  $[e_1, f_1] = \tau.$

Starting from dimension 5 there are more isomorphism classes of nilpotent Lie algebras than the ones appearing here. In dimension 6, the list of isomorphism classes of flat Lorentzian Lie algebras is:  $\mathbb{R}^6, L_{3,2} \oplus \mathbb{R}^3, L_{4,3} \oplus \mathbb{R}^2, L_{5,5} \oplus \mathbb{R}$  and  $L_{6,19}(1)$  with  $L_{6,19}(1)$  defined in the basis  $(\tau, e_1, f_1, e_2, f_2, \xi)$  by the following non-zero Lie brackets  $[\xi, f_1] = -e_1, [\xi, f_2] = -e_2, [e_1, f_1] = \tau$  and  $[e_2, f_2] = \tau$ . The nomenclature we have used here is the same as in [20].

**Proposition 1.3.11** ([31]). *Let  $\Gamma \leq \mathcal{E}(n, 1)$  be a virtually nilpotent crystallographic group. Then*

- *the group  $\Gamma$  is abstractly commensurable to one of the groups  $\Gamma_i(n+1, k, S)$ ,*
- *every group  $\Gamma_i(n+1, k, S)$  can be realised as a Lorentzian crystallographic group,*
- *the groups  $\Gamma_i(n+1, k, S)$  and  $\Gamma_j(n'+1, k', S')$  are abstractly commensurable if and only if  $i = j, n = n', k = k'$  and there exists  $\alpha \in \mathbb{Q}^*$  such that  $S$  and  $\alpha S'$  are equivalent over  $\mathbb{Q}$ .*

*Remark 1.3.12.* This shows that a simply transitive affine group can contain infinitely many lattices that are pairwise not abstractly commensurable.

*Remark 1.3.13.* Here are more details about how the groups  $\Gamma_i(n+1, k, S)$  act, more can be found in [31, Theorem 1.14]. Since every rational symmetric matrix  $S$  is equivalent to a diagonal matrix with diagonal coefficients  $m = (m_1, \dots, m_k) \in \mathbb{Z}^k$ , every group  $\Gamma_i(n+1, k, S)$  is abstractly commensurable to one of the following, given with the action on affine space. In order to define them let

$$L(\sqrt{m}) = \{(x_1\sqrt{m_1}, \dots, x_k\sqrt{m_k}) \mid x_1, \dots, x_k \in \mathbb{Z}\},$$

then

$$\Gamma_1(n+1, k, m) = \left\{ \begin{pmatrix} 1 & 0 & 0 & -x^\top & -\frac{1}{2}x^\top x & r \\ 0 & \text{Id}_{n-1-2k} & 0 & 0 & 0 & z \\ 0 & 0 & \text{Id}_k & 0 & 0 & x \\ 0 & 0 & 0 & \text{Id}_k & x & y \\ 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid r, s \in \mathbb{Z}, x, y \in L(\sqrt{m}), z \in \mathbb{Z}^{n-1-2k} \right\}$$

and, if  $k < \frac{n-1}{2}$ , let  $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^{n-1-2k}$ , define

$$\Gamma_2(n+1, k, m) = \left\{ \begin{pmatrix} 1 & -2se_1^\top & 0 & -x^\top & -\frac{1}{2}(4s^2 + x^\top x) & r \\ 0 & \text{Id}_{n-1-2k} & 0 & 0 & 2se_1 & z \\ 0 & 0 & \text{Id}_k & 0 & 0 & x \\ 0 & 0 & 0 & \text{Id}_k & x & y \\ 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{l} r, s \in \mathbb{Z}, x, y \in L(\sqrt{m}), \\ z \in \mathbb{Z}^{n-1-2k} \end{array} \right\}.$$

**Definition 1.3.14.** Let  $n \geq 1$  and let  $A \in \text{GL}(\mathbb{Z}^n)$  be diagonalisable with eigenvalues  $1, \lambda, \lambda^{-1}, a_i$  with  $\lambda \in \mathbb{R}_{>0}$  and  $|a_i| = 1$ . Call such a matrix of Lorentz type. Then write  $\Gamma(n+1, A) = \mathbb{Z} \rtimes_A \mathbb{Z}^n$ .

**Proposition 1.3.15** ([31]). *Let  $\Gamma \leq \mathcal{E}(n, 1)$  be a crystallographic group which is not virtually nilpotent. Then*

- the group  $\Gamma$  contains a finite index subgroup  $\Gamma_0$  such that  $\Gamma_0 \cong \Gamma(n+1, A)$  with  $A \in \text{GL}(\mathbb{Z}^n)$  of Lorentz type,
- every group  $\Gamma(n+1, A)$  can be realised as a Lorentzian crystallographic group,
- two such groups  $\Gamma(n+1, A)$  and  $\Gamma(n+1, A')$  are abstractly commensurable if and only if  $A^r$  is  $\text{GL}(\mathbb{Q}^n)$ -conjugate to  $A'^s$  for some  $r, s \in \mathbb{Z} \setminus \{0\}$ .

*Remark 1.3.16.* This last case is essentially the same for Hermite-Lorentz crystallographic groups, see Proposition 2.2.11.

*Remark 1.3.17.* The dichotomy that appears in Theorem 1.3.6 and that we will find again in the Hermite-Lorentz context in Theorem 2.2.7 does not always hold, even in the context of flat pseudo-Riemannian manifolds. See the following example.

**Example 1.3.18.** Here is an example of a flat compact pseudo-Riemannian manifold  $N$  such that its fundamental group is not abelian-by-cyclic. Indeed we construct  $N$  such that its fundamental group is nilpotent-by-cyclic, i.e.  $\pi_1(N) \cong \mathbb{Z} \rtimes \Gamma$  where  $\Gamma$  is a nilpotent group. This example is constructed as follows. Let  $\mathfrak{g}$  be a flat pseudo-Riemannian Lie algebra with a rational form  $\mathfrak{g}_{\mathbb{Q}}$ . Let furthermore  $\varphi$  be an automorphism of  $\mathfrak{g}$  that preserves the metric and such that in a rational basis for  $\mathfrak{g}_{\mathbb{Q}}$  is represented by an integer matrix. Denote by  $G$  the connected simply connected Lie group whose Lie algebra is  $\mathfrak{g}$  and by  $\Gamma$  the lattice in  $G$  associated to the rational form  $\mathfrak{g}_{\mathbb{Q}}$ . We can then consider the compact flat pseudo-Riemannian manifold  $M = \Gamma \backslash G$  and the automorphism of  $M$  induced by  $\varphi$ . The manifold  $N$  obtained as the suspension of  $M$  by  $\varphi$  will be a flat compact pseudo-Riemannian manifold with fundamental group  $\mathbb{Z} \rtimes_A \Gamma$  where  $A$  is the matrix representing  $\varphi$  in a rational basis. Explicitly let  $\mathfrak{g}$  be the free 2-step nilpotent Lie algebra on three generators. In a basis  $\beta = (x_1, \dots, x_6)$  the Lie brackets are as follows

$$[x_1, x_2] = x_4, \quad [x_1, x_3] = x_5 \quad \text{and} \quad [x_2, x_3] = x_6.$$

The following is a family of flat pseudo-Riemannian metrics of signature  $(3,3)$  defined on  $\mathfrak{g}$  with respect to the basis  $\beta$

$$q(r_1, \dots, r_6) = r_1 r_6 + \mu r_2 r_5 + (\mu - 1) r_3 r_4$$

with  $\mu \in \mathbb{R} \setminus \{0, 1\}$ , see [34]. Let  $\lambda$  and  $\lambda^{-1}$  be roots of  $x^2 - 2ax + 1 \in \mathbb{Z}[x]$  with  $a \geq 2$ , more explicitly  $\lambda = a + \sqrt{a^2 - 1}$  and  $\lambda^{-1} = a - \sqrt{a^2 - 1}$ . Then the following matrix, written with respect to the basis  $\beta$ ,

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}$$

represents an automorphism,  $\varphi$ , of  $\mathfrak{g}$  that preserves the metric  $q$ . Furthermore

$$\beta' = \left( x_1 + x_2, \sqrt{a^2 - 1}(x_1 - x_2), \frac{1}{\sqrt{a^2 - 1}}x_3, \sqrt{a^2 - 1}x_4, \frac{1}{\sqrt{a^2 - 1}}(x_5 + x_6), x_5 - x_6 \right)$$

is a basis for  $\mathfrak{g}$  with respect to which  $\varphi$  has the following integer expression

$$\begin{pmatrix} a & a^2 - 1 & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & a^2 - 1 \\ 0 & 0 & 0 & 0 & 1 & a \end{pmatrix}$$

and  $\mathfrak{g}$  has rational coefficients. Hence  $\varphi$  is the automorphism of  $\mathfrak{g}$  we were looking for.



## Chapter 2

# Flat compact Hermite-Lorentz manifolds

Let  $V$  be a complex vector space of dimension  $n + 1$  endowed with a Hermitian form  $h$  of signature  $(n, 1)$ ; such a form will be called a Hermite-Lorentz form. Denote by  $\mathfrak{a}(V)$  the affine space associated to  $V$ . Let furthermore  $U(n, 1)$  be the group of linear transformations of  $V$  that preserve  $h$  and  $\mathcal{H}(n, 1)$  the group of affine transformations of  $\mathfrak{a}(V)$  whose linear part is in  $U(n, 1)$ . We will denote by  $\mathfrak{u}(n, 1)$  the Lie algebra of  $U(n, 1)$ . Remember that we denote by  $L : \text{Aff}(V) \rightarrow \text{GL}(V)$  the homomorphism that associates to each affine transformation its linear part.

*Remark 2.0.1.* Let us notice that the pair  $(V, h)$ , where  $V$  is a complex vector space and  $h$  is a Hermite-Lorentz form defined on it, is equivalent to the triple  $(V(\mathbb{R}), J, \langle \cdot, \cdot \rangle)$ . Here  $V(\mathbb{R})$  is the associated real vector space,  $J : V(\mathbb{R}) \rightarrow V(\mathbb{R})$  is a complex structure on  $V(\mathbb{R})$ , i.e. a linear endomorphism such that  $J^2 = -\text{Id}$  and  $\langle \cdot, \cdot \rangle$  is a pseudo-Riemannian metric on  $V(\mathbb{R})$  of signature  $(2n, 2)$  such that, if  $\omega(u, v) := \langle u, Jv \rangle = -\langle Ju, v \rangle$ , we have

$$h(u, v) = \langle u, v \rangle + i\omega(u, v).$$

Let us also fix the following notation: for  $z \in \mathbb{C}$  we denote by  $\Re(z)$  and  $\Im(z)$  its real and imaginary part respectively.

We have seen in Theorem 1.2.35 that every Hermite Lorentz crystallographic group is virtually solvable. Hence in order to study them we can use Theorem 1.2.37. More precisely, in order to study crystallographic subgroups  $\Gamma \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$  we will study lattices in simply transitive affine groups  $H \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$ .

*Remark 2.0.2.* Let us remark that simply transitive subgroups of  $U(n, 1) \ltimes \mathbb{C}^{n+1}$  are real Lie groups which inherit an integrable left invariant complex structure from  $\mathfrak{a}(\mathbb{C}^{n+1})$ . The integrability condition, see for example [41] or Definition 3.1.15, is more general than the condition for a Lie group to be a complex Lie group. Indeed the Lie groups we are looking at are not complex Lie groups. The study of left invariant structures on Lie groups will be deepened in Chapter 3.

In Section 2.1 we give a presentation, valid in any dimension, of unipotent simply transitive Hermite-Lorentz groups and prove a proposition about their classification up to conjugacy. In Section

2.2 we reduce, as for the Lorentzian case, the study of crystallographic groups to the study of lattices in simply transitive unipotent Lie groups. In Section 2.3 we start the classification, up to isomorphism, of those  $H$  found in Section 2.1. We complete their classification for the dimensions 2 and 3. In dimension 4 we give the classification for some particular cases that we have called degenerate cases. In Section 2.4 we finish the classification of the unipotent simply transitive groups  $H$  in dimension 4. For this more general case we are left with a classification problem of 8-dimensional nilpotent Lie algebras defined by three parameters,  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$ , see Definition 2.4.23. Since there are no complete classifications of nilpotent Lie algebras in dimension bigger than 7 we introduce an ad-hoc method to study our particular family of Lie algebras. Using the fact that these Lie algebras are Carnot, see Subsection 2.4.2, we can identify their isomorphism classes with some orbits of the  $\mathrm{SL}(3, \mathbb{R})$ -action on the Grassmannian of 2-dimensional subspaces of  $\mathfrak{sl}(3, \mathbb{R})$  induced by the adjoint action. Hence studying the orbits for this action is the same as studying the isomorphism classes for the family of Lie algebras. Remember that in the Lorentzian case, for every fixed dimension, there are finitely many isomorphism classes of unipotent subgroups of  $\mathrm{O}(n, 1) \ltimes \mathbb{R}^{n+1}$  acting simply transitively on  $\mathfrak{a}(\mathbb{R}^{n+1})$  see Proposition 1.3.9. In contrast with this fact, we find that the Lie algebras  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  constitute an infinite family of pairwise non-isomorphic Lie algebras of Lie groups acting simply transitively on  $\mathfrak{a}(\mathbb{C}^{3+1})$  see Proposition 2.4.36. In Section 2.5 we study the Hermite-Lorentz crystallographic groups in the nilpotent case. Indeed we have reduced it to the study of lattices in simply transitive unipotent groups. Since in Sections 2.3 and 2.4 we have given the complete list of those simply transitive groups we just need to understand which of them admit lattices and what are their abstract commensurability classes. Finally in Section 2.6 we give some topological considerations about the manifolds  $\Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$  that are virtually nilpotent. More precisely we will see that they are finitely covered by torus bundles over tori.

## 2.1 Unipotent Hermite-Lorentz simply transitive groups

In this section we look at unipotent subgroups of  $\mathrm{U}(n, 1) \ltimes \mathbb{C}^{n+1}$  acting simply transitively on  $\mathfrak{a}(\mathbb{C}^{n+1})$ . The reason why we have the unipotent hypothesis, that simplifies our study, is that, as we will see in the following section, the study of crystallographic groups will be reduced to the study of lattices in unipotent simply transitive Lie groups.

**Lemma 2.1.1.** *Every unipotent subgroup of  $\mathrm{U}(n, 1)$  fixes a non-zero null vector for the Hermite-Lorentz form  $h$ .*

*Proof.* Let  $U$  be a unipotent subgroup of  $\mathrm{U}(n, 1)$ . From Engel's theorem we know that  $U$  fixes a vector  $v_0 \in V$ . If  $v_0$  is timelike, i.e.  $h(v_0) < 0$ , then  $U$  fixes  $v_0^\perp$  that is spacelike, i.e.  $h|_{v_0^\perp}$  is positive definite, hence it is contained in  $\mathrm{U}(n)$ , but a unipotent unitary matrix is trivial hence  $U$  would be trivial. If instead  $v_0$  were spacelike then, if  $U$  does not fix a null vector in  $v_0^\perp$ , it would have to fix a spacelike vector  $v_1$  in  $v_0^\perp$ . This implies that  $U$  preserves the Hermite-Lorentz space  $\mathrm{span}\{v_0, v_1\}^\perp$ .

By induction we get that then  $U$  is contained in  $U(1, 1)$  and we can see that a unipotent element in  $U(1, 1)$  is trivial.  $\square$

**Lemma 2.1.2.** *We can choose a basis for  $V$  with respect to which a maximal unipotent subgroup of  $U(n, 1)$  is written as*

$$\mathcal{U} = \left\{ \begin{pmatrix} 1 & -\bar{v}^\top & -\frac{1}{2}\bar{v}^\top v + ib \\ 0 & \text{Id}_{n-1} & v \\ 0 & 0 & 1 \end{pmatrix} \mid v \in \mathbb{C}^{n-1}, b \in \mathbb{R} \right\}.$$

*Proof.* Let  $\mathcal{U}$  be a maximal unipotent subgroup of  $U(n, 1)$ . We might choose the fixed null vector  $v_0$  of  $\mathcal{U}$  from Lemma 2.1.1 to be the first vector of a base of  $V$  and complete it to an orthonormal basis for  $V$  with respect to  $h$ . Let  $(r, u, s) \in \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}$  be the coordinates with respect to this basis. Then we write the quadratic form  $q$  associated to the Hermitian form  $h$  as

$$q(r, u, s) = r\bar{s} + \bar{r}s + u^\top \bar{u}.$$

With respect to this basis we can write an element of  $\mathcal{U}$  as

$$A = \begin{pmatrix} 1 & v^\top & a \\ 0 & M & w \\ 0 & 0 & 1 \end{pmatrix}$$

with  $v, w \in \mathbb{C}^{n-1}$ ,  $a \in \mathbb{C}$  and  $M$  an upper triangular matrix with 1's on the diagonal. Since  $A$  belongs to  $\mathcal{U}$ , subgroup of  $U(n, 1)$ , it must satisfy  $\bar{A}^\top H A = H$ , where  $H$  is the matrix associated to  $q$ . One can see that  $A$  has the desired form making this condition explicit.  $\square$

**Definition 2.1.3.** Remember that a subgroup  $\mathcal{B}$  of an algebraic group  $G$  is called a *Borel subgroup* if it is a maximal Zariski closed connected solvable algebraic subgroup. Any group  $\mathcal{P}$  between a Borel subgroup and the whole group  $G$  is called a *parabolic subgroup*.

**Lemma 2.1.4.** *Let  $\mathcal{P}$  be a parabolic subgroup of  $U(n, 1)$ . We can choose a basis for  $V$  with respect to which we write  $\mathcal{P}$  as*

$$\mathcal{P} = \left\{ \begin{pmatrix} \bar{\lambda} & -\bar{v}^\top & a \\ 0 & \sigma & \lambda^{-1}\sigma v \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^*, \sigma \in U(n-1), v \in \mathbb{C}^{n-1}, \Re(\bar{\lambda}a) = -\frac{1}{2}\bar{v}^\top v \right\}.$$

A minimal parabolic subgroup of  $U(n, 1)$ , that is a Borel subgroup  $\mathcal{B}$ , is given by elements in  $\mathcal{P}$  with  $\sigma = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}})$  and  $\theta_i \in \mathbb{R}$ .

*Proof.* By definition a Borel subgroup is a solvable subgroup of  $U(n, 1)$ . By Lie's theorem this implies that all the elements of  $\mathcal{B}$  have a common eigenvector. As in Lemma 2.1.1 we can see that this

common vector is isotropic and hence we may choose it to be the first vector of a basis and complete it to an orthonormal basis for  $V$ . With respect to this basis we write the quadratic form associated to  $h$  as  $q(r, u, s) = r\bar{s} + \bar{r}s + u^\top \bar{u}$  with  $(r, u, s) \in \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}$ . Let us apply again Lie's theorem to the orthogonal space of the first vector of the basis. After noticing that a solvable subgroup of  $U(n-1)$  is abelian and imposing the equation  $\bar{A}^\top H A = H$ , where  $H$  is the matrix associated to  $q$  and  $A$  is an element in  $\mathcal{B}$ , we find the desired form. Finally an element in a parabolic subgroup should have a block upper triangular form.  $\square$

**Proposition 2.1.5.** *Let  $U$  be a unipotent subgroup of  $\mathcal{H}(n, 1)$  that is acting simply transitively and affinely on  $\mathfrak{a}(V)$ . Then  $U = \exp(\mathfrak{u})$  where  $\mathfrak{u}$  is a nilpotent Lie subalgebra of  $\mathfrak{u}(n, 1) \ltimes \mathbb{C}^{n+1}$ . In suitable coordinates  $(r, u, s) \in V = \mathbb{C} \oplus W \oplus \mathbb{C}$ , where  $W = \mathbb{C}^{n-1}$ , the quadratic form  $q$  associated to the Hermitian form  $h$  is given by*

$$q(r, u, s) = r\bar{s} + \bar{r}s + u^\top \bar{u}$$

and  $\mathfrak{u} = \mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  has the expression

$$\left\{ \begin{pmatrix} 0 & -(\overline{\gamma_2(u)}^\top + \overline{\gamma_3(s)}^\top) & i(b_2(u) + b_3(s)) & r \\ 0 & 0 & \gamma_2(u) + \gamma_3(s) & u \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid (r, u, s)^\top \in \mathbb{C} \oplus W \oplus \mathbb{C} \right\}$$

where

1.  $\gamma_2 : W \rightarrow W$  is an  $\mathbb{R}$ -linear map such that

$$\text{Im } \gamma_2 \cap J \text{Im } \gamma_2 = \{0\}, \text{Im } \gamma_2 \oplus J \text{Im } \gamma_2 \subseteq \ker \gamma_2 \text{ and } \omega(\text{Im } \gamma_2, \text{Im } \gamma_2) = 0,$$

2.  $\gamma_3 : \mathbb{C} \rightarrow W$  is an  $\mathbb{R}$ -linear map such that  $\gamma_3(is) - J\gamma_3(s) \in \ker \gamma_2$  for all  $s \in \mathbb{C}$ ,

3.  $b_2 : W \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear map such that

$$b_2(s\gamma_2(u)) = 2\omega(\gamma_2(u), \gamma_3(s)) \text{ and}$$

$$b_2(\gamma_3(is) - J\gamma_3(s)) = 2\omega(\gamma_3(is), \gamma_3(s)) \text{ for all } u \in W \text{ and } s \in \mathbb{C},$$

4.  $b_3 : \mathbb{C} \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear map.

Furthermore every Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  is the Lie algebra of a unipotent subgroup of  $U(n, 1)$  that acts simply transitively on  $\mathfrak{a}(V)$ .

*Proof.* Since  $U$  is acting simply transitively on  $\mathfrak{a}(V)$  it is simply connected. Furthermore since  $U$  is unipotent  $U = \exp(\mathfrak{u})$  where  $\mathfrak{u}$  is its Lie algebra. The linear part of  $\mathfrak{u}$  can be conjugate to be as in Lemma 2.1.2. Equivalently we can find coordinates  $(r, u, s) \in V = \mathbb{C} \oplus W \oplus \mathbb{C}$ , where  $W = \mathbb{C}^{n-1}$ , with respect to which the quadratic form  $q$  associated to the Hermitian form  $h$  reads as

$$q(r, u, s) = r\bar{s} + \bar{r}s + u^\top \bar{u}$$

and the linear part  $L(\mathfrak{u})$  of  $\mathfrak{u}$  is contained in

$$\left\{ \begin{pmatrix} 0 & -\bar{\gamma}^\top & ib \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \mid \gamma \in W, b \in \mathbb{R} \right\}.$$

Since  $\mathfrak{u}$  is the Lie algebra of  $U$  that is acting simply transitively on  $\mathfrak{a}(V)$  there exists a linear isomorphism  $V \rightarrow \mathfrak{u}$ . To be more precise let us define a map  $g : \mathfrak{u} \rightarrow V$  as follows. If

$$X = \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{u}$$

we define  $g(X) = v$ . This map is clearly linear. Furthermore assume we have  $g(X_1) = g(X_2) = v$ , if we consider the element  $\exp(X_1 - X_2)$  its action on  $0 \in V$  is trivial. Being the action of  $U$  on  $\mathfrak{a}(V)$  simple this implies  $\exp(X_1 - X_2) = \text{Id}$  and hence  $X_1 = X_2$  and the map is injective. Finally since  $\dim \mathfrak{u} = \dim V$  the map  $g$  is a linear isomorphism. If we compose  $g^{-1}$  with the linear function  $L : \mathfrak{u} \rightarrow \text{End}(V)$  we obtain two  $\mathbb{R}$ -linear maps

$$\gamma : \mathbb{C} \oplus W \oplus \mathbb{C} \rightarrow \mathbb{C}^{n-1} \text{ and}$$

$$b : \mathbb{C} \oplus W \oplus \mathbb{C} \rightarrow \mathbb{R},$$

such that

$$\mathfrak{u} = \left\{ \begin{pmatrix} 0 & -\overline{\gamma(v)}^\top & ib(v) & r \\ 0 & 0 & \gamma(v) & u \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid v = (r, u, s) \in \mathbb{C} \oplus W \oplus \mathbb{C} \right\}.$$

Let us write  $(A(v), v)$  for an element in  $\mathfrak{u}$  where  $A(v)$  denotes the linear part and  $v$  the translation part. If we compute the commutator bracket of two elements in  $\mathfrak{u}$  we get

$$[(A(v), v), (A(v'), v')] = \begin{pmatrix} 0 & 0 & \overline{\gamma(v')}^\top \gamma(v) - \overline{\gamma(v)}^\top \gamma(v') & \overline{\gamma(v')}^\top u - \overline{\gamma(v)}^\top u' + i(s'b(v) - sb(v')) \\ 0 & 0 & 0 & s'\gamma(v) - s\gamma(v') \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since  $\mathfrak{u}$  is a Lie algebra we must have  $\gamma(v'') = 0$  and  $b(v'') = 2\Im(\overline{\gamma(v')}^\top \gamma(v))$  where

$$v'' = (\overline{\gamma(v')}^\top u - \overline{\gamma(v)}^\top u' + i(s'b(v) - sb(v')), s'\gamma(v) - s\gamma(v'), 0).$$

Let us write  $\gamma(v) = \gamma_1(r) + \gamma_2(u) + \gamma_3(s)$  and  $b(v) = b_1(r) + b_2(u) + b_3(s)$ .

**Claim 2.1.5.1.** We have  $\gamma_1 = 0$ ,  $\text{Im } \gamma_2 + J \text{Im } \gamma_2 \subseteq \ker \gamma_2$  and that property 2 holds.

*Proof.* Let  $s = s' = 0$  then  $\gamma(v'') = \gamma_1(\overline{\gamma(v')}^\top u - \overline{\gamma(v)}^\top u') = 0$ . Now let  $u' = 0$  we get  $\gamma(v'') = \gamma_1(\overline{\gamma_1(r')}^\top u) = 0$  for all  $r' \in \mathbb{C}, u \in W$ . Let us show that this implies that  $\gamma_1 = 0$ . Indeed either  $\gamma_1 = 0$  or there exists  $r_0 \in \mathbb{C}$  such that  $\gamma_1(r_0) \neq 0$ . In the second case, since  $h|_W$  is non-degenerate, there exists  $u_0 \in W$  such that  $\overline{\gamma_1(r_0)}^\top u_0 = z \in \mathbb{C}^*$ . Then for all  $\lambda \in \mathbb{C}$  we get  $\overline{\gamma_1(r_0)}^\top \lambda u_0 = \lambda z$ , that is, making  $\lambda$  vary, we can get all the elements in  $\mathbb{C}$  and then from  $\gamma_1(\overline{\gamma_1(r_0)}^\top \lambda u_0) = 0$  we can conclude that  $\gamma_1 = 0$ . Now take  $s = 0$  we have  $\gamma(v'') = \gamma_2(s'\gamma(v)) = \gamma_2(s'\gamma_2(u)) = 0$  hence if we let  $s'$  be real we obtain  $(\gamma_2)^2 = 0$  and if we let  $s'$  be purely imaginary we obtain  $\gamma_2(J\gamma_2) = 0$ . Now the equation  $\gamma(v'') = 0$  becomes  $\gamma_2(s'\gamma_3(s)) - \gamma_2(s\gamma_3(s')) = 0$ . Let  $s = is'$  and the claim follows.  $\square$

**Claim 2.1.5.2.** We have  $b_1 = 0$ ,  $\omega(\text{Im } \gamma_2, \text{Im } \gamma_2) = 0$ ,  $\text{Im } \gamma_2 \cap J \text{Im } \gamma_2 = \{0\}$  and that property 3 holds.

*Proof.* Let  $u = u' = 0$  then

$$b(v'') = b_1(i(s'b(v) - sb(v'))) + b_2(s'\gamma_3(s) - s\gamma_3(s')) = 2\Im(\overline{\gamma_3(s')}^\top \gamma_3(s)).$$

Now let  $s = 0$  we get  $b(v'') = b_1(is'b(v)) = b_1(is'b_1(r)) = 0$  hence  $b_1 = 0$ . Looking again at the equation  $b(v'') = 2\Im(\overline{\gamma(v')}^\top \gamma(v))$  and letting  $s = s' = 0$  we get  $\Im(\overline{\gamma_2(u')}^\top \gamma_2(u)) = 0$  for all  $u, u'$ , in other words  $\omega(\text{Im } \gamma_2, \text{Im } \gamma_2) = 0$ . Furthermore we have  $\text{Im } \gamma_2 \cap J \text{Im } \gamma_2 = \{0\}$ . This is because if  $v_1 = Jv_2$  with  $v_1, v_2 \in \text{Im } \gamma_2$  then  $\langle v_1, v_1 \rangle = \langle v_1, Jv_2 \rangle = \omega(v_1, v_2) = 0$  then  $v_1 = 0$ . Finally the equation  $b(v'') = 2\Im(\overline{\gamma(v')}^\top \gamma(v))$  becomes

$$b_2(s'\gamma_2(u) - s\gamma_2(u')) + s'\gamma_3(s) - s\gamma_3(s') = 2\Im(\overline{\gamma_2(u')}^\top \gamma_3(s) + \overline{\gamma_3(s')}^\top \gamma_2(u) + \overline{\gamma_3(s')}^\top \gamma_3(s)).$$

Now take  $s = 0$  we get  $b_2(s'\gamma_2(u)) = 2\Im(\overline{\gamma_3(s')}^\top \gamma_2(u))$ . Finally if we consider  $u = u' = 0$  we get  $b_2(s'\gamma_3(s) - s\gamma_3(s')) = 2\Im(\overline{\gamma_3(s')}^\top \gamma_3(s))$ , now if we let  $s = is'$  the claim follows.  $\square$

We can notice that the conditions given in Claim 2.1.5.1 and 2.1.5.2 are also sufficient in order to have  $\gamma(v'') = 0$  and  $b(v'') = 2\Im(\overline{\gamma(v')}^\top \gamma(v))$ .

Finally let us prove that every such Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  is the Lie algebra of a group acting simply transitively on  $\mathfrak{a}(V)$ . Indeed from the presentation we have of  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ , we see that the associated Lie group has an affine action on  $\mathfrak{a}(V)$ . It is left to prove that this action is simply transitive. It is sufficient then to look at the orbit of 0. The stabiliser of 0 is trivial since the linear parts of the elements in  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  depend linearly on the translation part. Finally the orbit of 0 is the whole space since the translation part of elements in  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  span the whole  $V$ .  $\square$

Because of Proposition 2.1.5, let us decompose  $W$  as a real vector space as follows

$$W = \text{Im } \gamma_2 \oplus J\text{Im } \gamma_2 \oplus S \oplus T \quad (2.1)$$

where  $S \oplus T$  is orthogonal to  $\text{Im } \gamma_2 \oplus J\text{Im } \gamma_2$  with respect to  $h|_W$ ,  $T$  is orthogonal to  $S$  with respect to  $\langle \cdot, \cdot \rangle|_W$  and

$$\ker \gamma_2 = \text{Im } \gamma_2 \oplus J\text{Im } \gamma_2 \oplus S.$$

Write  $\pi$  for the projection of  $\ker \gamma_2$  on  $S$  with respect to this decomposition.

**Proposition 2.1.6.** *We have an upper bound  $\text{rank}(\gamma_2) \leq \frac{2n-2}{3}$ .*

*Proof.* We have  $\text{Im } \gamma_2 \cap J\text{Im } \gamma_2 = \{0\}$ , see Proposition 2.1.5 condition 1, and since both  $\text{Im } \gamma_2$  and  $J\text{Im } \gamma_2$  are contained in  $\ker \gamma_2$ , we have  $2\text{rank}(\gamma_2) \leq \dim \ker \gamma_2$  then  $3\text{rank}(\gamma_2) \leq \text{rank}(\gamma_2) + \dim \ker \gamma_2 = 2n - 2$ .  $\square$

*Remark 2.1.7.* We can notice that  $\pi(\gamma_3(is) - J\gamma_3(s)) = 0$  for some  $s \in \mathbb{C}$ ,  $s \neq 0$ , if and only if  $\pi(\gamma_3(is) - J\gamma_3(s)) = 0$  for all  $s \in \mathbb{C}$ . Indeed it suffices to notice that  $\gamma_3(is) - J\gamma_3(s) = \bar{s}(\gamma_3(i) - J\gamma_3(1))$  for all  $s \in \mathbb{C}$ . Hence  $\gamma_3(is) - J\gamma_3(s) \in \text{Im } \gamma_2 \oplus J\text{Im } \gamma_2$  if and only if  $\gamma_3(i) - J\gamma_3(1)$  does.

**Proposition 2.1.8.** *Assume there exists  $s \in \mathbb{C}$ ,  $s \neq 0$ , such that  $\pi(\gamma_3(is) - J\gamma_3(s)) = 0$ . Then  $\text{Im } \gamma_3 \subseteq \text{Im } \gamma_2$  and hence  $b_2$  is 0 on  $\text{Im } \gamma_2 \oplus J\text{Im } \gamma_2$ .*

*Proof.* From Remark 2.1.7  $\pi(\gamma_3(is) - J\gamma_3(s)) = 0$  for some  $s \in \mathbb{C}$ ,  $s \neq 0$ , implies  $\pi(\gamma_3(i) - J\gamma_3(1)) = 0$ . Hence let  $w_0 = \gamma_3(i) - J\gamma_3(1) \in \ker \gamma_2$ . If  $\pi(w_0) = 0$  then write  $w_0 = w_1 + Jw_2$  with  $w_1, w_2 \in \text{Im } \gamma_2$ . Condition 3 of Proposition 2.1.5 implies that  $2\omega(\gamma_3(i), \gamma_3(1)) = b_2(w_0) = b_2(w_1 + Jw_2) = 2\omega(w_1, \gamma_3(1)) + 2\omega(w_2, \gamma_3(i))$ , hence  $\omega(w_1 - \gamma_3(i), \gamma_3(1)) + \omega(w_2, \gamma_3(i)) = 0$ . Since  $w_1 - \gamma_3(i) = -J(w_2 + \gamma_3(1))$  and  $\gamma_3(i) = w_1 + Jw_2 + J\gamma_3(1)$ , substituting them in the previous expression, we obtain  $\omega(J\gamma_3(1), \gamma_3(1)) + \omega(Jw_2, w_2) + 2\omega(Jw_2, \gamma_3(1)) = 0$ , i.e.  $\|\gamma_3(1)\|^2 + 2\langle w_2, \gamma_3(1) \rangle + \|w_2\|^2 = 0$ . Since  $h|_{\text{Im } \gamma_2 \oplus J\text{Im } \gamma_2}$  is non-degenerate, this implies  $\gamma_3(1) = -w_2$ , in other words  $\text{Im } \gamma_3 \subseteq \text{Im } \gamma_2$  hence  $b_2$  is 0 on  $\text{Im } \gamma_2 \oplus J\text{Im } \gamma_2$ .  $\square$

As an abstract Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  can be described as the real vector space  $\mathbb{C} \oplus W \oplus \mathbb{C}$ , with  $W = \mathbb{C}^{n-1}$ , with Lie brackets

$$[(r, u, s), (r', u', s')] = (h(u, \gamma(u', s')) - h(u', \gamma(u, s)) + i(b(u, s)s' - b(u', s')s), s'\gamma(u, s) - s\gamma(u', s'), 0)$$

and with  $\gamma: W \oplus \mathbb{C} \rightarrow W$  and  $b: W \oplus \mathbb{C} \rightarrow \mathbb{R}$  linear maps,  $\gamma = \gamma_2 + \gamma_3$  and  $b = b_2 + b_3$  satisfying the conditions of Proposition 2.1.5.

*Remark 2.1.9.* From Remark 2.1.7 it follows that the condition  $\pi(\gamma_3(is) - J\gamma_3(s)) = 0$  does not depend on the basis we have chosen for  $\mathbb{C}$  as a real vector space. Hence we will write this condition as  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$  where  $(\xi, i\xi)$  is any basis of  $\mathbb{C}$  as a real vector space.

**Definition 2.1.10.** We say that a Lie algebra  $\mathfrak{u}$  is  $k$ -step nilpotent if  $\mathcal{C}^{k+1}\mathfrak{u} = \{0\}$  where  $\mathcal{C}^k\mathfrak{u} = [\mathfrak{u}, \mathcal{C}^{k-1}\mathfrak{u}]$  and  $\mathcal{C}^1\mathfrak{u} = \mathfrak{u}$ . The smallest  $k$  for which  $\mathcal{C}^{k+1}\mathfrak{u} = \{0\}$  is called the nilpotency class.

**Proposition 2.1.11.** *The Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are 3-step nilpotent. The lower central series looks like*

$$\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \supseteq \mathcal{C}^2\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \supseteq \mathcal{C}^3\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \supseteq \{0\}$$

where  $\mathcal{C}^2\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \subseteq \mathbb{C} \oplus \mathbb{C}\text{Im}(\gamma_2) \oplus \mathbb{R}\pi(\gamma_3(i\xi) - J\gamma_3(\xi))$  and  $\mathcal{C}^3\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \subseteq \mathbb{C}$ .

*Proof.* Looking at the definition of the Lie brackets of  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  one can see directly that  $\mathcal{C}^2\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \subseteq \mathbb{C} \oplus \mathbb{C}\text{Im}(\gamma_2) \oplus \mathbb{R}\pi(\gamma_3(i\xi) - J\gamma_3(\xi))$ . From conditions 1 and 2 of Proposition 2.1.5 we get that  $\text{Im}(\gamma_2) \oplus \mathbb{R}\pi(\gamma_3(i\xi) - J\gamma_3(\xi))$  is contained in  $\ker \gamma_2$ . This implies that  $\mathcal{C}^3\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \subseteq \mathbb{C}$  and that the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are 3-step nilpotent.  $\square$

*Remark 2.1.12.* When  $\gamma_2 \neq 0$ , the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are nilpotent of class 3. When  $\gamma_2 = 0$  we have equalities  $\mathcal{C}^2\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) = \mathbb{C} \oplus \mathbb{C}\text{Im}(\gamma_2) \oplus \mathbb{R}\pi(\gamma_3(i\xi) - J\gamma_3(\xi))$  and  $\mathcal{C}^3\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) = \mathbb{C}$ . To see this let  $(\tau, i\tau, e_1, \dots, e_k, Je_1, \dots, Je_k, g_1, \dots, g_m, f_1, \dots, f_k, \xi, i\xi)$  be a basis for  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  with  $k = \text{rank}(\gamma_2)$ ,  $m = 2n - 2 - k$ ,  $(e_1, \dots, e_k)$  a basis for  $\text{Im} \gamma_2$  and  $\{f_1, \dots, f_k\}$  such that  $\gamma_2(f_i) = e_i$ . Then  $[e_i, f_i] = \tau$ ,  $[Je_i, f_i] = i\tau$ ,  $[f_i, \xi] = e_i \pmod{\mathbb{C}\tau}$ ,  $[f_i, i\xi] = Je_i \pmod{\mathbb{C}\tau}$  and  $[\xi, i\xi] = -\gamma_3(i\xi) - J\gamma_3(\xi) \pmod{\mathbb{C}\tau}$ . Instead, when  $\gamma_2 = 0$ , all nilpotency classes in  $\{1, 2, 3\}$  can occur, see Appendix A for the case  $n = 3$ .

### 2.1.1 Classification up to conjugation

We now write a classification up to conjugacy of the unipotent subgroup  $U(\gamma_2, \gamma_3, b_2, b_3) = \exp \mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  of  $\mathcal{H}(n, 1)$ . This will be useful for the proof of Proposition 2.2.6.

**Proposition 2.1.13.** *There exists  $g \in \mathcal{H}(n, 1)$  such that  $gU(\gamma_2, \gamma_3, b_2, b_3)g^{-1} = U(\gamma'_2, \gamma'_3, b'_2, b'_3)$  if and only if there exist  $\lambda \in \mathbb{C}^*$ ,  $\sigma \in U(n-1)$ ,  $v \in W$ ,  $s_1 \in \mathbb{C}$  such that*

1.  $\gamma'_2(u) = \lambda \sigma \gamma_2(\sigma^{-1}u)$ ,
2.  $\gamma'_3(s) = \lambda \sigma \gamma_3(\lambda s) + \lambda \sigma \gamma_2(\tilde{s})$ ,
3.  $b'_2(u) = |\lambda|^2 b_2(\sigma^{-1}u - \lambda s_1 \gamma_2(\sigma^{-1}u)) - 2\omega(\lambda \gamma_2(\sigma^{-1}u), v)$ ,
4.  $b'_3(s) = |\lambda|^2 b_3(\lambda s) + |\lambda|^2 b_2(\tilde{s} - \lambda s_1 \gamma_2(\tilde{s})) - 2\omega(\lambda(\gamma_2(\tilde{s}) + \gamma_3(\lambda s)), v)$

where  $\tilde{s} = \lambda s_1 \gamma_3(\lambda s) - sv$ .

*Proof.* Let  $g \in \mathcal{H}(n, 1)$  and  $h \in U(\gamma_2, \gamma_3, b_2, b_3)$ . Since a parabolic subgroup is self normalizing if  $L(g) \in U(n, 1)$  and  $L(h) \in \mathcal{P}$  such that  $L(ghg^{-1}) \in \mathcal{P}$  then  $L(g) \in \mathcal{P}$ . Hence we may assume that

$$g = \begin{pmatrix} \bar{\lambda} & -\bar{v}^\top & a & r_1 \\ 0 & \sigma & \lambda^{-1}\sigma v & u_1 \\ 0 & 0 & \lambda^{-1} & s_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with  $\lambda \in \mathbb{C}^*$ ,  $\sigma \in U(n-1)$ ,  $v \in W$ ,  $a \in \mathbb{C}$  such that  $\bar{\lambda}a + \lambda a = -\bar{v}^\top v$  and  $(r_1, u_1, s_1) \in \mathbb{C} \oplus W \oplus \mathbb{C}$ . Notice that the inverse of the linear part of  $g$  is of the form

$$L(g)^{-1} = \begin{pmatrix} \bar{\lambda}^{-1} & \bar{\lambda}^{-1} \bar{v}^\top \bar{\sigma}^\top & \bar{a} \\ 0 & \bar{\sigma}^\top & -v \\ 0 & 0 & \lambda \end{pmatrix}.$$

Since  $U(\gamma_2, \gamma_3, b_2, b_3) = \exp(\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3))$  and, for  $X \in \mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ ,

$$g \exp(X) g^{-1} = \exp(gXg^{-1})$$

we work on the level of the Lie algebra. Explicitly let

$$X = \begin{pmatrix} 0 & -(\overline{\gamma_2(u)}^\top + \overline{\gamma_3(s)}^\top) & i(b_2(u) + b_3(s)) & r \\ 0 & 0 & \gamma_2(u) + \gamma_3(s) & u \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3),$$

then the linear part of  $gXg^{-1}$  is

$$\begin{pmatrix} 0 & -\bar{\lambda}(\overline{\gamma_2(u)}^\top + \overline{\gamma_3(s)}^\top) \bar{\sigma}^\top & \bar{\lambda}(\overline{\gamma_2(u)}^\top + \overline{\gamma_3(s)}^\top) v + i|\lambda|^2(b_2(u) + b_3(s)) - \lambda \bar{v}^\top (\gamma_2(u) + \gamma_3(s)) \\ 0 & 0 & \lambda \sigma (\gamma_2(u) + \gamma_3(s)) \\ 0 & 0 & 0 \end{pmatrix}$$

and the translation part of  $gXg^{-1}$  is

$$\begin{pmatrix} * \\ -\lambda s_1 \sigma (\gamma_2(u) + \gamma_3(s)) + \sigma u + \lambda^{-1} s \sigma v \\ \lambda^{-1} s \end{pmatrix}.$$

Where  $*$  stands for a complicated expression that we will not need. Assume  $gXg^{-1} \in \mathfrak{u}(\gamma'_2, \gamma'_3, b'_2, b'_3)$  and let

$$(r', u', s') = (*, -\lambda s_1 \sigma (\gamma_2(u) + \gamma_3(s)) + \sigma u + \lambda^{-1} s \sigma v, \lambda^{-1} s).$$

When  $s' = 0$  then  $s = 0$  and we get  $\gamma'_2(\sigma u - \lambda s_1 \sigma (\gamma_2(u))) = \lambda \sigma \gamma_2(u)$ . Take  $u = \gamma_2(\tilde{u})$  for some  $\tilde{u} \in W$  then we get  $\gamma'_2(\sigma(\gamma_2(\tilde{u}))) = 0$  hence  $\gamma'_2(\sigma u) = \lambda \sigma \gamma_2(u)$  that is  $\gamma'_2 = \lambda \sigma \gamma_2 \sigma^{-1}$ .

When  $u' = 0$ ,  $u = \lambda s_1 \gamma_2(u) + \lambda s_1 \gamma_3(s) - \lambda^{-1} s v$  and we get  $\gamma'_3(\lambda^{-1} s) = \lambda \sigma \gamma_2(u) + \lambda \sigma \gamma_3(s)$ , hence  $\gamma_3(\lambda^{-1} s) = \lambda \sigma \gamma_3(s) + \lambda \sigma \gamma_2(\lambda s_1 \gamma_3(s) - \lambda^{-1} s v)$  that is

$$\gamma'_3(s) = \lambda \sigma \gamma_3(\lambda s) + \lambda \sigma \gamma_2(\lambda s_1 \gamma_3(\lambda s) - s v).$$

Now consider the equation

$$|\lambda|^2 b_2(u) + |\lambda|^2 b_3(s) - 2\Im(\lambda \bar{v}^\top (\gamma_2(u) + \gamma_3(s))) = b'_2(-\lambda s_1 \sigma (\gamma_2(u) + \gamma_3(s)) + \sigma u + \lambda^{-1} s \sigma v) + b'_3(\lambda^{-1} s).$$

Let  $s = 0$  then we are left with

$$|\lambda|^2 b_2(u) - 2\Im(\lambda \bar{v}^\top \gamma_2(u)) = b'_2(-\lambda s_1 \sigma \gamma_2(u) + \sigma u).$$

Take  $u = \gamma_2(\tilde{u})$  then we have  $|\lambda|^2 b_2(\gamma_2(\tilde{u})) = b'_2(\sigma \gamma_2(\tilde{u}))$ . Hence the equation becomes  $b'_2(\sigma u) = |\lambda|^2 b_2(u - \lambda s_1 \gamma_2(u)) - 2\Im(\lambda \bar{v}^\top \gamma_2(u))$ .

Let  $u = 0$  then the equation becomes

$$|\lambda|^2 b_3(s) - 2\Im(\lambda \bar{v}^\top \gamma_3(s)) = b'_2(-\lambda s_1 \sigma \gamma_3(s) + \lambda^{-1} s \sigma v) + b'_3(\lambda^{-1} s),$$

substituting  $b'_2$  we get

$$\begin{aligned} |\lambda|^2 b_3(s) - 2\Im(\lambda \bar{v}^\top \gamma_3(s)) &= |\lambda|^2 b_2(-\lambda s_1 \gamma_3(s) + \lambda^{-1} s v - \lambda s_1 \gamma_2(-\lambda s_1 \gamma_3(s) + \lambda^{-1} s v)) \\ &\quad - 2\Im(\lambda \bar{v}^\top \gamma_2(-\lambda s_1 \gamma_3(s) + \lambda^{-1} s v)) + b'_3(\lambda^{-1} s). \end{aligned}$$

Hence the proposition follows.  $\square$

*Remark 2.1.14.* We saw in Proposition 2.1.8 that if  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$  then  $b_2$  is 0 on  $\text{Im } \gamma_2 \oplus J\text{Im } \gamma_2$ . On the other hand when  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$ , if  $\gamma_2 \neq 0$ , using Proposition 2.1.13 we can conjugate the Lie group  $U(\gamma_2, \gamma_3, b_2, b_3)$  so that  $\gamma'_3(\xi) = i\gamma_3(i\xi) - i\gamma_2(v)$  for some  $v \in W$ . Hence we can suppose that  $\gamma_3(\xi)$  has 0 component in  $J\text{Im } \gamma_2$ . In other words, since  $b_2(\gamma_2(u)) = 2\omega(\gamma_2(u), \gamma_3(\xi))$ , we can always suppose that  $b_2$  is 0 on  $\text{Im } \gamma_2$ .

## 2.2 Hermite-Lorentz crystallographic groups

In this section we go back to our original question about crystallographic subgroups of  $U(n, 1) \times \mathbb{C}^{n+1}$  and prove the main result of this section, namely Theorem 2.2.7.

**Definition 2.2.1.** Consider the following subgroups of the Borel subgroup  $\mathcal{B}$  of  $U(n, 1)$

$$\mathcal{D} = \left\{ \begin{pmatrix} \bar{\lambda} & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \mid \sigma = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}) \right\}$$

and

$$\widehat{\mathcal{B}} = \left\{ \begin{pmatrix} \lambda & -\bar{v}^\top & a \\ 0 & \sigma & \lambda \sigma v \\ 0 & 0 & \lambda \end{pmatrix} \mid |\lambda|^2 = 1, \sigma = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}), \Re(\bar{\lambda}a) = -\frac{1}{2}\bar{v}^\top v \right\}.$$

**Definition 2.2.2.** The *unipotent radical* of an algebraic group  $G$  is the set of all unipotent elements in the largest connected solvable normal subgroup of  $G$ . It may be characterised as the largest connected unipotent normal subgroup of  $G$ . See [33].

**Definition 2.2.3.** The *nilradical*,  $N$ , of a Lie group  $G$  is the largest connected normal nilpotent subgroup of  $G$ . Analogously the nilradical,  $\mathfrak{n}$ , of a Lie algebra  $\mathfrak{g}$  is the largest nilpotent ideal of  $\mathfrak{g}$ . Then we have  $N = \exp(\mathfrak{n})$ .

**Lemma 2.2.4.** Let  $H \leq \mathcal{H}(n, 1)$  be a group acting simply transitively on  $\mathfrak{a}(V)$  and let  $N$  be its nilradical. Consider furthermore the Zariski closure of  $H$ , denoted by  $\tilde{H}$ , inside  $\mathcal{H}(n, 1)$  and let  $U$  be the unipotent radical of  $\tilde{H}$ . Then  $(H \cap U)^0 = N$ .

*Proof.* Clearly  $(H \cap U)^0 \subseteq N$ . On the other side from [7, Corollary III.3] we have that  $N \subseteq U$ , hence the lemma follows.  $\square$

**Proposition 2.2.5.** Let  $H \leq \mathcal{H}(n, 1)$  be a subgroup that acts simply transitively on  $\mathfrak{a}(V)$ . Then  $H$  is  $\mathcal{H}(n, 1)$ -conjugate to a subgroup whose linear part is in  $\mathcal{B}$ .

*Proof.* Since  $H$  acts simply transitively by Theorem 1.2.40,  $H$  is solvable. Then its linear part  $L(H) \leq U(n, 1)$  is solvable and the Zariski closure of  $L(H)$  is solvable as well. Thus  $H$  is conjugate in  $\mathcal{H}(n, 1)$  to a group whose linear part is a subgroup of the Borel group  $\mathcal{B}$ .  $\square$

**Proposition 2.2.6.** Let  $H \leq \mathcal{H}(n, 1)$  be a subgroup whose action on  $\mathfrak{a}(V)$  is simply transitive, such that  $L(H) \leq \mathcal{B}$ . Then either  $L(H) \leq \widehat{\mathcal{B}}$  or  $L(H)$  is  $\mathcal{B}$ -conjugate to a subgroup of  $\mathcal{D}$ .

*Proof.* Let  $U$  be the unipotent radical of the Zariski closure  $\tilde{H}$  of  $H$ , then by [7, Theorem III.1.]  $U$  also acts simply transitively on  $\mathfrak{a}(V)$ . Hence by Proposition 2.1.5 we have that  $U = U(\gamma_2, \gamma_3, b_2, b_3)$  for some  $\gamma_2, \gamma_3, b_2, b_3$ . By definition  $U$  is a normal subgroup of  $\tilde{H}$  hence  $H$  normalizes  $U$ . That is for all  $h \in H$  we have  $hU(\gamma_2, \gamma_3, b_2, b_3)h^{-1} = U(\gamma_2, \gamma_3, b_2, b_3)$ . Now for every fixed  $h \in H$  its linear part is of the form

$$\begin{pmatrix} \bar{\lambda} & -\bar{v}^\top & a \\ 0 & \sigma & \lambda^{-1} \sigma v \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

and it follows from Proposition 2.1.13 that  $\gamma_2(u) = \lambda \sigma \gamma_2(\sigma^{-1}u)$ , hence if  $\gamma_2 \neq 0$  then  $|\lambda|^2 = 1$ . If  $\gamma_2 = 0$  then  $\gamma_3(s) = \lambda \sigma \gamma_3(\lambda s)$  and  $b_2(u) = |\lambda|^2 b_2(\sigma^{-1}u)$ . So if  $b_2$  is non-zero then  $|\lambda|^2 = 1$  if instead  $\gamma_3$  is not 0 then  $\lambda^2 = 1$ . Finally if all of  $\gamma_2, \gamma_3, b_2$  are identically 0 then  $b_3(s) = |\lambda|^2 b_3(\lambda s)$  and hence, if  $b_3$  is non-zero,  $\lambda = 1$ . Therefore if at least one of  $\gamma_2, \gamma_3, b_2, b_3$  is non-zero then  $|\lambda|^2 = 1$  and

$L(H) \leq \widehat{\mathcal{B}}$ , otherwise all of  $\gamma_2, \gamma_3, b_2, b_3$  are 0 and hence  $L(U) = \{\text{Id}\}$ . Since  $L(U)$  is the unipotent radical of  $\widetilde{L(H)}$  then this implies that  $\widetilde{L(H)}$  is reductive and being solvable and connected it is a torus. Hence  $L(H)$  is  $\mathcal{B}$ -conjugate to a subgroup of  $\mathcal{D}$ .  $\square$

We are now ready to prove the main theorem of this section.

**Theorem 2.2.7.** *Let  $\Gamma$  be a subgroup of  $\mathcal{H}(n, 1)$  that acts properly discontinuously and cocompactly on  $\mathfrak{a}(V)$  then  $\Gamma$  is either virtually nilpotent or virtually abelian-by-cyclic.*

*Proof.* Since  $\Gamma \leq \mathcal{H}(n, 1)$  acts properly discontinuously and cocompactly on  $\mathfrak{a}(V)$  then  $\Gamma$  is virtually solvable by Theorem 1.2.35. Let  $H$  be a subgroup of  $\text{Aff}(V)$  acting simply transitively on  $\mathfrak{a}(V)$  coming from Theorem 1.2.37. Then we know that  $\Gamma \cap H$  has finite index in  $\Gamma$  and is a lattice in  $H$ . Hence, after replacing  $\Gamma$  with  $\Gamma \cap H$ , we can assume that  $\Gamma$  is a lattice in  $H$ . Notice that actually  $H$  is the connected component of the identity of a crystallographic hull of  $\Gamma$  and after [26, Theorem 1.4.] the Zariski closure of  $\Gamma$  is the same as the one of its crystallographic hull. Hence  $H$  is contained in the Zariski closure of  $\Gamma$  that lies inside  $\mathcal{H}(n, 1)$ , it follows that  $H \leq \mathcal{H}(n, 1)$ .

From Proposition 2.2.5  $H$  can be conjugate to a subgroup whose linear part is in  $\mathcal{B}$  and after Proposition 2.2.6 either  $L(H) \leq \widehat{\mathcal{B}}$  or  $L(H)$  is conjugate to a subgroup of  $\mathcal{D}$ .

Assume  $L(H) \leq \widehat{\mathcal{B}}$ . Notice that in this case there exists a unipotent subgroup,  $U$ , of  $\widehat{\mathcal{B}}$  such that the quotient is a compact group  $C$ . From [18, Prop 4.C.11] we know that the inclusion of  $U \times \mathbb{C}^{n+1} \hookrightarrow (C \times U) \times \mathbb{C}^{n+1} = \widehat{\mathcal{B}} \times \mathbb{C}^{n+1}$  is a quasi-isometry. Hence  $H$  is quasi-isometric to a nilpotent group. Furthermore since  $\Gamma$  is a lattice in  $H$ ,  $\Gamma$  is quasi-isometric to  $H$ , see [21, Corollary IV.24]. Nilpotent groups are of polynomial growth, [21, Theorem VII.26], and being the growth type an invariance by quasi-isometry, [21, Proposition VI.27],  $\Gamma$  has polynomial growth. From [47, Theorem 4.3] a finitely generated polycyclic group of polynomial growth is virtually nilpotent. This implies that  $\Gamma$  is virtually nilpotent.

We are left to treat the case  $L(H) \leq \mathcal{D}$ . We denote  $\widetilde{H}$  the Zariski closure of  $H$ . Then the unipotent radical of  $\widetilde{H}$  is equal to the group  $T$  of translations of  $\mathcal{H}(n, 1)$ , see the proof of Proposition 2.2.6. From Lemma 2.2.4 the nilradical of  $H$  is equal to  $T \cap H$ , the subgroup of translations that lies in  $H$ .

**Claim 2.2.7.1.** *Let  $H$  be a simply connected solvable Lie group and  $N$  its nilradical. Let  $\Gamma$  be a lattice in  $H$  then  $\Gamma N/N \subseteq H/N$  is discrete.*

*Proof.* From [6, Mostow Theorem] we know that  $\overline{\Gamma N}^0 = N$  hence  $N$  is open in  $\Gamma N$  and the claim follows.  $\square$

From Claim 2.2.7.1 the group  $\Gamma/\Gamma \cap N$  is isomorphic to  $\Gamma N/N$ , that is discrete in  $H/N$ . Being  $H/N$  isomorphic to  $L(H)$ , it is contained in  $\mathbb{C}^* \times \text{U}(n-1)$ . Let us call  $\Delta$  the group  $\Gamma/\Gamma \cap N$  for convenience. Since  $S^1 \times \text{U}(n-1)$  is compact,  $\Delta/(\Delta \cap (S^1 \times \text{U}(n-1)))$  is discrete in  $\mathbb{R}^*$  hence cyclic. Furthermore  $\Delta \cap (S^1 \times \text{U}(n-1))$  is discrete in  $S^1 \times \text{U}(n-1)$  with the induced topology hence finite. Then, up to take a finite index subgroup of  $\Gamma$ , we can assume that  $\Delta \cap (S^1 \times \text{U}(n-1))$  is trivial and so that  $\Delta = \Gamma/\Gamma \cap N$  is cyclic. Finally  $\Gamma \cap N$  is discrete in  $N$  which is some group of translations so

$\Gamma \cap N$  is isomorphic to  $\mathbb{Z}^m$  for some  $m$  and then for dimension reasons we have  $m = 2n + 1$  or  $\Gamma$  is abelian. So in this case  $\Gamma$  has a finite index subgroup that is abelian-by-cyclic.  $\square$

*Remark 2.2.8.* Let us remark that this theorem is the same as for the Lorentzian case, see Theorem 1.3.6. Furthermore let us remark again that this dichotomy between either virtually nilpotent or virtually abelian-by-cyclic is not true in general, see Example 1.3.18.

According to this theorem, the unipotent hypothesis we made in Section 2.1 only leaves out the easy abelian-by-cyclic case. We end this section with the classification of the latter case.

**Definition 2.2.9.** Let  $n \geq 1$  and  $A \in \text{GL}(2n + 1, \mathbb{Z})$  diagonalisable over  $\mathbb{C}$  with eigenvalues  $1$  and  $\{\lambda, \lambda^{-1}, a_1, \dots, a_{n-2}\}$  and their complex conjugates with  $\lambda, \lambda^{-1} \in \mathbb{C}^*$  and  $|a_i| = 1$ . Define the group  $\tilde{\Gamma}(2n + 2, A) = \mathbb{Z} \rtimes_A \mathbb{Z}^{2n+1}$ .

Then the following propositions provide a classification in the abelian-by-cyclic case.

**Proposition 2.2.10.** Let  $n \geq 1$  and  $A, A' \in \text{GL}(2n + 1, \mathbb{Z})$  then

- $\tilde{\Gamma}(2n + 2, A) \cong \tilde{\Gamma}(2n + 2, A')$  if and only if  $A$  is  $\text{GL}(2n + 1, \mathbb{Z})$ -conjugate to either  $A'$  or  $A'^{-1}$ .
- $\tilde{\Gamma}(2n + 2, A)$  is commensurable with  $\tilde{\Gamma}(2n + 2, A')$  if and only if  $A'$  is  $\text{GL}(2n + 1, \mathbb{Q})$ -conjugate to  $A'^s$  for some  $r, s \in \mathbb{Z} \setminus \{0\}$ .

*Proof.* Assume  $A \neq \text{Id}$  otherwise the proposition is trivial. For the first claim let us notice that  $\mathbb{Z}^{2n+1}$  is a maximal abelian subgroup of  $\tilde{\Gamma}(2n + 2, A)$ . Indeed if  $(r, u) \in \tilde{\Gamma}(2n + 2, A)$ , with  $r \neq 0$ , commutes with  $(0, u')$  for all  $u' \in \mathbb{Z}^{2n+1}$  then  $(A^r - \text{Id})u' = 0$  but then  $A$  would be nilpotent and being diagonalisable it is the identity which is a contradiction. Hence we can write an isomorphism between  $\tilde{\Gamma}(2n + 2, A)$  and  $\tilde{\Gamma}(2n + 2, A')$  as  $\varphi(r, u) = (\varphi_1(r), \varphi_2^1(r) + \varphi_2^2(u))$ . Then in order for  $\varphi$  to be a group homomorphism we need

$$\varphi_2^1(r + r') = A'^{\varphi_1(r)} \varphi_2^1(r') + \varphi_2^1(r) \text{ and } \varphi_2^2(A^r u' + u) = A'^{\varphi_1(r)} \varphi_2^2(u') + \varphi_2^2(u).$$

Taking then  $u = 0$  we get  $\varphi_2^2(A^r u') = A'^{\varphi_1(r)} \varphi_2^2(u')$ . Since  $\varphi_2^2 \in \text{GL}(2n + 1, \mathbb{Z})$  and, being  $\varphi_1 : \mathbb{Z} \rightarrow \mathbb{Z}$  an isomorphism,  $\varphi_1$  sends  $1$  to  $\pm 1$ , this means that  $A$  is  $\text{GL}(2n + 1, \mathbb{Z})$ -conjugate to  $A'$  or  $A'^{-1}$ .

For the second claim just notice that a finite index subgroup  $\Gamma_0$  of  $\tilde{\Gamma}(2n + 2, A)$  is of the form  $\mathbb{Z} \rtimes_B \mathbb{Z}^{2n+1}$  for some  $B \in \text{GL}(2n + 1, \mathbb{Z})$ . In order to see it as a finite index subgroup of  $\tilde{\Gamma}(2n + 2, A)$  we have to give an injective morphism  $\varphi : \mathbb{Z} \rtimes_B \mathbb{Z}^{2n+1} \rightarrow \tilde{\Gamma}(2n + 2, A)$ . Using the same notation as before this means that  $\varphi_1(r) = mr$  with  $m \in \mathbb{Z} \setminus \{0\}$  and if we denote by  $(v_i)_i$  a basis for  $\mathbb{Z}^{2n+1}$  we have  $\varphi_2^2(v_i) = m_i v_i$  i.e.  $\varphi_2^2 \in \text{GL}(2n + 1, \mathbb{Q})$ . As before being  $\varphi$  a morphism we have  $\varphi_2^2(B^r u') = A^{mr} \varphi_2^2(u')$ . That means that  $B$  is  $\text{GL}(2n + 1, \mathbb{Q})$ -conjugate to  $A^m$  for some  $m \in \mathbb{Z} \setminus \{0\}$ . Apply then the first part of the lemma.  $\square$

**Proposition 2.2.11.** Let  $n \geq 1$  then we have the following.

- If  $\Gamma \leq \mathcal{H}(n, 1)$  is a crystallographic group and  $\Gamma$  is not virtually nilpotent then  $\Gamma$  contains a subgroup of finite index  $\Gamma_0$  that is isomorphic to  $\tilde{\Gamma}(2n+2, A)$  for some  $A \in \text{GL}(2n+1, \mathbb{Z})$  as in Definition 2.2.9.
- Every  $\tilde{\Gamma}(2n+2, A)$  as in Definition 2.2.9 can be realised as a crystallographic group in  $\mathcal{H}(n, 1)$ .

*Proof.* The first claim follows from the proof of Theorem 2.2.7. Indeed we are in the case where  $\Gamma$  is a lattice in  $H$  with  $L(H) \leq \mathcal{D}$ . Hence

$$H = \left\{ \begin{pmatrix} \bar{\lambda}(r, u, s) & 0 & 0 & r \\ 0 & \sigma(r, u, s) & 0 & u \\ 0 & 0 & \lambda^{-1}(r, u, s) & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid (r, u, s) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} \right\}$$

and  $\lambda : \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} \rightarrow \mathbb{C}^*$ ,  $\sigma = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}) : \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} \rightarrow U(n-1)$ . The nilradical of  $H$  is then contained in the subgroup of translations of  $H$  and the action of  $\Gamma$  leaves invariant the subspace,  $F$ , associated to it, and hence its orthogonal direction. We will prove that either  $\lambda$  is trivial or  $F^\perp$  is spacelike. So that either  $\Gamma$  is abelian or the action of  $\Gamma/\Gamma \cap N$  on  $\Gamma \cap N$  is via Hermite-Lorentz transformations. In order to do this we will prove that either  $\lambda$  is trivial or  $\lambda$  and  $\sigma$  do not depend on  $r$  and  $s$ . Consider an element of the Lie algebra of  $H$ ,  $\mathfrak{h}$ ,

$$(r, u, s) = \begin{pmatrix} \bar{\tau}(r, u, s) & 0 & 0 & r \\ 0 & \rho(r, u, s) & 0 & u \\ 0 & 0 & -\tau(r, u, s) & s \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where  $\tau : \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} \rightarrow \mathbb{C}$ ,  $\rho = \text{diag}(i\theta_1, \dots, i\theta_{n-1}) : \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} \rightarrow \mathfrak{u}(n-1)$ . Computing the commutator we have

$$[(r, u, s), (r', u', s')] = \begin{pmatrix} 0 & 0 & 0 & \bar{\tau}(r, u, s)r' - \bar{\tau}(r', u', s')r \\ 0 & 0 & 0 & \rho(r, u, s)u' - \rho(r', u', s')u \\ 0 & 0 & 0 & -\tau(r, u, s)s' + \tau(r', u', s')s \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the fact that  $\mathfrak{h}$  is a Lie algebra we get the following relations

$$\bar{\tau}(\bar{\tau}(r, 0, 0)r' - \bar{\tau}(r', 0, 0)r, 0, 0) = 0, \quad (2.2)$$

$$\bar{\tau}(\bar{\tau}(0, u, 0)r', -\rho(r', 0, s')u, -\tau(0, u, 0)s') = 0. \quad (2.3)$$

Then, from equation (2.2), either  $\bar{\tau}(x, 0, 0) = 0$  for every  $x \in \mathbb{C}$  or  $\bar{\tau}(r, 0, 0)r' - \bar{\tau}(r', 0, 0)r = 0$ , which implies  $\bar{\tau}(r, 0, 0) = \bar{\tau}(1, 0, 0)r$  for all  $r \in \mathbb{C}$ . In this second case let us consider the following element

that is the product of two elements in  $H$

$$\begin{pmatrix} \bar{\lambda}(r, u, s)\bar{\lambda}(r', u', s') & 0 & 0 & \bar{\lambda}(r, u, s)r' + r \\ 0 & \sigma(r, u, s)\sigma(r', u', s') & 0 & \sigma(r, u, s)u' + u \\ 0 & 0 & \lambda^{-1}(r, u, s) & \lambda^{-1}(r, u, s)s' + s \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The fact that  $H$  is a group implies the following relations

$$\bar{\lambda}(\bar{\lambda}(r, 0, s)r' + r, 0, \lambda^{-1}(r, 0, s)s' + s) = \bar{\lambda}(r, 0, s)\bar{\lambda}(r', 0, s'), \quad (2.4)$$

$$\bar{\lambda}(r, u, s) = \bar{\lambda}(0, u, 0)\bar{\lambda}(\lambda^{-1}(0, u, 0)r, 0, \lambda(0, u, 0)s). \quad (2.5)$$

Since  $\bar{\tau}(r, 0, 0) = \bar{\tau}(1, 0, 0)r$  we have  $\lambda(r, 0, 0) = e^{\tau(1, 0, 0)r} = a^r$  for some  $a \in \mathbb{C}$ . Hence taking equation (2.4) for  $s = 0$  we get  $a^{r'r'+r} = a^r a^{r'}$  that implies that  $a^r = 1$ . Hence we have proved that  $\lambda(r, 0, 0) = 1$ . Similarly one can prove that  $\lambda(0, 0, s) = 1$ . Now, again from equation (2.4) with  $s = 0$ , we have  $\bar{\lambda}(r' + r, 0, s') = \bar{\lambda}(r', 0, s')$  which implies that  $\bar{\lambda}(r, 0, s) = 1$  for all  $r, s \in \mathbb{C}$ . From equation (2.5) we see that this implies that  $\lambda(r, u, s) = \lambda(0, u, 0)$  for all  $u \in \mathbb{C}^{n-1}$ . That is  $\lambda$  does not depend on  $r$  and  $s$ . Finally from equation (2.3), unless  $\lambda$  is trivial, we see that also  $\rho(r, 0, s) = 0$ , that is  $\sigma$  does not depend on  $r$  and  $s$ .

For the second claim we can just realise the group  $\tilde{\Gamma}(2n+2, A)$  as an affine group as follows

$$\left\{ \begin{pmatrix} A^{u_1} & 0 & v \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix} \mid v = (r_1, r_2, u_2, \dots, u_{2n-2}, s_1, s_2) \in \mathbb{Z}^2 \times \mathbb{Z}^{2n-3} \times \mathbb{Z}^2, u_1 \in \mathbb{Z} \right\}.$$

Then since  $A$  is diagonalisable with eigenvalues 1 and  $\{\lambda, \lambda^{-1}, a_i\}$  and their complex conjugates with  $\lambda, \lambda^{-1} \in \mathbb{C}^*$  and  $|a_i| = 1$  this means that we can extract a matrix from  $A$  that can be conjugated to a matrix belonging to  $U(n, 1)$  and hence conjugate the whole group to a subgroup of  $\mathcal{H}(n, 1)$ .  $\square$

### 2.3 Classification up to isomorphism, dimension 2 and 3, and degenerate cases in dimension 4

In this section and the next one we will address the problem of classification of the groups  $U(\gamma_2, \gamma_3, b_2, b_3)$  up to isomorphism. This classification translates to the classification of the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  up to isomorphism. At the beginning we will study the isomorphism classes in dimension 2 and 3. Still in this section we will study the isomorphism classes of some degenerate cases in dimension 4. Namely when  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$  or when  $\gamma_2 = 0$ . In the following section we will study the case  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$  and  $\gamma_2 \neq 0$ . Eventually we will prove the following.

**Theorem 2.3.1.** *The list given in Appendix A with  $\mathbb{K} = \mathbb{R}$  is a complete non-redundant list of isomorphism classes of unipotent simply transitive subgroups of  $U(3, 1) \ltimes \mathbb{C}^{3+1}$ .*

For the Lie algebras that appear in dimension 2 and 3 we will use the terminology of de Graaf in [20]. All the Lie algebras that appear in this section are defined in Appendix A where we also elucidate the correspondence with the terminology of [20] for isomorphism classes of nilpotent Lie algebras up to dimension 6 and of Gong, [30], for dimension 7. The notation used in [20] for the list of Lie algebras up to isomorphism is  $L_{i,j}$  where  $i$  denotes the dimension of the Lie algebra and  $j$  is just an index. While in [30] each Lie algebra is labelled by its upper central series dimensions plus an additional letter to distinguish non-isomorphic Lie algebras. Let us start with the classification in small dimensions. Recall that the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are abstractly described as  $\mathbb{C} \oplus W \oplus \mathbb{C}$ , with  $W = \mathbb{C}^{n-1}$ , with Lie brackets

$$[(r, u, s), (r', u', s')] = (h(u, \gamma(u', s')) - h(u', \gamma(u, s)) + i(b(u, s)s' - b(u', s')s), s'\gamma(u, s) - s\gamma(u', s'), 0)$$

and with  $\gamma: W \oplus \mathbb{C} \rightarrow W$  and  $b: W \oplus \mathbb{C} \rightarrow \mathbb{R}$  linear maps,  $\gamma = \gamma_2 + \gamma_3$  and  $b = b_2 + b_3$  satisfying the conditions of Proposition 2.1.5.

**Proposition 2.3.2.** *For  $n + 1 = 2$  the Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  is isomorphic to either  $\mathbb{R}^4$  or  $L_{3,2} \oplus \mathbb{R}$ .*

*Proof.* With respect to the basis  $(\tau, i\tau, \xi, i\xi)$  of  $\mathbb{C} \oplus \mathbb{C}$  the Lie brackets are given by  $[\xi, i\xi] = -b_3(\xi)\tau - b_3(i\xi)i\tau$ . And we can see that if  $b_3 \neq 0$  this Lie algebra is isomorphic to a Lie algebra that is the direct sum of the 3-dimensional Heisenberg Lie algebra,  $L_{3,2}$ , and a one-dimensional abelian ideal.  $\square$

**Proposition 2.3.3.** *For  $n + 1 = 3$  the Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  is isomorphic to either  $\mathbb{R}^6, L_{3,2} \oplus \mathbb{R}^3, L_{5,8} \oplus \mathbb{R}, L_{6,23}, L_{6,24}(\varepsilon)$  with  $\varepsilon \in \{0, \pm 1\}, L_{6,25}$  or  $L_{6,19}(0)$ .*

*Proof.* Notice that from Lemma 2.1.6 we have  $\gamma_2 = 0$ . Let  $(\tau, i\tau, g_1, g_2 = ig_1, \xi, i\xi)$  be a basis of the Lie algebra. Let us denote by  $w_0 := \gamma_3(i\xi) - J\gamma_3(\xi)$ .

Assume first  $w_0 = 0$ . Remember that from Proposition 2.1.8 we know that this implies that  $\gamma_3 = 0$ . Let furthermore  $b_2 = 0$ . Then the Lie algebra is either abelian if  $b_3 = 0$  or isomorphic to  $L_{3,2} \oplus \mathbb{R}^3$ . Let instead  $b_2 \neq 0$ . Then, using Proposition 2.1.13, we can conjugate the group in order to replace  $b_2$  by  $b'_2(u) = |\lambda|^2 b_2(\sigma^{-1}u)$  with  $\sigma \in U(1)$  and  $\lambda \in \mathbb{C}^*$  so that  $b_2(u) = \omega(u, g_1)$  for  $u \in W$ . Furthermore we can modify  $b_3$  by  $b'_3(s) = b_3(s) - b_2(ss)$  for some  $x \in \mathbb{C}$ , hence we may assume  $b_3 = 0$ . Hence the only non-zero brackets are  $[g_2, \xi] = i\tau$  and  $[g_2, i\xi] = -\tau$ , i.e. the Lie algebra is isomorphic to  $L_{5,8} \oplus \mathbb{R}$ .

Assume now  $w_0 \neq 0$ .

Assume that the real rank of  $\gamma_3$  is 1 and  $b_2 = 0$ . Then we might change  $\gamma_3$  by  $\gamma'_3(s) = \lambda \sigma \gamma_3(\lambda s)$  with  $\lambda \in \mathbb{C}^*$  and  $\sigma \in U(1)$  so that  $\gamma_3(\xi) = \varepsilon g_1$ , with  $\varepsilon \in \mathbb{R}$ , and  $\gamma_3(i\xi) = 0$ , notice that since

$\gamma_3 \neq 0$  we have  $\varepsilon \neq 0$ . Then the non-zero brackets are  $[g_1, \xi] = \varepsilon\tau$ ,  $[g_2, \xi] = \varepsilon i\tau$  and  $[\xi, i\xi] = -b_3(\xi)\tau - b_3(i\xi)i\tau - \varepsilon g_2$ . We can then see that in this case the Lie algebra is isomorphic to  $L_{6,25}$ .

Assume that the real rank of  $\gamma_3$  is 1 and  $b_2 \neq 0$ . As before we conjugate the group so that  $\gamma_3(\xi) = \varepsilon g_1$  and  $\gamma_3(i\xi) = 0$  with  $\varepsilon \neq 0$ . Since  $b_2(w_0) = 2\omega(\gamma_3(i\xi), \gamma_3(\xi)) = 0$ , we have that  $b_2(g_2) = 0$  and since  $b_2 \neq 0$  then  $b_2(g_1) \neq 0$ . Hence the Lie brackets are  $[g_1, \xi] = \varepsilon\tau + b_2(g_1)i\tau$ ,  $[g_2, \xi] = \varepsilon i\tau$ ,  $[g_1, i\xi] = -b_2(g_1)\tau$ ,  $[\xi, i\xi] = -b_3(\xi)\tau - b_3(i\xi)i\tau - \varepsilon g_2$ . It is easy to see that in this case the Lie algebra is isomorphic to  $L_{6,19}(0)$ .

Assume that the real rank of  $\gamma_3$  is 2. Notice that nevertheless  $\gamma_3(\xi)$  and  $\gamma_3(i\xi)$  are linearly dependent over  $\mathbb{C}$  hence  $\gamma_3(i\xi) = \bar{\lambda}\gamma_3(\xi)$  with  $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ . Let  $v_1 = \frac{\gamma_3(\xi)}{\|\gamma_3(\xi)\|^2} \in W$  so that  $h(v_1, \gamma_3(\xi)) = 1$  and  $h(v_1, \gamma_3(i\xi)) = \lambda$ . Let  $\alpha = \|\lambda_3(\xi)\|^2$  and  $(x, y)$  be a basis for  $W = \mathbb{C}$  as a real vector space with  $x = v_1$  and  $y = iv_1$ . We have  $w_0 = \alpha\lambda_1 x - \alpha(\lambda_2 + 1)y$ , then the Lie brackets are

$$\begin{aligned} [x, \xi] &= \tau + b_2(x)i\tau, & [y, \xi] &= (1 + b_2(y))i\tau, \\ [x, i\xi] &= (\lambda_1 - b_2(x))\tau + \lambda_2 i\tau, & [y, i\xi] &= (-\lambda_2 - b_2(y))\tau + \lambda_1 i\tau, \\ [\xi, i\xi] &= -b_3(\xi)\tau - b_3(i\xi)i\tau - w_0. \end{aligned}$$

Remember that  $b_2(w_0) = 2\omega(\gamma_3(i\xi), \gamma_3(\xi)) = -2\alpha\lambda_2$ . Hence we have

$$\begin{aligned} [w_0, \xi] &= \alpha\lambda_1\tau - \alpha(1 + 3\lambda_2)i\tau, \\ [w_0, i\xi] &= \alpha(\lambda_1^2 + \lambda_2^2 + 3\lambda_2)\tau - \alpha\lambda_1 i\tau. \end{aligned}$$

Now if the transformation  $\tau_1 = \alpha\lambda_1\tau - \alpha(1 + 3\lambda_2)i\tau$ ,  $\tau_2 = \alpha(\lambda_1^2 + \lambda_2^2 + 3\lambda_2)\tau - \alpha\lambda_1 i\tau$  has non-zero determinant, taking  $x_1 = \xi$ ,  $x_2 = i\xi$ ,  $x_3 = -b_3(\xi)\tau - b_3(i\xi)i\tau - w_0$ ,  $x_4 = y$ ,  $x_5 = \alpha\lambda_1\tau - \alpha(1 + 3\lambda_2)i\tau$  and  $x_6 = \alpha(\lambda_1^2 + \lambda_2^2 + 3\lambda_2)\tau - \alpha\lambda_1 i\tau$ , we can bring the Lie algebra to the following form

$$\begin{aligned} [x_1, x_2] &= x_3, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_4] &= ax_5 + bx_6, & [x_2, x_4] &= cx_5 + dx_6. \end{aligned}$$

Assuming  $c \neq 0$  and making the following change of variables  $x'_1 = x_1 + \frac{d-a}{2c}x_2$ ,  $x'_2 = x_2$ ,  $x'_3 = x_3$ ,  $x'_4 = \frac{1}{c}x_4 - \frac{d+a}{2c}x_3$ ,  $x'_5 = x_5 + \frac{d-a}{2c}x_6$  and  $x'_6 = x_6$  we see the isomorphism with one of the  $L_{6,24}(\varepsilon)$ . If  $c = 0$  in a similar way we can arrive to an isomorphism with  $L_{6,24}(\varepsilon)$ . If instead the determinant is 0 we can bring it to the standard form  $L_{6,23}$  or one of  $L_{6,25}$  or  $L_{6,19}(0)$  that we already encountered.  $\square$

Now we want to give representatives of the isomorphism classes of the groups  $U(\gamma_2, \gamma_3, b_2, b_3)$  in dimension 4 when either  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$  or  $\text{rank}(\gamma_2) = 0$ . The dimension 4 is the smallest dimension where  $\gamma_2$  can be non-trivial.

**Proposition 2.3.4.** *The list of Lie algebras  $L_j$  for  $j = 1, \dots, 5$  and  $L_6(1)$  given in Appendix A with  $\mathbb{K} = \mathbb{R}$  is a complete non-redundant list of isomorphism classes of the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  when  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$  and  $n + 1 = 4$ .*

*Proof.* Let us call  $w_0 := \gamma_3(i\xi) - J\gamma_3(\xi)$ . Since  $\pi(w_0) = 0$  we saw in Proposition 2.1.8 that  $\text{Im } \gamma_3 \subseteq \text{Im } \gamma_2$  and  $b_2 = 0$  on  $\text{Im } \gamma_2 \oplus J\text{Im } \gamma_2$ . If  $\gamma_2 = 0$  the result follows easily and we find the Lie algebras  $L_i$  with  $i = 1, 2, 3$  which are 2-step nilpotent. Hence let us suppose  $\text{rank}(\gamma_2) = 1$  and let  $(\tau, i\tau, e, Je, g, f, \xi, i\xi)$  be a basis for  $\mathbb{C} \oplus W \oplus \mathbb{C}$  that respects the decomposition (2.1). Then  $\gamma_2(f) = \delta e$ ,  $\gamma_3(\xi) = v e$  and  $\gamma_3(i\xi) = \mu e$  with  $\delta \in \mathbb{R}^*$  and  $v, \mu \in \mathbb{R}$  and  $w_0 = \mu e - vJe$ . If  $f' = \delta^{-1}f$  the Lie brackets become

$$\begin{aligned} [e, f'] &= \tau, & [e, \xi] &= v\tau, & [e, i\xi] &= \mu\tau, \\ [Je, f'] &= i\tau, & [Je, \xi] &= vi\tau, & [Je, i\xi] &= \mu i\tau, \\ [g, \xi] &= b_2(g)i\tau, & [g, i\xi] &= -b_2(g)\tau, \\ [f', \xi] &= b_2(f')i\tau + e, & [f', i\xi] &= -b_2(f')\tau + Je, \\ [\xi, i\xi] &= -b_3(\xi)\tau - b_3(i\xi)i\tau - w_0. \end{aligned}$$

After replacing  $e$  by  $b_2(f')i\tau + e$  and  $Je$  by  $-b_2(f')\tau + Je$  we can assume that  $[f', \xi] = e$  and  $[f', i\xi] = Je$ . Furthermore, letting  $\xi_1 = \xi - v f'$  and  $\xi_2 = i\xi - \mu f'$ , we have  $[e, \xi_1] = [e, \xi_2] = [Je, \xi_1] = [Je, \xi_2] = 0$  and  $[\xi_1, \xi_2] = b\tau + ci\tau$  for some  $b, c \in \mathbb{R}$ . Now, if  $b_2(g) = 0$  and  $b = c = 0$ , then we can find an isomorphism between the Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  and  $L_4$ . If instead  $b_2(g) = 0$  but  $b \neq 0$  then defining  $x_1 = f', x_2 = -(b\xi_1 + c\xi_2), x_3 = -b^{-1}\xi_2, x_4 = -(be + cJe), x_5 = -b^{-1}Je, x_6 = b\tau + ci\tau, x_7 = b^{-1}i\tau$  we see that the Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  is isomorphic to  $L_5$ . Finally, assuming  $b_2(g) \neq 0$ , we can define  $x_1 = f', x_2 = \xi + \frac{b}{b_2(g)}g, x_3 = i\xi + \frac{c}{b_2(g)}g, x_4 = e, x_5 = Je, x_6 = \frac{1}{b_2(g)}g, x_7 = -\tau$  and  $x_8 = -i\tau$  in order to see that  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  is isomorphic to  $L_6(1)$ .  $\square$

Next we give the classification in dimension 4 for the case  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$  and  $\gamma_2 = 0$ .

**Proposition 2.3.5.** *The list of Lie algebras  $N_j$  for  $j = 1, \dots, 13$  given in Appendix A with  $\mathbb{K} = \mathbb{R}$  is a complete non-redundant list of isomorphism classes of the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  when  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$ ,  $\gamma_2 = 0$  and  $n + 1 = 4$ .*

*Proof.* Let  $(\tau, i\tau, g_1, g_2, g_3, g_4, \xi, i\xi)$  be a basis for the real vector space  $\mathbb{C} \oplus W \oplus \mathbb{C}$ . Let us call  $w_0 := \gamma_3(i\xi) - J\gamma_3(\xi)$  then the Lie brackets of  $\mathfrak{u}(0, \gamma_3, b_2, b_3)$  read as

$$\begin{aligned} [g_j, \xi] &= \langle g_j, \gamma_3(\xi) \rangle \tau + (\omega(g_j, \gamma_3(\xi)) + b_2(g_j)) i\tau, \\ [g_j, i\xi] &= (\langle g_j, \gamma_3(i\xi) \rangle - b_2(g_j)) \tau + \omega(g_j, \gamma_3(i\xi)) i\tau, \\ [\xi, i\xi] &= -b_3(\xi)\tau - b_3(i\xi)i\tau - w_0. \end{aligned}$$

Moreover

$$\begin{aligned} [w_0, \xi] &= \langle \gamma_3(i\xi), \gamma_3(\xi) \rangle \tau + (3\omega(\gamma_3(i\xi), \gamma_3(\xi)) - \|\gamma_3(\xi)\|^2) i\tau \\ [w_0, i\xi] &= (\|\gamma_3(i\xi)\|^2 - 3\omega(\gamma_3(i\xi), \gamma_3(\xi))) \tau - \langle \gamma_3(i\xi), \gamma_3(\xi) \rangle i\tau. \end{aligned}$$

Let

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with  $a = \langle \gamma_3(i\xi), \gamma_3(\xi) \rangle$ ,  $b = (3\omega(\gamma_3(i\xi), \gamma_3(\xi)) - \|\gamma_3(\xi)\|^2)$  and  $c = (\|\gamma_3(i\xi)\|^2 - 3\omega(\gamma_3(i\xi), \gamma_3(\xi)))$ . First of all by redefining  $w_0$  we might assume that  $[\xi, i\xi] = w_0$ . Let  $v_0 \in W$  be a vector such that  $b_2(u) = \langle u, v_0 \rangle$ . Then we can notice that the restrictions of  $\text{ad}(\xi)$  and  $\text{ad}(i\xi)$  to  $W$  define two linear maps  $W \rightarrow \mathbb{C}$  and that taking their real and imaginary part we get four linear forms over the reals. Call these linear forms  $\alpha_i$ , explicitly  $\alpha_1 = \langle \cdot, \gamma_3(\xi) \rangle$ ,  $\alpha_2 = \langle \cdot, J\gamma_3(\xi) + v_0 \rangle$ ,  $\alpha_3 = \langle \cdot, \gamma_3(i\xi) - v_0 \rangle$ ,  $\alpha_4 = \langle \cdot, J\gamma_3(i\xi) \rangle$ . For  $i = 1, \dots, 4$  let us denote by  $v_i$  the vector associated with the linear form  $\alpha_i$ . Notice that, if  $\mathcal{Z}$  is the center of  $\mathfrak{u}(0, \gamma_3, b_2, b_3)$ , the center of  $\mathfrak{u}(0, \gamma_3, b_2, b_3)/\mathbb{C}\tau$  is  $W$ , hence  $\dim(W \cap \mathcal{Z})$  is an invariant and we have

$$\dim(\mathcal{Z} \cap W) = \dim\left(\bigcap_{i=1}^4 \ker \alpha_i\right) = \dim\left(\text{span}\{v_1, \dots, v_4\}^\perp\right).$$

**Claim 2.3.5.1.** If  $w_0 \in \mathcal{Z} \cap W$  then  $\dim(\text{span}\{v_1, \dots, v_4\}) \geq 3$ . If instead  $w_0 \notin \mathcal{Z} \cap W$  then we have  $\dim(\text{span}\{v_1, \dots, v_4\}) \geq 2$ . Hence in any case  $0 \leq \dim(\mathcal{Z} \cap W) \leq 2$ .

*Proof.* Notice first that since  $w_0 \neq 0$ ,  $\gamma_3(\xi)$  and  $\gamma_3(i\xi)$  cannot be both zero. Furthermore if one of them is zero, say  $\gamma_3(\xi)$ , then  $w_0 \notin \mathcal{Z} \cap W$  and  $\text{span}\{v_1, \dots, v_4\} = \text{span}\{v_0, \gamma_3(i\xi), J\gamma_3(i\xi)\}$  has dimension at least two hence the claim follows. Assume now that both  $\gamma_3(\xi)$  and  $\gamma_3(i\xi)$  are non-zero. Assume by contradiction that  $w_0 \notin \mathcal{Z} \cap W$  and  $\dim(\text{span}\{v_1, \dots, v_4\}) = 1$ . Since  $\gamma_3(\xi) \neq 0$  we can take it as generator of  $\text{span}\{v_1, \dots, v_4\}$ . But  $w_0 = \gamma_3(i\xi) - J\gamma_3(\xi) = -J(v_4 + v_1) = -\lambda J\gamma_3(\xi)$  with  $\lambda \in \mathbb{R}$ . Hence  $w_0$  is orthogonal to all of  $\text{span}\{v_1, \dots, v_4\}$  and hence  $w_0 \in \mathcal{Z} \cap W$  and we get a contradiction. Assume now that  $w_0 \in \mathcal{Z} \cap W$  then we claim that  $\gamma_3(\xi)$  and  $\gamma_3(i\xi)$  are linearly independent over  $\mathbb{C}$ . Indeed suppose that  $\gamma_3(i\xi) = \lambda \gamma_3(\xi)$  with  $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ . Since  $w_0$  is in the center, we must have  $\langle \gamma_3(i\xi), \gamma_3(\xi) \rangle = 0$  hence  $\lambda_1 = 0$ ,  $3\omega(\gamma_3(i\xi), \gamma_3(\xi)) - \|\gamma_3(\xi)\|^2 = 0$  hence  $3\lambda_2 - 1 = 0$ , finally  $\|\gamma_3(i\xi)\|^2 - 3\omega(\gamma_3(i\xi), \gamma_3(\xi)) = 0$  hence  $\lambda_2^2 - 3\lambda_2 = 0$  and we get a contradiction. Then  $\{\gamma_3(\xi), J\gamma_3(\xi) + \gamma_3(i\xi), J\gamma_3(i\xi)\}$  are linearly independent hence  $\dim(\text{span}\{v_1, \dots, v_4\}) \geq 3$ .  $\square$

Let us first assume  $\dim(\text{span}\{v_1, \dots, v_4\}) = 4$ , let  $(z_1, \dots, z_4)$  be the dual basis of  $(\alpha_1, \dots, \alpha_4)$  then

$$[z_1, \xi] = \tau, \quad [z_2, \xi] = i\tau, \quad [z_3, i\xi] = \tau, \quad [z_4, i\xi] = i\tau, \quad [\xi, i\xi] = w_0.$$

If  $w_0 = \sum a_i z_i$  then  $[[\xi, i\xi], \xi] = a_1 \tau + a_2 i\tau$  and  $[[\xi, i\xi], i\xi] = a_3 \tau + a_4 i\tau$ . Let us write the structure constants in a matrix

$$A' = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

that is conjugated to  $A$ , and let us represent a change of basis in  $\text{span}\{\xi, i\xi\}$  with  $P \in \text{GL}(2, \mathbb{R})$ . Then the matrix that represents the structure constants in the new basis is just  $\det(P)PA'$ . Hence, depending on the rank of the matrix  $A'$ , using the left action of  $\text{GL}(2, \mathbb{R})$  just defined, we can bring  $A'$  to one of the three normal forms  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . The first matrix, if we call  $z'_1 = z_2$ , leads to the following Lie brackets

$$\begin{aligned} [\xi, i\xi] &= z'_1, & [z'_1, \xi] &= \tau, \\ [z_2, \xi] &= i\tau, & [z_3, i\xi] &= \tau, & [z_4, i\xi] &= i\tau \end{aligned}$$

and one can easily see that this Lie algebra is isomorphic to  $N_{10}$ . From the second normal, after calling  $z'_1 = z_1 + z_4$ , we have the following Lie brackets

$$\begin{aligned} [\xi, i\xi] &= z'_1, & [z'_1, \xi] &= \tau, & [z'_1, i\xi] &= i\tau, \\ [z_2, \xi] &= i\tau, & [z_3, i\xi] &= \tau, & [z_4, i\xi] &= i\tau \end{aligned}$$

and this Lie algebra is isomorphic to the Lie algebra  $N_{11}$ .

Let us now assume that  $\dim(\text{span}\{v_1, \dots, v_4\}) = 3$ . If  $w_0 \in \mathcal{Z} \cap W$  suppose then that  $\{v_1, v_2, v_3\}$  are linearly independent, indeed the other combinations can be reduced to this case by a change of variables. Complete  $\{\alpha_1, \alpha_2, \alpha_3\}$  to  $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$  in order to have a basis for  $W^*$ . Let us take its dual basis, call it  $(z_1, \dots, z_4)$ , as basis for  $W$ . The Lie brackets expressed in this basis become

$$\begin{aligned} [z_1, \xi] &= \tau, & [z_1, i\xi] &= \alpha_4(z_1)i\tau, & [z_2, \xi] &= i\tau, & [z_2, i\xi] &= \alpha_4(z_2)i\tau, \\ [z_3, i\xi] &= \tau + \alpha_4(z_3)i\tau, & [\xi, i\xi] &= w_0 = z_4. \end{aligned}$$

After a change of basis we can arrive to the following form

$$[z_1, \xi_1] = \tau_1, \quad [z_2, \xi_1] = \tau_2, \quad [z_1, \xi_2] = \mu \tau_2, \quad [z_3, \xi_2] = \tau_1, \quad [\xi_1, \xi_2] = z_4$$

with  $\mu = \alpha_4(z_1) + \alpha_4(z_2)\alpha_4(z_3)$  and, depending on whether  $\mu$  is 0 or not, we find the normal forms  $N_{12}$  or  $N_{13}$ . If instead  $w_0 \notin \mathcal{Z} \cap W$  there exists  $x \in \mathcal{Z} \cap W$  such that  $\mathfrak{u}(0, \gamma_3, b_2, b_3) \cong \mathbb{R}x \oplus \mathfrak{g}'$  where  $\mathfrak{g}'$  is a 7-dimensional 3-nilpotent Lie algebra with a 2-dimensional center. Notice if  $\mathfrak{g}'$  decomposes as the sum of smaller Lie algebras then one of them must be abelian, but this would imply  $\dim(\mathcal{Z} \cap W) > 1$ , hence  $\mathfrak{g}'$  is indecomposable. Hence  $\mathfrak{g}'$  can be one of  $(257\delta)$  with  $\delta \in \{A, \dots, L\}$ , or some real form of them, in Gong's list. Some of them can be excluded since we know that  $W \oplus \mathbb{C}\tau$  is a 6-dimensional abelian subalgebra, so we get that the only possibilities are  $(257\delta)$  with  $\delta \in \{A, B, C, D, I, J, J_1\}$ .

Hence in this case  $u(0, \gamma_3, b_2, b_3)$  is isomorphic to one of the  $N_j$  with  $j = 5, \dots, 8$  or  $N_9(\varepsilon)$  with  $\varepsilon \in \{0, 1, -1\}$ .

Let us now assume that  $\dim(\text{span}\{v_1, \dots, v_4\}) = 2$ . It follows from Claim 2.3.5.1 that then  $w_0 \notin \mathcal{L} \cap W$ . Then there exist  $x, y \in \mathcal{L} \cap W$  such that  $u(0, \gamma_3, b_2, b_3) \cong \text{span}\{x, y\} \oplus \mathfrak{g}'$  where  $\mathfrak{g}'$  is a 6-dimensional 3-nilpotent Lie algebra with a center of dimension 2, hence one of  $L_{6,23}, L_{6,24}(\varepsilon), L_{6,25}, L_{6,27}$  or  $L_{5,5} \oplus \mathbb{R}$  in de Graaf's list. One can see that all of these Lie algebras, except  $L_{5,5} \oplus \mathbb{R}$ , appeared already in dimension 6 hence they can be realised. Instead let us see that  $L_{5,5} \oplus \mathbb{R}^3$  does not appear as Hermite-Lorentz Lie algebra. Indeed this Lie algebra is characterised by the fact that  $\dim(\text{span}\{v_1, \dots, v_4\}) = 2$  and that the dimension of  $\text{ad}(w_0)$  is 1. Notice that if  $v_1 - v_4 = 0$  then  $\gamma_3(\xi) = J\gamma_3(i\xi)$  but then the dimension of  $\text{ad}(w_0)$  being 1 would imply  $\gamma_3(i\xi) = 0$  and hence  $w_0 = 0$ . Hence  $v_1 - v_4 \neq 0$  and since  $v_2 + v_3 = J(v_1 - v_4)$  this implies that  $\dim \text{span}\{v_1 - v_4, v_2 + v_3\} = 2$  and we can take  $\{v_1 - v_4, v_2 + v_3\}$  as generators of  $\{v_1, \dots, v_4\}$ . Then we can write  $v_1$  as a linear combination of  $\{v_1 - v_4, v_2 + v_3\}$ , i.e.  $\gamma_3(\xi) = \lambda(\gamma_3(\xi) - J\gamma_3(i\xi)) + \mu(J\gamma_3(\xi) + \gamma_3(i\xi))$  for some  $\lambda, \mu \in \mathbb{R}$ . Hence  $\gamma_3(\xi)$  and  $\gamma_3(i\xi)$  are linearly dependent over  $\mathbb{C}$ . Hence  $\text{span}\{v_2 + v_3, v_1 - v_4\} = \text{span}\{\gamma_3(\xi), J\gamma_3(\xi)\}$ . Since  $v_2 = J\gamma_3(\xi) + v_0 \in \text{span}\{v_1 - v_4, v_2 + v_3\} = \text{span}\{\gamma_3(\xi), J\gamma_3(\xi)\}$  this implied that also  $v_0 \in \text{span}\{\gamma_3(\xi), J\gamma_3(\xi)\}$ . Then everything is contained in the plane  $\text{span}\{\gamma_3(\xi), J\gamma_3(\xi)\}$  and hence the Lie algebra  $L_{5,5} \oplus \mathbb{R}$  should have appeared in dimension 6. Hence  $u(0, \gamma_3, b_2, b_3)$  is isomorphic to one of  $N_j$  with  $j = 1, \dots, 4$ .

**Claim 2.3.5.2.** The Lie algebras  $N_5, \dots, N_{13}$  can all be realised.

*Proof.* We will give explicit realisations for each Lie algebra  $N_j$  for  $j = 5, \dots, 13$ . We refer to the setting of the beginning of the proof, let  $(\tau, i\tau, g_1, g_2, g_3, g_4, \xi, i\xi)$  be a basis for  $\mathbb{C} \oplus W \oplus \mathbb{C}$  with  $(g_1, \dots, g_4)$  orthonormal with respect to  $h|_W$  and such that  $Jg_1 = g_2$  and  $Jg_3 = g_4$ . Then we write  $\gamma_3(\xi) = \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 + \lambda_4 g_4$ ,  $\gamma_3(i\xi) = \mu_1 g_1 + \mu_2 g_2 + \mu_3 g_3 + \mu_4 g_4$  and  $b_2(g_j) = n_j$  for  $j = 1, \dots, 4$  and we let  $b_3 = 0$ . We have  $w_0 = \gamma_3(i\xi) - J\gamma_3(\xi) = (\mu_1 + \lambda_2)g_1 + (\mu_2 - \lambda_1)g_2 + (\mu_3 + \lambda_4)g_3 + (\mu_4 - \lambda_3)g_4$ . The only condition these parameters must satisfy is  $b_2(w_0) = 2\omega(\gamma_3(i\xi), \gamma_3(\xi))$ , that reads as

$$n_1(\mu_1 + \lambda_2) + n_2(\mu_2 - \lambda_1) + n_3(\mu_3 + \lambda_4) + n_4(\mu_4 - \lambda_3) = 2(-\mu_1\lambda_2 + \mu_2\lambda_1 - \mu_3\lambda_4 + \mu_4\lambda_3).$$

The coefficients of the matrix A read as

$$a = \mu_1\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_3 + \mu_4\lambda_4,$$

$$b = 3(-\mu_1\lambda_2 + \mu_2\lambda_1 - \mu_3\lambda_4 + \mu_4\lambda_3) - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2),$$

and

$$c = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 - 3(-\mu_1\lambda_2 + \mu_2\lambda_1 - \mu_3\lambda_4 + \mu_4\lambda_3).$$

Let us call

$$B = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ n_1 - \lambda_2 & n_2 + \lambda_1 & n_3 - \lambda_4 & n_4 + \lambda_3 \\ \mu_1 - n_1 & \mu_2 - n_2 & \mu_3 - n_3 & \mu_4 - n_4 \\ -\mu_2 & \mu_1 & -\mu_4 & \mu_3 \end{pmatrix}$$

the matrix constructed with  $v_1, \dots, v_4$  as lines. For each Lie algebra  $N_5, \dots, N_{13}$  we will now give a realisation.

The first Lie algebras we want to construct are  $N_{10}$  and  $N_{11}$ . They must satisfy  $\dim(\text{span}\{v_1, \dots, v_4\}) = 4$ , in other words  $\text{rank}(B) = 4$ .

First we also want  $\text{rank}(A) = 1$ . Hence let  $\gamma_3(\xi) = \lambda_2 g_2 + \lambda_3 g_3$ ,  $\gamma_3(i\xi) = 3\lambda_3 g_4$  with  $\lambda_2 \lambda_3 \in \mathbb{R}^*$  and  $n_2 = n_3 = 0$ ,  $n_1, n_4 \in \mathbb{R}^*$  with  $n_1 \lambda_2 + 2n_4 \lambda_3 = 6\lambda_3^2$  and  $n_4 \neq 3\lambda_3$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = \lambda_2 g_1 + 2\lambda_3 g_4$ , then we obtain

$$\begin{aligned} [x_1, x_2] &= x_3, [x_1, x_3] = (8\lambda_3^2 - \lambda_2^2)i\tau, [x_1, g_2] = -\lambda_2\tau, [x_1, g_3] = -\lambda_3\tau, [x_1, g_4] = -n_4i\tau \\ [x_2, g_3] &= -3\lambda_3i\tau, [x_2, g_4] = n_4\tau. \end{aligned}$$

This Lie algebra is isomorphic to  $N_{10}$ .

Secondly we want  $\text{rank}(A) = 2$ . Let  $\gamma_3(\xi) = \lambda_1 g_1$ ,  $\gamma_3(i\xi) = \mu_3 g_3$  with  $\lambda_1, \mu_3 \in \mathbb{R}^*$  and  $n_1 = n_2 = n_3 = n_4 = 0$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = -\lambda_1 g_2 + \mu_3 g_3$ , then we obtain

$$\begin{aligned} [x_1, x_2] &= x_3, [x_1, x_3] = -\lambda_1^2 i\tau, [x_1, g_1] = -\lambda_1\tau \\ [x_2, x_3] &= \mu_3^2 \tau, [x_2, g_3] = -\mu_3\tau, [x_2, g_4] = -\mu_3i\tau. \end{aligned}$$

This Lie algebra is isomorphic to  $N_{11}$ .

In order to construct all the other Lie algebras we need  $\dim(\text{span}\{v_1, \dots, v_4\}) = 3$ , in other words  $\text{rank}(B) = 3$ .

In order to construct  $N_{12}$  and  $N_{13}$  we need  $A = 0$ . Take  $\gamma_3(\xi) = -3g_2$ ,  $\gamma_3(i\xi) = g_1 + 2\sqrt{2}g_3$  and  $n_1 = -3, n_2 = n_3 = n_4 = 0$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_6 = -w_0$  with  $w_0 = -2g_1 + 2\sqrt{2}g_3$ , we obtain

$$\begin{aligned} [x_1, x_2] &= x_6, [x_1, g_2] = 3\tau, \\ [x_2, g_2] &= -i\tau, [x_2, g_3] = -2\sqrt{2}\tau, [x_2, g_4] = -2\sqrt{2}i\tau. \end{aligned}$$

This Lie algebra is isomorphic to  $N_{12}$ . For  $N_{13}$  take  $\gamma_3(\xi) = -3g_2$ ,  $\gamma_3(i\xi) = g_1 + 2\sqrt{2}g_3$  and  $n_2 = n_4 = 0$ ,  $n_3 \in \mathbb{R}^*$ ,  $n_2 \neq 2\sqrt{2}$  and  $n_1 = \sqrt{2}n_3 - 3$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_6 = -w_0$  with  $w_0 = -2g_1 + 2\sqrt{2}g_3$ , then we obtain

$$\begin{aligned} [x_1, x_2] &= x_6, [x_1, g_2] = 3\tau, [x_1, g_3] = -n_3i\tau \\ [x_2, g_2] &= -i\tau, [x_2, g_3] = (n_3 - 2\sqrt{2})\tau, [x_2, g_4] = -2\sqrt{2}i\tau. \end{aligned}$$

This Lie algebra is isomorphic to  $N_{13}$ .

In order to construct  $N_5, N_6, N_7$  and  $N_8$  we need to assume  $\text{rank}(A) = 1$ . Take  $\gamma_3(\xi) = \sqrt{2}g_2 + g_3$ ,  $\gamma_3(i\xi) = 2\sqrt{2}g_1 - g_4$  and  $n_1 = -3\sqrt{2}, n_2 = n_3 = 0, n_4 = -1$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = 3\sqrt{2}g_1 - 2g_4$ . Then we obtain

$$[x_1, x_2] = x_3, [x_1, x_3] = 6i\tau, [x_1, g_2] = -\sqrt{2}\tau, [x_1, g_3] = -\tau$$

$$[x_2, g_2] = -2\sqrt{2}i\tau, [x_2, g_3] = -i\tau.$$

This Lie algebra is isomorphic to  $N_5$ . Take now  $\gamma_3(\xi) = \lambda_1 g_1 + \lambda_2 g_2$ , with  $\lambda_1, \lambda_2 \in \mathbb{R}^*$ ,  $\gamma_3(i\xi) = 0$  and  $n_1 = n_2 = n_4 = 0, n_3 \in \mathbb{R}^*$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = \lambda_2 g_1 - \lambda_1 g_2$ . Then we obtain

$$[x_1, x_2] = x_3, [x_1, x_3] = -(\lambda_1^2 + \lambda_2^2)i\tau, [x_1, g_2] = -\lambda_2\tau - \lambda_1 i\tau, [x_1, g_3] = -n_3 i\tau$$

$$[x_2, g_3] = n_3\tau.$$

This Lie algebra is isomorphic to  $N_6$ . Take  $\gamma_3(\xi) = \lambda_2 g_2 + \lambda_3 g_3$ , with  $\lambda_2, \lambda_3 \in \mathbb{R}^*$ ,  $\lambda_2 \neq \pm 2\sqrt{2}\lambda_3$ ,  $\gamma_3(i\xi) = 3\lambda_3 g_4$  and  $n_1 = 0, n_2 = -\lambda_2, n_3 = -\lambda_3, n_4 = 3\lambda_3$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = -\lambda_2 g_1 + 2\lambda_3 g_4$ . Then we obtain

$$[x_1, x_2] = x_3, [x_1, x_3] = (8\lambda_3^2 - \lambda_2^2)i\tau, [x_1, g_2] = -\lambda_2\tau + \lambda_2 i\tau, [x_1, g_3] = -\lambda_3\tau + \lambda_3 i\tau,$$

$$[x_1, g_4] = -4\lambda_3 i\tau, [x_2, g_2] = -\lambda_2\tau, [x_2, g_3] = -\lambda_3\tau + 3\lambda_3 i\tau.$$

This Lie algebra is isomorphic to  $N_7$ . Take  $\gamma_3(\xi) = \lambda_1 g_1 + \lambda_2 g_2$ , with  $\lambda_1, \lambda_2 \in \mathbb{R}^*$ ,  $\gamma_3(i\xi) = -3\lambda_2 g_1 + 3\lambda_1 g_2$  and  $n_1 = -3\lambda_2, n_2 = 3\lambda_1, n_4 = 0, n_3 \in \mathbb{R}^*$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = -2\lambda_2 g_1 + 2\lambda_1 g_2$ . Then we obtain

$$[x_1, x_2] = x_3, [x_1, x_3] = 8(\lambda_1^2 + \lambda_2^2)i\tau, [x_1, g_2] = -\lambda_2\tau - 4\lambda_1 i\tau, [x_1, g_3] = -n_3 i\tau$$

$$[x_2, g_2] = -3\lambda_2 i\tau, [x_2, g_3] = n_3\tau.$$

This Lie algebra is isomorphic to  $N_8$ .

Finally if we want to construct  $N_9(\varepsilon)$  we need to assume  $\text{rank}(A) = 2$ . Take  $\gamma_3(\xi) = \lambda_2 g_2$ ,  $\gamma_3(i\xi) = \mu_1 g_1$ , with  $\lambda_2, \mu_1 \in \mathbb{R}^*$ ,  $\lambda_2 \neq -\mu_1$  and  $n_1$  such that  $(\lambda_2 + \mu_1)n_1 = -2\mu_1\lambda_2, n_2 = n_4 = 0$  and  $n_3 \in \mathbb{R}^*$ . Define  $x_1 = \xi, x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = (\mu_1 + \lambda_2)g_1$ . Then we obtain

$$[x_1, x_2] = x_3, [x_1, x_3] = -\lambda_2(\lambda_2 + 3\mu_1)i\tau, [x_1, g_2] = -\lambda_2\tau, [x_1, g_3] = -n_3 i\tau$$

$$[x_2, x_3] = \mu_1(\mu_1 + 3\lambda_2)\tau, [x_2, g_2] = -\mu_1 i\tau, [x_2, g_3] = n_3\tau.$$

This Lie algebra is isomorphic to  $N_9(\varepsilon)$  with  $\varepsilon = -\frac{\mu_1^2(\mu_1+3\lambda_2)}{\lambda_2^2(\lambda_2+3\mu_1)} \in \mathbb{R}^*$ . Finally take  $\gamma_3(\xi) = \lambda_1 g_1 - g_2 + 2g_3$ ,  $\gamma_3(i\xi) = 3g_1 - 2g_4$  with  $\lambda_1 = \frac{8\sqrt{7}}{5}$  and  $n_1 = -\frac{17}{9}$ ,  $n_2 = \frac{9\lambda_1}{\lambda_1^2+8}$ ,  $n_3 = \frac{6\lambda_1}{\lambda_1^2+8}$ ,  $n_4 = -2$ . Define  $x_1 = \xi$ ,  $x_2 = i\xi$  and  $x_3 = -w_0$  with  $w_0 = 2g_1 - \lambda_1 g_2 - 4g_4$ . Then we obtain

$$\begin{aligned} [x_1, x_2] &= x_3, [x_1, x_3] = 3\lambda_1 \tau - (\lambda_1^2 + 8)i\tau, [x_1, g_2] = \tau - (n_2 + \lambda_1)i\tau, [x_1, g_3] = -2\tau - n_3 i\tau \\ [x_2, x_3] &= -16\tau - 3\lambda_1 i\tau, [x_2, g_2] = n_2 \tau - 3i\tau, [x_2, g_3] = n_3 \tau + 2i\tau. \end{aligned}$$

We see it is isomorphic to  $N_9(0)$ . □

This finishes the proof of the proposition. □

## 2.4 Classification up to isomorphism, the non-degenerate case in dimension 4

In this section we will treat the most general case  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$  and  $\gamma_2 \neq 0$  in dimension 4. We will first see how we can bring these Lie algebras to a simpler form. Then we will introduce the concept of Carnot Lie algebras and see how the classification up to isomorphism is translated to the classification of the orbits of the  $SL(3, \mathbb{R})$ -adjoint action on  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ . Finally we will study separately a particular case, namely the case  $\alpha = 0$  before studying the most general case.

### 2.4.1 Reduction

Let us introduce the following family of Lie algebras to which the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are isomorphic in the case  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$  and  $\text{rank}(\gamma_2) = 1$  in dimension 4.

**Definition 2.4.1.** Let  $\alpha, a, b, c \in \mathbb{R}$  and define the family of Lie algebras  $\{\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)\}_{\alpha, a, b, c}$  by the following non-zero Lie brackets expressed in the basis  $(x_1, \dots, x_8)$  of  $\mathbb{R}^8$ :

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_4] &= x_7, & [x_1, x_5] &= x_8, \\ [x_2, x_4] &= \alpha x_7, & [x_2, x_5] &= -\alpha x_8, \\ [x_3, x_4] &= -\alpha x_8, & [x_3, x_5] &= 3\alpha x_7, \\ [x_2, x_6] &= ax_7 + bx_8, & [x_3, x_6] &= cx_7 - ax_8. \end{aligned}$$

*Remark 2.4.2.* If  $a = b = c = 0$  the center of the corresponding Lie algebras has dimension 3 and is generated by  $\{x_6, x_7, x_8\}$ , otherwise it is of dimension 2 and is generated by  $\{x_7, x_8\}$ .

**Proposition 2.4.3.** Let  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$ ,  $n + 1 = 4$  and  $\text{rank}(\gamma_2) = 1$ . Then the Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  is isomorphic to  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$  for some  $(\alpha, a, b, c) \in \mathbb{R}^4$ .

*Proof.* Let  $(\tau, i\tau, e, Je, g, f, \xi, i\xi)$  be a basis for  $\mathbb{C} \oplus W \oplus \mathbb{C}$  adapted to the decomposition (2.1) and let  $w_0 := \gamma_3(i\xi) - J\gamma_3(\xi)$ . The Lie brackets of  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  expressed in this basis read as

$$\begin{aligned} [e, f] &= h(e, \gamma_2(f))\tau, & [Je, f] &= h(Je, \gamma_2(f))\tau, \\ [e, \xi] &= (h(e, \gamma_3(\xi)) + ib_2(e))\tau, & [e, i\xi] &= (h(e, \gamma_3(i\xi)) - b_2(e))\tau, \\ [Je, \xi] &= (h(Je, \gamma_3(\xi)) + ib_2(Je))\tau, & [Je, i\xi] &= (h(Je, \gamma_3(i\xi)) - b_2(Je))\tau, \\ [g, \xi] &= (h(g, \gamma_3(\xi)) + ib_2(g))\tau, & [g, i\xi] &= (h(g, \gamma_3(i\xi)) - b_2(g))\tau, \\ [f, \xi] &= (h(f, \gamma_3(\xi)) + ib_2(f))\tau + \gamma_2(f), & [f, i\xi] &= (h(f, \gamma_3(i\xi)) - b_2(f))\tau + J\gamma_2(f), \\ [\xi, i\xi] &= (-b_3(\xi) - ib_3(i\xi))\tau - w_0. \end{aligned}$$

First of all, as  $e \in \text{Im } \gamma_2$ , accordingly to Remark 2.1.14 we can assume that  $b_2(e) = 0$  i.e.  $\langle e, J\gamma_3(\xi) \rangle = 0$ . Furthermore, using again Proposition 2.1.13, we can conjugate the group  $U(\gamma_2, \gamma_2, b_2, b_3)$  by an element  $g \in \mathcal{P}$  that stabilises  $b_2(e)$  and makes  $b_2(f) = 0$ . To achieve this, with the notation of Proposition 2.1.13, take  $\lambda = 1, \sigma = \text{Id}, s_1 = 0$  and  $v \in W$  such that  $b_2(f) - 2\omega(e, v) = 0$ .

If  $\gamma_2(f) = \delta e$ , with  $\delta \in \mathbb{R}^*$ , making the following change of variables

$$\begin{cases} \xi_1 = \xi - \frac{1}{\delta} (\langle e, \gamma_3(\xi) \rangle - \langle Je, \gamma_3(i\xi) \rangle) f \\ \xi_2 = i\xi - \frac{1}{\delta} \langle e, \gamma_3(i\xi) \rangle f \end{cases}$$

we have that  $\gamma_3(\xi_1) = \gamma_3(\xi) - (\langle e, \gamma_3(\xi) \rangle - \langle Je, \gamma_3(i\xi) \rangle)e$  and  $\gamma_3(\xi_2) = \gamma_3(i\xi) - \langle e, \gamma_3(i\xi) \rangle e$ , in other words  $\langle \gamma_3(\xi_2), e \rangle = 0$  and  $\langle \gamma_3(\xi_1), e \rangle = \langle \gamma_3(\xi_2), Je \rangle$ . Then  $[\xi_1, \xi_2] = -b_3(\xi_1)\tau - b_3(\xi_2)i\tau - w'_0$  with  $w'_0 = \gamma_3(\xi_2) - J\gamma_3(\xi_1)$  and  $\langle w'_0, e \rangle = \langle w'_0, Je \rangle = 0$ , hence  $w'_0 = \beta g$  for some  $\beta \in \mathbb{R}^*$ . Defining now  $x_1 = \delta^{-1}f = f', x_2 = \xi_1$  and  $x_3 = \xi_2$  we have

$$\begin{aligned} [x_1, x_2] &= h(f', \gamma_3(\xi_1))\tau + e, \\ [x_1, x_3] &= h(f', \gamma_3(\xi_2))\tau + Je, \\ [x_2, x_3] &= -b_3(\xi_1)\tau - b_3(\xi_2)i\tau - w'_0. \end{aligned}$$

Hence let us define  $x_4 = h(f', \gamma_3(\xi_1))\tau + e, x_5 = h(f', \gamma_3(\xi_2))\tau + Je, x_6 = -b_3(\xi_1)\tau - b_3(\xi_2)i\tau - w'_0, x_7 = -\tau$  and  $x_8 = -i\tau$ . Then, remembering that  $b_2(Je) = 2\langle e, J\gamma_3(\xi_2) \rangle = -2\langle \gamma_3(\xi_2), Je \rangle$  and  $b_2(w'_0) = 2\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle$ , we have

$$\begin{aligned} [x_1, x_4] &= x_7, & [x_2, x_4] &= \langle \gamma_3(\xi_2), Je \rangle x_7, & [x_3, x_4] &= -\langle \gamma_3(\xi_2), Je \rangle x_8, \\ [x_1, x_5] &= x_8, & [x_2, x_5] &= -\langle \gamma_3(\xi_2), Je \rangle x_8, & [x_3, x_5] &= 3\langle \gamma_3(\xi_2), Je \rangle x_7, \\ [x_2, x_6] &= -(\langle \gamma_3(\xi_2), \gamma_3(\xi_1) \rangle x_7 + (3\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle - \|\gamma_3(\xi_1)\|^2) x_8), \end{aligned}$$

$$[x_3, x_6] = -((\|\gamma_3(\xi_2)\|^2 - 3\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle)x_7 - \langle \gamma_3(\xi_2), \gamma_3(\xi_1) \rangle x_8).$$

Calling  $\alpha = \langle \gamma_3(\xi_2), Je \rangle$ ,

$$\begin{aligned} a &= -\langle \gamma_3(\xi_2), \gamma_3(\xi_1) \rangle, \\ b &= -(3\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle - \|\gamma_3(\xi_1)\|^2), \\ c &= -(\|\gamma_3(\xi_2)\|^2 - 3\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle) \end{aligned}$$

we can see the isomorphism with  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$ .  $\square$

Notice that all the Lie algebras  $\mathfrak{g}(\alpha, a, b, c)$  are obtained, indeed we have the following.

**Proposition 2.4.4.** *Every Lie algebra  $\mathfrak{g}(\alpha, a, b, c)$  is isomorphic to, at least one, Lie algebra of the form  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ .*

*Proof.* We will prove that  $\mathfrak{g}(\alpha, a, b, c)$  is isomorphic to one Lie algebra of the form  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  if the following polynomial has a real solution

$$F(x) = 8x^4 + (4\alpha^2 + 9b + 7c)x^2 - 4\alpha^4 + 4\alpha^2c - 9a^2 - c^2.$$

Notice that the proposition follows because the polynomial  $F(x)$  has always a real solution since  $F(0) < 0$  and  $F(x)$  tends to  $+\infty$  as  $x$  tends to infinity.

We are in the situation of Proposition 2.4.3 so let  $\gamma_3(\xi_1) = \lambda_1 e + \lambda_3 g + \lambda_4 f$ ,  $\gamma_3(\xi_2) = \lambda_1 Je + \mu_3 g + \mu_4 f$  with  $\lambda_1, \lambda_3, \lambda_4, \mu_3, \mu_4 \in \mathbb{R}$ . Then the relations between  $\alpha, a, b, c$  and  $\lambda_1, \lambda_3, \lambda_4, \mu_3, \mu_4$  are the following

$$\begin{aligned} \alpha &= \lambda_1, \\ a &= -\lambda_3\mu_3 - \lambda_4\mu_4, \\ b &= \lambda_3^2 + \lambda_4^2 - 2\lambda_1^2 - 3\mu_4\lambda_3 + 3\lambda_4\mu_3, \\ c &= 2\lambda_1^2 - \mu_3^2 - \mu_4^2 + 3\mu_4\lambda_3 - 3\lambda_4\mu_3. \end{aligned}$$

If  $\mu_3 \neq 0$  then  $\lambda_3 = \frac{-a - \lambda_4\mu_4}{\mu_3}$  and

$$b = \frac{1}{\mu_3^2}(a^2 + \lambda_4^4\mu_4^2 + 2a\lambda_4\mu_4) + \lambda_4^2 - 2\lambda_1^2 + \frac{3\mu_4(a + \lambda_4\mu_4)}{\mu_3} + 3\lambda_4\mu_3 \quad (2.6)$$

$$c = 2\lambda_1^2 - \mu_3^2 - \mu_4^2 - \frac{3\mu_4(a + \lambda_4\mu_4)}{\mu_3} - 3\lambda_4\mu_3 \quad (2.7)$$

From equation (2.7) we find that  $\lambda_4 = \frac{2a^2\mu_3 - \mu_3^3 - \mu_4^2\mu_3 - 3\mu_4b - c\mu_3}{3(\mu_3^2 + \mu_4^2)}$  and substituting this in (2.6) we find that  $F(\mu_3, \mu_4) = 0$  where

$$F(x, y) = 8x^4 + (4\alpha^2 + 16y^2 + 9b + 7c)x^2 - 4\alpha^4 + 4\alpha^2c + 4\alpha^2y^2 + 9a^2 + 9by^2 - c^2 + 7cy^2 + 8y^4.$$

Notice that  $F(x, y) = F(y, x)$ . Furthermore, looking at the discriminant of  $F(x, y)$  seen as a polynomial in  $x$ , one can see that the value of  $y$  does not affect the existence of real roots. In particular we can look for solutions with  $\mu_4 = 0$  and by setting  $\mu_3$  to be a real solution of  $F(x, 0)$  we find a tuple  $(\lambda_1, \lambda_3, \lambda_4, \mu_3, 0)$  giving the corresponding  $(\alpha, a, b, c)$ .  $\square$

## 2.4.2 Carnot Lie algebras

**Definition 2.4.5.** A *Carnot grading* on a Lie algebra  $\mathfrak{g}$  is an algebra grading of  $\mathfrak{g}$ , i.e.  $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$  with  $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ , such that  $\mathfrak{g}$  is generated by  $\mathfrak{g}_1$ . A Lie algebra is *Carnot graded* if it is endowed with a Carnot grading and *Carnot* if it admits a Carnot grading.

**Definition 2.4.6.** Given a Lie algebra  $\mathfrak{g}$  consider the direct sum  $\text{Car}(\mathfrak{g}) = \bigoplus_{i \geq 1} \mathfrak{v}_i$ , where  $\mathfrak{v}_i = \mathfrak{g}^i / \mathfrak{g}^{i+1}$  and  $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$ . The Lie algebra obtained by endowing  $\text{Car}(\mathfrak{g})$  with the Lie brackets induced on each quotient by the ones of the Lie algebra  $\mathfrak{g}$  is called the *associated Carnot-graded Lie algebra* to the Lie algebra  $\mathfrak{g}$ .

**Proposition 2.4.7** ([17, Proposition 3.5.]). *A Lie algebra is Carnot if and only if it is isomorphic, as a Lie algebra, to its associated Carnot-graded Lie algebra. Furthermore if these conditions hold, then:*

- *for any Carnot grading on  $\mathfrak{g}$ , the graded Lie algebras  $\mathfrak{g}$  and  $\text{Car}(\mathfrak{g})$  are isomorphic,*
- *for any two Carnot gradings on  $\mathfrak{g}$ , there is a unique automorphism mapping the first to the second and inducing the identity modulo  $[\mathfrak{g}, \mathfrak{g}]$ .*

**Corollary 2.4.8** ([17, Corollary 3.6.]). *Let  $\mathfrak{g}$  be a Carnot graded Lie algebra. Denote by  $\text{Aut}(\mathfrak{g})$  its automorphism group as a Lie algebra and  $\text{Aut}(\mathfrak{g})_0$  its automorphism group as a graded Lie algebra. Let  $\text{Aut}(\mathfrak{g})_{\geq 1}$  be the group of automorphism of the Lie algebra  $\mathfrak{g}$  inducing the identity on  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ . Then  $\text{Aut}(\mathfrak{g})_{\geq 1}$  is a normal subgroup and*

$$\text{Aut}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})_0 \ltimes \text{Aut}(\mathfrak{g})_{\geq 1}$$

**Example 2.4.9.** Here is an example of a nilpotent Lie algebra that is not Carnot. It is given by the following non-zero Lie brackets on the basis  $(x_1, \dots, x_5)$

$$[x_1, x_3] = x_4,$$

$$[x_1, x_4] = [x_2, x_3] = x_5.$$

Another one is the Lie algebra  $L_5$  of Appendix A.

**Remark 2.4.10.** Each Lie algebra  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$  is Carnot with grading  $\mathfrak{g}_1 = \text{span}_{\mathbb{R}}\{x_1, x_2, x_3\}$ ,  $\mathfrak{g}_2 = \text{span}_{\mathbb{R}}\{x_4, x_5, x_6\}$  and  $\mathfrak{g}_3 = \text{span}_{\mathbb{R}}\{x_7, x_8\}$ .

**Example 2.4.11.** Let us remember that the free  $k$ -step nilpotent Lie algebra of rank  $n$ , denoted by  $F_{k,n}$ , is constructed by taking the quotient of the free Lie algebra on  $n$  generators,  $F_n$ , by its  $(k+1)$ -derived Lie algebra,  $F_n^{k+1}$ . Every free  $k$ -step nilpotent Lie algebra of rank  $n$  is Carnot.

**Proposition 2.4.12.** *Every quotient of the free  $k$ -step nilpotent Lie algebra of rank  $n$  by a homogeneous ideal is a Carnot nilpotent Lie algebra. Let us recall that an homogeneous ideal of a graded Lie algebra is an ideal  $\mathfrak{h}$  such that, for all  $x \in \mathfrak{h}$ , all the homogeneous components of  $x$  belong to  $\mathfrak{h}$ .*

*Proof.* Denote by  $F_{k,n}$  the free  $k$ -step nilpotent Lie algebra of rank  $n$  and let  $\mathfrak{h}$  be a homogeneous ideal of  $F_{k,n}$ , if  $F_{k,n} = \bigoplus_{i=1}^k F_i$  we have  $\mathfrak{h} = \bigoplus_i (\mathfrak{h} \cap F_i) = \bigoplus_i \mathfrak{h}_i$ . Then the quotient  $\mathfrak{g} = F_{k,n}/\mathfrak{h}$  inherits the grading and it will be generated by  $F_1/\mathfrak{h}_1$  since  $\mathfrak{h}$  is an ideal. Finally the quotient will be a nilpotent Lie algebra.  $\square$

Let us introduce a general setting in which we will see our family of Lie algebras  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$ .

**Definition 2.4.13.** Let  $F$  be the free 3-step nilpotent Lie algebra on 3 generators. Then with respect to the basis  $(y_1, \dots, y_{14})$  of  $\mathbb{R}^{14}$  the Lie brackets are as follows

$$\begin{aligned} [y_1, y_2] &= y_4, & [y_1, y_3] &= y_5, & [y_2, y_3] &= y_6, \\ [y_1, y_4] &= y_7, & [y_1, y_5] &= y_8, & [y_1, y_6] &= y_9, \\ [y_2, y_4] &= y_{10}, & [y_2, y_5] &= y_{11}, & [y_2, y_6] &= y_{12}, \\ [y_3, y_4] &= y_{11} - y_9, & [y_3, y_5] &= y_{13}, & [y_3, y_6] &= y_{14}. \end{aligned}$$

We also have a grading of  $F$  as  $F \cong \bigoplus_{i=1}^3 F_i$  where  $F_i = F^i/F^{i+1}$ . Notice that  $F_1 = \text{span}_{\mathbb{R}}\{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ ,  $F_2 = \text{span}_{\mathbb{R}}\{\bar{y}_4, \bar{y}_5, \bar{y}_6\}$  and  $F_3 = \text{span}_{\mathbb{R}}\{\bar{y}_7, \dots, \bar{y}_{14}\}$ , where  $\bar{y}_i$  is the equivalence class represented by  $y_i$ .

**Proposition 2.4.14.** *There exists a unique action of  $\text{GL}(3, \mathbb{R})$  as graded Lie algebra automorphisms on  $F$  that extends the linear action on  $F_1 \cong \mathbb{R}^3$ .*

*Proof.* Let us fix a basis  $\{y_1, y_2, y_3\}$  for  $F_1$ . The set  $\{[y_1, y_2], [y_1, y_3], [y_2, y_3]\}$  is a basis for  $F_2$  and we define the action of  $M \in \text{GL}(3, \mathbb{R})$  by  $M * [y_i, y_j] = [My_i, My_j]$ . Similarly there is a basis of  $F_3$  whose elements are all of the form  $[y_i, [y_j, y_k]]$  and we define the action of  $M \in \text{GL}(3, \mathbb{R})$  by  $M * [y_i, [y_j, y_k]] = [My_i, [My_j, My_k]]$ . This action on  $F$  is the unique extension of the one on  $F_1$  that preserves the Lie algebra structure and it is graded.  $\square$

*Remark 2.4.15.* In particular Proposition 2.4.14 gives an action of  $\text{GL}(3, \mathbb{R})$  on  $F_3$ . We have hence a representation  $\rho : \text{GL}(3, \mathbb{R}) \rightarrow \text{GL}(F_3)$ , and by abuse of notation let us call  $\rho$  also the induced representation  $\rho : \text{SL}(3, \mathbb{R}) \rightarrow \text{GL}(F_3)$ .

**Example 2.4.16.** For example the action of a diagonal element  $g = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \in \text{GL}(3, \mathbb{R})$  on  $y_7 = [y_1, [y_1, y_2]]$  is  $g * y_7 = [gy_1, [gy_1, gy_2]] = [\lambda_1 y_1, [\lambda_1 y_1, \lambda_2 y_2]] = \lambda_1^2 \lambda_2 y_7$ .

**Proposition 2.4.17.** *There exists an  $\mathrm{SL}(3, \mathbb{R})$ -equivariant isomorphism  $\varphi : \mathfrak{sl}(3, \mathbb{R}) \rightarrow F_3$ , where  $\mathrm{SL}(3, \mathbb{R})$  acts on  $\mathfrak{sl}(3, \mathbb{R})$  via the adjoint action and on  $F_3$  via  $\rho$ .*

*Proof.* Let us denote by  $E_{ij}$  the  $3 \times 3$  matrix whose  $(i, j)$ -th entry is 1 and all the rest is 0. Then the matrices  $\{E_1, \dots, E_8\}$ , where  $E_1 = E_{11} - E_{22}, E_2 = E_{22} - E_{33}, E_3 = E_{12}, E_4 = E_{13}, E_5 = E_{21}, E_6 = E_{23}, E_7 = E_{31}$  and  $E_8 = E_{32}$ , form a basis of  $\mathfrak{sl}(3, \mathbb{R})$ . The isomorphism  $\varphi : \mathfrak{sl}(3, \mathbb{R}) \rightarrow F_3$  is given by  $\varphi(E_1) = y_9 + y_{11}, \varphi(E_2) = y_9 - 2y_{11}, \varphi(E_3) = -y_8, \varphi(E_4) = y_7, \varphi(E_5) = y_{12}, \varphi(E_6) = y_{10}, \varphi(E_7) = y_{14}, \varphi(E_8) = -y_{13}$ . In order to check the equivariance of  $\varphi$  it is sufficient to look at the level of the Lie algebras. Let  $\rho^*$  be the induced representation  $\rho^* : \mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathrm{End}(F_3)$  defined by  $\rho^*(E_i)v = \frac{d}{dt}|_{t=0} \rho(\exp(tE_i))v$ . Then the equivariant condition becomes  $\varphi(\mathrm{ad}(E_i)E_j) = \rho^*(E_i)\varphi(E_j)$  for all  $i, j = 1, \dots, 8$ . This can be verified by easy calculations, for example  $\varphi(\mathrm{ad}(E_1)E_4) = \varphi(E_4) = y_7$  and  $\rho^*(E_1)y_7 = \frac{d}{dt}|_{t=0} [e^T y_1, [e^T y_1, e^{-T} y_2]] = y_7$ .  $\square$

**Definition 2.4.18.** Let  $Q \in \mathrm{Gr}(6, F_3)$  and define the Lie algebra  $\mathfrak{g}(Q)$  to be the vector space  $F_1 \oplus F_2 \oplus F_3/Q$  with the Lie algebra structure induced by the one of  $F$ . Let us also define  $P \in \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  by  $P := \varphi^{-1}(Q)^\perp$  where the orthogonal space is taken with respect to the Killing form on  $\mathfrak{sl}(3, \mathbb{R})$ .

*Remark 2.4.19.* The Lie algebra  $\mathfrak{g}(Q)$  is a 8-dimensional 3-step nilpotent Lie algebra that is Carnot.

**Proposition 2.4.20.** *We have a bijection*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{the Lie algebras } \mathfrak{g}(Q) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Orbits of elements } P := \varphi^{-1}(Q)^\perp \in \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \\ \text{under the adjoint action} \end{array} \right\}.$$

*Proof.* As we have seen, being the Lie algebras  $\mathfrak{g}(Q)$  Carnot, each isomorphism between them is just induced by a linear isomorphism between their homogeneous parts of degree 1. So, after fixing a basis for the two Lie algebras, an isomorphism between  $\mathfrak{g}(Q)$  and  $\mathfrak{g}(Q')$  is induced by an element of  $\mathrm{GL}(3, \mathbb{R})$  that sends  $Q$  to  $Q'$ . Since scalar multiples of the identity act trivially on  $\mathrm{Gr}(6, F_3)$  and every element in  $\mathbb{R}$  is a cube, the  $\mathrm{GL}(3, \mathbb{R})$ -orbits in  $\mathrm{Gr}(6, F_3)$  are the same as those of  $\mathrm{SL}(3, \mathbb{R})$ . Furthermore the isomorphism  $\varphi$  of Proposition 2.4.17 gives a  $\mathrm{SL}(3, \mathbb{R})$ -equivariant isomorphism  $\mathrm{Gr}(6, F_3) \cong \mathrm{Gr}(6, \mathfrak{sl}(3, \mathbb{R}))$  and using the Killing form of  $\mathfrak{sl}(3, \mathbb{R})$  we have a  $\mathrm{SL}(3, \mathbb{R})$ -equivariant isomorphism  $\mathrm{Gr}(6, \mathfrak{sl}(3, \mathbb{R})) \cong \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ . Hence, using these identifications, the orbit of each  $P \in \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  under the  $\mathrm{SL}(3, \mathbb{R})$ -adjoint action represents the isomorphism class of the associated Lie algebra.  $\square$

### 2.4.3 The case $\alpha = 0$

When  $\alpha = 0$  the family of Lie algebras  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$  is quite easy to classify. Indeed we have the following.

**Proposition 2.4.21.** *The isomorphism classes of  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  are represented by one of the following pairwise non-isomorphic Lie algebras*

- $\mathfrak{g}_{\mathbb{R}}(0, 1, 0, 0)$  if  $a^2 + bc > 0$ ,
- $\mathfrak{g}_{\mathbb{R}}(0, 0, 1, -1)$  if  $a^2 + bc < 0$ ,
- $\mathfrak{g}_{\mathbb{R}}(0, 0, 0, 1)$  if  $a^2 + bc = 0$  but either  $b$  or  $c \neq 0$ ,
- $\mathfrak{g}_{\mathbb{R}}(0, 0, 0, 0)$  if  $a = b = c = 0$ .

*Proof.* Since we know that the Lie algebras  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  are Carnot an isomorphism between them is induced by a linear isomorphism between their homogenous parts of degree 1, hence by an element of  $\mathrm{GL}(3, \mathbb{R})$ . Since every element in  $\mathbb{R}$  is a cube, the group  $\mathrm{GL}(3, \mathbb{R})$  is generated by  $\mathrm{SL}(3, \mathbb{R})$  and the scalar matrices that have trivial action. Hence we classify the Lie algebras  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  up to isomorphism by looking at the action of  $\mathrm{SL}(3, \mathbb{R})$ . Then  $g \in \mathrm{SL}(3, \mathbb{R})$  is such that  $g \cdot \mathfrak{g}_{\mathbb{R}}(0, a, b, c) = \mathfrak{g}_{\mathbb{R}}(0, a', b', c')$  if and only if  $g = \begin{pmatrix} \frac{1}{\Delta} & 0 \\ 0 & \tilde{g} \end{pmatrix}$ , with  $\tilde{g} = \begin{pmatrix} \lambda & \mu \\ \delta & \rho \end{pmatrix}$  and  $\Delta = \det \tilde{g}$ , and

$$a' = \Delta^2(\rho(a\lambda + c\delta) - \mu(b\lambda - a\delta)),$$

$$b' = \Delta^2(b\lambda^2 - 2a\delta\lambda - c\delta^2),$$

$$c' = \Delta^2(-b\mu^2 + 2a\rho\mu + c\rho^2).$$

Another way to see this is the following. Consider the injection  $\iota : \mathrm{GL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(3, \mathbb{R})$  that associates to  $\tilde{g} \in \mathrm{GL}(2, \mathbb{R})$  the matrix

$$\begin{pmatrix} \frac{1}{\det \tilde{g}} & 0 \\ 0 & \tilde{g} \end{pmatrix}.$$

Then the map

$$\begin{aligned} \psi : \mathfrak{sl}(2, \mathbb{R}) &\rightarrow \{\mathfrak{g}_{\mathbb{R}}(0, a, b, c)\} \\ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} &\mapsto \mathfrak{g}_{\mathbb{R}}(0, a, b, c) \end{aligned}$$

intertwines the two actions, i.e for  $A \in \mathfrak{sl}(2, \mathbb{R})$  we have  $\Delta^3 \psi(\tilde{g}A\tilde{g}^{-1}) = \iota(\tilde{g})^T \psi(A)$ .

Depending on the sign of  $a^2 + bc$  we can bring the matrix in  $\mathfrak{sl}(2, \mathbb{R})$  to one of the following normal forms

$$\begin{pmatrix} \sqrt{a^2 + bc} & 0 \\ 0 & -\sqrt{a^2 + bc} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-(a^2 + bc)} \\ -\sqrt{-(a^2 + bc)} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ or } 0.$$

Finally, since  $A \in \mathfrak{sl}(2, \mathbb{R})$  and  $\lambda A$  with  $\lambda \in \mathbb{R}^*$  define the same isomorphism class, we might assume the coefficients to be 1.  $\square$

*Remark 2.4.22.* We can present the representatives of the isomorphism classes of  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  in a more compact form as follows. The Lie algebras  $\mathfrak{g}_{\mathbb{R}}(0, 0, 0, 0)$  and  $\mathfrak{g}_{\mathbb{R}}(0, 0, \varepsilon, 1)$  with  $\varepsilon \in \mathbb{R}$  represent

the isomorphism classes of  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$ . With  $\mathfrak{g}_{\mathbb{R}}(0, 0, \varepsilon', 1) \cong \mathfrak{g}_{\mathbb{R}}(0, 0, \varepsilon, 1)$  if and only if there exists  $\alpha \in \mathbb{R}^*$  such that  $\varepsilon' = \alpha^2 \varepsilon$ . Indeed notice that  $\mathfrak{g}_{\mathbb{R}}(0, 1, 0, 0) \cong \mathfrak{g}_{\mathbb{R}}(0, 0, 1, 1)$ .

#### 2.4.4 The case $\alpha \neq 0$

When  $\alpha \neq 0$  let us write the family  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$  as follows.

**Definition 2.4.23.** When  $\alpha \neq 0$  a change of variables induced by  $x'_1 = \alpha x_1$  brings our family  $\{\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)\}_{\alpha, a, b, c}$  to one denoted by  $\{\mathfrak{g}_{\mathbb{R}}(a, b, c)\}_{a, b, c}$ , where the non-zero Lie brackets, expressed in the basis  $(x_1, \dots, x_8)$  of  $\mathbb{R}^8$  are:

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_4] &= x_7, & [x_1, x_5] &= x_8, \\ [x_2, x_4] &= x_7, & [x_2, x_5] &= -x_8, \\ [x_3, x_4] &= -x_8, & [x_3, x_5] &= 3x_7, \\ [x_2, x_6] &= ax_7 + bx_8, & [x_3, x_6] &= cx_7 - ax_8. \end{aligned}$$

In other words  $\mathfrak{g}_{\mathbb{R}}(a, b, c) = \mathfrak{g}_{\mathbb{R}}(1, a, b, c)$ .

**Proposition 2.4.24.** No Lie algebra  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  is isomorphic to  $\mathfrak{g}_{\mathbb{R}}(\alpha, a', b', c')$  with  $\alpha \neq 0$ .

*Proof.* In the family  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$  we denote  $\mathfrak{g}_i$  with  $i = 1, 2, 3$  the grading. Let us consider for  $x \in \mathfrak{g}_1$  the rank  $r(x)$  of  $ad(x)$  considered as a morphism from  $\mathfrak{g}_2$  to  $\mathfrak{g}_3$ . Then for  $x = t_1 x_1 + t_2 x_2 + t_3 x_3 \in \mathfrak{g}_1$  one minor of  $ad(x)$  is  $t_1^3 - \alpha^2 t_2^4 - 3\alpha^2 t_3^2$ . Hence if  $\alpha = 0$  we have  $r(x) \leq 1$  if and only if  $t_1 = 0$ . Instead if  $\alpha \neq 0$ , and hence  $\alpha = 1$ , we have that  $r(x) \leq 1$  implies  $t_1^3 - t_2^4 - 3t_3^2 = 0$  and this hypersurface does not contain any hyperplane. Hence the subvariety  $\{x \in \mathfrak{g}_1 \mid r(x) \leq 1\}$  is a hyperplane in  $\mathfrak{g}_1$  if and only if  $\alpha = 0$ . This proves that no Lie algebra  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  is isomorphic to  $\mathfrak{g}_{\mathbb{R}}(\alpha, a', b', c')$  with  $\alpha \neq 0$  as graded Lie algebras and hence as Lie algebras since all these Lie algebras are Carnot.  $\square$

Our family of Lie algebras  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  is a particular case of the Lie algebras  $\mathfrak{g}(Q)$  defined in Definition 2.4.18. We will now find explicitly the submanifold of  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  to which it corresponds.

**Definition 2.4.25.** Let us consider the following 6-dimensional subspace of  $F_3$

$$Q(a, b, c) = \text{span}_{\mathbb{R}}\{y_9, y_{10} - y_7, y_{11} + y_8, y_{13} - 3y_7, y_{12} - ay_7 - by_8, y_{14} - cy_7 + ay_8\} \subseteq F_3.$$

**Proposition 2.4.26.** For all  $a, b, c \in \mathbb{R}$  we have an isomorphism of Lie algebras between  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  and the Lie algebra  $\mathfrak{g}(Q(a, b, c))$ .

*Proof.* By imposing the conditions

$$\begin{aligned} [x_1, x_4] &= x_7 = y_7, & [x_1, x_5] &= x_8 = y_8, & [x_1, x_6] &= 0 = y_9, \\ [x_2, x_4] &= x_7 = y_{10}, & [x_2, x_5] &= -x_8 = y_{11}, & [x_2, x_6] &= ax_7 + bx_8 = y_{12}, \\ [x_3, x_5] &= 3x_7 = y_{13}, & [x_3, x_6] &= cx_7 - ax_8 = y_{14} \end{aligned}$$

we can find the generators for  $Q(a, b, c)$  and hence define a natural isomorphism.  $\square$

*Remark 2.4.27.* With this point of view we see our family of Lie algebras as a submanifold of the Grassmannian  $\text{Gr}(6, F_3)$ . Furthermore we can see that this submanifold is contained in  $\{Q \in \text{Gr}(6, F_3) \mid Q_0 \subseteq Q\} \cong \text{Gr}(2, F_3/Q_0) \cong \text{Gr}(2, 4)$  where  $Q_0 = \text{span}_{\mathbb{R}}\{y_9, y_{10} - y_7, y_{11} + y_8, y_{13} - 3y_7\}$  and it corresponds actually to just one chart of  $\text{Gr}(2, 4)$ . This will be developed more in Section 3.2.

*Remark 2.4.28.* Under  $\varphi$ , the isomorphism between  $\mathfrak{sl}(3, \mathbb{R})$  and  $F_3$  defined in Proposition 2.4.17, the subspace  $Q(a, b, c)$  of  $F_3$  corresponds to the subspace of  $\mathfrak{sl}(3, \mathbb{R})$ , spanned by  $\{2E_1 + E_2, -E_4 + E_6, E_1 - E_2 - 3E_3, bE_3 - aE_4 + E_5, -3E_4 - E_8, -aE_3 - cE_4 + E_7\}$ . Using the Killing form on  $\mathfrak{sl}(3, \mathbb{R})$  we can identify this subspace with a 2-dimensional subspace  $P(a, b, c)$  of  $\mathfrak{sl}(3, \mathbb{R})$  spanned by  $\{-E_2 - bE_3 + aE_4 + E_5, aE_3 + cE_4 - 3E_6 + E_7 + E_8\}$ .

**Definition 2.4.29.** We choose as basis of  $\mathfrak{sl}(3, \mathbb{R})$ , as in Proposition 2.4.17, the matrices  $\{E_1, \dots, E_8\}$ , where  $E_1 = E_{11} - E_{22}, E_2 = E_{22} - E_{33}, E_3 = E_{12}, E_4 = E_{13}, E_5 = E_{21}, E_6 = E_{23}, E_7 = E_{31}$  and  $E_8 = E_{32}$ . Here  $E_{ij}$  is the  $3 \times 3$  matrix whose  $(i, j)$ -th entry is 1 and all the rest is 0.

Let us define  $P := P(a, b, c) \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  the 2-dimensional subspace spanned by

$$u := u(a, b, c) = -E_2 - bE_3 + aE_4 + E_5 = \begin{pmatrix} 0 & -b & a \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$v := v(a, b, c) = aE_3 + cE_4 - 3E_6 + E_7 + E_8 = \begin{pmatrix} 0 & a & c \\ 0 & 0 & -3 \\ 1 & 1 & 0 \end{pmatrix}.$$

The following is just a reformulation of Proposition 2.4.20 in our particular case.

**Proposition 2.4.30.** *We have a bijection*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{the Lie algebras } \mathfrak{g}_{\mathbb{R}}(a, b, c) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Orbits of elements } P(a, b, c) \\ \text{under the adjoint action} \end{array} \right\}.$$

**The  $\text{SL}(3, \mathbb{R})$ -action on  $\{\mathfrak{g}_{\mathbb{R}}(a, b, c)\} \subseteq \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$**

In order to better understand the action we will embed the Grassmannian into a projective space and decompose it in  $\text{SL}(3, \mathbb{R})$ -invariant subspaces. We embed  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  in projective space by the

Plücker embedding given by:

$$\iota : \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \rightarrow \mathbb{P} \left( \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \right).$$

Fixing  $(E_i)_{i=1}^8$  as basis for  $\mathfrak{sl}(3, \mathbb{R})$  we can represent a generic element  $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  as a  $2 \times 8$  matrix whose lines are the vectors spanning it, then we have that  $\iota(P) = [a_{ij}]$  where the Plücker coordinates  $a_{ij}$  are the minors of the  $2 \times 2$  submatrix of  $P$  obtained taking the  $i$ -th and  $j$ -th columns.

From classical representation theory, or from what we will show later, we have the following decomposition of  $\mathfrak{sl}(3, \mathbb{R})$ -representations

$$\bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) = \mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^3(\mathbb{R}^{3*}),$$

where  $S^3(\mathbb{R}^3)$  is the 3-rd symmetric power of  $\mathbb{R}^3$ . We denote by  $\odot$  the symmetric tensor product on  $\mathbb{R}^3$ . The group  $\text{SL}(3, \mathbb{R})$  acts on  $\mathfrak{sl}(3, \mathbb{R})$  by the adjoint action, i.e. for  $g \in \text{SL}(3, \mathbb{R})$  and  $u \in \mathfrak{sl}(3, \mathbb{R})$  we have  $g \cdot u = gug^{-1}$ . The action on  $S^3(\mathbb{R}^3)$  and  $S^3(\mathbb{R}^{3*})$  is given by

$$g \cdot (x \odot y \odot z) = gx \odot gy \odot gz$$

and

$$g \cdot (x^* \odot y^* \odot z^*) = g^{\top-1} x^* \odot g^{\top-1} y^* \odot g^{\top-1} z^*$$

respectively with  $g \in \text{SL}(3, \mathbb{R})$ ,  $x, y, z \in \mathbb{R}^3$  and  $x^*, y^*, z^* \in \mathbb{R}^{3*}$ . This explains the difference between  $S^3(\mathbb{R}^3)$  and  $S^3(\mathbb{R}^{3*})$ . Let us call  $\pi_1$  the projection to the first factor of the decomposition,

$$\begin{aligned} \pi_1 : \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) &\longrightarrow \mathfrak{sl}(3, \mathbb{R}) \\ u_1 \wedge u_2 &\mapsto [u_1, u_2]. \end{aligned}$$

**Lemma 2.4.31.** *The projection just defined,  $\pi_1$ , is a morphism of  $\mathfrak{sl}(3, \mathbb{R})$ -representations.*

*Proof.* Take  $u_1, u_2 \in \mathfrak{sl}(3, \mathbb{R})$  and  $x \in \mathfrak{sl}(3, \mathbb{R})$  we have  $\pi_1(x \cdot (u_1 \wedge u_2)) = \pi_1([x, u_1] \wedge u_2 + u_1 \wedge [x, u_2]) = [[x, u_1], u_2] + [u_1, [x, u_2]] = [x, [u_1, u_2]] = x \cdot \pi_1(u_1 \wedge u_2)$ .  $\square$

In order to define the projection to the second and third factor of the decomposition let us denote by  $\times$  the standard cross product on  $\mathbb{R}^3$ , by  $(e_i)_{i=1}^3$  the standard basis for  $\mathbb{R}^3$ . Let us define then  $\pi_2$  as

$$\begin{aligned} \pi_2 : \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) &\longrightarrow S^3(\mathbb{R}^3) \\ u_1 \wedge u_2 &\mapsto \sum_{i,j=1}^3 u_1 e_i \odot u_2 e_j \odot (e_i \times e_j). \end{aligned}$$

**Lemma 2.4.32.** *The second projection,  $\pi_2$ , is a morphism of  $\mathfrak{sl}(3, \mathbb{R})$ -representations.*

*Proof.* It is sufficient to check it for  $u_1 = e_i^* \otimes u_1(e_i)$  and  $u_2 = e_j^* \otimes u_2(e_j) \in \mathfrak{sl}(3, \mathbb{R})$ . Let  $x \in \mathfrak{sl}(3, \mathbb{R})$  then on one side we have

$$\begin{aligned} x \cdot \pi_2(u_1 \wedge u_2) &= x \cdot (u_1(e_i) \odot u_2(e_j) \odot (e_i \times e_j)) = \\ &= x(u_1(e_i)) \odot u_2(e_j) \odot (e_i \times e_j) + u_1(e_i) \odot x(u_2(e_j)) \odot (e_i \times e_j) + u_1(e_i) \odot u_2(e_j) \odot x(e_i \times e_j). \end{aligned}$$

On the other hand

$$\pi_2(x \cdot (u_1 \wedge u_2)) = \pi_2((x \cdot u_1) \wedge u_2 + u_1 \wedge (x \cdot u_2))$$

but  $x \cdot u_1 = x \cdot (e_i^* \otimes u_1(e_i)) = -e_i^* x \otimes u_1(e_i) + e_i^* \otimes x u_1(e_i)$ , hence

$$\begin{aligned} \pi_2(-e_i^* x \otimes u_1(e_i) \wedge e_j^* \otimes u_2(e_j) + e_i^* \otimes x u_1(e_i) \wedge e_j^* \otimes u_2(e_j) - e_i^* \otimes u_1(e_i) \wedge e_j^* x \otimes u_2(e_j) + \\ e_i^* \otimes u_1(e_i) \wedge e_j^* \otimes x u_2(e_j)) = -u_1(e_i) \odot u_2(e_j) \odot (x^\top e_i \times e_j) + \\ x u_1(e_i) \odot u_2(e_j) \odot (e_i \times e_j) - u_1(e_i) \odot u_2(e_j) \odot (e_i \times x^\top e_j) + u_1(e_i) \odot x u_2(e_j) \odot (e_i \times e_j). \end{aligned}$$

So it is left to prove that  $x(e_i \times e_j) = -x^\top e_i \times e_j - e_i \times x^\top e_j$ . For this consider a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^3$  and then by definition of cross product we have  $\det(e_i, e_j, v) = \langle e_i \times e_j, v \rangle$  for any  $e_i, e_j, v \in \mathbb{R}^3$ . For any  $g \in \text{SL}(3, \mathbb{R})$  we have  $\det(g^\top e_i, g^\top e_j, g^\top v) = c$  for some constant  $c \in \mathbb{R}$  that does not depend on  $g$ . Hence differentiating this expression at the identity and letting  $x \in \mathfrak{sl}(3, \mathbb{R})$  we obtain that  $\det(x^\top e_i, e_j, v) + \det(e_i, x^\top e_j, v) + \det(e_i, e_j, x^\top v) = 0$  hence

$$\langle x^\top e_i \times e_j, v \rangle + \langle e_i \times x^\top e_j, v \rangle + \langle e_i \times e_j, x^\top v \rangle = 0$$

for any  $v \in \mathbb{R}^3$  and the result follows.  $\square$

We also define

$$\begin{aligned} \pi_3 : \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) &\longrightarrow S^3(\mathbb{R}^{3*}) \\ u_1 \wedge u_2 &\mapsto \sum_{i,j=1}^3 e_i^* \odot e_j^* \odot (u_1 e_i \times u_2 e_j)^*. \end{aligned}$$

*Remark 2.4.33.* Notice that  $\pi_3(u_1 \wedge u_2) = \pi_2(u_1^\top \wedge u_2^\top)$ . Hence since  $\pi_2$  is  $\text{SL}(3, \mathbb{R})$ -equivariant we get that  $\pi_3$  is  $\text{SL}(3, \mathbb{R})$ -equivariant as well. Indeed for  $g \in \text{SL}(3, \mathbb{R})$  we have

$$\pi_3(g u_1 g^{-1} \wedge g u_2 g^{-1}) = \pi_2(g^{-1\top} u_1^\top g^\top \wedge g^{-1\top} u_2^\top g^\top) = g^{-1\top} \pi_2(u_1^\top \wedge u_2^\top) = g^{-1\top} \pi_3(u_1 \wedge u_2).$$

*Remark 2.4.34.* Under the identification  $\mathbb{P}(\bigwedge^2 \mathfrak{sl}(3, \mathbb{R})) = \mathbb{P}(\mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^3(\mathbb{R}^{3*}))$  we write the element  $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  as  $[(\pi_1(P), \pi_2(P), \pi_3(P))]$ . Let us identify furthermore the elements

of both  $S^3(\mathbb{R}^3)$  and  $S^3(\mathbb{R}^{3*})$  with ternary cubics in the variables  $x, y, z$  for convenience. Explicitly each projection reads in the Plücker coordinates as follows

$$\pi_1(P) = \begin{pmatrix} a_{35} + a_{47} & 2a_{13} - a_{23} + a_{48} & a_{14} + a_{24} + a_{36} \\ -2a_{15} + a_{25} + a_{67} & -a_{35} + a_{68} & -a_{16} + 2a_{26} - a_{54} \\ -a_{17} - a_{27} - a_{58} & a_{18} - 2a_{28} - a_{37} & -a_{47} - a_{68} \end{pmatrix},$$

$$\begin{aligned} \pi_2(P) = & a_{34}x^3 + ((-2a_{14} + a_{24} + a_{36})y + (a_{13} + a_{23} - a_{48})z)x^2 + \\ & ((-2a_{16} + a_{26} + a_{45})y^2 + (3a_{12} - a_{35} + a_{47} - a_{68})yz + (a_{18} + a_{28} - a_{37})z^2)x \\ & - a_{56}y^3 + (a_{15} - 2a_{25} + a_{67})y^2z + (a_{17} - 2a_{27} + a_{58})yz^2 + a_{78}z^3, \end{aligned}$$

$$\begin{aligned} \pi_3(P) = & a_{57}x^3 + ((-2a_{17} + a_{27} + a_{58})y + (a_{15} + a_{25} + a_{67})z)x^2 + \\ & ((-2a_{18} + a_{28} - a_{37})y^2 + (3a_{12} + a_{35} - a_{47} + a_{68})yz + (a_{16} + a_{26} + a_{45})z^2)x \\ & - a_{38}y^3 + (a_{13} - 2a_{23} - a_{48})y^2z + (a_{14} - 2a_{24} + a_{36})yz^2 + a_{46}z^3. \end{aligned}$$

We will fix also the notations  $\pi_2(P) = a_{34}x^3 + p_1(y, z)x^2 + p_2(y, z)x + p_3(y, z)$  and  $\pi_3(P) = a_{57}x^3 + q_1(y, z)x^2 + q_2(y, z)x + q_3(y, z)$ .

Using the explicit projections one can see that we have the following.

**Proposition 2.4.35.** *The morphism  $(\pi_1, \pi_2, \pi_3) : \Lambda^2 \mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^3(\mathbb{R}^{3*})$  is an  $\mathfrak{sl}(3, \mathbb{R})$ -equivariant isomorphism.*

We can now use the identification of Proposition 2.4.30 and the decomposition of  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  in invariant subspaces under the  $\text{SL}(3, \mathbb{R})$ -action in order to classify the family of Lie algebras  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$ .

**Proposition 2.4.36.** *The Lie algebras  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  and  $\mathfrak{g}_{\mathbb{R}}(a', b', c')$  are isomorphic if and only if  $a' = \pm a, b' = b$  and  $c' = c$ .*

*Proof.* Under the identification  $\mathbb{P}(\Lambda^2 \mathfrak{sl}(3, \mathbb{R})) = \mathbb{P}(\mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^3(\mathbb{R}^{3*}))$  we write the elements  $P(a, b, c)$  as  $[(M(a, b, c), f_1(a, b, c), f_2(a, b, c))]$  where

$$M(a, b, c) = \begin{pmatrix} 0 & 2a & -c + 3b \\ 0 & a & c + 6 \\ 0 & b + 2 & -a \end{pmatrix},$$

$$\begin{aligned} f_1(a, b, c) = & (-a^2 - bc)e_1 \odot e_1 \odot e_1 + (3b - c)e_1 \odot e_1 \odot e_2 - 2ae_1 \odot e_1 \odot e_3 \\ & + (3 - c)e_1 \odot e_2 \odot e_2 + 2ae_1 \odot e_2 \odot e_3 + (b - 1)e_1 \odot e_3 \odot e_3 + 3e_2 \odot e_2 \odot e_2 + 3e_2 \odot e_3 \odot e_3 \end{aligned}$$

and

$$f_2(a, b, c) = e_1^* \odot e_1^* \odot e_1^* + (b-1)e_1^* \odot e_2^* \odot e_2^* - 2ae_1^* \odot e_2^* \odot e_3^* \\ + (3-c)e_1^* \odot e_3^* \odot e_3^* + be_2^* \odot e_2^* \odot e_2^* + ae_2^* \odot e_2^* \odot e_3^* + (2c+3b)e_2^* \odot e_3^* \odot e_3^* - 3ae_3^* \odot e_3^* \odot e_3^*.$$

Let us identify furthermore the elements of both  $S^3(\mathbb{R}^3)$  and  $S^3(\mathbb{R}^{3*})$  with ternary cubics in the variables  $x, y, z$  for convenience. Now an element  $g$  of  $\mathrm{SL}(3, \mathbb{R})$  that sends  $P(a, b, c)$  to  $P(a', b', c')$  should preserve the kernel of  $M(a, b, c)$ .

If  $M(a, b, c)$  is of rank 2 then  $\ker M(a, b, c) = \mathbb{R}e_1$ . Hence  $g$  has the form

$$\begin{pmatrix} (\sigma_1\sigma_4 - \sigma_2\sigma_3)^{-1} & \mu & \nu \\ 0 & \sigma_1 & \sigma_2 \\ 0 & \sigma_3 & \sigma_4 \end{pmatrix}$$

and we are left with an action of the group  $\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^2$ . Let us notice that this group induces an action of  $\mathrm{GL}(2, \mathbb{R})$  on  $S^3(\mathbb{R}^2)$ , the binary cubics in two variables  $y$  and  $z$ . More precisely, if  $P \in \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  and  $\pi_2(P) = p_0x^3 + p_1(y, z)x^2 + p_2(y, z)x + p_3(y, z)$  let us call  $\mathrm{pr}(P) = p_3(y, z)$ . Then if  $g = (h, v) \in \mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^2$  we have  $\pi_2(g \cdot P) = h \cdot \mathrm{pr}(P)$ . Let us notice that for all  $a, b, c$  we have  $\mathrm{pr}(P(a, b, c)) = [3(y^3 + yz^2)]$ . Since we search for the  $g \in \mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^2$  such that  $g \cdot f_1(a, b, c) = f_1(a', b', c')$  then in particular we want  $g$  such that  $[h \cdot \mathrm{pr}(P(a, b, c))] = [3(y^3 + yz^2)]$ . Hence since

$$h \cdot \mathrm{pr}(P(a, b, c)) = 3\sigma_1(\sigma_1^2 + \sigma_2^2)y^3 + 3(3\sigma_1^2\sigma_3 + 2\sigma_1\sigma_2\sigma_4 + \sigma_3\sigma_2^2)y^2z \\ + 3(3\sigma_1\sigma_3^2 + 2\sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_4^2)yz^2 + 3\sigma_3(\sigma_3^2 + \sigma_4^2)z^3$$

then we should have  $3\sigma_3(\sigma_3^2 + \sigma_4^2) = 0$ , i.e.  $\sigma_3 = 0$  and  $3(3\sigma_1^2\sigma_3 + 2\sigma_1\sigma_2\sigma_4 + \sigma_3\sigma_2^2) = 6\sigma_1\sigma_2\sigma_4 = 0$ , i.e.  $\sigma_2 = 0$ , also  $3\sigma_1(\sigma_1^2 + \sigma_2^2) = 3(3\sigma_1\sigma_3^2 + 2\sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_4^2)$  that is  $\sigma_1^2 = \sigma_4^2$ . In a similar way we want  $g$  such that  $g \cdot f_2(a, b, c) = f_2(a', b', c')$ . We can compare the terms with at least one  $x^2$  in  $f_2(a, b, c)$ , that is just  $x^3$ , and in  $g \cdot f_2(a, b, c)$  that are

$$(\sigma_1\sigma_4)^3x^3 - 3(\sigma_1\sigma_4)^2\mu\sigma_4x^2y - 3(\sigma_1\sigma_4)^2\nu\sigma_1x^2z.$$

Then we must have  $\sigma_1^2\sigma_4^3\mu = 0$ , i.e.  $\mu = 0$  and  $\sigma_1^3\sigma_4^2\nu = 0$ , i.e.  $\nu = 0$ . Finally the coefficient of  $y^3$  in  $h \cdot \mathrm{pr}(P(a, b, c))$  is 3 times the one of  $x^3$  in  $g \cdot f_2(a, b, c)$  hence  $3\sigma_1^3 = 3(\sigma_1\sigma_4)^3$ , that implies  $\sigma_1^3 = (\pm\sigma_1^2)^3$  so  $\sigma_1 = \pm 1$ . Hence  $g \in \mathrm{SL}(3, \mathbb{R})$  is such that  $g \cdot P(a, b, c) = P(a', b', c')$  if and only if  $g = \mathrm{Id}$  or  $g = g_1$  where

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The element  $g_1$  sends  $P(a, b, c)$  to  $P(-a, b, c)$ .

Finally let us treat the case where the rank of  $M(a, b, c)$  is strictly less than 2, i.e. when  $a = 0$  and  $b = -2$ . If  $c = -6$  the matrix  $M(0, -2, -6)$  is 0 then of course  $P(0, -2, -6)$  is in the same orbit of no other point. If instead  $c \neq -6$  let us act on the point  $P(0, -2, c)$  in order to have  $M(0, -2, c)$  in a normal form. Indeed take

$$g = \sqrt[3]{c+6} \begin{pmatrix} 0 & (c+6)^{-1} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R}),$$

its action on  $P(0, -2, c)$  gives

$$[g \cdot P(0, -2, c)] = [(N, p_1, p_2)] = \left[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{3}{(c+6)^2}x^3 + \frac{12-c}{c+6}x^2z + 3xy^2 - 6xyz - 3(c-4)xz^2, \right. \\ \left. 3(c-4)xy^2 - 6xyz - 3xz^2 + \frac{4-c}{c+6}y^3 + \frac{6-c}{c+6}y^2z + \frac{3}{c+6}yz^2 + \frac{1}{c+6}z^3 \right].$$

A point in the same orbit should have the same coefficients of the cubics modulo the actions of the stabiliser of  $N$  that is

$$\mathrm{Stab}_{\mathrm{SL}(3, \mathbb{R})} \left( \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ 0 & \sigma_1 & 0 \\ 0 & \sigma_4 & \sigma_1^{-2} \end{pmatrix} \mid \sigma_1 \in \mathbb{R}^*, \sigma_2, \sigma_3, \sigma_4 \in \mathbb{R} \right\}.$$

Taking  $g = \begin{pmatrix} \sigma_1 & -\frac{\sigma_1}{3(c+6)} & -\frac{\sigma_1}{3(c+6)} \\ 0 & \sigma_1 & 0 \\ 0 & \sigma_1^{-2} & \sigma_1^{-2} \end{pmatrix} \in \mathrm{Stab}_{\mathrm{SL}(3, \mathbb{R})}(N)$  we obtain

$$[g \cdot (N, p_1, p_2)] = \left[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, zx^2 + \left( 3\sigma_1^3y^2 + \frac{9-3c}{\sigma_1^3}z^2 \right) x, \left( \frac{3c-9}{\sigma_1^3}y^2 - 3\sigma_1^3z^2 \right) x \right].$$

The parameter  $c$  is then an invariant and all the points  $P(0, -2, c)$  are pairwise not isomorphic.  $\square$

Putting together Proposition 2.3.4, 2.3.5, 2.4.21 and 2.4.36 we see that we have proved Theorem 2.3.1.

## 2.5 Nilpotent Hermite-Lorentz crystallographic groups in dimension 4

In this section we want to prove the following.

**Theorem 2.5.1.** *The list given in Appendix A with  $\mathbb{K} = \mathbb{Q}$  is a complete list without repetition of abstract commensurability classes of nilpotent crystallographic subgroups of  $U(3, 1) \times \mathbb{C}^{3+1}$ .*

In Section 2.2 we looked at crystallographic groups  $\Gamma$  and studied the abelian-by-cyclic case. We are then left with the virtually nilpotent case. We know that nilpotent Hermite-Lorentz crystallographic groups are lattices in unipotent simply transitive subgroups of  $\mathcal{H}(3, 1)$ . Since we have listed all the possible unipotent Lie subgroups of  $\mathcal{H}(3, 1)$  that act simply transitively on  $\mathfrak{a}(V)$  we are interested in studying their lattices. Remember that in Section 1.2.1 we have recalled the theory of lattices in nilpotent Lie groups. We will start by studying  $\mathbb{Q}$ -isomorphism classes of  $\mathbb{Q}$ -forms of the Lie algebras  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$ .

**Proposition 2.5.2.** *Every  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  is isomorphic to one of the Lie algebras  $\mathfrak{g}_{\mathbb{Q}}(0, -2, c)$  or  $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$  of Appendix A. Moreover every Lie algebra  $\mathfrak{g}_{\mathbb{Q}}(0, -2, c)$  or  $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$  as in Appendix A is obtained as  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  for some  $a, b, c$ . Furthermore in Appendix A we give representatives of  $\mathbb{Q}$ -isomorphism classes of the families  $\mathfrak{g}_{\mathbb{Q}}(0, -2, c)$  and  $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$ .*

*Proof.* Let  $\mathfrak{h}$  be a  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$ . Let  $P(a, b, c) \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ , see Definition 2.4.29, be the plane associated to  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  under the bijection of Proposition 2.4.30. By definition  $\mathfrak{h} \otimes \mathbb{R} \cong \mathfrak{g}_{\mathbb{R}}(a, b, c)$ . By [17, Theorem 3.15]  $\mathfrak{h}$  is Carnot over  $\mathbb{Q}$  if and only if  $\mathfrak{h} \otimes \mathbb{R}$  is Carnot over  $\mathbb{R}$ . Hence  $\mathfrak{h} = \mathfrak{g}(\mathbb{Q})$ , see Definition 2.4.18, to which we associate an element  $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{Q}))$  by Proposition 2.4.30. Furthermore  $P$  is in the same real orbit of  $P(a, b, c)$  under the action of  $\text{SL}(3, \mathbb{R})$ . We will use the same terminology as in Section 2.4.4 and in particular of Remark 2.4.34. The idea of the proof is the same as in Proposition 2.4.36 but over  $\mathbb{Q}$ . Observe that the  $\text{SL}(3, \mathbb{R})$ -action that we had over  $\mathbb{R}$  is now replaced by an action of  $\text{PGL}(3, \mathbb{Q}) \not\cong \text{SL}(3, \mathbb{Q})$ . For a subgroup  $H$  of  $\text{GL}(3, \mathbb{Q})$  we will denote by  $P(H)$  its image in  $\text{PGL}(3, \mathbb{Q})$ . Knowing that  $P$  is in the same real orbit of  $P(a, b, c)$  implies that 0 is an eigenvalue of  $\pi_1(P)$ . We can then conjugate in order to have

$$\pi_1(P) = \left[ \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right].$$

Assuming  $P$  is such that  $\pi_1(P)$  has rank 2 allows us to reduce the action to the one of  $P((\text{GL}(1, \mathbb{Q}) \times \text{GL}(2, \mathbb{Q})) \times \mathbb{Q}^2) \cong \text{GL}(2, \mathbb{Q}) \times \mathbb{Q}^2$ , where we have normalised the up-left coefficient to be 1. Using the action of  $\mathbb{Q}^2$  we can also achieve that  $q_1(y, z) = 0$  for  $\pi_3(P)$ . Hence we reduce the action to the one of the stabiliser of  $q_1(y, z) = 0$  in  $\text{GL}(2, \mathbb{Q}) \times \mathbb{Q}^2$ , that is  $\text{GL}(2, \mathbb{Q})$ . The subvariety defined by these conditions is defined by the following equations in the Plücker coordinates  $\{a_{35} + a_{47} = 0, -2a_{15} + a_{25} + a_{67} = 0, a_{17} + a_{27} + a_{58} = 0, a_{15} + a_{25} + a_{67} = 0, -2a_{17} + a_{27} + a_{58} = 0\}$ , compare with 2.4.34. The Lie algebra  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  has also the property that  $a_{57} = \text{coeff}(\pi_3(P(a, b, c)), x^3) \neq 0$ ,

hence the same is true for  $P$ , which is in the same real orbit. Working in the affine chart of the Grassmannian defined by  $a_{57} \neq 0$  and solving the above equations we find that  $P$  is in the orbit of

$$\begin{pmatrix} 0 & -e & -k & j & 1 & -g & 0 & -f \\ 0 & -g & j & l & 0 & -h & 1 & e \end{pmatrix}. \quad (2.8)$$

We still have the action of  $\mathrm{GL}(2, \mathbb{Q})$ . This action induces an action over the binary cubics in the variables  $y, z$  of  $\pi_2(P)$ :  $p_3(y, z) = hy^3 - 3gy^2z + 3eyz^2 + fz^3 \in \mathbb{Q}[y, z]$ . This cubic is in the same real orbit of the corresponding cubic of  $\pi_2(P(a, b, c))$ , that is  $3y^3 + 3yz^2$ , hence it is a cubic with negative discriminant. We are then interested in finding normal forms for the action of  $\mathrm{GL}(2, \mathbb{Q})$  over binary cubics with negative discriminant. We can proceed as follows. First of all we can assume without loss of generality that  $h \neq 0$ , using the action of  $\mathrm{GL}(2, \mathbb{Q})$ . We consider the associated univariate polynomial  $p_3(y, 1)$ . Since the discriminant is negative  $p_3(y, 1)$  has only one real root,  $\rho \in \mathbb{R}$ . If  $\rho$  is rational, using the action of  $\mathrm{GL}(2, \mathbb{Q})$  on  $\rho$  by homographies, we can bring  $\rho$  to 0 and reduce  $p_3(y, z)$  to  $y(hy^2 - 3gyz + 3ez^2)$ . Reducing further the quadratic polynomial we find that the normal forms for  $p_3(y, z)$  over  $\mathbb{Q}$  is  $y(hy^2 + ez^2)$  with  $y(hy^2 + ez^2)$  in the same orbit as  $y(h'y^2 + e'z^2)$  if and only if there exists  $\sigma_1, \sigma_2 \in \mathbb{Q}^*$  such that  $h' = \sigma_1^3 h$  and  $e' = \sigma_1 \sigma_2^2 e$ . If  $p_3(y, 1)$  is irreducible over the rationals then we can consider the field  $K$ , a cubic extension of  $\mathbb{Q}$ , obtained by adjoining the root  $\rho$  to  $\mathbb{Q}$ . Cubic fields have been classified in [40]. Their result, [40, Corollary 1.3], is that in every such extension one can find an element  $\theta$  whose minimal polynomial is either

$$y^3 - t \quad (2.9)$$

or

$$y^3 - 3y - t \quad (2.10)$$

with  $t \in \mathbb{Q}$ . A polynomial as in (2.9) has always negative discriminant and is irreducible over  $\mathbb{Q}$  if and only if  $t \notin \mathbb{Q}^3$ . A polynomial as in (2.10) has negative discriminant if and only if  $t^2 - 4 > 0$  and is irreducible over  $\mathbb{Q}$  if and only if  $t$  does not belong to the image of the function  $x^3 - 3x$  defined over the rational. Furthermore two polynomials in the form (2.9),  $y^3 - t_1$  and  $y^3 - t_2$ , generate the same extension if and only if there exists  $\mu \in \mathbb{Q}$  such that  $t_2 = \mu^3 t_1^j$  with  $j = 1, 2$ . Two polynomials in the form (2.9),  $y^3 - 3y - t_1$  and  $y^3 - 3y - t_2$ , generate the same extension if and only if  $t_2 = -3t_1 \alpha^2 \beta + t_1 \beta^3 + 6\alpha + \alpha^3 t_1^2 - 8\alpha^3$  with  $\alpha, \beta \in \mathbb{Q}$  such that  $\alpha^2 + t_1 \alpha \beta + \beta^2 = 1$ . Finally, a polynomial as in (2.10) generates the same extension as one in the form (2.9) if the polynomial  $x^2 + tx + 1$  has a root in  $\mathbb{Q}$ , i.e if  $t^2 - 4 \in \mathbb{Q}^2$ . So  $\rho$  can be written as  $m\theta^2 + n\theta + r$  with  $m, n, r \in \mathbb{Q}$  and  $\theta$  satisfying (2.9) or (2.10). Observe that for  $g \in \mathrm{GL}(2, \mathbb{Q})$  the element  $g \cdot p_3(y, z)$  equals a minimal polynomial of  $g \cdot \rho$  where again we act on  $\rho$  by homographies. Via this action we can bring  $\rho = m\theta^2 + n\theta + r$  to either  $m\theta^2 + \theta$  or  $\theta^2$ . These two cases correspond to different orbits and different values of  $m$  also correspond to different orbits. We are left with finding a minimal polynomial

for  $m\theta^2 + \theta$  and  $\theta^2$ . If  $\theta$  satisfies (2.9) the element  $m\theta^2 + \theta$  has minimal polynomial

$$y^3 - 3mty - m^3t^2 - t \quad (2.11)$$

and the element  $\theta^2$  has minimal polynomial

$$y^3 - t^2. \quad (2.12)$$

If instead  $\theta$  is a root of (2.10) then  $m\theta^2 + \theta$  has minimal polynomial

$$y^3 - 6my^2 + (9m^2 - 3tm - 3)y + 3tm^2 - m^3t^2 - t \quad (2.13)$$

and  $\theta^2$  has minimal polynomial

$$y^3 - 6y^2 + 9y - t^2. \quad (2.14)$$

The way to find the minimal polynomial of a sum of algebraic numbers is classical and uses the resultant between the two minimal polynomials of the corresponding algebraic numbers. We declare a plane as in (2.8) in normal form if  $p_3(y, z)$  is in one of the normal forms (2.11), (2.12), (2.13) or (2.14). We still have the action by homotheties  $\text{diag}(1, \mu, \mu) \in P(\text{GL}(3, \mathbb{Q}))$  with  $\mu \in \mathbb{Q}^*$  that does not change  $\rho$  but indeed transform  $P$ . Hence two normal forms are the same if they differ by multiplication by  $\mu^3$  for some  $\mu \in \mathbb{Q}^*$ . Then notice that once  $p_3(y, z)$  is in a normal form the other coefficients of  $P$  cannot be reduced anymore and they describe different Lie algebras. This is because the stabiliser of a binary cubic with non-zero discriminant is trivial. Calculating the Lie algebra associated to (2.8) we find the families of Lie algebras  $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$  as in Appendix A. Furthermore each of the representative of  $\mathbb{Q}$ -isomorphism classes in Appendix A corresponds to one of the normal forms that we have just constructed.

Assume now that  $\mathfrak{h}$  is a  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(0, -2, c)$  with  $c \neq -6$ . This family of Lie algebras is characterised by the fact that  $\pi_1(P(0, -2, c))$  has rank 1. Remember from Proposition 2.4.36 that we can bring  $P(0, -2, c)$  via the action to

$$\left[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z^2 + (3y^2 + (9 - 3c)z^2)x, ((3c - 9)y^2 - 3z^2)x \right]. \quad (2.15)$$

We can then assume that the plane  $P$  associated to  $\mathfrak{h}$  is such that  $\pi_1(P)$  is the same as in (2.15). Then we are left with the action of the stabiliser of such a matrix that is a subgroup of  $P((\text{GL}(1, \mathbb{Q}) \times \text{GL}(2, \mathbb{Q})) \ltimes \mathbb{Q}^2)$ , see proof of Proposition 2.4.36. Since for  $\mathfrak{g}_{\mathbb{R}}(0, -2, c)$  we have  $p_3(y, z) = 0$ ,  $a_{57} = 0$  and  $q_1(y, z) = 0$ , then we must have the same for  $P$ . Furthermore there exists  $g$  in the stabiliser of  $\pi_1(P)$  such that  $g \cdot P$  satisfies  $\text{coeff}(p_1(y, z), y) = \text{coeff}(p_2(y, z), yz) = \text{coeff}(q_3(y, z), z^3) = 0$ . Solving

these equations we see that

$$g \cdot P = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & c \end{pmatrix}.$$

One can see that  $[\pi_1(P), \pi_2(P), \pi_3(P)]$  is in the same form as (2.15), hence the associated Lie algebra is  $\mathfrak{g}_{\mathbb{Q}}(0, -2, c)$ .

Finally if  $\mathfrak{h}$  is a  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  such that  $\pi_1(P(a, b, c)) = 0$  then  $a = 0, b = -2$  and  $c = -6$ . Then  $P(0, -2, -6)$  is generated by the following two matrices

$$u = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -9 \\ 0 & 1 & 0 \end{pmatrix}.$$

As  $P$  is a  $\mathbb{Q}$ -form of  $P(0, -2, -6)$ , for all  $X \in P$  there exist  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha u + \beta v = X$ . Since  $X$  is a rational matrix,  $\alpha$  and  $\beta$  must be rational. Hence  $P$  is generated by  $u$  and  $v$  over  $\mathbb{Q}$ . That is  $\mathfrak{h}$  is isomorphic to  $\mathfrak{g}_{\mathbb{Q}}(0, -2, -6)$ .

Last let us say a few words about why every Lie algebra  $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$  or  $\mathfrak{g}_{\mathbb{Q}}(0, -2, c)$  as in Appendix A is obtained as  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  for some  $a, b, c$ . Let us consider a Lie algebra  $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$  with  $3g^2e^2 - 6efgh + 4g^3f - 4e^3h - h^2f^2 < 0$ . The element in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{Q}))$  that corresponds to  $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$  is

$$P = \begin{pmatrix} 0 & -e & -k & j & 1 & -g & 0 & -f \\ 0 & -g & j & l & 0 & -h & 1 & e \end{pmatrix}.$$

One can compute that the binary cubic in the variables  $y, z$  of  $\pi_2(P)$  is  $p_3(y, z) = hy^3 - 3gy^2z + 3eyz^2 + fz^3 \in \mathbb{Q}[y, z]$ . The condition  $3g^2e^2 - 6efgh + 4g^3f - 4e^3h - h^2f^2 < 0$  guarantees that  $p_3(y, z)$  has negative discriminant, hence, over  $\mathbb{R}$ , is in the same orbit of the corresponding cubic of  $\pi_2(P(a, b, c))$ , that is  $3y^3 + 3yz^2$ . Hence there exist  $g \in \text{GL}(2, \mathbb{R})$  sending  $p_3(y, z)$  to  $3y^3 + 3yz^2$  and hence bringing  $P$  to

$$\begin{pmatrix} 0 & -1 & -k' & j' & 1 & 0 & 0 & 0 \\ 0 & 0 & j' & l' & 0 & -3 & 1 & 1 \end{pmatrix}.$$

This element is precisely  $P(j', k', l')$  corresponding to  $\mathfrak{g}_{\mathbb{R}}(j', k', l')$ . □

*Remark 2.5.3.* A part from the family  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$ , it is clear from the presentation we have given of representatives of the isomorphism classes of the other nilpotent Lie algebras, that all the others admit  $\mathbb{Q}$ -forms. Furthermore from the simple remark that there are uncountably many non-isomorphic Lie algebras  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  but that there are countably many  $\mathbb{Q}$ -forms we know that there are certain Lie algebras  $\mathfrak{g}_{\mathbb{R}}(a, b, c)$  that do not admit  $\mathbb{Q}$ -forms.

The following proposition answers the question of  $\mathbb{Q}$ -isomorphism classes of  $\mathbb{Q}$ -forms in the Lie algebras  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$ .

**Proposition 2.5.4.** *The  $\mathbb{Q}$ -isomorphism classes of  $\mathbb{Q}$ -forms of the family  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  are  $\mathfrak{g}_{\mathbb{Q}}(0, 0, 0, 0)$  and  $\mathfrak{g}_{\mathbb{Q}}(0, 0, \varepsilon, 1)$  with  $\varepsilon \in \mathbb{Q}$ , where  $\mathfrak{g}_{\mathbb{Q}}(0, 0, \varepsilon', 1) \cong \mathfrak{g}_{\mathbb{Q}}(0, 0, \varepsilon, 1)$  if and only if there exists  $\alpha \in \mathbb{Q}^*$  such that  $\varepsilon' = \alpha^2 \varepsilon$ .*

*Proof.* We will first proceed as in the previous proposition and prove that if  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  admits a  $\mathbb{Q}$ -form  $\mathfrak{h}$ , then  $\mathfrak{h} \cong \mathfrak{g}_{\mathbb{Q}}(0, a', b', c')$  with  $a', b', c' \in \mathbb{Q}$ . Secondly we will study the isomorphism classes of  $\mathfrak{g}_{\mathbb{Q}}(0, a', b', c')$ . Let  $\mathfrak{h}$  be a  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$ . Then by definition  $\mathfrak{h} \otimes \mathbb{R} \cong \mathfrak{g}_{\mathbb{R}}(0, a, b, c)$ . Hence  $\mathfrak{h} = \mathfrak{g}(Q)$ , see Definition 2.4.18. One can see that the 2-dimensional plane associated with  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  is

$$P(0, a, b, c) = \begin{pmatrix} 0 & 0 & -b & a & 1 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & 0 & 1 & 0 \end{pmatrix},$$

whose projections are

$$\left[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & b & -a \end{pmatrix}, -(a^2 + bc)x^3 + (-cy^2 + 2ayz + by^2)x, x^3 + (by^2 - 2ayz - cz^2)x \right].$$

Let  $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{Q}))$  be the 2-plane in  $\mathfrak{sl}(3, \mathbb{Q})$  corresponding to  $Q$  such that  $\mathfrak{h} = \mathfrak{g}(Q)$ , then  $P$  is in the same real orbit as  $P(0, a, b, c)$ . This implies that 0 is an eigenvalue of  $\pi_1(P)$ . We can then conjugate in order to have

$$\pi_1(P) = \left[ \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right].$$

Assuming  $P$  is such that  $\pi_1(P)$  has rank 2 allows us to reduce the action to  $\text{GL}(2, \mathbb{Q}) \times \mathbb{Q}^2$ . The plane  $P(0, a, b, c)$  has the further property that  $p_3(y, z) = 0$  and  $a_{57} = \text{coeff}(\pi_3(P(0, a, b, c)), x^3) \neq 0$ , hence the same is true for  $P$ . Furthermore as in the previous proposition we can use the action to achieve  $p_1(y, z) = 0$  for  $\pi_2(P)$ . Working in the affine chart of the Grassmannian defined by  $a_{57} \neq 0$  and solving the above equations we find that  $P$  is in the orbit of

$$\begin{pmatrix} 0 & 0 & b' & a' & 1 & 0 & 0 & 0 \\ 0 & 0 & a' & c' & 0 & 0 & 1 & 0 \end{pmatrix} \quad (2.16)$$

with  $a', b', c' \in \mathbb{Q}$ . This means that  $\mathfrak{h} \cong \mathfrak{g}_{\mathbb{Q}}(0, a', b', c')$ .

Suppose now that  $\pi_1(P)$  has rank 1. We have seen in Proposition 2.4.21 that in this case  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  is isomorphic to  $\mathfrak{g}_{\mathbb{R}}(0, 0, 0, 1)$ . We can then bring the associated plane via the  $\mathbb{R}$ -action to

$$\left[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, -zx^2, -y^2z - z^3 \right]. \quad (2.17)$$

We can then assume that the plane  $P$  associated to  $\mathfrak{h}$  is such that  $\pi_1(P)$  is the same as in (2.17). Then we are left with the action of the stabiliser of such matrix, that is included in  $\mathrm{GL}(2, \mathbb{Q}) \times \mathbb{Q}^2$ . Since for  $\mathfrak{g}_{\mathbb{R}}(0, 0, 0, 1)$  we have  $p_3(y, z) = p_2(y, z) = 0$ ,  $a_{57} = 0$  and  $q_1(y, z) = q_2(y, z) = 0$ , and since the stabiliser acts separately on each of these terms, we must have the same also for  $P$ . Furthermore using the action we may put  $\mathrm{coeff}(p_1(y, z), y) = \mathrm{coeff}(\pi_2(P), x^3) = \mathrm{coeff}(q_3(y, z), yz^2) = 0$ . Then

$$[\pi_1(P), \pi_2(P), \pi_3(P)] = \left[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, -zx^2, -y^2z - \frac{1}{c}z^3 \right].$$

This means that  $\mathfrak{h} \cong \mathfrak{g}_{\mathbb{Q}}(0, 0, 0, \frac{1}{c})$ .

Finally if  $\pi_1(P) = 0$  then  $\mathfrak{h}$  is a  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(0, 0, 0, 0)$  that is generated by the following two matrices

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This means that  $\mathfrak{h}$  is isomorphic to  $\mathfrak{g}_{\mathbb{Q}}(0, 0, 0, 0)$ . In conclusion we have proved that  $\mathfrak{h} \cong \mathfrak{g}_{\mathbb{Q}}(0, a, b, c)$ . As we have seen in Proposition 2.4.21, finding representatives for the isomorphism classes corresponds to finding normal forms for the adjoint action of  $\mathrm{GL}_2(\mathbb{Q})$  over  $\mathbb{P}(\mathfrak{sl}(2, \mathbb{Q}))$ . Using the theory of rational canonical forms we can see that the normal forms are

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \text{ with } \varepsilon \notin \mathbb{Q}^2$$

or the 0 matrix. Hence as representatives of the  $\mathbb{Q}$ -isomorphism classes of the family  $\mathfrak{g}_{\mathbb{Q}}(0, a, b, c)$  we have  $\mathfrak{g}_{\mathbb{Q}}(0, 0, 0, 0)$  and  $\mathfrak{g}_{\mathbb{Q}}(0, 0, \varepsilon, 1)$ , noticing that  $\mathfrak{g}_{\mathbb{Q}}(0, 0, \varepsilon, 1) \cong \mathfrak{g}_{\mathbb{Q}}(0, 1, 0, 0)$  if  $\varepsilon \in \mathbb{Q}^2$ . Finally it is clear that  $\mathfrak{g}_{\mathbb{Q}}(0, 0, \varepsilon', 1) \cong \mathfrak{g}_{\mathbb{Q}}(0, 0, \varepsilon, 1)$  if and only if the ratio of  $\varepsilon$  and  $\varepsilon'$  is the square of a rational number.  $\square$

*Remark 2.5.5.* From the proof of Proposition 2.5.4 we see in particular that every  $\mathbb{Q}$ -form of  $\mathfrak{g}_{\mathbb{R}}(0, a, b, c)$  is of the form  $\mathfrak{g}_{\mathbb{Q}}(0, a', b', c')$  for some  $(a', b', c') \in \mathbb{Q}^3$ , which was not clear a priori. Furthermore  $\mathfrak{g}_{\mathbb{Q}}(0, a_1, b_1, c_1) \cong \mathfrak{g}_{\mathbb{Q}}(0, a_2, b_2, c_2)$  if and only if  $(0, a_1, b_1, c_1) = (0, 0, 0, 0) = (0, a_2, b_2, c_2)$  or  $(0, a_1, b_1, c_1) \neq (0, 0, 0, 0) \neq (0, a_2, b_2, c_2)$  and  $a_1^2 + b_1c_1$  and  $a_2^2 + b_2c_2$  differ by multiplication by a non zero rational number.

### Skjelbred and Sund method

Now that the questions of commensurability classes of lattices in the Lie groups associated with the Lie algebras  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$  is settled we want to study the commensurability question for the Lie groups associated to the Lie algebras  $L_j$  and  $N_j$ . We have seen that, in order to study this question, we have to understand the  $\mathbb{Q}$ -isomorphism classes of rational subalgebras of these Lie algebras. To do so

we introduce a method that was developed by Skjelbred and Sund in [44] and applied by Gong in his thesis [30] in order to classify 7-dimensional real nilpotent Lie algebras. To state the method we need to introduce some terminology.

**Definition 2.5.6.** Let  $\mathbb{K}$  be a field and  $\mathfrak{g}$  a Lie algebra over  $\mathbb{K}$ . A map  $B : \wedge^2 \mathfrak{g} \rightarrow \mathbb{K}$  such that

$$B([x_1, x_2], x_3) + B([x_2, x_3], x_1) + B([x_3, x_1], x_2) = 0$$

is called a *cocycle* and the space of cocycles is denoted by  $Z^2(\mathfrak{g}, \mathbb{K})$ . A map  $B : \wedge^2 \mathfrak{g} \rightarrow \mathbb{K}$  such that there exists  $g \in \text{Hom}(\mathfrak{g}, \mathbb{K})$  and

$$B(x, y) = g([x, y])$$

is called a *coboundary* and the space of coboundaries is denoted by  $B^2(\mathfrak{g}, \mathbb{K})$ . It can be noticed that the coboundaries form a subspace of the cocycles and finally the quotient space

$$H^2(\mathfrak{g}, \mathbb{K}) = Z^2(\mathfrak{g}, \mathbb{K})/B^2(\mathfrak{g}, \mathbb{K})$$

is called the *2-nd cohomology group* of  $\mathfrak{g}$  with coefficients in  $\mathbb{K}$ .

**Definition 2.5.7.** For  $B : \wedge^2 \mathfrak{g} \rightarrow \mathbb{K}$  we can define the set

$$\mathfrak{g}_B^\perp = \{x \in \mathfrak{g} \mid B(x, \mathfrak{g}) = 0\}.$$

And if  $B = (B_1, \dots, B_k) : \wedge^2 \mathfrak{g} \rightarrow \mathbb{K}^k$  we can define  $\mathfrak{g}_B^\perp = \mathfrak{g}_{B_1}^\perp \cap \dots \cap \mathfrak{g}_{B_k}^\perp$ .

**Definition 2.5.8.** Let us denote by  $\text{Gr}(k, H^2(\mathfrak{g}, \mathbb{K}))$  the  $k$ -th Grassmannian of  $H^2(\mathfrak{g}, \mathbb{K})$ . Furthermore if  $B$  is a cocycle let us denote by  $\tilde{B}$  its image in cohomology. Thus we define a subspace of the  $k$ -th Grassmannian of the cohomology of  $\mathfrak{g}$  as follows

$$U_k(\mathfrak{g}) = \{\tilde{B}_1 \mathbb{K} \oplus \dots \oplus \tilde{B}_k \mathbb{K} \in \text{Gr}(k, H^2(\mathfrak{g}, \mathbb{K})) \mid \mathfrak{g}_{B=(B_1, \dots, B_k)}^\perp \cap \mathcal{Z}(\mathfrak{g}) = 0\}$$

where  $\mathcal{Z}(\mathfrak{g})$  is the center of  $\mathfrak{g}$ .

If we call  $\text{Aut}(\mathfrak{g})$  the automorphism group of the Lie algebra  $\mathfrak{g}$  we can notice that we have an action of  $\text{Aut}(\mathfrak{g})$  on  $H^2(\mathfrak{g}, \mathbb{K})$  induce by the following action on cocycles:

$$\text{if } \varphi \in \text{Aut}(\mathfrak{g}) \text{ and } B \in Z^2(\mathfrak{g}, \mathbb{K}) \text{ then } \varphi \cdot B(x, y) = B(\varphi(x), \varphi(y)).$$

Furthermore it can be proved that this action induces an action on  $U_k(\mathfrak{g})$ . We are now ready to state the theorem of Skjelbred and Sund.

**Theorem 2.5.9** ([44, Theorem 3.5.]). *Let  $\mathfrak{g}$  be a Lie algebra over a field  $\mathbb{K}$ . The isomorphism classes of Lie algebras  $\tilde{\mathfrak{g}}$ , with center  $\tilde{\mathfrak{Z}}$  of dimension  $k$ , with  $\tilde{\mathfrak{g}}/\tilde{\mathfrak{Z}} \cong \mathfrak{g}$  and without abelian factors are in bijective correspondence with elements in  $U_k(\mathfrak{g})/\text{Aut}(\mathfrak{g})$ .*

*Remark 2.5.10.* Let  $B \in U_k(\mathfrak{g})/\text{Aut}(\mathfrak{g})$  be a representative of one orbit. Then the corresponding  $k$ -dimensional central extension of  $\mathfrak{g}$  is defined as the vector space  $\mathfrak{g}(B) = \mathfrak{g} \oplus \mathbb{K}^k$  with Lie brackets

$$[(x, u), (y, v)] = ([x, y], B(x, y)).$$

We will now apply this method to the aforementioned Lie algebras.

**Proposition 2.5.11.** *All the Lie algebras  $L_i$  and  $N_j$  have just one  $\mathbb{Q}$ -form, up to  $\mathbb{Q}$ -isomorphism, except  $L_6^{\mathbb{R}}(\varepsilon), N_3^{\mathbb{R}}(\varepsilon)$  and  $N_9^{\mathbb{R}}(\varepsilon)$ . For these Lie algebras their  $\mathbb{Q}$ -forms are  $L_6^{\mathbb{Q}}(\varepsilon), N_3^{\mathbb{Q}}(\varepsilon)$  and  $N_9^{\mathbb{Q}}(\varepsilon)$  respectively, as defined in Appendix A.*

*Proof.* First consider the Lie algebras presented in Appendix A that are the sum of a 6-dimensional Lie algebra and an abelian ideal. Since in [20] all nilpotent 6-dimensional Lie algebras over any field of characteristic different from 2 are classified up to isomorphism the result follows from their analysis.

Now consider the Lie algebras in Appendix A that are the sum of a 7-dimensional Lie algebra and an abelian ideal. Nilpotent Lie algebras of dimension 7 over  $\mathbb{R}$  are classified by Gong in [30]. Hence we can follow his classification for the isomorphism classes that concern us. For almost all the Lie algebras in dimension 7 the analysis that is done for  $\mathbb{R}$  in [30] works for  $\mathbb{Q}$  without any problem so we will point out only the cases where there is a difference. Let again fix some terminology. If  $(e_i)_i$  is a basis for  $\mathfrak{g}$  we let  $\Delta_{ij} = (e_i \wedge e_j)^*$  be the elements of the basis for  $(\wedge^2 \mathfrak{g})^*$ . We will write the elements of the cohomology group  $H^2(\mathfrak{g}, \mathbb{K})$  simply as cocycles thinking them as equivalence classes. Finally for the action of  $\text{Aut}(\mathfrak{g})$  on  $H^2(\mathfrak{g}, \mathbb{K})$  if  $g \cdot (\sum \alpha_{ij} \Delta_{ij}) = \sum \alpha'_{ij} \Delta_{ij}$  we will write  $\alpha_{ij} \mapsto \alpha'_{ij}$ .

The Lie algebras  $N_9^{\mathbb{R}}(\varepsilon)$  are 2-dimensional central extensions of the Lie algebra  $\mathfrak{g}$  defined on the basis  $(x_1, \dots, x_5)$  by  $[x_1, x_2] = x_3$ . An element of the cohomology group of  $\mathfrak{g}$  reads as

$$a\Delta_{13} + b\Delta_{14} + c\Delta_{15} + d\Delta_{23} + e\Delta_{24} + f\Delta_{25} + g\Delta_{45}$$

and we will just write  $B = [a, b, c, d, e, f, g]$ . The action of the automorphism group of  $\mathfrak{g}$  on it is as follows:

$$\begin{aligned} a &\mapsto aa_{11}\delta + da_{21}\delta; \\ b &\mapsto a_{11}(aa_{34} + ba_{44} + ca_{54}) + a_{21}(da_{34} + ea_{44} + fa_{54}) + g(a_{41}a_{54} - a_{51}a_{44}); \\ c &\mapsto a_{11}(aa_{35} + ba_{45} + ca_{55}) + a_{21}(da_{35} + ea_{45} + fa_{55}) + g(a_{41}a_{55} - a_{51}a_{45}); \\ d &\mapsto aa_{12}\delta + da_{22}\delta; \\ e &\mapsto a_{12}(aa_{34} + ba_{44} + ca_{54}) + a_{22}(da_{34} + ea_{44} + fa_{54}) + g(a_{42}a_{54} - a_{52}a_{44}); \\ f &\mapsto a_{12}(aa_{35} + ba_{45} + ca_{55}) + a_{22}(da_{35} + ea_{45} + fa_{55}) + g(a_{42}a_{55} - a_{52}a_{45}); \\ g &\mapsto g(a_{44}a_{55} - a_{54}a_{45}) \end{aligned}$$

where  $\delta = a_{11}a_{22} - a_{12}a_{21} = a_{33}$ . Looking at the definition of the family of Lie algebras  $N_9^{\mathbb{R}}(\varepsilon)$  in Appendix A we can see that it corresponds to the 2-dimensional subspace of the cohomology of  $\mathfrak{g}$  generated by  $(\Delta_{13} + \Delta_{14} + \varepsilon\Delta_{25}) \wedge (\Delta_{15} + \Delta_{23})$  with  $\varepsilon \in \{0, 1, -1\}$ . Let  $V_1$  be the subspace of  $H^2(\mathfrak{g}, \mathbb{R})$  generated by  $\{\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}\}$ ,  $V_2$  the subspace generated by  $\{\Delta_{14}, \Delta_{15}, \Delta_{24}, \Delta_{25}, \Delta_{45}\}$  and  $V_3 = V_1 \cap V_2$ . Notice that  $V_1, V_2$  and  $V_3$  are submodules for the group action. If  $L_\varepsilon$  is a 2-dimensional subspace of  $H^2(\mathfrak{g}, \mathbb{R})$  associated to  $N_9^{\mathbb{R}}(\varepsilon)$  then in Gong's analysis  $L$  is characterised by  $L_\varepsilon \subseteq V_1$  and  $L_\varepsilon \cap V_3 = 0$ . Let us now consider a  $\mathbb{Q}$ -form  $\mathfrak{h}$  of  $N_9^{\mathbb{R}}(\varepsilon)$ . Then  $\mathfrak{h}$  is a 2-dimensional central extension of  $\mathfrak{g}$  and let  $P$  be the 2-dimensional subspace of  $H^2(\mathfrak{g}, \mathbb{Q})$  associated to it. We can then assume  $P \subseteq V_1$  and  $P \cap V_3 = 0$  so that the elements of a basis for  $P$  are  $A = [1, b, c, d, e, f, 0]$  and  $B = [0, b_1, c_1, 1, e_1, f_1, 0]$ . Following Gong's analysis we can bring the second element of the basis to either  $B_1 = [0, 0, 1, 1, 0, 0, 0]$  or  $B_2 = [0, 0, 0, 1, 0, 0, 0]$ . For the case  $B_1 = [0, 0, 1, 1, 0, 0, 0]$  we choose  $a_{21} = a_{34} = a_{54} = 0$ ,  $a_{35} = -\frac{a_{12}a_{55}}{a_{22}}$ ,  $a_{11}a_{55} = a_{11}a_{22}^2 = 1$  in order to fix  $B_1$ . Considering then the action on  $A$  we have

$$\begin{aligned} a &\mapsto a_{11}\delta; b \mapsto a_{11}ba_{44}; c \mapsto a_{11}(a_{35} + ba_{45} + ca_{55}); d \mapsto a_{12}\delta; \\ e &\mapsto a_{12}ba_{44} + a_{22}ea_{44}; f \mapsto a_{12}(a_{35} + ba_{45} + ca_{55}) + a_{22}(ea_{45} + fa_{55}); g \mapsto 0. \end{aligned}$$

Notice that  $b$  and  $e$  cannot be both 0 otherwise there will be a non-trivial element in the intersection  $\mathfrak{g}_{(A,B)}^\perp \cap \mathcal{L}(\mathfrak{g})$ . Now assuming  $b \neq 0$  we can make it 1 taking  $a_{44} = \frac{1}{ba_{11}}$  and solving for  $a_{12}$  make  $e$  equal 0. Solving for  $a_{45}$  we can make  $c = d$  and then subtracting a multiple of  $B$  make them equal 0. Then taking  $a_{12} = a_{35} = a_{45} = 0$  we obtain

$$a = 1 \mapsto a_{11}\delta; b = 1 \mapsto a_{11}a_{44}; c = 0 \mapsto 0; d = 0 \mapsto 0; e = 0 \mapsto 0; f \mapsto a_{22}fa_{55}; g \mapsto 0.$$

The representatives of the orbits are then  $A_\varepsilon = [1, 1, 0, 0, 0, \varepsilon, 0]$  with  $\varepsilon \in \mathbb{Q}$  and  $A_{\varepsilon'}$  is in the same orbit of  $A_\varepsilon$  if and only if there exists  $\alpha \in \mathbb{Q}^*$  such that  $\varepsilon' = \alpha^6\varepsilon$ . For the case  $B_2 = [0, 0, 0, 1, 0, 0, 0]$  we can then assume  $A = [1, b, c, 0, e, f, 0]$ . In order to fix  $B_2$  we need  $a_{21} = a_{34} = a_{35} = 0$  and  $a_{11}a_{22}^2 = 1$ . Then the action on  $A$  is as follows

$$\begin{aligned} a &\mapsto a_{11}\delta; b \mapsto a_{11}(ba_{44} + ca_{54}); c \mapsto a_{11}(ba_{45} + ca_{55}); d \mapsto a_{12}\delta; \\ e &\mapsto a_{12}(ba_{44} + ca_{54}) + a_{22}(ea_{44} + fa_{54}); f \mapsto a_{12}(ba_{45} + ca_{55}) + a_{22}(ea_{45} + fa_{55}); g \mapsto 0. \end{aligned}$$

Again one of  $b$  or  $e$  should be non-zero. Make then  $b = 1$  and  $e = 0$ . Make also  $c = 0$  by solving for  $a_{45}$ . Then  $f \neq 0$ , since otherwise the 2-cocycle will contain a non-trivial element of the center in its kernel, and we get  $A = [1, 1, 0, 0, 0, 1, 0]$ . Then one can see that  $[1, 1, 0, 0, 0, 1, 0] \wedge B_2$  is in the same orbit as  $A_0 \wedge B_1$ .

We are now left to consider the Lie algebras in Appendix A that are not decomposable, namely  $L_6^{\mathbb{R}}(1), N_{10}, N_{11}, N_{12}$  and  $N_{13}$ . The Lie algebra  $L_6^{\mathbb{R}}(1)$  is a 2-dimensional central extension of the Lie algebra defined on the basis  $(x_1, \dots, x_6)$  by  $[x_1, x_2] = x_4, [x_1, x_3] = x_5$ . The elements in its cohomology

group can be represented by  $a\Delta_{14} + b\Delta_{15} + c\Delta_{16} + d\Delta_{23} + e\Delta_{24} + f(\Delta_{25} + \Delta_{34}) + g\Delta_{26} + h\Delta_{35} + i\Delta_{36}$ . The action of the automorphism group is as follows:

$$\begin{aligned}
a &\mapsto aa_{11}^2 a_{22} + ba_{11}^2 a_{32} + ea_{11} a_{22} a_{21} + fa_{11} (a_{21} a_{32} + a_{31} a_{22}) + ha_{11} a_{31} a_{32}; \\
b &\mapsto aa_{11}^2 a_{23} + ba_{11}^2 a_{33} + ea_{11} a_{21} a_{23} + fa_{11} (a_{21} a_{33} + a_{31} a_{23}) + ha_{11} a_{31} a_{33}; \\
c &\mapsto aa_{11} a_{46} + ba_{11} a_{56} + ca_{11} a_{66} + ea_{21} a_{46} + f(a_{21} a_{56} + a_{31} a_{46}) + ga_{21} a_{66} + ha_{31} a_{56} + ia_{31} a_{66}; \\
d &\mapsto d(a_{22} a_{33} - a_{23} a_{32}) + e(a_{22} a_{43} - a_{23} a_{42}) + f(a_{22} a_{53} - a_{23} a_{52} + a_{32} a_{43} - a_{33} a_{42}) \\
&\quad + g(a_{22} a_{63} - a_{23} a_{62}) + h(a_{32} a_{53} - a_{33} a_{52}) + i(a_{32} a_{63} - a_{33} a_{62}); \\
e &\mapsto ea_{11} a_{22}^2 + 2fa_{11} a_{22} a_{32} + ha_{11} a_{32}^2; \\
f &\mapsto ea_{11} a_{22} a_{23} + f(a_{11} a_{22} a_{33} + a_{11} a_{32} a_{23}) + ha_{11} a_{32} a_{33}; \\
g &\mapsto ea_{22} a_{46} + f(a_{22} a_{56} + a_{32} a_{46}) + ga_{22} a_{66} + ha_{32} a_{56} + ia_{32} a_{66}; \\
h &\mapsto ea_{11} a_{23}^2 + 2fa_{11} a_{23} a_{33} + ha_{11} a_{33}^2; \\
i &\mapsto ea_{23} a_{46} + f(a_{23} a_{56} + a_{33} a_{46}) + ga_{23} a_{66} + ha_{33} a_{56} + ia_{33} a_{66}.
\end{aligned}$$

The family  $L_6^{\mathbb{R}}(1)$  corresponds to  $(\Delta_{14} - \Delta_{36}) \wedge (\Delta_{15} + \Delta_{26})$ . Call  $V_1$  the subspace of  $H^2(\mathfrak{g}, \mathbb{R})$  generated by  $\{\Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{23}, \Delta_{26}, \Delta_{36}\}$  and  $V_2$  the one generated by  $\{\Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{23}\}$ . Notice that  $V_1$  and  $V_2$  are submodules under the action of  $\text{Aut}(\mathfrak{g})$ . If  $L$  is a 2-dimensional subspace of  $H^2(\mathfrak{g}, \mathbb{R})$  associated to  $L_6^{\mathbb{R}}(1)$  then  $L$  is characterised by  $L \subseteq V_1$  and  $L \not\subseteq V_2$ . Hence the two generators of  $L$  have the form  $A = [a_1, b_1, c_1, d_1, 0, 0, g_1, 0, i_1]$  with  $i_1 \neq 0$  and  $B = [a, b, c, d, 0, 0, g, 0, i]$ . Furthermore we must have either  $a$  or  $a_1$  not 0 since otherwise we would have a non-trivial element in  $\mathfrak{g}_{(A,B)}^\perp \cap \mathcal{L}(\mathfrak{g})$ . Then we can bring the first generator of a 2-dimensional subspace of  $H^2(\mathfrak{g}, \mathbb{Q})$  related to a  $\mathbb{Q}$ -form of  $L_6^{\mathbb{R}}(1)$  to  $A = [1, 0, 0, 0, 0, 0, 0, 0, -1]$ . In order to leave  $A$  stable we need  $a_{23} = a_{32} = a_{62} = 0, a_{46} = \frac{a_{31} a_{66}}{a_{11}}, a_{11}^2 a_{22} = a_{33} a_{66} = 1$ . Then the action on  $B = [0, b, c, d, 0, 0, g, 0, i]$  is as follows

$$\begin{aligned}
a &\mapsto 0; b \mapsto ba_{11}^2 a_{33}; c \mapsto ba_{11} a_{56} + ca_{11} a_{66} + ga_{21} a_{66} + ia_{31} a_{66}; d \mapsto da_{22} a_{33} + ga_{22} a_{63}; \\
e &\mapsto 0; f \mapsto 0; g \mapsto ga_{22} a_{66}; h \mapsto 0; i \mapsto i.
\end{aligned}$$

Notice that  $i$  should be 0 since otherwise the corresponding Lie algebra over  $\mathbb{R}$  is not in the same isomorphism class of  $L_6^{\mathbb{R}}(1)$ . Notice furthermore that  $b$  and  $g$  should be non-zero since otherwise we would have a non-trivial element in  $\mathfrak{g}_{(A,B)}^\perp \cap \mathcal{L}(\mathfrak{g})$ . Then solving for  $a_{63}, a_{56}$  put  $d = c = 0$ . Then taking  $a_{63} = a_{56} = a_{21} = 0$  we are left with

$$a \mapsto 0; b \mapsto ba_{11}^2 a_{33}; c \mapsto 0; d \mapsto 0; e \mapsto 0; f \mapsto 0; g \mapsto ga_{22} a_{66}; h \mapsto 0; i \mapsto 0.$$

Then a representative of the orbit is  $B_\varepsilon = [0, 1, 0, 0, 0, 0, \varepsilon, 0, 0]$  and  $B_{\varepsilon'}$  is in the same orbit as  $B_\varepsilon$  if and only if there exists  $\alpha \in \mathbb{Q}^*$  such that  $\varepsilon' = \alpha^2 \varepsilon$ . Since over  $\mathbb{R}$  the Lie algebra associated to  $A \wedge B_\varepsilon$  should be isomorphic to  $L_6^{\mathbb{R}}(1)$  then  $\varepsilon > 0$ .

The Lie algebras  $N_{10}$  and  $N_{11}$  are 2-dimensional central extensions of the Lie algebra defined on the basis  $(x_1, \dots, x_6)$  by  $[x_1, x_2] = x_3$ . An element of the cohomology group of  $\mathfrak{g}$  reads as  $a\Delta_{13} + b\Delta_{14} + c\Delta_{15} + d\Delta_{16} + e\Delta_{23} + f\Delta_{24} + g\Delta_{25} + h\Delta_{26} + i\Delta_{45} + l\Delta_{46} + m\Delta_{56}$ . The action of the automorphism group of  $\mathfrak{g}$  on it is as follows:

$$\begin{aligned}
a &\mapsto aa_{11}\delta + ea_{21}\delta; \\
b &\mapsto a_{11}(aa_{34} + ba_{44} + ca_{54} + da_{64}) + a_{21}(ea_{34} + fa_{44} + ga_{54} + ha_{64}) \\
&\quad + i(a_{41}a_{54} - a_{51}a_{44}) + l(a_{41}a_{64} - a_{44}a_{61}) + m(a_{51}a_{64} - a_{54}a_{61}); \\
c &\mapsto a_{11}(aa_{35} + ba_{45} + ca_{55} + da_{65}) + a_{21}(ea_{35} + fa_{45} + ga_{55} + ha_{65}) \\
&\quad + i(a_{41}a_{55} - a_{51}a_{45}) + l(a_{41}a_{65} - a_{45}a_{61}) + m(a_{51}a_{65} - a_{55}a_{61}); \\
d &\mapsto a_{11}(aa_{36} + ba_{46} + ca_{56} + da_{66}) + a_{21}(ea_{36} + fa_{46} + ga_{56} + ha_{66}) \\
&\quad + i(a_{41}a_{56} - a_{51}a_{46}) + l(a_{41}a_{66} - a_{46}a_{61}) + m(a_{51}a_{66} - a_{56}a_{61}); \\
e &\mapsto aa_{12}\delta + ea_{22}\delta; \\
f &\mapsto a_{12}(aa_{34} + ba_{44} + ca_{54} + da_{64}) + a_{22}(ea_{34} + fa_{44} + ga_{54} + ha_{64}) \\
&\quad + i(a_{42}a_{54} - a_{52}a_{44}) + l(a_{42}a_{64} - a_{44}a_{62}) + m(a_{52}a_{64} - a_{54}a_{62}); \\
g &\mapsto a_{12}(aa_{35} + ba_{45} + ca_{55} + da_{65}) + a_{22}(ea_{35} + fa_{45} + ga_{55} + ha_{65}) \\
&\quad + i(a_{42}a_{55} - a_{52}a_{45}) + l(a_{42}a_{65} - a_{45}a_{62}) + m(a_{52}a_{65} - a_{55}a_{62}); \\
h &\mapsto a_{12}(aa_{36} + ba_{46} + ca_{56} + da_{66}) + a_{22}(ea_{36} + fa_{46} + ga_{56} + ha_{66}) \\
&\quad + i(a_{42}a_{56} - a_{52}a_{46}) + l(a_{42}a_{66} - a_{46}a_{62}) + m(a_{52}a_{66} - a_{56}a_{62}); \\
i &\mapsto i(a_{44}a_{55} - a_{54}a_{45}) + l(a_{44}a_{65} - a_{45}a_{64}) + m(a_{54}a_{65} - a_{55}a_{64}); \\
l &\mapsto i(a_{44}a_{56} - a_{54}a_{46}) + l(a_{44}a_{66} - a_{46}a_{64}) + m(a_{54}a_{66} - a_{56}a_{64}); \\
m &\mapsto i(a_{45}a_{56} - a_{55}a_{46}) + l(a_{45}a_{66} - a_{46}a_{65}) + m(a_{55}a_{66} - a_{56}a_{65})
\end{aligned}$$

where  $\delta = a_{11}a_{22} - a_{12}a_{21}$ . The Lie algebras  $N_{10}$  and  $N_{11}$  correspond to  $(\Delta_{13} + \Delta_{25}) \wedge (\Delta_{14} + \Delta_{26})$  and  $(\Delta_{13} + \Delta_{25}) \wedge (\Delta_{14} + \Delta_{23} + \Delta_{26})$  respectively. Call  $V_1$  the subspace of  $H^2(\mathfrak{g}, \mathbb{R})$  generated by  $\{\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{23}, \Delta_{24}, \Delta_{25}, \Delta_{26}\}$  and  $V_2$  the one generated by  $\{\Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{24}, \Delta_{25}, \Delta_{26}\}$ . Notice that  $V_1$  and  $V_2$  are submodules under the action of  $\text{Aut}(\mathfrak{g})$ . If  $L_1$  and  $L_2$  are 2-dimensional subspaces of  $H^2(\mathfrak{g}, \mathbb{R})$  associated to  $N_{10}$  and  $N_{11}$  respectively then they are both characterised by  $L_i \subseteq V_1$  and  $L_i \not\subseteq V_2$ . Hence the two generators of the two-dimensional subspaces of  $H^2(\mathfrak{g}, \mathbb{R})$  associated to a  $\mathbb{Q}$ -form of both  $N_{10}$  and  $N_{11}$  should have the form  $A = [a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, 0, 0, 0]$  and  $B = [a, b, c, d, e, f, g, h, 0, 0, 0]$  with either  $a$  or  $a_1$  not 0. Then under the group action we can bring  $A$  to

$A = [1, 0, 0, 0, 0, 1, 0, 0, 0, 0]$ . In order to stabilise it we need  $a_{12} = a_{54} = a_{56} = a_{34} = a_{36} = 0, a_{35} = -\frac{a_{21}a_{55}}{a_{11}}, a_{11}^2 a_{22} = 1, a_{22}a_{55} = 1$ . Then the action on  $B = [0, b, c, d, e, f, g, h, 0, 0, 0]$  is as follows

$$\begin{aligned} a &\mapsto ea_{21}\delta; b \mapsto a_{11}(ba_{44} + da_{64}) + a_{21}(fa_{44} + ha_{64}); \\ c &\mapsto a_{11}(ba_{45} + ca_{55} + da_{65}) + a_{21}(ea_{35} + fa_{45} + ga_{55} + ha_{65}); \\ d &\mapsto a_{11}(ba_{46} + da_{66}) + a_{21}(fa_{46} + ha_{66}); e \mapsto ea_{22}\delta; f \mapsto a_{22}(fa_{44} + ha_{64}); \\ g &\mapsto a_{22}(ea_{35} + fa_{45} + ga_{55} + ha_{65}); h \mapsto a_{22}(fa_{46} + ha_{66}); i \mapsto 0; l \mapsto 0; m \mapsto 0. \end{aligned}$$

Since we cannot have all  $b, c, d, e, f, g, h$  equal 0 let us suppose that  $b \neq 0$  then solving for  $a_{45}$  and  $a_{46}$  we can put  $c = d = 0$ . Putting now  $a_{45} = a_{46} = a_{21} = 0$  we are left with

$$\begin{aligned} a &\mapsto 0; b \mapsto a_{11}ba_{44}; c \mapsto 0; d \mapsto 0; e \mapsto ea_{22}\delta; f \mapsto a_{22}(fa_{44} + ha_{64}); \\ g &\mapsto a_{22}(ga_{55} + ha_{65}); h \mapsto a_{22}ha_{66}; i \mapsto 0; l \mapsto 0; m \mapsto 0. \end{aligned}$$

Now if  $h$  is 0 the Lie algebra is not isomorphic to a  $\mathbb{Q}$ -form of neither  $N_{10}$  nor  $N_{11}$ . Then solving for  $a_{64}$  and  $a_{65}$  we can put  $f = g = 0$ . Taking now  $a_{64} = a_{65} = 0$  we have

$$a \mapsto 0; b \mapsto a_{11}ba_{44}; c \mapsto 0; d \mapsto 0; e \mapsto ea_{22}\delta; f \mapsto 0; g \mapsto 0; h \mapsto a_{22}ha_{66}; i \mapsto 0; l \mapsto 0; m \mapsto 0.$$

Then depending on whether  $e$  is 0 or not we can arrive to the standard form  $B_1 = [0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0]$  that corresponds to  $L_{10}$  or  $B_2 = [0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0]$  that corresponds to  $L_{11}$ .

The Lie algebras  $N_{12}$  and  $N_{13}$  are 3-dimensional extensions of the abelian Lie algebra of dimension 5,  $\mathbb{R}^5$ . Elements of the cohomology group of  $\mathbb{R}^5$  can be represented as  $B = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{15} + e\Delta_{23} + f\Delta_{24} + g\Delta_{25} + h\Delta_{34} + i\Delta_{35} + l\Delta_{45}$ . If we denote by  $g = (a_{ij})$  an element of the automorphism group of  $\mathbb{R}^5$ , that in this case is just  $GL(5, \mathbb{R})$ , then its action on the cohomology group is as follows:

$$\begin{aligned} a &\mapsto a\Sigma_{12}^{12} + b\Sigma_{13}^{12} + c\Sigma_{14}^{12} + d\Sigma_{15}^{12} + e\Sigma_{23}^{12} + f\Sigma_{24}^{12} + g\Sigma_{25}^{12} + h\Sigma_{34}^{12} + i\Sigma_{35}^{12} + l\Sigma_{45}^{12}; \\ b &\mapsto a\Sigma_{12}^{13} + b\Sigma_{13}^{13} + c\Sigma_{14}^{13} + d\Sigma_{15}^{13} + e\Sigma_{23}^{13} + f\Sigma_{24}^{13} + g\Sigma_{25}^{13} + h\Sigma_{34}^{13} + i\Sigma_{35}^{13} + l\Sigma_{45}^{13}; \\ c &\mapsto a\Sigma_{12}^{14} + b\Sigma_{13}^{14} + c\Sigma_{14}^{14} + d\Sigma_{15}^{14} + e\Sigma_{23}^{14} + f\Sigma_{24}^{14} + g\Sigma_{25}^{14} + h\Sigma_{34}^{14} + i\Sigma_{35}^{14} + l\Sigma_{45}^{14}; \\ d &\mapsto a\Sigma_{12}^{15} + b\Sigma_{13}^{15} + c\Sigma_{14}^{15} + d\Sigma_{15}^{15} + e\Sigma_{23}^{15} + f\Sigma_{24}^{15} + g\Sigma_{25}^{15} + h\Sigma_{34}^{15} + i\Sigma_{35}^{15} + l\Sigma_{45}^{15}; \\ e &\mapsto a\Sigma_{12}^{23} + b\Sigma_{13}^{23} + c\Sigma_{14}^{23} + d\Sigma_{15}^{23} + e\Sigma_{23}^{23} + f\Sigma_{24}^{23} + g\Sigma_{25}^{23} + h\Sigma_{34}^{23} + i\Sigma_{35}^{23} + l\Sigma_{45}^{23}; \\ f &\mapsto a\Sigma_{12}^{24} + b\Sigma_{13}^{24} + c\Sigma_{14}^{24} + d\Sigma_{15}^{24} + e\Sigma_{23}^{24} + f\Sigma_{24}^{24} + g\Sigma_{25}^{24} + h\Sigma_{34}^{24} + i\Sigma_{35}^{24} + l\Sigma_{45}^{24}; \\ g &\mapsto a\Sigma_{12}^{25} + b\Sigma_{13}^{25} + c\Sigma_{14}^{25} + d\Sigma_{15}^{25} + e\Sigma_{23}^{25} + f\Sigma_{24}^{25} + g\Sigma_{25}^{25} + h\Sigma_{34}^{25} + i\Sigma_{35}^{25} + l\Sigma_{45}^{25}; \\ h &\mapsto a\Sigma_{12}^{34} + b\Sigma_{13}^{34} + c\Sigma_{14}^{34} + d\Sigma_{15}^{34} + e\Sigma_{23}^{34} + f\Sigma_{24}^{34} + g\Sigma_{25}^{34} + h\Sigma_{34}^{34} + i\Sigma_{35}^{34} + l\Sigma_{45}^{34}; \\ i &\mapsto a\Sigma_{12}^{35} + b\Sigma_{13}^{35} + c\Sigma_{14}^{35} + d\Sigma_{15}^{35} + e\Sigma_{23}^{35} + f\Sigma_{24}^{35} + g\Sigma_{25}^{35} + h\Sigma_{34}^{35} + i\Sigma_{35}^{35} + l\Sigma_{45}^{35}; \\ l &\mapsto a\Sigma_{12}^{45} + b\Sigma_{13}^{45} + c\Sigma_{14}^{45} + d\Sigma_{15}^{45} + e\Sigma_{23}^{45} + f\Sigma_{24}^{45} + g\Sigma_{25}^{45} + h\Sigma_{34}^{45} + i\Sigma_{35}^{45} + l\Sigma_{45}^{45} \end{aligned}$$

where  $\Sigma_{ij}^{st} = a_{is}a_{jt} - a_{it}a_{js}$ . The Lie algebras  $N_{12}$  and  $N_{13}$  correspond to the three-dimensional subspace of the cohomology group of  $\mathbb{R}^5$  represented by  $\Delta_{12} \wedge \Delta_{13} \wedge (\Delta_{14} + \Delta_{25})$  and  $\Delta_{12} \wedge (\Delta_{13} + \Delta_{24}) \wedge (\Delta_{14} + \Delta_{25})$  respectively. We can notice that for both subspaces two of the generators are such that one is a symplectic form when restricted to the subspace  $\text{span}\{e_1, e_2, e_4, e_5\}$  with  $(e_i)$  the standard basis for  $\mathbb{R}^5$  and the other is a Lagrangian subspace with respect to the first. Then if  $P$  is the three-dimensional subspace of  $H^2(\mathbb{Q}^4, \mathbb{Q})$  corresponding to a  $\mathbb{Q}$ -form  $\mathfrak{h}$  of  $N_{12}$  or  $N_{13}$  we can bring two of the generators to  $A = [1, 0, 0, 0, 0, 0, 0, 0, 0]$  and  $C = [0, 0, 1, 0, 0, 0, 1, 0, 0, 0]$ . Now in order to fix  $A$  and  $C$  we need  $a_{12} = a_{13} = a_{14} = a_{15} = a_{23} = a_{24} = a_{25} = a_{53} = a_{54} = a_{43} = 0$ ,  $a_{45} = \frac{-a_{21}a_{55}}{a_{11}}$  and  $a_{11}a_{22} = a_{11}a_{44} = a_{22}a_{55} = 1$ . Then if  $B = [0, b, 0, d, e, f, g, h, i, l]$  we get

$$\begin{aligned} a &\mapsto 0; b \mapsto ba_{11}a_{33} + ea_{21}a_{33} - ha_{33}a_{41} - ia_{33}a_{51}; \\ c &\mapsto ba_{11}a_{34} + ca_{11}a_{44} - da_{21}a_{45}a_{55} + ea_{21}a_{34} + fa_{21}a_{44} - g\frac{a_{21}^2a_{55}}{a_{11}} + h\Sigma_{34}^{14} - ia_{34}a_{51} - la_{44}a_{51}; \\ d &\mapsto ba_{11}a_{35} + da_{11}a_{55} + ea_{21}a_{35} - f\frac{a_{21}^2a_{55}}{a_{11}} + ga_{21}a_{55} + h\Sigma_{34}^{15} + i\Sigma_{35}^{15} + l\Sigma_{45}^{15}; \\ e &\mapsto ea_{22}a_{33} - ha_{33}a_{42} - ia_{33}a_{52}; f \mapsto ea_{22}a_{34} + fa_{22}a_{44} + h\Sigma_{34}^{24} - ia_{34}a_{52} - la_{44}a_{52}; \\ g &\mapsto ea_{22}a_{35} - f\frac{a_{21}a_{22}a_{55}}{a_{11}} + ga_{22}a_{55} + h\Sigma_{34}^{25} + i\Sigma_{35}^{25} + l\Sigma_{45}^{25}; \\ h &\mapsto ha_{33}a_{44}; i \mapsto -h\frac{a_{21}a_{33}a_{55}}{a_{11}} + ia_{33}a_{55}; l \mapsto h\Sigma_{34}^{45} + ia_{34}a_{55} + la_{44}a_{55}. \end{aligned}$$

Now since the subspace spanned by  $\{\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}\}$  is a submodule for the action of the stabiliser of  $A$  and  $C$  we have  $h = i = l = 0$ . Then  $e$  should be 0 as well otherwise the Lie algebra is isomorphic to a  $\mathbb{Q}$ -form of a Lie algebra that does not interest us. Then  $b$  should be non-zero otherwise the cocycle will contain a non-trivial element of the center in its kernel. Now solving for  $a_{34}$  make  $c = g$  and then by subtracting a scalar multiple of  $C$  make them equal 0. Furthermore solving for  $a_{35}$  make  $d = 0$ . Taking now  $a_{34} = a_{35} = a_{21} = 0$  we are left with

$$a \mapsto 0; b \mapsto ba_{11}a_{33}; c \mapsto 0; d \mapsto 0; e \mapsto 0; f \mapsto fa_{22}a_{44}; g \mapsto 0; h \mapsto 0; i \mapsto 0; l \mapsto 0.$$

Hence depending on whether  $f$  is 0 or not, a representative of the  $\mathbb{Q}$ -orbit is either the element  $B_1 = [0, 1, 0, 0, 0, 0, 0, 0, 0]$  or  $B_2 = [0, 1, 0, 0, 0, 1, 0, 0, 0, 0]$  that correspond to  $N_{12}$  and  $N_{13}$  defined over  $\mathbb{Q}$ .  $\square$

Putting together Proposition 2.5.2, 2.5.4 and 2.5.11 we see that the Proof of Theorem 2.5.1 is completed.

*Remark 2.5.12.* A priori the method just presented could have been applied also to classify the family of Lie algebras that we found in the non-degenerate case, namely  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$ . Nevertheless, the method was not that easy to apply for that case hence we decided to use an ad-hoc method. Indeed

also in [30] for the case of central extensions of the free Lie algebra of rank 2 over 3 generators the author uses another method.

## 2.6 Topological considerations

Let  $\Gamma$  be a subgroup of  $\mathcal{H}(n, 1)$  acting properly discontinuously and cocompactly on  $\mathfrak{a}(\mathbb{C}^{n+1})$ . From a theorem of Selberg every finitely generated linear group contains a torsion-free subgroup of finite index. Hence, up to replacing  $\Gamma$  by a finite index subgroup, we can consider  $M = \Gamma \backslash \mathfrak{a}(\mathbb{C}^{n+1})$  to be a compact flat Hermite-Lorentz manifold. From Theorem 1.2.37 there exists a subgroup  $H \leq \mathcal{H}(n, 1)$  that acts simply transitively on  $\mathfrak{a}(\mathbb{C}^{n+1})$  and  $\Gamma \cap H$  has finite index in  $\Gamma$  and it is a lattice in  $H$ . Hence, up to finite cover,  $M$  is diffeomorphic to  $(\Gamma \cap H) \backslash H$ . Let us suppose that  $H = U(\gamma_2, \gamma_3, b_2, b_3)$  is a unipotent group so that it has the form of Proposition 2.1.5.

**Proposition 2.6.1.** *Let  $\Gamma$  be a lattice in the group  $U := U(\gamma_2, \gamma_3, b_2, b_3)$ . The manifold  $\Gamma \backslash U$  is a fiber bundle over a real torus  $\Gamma / (\Gamma \cap \mathcal{C}^2 U) \backslash U / \mathcal{C}^2 U$  of dimension  $\frac{2n+1}{3} \leq p \leq 2n+2$  with fibers that are real tori of dimension  $q = 2n+2-p$ . Furthermore this fibration split into two fiber bundles as follows*

$$\Gamma \backslash U \longrightarrow \Gamma / (\Gamma \cap \mathcal{C}^3 U) \backslash U / \mathcal{C}^3 U \longrightarrow \Gamma / (\Gamma \cap \mathcal{C}^2 U) \backslash U / \mathcal{C}^2 U.$$

*Proof.* Let  $\mathcal{C}^i U$ , with  $i \leq 3$ , be the elements of the lower central series, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}^3 U & \longrightarrow & U & \xrightarrow{\pi_1} & U / \mathcal{C}^3 U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{Id} & & \downarrow \pi_2 & & \\ 0 & \longrightarrow & \mathcal{C}^2 U & \longrightarrow & U & \xrightarrow{\pi} & U / \mathcal{C}^2 U & \longrightarrow & 0. \end{array}$$

From [39, Corollary 1 of Theorem 2.3.] if  $\Gamma$  is a lattice in  $U$  nilpotent simply connected Lie group then  $\Gamma \cap \mathcal{C}^i U$  is a lattice in  $\mathcal{C}^i U$ . Hence considering the maps induced by  $\pi_1$  and  $\pi_2$  on the quotients by  $\Gamma$  we have that  $\Gamma \backslash U \rightarrow \Gamma / (\Gamma \cap \mathcal{C}^2 U) \backslash U / \mathcal{C}^2 U$  can be seen as a sequence of fiber bundles. Furthermore the derived group  $\mathcal{C}^2 U$  is abelian of dimension  $q$  with  $0 \leq q \leq 2 + 2\text{rank}(\gamma_2) + 1$ . From Lemma 2.1.6 we then have  $q \leq \frac{4n+5}{3}$ . Hence the quotient  $U / \mathcal{C}^2 U$  is an abelian Lie group of dimension  $p$  with  $\frac{2n+1}{3} \leq p \leq 2n+2$ . Then  $\mathcal{C}^2 U$  is isomorphic to  $\mathbb{R}^q$  and  $U / \mathcal{C}^2 U$  is isomorphic to  $\mathbb{R}^p$ . Since, as we have seen,  $\Gamma$  intersects  $\mathcal{C}^2 U$  in a lattice we have that  $\Gamma \cap \mathcal{C}^2 U$  is isomorphic to  $\mathbb{Z}^q$  and  $\pi(\Gamma)$  is isomorphic to  $\mathbb{Z}^p$ . Then finally the manifold  $\Gamma / (\Gamma \cap \mathcal{C}^2 U) \backslash U / \mathcal{C}^2 U$  is a torus and the fibers  $\Gamma \cap \mathcal{C}^2 U \backslash \mathcal{C}^2 U$  of the fiber bundle induced by  $\pi$  are real tori.  $\square$

*Remark 2.6.2.* Notice that the above proposition is just a translation of the fact that the group  $U(\gamma_2, \gamma_3, b_2, b_3)$  is 3-step nilpotent.



## Chapter 3

# Flat Pseudo-Riemannian Lie algebras

In Chapter 2 we have studied, among other things, simply transitive unipotent Hermite-Lorentz groups,  $U(\gamma_2, \gamma_3, b_2, b_3)$ , and we have given their classification, up to isomorphism, in dimension 4. We can notice that these Lie groups can be seen as real Lie groups that are endowed with a left invariant flat Hermite-Lorentz metric. Hence they are endowed with three different left invariant structures. A left invariant affine structure, a pseudo-Riemannian structure compatible with the affine one and, finally, a complex structure compatible with all the previous ones. In this chapter we will mostly adopt the point of view of left invariant structures on Lie groups. There is a vast literature that treats related questions and in Section 3.1 we will compare our results with results that already exist. Construction results for Lorentzian and pseudo-Riemannian nilpotent Lie algebras of signature  $(n-2, 2)$  are given in [5] and [12]. More precisely they construct these Lie algebras from Lie algebras of smaller dimension using a double extension process. In Section 3.1.1 we will see how to recognise that the Lie algebras  $u(\gamma_2, \gamma_3, b_2, b_3)$  are Lie algebras obtained with a double extension process and deduce a criterion for Lie algebras constructed with a double extension process and with a metric of signature  $(n-2, 2)$  to have a Hermite-Lorentz structure. Finally, let us remember that, in dimension 4 and in the non-degenerate case, the Lie algebras  $u(\gamma_2, \gamma_3, b_2, b_3)$  are isomorphic to the ones we called  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$ . These Lie algebras naturally live in the more general family of 8-dimensional nilpotent Lie algebras,  $\mathfrak{g}(Q)$ , defined in Definition 2.4.18. With the same methods used in Section 2.4 for  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$ , we can give a general procedure in order to classify all the Lie algebras  $\mathfrak{g}(Q)$ , see Appendix B for the list of isomorphism classes. In the last part of Section 3.2, starting from the metric and complex structures we have on the subfamily  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$ , we study how this can be extended to metric and complex structures on the other Lie algebras of the family  $\mathfrak{g}(Q)$ . This is also the occasion to compare the affine structure we have on  $u(\gamma_2, \gamma_3, b_2, b_3)$  with the one defined by Scheuneman, [42], and see that in general they differ.

### 3.1 General facts and notations

Let  $V$  be a vector space of dimension  $n$ , denote by  $\mathfrak{a}(V)$  the affine space associated to  $V$  and by  $\text{Aff}(V)$  the group of affine transformations of  $\mathfrak{a}(V)$ .

**Definition 3.1.1.** An affine structure, as introduced in Section 1.2, on a Lie group  $G$  is called *left invariant* if for all  $g \in G$  the map of left multiplication,  $L_g : G \rightarrow G$ , is an automorphism of the affine structure. In other words, in affine coordinates  $L_g$  is expressed by an affine map.

**Definition 3.1.2.** Let  $\rho : G \rightarrow \text{Aff}(V)$  be an affine representation. Then  $\rho$  is called an *étale representation* if there exists  $x \in \mathfrak{a}(V)$  such that  $O_x : G \rightarrow \mathfrak{a}(V)$ , the orbit map, is a local diffeomorphism onto an open subset of  $\mathfrak{a}(V)$ . The representation is called *simply transitive* if  $O_x$  is a diffeomorphism for one, hence all,  $x \in \mathfrak{a}(V)$ .

**Proposition 3.1.3** ([10, Proposition 5.2.]). *Let  $G$  be a simply connected Lie group. Then the following are equivalent:*

- $G$  has a left invariant affine structure,
- there exists an étale affine representation of  $G$ .

Moreover the affine structure is complete if and only if  $\rho$  is simply transitive.

*Proof.* Let  $G$  be endowed with a left invariant affine structure. Then the developing map  $D : G \rightarrow \mathfrak{a}(V)$  is such that for all  $g \in G$  there exists a unique automorphism  $\alpha(g)$  of  $\mathfrak{a}(V)$  such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{D} & \mathfrak{a}(V) \\ \downarrow L_g & & \downarrow \alpha(g) \\ G & \xrightarrow{D} & \mathfrak{a}(V). \end{array}$$

Then  $\alpha : G \rightarrow \text{Aff}(V)$  is an affine representation. In particular  $\alpha(G)$  preserves the connected open subset  $D(G)$  of  $\mathfrak{a}(V)$  and acts transitively on it. Hence the representation is étale. On the other hand, let  $\alpha : G \rightarrow \text{Aff}(V)$  an étale affine representation having an open orbit  $U = D(G)x$  for some  $x \in \mathfrak{a}(V)$ . Then there exists a unique left invariant affine structure on  $G$  whose developing map  $D : G \rightarrow U \subseteq \mathfrak{a}(V)$  is given by  $D(g) = \alpha(g)x$ . Notice that when  $x = 0$  then this is just the translational part of the representation.  $\square$

**Definition 3.1.4.** Let  $\rho : G \rightarrow \text{Aff}(V)$  be an affine representation and let  $\rho_* : \mathfrak{g} \rightarrow \mathfrak{aff}(V)$  be the induced Lie algebra representation. For  $x \in \mathfrak{a}(V)$  call  $O_x : \text{Aff}(V) \rightarrow \mathfrak{a}(V)$  the orbit map and  $t_x : \mathfrak{aff}(V) \rightarrow V$  its derivative at the identity. Choose a basis  $(X_1, \dots, X_n)$  for  $\mathfrak{g}$  and  $x \in \mathfrak{a}(V)$ . Choose also a basis for  $V$

with respect to which we define the following function

$$\begin{aligned} \tau_x(\rho) : V &\longrightarrow V \\ (\alpha_1, \dots, \alpha_n) &\mapsto t_x \left( \sum_i \alpha_i \rho_*(X_i) \right). \end{aligned}$$

*Remark 3.1.5.* The function  $\tau_x(\rho)$  is an isomorphism if and only if  $O_x$  is non-singular, i.e. if  $\rho$  is étale at  $x$ .

**Definition 3.1.6.** Define the polynomial function

$$\begin{aligned} \delta := \delta(\rho) : \mathfrak{a}(V) &\longrightarrow \mathbb{R} \\ x &\mapsto \delta(x) = \det \tau_x(\rho). \end{aligned}$$

When  $\rho$  is an étale representation, i.e. when the function  $\delta$  is not constantly 0, it is called the *relative invariant* of the representation. It satisfies the transformation rule

$$\delta(\rho(g)x) = \det \text{Ad}(g)^{-1} \det L(\rho(g)) \delta(x) \quad (3.1)$$

for  $g \in G$ ,  $x \in \mathfrak{a}(V)$  and  $\text{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}$  the adjoint action. The character associated to the relative invariant is

$$\begin{aligned} \chi := \chi(\rho) : G &\longrightarrow \mathbb{R}^{>0} \\ g &\mapsto \det \text{Ad}(g)^{-1} \det L(\rho(g)). \end{aligned}$$

**Theorem 3.1.7** ([10, Theorem 5.19]). *Let  $G$  be a Lie group and  $\rho : G \rightarrow \text{Aff}(V)$  an étale affine representation. The following are equivalent:*

- $\rho$  is simply transitive,
- the relative invariant  $\delta(\rho)$  is a non-zero constant polynomial function.

*Proof.* If  $\delta(\rho)$  is everywhere non-zero all orbits of  $G$  are open, hence  $G$  has only one orbit and the action is simply transitive. Conversely assume  $\rho$  is simply transitive. We know then that there exists an  $x \in \mathfrak{a}(V)$  such that  $\delta(x) \neq 0$ . Let  $\tilde{G}$  be the Zariski closure of  $G$  inside  $\text{Aff}(V)$ . Then  $\delta(\rho)$  remains a relative invariant for some polynomial homomorphism  $\tilde{\chi} : \tilde{G} \rightarrow \mathbb{R}^{>0}$  that extends  $\chi$ . The unipotent radical,  $U$ , of  $\tilde{G}$  also acts simply transitively on  $\mathfrak{a}(V)$ . Since every algebraic multiplicative character of an unipotent group is trivial,  $\tilde{\chi}$  is trivial on  $U$ . Then, being  $\mathfrak{a}(V)$  a unique orbit under  $U$ , by (3.1)  $\delta(\rho)$  must be constant.  $\square$

A left invariant affine structure on a simply connected Lie group  $G$  can also be seen in more algebraic terms. Indeed the affine structure endows  $G$  with a flat, left invariant, torsion-free connection

$\nabla$  coming from the affine space  $\mathfrak{a}(V)$ . Then for any  $x, y \in \mathfrak{g}$  we can define  $x \cdot y := \nabla_{x^+} y^+(\text{Id})$ , where  $x^+$  is the left invariant vector field associated with  $x \in \mathfrak{g}$ . Hence the following natural definition translates on the Lie algebra level the properties of the connection. Indeed (3.2) is the torsion-free condition and (3.3) is flatness.

**Definition 3.1.8.** Let  $\mathfrak{g}$  be a Lie algebra. Consider on it a product  $L_x y = R_y x = x \cdot y$ , compatible with the Lie algebra structure, i.e.

$$[x, y] = x \cdot y - y \cdot x, \quad (3.2)$$

and left symmetric, i.e.

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z). \quad (3.3)$$

Then we will call  $(\mathfrak{g}, \cdot)$  an *affine Lie algebra*. If  $G$  is the simply connected Lie group whose Lie algebra is  $\mathfrak{g}$  we will call  $(G, \nabla)$  an affine Lie group.

*Remark 3.1.9.* An equivalent point of view for a Lie algebra  $\mathfrak{g}$ , of dimension  $n$ , to have an affine structure is to have a Lie algebra representation  $\rho : \mathfrak{g} \rightarrow \text{aff}(\mathbb{R}^n)$  such that the translation part of  $\rho(\mathfrak{g})$  spans the whole  $\mathbb{R}^n$ . Then the natural notion of isomorphism one has for affine Lie algebras translates to an isomorphism of representations.

In [42] Scheuneman studied affine structures on 3-step nilpotent Lie algebras. His result is the following.

**Proposition 3.1.10** ([42]). *Every 3-step nilpotent Lie algebra admits an affine structure.*

*Proof.* Let  $\mathfrak{g}$  be a 3-step nilpotent Lie algebra of dimension  $n$ . We will construct an affine representation, such that the translation part of the image of the representation span the whole  $\mathbb{R}^n$ , deforming the adjoint representation. Call  $W = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$  and choose a complement  $V$  to  $W$  in  $[\mathfrak{g}, \mathfrak{g}]$  and a complement  $U$  to  $[\mathfrak{g}, \mathfrak{g}]$  in  $\mathfrak{g}$ . Then we have  $\mathfrak{g} = W \oplus V \oplus U$  as vector space. Let  $(w_1, \dots, w_m, v_1, \dots, v_q, u_1, \dots, u_p)$  be a basis adapted to the decomposition. Then

$$[u_i, u_j] = \sum_k a_{ikj} v_k + \sum_k c_{ikj} w_k \quad \text{and} \quad [u_i, v_j] = \sum_k b_{ikj} w_k$$

with  $a_{ikj}, b_{ikj}$  and  $c_{ikj} \in \mathbb{R}$ . The matrices of the adjoint representation, written with respect to the above basis, are as follows

$$\text{adu}_i = \begin{pmatrix} 0 & B_i & C_i \\ 0 & 0 & A_i \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{adv}_i = \begin{pmatrix} 0 & 0 & D_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \text{adw}_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where  $B_i = (b_{ikj})_{kj}$ ,  $C_i = (c_{ikj})_{kj}$ ,  $A_i = (a_{ikj})_{kj}$  and  $D_i = (-b_{jki})_{kj}$ . Then we can define a linear map  $f : \mathfrak{g} \rightarrow \mathfrak{aff}(\mathbb{R}^n)$  by

$$u_i \mapsto U_i = \begin{pmatrix} 0 & bB_i & cC_i & 0 \\ 0 & 0 & aA_i & 0 \\ 0 & 0 & 0 & u_i \\ 0 & 0 & 0 & 0 \end{pmatrix}, v_i \mapsto V_i = \begin{pmatrix} 0 & 0 & dD_i & 0 \\ 0 & 0 & 0 & rv_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, w_i \mapsto W_i = \begin{pmatrix} 0 & 0 & 0 & sw_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with  $a, b, c, d, r, s \in \mathbb{R}$ . It remains to prove that we can choose  $a, b, c, d, r$  and  $s$  so that  $r$  and  $s$  are not 0, in order for the representation to be transitive, and  $f$  is a Lie algebra homomorphism. Making the calculation we find that  $a, b, c, d, r$  and  $s$  should satisfy the conditions  $d = ab, s = 2c, r = 2a$  and  $s = br + ab$ . Choosing for example  $a = b = c = \frac{2}{3}, d = \frac{4}{9}$  and  $r = s = \frac{4}{3}$  we see that such constants exist.  $\square$

*Remark 3.1.11.* Scheuneman's work was generalised by [22] where they put an affine structure on a class of 4-step nilpotent Lie algebras.

We will also be interested in putting other structures on the Lie algebra  $\mathfrak{g}$ . Let us give the following definitions.

**Definition 3.1.12.** Let  $(\mathfrak{g}, \cdot)$  be an affine Lie algebra. Consider on it a non-degenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  that is compatible with the connection, i.e.

$$\langle x \cdot y, z \rangle = -\langle y, x \cdot z \rangle. \quad (3.4)$$

Then we will call  $(\mathfrak{g}, \cdot, \langle \cdot, \cdot \rangle)$  a *flat pseudo-Riemannian Lie algebra*.

*Remark 3.1.13.* From (3.2) and (3.4) we get the following very useful formula

$$\langle x \cdot y, z \rangle = \frac{1}{2} \{ \langle [x, y], z \rangle - \langle [y, z], x \rangle + \langle [z, x], y \rangle \}. \quad (3.5)$$

*Remark 3.1.14.* If  $G$  is a Lie group with a left-invariant flat pseudo-Riemannian metric then its Lie algebra is naturally a flat pseudo-Riemannian Lie algebra.

For complex structures on nilpotent Lie algebras see also [41].

**Definition 3.1.15.** An *almost complex structure* on  $\mathfrak{g}$  is a linear endomorphism  $J : \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $J^2 = -1$ . The pair  $(\mathfrak{g}, J)$  is called an almost complex Lie algebra. The almost complex structure is called *integrable* if for all  $x, y \in \mathfrak{g}$  we have

$$[Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy]. \quad (3.6)$$

The pair  $(\mathfrak{g}, J)$  with  $J$  an integrable almost complex structure on  $\mathfrak{g}$  is called a Lie algebra endowed with a complex structure. In this case  $J$  defines a left-invariant  $(1, 1)$ -tensor field on  $G$ , the Lie group associated to  $\mathfrak{g}$ , and makes it a complex manifold.

*Remark 3.1.16.* Notice that a Lie algebra endowed with a complex structure is usually not a complex Lie algebra.

**Definition 3.1.17.** Consider an almost complex Lie algebra,  $(\mathfrak{g}, J)$ , and a non-degenerate symmetric bilinear form,  $\langle \cdot, \cdot \rangle$ , on it. We call  $\langle \cdot, \cdot \rangle$  *pseudo-Hermitian* if

$$\langle Jx, Jy \rangle = \langle x, y \rangle \quad \text{for all } x, y \in \mathfrak{g}.$$

In this context we can define the *associated fundamental 2-form*

$$\omega(x, y) = \langle x, Jy \rangle,$$

a non-degenerate, skew-symmetric bilinear form.

**Definition 3.1.18.** If  $\langle \cdot, \cdot \rangle$  is pseudo-Hermitian then we can define  $h = \langle \cdot, \cdot \rangle + i\omega$ . Then  $h$  is an Hermitian form in the classical sense and the *pseudo* stands for the fact that it might not be positive definite. We will call  $\langle \cdot, \cdot \rangle$  *Hermitic-Lorentz* if  $h$  is of signature  $(n, 1)$ .

**Definition 3.1.19.** Consider a Lie algebra  $\mathfrak{g}$  endowed with an almost complex structure  $J$  and a pseudo-Hermitian metric  $\langle \cdot, \cdot \rangle$ . Define a product on  $\mathfrak{g}$  as follows: for every  $x, y \in \mathfrak{g}$ ,  $x \cdot y := \nabla_{x^+} y^+(\text{Id})$ , where  $x^+$  is the left invariant vector field associated with  $x \in \mathfrak{g}$  and  $\nabla$  is the Levi-Civita connection associated with  $\langle \cdot, \cdot \rangle$ . The pseudo-Hermitian metric  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  is called *pseudo-Kähler* if we have

$$x \cdot Jy = J(x \cdot y) \quad \text{for all } x, y \in \mathfrak{g}. \quad (3.7)$$

*Remark 3.1.20.* From (3.2) and (3.7) we have

$$Jx \cdot y = Jx \cdot y - J[x, y] + [Jx, y]. \quad (3.8)$$

*Remark 3.1.21.* Condition (3.7) express algebraically the fact the the complex structure is parallel with respect to the Levi-Civita connection defined by  $\langle \cdot, \cdot \rangle$ . A tensor being parallel means that its covariant derivative is 0.

**Proposition 3.1.22** ([35, Theorem 4.3. of Chapter IX]). *Given a Lie group  $G$  endowed with an almost complex structure and a pseudo-Hermitian metric on it  $\langle \cdot, \cdot \rangle$  the following are equivalent:*

- *the metric  $\langle \cdot, \cdot \rangle$  is pseudo-Kähler;*
- *the almost complex structure  $J$  is integrable and the fundamental 2-form  $\omega$  is closed.*

*Remark 3.1.23.* From Clifford-Klein Theorem [16, Theorem 5.2.] a flat complete pseudo-Riemannian simply connected manifold is isometric to  $\mathbb{R}^n$  with the standard metric, where  $n$  is the dimension of the manifold. If furthermore the manifold is pseudo-Hermitian this means that it is endowed with an almost complex structure  $J$  that is an isometry with respect to the pseudo-Riemannian metric.

Hence the almost complex structure should be linear. Then  $J$  is parallel with respect to the Levi-Civita connection. It follows that a flat complete simply connected pseudo-Hermitian manifold is automatically pseudo-Kähler.

### 3.1.1 Double extension

In [5] Aubert and Medina study flat pseudo-Riemannian nilpotent Lie algebras. They introduce the concept of double extension of a flat pseudo-Riemannian nilpotent Lie algebra by a line and use this to characterise the Lorentzian ones. In [12] the concept of double extension is used to study flat pseudo-Riemannian nilpotent Lie algebras of signature  $(n-2, 2)$ . Let us introduce this concept.

**Definition 3.1.24.** Let  $(B_0, [\cdot, \cdot]_0, \langle \cdot, \cdot \rangle_0)$  be a flat pseudo-Riemannian nilpotent Lie algebra. Remember that we have the Levi-Civita connection, associated to the pseudo-Riemannian metric, which defines a left symmetric product  $x \cdot y$ . Let  $u \in Z^1(B_0, B_0)$  be a cocycle, i.e.  $u([a, b]) = a \cdot u(b) - b \cdot u(a)$  for all  $a, b \in B_0$ , and  $D \in \text{Der}(B_0)$  be a derivation such that

$$a \cdot u(b) - u(a \cdot b) = D(a) \cdot b + a \cdot D(b) - D(a \cdot b) \quad \text{for all } a, b \in B_0 \quad (3.9)$$

and  $(D - u) \in \mathfrak{o}(B_0, \langle \cdot, \cdot \rangle_0)$ , the Lie algebra of the orthogonal group of  $\langle \cdot, \cdot \rangle_0$ . Furthermore let  $\mu \in \mathbb{R}$  and  $b_0 \in B_0$  such that

$$[D, u] = u^2 - \mu u - R_{b_0} \quad (3.10)$$

where  $R_{b_0} : B_0 \rightarrow B_0$  is right multiplication by  $b_0$ . The *double extension* of  $B_0$  following  $(u, D, \mu, b_0)$  is the flat pseudo-Riemannian Lie algebra given by  $\mathbb{R}e \oplus B_0 \oplus \mathbb{R}d$  with Lie brackets

$$[d, e] = \mu e,$$

$$[d, a] = D(a) - \langle b_0, a \rangle_0 e,$$

$$[a, b] = [a, b]_0 + \langle (u - u^*)(a), b \rangle_0 e$$

for  $a, b \in B_0$ . The non-degenerate bilinear form  $\langle \cdot, \cdot \rangle$  defined on it equals  $\langle \cdot, \cdot \rangle_0$  on  $B_0$  and is such that  $\text{span}\{e, d\}$  is an hyperbolic plane orthogonal to  $B_0$ .

*Remark 3.1.25.* This definition is justified from [5, Proposition 3.1 and Proposition 3.2] where they prove that if we start with a flat pseudo-Riemannian nilpotent Lie algebra  $(B_0, \langle \cdot, \cdot \rangle)$  of signature  $(p, q)$ , the double extension process creates a new flat pseudo-Riemannian nilpotent Lie algebra with metric of signature  $(p+1, q+1)$ . Conversely they also prove that if we start with a flat pseudo-Riemannian nilpotent Lie algebra with a one-dimensional isotropic ideal,  $I$ , such that  $I^\perp$  is an ideal as well, then the Lie algebra is constructed by a process of double extension.

For the Lorentzian case the following more precise statement holds.

**Proposition 3.1.26** ([5, Théorème 3.1]). *Let  $G$  be a nilpotent Lie group. Then  $G$  admits a flat Lorentzian metric if and only if its Lie algebra is either abelian or a double extension of an abelian Riemannian Lie algebra following  $(u = D, D, \mu = 0, b_0)$  with  $D^2 = 0$  and  $\text{Im}D \subseteq \mathbb{R}b_0^\perp$ .*

For the pseudo-Riemannian nilpotent Lie algebras of signature  $(n - 2, 2)$  the following result holds.

**Proposition 3.1.27** ([12, Theorem 4.2.]). *Let  $\mathfrak{g}$  be a flat pseudo-Riemannian nilpotent Lie algebra of signature  $(n - 2, 2)$ . Then  $\mathfrak{g}$  is obtained by double extension of a flat Lorentzian nilpotent Lie algebra following  $(u, D, \mu = 0, b)$  with  $D$  a nilpotent endomorphism.*

The Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are in particular flat nilpotent pseudo-Riemannian Lie algebras of signature  $(n, 2)$ . Hence let us first see how they are built as two times a double extension. Secondly, let us characterise them between the ones obtained by a double extension process.

**Proposition 3.1.28.** *The Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are obtained by two times a double extension process from an abelian Riemannian Lie algebra  $W = \mathbb{R}^{2n-2}$ . More precisely decompose the vector space associated to  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  as  $\mathbb{R}\tau \oplus \mathbb{R}i\tau \oplus W \oplus \mathbb{R}i\xi \oplus \mathbb{R}\xi$ . The first double extension is obtained from the abelian Riemannian Lie algebra  $W$  by the line  $\mathbb{R}i\tau$  following  $(D = -J\gamma_2, b_0 = J\gamma_3(i\xi))$ . The second double extension is obtained from the Lie algebra  $\mathbb{R}i\tau \oplus W \oplus \mathbb{R}i\xi$  by the line  $\mathbb{R}\tau$  following  $(u_1, D_1, b_1)$  where  $b_1 = b_3(\xi)i\tau + \gamma_3(\xi)$  and, if  $b_2(a) = \langle a, v_0 \rangle$  with  $v_0 \in W$ ,*

$$\begin{cases} D_1(i\tau) = 0 \\ D_1(a) = -\langle a, J\gamma_3(\xi) + v_0 \rangle i\tau - \gamma_2(a) \\ D_1(i\xi) = -b_3(i\xi)i\tau - \gamma_3(i\xi) + J\gamma_3(\xi) \end{cases} \quad \text{and} \quad \begin{cases} u_1(i\tau) = 0 \\ u_1(a) = -\langle a, v_0 \rangle i\tau - \gamma_2(a) \\ u_1(i\xi) = -b_3(i\xi)i\tau - \gamma_3(i\xi) \end{cases}$$

for any  $a \in W$ .

*Proof.* Decompose the Lie algebra  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  as a vector space as  $\mathbb{R}\tau \oplus \mathbb{R}i\tau \oplus W \oplus \mathbb{R}i\xi \oplus \mathbb{R}\xi$ . Assume furthermore that  $b_2(a) = \langle a, v_0 \rangle$  with  $v_0 \in W$ . Let  $\mathfrak{g}$  be a Lie algebra obtained by two times a double extension process. The first one from a Riemannian Lie algebra following  $(u = D, D, \mu = 0, b_0)$  to obtain a Lorentzian Lie algebra. The second one following  $(u_1, D_1, \mu_1 = 0, b_1)$  to obtain a Lie algebra of signature  $(n, 2)$ . Then just comparing the Lie brackets we must have  $D(a) = -J\gamma_2(a)$ ,  $b_0 = J\gamma_3(i\xi)$ ,  $D_1(i\tau) = 0$ ,  $D_1(a) = -\langle a, J\gamma_3(\xi) + v_0 \rangle i\tau - \gamma_2(a)$ ,  $D_1(i\xi) = -b_3(i\xi)i\tau - \gamma_3(i\xi) + J\gamma_3(\xi)$  and  $b_1 = b_3(\xi)i\tau + \gamma_3(\xi)$ . Using furthermore the fact that  $(D_1 - u_1)$  is antisymmetric with respect to  $\langle \cdot, \cdot \rangle$  we also find the desired form for  $u_1$ . Then one can check that they satisfy the required conditions of Definition 3.1.24.  $\square$

As a consequence of Proposition 2.1.5 we have the following.

**Lemma 3.1.29.** *Let  $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$  be a flat Hermite-Lorentz nilpotent Lie algebra then there exists an isotropic complex line in its center.*

**Proposition 3.1.30.** *Let  $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$  be a flat pseudo-Riemannian nilpotent Lie algebra of signature  $(n, 2)$ . Let  $(D, b_0, D_1, u_1, b_1)$  be the elements that define the two times double extension process. Then  $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$  is a flat Hermite-Lorentz Lie algebra if and only if there exists a complex structure  $J$  on  $\mathfrak{g}$  compatible with  $\langle \cdot, \cdot \rangle$  such that, if  $\mathfrak{g}$  decomposes as  $\mathfrak{g} = \mathbb{R}e_1 \oplus \mathbb{R}e \oplus B_0 \oplus \mathbb{R}d \oplus \mathbb{R}d_1$  with  $Je_1 = e$  and  $Jd_1 = d$  we have for all  $a, b \in B_0$*

1.  $D_1(e) = u_1(e) = 0$ ;
2.  $\text{Im}D_1, \text{Im}u_1, b_1 \subseteq \mathbb{R}e^\perp$ ;
3.  $\langle JD_1(a), b \rangle = \langle D(a), b \rangle$ ;
4.  $\langle (D_1 - u_1)(a), b \rangle = 0$ ;
5.  $\langle Ju_1(d), a \rangle = -\langle b_0, a \rangle$ ;
6.  $\langle Jb_1, a \rangle = \langle (D_1 - u_1)(d), a \rangle$ .

*Proof.* Let  $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$  be a flat pseudo-Riemannian nilpotent Lie algebra of signature  $(n, 2)$  that is Hermite-Lorentz. Then  $\mathfrak{g}$  is endowed with a complex integrable structure  $J$  such that (3.7) holds. Furthermore from Lemma 3.1.29 we can assume that the decomposition given by the double extension process is  $\mathfrak{g} = \mathbb{R}e_1 \oplus \mathbb{R}e \oplus B_0 \oplus \mathbb{R}d \oplus \mathbb{R}d_1$  with  $Je_1 = e, Jd_1 = d$  and  $\text{span}\{e_1, e\} \subseteq \mathcal{Z}(\mathfrak{g})$ , the center of  $\mathfrak{g}$ . Then, being  $e$  in the center, we must have  $D_1(e) = 0, b_1 \in \mathbb{R}e^\perp$ ,

$$\langle (u_1 - u_1^*)(d), e \rangle = 0 \text{ and } \langle (u_1 - u_1^*)(a), e \rangle = 0 \quad (3.11)$$

for all  $a \in B_0$ . If  $z \in \mathcal{Z}(\mathfrak{g})$  from (3.5) we have  $\langle x \cdot y, z \rangle = \frac{1}{2} \langle [x, y], z \rangle$  for all  $x, y \in \mathfrak{g}$ . Hence, since by construction  $e_1 \in \mathcal{Z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})^\perp$ , we have  $\langle x \cdot y, e_1 \rangle = 0$ . Then, knowing that  $e$  also belongs to the center, and from  $\langle x \cdot y, e \rangle = \langle x \cdot y, Je_1 \rangle = -\langle x \cdot Jy, e_1 \rangle = 0$  we also have  $e \in \mathcal{Z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})^\perp$ . This implies that  $\text{Im}D_1 \subseteq \mathbb{R}e^\perp$ . Then for any  $x \in \mathbb{R}e \oplus B_0 \oplus \mathbb{R}d$  we have  $-\langle u_1(x), e \rangle = \langle (D_1 - u_1)(x), e \rangle = -\langle x, (D_1 - u_1)(e) \rangle = \langle x, u_1(e) \rangle$ . Using (3.11) we have  $\langle u_1(x), e \rangle = 0$  for all  $x \in \mathbb{R}e \oplus B_0 \oplus \mathbb{R}d$  hence  $u_1(e) = 0$  and  $\text{Im}u_1 \subseteq \mathbb{R}e^\perp$ . And we have proved 1 and 2. From (3.7) we have  $\langle Jd_1 \cdot Ja, d_1 \rangle = -\langle Jd_1 \cdot a, Jd_1 \rangle$  for all  $a \in B_0$ . Using (3.5) we have that  $\langle Ju_1(d), a \rangle = -\langle b_0, a \rangle$  and we have proved 5. Similarly from  $\langle d_1 \cdot Ja, d_1 \rangle = -\langle d_1 \cdot a, Jd_1 \rangle$  we get  $\langle Jb_1, a \rangle = \langle (D_1 - u_1)(d), a \rangle$  and we have proved 6. Again from  $\langle a \cdot Jb, d_1 \rangle = -\langle a \cdot b, Jd_1 \rangle$  we get  $\langle D(a), b \rangle = \langle Ju_1(a), b \rangle$ . Finally from the fact that  $[D_1, u_1] = u_1^2 - R_{b_1}$  we have  $\langle [D_1, u_1](a), Jb \rangle = \langle u_1^2(a), Jb \rangle - \langle R_{b_1}(a), Jb \rangle = -\langle D^2(a), Jb \rangle = 0$  hence  $\langle [D_1, u_1](a), Jb \rangle = 0$ .

**Claim 3.1.30.1.** Suppose we have two endomorphism  $A, B : V \rightarrow V$  such that  $A^2 = 0, [A, B] = 0$  and  $A - B$  is skew symmetric with respect to a positive definite scalar product. Then  $A = B$ .

*Proof.* Since  $A$  and  $B$  commute, if the Jordan decomposition of  $B$  is  $D + N$ , then the Jordan decomposition of  $A - B$  is  $-D + (A - N)$ . But for all  $x \in V$  we have  $0 = \langle (A - B)(x), x \rangle = \langle -D(x), x \rangle + \langle (A -$

$N)(x, x) = -\langle D(x), x \rangle$  hence  $D = 0$ . Now notice that if  $A - B$  is nilpotent and skew symmetric then it is the 0 matrix.  $\square$

From the claim and the fact that  $\langle u_1(a), Jb \rangle = -\langle D(a), b \rangle$  and  $D^2 = 0$  we get that  $\langle (D_1 - u_1)(a), b \rangle = 0$ . Hence we have proved 3 and 4. On the other hand assume  $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$  is constructed with a double extension process following  $(D, b_0, u_1, D_1, b_1)$  satisfying the conditions (1), ..., (6). Then we can relate them with  $\gamma_2, \gamma_3, b_2, b_3$  in the following way  $D = -J\gamma_2, b_0 = J\gamma_3(i\xi)$  and  $b_1 = b_3(\xi)i\tau + \gamma_3(\xi)$ ,

$$\begin{cases} D_1(i\tau) = 0 \\ D_1(a) = -\langle a, J\gamma_3(\xi) + v_0 \rangle i\tau - \gamma_2(a) \\ D_1(i\xi) = -b_3(i\xi)i\tau - \gamma_3(i\xi) + J\gamma_3(\xi) \end{cases} \quad \text{and} \quad \begin{cases} u_1(i\tau) = 0 \\ u_1(a) = -\langle a, v_0 \rangle i\tau - \gamma_2(a) \\ u_1(i\xi) = -b_3(i\xi)i\tau - \gamma_3(i\xi) \end{cases} .$$

So that we can see that  $\mathfrak{g}$  is isomorphic to  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  and hence it is endowed with a Hermite-Lorentz structure.  $\square$

**Example 3.1.31.** In [12] it is proven that in dimension 4 all the 4-dimensional nilpotent Lie algebras admit a flat pseudo-Riemannian metric of signature  $(2, 2)$ . These are, up to isomorphism,  $\mathbb{R}^4, H_3 \oplus \mathbb{R}$  and the filiform Lie algebra. With  $H_3$  being the Heisenberg algebra and the filiform algebra being the Lie algebra defined on the basis  $(x_1, \dots, x_4)$  by  $[x_1, x_2] = x_3, [x_1, x_3] = x_4$ . Among these only  $\mathbb{R}^4$  and  $H_3 \oplus \mathbb{R}$  admit a Hermite-Lorentz structure.

## 3.2 Carnot 3-step 8-dimensional nilpotent Lie algebras

We will now focus our attention on a family of 8-dimensional 3-step nilpotent Lie algebras. This family was already defined in Section 2.4.2 where it appeared as a natural framework where some of the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  were living. The goal will be to spread some of the structures we have on  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ , for example the affine, complex and metric ones, to other Lie algebras of the family. For simplicity we recall the definition.

**Definition 3.2.1.** For any  $Q \in \text{Gr}(6, F_3)$  let us define the Lie algebra  $\mathfrak{g}(Q)$  to be the vector space

$$\mathfrak{g}(Q) = F_1 \oplus F_2 \oplus F_3 / Q$$

with the Lie algebra structure induced by the one of  $F$ . Here  $F$  is the free 3-step nilpotent Lie algebra on 3 generators and, if we denote the elements of the lower central series as  $F^{i+1} = [F, F^i]$ , the grading is defined as  $F_i = F^i / F^{i+1}$ . We will write the grading on  $\mathfrak{g}(Q)$  also as  $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ . Explicitly, in the basis  $(x_1, \dots, x_8)$  adapted to the decomposition, the non-zero Lie brackets of  $\mathfrak{g}(Q)$  are as follows

$$[x_1, x_2] = x_4, \quad [x_1, x_3] = x_5, \quad [x_2, x_3] = x_6$$

$$[x_i, x_j] = \alpha_{ij}x_7 + \beta_{ij}x_8 \quad i \in \{1, 2, 3\} \text{ and } j \in \{4, 5, 6\},$$

with the coefficients  $\alpha_{ij}$  and  $\beta_{ij} \in \mathbb{R}$  satisfying  $\alpha_{34} = \alpha_{25} - \alpha_{16}$  and  $\beta_{34} = \beta_{25} - \beta_{16}$ .

Let us also define  $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  by  $P := \varphi^{-1}(Q)^\perp$  where the orthogonal space is taken with respect to the Killing form on  $\mathfrak{sl}(3, \mathbb{R})$  and  $\varphi : \mathfrak{sl}(3, \mathbb{R}) \rightarrow F_3$  is the  $\text{SL}(3, \mathbb{R})$ -equivariant isomorphism defined in Proposition 2.4.17.

*Remark 3.2.2.* Every Carnot 3-step nilpotent Lie algebra of dimension 8 having the free 2-step nilpotent Lie algebra on 3 generators as quotient of the third stratum is isomorphic to a Lie algebra  $\mathfrak{g}(Q)$  for some  $Q \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ .

Let us also give a name and define two particular subfamilies. The first one is a generalisation of the family of Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  when  $n = 3$ ,  $\text{rank} \gamma_2 = 1$  and  $\pi(w_0) \neq 0$ , see Proposition 2.4.3. Indeed the Lie algebras  $\mathfrak{g}_{\mathbb{R}}(\alpha, a, b, c)$  defined in Definition 2.4.1 coincide with  $\mathfrak{g}(\alpha, a, b, c, -a)$  of the following definition.

**Definition 3.2.3.** Denote by  $\mathfrak{g}(\alpha, a, b, c, d)$  the Lie algebra whose non-zero Lie brackets, expressed in the basis  $(x_1, \dots, x_8)$  of  $\mathbb{R}^8$  are:

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_4] &= x_7, & [x_1, x_5] &= x_8, \\ [x_2, x_4] &= \alpha x_7, & [x_2, x_5] &= -\alpha x_8, \\ [x_3, x_4] &= -\alpha x_8, & [x_3, x_5] &= 3\alpha x_7, \\ [x_2, x_6] &= ax_7 + bx_8, & [x_3, x_6] &= cx_7 + dx_8. \end{aligned}$$

When  $\alpha \neq 0$  a change of basis brings  $\mathfrak{g}(\alpha, a, b, c, d)$  to the form  $\mathfrak{g}(1, \frac{a}{\alpha^2}, \frac{b}{\alpha^2}, \frac{c}{\alpha^2}, \frac{d}{\alpha^2})$  that we will denote by  $\mathfrak{g}(a, b, c, d)$ .

*Remark 3.2.4.* We can see that the family  $\mathfrak{g}(a, b, c, d)$  has a natural compactification. Indeed it corresponds to a subvariety of  $\text{Gr}(6, F_3)$  defined by the 6-dimensional subspaces of  $F_3$  of the form

$$Q(a, b, c, d) = \text{span}_{\mathbb{R}}\{y_9, y_{10} - y_7, y_{11} + y_8, y_{13} - 3y_7 - ay_7 - by_8, y_{14} - cy_7 - dy_8\} \subseteq F_3.$$

All these subspaces  $Q(a, b, c, d)$  contain the 4-dimensional subspace of  $F_3$ ,  $\tilde{Q} = \text{span}_{\mathbb{R}}\{y_9, y_{10} - y_7, y_{11} + y_8, y_{13} - 3y_7\}$ . Hence every  $Q(a, b, c, d)$  is contained in the subvariety of  $\text{Gr}(6, F_3)$  of all 6-dimensional subspaces of  $F_3$  containing  $\tilde{Q}$ . This latter subvariety is isomorphic to  $\text{Gr}(2, F_3/\tilde{Q}) \cong \text{Gr}(2, 4)$ . Hence via the identification of  $\mathfrak{g}(a, b, c, d)$  with  $Q(a, b, c, d)$  we see the family of Lie algebras  $\mathfrak{g}(a, b, c, d)$  as a subvariety of  $\text{Gr}(2, F_3/\tilde{Q})$ . Finally the family  $\mathfrak{g}(a, b, c, d)$  corresponds actually to just one chart of  $\text{Gr}(2, F_3/\tilde{Q})$ . Indeed, let  $(\bar{y}_7, \bar{y}_8, \bar{y}_{12}, \bar{y}_{14})$  be a basis for  $F_3/\tilde{Q}$ , where by  $\bar{y}_i$  we mean the equivalence class represented by  $y_i$ . We can represent elements of the Grassmannian  $\text{Gr}(2, F_3/\tilde{Q})$  as

matrices whose lines are the vectors spanning them. Then the Lie algebra  $\mathfrak{g}(a, b, c, d)$  corresponds to the matrix

$$\begin{pmatrix} -a & -b & 1 & 0 \\ -c & -d & 0 & 1 \end{pmatrix}$$

and we see that it is exactly an affine chart of  $\text{Gr}(2, F_3/\tilde{Q})$ . The Lie algebras in the boundary of  $\mathfrak{g}(a, b, c, d)$  inside  $\text{Gr}(2, F_3/\tilde{Q})$  can be presented as follows.

The Lie algebras in the chart  $U_{24} \setminus U_{34}$  correspond to elements  $\begin{pmatrix} 0 & 0 & -1 & d & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & -c & a & -3 & 1 & 1 \end{pmatrix}$  in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  and are defined by the following non-zero Lie brackets

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_4] &= x_7, & [x_1, x_5] &= -ax_7, \\ [x_2, x_4] &= x_7, & [x_2, x_5] &= ax_7, & [x_2, x_6] &= x_8, \\ [x_3, x_4] &= ax_7, & [x_3, x_5] &= 3x_7, & [x_3, x_6] &= -cx_7 - dx_8 \end{aligned}$$

with  $a, c, d \in \mathbb{R}$ .

The ones in  $U_{23} \setminus (U_{24} \cup U_{34})$  correspond to  $\begin{pmatrix} 0 & -a & -c & 0 & a & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$  in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  and are defined by

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_4] &= x_7, & [x_1, x_5] &= -ax_7 \\ [x_2, x_4] &= x_7, & [x_2, x_5] &= ax_7, & [x_2, x_6] &= -cx_7 \\ [x_3, x_4] &= ax_7, & [x_3, x_5] &= 3x_7, & [x_3, x_6] &= x_8 \end{aligned}$$

with  $a, c \in \mathbb{R}$ .

The ones in  $U_{14} \setminus (U_{23} \cup U_{24} \cup U_{34})$  correspond to  $\begin{pmatrix} 0 & -1 & 0 & c & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d & 0 & 0 & 0 & 0 \end{pmatrix}$  in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  and are defined by

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_5] &= x_7, & [x_2, x_5] &= -x_7, & [x_2, x_6] &= x_8, \\ [x_3, x_4] &= -x_7, & [x_3, x_6] &= -cx_7 - dx_8 \end{aligned}$$

with  $c, d \in \mathbb{R}$ .

The ones in  $U_{13} \setminus (U_{14} \cup U_{23} \cup U_{24} \cup U_{34})$  correspond to  $\begin{pmatrix} 0 & -1 & c & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$  in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  and are defined by

$$[x_1, x_2] = x_4, \quad [x_1, x_3] = x_5, \quad [x_2, x_3] = x_6,$$

$$\begin{aligned} [x_1, x_5] &= x_7, & [x_2, x_5] &= -x_7, & [x_2, x_6] &= -cx_7, \\ [x_3, x_4] &= -x_7, & [x_3, x_6] &= x_8 \end{aligned}$$

with  $c \in \mathbb{R}$ .

The one in  $U_{12} \setminus (U_{13} \cup U_{14} \cup U_{23} \cup U_{24} \cup U_{34})$  corresponds to  $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$  in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  and is defined by

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_2, x_6] &= x_7, & [x_3, x_6] &= x_8. \end{aligned}$$

*Remark 3.2.5.* Remember from Proposition 2.4.14 that  $\text{GL}(3, \mathbb{R})$  acts on  $F_3$ . Then it acts on the Grassmannian of  $k$ -subspaces of  $F_3$ . Certainly we can extend all the scalars to  $\mathbb{C}$  and act with  $g = \text{diag}(1, 1, i)$  on  $\tilde{Q}$ . We then obtain  $g \cdot \tilde{Q} = \tilde{Q}' = \text{span}_{\mathbb{R}}\{y_9, y_{10} - y_7, y_{11} + y_8, y_{13} + 3y_7\}$  that is defined over the real numbers. Hence we might consider the family of Lie algebras corresponding to one chart of  $\text{Gr}(2, F_3/\tilde{Q}') \cong \text{Gr}(2, 4)$ , call it  $\mathfrak{g}'(a, b, c, d)$ . Explicitly  $\mathfrak{g}'(a, b, c, d)$  is defined as follows.

**Definition 3.2.6.** Denote by  $\mathfrak{g}'(a, b, c, d)$  the Lie algebra whose non-zero Lie brackets, expressed in the basis  $(x_1, \dots, x_8)$  of  $\mathbb{R}^8$  are:

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\ [x_1, x_4] &= x_7, & [x_1, x_5] &= x_8, \\ [x_2, x_4] &= x_7, & [x_2, x_5] &= -x_8, \\ [x_3, x_4] &= -x_8, & [x_3, x_5] &= -3x_7, \\ [x_2, x_6] &= ax_7 + bx_8, & [x_3, x_6] &= cx_7 + dx_8. \end{aligned}$$

### 3.2.1 GIT and classification "à la main"

In this section we will be interested in classifying the Lie algebras  $\mathfrak{g}(Q)$  up to isomorphism. In Proposition 2.4.20 we saw that the isomorphism classes of the Lie algebras  $\mathfrak{g}(Q)$  are in bijection with the orbits of elements of  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  under the  $\text{SL}(3, \mathbb{R})$ -action. To understand quotients in algebraic geometry there is a classical theory called GIT, geometric invariant theory, that has been developed. We started with this approach searching for a full set of invariants. We couldn't find all of them, see Remark 3.2.13, hence we used an ad-hoc method. Notice that in any case the GIT approach would not distinguish all the orbits. We will first give some details of the GIT theory, explain what we did in this direction and finally explain the classification of the Lie algebras  $\mathfrak{g}(Q)$  more by hand. The classification we get is a topological description of the quotient of  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  by  $\text{SL}(3, \mathbb{R})$ . Let us first give some highlights of the GIT.

**Definition 3.2.7.** An action of a reductive group  $G$  on a projective variety  $X \subseteq \mathbb{P}^n$  is said to be *linear* if  $G$  acts via a homomorphism  $G \rightarrow \mathrm{GL}(n+1)$ .

*Remark 3.2.8.* If we have a linear action of  $G$  on  $X \subseteq \mathbb{P}^n$  then  $G$  acts on the affine cones  $\mathbb{A}^{n+1}$  and  $\tilde{X}$  over  $\mathbb{P}^n$  and  $X$ . Hence  $G$  acts on  $R = A(\tilde{X}) = R(X)$ , the coordinate ring of  $\tilde{X}$  that is the same as the homogeneous coordinate ring of  $X$ . A theorem by Nagata says that if  $G$  is reductive then  $R^G$ , the ring of  $G$ -invariant regular functions, is finitely generated so that we can consider  $\mathrm{Proj}(R^G)$ , the projective variety associated to  $R^G$ .

**Definition 3.2.9.** For a linear action of a reductive group  $G$  on a projective variety  $X \subseteq \mathbb{P}^n$  let  $X//G$  denote the projective variety  $\mathrm{Proj}(R^G)$  associated to the finitely generated graded algebra  $R^G$  of  $G$ -invariant functions where  $R = R(X)$  is the homogeneous coordinate ring of  $X$ . The inclusion  $R^G \hookrightarrow R$  defines a rational map

$$X \dashrightarrow X//G$$

which is undefined on the null cone  $N_{RG}(X) = \{x \in X \mid f(x) = 0 \forall f \in R_+^G\}$ . Define then the semistable locus  $X^{ss} = X \setminus N_{RG}(X)$  to be the complement of the null cone. Then the projective GIT quotient for the linear action of  $G$  on  $X \subseteq \mathbb{P}^n$  is the morphism

$$X^{ss} \rightarrow X//G.$$

For details about the definitions of concepts appearing in the following theorem we invite the reader to look at the fundamental book on geometric invariant theory, [23]. Essentially the theorem states that the definition just given for the GIT quotient is a good notion of quotient in algebraic geometry.

**Theorem 3.2.10** ([23, Theorem 8.1.]). *Let  $G$  be a reductive group acting linearly on a projective variety  $X \subseteq \mathbb{P}^n$  then:*

- the GIT quotient

$$\pi : X^{ss} \rightarrow Y := X//G$$

*is a good quotient and a categorical quotient and  $Y$  is a projective variety,*

- $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset \Leftrightarrow \pi(x_1) = \pi(x_2)$ ,
- there exists an opens subset  $Y^s \subseteq Y$  such that  $\pi^{-1}(Y^s) = X^s$  and  $\pi : X^s \rightarrow Y^s$  is a geometric quotient.

Hence to study the projective GIT one needs to study invariant homogeneous polynomials.

*Remark 3.2.11.* Remember, as it was explained in Section 2.4.4, that we can use the Plücker embedding of  $\mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  in projective space

$$\iota : \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \rightarrow \mathbb{P} \left( \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \right).$$

Furthermore fixing  $(E_i)_{i=1}^8$  as basis for  $\mathfrak{sl}(3, \mathbb{R})$  we can represent an element  $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  as a  $2 \times 8$  matrix whose lines are the vectors spanning it, then we have that the Plücker coordinates  $a_{ij}$  are the minors of the  $2 \times 2$  submatrix of  $P$  obtained taking the  $i$ -th and  $j$ -th columns. See Remark 2.4.34 for the details of the projection.

In the study of invariants it will be convenient to introduce the following polynomials in the variables  $(a, b, c, d)$ :

- $c_1(a, b, c, d) = 12 + 6b + 2c + a^2 + ad + bc + d^2,$
- $c_2(a, b, c, d) = (a + d)(12 + 6b + 2c - ad + bc),$
- $c_3(a, b, c, d) = 144 - 96(3b + c) - 8(69a^2 + 186ad - 234b^2 - 204bc - 26c^2 + 69d^2) - 8(21a^2b - 11a^2c - 204abd - 32acd + 54b^3 + 138b^2c + 64bc^2 + 21bd^2 - 11cd^2) + (a^2 - 2ad + 4bc + d^2)^2,$
- $c_4(a, b, c, d) = -1728 + 5184b + 1728c - 21168a^2 - 35424ad + 57024b^2 + 31104bc + 6336c^2 - 21168d^2 + 24192a^2b + 5472a^2c + 68256abd + 27936acd - 52704b^3 - 95904b^2c - 29376bc^2 + 24192bd^2 - 2240c^3 + 5472cd^2 + 828a^4 + 16128a^3d + 9936a^2b^2 - 19152a^2bc - 3216a^2c^2 + 36072a^2d^2 - 136512ab^2d - 101664abcd - 6528ac^2d + 16128ad^3 + 38880b^4 + 91584b^3c + 70848b^2c^2 + 9936b^2d^2 + 13056bc^3 - 19152bcd^2 - 3216c^2d^2 + 828d^4 + 252a^4b - 132a^4c + 4824a^3bd - 120a^3cd + 648a^2b^3 - 5112a^2b^2c + 240a^2bc^2 - 10152a^2bd^2 + 504a^2cd^2 + 6480ab^3d + 18000ab^2cd + 4704abc^2d + 4824abd^3 - 120acd^3 - 5184b^4c - 8928b^3c^2 + 648b^3d^2 - 4704b^2c^3 - 5112b^2cd^2 + 240bc^2d^2 + 252bd^4 - 132cd^4 - (a^2 - 2ad + 4bc + d^2)^3,$
- $c_5(a, b, c, d) = 12 - 16(3b + c) + a^2 - 8ad + 10bc + d^2,$
- $c_6(a, b, c, d) = (a + d)(60 + 28(3b + c) + 5a^2 + 14ad - 4bc + 5d^2)(12 + 2(3b + c) - ad + bc),$
- $c_7(a, b, c, d) = 62208b + 20736c + 24192a^2 + 69120ad + 108864b^2 + 51840bc + 12096c^2 + 24192d^2 + 63504a^2b + 15984a^2c + 106272abd + 45792acd + 73872b^3 + 47952b^2c + 21168bc^2 + 63504bd^2 + 2160c^3 + 15984cd^2 + 2664a^4 + 10440a^3d + 33696a^2b^2 + 35640a^2bc + 2232a^2c^2 + 11448a^2d^2 + 57672ab^2d + 56160abcd + 9432ac^2d + 6552ad^3 + 25272b^4 + 20088b^3c + 14040b^2c^2 + 51192b^2d^2 + 3240bc^3 + 23976bcd^2 + 144c^4 + 4176c^2d^2 + 4608d^4 + 5364a^4b + 492a^4c + 8928a^3bd + 2760a^3cd + 6912a^2b^3 + 14904a^2b^2c + 5940a^2bc^2 + 4212a^2bd^2 - 68a^2c^3 + 4428a^2cd^2 + 12528ab^3d$

$$\begin{aligned}
& +21384ab^2cd + 10584abc^2d + 6984abd^3 + 512ac^3d + 2112acd^3 + 4860b^5 + 3564b^4c + 4428b^3c^2 \\
& +15660b^3d^2 + 1620b^2c^3 + 11988b^2cd^2 + 216bc^4 + 4968bc^2d^2 + 6336bd^4 + 256c^3d^2 + 816cd^4 - 18a^6 \\
& -30a^5d + 774a^4b^2 + 1410a^4bc - 22a^4c^2 + 222a^4d^2 + 1260a^3b^2d + 2328a^3bcd + 158a^3c^2d + 630a^3d^3 \\
& +1458a^2b^4 + 540a^2b^3c + 2808a^2b^2c^2 + 3564a^2b^2d^2 + 78a^2bc^3 + 270a^2bcd^2 + 576a^2c^2d^2 + 60a^2d^4 \\
& +810ab^4d + 2160ab^3cd + 3510ab^2c^2d + 2718ab^2d^3 + 480abc^3d + 1032abcd^3 + 320ac^2d^3 - 192ad^5 \\
& +486b^6 + 162b^5c + 702b^4c^2 + 1458b^4d^2 + 270b^3c^3 + 1998b^3cd^2 + 108b^2c^4 + 1836b^2c^2d^2 + 2232b^2d^4 \\
& +240bc^3d^2 + 600bcd^4 + 140c^2d^4 + 144d^6 + 72a^6b - 12a^6c + 120a^5bd - 32a^5cd + 108a^4b^3 + 96a^4b^2c \\
& +68a^4bc^2 + 30a^4bd^2 + 28a^4cd^2 + 432a^3b^3d - 12a^3b^2cd + 176a^3bc^2d - 144a^3bd^3 + 132a^3cd^3 + 198a^2b^3c^2 \\
& +702a^2b^3d^2 + 48a^2b^2c^3 + 162a^2bc^2d^2 + 30a^2bd^4 + 28a^2cd^4 - 108ab^4cd + 288ab^3c^2d + 432ab^3d^3 \\
& +96ab^2c^3d - 12ab^2cd^3 + 176abc^2d^3 + 120abd^5 - 32acd^5 + 54b^5c^2 + 18b^3c^4 + 198b^3c^2d^2 + 108b^3d^4 \\
& +48b^2c^3d^2 + 96b^2cd^4 + 68bc^2d^4 + 72bd^6 - 12cd^6 + 6a^7d - 6a^6bc + 35a^6d^2 - 34a^5bcd + 90a^5d^3 - a^4b^2c^2 \\
& -94a^4bcd^2 + 122a^4d^4 + 8a^3b^2c^2d - 132a^3bcd^3 + 90a^3d^5 - 4a^2b^3c^3 + 18a^2b^2c^2d^2 - 94a^2bcd^4 + 35a^2d^6 \\
& -8ab^3c^3d + 8ab^2c^2d^3 - 34abcd^5 + 6ad^7 - 4b^3c^3d^2 - b^2c^2d^4 - 6bcd^6.
\end{aligned}$$

**Proposition 3.2.12.** *We can find seven homogenous  $\mathrm{SL}(3, \mathbb{R})$ -invariant polynomials on  $\mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  whose expression on  $\mathfrak{g}(a, b, c, d)$  are the polynomials  $\{c_1, \dots, c_7\}$  defined above. The polynomials  $\{c_1, c_2, c_3, c_5, c_7\}$  are generically algebraically independent on the set of the  $\mathfrak{g}(a, b, c, d)$ .*

*Proof.* Let us identify  $\mathfrak{g}(a, b, c, d)$  with  $P(a, b, c, d) = [u \wedge v] \in \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  with  $u = E_2 + bE_3 + dE_4 - E_5$  and  $v = aE_3 + cE_4 - 3E_6 + E_7 + E_8$ . Using the Plücker embedding, we can see  $\{P(a, b, c, d)\}$  as a submanifold of the projective space  $\mathbb{P}(\wedge^2 \mathfrak{sl}(3, \mathbb{R}))$ . Since the invariants associated to each of the irreducible representation appearing in the decomposition of  $\wedge^2 \mathfrak{sl}(3, \mathbb{R})$  are well understood we will use them in order to find invariants for the family of Lie algebras. The first projection restricted to  $\mathfrak{g}(a, b, c, d)$  is

$$\pi_1(u \wedge v) = (a + d)E_1 + dE_2 + (d - a)E_3 + (c - 3b)E_4 - (c + 6)E_6 - (b + 2)E_8.$$

The algebra of invariants of the  $\mathrm{SL}(3, \mathbb{R})$ -action on  $\mathfrak{sl}(3, \mathbb{R})$  is generated by the coefficients of the characteristic polynomial of the associated matrix, that, in our case, looks like

$$\lambda^3 - (12 + 6b + 2c + a^2 + ad + bc + d^2)\lambda + (a + d)(12 + 6b + 2c - ad + bc).$$

Then let us define the two invariants polynomials  $c_1, c_2 \in \mathbb{R}[a, b, c, d]$  as

$$c_1 = c_1(a, b, c, d) = 12 + 6b + 2c + a^2 + ad + bc + d^2 \quad \text{and}$$

$$c_2 = c_2(a, b, c, d) = (a + d)(12 + 6b + 2c - ad + bc).$$

These polynomials are well defined up to scalar multiple  $\lambda^2 c_1$  and  $\lambda^3 c_2$  with  $\lambda \in \mathbb{R}^*$ . Restricting the second and third projection to the family of lie algebra we get

$$\begin{aligned} \pi_2(u \wedge v) = & \\ (bc - ad)e_1 \odot e_1 \odot e_1 + (c - 3b)e_1 \odot e_1 \odot e_2 + (a - d)e_1 \odot e_1 \odot e_3 + (c - 3)e_1 \odot e_2 \odot e_2 & \\ + (d - a)e_1 \odot e_2 \odot e_3 + (1 - b)e_1 \odot e_3 \odot e_3 - 3e_2 \odot e_2 \odot e_2 - 3e_2 \odot e_3 \odot e_3 & \end{aligned}$$

and

$$\begin{aligned} \pi_3(u \wedge v) = & \\ -e_1^* \odot e_1^* \odot e_1^* + (1 - b)e_1^* \odot e_2^* \odot e_2^* + (a - d)e_1^* \odot e_2^* \odot e_3^* + (c - 3)e_1^* \odot e_3^* \odot e_3^* & \\ -be_2^* \odot e_2^* \odot e_2^* - (2a + d)e_2^* \odot e_2^* \odot e_3^* - (2c + 3b)e_2^* \odot e_3^* \odot e_3^* - 3de_3^* \odot e_3^* \odot e_3^*. & \end{aligned}$$

Now if we interpret  $S^3(\mathbb{R}^3)$  and  $S^3(\mathbb{R}^{3*})$  as homogeneous polynomials of degree 3 in 3 variables it is known, [45], that the ring of invariants of the  $SL(3, \mathbb{R})$ -action is generated by two homogeneous polynomials  $S$  and  $T$ , of degree 4 and 6 respectively. Let us call  $S(a, b, c, d)$  and  $T(a, b, c, d)$  their evaluation on  $\pi_2(P(a, b, c, d))$  and  $S^*(a, b, c, d)$  and  $T^*(a, b, c, d)$  the one on  $\pi_3(P(a, b, c, d))$ , then we find that

$$S^*(a, b, c, d) = S(a, b, c, d) \quad \text{and} \quad T^*(a, b, c, d) = T(a, b, c, d).$$

Then  $c_3 = c_3(a, b, c, d) = S(a, b, c, d)$  and  $c_4 = c_4(a, b, c, d) = T(a, b, c, d)$ . These polynomials are well defined up to a scalar multiple  $\lambda^4 c_3, \lambda^6 c_4$  with  $\lambda \in \mathbb{R}^*$ . Let us notice that we can produce other invariants. If we consider the point  $(M, v, \varphi) \in \mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^3(\mathbb{R}^{3*})$  and we make an identification  $S^3(\mathbb{R}^{3*}) \cong S^3(\mathbb{R}^3)^*$  by setting

$$e_{i_1}^* \odot e_{i_2}^* \odot e_{i_3}^*(e_{j_1} \odot e_{j_2} \odot e_{j_3}) = \frac{1}{6} \sum_{\sigma \in S_3} e_{i_{\sigma(1)}}^*(e_{j_1}) e_{i_{\sigma(2)}}^*(e_{j_2}) e_{i_{\sigma(3)}}^*(e_{j_3})$$

then if we think  $\varphi$  as a linear form we have that  $\varphi(v)$ ,  $\varphi(Mv)$  and  $\varphi(M^2v)$  are invariant under the  $SL(3, \mathbb{R})$ -action. If we calculate these function on the subvariety  $\{\mathfrak{g}(a, b, c, d)\}$  we obtain, again up to scalars, the polynomials  $c_5(a, b, c, s)$ ,  $c_6(a, b, c, d)$  and  $c_7(a, b, c, d)$ . These polynomials are well defined up to a scalar multiple  $\lambda^2 c_5, \lambda^5 c_6$  and  $\lambda^8 c_7$  with  $\lambda \in \mathbb{R}^*$ . Now it can be computed that the rank of the Jacobian of  $(c_1, c_2, c_3, c_5, c_7)$  is 5 hence these polynomial are generically algebraically independent.  $\square$

*Remark 3.2.13.* The invariants we have found,  $\{c_1, \dots, c_7\}$ , do not distinguish all the orbits, hence at least an invariant is missing. Indeed let us consider the point  $p_0 = (1, -3, -1, -1)$  inside  $\mathfrak{g}(a, b, c, d)$ . When we consider  $\pi_1(p_0)$  we see that  $a + d = 0$  is the only real eigenvalue. Hence  $a + d$  becomes a real invariant and we can rewrite the invariants in order to get a simpler form as follows  $\{q_1 =$

$\lambda(a+d), q_2 = \lambda^2(6-ad+bc), q_3 = \lambda(3b+c+3), q_4 = \lambda^4 c_3, q_5 = \lambda^8 c_7\}$ , with  $\lambda \in \mathbb{R}^*$ . With this simpler form we are able to solve the system  $\{q_1(p) = 0, q_2(p) = 10, q_3(p) = -7, c_3(p) = 52816, c_7(p) = 12144\}$  and find all points in  $\mathfrak{g}(a, b, c, d)$  on which the invariants take the same values as on  $p_0$ . This calculation shows that there are 12 points in  $\mathfrak{g}(a, b, c, d)$  that have the same invariants. Consider one of them,  $p_1 = (\frac{1}{2} + 2i, -1 + \frac{1}{2}i, -7 - \frac{3}{2}i, -\frac{1}{2} - 2i)$ . One can check that  $p_1$  is not in the same orbit of  $p_0$  as follows. We act both on  $p_0$  and  $p_1$ , with  $g_0$  and  $g_1 \in \mathrm{SL}(3, \mathbb{R})$  respectively, in order to have their first projections in a diagonal form. Then  $p_1$  will be in the orbit of  $p_0$  if there exists a diagonal matrix in  $\mathrm{SL}(3, \mathbb{R})$  that sends the second and third projection of  $g_1 \cdot p_1$  to the second and third projection of  $g_0 \cdot p_0$ . One can check that it is not the case here.

We now give the ad-hoc method of classification.

**Theorem 3.2.14.** *Each Lie algebra  $\mathfrak{g}(Q)$  belongs to one of the non-isomorphic families represented in Appendix B.*

*Proof.* As noticed in Proposition 2.4.20 the orbit of each  $P$  under the  $\mathrm{SL}(3, \mathbb{R})$ -adjoint action represents the isomorphism class of the associated Lie algebra  $\mathfrak{g}(Q)$ , with  $P = \varphi^{-1}(Q)^\perp \in \mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ . Remember from Remark 2.4.34 the expressions of the projections  $\pi_i$  on the factors of the decomposition of  $\mathbb{P}(\wedge^2 \mathfrak{sl}(3, \mathbb{R}))$ . Hence we will write  $\pi_2(P) = [a_{34}x^3 + p_1(y, z)x^2 + p_2(y, z)x + p_3(y, z)]$  and  $\pi_3(P) = [a_{57}x^3 + q_1(y, z)x^2 + q_2(y, z)x + q_3(y, z)]$ . Now since every matrix in  $\mathfrak{sl}(3, \mathbb{R})$  has at least one real eigenvalue  $\lambda_0 \in \mathbb{R}$ , in the orbit of every element in  $\mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  there exists an element  $P$  such that

$$\pi_1(P) = \left[ \begin{pmatrix} \lambda_0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right]. \quad (3.12)$$

In order to get a non-redundant classification let us notice that, if  $\pi_1(P)$  is not 0, we can always choose the first vector of the basis with respect to which we write  $\pi_1(P)$  as (3.12) in a unique way, up to scalar multiplication. Indeed let  $S$  be the set  $\{\lambda \in \mathbb{R} \mid \lambda \text{ is a real eigenvalue of } \pi_1(P) \text{ of multiplicity } 1\}$ . If  $\pi_1(P)$  is not nilpotent let  $\lambda_0$  be the minimum of the absolute value function on  $S$  and choose as first vector of the basis the corresponding eigenvector. If  $\pi_1(P)$  is nilpotent of rank 1 then we choose a spanning vector for the image of  $\pi_1(P)$  and if it has rank 2 then we choose a spanning vector for the kernel of  $\pi_1(P)$ . We will treat the case  $\pi_1(P) = 0$  separately at the end of the proof. If we let  $X$  be the subvariety of  $\mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  such that the first projection is of the form (3.12), we can reduce the  $\mathrm{SL}(3, \mathbb{R})$ -action on  $\mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  to the  $\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^2$ -action on  $X$ . Let us notice that this group induces an action of  $\mathrm{GL}(2, \mathbb{R})$  on  $\mathcal{S}^3(\mathbb{R}^2)$ , the cubics in two variables  $y$  and  $z$ . More precisely, if  $g = (h, v) \in \mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^2$  we have  $[p_3(\pi_2(g \cdot P))] = [h \cdot p_3(y, z)]$ . If  $p_3(y, z) = p_0y^3 + 3p_1y^2z + 3p_2yz^2 + p_3z^3$  is a binary cubic let  $\Delta$  be its discriminant,  $\Delta(p_3(y, z)) = 3p_1^2p_2^2 + 6p_0p_1p_2p_3 - 4p_1^3p_3 - 4p_2^3p_0 - p_0^2p_3^2$ , and  $H$  its Hessian. The discriminant and the Hessian are respectively an invariant and a covariant for the action of  $\mathrm{GL}(2, \mathbb{R})$  over the binary cubics, where here an invariant is in the large sense of a polynomial that gets multiplied by powers of the determinant of the matrix of the action. Over  $\mathbb{R}$  the normal forms are  $y^3 + yz^2, y^3 - yz^2, y^2z, y^3$  and 0, depending

respectively on whether  $\Delta < 0$ ,  $\Delta > 0$  or  $\Delta = 0$  and either  $H \neq 0$  or  $H = 0$ . The sign of the discriminant  $\Delta$  of the binary cubic  $p_3(y, z)$  becomes then an invariant for the  $\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^2$ -action on  $X$ . Hence we start our classification bringing  $p_3(y, z)$  into a normal form depending on the value of  $\Delta(p_3(y, z))$ . When  $\Delta(p_3(y, z)) < 0$ , we may suppose  $p_3(y, z)$  is in the normal form  $-3y^3 - 3yz^2$ , up to a non-zero scalar  $\alpha \in \mathbb{R}^*$ , that is the same as  $y^3 + yz^2$ . Then, using Plücker coordinates,  $P$  has to be in the zero locus of  $\{a_{56} - 3\alpha, a_{15} - a_{25}, a_{27} - \alpha, a_{78}, a_{67} - a_{25}, a_{17} + a_{27} + a_{58}\}$ . Hence we can write it as

$$P = \begin{pmatrix} -1 - n_5 m_8 & 1 & n_3 & n_4 & n_5 & m_5 & 0 & 0 \\ \frac{m_5(-n_5 m_8 - 2)}{n_5} & 0 & m_3 & m_4 & m_5 & \frac{3 + m_5^2}{n_5} & 1 & m_8 \end{pmatrix}$$

with  $n_i, m_j \in \mathbb{R}$ . We are left with the action of  $\mathrm{Stab}_{\mathrm{GL}(2, \mathbb{R})}(-3y^3 - 3yz^2) \times \mathbb{R}^2$ , where one can see that  $\mathrm{Stab}_{\mathrm{GL}(2, \mathbb{R})}(-3y^3 - 3yz^2) = \mathrm{diag}(\lambda, \pm\lambda)$ . Hence we find the normal form

$$P = \begin{pmatrix} 0 & 1 & b & \varepsilon d & -1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon a & c & 0 & -3 & 1 & 1 \end{pmatrix}$$

with  $\varepsilon = \pm 1$ , that corresponds to  $\mathfrak{g}(a, b, c, d)$ . When  $\Delta(p_3(y, z)) > 0$  in the same way we find the normal form

$$P = \begin{pmatrix} 0 & 1 & b & \varepsilon d & -1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon a & c & 0 & 3 & 1 & 1 \end{pmatrix}$$

with  $\varepsilon = \pm 1$ , that corresponds to  $\mathfrak{g}'(a, b, c, d)$ . When  $\Delta(p_3(y, z)) = 0$  and assuming  $p_3(y, z)$  is in the normal form  $p_3(y, z) = yz^2$ , then  $P$  has to be in the zero locus of  $\{a_{56}, a_{15} - a_{25}, a_{27} - \alpha, a_{78}, a_{67} - a_{25}, a_{17} + a_{27} + a_{58}\}$ . Then we can write  $P$  as

$$P = \begin{pmatrix} -1 & 1 & n_3 & n_4 & 0 & 0 & 0 & 0 \\ m_1 & 0 & m_3 & m_4 & 0 & m_6 & 1 & m_8 \end{pmatrix}$$

or

$$P = \begin{pmatrix} -1 - n_5 m_8 & 1 & n_3 & n_4 & n_5 & m_5 & 0 & 0 \\ \frac{m_5(-n_5 m_8 - 2)}{n_5} & 0 & m_3 & m_4 & m_5 & \frac{m_5^2}{n_5} & 1 & m_8 \end{pmatrix}.$$

We are hence left with the action of  $\mathrm{Stab}_{\mathrm{GL}(2, \mathbb{R})}(yz^2) \times \mathbb{R}^2$ , where one can see that  $\mathrm{Stab}_{\mathrm{GL}(2, \mathbb{R})}(yz^2) = \mathrm{diag}(\lambda, \mu)$ . Hence we find the normal form

$$P = \begin{pmatrix} -1 & 1 & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & e & 1 & 0 \end{pmatrix}$$

with still the action of  $\mathrm{diag}(\lambda^{-1}\mu^{-1}, \lambda, \mu)$ . When  $\Delta(p_3(y, z)) = H(p_3(y, z)) = 0$  and assuming  $p_3(y, z)$  is in the normal form  $p_3(y, z) = y^3$ , then  $P$  has to be in the zero locus of  $\{a_{56} - \alpha, a_{15} -$

$a_{25}, a_{27}, a_{78}, a_{67} - a_{25}, a_{17} + a_{58}$ . Then we can write  $P$  as

$$P = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 & 1 & 0 & 0 & n_8 \\ 0 & 0 & m_3 & m_4 & 0 & 1 & 0 & 0 \end{pmatrix}$$

or

$$P = \begin{pmatrix} \frac{m_5^2 - m_8}{m_7} & \frac{m_2^2}{m_7} & n_3 & n_4 & 1 & 0 & m_2 & \frac{m_2 m_8}{m_7} \\ m_2 & m_2 & m_3 & m_4 & 0 & 1 & m_7 & m_8 \end{pmatrix}.$$

We are hence left with the action of  $\text{Stab}_{\text{GL}(2, \mathbb{R})}(y^3) \times \mathbb{R}^2$ , where one can see that  $\text{Stab}_{\text{GL}(2, \mathbb{R})}(y^3) = \begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix}$ . Hence we find the normal forms

$$P = \begin{pmatrix} 0 & 0 & a & b & 1 & 0 & 0 & 1 \\ 0 & 0 & c & d & 0 & 1 & 0 & 0 \end{pmatrix}$$

with still the action of  $\text{diag}(\lambda_2^{-1}, 1, \lambda_2)$ , or

$$P = \begin{pmatrix} a & 0 & 0 & 0 & b - 2a^2 & 1 & c & b \\ 0 & 0 & 0 & 0 & 1 & 0 & 2a & 1 \end{pmatrix}$$

with still the action of  $\text{diag}(\lambda_2^{-1}, 1, \lambda_2)$ , or finally

$$P = \begin{pmatrix} 0 & 0 & a & b & 1 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 1 & 1 & 0 \end{pmatrix}$$

with still the action of  $\text{diag}(\lambda_1^{-1}, \lambda_1, 1)$ . When  $p_3(y, z) = 0$  we will then distinguish the normal form on whether  $a_{57}$  is 0 or not. The second being represented by the family  $\mathfrak{g}(0, a, b, c, d)$  and the first leading to a subsequent repartition of the normal forms using the discriminant of  $q_2(y, z)$ . This algorithm goes on in a similar matter and allows us to present normal forms for the planes  $P$  under the  $\text{SL}(3, \mathbb{R})$ -action.

Let us say two words on the case  $\pi_1(P) = 0$ . Let  $u$  and  $v$  be a basis for  $P$ , then  $[u, v] = 0$ . Assume first that  $u$  and  $v$  are both diagonalisable over  $\mathbb{R}$ . Let  $(\rho_1, v_1 - \rho_1, -v_1)$  and  $(\rho_2, v_2 - \rho_2, -v_2)$  be the spectrum of  $u$  and  $v$  respectively. Since  $u$  and  $v$  are linearly independent  $(\rho_1, -v_1)$  and  $(\rho_2, -v_2)$  span  $\mathbb{R}^2$ . Hence we can find generators of  $P$  such that their spectrum is  $(1, -1, 0)$  and  $(0, 1, -1)$ , i.e.

$$u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If  $u$  is diagonalisable over  $\mathbb{R}$  and  $v$  diagonalisable over  $\mathbb{C}$  one can see that  $u$  should have an eigenvalue of multiplicity 2 and that we can pick generators of the form

$$u = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

If  $u$  is diagonalisable over  $\mathbb{R}$  and  $v$  is not diagonalisable then, since they commute, one can see that  $u$  must have an eigenvalue of multiplicity 2. Hence we can pick generators of the form

$$u = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can check that if  $u$  is diagonalisable over  $\mathbb{R}$  and  $v$  is nilpotent then they can commute if and only if the rank of  $v$  is 1, but in this case we can find generators as the one just mentioned. Remark also that either the plane  $P$  is made only of nilpotent elements or we can always find an element in  $P$  that is diagonalisable over  $\mathbb{R}$ , hence the only case left is when  $P$  is made only of nilpotent elements. If we can find  $u$  in  $P$  nilpotent of rank 2 and  $v$  commutes with  $u$  then  $v$  must be of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we can find generators of the form

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally if  $P$  is made only of nilpotent matrices of rank 1 then one can see that we can find generators of the form

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This finishes the classification. □

**Corollary 3.2.15.** *The union of  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  is an open dense subset of the topological quotient of  $\mathrm{SL}(3, \mathbb{R})$  acting on  $\mathrm{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ .*

*Proof.* Indeed the union of  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  is the non-zero locus of  $\Delta(p_3(y, z))$ . Where  $\Delta(p_3(y, z))$  is the discriminant of the binary cubic  $p_3(y, z)$  from the second projection  $\pi_2(P)$ . Hence

it is an open set. Furthermore the set where  $\Delta(p_3(y, z))$  is zero, from the analysis coming from the previous proposition, has empty interior. Hence the union of  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  is an open dense subset.  $\square$

*Remark 3.2.16.* Let us say a few words about the respective position of  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$ . During the proof of the previous proposition we have seen that  $\mathfrak{g}(a, b, c, d)$  corresponds to the element of  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$

$$\begin{pmatrix} 0 & 1 & b & d & -1 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & -3 & 1 & 1 \end{pmatrix}$$

and that  $\mathfrak{g}'(a, b, c, d)$  corresponds to

$$\begin{pmatrix} 0 & 1 & b & d & -1 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & 3 & 1 & 1 \end{pmatrix}.$$

Hence we can notice that no element can be in both  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$ . Consider the two Grassmannians  $\text{Gr}(2, F_3/\tilde{Q})$  and  $\text{Gr}(2, F_3/\tilde{Q}')$  in which  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  respectively sit. Since  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  do not intersect, we can deduce that the intersection of  $\text{Gr}(2, F_3/\tilde{Q})$  and  $\text{Gr}(2, F_3/\tilde{Q}')$ , that is  $\text{Gr}(2, F_3/(\tilde{Q} + \tilde{Q}')) \cong \mathbb{P}^2(\mathbb{R})$ , is contained in the boundary of  $\mathfrak{g}(a, b, c, d)$  inside  $\text{Gr}(2, F_3/\tilde{Q})$ . Furthermore the element  $g = \text{diag}(1, 1, i)$  defines a complex isomorphism between the complexification of the two Lie algebras  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$ . More precisely we have  $\mathfrak{g}_{\mathbb{C}}(a, b, c, d) \cong \mathfrak{g}'_{\mathbb{C}}(ia, b, -c, id)$ . Hence when  $a = d = 0$ ,  $\mathfrak{g}(0, b, c, 0)$  and  $\mathfrak{g}(0, b, -c, 0)$  represent two real orbits inside the same complex orbit. Indeed the two real orbits are different since  $\Delta(\mathfrak{g}(0, b, c, 0)) < 0$  and  $\Delta(\mathfrak{g}'(0, b, -c, 0)) > 0$ . Finally notice that when writing the Lie algebras  $\mathfrak{g}(Q)$  in normal form we have made a choice of an eigenvalue for  $\pi_1(P)$ , hence it is not clear whether in general  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  have disjoint orbits.

*Remark 3.2.17.* The Lie algebras in the boundary of  $\mathfrak{g}(a, b, c, d)$  inside  $\text{Gr}(2, F_3/\tilde{Q})$  are all characterised by the fact that  $p_3(y, z) = 0$  and  $a_{57} = 0$ .

From the classification up to isomorphism of  $\mathfrak{g}(Q)$  we have seen that  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  essentially fill up all the space of the quotient. The idea is then to try to extend the particular structures we have on  $\mathfrak{g}(\alpha, a, b, c, -a)$ , i.e. the complex structure and the pseudo-Riemannian metric to the other Lie algebras of the family  $\mathfrak{g}(Q)$  in which  $\mathfrak{g}(\alpha, a, b, c, -a)$  lives.

### 3.2.2 Complex structures

In this section we show that, under some assumptions, if an integrable complex structure exists on the Lie algebra  $\mathfrak{g}(Q)$  then we can bring it to a standard form. Indeed we have the following.

**Proposition 3.2.18.** *Let  $\mathfrak{g}(Q) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$  and  $J$  be a complex structure on  $\mathfrak{g}(Q)$ . Assume that the center  $\mathcal{Z}$  of  $\mathfrak{g}(Q)$  equals  $\mathfrak{g}_3$ . Then  $\mathcal{Z}$  is  $J$ -invariant.*

*Proof.* Remember that we have a grading  $\mathfrak{g}(Q) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$  and we are assuming  $\mathcal{Z} = \mathfrak{g}_3$ . Let us notice that if we establish that  $\mathcal{Z} \cap J\mathcal{Z} \neq \{0\}$  then we can deduce that  $\mathcal{Z}$  is  $J$ -invariant. Let  $z_1$  and  $z_2$  be two linearly independent elements of  $\mathcal{Z}$ , that we know has dimension 2. Then it follows from (3.6) that  $[Jz_1, Jz_2] = 0$ . Hence if we write  $Jz_1 = x_1 + x_2 + x_3$  and  $Jz_2 = y_1 + y_2 + y_3$ , with respect to the grading, we have in particular  $[x_1, y_1] = 0$ . This implies that  $x_1$  and  $y_1$  are linearly dependent so that there exists  $c \in \mathbb{R}$  such that  $Jz_1 - cJz_2 \in \mathfrak{g}_2 \oplus \mathfrak{g}_3$ . So we have found an element  $z_1 - cz_2 \in \mathcal{Z}$  such that  $J(z_1 - cz_2) \in \mathfrak{g}_2 \oplus \mathfrak{g}_3$ . Now suppose we have an element  $z \in \mathcal{Z}$  such that  $Jz$  is not in  $\mathcal{Z}$  but  $Jz \in \mathfrak{g}_2 \oplus \mathfrak{g}_3$ , write  $Jz = x_2 + x_3$  with  $x_2 \in \mathfrak{g}_2$  and  $x_3 \in \mathfrak{g}_3$ . Since  $Jz$  is not in the center there exists  $y \in \mathfrak{g}$  such that  $[Jz, y] \neq 0$ , so we have found a non-zero element  $\tilde{z} = [Jz, y] = [x_2, y] \in \mathfrak{g}_3 = \mathcal{Z}$  such that  $J\tilde{z} = J[Jz, y] = [Jz, Jy] = [x_2, Jy] \in \mathfrak{g}_3 = \mathcal{Z}$ .  $\square$

**Proposition 3.2.19.** *Let  $\mathfrak{g}(Q) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$  with  $\mathcal{Z}$  its center and  $J$  a complex structure on  $\mathfrak{g}(Q)$ . Assume that  $\mathcal{Z} = \mathfrak{g}_3$ . Then we can always find a basis  $(x_1, x_2, x_3, x_4, x_5, x_6)$  of  $\mathfrak{g}(Q)/\mathfrak{g}^3$  such that the Lie brackets are of the form*

$$[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6$$

and the complex structure is such that  $Jx_1 = x_6, Jx_2 = x_3, Jx_4 = x_5$ .

*Proof.* Notice that there exists  $w \in \mathfrak{g}_1$  such that  $Jw \not\equiv 0 \pmod{\mathfrak{g}^2}$  and furthermore they can be chosen so that  $w$  and  $Jw \pmod{\mathfrak{g}^2}$  are linearly independent. Indeed assume that for all  $w \in \mathfrak{g}_1$  we have  $Jw \in \mathfrak{g}^2$ . Taking any two  $w_1, w_2 \in \mathfrak{g}_1$  we get  $[Jw_1, Jw_2] = [w_1, w_2] + J[Jw_1, w_2] + J[w_1, Jw_2]$ . Since  $[Jw_1, Jw_2] = 0$  and  $J[Jw_1, w_2] + J[w_1, Jw_2] \in \mathfrak{g}^3$  this would imply that  $[w_1, w_2] \in \mathfrak{g}^3$  and hence that the Lie algebra  $\mathfrak{g}(Q)$  is not Carnot, which is a contradiction. Consider then a basis  $(w, Jw, u)$  for  $\mathfrak{g}_1$  and let  $[w, Jw] = v_1, [u, w] = v_2$  and  $[u, Jw] = v_3$ . For dimension reasons there exists one of  $v_1, v_2, v_3$  such that  $Jv_i \not\equiv 0 \pmod{\mathfrak{g}^2}$ , let  $v$  be the one with this property so that  $Jv = \alpha w + \beta Jw + \gamma u \pmod{\mathfrak{g}^2}$ , with one between  $\alpha, \beta, \gamma$  not 0. Notice that if  $\gamma = 0$  then we have  $[Jw, Jv] \equiv \alpha[Jw, w] \pmod{\mathcal{Z}}$  and  $[Jw, Jv] \equiv J[w, Jv] \equiv \beta J[w, Jw] \pmod{\mathcal{Z}}$  so that  $\beta J[w, Jw] \equiv \alpha[Jw, w] \pmod{\mathcal{Z}}$  which is a contradiction. Hence  $\gamma \neq 0$  and let  $u' = -Jv$  so we have constructed a new basis  $(w, Jw, u')$  for  $\mathfrak{g}_1$  such that  $Ju' \in \mathfrak{g}^2$ , this implies that from the equation  $[Ju', Jw] = [u', w] + J[Ju', w] + J[u', Jw]$  we get that  $J[u', Jw] \equiv -[u', w] \pmod{\mathcal{Z}}$ . Hence we have  $[w, Jw] = v_1, [u', w] = v'_2, [u', Jw] = v'_3$  and by construction  $Ju' = v_1$  and  $Jv'_2 = v'_3 \pmod{\mathcal{Z}}$ . Hence the proposition follows.  $\square$

*Remark 3.2.20.* Let  $J$  be a complex structure as in Proposition 3.2.19. In order for  $J$  to be integrable we need the following three conditions

$$Jx_4 = x_5 - [x_2, x_6] - J[x_3, x_6],$$

$$J[x_1, x_4] = [x_1, x_5],$$

and

$$[x_3, x_5] = [x_2, x_4] + 2J[x_2, x_5] - J[x_1, x_6].$$

Hence we can see that on  $\mathfrak{g}(a, b, c, d)$  a complex structure always exists. Whether it is the case for the other Lie algebras  $\mathfrak{g}(Q)$  it is not clear. Furthermore on  $\mathfrak{g}(0, 0, 0, 0)$  a complex structure exists but the center has dimension 3 so Proposition 3.2.18 cannot be true in general.

### 3.2.3 Metric structures

Starting from the Hermite-Lorentz pseudo-Kähler structure we have on  $\mathfrak{g}(\alpha, a, b, c, -a)$ , coming from  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ , we will try to see to what extent we can get metric structures on the other Lie algebras of the family  $\mathfrak{g}(Q)$ . Consider the simply transitive pseudo-Riemannian representations we can define on the Lie algebras  $\mathfrak{g}(a, b, c, -a)$ , and which induce the flat metric of Remark 3.2.23. We will deform it in order to define a flat pseudo-Riemannian metric of signature  $(6, 2)$  on both  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$ . Finally we will use the tool coming from Definition 3.1.6 to show that these metrics cannot be extended to the boundary of  $\mathfrak{g}(a, b, c, d)$  in  $\text{Gr}(6, F_3/\tilde{Q})$ . En passant we will also see that the affine structure we get from this representation is not the same as the one defined by Scheuneman on any 3-step nilpotent Lie algebra.

**Proposition 3.2.21.** *We can define a simply transitive flat pseudo-Riemannian representation  $\rho : \mathfrak{g}(a, b, c, d) \rightarrow \mathfrak{o}(6, 2) \times \mathbb{R}^{6+2}$  that coincides with the natural one on  $\mathfrak{g}(\alpha, a, b, c, -a)$ .*

*Proof.* Let us first make explicit what is the representation we have on  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  in the case of complex dimension 4,  $\text{rank } \gamma_2 = 1$  and  $\pi(w_0) \neq 0$ . Fix  $(\tau, i\tau, e, Je, g, f, \xi, i\xi)$  as basis for the vector space  $\mathbb{C} \oplus W \oplus \mathbb{C}$  underlying  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ . Then let  $\gamma_2(f) = ve, \pi(w_0) = \beta g$ , with  $v, \beta \in \mathbb{R}^*$ , and  $b_2(e) = b_2(f) = 0$ . From the fact that  $b_2(e) = 0$  we have  $\langle e, J\gamma_3(\xi) \rangle = 0$ . Let  $\gamma_3(\xi) = \lambda_1 e + \lambda_3 g + \lambda_4 f$  and  $\gamma_3(i\xi) = \mu_1 e + \mu_2 Je + \mu_3 g + \lambda_3 f$ , with  $\lambda_i, \mu_i \in \mathbb{R}$  and  $\mu_3 + \lambda_4 = \beta$ . If  $w = (x_1, y_1, x_2, y_2)$ , the representation reads as follows

$$\rho : \mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \longrightarrow \mathfrak{u}(3, 1) \times \mathbb{C}^{3+1} \subseteq \mathfrak{o}(6, 2) \times \mathbb{R}^{6+2}$$

$$(r_1, r_2, w, s_1, s_2) \mapsto \begin{pmatrix} 0 & 0 & -v & 0 & -b & r_1 \\ 0 & 0 & -Jv & b & 0 & r_2 \\ 0 & 0 & 0 & v^\top & Jv^\top & w^\top \\ 0 & 0 & 0 & 0 & 0 & s_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$v = (vy_2 + \lambda_1 s_1 + \mu_1 s_2, \mu_2 s_2, \lambda_3 s_1 + \mu_3 s_2, \lambda_4 s_1 + \lambda_3 s_2)$$

and

$$b = -2\mu_2 y_1 + \frac{2(\mu_2^2 + \lambda_3^2 + \lambda_4^2 - \beta\lambda_4)}{\beta} x_2 + b_3(\xi)s_1 + b_3(i\xi)s_2.$$

Since the Lie algebra  $\mathfrak{g}(\alpha, a, b, c, -a)$  is isomorphic to  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ , under the hypothesis previously mentioned, we will use the above representation to define one on  $\mathfrak{g}(\alpha, a, b, c, -a)$ . The basis  $(x_1, \dots, x_8)$  in which we define  $\mathfrak{g}(\alpha, a, b, c, -a)$  is obtained from  $(\tau, i\tau, e, Je, g, f, \xi, i\xi)$  by the following change of basis

$$P = v^{-1} \begin{pmatrix} 0 & 0 & 0 & \lambda_4 & \lambda_3 & -\lambda_3(\lambda_1 - \mu_2) + \lambda_4\mu_1 - vb_3(\xi) & -v & 0 \\ 0 & 0 & 0 & \lambda_3 & \mu_3 & -\mu_3(\lambda_1 - \mu_2) + \lambda_3\mu_1 - vb_3(i\xi) & 0 & -v \\ 0 & 0 & 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -v\beta & 0 & 0 \\ 1 & -(\lambda_1 - \mu_2) & -\mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence if we let  $(x^1, \dots, x^8)$  be the coordinates with respect to the basis  $(x_1, \dots, x_8)$  of the Lie algebra  $\mathfrak{g}(\alpha, a, b, c, -a)$ , the representation  $\rho' : \mathfrak{g}(\alpha, a, b, c, -a) \rightarrow \mathfrak{u}(3, 1) \times \mathbb{C}^{3+1}$  is defined as follows. The translation part  $T \circ \rho'(x^1, \dots, x^8) = P(x^1, \dots, x^8)^T$  and the linear part

$$L \circ \rho'(x^1, \dots, x^8) = \begin{pmatrix} 0 & 0 & -v' & 0 & -b' \\ 0 & 0 & -Jv' & b' & 0 \\ 0 & 0 & 0 & v'^T & Jv'^T \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$v' = (x^1 + \mu_2 x^2, \mu_2 x^3, \lambda_3 x^2 + \mu_3 x^3, \lambda_4 x^2 + \lambda_3 x^3)$$

and

$$b' = -2\mu_2 x^5 - 2(\mu_2^2 + \lambda_3^2 + \lambda_4^2 - \beta\lambda_4)x^6 + b_3(\xi)x^2 + b_3(i\xi)x^3.$$

The relations between the coefficients that define  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  and the ones of  $\mathfrak{g}(\alpha, a, b, c, -a)$  are the following

$$\begin{cases} \alpha = \mu_2 \\ a = -\lambda_3\beta \\ b = -(2\mu_2^2 + 2\lambda_3^2 + 2\lambda_4^2 - 3\lambda_4\beta) \\ c = 2\mu_2^2 + 2\lambda_3^2 + 2\lambda_4^2 - \lambda_4\beta - \beta^2 \end{cases} \quad \text{i.e.} \quad \begin{cases} \mu_2 = \alpha \\ \lambda_3 = \frac{-a}{\beta} \\ \lambda_4 = \frac{\beta^2 + b + c}{2\beta} \\ \mu_3 = \frac{\beta^2 - b - c}{2\beta} \end{cases}$$

with  $\beta$  solution of  $2x^4 + (-4\alpha^2 - b + c)x^2 - 4a^2 - b^2 - 2bc - c^2 = 0$ . Notice that  $v, \lambda_1, \mu_1, b_3(\xi)$  and  $b_3(i\xi)$  are not related to  $\alpha, a, b, c$  and can be used to deform the representation. It is easy to see that the relative invariant for this representation is  $\delta(\rho') = \det P = \frac{\beta}{v}$ . We can now deform the above

representation  $\rho'$  to get a representation,  $\widehat{\rho}' : \mathfrak{g}(a, b, c, d) \rightarrow \mathfrak{o}(6, 2) \times \mathbb{R}^{6+2}$ , on  $\mathfrak{g}(a, b, c, d)$ . Define it as with same translation part as  $\rho'$  and linear part

$$L \circ \widehat{\rho}'(x^1, \dots, x^8) = \begin{pmatrix} 0 & 0 & -v' & 0 & -\widehat{b}' \\ 0 & 0 & -v'' & \widehat{b}' & 0 \\ 0 & 0 & 0 & v'^T & v''^T \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$v'' = (-\mu_2 x^3, x^1 + \mu_2 x^2, -\lambda_4 x^2 + \sigma x^3, \lambda_3 x^2 + \mu_3 x^3)$$

and

$$\widehat{b}' = -2\mu_2 x^5 - (2\mu_2^2 + \lambda_3^2 + 2\lambda_4^2 - 2\beta\lambda_4 - \lambda_3\sigma)x^6 + b_3(\xi)x^2 + b_3(i\xi)x^3.$$

The relations between the coefficients that define the representation and the ones of  $\mathfrak{g}(a, b, c, d)$  are the following

$$\begin{cases} \alpha = \mu_2 \\ a = -\lambda_3\beta \\ b = -(2\mu_2^2 + \lambda_3^2 + 2\lambda_4^2 - 3\lambda_4\beta - \lambda_3\sigma) \\ c = 2\mu_2^2 + \lambda_3^2 + 2\lambda_4^2 - \lambda_4\beta - \beta^2 - \lambda_3\sigma \\ d = -\sigma\beta \end{cases} \quad \text{i.e.} \quad \begin{cases} \mu_2 = \alpha \\ \lambda_3 = \frac{-a}{\beta} \\ \sigma = \frac{-d}{\beta} \\ \lambda_4 = \frac{\beta^2 + b + c}{2\beta} \\ \mu_3 = \frac{\beta^2 - b - c}{2\beta} \end{cases}$$

with  $\beta$  solution of  $2x^4 + (-4\alpha^2 - b + c)x^2 - 2a^2 + 2ad - b^2 - 2bc - c^2 = 0$ . We can see that the relative invariant has not changed  $\delta(\widehat{\rho}') = \delta(\rho') = \frac{\beta}{v}$  and  $\widehat{\rho}'$  is a simply transitive representation of  $\mathfrak{g}(a, b, c, d)$ .  $\square$

Let us remember now that the Lie algebras  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  are 3-step nilpotent. Hence we can wonder if the affine structure these Lie algebras naturally have coincides with the one constructed by Scheuneman. This is not the case in general, indeed we have the following.

**Proposition 3.2.22.** *The affine structure defined on  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$  when  $n = 3$ ,  $\text{rank } \gamma_2 = 1$  and  $\pi(w_0) \neq 0$ , by the representation  $\rho : \mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3) \rightarrow \mathfrak{u}(3, 1) \times \mathbb{C}^{3+1} \subseteq \text{aff}(\mathbb{R}^8)$  as in Proposition 3.2.21 is not isomorphic to the one defined by Scheuneman in Proposition 3.1.10.*

*Proof.* Let  $\widetilde{ad} : \mathfrak{g}(\alpha, a, b, c, -a) \rightarrow \text{aff}(\mathbb{R}^8)$  be the affine representation defined by Scheuneman as in Proposition 3.1.10. Let furthermore  $\rho' : \mathfrak{g}(\alpha, a, b, c, -a) \rightarrow \mathfrak{u}(3, 1) \times \mathbb{C}^{3+1}$  be the representation defined in Proposition 3.2.21 on  $\mathfrak{g}(\alpha, a, b, c, -a)$  isomorphic to  $\mathfrak{u}(\gamma_2, \gamma_3, b_2, b_3)$ . The two affine structures are the same if the two representations are conjugate in the affine group, i.e. if there exists  $g \in \text{Aff}(\mathbb{R}^8)$  such that  $\forall x \in \mathfrak{g}(\alpha, a, b, c, -a)$  we have  $\rho(x) = g\widetilde{ad}(x)g^{-1}$ . Consider  $(x_1, \dots, x_8)$ , the standard basis for  $\mathfrak{g}(\alpha, a, b, c, -a)$ , and  $L$  is the morphism that takes the linear part of an affine



**Proposition 3.2.24.** *We can put a flat pseudo-Riemannian metric of signature  $(6, 2)$  on both  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$ . More precisely, the symmetric bilinear form written in the basis  $(x_1, \dots, x_8)$  of  $\mathfrak{g}(Q)$*

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{24} & s_{25} & 0 & -1 & 0 \\ 0 & 0 & 0 & s_{34} & s_{35} & 0 & 0 & -1 \\ 0 & s_{24} & s_{34} & 1 & 0 & 0 & 0 & 0 \\ 0 & s_{25} & s_{35} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s^2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

that is non-degenerate when  $s \neq 0$ , defines a flat metric on  $\mathfrak{g}(Q)$  if and only if

$$\begin{cases} \alpha_{16} = \beta_{16} = 0, \\ \alpha_{14} = 1, \beta_{14} = 0, \\ \alpha_{15} = 0, \beta_{15} = 1 \end{cases}$$

and

$$3s^4 + (-\alpha_{25}^2 + 4s_{35}s_{24} - s_{25}^2 - 2s_{34}s_{25} - s_{34}^2 + (-2\beta_{24} + 4\beta_{35})\alpha_{25} - \beta_{24}^2 + 2\alpha_{36} - \beta_{25}^2 + (4\alpha_{24} - 2\alpha_{35})\beta_{25} - \alpha_{35}^2 - 2\beta_{26})s^2 - \alpha_{36}^2 - 2\beta_{26}\alpha_{36} - \beta_{26}^2 + 4\beta_{36}\alpha_{26} = 0. \quad (3.13)$$

*Proof.* Let  $\langle \cdot, \cdot \rangle$  be the metric on  $\mathfrak{g}(Q)$  defined by  $K$ . In order for  $\langle \cdot, \cdot \rangle$  to be a flat metric on  $\mathfrak{g}(Q)$  we need it to satisfy (3.3) with  $x \cdot y$  defined by (3.5). We start calculating the left symmetric product  $x \cdot y$ . For example

$$\begin{aligned} \langle x_1 \cdot x_4, x_1 \rangle &= \langle [x_1, x_4], x_1 \rangle = 0, \\ \langle x_1 \cdot x_4, x_2 \rangle &= \frac{1}{2} \{ \langle [x_1, x_4], x_2 \rangle - \langle [x_4, x_2], x_1 \rangle + \langle [x_2, x_1], x_4 \rangle \} = \frac{1}{2} \{ -\alpha_{14} - 1 \}, \\ \langle x_1 \cdot x_4, x_3 \rangle &= \frac{1}{2} \{ \langle [x_1, x_4], x_3 \rangle - \langle [x_4, x_3], x_1 \rangle + \langle [x_3, x_1], x_4 \rangle \} = \frac{1}{2} \{ -\beta_{14} \} \end{aligned}$$

and for  $y$  any other element of the basis  $(x_1, \dots, x_8)$  of  $\mathfrak{g}(Q)$  we have  $\langle x_1 \cdot x_4, y \rangle = 0$ . We obtain then  $x_1 \cdot x_4 = \frac{1+\alpha_{14}}{2}x_7 + \frac{\beta_{14}}{2}x_8$ . With a similar calculation we can see that

$$\begin{aligned} x_1 \cdot x_5 &= \frac{\alpha_{15}}{2}x_7 + \frac{1+\beta_{15}}{2}x_8, \\ x_1 \cdot x_6 &= \frac{\alpha_{16}}{2}x_7 + \frac{\beta_{16}}{2}x_8, \\ x_2 \cdot x_4 &= \frac{1-\alpha_{14}}{2}x_1 + \alpha_{24}x_7 + \frac{\beta_{24} + \alpha_{25} - \alpha_{16}}{2}x_8, \end{aligned}$$

$$\begin{aligned}
x_2 \cdot x_5 &= -\frac{\alpha_{15}}{2}x_1 + \alpha_{25}x_7 + \frac{\alpha_{35} + \beta_{25}}{2}x_8, \\
x_2 \cdot x_6 &= -\frac{\alpha_{16}}{2}x_1 + \alpha_{26}x_7 + \frac{s^2 + \alpha_{36} + \beta_{26}}{2}x_8, \\
x_3 \cdot x_4 &= -\frac{\beta_{14}}{2}x_1 + \frac{\alpha_{25} - \alpha_{16} + \beta_{24}}{2}x_7 + (\beta_{25} - \beta_{16})x_8, \\
x_3 \cdot x_5 &= \frac{1 - \beta_{15}}{2}x_1 + \frac{\alpha_{35} + \beta_{25}}{2}x_7 + \beta_{35}x_8, \\
x_3 \cdot x_6 &= -\frac{\beta_{16}}{2}x_1 + \frac{-s^2 + \beta_{26} + \alpha_{36}}{2}x_7 + \beta_{36}x_8.
\end{aligned}$$

Also notice that  $y \cdot x_7 = y \cdot x_8 = 0$  for any  $y \in \mathfrak{g}(Q)$  and  $x \cdot y = 0$  for  $x, y \in \mathfrak{g}^2$ . Hence the equation (3.3) implies in particular that

$$(x_2 \cdot x_4) \cdot x_4 - x_2 \cdot (x_4 \cdot x_4) - (x_4 \cdot x_2) \cdot x_4 + x_4 \cdot (x_2 \cdot x_4).$$

Substituting what we have calculated we get  $\frac{1-\alpha_{14}}{2}(x_4 \cdot x_1) = \frac{1-\alpha_{14}}{2}(\frac{1-\alpha_{14}}{2}x_7 - \frac{\beta_{14}}{2}x_8) = 0$  that implies  $\alpha_{14} = 1$ . Similarly from

$$(x_3 \cdot x_4) \cdot x_4 - x_3 \cdot (x_4 \cdot x_4) - (x_4 \cdot x_3) \cdot x_4 + x_4 \cdot (x_3 \cdot x_4)$$

we get  $-\frac{\beta_{14}}{2}(x_4 \cdot x_1) = -\frac{\beta_{14}}{2}(\frac{1-\alpha_{14}}{2}x_7 - \frac{\beta_{14}}{2}x_8) = 0$  that implies  $\beta_{14} = 0$ . From  $(x_2 \cdot x_5) \cdot x_5 - x_2 \cdot (x_5 \cdot x_5) - (x_5 \cdot x_2) \cdot x_5 + x_5 \cdot (x_2 \cdot x_5)$  we get  $\alpha_{15} = 0$  and from  $(x_3 \cdot x_5) \cdot x_5 - x_3 \cdot (x_5 \cdot x_5) - (x_5 \cdot x_3) \cdot x_5 + x_5 \cdot (x_3 \cdot x_5)$  we get  $\beta_{15} = 1$ . Furthermore  $(x_2 \cdot x_6) \cdot x_6 - x_2 \cdot (x_6 \cdot x_6) - (x_6 \cdot x_2) \cdot x_6 + x_6 \cdot (x_2 \cdot x_6)$  and  $(x_3 \cdot x_6) \cdot x_6 - x_3 \cdot (x_6 \cdot x_6) - (x_6 \cdot x_3) \cdot x_6 + x_6 \cdot (x_3 \cdot x_6)$  imply  $\alpha_{16} = \beta_{16} = 0$ . Finally from the equation  $(x_2 \cdot x_3) \cdot x_2 - x_2 \cdot (x_3 \cdot x_2) - (x_3 \cdot x_2) \cdot x_2 + x_3 \cdot (x_2 \cdot x_2)$  we get condition (3.13). Notice that this condition, that for any fixed Lie algebra  $\mathfrak{g}(Q)$  can be seen as an equation in  $s$ , is of the form  $3s^4 + p(s_{24}, s_{34}, s_{25}, s_{35})s^2 + q = 0$  with  $p(s_{24}, s_{34}, s_{25}, s_{35})$  a polynomial function and  $q \in \mathbb{R}$ . Hence choosing properly  $s_{35}$  and  $s_{24}$  we can guarantee that  $p(s_{24}, s_{34}, s_{25}, s_{35})^2 - 12q > 0$  and that  $-p(s_{24}, s_{34}, s_{25}, s_{35}) - \sqrt{p(s_{24}, s_{34}, s_{25}, s_{35})^2 - 12q} > 0$ , so that there exists a real solution. This implies that on the family of Lie algebras  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  the conditions are satisfied and  $K$  defines a flat metric of signature  $(6, 2)$  on them.  $\square$

*Remark 3.2.25.* If  $\mathfrak{g}(Q)$  is a Lie algebra that belongs to  $\text{Gr}(6, F_3/\tilde{Q})$  or  $\text{Gr}(6, F_3/\tilde{Q}')$  then the condition  $\{\alpha_{14} = 1, \beta_{14} = 0, \alpha_{15} = 0, \beta_{15} = 1\}$  is equivalent to the fact that  $\mathfrak{g}(Q)$  belongs to either  $\mathfrak{g}(a, b, c, d)$  or  $\mathfrak{g}'(a, b, c, d)$ .

**Proposition 3.2.26.** *The representation defined in Proposition 3.2.21 cannot be extended to a simply transitive representation on the boundary of  $\mathfrak{g}(a, b, c, d)$  in  $\text{Gr}(6, F_3/W_0)$ .*

*Proof.* Going to the boundary of  $\mathfrak{g}(a, b, c, d)$  in  $\text{Gr}(6, F_3/W_0)$  is equivalent to changing the basis in the plane  $\text{span}\{x_7, x_8\}$  and then make the determinant of the matrix of change of basis,  $A$ , go to

0. But then if we write the representation in the new basis it will have translation part given by  $P'(x^1, \dots, x^8)^\top$  where  $P'$  is constructed from  $P$  substituting  $A$  in the block of the plane  $\text{span}\{x_7, x_8\}$ . Hence  $\delta(\widehat{\rho}') = \det P' = \frac{\beta}{v} \det A \rightarrow 0$  on the boundary. If we could have extended the representation to a simply transitive representation on the boundary the relative invariant would have tend to a non-zero constant polynomial function. This implies that the possible extensions of the representation  $\widehat{\rho}'$  to the boundary are not simply transitive.  $\square$

**Proposition 3.2.27.** *The condition  $[x_1, x_6] = 0$ , i.e.  $\alpha_{16} = \beta_{16} = 0$  can always be satisfied up to the  $\text{SL}(3, \mathbb{R})$ -action on  $\mathfrak{g}(Q)$ .*

*Proof.* The condition  $[x_1, x_6] = 0$  in  $\mathfrak{g}(Q)$  is equivalent to the fact that  $\text{span}_{\mathbb{R}}\{2E_1 + E_2\} \subseteq W$  where  $W \in \text{Gr}(6, \mathfrak{sl}(3, \mathbb{R}))$  and, if  $\varphi$  is the isomorphism between  $\mathfrak{sl}(3, \mathbb{R})$  and  $F_3$ ,  $\varphi(W) = Q$ . Hence to prove the proposition we should prove that the  $\text{SL}(3, \mathbb{R})$ -orbit of the line  $\text{span}_{\mathbb{R}}\{2E_1 + E_2\}$  and  $W$  intersect in  $\mathfrak{sl}(3, \mathbb{R})$ . Consider the map  $f : \mathbb{P}(\mathfrak{sl}(3, \mathbb{R}))^{ss} \rightarrow \mathbb{P}^1$  defined by  $f([M]) = [p_1 : p_2]$  where the characteristic polynomial associated to  $M$  is  $p(M, \lambda) = \lambda^3 - p_1\lambda - p_2$  and  $\mathbb{P}(\mathfrak{sl}(3, \mathbb{R}))^{ss}$  means the semistable points in  $\mathbb{P}(\mathfrak{sl}(3, \mathbb{R}))$ , i.e. the non-nilpotent matrices. Since the matrix  $2E_1 + E_2$  has eigenvalues  $2, -1, -1$  we can see that  $f([2E_1 + E_2]) = [3 : 2]$  and, being this a regular point, we have that  $f^{-1}([3 : 2])$  is a subvariety of dimension 6. Hence for dimension reasons  $\dim(\mathbb{P}(W) \cap f^{-1}([3 : 2])) \geq 4$ . But inside  $f^{-1}([3 : 2])$  there are also non-diagonalisable matrices with characteristic polynomial  $p(M, \lambda) = \lambda^3 - 3\lambda - 2$ . Take then  $[M] \in \mathbb{P}(W) \cap f^{-1}([3 : 2])$  and consider its eigenspace for the eigenvalue  $-1$ . In order for  $M$  to be diagonalisable we need  $\text{rank}(M + \text{Id}) = 1$  and we know that  $\text{rank}(M + \text{Id}) \leq 2$ . Hence we just need to impose 2 equations in order to ensure the rank to be 1. These equations correspond to impose two minors equal 0. We are then left with a subspace of dimension at least 2 of diagonalisable matrices in  $W$  with eigenvalues  $2, -1, -1$ .  $\square$

Using the complex isomorphism from  $\mathfrak{g}(0, b, c, 0)$  to  $\mathfrak{g}'(0, b, -c, 0)$  we can transport the pseudo-Kähler structure we have on  $\mathfrak{g}(0, b, c, 0)$  to  $\mathfrak{g}'(0, b, -c, 0)$  and we get what is called a para-Kähler structure.

**Definition 3.2.28.** A *para-Kähler manifold* is a pseudo-Riemannian manifold  $(M, \langle \cdot, \cdot \rangle)$  together with a skew-symmetric para-complex structure  $L$  which is parallel with respect to the Levi-Civita connection. Where the para-complex structure is a  $(1, 1)$ -tensor field such that  $L^2 = \text{Id}$ . If  $\mathfrak{g}$  is the Lie algebra of a para-Kähler Lie group  $G$  we will say that  $\mathfrak{g}$  has a para-Kähler structure.

**Proposition 3.2.29.** *On  $\mathfrak{g}'(0, b, c, 0)$  we have a flat para-Kähler structure.*

*Proof.* If we pullback on  $\mathfrak{g}'(0, b, -c, 0)$  the metric  $G$ , defined on  $\mathfrak{g}(0, b, c, 0)$  as in Remark 3.2.23, via the isomorphism defined by  $g = \text{diag}(1, 1, i)$  we get the following non-degenerate symmetric matrix

$$G' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta^2+b+c}{2\beta} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{-\beta^2+b+c}{2\beta} & 0 & 0 & 1 \\ 0 & \frac{\beta^2+b+c}{2\beta} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\beta^2+b+c}{2\beta} & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta^2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

of signature  $(4, 4)$  that represents a flat pseudo-Riemannian metric on the corresponding Lie group. Furthermore, transporting the complex structure  $J$  defined on  $\mathfrak{g}(0, b, c, 0)$  we get  $g^{-1}Jg = iL$  with  $L$  and endomorphism of  $\mathfrak{g}'(0, b, -c, 0)$  such that  $L^2 = \text{Id}$  and skew-symmetric with respect to  $G'$ . Hence the pair  $(G', L)$  defines a para-Kähler structure on  $\mathfrak{g}'(0, b, -c, 0)$ .  $\square$

*Remark 3.2.30.* In the same way as in Proposition 3.2.24 we can deform the metric defined by  $G'$  to get a flat pseudo-Riemannian metric of signature  $(4, 4)$  on some Lie algebras  $\mathfrak{g}(Q)$ .

**Proposition 3.2.31.** *On  $\mathfrak{g}(a, b, c, d)$  and  $\mathfrak{g}'(a, b, c, d)$  we can define a flat  $(4, 4)$  pseudo-Riemannian metric. More generally*

$$K' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{24} & s_{25} & 0 & -1 & 0 \\ 0 & 0 & 0 & s_{34} & s_{35} & 0 & 0 & 1 \\ 0 & s_{24} & s_{34} & 1 & 0 & 0 & 0 & 0 \\ 0 & s_{25} & s_{35} & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -s^2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

that is non-degenerate if  $s \neq 0$ , defines a flat pseudo-Riemannian metric of signature  $(4, 4)$  on  $\mathfrak{g}(Q)$  if and only if

$$\begin{cases} \alpha_{16} = \beta_{16} = 0, \\ \alpha_{14} = 1, \beta_{14} = 0, \\ \alpha_{15} = 0, \beta_{15} = 1 \end{cases}$$

and

$$3s^4 + (-4s_{24}s_{35} + s_{25}^2 + 2s_{34}s_{25} + s_{34}^2 + \alpha_{25}^2 + (-2\beta_{24} - 4\beta_{35})\alpha_{25} + \beta_{24}^2 - 2\alpha_{36} - \beta_{25}^2 + (4\alpha_{24} + 2\alpha_{35})\beta_{25} - \alpha_{35}^2 - 2\beta_{26})s^2 - \alpha_{36}^2 + 2\beta_{26}\alpha_{36} - \beta_{26}^2 - 4\beta_{36}\alpha_{26} = 0.$$

*Remark 3.2.32.* As in Remark 3.2.25 if a Lie algebra  $\mathfrak{g}(Q)$  belongs to either  $\text{Gr}(6, F_3/\tilde{Q})$  or  $\text{Gr}(6, F_3/\tilde{Q}')$  then the condition  $\{\alpha_{14} = 1, \beta_{14} = 0, \alpha_{15} = 0, \beta_{15} = 1\}$  is equivalent to the fact that  $\mathfrak{g}(Q)$  belongs to either  $\mathfrak{g}(a, b, c, d)$  or  $\mathfrak{g}'(a, b, c, d)$  hence this family of flat metric of signature  $(4, 4)$  cannot be extended to the boundary of neither  $\mathfrak{g}(a, b, c, d)$  nor  $\mathfrak{g}'(a, b, c, d)$ .

## Appendix A

# List of nilpotent Hermite-Lorentz Lie algebras in dimension 4

For  $\mathbb{K} = \mathbb{R}$  the following is a non-redundant list, up to isomorphism, of the 8-dimensional nilpotent Lie algebras that appear as Lie algebras of unipotent simply transitive subgroups of  $U(3, 1) \ltimes \mathbb{C}^{3+1}$ . They are found putting together Proposition 2.3.4, 2.3.5, 2.4.21 and 2.4.36. Furthermore taking  $\mathbb{K} = \mathbb{Q}$  this is also the complete non-redundant list of the  $\mathbb{Q}$ -isomorphism classes of  $\mathbb{Q}$ -forms in the aforementioned Lie algebras and hence of the abstract commensurability classes of nilpotent crystallographic subgroups of  $U(3, 1) \ltimes \mathbb{C}^{3+1}$ . They are found putting together Proposition 2.5.2, 2.5.4 and 2.5.11. We present these Lie algebras defined over the field  $\mathbb{K}$  in a compact version that is valid for both  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{Q}$ . The presentation is given in the basis  $(x_1, \dots, x_8)$  and we will write only the non-zero Lie brackets. For the Lie algebras that decompose as a direct sum of an abelian ideal and a smaller dimensional Lie algebra we have written in brackets the corresponding names in the lists of de Graaf [20] for dimension up to 6 and of Gong [30] for dimension 7. For the notations used here see Section 2.1 and in particular Proposition 2.1.5.

For the case  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$ , see Proposition 2.3.4, we have

- $L_1$ : abelian,
- $L_2$ :  $[x_1, x_2] = x_3$  ( $L_{3,2} \oplus \mathbb{R}^5$ ),
- $L_3$ :  $[x_1, x_2] = x_4, [x_1, x_3] = x_5$  ( $L_{5,8} \oplus \mathbb{R}^3$ ),
- $L_4$ :  $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_1, x_5] = x_7$  ( $247A \oplus \mathbb{R}$ ),
- $L_5$ :  $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_1, x_5] = x_7, [x_2, x_3] = x_6$  ( $247L \oplus \mathbb{R}$ ),
- $L_6^{\mathbb{K}}(\varepsilon)$ :  $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_6] = \varepsilon x_8, [x_3, x_6] = -x_7$  with  $\varepsilon \in \mathbb{K}_{>0}$  and  $L_6(\varepsilon) \cong L_6(\varepsilon')$  if and only if there exists  $\alpha \in \mathbb{K}^*$  such that  $\varepsilon' = \alpha^2 \varepsilon$ .

For the case  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$  and  $\gamma_2 = 0$ , see Proposition 2.3.5, we have

- $N_1: [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_6 \quad (L_{6,19}(0) \oplus \mathbb{R}^2),$
- $N_2: [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6 \quad (L_{6,25} \oplus \mathbb{R}^2),$
- $N_3^{\mathbb{K}}(\varepsilon): [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = \varepsilon x_6, [x_2, x_3] = x_6, [x_2, x_4] = x_5$  with  $\varepsilon \in \mathbb{K}$  and  $N_3(\varepsilon) \cong N_3(\varepsilon')$  if and only if there exists  $\alpha \in \mathbb{K}^*$  such that  $\varepsilon' = \alpha^2 \varepsilon \quad (L_{6,24}(\varepsilon) \oplus \mathbb{R}^2),$
- $N_4: [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_2, x_4] = x_5 \quad (L_{6,23} \oplus \mathbb{R}^2),$
- $N_5: [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_5] = x_7, [x_2, x_4] = x_6 \quad (257A \oplus \mathbb{R}),$
- $N_6: [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_5] = x_7 \quad (257B \oplus \mathbb{R}),$
- $N_7: [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_2, x_4] = x_6, [x_2, x_5] = x_7 \quad (257C \oplus \mathbb{R}),$
- $N_8: [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_4] = x_6, [x_2, x_5] = x_7 \quad (257D \oplus \mathbb{R}),$
- $N_9^{\mathbb{K}}(\varepsilon): [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_6, [x_1, x_5] = x_7, [x_2, x_3] = x_7, [x_2, x_5] = \varepsilon x_6$  with  $\varepsilon \in \mathbb{K}$  and  $N_9(\varepsilon) \cong N_9(\varepsilon')$  if and only if there exists  $\alpha \in \mathbb{K}^*$  such that  $\varepsilon' = \alpha^6 \varepsilon$ . Over  $\mathbb{R}$  to classify equivalence classes the parameter  $\varepsilon$  can then take three values  $\varepsilon \in \{0, 1, -1\}$  that correspond to  $(257I \oplus \mathbb{R}), (257J_1 \oplus \mathbb{R})$  and  $(257J \oplus \mathbb{R})$  respectively,
- $N_{10}: [x_1, x_2] = x_3, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_5] = x_7, [x_2, x_6] = x_8$
- $N_{11}: [x_1, x_2] = x_3, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_3] = x_8, [x_2, x_5] = x_7, [x_2, x_6] = x_8$
- $N_{12}: [x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_5] = x_8$
- $N_{13}: [x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_4] = x_7, [x_2, x_5] = x_8.$

For the case  $\pi(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0$  and  $\gamma_2 \neq 0$ , see Proposition 2.4.21, 2.4.36, 2.5.2 and 2.5.4, we have

- $\mathfrak{g}_{\mathbb{K}}(0, 0, 0, 0): [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8$
- $\mathfrak{g}_{\mathbb{K}}(0, 0, \varepsilon, 1): [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_6] = \varepsilon x_8, [x_3, x_6] = x_7$  with  $\varepsilon \in \mathbb{K}$  such that  $\mathfrak{g}_{\mathbb{K}}(0, 0, \varepsilon', 1) \cong \mathfrak{g}_{\mathbb{K}}(0, 0, \varepsilon, 1)$  if and only if there exists  $\alpha \in \mathbb{K}^*$  such that  $\varepsilon' = \alpha^2 \varepsilon$
- $\mathfrak{g}_{\mathbb{R}}(a, b, c): [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_4] = x_7, [x_2, x_5] = -x_8, [x_3, x_4] = -x_8, [x_3, x_5] = 3x_7, [x_2, x_6] = ax_7 + bx_8, [x_3, x_6] = cx_7 - ax_8$  with  $a, b, c \in \mathbb{R}, (a, b) \neq (0, -2)$  and  $\mathfrak{g}_{\mathbb{R}}(a, b, c) \cong \mathfrak{g}_{\mathbb{R}}(-a, b, c)$
- $\mathfrak{g}_{\mathbb{Q}}(e, f, g, h, j, k, l)$  with  $e, f, g, h, j, k, l \in \mathbb{Q}$ . This is a general family of Lie algebras of the form:  
 $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_4] = fx_7 + hx_8, [x_2, x_5] = -gx_7 - fx_8, [x_3, x_4] = -gx_7 - fx_8, [x_3, x_5] = ex_7 + gx_8, [x_2, x_6] = jx_7 + kx_8, [x_3, x_6] = lx_7 - jx_8.$

Each isomorphism class of this family is represented by one of the Lie algebras of the following lists. For all of them we will ask

$$3g^2e^2 - 6efgh + 4g^3f - 4e^3h - h^2f^2 < 0 \text{ and } \begin{pmatrix} hl + 2gj - fk \\ -eh + fg + j \\ 2g^2 - 2fh + k \end{pmatrix} \wedge \begin{pmatrix} -gl + ek - 2fj \\ 2eg - 2f^2 + l \\ eh - fg - j \end{pmatrix} \neq 0.$$

- $\mathfrak{h}_{\mathbb{Q}}^1([e], [h], j, k, l)$  : with  $j, k, l \in \mathbb{Q}$  and  $(e, h) \in \mathbb{Q}^2$  representatives of the equivalence class defined by  $(e, h) \sim (e', h')$  if and only if there exists  $\sigma_1, \sigma_2 \in \mathbb{Q}^*$  such that  $h' = \sigma_1^3 h$  and  $e' = \sigma_1 \sigma_2^2 e$ . This family is defined by  $f = g = 0$ .
- $\mathfrak{h}_{\mathbb{Q}}^2([h], m, j, k, l, [t])$  : with  $j, k, l \in \mathbb{Q}$ ,  $h \in \mathbb{Q}^*$  a representative of the equivalence class defined by  $h \sim h'$  if and only if there exists  $\mu \in \mathbb{Q}^*$  such that  $h' = \mu^3 h$ ,  $m \in \mathbb{Q} \cup \{\infty\}$  and  $t \in \mathbb{Q} \setminus \mathbb{Q}^3$  a representative of the equivalence class defined by  $t \sim t'$  if and only if there exists  $\mu \in \mathbb{Q}$  such that  $t' = \mu^3 t^j$  with  $j = 1, 2$ .
  - \* if  $m \in \mathbb{Q}$ : this family is defined by  $g = 0, e = -hmt$  and  $f = -ht(m^3 t + 1)$ ,
  - \* if  $m = \infty$ : this family is defined by  $g = 0, e = 0$  and  $f = -ht^2$ .
- $\mathfrak{h}_{\mathbb{Q}}^3([h], m, j, k, l, [t])$  : with  $j, k, l \in \mathbb{Q}$ ,  $h \in \mathbb{Q}^*$  a representative of the equivalence class defined by  $h \sim h'$  if and only if there exists  $\mu \in \mathbb{Q}^*$  such that  $h' = \mu^3 h$ ,  $m \in \mathbb{Q} \cup \{\infty\}$  and  $t \in \mathbb{Q}$  such that  $t^2 - 4 > 0$ ,  $t^2 - 4 \notin \mathbb{Q}^2$  and  $t$  not in the image of the function  $f(x) = x^3 - 3x$  over the rational. Finally we choose  $t$  as a representative of the equivalence class defined by  $t \sim t'$  if and only if there exists  $\alpha, \beta \in \mathbb{Q}$  with  $\alpha^2 + t\alpha\beta + \beta^2 = 1$  such that  $t' = -3t\alpha^2\beta + t\beta^3 + 6\alpha + \alpha^3 t^2 - 8\alpha^3$ .
  - \*  $m \in \mathbb{Q}$ : this family is defined by  $g = 2hm, e = h(3m^2 - tm - 1)$  and  $f = ht(3m^2 - m^3 t - 1)$ ,
  - \*  $m = \infty$ : this family is defined by  $g = 2h, e = 3h$  and  $f = -ht^2$ .

Let us remark that the presentation of the Lie algebras depend on the choice of representative of the class  $[t]$ .

- $\mathfrak{g}_{\mathbb{K}}(0, -2, c)$  :  $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_4] = x_7, [x_2, x_5] = -x_8, [x_3, x_4] = -x_8, [x_3, x_5] = 3x_7, [x_2, x_6] = -2x_8, [x_3, x_6] = cx_7$  with  $c \in \mathbb{K}$ .



## Appendix B

# List of Carnot 3-step 8-dimensional nilpotent Lie algebras

Here we give a list of representatives for the isomorphism classes of the Lie algebras  $\mathfrak{g}(Q)$ , see Definition 2.4.2. Remember from Theorem 3.2.14 that isomorphism classes of the Lie algebras  $\mathfrak{g}(Q)$  are in bijection with orbits of elements in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  under the  $\text{SL}(3, \mathbb{R})$ -adjoint action. Hence, in this list, we will write a representative of the isomorphism class as the corresponding element in  $\text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ . Fixing  $(E_i)_{i=1}^8$  as basis for  $\mathfrak{sl}(3, \mathbb{R})$  we can represent a generic element  $P = \text{span}_{\mathbb{R}}\{u, v\} \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  as a  $2 \times 8$  matrix whose lines are the vectors spanning it. Remember that we have the identification  $\mathbb{P}(\wedge^2 \mathfrak{sl}(3, \mathbb{R})) = \mathbb{P}(\mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^3(\mathbb{R}^{3*}))$  and that we identify  $S^3(\mathbb{R}^3)$  and  $S^3(\mathbb{R}^{3*})$  with ternary cubics in the variables  $x, y, z$ . Hence an element  $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$  is identified with  $[(\pi_1(P), \pi_2(P), \pi_3(P))]$ , with  $\pi_2(P) = [a_{34}x^3 + p_1(y, z)x^2 + p_2(y, z)x + p_3(y, z)]$  and  $\pi_3(P) = [a_{57}x^3 + q_1(y, z)x^2 + q_2(y, z)x + q_3(y, z)]$ . Remember finally that for a binary cubic  $f = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3$  we have the invariant  $\Delta(f) = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_1^3a_3 - 4a_2^3a_0 - a_0^2a_3^2$  and the covariant  $H(f) = 36((a_0a_2 - a_1^2)x_1^2 + (a_1a_3 - a_2^2)x_1x_2 + (a_1a_3 - a_2^2)x_2^2)$  that completely classify them. And that for a binary quadric  $g = a_0x_1^2 + 2a_1x_1x_2 + a_3x_2^2$  the classical invariant  $\tilde{\Delta}(g) = a_0a_2 - a_1^2$  suffices.

For each representative one should also impose the condition, that we have not explicitly written here, that the first vector of the basis with respect to which we write  $\pi_1(P)$  have been chosen as indicated in Theorem 3.2.14. Finally in some cases the group of diagonal matrices in  $\text{SL}(3, \mathbb{R})$  still have an action on the representatives that we have kept more general in order to avoid too many sub-cases. In general this action is given as follows

$$\text{diag}(\lambda, \mu, \lambda^{-1}\mu^{-1}) \cdot \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \lambda\mu^{-1}u_3 & \lambda^2\mu u_4 & \lambda^{-1}\mu u_5 & \lambda\mu^2u_6 & \lambda^{-2}\mu^{-1}u_7 & \lambda^{-1}\mu^{-2}u_8 \\ v_1 & v_2 & \lambda\mu^{-1}v_3 & \lambda^2\mu v_4 & \lambda^{-1}\mu v_5 & \lambda\mu^2v_6 & \lambda^{-2}\mu^{-1}v_7 & \lambda^{-1}\mu^{-2}v_8 \end{pmatrix}$$

Invariants	Representative	Action	$\pi_1(P)$
$\Delta(p_3(y,z)) < 0$	$\begin{pmatrix} 0 & 1 & b & d & -1 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & -3 & 1 & 1 \end{pmatrix}$ $(a,b,c,d) \neq (0,-2,-6,0)$	$\text{diag}(-1,-1,1)$	$\begin{pmatrix} a+d & d-a & c-3b \\ 0 & -a & -c-6 \\ 0 & -b-2 & -d \end{pmatrix}$
$\Delta(p_3(y,z)) > 0$	$\begin{pmatrix} 0 & 1 & b & d & -1 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & 3 & 1 & 1 \end{pmatrix}$ $(a,b,c,d) \neq (0,-2,6,0)$	$\text{diag}(-1,-1,1)$	$\begin{pmatrix} a+d & d-a & c+3b \\ 0 & -a & -c+6 \\ 0 & -b-2 & -d \end{pmatrix}$

$\Delta(p_3(y,z)) = 0$		
Invariants	Representative	$\pi_1(P)$
$H(p_3(y,z)) \neq 0$	$\begin{pmatrix} -1 & 1 & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & e & 1 & 0 \end{pmatrix}$ $(a,b,c,e) \neq (0,0,0,0)$	$\begin{pmatrix} b & -3c & ae \\ 0 & 0 & 3e \\ 0 & -a & -b \end{pmatrix}$
	$\begin{pmatrix} -1 & 1 & a & b & 1 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 & 1 & 0 \end{pmatrix}$ $(a,b,c,d) \neq (0,0,0,0)$	$\begin{pmatrix} b-c & -3c & 0 \\ 0 & c & d \\ 0 & -a & -b \end{pmatrix}$
$H(p_3(y,z)) = 0,$ $p_3(y,z) \neq 0$	$\begin{pmatrix} 0 & 0 & a & b & 1 & 0 & 0 & 1 \\ 0 & 0 & c & d & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -c & -d & a \\ 0 & c-1 & d \\ 0 & 0 & 1 \end{pmatrix}$
	$\begin{pmatrix} a & 0 & 0 & 0 & b-2a^2 & 1 & c & b \\ 0 & 0 & 0 & 0 & 1 & 0 & 2a & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & -1 \end{pmatrix}$
	$\begin{pmatrix} 0 & 0 & a & b & 1 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 1 & 1 & 0 \end{pmatrix}$ $(a,b,c,d) \neq (0,0,0,0)$	$\begin{pmatrix} b-c & 0 & a \\ 0 & c & d \\ 0 & -a & -b \end{pmatrix}$
$p_3(y,z) = 0, a_{57} \neq 0$	$\begin{pmatrix} 0 & 0 & a & b & 1 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 & 1 & 0 \end{pmatrix}$ $(a,b,c,d) \neq (0,0,0,0)$	$\begin{pmatrix} b-c & 0 & 0 \\ 0 & c & d \\ 0 & -a & -b \end{pmatrix}$

$p_3(y, z) = 0, a_{57} = 0$		
Invariants	Representative	$\pi_1(P)$
$\tilde{\Delta}(q_2(y, z)) > 0$	$\begin{pmatrix} a & 1 & b & 0 & 0 & 0 & 0 & -3c \\ c & 0 & d & 0 & 0 & 1 & 0 & \frac{-6c^2-a-1}{2a-1} \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & a & 0 & 0 & 0 & 0 & 3b \\ 0 & b & c & 0 & 0 & 1 & 0 & \frac{3b^2+1}{-2} \end{pmatrix}$	$\begin{pmatrix} 0 & (2a-1)d-2bc & b \\ 0 & 3c & 2-a \\ 0 & \frac{9c^2+a+2-a^2}{2a-1} & -3c \end{pmatrix}$ $\begin{pmatrix} 0 & ab+2c & a \\ 0 & -3c & -1 \\ 0 & \frac{9b^2-1}{2} & 3b \end{pmatrix}$
$\tilde{\Delta}(q_2(y, z)) < 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -a & 0 & -c & -c & 0 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} a & 2a & b & 0 & 0 & 1 & 0 & 0 \\ c & -c & 0 & d & 0 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & b & c & 0 & a & 0 & 0 \end{pmatrix}$ $(a, b, c) \neq (0, 0, 0)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -a \\ 0 & 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 0 & -3bc & 3ad \\ 0 & 1 & 3c \\ 0 & -3a & -1 \end{pmatrix}$ $\begin{pmatrix} 0 & 2b & c-ab \\ 0 & 0 & -3a \\ 0 & 0 & 0 \end{pmatrix}$
$\tilde{\Delta}(q_2(y, z)) = 0,$ $q_2(y, z) \neq 0$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a & -a & b & 0 & 1 & d & 0 & c \end{pmatrix}$ $c \neq 0$ $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2a & a & b & -1 & 0 & 0 & c & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & a & 0 & b & 0 & -3c & 0 & 0 \\ 0 & c & 0 & d & 0 & \frac{-3c^2}{a+1} & 0 & 1 \end{pmatrix}$ $a \neq 1, (a, b, c, d) \neq (\frac{1}{2}, 0, 0, 0)$ $\begin{pmatrix} 1 & 1 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & c & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 3a & 0 & 0 \\ a & 0 & b & c & 0 & 3a^2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & c & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & b & ad-bc+d \\ 0 & -3c & \frac{9c^2}{a+1} \\ 0 & -2a+1 & 3c \end{pmatrix}$ $\begin{pmatrix} 0 & a & 2b \\ 0 & 0 & c \\ 0 & -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -b & -3ab+c \\ 0 & 3a & 9a^2 \\ 0 & -2 & -3a \end{pmatrix}$

$p_3(y, z) = 0, a_{57} = 0, q_2(y, z) = 0$		
Invariants	Representative	$\pi_1(P)$
$\Delta(q_3(y, z)) < 0$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2a & a & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -3a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 1 & -1 & 0 & -\frac{1}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$
$\Delta(q_3(y, z)) > 0$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2a & a & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -3a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2a-1 & a & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 3-3a & 1-3a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 2a+1 & a & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -3-3a & 1+3a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 1 & -1 & 0 & -\frac{1}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 3 & 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & -3 & 3 \\ 0 & 0 & 9 \\ 0 & -9 & 0 \end{pmatrix}$
	$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & -3 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 3 & 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 3 \\ 0 & -9 & 9 \\ 0 & -9 & 9 \end{pmatrix}$

$p_3(y,z) = 0, a_{57} = 0, q_2(y,z) = 0, \Delta(q_3(y,z)) = 0$		
Invariants	Representative	$\pi_1(P)$
$H(q_3(y,z)) \neq 0$	$\begin{pmatrix} -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1-a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $a \neq -1$ $\begin{pmatrix} 1 & -1 & 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & a & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ a & -a & 0 & 0 & 0 & \frac{3a^2}{4} & 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 2a & a & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & a+1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & -3a \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$H(q_3(y,z)) = 0,$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$q_3(y,z) \neq 0$	$\begin{pmatrix} -2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$q_3(y,z) = 0, p_2(y,z) \neq 0$	$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$q_3(y,z) = 0, p_2(y,z) = 0,$ $p_1(y,z) = 0, a_{34} = 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\pi_1(P) = 0$		
Invariants	Representative	$\pi_1(P)$
$u$ and $v$ diagonalisable over $\mathbb{R}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	0
$u$ diagonalisable over $\mathbb{R}$ and $v$ diagonalisable over $\mathbb{C}$	$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$	0
$u$ diagonalisable over $\mathbb{R}$ and $v$ not diagonalisable	$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	0
$u$ nilpotent of rank 2 and $v$ nilpotent of rank 1	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	0
$u$ and $v$ nilpotent of rank 1	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	0

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