Diffusive processes on the Wasserstein space: Coalescing models, Regularization properties and McKean-Vlasov equations

Victor Marx

UNIVERSITÉ CÔTE D'AZUR

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Processus de diffusion sur l'espace de Wasserstein : modèles coalescents, propriétés de régularisation et équations de McKean-Vlasov

Présentée et soutenue par :

Victor Marx

Laboratoire de Mathématiques J.A. Dieudonné (LJAD)

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Thèse dirigée par

François Delarue

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Devant le jury composé de :

Mireille Bossy	Directrice de recherche, INRIA Sophia-Antipolis	examinatrice
Djalil Chafaï	Professeur des Universités, Université Paris-Dauphine	examinateur
François Delarue	Professeur des Universités, Université Côte d'Azur	directeur de thèse
Arnaud Guillin	Professeur des Universités, Université Clermont-Auvergne	rapporteur
Benjamin Jourdain	Ingénieur en Chef des Ponts, HdR, Ecole des Ponts ParisTech	examinateur
Max von Renesse	Professeur des Universités, Universität Leipzig, Allemagne	rapporteur

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Examinatrice/teurs

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Directeur de thèse

- François Delarue, Professeur des Universités, Université Côte d'Azur.

Titre: Processus de diffusion sur l'espace de Wasserstein : modèles coalescents, propriétés de régularisation et équations de McKean-Vlasov.

Résumé: La thèse vise à étudier une classe de processus stochastiques à valeurs dans l'espace des mesures de probabilité sur la droite réelle, appelé espace de Wasserstein lorsqu'il est muni de la métrique de Wasserstein W_2. Ce travail aborde principalement les questions suivantes : comment construire effectivement des processus stochastiques vérifiant des propriétes diffusives à valeurs dans un espace de dimension infinie ? existe-t-il une forme d'unicité, forte ou faible, satisfaite par certains processus ainsi construits ? peut-on établir des propriétés régularisantes de ces diffusions, en particulier le forçage stochastique d'équations de McKean-Vlasov ou des formules d'intégration par parties de Bismut-Elworthy ?

Le chapitre I propose une construction alternative, par approximations lisses, du système de particules défini par Konarovskyi et von Renesse, et appelé ci-après modèle coalescent. Le modèle coalescent est un processus aléatoire à valeurs dans l'espace de Wasserstein, satisfaisant une formule de type Itô sur cet espace et dont les déviations en temps petit sont régies par la métrique de Wasserstein, par analogie avec les déviations en temps court du mouvement brownien standard gouvernées par la métrique euclidienne. L'approximation régulière construite dans cette thèse partage ces propriétés diffusives et est obtenue par lissage des coefficients de l'équation différentielle stochastique satisfaite par le modèle coalescent. Cette variante présente l'avantage principal de satisfaire des résultats d'unicité demeurant ouverts pour le modèle coalescent. De plus, à de petites modifications de sa structure près, cette diffusion lissée possède des propriétés régularisantes : c'est précisément l'objet de l'étude des chapitres II à IV. Dans le chapitre II, on perturbe une équation de McKean-Vlasov mal posée par une de ces versions lissées du modèle coalescent, afin d'en restaurer l'unicité. Le lien est fait avec les résultats récents (Jourdain, Mishura-Veretennikov, Chaudru de Raynal-Frikha, Lacker, Röckner-Zhang) où l'unicité d'une solution est démontrée lorsque le bruit est de dimension finie et le coefficient de dérive est lipschitzien en distance de variation totale en la variable de mesure. Dans notre cas, la diffusion sur l'espace de Wasserstein permet de régulariser le champ de vitesse en l'argument de mesure et ainsi de traiter des fonctions de dérive de faible régularité à la fois en la variable d'espace et de mesure. Enfin, les chapitres III et IV étudient, pour une diffusion définie sur l'espace de Wasserstein du cercle, les propriétés de régularisation du semi-groupe associé. Utilisant dans le chapitre III le calcul différentiel sur l'espace de Wasserstein introduit par Lions, on établit une inégalité de Bismut-Elworthy, contrôlant le gradient du semi-groupe aux points de l'espace des mesures de probabilité qui ont une densité assez régulière. Dans le chapitre IV, la vitesse d'explosion lorsqu'on fait tendre la variable temporelle vers zéro est améliorée sous certaines conditions de régularité supplémentaires. On déduit de ces résultats des estimations *a priori* pour une EDP posée sur l'espace de Wasserstein et dirigée par la diffusion sur le tore mentionnée ci-dessus, dans le cas homogène (chapitre III) et avec un terme source non trivial (chapitre IV).

Mots-clés: diffusion de Wasserstein, système de particules en interaction, particules coalescentes, propriétés de régularisation, équation de McKean-Vlasov, équation de Fokker-Planck, restauration de l'unicité, notion de solution faible, formule de Bismut-Elworthy.

Title: Diffusive processes on the Wasserstein space: Coalescing models, Regularization properties and McKean-Vlasov equations.

Abstract: The aim of this thesis is to study a class of diffusive stochastic processes with values in the space of probability measures on the real line, called Wasserstein space if it is endowed with the Wasserstein metric W_2. The following issues are mainly addressed in this work: how can we effectively construct a stochastic process satisfying diffusive properties with values in a space of infinite dimension? is there a form of uniqueness, in a strong or a weak sense, satisfied by some of those processes? do those diffusions own smoothing properties, e.g. regularization by noise of McKean-Vlasov equations or e.g. Bismut-Elworthy integration by parts formulæ?

Chapter I introduces an alternative construction, by smooth approximations, of the particle system defined by Konarovskyi and von Renesse, hereinafter designed by coalescing model. The coalescing model is a random process with values in the Wasserstein space, following an Itô-like formula on that space and whose short-time deviations are governed by the Wasserstein metric, by analogy with the short-time deviations of the standard Brownian motion governed by the Euclidean metric. The regular approximation constructed in this thesis shares those diffusive properties and is obtained by smoothing the coefficients of the stochastic differential equation satisfied by the coalescing model. The main benefit of this variant is that it satisfies uniqueness results which are still open for the coalescing model. Moreover, up to small modifications of its structure, that smooth diffusion owns regularizing properties: this is precisely the object of study of chapters II to IV. In chapter II, an ill-posed McKean-Vlasov equation is perturbed by one of those smooth versions of the coalescing model, in order to restore uniqueness. A connection is made with recent results (Jourdain, Mishura-Veretennikov, Chaudru de Raynal-Frikha, Lacker, Röckner-Zhang) where uniqueness of a solution is proved when the noise is finite dimensional and the drift coefficient is Lipschitz-continuous in total variation distance in its measure argument. In our case, the diffusion on the Wasserstein space allows to mollify the velocity field in its measure argument and so to handle with drift functions having low regularity in both space and measure variables. Lastly, chapters III and IV are dedicated to the study, for a diffusion defined on the Wasserstein space of the circle, of the smoothing properties of the associated semi-group. Applying in chapter III the differential calculus on the Wasserstein space introduced by Lions, a Bismut-Elworthy inequality is obtained, controlling the gradient of the semi-group at those points of the space of probability measures that have a sufficiently smooth density. In chapter IV, a better explosion rate when time tends to zero is established under additional regularity conditions. This leads to *a priori* estimates for a PDE defined on the Wasserstein space and governed by the diffusion on the torus mentioned above, in the homogeneous case (chapter III) and in the case of a non-trivial source term (chapter IV).

Keywords: Wasserstein diffusion, interacting particle system, coalescing particles, regularization properties, McKean-Vlasov equation, Fokker-Planck equation, restoration of uniqueness, notion of weak solution, Bismut-Elworthy formula.

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À ces gens qui ce soir-là
ne voulaient que s'émerveiller
devant un feu d'artifice,

Contents

Chapter 0

Introduction

Outline of the manuscript

The purpose of this manuscript is to address a class of stochastic processes with values in the space of probability measures on the real line and to investigate some of their main features, including possible routes to the existence and the effective construction of such processes, various forms of uniqueness and related smoothing diffusive properties. All these questions have been studied in details for finite-dimensional diffusion processes since the early days of the theory of stochastic analysis (which certainly goes back to the earlier works of Kolmogorov himself and maybe even before) and have branched into a wide collection of topics and of techniques, the exhaustive description of which would be out of scope of any reasonable introduction. Let us say, to draw a simple parallel with the finite dimensional setting and to illustrate in a quite comprehensive manner the spirit of the thesis, that our original desire was to investigate the analogue of a Brownian motion but on the set of probability measures and to understand, not only the pathwise behaviour of it, but also its fine statistical properties among which the regularization properties it may have.

In finite dimension, the Brownian motion (or Wiener process) is a central object in stochastic analysis, as it directly appears as the limit of a standard simple random walk with independent and identically distributed increments. Meanwhile, it is directly connected, through Kolmogorov's formula, to the heat equation. As a matter of fact, the regularizing effect of the Brownian motion and more generally of various forms of diffusive processes may be tackled by many ways, which include among others: semi-group theory in functional analysis, regularity estimates and parametrix method in PDE theory, Itô and Malliavin calculus in probability theory. From the sole purely probabilistic point of view, many people have contributed to the field, but to make the picture clear, we may choose, among all the applications and refinements, one which we feel very striking (and which remains, even after the numerous papers that have appeared on the subject, absolutely fascinating): uniqueness of solutions to an ill-posed ordinary differential equation (ODE) in dimension d may be restored by forcing the dynamics with a Brownian motion of the same dimension as the ambiant space. This phenomenon is called restoration of uniqueness and the reader may find a very nice overview in the textbook of Flandoli [Fla11]. As we have just alluded to, this phenomenon is in fact deeply correlated to the properties of the heat equation itself or, equivalently, to the properties of the semi-group $(P_t)_{t\in[0,T]}$ generated by the finite-dimensional Brownian motion: in short, for a bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}^d$, $P_t f$ is a $\mathcal{C}^\infty$-function for every $t > 0$. In fact, restoration of uniqueness is continually switching back-and-forth between pathwise and statistical analysis. We will face similar features in the manuscript, including cases where uniqueness eventually holds in a statistical sense only: this is another facet of the finite-dimensional theory, which leads to the notion of weak solutions to stochastic differential equations (SDEs) and the analysis of which has been carried out in an

elegant manner by Stroock and Varadhan in their seminal monograph [SV79].

Although it comes as a quite natural object in the limit form of a simple random walk, the finite-dimensional Brownian motion is in fact intrinsically connected to the underlying Euclidean structure. Extensions to infinite dimensional cases (and in particular to Hilbert spaces) have been also intensively studied, see for instance the textbook of Da Prato [DP04], but our desire to handle a similar object on the space of probability measures faces a prior choice: which topology do we consider on the space of probability measures on $\mathbb{R}$? Several distances between probability measures may be indeed used in practice. To quote a few of them: Lévy-Prokhorov, Wasserstein, total variation distances... Actually, as the title of the manuscript suggests it, we will make an intense use below of the L_2-Wasserstein metric (even though, at some point, we will also need other distances). We will denote by $\mathcal{P}_2(\mathbb{R})$ the L_2-Wasserstein space on the real line, which is the space of probability measures with a finite second-order moment endowed with the L_2-Wasserstein distance. There are many reasons for explaining our main interest in the L_2-Wasserstein distance: inspired by the aforementioned restoration of uniqueness problem in finite dimension, we may wonder about the analogue of an ordinary differential equation but on the space of probability measures. It turns out that a corresponding object is, from the analytic point of view, a Fokker-Planck partial differential equation (PDE) and, from the probabilistic point of view, a non-linear McKean-Vlasov equation. As explained in the review we provide below on the topic, the Wasserstein distance is a natural distance when addressing these kinds of dynamics. Another reason to justify our need for the Wasserstein distance is that our desire to study the regularizing properties of a possible Brownian motion with values in the space of probability measures may lead us, exactly as in the finite dimensional setting, to handle PDEs on the set of probability measures. Again, it turns out that the differential calculus that has been used on the space of probability measures in optimal transportation theory, in the analysis of gradient flows or more recently in the theory of mean-field games is directly connected to the geometric properties of the space of probability measures when equipped with the L_2-Wasserstein metric. We use below the so-called Lions' approach to the Wasserstein differential calculus, which is based on a lifting procedure from $\mathcal{P}_2(\mathbb{R})$ onto an L_2-space defined on a rich enough probability space carrying random variables with any possible distribution on $\mathbb{R}$ (see Section 0.4 for a brief introduction to it). In this regard, it must be stressed that our manuscript is restricted to the study of diffusion processes taking values within the space of probability measures on the real line (so, $d = 1$) or (to avoid any boundary conditions) on the torus (see for instance Chapters III and IV).

Actually, several diffusions on the L_2-Wasserstein space have been constructed recently, see Section 0.1 for an overview. Our work is based on one of them: we indeed decided to work with the model due to Konarovskyi, see [Kon11, Kon17b, Kon17a, KvR18]. The latter describes a system of coalescing massive particles starting (say) from a uniform distribution on $[0, 1]$. It shares some of the features of other Wasserstein diffusion processes: the large deviations in small time are given by the Wasserstein distance and the martingale term that arises when expanding any smooth function φ of the measure argument along the coalescing process has exactly the square norm of the Wasserstein gradient of φ as local quadratic variation. In words, the objective of the thesis is to understand some qualitative and quantitative properties of this form of Wasserstein diffusion or of related $\mathcal{P}_2(\mathbb{R})$-valued processes of a similar type. As a starter, we provide an alternative construction of Konarovskyi's model in Chapter I. As explained therein, uniqueness in law remains an open question but the alternative construction we provide below shows that uniqueness holds for a variant of it, in which the particles may get really close one to each other but never meet.

Our variant of Konarovskyi's coalescing model will serve as a basis for constructing other $\mathcal{P}_2(\mathbb{R})$-valued processes sharing a similar structure but featuring additional interesting properties. Most of those additional properties are in fact related with a possible smoothing effect of

the Wasserstein diffusion. In this regard, it is worth mentioning that our motivation to pursue our research in this direction was twofold. Despite we are not able to prove any form of uniqueness for the original Konarovskyi model, our initial guess was that, to do so, a convenient weak approach *à la* Stroock-Varadhan would have been needed. Recall indeed that uniqueness in the aforementioned Stroock-Varadhan martingale problem relies in fact on non-trivial quantitive results on the smoothing effect of parabolic equations. Addressing the analogue for the Wasserstein diffusion was our first motivation to investigate the regularization properties of Konarovskyi's process or, more precisely, of the variant that is constructed in Chapter I. Another motivation was to use the Wasserstein diffusion as a random forcing acting on McKean-Vlasov dynamics or, equivalently, acting on Fokker-Planck dynamics. To make it clear, we are indeed interested below in restoring well-posedness of ill-posed dynamics defined on $\mathcal{P}_2(\mathbb{R})$ by means of a random forcing inspired from the Wasserstein diffusion introduced in Chapter I. This is part of the thesis to explain why we cannot do so with the original Konarovskyi's model nor with the variant we introduce in the first chapter. So, the last three chapters are dedicated to the analysis of new modifications of the original model: these modifications are used to restore uniqueness to ill-posed McKean-Vlasov equations or to obtain quantitative semi-group gradient estimates.

Adopting the particle point of view, the common feature of all the $\mathcal{P}_2(\mathbb{R})$-valued processes that we will study in this manuscript is that they are constructed in the form of an infinite dimensional particle system driven by a common infinite dimensional noise. In particular, the latter is in contrast to idiosyncratic noises acting independently on each particle like it is often the case in standard McKean-Vlasov equations. Put it differently, we are looking for $\mathcal{P}_2(\mathbb{R})$-valued random processes $(\mu_t)_{t\in[0,T]}$ through their quantile (or increasing rearrangement) functions $(x_t)_{t\in[0,T]}$: for each $u \in [0,1]$, $(x_t(u))_{t\in[0,T]}$ is then required to be a continuous real-valued process satisfying a one-dimensional SDE, interpreted as the trajectory of the random particle with label u on the real line. Hence, the process $(\mu_t)_{t\in[0,T]}$ represents the time-evolution of the distribution of a random system of infinitely many particles, each one being indexed by a number $u \in [0,1]$. This point is very much in the spirit of Lions' approach to the differential calculus on the Wasserstein space: $u \mapsto x_t(u)$ is here understood as a lift of μ_t, but, importantly enough (and this is in fact our guideline throughout the manuscript), this lift is *canonical* in the sense that, once again, $x_t(\cdot)$ is precisely *the* quantile function of μ_t.

In the following sections, we give an overview of prior results related to the aforementioned problems. In Section 0.1, we recall the definition of the Wasserstein distance on $\mathcal{P}_2(\mathbb{R})$ and we give a short description of the topics where this metric plays a significant role. In particular, we focus on some models of diffusions on the Wasserstein space, constructed and investigated by von Renesse, Sturm and Konarovskyi among others. In Section 0.2, we introduce some regularization properties of stochastic differential equations, starting by fundamental results in finite dimension. Some recently obtained extensions to infinite dimensional spaces are also presented in this section, in particular regarding McKean-Vlasov SDEs. In Section 0.3, we describe, chapter by chapter, the contents of this thesis and we state the main results obtained in this manuscript, with some comments on remaining open problems. Finally, we give in Section 0.4 a toolbox recalling a few basic definitions and fundamental properties that we will later use on two notions of differential calculus on $\mathcal{P}_2(\mathbb{R})$.

0.1 Diffusion processes on the Wasserstein space

Let us start by introducing the L_2-Wasserstein space on $\mathbb{R}$. It is the set of the probability measures μ on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x|^2 \mathrm{d}\mu(x)$ is finite. This space is usually denoted by $\mathcal{P}_2(\mathbb{R})$. Let us define on $\mathcal{P}_2(\mathbb{R})$ the following distance; for every probability measures $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$,

$$W_2(\mu,\nu) := \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2} |x-y|^2 \mathrm{d}\pi(x,y) \right)^{1/2}, \tag{1}$$

where $\Pi(\mu,\nu)$ is the set of the probability measures π on $\mathbb{R}^2$ with marginals μ and ν, that is such that for every Borel set A of $\mathbb{R}$, $\pi(A \times \mathbb{R}) = \mu(A)$ and $\pi(\mathbb{R} \times A) = \nu(A)$. W_2 is called L_2-Wasserstein distance, sometimes also called L_2-Kantorovich distance.

Wasserstein space and optimal transportation

The Wasserstein distance W_2 has an interpretation in terms of optimal transportation. The Monge-Kantorovich problem addresses the optimal transportation of a measure μ into a measure ν, where the optimization is relative to a cost function $c(x,y)$ which represents the cost of transporting a unit mass from x to y. If the cost function is quadratic, *i.e.* if $c(x,y) = |x-y|^2$, then Monge-Kantorovich problem is equivalent to determine a measure π minimizing the right-hand-side of (1). Benamou-Brenier's formula (stated e.g. in [BB00]) states that the optimal transportation map is the gradient of a convex function and

$$W_2(\mu_0,\mu_1)^2 = \inf \int_0^1 \int_{\mathbb{R}} |\nabla_x \xi_t|^2 \mathrm{d}\mu_t \mathrm{d}t,$$

where the infimum is taken over every $\xi : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that there is a continuous function $\mu : [0,1] \to \mathcal{P}_2(\mathbb{R})$ satisfying $\mu(0) = \mu_0$, $\mu(1) = \mu_1$ and

$$\partial_t \mu_t + \mathrm{div}(\mu_t \nabla_x \xi_t) = 0,$$

the latter being interpreted as a first-order Fokker-Planck equation and pointing out, if needed, the various uses of the Wasserstein distance in the analysis of Fokker-Planck equations and *vice versa*. Otto (in [Ott01] among others) has given a Riemannian interpretation of Benamou-Brenier's formula. Furthermore, Jordan, Kinderlehrer and Otto [JKO98] have proved that the diffusion equation $\partial_t \mu = \Delta \mu$ is the gradient flow equation for the relative entropy H in $(\mathcal{P}_2(\mathbb{R}), W_2)$, where $H(\mu) := \int_{\mathbb{R}} \rho \ln \rho$ if $\mathrm{d}\mu(x) = \rho(x)\mathrm{d}x$ and $H(\mu) := +\infty$ if μ is not absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. The result of [JKO98] states more generally that the gradient flow for the free energy functional associated to some potential ψ writes in the form of a certain Fokker-Planck equation, where the gradient is understood relatively to the L_2-Wasserstein distance. The interested reader may find further developments of the metric theory of gradient flows in the book of Ambrosio, Gigli, Savare [AGS05]. See also Villani's book [Vil09] for an extensive overview of the problems in optimal transportation.

Therefore, dynamics on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ have become an important issue of research in recent years. Along with the subjects of gradient flows and optimal transportation, the theory of Mean-Field Games (MFG) has also led to research around stochastic differential equations on $\mathcal{P}_2(\mathbb{R})$. Pioneer work were simultaneously released by Lasry and Lions [LL06a, LL06b, LL07] and by Huang, Caines and Malhamé [HMC06] on this topic. In short, the MFG theory describes Nash equilibria in games with a continuum of players that arise as the limit of games with finitely many particles, called players, interacting with one another through the distribution of the whole population. In order to study these systems, Lions has introduced a suitable approach to the differential calculus on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$, see Section 0.4. In MFG theory, the differential calculus on $\mathcal{P}_2(\mathbb{R})$ permits to define properly the Nash system, namely the system satisfied by the equilibrium values of the players.

A first example of a Wasserstein diffusion: von Renesse and Sturm's model

In this thesis, we are interested in stochastic processes on $\mathcal{P}_2(\mathbb{R})$ that are infinite dimensional counterparts to the Brownian motion on $\mathbb{R}^d$. There is not a unique way to construct this kind of diffusions on the Wasserstein space. Let us introduce two different constructions which have led to two diffusions having very different features. On the one hand, von Renesse and Sturm [vRS09] constructed a Wasserstein diffusion on $\mathcal{P}_2(\mathbb{R})$ with the aim, among others, to improve the understanding on the Riemannian structure of $\mathcal{P}_2(\mathbb{R})$ and to construct a kind of Riemannian volume measure on that space. The construction is based on the machinery of Dirichlet forms (see [FOT11]) on $L_2(\mathcal{P}, \mathbb{P})$, where $\mathcal{P}$ is a set of probability measures, either $\mathcal{P}([0,1])$ or $\mathcal{P}(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus (or unit circle), and $\mathbb{P}$ is a certain entropic measure on that space. This theory leads to the construction of a Markovian stochastic process $(\mu_t)_{t\in[0,T]}$, either on $\mathcal{P}([0,1])$ or on $\mathcal{P}(\mathbb{T})$, which is reversible with respect to the entropic measure. As it is shown in [vRS09], the dynamics of $(\mu_t)_{t\in[0,T]}$ have similar features as the dynamics of a Brownian motion. First, they satisfy an Itô-like formula: for a certain class of sufficiently smooth functions $u : \mathcal{P}([0,1]) \to \mathbb{R}$ (or $\mathcal{P}(\mathbb{T}) \to \mathbb{R}$),

$$\left(u(\mu_t) - u(\mu_0) - \frac{1}{2}\int_0^t \mathbb{L}u(\mu_s)\mathrm{d}s\right)_{t\in[0,T]} \tag{2}$$

is a martingale, where $\mathbb{L}$ is a certain second-order differential operator. Moreover, the quadratic variation of the martingale is given, as for the Euclidean Brownian motion, by the square of the gradient of u, where gradient is understood here relatively to the Wasserstein metric on $\mathcal{P}_2(\mathbb{R})$. Second, von Renesse and Sturm also stated a Varadhan-like formula for the Markov process $(\mu_t)_{t\in[0,T]}$: for any Borel subset A of $\mathcal{P}_2(\mathbb{R})$

$$\lim_{\varepsilon\to 0^+} \varepsilon \ln \mathbb{P}\left[\mu_{t+\varepsilon} \in A | \mu_t\right] = -\frac{1}{2} W_2(\mu_t, A)^2, \tag{3}$$

which shows that the Wasserstein distance W_2 plays a similar role for the Wasserstein diffusion as the Euclidean distance for the finite-dimensional Brownian motion.

Von Renesse and Sturm's contruction of a Wasserstein diffusion is quite abstract, but some recent papers improve the understanding of this process. Finite-dimensional approximations of the Wasserstein diffusion has been proposed, in terms of a system of particles [AvR10] or interacting systems of Brownian motions [Stu14]. The geometry of the entropic measure has been studied in [Cho12, DS09]. One of the reasons why it is difficult to deal with this diffusion is that, in the case where μ_0 is the Lebesgue measure on $[0,1]$, then almost surely for every $t > 0$, μ_t is a probability measure with no continuous part and no discrete part. That makes a big difference with the second model introduced in next paragraph, which is constructed starting from a particle system and for which μ_t is a discrete measure almost surely for every $t > 0$.

The starting point of our work: Konarovskyi's model

In this thesis, our starting point is a diffusion model on $\mathcal{P}_2(\mathbb{R})$ that has the benefit of having a nice representation in terms of its quantile function (or inverse distribution function or increasing rearrangement function), which stems from the fact that it is constructed starting from a system of particles on the real line. That model has been proposed by Konarovskyi in a series of papers [Kon11, Kon17b, Kon17a, KvR18]. Konarovskyi's model is inspired from the Arratia flow (see [Arr79]), sometimes also called Coalescing Brownian flow, which is a system of Brownian motions that coalesce, that is that they stick together after collision and move together like a single Brownian particle. Konarovskyi's diffusion, also called modified massive Arratia flow, introduces a mass to the model.

Let us briefly introduce Konarovskyi's construction. First, fix $N \geqslant 1$ and fix N distinct starting points $x_0^{1,N} < \cdots < x_0^{N,N}$ on the real line ranged in an increasing order. Then construct the following system of N coalescing particles $(x_t^{k,N})_{t\geqslant 0,\, 1\leqslant k\leqslant N}$:

1) start by considering N independent standard Brownian motions $(\beta_t^k)_{t\geqslant 0}$, for $k = 1, \ldots, N$, and denote for each k by $(x_t^{k,N})_{t\geqslant 0}$ the process $x_t^{k,N} := x_0^{k,N} + \frac{1}{\sqrt{N}}\beta_t^k$; associate to each particle a mass $m^{k,N} = \frac{1}{N}$;

2) define the stopping time τ as the first (random) time where two processes $x_t^{k_0,N}$ and $x_t^{k_0+1,N}$ collide: $\tau := \inf\{t \geqslant 0 : \exists k, l \in \{1, \ldots, N\}, x_t^{k,N} = x_t^{l,N}\}$;

3) replace, after time τ, the trajectories $(x_t^{k_0,N})_{t\geqslant\tau}$ and $(x_t^{k_0+1,N})_{t\geqslant\tau}$ of the two Brownian motions which have collided by their mean trajectory

$$\left(\frac{m^{k_0,N}}{m^{k_0,N} + m^{k_0+1,N}}\, x_t^{k_0,N} + \frac{m^{k_0+1,N}}{m^{k_0,N} + m^{k_0+1,N}}\, x_t^{k_0+1,N}\right)_{t\geqslant\tau},$$

and give a mass $m^{k_0,N} + m^{k_0+1,N}$ to that new particle;

4) repeat the same procedure with the remaining $N - 1$ particles after time τ: find the next collision time, replace the trajectories of the colliding particles after that time by their weighted average, etc...

5) stop the procedure when there remains only one particle.

These particles have coalescing trajectories: when two particles collide, they remain together and move like a single particle, but with a different diffusion rate. Indeed, the mean of two Wiener processes behaves like a Wiener process divided by $\sqrt{2}$. That is the main difference with Arratia's model, in which the diffusion rate was constant for each particle. A noticeable feature of this coalescing model is that the fluctuations of each particle can be described by its mass: initially, every particle has a mass of $\frac{1}{N}$ and when two particles collide, the resulting particle get a mass equal to the sum the masses of both incident particles. With this representation, the quadratic variation of every particle increases like the inverse of its mass.

Moreover, we associate to that system of particles the process of empirical measures:

$$\mu_t^N := \frac{1}{N}\sum_{k=1}^{N} \delta_{x_t^{k,N}},$$

where $\delta_{x_t^{k,N}}$ denotes the Dirac measure at the (random) position $x_t^{k,N}$. Thus for each time t, μ_t^N is a random probability measure. We associate to μ_t^N its quantile function $y_t^N : [0,1] \to \mathbb{R}$, defined by

$$y_t^N(u) = \begin{cases} \inf\{x \in \mathbb{R} : \mu_t^N((-\infty, x]) > u\} \text{ if } u \in [0,1); \\ \inf\{x \in \mathbb{R} : \mu_t^N((-\infty, x]) \geqslant u\} \text{ if } u = 1. \end{cases}$$

With this convention, y_t^N is a right-continuous function with left limits (càdlàg) and is continuous at point $u = 1$. Moreover, for each time t, y_t^N is a step function, equal to $x_t^{k,N}$ on every $[\frac{k}{N}, \frac{k+1}{N})$. Reversely, $\mu_t^N = \mathrm{Leb}_{[0,1]} \circ (y_t^N)^{-1}$.

The introduction of a mass to the model allows to obtain nice properties for the limit process when N tends to infinity. To fix ideas, let us for instance assume that for each N, the initial points are regularly distributed on the unit interval $[0,1]$: $x_0^{i,N} = \frac{i}{N}$. In this case, the sequence of probability measures $(\mu_0^N)_{N\geqslant 1}$ tends for the L_2-Wasserstein distance to $\mu_0 = \mathrm{Leb}_{[0,1]}$. In [Kon17b],

Konarovskyi proves that the sequence of processes $((y_t^N)_{t\geqslant 0})_{N\geqslant 1}$ is tight in $\mathcal{D}([0,1],\mathcal{C}[0,T])$ and consequently that a subsequence converges to $(y_t)_{t\geqslant 0}$, and proves that almost surely for each time $t>0$, the càdlàg function $u\mapsto y_t(u)$ is a step function. In other words, the probability measure $\mu_t = \mathrm{Leb}_{[0,1]}\circ(y_t)^{-1}$ is an atomic measure, with a finite number of atoms. The behaviour when t tends to $+\infty$ is not interesting for our study, since almost surely, there is a time from which μ_t is a Dirac mass, or equivalently y_t is a constant function, or equivalently there remains only one Brownian particle. Thus we fix in all our work a final time T and we study the processes on $[0,T]$.

In [KvR18], Konarovskyi and von Renesse proved that the process $(\mu_t)_{t\in[0,T]}$ introduced above enjoys similar properties to the Wasserstein diffusion. In particular, it also satisfies an Itô-like formula, like (2) and a Varadhan-like formula, like (3). Moreover, the flow of that process preserves the monotonicity of the initial quantile function, so that $(\mu_t)_{t\in[0,T]}$ has a canonical representation as a process of quantile functions $(y_t)_{t\in[0,T]}$. Most of the time, we will take the point of view of the quantile function representation, because it has an interpretation as a particle system and it is more convenient to work on $L_2[0,1]$ than on $\mathcal{P}_2(\mathbb{R})$. As we already mentioned, the latter is very much in the spirit of Lions' approach to the differential calculus on $\mathcal{P}_2(\mathbb{R})$.

Nevertheless, Konarovskyi's model remains partially misunderstood. This may be rather surprising because the dynamics look quite simple at first sight. In particular, the question of uniqueness (not only pathwise uniqueness but also uniqueness in law) remains an open question. The major difficulty is that it suffers from a tough singularity at time $t=0^+$ since the solution may pass instantaneously from a probability with a density with respect to the Lebesgue measure on the real line to a discrete measure with a finite weighted sum of Dirac masses. Coming back to the previous example, this is precisely what happens if $\mu_0=\mathrm{Leb}_{[0,1]}$, but this feature is actually general: it happens for each initial absolutely continuous measure with respect to the Lebesgue measure. We believe that any attempt to prove uniqueness would require a better understanding of the way the particles coalesce at $t=0^+$. For sure, this is part of our research program for the future to (re)attack this problem, see the list of open questions on page 23. As for the rest of the thesis, we overcome lack of uniqueness (obviously, it is a serious drawback for addressing more quantitative questions) by modifying Konarovskyi's original model, see Section 0.3 and Chapter I. Among others, the new model does not coalesce and, whenever it starts from a regular density, the solution itself remains a density. Subsequent variants are addressed in the last three chapters, see also Section 0.3 for a first overview.

Recent new developments

Several classes of stochastic partial differential equations (SPDEs) for measure-valued processes have been recently studied. For instance, the interested reader may find in the work of Xiong [Xio13], together with the references that are cited therein, a detailed analysis of SPDEs arising in the study of the super-Brownian motion and of the Fleming-Viot process. More recently, Konarovskyi and von Renesse have proposed a new model of reversible Coalescing-Fragmentating Wasserstein dynamics [KvR, Kon]. Instead of coalescing, the particles of that system stick together when they collide and remain together during a certain (random) time, before fragmentating. It is reversible in time, by opposition with Konarovskyi's modified massive Arratia flow where the number of particles decreases with time. Moreover, Konarovskyi, Lehmann and von Renesse [KvR18, KLvR19, KLvR] have shown that every processes described above can be considered as a solution of a certain Dean-Kawasaki PDE of the form

$$\partial_t\mu = \alpha\Delta\mu + \Xi(\mu) + \mathrm{div}(\sqrt{\mu}\dot{W}),$$

where Ξ is a non-linear operator and W is space-time white noise. The Wasserstein diffusion of von Renesse and Sturm corresponds to the case $\alpha>0$, while the modified massive Arratia

flow and the Coalescing-Fragmentating Wasserstein process correspond to the case $\alpha = 0$; more precisely, the two latter processes satisfy exactly the same equation, with $\alpha = 0$ and

$$\Xi(\mu) = \sum_{z \in \mathrm{Supp}(\mu)} (\delta_z)''.$$

This shows that the martingale problem associated to that equation is not well-posed. According to [KLvR19], the correction term Ξ is necessary to get non-trivial solutions of Dean-Kawasaki equation.

Of course, the above SPDE should be regarded as a stochastic Fokker-Planck equation, the stochasticity accounting for the fact that W here plays the role of a common noise according to the terminology we introduced in the outline of the thesis. Although we do not make any real use of the SPDE approach in the manuscript (somehow, our point of view is mostly Lagrangian), it must be stressed that we use quite systematically a similar white noise W to drive the various particle systems at hand. In particular, we spend some time in Chapter I explaining how W shows up in the original Konarovskyi model. As for the Laplace term of viscosity α, we introduce at the end of Chapter II and in Chapters III and IV variants of the Konarovskyi process featuring a similar diffusive behavior: in short, our strategy therein is to introduce, in addition to the common noise W, an independent idiosyncratic noise β, the effect of which manifests precisely in the corresponding Fokker-Planck equation in the form of a Laplacian.

0.2 Smoothing properties of stochastic diffusive processes

One of our purposes in this thesis is to study regularization properties of $\mathcal{P}_2(\mathbb{R})$-valued processes. Let us introduce some fundamental results on this topic, first in a finite dimensional framework and then for processes defined on infinite dimensional spaces, e.g. on Hilbert spaces. We will particularly focus on equations with mean-field interaction, in particular McKean-Vlasov SDEs.

In fact, regularizing effects of stochastic processes may manifest in several ways, but certainly the phenomenon may be easily understood when considering the semi-group generated by the underlying process. Basically, the semi-group has some smoothing effect if it maps, in positive time, a class of input functions into a class of more regular output functions. For instance, the semi-group may map bounded measurable functions onto Hölder-continuous functions, which is for instance what happens in the so-called Krylov-Safonov theory [KS79] for (finite-dimensional) diffusion processes with coefficients of lower regularity. As made clear below, stronger forms of regularization may occur: for example, the semi-group may send bounded measurable functions onto differentiable functions. All and all, these kinds of properties are deeply connected with regularity results for solutions to parabolic or elliptic PDEs, in finite or infinite dimension depending on the state space of the underlying process itself. As we already alluded to in the outline of the manuscript and as we explain in a more detailed fashion below, the regularity properties of the semi-group have many applications, one of which is remarkable for us: it is widely used for restoration of uniqueness results, a review of which we also provide below. In this regard, it must be stressed (and this is a key fact in the thesis) that, although we prove below some restoration of uniqueness for McKean-Vlasov equations, we do not make use of any semi-group approach to do so: instead we make a systemic use of Girsanov's transformation. We will pay some price for it. This forces us to work with weak instead of strong uniqueness. Anyway, using Girsanov's transformation should not come as a surprise since it is in fact one of the basic ingredient to address regularity of semi-groups by probabilistic arguments. So, to sum-up, the results of restoration of uniqueness we show below do not rely on any semi-group estimates but they do rely on Girsanov's Theorem, see for instance Chapter II, and the same Girsanov transformation is widely used in Chapter III to address semi-group regularity. We make a detailed review of our results in Section 0.3 below, and we explain in this context the

role of the Girsanov transformation. For sure, other forms of regularization could be addressed, but we won't tackle them in the text; say for instance: regularity of the transition kernels of the stochastic processes under study, long time behavior and in particular ergodicity of those stochastic processes...

Regularization results in finite dimension

Smoothing properties of diffusive processes and in particular of the Brownian motion itself is a key ingredient in stochastic calculus and stochastic analysis. In particular, the semi-group associated to the standard Brownian motion in $\mathbb{R}^d$ is the heat semi-group and thus this process inherits from the regularizing effect of the Laplace operator.

As for the aforementioned application to restoration of uniqueness, one of the first breakthrough is due to Zvonkin [Zvo74] in the 1970s, who showed that adding, in dimension 1, a small Brownian perturbation to an ill-posed ordinary differential equation (ODE) was sufficient to guarantee strong existence and uniqueness of a solution. To clarify, let us consider the following ODE in dimension 1:

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = b(t, x_t); \quad x_0 = a_0 \in \mathbb{R}. \tag{4}$$

Cauchy-Lipschitz theory states that existence and uniqueness of a solution to the Cauchy problem (4) hold if $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function. Peano's Theorem adds that local existence still holds even if b is only continuous. Nevertheless, uniqueness does no more necessarily hold out of Cauchy-Lipschitz's assumptions. Indeed, in the case of Peano's counter-example $b(t, x) = 3\,\mathrm{sign}(x)|x|^{2/3}$, where $\mathrm{sign}(x) = 1$ if $x \geqslant 0$ and $\mathrm{sign}(x) = -1$ if $x < 0$, the Cauchy problem (here with initial condition $x_0 = 0$) has infinitely many solutions, among which $x_t \equiv 0$, $x_t \equiv t^3$, $x_t \equiv \max(t-1, 0)^3$. In [Zvo74], Zvonkin proved that for a certain class of functions b, including Peano counter-example, adding to the equation a Brownian motion $(W_t)_{t\in[0,T]}$ allows to restore uniqueness: as we alluded to above, restoration of uniqueness happens when uniqueness does not hold for the deterministic equation and is satisfied by the stochastically perturbed equation. More precisely, Zvonkin proved that pathwise uniqueness, meaning that the trajectories of two solutions are indistinguishable, holds for the following SDE in dimension 1:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \tag{5}$$

where $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded and where $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ is bounded by above and by below (far from 0) and is Hölder-continuous. Zvonkin's result has been generalized to higher dimensions by Veretennikov [Ver80] and later to the case of drift functions b which are no longer bounded but which are sufficiently integrable in time and space, the right exponents of integrability depending upon the dimension of the state space (see Krylov-Röckner [KR05]). The basic proof of all these results is to solve a parabolic PDE, driven by the generator of (5), with the drift itself as source term. Expanding by Itô's formula the resulting solution of the PDE along any solution of (5), this permits to get rid of the drift, at the price of getting a more intricate diffusion coefficient. Hopefully, this diffusion coefficient remains sufficiently regular to get pathwise uniqueness if the solution of the PDE itself is sufficiently regular: this is a typical instance of the fact that regularity and uniqueness are correlated.

In the case where the diffusion matrix σ has less regularity, Tanaka's example

$$\mathrm{d}X_t = \mathrm{sign}(X_t)\mathrm{d}W_t,$$

shows that two solutions may share the same distribution (uniqueness in law) even though there is no pathwise uniqueness; also there may be existence of a weak solution, but the solution may not be strong, *i.e.* it may not be measurable with respect to the filtration generated by

the Brownian motion W. Of course, it strongly suggests that restoration of uniqueness may be obtained in a weak sense under weaker assumptions on the coefficients, and in particular on the diffusion term σ, which is in fact the precise purpose of the Stroock and Varadhan theory: in the case of Itô's SDE (5) in $\mathbb{R}^d$, Stroock and Varadhan [SV69] proved that weak existence and uniqueness in law hold when b is bounded and measurable and when σ is continuous, bounded and uniformly elliptic. Furthermore, Stroock and Varadhan developed in [SV79] the correspondance between PDEs and so-called martingale problems, showing that properties of the SDEs are closely related to properties of PDEs driven by the generator of (5): in short, those properties are weaker than those needed in the pathwise uniqueness results; somehow, pathwise uniqueness requires pointwise estimates on the derivatives of the solution of the PDE whilst Stroock and Varadhan theory allows to work with Sobolev solutions. Since these pioneer works, restoration of uniqueness results have been obtained in various frameworks, for noises less regular than Wiener processes and is still a source of active research.

This prompts us to say a few words on the regularity properties of the semi-group itself (assuming to simplify that existence and uniqueness hold in a strong sense). So, let us denote by $(X_t^x)_{t\in[0,T]}$ the solution to (5) with initial condition $X_0^x = x$. For any bounded and continuous $\phi : \mathbb{R} \to \mathbb{R}$ and for any time $t \in [0,T]$, let us denote by $P_t\phi$ the following function:

$$P_t\phi(x) = \mathbb{E}\left[\phi(X_t^x)\right].$$

The collection $(P_t)_{t\in[0,T]}$ is called semi-group associated to the stochastic process $(X_t^x)_{t\in[0,T],x\in\mathbb{R}}$, since it owns the properties $P_t \circ P_s = P_{t+s}$ and $P_0 = \mathrm{id}$ by Markovian properties of the solution $(X_t^x)_{t\in[0,T]}$. A regularization effect is characterized by the fact that for each time $t > 0$, $P_t\phi$ is more regular than ϕ. For instance, in the case of the standard Wiener process ($b \equiv 0$ and $\sigma \equiv 1$), $P_t\phi = \phi * g_t$, where g_t is the normal density $g_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-\|x\|^2/2t}$; thus $P_t\phi$ is a $\mathcal{C}^\infty$-function for every $t > 0$ and for every bounded and continuous ϕ.

Smoothing effects of Itô's SDEs have been intensively studied, in particular using Malliavin calculus, especially in the hypoelliptic setting, that is when the diffusion rate is degenerate (for an introduction to Malliavin calculus, see [Nor86, Nua06]). In this thesis, we will focus on a parallel method, proposed by Bismut in [Bis81], relying on Girsanov's Theorem and for which it is not necessary to apply the machinery of Malliavin calculus. This method leads to an integration by parts formula, often called Bismut-Elworthy formula, or Bismut-Elworthy-Li formula, after the studies [Bis81, Elw92, EL94]. It can be written in the form of an integration by parts (very much in the spirit of Malliavin calculus):

$$d(P_t\phi)_{x_0}(v_0) = \frac{1}{t}\mathbb{E}\left[\phi(X_t)\int_0^t \langle V_s, \sigma(X_s)\mathrm{d}W_s\rangle\right], \tag{6}$$

where V_s is a certain stochastic process starting at v_0 (see [EL94]). It follows from Bismut-Elworthy formula (6) an inequality on the gradient of $P_t\phi$: more precisely, under some boundedness and regularity assumptions on the coefficients b and σ, the gradient of $P_t\phi$ is uniformly bounded by $\frac{1}{\sqrt{t}}\|\phi\|_{L_\infty}$, for every bounded and continuous ϕ. Of course, since ϕ is not necessarily locally Lipschitz, the control tends to infinity when t tends to 0 by positive values. But that gradient estimate provides a diffusive rate of explosion $t^{-1/2}$, typical of Brownian fluctuations, which is important for applications. For instance, gradient estimates deriving from integration by parts formulae are used in finance for computing price sensitivies, called Greeks (see e.g. [FLL$^+$99, MT06]). In the case of finite-dimension, Bismut-Elworthy formula has important applications in geometry, in particular concerning gradient estimates of PDEs on manifolds (see [Tha97, TW98, ATW06]) and in the probabilistic analysis of non-linear PDE, like Kolmorogorov backward equation and Forward-Backward SDEs [Del03, Zha05].

Regularization results in infinite dimension

The zoology of stochastic processes in infinite dimension is really wide. The $\mathcal{P}_2(\mathbb{R})$-valued processes that we address in this manuscript are infinite-dimensional processes and hence fall within this class. Obviously, we cannot provide an exhaustive overview of the field and we just give a short account of it here, pointing out some connections with the content of the manuscript. In particular, regularization results for equations on infinite-dimensional spaces have been carefully studied for several years and for various types of equations. Let us cite among others some works on Kolmogorov equations on Hilbert spaces by Da Prato, Elworthy, Zabczyk and Cerrai [DPEZ95, Cer01, DP04].

Restoration of uniqueness of PDEs is a more recent subject. The first results have been obtained for transport equations by Flandoli, Gubinelli and Priola [FGP10]. For Hölder-continuous functions b, the transport equation on $[0,T]\times\mathbb{R}^d$

$$\mathrm{d}_t u(t,x) = (b(t,x)\cdot Du(t,x))\,\mathrm{d}t$$

is not necessarily well-posed (following [FGP10], $b(t,x) = \frac{1}{1-\gamma}([x]\wedge R)^\gamma$ for a fixed $R>0$ and for a fixed $\gamma\in(0,1)$ is a counter-example). By adding a noise, uniqueness can be restored, as Flandoli, Gubinelli and Priola showed for the following equation (where the stochastic integral is here considered in the sense of Stratonovich):

$$\mathrm{d}_t u(t,x) = (b(t,x)\cdot Du(t,x))\mathrm{d}t + \sum_{i=1}^{d} e_i\cdot Du(t,x)\circ dW_t^i.$$

Let us mention that the noise is called here multiplicative, since it depends on the value $u(t,x)$ of the solution (by opposition with an additive noise that depends only on t and x but not on $u(t,x)$). Many further investigations have been made for SDEs on Hilbert spaces. In a series of papers [DPF10, DPFPR13, DPFPR15], Da Prato, Flandoli, Priola and Röckner proved that pathwise uniqueness holds for an SDE on a Hilbert space H of the form

$$dX_t = (AX_t + B(X_t))\mathrm{d}t + \mathrm{d}W_t, \tag{7}$$

for a certain class of self-adjoint, negative definite operators $A: D(A)\subset H\to H$, for W a cylindrical Wiener process on H and for $B: H\to H$ only measurable and locally bounded. For an interesting introduction and a survey of results on regularization by noise phenomena, see Flandoli's seminal lecture notes [Fla11]. Various other equations in infinite-dimension have also been studied, like e.g. kinetic equations [FFPV17].

Interestingly enough, we quote in this context the recent result of Delarue [Del19] in which some of the above results are used to restore uniqueness to a mean-field game by means of an infinite dimensional common noise of Ornstein-Uhlenbeck type. Although this work shares some motivation with ours, it must be stressed that the dynamics of the particle therein obey an operator A similar to the one that appears in (7). Equivalently, this says that uniqueness is restored but at the price of an extra layer of interactions which is, in contrast to the mean-field one, purely local, arising from the Ornstein-Uhlenbeck noise. The models that we address in the rest of the text do not have the latter feature.

Regularization results for systems with mean-field interaction

Let us focus now in a more detailed fashion on a certain class of equations, namely the SDEs with mean-field interaction called McKean-Vlasov equations. Let $b:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ be a drift function, $\sigma:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times m}$ be a diffusion matrix and $(B_t)_{t\in[0,T]}$ be a Brownian motion in $\mathbb{R}^m$. McKean-Vlasov equation reads as follows

$$\begin{cases} \mathrm{d}X_t = b(t,X_t,\mu_t)\mathrm{d}t + \sigma(t,X_t,\mu_t)\mathrm{d}B_t, \\ \mu_t = \mathcal{L}(X_t), \end{cases} \tag{8}$$

where $\mathcal{L}(X_t)$ denotes the law of X_t. The coefficients in the stochastic differential equation (8) depend on the distribution of the solution X_t. That dependence is called mean-field interaction, due to the link with a particle system. Indeed, the McKean-Vlasov equation should be regarded as the limit when $N \to +\infty$ of a system of particles of the following form:

$$\mathrm{d}X_t^i = b(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}t + \sigma(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}B_t^i, \quad i = 1, \dots, N, \tag{9}$$

where $\overline{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$ and $(B_t^i)_{t\in[0,T],\, 1\leqslant i\leqslant N}$ are independent Brownian motions, the latter being usually referred to as idiosyncratic noises in order to stress the fact that there are somehow proper to a given particle. The trajectory of each particle $(X_t^i)_{t\in[0,T]}$ depends on time, on the current position of the particles and also on the positions of the other particles, but only via the empirical distribution $\overline{\mu}_t^N$; that is why this system is called mean-field.

Well-posedness of McKean-Vlasov SDEs has been widely studied. We here provide a tiny example of all the existing references in the field. Generally speaking, existence and uniqueness may be proved by a Picard fixed point argument on the process $(\mu_t)_{t\in[0,T]}$ provided that the coefficients are sufficiently regular, say for instance that they are Lipschitz-continuous in both variables, Lipschitz-continuity with respect to the measure argument being understood with respect to the L_2-Wasserstein distance. This strategy is made clear in the seminal lecture notes of Sznitman [Szn91]. Variants may be found, see for example (to quote earlier ones) the works of Funaki [Fun84], Gärtner [G88] or Oelschläger [Oel84]. Interestingly enough, the proof of existence and uniqueness extends to models with a common noise of the form

$$\mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X_t, \mu_t)\mathrm{d}B_t + \sigma_0(t, X_t, \mu_t)\mathrm{d}W_t, \tag{10}$$

with the constraint that μ_t now matches the conditional law of X_t given the realization of W, where $(W_t)_{t\in[0,T]}$ is a new Brownian motion, independent of $(B_t)_{t\in[0,T]}$ and of dimension m_0, and σ_0 stands for a new volatility coefficient defined in the same manner as σ. Importantly, μ_t becomes random under the presence of W. The terminology *common noise* is better understood when we write down the analogue of (9), which reads:

$$\mathrm{d}X_t^i = b(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}t + \sigma(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}B_t^i + \sigma_0(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}W_t, \quad i = 1, \dots, N. \tag{11}$$

The key fact here is that all the particles are driven by the same noise $(W_t)_{t\in[0,T]}$, which is of course assumed to be independent of the collection $(B_t^i)_{t\in[0,T],\, 1\leqslant i\leqslant N}$. The reader may have a look at the works of Vaillancourt [Vai88], Dawson and Vaillancourt [DV95], Kurtz and Xiong [KX99, KX04] or Coghi and Flandoli [CF16] for more details on (10) and (11). We will also come back to it later on in the text: in particular, McKean-Vlasov equations with a common noise are addressed with care in Chapter II.

The passage from (9) to (8) should be understood in the following way. Assume that the initial positions of the particles $X_0^1, \dots, X_0^N$ in (9) are independently distributed with the same law on $\mathbb{R}^d$. Of course, due to the mean-field interaction, $X_t^1, \dots, X_t^N$ are not necessarily independent at a fixed time $t > 0$. Nevertheless, if $k \in \mathbb{N}\backslash\{0\}$ is fixed, then the distribution of $(X_t^1, \dots, X_t^k)_{t\in[0,T]}$ is expected to converge in law when N tends to $+\infty$ to $(\mu_t^{\otimes k})_{t\in[0,T]}$, where $(\mu_t)_{t\in[0,T]}$ is the law of the solution to McKean-Vlasov equation (8) (at least whenever it is uniquely solvable). In other words, asymptotically when the number of particles tends to infinity, two given particles (or more, as long as the number is fixed) have approximatively independent trajectories. This phenomenon is called propagation of chaos.

The concept of propagation of chaos for McKean-Vlasov equation (8) has been introduced in the early works of Kac and McKean [Kac56, McK66, McK67] and is proved in the case where $b(t, x, \mu) = \int B(t, x, y)\mathrm{d}\mu(y)$ among others in Sznitman's course [Szn91]. Along with propagation of chaos results, authors are also interested by the rate of convergence of the particle system to the limit. For instance, this question was addressed by Sznitman in [Szn91] using a coupling

argument under the same Lipschitz assumptions as for proving existence and uniqueness. Applications and instances are numerous, including models from fluid mechanics or statistical physics that go beyond the scope of Sznitman's coupling result: for instance, Bossy and Talay [BT97] also obtained a rate of convergence for a particle system converging to Burgers' equation. More recently, Guillin and Fournier [FG17] proved a quantitative result of convergence of a conservative particle system to the solution of the homogeneous Landau equation for hard potentials; this includes cases where propagation of chaos is shown to be uniform in time. The rate of convergence may be also addressed in the form of a central limit theorem or of a large deviation principle, see for instance [TH81, HM86, DG87, JM98, Fis14]. Lastly, notice that propagation of chaos may be also established but without any rate by using compactness arguments, see for instance Méléard [Mé96].

Let us describe restoration of uniqueness phenomena for McKean-Vlasov equations. Well-posedness may fail to be true for the "deterministic" equation (i.e. $\sigma \equiv 0$ in (8) in which case the randomness only comes from the initial condition) when the drift term b is not regular enough. Let us consider a counter-example that is intrinsically finite-dimensional:

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t, \\ \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}, \end{cases}$$

with again Peano counter-example: $b(y,\mu) := 3\,\mathrm{sign}(m)m^{2/3}$ and $m = \int_{\mathbb{R}} x\mathrm{d}\mu(x)$. Then the Cauchy problem with initial condition $\mu_0 = \delta_0$ has infinitely many solutions, e.g. $\mu_t \equiv \delta_0$ or $\mu_t \equiv \delta_{t^3}$. In this framework, the noise B has been shown to help for uniqueness, but in some cases only, which leave the latter counter-example out of reach. Typically, the drift b may have low regularity in the space variable x, which is consistent with the results we recalled before for standard diffusion processes. However, as for the regularity in the measure argument, b cannot be reasonably expected to be merely measurable in μ. Indeed, the noise B is finite-dimensional only, whereas the measure argument is of infinite dimension. Anyhow, existing results in the field show that it is possible to require b to be Lipschitz-continuous but with respect to the finer topology generated by the total variation distance $\mathrm{d_{TV}}$, defined by

$$\mathrm{d_{TV}}(\mu,\nu) := \sup\left\{ \int_{\mathbb{R}^d} f\mathrm{d}\mu - \int_{\mathbb{R}^d} f\mathrm{d}\nu \,;\, f : \mathbb{R}^d \to \mathbb{R} \text{ measurable}, \|f\|_{L_\infty} \leqslant 1 \right\}.$$

In particular, Jourdain [Jou97] has proved that restoration of uniqueness holds in a weak sense for McKean-Vlasov equation in a case where $\sigma \equiv 1$ and b is bounded, measurable and Lipschitz-continuous with respect to its measure variable, with respect to the distance in total variation. Recently, several papers have improved the results, proving well-posedness for more general coefficients σ, but in cases where σ does not depend on μ [MV, Lac18, CdRF, RZ]. In [MV], Mishura and Veretennikov have in particular shown pathwise uniqueness under Lipschitz-continuity assumptions on b with respect to the measure variable and on σ with respect to the space variable. Röckner and Zhang [RZ] have extended the results to the case of unbounded coefficients with suitable integrable properties, in the sense of Krylov-Röckner [KR05]. In all these results, b is assumed to be Lipschitz-continuous with respect to the measure-variable in distance of total variation. Importantly, we will revisit these models in Chapter II when addressing restoration of uniqueness under the action of a forcing by a Wasserstein diffusion.

Moreover, let us describe some results on semi-group regularity, in particular gradient estimates. Denoting by X_t^θ the solution to the SDE (8) starting at θ, let us introduce the so-called decoupled equation associated to (8):

$$\begin{cases} \mathrm{d}X_t^{x,[\theta]} = b(t, X_t^{x,[\theta]}, \mathcal{L}(X_t^\theta))\mathrm{d}t + \sigma(t, X_t^{x,[\theta]}, \mathcal{L}(X_t^\theta))\mathrm{d}W_t, \\ X_0^{x,[\theta]} = x. \end{cases} \tag{12}$$

Here, the notation $[\theta]$ is used to indicate that the solution only depends on the law of θ. As for x, it is implicitly required to be deterministic; anyhow, it may be easily randomized, in which case we may obtain the identity $X_t^{\theta,[\theta]} = X_t^\theta$ by choosing $x = \theta$. We say that the system is decoupled because we first solve for the McKean-Vlasov equation (8) and then we use the law of the solution to freeze in (12) the dependence of the coefficients b and σ with respect to the measure variable. Therefore, (12) becomes a simple Itô's SDE once the solution of (8) is known. In practice, both $X^{x,[\theta]}$ and X^θ are useful. For instance, Buckdahn, Li, Peng and Rainer [BLPR17] determined that the semi-group of the pair $(X^{x,[\theta]}, X^\theta)$ satisfies a second order PDE, involving the first and second derivatives with respect to the space and the measure variables, in the sense of Lions' differential calculus (see section 0.4). This result is extended to nonlinear equations as those arising in mean-field games in the work of Chassagneux, Crisan and Delarue [CCD19]. From a different prospect, Crisan and McMurray [CM18] obtained an integration by parts formula, like Bismut-Elworthy formula, for the derivatives of the expectation of functions of $X_t^{x,[\theta]}$ both in the direction of variable x and in the direction of the law of θ. They deduced estimates for the derivatives of the density associated to the law of the solution to (12). Similar forms of integration by parts have been also obtained by Baños [Bn18]. Moreover, Chaudru de Raynal [CdR19] and Chaudru de Raynal and Frikha [CdRF] implemented a parametrix method to address the regularity properties of the transition kernel of the solution to (12) under various assumptions, including those used for the aforementioned result of restoration of uniqueness. Related results were recently obtained by Röckner and Zhang [RZ], under integrable properties.

0.3 Contents of the chapters

Let us now introduce the main results of this thesis.

Chapter I - A new approach for the construction of a Wasserstein diffusion

Recall that we are interested in Konarovskyi's model of coalescing particles on the real line, which has been introduced in [Kon11] and [Kon17b] and whose diffusive properties have been identified in [KvR18]. Recall also that this model is intrinsically very singular, due to the fact that two colliding particles switch instantaneously from having independent trajectories to sharing the same trajectory. The aim of Chapter I is to construct on the Wasserstein space a close relative to Konarovskyi's diffusion $(\mu_t)_{t\in[0,T]}$ with the following two constraints: it should be more regular and it should also admit a canonical representation in the form of a tractable process of quantile functions. Regularization properties of that new model will be studied in the subsequent chapters.

In Chapter I, we introduce a smooth approximation of Konarovskyi's model, consisting in a system of particles interacting at short range. In that model, two given particles have independent trajectories up to the moment where their relative distance becomes smaller than a fixed $\sigma > 0$; then the closer to zero the relative distance is, the bigger the correlations between both trajectories are. Two particles never collide in that model: when they are very close one to another, they have almost parallel trajectories, although with a small probability, two particles that are very close one to each other may move away again. We will prove in this Chapter I that our model converges in some sense to Konarovskyi's model when the parameter σ tends to zero (see a numerical simulation on figure 1).

To fix ideas, let us introduce more precisely Konarovskyi's process. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. Let us denote by $\mathcal{L}_{2+}^{\uparrow}[0,1]$ the space of càdlàg and non-decreasing functions $g : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ such that there is $p > 2$ for which g belongs to $L_p[0,1]$. It describes the space of initial conditions. Konarovskyi has constructed (see [Kon17a]) a stochastic process $(y(u,t))_{u\in[0,1],t\in[0,T]}$ starting from a fixed $g \in \mathcal{L}_{2+}^{\uparrow}[0,1]$ and belonging to $\mathcal{D}([0,1], \mathcal{C}[0,T])$, the

space of càdlàg functions from $[0,1]$ with values in the space of continuous functions $\mathcal{C}[0,T]$, satisfying the following properties:

(*i*) for every $u \in [0,1]$, $y(u,0) = g(u)$;

(*ii*) for every $u \leqslant v$ and for every $t \in [0,T]$, $y(u,t) \leqslant y(v,t)$;

(*iii*) for every $u \in [0,1]$, the process $(y(u,t))_{t\in[0,t]}$ is a square-integrable martingale relatively to the filtration generated by the process y;

(*iv*) for every $u, u' \in [0,1]$, the quadratic variation is given by

$$\langle y(u,\cdot), y(u',\cdot)\rangle_t = \int_{\tau_{u,u'}}^{t\wedge\tau_{u,u'}} \frac{1}{m(u,s)}\mathrm{d}s,$$

where $\tau_{u,u'}$ is the collision time between u and u', defined by $\tau_{u,u'} = \inf\{t \geqslant 0 : y(u,t) = y(u',t)\} \wedge T$, and $m(u,s) = \mathrm{Leb}\{v \in [0,1] : \tau_{u,v} \leqslant s\}$.

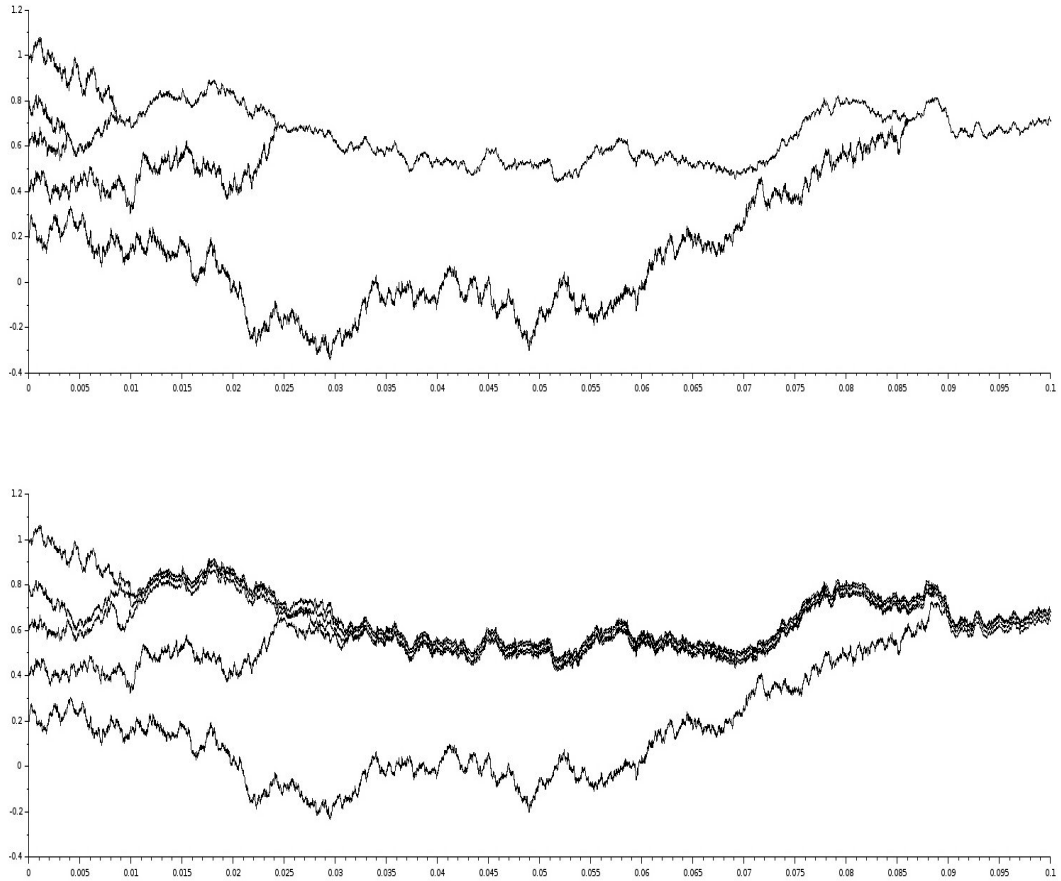

Figure 1: On top, a numerical simulation of Konarovskyi's model with five initial particles. On bottom, a simulation of a particle system with interaction at range σ, with the same underlying noise. The horizontal axis represents time, the vertical axis represents the positions of the particle on $\mathbb{R}$.

It follows from properties (*iii*) and (*iv*) that two distinct particles behave like independent Wiener processes with diffusion rates given by the inverse of their respective mass m, up to the moment of collision, from which their trajectories become equal. By property (*ii*), for a fixed time t, the map $u \mapsto y(u,t)$ can be seen as the quantile function (or increasing rearrangement function) associated to a probability measure μ_t defined by $\mu_t = \mathrm{Leb}_{[0,1]} \circ y(\cdot,t)^{-1}$. Konarovskyi

and von Renesse [KvR18] showed an Itô-like formula for the process $(\mu_t)_{t\in[0,T]}$. In an appendix to Chapter I, we extend this formula to a more general class of twice differentiable functions on $\mathcal{P}_2(\mathbb{R})$, using Lions' differential calculus.

We show in Chapter I that the process $(y(u,t))_{u\in[0,1],t\in[0,T]}$ satisfies a stochastic differential equation, where the noise is written in the form of a Brownian sheet w on $[0,1]\times[0,T]$. Given a space-time white noise W on $[0,1]\times[0,T]$, w is defined by $w(u,t) = W((0,u]\times(0,t])$. Denoting by $(\mathcal{G}_t)_{t\in[0,T]}$ the filtration generated by w, namely $\mathcal{G}_t := \sigma(w(u,s), u\in[0,1], s\leqslant t)$, the Brownian sheet w satisfies the following properties:

- for each $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable function f defined on $[0,1]\times[0,T]$ such that $\int_0^T\int_0^1 f^2(u,s)\mathrm{d}u\mathrm{d}s < +\infty$ almost surely, the process $\left(\int_0^t\int_0^1 f(u,s)\mathrm{d}w(u,s)\right)_{t\in[0,T]}$ is a local martingale ;
- for each f_1 and f_2 satisfying the same conditions as f,

$$\langle\int_0^\cdot\int_0^1 f_1(u,s)\mathrm{d}w(u,s), \int_0^\cdot\int_0^1 f_2(u,s)\mathrm{d}w(u,s)\rangle_t = \int_0^t\int_0^1 f_1(u,s)f_2(u,s)\mathrm{d}u\mathrm{d}s.$$

Let us state our result:

Theorem 1. *Let $g\in\mathcal{L}^{\uparrow}_{2+}[0,1]$. Let y satisfy $(i)-(iv)$ and m be defined as in (iv) above. Then there exists a Brownian sheet w on $[0,1]\times[0,T]$ such that almost surely for all $u\in(0,1]$ and for all $t\in[0,T]$,*

$$y(u,t) = g(u) + \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}w(u',s). \tag{13}$$

Therefore, Konarovskyi's process y is a weak solution to SDE (13). Nevertheless, uniqueness of solutions to (13) is an open problem. There are two difficulties. The first one is that the inverse of the mass is not bounded. In particular, the mass $m(u,s)$ tends to 0 for each fixed $u\in(0,1]$ when $s\to 0^+$. The second problem comes from the indicator function, which is not smooth enough to apply a classical Lipschitz theory.

In order to overcome these two issues, we introduce a mollification based upon two parameters $\sigma>0$ and $\varepsilon>0$. Let $\varphi_\sigma:\mathbb{R}\to\mathbb{R}$ be a $\mathcal{C}^\infty$-function approximating $\mathbb{1}$: assume that φ_σ is bounded by 1, equal to 1 on $[-\frac{\sigma}{3},\frac{\sigma}{3}]$ and with support included in $[-\frac{\sigma}{2},\frac{\sigma}{2}]$. Let us define the following SDE:

$$y_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t\int_0^1 \frac{\varphi_\sigma(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(u',s))}{\varepsilon + m_{\sigma,\varepsilon}(u,s)}\mathrm{d}w(u',s), \tag{14}$$

where $m_{\sigma,\varepsilon}(u,s) := \int_0^1 \varphi_\sigma^2(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s))\mathrm{d}v$. The map φ_σ is Lipschitz-continuous and the inverse of the mass is bounded by $\frac{1}{\varepsilon}$, so we prove by a standard fixed-point method that there is a unique solution to (14) in $L_2([0,1],\mathcal{C}[0,T])$ and that for each time t, $u\mapsto y_{\sigma,\varepsilon}(u,t)$ is non-increasing. In other words, equation (14) preserves the monotonicity of the initial condition. Therefore, for each time $t\in[0,T]$, we can regard $u\mapsto y_{\sigma,\varepsilon}(u,t)$ as the quantile function of the probability measure $\mu_t^{\sigma,\varepsilon} := \mathrm{Leb}_{[0,1]}\circ y_{\sigma,\varepsilon}(\cdot,t)^{-1}$ and we get a stochastic process $(\mu_t^{\sigma,\varepsilon})_{t\in[0,T]}$ on the Wasserstein space.

Remark that studying the convergence in W_2-distance of a sequence of measures is equivalent to studying the convergence in L_2 of the associated quantile functions (see e.g. [vRS09, Prop. 2.1]). We will then focus on proving convergence of $(y_{\sigma,\varepsilon})_{\sigma,\varepsilon\in\mathbb{Q}\cap(0,+\infty)}$ in $L_2([0,1],\mathcal{C}[0,T])$. We prove tightness of that sequence, which lies mainly on the following control of the inverse of the

mass. Let $p > 2$ be such that g belongs to $L_p[0,1]$. Let $\beta \in (0, \frac{3}{2} - \frac{1}{p})$. Then there is $C > 0$ depending on β and g but independent of σ and ε such that for every $t \in [0,T]$,

$$\mathbb{E}\left[\int_0^t \int_0^1 \frac{1}{M_{\sigma,\varepsilon}(u,s)^\beta} \mathrm{d}u\mathrm{d}t\right] \leqslant C\sqrt{t}, \tag{15}$$

where $\frac{1}{M_{\sigma,\varepsilon}} = \frac{m_{\sigma,\varepsilon}}{(\varepsilon + m_{\sigma,\varepsilon})^2}$. Inequality (15) is similar to the inequality given in [Kon17b, Prop 4.3] for the coalescing process.

Let us state the main result of Chapter I.

Theorem 2. *Let $g \in \mathcal{L}_{2+}^{\uparrow}[0,1]$. For each $\sigma > 0$ and for each $\varepsilon > 0$, there is a unique solution $y_{\sigma,\varepsilon}$ to equation (14) in $L_2([0,1], \mathcal{C}[0,T])$ and almost surely $u \mapsto y_{\sigma,\varepsilon}(u,t)$ is non-decreasing and càdlàg for each time $t \in [0,T]$.*

Furthermore, there is a subsequence of $(y_{\sigma,\varepsilon})_{\sigma,\varepsilon \in \mathbb{Q} \cap (0,+\infty)}$ that converges in distribution to a limit y in $L_2([0,1], \mathcal{C}[0,T])$. The limit process y satisfies properties $(i) - (iv)$, belongs to $\mathcal{D}((0,1), \mathcal{C}[0,T])$ and for each time $t \in (0,T]$, $u \mapsto y(u,t)$ is a step function.

Recall that it is equivalent to say that the quantile function $y(\cdot, t)$ is a step function and that the associated measure μ_t is a finite weighted sum of Dirac masses. Thus for every positive time t, the limit process consists of a finite (and random) number of particles. Therefore, the limit system satisfies the same features as Konarovskyi's process, thus by Theorem 1, it satisfies equation (13). In other words, Theorem 2 provides an alternative construction of Konarovskyi's model of coalescing particles: Konarovskyi's construction is based on a particle system, ours is based on an infinite dimensional SDE with a mollified diffusion coefficient. Of course, we cannot state that both constructions lead to the same process, since we have not proved uniqueness of the solution to SDE (13).

Open problems related to Chapter I. Several properties of the smooth approximation of Konarovskyi's model introduced in Chapter I will be studied in Chapters II-IV, in particular regularization results. There remains some properties that we would be interested in studying. Among others, we expect the large deviations in short time of our process to have the same behaviour, up to a possible correction term, as those of the coalescing particle system, the large deviations being given by the Wasserstein metric (see Varadhan formula (3) and Konarovskyi-von Renesse [KvR18]).

Trying to prove (even in a weak sense) uniqueness of a process satisfying conditions $(i) - (iv)$ remains a challenging question. Actually, this is precisely lack of uniqueness that prompted us to introduce a smooth approximation of Konarovskyi's process. Of course, there is no reason why well-posedness of any approximation of $(i) - (iv)$ should guarantee well-posedness of $(i) - (iv)$; note that the particle system used by Konarovskyi himself to construct a solution to $(i) - (iv)$ may be also regarded as a well-posed approximation. Still, it is worth recalling that the Stroock and Varadhan approach to the martingale is precisely based upon a similar picture: weak uniqueness for an SDE with coefficients of lower regularity is checked locally by regarding this latter SDE as the perturbation of a well-posed model, but the machinery therein works at the price of having additional properties on the well-posed model, including regularization properties. We tried to implement similar ideas, but, unfortunately, our attempts to derive uniqueness to $(i) - (iv)$ from the well-posed approximation failed. Certainly, we paid a high price for the fact that the model is in infinite dimension. Let us briefly explain one of our ideas: given two solutions of (13) starting from the same initial condition, let us say $\mu_0 = \mathrm{Leb}_{[0,1]}$, we tried to add a drift term to one of these two solutions (which we call solution #2, the other one being referred to as solution #1) in order to force the coalescences of solution #2 to happen identically (or at least very closely) to the coalescences of solution #1. In other words, we wanted that at a fixed time $t > 0$, both processes have approximatively the same number of particles, and that the particles

of both processes have almost the same position and the same mass. Obviously, the key tool to modify the drift in the dynamics of the solution #2 is Girsanov's transformation. This should not come as a big surprise: as we recalled earlier in the text, Girsanov's transformation indeed plays a key role in the analysis of weak existence and uniqueness to SDEs. Nevertheless, in our own framework, our attempts to apply Girsanov's Theorem failed because we were not able to obtain sufficient integrability for the additional drift. At the end of the day, uniqueness for (13) seems to be a quite difficult problem and we are not sure at that stage that it actually holds. On a different prospect, we will see that Girsanov's Theorem is at the core of the subsequent chapters.

It should be noticed that uniqueness for Konarovskyi's process is known when the initial condition is a finite weighted sum of Dirac masses (see [Kon11]). In the general case, for each time $t > 0$, there remains only a finite number of particles. So the issue to prove uniqueness is to understand what happens between time $t = 0$ and $t = 0^+$. A new approach to tackle this problem could be to reverse the time and to understand the law of the reverse process conditioned to be equal to $\mathrm{Leb}_{[0,1]}$ at final time. We expect this reverse process to consist of some Bessel processes fragmenting at random times uniformly distributed on the remaining time.

Chapter II - Restoration of uniqueness for a Fokker-Planck equation with irregular drift coefficient

In Chapters II-III-IV, we want to prove smoothing properties of some diffusion processes constructed like the smooth approximation of Konarovskyi's diffusion described in Chapter I. In Chapter II, we will focus on a restoration of uniqueness phenomenon. We prove a well-posedness result, in the weak sense, for a certain kind of McKean-Vlasov SDE with irregular drift coefficients. In contrast with the equations studied by Jourdain [Jou97] and more recently by [MV, CdRF, Lac18, RZ], where the noise is finite-dimensional, we use here an infinite-dimensional noise derived from the $\mathcal{P}_2(\mathbb{R})$-valued diffusion constructed in the first chapter. This allows us to obtain restoration of uniqueness for a drift coefficient b that has strictly less regularity in the measure variable than Lipschitz-continuity with respect to distance in total variation, which is the typical assumptions used in the above-mentioned papers; the connection between our own results and the latter ones is made at the end of Chapter II.

Let us consider the following McKean-Vlasov equation:

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t; \\ \quad \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}; \\ \quad y_0 = g. \end{cases} \tag{16}$$

The unknown is the quantile function process $(y_t)_{t\in[0,T]}$ or equivalently the measure-valued process $(\mu_t)_{t\in[0,T]}$, which satisfies the following non-linear Fokker-Planck equation, the unknown taking values in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$:

$$\partial_t \mu_t + \mathrm{div}(\mu_t\, b(\cdot, \mu_t)) = 0. \tag{17}$$

As we already accounted for, a standard fixed-point method shows that both equations are well-posed if $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is Lipschitz-continuous (the proof is completely similar to [Szn91, Thm 1.1]). Recall that by Peano's counter-example, uniqueness for (16) does not necessarily hold for too irregular functions b. Inspired by our preliminary analysis in Chapter I, we will add a stochastic diffusive process to equation (16) in order to restore uniqueness. To make it clear, let us introduce the following equation defining the diffusion:

$$y_t(u) = g(u) + \int_0^t \frac{1}{\left(\int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v\right)^{1/2}} \int_{\mathbb{R}} f(k)\Re(e^{-iky_s(u)}\mathrm{d}w(k,s)), \tag{18}$$

for $u \in [0,1]$, where w is a complex-valued Brownian sheet and $\Re$ denotes the real part of a complex number, namely $\Re(e^{-iky_s(u)}\mathrm{d}w(k,s)) = \cos(ky_s(u))\mathrm{d}w^{\Re}(k,s) + \sin(ky_s(u))\mathrm{d}w^{\Im}(k,s)$, or equivalently $w = w^{\Re} + iw^{\Im}$, where $w^{\Re}$ and $w^{\Im}$ are two independent (real-valued) Brownian sheets and $i = \sqrt{-1}$. That equation is slightly different from the smooth approximation of the coalescing particle system introduced in Chapter I. Let us describe the different coefficients appearing in (18), explaining why we made some modifications with respect to diffusion (14) of Chapter I and comparing those two diffusions in order to show that they share a lot of common properties.

We refer to Section II.1 for a detailed introduction to the different functions appearing in (18). Let us just mention that $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently integrable function, a typical instance of which is $f_\alpha(k) = \frac{1}{(1+k^2)^{\alpha/2}}$ for α large enough; φ is a bounded smooth function with values in $(0, +\infty)$, as the Gaussian density $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ or the constant function $\varphi \equiv 1$. The initial condition g is a strictly increasing $\mathcal{C}^1$-function. If $\alpha > \frac{3}{2}$, we prove that equation (18) has a unique strong solution. Furthermore, the order of regularity of the solution y depends on the regularity of the initial condition g and on the integrability of f: if g is a $\mathcal{C}^j$-function with a derivative of order j that is θ-Hölder continuous for a certain $\theta > 0$, and if $\alpha > j + \frac{1}{2}$, then almost surely, for every time t, $u \mapsto y_t(u)$ is a $\mathcal{C}^j$-function. The solution to equation (18) is a close relative to the smooth approximation of Konarovskyi's model introduced in (14). It shares in particular two important properties: on the one hand, almost surely for every time t, $u \mapsto y_t(u)$ is strictly increasing and, on the other hand, $(y_t(u))_{t\in[0,T]}$ is a martingale with quadratic variation proportional to the inverse of the mass m, defined by $m_s(u) = \int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v$. In words, the role of φ is here to tune the local variance of the particle. This is similar to the model presented in Chapter I except for the fact that the function φ therein also drives the interaction kernel between the particles; *i.e.* φ also appears in the numerator of the integrand of the martingale part of y_t in (14). In order to make the comparison more precise, we may compute the local covariation field of the martingale component in (18), namely, for any two $u, u' \in [0,1]$,

$$\begin{aligned}\mathrm{d}\big\langle y_\cdot(u), y_\cdot(u')\big\rangle_t &= \frac{1}{m_t(u)^{1/2}m_t(u')^{1/2}}\int_{\mathbb{R}} f^2(k)\cos\big(k(y_t(u) - y_t(u'))\big)\mathrm{d}k\,\mathrm{d}t \\ &= \frac{\Re\big(\widehat{f^2}\big)\big(y_t(u) - y_t(u')\big)}{m_t(u)^{1/2}m_t(u')^{1/2}}\,\mathrm{d}t, \end{aligned} \tag{19}$$

where $\widehat{f^2}$ stands for the Fourier transform of f^2. For sure, we recover $\mathrm{d}\big\langle y_\cdot(u), y_\cdot(u)\big\rangle_t = \frac{1}{m_t(u)}\mathrm{d}t$ by choosing $u = u'$. Interestingly enough, formula (19) may be compared with (iv) in the definition of Konarovskyi's process on page 21. For instance, whenever (say to make it simple) $f(k) = f_1(k) = \frac{1}{(1+k^2)^{1/2}}$, $\widehat{f^2}(x)$ behaves like $\exp(-|x|)$, which shows that the range of interaction in (18) is infinite but decays exponentially fast. By computing the Fourier transform with a residue formula, the latter may be shown to remain true whenever $f(k) = f_n(k)$ for any integer $n \geqslant 1$, which proves that this new model shares some of the features of (14) but has a longer interaction range. Anyway, we must admit that, in the rest of the text, we do not spend more effort in tuning φ and f accordingly to make the local covariation of (18) be a good approximation of (iv) on page 21. Most of the reason is that the computations we perform below on the smoothing effect of (18) do not seem subtle enough to capture finely the joint effect of φ and f. Instead we focus now on the reasons why we decided to switch from (14) to (18).

Indeed, following our program, let us consider the McKean-Vlasov equation (16) perturbed by the diffusion introduced in (18):

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t + \dfrac{1}{\big(\int_0^1 \varphi(y_t(u) - y_t(v))\mathrm{d}v\big)^{1/2}}\displaystyle\int_{\mathbb{R}} f(k)\Re(e^{-iky_t(u)}\mathrm{d}w(k,t)); \\ \quad \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}, \end{cases} \tag{20}$$

with initial condition $y_0 = g$. The process $(\mu_t)_{t\in[0,T]}$ satisfies the following SPDE:

$$\mathrm{d}\mu_t = \frac{C_f}{2}\,\partial^2_{xx}\left(\frac{\mu_t}{\varphi * \mu_t}\right)\mathrm{d}t - \partial_x\Big(b(\cdot,\mu_t)\;\mu_t\Big)\mathrm{d}t - \partial_x\left(\frac{\mu_t}{(\varphi*\mu_t)^{1/2}}\int_{\mathbb{R}} f(k)\Re\left(e^{-ik\,\cdot}\,\mathrm{d}w(k,t)\right)\right) \tag{21}$$

where $C_f = \int_{\mathbb{R}} f(k)^2\mathrm{d}k$ and $\varphi * \mu_t(x) = \int_{\mathbb{R}} \varphi(x-y)\mathrm{d}\mu_t(y)$. In the same vein as before, let us remark that if φ is close to a Dirac mass at zero, then $\frac{\mu_t}{(\varphi*\mu_t)^{1/2}}$ is close to $\sqrt{\mu_t}$. This should be compared with the SPDE obtained by Konarovskyi and von Renesse for their model [KvR18] without drift b:

$$\mathrm{d}\mu_t = \Gamma(\mu_t)\mathrm{d}t + \mathrm{div}(\sqrt{\mu_t}dW_t),$$

where Γ is defined as follows:

$$\langle f, \Gamma(\nu)\rangle := \frac{1}{2}\sum_{x\in\mathrm{Supp}(\nu)} f''(x).$$

The main result of this chapter is the following theorem stating restoration of uniqueness for equation (20). The assumptions on the velocity field b are simplified, we refer to Paragraph II.3.1 for a complete discussion on the class of admissible drift functions. We denote by $\partial_1^{(j)}b$, $j = 1, 2$, the first two derivatives of $x \mapsto b(x,\mu)$ at fixed μ.

Theorem 3. *Let g be a strictly increasing $\mathcal{C}^1$-function. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(k) := \frac{1}{(1+k^2)^{\alpha/2}}$, with $\alpha > \frac{3}{2}$. Let $b : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a bounded measurable function such that for each $\mu \in \mathcal{P}_2(\mathbb{R})$, $x \mapsto b(x,\mu)$ is twice continuously differentiable and $\partial_1^{(j)}b$, for $j = 1, 2$, are uniformly bounded on $\mathbb{R}\times\mathcal{P}_2(\mathbb{R})$. Then there is a unique weak solution to equation* (20).

The proof of Theorem 3 relies on Girsanov's Theorem. The main issue is to write the drift term b as a perturbation of the noise. To achieve this goal, we have to inverse the diffusion coefficient; more precisely, we will resolve the following equation: find a complex-valued process $(h_t)_{t\in[0,T]}$ satisfying

$$b(x,\mu_t) = \frac{1}{\left(\int_0^1 \varphi(x - y_t(v))\mathrm{d}v\right)^{1/2}}\int_{\mathbb{R}} f(k)\Re(e^{-ikx}h_t(k))\mathrm{d}k.$$

Thanks to the fact (and this is our rationale for it) that we chose an interaction kernel in the diffusion term of (18) in a Fourier-like shape, h can be defined as a Fourier transform. To apply Girsanov's Theorem, h should belong to $L_2(\mathbb{R};\mathbb{C})$; that is why we have to assume some regularity of b with respect to the variable x in Theorem 3. Remark that the higher α is, the more difficult it is to inverse the kernel. It highlights a balance between the regularity of the process y and the integrability of the Fourier inverse of h. We refer to the introduction of Chapter II for a more detailed discussion on the method of proof and on the reasons why we chose to modify the shape of the diffusive term.

Theorem 3 gives a result of restoration of uniqueness for a McKean-Vlasov equation with an infinite-dimensional noise and with a drift coefficient that is only bounded and measurable with respect to the measure variable. This makes a clear difference with the results of [Jou97, MV, CdRF, Lac18, RZ] in which b is assumed to be Lipschitz-continuous in total variation with respect to the measure variable. However, in all these references, b is just assumed to be bounded and measurable in the space variable x, whereas we need it to be of class $\mathcal{C}^2$ in x. It seems that, with our approach, a minimal regularity is needed to ensure a restoration of the uniqueness property. To go beyond and to reach the results obtained in [Jou97, MV, CdRF, Lac18, RZ],

we prove in Section II.5 a new result of restoration of uniqueness for a continuum of admissible drift functions b that somehow interpolates the assumptions of [Jou97, MV, CdRF, Lac18, RZ] and ours in Theorem 3. Importantly, we succeed to do so at the price of relaxing in a dramatic manner the structure of the noise in hand and of the related notion of solution. In particular, as we explain below, the diffusion used to obtain the latter interpolation result does not fit the main features of the other models addressed in the thesis. Among others, we lose here the underlying property of monotonicity, meaning that the solution can no longer be seen as the quantile function of the associated measure-valued process. Nevertheless, we feel that this interpolation argument is important to make the connection between the regularity assumptions of Theorem 3 and those used in the pre-existing literature. Also, it has also some common features with the diffusion studied in Chapters III and IV (although the latter enjoys the property of monotonicity): in particular, for both models, we introduce in addition to the common noise w a new idiosyncratic noise denoted by β.

Here is our new model. Let $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ be an initial condition and ξ a random variable with law μ_0. Let $(\beta_t)_{t\in[0,T]}$ be a Brownian motion independent of (w, ξ). Let us consider the following McKean-Vlasov SDE with constant mass:

$$\begin{cases} \mathrm{d}z_t = b(z_t, \mu_t)\mathrm{d}t + \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t}dw(k,t)) + \mathrm{d}\beta_t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^{\mu,W}), \quad (\mu, w) \perp\!\!\!\perp (\beta, \xi) \\ z_0 = \xi, \quad \mathcal{L}^{\mathbb{P}}(\xi) = \mu_0, \end{cases} \tag{22}$$

where $(\mathcal{G}_t^{\mu,W})_{t\in[0,T]}$ is the filtration generated by the Brownian sheet w and by the measure-valued process $(\mu_t)_{t\in[0,T]}$ itself. The Brownian motion β is seen here as an idiosyncratic source of randomness and μ_t can be seen as the law of z_t with respect to the randomness carrying both the initial condition and the idiosyncratic noise. The addition of the new source of randomness β is easily understood: similar to the Brownian motion in standard SDEs, it allows to mollify the drift in the space variable x. As for the conditioning in the identity $\mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^{\mu,W})$, it must be compared with our presentation of McKean-Vlasov equations with a common noise, see (10). The main difference between both is that the conditioning now involves μ itself: this comes from the fact we will allow for weak solutions, namely for solutions for which μ may not be adapted with respect to the common noise w. In fact, the latter causes some technical difficulties in the proof. In particular, it requires to work with solutions that satisfy an additional compatibility condition, in the sense that the observation of z cannot bias the future realizations of μ, w and β. We feel better to postpone to Section II.5 the discussion on the precise role of this compatibility condition.

Here is our result. Let $\eta > 0$ and $\delta \in [0, 1]$. Let us consider the drift function $b : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ in the class $(H^\eta, \mathcal{C}^\delta)$. The definition of that class of admissible drift functions is given in Paragraph II.5.1, but roughly speaking, it contains functions b such that for every fixed μ, $x \mapsto b(x, \mu)$ belongs to the Sobolev space $H^\eta(\mathbb{R})$ with a Sobolev norm uniform in μ, and for every fixed x, $\mu \mapsto b(x, \mu)$ is δ-Hölder continuous in total variation distance, the Hölder norm being uniform in x. Then, we have the following statement:

Theorem 4. *Let $\eta > 0$ and $\delta \in [0, 1]$ be such that $\eta > \frac{3}{2}(1-\delta)$ and let b be of class $(H^\eta, \mathcal{C}^\delta)$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(k) := \frac{1}{(1+k^2)^{\alpha/2}}$, with $\frac{3}{2} < \alpha \leqslant \frac{\eta}{1-\delta}$. Then existence and uniqueness of a weak compatible solution to equation (22) hold.*

The condition $\eta > \frac{3}{2}(1 - \delta)$ quantifies the minimal regularity that is needed, with our approach, to restore uniqueness. If b is Lipschitz-continuous in total variation distance with respect to μ ($\delta = 1$), then almost no regularity of b in x is needed ($\eta > 0$): it is close to the assumptions of [Jou97, MV, CdRF, Lac18, RZ]. If b is only uniformly bounded in μ ($\delta = 0$), then $x \mapsto b(x, \mu)$ should belong to $H^\eta(\mathbb{R})$ for some $\eta > \frac{3}{2}$, which is slightly stronger than the assumption made in Theorem 3.

Open problems related to Chapter II. In the proof of Theorem 4, there are two different applications of Girsanov's Theorem: the first one with respect to the noise w in order to deal with the lack of regularity of the drift b in the variable μ; the second one with respect to the noise β in order to deal with the lack of regularity of the drift b in the variable x. That second application of a Girsanov transformation destroys the monotonicity property of the diffusion. Obtaining a regularization result for a drift of class $(H^\eta, \mathcal{C}^\delta)$ with a diffusion of the same type as Konarovskyi's model is maybe possible if we find an alternative way to add an idiosyncratic source of randomness that does not break the monotonous structure of the diffusion.

Another question lies in the importance of the mass $m_t(u) = \int_0^1 \varphi(y_t(u) - y_t(v))\mathrm{d}v$ in equation (20). In Konarovskyi's model, the normalization by the mass allows to reduce the fluctuations of *heavy* particles and to pass to the limit when N tends to infinity. In the argument of restoration of uniqueness presented in Chapter II, the mass has a minor role, since the proofs work in the case where $\varphi \equiv 1$, that is in the case where the mass is constant. However, it could be that the presence of mass helps to improve our results. Indeed, the mass can be rewritten as $m_t(u) = \int_{\mathbb{R}} \varphi(y_t(u) - x)p_t(x)\mathrm{d}x = (\varphi * p_t)(y_t(u))$, where p_t is the density of the law of $v \mapsto y_t(v)$. In the case of Konarovskyi's coalescing system, φ is replaced by a Dirac mass in zero, thus $\frac{1}{m_t(u)} = \frac{1}{p_t(y_t(u))} = \partial_u y_t(u)$. The presence of a derivative in the equation might bring some additional regularity, but so far, we do not know how to deal with the convolution by a smooth function φ.

Moreover, one of our original motivations for looking for regularization results of smooth approximation of Konarovskyi's model was to tackle the problem of uniqueness of equation (13) through a new point of view. So far, uniqueness remains unreachable, but we hope that a better understanding of the smoothing properties of diffusions on the Wasserstein space will lead to some progress in the problem of uniqueness. A better identification of the role of the mass in the regularizing effect would surely be a first step in that direction.

Chapter III - Bismut-Elworthy inequality for a Wasserstein diffusion on the torus

Following our original objective, we study in Chapter III further regularization properties of the diffusion introduced in Chapters I and II, or more precisely of a new variant of it, all the modifications that we introduce being justified by our desire to obtain other forms of smoothing effect. Precisely, we give a gradient estimate, obtained in the form of a Bismut-Elworthy-like formula, for the semi-group generated by a $\mathcal{P}_2(\mathbb{R})$-valued stochastic process that should be regarded as another smooth approximation of the original Wasserstein diffusion introduced in Chapter I. In comparison with the SDE (18) introduced in Chapter II, the process addressed in this chapter enjoys two modifications. First, we work in Chapter III on the one-dimensional torus instead of the real line and second, we add to the equation an idiosyncratic source of randomness β very much in the spirit of (22). Despite these changes, this diffusion $(\mu_t)_{t\in[0,T]}$ shares with the previously constructed processes a similar representation, in the sense that the representation we provide in the form of a particle system preserves the respective order of the particles, here on the torus.

Let us introduce the stochastic differential equation studied in Chapter III: for every $u \in \mathbb{R}$ and for every $t \in [0, T]$,

$$x_t^g(u) = g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \cos(kx_s^g(u))\mathrm{d}W_s^{\Re,k} + \sum_{k\in\mathbb{Z}} f_k \int_0^t \sin(kx_s^g(u))\mathrm{d}W_s^{\Im,k} + \beta_t,$$

which can be rewritten as follows,

$$x_t^g(u) = g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re(e^{-ikx_s^g(u)}\mathrm{d}W_s^k) + \beta_t, \tag{23}$$

where for every $k \in \mathbb{Z}$, $W^k := W^{\Re,k} + iW^{\Im,k}$. Up to the introduction of the additional noise β, equation (23) is exactly the counterpart on the torus of equation (18) on $\mathbb{R}$, when the mass function is constant. In that equation, $((W^{\Re,k})_{k\in\mathbb{Z}}, (W^{\Im,k})_{k\in\mathbb{Z}}, \beta)$ is a collection of independent standard Brownian motions on $\mathbb{R}$; more precisely, $(W^k)_{k\in\mathbb{Z}}$ is defined on a probability space $(\Omega^W, \mathcal{G}^W, \mathbb{P}^W)$, β is defined on a probability space $(\Omega^\beta, \mathcal{G}^\beta, \mathbb{P}^\beta)$ and equation (23) is defined on the product of these two spaces. The sequence $(f_k)_{k\in\mathbb{Z}}$ is typically of the form $f_k = \frac{C_\alpha}{(1+k^2)^{\alpha/2}}$, with C_α chosen so that $\sum_{k\in\mathbb{Z}} f_k^2 = 1$. The initial function $g : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$ and satisfies, for every $u \in \mathbb{R}$, $g'(u) > 0$ and the pseudo-periodic assumption

$$g(u+1) = g(u) + 2\pi, \tag{24}$$

so that its restriction to $[0,1]$ can be seen as a function from $[0,1]$ into the one-dimensional torus $\mathbb{T}$, identified with the interval $[0, 2\pi]$. Moreover, for every time t, x_t^g will be regarded as the quantile function associated with a certain probability measure on the torus $\mathbb{T}$; this notion of quantile function for probability measures on the torus will be explained in Paragraph III.2.1, since the notion of increasing rearrangement is *a priori* unclear on the circle $\mathbb{T}$.

We will prove the following properties for the SDE (23). If $\alpha > \frac{3}{2}$, there exists a unique (strong) solution $(x_t^g)_{t\in[0,T]}$ on $\mathcal{C}(\mathbb{R}\times[0,T])$. Moreover, this solution satisfies for every time t the pseudo-periodic property (24) and is strictly increasing. As for the solution to equation (18), the regularity of the solution depends on the order α of summability of the sequence $(f_k)_{k\in\mathbb{Z}}$. Indeed, if $\alpha > j + \frac{1}{2}$ and if g is j-times differentiable with $\partial_u^{(j)} g$ being θ-Hölder-continuous for some $\theta > 0$, then $u \mapsto x_t^g(u)$ is j-times continuously differentiable. In particular, the fact that for each $u \in [0,1]$, $\partial_u x_t^g(u) > 0$ is equivalent to say that the density of the associated probability measure is strictly positive everywhere on the torus. We take here benefit of the compactness of the torus to consider probability measures with both a compact support and a non-vanishing density.

For all the above-mentioned properties, the addition of the Brownian motion β does not play any substantial role since β just acts on the whole system in the form of a translation independent of u. However, the role of β becomes crucial in the study of the semi-group associated to (23). In the previous chapters, we associated to x_t^g the pull-back measure $\nu_t^g = \mathrm{Leb}_{[0,1]} \circ (x_t^g)^{-1}$. Now, we include an extra layer of averaging in the definition of the measure of interest. Precisely, we consider the law of x_t^g with respect to the randomness that carries both the initial condition and the idiosyncratic noise, hence letting $\mu_t^g = (\mathrm{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ (x_t^g)^{-1}$. In other words, since we assume that β and $(W^k)_{k\in\mathbb{Z}}$ are independent, it is equivalent to say that μ_t^g is the conditional law of x_t^g given $(W^k)_{k\in\mathbb{Z}}$. Furthermore, the process $(\mu_t^g)_{t\in[0,T]}$ satisfies the following SPDE:

$$\begin{cases} \mathrm{d}\mu_t^g - \dfrac{C_f+1}{2}\partial_{xx}^2(\mu_t^g)\mathrm{d}t + \partial_x\left(\displaystyle\sum_{k\in\mathbb{Z}} f_k \Re(e^{-ik\,\cdot}\,\mathrm{d}W_t^k)\ \mu_t^g\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \mu_t^g\Big|_{t=0} = \mu_0^g, \end{cases} \tag{25}$$

which is the equivalent on the torus of the SPDE (21), in the case where the drift b is zero and the mass is constant $\varphi \equiv 1$. In (25), $C_f = \sum_{k\in\mathbb{Z}} f_k^2$, which may be in fact chosen as 1, as explained before. The idiosyncratic noise β manifests in the additional term $\frac{1}{2}$ in front of $\partial_{xx}^2(\mu_t^g)$; as we will see hereafter, this extra $\frac{1}{2}$ is crucial to get a regularization result.

Let us associate to SPDE (25) a semi-group $(P_t)_{t\in[0,T]}$. Let us consider a bounded and uniformly continuous function $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ with respect to the Wasserstein distance, such that $\phi(\mu) = \phi(\nu)$ if $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ have the same trace on the torus, *i.e.* if for every Borel set A of $[0, 2\pi]$, $\mu(A + 2\pi\mathbb{Z}) = \nu(A + 2\pi\mathbb{Z})$. Let us call that last property $\mathbb{T}$-stability of ϕ. We will prove a correspondance between bounded, uniformly continuous and $\mathbb{T}$-stable functions $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ on the one hand, and bounded functions $\phi : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ that are continuous with respect to

the L_2-Wasserstein distance on the torus on the other hand. Let us consider the semi-group $(P_t)_{t\in[0,T]}$ on that class of functions ϕ, defined as follows:

$$P_t\phi(\mu_0^g) := \mathbb{E}^W\left[\phi(\mu_t^g)\right],$$

where $\mathbb{E}^W$ is the expectation associated to the probability measure $\mathbb{P}^W$.

The introduction of the idiosyncratic noise β may seem rather artificial, but actually it is consistent with Theorem 4 and with the standard smoothing effects of finite-dimensional SDEs, see Section 0.2. It is also consistent with earlier results in SPDE theory. Indeed, we can draw a comparison with the results of Denis, Matoussi and Stoica [DS04, DMS05] on nonlinear SPDEs. Recall that $\sum_{k\in\mathbb{Z}} f_k^2 = 1$ and denote by p_t^g the density associated to the quantile function $u \in [0,1] \mapsto x_t^g(u)$, where $(x_t^g)_{t\in[0,T]}$ is solution to equation (23). Then, rewriting (25), $(p_t^g)_{t\in[0,T]}$ satisfies the following SDE:

$$\mathrm{d}p_t^g(v) = -\partial_v\left(p_t(v)\sum_{k\in\mathbb{Z}} f_k\Re(e^{-ikv}\mathrm{d}W_t^k)\right) + \lambda p_t''(v)\mathrm{d}t,$$

with $\lambda = 1$. If we consider now equation (23) *without* β-term, then the corresponding $(p_t^g)_{t\in[0,T]}$ satisfies the same equation, but with $\lambda = \frac{1}{2}$. According to [DS04, DMS05], $\lambda > \frac{1}{2}$ is actually the critical threshold for SPDEs of the above type to get an energy estimate of the solutions.

The two main results of Chapter III describe a regularizing effect of the semi-group $(P_t)_{t\in[0,T]}$. First, let us assume that $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ has some additional regularity: let us consider $(\Omega, \mathcal{F}, \mathbb{P})$ any probability space such that Ω is Polish and $\mathbb{P}$ is atomless; denote by H the Hilbert space $L_2(\Omega, \mathcal{F}, \mathbb{P})$; we say that ϕ satisfies the ϕ-assumptions if ϕ is $\mathbb{T}$-stable and if the map $X \mapsto \phi(\mathcal{L}(X))$ defined on H belongs to $\mathcal{C}_b^{1,1}(H)$, the class of bounded $\mathcal{C}^1$-functions on H with bounded and Lipschitz-continuous derivatives on H. We denote by $\mathcal{C}^{j+\theta}$, for some integer j and $\theta \in [0,1)$ the class of $\mathcal{C}^j$-functions whose derivative of order j is θ-Hölder continuous. We also denote by $\mathbb{V}^W$ the variance associated to the probability measure $\mathbb{P}^W$. The first statement addresses directional derivatives of the semi-group:

Theorem 5. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfy the ϕ-assumptions. Let $\theta \in (0,1)$ and let $f_k = \frac{C_\alpha}{(1+k^2)^{\alpha/2}}$, with $\alpha > \frac{7}{2} + \theta$. Assume that $g : \mathbb{R} \to \mathbb{R}$ satisfies the pseudo-periodic assumption (24), is strictly increasing and is of class $\mathcal{C}^{3+\theta}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a 1-periodic and $\mathcal{C}^1$-function. Then $\rho \mapsto P_t\phi(\mu_0^{g+\rho h})$ is differentiable at $\rho = 0$ and there is $C > 0$ independent of h such that for every $t \in (0,T]$,*

$$\left|\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h})\right| \leqslant C\frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{t^{2+\theta}}\|h\|_{\mathcal{C}^1}.$$

Moreover, C depends polynomially on $\|g'''\|_{L_\infty}$, $\|g''\|_{L_\infty}$, $\|g'\|_{L_\infty}$ and $\|\frac{1}{g'}\|_{L_\infty}$.

The direction of perturbation h should be at least $\mathcal{C}^1$ in order to preserve the class of initial conditions that are strictly increasing: for values of ρ close to zero, $g' + \rho h > 0$. Accordingly, the above result reads as an upper bound for the derivative of the semi-group along certain directions h and at some initial conditions g. Although the latter ones do not cover the entire $\mathcal{P}(\mathbb{T})$, observe anyhow that the admissible initial conditions are dense in $\mathcal{P}(\mathbb{T})$. Also, the fact that the bound just holds in sufficiently smooth directions h is reminiscent of the fact that the process x^g in (23) remains regular in the variable u, see for instance further comments in (26) below.

To state our second result, let us now assume that $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is bounded, uniformly continuous and $\mathbb{T}$-stable. Then, the next theorem provides another point of view on the smoothing

effect: for each time $t > 0$, $\rho \mapsto P_t\phi(\mu_0^{g+\rho h})$ is locally Lipschitz at $\rho = 0$, if we assume that the quantile functions g on the torus is sufficiently smooth and that the direction of perturbation h is also sufficiently smooth. Furthermore, we have an explicit Lipschitz bound.

Theorem 6. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a bounded, uniformly continuous and $\mathbb{T}$-stable function. Let $\theta \in (0,1)$ and let $f_k = \frac{C_\alpha}{(1+k^2)^{\alpha/2}}$, with $\alpha > \frac{7}{2} + \theta$. Assume that $g : \mathbb{R} \to \mathbb{R}$ satisfies the pseudo-periodic assumption (24), is strictly increasing and is of class $\mathcal{C}^{3+\theta}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a 1-periodic and $\mathcal{C}^{3+\theta}$-function. Then there is $\rho_0 > 0$ and $C > 0$, depending on g and h such that for every $\rho \in (-\rho_0, \rho_0)$ and every $t \in (0, T]$,*

$$\left|P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g)\right| \leqslant C|\rho|\frac{\|\phi\|_{L_\infty}}{t^{2+\theta}}.$$

Moreover, C depends polynomially on $\|g'''\|_{L_\infty}$, $\|g''\|_{L_\infty}$, $\|g'\|_{L_\infty}$, $\|\frac{1}{g'}\|_{L_\infty}$ and $\|h\|_{\mathcal{C}^3}$.

The method of proof of both results is based on a Bismut-Elworthy like integration by parts, which in turn relies on Kunita's expansion of processes with time-dependent initial condition, see [Kun90, Chap III, Thm 3.3.1], and on Girsanov's Theorem. As in Chapter II, the main idea consists in inverting the interaction coefficient in front of the noise $(W^k)_{k\in\mathbb{Z}}$; due to the discrete setting of equation (23), the inverse is now based on Fourier series instead of a Fourier transformation. More precisely, we need to find a collection of $\mathbb{C}$-valued adapted processes $((\lambda_t^k)_{t\in[0,T]})_{k\in\mathbb{Z}}$ such that $\sum_{k\in Z}\int_0^T |\lambda_s^k|^2 \mathrm{d}s$ is almost surely finite and such that for every $t \in [0,T]$ and for every $x \in \mathbb{R}$,

$$\partial_u x_t^g(u)h(u) = \sum_{k\in\mathbb{Z}} f_k \Re\Big(e^{-ikx_t^g(u)}\lambda_t^k\Big). \tag{26}$$

We refer to the introduction of Chapter III for a more precise explanation on the derivation of (26). In order to satisfy the square-integrability condition on $(\lambda^k)_{k\in\mathbb{Z}}$, we need some regularity on the left-hand side of (26). As in the previous chapter, the bigger α is, the smoother the solution $(x_t^g)_{t\in[0,T]}$ is, but the more difficult it becomes to get L_2-integrability of $(\lambda^k)_{k\in\mathbb{Z}}$ because of the division by f_k. In Theorem 5, we assume that $h \in \mathcal{C}^1$: it is not smooth enough to obtain the aforementioned integrability. Thus, our strategy is to regularize the left-hand side of (26) by a convolution kernel, to get the desired result for the regularized term and then to deal with the remainder term. The latter will be shown to be small with the regularization parameter: this claim will be checked by using another Girsanov's transformation, that time using the additional noise β.

Lastly, it is worth mentioning that the rate of explosion in the upper bound is $t^{-(2+\theta)}$ in both theorems, which is much worse than the usual rate of the heat semi-group, which is $t^{-1/2}$. Obviously, it is not integrable, which may be a serious drawback for practical applications. Still, it should not come as a surprise: as alluded to in the discussion of equation (26) above, our model is not elliptic; in this regard, the non-diffusive rate obtained in Theorem 5 is reminiscent of the rates that may be observed in finite-dimensional hypoelliptic SDEs, see for instance [KS84, KS85, KS87]. Notice that the bound will be improved in Chapter IV under stronger conditions on g and h.

Chapter IV - Gradient estimate for an inhomogeneous SPDE

Chapter IV is in fact the continuation of Chapter III and consists in improving the explosion rate obtained in Theorem 5, when g and h are more regular. Let us state the main result of Chapter IV.

Theorem 7. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfy the ϕ-assumptions. Let $\theta \in (0, \frac{1}{2})$ and assume that f is of the form $f(k) = \frac{1}{(1+k^2)^{\alpha/2}}$, with $\alpha = \frac{7}{2} + \theta$. Assume that $g : \mathbb{R} \to \mathbb{R}$ satisfies the*

pseudo-periodic assumption (24)*, is strictly increasing and is of class* $\mathcal{C}^{4+\theta}$*. Let* $h : \mathbb{R} \to \mathbb{R}$ *be a* 1*-periodic and* $\mathcal{C}^{4+\theta}$*-function. Then there is* $C > 0$ *depending on* g *and independent of* h*,* ϕ*,* θ *and* t *such that for every* $t \in (0, T]$*,*

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) \right| \leqslant C \frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{t^{1+3\theta}} \|h\|_{\mathcal{C}^{4+\theta}}. \tag{27}$$

Assuming more regularity on h helps to increase the regularity of the left-hand side of (26). Nevertheless, we will show that if $\alpha = \frac{7}{2}+\theta$, the left-hand side of (26) has to be of $\mathcal{C}^{4+\theta}$-regularity in order to achieve the inversion and to prove that $(\lambda^k)_{k\in\mathbb{Z}}$ is square-integrable. But at the same time, if $\alpha = \frac{7}{2} + \theta$, the map $u \mapsto \partial_u x_t^g(u)$ is only of order $\mathcal{C}^{2+\theta'}$, for some $\theta' < \theta$. Therefore, we will need a new interpolation method in order to regularize the derivative. Using the following explicit form of the derivative:

$$\partial_u x_t^g(u) = g'(u) \exp\Big(\sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(-ike^{-ikx_s^g(u)} \mathrm{d}W_s^k\right) - \frac{t}{2} \sum_{k\in\mathbb{Z}} f_k^2 k^2 \Big), \tag{28}$$

we will replace it by the following function:

$$u \mapsto g'(u) \exp\Big(\sum_{|k|\leqslant N_t} f_k \int_0^t \Re\left(-ike^{-ikg(u)} \mathrm{d}W_s^k\right) - \frac{t}{2} \sum_{k\in\mathbb{Z}} f_k^2 k^2 \Big), \tag{29}$$

where N_t is some fixed integer depending only on t. Notice that we also replaced x_s^g by g in the exponential. We will prove that the function (29) is of class $\mathcal{C}^{4+\theta}$, which will allow us to apply the inversion (26). We will then have to deal with the difference between (28) and (29) and interpolate the two terms.

Theorem 7 is interesting since it improves our regularization result for the semi-group $(P_t)_{t\in[0,T]}$ associated to the diffusion $(\mu_t)_{t\in[0,T]}$ that solves (25). Moreover, we easily deduce from that result a gradient estimate for an inhomogeneous SPDE of the form

$$\begin{cases} \mathrm{d}\mu_t^g - \dfrac{C_f+1}{2}\partial_{xx}^2(\mu_t^g)\mathrm{d}t + \partial_x\Big(\displaystyle\sum_{k\in\mathbb{Z}} f_k \Re(e^{-ik\,\cdot}\,\mathrm{d}W_t^k)\ \mu_t^g \Big) = F(t, \mu_t^g), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu_t^g\Big|_{t=0} = \mu_0^g. \end{cases} \tag{30}$$

Let $(Q_t)_{t\in[0,T]}$ denote the semi-group associated with equation (30). We consider the following regularity-assumption on the source term $F : [0, T] \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$: there is $C > 0$ such that for every $t \in [0, T]$ and for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$,

$$|F(t, \mu) - F(t, \nu)| \leqslant C W_2(\mu, \nu)^\eta. \tag{31}$$

for some $\eta > 0$.

We will prove the following gradient estimate:

Theorem 8. *Let* $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ *satisfy the* ϕ*-assumptions. Let* $\theta \in (0, \frac{1}{8})$ *and assume that* f *is of the form* $f(k) = \frac{1}{(1+k^2)^{\alpha/2}}$*, with* $\alpha = \frac{7}{2} + \theta$*. Assume that* $g : \mathbb{R} \to \mathbb{R}$ *satisfies the pseudo-periodic assumption* (24)*, is strictly increasing and is of class* $\mathcal{C}^{4+\theta}$*. Let* $h : \mathbb{R} \to \mathbb{R}$ *be a* 1*-periodic and* $\mathcal{C}^{4+\theta}$*-function. Assume that* F *satisfies* (31) *with* $\eta = 8\theta$*. Then there is* $C > 0$ *depending on* g*,* ϕ *and* F *and independent of* h*,* θ *and* t *such that for every* $t \in (0, T]$*,*

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} Q_t\phi(\mu_0^{g+\rho h}) \right| \leqslant \frac{C}{t^{1+3\theta}} \|h\|_{\mathcal{C}^{4+\theta}}.$$

The regularity assumption on F is connected with the fact that we need the function $s \mapsto \frac{1}{(t-s)^{1+3\theta}} \mathbb{V}^W \left[F(s, \mu^g_{t-s})\right]^{1/2}$ to be integrable on $[0,t]$ for every $t \in [0,T]$. Thus, the Hölder-regularity of F compensates the fact that $t \mapsto t^{-(1+3\theta)}$ is not integrable on $[0,T]$.

Open problems related to Chapters III and IV. In Chapters III and IV, we give results for a stochastic diffusive process with a constant mass. A natural question is to determine the influence of a non-constant mass on the model. The *massive* version of the model would write

$$x^g_t(u) = g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \frac{1}{m^g_s(u)^{1/2}} \Re(e^{-ikx^g_s(u)} \mathrm{d}W^k_s) + \beta_t,$$

with $m^g_s(u) = \int_{\mathbb{R}} \varphi(x^g_s(u) - y)\mu_t(\mathrm{d}y) = \mathbb{E}^\beta \left[\int_0^1 \varphi(x^g_s(u) - x^g_s(v))\mathrm{d}v\right]$, where $\varphi : \mathbb{R} \to (0, +\infty)$ is a 2π-periodic smooth function, taking its maximal value at points of $2\pi\mathbb{Z}$ and its minimal value at points of $\pi + 2\pi\mathbb{Z}$. Nevertheless, like in Chapter II, our preliminary intuition is that the mass does not seem to have any effect on the result. Hence, this question remains open.

As demonstrated in [CD18a, Chapter 5], regularity properties of the semi-group in the style of Theorems 5 and 7 may be helpful to obtain propagation of chaos with rates for finer topologies (or equivalently for functionals of the empirical measure of lower regularity). Here, we are particularly interested in obtaining a convergence rate for the system of particles associated with equation (23). The main difficulty arising in that direction is the fact that we only obtained a gradient estimate at smooth points g, whereas we would need locally Lipschitz bounds at points g associated to empirical measures; these functions g are step functions and of course, they are neither strictly increasing nor continuous.

We are also interested in improving the rate obtained in Theorem 7, since it remains far from the Gaussian case, where the rate is of order $t^{-1/2}$. Obtaining a rate of order $t^{-(1-\theta)}$ would allow to extend the application given in Theorem 8 to source terms F which are only bounded and continuous. Furthermore, we also need a better rate if we want to extend the results obtained in Chapters III and IV to the case of a drifted diffusion, as we studied in Chapter II. Let us introduce the following SDE:

$$y^g_t(u) = g(u) + \int_0^t b(y^g_s(u), \nu^g_s)\mathrm{d}s + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re(e^{-iky^g_s(u)} \mathrm{d}W^k_s) + \beta_t, \tag{32}$$

where $\nu^g_s = (\mathrm{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ (y^g_s)^{-1}$ and where the drift function $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is of lower regularity. Let us explain a possible strategy to tackle this problem. As we have already done in Chapter II, we first apply Girsanov's Theorem to compare the solution $(x^g_t)_{t\in[0,T]}$ of the equation with zero drift to the solution $(y^g_t)_{t\in[0,T]}$. Then we prove the existence of a semi-group $(R_t)_{t\in[0,T]}$ associated to (32); probably, we need here to replace b by a smooth approximation b^n in order to ensure that equation (32) is strongly solvable. Using the Girsanov transformation, we expect to be able to show an energy estimate, in other words to prove that the gradient of $(R_t)_{t\in[0,T]}$ applied to the process $(x^g_t)_{t\in[0,T]}$ (with $b = 0$) is in $L_2(\Omega \times [0,T])$. This energy estimate would then be used in a Duhamel formula to deduce a Bismut-Elworthy-like inequality for $(R_t)_{t\in[0,T]}$. Nevertheless, that strategy seems unreachable with a rate of order $t^{-(1+\theta)}$ in inequality (27). Indeed, we need a rate which is square-integrable on $[0,T]$. With a rate of order to $t^{-(1/2+\theta)}$, it would be possible to get a result, since we could apply the following improvement of the aforementioned strategy: we start by extending Theorem 7 to the case of a drift $b(g_0(u), \mu^{g_0}_0)$, where g_0 is a fixed frozen function, so that the drift is constant. Then, if we assume in addition some Hölder-continuity assumption on b, we can take benefit from the fact that for s close to zero, $b(x^{g_0}_s(u), \mu^{g_0}_s) - b(g_0(u), \mu^{g_0}_0)$ is small with s: this could help in Duhamel formula to compensate for the fact that $s^{-(1/2+\theta)}$ is not square-integrable. Nevertheless, there remains an important gap in this strategy, since we are not able for now to improve the rate of Theorem 7 to $t^{(-1/2+\theta)}$.

Other extensions concern the d-dimensional case, although it is not entirely clear what we would mean in such a case by a canonical representation of an SPDE of the type (25) in the form of an infinite particle system. Anyway, depending on the possible applications of the results obtained in these two chapters, having gradient estimates for the semi-group would be certainly interesting. In this respect, another route would be to address the SPDE itself directly. Notice that this would be complementary to what we have exposed in this introduction, as all the subsequent proofs are purely probabilistic.

0.4 Lions' differential calculus on the Wasserstein space

This is a toolbox section recalling important definitions and properties of the differential calculus introduced by Lions on the L_2-Wasserstein space $(\mathcal{P}_2(\mathbb{R}), W_2)$ at his lectures at Collège de France [Lio]. Notes on those lectures can be found on Cardaliaguet's webpage [Car13]. This approach to a differential calculus has been developed by Lions since it is particularly appropriate in the study of mean field games and more generally of any particle system with mean field interaction. Indeed the point is that Lions' derivative at $\mu \in \mathcal{P}_2(\mathbb{R})$ of a function $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ has a nice representation if μ is an empirical measure, *i.e.* if μ is of the form

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \tag{33}$$

where δ_{x_i} is the Dirac measure at point $x_i \in \mathbb{R}$.

Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a continuous function (with respect to W_2-distance). In order to write Itô's formula for processes on $\mathcal{P}_2(\mathbb{R})$ or in order to study regularity properties of associated semi-groups, we have to differentiate ϕ. There is no a single way to do this. We will describe in this paragraph two notions of differentiability of ϕ. In fact, there are other notions, see e.g. [CD18a, Chapter 5], but, at the end, we know how to connect all of them!

Remark. The results shortly presented below are taken from Cardaliaguet's notes [Car13] and from Carmona and Delarue's book [CD18a, Chapter 5]. We refer to those works for the proofs of the propositions stated below and for a complete overview of differential calculus on $\mathcal{P}_2(\mathbb{R})$.

L-derivative (or Lions-derivative)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In order to insure that the probability space is rich enough, we assume Ω to be a Polish space and $\mathbb{P}$ to be atomless (or equivalently since Ω is Polish, assume that every singleton has zero measure). Let $L_2(\Omega) := L_2(\Omega, \mathcal{F}, \mathbb{P})$ be the space of square-integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the L_2-Wasserstein distance, defined by (1), is also given by

$$W_2(\mu, \nu) = \inf \mathbb{E}\left[|X - Y|^2\right]^{1/2},$$

where the infimum is taken over every couple (X, Y) of elements of $L_2(\Omega)$ such that X has distribution μ and Y has distribution ν. Let us define the lifted function $\widetilde{\phi} : L_2(\Omega) \to \mathbb{R}$ by

$$\widetilde{\phi}(X) := \phi(\mathcal{L}(X)),$$

where $\mathcal{L}(X)$ denotes the law of X. On $L_2(\Omega)$, we will take advantage of the differential calculus given by the Hilbert structure of that space.

Definition. A function $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is said to be L-differentiable at $\mu \in \mathcal{P}_2(\mathbb{R})$ if for every $X \in L_2(\Omega)$ with law μ, the lifted function $\widetilde{\phi} : L_2(\Omega) \to \mathbb{R}$ is Fréchet differentiable at X.

If the derivative exists, an important property is that the law of $(X, D\widetilde{\phi}(X))$ does not depend on the random variable X as long as $\mathcal{L}(X) = \mu$. Let us say that ϕ is $\mathcal{C}^1$ on $\mathcal{P}_2(\mathbb{R})$ if the Fréchet derivative $D\widetilde{\phi} : L_2(\Omega) \to L_2(\Omega)$ is continuous.

Proposition-Definition. Let ϕ be a $\mathcal{C}^1$ function on $\mathcal{P}_2(\mathbb{R})$. Then for any $\mu \in \mathcal{P}_2(\mathbb{R})$, there exists a measurable function $\xi_\mu : \mathbb{R} \to \mathbb{R}$ such that for every $X \in L_2(\Omega)$ with law μ, we have $D\widetilde{\phi}(X) = \xi_\mu(X)$ almost surely, or more precisely for almost every $\omega \in \Omega$, $[D\widetilde{\phi}(X)](\omega) = \xi_\mu(X(\omega))$. The map ξ_μ will be denoted by $\partial_\mu\phi(\mu)$ and be called L-derivative (or Lions-derivative) of ϕ at μ.

Remark that this proposition implies that the L-derivative of ϕ at μ is independent of the choice of the space $(\Omega, \mathcal{F}, \mathbb{P})$ and of the random variable X chosen to express μ. Therefore, the L-derivative of ϕ has a meaning independently of the chosen lifted function.

Let us give some examples:

Example 1. Let us consider

$$\phi(\mu) = \int_{\mathbb{R}} h(x)\mathrm{d}\mu(x),$$

with $h \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$, h' being at most of linear growth. Then for every $\mu \in \mathcal{P}_2(\mathbb{R})$, the L-derivative $\partial_\mu\phi(\mu)$ is equal to h'. In particular, the L-derivative of ϕ is constant in this case.

Example 2. Let us consider

$$\phi(\mu) = \int_{\mathbb{R}} (h * \mu)(x)\mathrm{d}\mu(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} h(x-y)\mathrm{d}\mu(y)\mathrm{d}\mu(x),$$

where $h \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$, h' being at most of linear growth. Then for every $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$\partial_\mu\phi(\mu) = (h' + \overline{h'}) * \mu,$$

where $\overline{h'}(x) = -h'(-x)$.

Example 3. Let us consider

$$\phi(\mu) = \int_{\mathbb{R}} \psi(x,\mu)\mathrm{d}\mu(x),$$

where $\psi : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is continuous, differentiable in $x \in \mathbb{R}$ for every fixed μ, with a bounded derivative jointly continuous in (x,μ) and L-differentiable in μ for every fixed x, such that $(x, x', \mu) \mapsto \partial_\mu\psi(x,\mu)(x')$ is measurable, bounded and continuous. Then

$$\partial_\mu\phi(\mu)(\cdot) = \partial_x\psi(\cdot,\mu) + \int_{\mathbb{R}} \partial_\mu\psi(x',\mu)(\cdot)\mathrm{d}\mu(x').$$

We refer to [CD18a, Paragraph 5.2.2] for a list of further examples and for the proofs of the derivatives computed above.

As we already mentioned, the L-derivative is particularly well suited in the case of functions of measures of the form (33). More precisely, given $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, let us define $\phi^N : \mathbb{R}^N \to \mathbb{R}$ by

$$\phi^N(x_1, \dots, x_N) \mapsto \phi\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}\right).$$

Let us state the following relation between the partial derivatives of ϕ^N (for the classical differential calculus on $\mathbb{R}^N$) and the L-derivative of ϕ:

Proposition. *Let ϕ be $\mathcal{C}^1$-function on $\mathcal{P}_2(\mathbb{R})$. Then ϕ^N is differentiable on $\mathbb{R}^N$ and for every $i = 1, \dots, N$,*

$$\partial_{x_i}\phi^N(x_1, \dots, x_N) = \frac{1}{N}\partial_\mu\phi\left(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right)(x_i).$$

Moreover, following [CD18a], we say that ϕ is fully-$\mathcal{C}^2$ on $\mathcal{P}_2(\mathbb{R})$ if

(A1) ϕ is a $\mathcal{C}^1$-function on $\mathcal{P}_2(\mathbb{R})$ and $(\mu, v) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \mapsto \partial_\mu\phi(\mu)(v) \in \mathbb{R}$ is jointly continuous.

(A2) for each fixed μ, the version $v \in \mathbb{R} \mapsto \partial_\mu\phi(\mu)(v) \in \mathbb{R}$ used in (A1) is differentiable and its derivative $(\mu, v) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \mapsto \partial_v\partial_\mu\phi(\mu)(v) \in \mathbb{R}$ is jointly continuous.

(A3) for each fixed v, the version $\mu \in \mathcal{P}_2(\mathbb{R}) \mapsto \partial_\mu\phi(\mu)(v) \in \mathbb{R}$ used in (A1) is $\mathcal{C}^1$ on $\mathcal{P}_2(\mathbb{R})$, and its derivative given by $(\mu, v, v') \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \mapsto \partial^2_\mu\phi(\mu)(v, v')$ is jointly continuous.

Then, we have the following computation for the second-order derivatives of ϕ^N:

Proposition. *Let ϕ be fully-$\mathcal{C}^2$ on $\mathcal{P}_2(\mathbb{R})$. Then ϕ^N is $\mathcal{C}^2$ on $\mathbb{R}^N$ and for every $i = 1, \dots, N$,*

$$\partial^2_{x_i,x_j}\phi^N(x_1, \dots, x_N) = \frac{1}{N}\partial_v\partial_\mu\phi\left(\frac{1}{N}\sum_{l=1}^N \delta_{x_l}\right)(x_i)\mathbb{1}_{i=j} + \frac{1}{N^2}\partial^2_\mu\phi\left(\frac{1}{N}\sum_{l=1}^N \delta_{x_l}\right)(x_i, x_j).$$

Flat derivative (or Linear Functional derivative)

As announced, we may compare the notion of L-derivative with another notion of differentiability for $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$: the linear functional (or flat) derivative. Basically, it is nothing but the notion of differentiability we would use for $\phi : \mathcal{M}(\mathbb{R}) \to \mathbb{R}$ if it were defined on the whole $\mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ is the linear space of signed measures on $\mathbb{R}$. Note that a subset K of $\mathcal{P}_2(\mathbb{R})$ is said to be bounded if there is M such that for every $\mu \in K$, $\int_{\mathbb{R}} |x|^2 \mathrm{d}\mu(x) \leqslant M$.

Definition. A function $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is said to have a linear functional (or flat) derivative if there exists a function

$$\frac{\delta\phi}{\delta m} : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$$
$$(m, x) \mapsto \frac{\delta\phi}{\delta m}(m)(x),$$

jointly continuous in (m, x), such that for any bounded subset K of $\mathcal{P}_2(\mathbb{R})$, the function $x \mapsto \frac{\delta\phi}{\delta m}(m)(x)$ is at most of quadratic growth in x uniformly in m for $m \in K$, and such that for all $m, m' \in \mathcal{P}_2(\mathbb{R})$,

$$\phi(m') - \phi(m) = \int_0^1 \int_{\mathbb{R}} \frac{\delta\phi}{\delta m}(\lambda m' + (1-\lambda)m)(x)\ \mathrm{d}(m' - m)(x)\mathrm{d}\lambda.$$

Note that $\frac{\delta\phi}{\delta m}$ is uniquely defined up to an additive constant only.

In the case of example 1 given above, $\frac{\delta\phi}{\delta m}(m)(x) = h(x)$ for every $m \in \mathcal{P}_2(\mathbb{R})$ and for every $x \in \mathbb{R}$. This highlights the fact that the two notions of differentiability are not equivalent. Nevertheless, we can give the following relation between the two notions:

Proposition. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be L-differentiable on $\mathcal{P}_2(\mathbb{R})$, such that the Fréchet derivative of its lifted function $D\tilde{\phi} : L_2(\Omega) \to L_2(\Omega)$ is uniformly Lipschitz. Assume also that for each $\mu \in \mathcal{P}_2(\mathbb{R})$, there is a version of $v \in \mathbb{R} \mapsto \partial_\mu\phi(\mu)(v)$ such that the map $(v, \mu) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \partial_\mu\phi(\mu)(v)$ is continuous.*

Then ϕ has a linear functional derivative and for every $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$\partial_\mu\phi(\mu)(\cdot) = \partial_x\left\{\frac{\delta\phi}{\delta m}(\mu)\right\}(\cdot).$$

Chapter I

A new approach for the construction of a Wasserstein diffusion

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Abstract. We propose in this chapter a construction of a diffusion process on the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with a second-order moment. This process was introduced in several papers by Konarovskyi (see e.g. [Kon17b]) and consists of the limit as N tends to $+\infty$ of a system of N coalescing and mass-carrying particles. It has properties analogous to those of a standard Euclidean Brownian motion, in a sense that we will specify in this chapter. We also compare it to the Wasserstein diffusion on $\mathcal{P}_2(\mathbb{R})$ constructed by von Renesse and Sturm in [vRS09]. We obtain that process by the construction of a system of particles having short-range interactions and by letting the range of interactions tend to zero. This construction can be seen as an approximation of the singular process of Konarovskyi by a sequence of smoother processes.

I.1 Introduction

This chapter introduces a new approach to construct the stochastic diffusion process studied by Konarovskyi (see [Kon11, Kon17a, Kon17b, KvR18]). It is a close relative to the Wasserstein diffusion, introduced by von Renesse and Sturm [vRS09]. Our interest is to construct an analogous process to the Euclidean Brownian motion taking values on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$, defined as the set of probability measures on $\mathbb{R}$ having a second-order moment.

In [vRS09], von Renesse and Sturm construct a strong Markov process called *Wasserstein diffusion* on $\mathcal{P}_2(M)$, for M equal either to the interval $[0,1]$ or to the circle $\mathbb{S}^1$. Two major features of that process illustrate the analogy with the standard Brownian motion on a Euclidean space. First, the energy of the martingale part of the Wasserstein diffusion has the same form as that of a k-dimensional standard Brownian motion, up to replacing the Euclidean norm on $\mathbb{R}^k$ by the L_2-Wasserstein distance:

$$W_2(\mu,\nu) = \inf \mathbb{E}\left[|X-Y|^2\right]^{1/2},$$

where the infimum is taken over all couplings of two random variables X and Y such that X (resp. Y) has law μ (resp. ν). It should be noticed that the geometry of $\mathcal{P}_2(M)$, equipped

with the Wasserstein distance, for M a Euclidean space, was the subject of fundamental studies conducted by Ambrosio, Gigli, Savare, Villani, Lions and many others (see [AGS05, Car13, Lio, Vil03, Vil09]), which led to important improvements in optimal transport theory. Second, the transition costs of the Wasserstein diffusion are given by a Varadhan formula (see [vRS09], Corollary 7.19). The formula is identical to the Euclidean case, up to the replacement of the Euclidean norm by W_2.

Although the existence of a Wasserstein diffusion was initially proven by von Renesse and Sturm using Dirichlet processes and the theory of Dirichlet forms (see [FOT11]), it can also be obtained as a limit of finite-dimensional systems of interacting particles, see [AvR10, Stu14]. Nevertheless, we will focus in this chapter on a construction of a system of particles which seems more natural and simpler and which is due to Konarovskyi in [Kon11, Kon17b].

I.1.1 Konarovskyi's model

In [Kon17b], Konarovskyi studies a simple system of N interacting and coalescing particles and proves its convergence to an infinite-dimensional process which has the features of a diffusion on the L_2-Wasserstein space of probability measures (see also [Kon11, Kon17a, KvR18]). However, even if it has common properties with the diffusion of von Renesse and Sturm, there are also important differences between the two processes. An outstanding property of Konarovskyi's process is the fact that, for a large family of initial measures, it takes values in the set of measures with finite support for each time $t > 0$ (see [Kon17a]), whereas the values of the Wasserstein diffusion of von Renesse and Sturm are probability measures on $[0,1]$ with no absolutely continuous part and no discrete part.

The model introduced by Konarovskyi is a modification of the Arratia flow, also called Coalescing Brownian flow, introduced by Arratia [Arr79] and subject of many interest, among others in [Dor04, LJR04, NT15, Pit98]. It consists of Brownian particles starting at discrete points of the real line and moving independently until they meet another particle: when they meet, they stick together to form a single Brownian particle.

In his model (see [Kon17b]), Konarovskyi adds a mass to every particle: at time $t = 0$, N particles, denoted by $(x_k(t))_{k\in\{1,\dots,N\}}$, start from N points regularly distributed on the unit interval $[0,1]$, and each particle has a mass equal to $\frac{1}{N}$. When two particles stick together, they form as in the standard Arratia flow a unique particle, but with a mass equal to the sum of the two incident particles. Furthermore, the quadratic variation process of each particle is assumed to be inversely proportional to its mass. In other words, the heavier a particle is, the smaller its fluctuations are.

Konarovskyi constructs an associated process $(y^N(u,t))_{u\in[0,1],t\in[0,T]}$ in $\mathcal{D}([0,1],\mathcal{C}[0,T])$, the set of càdlàg functions on $[0,1]$ taking values in $\mathcal{C}[0,T]$, by defining:

$$y^N(u,t) := \sum_{k=1}^{N} x_k(t)\mathbb{1}_{\{u\in[\frac{k-1}{N},\frac{k}{N})\}} + x_N(t)\mathbb{1}_{\{u=1\}}.$$

In other words, $y^N(\cdot,t)$ denotes the quantile function associated to the empirical measure $\frac{1}{N}\sum_{k=1}^{N}\delta_{x_k(t)}$. Konarovskyi showed in [Kon17b] that the sequence $(y^N)_{N\geqslant 1}$ is tight in the space $\mathcal{D}([0,1],\mathcal{C}[0,T])$. Hence, by passing to the limit upon a subsequence, there exists a process $(y(u,t))_{u\in[0,1],t\in[0,T]}$ belonging to $\mathcal{D}([0,1],\mathcal{C}[0,T])$ and satisfying the following four properties:

(i_0) for all $u\in[0,1]$, $y(u,0) = u$;

(ii) for all $u\leqslant v$, for all $t\in[0,T]$, $y(u,t)\leqslant y(v,t)$;

(iii) for all $u\in[0,1]$, $y(u,\cdot)$ is a square integrable continuous martingale relatively to the filtration $(\mathcal{F}_t)_{t\in[0,T]} := (\sigma(y(v,s), v\in[0,1], s\leqslant t))_{t\in[0,T]}$;

(iv) for all $u, u' \in [0,1]$,

$$\langle y(u,\cdot), y(u',\cdot)\rangle_t = \int_0^t \frac{\mathbb{1}_{\{\tau_{u,u'}\leqslant s\}}}{m(u,s)}\mathrm{d}s,$$

where $m(u,t) := \int_0^1 \mathbb{1}_{\{\exists s\leqslant t:\, y(u,s)=y(v,s)\}}\mathrm{d}v$; $\tau_{u,u'} := \inf\{t \geqslant 0 : y(u,t) = y(u',t)\} \wedge T$.

By transporting the Lebesgue measure on $[0,1]$ by the map $y(\cdot,t)$, we obtain a measure-valued process $(\mu_t)_{t\in[0,T]}$ defined by: $\mu_t := \mathrm{Leb}\,|_{[0,1]} \circ y(\cdot,t)^{-1}$. In other words, $u \mapsto y(u,t)$ is the quantile function associated to μ_t. An important feature of this process is that for each positive t, μ_t is an atomic measure with a finite number of atoms, or in other words that $y(\cdot,t)$ is a step function.

More generally, Konarovskyi proves in [Kon17a] that this construction also holds for a greater family of initial measures μ_0. He constructs a process y^g in $\mathcal{D}([0,1],\mathcal{C}[0,T])$ satisfying $(ii)-(iv)$ and:

(i) for all $u \in [0,1]$, $y^g(u,0) = g(u)$,

for every non-decreasing càdlàg function g from $[0,1]$ into $\mathbb{R}$ such that there exists $p > 2$ satisfying $\int_0^1 |g(u)|^p\mathrm{d}u < \infty$. In other words, he generalizes the construction of a diffusion starting from any probability measure μ_0 satisfying $\int_{\mathbb{R}} |x|^p\mathrm{d}\mu_0(x) < \infty$ for a certain $p > 2$, where $\mu_0 = \mathrm{Leb}\,|_{[0,1]} \circ g^{-1}$, which means that g is the quantile function of the initial measure. The property that $y^g(\cdot,t)$ is a step function for each $t > 0$ remains true for this larger class of functions g.

The process y^g is said to be *coalescent*: almost surely, for every $u, v \in [0,1]$ and for every $t \in (\tau_{u,v}, T]$, we have $y^g(u,t) = y^g(v,t)$ (recall that $\tau_{u,v} = \inf\{t \geqslant 0 : y^g(u,t) = y^g(v,t)\} \wedge T$). This property is a consequence of (ii), (iii) and of the fact that for each $t > 0$, $y^g(\cdot,t)$ is a step function (see [KvR18, p.11]). Therefore, we can rewrite the formula for the mass as follows:

$$m^g(u,t) = \int_0^1 \mathbb{1}_{\{\exists s\leqslant t:\, y^g(u,s)=y^g(v,s)\}}\mathrm{d}v = \int_0^1 \mathbb{1}_{\{y^g(u,t)=y^g(v,t)\}}\mathrm{d}v.$$

Moreover, we can compare the diffusive properties of the process $(\mu_t)_{t\in[0,T]}$ in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ with the Wasserstein diffusion of von Renesse and Sturm. To that extent and thanks to Lions' differential calculus on $\mathcal{P}_2(\mathbb{R})$ ([Lio, Car13]), we give in Appendix I.6 an Itô formula on $\mathcal{P}_2(\mathbb{R})$ for the process $(\mu_t)_{t\in[0,T]}$ in order to describe the energy of the martingale part of this diffusion. Appendix I.6 also contains a small introduction to the differentiability on $\mathcal{P}_2(\mathbb{R})$ in the sense of Lions.

I.1.2 Approximation of a Wasserstein diffusion

In this chapter, we propose a new method to construct a process y satisfying properties (i)-(iv), by approaching y by a sequence of smooth processes. Finding smooth approximations of processes having singularities has already led to interesting results, typically in the case of the Arratia flow. Piterbarg [Pit98] shows that the Coalescing Brownian flow is the weak limit of isotropic homeomorphic flows in some space of discontinuous functions, and deduces from the properties of the limit process a careful description of contraction and expansion regions of homeomorphic flows. Dorogovtsev's approximation [Dor04] is based on a representation of the Arratia flow with a Brownian sheet.

We propose an adaptation of Dorogovtsev's idea in the case of Wasserstein diffusions. First, we show that a process y satisfying (i)-(iv) admits a representation in terms of a Brownian sheet; we refer to the lectures of Walsh [Wal86] for a complete introduction to Brownian sheet and to Section I.2 for the characterization of Brownian sheet which we use in this chapter.

Theorem I.1. *Let $g : [0,1] \to \mathbb{R}$ be a non-decreasing and càdlàg function such that there exists $p > 2$ satisfying $\int_0^1 |g(u)|^p \mathrm{d}u < +\infty$. Let y be a process in $L_2([0,1], \mathcal{C}[0,T])$ that satisfies conditions (i), (ii), (iii) and (iv). There exists a Brownian sheet w on $[0,1] \times [0,T]$ such that for all $u \in [0,1]$ and $t \in [0,T]$:*

$$y(u,t) = g(u) + \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)} \mathrm{d}w(u',s), \tag{I.1}$$

where $m(u,s) = \displaystyle\int_0^1 \mathbb{1}_{\{y(u,s)=y(v,s)\}} \mathrm{d}v$.

Remark I.2. We refer to Appendix I.6 to justify the use of the term "Wasserstein diffusion" for a process satisfying equation (I.1). Indeed, we can write an Itô formula for this process for a smooth function $u : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$. As in the case of the standard Euclidean Brownian motion, the quadratic variation of the martingale term is proportional to the square of the gradient of u, in the sense of Lions' differential calculus on $\mathcal{P}_2(\mathbb{R})$, which is the same as the differential calculus on the Wasserstein space (see [CD18a, Section 5.4]).

The aim of this chapter is to construct a sequence of smooth processes approaching y in the space $L_2([0,1], \mathcal{C}[0,T])$. Therefore, we use the representation (I.1) in terms of a Brownian sheet of y and, given a positive parameter σ, we replace in the latter representation the indicator functions by a smooth function φ_σ equal to 1 in the neighbourhood of 0 and whose support is included in the interval $\left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right]$ of small diameter σ. Fix $\sigma > 0$ and $\varepsilon > 0$. Given a Brownian sheet w on $[0,1] \times [0,T]$, we prove the existence of a process $y_{\sigma,\varepsilon}$ satisfying:

$$y_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t \int_0^1 \frac{\varphi_\sigma(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(u',s))}{\varepsilon + m_{\sigma,\varepsilon}(u,s)} \mathrm{d}w(u',s), \tag{I.2}$$

where $m_{\sigma,\varepsilon}(u,s) := \int_0^1 \varphi_\sigma^2(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s))\mathrm{d}v$ can be seen as a kind of mass of particle $y_{\sigma,\varepsilon}(u)$ at time s. Remark that, due to the fact that the support of φ_σ is small, only the particles located at a distance lower than $\frac{\sigma}{2}$ of particle u at time s are taken into account in the computation of the mass $m_{\sigma,\varepsilon}(u,s)$.

The smooth process $(y_{\sigma,\varepsilon}(u,t))_{u\in[0,1],t\in[0,T]}$ offers several advantages. First, we are able to construct a strong solution $(y_{\sigma,\varepsilon}, w)$ to equation (I.2), whereas in equation (I.1), we do not know if, given a Brownian sheet w, there exists an adapted solution y. Second, in Konarovskyi's process, the question of uniqueness of a solution to (1), even in the weak sense, or equivalently the question of uniqueness of a process in $L_2([0,1], \mathcal{C}[0,T])$ satisfying conditions (i)-(iv), remains open. Here, pathwise uniqueness holds for equation (I.2). Moreover, the measure-valued process $(\mu_t^{\sigma,\varepsilon})_{t\in[0,T]}$ associated to the process of quantile functions $(y_{\sigma,\varepsilon}(\cdot,t))_{t\in[0,T]}$ does generally no longer consist of atomic measures. For example, if $g(u) = u$, $(\mu_t^{\sigma,\varepsilon})_{t\in[0,T]}$ is a process of absolutely continuous measures with respect to the Lebesgue measure.

Let $L_2[0,1]$ be the usual space of square integrable functions from $[0,1]$ to $\mathbb{R}$, and $(\cdot,\cdot)_{L_2}$ the usual scalar product. We denote by $L_2^\uparrow[0,1]$ the set of functions $f \in L_2[0,1]$ such that there exists a non-decreasing and therefore *càdlàg* (*i.e.* right-continuous with left limits everywhere) element in the equivalence class of f. Let $\mathcal{D}((0,1), \mathcal{C}[0,T])$ be the space of right-continuous $\mathcal{C}[0,T]$-valued functions with left limits, equipped with the Skorohod metric.

We follow the definition given in [GM11, p.21]:

Definition I.3. An $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process M is called $L_2^\uparrow[0,1]$-*valued* $(\mathcal{F}_t)_{t\in[0,T]}$-*martingale* if M_t belongs to $L_2^\uparrow[0,1]$ for each $t \in [0,T]$, if $\mathbb{E}\left[\|M_t\|_{L_2}\right] < \infty$ and if for each $h \in L_2[0,1]$, $(M_t, h)_{L_2}$ is a real-valued $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. The martingale is said to be *square integrable* if for each $t \in [0,T]$, $\mathbb{E}\left[\|M_t\|_{L_2}^2\right] < +\infty$, and *continuous* if the process $t \mapsto M_t$ is a continuous function from $[0,T]$ to $L_2[0,1]$.

Let us denote by $\mathcal{L}_{2+}^{\uparrow}[0,1]$ the set of all non-decreasing and càdlàg functions $g : [0,1] \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, such that there exists $p > 2$ for which $\int_0^1 |g(u)|^p \mathrm{d}u < +\infty$. Let $\mathbb{Q}_+ = \mathbb{Q} \cap [0,1]$. The following theorem states the convergence of the mollified sequence $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ to a limit process satisfying properties $(i) - (iv)$. It uses the framework introduced by Konarovskyi in [Kon17a]:

Theorem I.4. *Let $g \in \mathcal{L}_{2+}^{\uparrow}[0,1]$. For each positive σ and ε, there exists a solution $y_{\sigma,\varepsilon}$ to equation* (I.2) *such that $(y_{\sigma,\varepsilon}(u,t))_{u\in[0,1],t\in[0,T]}$ belongs to $L_2([0,1],\mathcal{C}[0,T])$ and almost surely, for each $t \in [0,T]$, $y_{\sigma,\varepsilon}(\cdot,t) \in L_2^{\uparrow}[0,1]$.*

Furthermore, up to extracting a subsequence, the sequence $(y_{\sigma,\varepsilon})_{\varepsilon>0}$ converges in distribution in $L_2([0,1],\mathcal{C}[0,T])$ for every $\sigma \in \mathbb{Q}_+$ as ε tends to 0 to a limit y_σ and the sequence $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges in distribution in $L_2([0,1],\mathcal{C}[0,T])$ as σ tends to 0 to a limit y. Let $Y(t) := y(\cdot,t)$. Then $(Y(t))_{t\in[0,T]}$ is a $L_2^{\uparrow}[0,1]$-valued process such that:

$(C1)$ $Y(0) = g$;

$(C2)$ $(Y(t))_{t\in[0,T]}$ *is a square integrable continuous $L_2^{\uparrow}[0,1]$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-martingale, where $\mathcal{F}_t := \sigma(Y(s), s \leqslant t)$;*

$(C3)$ *almost surely, for every $t > 0$, $Y(t)$ is a step function,* i.e. *there exist $n \geqslant 1$, $0 = a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$ and $z_1 < z_2 < \cdots < z_n$ such that for all $u \in [0,1]$*

$$Y(t)(u) = y(u,t) = \sum_{k=1}^{n} z_k \mathbb{1}_{\{u\in[a_k,a_{k+1})\}} + z_n \mathbb{1}_{\{u=1\}};$$

$(C4)$ *y belongs to $\mathcal{D}((0,1),\mathcal{C}[0,T])$ and for every $u \in (0,1)$, $y(u,\cdot)$ is a square integrable and continuous $(\mathcal{F}_t)_{t\in[0,T]}$-martingale and*

$$\mathbb{P}\left[\forall u, v \in (0,1), \forall s \in [0,T], y(u,s) = y(v,s) \text{ implies } \forall t \geqslant s, y(u,t) = y(v,t)\right] = 1;$$

$(C5)$ *for each u and u' in $(0,1)$,*

$$\langle y(u,\cdot), y(u',\cdot)\rangle_t = \int_0^t \frac{\mathbb{1}_{\{\tau_{u,u'}\leqslant s\}}}{m(u,s)} \mathrm{d}s,$$

where $m(u,s) = \displaystyle\int_0^1 \mathbb{1}_{\{y(u,s)=y(v,s)\}} \mathrm{d}v$ and $\tau_{u,u'} = \inf\{t \geqslant 0 : y(u,t) = y(u',t)\} \wedge T$.

Remark I.5. More precisely, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is given by:

$$\mathcal{F}_t = \sigma((Y(s),h)_{L_2}, s \leqslant t, h \in L_2[0,1]).$$

Remark I.6. By property $(C4)$, the limit process y is said to be coalescent: if for a certain time t_0, two particles $y(u,t_0)$ and $y(v,t_0)$ coincide, then they move together forever, *i.e.* $y(u,t) = y(v,t)$ for every $t \geqslant t_0$.

It is interesting to wonder how the coalescence property of the process y translates to its smooth approximation $y_{\sigma,\varepsilon}$: two paths $(y_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ and $(y_{\sigma,\varepsilon}(v,t))_{t\in[0,T]}$, starting from two distinct points $g(u)$ and $g(v)$, do not meet, which means that $y_{\sigma,\varepsilon}(\cdot,t)$ is non-decreasing for each fixed t. If $y_{\sigma,\varepsilon}(u,\cdot)$ and $y_{\sigma,\varepsilon}(v,\cdot)$ get close enough, at distance smaller than σ, they begin to interact and to move together, whereas as long as they remain at distance greater than σ, they move "independently": more precisely, the covariation $\langle y_{\sigma,\varepsilon}(u,\cdot), y_{\sigma,\varepsilon}(v,\cdot)\rangle_t$ is equal to zero for every time $t \leqslant \tau_{u,v}^{\sigma} := \inf\{s \geqslant 0 : |y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)| \leqslant \sigma\}$ (see figure 1 in the introduction).

Organisation of the chapter

We begin in Section I.2 by proving Theorem I.1, which states that a process y satisfying properties (i)-(iv) admits a representation in terms of a Brownian sheet. In Section I.3, given a two-dimensional Brownian sheet, we prove the existence of a smooth process in the space $L_2([0,1],\mathcal{C}[0,T])$ intended to approach Konarovskyi's process of coalescing particles; this smooth process can be seen as a cloud of point-particles interacting with all the particles at a distance smaller than σ, and in which two particles have independent trajectories conditionally to the fact that the distance between them is greater than σ. When the distance becomes smaller than σ, both trajectories are correlated, mimicking the coalescence property.

Section I.4 is devoted to the proof of convergence when the parameter ε and the range of interaction σ tend to zero, using a tightness criterion in $L_2([0,1],\mathcal{C}[0,T])$. In Section I.5, we study the stochastic properties of the limit process, including the convergence of the mass process. The aim of this final part is to prove that the limit process y satisfies properties $(C1)$-$(C5)$ of Theorem I.4, in other words that our sequence of short-range interaction processes converges in distribution to the process of coalescing particles.

In Appendix I.6, we give an Itô formula in the Wasserstein space for the limit process y, after having recalled some basic definitions and properties of Lions' differential calculus on $\mathcal{P}_2(\mathbb{R})$.

I.2 Singular representation of the process y

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let us consider on $(\Omega,\mathcal{F},\mathbb{P})$ a random process $y \in L_2((0,1),\mathcal{C}[0,T])$ satisfying properties (i)-(iv). We refer to [Kon17a] for a comprehensive construction of y. We will give another one later in this chapter.

The aim of this paragraph is to prove Theorem I.1. Before that, we recall the definition of a Brownian sheet given by Walsh in [Wal86, p.269]. Let $(E,\mathcal{E},\nu)$ be a Euclidean space equipped with Lebesgue measure. A white noise based on ν is a random set function W on the sets $A \in \mathcal{E}$ of finite ν-measure such that

- $W(A)$ is a $\mathcal{N}(0,\nu(A))$ random variable,
- if $A \cap B = \varnothing$, then $W(A)$ and $W(B)$ are independent and $W(A \cap B) = W(A) + W(B)$.

Let $T > 0$. Consider $E = [0,1] \times [0,T]$ and ν the associated Lebesgue measure. The *Brownian sheet* w on $[0,1]\times[0,T]$ associated to the white noise W is the process $(w(u,t))_{u\in[0,1]\times[0,T]}$ defined by $w(u,t) := W((0,u] \times (0,t])$.

Define the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ by $\mathcal{G}_t := \sigma(w(u,s), u \in [0,1], s \leqslant t)$. Then in particular,

(i) for each $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable function f defined on $[0,1] \times [0,T]$ such that $\int_0^T \int_0^1 f^2(u,s)\mathrm{d}u\mathrm{d}s < +\infty$ almost surely, the process $\left(\int_0^t \int_0^1 f(u,s)\mathrm{d}w(u,s)\right)_{t\in[0,T]}$ is a local martingale (we often write $\mathrm{d}w(u,s)$ instead of $w(\mathrm{d}u,\mathrm{d}s)$);

(ii) for each f_1 and f_2 satisfying the same conditions as f,

$$\langle \int_0^{\cdot}\int_0^1 f_1(u,s)\mathrm{d}w(u,s), \int_0^{\cdot}\int_0^1 f_2(u,s)\mathrm{d}w(u,s)\rangle_t = \int_0^t\int_0^1 f_1(u,s)f_2(u,s)\mathrm{d}u\mathrm{d}s.$$

By Lévy's characterization of the Brownian motion, a process w satisfying (i) and (ii) is a Brownian sheet. Let us now prove Theorem I.1.

Proof (Theorem I.1). We take a Brownian sheet η on $[0,1] \times [0,T]$ independent of the process y, constructed by possibly extending the probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then, we define

$(w(u,t))_{u\in[0,1],t\in[0,T]}$ by $w(0,\cdot)\equiv 0$, $w(\cdot,0)\equiv 0$ and:

$$w(\mathrm{d}u,\mathrm{d}t)=\eta(\mathrm{d}u,\mathrm{d}t)+y(u,\mathrm{d}t)\mathrm{d}u-\frac{1}{m(u,t)}\int_0^1 \mathbb{1}_{\{y(u,t)=y(u',t)\}}\eta(\mathrm{d}u',\mathrm{d}t)\mathrm{d}u.$$

We denote by $\mathcal{H}_t$ the filtration $\sigma((y(u,s))_{u\in[0,1],s\leqslant t},(\eta(u,s))_{u\in[0,1],s\leqslant t})$.

In order to prove that w is an $(\mathcal{H}_t)_{t\in[0,T]}$-Brownian sheet on $[0,1]\times[0,T]$, let us consider two $(\mathcal{H}_t)_{t\in[0,T]}$-progressively measurable functions f_1 and f_2 and compute, using independence of η and y:

$$\langle\int_0^{\cdot}\int_0^1 f_1(u,s)\mathrm{d}w(u,s),\int_0^{\cdot}\int_0^1 f_2(v,s)\mathrm{d}w(v,s)\rangle_t=V_1+V_2-V_3-V_4+V_5,$$

where

$$V_1:=\langle\int_0^{\cdot}\int_0^1 f_1(u,s)\mathrm{d}\eta(u,s),\int_0^{\cdot}\int_0^1 f_2(v,s)\mathrm{d}\eta(v,s)\rangle_t=\int_0^t\int_0^1 f_1(u,s)f_2(u,s)\mathrm{d}u\mathrm{d}s,$$

since η is an $(\mathcal{H}_t)_{t\in[0,T]}$-Brownian sheet;

$$\begin{aligned}V_2&:=\langle\int_0^{\cdot}\int_0^1 f_1(u,s)\mathrm{d}y(u,s)\mathrm{d}u,\int_0^{\cdot}\int_0^1 f_2(v,s)\mathrm{d}y(v,s)\mathrm{d}v\rangle_t\\&=\int_0^t\int_0^1\int_0^1 f_1(u,s)f_2(v,s)\frac{\mathbb{1}_{\{y(u,s)=y(v,s)\}}}{m(u,s)}\mathrm{d}u\mathrm{d}v\mathrm{d}s,\end{aligned}$$

using property (iv) of process y;

$$\begin{aligned}V_3&:=\langle\int_0^{\cdot}\int_0^1 f_1(u,s)\mathrm{d}\eta(u,s),\int_0^{\cdot}\int_0^1\frac{f_2(v,s)}{m(v,s)}\int_0^1\mathbb{1}_{\{y(v,s)=y(v',s)\}}\mathrm{d}\eta(v',s)\mathrm{d}v\rangle_t\\&=\int_0^t\int_0^1\int_0^1\frac{f_1(u,s)f_2(v,s)}{m(v,s)}\mathbb{1}_{\{y(v,s)=y(u,s)\}}\mathrm{d}u\mathrm{d}v\mathrm{d}s=V_2,\end{aligned}$$

since $m(u,s)=m(v,s)$ whenever $y(u,s)$ is equal to $y(v,s)$. By similar computations,

$$V_4:=\langle\int_0^{\cdot}\int_0^1\frac{f_1(u,s)}{m(u,s)}\int_0^1\mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathrm{d}\eta(u',s)\mathrm{d}u,\int_0^{\cdot}\int_0^1 f_2(v,s)\mathrm{d}\eta(v,s)\rangle_t=V_2,$$

and

$$\begin{aligned}V_5:=&\langle\int_0^{\cdot}\int_0^1\frac{f_1(u,s)}{m(u,s)}\int_0^1\mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathrm{d}\eta(u',s)\mathrm{d}u,\\&\qquad\int_0^{\cdot}\int_0^1\frac{f_2(v,s)}{m(v,s)}\int_0^1\mathbb{1}_{\{y(v,s)=y(v',s)\}}\mathrm{d}\eta(v',s)\mathrm{d}v\rangle_t\\=&\int_0^t\int_0^1\int_0^1\int_0^1\frac{f_1(u,s)f_2(v,s)}{m(u,s)m(v,s)}\mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathbb{1}_{\{y(v,s)=y(u',s)\}}\mathrm{d}u'\mathrm{d}u\mathrm{d}v\mathrm{d}s\\=&\int_0^t\int_0^1\int_0^1\frac{f_1(u,s)f_2(v,s)}{m(u,s)^2}\left(\int_0^1\mathbb{1}_{\{y(u,s)=y(u',s)\}}\mathrm{d}u'\right)\mathbb{1}_{\{y(u,s)=y(v,s)\}}\mathrm{d}u\mathrm{d}v\mathrm{d}s\\=&\int_0^t\int_0^1\int_0^1\frac{f_1(u,s)f_2(v,s)}{m(u,s)}\mathbb{1}_{\{y(u,s)=y(v,s)\}}\mathrm{d}u\mathrm{d}v\mathrm{d}s\\=&V_2.\end{aligned}$$

To sum up,

$$\langle\int_0^{\cdot}\int_0^1 f_1(u,s)\mathrm{d}w(u,s),\int_0^{\cdot}\int_0^1 f_2(v,s)\mathrm{d}w(v,s)\rangle_t=V_1=\int_0^t\int_0^1 f_1(u,s)f_2(u,s)\mathrm{d}u\mathrm{d}s,$$

whence w is an $(\mathcal{H}_t)_{t\in[0,T]}$-Brownian sheet. Finally, we show that (y, w) satisfies equation (I.1):

$$\int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}w(u',s) = \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}\eta(u',s) \qquad (=: V_6)$$
$$+ \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}y(u',s)\mathrm{d}u' \qquad (=: V_7)$$
$$- \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\int_0^1 \frac{\mathbb{1}_{\{y(u',s)=y(v,s)\}}}{m(u',s)}\mathrm{d}\eta(v,s)\mathrm{d}u'. \qquad (=: V_8)$$

The result follows from the two below equalities:

$$V_7 = \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\mathrm{d}y(u,s)\mathrm{d}u' = \int_0^t \mathrm{d}y(u,s) = y(u,t) - y(u,0) = y(u,t) - g(u);$$
$$V_8 = \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(u',s)\}}}{m(u,s)}\int_0^1 \frac{\mathbb{1}_{\{y(u',s)=y(v,s)\}}}{m(v,s)}\mathrm{d}\eta(v,s)\mathrm{d}u'$$
$$= \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(v,s)\}}}{m(u,s)m(v,s)}\left(\int_0^1 \mathbb{1}_{\{y(u',s)=y(v,s)\}}\mathrm{d}u'\right)\mathrm{d}\eta(v,s)$$
$$= \int_0^t\int_0^1 \frac{\mathbb{1}_{\{y(u,s)=y(v,s)\}}}{m(u,s)}\mathrm{d}\eta(v,s),$$

which implies that $V_8 = V_6$ and consequently equation (I.1). □

Therefore, every solution of the martingale problem (i)-(iv) has a representation in terms of a Brownian sheet. In the next section, we will construct, given a Brownian sheet, an approximation of the process y.

I.3 Construction of a process with short-range interactions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, on which we define a Brownian sheet w on $[0,1]\times[0,T]$. We associate to that process the filtration $\mathcal{G}_t := \sigma(w(u,s), u \in [0,1], s \leqslant t)$. Up to completing the filtration, we assume that $\mathcal{G}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and that the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ is right-continuous.

Fix $\sigma > 0$ and $\varepsilon > 0$. Let φ_σ denote a smooth and even function, bounded by 1, equal to 1 on $[0, \frac{\sigma}{2}]$ and equal to 0 on $[\frac{\sigma}{2}, +\infty)$. Recall that $\mathcal{L}^{\uparrow}_{2+}[0,1]$ represents the set of non-decreasing and càdlàg functions $g : [0,1] \to \mathbb{R}$ such that there exists $p > 2$ satisfying $\int_0^1 |g(u)|^p \mathrm{d}u < +\infty$. The aim of this section is to construct, for each initial quantile function $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$, a square integrable random variable $y^g_{\sigma,\varepsilon}$ taking values in $L_2([0,1], \mathcal{C}[0,T])$ such that almost surely, for every $t \in [0,T]$, the following equality holds in $L_2[0,1]$:

$$y^g_{\sigma,\varepsilon}(\cdot,t) = g + \int_0^t\int_0^1 \frac{\varphi_\sigma(y^g_{\sigma,\varepsilon}(\cdot,s) - y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon + \int_0^1 \varphi^2_\sigma(y^g_{\sigma,\varepsilon}(\cdot,s) - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}\mathrm{d}w(u',s). \tag{I.3}$$

Remark I.7. We add the parameter ε to the denominator in order to ensure that it is bounded from below. We also point out that relation (I.3) has to be compared with equation (I.1), where $x \mapsto \mathbb{1}_{\{x=0\}}$ is replaced by the function φ_σ.

More precisely, we will prove the following proposition. Recall that $L^{\uparrow}_2[0,1]$ represents the set of functions $f \in L_2[0,1]$ such that there is a non-decreasing and càdlàg element in the equivalence class of f.

Proposition I.8. *Let $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$.*

There is an $L^{\uparrow}_2[0,1]$-valued process $(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]} = (y^g_{\sigma,\varepsilon}(\cdot,t))_{t\in[0,T]}$ such that:

(*A*1) $Y^g_{\sigma,\varepsilon}(0) = g$;

(*A*2) $Y^g_{\sigma,\varepsilon}$ *is a square integrable continuous* $L^{\uparrow}_2[0,1]$*-valued* $(\mathcal{F}^{\sigma,\varepsilon}_t)_{t\in[0,T]}$*-martingale, where* $\mathcal{F}^{\sigma,\varepsilon}_t := \sigma(Y^g_{\sigma,\varepsilon}(s), s \leqslant t)$;

(*A*3) *for every* $h, k \in L_2[0,1]$,

$$\langle (Y^g_{\sigma,\varepsilon}, h)_{L_2}, (Y^g_{\sigma,\varepsilon}, k)_{L_2}\rangle_t = \int_0^t\int_0^1\int_0^1 h(u)k(u')\frac{m^g_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon + m^g_{\sigma,\varepsilon}(u,s))(\varepsilon + m^g_{\sigma,\varepsilon}(u',s))}\mathrm{d}u\mathrm{d}u'\mathrm{d}s,$$

where $m^g_{\sigma,\varepsilon}(u,u',s) = \int_0^1 \varphi_\sigma(y^g_{\sigma,\varepsilon}(u,s) - y^g_{\sigma,\varepsilon}(v,s))\varphi_\sigma(y^g_{\sigma,\varepsilon}(u',s) - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v$ *and* $m^g_{\sigma,\varepsilon}(u,s) = \int_0^1 \varphi^2_\sigma(y^g_{\sigma,\varepsilon}(u,s) - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v$.

I.3.1 Existence of an approximate solution

Denote by $\mathcal{M}$ the set of random variables $z \in L_2(\Omega, \mathcal{C}([0,T], L_2(0,1)))$ such that $(z(\omega,\cdot,t))_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable process with values in $L_2(0,1)$. We consider the following norm on $\mathcal{M}$:

$$\|z\|_{\mathcal{M}} = \mathbb{E}\left[\sup_{t\leqslant T}\int_0^1 |z(u,t)|^2\mathrm{d}u\right]^{1/2}. \tag{I.4}$$

Throughout this section, σ and ε are two *fixed* positive numbers. To begin, we want to prove that the map $\psi : \mathcal{M} \to \mathcal{M}$, defined below, admits a unique fixed point. Fix $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$ an initial quantile function. For all $z \in \mathcal{M}$, define:

$$\psi(z)(\omega,u,t) := g(u) + \int_0^t\int_0^1 \frac{\varphi_\sigma(z(\omega,u,s) - z(\omega,u',s))}{\varepsilon + m_\sigma(\omega,u,s)}\mathrm{d}w(\omega,u',s), \tag{I.5}$$

where $m_\sigma(\omega,u,s) = \int_0^1 \varphi^2_\sigma(z(\omega,u,s) - z(\omega,v,s))\mathrm{d}v$. We start by making sure that ψ is well-defined.

Proposition I.9. *For all* $z \in \mathcal{M}$, $\psi(z)$ *belongs to* $\mathcal{M}$. *Furthermore,* $(\psi(z)(\cdot,t))_{t\in[0,T]}$ *is an* $L_2(0,1)$*-valued continuous* $(\mathcal{G}_t)_{t\in[0,T]}$*-martingale.*

Remark I.10. The definition of an $L^{\uparrow}_2[0,1]$-valued martingale was given in Definition I.3. Up to replacing $L^{\uparrow}_2$ by L_2, the definition of an $L_2(0,1)$-valued martingale is exactly the same.

Proof. We want to prove that $(\psi(z)(\cdot,t))_{t\in[0,T]}$ is an $L_2(0,1)$-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. Since z belongs to $\mathcal{M}$, the process $(z(\cdot,t))_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable. Therefore $(m_\sigma(\cdot,t))_{t\in[0,T]}$ is also $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable and we deduce that $(\psi(z)(\cdot,t))_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable.

Then, we check that for each $t \in [0,T]$, $\psi(z)(\cdot,t) \in L_2(0,1)$ and $\mathbb{E}\left[\|\psi(z)(\cdot,t)\|_{L_2}\right] < \infty$. We deduce this statement by recalling that $\|g\|_{L_2} < +\infty$, because $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$, and by computing:

$$\mathbb{E}\left[\left\|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(\cdot,s) - z(u',s))}{\varepsilon + m_\sigma(\cdot,s)}\mathrm{d}w(u',s)\right\|_{L_2}\right]^2 \leqslant \mathbb{E}\left[\left\|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(\cdot,s) - z(u',s))}{\varepsilon + m_\sigma(\cdot,s)}\mathrm{d}w(u',s)\right\|^2_{L_2}\right],$$

and

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(\cdot,s)-z(u',s))}{\varepsilon+m_\sigma(\cdot,s)}\mathrm{d}w(u',s)\right\|_{L_2}^2\right] &= \mathbb{E}\left[\int_0^1\left|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]\\ &= \int_0^1\mathbb{E}\left[\left|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\right]\mathrm{d}u\\ &= \int_0^1\mathbb{E}\left[\int_0^t\int_0^1\left(\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\right)^2\mathrm{d}u'\mathrm{d}s\right]\mathrm{d}u\\ &\leqslant \frac{\|\varphi_\sigma\|_\infty^2 t}{\varepsilon^2} = \frac{t}{\varepsilon^2}.\end{aligned}$$

Thus

$$\mathbb{E}\left[\left\|\int_0^t\int_0^1 \frac{\varphi_\sigma(z(\cdot,s)-z(u',s))}{\varepsilon+m_\sigma(\cdot,s)}\mathrm{d}w(u',s)\right\|_{L_2}\right]^2 \leqslant \frac{t}{\varepsilon^2} < +\infty. \tag{I.6}$$

Furthermore, for each $h \in L_2[0,1]$,

$$(\psi(z)(\cdot,t),h)_{L_2} = (g,h)_{L_2} + \int_0^t\int_0^1\int_0^1 h(u)\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}u\mathrm{d}w(u',s)$$

is a $(\mathcal{G}_t)_{t\in[0,T]}$-local martingale. Then, we compute the quadratic variation:

$$\begin{aligned}&\mathbb{E}\left[\langle(\psi(z),h)_{L_2},(\psi(z),h)_{L_2}\rangle_t\right]\\ &\quad=\int_0^t\int_0^1\int_0^1\int_0^1 h(u_1)h(u_2)\frac{\varphi_\sigma(z(u_1,s)-z(u',s))\varphi_\sigma(z(u_2,s)-z(u',s))}{(\varepsilon+m_\sigma(u_1,s))(\varepsilon+m_\sigma(u_2,s))}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u'\mathrm{d}s\\ &\quad\leqslant \frac{t}{\varepsilon^2}\|h\|_{L_2}^2.\end{aligned}$$

Since it is finite, the local martingale is actually a martingale.

Moreover, by Doob's inequality (see Theorem 2.2 in [GM11, p.22])

$$\begin{aligned}\|\psi(z)\|_{\mathcal{M}} &= \mathbb{E}\left[\sup_{t\leqslant T}\int_0^1|\psi(z)(u,t)|^2\mathrm{d}u\right]^{1/2}\\ &\leqslant \|g\|_{L_2} + \mathbb{E}\left[\sup_{t\leqslant T}\int_0^1\left|\int_0^t\int_0^1\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]^{1/2}\\ &\leqslant \|g\|_{L_2} + 2\mathbb{E}\left[\int_0^1\left|\int_0^T\int_0^1\frac{\varphi_\sigma(z(u,s)-z(u',s))}{\varepsilon+m_\sigma(u,s)}\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]^{1/2}.\end{aligned}$$

The last term is finite by (I.6). Thus $\|\psi(z)\|_{\mathcal{M}}$ is finite and $\psi(z)$ belongs to $\mathcal{M}$, which completes the proof. □

Let us now prove that ψ has a unique fixed point:

Proposition I.11. *Let $\sigma > 0$ and $\varepsilon > 0$. Then the map $\psi : \mathcal{M} \to \mathcal{M}$ defined by (I.5) has a unique fixed point in $\mathcal{M}$, denoted by $y^g_{\sigma,\varepsilon}$.*

Proof. For all $n \in \mathbb{N}$, denote by ψ^n the n-fold composition of ψ, where ψ^0 denotes the identity function of $\mathcal{M}$. We want to prove that ψ^n is a contraction for n large enough.

Let z_1 and z_2 be two elements of $\mathcal{M}$. We define

$$h_n(t) := \mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 |\psi^n(z_1)(u,s)-\psi^n(z_2)(u,s)|^2\mathrm{d}u\right].$$

Let us remark that $h_n(T) = \|\psi^n(z_1)-\psi^n(z_2)\|^2_{\mathcal{M}}$ and recall that, by Proposition I.9, $(\psi(z_1)(\cdot,t)-\psi(z_2)(\cdot,t))_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. We denote by $m_{\sigma,1}$ and $m_{\sigma,2}$ the masses associated respectively to z_1 and z_2. By Doob's inequality, we have:

$$\begin{aligned}
h_1(t) &= \mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 |\psi(z_1)(u,s)-\psi(z_2)(u,s)|^2\mathrm{d}u\right]\\
&= \mathbb{E}\left[\sup_{s\leqslant t}\int_0^1\left|\int_0^s\int_0^1\left(\frac{\varphi_\sigma(z_1(u,r)-z_1(u',r))}{\varepsilon+m_{\sigma,1}(u,r)}-\frac{\varphi_\sigma(z_2(u,r)-z_2(u',r))}{\varepsilon+m_{\sigma,2}(u,r)}\right)\mathrm{d}w(u',r)\right|^2\mathrm{d}u\right]\\
&\leqslant 4\mathbb{E}\left[\int_0^1\int_0^t\int_0^1\left|\frac{\varphi_\sigma(z_1(u,s)-z_1(u',s))}{\varepsilon+m_{\sigma,1}(u,s)}-\frac{\varphi_\sigma(z_2(u,s)-z_2(u',s))}{\varepsilon+m_{\sigma,2}(u,s)}\right|^2\mathrm{d}u'\mathrm{d}s\mathrm{d}u\right].
\end{aligned}$$

Furthermore, we compute:

$$\begin{aligned}
&\left|\frac{\varphi_\sigma(z_1(u,s)-z_1(u',s))}{\varepsilon+m_{\sigma,1}(u,s)}-\frac{\varphi_\sigma(z_2(u,s)-z_2(u',s))}{\varepsilon+m_{\sigma,2}(u,s)}\right|^2\\
&\qquad\leqslant 2\Bigg(\left|\frac{\varphi_\sigma(z_1(u,s)-z_1(u',s))-\varphi_\sigma(z_2(u,s)-z_2(u',s))}{\varepsilon+m_{\sigma,1}(u,s)}\right|^2\\
&\qquad\qquad+\left|\frac{\varphi_\sigma(z_2(u,s)-z_2(u',s))}{(\varepsilon+m_{\sigma,1}(u,s))(\varepsilon+m_{\sigma,2}(u,s))}\,(m_{\sigma,1}(u,s)-m_{\sigma,2}(u,s))\right|^2\Bigg).
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
|m_{\sigma,1}(u,s)-m_{\sigma,2}(u,s)| &\leqslant \int_0^1 |\varphi^2_\sigma(z_1(u,s)-z_1(v,s))-\varphi^2_\sigma(z_2(u,s)-z_2(v,s))|\mathrm{d}v\\
&\leqslant \mathrm{Lip}(\varphi^2_\sigma)\int_0^1 |(z_1(u,s)-z_1(v,s))-(z_2(u,s)-z_2(v,s))|\mathrm{d}v\\
&\leqslant \mathrm{Lip}(\varphi^2_\sigma)\left(|z_1(u,s)-z_2(u,s)|+\int_0^1|z_1(v,s)-z_2(v,s)|\mathrm{d}v\right).
\end{aligned}$$

We obtain the following upper bound:

$$\begin{aligned}
&\left|\frac{\varphi_\sigma(z_1(u,s)-z_1(u',s))}{\varepsilon+m_{\sigma,1}(u,s)}-\frac{\varphi_\sigma(z_2(u,s)-z_2(u',s))}{\varepsilon+m_{\sigma,2}(u,s)}\right|^2\\
&\qquad\leqslant\left(4\left(\frac{\mathrm{Lip}\,\varphi_\sigma}{\varepsilon}\right)^2+4\left(\frac{\mathrm{Lip}(\varphi^2_\sigma)}{\varepsilon^2}\right)^2\right)\Big(|z_1(u,s)-z_2(u,s)|^2\\
&\qquad\qquad+|z_1(u',s)-z_2(u',s)|^2+\int_0^1|z_1(v,s)-z_2(v,s)|^2\mathrm{d}v\Big).
\end{aligned}$$

Finally, we deduce that there is a constant $C_{\sigma,\varepsilon}$ depending only on σ and ε such that

$$\begin{aligned}
h_1(t) &\leqslant C_{\sigma,\varepsilon}\mathbb{E}\left[\int_0^t\int_0^1|z_1(u,s)-z_2(u,s)|^2\mathrm{d}u\mathrm{d}s\right]\\
&\leqslant C_{\sigma,\varepsilon}\int_0^t\mathbb{E}\left[\sup_{r\leqslant s}\int_0^1|z_1(u,r)-z_2(u,r)|^2\mathrm{d}u\right]\mathrm{d}s = C_{\sigma,\varepsilon}\int_0^t h_0(s)\mathrm{d}s.
\end{aligned}$$

Applied to $\psi^n(z_1)$ and $\psi^n(z_2)$ instead of z_1 and z_2, those computations show that for every $t \in [0,T]$, $h_{n+1}(t) \leqslant C_{\sigma,\varepsilon} \int_0^t h_n(s)\mathrm{d}s$. Using the fact that h_0 is non-decreasing with respect to t, it follows that $h_n(T) \leqslant \frac{(C_{\sigma,\varepsilon}T)^n}{n!} h_0(T)$, whence we have:

$$\|\psi^n(z_1) - \psi^n(z_2)\|^2_{\mathcal{M}} \leqslant \frac{(C_{\sigma,\varepsilon}T)^n}{n!} \|z_1 - z_2\|^2_{\mathcal{M}}.$$

Thus, for n large enough, the map ψ^n is a contraction. By completeness of $\mathcal{M}$ under the norm $\|\cdot\|_{\mathcal{M}}$ (remark that $\mathcal{M}$ is a closed subset of $L_2(\Omega, \mathcal{C}([0,T], L_2(0,1)))$, it follows that ψ has a unique fixed point in $\mathcal{M}$. □

We denote by $y^g_{\sigma,\varepsilon}$ the unique fixed point of ψ. Remark that by construction it satisfies equation (I.3) almost surely and for every $t \in [0,T]$.

I.3.2 Non-decreasing property

Define, for each $t \in [0,T]$, $Y^g_{\sigma,\varepsilon}(t) := y^g_{\sigma,\varepsilon}(\cdot, t)$. So far, by Proposition I.11, we have established that $(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]}$ is an $L_2[0,1]$-valued process, satisfying property $(A1)$ of Proposition I.8. Since $Y^g_{\sigma,\varepsilon}$ belongs to $\mathcal{M}$, and by Proposition I.9, $(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]}$ is a square integrable continuous $L_2[0,1]$-valued martingale, with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T]}$. Therefore, it is also an $(\mathcal{F}^{\sigma,\varepsilon}_t)_{t\in[0,T]}$-martingale, where $\mathcal{F}^{\sigma,\varepsilon}_t := \sigma(Y^g_{\sigma,\varepsilon}(s), s \leqslant t)$.

In order to obtain property $(A2)$, it remains to prove the following statement:

Proposition I.12. *$(Y^g_{\sigma,\varepsilon}(t))_{t\in[0,T]}$ is an $L_2^{\uparrow}[0,1]$-valued process.*

We will start by proving three lemmas and then we will conclude the proof of Proposition I.12. For every $x \in \mathbb{R}$, we consider the following stochastic differential equation:

$$z(x,t) = x + \int_0^t \int_0^1 \frac{\varphi_\sigma(z(x,s) - y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon + \int_0^1 \varphi^2_\sigma(z(x,s) - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v} \mathrm{d}w(u',s), \tag{I.7}$$

where $y^g_{\sigma,\varepsilon}$ is the unique solution of equation (I.3).

Lemma I.13. *Let $x \in \mathbb{R}$. For almost every $\omega \in \Omega$, equation* (I.7) *has a unique solution in $\mathcal{C}[0,T]$, denoted by $(z(\omega,x,t))_{t\in[0,T]}$. Moreover, $(z(x,t))_{t\in[0,T]}$ is a real-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale.*

Proof. We get existence and uniqueness of the solution by applying a fixed-point argument. The proof is the same as the proof of Proposition I.11. We obtain the martingale property by the same argument as in Proposition I.9. □

Then, take x_1, $x_2 \in \mathbb{R}$. After some computations similar to those of the proof of Proposition I.11, we have for every $t \in [0,T]$:

$$\mathbb{E}\left[\sup_{s\leqslant t} |z(x_1,s) - z(x_2,s)|^2\right] \leqslant 2|x_1 - x_2|^2 + C_{\sigma,\varepsilon} \int_0^t \mathbb{E}\left[\sup_{r\leqslant s} |z(x_1,r) - z(x_2,r)|^2\right] \mathrm{d}s.$$

By Gronwall's Lemma, we deduce that:

$$\mathbb{E}\left[\sup_{t\leqslant T} |z(x_1,t) - z(x_2,t)|^2\right] \leqslant C_{\sigma,\varepsilon}|x_1 - x_2|^2.$$

By Kolmogorov's Lemma, there is a modification $\widetilde{z}$ of z in $\mathcal{C}(\mathbb{R}\times[0,T])$. We define $\widetilde{y}^g_{\sigma,\varepsilon}(u,t) := \widetilde{z}(g(u),t)$. In particular, $u \mapsto \widetilde{y}^g_{\sigma,\varepsilon}(u,\cdot)$ is measurable and, since g is a càdlàg function, $\widetilde{y}^g_{\sigma,\varepsilon}$ belongs to $\mathcal{D}((0,1), \mathcal{C}[0,T])$.

Remark I.14. In the case where g is continuous, it is straightforward to see that $\widetilde{y}^g_{\sigma,\varepsilon}$ belongs to $\mathcal{C}([0,1]\times[0,T])$.

Furthermore, $\widetilde{y}^g_{\sigma,\varepsilon}$ belongs to $\mathcal{M}$. Indeed,

$$\mathbb{E}\left[\sup_{t\leqslant T}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u\right]\leqslant\mathbb{E}\left[\int_0^1\sup_{t\leqslant T}\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u\right]=\int_0^1\mathbb{E}\left[\sup_{t\leqslant T}\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2\right]\mathrm{d}u.$$

By Lemma I.13, for every $u\in[0,1]$, $(\widetilde{y}^g_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is a martingale, we have by Doob's inequality:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T}\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)\right|^2\right]&\leqslant C\mathbb{E}\left[\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,T)\right|^2\right]\\&\leqslant 2Cg(u)^2+2C\mathbb{E}\left[\int_0^T\int_0^1\left|\frac{\varphi_\sigma(\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi^2_\sigma(\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}\right|^2\mathrm{d}u'\mathrm{d}s\right]\\&\leqslant 2Cg(u)^2+2C\frac{T}{\varepsilon^2}.\end{aligned}$$

Therefore, $\|\widetilde{y}^g_{\sigma,\varepsilon}\|_{\mathcal{M}}\leqslant 2C\|g\|^2_{L_2}+2C\frac{T}{\varepsilon^2}<+\infty$. Moreover, $(\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t))_{t\in[0,T]}$ is an $L_2[0,1]$-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. Indeed, for every $h\in L_2[0,1]$, for every $t\in[0,T]$, the expectation $\mathbb{E}\left[(\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t),h)_{L_2}\right]$ is finite. Fix $0\leqslant s\leqslant t\leqslant T$, and $A_s\in\mathcal{G}_s$. We have:

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1\widetilde{y}^g_{\sigma,\varepsilon}(u,t)h(u)\mathrm{d}u-\int_0^1\widetilde{y}^g_{\sigma,\varepsilon}(u,s)h(u)\mathrm{d}u\right)\mathbb{1}_{A_s}\right]&\\=\int_0^1\mathbb{E}\left[(\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-\widetilde{y}^g_{\sigma,\varepsilon}(u,s))\mathbb{1}_{A_s}\right]h(u)\mathrm{d}u=0.&\end{aligned}$$

Lemma I.15. *We have* $\mathbb{E}\left[\sup_{t\leqslant T}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-y^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u\right]=0$. *Therefore,* $\widetilde{y}^g_{\sigma,\varepsilon}=y^g_{\sigma,\varepsilon}$ *in* $\mathcal{M}$.

Proof. Since $(\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t)-y^g_{\sigma,\varepsilon}(\cdot,t))_{t\in[0,T]}$ is an $L_2[0,1]$-valued martingale, then by [GM11, p.21-22] $\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-y^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u$ is a real-valued submartingale. By Doob's inequality,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2\mathrm{d}u\right]&\leqslant C\mathbb{E}\left[\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,t)-y^g_{\sigma,\varepsilon}(u,t)\right|^2\mathrm{d}u\right]\\&\leqslant C\mathbb{E}\left[\int_0^1\left|\int_0^t\int_0^1(\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),u',s)-\theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u,s),u',s))\mathrm{d}w(u',s)\right|^2\mathrm{d}u\right]\\&\leqslant C\mathbb{E}\left[\int_0^1\int_0^t\int_0^1\left|\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),u',s)-\theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u,s),u',s)\right|^2\mathrm{d}u'\mathrm{d}s\mathrm{d}u\right],\end{aligned}$$

where $\theta_{\sigma,\varepsilon}(x,u',s)=\frac{\varphi_\sigma(x-y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi^2_\sigma(x-y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}$. Using the same constant $C_{\sigma,\varepsilon}$ as in the proof of Proposition I.11, we have:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2\mathrm{d}u\right]&\leqslant C_{\sigma,\varepsilon}\mathbb{E}\left[\int_0^1\int_0^t\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2\mathrm{d}s\mathrm{d}u\right]\\&\leqslant C_{\sigma,\varepsilon}\int_0^t\mathbb{E}\left[\sup_{r\leqslant s}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,r)-y^g_{\sigma,\varepsilon}(u,r)\right|^2\mathrm{d}u\right]\mathrm{d}s.\end{aligned}$$

By Gronwall's Lemma, we deduce that $\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1\left|\widetilde{y}^g_{\sigma,\varepsilon}(u,s)-y^g_{\sigma,\varepsilon}(u,s)\right|^2\mathrm{d}u\right]=0$ for every $t\in[0,T]$. This implies the statement of the lemma. □

Lemma I.16. *Almost surely, for every $u_1, u_2 \in \mathbb{Q}$ such that $u_1 < u_2$, we have for every $t \geqslant 0$, $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t) \leqslant \widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$. Furthermore, if $g(u_1) < g(u_2)$ (resp. $g(u_1) = g(u_2)$), then for every $t \geqslant 0$, $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t) < \widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$ (resp. $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t) = \widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$).*

Proof. Let $(u_1,u_2) \in \mathbb{Q}^2$ such that $0 \leqslant u_1 < u_2 \leqslant 1$. For $u = u_1, u_2$, we have:

$$\widetilde{y}^g_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t \int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s), u', s)\mathrm{d}w(u',s),$$

where $\theta_{\sigma,\varepsilon}(x,u',s) = \frac{\varphi_\sigma(x - y^g_{\sigma,\varepsilon}(u',s))}{\varepsilon + \int_0^1 \varphi^2_\sigma(x - y^g_{\sigma,\varepsilon}(v,s))\mathrm{d}v}$. Therefore, we have (writing $\widetilde{y}$ instead of $\widetilde{y}^g_{\sigma,\varepsilon}$ and θ instead of $\theta_{\sigma,\varepsilon}$):

$$\begin{aligned}\widetilde{y}(u_2,t) - \widetilde{y}(u_1,t) &= g(u_2) - g(u_1) + \int_0^t \int_0^1 (\theta(\widetilde{y}(u_2,s),u',s) - \theta(\widetilde{y}(u_1,s),u',s))\mathrm{d}w(u',s) \\ &= g(u_2) - g(u_1) + \int_0^t (\widetilde{y}(u_2,s) - \widetilde{y}(u_1,s))\mathrm{d}M_s \end{aligned} \tag{I.8}$$

where $M_t = \int_0^t \int_0^1 \mathbb{1}_{\{\widetilde{y}(u_2,s) \neq \widetilde{y}(u_1,s)\}} \frac{\theta(\widetilde{y}(u_2,s),u',s) - \theta(\widetilde{y}(u_1,s),u',s)}{\widetilde{y}(u_2,s) - \widetilde{y}(u_1,s)} \mathrm{d}w(u',s)$. Observe that:

$$\theta(\widetilde{y}(u_2,s),u',s) - \theta(\widetilde{y}(u_1,s),u',s) = \int_{\widetilde{y}(u_1,s)}^{\widetilde{y}(u_2,s)} \partial_x \theta(x,u',s)\mathrm{d}x,$$

and that $\partial_x \theta(x,u',s) = \frac{\varphi'_\sigma(x - y(u',s))}{\varepsilon + \int_0^1 \varphi^2_\sigma(x - y(v,s))\mathrm{d}v} - \frac{\varphi_\sigma(x - y(u',s)) \int_0^1 (\varphi^2_\sigma)'(x - y(v,s))\mathrm{d}v}{(\varepsilon + \int_0^1 \varphi^2_\sigma(x - y(v,s))\mathrm{d}v)^2}$. Therefore, $\partial_x \theta$ is bounded uniformly in $(x,u',s) \in \mathbb{R} \times [0,1] \times [0,T]$ by $C_{\sigma,\varepsilon} := \frac{\|\varphi'_\sigma\|_{L^\infty}}{\varepsilon} + \frac{\|\varphi_\sigma\|_{L^\infty} \|(\varphi^2_\sigma)'\|_{L^\infty}}{\varepsilon^2}$. We deduce that

$$\begin{aligned}\mathbb{E}\left[\langle M, M\rangle_T\right] &= \mathbb{E}\left[\int_0^T \int_0^1 \mathbb{1}_{\{\widetilde{y}(u_2,s) \neq \widetilde{y}(u_1,s)\}} \left(\frac{\theta(\widetilde{y}(u_2,s),u',s) - \theta(\widetilde{y}(u_1,s),u',s)}{\widetilde{y}(u_2,s) - \widetilde{y}(u_1,s)}\right)^2 \mathrm{d}u'\mathrm{d}s\right] \\ &\leqslant \mathbb{E}\left[\int_0^T \int_0^1 \left(C_{\sigma,\varepsilon}\right)^2 \mathrm{d}u'\mathrm{d}s\right] \leqslant T\left(C_{\sigma,\varepsilon}\right)^2,\end{aligned}$$

and thus M is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale on $[0,T]$. We resolve the stochastic differential equation (I.8): $\widetilde{y}^g_{\sigma,\varepsilon}(u_2,t) - \widetilde{y}^g_{\sigma,\varepsilon}(u_1,t) = (g(u_2) - g(u_1)) \exp\left(M_t - \frac{1}{2}\langle M, M\rangle_t\right)$. If $g(u_1) < g(u_2)$ (resp. $g(u_1) = g(u_2)$), then almost surely for every $t \in [0,T]$, $\widetilde{y}^g_{\sigma,\varepsilon}(u_1,t) < \widetilde{y}^g_{\sigma,\varepsilon}(u_2,t)$ (resp. $=$). Thus it is true almost surely for every $(u_1,u_2) \in \mathbb{Q}^2$ such that $u_1 < u_2$. □

Therefore the proof of Proposition I.12 is complete:

Proof (Proposition I.12). For each $t \in [0,T]$, $Y^g_{\sigma,\varepsilon}(t) = y^g_{\sigma,\varepsilon}(\cdot,t)$ has a modification $\widetilde{y}^g_{\sigma,\varepsilon}(\cdot,t)$ belonging to $L^\uparrow_2[0,1]$. □

We specify the properties of $\widetilde{y}^g_{\sigma,\varepsilon}$ in the following corollary, which derives directly from Proposition I.12. From now on, we will always use this version of the process.

Corollary I.17. *The following two statements hold:*

- *for almost every $u \in (0,1)$, $(\widetilde{y}^g_{\sigma,\varepsilon}(\omega,u,t))_{t\in[0,T]}$ is a $(\mathcal{F}^{\sigma,\varepsilon}_t)_{t\in[0,T]}$-martingale, and it is continuous for almost every $(u,\omega) \in (0,1) \times \Omega$.*
- *almost surely, for every $t \in [0,T]$, $u \mapsto \widetilde{y}^g_{\sigma,\varepsilon}(u,t)$ is càdlàg and non-decreasing.*

We complete the proof of Proposition I.8.

Proof (Proposition I.8). Thanks to Proposition I.12, the proof of properties $(A1)$ and $(A2)$ has been completed. It remains to compute the quadratic variation. Recall that for every $u \in [0,1]$, $(\widetilde{y}^g_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale and that

$$\widetilde{y}^g_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),u',s)\mathrm{d}w(u',s).$$

Therefore, for every $u,u' \in [0,1]$,

$$\begin{aligned}\langle \widetilde{y}^g_{\sigma,\varepsilon}(u,\cdot),\widetilde{y}^g_{\sigma,\varepsilon}(u',\cdot)\rangle_t &= \langle \int_0^{\cdot}\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),v,s)\mathrm{d}w(v,s), \int_0^{\cdot}\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}w(v,s)\rangle_t \\ &= \int_0^t\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),v,s)\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}v\mathrm{d}s.\end{aligned}$$

Therefore, for every $h,k \in L_2[0,1]$,

$$\begin{aligned}&\langle (Y^g_{\sigma,\varepsilon},h)_{L_2},(Y^g_{\sigma,\varepsilon},k)_{L_2}\rangle_t \\ &\qquad= \int_0^t\int_0^1\int_0^1 h(u)k(u')\int_0^1 \theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u,s),v,s)\theta_{\sigma,\varepsilon}(\widetilde{y}^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}v\mathrm{d}u\mathrm{d}u'\mathrm{d}s \\ &\qquad= \int_0^t\int_0^1\int_0^1 h(u)k(u')\int_0^1 \theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u,s),v,s)\theta_{\sigma,\varepsilon}(y^g_{\sigma,\varepsilon}(u',s),v,s)\mathrm{d}v\mathrm{d}u\mathrm{d}u'\mathrm{d}s \\ &\qquad= \int_0^t\int_0^1\int_0^1 h(u)k(u')\frac{m^g_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon+m^g_{\sigma,\varepsilon}(u,s))(\varepsilon+m^g_{\sigma,\varepsilon}(u',s))}\mathrm{d}u\mathrm{d}u'\mathrm{d}s,\end{aligned}$$

which completes the proof. □

We conclude this section with a property on the quadratic variation of two fixed particles, which will be useful to obtain lower bounds on the mass in the next section.

Corollary I.18. *For almost every $u,u' \in [0,1]$,*

$$\langle \widetilde{y}^g_{\sigma,\varepsilon}(u,\cdot),\widetilde{y}^g_{\sigma,\varepsilon}(u',\cdot)\rangle_t = \int_0^t\int_0^1 \frac{m^g_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon+m^g_{\sigma,\varepsilon}(u,s))(\varepsilon+m^g_{\sigma,\varepsilon}(u',s))}\mathrm{d}v\mathrm{d}s. \tag{I.9}$$

Proof. This statement follows clearly from the proof of Proposition I.8, from the fact that for almost every $u \in (0,1)$, $(\widetilde{y}^g_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is a continuous martingale. □

I.4 Convergence of the process $(y^g_{\sigma,\varepsilon})_{\sigma,\varepsilon\in\mathbb{Q}_+}$

From now on, for the sake of simplicity, we fix a function g in $\mathcal{L}^{\uparrow}_{2+}[0,1]$ and $y_{\sigma,\varepsilon}$ will denote the version $\widetilde{y}^g_{\sigma,\varepsilon}$ starting from g. We denote by p a number such that $p > 2$ and $g \in L_p(0,1)$.

We begin by proving the tightness of the sequence $(y_{\sigma,\varepsilon})_{\sigma,\varepsilon\in\mathbb{Q}_+}$ in $L_2([0,1],\mathcal{C}[0,T])$ in Paragraph I.4.1. We will then pass to the limit in distribution, first when $\varepsilon \to 0$ and then when $\sigma \to 0$ and prove, in Paragraph I.4.5, that the limit process is also a martingale.

I.4.1 Tightness of the collection $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ in $L_2([0,1],\mathcal{C}[0,T])$

Recall that for all $\sigma > 0$, the map φ_σ is smooth, even, bounded by 1, equal to 1 on $\left[0,\frac{\sigma-\eta}{2}\right]$ and equal to 0 on $\left[\frac{\sigma}{2},+\infty\right)$, where η is chosen so that $\eta < \frac{\sigma}{3}$. Recall that $y_{\sigma,\varepsilon}$ is solution of the following equation:

$$y_{\sigma,\varepsilon}(u,t) = g(u) + \int_0^t\int_0^1 \frac{\varphi_\sigma(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(u',s))}{\varepsilon+\int_0^1\varphi^2_\sigma(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s))\mathrm{d}v}\mathrm{d}w(u',s).$$

We begin by proving that the collection $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ satisfies a compactness criterion in the space $L_2([0,1],\mathcal{C}[0,T])$. We recall the following criterion (see [Sim87, Theorem 1, p.71]):

Proposition I.19. *Let K be a subset of $L_2([0,1],\mathcal{C}[0,T])$.*

K is relatively compact in $L_2([0,1],\mathcal{C}[0,T])$ if and only if:

$(H1)$ *for every $0 \leqslant u_1 < u_2 \leqslant 1$, $\left\{\int_{u_1}^{u_2} f(u,\cdot)\mathrm{d}u, f \in K\right\}$ is relatively compact in $C[0,T]$,*

$(H2)$ $\lim_{h\to 0^+} \sup_{f\in K} \int_0^{1-h} \|f(u+h,\cdot) - f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u = 0$.

By Ascoli's Theorem, $(H1)$ is satisfied if and only if for every $0 \leqslant u_1 < u_2 \leqslant 1$,

- for every $t \in [0,T]$, $\int_{u_1}^{u_2} f(u,t)\mathrm{d}u$ is uniformly bounded,

- $\lim_{\eta\to 0^+} \sup_{f\in K} \sup_{|t_2-t_1|<\eta} \left|\int_{u_1}^{u_2}(f(u,t_2) - f(u,t_1))\mathrm{d}u\right| = 0$.

In order to prove tightness for the collection $(y_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$, we will prove the following Proposition:

Proposition I.20. *Let $\delta > 0$. The following statements hold:*

$(K1)$ *there exists $M > 0$ such that for all $\sigma, \varepsilon > 0$, $\mathbb{P}\left[\int_0^1 \|y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \leqslant M\right] \geqslant 1-\delta$,*

$(K2)$ *for all $k \geqslant 1$, there exists $\eta_k > 0$ such that for all $\sigma > 0$, $\varepsilon > 0$,*

$$\mathbb{P}\left[\int_0^1 \sup_{|t_2-t_1|<\eta_k} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u \leqslant \frac{1}{k}\right] \geqslant 1 - \frac{\delta}{2^k},$$

$(K3)$ *for all $k \geqslant 1$, there exists $h_k > 0$ such that for all $\sigma > 0$, $\varepsilon > 0$,*

$$\mathbb{P}\left[\forall h \in (0,h_k), \int_0^{1-h} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \leqslant \frac{1}{k}\right] \geqslant 1 - \frac{\delta}{2^k}.$$

Proposition I.20 will be proved in Paragraph I.4.3. It implies tightness of $(y^g_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ in $L_2([0,1],\mathcal{C}[0,T])$:

Corollary I.21. *For all $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$, the collection $(y^g_{\sigma,\varepsilon})_{\sigma>0,\varepsilon>0}$ is tight in $L_2([0,1],\mathcal{C}[0,T])$.*

Proof (Corollary I.21). Let $\delta > 0$. Let M, $(h_k)_{k\geqslant 1}$, $(\eta_k)_{k\geqslant 1}$ be such that the statements of Proposition I.20 hold with δ.

Denote K_δ the closed set of all functions $f \in L_2([0,1],\mathcal{C}[0,T])$ satisfying:

$(L1)$ $\int_0^1 \|f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \leqslant M$.

$(L2)$ for all $k \geqslant 1$, $\int_0^1 \sup_{|t_2-t_1|<\eta_k} |f(u,t_2) - f(u,t_1)|\mathrm{d}u \leqslant \frac{1}{k}$.

$(L3)$ for all $k \geqslant 1$, $\forall h \in (0,h_k)$, $\int_0^{1-h} \|f(u+h,\cdot) - f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \leqslant \frac{1}{k}$.

Let $0 \leqslant u_1 < u_2 \leqslant 1$. We deduce from $(L1)$ that for every $t \in [0,T]$, and every $f \in K_\delta$, $\left|\int_{u_1}^{u_2} f(u,t)\mathrm{d}u\right| \leqslant \left(\int_{u_1}^{u_2} f(u,t)^2\mathrm{d}u\right)^{1/2} \leqslant \left(\int_0^1 \|f(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\right)^{1/2} \leqslant \sqrt{M}$. We deduce from $(L2)$ that for every $k \geqslant 1$,

$$\sup_{f\in K_\delta}\sup_{|t_2-t_1|<\eta_k}\left|\int_{u_1}^{u_2}(f(u,t_2)-f(u,t_1))\mathrm{d}u\right| \leqslant \sup_{f\in K_\delta}\int_0^1 \sup_{|t_2-t_1|<\eta_k}|f(u,t_2)-f(u,t_1)|\mathrm{d}u \leqslant \frac{1}{k}.$$

Therefore, by Ascoli's Theorem, condition $(H1)$ of Proposition I.19 is satisfied.

Furthermore, by $(L3)$, condition $(H2)$ is also satisfied uniformly for $f \in K_\delta$. Therefore, K_δ is compact in $L_2([0,1],\mathcal{C}[0,T])$. By Proposition I.20, for all $\sigma > 0$, $\varepsilon > 0$, $\mathbb{P}\left[y_{\sigma,\varepsilon} \in K_\delta\right] \geqslant 1 - 3\delta$. This completes the proof. □

To prove Proposition I.20, we will first give in the next paragraph an estimation of the inverse of the mass function (see Lemma I.24). This lemma is an equivalent in our case of short-range interacting particles of Lemma 2.16 in [Kon17b], stated in the case of a system of coalescing particles.

I.4.2 Estimation of the inverse of mass

Recall that $m_{\sigma,\varepsilon}(u,t) = \int_0^1 \varphi_\sigma^2(y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(v,t))\mathrm{d}v$. We define a modified mass

$$M_{\sigma,\varepsilon}(u,t) := \begin{cases} \dfrac{(\varepsilon + m_{\sigma,\varepsilon})^2}{m_{\sigma,\varepsilon}}(u,t) \text{ if } m_{\sigma,\varepsilon}(u,t) > 0, \\ +\infty \text{ otherwise.} \end{cases}$$

Clearly, $M_{\sigma,\varepsilon}(u,t) \geqslant m_{\sigma,\varepsilon}(u,t)$ for every $u \in [0,1]$ and $t \in [0,T]$.

By Corollary I.17, there exists a (non-random) Borel set $\mathcal{A}$ in $[0,1]$, $\mathrm{Leb}(\mathcal{A}) = 1$, such that for all $u \in \mathcal{A}$, $(y_{\sigma,\varepsilon}(u,t))_{t\in[0,T]}$ is almost surely a continuous $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-martingale. Recall also that almost surely, for every $t \in [0,T]$, $u \mapsto y_{\sigma,\varepsilon}(u,t)$ is càdlàg and non-decreasing. Moreover, we assume that for every $u, u' \in \mathcal{A}$, equality (I.9) holds.

Lemma I.22. *There exist $C > 0$ and $\gamma \in (0,1)$ such that for each $\sigma, \varepsilon > 0$, $t \in (0,T]$ and for every $u \in \mathcal{A}$ and every $h > 0$ satisfying $u - h \in (0,1)$,*

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,s)<\gamma h\}}\mathrm{d}s \geqslant t\right] \leqslant C\left[g(u) - g(u-h)\right]\sqrt{\frac{h}{t}}. \tag{I.10}$$

Proof. Fix $\sigma > 0$ and $\varepsilon > 0$. Let $h > 0$ be such that $u-h$ belongs to $\mathcal{A}$. If $g(u-h) = g(u)$, then for every $t \in [0,T]$, $y_{\sigma,\varepsilon}(u-h,t) = y_{\sigma,\varepsilon}(u,t)$. By the non-decreasing and càdlàg property, for every $v \in (u-h,u)$, we have $y_{\sigma,\varepsilon}(v,t) = y_{\sigma,\varepsilon}(u,t)$. We deduce that $m_{\sigma,\varepsilon}(u,t) \geqslant \int_{u-h}^u \varphi_\sigma^2(y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(v,t))\mathrm{d}v = \int_{u-h}^u \varphi_\sigma^2(0)\mathrm{d}v = h$. Therefore, $M_{\sigma,\varepsilon}(u,t) \geqslant h \geqslant \gamma h$ for every $t \in [0,T]$, and (I.10) is satisfied.

Consider now the case where $g(u-h) < g(u)$. Choose k in $(\frac{h}{3}, \frac{2h}{3})$ such that $u - k \in \mathcal{A}$. Denote by N and $\widetilde{N}$ the following two $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-martingales:

$$N_t = y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(u-h,t),$$
$$\widetilde{N}_t = y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(u-k,t).$$

Denote by G_s and H_s respectively the events $\left\{M_{\sigma,\varepsilon}(u,s) < \frac{h}{2^6}\right\}$ and $\{\widetilde{N}_s > \frac{\sigma+\eta}{2}\}$. We want to prove the existence of a constant C_1 independent of h and u such that for all $\sigma > 0$, $\varepsilon > 0$ and

$t > 0$,

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\}}\mathrm{d}s \geqslant t\right] \leqslant C_1\left[g(u) - g(u-h)\right]\sqrt{\frac{h}{t}}. \tag{I.11}$$

Decompose this probability in two terms:

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\}}\mathrm{d}s \geqslant t\right] \leqslant \mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{t}{2}\right] + \mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s^{\complement}\}}\mathrm{d}s \geqslant \frac{t}{2}\right], \tag{I.12}$$

where $H_s^{\complement}$ denotes the complement of the event H_s.

- **First step:** Study of $G_s \cap H_s$.

  Fix $s \in [0,T]$. Under $G_s \cap H_s$, we have $M_{\sigma,\varepsilon}(u,s) < \frac{h}{2^6}$ and $\widetilde{N}_s > \frac{\sigma+\eta}{2}$. We want to show that it implies the following inequality:

  $$\frac{2m_{\sigma,\varepsilon}(u,u-h,s)}{(\varepsilon + m_{\sigma,\varepsilon}(u,s))(\varepsilon + m_{\sigma,\varepsilon}(u-h,s))} \leqslant \frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4}M_{\sigma,\varepsilon}(u-h,s)^{1/4}}. \tag{I.13}$$

  Assume, by contradiction, that (I.13) is false. By Cauchy-Schwarz inequality, $m_{\sigma,\varepsilon}(u,u-h,s) \leqslant m_{\sigma,\varepsilon}(u,s)^{1/2}m_{\sigma,\varepsilon}(u-h,s)^{1/2}$ and we would deduce that:

  $$\frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4}M_{\sigma,\varepsilon}(u-h,s)^{1/4}} \leqslant \frac{2}{M_{\sigma,\varepsilon}(u,s)^{1/2}M_{\sigma,\varepsilon}(u-h,s)^{1/2}},$$

  and thus $M_{\sigma,\varepsilon}(u-h,s) \leqslant 2^4 M_{\sigma,\varepsilon}(u,s)$. Using the fact that $M_{\sigma,\varepsilon} \geqslant m_{\sigma,\varepsilon}$, we can deduce that

  $$m_{\sigma,\varepsilon}(u,s) + m_{\sigma,\varepsilon}(u-h,s) \leqslant M_{\sigma,\varepsilon}(u,s) + M_{\sigma,\varepsilon}(u-h,s) \leqslant (1+2^4)\tfrac{h}{2^6} < \tfrac{h}{3}. \tag{I.14}$$

  We distinguish three cases depending on the value of $N_s = y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(u-h,s)$.

  - $N_s \leqslant \sigma - \eta$**:** For each $v \in [u-h,u]$, one of the two terms $y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)$ and $y_{\sigma,\varepsilon}(v,s) - y_{\sigma,\varepsilon}(u-h,s)$ is lower than $\frac{\sigma-\eta}{2}$, which means that one of those terms belongs to the preimage of 1 by the function φ_σ. Hence

    $$\begin{aligned} m_{\sigma,\varepsilon}(u,s) &+ m_{\sigma,\varepsilon}(u-h,s) \\ &= \int_0^1 \left(\varphi_\sigma^2(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)) + \varphi_\sigma^2(y_{\sigma,\varepsilon}(u-h,s) - y_{\sigma,\varepsilon}(v,s))\right)\mathrm{d}v \\ &\geqslant \int_{u-h}^u \mathrm{d}v = h. \end{aligned}$$

    This is in contradiction with (I.14). Therefore inequality (I.13) is satisfied in this case.

  - $N_s \in (\sigma-\eta, \sigma)$**:** Introduce $\mathrm{Med} := \{v : y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) \in [\frac{\sigma-\eta}{2}, \frac{\sigma+\eta}{2}]\}$, which is a set of particles more or less at half distance between particle u and particle $u-h$. Since $\eta < \frac{\sigma}{3}$, we have $N_s > \sigma - \eta \geqslant \frac{\sigma+\eta}{2}$ and thus $\mathrm{Med} \subset [u-h,u]$. Let $v \in [u-h,u]$. We distinguish three new cases:
    - if $y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) < \frac{\sigma-\eta}{2}$, then $\varphi_\sigma(y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s)) = 1$.
    - if $y_{\sigma,\varepsilon}(u,s) - y_{\sigma,\varepsilon}(v,s) > \frac{\sigma+\eta}{2}$, and since $N_s \leqslant \sigma$, $y_{\sigma,\varepsilon}(v,s) - y_{\sigma,\varepsilon}(u-h,s)$ is lower than $\frac{\sigma-\eta}{2}$ and thus $\varphi_\sigma(y_{\sigma,\varepsilon}(u-h,s) - y_{\sigma,\varepsilon}(v,s)) = 1$.
    - otherwise, v belongs to Med.

It follows that:

$$\begin{aligned} h &= \int_{u-h}^{u} (\mathbb{1}_{\{y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s)<\frac{\sigma-\eta}{2}\}} + \mathbb{1}_{\{y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s)>\frac{\sigma+\eta}{2}\}} + \mathbb{1}_{\{v\in \mathrm{Med}\}})\mathrm{d}v \\ &\leqslant \int_{u-h}^{u} (\varphi_\sigma^2(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s)) + \varphi_\sigma^2(y_{\sigma,\varepsilon}(u-h,s)-y_{\sigma,\varepsilon}(v,s)) + \mathbb{1}_{\{v\in \mathrm{Med}\}})\mathrm{d}v \\ &\leqslant m_{\sigma,\varepsilon}(u,s) + m_{\sigma,\varepsilon}(u-h,s) + \mathrm{Leb}(\mathrm{Med}). \end{aligned}$$

By inequality (I.14), we deduce that $\mathrm{Leb}(\mathrm{Med}) > \frac{2h}{3}$. As Med is an interval included in $[u-h,u]$ and since $k \in (\frac{h}{3}, \frac{2h}{3})$ we deduce that $u-k \in \mathrm{Med}$, *i.e.* $\widetilde{N}_s \in [\frac{\sigma-\eta}{2}, \frac{\sigma+\eta}{2}]$, which is in contradiction with the hypothesis $\widetilde{N}_s > \frac{\sigma+\eta}{2}$. Thus inequality (I.13) is also true in this case.

- $N_s \geqslant \sigma$**:** In this case, the two particles u and $u-h$ do not have any interaction. In other words, since the support of φ_σ is included in $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$, $\varphi_\sigma(y_{\sigma,\varepsilon}(u,s)-y_{\sigma,\varepsilon}(v,s))$ and $\varphi_\sigma(y_{\sigma,\varepsilon}(u-h,s)-y_{\sigma,\varepsilon}(v,s))$ can not be simultaneously non-zero, whence we deduce that $m_{\sigma,\varepsilon}(u,u-h,s)=0$. Inequality (I.13) follows clearly.

Therefore, inequality (I.13) is proved. By Corollary I.18, it follows that, on $G_s \cap H_s$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s &= \frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u-h,s)} - \frac{2m_{\sigma,\varepsilon}(u,u-h,s)}{(\varepsilon+m_{\sigma,\varepsilon}(u,s))(\varepsilon+m_{\sigma,\varepsilon}(u-h,s))} \\ &\geqslant \frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u-h,s)} - \frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4}M_{\sigma,\varepsilon}(u-h,s)^{1/4}} \\ &\geqslant \frac{1}{4M_{\sigma,\varepsilon}(u,s)} + \frac{3}{4M_{\sigma,\varepsilon}(u-h,s)} \geqslant \frac{1}{4M_{\sigma,\varepsilon}(u,s)} \geqslant \frac{2^4}{h}, \end{aligned}$$

where we have applied a convexity inequality: $\forall a,b>0, a^{3/4}b^{1/4} \leqslant \frac{3a}{4} + \frac{b}{4}$.

To sum up, we showed that $G_s \cap H_s$ implies $\frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s \geqslant \frac{2^4}{h}$. If $\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{t}{2}$, we get

$$\langle N,N\rangle_T = \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s \mathrm{d}s \geqslant \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\langle N,N\rangle_s \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{2^4}{h}\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{2^3 t}{h}.$$

Hence, since N is a continuous square integrable $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-martingale, there exists a standard $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-Brownian motion β such that we have the relation $N_t = g(u) - g(u-h) - \beta(\langle N,N\rangle_t)$. Since N remains positive on $[0,T]$ by Lemma I.16 (because $g(u-h) < g(u)$), we deduce that $\sup_{[0,\langle N,N\rangle_T]}\beta \leqslant g(u)-g(u-h)$. Therefore,

$$\begin{aligned} \mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s\cap H_s\}}\mathrm{d}s \geqslant \frac{t}{2}\right] &\leqslant \mathbb{P}\left[\sup_{[0,\frac{2^3t}{h}]}\beta \leqslant g(u)-g(u-h)\right] \\ &= \mathbb{P}\left[\sqrt{\frac{2^3}{h}}\sup_{[0,t]}\widehat{\beta} \leqslant g(u)-g(u-h)\right] \leqslant C_2\left[g(u)-g(u-h)\right]\sqrt{\frac{h}{t}}, \end{aligned} \tag{I.15}$$

where $\widehat{\beta}$ is a rescaled Brownian motion and C_2 does not depend on u, h, σ, ε and t.

- **Second step:** Study of $G_s \cap H_s^{\complement}$.

Under this event, we have $M_{\sigma,\varepsilon}(u,s) < \frac{h}{2^6}$ and $\widetilde{N}_s \leqslant \frac{\sigma+\eta}{2}$. In particular, by the assumption $\eta < \frac{\sigma}{3}$, we have $\widetilde{N}_s \leqslant \sigma - \eta$. We claim that the following inequality holds true:

$$\frac{2m_{\sigma,\varepsilon}(u,u-k,s)}{(\varepsilon+m_{\sigma,\varepsilon}(u,s))(\varepsilon+m_{\sigma,\varepsilon}(u-k,s))} \leqslant \frac{1}{M_{\sigma,\varepsilon}(u,s)^{3/4}M_{\sigma,\varepsilon}(u-k,s)^{1/4}}. \tag{I.16}$$

To prove it, it is sufficient to imitate the proof of the case $N_s \leqslant \sigma - \eta$ of the previous step. We should notice that we did not use the hypothesis $\widetilde{N}_s > \frac{\sigma+\eta}{2}$ in that case.

Using inequality (I.16) as in the first step, we show that $\frac{\mathrm{d}}{\mathrm{d}s}\langle \widetilde{N}, \widetilde{N}\rangle_s \geqslant \frac{2^4}{h}$. Therefore, $\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s \cap H_s^{\complement}\}}\mathrm{d}s \geqslant \frac{t}{2}\right] \leqslant \mathbb{P}\left[\langle \widetilde{N}, \widetilde{N}\rangle_T \geqslant \frac{2^3 t}{h}\right]$. There exists a $(\mathcal{F}_t^{\sigma,\varepsilon})_{t\in[0,T]}$-Brownian motion $\widetilde{\beta}$ such that $\widetilde{N}_t = g(u) - g(u-k) - \widetilde{\beta}(\langle \widetilde{N}, \widetilde{N}\rangle_t)$. Finally, we obtain the existence of a constant C_3 independent of u, h, k, σ, ε and t such that:

$$\begin{aligned}\mathbb{P}\left[\int_0^T \mathbb{1}_{\{G_s \cap H_s^{\complement}\}}\mathrm{d}s \geqslant \frac{t}{2}\right] &\leqslant \mathbb{P}\left[\sup_{[0,\frac{2^3 t}{h}]} \widetilde{\beta} \leqslant g(u) - g(u-k)\right] \\ &\leqslant C_3\left[g(u) - g(u-k)\right]\sqrt{\frac{h}{t}} \\ &\leqslant C_3\left[g(u) - g(u-h)\right]\sqrt{\frac{h}{t}}. \end{aligned} \tag{I.17}$$

Putting together inequality (I.12) and inequalities (I.15) and (I.17), we conclude the proof of inequality (I.11). Thus inequality (I.10) is proved for every h such that $u - h \in \mathcal{A}$. Let $h > 0$ be such that $u - h \in (0,1)$. Let $h_1 \in (\frac{h}{2}, h)$ be such that $u - h_1 \in \mathcal{A}$.

$$\begin{aligned}\mathbb{P}\left[\int_0^T \mathbb{1}_{\left\{M_{\sigma,\varepsilon}(u,s) < \frac{\gamma h}{2}\right\}}\mathrm{d}s \geqslant t\right] &\leqslant \mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,s) < \gamma h_1\}}\mathrm{d}s \geqslant t\right] \\ &\leqslant C\left[g(u) - g(u-h_1)\right]\sqrt{\frac{h_1}{t}} \\ &\leqslant C\left[g(u) - g(u-h)\right]\sqrt{\frac{h}{t}}. \end{aligned}$$

Up to replacing γ by $\frac{\gamma}{2}$, inequality (I.10) follows for every $h > 0$ such that $u - h \in (0,1)$. □

Remark I.23. Similarly, there exist $C > 0$ and $\gamma \in (0,1)$ such that for each $\sigma, \varepsilon > 0$, $t \in (0,T]$ and for every $u \in \mathcal{A}$ and every $h > 0$ satisfying $u + h \in (0,1)$,

$$\mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,s) < \gamma h\}}\mathrm{d}s \geqslant t\right] \leqslant C\left[g(u+h) - g(u)\right]\sqrt{\frac{h}{t}}.$$

Thanks to Lemma I.22 and to the above remark, we obtain the following result, which has to be compared with Proposition 4.3 in [Kon17a]:

Lemma I.24. *Let $g \in L_p(0,1)$. For all $\beta \in (0, \frac{3}{2} - \frac{1}{p})$, there is a constant $C > 0$ depending only on β and $\|g\|_{L_p}$ such that for all $\sigma, \varepsilon > 0$ and $0 \leqslant s < t \leqslant T$, we have the following inequality:*

$$\mathbb{E}\left[\int_s^t \int_0^1 \frac{1}{M_{\sigma,\varepsilon}^{\beta}(u,r)}\mathrm{d}u\mathrm{d}r\right] \leqslant C\sqrt{t-s}. \tag{I.18}$$

Remark I.25. Observe that by the assumption $p > 2$, made at the beginning of Section I.4, there exists some $\beta > 1$ such that (I.18) holds.

Proof. By Fubini-Tonelli Theorem, we have:

$$\begin{aligned}\mathbb{E}\left[\int_s^t\int_0^1 \frac{\mathrm{d}u\mathrm{d}r}{M^{\beta}_{\sigma,\varepsilon}(u,r)}\right] &= \int_0^1 \mathbb{E}\left[\int_s^t\int_0^{+\infty} \mathbb{1}_{\{M^{-\beta}_{\sigma,\varepsilon}(u,r)>x\}}\mathrm{d}x\mathrm{d}r\right]\mathrm{d}u\\ &\leqslant 2^{\beta}(t-s) + \int_0^1\int_{2^{\beta}}^{+\infty}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<x^{-1/\beta}\}}\mathrm{d}r\right]\mathrm{d}x\mathrm{d}u\\ &\leqslant 2^{\beta}\sqrt{T}\sqrt{t-s} + \int_0^1\int_{2^{\beta}\gamma^{\beta}}^{+\infty}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r\right]\gamma^{-\beta}\mathrm{d}x\mathrm{d}u.\end{aligned}$$

Furthermore, we compute:

$$\begin{aligned}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r\right] &= \int_0^{t-s}\mathbb{P}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r > \alpha\right]\mathrm{d}\alpha\\ &\leqslant \int_0^{t-s}\mathbb{P}\left[\int_0^T \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r > \alpha\right]\mathrm{d}\alpha.\end{aligned}$$

Using Lemma I.22, we obtain a constant C_1 independent of σ and ε such that for all $x > 2^{\beta}$:

$$\begin{aligned}\int_{\frac12}^1 \mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<\gamma x^{-1/\beta}\}}\mathrm{d}r\right]\mathrm{d}u &\leqslant \int_{\frac12}^1\int_0^{t-s} C_1\left[g(u)-g(u-x^{-1/\beta})\right]\sqrt{\frac{x^{-1/\beta}}{\alpha}}\mathrm{d}\alpha\mathrm{d}u\\ &\leqslant 2C_1\frac{\int_{1/2}^1(g(u)-g(u-x^{-1/\beta}))\mathrm{d}u}{x^{1/(2\beta)}}\sqrt{t-s}.\end{aligned}$$

Moreover, we have for each $x > 2^{\beta}$, using Hölder's inequality:

$$\begin{aligned}\int_{\frac12}^1\left(g(u)-g(u-x^{-1/\beta})\right)\mathrm{d}u &= \int_0^1\left(\mathbb{1}_{[\frac12,1]}(u)-\mathbb{1}_{[\frac12-x^{-1/\beta},1-x^{-1/\beta}]}(u)\right)g(u)\mathrm{d}u\\ &\leqslant \|g\|_{L_p}(2x^{-1/\beta})^{1-\frac1p}. \qquad \text{(I.19)}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\frac12}^1\int_{2^{\beta}\gamma^{\beta}}^{+\infty}\mathbb{E}\left[\int_s^t \mathbb{1}_{\{M_{\sigma,\varepsilon}(u,r)<x^{-1/\beta}\}}\mathrm{d}r\right]\mathrm{d}x\mathrm{d}u &\leqslant C_2\int_{2^{\beta}\gamma^{\beta}}^{+\infty}\frac{\|g\|_{L_p}\sqrt{t-s}}{x^{\frac{1}{2\beta}}x^{\frac1\beta(1-\frac1p)}}\mathrm{d}x\\ &\leqslant C_3\|g\|_{L_p}\sqrt{t-s},\end{aligned}$$

where C_2 and C_3 are independent of σ, ε, and t. The last inequality holds because $\frac1\beta\left(\frac32-\frac1p\right) > 1$.

We conclude the proof of the lemma by using a similar argument for u belonging to $[0,\frac12]$ and using $g(u+x^{-1/\beta})-g(u)$ instead of $g(u)-g(u-x^{-1/\beta})$. □

Corollary I.26. *There is a constant C such that for every $t\in[0,T]$ and for every $\sigma,\varepsilon>0$,*

$$\mathbb{E}\left[\int_0^1 y^2_{\sigma,\varepsilon}(u,t)\mathrm{d}u\right]\leqslant C.$$

Proof. We have:

$$\mathbb{E}\left[\int_0^1 y^2_{\sigma,\varepsilon}(u,t)\mathrm{d}u\right]^{1/2}\leqslant \mathbb{E}\left[\int_0^1 g(u)^2\mathrm{d}u\right]^{1/2}+\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t)-g(u))^2\mathrm{d}u\right]^{1/2}.$$

Since g belongs to $L_2(0,1)$, the first term of the right hand side is bounded. Furthermore, by Corollary I.18 and Fubini-Tonelli Theorem:

$$\begin{aligned}\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t)-g(u))^2\mathrm{d}u\right] &= \int_0^1\mathbb{E}\left[\langle y_{\sigma,\varepsilon}(u,\cdot),y_{\sigma,\varepsilon}(u,\cdot)\rangle_t\right]\mathrm{d}u = \int_0^1\mathbb{E}\left[\int_0^t\frac{1}{M_{\sigma,\varepsilon}(u,s)}\mathrm{d}s\right]\mathrm{d}u\\ &\leqslant C\sqrt{t},\end{aligned}$$

by Lemma I.24. □

I.4.3 Proof of Proposition I.20

We will now use Lemma I.24 and its Corollary I.26 to prove Proposition I.20. We start by $(K1)$:

Proposition I.27. *Let $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$ and δ be positive. Then there exists $M > 0$ such that for all $\sigma > 0$ and $\varepsilon > 0$, $\mathbb{P}\left[\int_0^1 \|y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \geqslant M\right] \leqslant \delta$.*

Proof. Using again Fubini-Tonelli Theorem,

$$\mathbb{E}\left[\int_0^1 \sup_{t\leqslant T}|y_{\sigma,\varepsilon}(u,t)|^2\mathrm{d}u\right] = \int_0^1 \mathbb{E}\left[\sup_{t\leqslant T}|y_{\sigma,\varepsilon}(u,t)|^2\right]\mathrm{d}u.$$

Moreover, for almost every $u \in [0,1]$, $y_{\sigma,\varepsilon}(u,\cdot)$ is a $(\mathcal{F}^{\sigma,\varepsilon}_t)_{t\in[0,T]}$-martingale. Hence by Doob's inequality, there is a constant C_1 independent of u, σ and ε such that:

$$\mathbb{E}\left[\sup_{t\leqslant T}|y_{\sigma,\varepsilon}(u,t)|^2\right] \leqslant C_1\mathbb{E}\left[|y_{\sigma,\varepsilon}(u,T)|^2\right].$$

Therefore, by Corollary I.26,

$$\mathbb{E}\left[\int_0^1 \sup_{t\leqslant T}|y_{\sigma,\varepsilon}(u,t)|^2\mathrm{d}u\right] \leqslant C_1\int_0^1 \mathbb{E}\left[|y_{\sigma,\varepsilon}(u,T)|^2\right]\mathrm{d}u \leqslant C_2, \tag{I.20}$$

where C_2 is independent of σ and ε. We conclude by Markov's inequality: there is a constant $C > 0$ such that for all $\sigma, \varepsilon > 0$,

$$\mathbb{P}\left[\int_0^1 \|y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \geqslant M\right] \leqslant \frac{\mathbb{E}\left[\int_0^1 \sup_{t\leqslant T}|y_{\sigma,\varepsilon}(u,t)|^2\mathrm{d}u\right]}{M} \leqslant \frac{C}{M}.$$

For M large enough, that last quantity is smaller than δ. □

Then, we show criterion $(K2)$:

Proposition I.28. *Let $g \in L_p[0,1]$ and $\delta > 0$. Then for all $k \geqslant 1$, there exists $\eta_k > 0$ such that for every $\sigma, \varepsilon > 0$,*

$$\mathbb{P}\left[\int_0^1 \sup_{|t_2-t_1|<\eta_k}|y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u \geqslant \frac{1}{k}\right] \leqslant \frac{\delta}{2^k}.$$

Proof. By Markov's inequality, it is sufficient to prove that:

$$\lim_{\eta\to 0^+}\sup_{\sigma>0,\varepsilon>0}\mathbb{E}\left[\int_0^1 \sup_{|t_2-t_1|<\eta}|y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u\right] = 0. \tag{I.21}$$

Fix $\delta > 0$ and $\beta \in (1, \frac{3}{2} - \frac{1}{p})$. For every $u \in (0,1)$, define

$$K_1(u) := \mathbb{E}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]}\right],$$
$$K_2(u) := \mathbb{E}\left[\int_0^T \frac{1}{M^{\beta}_{\sigma,\varepsilon}(u,s)}\mathrm{d}s\right].$$

Since $y_{\sigma,\varepsilon}$ is uniformly bounded for $\sigma > 0$ and $\varepsilon > 0$ in $L_2([0,1],\mathcal{C}[0,T])$ (see inequality (I.20)) and by Lemma I.24, $\int_0^1 K_1(u)\mathrm{d}u$ and $\int_0^1 K_2(u)\mathrm{d}u$ are uniformly bounded for $\sigma > 0$ and $\varepsilon > 0$.

Therefore, there exists $C > 0$ such that $\int_0^1 \mathbb{1}_{\{K_1(u)\geqslant C\}}\mathrm{d}u \leqslant \delta$ and $\int_0^1 \mathbb{1}_{\{K_2(u)\geqslant C\}}\mathrm{d}u \leqslant \delta$. We define:

$$K_1 := \{u \in (0,1) : K_1(u) \leqslant C\},$$
$$K_2 := \{u \in (0,1) : K_2(u) \leqslant C\}.$$

The collection $(y_{\sigma,\varepsilon}(u,\cdot))_{\sigma>0,\varepsilon>0,u\in K_1\cap K_2}$ is tight in $\mathcal{C}[0,T]$. We use Aldous' tightness criterion to prove this claim (see [Bil99, Theorem 16.10]). We prove the following two statements:

- $\lim_{a\to\infty}\sup_{\sigma>0,\varepsilon>0,u\in K_1\cap K_2}\mathbb{P}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]}\geqslant a\right] = 0$.
- for all $\alpha > 0$ and $r > 0$, there is η_0 such that for all $\eta \in (0,\eta_0)$, for all $\sigma > 0$, $\varepsilon > 0$ and $u \in K_1 \cap K_2$, if τ is a stopping time for $y_{\sigma,\varepsilon}(u,\cdot)$ such that $\tau \leqslant T$, then $\mathbb{P}\left[|y_{\sigma,\varepsilon}(u,\tau+\eta) - y_{\sigma,\varepsilon}(u,\tau)| \geqslant r\right] \leqslant \alpha$.

By Markov's inequality, for all $a > 0$, $\sigma > 0$, $\varepsilon > 0$ and $u \in K_1 \cap K_2$,

$$\mathbb{P}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]}\geqslant a\right] \leqslant \frac{1}{a}\mathbb{E}\left[\|y_{\sigma,\varepsilon}(u,\cdot)\|_{\mathcal{C}[0,T]}\right] = \frac{K_1(u)}{a} \leqslant \frac{C}{a},$$

whence we obtain the first statement. Moreover, for all $u \in K_1 \cap K_2$, by Hölder's inequality,

$$\mathbb{E}\left[|y_{\sigma,\varepsilon}(u,\tau+\eta) - y_{\sigma,\varepsilon}(u,\tau)|^2\right] = \mathbb{E}\left[\int_\tau^{\tau+\eta}\frac{1}{M_{\sigma,\varepsilon}(u,s)}\mathrm{d}s\right] \leqslant K_2(u)^{\frac{1}{\beta}}\eta^{1-\frac{1}{\beta}} \leqslant C^{\frac{1}{\beta}}\eta^{1-\frac{1}{\beta}},$$

whence we obtain the second statement.

By Aldous' tightness criterion, there exists a compact L of the set $\mathcal{D}[0,T]$ of càdlàg functions on $[0,T]$ such that for all $\sigma > 0$, $\varepsilon > 0$ and $u \in K_1 \cap K_2$, $\mathbb{P}\left[y_{\sigma,\varepsilon}(u,\cdot) \in L\right] \geqslant 1-\delta$. Since $\mathcal{C}[0,T]$ is closed in $\mathcal{D}[0,T]$ with respect to Skorohod's topology, and $y_{\sigma,\varepsilon}(u,\cdot) \in \mathcal{C}[0,T]$ almost surely, we may suppose that L is a compact set of $\mathcal{C}[0,T]$.

Back to (I.21), we have:

$$\begin{aligned}\mathbb{E}\left[\int_0^1 \sup_{|t_2-t_1|<\eta}|y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u\right] &= \int_0^1 \mathbb{E}\left[\sup_{|t_2-t_1|<\eta}|y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right]\mathrm{d}u \\ &= \int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in K_1\cap K_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}^\complement}\sup_{|t_2-t_1|<\eta}|y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right]\mathrm{d}u \\ &\quad + \int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in K_1\cap K_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}}\sup_{|t_2-t_1|<\eta}|y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right]\mathrm{d}u.\end{aligned} \tag{I.22}$$

The first term on the right hand side of (I.22) is bounded by:

$$\left(\int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in K_1\cap K_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}^\complement}\right]\mathrm{d}u\right)^{1/2}\left(\int_0^1 \mathbb{E}\left[\sup_{|t_2-t_1|<\eta}|y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|^2\right]\mathrm{d}u\right)^{1/2}.$$

We have:

$$\begin{aligned}\int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in K_1\cap K_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}^\complement}\right]\mathrm{d}u &\leqslant \int_0^1 \mathbb{1}_{\{u\in K_1\cap K_2\}}\mathbb{P}\left[y_{\sigma,\varepsilon}(u,\cdot)\notin L\right]\mathrm{d}u \\ &\quad + \int_0^1 \mathbb{1}_{\{K_1(u)\geqslant C\}}\mathrm{d}u + \int_0^1 \mathbb{1}_{\{K_2(u)\geqslant C\}}\mathrm{d}u \\ &\leqslant 3\delta.\end{aligned}$$

Moreover,

$$\int_0^1 \mathbb{E}\left[\sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|^2\right] \mathrm{d}u \leqslant 4\int_0^1 \mathbb{E}\left[\sup_{t\leqslant T} |y_{\sigma,\varepsilon}(u,t)|^2\right] \mathrm{d}u \leqslant 4M,$$

where M is a constant independent of $\sigma > 0$ and $\varepsilon > 0$ by inequality (I.20).

It remains to handle the second term on the right hand side of (I.22). Since L is a compact set of $\mathcal{C}[0,T]$, there exists $\eta > 0$ such that for every $f \in L$, $\omega_f(\eta) := \sup_{|t-s|<\eta} |f(t) - f(s)| < \delta$. Therefore, there exists $\eta > 0$ such that:

$$\int_0^1 \mathbb{E}\left[\mathbb{1}_{\{u\in K_1\cap K_2, y_{\sigma,\varepsilon}(u,\cdot)\in L\}} \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\right] \mathrm{d}u \leqslant \delta.$$

Back to equality (I.22), we have proved that there is $\eta > 0$ such that for every $\sigma > 0$ and $\varepsilon > 0$:

$$\mathbb{E}\left[\int_0^1 \sup_{|t_2-t_1|<\eta} |y_{\sigma,\varepsilon}(u,t_2) - y_{\sigma,\varepsilon}(u,t_1)|\mathrm{d}u\right] \leqslant \delta + \sqrt{12\delta M}.$$

This proves convergence (I.21) and thus completes the proof of the proposition. □

Then, to obtain criterion $(K3)$, we state the following Proposition:

Proposition I.29. *Let $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$ and $\delta > 0$. Then for all $k \geqslant 1$, there is $h_k > 0$ such that for all $\sigma, \varepsilon > 0$,*

$$\mathbb{P}\left[\int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \geqslant \frac{1}{k}\right] \leqslant \frac{\delta}{2^k}.$$

If $\int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \leqslant \frac{1}{k}$, we deduce by monotonicity of $u \mapsto y_{\sigma,\varepsilon}(u,t)$ for every $t \in [0,T]$ that for every $h \in (0, h_k)$,

$$\begin{aligned}
&\int_0^{1-h} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \\
&\leqslant \int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u + \int_{1-2h_k+h}^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u+h_k-h,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \\
&\leqslant 2\int_0^{1-h_k} \|y_{\sigma,\varepsilon}(u+h_k,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \leqslant \frac{2}{k}.
\end{aligned}$$

Therefore, the latter proposition implies the following corollary, which is equivalent to criterion $(K3)$:

Corollary I.30. *Let $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$ and $\delta > 0$. Then for all $k \geqslant 1$, there is $h_k > 0$ such that for all $\sigma, \varepsilon > 0$,*

$$\mathbb{P}\left[\forall h \in (0,h_k), \int_0^{1-h} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u \leqslant \frac{2}{k}\right] \geqslant 1 - \frac{\delta}{2^k}.$$

Proof (Proposition I.29). Let $h \in (0,1)$. By Corollary I.17, for almost every $u \in (0, 1-h)$, $N_{u,t} := y_{\sigma,\varepsilon}(u+h,t) - y_{\sigma,\varepsilon}(u,t)$ is a martingale. By Fubini-Tonelli Theorem and Doob's inequality, we have:

$$\mathbb{E}\left[\int_0^{1-h} \|N_{u,\cdot}\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\right] = \int_0^{1-h} \mathbb{E}\left[\|N_{u,\cdot}\|^2_{\mathcal{C}[0,T]}\right] \mathrm{d}u \leqslant C\int_0^{1-h} \mathbb{E}\left[N^2_{u,T}\right] \mathrm{d}u. \tag{I.23}$$

Let us split $\mathbb{E}\left[N^2_{u,T}\right]$ in two terms $\mathbb{E}\left[N^2_{u,T}\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right] + \mathbb{E}\left[N^2_{u,T}\mathbb{1}_{\{N_{u,T}>1\}}\right]$.

Study of $\int_0^{1-h}\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right]\mathrm{d}u$. Let $u\in(0,1-h)$ be such that $N_{u,\cdot}$ is a martingale. By Lemma I.16, if $g(u+h)-g(u)=0$, then $N_{u,T}=0$ almost surely, thus $\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{N_{u,T}\leqslant 1}\right]=0$. From now on, we suppose that $g(u+h)-g(u)>0$. $N_{u,\cdot}$ is a square integrable continuous martingale, starting from $g(u+h)-g(u)>0$ and positive by Lemma I.16. Therefore, there exists a standard Brownian motion β_u such that $N_{u,t}=N_{u,0}+\beta_u(\langle N_{u,\cdot},N_{u,\cdot}\rangle_t)$. Recall that $N_{u,0}=g(u+h)-g(u)$ is a deterministic quantity. If $N_{u,0}\geqslant 1$, then the inequality $\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right]\leqslant N_{u,0}$ is obvious. Otherwise, we have

$$\begin{aligned}\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right]=\int_0^{+\infty}\mathbb{P}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\geqslant\lambda\right]\mathrm{d}\lambda&\leqslant\int_0^1\mathbb{P}\left[N_{u,T}^2\geqslant\lambda\right]\mathrm{d}\lambda\\&\leqslant N_{u,0}^2+\int_{N_{u,0}^2}^1\mathbb{P}\left[N_{u,T}\geqslant\lambda^{1/2}\right]\mathrm{d}\lambda.\quad\text{(I.24)}\end{aligned}$$

Let us estimate $\mathbb{P}\left[N_{u,T}\geqslant\kappa\right]$ for a real number $\kappa>N_{u,0}$. We define the following stopping times:

$$\begin{aligned}\tau_{-N_{u,0}}&:=\inf\{t\geqslant 0:N_{u,0}+\beta_u(t)\leqslant 0\};\\\tau_{\kappa-N_{u,0}}&:=\inf\{t\geqslant 0:N_{u,0}+\beta_u(t)\geqslant\kappa\};\\\tau&:=\inf\{t\geqslant 0:N_{u,t}\geqslant\kappa\}\wedge T.\end{aligned}$$

On the first hand, we know that almost surely, for all $t\in[0,T]$, $N_{u,t}>0$, hence $\tau_{-N_{u,0}}\geqslant\langle N_{u,\cdot},N_{u,\cdot}\rangle_T$. On the other hand, if $N_{u,T}\geqslant\kappa$, $N_{u,\tau}$ is equal to κ by continuity of $N_{u,\cdot}$, hence $\langle N_{u,\cdot},N_{u,\cdot}\rangle_\tau\geqslant\tau_{\kappa-N_{u,0}}$. It follows from both inequalities that $\tau_{\kappa-N_{u,0}}\leqslant\tau_{-N_{u,0}}$. Therefore,

$$\mathbb{P}\left[N_{u,T}\geqslant\kappa\right]\leqslant\mathbb{P}\left[\tau_{\kappa-N_{u,0}}\leqslant\tau_{-N_{u,0}}\right]=\frac{N_{u,0}}{\kappa},\qquad\text{(I.25)}$$

by a usual martingale equality. Using inequality (I.24) and $N_{u,0}\leqslant 1$, we obtain:

$$\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right]\leqslant N_{u,0}^2+\int_{N_{u,0}^2}^1\frac{N_{u,0}}{\lambda^{1/2}}\mathrm{d}\lambda\leqslant N_{u,0}^2+2N_{u,0}\leqslant 3N_{u,0}.$$

Therefore, we have: $\int_0^{1-h}\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}\leqslant 1\}}\right]\mathrm{d}u\leqslant 3\int_0^{1-h}N_{u,0}\mathrm{d}u$.

Study of $\int_0^{1-h}\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}>1\}}\right]\mathrm{d}u$. Recall that g belongs to $L_p(0,1)$ for some $p>2$. Fix $\beta\in(1,\frac{3}{2}-\frac{1}{p})$. We compute:

$$\begin{aligned}&\int_0^{1-h}\mathbb{E}\left[N_{u,T}^2\mathbb{1}_{\{N_{u,T}>1\}}\right]\mathrm{d}u\\&\quad\leqslant 2\int_0^{1-h}\mathbb{E}\left[(N_{u,T}-N_{u,0})^2\mathbb{1}_{\{N_{u,T}>1\}}\right]\mathrm{d}u+2\int_0^{1-h}\mathbb{E}\left[N_{u,0}^2\mathbb{1}_{\{N_{u,T}>1\}}\right]\mathrm{d}u\\&\quad\leqslant 2\left(\int_0^{1-h}\mathbb{E}\left[(N_{u,T}-N_{u,0})^{2\beta}\right]\mathrm{d}u\right)^{\frac{1}{\beta}}\left(\int_0^{1-h}\mathbb{P}\left[N_{u,T}>1\right]\mathrm{d}u\right)^{1-\frac{1}{\beta}}+2\int_0^{1-h}N_{u,0}^2\mathrm{d}u.\end{aligned}$$

Furthermore, we have $\mathbb{P}\left[N_{u,T}>1\right]\leqslant N_{u,0}$: that inequality is obvious if $N_{u,0}\geqslant 1$ and otherwise, it is a consequence of inequality (I.25).

Then, we are willing to give an upper bound for $\mathbb{E}\left[(N_{u,T}-N_{u,0})^{2\beta}\right]$. Using Burkholder-Davis-Gundy inequality, there is C_β such that $\mathbb{E}\left[(N_{u,T}-N_{u,0})^{2\beta}\right]\leqslant C_\beta\mathbb{E}\left[\langle N_{u,\cdot},N_{u,\cdot}\rangle_T^\beta\right]$.

We compute the quadratic variation of the martingale $N_{u,t} = y_{\sigma,\varepsilon}(u+h,t) - y_{\sigma,\varepsilon}(u,t)$:

$$\mathbb{E}\left[\langle N_{u,\cdot}, N_{u,\cdot}\rangle_T^\beta\right] = \mathbb{E}\Bigg[\bigg|\int_0^T \Big(\frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u+h,s)} - \frac{2m_{\sigma,\varepsilon}(u,u+h,s)}{(\varepsilon + m_{\sigma,\varepsilon}(u,s))(\varepsilon + m_{\sigma,\varepsilon}(u+h,s))}\Big)\mathrm{d}s\bigg|^\beta\Bigg].$$

By Cauchy-Schwarz inequality $m_{\sigma,\varepsilon}(u,u+h,s) \leqslant m_{\sigma,\varepsilon}^{1/2}(u,s) m_{\sigma,\varepsilon}^{1/2}(u+h,s)$, we deduce that the sum of the three terms in the integral is non-negative and thus that it is bounded by $\frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u+h,s)}$, whence we obtain:

$$\begin{aligned}\mathbb{E}\left[\langle N_{u,\cdot}, N_{u,\cdot}\rangle_T^\beta\right] &\leqslant T^{\beta-1}\mathbb{E}\left[\int_0^T \left|\frac{1}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{M_{\sigma,\varepsilon}(u+h,s)}\right|^\beta \mathrm{d}s\right] \\ &\leqslant C_{\beta,T}\left(\mathbb{E}\left[\int_0^T \frac{\mathrm{d}s}{M_{\sigma,\varepsilon}^\beta(u,s)}\right] + \mathbb{E}\left[\int_0^T \frac{\mathrm{d}s}{M_{\sigma,\varepsilon}^\beta(u+h,s)}\right]\right).\end{aligned}$$

By Lemma I.24, we deduce that $\int_0^{1-h} \mathbb{E}\left[\langle N_{u,\cdot}, N_{u,\cdot}\rangle_T^\beta\right] \mathrm{d}u$ is bounded, because $\beta < \frac{3}{2} - \frac{1}{p}$. Therefore, we can conclude that there is a constant $C_{T,\beta}$ such that:

$$\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2 \mathbb{1}_{\{N_{u,T}>1\}}\right] \mathrm{d}u \leqslant 2C_{T,\beta}\left(\int_0^{1-h} N_{u,0}\,\mathrm{d}u\right)^{1-1/\beta} + 2\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u.$$

Conclusion: Putting together the studies of both cases, we have proved that there is a positive constant C satisfying, for all σ, ε and $h \in (0,1)$:

$$\int_0^{1-h} \mathbb{E}\left[N_{u,T}^2\right] \mathrm{d}u \leqslant C\int_0^{1-h} N_{u,0}\,\mathrm{d}u + C\left(\int_0^{1-h} N_{u,0}\,\mathrm{d}u\right)^{1-1/\beta} + C\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u. \tag{I.26}$$

Recall that there is $p > 2$ such that $g \in L_p(0,1)$. As for inequality (I.19), we get:

$$\int_0^{1-h} N_{u,0}\,\mathrm{d}u = \int_0^{1-h} (g(u+h) - g(u))\mathrm{d}u \leqslant \|g\|_{L_p}(2h)^{1-\frac{1}{p}}.$$

Furthermore, define $\alpha := \frac{p-2}{p-1} \in (0,1)$. We have

$$\begin{aligned}\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u &= \int_0^{1-h} (g(u+h) - g(u))^\alpha (g(u+h) - g(u))^{2-\alpha}\mathrm{d}u \\ &\leqslant \left(\int_0^{1-h} (g(u+h) - g(u))\mathrm{d}u\right)^\alpha \left(\int_0^{1-h} (g(u+h) - g(u))^{\frac{2-\alpha}{1-\alpha}}\mathrm{d}u\right)^{1-\alpha} \\ &\leqslant \left(\|g\|_{L_p}(2h)^{1-\frac{1}{p}}\right)^\alpha \left(C_p\|g\|_{L_p}\right)^{1-\alpha},\end{aligned}$$

because $\frac{2-\alpha}{1-\alpha} = p$. Therefore

$$\int_0^{1-h} N_{u,0}^2\,\mathrm{d}u = \int_0^{1-h} (g(u+h) - g(u))^2\mathrm{d}u \leqslant C_p^{1-\alpha}\|g\|_{L_p} h^{\frac{p-2}{p}}. \tag{I.27}$$

It follows from (I.26) that there is C_β such that for each $\sigma, \varepsilon > 0$,

$$\int_0^{1-h} \mathbb{E}\left[(y_{\sigma,\varepsilon}(u+h,T) - y_{\sigma,\varepsilon}(u,T))^2\right] \mathrm{d}u \leqslant C_\beta\|g\|_{L_p}\left(h^{\frac{p-1}{p}} + h^{\frac{p-1}{p}(1-\frac{1}{\beta})} + h^{\frac{p-2}{p}}\right),$$

for every $\beta < \frac{3}{2} - \frac{1}{p}$, *i.e.* such that $0 < 1 - \frac{1}{\beta} < \frac{p-2}{3p-2}$. Thus, there is $q > 0$ depending on p (e.g. $q = \frac{(p-1)(p-2)}{2p(3p-2)}$ by choosing $1 - \frac{1}{\beta} = \frac{p-2}{2(3p-2)}$) and a constant C such that for each $\sigma, \varepsilon > 0$,

$$\int_0^{1-h} \mathbb{E}\left[(y_{\sigma,\varepsilon}(u+h,T) - y_{\sigma,\varepsilon}(u,T))^2\right] \mathrm{d}u \leqslant C\|g\|_{L_p} h^q. \tag{I.28}$$

Therefore, by (I.23) and Markov's inequality, there is C such that for each $\sigma, \varepsilon > 0$,

$$\mathbb{P}\left[\int_0^{1-h} \|y_{\sigma,\varepsilon}(u+h,\cdot) - y_{\sigma,\varepsilon}(u,\cdot)\|^2_{\mathcal{C}[0,T]} \mathrm{d}u \geqslant \frac{1}{k}\right] \leqslant kC\|g\|_{L_p} h^q,$$

whence it is sufficient to choose h_k so that $kC\|g\|_{L_p} h_k^q < \frac{\delta}{2^k}$. □

I.4.4 Convergence when $\varepsilon \to 0$

Fix $\sigma \in \mathbb{Q}_+$. By Prokhorov's Theorem, it follows from Corollary I.21 that the collection of laws of the sequence $(y_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ is relatively compact in $\mathcal{P}(L_2([0,1],\mathcal{C}[0,T]))$. In particular, up to extracting a subsequence, we may suppose that $(y_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ converges in distribution in $L_2([0,1],\mathcal{C}[0,T])$ to a limit, denoted by y_σ.

For every $t \in [0,T]$, let us denote by $e_t(f) := f(\cdot,t)$ the continuous evaluation function: $L_2([0,1],\mathcal{C}[0,T]) \to L_2[0,1]$. We define $Y_\sigma(t) := e_t(y_\sigma) = y_\sigma(\cdot,t)$. Under the same model as Proposition I.8, we obtain:

Proposition I.31. *Fix $\sigma \in \mathbb{Q}_+$. Suppose that $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$. $(Y_\sigma(t))_{t\in[0,T]}$ is a $L^{\uparrow}_2[0,1]$-valued process such that:*

(B1) $Y_\sigma(0) = g$;

(B2) *$(Y_\sigma(t))_{t\in[0,T]}$ is a square integrable continuous $L^{\uparrow}_2[0,1]$-valued martingale relatively to the filtration $(\mathcal{F}^\sigma_t)_{t\in[0,T]}$, where $\mathcal{F}^\sigma_t = \sigma(Y_\sigma(s), s \leqslant t)$;*

(B3) *for every $h, k \in L_2[0,1]$,*

$$\langle (Y_\sigma,h)_{L_2}, (Y_\sigma,k)_{L_2}\rangle_t = \int_0^t\int_0^1\int_0^1 h(u)k(u')\frac{m_\sigma(u,u',s)}{m_\sigma(u,s)m_\sigma(u',s)}\mathrm{d}u\mathrm{d}u'\mathrm{d}s,$$

where $m_\sigma(u,u',s) = \int_0^1 \varphi_\sigma(y_\sigma(u,s) - y_\sigma(v,s))\varphi_\sigma(y_\sigma(u',s) - y_\sigma(v,s))\mathrm{d}v$ and $m_\sigma(u,s) = \int_0^1 \varphi^2_\sigma(y_\sigma(u,s) - y_\sigma(v,s))\mathrm{d}v$.

Proof. Fix $t \in [0,T]$. We want to prove that $Y_\sigma(t)$ belongs to $L^{\uparrow}_2[0,1]$. For each $\varepsilon \in \mathbb{Q}_+$, $Y_{\sigma,\varepsilon}(t)$ belongs with probability 1 to the set $\mathcal{K} :=$

$$\left\{f \in L_2(0,1) : \forall u,u',\forall r,r', \text{if } 0 < u < u+r < u' < u'+r' < 1, \text{then } \frac{1}{r}\int_u^{u+r} f \leqslant \frac{1}{r'}\int_{u'}^{u'+r'} f\right\}$$

which is closed in $L_2(0,1)$. Recall that the sequence $(y_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ converges in distribution to y_σ in $L_2([0,1],\mathcal{C}[0,T])$. Therefore, $(Y_{\sigma,\varepsilon}(t))_{\varepsilon\in\mathbb{Q}_+}$ converges in distribution to $Y_\sigma(t)$ in $L_2[0,1]$. Because $\mathcal{K}$ is closed, the limit $Y_\sigma(t)$ also belongs to $\mathcal{K}$ with probability 1.

Therefore, almost surely, for every $t \in [0,T] \cap \mathbb{Q}$, $Y_\sigma(t) \in \mathcal{K}$. Let $\omega \in \Omega'$, where Ω' is such that $\mathbb{P}[\Omega'] = 1$ and for every $\omega \in \Omega'$, $\int_0^1 \sup_{s\leqslant T} |y_\sigma(v,s)|^2(\omega)\mathrm{d}v < +\infty$ and for every $t \in [0,T] \cap \mathbb{Q}$, $Y_\sigma(t)(\omega) \in \mathcal{K}$. Let $t \in [0,T]$ and (t_n) be a sequence in $[0,T] \cap \mathbb{Q}$ tending to t. For every $n \in \mathbb{N}$ and each u,u',r,r' such that $0 < u < u+r < u' < u'+r' < 1$, $\frac{1}{r}\int_u^{u+r} y_\sigma(v,t_n)(\omega)\mathrm{d}v \leqslant \frac{1}{r'}\int_{u'}^{u'+r'} y_\sigma(v,t_n)(\omega)\mathrm{d}v$. Since $y_\sigma(\omega)$ belongs to $L_2([0,1],\mathcal{C}[0,T])$, and since $\int_u^{u+r} y_\sigma(v,t_n)^2(\omega)\mathrm{d}v \leqslant \int_0^1 \sup_{s\leqslant T} |y_\sigma(v,s)|^2(\omega)\mathrm{d}v < +\infty$, $\frac{1}{r}\int_u^{u+r} y_\sigma(v,t_n)(\omega)\mathrm{d}v$ tends to

$\frac{1}{r}\int_u^{u+r} y_\sigma(v,t)(\omega)\mathrm{d}v$ (and the same is true for u' and r'). Thus almost surely $Y_\sigma(t)$ belongs to $\mathcal{K}$ for every $t \in [0,T]$. It remains to prove that it implies that $Y_\sigma(t)$ belongs to $L_2^\uparrow[0,1]$.

Let $f \in \mathcal{K}$. Define, for each $u \in (0,1)$, $\widehat{f}(u) := \liminf_{h\to 0^+} \frac{1}{h}\int_u^{(u+h)\wedge 1} f(v)\mathrm{d}v$. First, remark that $\widehat{f}$ is non-decreasing. Then, since $h \mapsto \frac{1}{h}\int_u^{u+h} f$ is non-increasing, we have $\widehat{f}(u) = \lim_{h\to 0^+} \frac{1}{h}\int_u^{(u+h)\wedge 1} f(v)\mathrm{d}v$. Choose a sequence $(u_n) \searrow u$. By monotonicity, $\widehat{f}(u) \leqslant \widehat{f}(u_n)$. Fix $\delta > 0$. There exists $h > 0$ such that $u + h < 1$ and $|\widehat{f}(u) - \frac{1}{h}\int_u^{u+h} f| < \delta$. Since $f \in L_2$, there exists N such that for all $n \geqslant N$, $|\frac{1}{h}\int_{u_n}^{u_n+h} f - \frac{1}{h}\int_u^{u+h} f| < \delta$. Therefore, $\widehat{f}(u_n) \leqslant \frac{1}{h}\int_{u_n}^{u_n+h} f \leqslant \widehat{f}(u) + 2\delta$ for all $n \geqslant N$. Thus $\widehat{f}(u_n) \to \widehat{f}(u)$. In addition, $\widehat{f}$ has left limits because of its monotonicity. Hence $\widehat{f}$ is a càdlàg function.

Furthermore, $\widehat{f} = f$ almost everywhere. Indeed, for every $\delta > 0$, there exists $F \in \mathcal{C}[0,1]$ such that $\|f - F\|_{L_1(0,1)} < \delta$. Define $\widehat{F}(u) = \lim_{h\to 0^+} \frac{1}{h}\int_u^{(u+h)\wedge 1} F(v)\mathrm{d}v$. By continuity of F, $F(u) = \widehat{F}(u)$ for every $u \in (0,1)$. Thus we have:

$$\begin{aligned}\int_0^1 |f(u) - \widehat{f}(u)|\mathrm{d}u &\leqslant \int_0^1 |f(u) - F(u)|\mathrm{d}u + \int_0^1 |\widehat{f}(u) - \widehat{F}(u)|\mathrm{d}u \\ &\leqslant \delta + \int_0^1 \lim_{h\to 0^+} \frac{1}{h}\int_u^{(u+h)\wedge 1} |f(v) - F(v)|\mathrm{d}v\mathrm{d}u \\ &\leqslant \delta + \liminf_{h\to 0^+} \int_0^1 |f(v) - F(v)|\mathrm{d}v \leqslant 2\delta,\end{aligned}$$

where we used Fatou's Lemma to obtain the last line. Thus $\int_0^1 |f(u) - \widehat{f}(u)|\mathrm{d}u = 0$, whence $\widehat{f} = f$ almost everywhere. Thus f belongs to $L_2^\uparrow[0,1]$: Y_σ is a $L_2^\uparrow[0,1]$-valued process.

Property $(B1)$**.** $(Y_{\sigma,\varepsilon}(0))_{\varepsilon\in\mathbb{Q}_+}$ converges in law to $Y_\sigma(0)$ in $L_2[0,1]$. Therefore, $Y_\sigma(0) = g$.

Property $(B2)$**.** By inequality (I.20), $\mathbb{E}\left[\|Y_{\sigma,\varepsilon}\|^2_{L_2([0,1],\mathcal{C}[0,T])}\right]$ is bounded uniformly in $\varepsilon \in \mathbb{Q}_+$. We deduce that for every $t \in [0,T]$, $\mathbb{E}\left[\|Y_\sigma(t)\|^2_{L_2([0,1])}\right] < +\infty$, thus the process Y_σ is square integrable.

Furthermore, Y_σ is a continuous $L_2^\uparrow[0,1]$-valued process. Indeed, for each sequence $(t_n)_{n\geqslant 0}$ converging to a time t, $\|Y_\sigma(t_n) - Y_\sigma(t)\|^2_{L_2} = \int_0^1 (y_\sigma(u,t_n) - y_\sigma(u,t))^2\mathrm{d}u \underset{n\to\infty}{\longrightarrow} 0$ by dominated convergence Theorem, since for almost every $u \in (0,1)$, $y_\sigma(u,\cdot)$ is continuous at time t, and $(y_\sigma(u,t_n) - y_\sigma(u,t))^2 \leqslant 4\sup_{t\leqslant T} |y_\sigma(u,t)|^2$ which is almost surely integrable.

Moreover, we know from property $(A2)$ that for each $h \in L_2(0,1)$, each $l \geqslant 1$, $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t$ and each bounded and continuous function $f_l : (L_2(0,1))^l \to \mathbb{R}$:

$$\mathbb{E}\left[\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(u,s))\mathrm{d}u\ f_l(y_{\sigma,\varepsilon}(\cdot,s_1),\ldots,y_{\sigma,\varepsilon}(\cdot,s_l))\right] = 0. \tag{I.29}$$

Since $\left|\int_0^1 h(u)b(u,t)\mathrm{d}u\right| \leqslant \|h\|_{L_2}\left(\int_0^1 \sup_{[0,T]} |b(u,\cdot)|^2\mathrm{d}u\right)^{1/2}$ for every $b \in L_2([0,1],\mathcal{C}[0,T])$, the function $\varphi : b \in L_2([0,1],\mathcal{C}[0,T]) \mapsto \int_0^1 h(u)(b(u,t) - b(u,s))\mathrm{d}u\ f_l(b(\cdot,s_1),\ldots,b(\cdot,s_l))$ is continuous. Furthermore, we prove that $(\varphi(y_{\sigma,\varepsilon}))_{\varepsilon\in\mathbb{Q}_+}$ is bounded in L_2:

$$\begin{aligned}\mathbb{E}\left[\varphi(y_{\sigma,\varepsilon})^2\right] &\leqslant \|f_l\|^2_\infty \|h\|^2_{L_2}\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t) - y_{\sigma,\varepsilon}(u,s))^2\mathrm{d}u\right] \\ &\leqslant C\|f_l\|^2_\infty\|h\|^2_{L_2},\end{aligned}$$

where C is independent of ε by Corollary I.26. We deduce that $(\varphi(y_{\sigma,\varepsilon}))_{\varepsilon\in\mathbb{Q}_+}$ is uniformly integrable. By continuity of φ and since $(y_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ converges in law to y_σ in $L_2([0,1],\mathcal{C}[0,T])$,

we get: $\mathbb{E}\left[\varphi(y_{\sigma,\varepsilon})\right] \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}\left[\varphi(y_\sigma)\right]$. Since by equality (I.29), $\mathbb{E}\left[\varphi(y_{\sigma,\varepsilon})\right] = 0$ for each $\varepsilon \in \mathbb{Q}_+$, we have:

$$\mathbb{E}\left[\int_0^1 h(u)(y_\sigma(u,t) - y_\sigma(u,s))\mathrm{d}u f_l(Y_\sigma(s_1),\ldots,Y_\sigma(s_l))\right] = 0. \tag{I.30}$$

Therefore, $Y_\sigma(\cdot)$ is a square integrable continuous $(\mathcal{F}_t^\sigma)_{t\in[0,T]}$-martingale.

Property $(B3)$**.** We know, by property $(A3)$, that for every $l \geqslant 1$, for every $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t$, for every bounded and continuous $f_l : (L_2(0,1))^l \to \mathbb{R}$ and for every h and k in $L_2(0,1)$:

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^1\int_0^1 h(u)k(u')[(y_{\sigma,\varepsilon}(u,t) - g(u))(y_{\sigma,\varepsilon}(u',t) - g(u')) \\
&\qquad - (y_{\sigma,\varepsilon}(u,s) - g(u))(y_{\sigma,\varepsilon}(u',s) - g(u'))]\mathrm{d}u\mathrm{d}u' f_l(Y_{\sigma,\varepsilon}(s_1),\ldots,Y_{\sigma,\varepsilon}(s_l))\Big] \\
&= \mathbb{E}\Big[\int_0^1\int_0^1 h(u)k(u')\int_s^t \frac{m_{\sigma,\varepsilon}(u,u',r)\ \mathrm{d}r\mathrm{d}u\mathrm{d}u'}{(\varepsilon + m_{\sigma,\varepsilon}(u,r))(\varepsilon + m_{\sigma,\varepsilon}(u',r))} f_l(Y_{\sigma,\varepsilon}(s_1),\ldots,Y_{\sigma,\varepsilon}(s_l))\Big].
\end{aligned} \tag{I.31}$$

First, we want to obtain the convergence of the left hand side of (I.31). We proceed in the same way as for the proof of equality (I.30); to get a uniform integrability property, we have now to prove the existence of $\beta > 1$ such that

$$\sup_{\varepsilon\in\mathbb{Q}_+} \mathbb{E}\left[\left(\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t) - g(u))\mathrm{d}u \int_0^1 k(u')(y_{\sigma,\varepsilon}(u',t) - g(u'))\mathrm{d}u'\right)^\beta\right] \tag{I.32}$$

is finite. Therefore, it is sufficient to prove the existence of $\beta > 1$ such that

$$\sup_{\varepsilon\in\mathbb{Q}_+} \mathbb{E}\left[\left(\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t) - g(u))\mathrm{d}u\right)^{2\beta}\right]$$

is finite for every $h \in L_2[0,1]$. By Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1 h(u)(y_{\sigma,\varepsilon}(u,t) - g(u))\mathrm{d}u\right)^{2\beta}\right] &\leqslant \mathbb{E}\left[\|h\|_{L_2}^{2\beta}\left(\int_0^1 (y_{\sigma,\varepsilon}(u,t) - g(u))^2\mathrm{d}u\right)^\beta\right] \\
&\leqslant \|h\|_{L_2}^{2\beta}\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t) - g(u))^{2\beta}\mathrm{d}u\right].
\end{aligned} \tag{I.33}$$

We deduce by Burkholder-Davis-Gundy inequality and Fubini's Theorem that there are some constants independent of ε such that

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,t) - g(u))^{2\beta}\mathrm{d}u\right] &\leqslant C_1\int_0^1 \mathbb{E}\left[\langle y_{\sigma,\varepsilon}(u,\cdot), y_{\sigma,\varepsilon}(u,\cdot)\rangle_t^\beta\right]\mathrm{d}u \\
&\leqslant C_2\mathbb{E}\left[\int_0^1\int_0^t \frac{1}{M_{\sigma,\varepsilon}^\beta(u,r)}\mathrm{d}r\mathrm{d}u\right].
\end{aligned}$$

By Lemma I.24, there exists $\beta > 1$ such that $\mathbb{E}\left[\int_0^1\int_0^t \frac{1}{M_{\sigma,\varepsilon}^\beta(u,r)}\mathrm{d}r\mathrm{d}u\right]$ is bounded uniformly for $\varepsilon \in \mathbb{Q}_+$. Thus (I.32) is finite. It is also finite if we replace t by s.

To obtain the convergence of the right hand side of (I.31), we start by using Skorohod's representation Theorem[1]: there exists a sequence $(\widehat{y}_{\sigma,\varepsilon})_{\varepsilon\in\mathbb{Q}_+}$ defined on a common probability

[1] $L_2([0,1],\mathcal{C}[0,T])$ is a Polish space. Its separability can be proved using the separability of $C([0,1]\times[0,T])$ and the density of $C([0,1]\times[0,T])$ in $L_2([0,1],\mathcal{C}[0,T])$.

space $(\widehat{\Omega},\widehat{\mathbb{P}})$ that converges to $\widehat{y}_\sigma$ in $L_2([0,1],\mathcal{C}[0,T])$ almost surely, where $\widehat{y}_{\sigma,\varepsilon}$ (resp. $\widehat{y}_\sigma$) has same distribution as $y_{\sigma,\varepsilon}$ (resp. y_σ). We denote by $\widehat{m}_{\sigma,\varepsilon}$ (resp. $\widehat{m}_\sigma$) the mass associated to $\widehat{y}_{\sigma,\varepsilon}$ (resp. $\widehat{y}_\sigma$).

Furthermore, on the probability space $(\widehat{\Omega}\times[0,1],\widehat{\mathbb{P}}\otimes \mathrm{Leb}\,|_{[0,1]})$, $\widehat{y}_{\sigma,\varepsilon}$ converges in probability in the space $\mathcal{C}[0,T]$ to $\widehat{y}_\sigma$. Indeed, for every $\delta>0$, we have:

$$\begin{aligned}\widehat{\mathbb{P}}\otimes \mathrm{Leb}\,|_{[0,1]}\{(\omega,u) : \|(\widehat{y}_{\sigma,\varepsilon}-\widehat{y}_\sigma)(\omega,u)\|_{\mathcal{C}[0,T]}\geqslant\delta\}&\\ = \widehat{\mathbb{E}}\left[\mathrm{Leb}\{u : \|(\widehat{y}_{\sigma,\varepsilon}-\widehat{y}_\sigma)(\omega,u)\|_{\mathcal{C}[0,T]}\geqslant\delta\}\right]&\\ \leqslant \widehat{\mathbb{E}}\left[1\wedge\frac{1}{\delta^2}\int_0^1\|(\widehat{y}_{\sigma,\varepsilon}-\widehat{y}_\sigma)(\omega,u)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\right].&\end{aligned}$$

We know that, for every fixed $\delta>0$, $1\wedge\frac{1}{\delta^2}\int_0^1\|(\widehat{y}_{\sigma,\varepsilon}-\widehat{y}_\sigma)(\omega,u)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u$ converges to 0 almost surely, and it is bounded by 1, so we deduce that the latter term tends to 0. We deduce from the convergence in probability that there exists a subsequence $(\varepsilon_n)_n$, $\varepsilon_n\to0$, such that for almost every $(\omega,u)\in\widehat{\Omega}\times[0,1]$, $\|(\widehat{y}_{\sigma,\varepsilon_n}-\widehat{y}_\sigma)(\omega,u)\|_{\mathcal{C}[0,T]}\to0$.

We want to prove that,

$$\begin{aligned}\mathbb{E}\left[\int_0^1\int_0^1 h(u)k(u')\int_s^t\frac{\widehat{m}_{\sigma,\varepsilon_n}(u,u',r)\ \mathrm{d}r\mathrm{d}u\mathrm{d}u'}{(\varepsilon_n+\widehat{m}_{\sigma,\varepsilon_n}(u,r))(\varepsilon_n+\widehat{m}_{\sigma,\varepsilon_n}(u',r))}f_l(\widehat{Y}_{\sigma,\varepsilon_n}(s_1),\dots,\widehat{Y}_{\sigma,\varepsilon_n}(s_l))\right]&\\ \underset{n\to\infty}{\longrightarrow}\mathbb{E}\left[\int_0^1\int_0^1 h(u)k(u')\int_s^t\frac{\widehat{m}_\sigma(u,u',r)}{\widehat{m}_\sigma(u,r)\widehat{m}_\sigma(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u'f_l(\widehat{Y}_\sigma(s_1),\dots,\widehat{Y}_\sigma(s_l))\right].&\quad\text{(I.34)}\end{aligned}$$

On the one hand, almost surely and for almost every $u\in(0,1)$, $\widehat{y}_{\sigma,\varepsilon_n}(u,\cdot)\to\widehat{y}_\sigma(u,\cdot)$ in $\mathcal{C}[0,T]$. Then for almost every $u,u'\in(0,1)$,

$$\begin{aligned}\widehat{m}_{\sigma,\varepsilon_n}(u,u',r)&=\int_0^1\varphi_\sigma(\widehat{y}_{\sigma,\varepsilon_n}(u,r)-\widehat{y}_{\sigma,\varepsilon_n}(v,r))\varphi_\sigma(\widehat{y}_{\sigma,\varepsilon_n}(u',r)-\widehat{y}_{\sigma,\varepsilon_n}(v,r))\mathrm{d}v\\ &\underset{n\to\infty}{\longrightarrow}\widehat{m}_\sigma(u,u',r),&\text{(I.35)}\\ \varepsilon_n+\widehat{m}_{\sigma,\varepsilon_n}(u,r)&=\varepsilon_n+\int_0^1\varphi^2_\sigma(\widehat{y}_{\sigma,\varepsilon_n}(u,r)-\widehat{y}_{\sigma,\varepsilon_n}(v,r))\mathrm{d}v\underset{n\to\infty}{\longrightarrow}\widehat{m}_\sigma(u,r).&\text{(I.36)}\end{aligned}$$

Therefore, in order to obtain (I.34), it remains to justify that there exists $\beta>1$ such that:

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\left(\int_0^1\int_0^1 h(u)k(u')\int_s^t\frac{\widehat{m}_{\sigma,\varepsilon_n}(u,u',r)}{(\varepsilon_n+\widehat{m}_{\sigma,\varepsilon_n}(u,r))(\varepsilon_n+\widehat{m}_{\sigma,\varepsilon_n}(u',r))}\mathrm{d}r\mathrm{d}u\mathrm{d}u'\right)^\beta\right]$$

is finite. By Cauchy-Schwarz inequality, $\widehat{m}_{\sigma,\varepsilon_n}(u,u',r)\leqslant\widehat{m}^{1/2}_{\sigma,\varepsilon_n}(u,r)\widehat{m}^{1/2}_{\sigma,\varepsilon_n}(u',r)$, so that it is sufficient to prove that there is $\beta>1$ such that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\left(\int_0^1\int_0^1 h(u)k(u')\int_s^t\frac{1}{\widehat{M}^{1/2}_{\sigma,\varepsilon_n}(u,r)\widehat{M}^{1/2}_{\sigma,\varepsilon_n}(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u'\right)^\beta\right]$$

is finite, and thus that $\sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^1\int_s^t\frac{1}{\widehat{M}^\beta_{\sigma,\varepsilon_n}(u,r)}\mathrm{d}r\mathrm{d}u\right]$ is finite, using Cauchy-Schwarz inequality as in the proof of (I.33). By Lemma I.24, this statement holds. We conclude that we have the following equality:

$$\begin{aligned}\mathbb{E}\Big[\int_0^1\int_0^1 h(u)k(u')[(y_\sigma(u,t)-g(u))(y_\sigma(u',t)-g(u'))&\\ -(y_\sigma(u,s)-g(u))(y_\sigma(u',s)-g(u'))]\mathrm{d}u\mathrm{d}u'f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\Big]&\\ =\mathbb{E}\Big[\int_0^1\int_0^1 h(u)k(u')\int_s^t\frac{m_\sigma(u,u',r)}{m_\sigma(u,r)m_\sigma(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u'f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\Big],&\quad\text{(I.37)}\end{aligned}$$

whence we obtain property $(B3)$, since $\int_0^1\int_0^1 h(u)k(u')\int_0^t \frac{m_\sigma(u,u',r)}{m_\sigma(u,r)m_\sigma(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u'$ is $(\mathcal{F}_t^\sigma)_{t\in[0,T]}$-measurable. □

Property $(B3)$ implies the following Corollary:

Corollary I.32. *Let ψ be a non-negative and bounded map: $[0,1]\to\mathbb{R}$. Then for every $l\in\mathbb{N}\backslash\{0\}$, $0\leqslant s_1\leqslant s_2\leqslant\ldots\leqslant s_l\leqslant s\leqslant t$ and for every bounded and continuous function $f_l: L_2[0,1]^l\to\mathbb{R}$, we have:*

$$\mathbb{E}\Big[\int_0^1\psi(u)\left((y_\sigma(u,t)-g(u))^2-(y_\sigma(u,s)-g(u))^2-\int_s^t\frac{1}{m_\sigma(u,r)}\mathrm{d}r\right)\mathrm{d}u \\ f_l(Y_\sigma(s_1),\ldots,Y_\sigma(s_l))\Big]=0.$$

Proof. Let us introduce the following notations. First $z(u,\cdot):=y_\sigma(u,\cdot)-g(u)$ and second $F_l=f_l(Y_\sigma(s_1),\ldots,Y_\sigma(s_l))$. Let us consider an orthonormal basis $(e_i)_{i\geqslant1}$ in the Hilbert space $L_2(\psi(x)\mathrm{d}x)$. We denote by $[\cdot,\cdot]_{L_2(\psi)}$ the scalar product of $L_2(\psi(x)\mathrm{d}x)$: $[h,k]_{L_2(\psi)}=\int_0^1 hk\psi$. By Parseval's formula, we have:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\psi(u)(z(u,t)^2-z(u,s)^2)\mathrm{d}uF_l\right]&=\mathbb{E}\left[\sum_{i\geqslant1}([z(\cdot,t),e_i]^2_{L_2(\psi)}-[z(\cdot,s),e_i]^2_{L_2(\psi)})F_l\right]\\&=\sum_{i\geqslant1}\mathbb{E}\left[((z(\cdot,t),e_i\psi)^2_{L_2}-(z(\cdot,s),e_i\psi)^2_{L_2})F_l\right]\\&=\sum_{i\geqslant1}\mathbb{E}\left[\int_0^1\int_0^1 e_i(u)\psi(u)e_i(u')\psi(u')\int_s^t\frac{m_\sigma(u,u',r)}{m_\sigma(u,r)m_\sigma(u',r)}\mathrm{d}r\mathrm{d}u\mathrm{d}u'F_l\right],\end{aligned}$$

by applying equality (I.37) with $h=k=e_i$. By definition of $m_\sigma(u,u',r)$, we have:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\psi(u)(z(u,t)^2-z(u,s)^2)\mathrm{d}uF_l\right]&=\mathbb{E}\left[\int_s^t\int_0^1\sum_{i\geqslant1}\left[\frac{\varphi_\sigma(y_\sigma(\cdot,r)-y_\sigma(v,r))}{m_\sigma(\cdot,r)},e_i\right]^2_{L_2(\psi)}\mathrm{d}v\mathrm{d}rF_l\right]\\&=\mathbb{E}\left[\int_s^t\int_0^1\int_0^1\frac{\varphi^2_\sigma(y_\sigma(u,r)-y_\sigma(v,r))}{m^2_\sigma(u,r)}\psi(u)\mathrm{d}u\mathrm{d}v\mathrm{d}rF_l\right]\\&=\mathbb{E}\left[\int_0^1\int_s^t\frac{1}{m_\sigma(u,r)}\mathrm{d}r\psi(u)\mathrm{d}uF_l\right],\end{aligned}$$

since $m_\sigma(u,r)=\int_0^1\varphi^2_\sigma(y_\sigma(u,r)-y_\sigma(v,r))\mathrm{d}v$. □

We deduce the following estimation, by analogy with Lemma I.24:

Lemma I.33. *For all $\beta\in(0,\frac{3}{2}-\frac{1}{p})$, there is a constant $C>0$ such that for all $\sigma>0$ and $0\leqslant s<t\leqslant T$, we have the following inequality:*

$$\mathbb{E}\left[\int_s^t\int_0^1\frac{1}{m^\beta_\sigma(u,r)}\mathrm{d}u\mathrm{d}r\right]\leqslant C\sqrt{t-s}.$$

Proof. We use again the sequence $(\widehat{y}_{\sigma,\varepsilon_n})_{n\in\mathbb{N}}$ obtained by Skorohod's representation Theorem, as in the proof of convergence (I.34). Therefore, by Fatou's Lemma,

$$\mathbb{E}\left[\int_s^t\int_0^1\frac{1}{\widehat{m}^\beta_\sigma(u,r)}\mathrm{d}u\mathrm{d}r\right]\leqslant\liminf_{n\to\infty}\ \mathbb{E}\left[\int_s^t\int_0^1\frac{1}{\widehat{M}^\beta_{\sigma,\varepsilon_n}(u,r)}\mathrm{d}u\mathrm{d}r\right]\leqslant C\sqrt{t-s},$$

where C is obtained thanks to Lemma I.24. □

By Burkholder-Davis-Gundy inequality, we obtain immediately the following Corollary:

Corollary I.34. *For each $\beta\in(0,\frac{3}{2}-\frac{1}{p})$, $\sup_{\sigma\in\mathbb{Q}_+}\sup_{t\leqslant T}\mathbb{E}\left[\int_0^1(y_\sigma(u,t)-g(u))^{2\beta}\mathrm{d}u\right]<+\infty$.*

I.4.5 Convergence when $\sigma \to 0$

Recall that by Corollary I.21 and Prokhorov's Theorem, the collection of laws of the sequence $(y_{\sigma,\varepsilon})_{\sigma,\varepsilon\in\mathbb{Q}_+}$ is relatively compact in $\mathcal{P}(L_2([0,1],\mathcal{C}[0,T]))$. By construction, the collection of laws of the sequence $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ inherits the same property.

Thus, up to extracting a subsequence, we may suppose that $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges in distribution to a limit, denoted by y, in $L_2([0,1],\mathcal{C}[0,T])$. As before, we define $Y(t) := y(\cdot,t)$. We state the first part of Theorem I.4 in the following Proposition:

Proposition I.35. *Suppose that $g \in \mathcal{L}^{\uparrow}_{2+}[0,1]$. $(Y(t))_{t\in[0,T]}$ is a $L^{\uparrow}_2[0,1]$-valued process such that:*

(C1) $Y(0) = g$;

(C2) $(Y(t))_{t\in[0,T]}$ *is a square integrable continuous $L^{\uparrow}_2[0,1]$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-martingale, where $\mathcal{F}_t = \sigma(Y(s), s \leqslant t)$.*

Proof. We refer to the proof of Proposition I.31. □

Remark I.36. It should be noticed at this point that a new difficulty arises when we want to obtain a property analogous to $(B3)$. Indeed, whereas it was straightforward to prove (I.35) and (I.36), the convergence of the mass $m_\sigma(u,t) = \int_0^1 \varphi^2_\sigma(y_\sigma(u,t) - y_\sigma(v,t))\mathrm{d}v$ to $m(u,t) = \int_0^1 \mathbb{1}_{\{y(u,t)=y(v,t)\}}\mathrm{d}v$ is not obvious, due to the singularity of the indicator function. It will be the main goal of the next section to prove this convergence.

In Section I.5, we will study the martingale properties of the limit process Y and compute its quadratic variation (property $(C5)$ of Theorem I.4). To obtain this, we will first prove that for every *positive* t, $Y(t)$ is a step function (see property $(C3)$). It implies that y has a version in $\mathcal{D}((0,1),\mathcal{C}[0,T])$ (see property $(C4)$) by an argument given in ([Kon17a, Proposition 2.3]).

I.5 Properties of the limit process Y

The aim of this section is to complete the proof of Theorem I.4. Properties $(C3)$ and $(C4)$ will be proved in Paragraph I.5.1 and property $(C5)$ will be proved in two steps in Paragraph I.5.2 and Paragraph I.5.3.

I.5.1 Coalescence properties and step functions

In this paragraph, we will prove the following Proposition:

Proposition I.37. *Almost surely, for every $t > 0$, $Y(t)$ is a step function.*

Recall that $Y(0) = g$ is not necessarily a step function, since g can be chosen arbitrarily in $\mathcal{L}^{\uparrow}_{2+}[0,1]$. If we denote for each $t \in [0,T]$ by μ_t the measure associated to the quantile function $Y(t)$, that is $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ Y(t)^{-1}$, Proposition I.37 means that for every *positive* time t, μ_t is a finite weighted sum of Dirac measures. We begin by the following lemma. Recall the definition of the mass: $m(u,t) = \int_0^1 \mathbb{1}_{\{y(u,t)=y(v,t)\}}\mathrm{d}v$.

Lemma I.38. *There exists a probability space $(\widetilde{\Omega},\widetilde{\mathbb{P}})$ on which the sequence $(\widetilde{y}_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges almost surely to $\widetilde{y}$ in $L_2([0,1],\mathcal{C}[0,T])$ and where, for each $\sigma \in \mathbb{Q}_+$, $\widetilde{y}_\sigma$ (resp. $\widetilde{y}$) has same law as y_σ (resp. y). Furthermore, there is a subsequence $(\sigma_n)_n$, $\sigma_n \to 0$, such that for almost every $(\omega,u) \in \Omega \times (0,1)$ and for every time $t \in [0,T]$,*

$$\limsup_{n\to\infty} \widetilde{m}_{\sigma_n}(u,t) \leqslant \widetilde{m}(u,t).$$

Proof. Recall that $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges in distribution in $L_2([0,1],\mathcal{C}[0,T])$ to y. By Skorohod's representation Theorem, we deduce that there exists a sequence $(\widetilde{y}_\sigma)_{\sigma\in\mathbb{Q}_+}$ and a random variable $\widetilde{y}$ defined on a common probability space $(\widetilde{\Omega},\widetilde{\mathbb{P}})$ such that for every $\sigma\in\mathbb{Q}_+$, the laws of $\widetilde{y}_\sigma$ and y_σ are the same, the laws of $\widetilde{y}$ and y are also equal and the sequence $(\widetilde{y}_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges almost surely to $\widetilde{y}$ in $L_2([0,1],\mathcal{C}[0,T])$.

For every $\varepsilon>0$, we get by Markov's inequality:

$$\begin{aligned}\widetilde{\mathbb{P}}\otimes\mathrm{Leb}\{(\omega,u):\|(\widetilde{y}_\sigma-\widetilde{y})(\omega,u)\|_{\mathcal{C}[0,T]}\geqslant\varepsilon\}&=\widetilde{\mathbb{E}}\left[\mathrm{Leb}\{u:\|(\widetilde{y}_\sigma-\widetilde{y})(\omega,u)\|_{\mathcal{C}[0,T]}\geqslant\varepsilon\}\right]\\&\leqslant\widetilde{\mathbb{E}}\left[1\wedge\frac{1}{\varepsilon^2}\int_0^1\|(\widetilde{y}_\sigma-\widetilde{y})(\omega,u)\|^2_{\mathcal{C}[0,T]}\mathrm{d}u\right].\end{aligned}\tag{I.38}$$

Since $(\widetilde{y}_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges almost surely to $\widetilde{y}$ in $L_2([0,1],\mathcal{C}[0,T])$, the right hand side tends to 0. Therefore, $(\widetilde{y}_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges in probability to $\widetilde{y}$ in $\mathcal{C}[0,T]$ on the probability space $(\widetilde{\Omega}\times[0,1],\widetilde{\mathbb{P}}\otimes\mathrm{Leb})$. Thus there exists a subsequence $(\sigma_n)_n$ tending to 0 along which $\widetilde{y}_{\sigma_n}$ converges on an almost sure event of $\widetilde{\Omega}\times[0,1]$ to $\widetilde{y}$ in $\mathcal{C}[0,T]$. Therefore, there is Ω', $\widetilde{\mathbb{P}}[\Omega']=1$, such that for every $\omega\in\Omega'$, there exists a Borel set $\mathcal{A}=\mathcal{A}(\omega)$ in $[0,1]$, $\mathrm{Leb}(\mathcal{A})=1$, such that for all $u\in\mathcal{A}$, $\|\widetilde{y}_{\sigma_n}(u,\cdot)-\widetilde{y}(u,\cdot)\|_{\mathcal{C}[0,T]}$ tends to zero. Remark that the extraction $(\sigma_n)_n$ does not depend on ω. From now on, we forget the tildes and the extraction in our notation.

Let $\omega\in\Omega$. Fix $u\in\mathcal{A}(\omega)$ and $t\in[0,T]$. We set $v\in\mathcal{A}$ such that $y(v,t)\neq y(u,t)$. Then there exist $\sigma_0>0$ and $\delta>0$ such that for all $\sigma\in(0,\sigma_0)\cap\mathbb{Q}_+$, $|y_\sigma(v,t)-y_\sigma(u,t)|\geqslant\delta$. For all $\sigma\leqslant\min(\sigma_0,\delta)$, we have $|y_\sigma(v,t)-y_\sigma(u,t)|\geqslant\sigma$ and thus $\varphi_\sigma(y_\sigma(v,t)-y_\sigma(u,t))=0$. Hence, $\lim_{\sigma\to0}\left(1-\varphi^2_\sigma(y_\sigma(v,t)-y_\sigma(u,t))\right)=1$. Thus we have shown that for all $v\in\mathcal{A}$,

$$\mathbb{1}_{\{y(v,t)\neq y(u,t)\}}\leqslant\liminf_{\sigma\to0}\left(1-\varphi^2_\sigma(y_\sigma(v,t)-y_\sigma(u,t))\right),$$

since $1-\varphi^2_\sigma$ is non-negative. By Fatou's Lemma and since $\mathrm{Leb}(\mathcal{A})=1$, we deduce that:

$$1-m(u,t)=\int_0^1\mathbb{1}_{\{y(v,t)\neq y(u,t)\}}\mathrm{d}v\leqslant\liminf_{\sigma\to0}\int_0^1\left(1-\varphi^2_\sigma(y_\sigma(v,t)-y_\sigma(u,t))\right)\mathrm{d}v,$$

whence for all $u\in\mathcal{A}$ and $t\in[0,T]$, $\limsup_{n\to\infty}m_{\sigma_n}(u,t)\leqslant m(u,t)$. □

We deduce from Lemma I.38 the following corollary. Set $N(t):=\int_0^1\frac{\mathrm{d}u}{m(u,t)}$. By a classical combinatorial argument, $N(t)$ is the number of equivalence classes at time t relatively to the equivalence relation $u\underset{t}{\sim}v\iff y(u,t)=y(v,t)$. In other words, if $N(t)<\infty$, $Y(t)$ is a càdlàg step function taking $N(t)$ distinct values: there exist $0=a_1<a_2<\cdots<a_{N(t)}<a_{N(t)+1}=1$ and $y_1<y_2<\cdots<y_{N(t)}$ such that for all $u\in[0,1]$

$$Y(t)(u)=\sum_{k=1}^{N(t)}y_k\mathbb{1}_{\{u\in[a_k,a_{k+1})\}}+y_{N(t)}\mathbb{1}_{\{u=1\}}.$$

Corollary I.39. *For every time $t\in[0,T]$, $\mathbb{E}\left[\int_0^tN(s)\mathrm{d}s\right]$ is finite.*

Proof. By Lemma I.38, there is a subsequence (σ_n) such that almost surely, for every $t\in[0,T]$ and for almost every $u\in[0,1]$, $\limsup_{n\to\infty}m_{\sigma_n}(u,t)\leqslant m(u,t)$. Therefore, $\frac{1}{m(u,t)}\leqslant\liminf_{n\to\infty}\frac{1}{m_{\sigma_n}(u,t)}$. By Fatou's Lemma, we deduce that:

$$\mathbb{E}\left[\int_0^tN(s)\mathrm{d}s\right]\leqslant\mathbb{E}\left[\int_0^t\int_0^1\liminf_{n\to\infty}\frac{1}{m_{\sigma_n}(u,t)}\mathrm{d}u\mathrm{d}s\right]\leqslant\liminf_{n\to\infty}\mathbb{E}\left[\int_0^t\int_0^1\frac{\mathrm{d}u\mathrm{d}s}{m_{\sigma_n}(u,s)}\right]\leqslant C\sqrt{t},$$

by Lemma I.33. □

Corollary I.40. *Almost surely, for every $t > 0$, $N(t)$ is finite and $t \mapsto N(t)$ is non-increasing on $(0, T]$.*

Proof. We begin by proving the coalescence property. Let u_1, u_2, $h \in \mathbb{Q}$ be such that $0 < u_1 < u_1 + h < u_2 < u_2 + h < 1$. Define $y^h(u_1, t) = \frac{1}{h}\int_{u_1}^{u_1+h} y(v,t)\mathrm{d}v = (Y(t), \frac{1}{h}\mathbb{1}_{(u_1,u_1+h)})_{L_2}$ and $y^h(u_2, t) = (Y(t), \frac{1}{h}\mathbb{1}_{(u_2,u_2+h)})_{L_2}$. By Proposition I.35, $Z(t) = y^h(u_2, t) - y^h(u_1, t)$ is a continuous $\mathbb{R}$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-martingale, almost surely non-negative. As a consequence, $Z(t) = 0$ for every $t \geqslant \tau_0 = \inf\{s \geqslant 0, Z(s) = 0\}$. In other terms, the following coalescence property holds: for every u_1, u_2, $h \in \mathbb{Q}$ such that $0 < u_1 < u_1 + h < u_2 < u_2 + h < 1$, $y^h(u_1, t_0) = y^h(u_2, t_0)$ implies $y^h(u_1, t) = y^h(u_2, t)$ for every $t \geqslant t_0$ almost surely.

On a full event Ω' of $(\Omega, \mathbb{P})$, the latter statement is true and $\int_0^T N(s)\mathrm{d}s$ is finite (by Corollary I.39). Fix $\omega \in \Omega'$. In particular, for almost every $t \in (0, T)$, $N(t)$ is finite. Let $t_0 \in (0, T)$ be such that $N(t_0) < +\infty$. There exist $0 = a_1 < a_2 < \cdots < a_{N(t_0)} < a_{N(t_0)+1} = 1$ and $z_1 < z_2 < \cdots < z_{N(t_0)}$, depending on ω, such that for all $u \in [0, 1]$,

$$Y(t_0)(u) = \sum_{k=1}^{N(t_0)} z_k \mathbb{1}_{\{u\in[a_k,a_{k+1})\}} + z_{N(t_0)}\mathbb{1}_{\{u=1\}}.$$

Fix $k \in \{1, \ldots, N(t_0)\}$. By the coalescence property, almost surely, for all u_1, u_2, $h \in \mathbb{Q}$ such that $a_k < u_1 < u_1 + h < u_2 < u_2 + h < a_{k+1}$, since $y^h(u_1, t_0) = z_k = y^h(u_2, t_0)$, we have $y^h(u_1, t) = y^h(u_2, t)$ for every $t \geqslant t_0$. Fix $t \geqslant t_0$. By monotonicity of $Y(t)$, we deduce that $Y(t)$ is constant on $(u_1, u_2 + h)$. Thus $Y(t)$ is constant on (a_k, a_{k+1}). Therefore, since $Y(t)$ is càdlàg, there exist $\widetilde{z}_1 \leqslant \widetilde{z}_2 \leqslant \ldots \leqslant \widetilde{z}_{N(t_0)}$, depending on ω, such that for all $u \in [0, 1]$,

$$Y(t)(u) = \sum_{k=1}^{N(t_0)} \widetilde{z}_k \mathbb{1}_{\{u\in[a_k,a_{k+1})\}} + \widetilde{z}_{N(t_0)}\mathbb{1}_{\{u=1\}}.$$

We deduce that $N(t) \leqslant N(t_0) < +\infty$, for every $t \geqslant t_0$. Therefore, for every $\omega \in \Omega'$, $t \mapsto N(t)$ is finite and non-increasing on $(0, T]$. This completes the proof of the lemma. □

Therefore, Corollary I.40 completes the proof of Proposition I.37. Then, Proposition I.35 and Proposition I.37 imply the following property, by applying Proposition 2.3 of [Kon17a]:

Proposition I.41. *There exists a modification $\widetilde{y}$ of y in $L_2([0, 1], \mathcal{C}[0, T])$ such that $\widetilde{y}$ belongs to $\mathcal{D}((0, 1), \mathcal{C}[0, T])$. In particular, for every $t \in [0, T]$, $y(\cdot, t)$ and $\widetilde{y}(\cdot, t)$ are equal in $L_2[0, 1]$ almost surely. Moreover, for every $u \in (0, 1)$, $\widetilde{y}(u, \cdot)$ is a square integrable and continuous $(\mathcal{F}_t)_{t\in[0,T]}$-martingale and*

$$\mathbb{P}\left[\forall u, v \in (0, 1), \forall s \in [0, T], \widetilde{y}(u, s) = \widetilde{y}(v, s) \text{ implies } \forall t \geqslant s, \widetilde{y}(u, t) = \widetilde{y}(v, t)\right] = 1.$$

From now on, we denote by y (instead of $\widetilde{y}$) the version of the limit process in the space $\mathcal{D}((0, 1), \mathcal{C}[0, T])$.

Remark I.42. The proof can be found in Appendix B of [Kon17a]. It should be noticed that the difficult part of the proof relies on the construction of a version $\widetilde{y}$ such that for every $u \in (0, 1)$, $\widetilde{y}(u, \cdot)$ is continuous at time $t = 0$.

This completes the proof of properties $(C3)$ and $(C4)$ of Theorem I.4. The aim of the next two Paragraphs is to prove property $(C5)$, in two steps.

I.5.2 Quadratic variation of $y(u,\cdot)$

The following proposition shows that the quadratic variation of a particle is proportional to the inverse of its mass:

Proposition I.43. *Let y be the version in $\mathcal{D}((0,1),\mathcal{C}[0,T])$ of the limit process given by Proposition I.41. For every $u\in(0,1)$,*

$$\langle y(u,\cdot),y(u,\cdot)\rangle_t=\int_0^t\frac{1}{m(u,s)}\mathrm{d}s,$$

where $m(u,s)=\int_0^1\mathbb{1}_{\{y(u,s)=y(v,s)\}}\mathrm{d}v$.

Proof. By Corollary I.32, for every positive $\psi\in L_\infty(0,1)$, we have:

$$\mathbb{E}\left[\int_0^1\psi(u)[(y_\sigma(u,t)-g(u))^2-(y_\sigma(u,s)-g(u))^2]f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\mathrm{d}u\right]$$
$$=\mathbb{E}\left[\int_0^1\psi(u)\int_s^t\frac{1}{m_\sigma(u,r)}\mathrm{d}r\ f_l(Y_\sigma(s_1),\dots,Y_\sigma(s_l))\mathrm{d}u\right]. \quad \text{(I.39)}$$

To obtain the convergence of the left hand side of (I.39), we proceed in the same way as for the proof of equality (I.37). The uniform integrability property follows from Corollary I.34. Therefore, the left hand side of (I.39) converges when $\sigma\to0$ to

$$\mathbb{E}\left[\int_0^1\psi(u)[(y(u,t)-g(u))^2-(y(u,s)-g(u))^2]f_l(Y(s_1),\dots,Y(s_l))\mathrm{d}u\right].$$

We also get a uniform integrability property for the right hand side of (I.39) by the same argument as in the proof of property $(B3)$ (see Proposition I.31). Assume that there exists a sequence (σ_n) of rational numbers tending to 0, a probability space $(\widehat{\Omega},\widehat{\mathbb{P}})$, a modification $(\widehat{m}_{\sigma_n},\widehat{y}_{\sigma_n})_{n\in\mathbb{N}}$ of $(m_{\sigma_n},y_{\sigma_n})_{n\in\mathbb{N}}$ on $L_1([0,1],\mathcal{C}[0,T])\times L_2([0,1],\mathcal{C}[0,T])$ and a modification $(\widehat{m},\widehat{y})$ of (m,y) on the same space such that for almost each $\omega\in\Omega$ and almost every $(u,t)\in[0,1]\times[0,T]$, the sequence $(\widehat{m}_{\sigma_n}(\omega,u,t),\ \widehat{y}_{\sigma_n}(\omega))_{n\in\mathbb{N}}$ converges to $(\widehat{m}(\omega,u,t),\widehat{y}(\omega))$ in $\mathbb{R}\times L_2([0,1],\mathcal{C}[0,T])$. This will be proved in Lemma I.44.

It follows that for every $\psi\in L_\infty(0,1)$:

$$\mathbb{E}\left[\int_0^1\psi(u)\left[(\widehat{y}(u,t)-g(u))^2-(\widehat{y}(u,s)-g(u))^2-\int_s^t\frac{\mathrm{d}r}{\widehat{m}(u,r)}\right]f_l(\widehat{Y}(s_1),\dots,\widehat{Y}(s_l))\mathrm{d}u\right]=0.$$

By Fubini's Theorem, we deduce that for almost every $u\in(0,1)$,

$$\mathbb{E}\left[\left((\widehat{y}(u,t)-g(u))^2-(\widehat{y}(u,s)-g(u))^2-\int_s^t\frac{\mathrm{d}r}{\widehat{m}(u,r)}\right)f_l(\widehat{Y}(s_1),\dots,\widehat{Y}(s_l))\right]=0. \quad \text{(I.40)}$$

We want to prove that (I.40) holds with every $u\in(0,1)$. Let $u\in(0,1)$. Choose $\delta>0$ such that $u\in(\delta,1-\delta)$. Let $(u_p)_{p\in\mathbb{N}}$ be a decreasing sequence in $(\delta,1-\delta)$ converging to u such that for every $p\in\mathbb{N}$, equality (I.40) holds at point u_p, $(y_{\sigma_n,\varepsilon}(u_p,t))_{t\in[0,T]}$ is a square integrable continuous $(\mathcal{F}_t^{\sigma_n,\varepsilon})_{t\in[0,T]}$-martingale for every $n\in\mathbb{N}$ and $\varepsilon\in\mathbb{Q}_+$ and $\limsup_{n\to\infty}\widehat{m}_{\sigma_n}(u_p,t)\leqslant\widehat{m}(u_p,t)$ almost surely for all $t\in[0,T]$. Such a sequence exists by Corollary I.17 and Lemma I.38. We will use these different properties later in this proof.

Almost surely, for every $r\in(0,T]$, $\widehat{y}(\cdot,r)$ is right-continuous at point u and is a step function. Therefore, $\widehat{m}(\cdot,r)=\int_0^1\mathbb{1}_{\{\widehat{y}(\cdot,r)=\widehat{y}(v,r)\}}\mathrm{d}v$ is also right continuous at point u for every positive time r. In order to prove (I.40) at point u, it is thus sufficient to show the following uniform integrability property: there exists $\beta>1$ such that

$$\sup_{p\in\mathbb{N}}\mathbb{E}\left[\left((\widehat{y}(u_p,t)-g(u_p))^2-(\widehat{y}(u_p,s)-g(u_p))^2-\int_s^t\frac{\mathrm{d}r}{\widehat{m}(u_p,r)}\right)^\beta\right]<+\infty. \quad \text{(I.41)}$$

First, by monotonicity, for all $p \in \mathbb{N}$, $\mathbb{E}\left[g(u_p)^{2\beta}\right] \leqslant g(\delta)^{2\beta} + g(1-\delta)^{2\beta}$. Then, the following statement holds: there exists $\beta > 1$ such that for every $t \in [0,T]$, $\sup_{p\in\mathbb{N}} \mathbb{E}\left[\widehat{y}(u_p,t)^{2\beta}\right] < +\infty$. Indeed, for every $p \in \mathbb{N}$, by monotonicity,

$$\frac{1}{\delta}\int_0^\delta \widehat{y}(v,t)\mathrm{d}v \leqslant \widehat{y}(u_p,t) \leqslant \frac{1}{\delta}\int_{1-\delta}^1 \widehat{y}(v,t)\mathrm{d}v.$$

Therefore, we have:

$$\begin{aligned}\mathbb{E}\left[\widehat{y}(u_p,t)^{2\beta}\right] &\leqslant \mathbb{E}\left[\left(\frac{1}{\delta}\int_0^\delta \widehat{y}(v,t)\mathrm{d}v\right)^{2\beta}\right] + \mathbb{E}\left[\left(\frac{1}{\delta}\int_{1-\delta}^1 \widehat{y}(v,t)\mathrm{d}v\right)^{2\beta}\right] \\ &\leqslant \frac{2}{\delta}\mathbb{E}\left[\int_0^1 \widehat{y}(v,t)^{2\beta}\mathrm{d}v\right], \end{aligned} \tag{I.42}$$

by Hölder's inequality. By Fatou's Lemma

$$\mathbb{E}\left[\int_0^1 \widehat{y}(v,t)^{2\beta}\mathrm{d}v\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[\int_0^1 \widehat{y}_{\sigma_n}(v,t)^{2\beta}\mathrm{d}v\right],$$

which is finite by Corollary I.34, for a β chosen in $(1, \frac{3}{2} - \frac{1}{p})$.

Let us keep the same exponent $\beta \in (1, \frac{3}{2} - \frac{1}{p})$. It remains to show that for every $t \in [0,T]$, $\sup_{p\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^t \frac{\mathrm{d}r}{\widehat{m}(u_p,r)}\right)^\beta\right] < +\infty$. Since $\limsup_{n\to\infty} \widehat{m}_{\sigma_n}(u_p,t) \leqslant \widehat{m}(u_p,t)$ and by Fatou's Lemma,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \frac{\mathrm{d}r}{\widehat{m}(u_p,r)}\right|^\beta\right] \leqslant \mathbb{E}\left[\left|\int_0^t \liminf_{n\to\infty} \frac{\mathrm{d}r}{\widehat{m}_{\sigma_n}(u_p,r)}\right|^\beta\right] &\leqslant \liminf_{n\to\infty} \mathbb{E}\left[\left|\int_0^t \frac{\mathrm{d}r}{\widehat{m}_{\sigma_n}(u_p,r)}\right|^\beta\right] \\ &\leqslant \liminf_{\substack{n\to\infty \\ \varepsilon\in\mathbb{Q}_+}} \mathbb{E}\left[\left|\int_0^t \frac{\mathrm{d}r}{\widehat{M}_{\sigma_n,\varepsilon}(u_p,r)}\right|^\beta\right].\end{aligned}$$

Because $(\widehat{y}_{\sigma_n,\varepsilon}(u_p,t))_{t\in[0,T]}$ is a square integrable martingale relatively to $(\mathcal{F}_t^{\sigma_n,\varepsilon})_{t\in[0,T]}$ and $\langle \widehat{y}_{\sigma_n,\varepsilon}(u_p,\cdot), \widehat{y}_{\sigma_n,\varepsilon}(u_p,\cdot)\rangle_t = \int_0^t \frac{\mathrm{d}r}{\widehat{M}_{\sigma_n,\varepsilon}(u_p,r)}$, we obtain by Burkholder-Davis-Gundy inequality:

$$\mathbb{E}\left[\left(\int_0^t \frac{\mathrm{d}r}{\widehat{M}_{\sigma_n,\varepsilon}(u_p,r)}\right)^\beta\right] \leqslant C\mathbb{E}\left[(\widehat{y}_{\sigma_n,\varepsilon}(u_p,t) - g(u_p))^{2\beta}\right].$$

We have already seen that $\mathbb{E}\left[g(u_p)^{2\beta}\right]$ is uniformly bounded for $p \in \mathbb{N}$. By the same argument as for inequality (I.42), $\mathbb{E}\left[\widehat{y}_{\sigma_n,\varepsilon}(u_p,t)^{2\beta}\right] \leqslant \frac{2}{\delta}\mathbb{E}\left[\int_0^1 \widehat{y}_{\sigma_n,\varepsilon}(v,t)^{2\beta}\mathrm{d}v\right]$, which is uniformly bounded for $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{Q}_+$. This completes the proof of (I.41).

Therefore, equality (I.40) holds with every $u \in (0,1)$, for every bounded and continuous f_l and for every $0 \leqslant s_1 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t$. Thus for every $u \in (0,1)$, the process $\left((\widehat{y}(u,t)-g(u))^2 - \int_0^t \frac{\mathrm{d}s}{\widehat{m}(u,s)}\right)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. This completes the proof of the proposition. □

In the proof of Proposition I.43, we used the following Lemma:

Lemma I.44. *There exists a sequence (σ_n) of rational numbers tending to 0, a sequence of processes $(\widehat{m}_{\sigma_n}, \widehat{y}_{\sigma_n})_{n\in\mathbb{N}}$ and a process $(\widehat{m}, \widehat{y})$ defined on the same probability space such that*

- *for all $n \in \mathbb{N}$, $(\widehat{m}_{\sigma_n}, \widehat{y}_{\sigma_n})$ and $(m_{\sigma_n}, y_{\sigma_n})$ (resp. $(\widehat{m}, \widehat{y})$ and (m,y)) have same law on $L_1([0,1], \mathcal{C}[0,T]) \times L_2([0,1], \mathcal{C}[0,T])$.*

- *for almost each* $\omega \in \Omega$ *and for almost every* (u,t) *in* $[0,1]\times[0,T]$*, the sequence* $(\widehat{m}_{\sigma_n}(\omega,u,t), \widehat{y}_{\sigma_n}(\omega))_{n\in\mathbb{N}}$ *converges to* $(\widehat{m}(\omega,u,t),\widehat{y}(\omega))$ *in* $\mathbb{R}\times L_2([0,1],\mathcal{C}[0,T])$*.*

Remark I.45. The Borel subset of $[0,1]\times[0,T]$ on which we have the convergence can depend on ω.

Before giving the proof of Lemma I.44, we give the following definition and state the following Lemma, which will be useful in the proof. Let us define in $L_1([0,1]\times[0,1],\mathcal{C}[0,T])$:

$$C_\sigma(u_1,u_2,t) := \int_0^t \left(\frac{1}{m_\sigma(u_1,s)} + \frac{1}{m_\sigma(u_2,s)} - \frac{2m_\sigma(u_1,u_2,s)}{m_\sigma(u_1,s)m_\sigma(u_2,s)}\right)\mathrm{d}s.$$

Lemma I.46. *There exists a sequence* (σ_n) *in* $\mathbb{Q}_+$ *tending to* 0 *such that* $(y_{\sigma_n}, C_{\sigma_n})_{n\in\mathbb{N}}$ *converges in distribution to* (y,C) *in* $L_2([0,1],\mathcal{C}[0,T])\times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$*. For almost every* $u_1,u_2\in[0,1]$*, the limit process* $C(u_1,u_2,\cdot)$ *is the quadratic variation of* $y(u_1,\cdot)-y(u_2,\cdot)$ *relatively to the filtration generated by* Y *and* C*.*

We start by giving the proof of Lemma I.44 and then we give the proof of Lemma I.46.

Proof (Lemma I.44). By Skorohod's representation Theorem, it follows from Lemma I.46 that there exists a sequence $(\widehat{y}_{\sigma_n},\widehat{C}_{\sigma_n})_n$ and a random variable $(\widehat{y},\widehat{C})$ defined on the same probability space such that

- for all $n\in\mathbb{N}$, $(\widehat{y}_{\sigma_n},\widehat{C}_{\sigma_n})$ and $(y_{\sigma_n},C_{\sigma_n})$ (resp. $(\widehat{y},\widehat{C})$ and (y,C)) have same law,
- the sequence $(\widehat{y}_{\sigma_n},\widehat{C}_{\sigma_n})_n$ converges almost surely to $(\widehat{y},\widehat{C})$ in $L_2([0,1],\mathcal{C}[0,T])\times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$.

We apply to $(\widehat{y}_{\sigma_n})_n$ the argument in the proof of Lemma I.38 and we prove that, up to extracting another subsequence (independent of ω), for almost every $u\in[0,1]$ and almost surely, $\limsup_{n\to\infty}\widehat{m}_{\sigma_n}(u,t)\leqslant\widehat{m}(u,t)$ for every $t\in[0,T]$.

For each $t\in[0,T]$, we may suppose that for each $n\in\mathbb{N}$, $\widehat{y}_{\sigma_n}(\cdot,t)$ is a càdlàg function, so that for every $u\in(0,1)$,

$$\begin{aligned}\widehat{m}_{\sigma_n}(u,t) &= \int_0^1 \varphi^2_{\sigma_n}(\widehat{y}_{\sigma_n}(u,t)-\widehat{y}_{\sigma_n}(v,t))\mathrm{d}v\\ &= \lim_{p\to\infty} p\int_u^{(u+\frac{1}{p})\wedge 1}\int_0^1 \varphi^2_{\sigma_n}(\widehat{y}_{\sigma_n}(u',t)-\widehat{y}_{\sigma_n}(v,t))\mathrm{d}v\mathrm{d}u'\end{aligned}$$

is a measurable function with respect to $\widehat{y}_{\sigma_n}(\cdot,t)$. We deduce that $(\widehat{m}_{\sigma_n}(u,t),\widehat{y}_{\sigma_n})$ has the same law as $(m_{\sigma_n}(u,t),y_{\sigma_n})$ for every $u\in(0,1)$.

From now on, we forget the hats in our notation. We may suppose that y is the version in $\mathcal{D}((0,1),\mathcal{C}[0,T])$ given by Proposition I.41. Let Ω' be such that $\mathbb{P}[\Omega']=1$ and for all $\omega\in\Omega'$, we have the following convergences in $\mathbb{R}$:

$$\int_0^1 \sup_{t\leqslant T}|y_{\sigma_n}(u,t)-y(u,t)|^2(\omega)\mathrm{d}u \underset{n\to\infty}{\longrightarrow} 0, \tag{I.43}$$

$$\int_0^1\int_0^1 \sup_{t\leqslant T}|C_{\sigma_n}(u_1,u_2,t)-C(u_1,u_2,t)|(\omega)\mathrm{d}u_1\mathrm{d}u_2 \underset{n\to\infty}{\longrightarrow} 0. \tag{I.44}$$

Fix $\omega\in\Omega'$. Thanks to (I.43), we already have the convergence of $(y_{\sigma_n}(\omega))_n$ to $y(\omega)$ in $L_2([0,1],\mathcal{C}[0,T])$. It remains to show that for almost every $(u,t)\in[0,1]\times[0,T]$, $(m_{\sigma_n}(\omega,u,t))_n$ converges to $m(\omega,u,t)=\int_0^1 \mathbb{1}_{\{y(u,t)=y(v,t)\}}(\omega)\mathrm{d}v$. We already know that for every $\omega\in\Omega'$, every $t\in[0,T]$ and almost every $u\in(0,1)$, $\limsup_{n\to\infty} m_{\sigma_n}(\omega,u,t)\leqslant m(\omega,u,t)$.

Proof of inequality: $\liminf_{n\to\infty} m_{\sigma_n}(\omega,u,t) \geqslant m(\omega,u,t)$.

By the coalescence property given by Proposition I.41, for every u_1,u_2 and for all $t > \tau_{u_1,u_2}$, $y(u_1,t) = y(u_2,t)$. Therefore, since $C(u_1,u_2,\cdot)$ is the quadratic variation of $y(u_1,\cdot) - y(u_2,\cdot)$, $t \mapsto C(u_1,u_2,t)$ remains constant on (τ_{u_1,u_2},T). Thus we obtain:

$$\begin{aligned}
&\left|\int_0^1\int_0^1\int_{\tau_{u_1,u_2}}^T \left(\frac{1}{m_{\sigma_n}(u_1,t)} + \frac{1}{m_{\sigma_n}(u_2,t)} - \frac{2m_{\sigma_n}(u_1,u_2,t)}{m_{\sigma_n}(u_1,t)m_{\sigma_n}(u_2,t)}\right)\mathrm{d}t\mathrm{d}u_1\mathrm{d}u_2\right| \\
&\quad = \left|\int_0^1\int_0^1 (C_{\sigma_n}(u_1,u_2,T) - C_{\sigma_n}(u_1,u_2,\tau_{u_1,u_2}))\mathrm{d}u_1\mathrm{d}u_2\right| \\
&\quad = \left|\int_0^1\int_0^1 (C_{\sigma_n}(u_1,u_2,T) - C(u_1,u_2,T) + C(u_1,u_2,\tau_{u_1,u_2}) - C_{\sigma_n}(u_1,u_2,\tau_{u_1,u_2}))\mathrm{d}u_1\mathrm{d}u_2\right| \\
&\quad \leqslant 2\int_0^1\int_0^1 \sup_{t\leqslant T} |C_{\sigma_n}(u_1,u_2,t) - C(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2.
\end{aligned}$$

By (I.44), the latter term tends to 0. We also recall that

$$\begin{aligned}
&\frac{1}{m_{\sigma_n}(u_1,t)} + \frac{1}{m_{\sigma_n}(u_2,t)} - \frac{2m_{\sigma_n}(u_1,u_2,t)}{m_{\sigma_n}(u_1,t)m_{\sigma_n}(u_2,t)} \\
&\qquad = \frac{\int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_1,t_0) - y_{\sigma_n}(v,t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_2,t_0) - y_{\sigma_n}(v,t_0))|^2\,\mathrm{d}v}{m_{\sigma_n}(u_1,t_0)m_{\sigma_n}(u_2,t_0)}
\end{aligned}$$

is non-negative.

We define $f_{\sigma_n}(t,u_1,u_2) := \left(\frac{1}{m_{\sigma_n}(u_1,t)} + \frac{1}{m_{\sigma_n}(u_2,t)} - \frac{2m_{\sigma_n}(u_1,u_2,t)}{m_{\sigma_n}(u_1,t)m_{\sigma_n}(u_2,t)}\right)\mathbb{1}_{\{t\geqslant\tau_{u_1,u_2}\}}$. For every $\omega\in\Omega'$, $\int_0^T\int_0^1\int_0^1 f_{\sigma_n}(t,u_1,u_2)(\omega)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}t \underset{n\to\infty}{\longrightarrow} 0$. Therefore, for every $\varepsilon>0$, using Markov's inequality as in (I.38), and since $f_{\sigma_n}\geqslant 0$:

$$\begin{aligned}
&\mathbb{P}\otimes\frac{1}{T}\operatorname{Leb}|_{[0,T]}\otimes\operatorname{Leb}|_{[0,1]}\otimes\operatorname{Leb}|_{[0,1]}\left\{(\omega,t,u_1,u_2) : f_{\sigma_n}(t,u_1,u_2)(\omega)\geqslant\varepsilon\right\} \\
&\qquad\qquad \leqslant \mathbb{E}\left[1\wedge\frac{1}{\varepsilon T}\int_0^T\int_0^1\int_0^1 f_{\sigma_n}(t,u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}t\right],
\end{aligned}$$

which tends to 0 when $n\to\infty$, whence we obtain a convergence in probability with respect to the probability space $\Omega\times[0,T]\times[0,1]\times[0,1]$. Up to extracting another subsequence (independent of the choice of ω), we deduce the existence of an almost sure event on which (f_{σ_n}) converges to 0.

Let Ω'', $\mathbb{P}[\Omega'']=1$, be such that for every $\omega\in\Omega''$, we have $f_{\sigma_n}(t,u_1,u_2)(\omega)\to 0$ for almost every $(t,u_1,u_2)\in[0,T]\times[0,1]\times[0,1]$. Fix $\omega\in\Omega''$. Let us consider a Borel set $\mathcal{B}=\mathcal{B}(\omega)$ in $[0,T]$, $\operatorname{Leb}(\mathcal{B})=T$, such that for every $t\in\mathcal{B}$, $f_{\sigma_n}(t,u_1,u_2)\to 0$ for almost every $(u_1,u_2)\in[0,1]\times[0,1]$.

Let $t_0\in\mathcal{B}$. Let us consider a Borel set $\mathcal{A}$ (depending on ω and t_0) of measure 1 such that for all $u_1,u_2\in\mathcal{A}$,

$$f_{\sigma_n}(t_0,u_1,u_2) \underset{n\to\infty}{\longrightarrow} 0. \tag{I.45}$$

Let $u\in\mathcal{A}$. We want to prove that $\liminf_{n\to\infty} m_{\sigma_n}(u,t_0)\geqslant m(u,t_0)$. Define $u_{\sup}=\sup\{v\in[0,1] : y(v,t_0)=y(u,t_0)\}$ and $u_{\inf}$ the infimum of that set. Since $v\mapsto y(v,t_0)$ is non-decreasing, $m(u,t_0)=u_{\sup}-u_{\inf}$. If $m(u,t_0)=0$, then we clearly have:

$$\liminf_{n\to\infty} m_{\sigma_n}(u,t_0)\geqslant m(u,t_0).$$

Suppose now that $m(u,t_0)>0$. Choose $\delta>0$ such that $\delta<\frac{u_{\sup}-u_{\inf}}{6}$. Let $u_{\max}\in\mathcal{A}\cap(u_{\sup}-\delta,u_{\sup})$, $u_{\min}\in\mathcal{A}\cap(u_{\inf},u_{\inf}+\delta)$ and $u_{\mathrm{med}}\in\mathcal{A}\cap\left(\frac{u_{\min}+u_{\max}}{2}-\delta,\frac{u_{\min}+u_{\max}}{2}+\delta\right)$. We have:

$u_{\max} - u_{\min} \geqslant u_{\sup} - u_{\inf} - 2\delta = m(u,t_0) - 2\delta$ and by definition of $u_{\sup}$ and $u_{\inf}$ and since $u_{\max}$, $u_{\min}$ and u_{med} belongs to $(u_{\inf}, u_{\sup})$, we have $t_0 \geqslant \tau_{u_1,u_2}$ for $(u_1,u_2) = (u,u_{\max})$, $(u,u_{\min})$, $(u_{\max},u_{\min})$ and (u,u_{med}).

We deduce from (I.45) and the fact that $u, u_{\max}, u_{\min}, u_{\mathrm{med}}$ belongs to $\mathcal{A}$ that there exists N such that for each $n \geqslant N$, $f_{\sigma_n}(t_0,u_1,u_2) \leqslant \delta$ for $(u_1,u_2) = (u,u_{\max})$, $(u,u_{\min})$, $(u_{\max},u_{\min})$ and (u,u_{med}). It implies that for each $n \geqslant N$,

$$\frac{\int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_1,t_0) - y_{\sigma_n}(v,t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_2,t_0) - y_{\sigma_n}(v,t_0))|^2 \,\mathrm{d}v}{m_{\sigma_n}(u_1,t_0) m_{\sigma_n}(u_2,t_0)} = f_{\sigma_n}(t_0,u_1,u_2) \leqslant \delta. \tag{I.46}$$

Since the mass m_{σ_n} is bounded by 1, we deduce in particular that for all $n \geqslant N$,

$$\int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_1,t_0) - y_{\sigma_n}(v,t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_2,t_0) - y_{\sigma_n}(v,t_0))|^2 \,\mathrm{d}v \leqslant \delta. \tag{I.47}$$

Inequalities (I.46) and (I.47) are satisfied for $(u_1,u_2) = (u,u_{\max})$, $(u,u_{\min})$, $(u_{\max},u_{\min})$ and (u,u_{med}).

Let $n \geqslant N$ and $d := y_{\sigma_n}(u_{\max},t_0) - y_{\sigma_n}(u_{\min},t_0) \geqslant 0$. We distinguish three cases:

- $d \geqslant \sigma_n$**:** Recall that φ_{σ_n} is equal to 0 on $[\frac{\sigma_n}{2}, +\infty)$. Thus for every $v \in [0,1]$, the terms $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max},t_0) - y_{\sigma_n}(v,t_0))$ and $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\min},t_0) - y_{\sigma_n}(v,t_0))$ can not be simultaneously different from 0, because $d \geqslant \sigma_n$. Therefore, selecting $(u_1,u_2) = (u_{\max},u_{\min})$, inequality (I.46) implies:

  $$\frac{\int_0^1 \varphi^2_{\sigma_n}(y_{\sigma_n}(u_{\max},t_0) - y_{\sigma_n}(v,t_0))\mathrm{d}v + \int_0^1 \varphi^2_{\sigma_n}(y_{\sigma_n}(u_{\min},t_0) - y_{\sigma_n}(v,t_0))\mathrm{d}v}{m_{\sigma_n}(u_{\max},t_0) m_{\sigma_n}(u_{\min},t_0)} \leqslant \delta,$$

  that is:

  $$\frac{1}{m_{\sigma_n}(u_{\min},t_0)} + \frac{1}{m_{\sigma_n}(u_{\max},t_0)} \leqslant \delta.$$

  Thus, we obtain $\delta \geqslant 2$, which is excluded by definition of δ.

- $d \leqslant \sigma_n - \eta$**:** Recall that η is chosen so that $\eta < \frac{\sigma_n}{3}$. Define the following two sets

  $$V_{\max} = \{v \in [u_{\min},u_{\max}] : y_{\sigma_n}(u_{\max},t_0) - y_{\sigma_n}(v,t_0) \leqslant \tfrac{\sigma_n - \eta}{2}\},$$
  $$V_{\min} = \{v \in [u_{\min},u_{\max}] : y_{\sigma_n}(u_{\max},t_0) - y_{\sigma_n}(v,t_0) > \tfrac{\sigma_n - \eta}{2}\}.$$

  Clearly, we have: $\mathrm{Leb}(V_{\max}) + \mathrm{Leb}(V_{\min}) = u_{\max} - u_{\min} \geqslant m(u,t_0) - 2\delta$. Recall that φ_{σ_n} is equal to 1 on $[0, \frac{\sigma_n - \eta}{2}]$. Thus, for each $v \in V_{\max}$, $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max},t_0) - y_{\sigma_n}(v,t_0)) = 1$, and for each $v \in V_{\min}$, using $d \leqslant \sigma_n - \eta$, $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\min},t_0) - y_{\sigma_n}(v,t_0)) = 1$. We have

  $$m_{\sigma_n}(u,t_0) \geqslant \textstyle\int_{V_{\max}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u,t_0) - y_{\sigma_n}(v,t_0))\mathrm{d}v + \int_{V_{\min}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u,t_0) - y_{\sigma_n}(v,t_0))\mathrm{d}v. \tag{I.48}$$

  We can deduce from inequality (I.47) applied to $(u_1,u_2) = (u,u_{\max})$ that:

  $$\textstyle\int_{V_{\max}} |\varphi_{\sigma_n}(y_{\sigma_n}(u,t_0) - y_{\sigma_n}(v,t_0)) - \varphi_{\sigma_n}(y_{\sigma_n}(u_{\max},t_0) - y_{\sigma_n}(v,t_0))|^2 \,\mathrm{d}v \leqslant \delta.$$

  By Minkowski's inequality $|\|f_1\|_{L_2} - \|f_2\|_{L_2}| \leqslant \|f_1 - f_2\|_{L_2}$, we obtain:

  $$\left| \left(\int_{V_{\max}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u,t_0) - y_{\sigma_n}(v,t_0))\mathrm{d}v \right)^{1/2} - \mathrm{Leb}(V_{\max})^{1/2} \right| \leqslant \sqrt{\delta},$$

whence

$$\left|\int_{V_{\max}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u,t_0)-y_{\sigma_n}(v,t_0))\mathrm{d}v - \mathrm{Leb}(V_{\max})\right| \leqslant (m^{1/2}_{\sigma_n}(u,t_0)+\mathrm{Leb}(V_{\max})^{1/2})\sqrt{\delta} \leqslant 2\sqrt{\delta}.$$

Similarly, applying inequality (I.47) to $(u,u_{\min})$, we obtain:

$$\left|\int_{V_{\min}} \varphi^2_{\sigma_n}(y_{\sigma_n}(u,t_0)-y_{\sigma_n}(v,t_0))\mathrm{d}v - \mathrm{Leb}(V_{\min})\right| \leqslant 2\sqrt{\delta}.$$

Thus, by inequality (I.48), we conclude:

$$\begin{aligned} m_{\sigma_n}(u,t_0) &\geqslant \mathrm{Leb}(V_{\max})+\mathrm{Leb}(V_{\min})-4\sqrt{\delta} \\ &\geqslant m(u,t_0)-2\delta-4\sqrt{\delta}. \end{aligned}$$

- $d\in(\sigma_n-\eta,\sigma_n)$**:** We now define three distinct sets

$$\begin{aligned} V_{\max} &= \{v\in[u_{\min},u_{\max}] : y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(v,t_0) < \tfrac{\sigma_n-\eta}{2}\}, \\ V_{\mathrm{med}} &= \{v\in[u_{\min},u_{\max}] : y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(v,t_0) \in [\tfrac{\sigma_n-\eta}{2},\tfrac{\sigma_n+\eta}{2}]\}, \\ V_{\min} &= \{v\in[u_{\min},u_{\max}] : y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(v,t_0) > \tfrac{\sigma_n+\eta}{2}\}. \end{aligned}$$

By definition of those sets, and since $d\in(\sigma_n-\eta,\sigma_n)$, we have

$$\begin{aligned} &\forall v\in V_{\max},\ \varphi_{\sigma_n}(y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(v,t_0))=1, \\ &\forall v\in V_{\min},\ \varphi_{\sigma_n}(y_{\sigma_n}(u_{\min},t_0)-y_{\sigma_n}(v,t_0))=1. \end{aligned}$$

Moreover, we have $y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(u_{\mathrm{med}},t_0)\in[\frac{\sigma_n-\eta}{2},\frac{\sigma_n+\eta}{2}]$.

Indeed, if $y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(u_{\mathrm{med}},t_0)$ was greater than $\frac{\sigma_n+\eta}{2}$, we would have, for all $v\in[u_{\min},u_{\mathrm{med}}]$, $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(v,t_0))=0$ and $\varphi_{\sigma_n}(y_{\sigma_n}(u_{\min},t_0)-y_{\sigma_n}(v,t_0))=1$. By inequality (I.47) applied to $(u_1,u_2)=(u_{\max},u_{\min})$, we would deduce that:

$$\begin{aligned} \delta &\geqslant \int_0^1 |\varphi_{\sigma_n}(y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(v,t_0))-\varphi_{\sigma_n}(y_{\sigma_n}(u_{\min},t_0)-y_{\sigma_n}(v,t_0))|^2\,\mathrm{d}v \\ &\geqslant \int_{u_{\min}}^{u_{\mathrm{med}}} \mathrm{d}v = u_{\mathrm{med}}-u_{\min} \geqslant \frac{u_{\max}-u_{\min}}{2}-\delta. \end{aligned}$$

However, since $\delta<\frac{u_{\sup}-u_{\inf}}{6}$ and $u_{\max}-u_{\min}\geqslant u_{\sup}-u_{\inf}-2\delta$, we have $u_{\max}-u_{\min}>4\delta$, which is in contradiction with the above inequality. Similarly, $y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(u_{\mathrm{med}},t_0)$ can not be smaller than $\frac{\sigma_n-\eta}{2}$, otherwise $y_{\sigma_n}(u_{\mathrm{med}},t_0)-y_{\sigma_n}(u_{\min},t_0)$ would be greater than $\frac{\sigma_n+\eta}{2}$ and we would obtain the same contradiction. Therefore, $y_{\sigma_n}(u_{\max},t_0)-y_{\sigma_n}(u_{\mathrm{med}},t_0)\in[\frac{\sigma_n-\eta}{2},\frac{\sigma_n+\eta}{2}]$, which implies that $u_{\mathrm{med}}\in V_{\mathrm{med}}$ and in particular that

$$\forall v\in V_{\mathrm{med}},\ \varphi_{\sigma_n}(y_{\sigma_n}(u_{\mathrm{med}},t_0)-y_{\sigma_n}(v,t_0))=1.$$

As in the previous case, we deduce that

$$\begin{aligned} m_{\sigma_n}(u,t_0) &\geqslant \mathrm{Leb}(V_{\max})+\mathrm{Leb}(V_{\mathrm{med}})+\mathrm{Leb}(V_{\min})-6\sqrt{\delta} \\ &= u_{\max}-u_{\min}-6\sqrt{\delta} \\ &\geqslant m(u,t_0)-2\delta-6\sqrt{\delta}. \end{aligned}$$

Actually, putting all the cases together, we have proved that for each $n \geqslant N$, $m_{\sigma_n}(u,t_0) \geqslant m(u,t_0) - 2\delta - 6\sqrt{\delta}$. Hence, for all $\delta < \frac{u_{\sup}-u_{\inf}}{6}$, we have:

$$\liminf_{n\to\infty} m_{\sigma_n}(u,t_0) \geqslant m(u,t_0) - 2\delta - 6\sqrt{\delta}.$$

By letting δ converge to 0, we have for every $t_0 \in \mathcal{B}$, $\liminf_{n\to\infty} m_{\sigma_n}(u,t_0) \geqslant m(u,t_0)$ for every $u \in \mathcal{A}$. Therefore, there exists a subsequence (σ_n) such that for almost every ω, for almost every $t \in [0,T]$ and almost every $u \in [0,1]$, $m_{\sigma_n}(\omega,u,t) \to_{n\to\infty} m(\omega,u,t)$. □

It remains to give the proof of Lemma I.46.

Proof (Lemma I.46). The first step will be to prove that the sequence $(y_\sigma, C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is tight in $L_2([0,1],\mathcal{C}[0,T]) \times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$. We have already proved that $(y_\sigma)_{\sigma\in\mathbb{Q}_+}$ is tight in $L_2([0,1],\mathcal{C}[0,T])$. We will use a tightness criterion to prove that the sequence $(C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is tight in $L_1([0,1]\times[0,1],\mathcal{C}[0,T])$. The space changed in comparison with $L_2([0,1],\mathcal{C}[0,T])$, but the criterion remains very semilar to the one of Proposition I.20.

We have, similarly to Proposition I.20, three criteria to prove. We want to show the following criterion:

First criterion: Let $\delta > 0$. There is $M > 0$ such that for every σ in $\mathbb{Q}_+$, $\mathbb{P}\left[\|C_\sigma\| \geqslant M\right] \leqslant \delta$, where $\|C_\sigma\| := \int_0^1\int_0^1 \sup_{t\leqslant T} |C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2$.

That statement follows from Markov's inequality and the existence of a constant C independent of σ such that:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\int_0^1 \sup_{t\leqslant T}|C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\right] &\leqslant 2\mathbb{E}\left[\int_0^1\int_0^T \frac{\mathrm{d}t\mathrm{d}u_1}{m_\sigma(u_1,t)}\right] + 2\mathbb{E}\left[\int_0^1\int_0^T \frac{\mathrm{d}t\mathrm{d}u_2}{m_\sigma(u_2,t)}\right]\\ &\leqslant C.\end{aligned}$$

The existence of C is a consequence of Lemma I.33.

Then, we prove the following criterion:

Second criterion: Let $\delta > 0$. For each $k \geqslant 1$, there exists $\eta_k > 0$ such that for all σ in $\mathbb{Q}_+$,

$$\mathbb{P}\left[\int_0^1\int_0^1 \sup_{|t_2-t_1|<\eta_k} |C_\sigma(u_1,u_2,t_2) - C_\sigma(u_1,u_2,t_1)|\mathrm{d}u_1\mathrm{d}u_2 \geqslant \frac{1}{k}\right] \leqslant \frac{\delta}{2^k}.$$

The proof is very close to Proposition I.28. We start by defining for every u_1, $u_2 \in (0,1)$: $K_1(u_1,u_2) := \mathbb{E}\left[\|C_\sigma(u_1,u_2,\cdot)\|_{\mathcal{C}[0,T]}\right]$ and $K_2(u_1) := \mathbb{E}\left[\int_0^T \frac{1}{m_\sigma^\beta(u_1,s)}\mathrm{d}s\right]$. Fix $\delta > 0$. There exists $C > 0$ such that $\int_0^1\int_0^1 \mathbb{1}_{\{K_1(u_1,u_2)\geqslant C\}}\mathrm{d}u_1\mathrm{d}u_2 \leqslant \delta$ and $\int_0^1 \mathbb{1}_{\{K_2(u)\geqslant C\}}\mathrm{d}u \leqslant \delta$. Define the following set $K := \{(u_1,u_2) : K_1(u_1,u_2) \leqslant C, K_2(u_1) \leqslant C, K_2(u_2) \leqslant C\}$.

By Aldous' tightness criterion, the collection $(C_\sigma(u_1,u_2,\cdot))_{\sigma\in\mathbb{Q}_+,(u_1,u_2)\in K}$ is tight in $\mathcal{C}[0,T]$. This fact relies on the following inequality, where $\eta > 0$ and τ is a stopping time for $C_\sigma(u_1,u_2,\cdot)$:

$$\begin{aligned}&\mathbb{E}\left[|C_\sigma(u_1,u_2,\tau+\eta) - C_\sigma(u_1,u_2,\tau)|\right]\\ &\qquad= \mathbb{E}\left[\left|\int_\tau^{\tau+\eta}\left(\frac{1}{m_\sigma(u_1,s)} + \frac{1}{m_\sigma(u_2,s)} - \frac{2m_\sigma(u_1,u_2,s)}{m_\sigma(u_1,s)m_\sigma(u_2,s)}\right)\mathrm{d}s\right|\right]\\ &\qquad\leqslant 2\mathbb{E}\left[\int_\tau^{\tau+\eta}\left(\frac{1}{m_\sigma(u_1,s)} + \frac{1}{m_\sigma(u_2,s)}\right)\mathrm{d}s\right],\end{aligned}$$

and the rest of the proof is an adaptation of the proof of Proposition I.28.

Finally we show the third criterion:

Third criterion: Let $\delta > 0$. For each $k \geqslant 1$, there is $H > 0$ such that for all σ in $\mathbb{Q}_+$,

$$\mathbb{P}\Bigg[\forall h = (h_1, h_2), 0 < h_1 < H, 0 < h_2 < H, \\ \int_0^{1-h_1}\int_0^{1-h_2} \sup_{t\leqslant T} |C_\sigma(u_1+h_1, u_2+h_2, t) - C_\sigma(u_1, u_2, t)| \mathrm{d}u_1 \mathrm{d}u_2 \leqslant \frac{1}{k}\Bigg] \geqslant 1 - \frac{\delta}{2^k}. \quad \text{(I.49)}$$

Let $h_1 > 0$ and begin by estimating

$$E_\sigma := \mathbb{E}\left[\int_0^{1-h_1}\int_0^1 \sup_{t\leqslant T} |C_\sigma(u_1+h_1, u_2, t) - C_\sigma(u_1, u_2, t)| \mathrm{d}u_1 \mathrm{d}u_2\right].$$

We compute (for the sake of simplicity, we will write from now on $y_\sigma(u)$ instead of $y_\sigma(u, \cdot)$ if there is no possibility of confusion):

$$\begin{aligned} C_\sigma(u_1+h_1, u_2, t) - C_\sigma(u_1, u_2, t) &= \langle y_\sigma(u_1+h_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t \\ &\quad - \langle y_\sigma(u_1) - y_\sigma(u_2), y_\sigma(u_1) - y_\sigma(u_2)\rangle_t \\ &= \langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t \\ &\quad + \langle y_\sigma(u_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_t. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{t\leqslant T} |C_\sigma(u_1+h_1, u_2, t) - C_\sigma(u_1, u_2, t)| \\ &\qquad \leqslant \sup_{t\leqslant T} |\langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t| \\ &\qquad\quad + \sup_{t\leqslant T} |\langle y_\sigma(u_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_t|. \end{aligned} \quad \text{(I.50)}$$

Then, we use Kunita-Watanabe's inequality on the first term of the right hand side:

$$\begin{aligned} &|\langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t| \\ &\qquad \leqslant |\langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_t|^{\frac{1}{2}} \\ &\qquad\qquad |\langle y_\sigma(u_1+h_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_t|^{\frac{1}{2}} \\ &\qquad \leqslant |\langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_T|^{\frac{1}{2}} \\ &\qquad\qquad |\langle y_\sigma(u_1+h_1) - y_\sigma(u_2), y_\sigma(u_1+h_1) - y_\sigma(u_2)\rangle_T|^{\frac{1}{2}}. \end{aligned}$$

By doing the same computation on the second term of the right hand side of (I.50), by Cauchy-Schwarz inequality and by the substitution of $u_1 + h_1$ by u_1, we obtain:

$$\begin{aligned} E_\sigma &\leqslant 2\mathbb{E}\left[\int_0^{1-h_1}\int_0^1 \langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_T \mathrm{d}u_1 \mathrm{d}u_2\right]^{1/2} \\ &\quad \times \mathbb{E}\left[\int_0^1\int_0^1 \langle y_\sigma(u_1) - y_\sigma(u_2), y_\sigma(u_1) - y_\sigma(u_2)\rangle_T \mathrm{d}u_1 \mathrm{d}u_2\right]^{1/2} \\ &\leqslant 2\mathbb{E}\left[\int_0^{1-h_1} \langle y_\sigma(u_1+h_1) - y_\sigma(u_1), y_\sigma(u_1+h_1) - y_\sigma(u_1)\rangle_T \mathrm{d}u_1\right]^{1/2} C^{1/2}, \end{aligned}$$

where C is the same constant as the one in the first criterion. By Fubini's Theorem:

$$\begin{aligned}E_\sigma &\leqslant 2C^{1/2}\mathbb{E}\left[\int_0^{1-h_1}(y_\sigma(u_1+h_1,T)-y_\sigma(u_1,T)+g(u_1)-g(u_1+h_1))^2\mathrm{d}u_1\right]^{1/2}\\ &\leqslant 2C^{1/2}\mathbb{E}\left[\int_0^{1-h_1}(y_\sigma(u_1+h_1,T)-y_\sigma(u_1,T))^2\mathrm{d}u_1\right]^{1/2}\\ &\quad+2C^{1/2}\mathbb{E}\left[\int_0^{1-h_1}(g(u_1+h_1)-g(u_1))^2\mathrm{d}u_1\right]^{1/2}.\end{aligned}$$

We recall inequalities (I.27) and (I.28). Therefore, there are $\alpha>0$ and $C>0$ such that for each $\sigma\in\mathbb{Q}_+$ and each $h_1>0$,

$$E_\sigma\leqslant Ch_1^\alpha.$$

We deduce that for each $n\in\mathbb{N}$, by Markov's inequality,

$$\begin{aligned}p_n:=\mathbb{P}\left[\int_0^{1-\frac{1}{2^n}}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+\tfrac{1}{2^n},u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\geqslant\tfrac{1}{2^{\frac{n\alpha}{2}}}\right]&\leqslant 2^{\frac{n\alpha}{2}}C\left(\frac{1}{2^n}\right)^\alpha\\ &=\frac{C}{2^{\frac{n\alpha}{2}}}.\end{aligned}$$

Since $\alpha>0$, $\sum_{n\geqslant 0}p_n$ converges. By Borel-Cantelli's Lemma, for each $k\geqslant 1$, there is $n_0\geqslant 0$ such that, with probability greater than $1-\frac{\delta}{2^k}$, for all $n\geqslant n_0$,

$$\int_0^{1-\frac{1}{2^n}}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+\tfrac{1}{2^n},u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\leqslant\frac{1}{2^{\frac{n\alpha}{2}}}.$$

Furthermore, up to choosing a greater n_0, we can suppose that for all $n\geqslant n_0$, we also have:

$$\int_0^1\int_0^{1-\frac{1}{2^n}}\sup_{t\leqslant T}|C_\sigma(u_1,u_2+\tfrac{1}{2^n},t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\leqslant\frac{1}{2^{\frac{n\alpha}{2}}}.$$

We will now extend these estimations to more general perturbations. Let $h=(h_1,h_2)$ be such that $0<h_1<\frac{1}{2^{n_0}}$, $0<h_2<\frac{1}{2^{n_0}}$. We decompose:

$$\begin{aligned}&\int_0^{1-h_1}\int_0^{1-h_2}\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2+h_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\\ &\qquad\leqslant\int_0^{1-h_1}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\\ &\qquad\quad+\int_0^1\int_0^{1-h_2}\sup_{t\leqslant T}|C_\sigma(u_1,u_2+h_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2.\end{aligned}\tag{I.51}$$

Suppose $h_1\geqslant 0$. Since $h_1<\frac{1}{2^{n_0}}$, there exists a sequence $(\varepsilon_n)_{n>n_0}$ with values in $\{0,1\}$ such that $h_1=\sum_{n\geqslant n_0+1}\frac{\varepsilon_n}{2^n}$. Moreover, we have for every $q\geqslant 1$:

$$\begin{aligned}&\int_0^{1-h_1}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2,t)-C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\\ &\leqslant\sum_{k=1}^{q-1}\int_0^{1-h_1}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+k}\frac{\varepsilon_n}{2^n},u_2,t)-C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+k+1}\frac{\varepsilon_n}{2^n},u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\\ &\leqslant\sum_{k=1}^{q-1}\int_0^{1-\frac{1}{2^{n_0+k}}}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+\tfrac{1}{2^{n_0+k}},u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\leqslant\sum_{k=1}^{q-1}\frac{1}{2^{(n_0+k)\frac{\alpha}{2}}}.\end{aligned}\tag{I.52}$$

We want to let q tend to $+\infty$ in (I.52). To do that, we prove that:

$$\int_0^{1-h_1}\int_0^1 \sup_{t\leqslant T}|C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2 \underset{q\to+\infty}{\longrightarrow} 0. \tag{I.53}$$

By definition of C_σ,

$$\begin{aligned}
&\int_0^{1-h_1}\int_0^1 \sup_{t\leqslant T}|C_\sigma(u_1+\textstyle\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\\
&\quad\leqslant\int_0^{1-h_1}\int_0^T\left|\frac{1}{m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},s)}-\frac{1}{m_\sigma(u_1,s)}\right|\mathrm{d}s\mathrm{d}u_1\\
&\quad+\int_0^{1-h_1}\int_0^1\int_0^T\frac{2}{m_\sigma(u_2,s)}\left|\frac{m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},u_2,s)}{m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},s)}-\frac{m_\sigma(u_1,u_2,s)}{m_\sigma(u_1,s)}\right|\mathrm{d}s\mathrm{d}u_1\mathrm{d}u_2.
\end{aligned} \tag{I.54}$$

For each $s\in[0,T]$, $m_\sigma(\cdot,s)$ is right-continuous. Therefore, $m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},s)$ converges to $m_\sigma(u_1,s)$ when $q\to+\infty$. Furthermore, there is $\beta>1$ such that almost surely,

$$\sup_{u\in\left[0,\frac{1}{2^{n_0-1}}\right]}\int_0^{1-u}\int_0^T\left|\frac{1}{m_\sigma(u_1+u,s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta\mathrm{d}s\mathrm{d}u_1<+\infty.$$

Indeed,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{u\in\left[0,\frac{1}{2^{n_0-1}}\right]}\int_0^{1-u}\int_0^T\left|\frac{1}{m_\sigma(u_1+u,s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta\mathrm{d}s\mathrm{d}u_1\right]\\
&\qquad\qquad\leqslant C_\beta\mathbb{E}\left[\int_0^1\int_0^T\frac{1}{m_\sigma(u_1,s)^\beta}\mathrm{d}s\mathrm{d}u_1\right]<+\infty,
\end{aligned}$$

by Lemma I.24. Therefore, since $\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n}\leqslant h_1<\frac{1}{2^{n_0-1}}$ for every $q\geqslant 1$,

$$\begin{aligned}
&\int_0^{1-h_1}\int_0^T\left|\frac{1}{m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta\mathrm{d}s\mathrm{d}u_1\\
&\quad\leqslant\int_0^{1-\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n}}\int_0^T\left|\frac{1}{m_\sigma(u_1+\sum_{n\geqslant n_0+q}\frac{\varepsilon_n}{2^n},s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta\mathrm{d}s\mathrm{d}u_1\\
&\quad\leqslant\sup_{u\in\left[0,\frac{1}{2^{n_0-1}}\right]}\int_0^{1-u}\int_0^T\left|\frac{1}{m_\sigma(u_1+u,s)}-\frac{1}{m_\sigma(u_1,s)}\right|^\beta\mathrm{d}s\mathrm{d}u_1,
\end{aligned}$$

which is almost surely finite. Thus the first term of the right hand side of (I.54) tends almost surely to 0 for every $h_1<\frac{1}{2^{n_0}}$. A similar argument shows that the second term of the right hand side of (I.54) also converges to 0. Hence we have justified convergence (I.53).

When $q\to\infty$ in inequality (I.52), we obtain:

$$\int_0^{1-h_1}\int_0^1\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\leqslant\sum_{k=1}^{+\infty}\frac{1}{2^{(n_0+k)\frac{\alpha}{2}}}\leqslant\frac{C_\alpha}{2^{\frac{n_0\alpha}{2}}}.$$

Then, we proceed similarly for the second term of the right hand side of (I.51) and we finally obtain, for each $h=(h_1,h_2)$ such that $0<h_1<\frac{1}{2^{n_0}}$ and $0<h_2<\frac{1}{2^{n_0}}$,

$$\int_0^{1-h_1}\int_0^{1-h_2}\sup_{t\leqslant T}|C_\sigma(u_1+h_1,u_2+h_2,t)-C_\sigma(u_1,u_2,t)|\mathrm{d}u_1\mathrm{d}u_2\leqslant\frac{C}{2^{\frac{n_0\alpha}{2}}}.$$

Choosing $H=\frac{1}{2^{n_0}}$ such that $CH^{\alpha/2}\leqslant\frac{1}{k}$, we get (I.49) for each σ in $\mathbb{Q}_+$.

Conclusion of the proof. By Simon's tightness criterion on $L_1([0,1]\times[0,1],\mathcal{C}[0,T])$, the collection of laws of $(C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is relatively compact in $\mathcal{P}(L_1([0,1]\times[0,1],\mathcal{C}[0,T]))$. Thus the collection of laws of $(y_\sigma, C_\sigma)_{\sigma\in\mathbb{Q}_+}$ is also relatively compact in $\mathcal{P}(L_2([0,1],\mathcal{C}[0,T])\times L_1([0,1]\times[0,1],\mathcal{C}[0,T]))$. Thus there is a subsequence, $(y_{\sigma_n}, C_{\sigma_n})_{n\geqslant 1}$, which converges in distribution in $L_2([0,1],\mathcal{C}[0,T])\times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$. We denote by (y,C) the limit. We want to prove that for almost every $u_1,u_2\in[0,1]$, $C(u_1,u_2,\cdot)$ is the quadratic variation of $y(u_1,\cdot)-y(u_2,\cdot)$ relatively to the filtration generated by Y and C.

Let $l\geqslant 1$, $0\leqslant s_1\leqslant s_2\leqslant\ldots\leqslant s_l\leqslant s\leqslant t$ and $f_l:(L_2(0,1))^l\times L_1([0,1]\times[0,1])^l\to\mathbb{R}$ be a bounded and continuous function. For every non-negative $\psi_1,\psi_2\in L_\infty(0,1)$, we have for every $n\geqslant 1$:

$$\mathbb{E}\Bigg[\int_0^1\int_0^1\psi_1(u_1)\psi_2(u_2)\Big((y_{\sigma_n}(u_1,t)-y_{\sigma_n}(u_2,t)-g(u_1)+g(u_2))^2 \\ -(y_{\sigma_n}(u_1,s)-y_{\sigma_n}(u_2,s)-g(u_1)+g(u_2))^2-C_{\sigma_n}(u_1,u_2,t)+C_{\sigma_n}(u_1,u_2,s)\Big)\mathrm{d}u_1\mathrm{d}u_2 \\ f_l(Y_{\sigma_n}(s_1),\ldots,Y_{\sigma_n}(s_l),C_{\sigma_n}(s_1),\ldots,C_{\sigma_n}(s_l))\Bigg]=0,$$

since the process $(C_{\sigma_n}(t))_{t\in[0,T]}:=(C_{\sigma_n}(\cdot,\cdot,t))_{t\in[0,T]}$ is $(\mathcal{F}_t^{\sigma_n})_{t\in[0,T]}$-adapted. By the convergence in distribution, we obtain when n goes to ∞:

$$\mathbb{E}\Bigg[\int_0^1\int_0^1\psi_1(u_1)\psi_2(u_2)\Big((y(u_1,t)-y(u_2,t)-g(u_1)+g(u_2))^2 \\ -(y(u_1,s)-y(u_2,s)-g(u_1)+g(u_2))^2-C(u_1,u_2,t)+C(u_1,u_2,s)\Big)\mathrm{d}u_1\mathrm{d}u_2 \\ f_l(Y(s_1),\ldots,Y(s_l),C(s_1),\ldots,C(s_l))\Bigg]=0.$$

By Fubini's Theorem, we obtain that for almost every $u_1,u_2\in(0,1)$, for all rational numbers $(s_1,\ldots,s_l,s,t)$ such that $0\leqslant s_1\leqslant s_2\leqslant\ldots\leqslant s_l\leqslant s\leqslant t$:

$$\mathbb{E}\Big[\Big((y(u_1,t)-y(u_2,t)-g(u_1)+g(u_2))^2-(y(u_1,s)-y(u_2,s)-g(u_1)+g(u_2))^2 \\ -C(u_1,u_2,t)+C(u_1,u_2,s)\Big)f_l(Y(s_1),\ldots,Y(s_l),C(s_1),\ldots,C(s_l))\Big]=0.$$

By continuity in time, the latter equality remains true for every $0\leqslant s_1\leqslant s_2\leqslant\ldots\leqslant s_l\leqslant s\leqslant t$. Furthermore, for almost every u_1,u_2, $(C_{\sigma_n}(u_1,u_2,t))_{t\in[0,T]}$ is a non-decreasing bounded variation process. This remains true for the limit $(C(u_1,u_2,t))_{t\in[0,T]}$. Therefore, we deduce that

$$C(u_1,u_2,t)=\langle y(u_1)-y(u_2),y(u_1)-y(u_2)\rangle_t,$$

for almost every $u_1,u_2\in(0,1)$, with respect to the filtration generated by (Y,C). □

We conclude this paragraph by using Fatou's Lemma to extend the statement of Lemma I.33 to the limit process:

Proposition I.47. *Let $g\in L_p(0,1)$. For all $\beta\in(0,\frac{3}{2}-\frac{1}{p})$, there is a constant $C>0$ depending only on β and $\|g\|_{L_p}$ such that for all $0\leqslant s<t\leqslant T$, we have the following inequality:*

$$\mathbb{E}\left[\int_s^t\int_0^1\frac{1}{m(u,r)^\beta}\mathrm{d}u\mathrm{d}r\right]\leqslant C\sqrt{t-s}.$$

By Burkholder-Davis-Gundy inequality, we deduce the following estimation:

Corollary I.48. *For each* $\beta \in (0, \frac{3}{2} - \frac{1}{p})$, $\sup_{t \leqslant T} \mathbb{E}\left[\int_0^1 (y(u,t) - g(u))^{2\beta} \mathrm{d}u\right] < +\infty$.

I.5.3 Covariation of $y(u,\cdot)$ and $y(u',\cdot)$

In this paragraph, we want to complete the proof of property $(C5)$ of Theorem I.4. It remains to prove the following Proposition:

Proposition I.49. *Let* y *be the version in* $\mathcal{D}((0,1), \mathcal{C}[0,T])$ *of the limit process given by Proposition I.41. For every* $u, u' \in (0,1)$,

$$\langle y(u,\cdot), y(u',\cdot)\rangle_{t \wedge \tau_{u,u'}} = 0, \tag{I.55}$$

where $\tau_{u,u'} = \inf\{t \geqslant 0 : y(u,t) = y(u',t)\} \wedge T$.

As in the previous paragraph, we will need to prove the convergence of the joint law of y_σ and a quadratic covariation. More precisely, define:

$$K_\sigma(u,u',t) := \int_0^t \frac{m_\sigma(u,u',s)}{m_\sigma(u,s) m_\sigma(u',s)} \mathrm{d}s.$$

We state the following result:

Lemma I.50. *For every sequence* $(\sigma_n)_n$ *of rational numbers tending to* 0, *we can extract a subsequence* $(\widetilde{\sigma}_n)_n$ *such that the sequence* $(y_{\widetilde{\sigma}_n}, K_{\widetilde{\sigma}_n})_{n\to\infty}$ *converges in distribution to* (y,K) *in* $L_2([0,1], \mathcal{C}[0,T]) \times L_1([0,1]\times[0,1], \mathcal{C}[0,T])$, *where*

$$K(u,u',t) := \langle y(u,\cdot), y(u',\cdot)\rangle_t.$$

Proof (Lemma I.50). We follow the same structure as in the proof of Lemma I.46. First, we define $K_{\sigma,\varepsilon} = \langle y_{\sigma,\varepsilon}(u,\cdot), y_{\sigma,\varepsilon}(u',\cdot)\rangle_t = \int_0^t \frac{m_{\sigma,\varepsilon}(u,u',s)}{(\varepsilon + m_{\sigma,\varepsilon}(u,s))(\varepsilon + m_{\sigma,\varepsilon}(u',s))} \mathrm{d}s$. We show that $K_{\sigma,\varepsilon}$ satisfies the three criteria of tightness in $L_1([0,1]\times[0,1], \mathcal{C}[0,T])$. For the first criterion, we want to bound

$$\mathbb{E}\left[\int_0^1 \int_0^1 \sup_{t\leqslant T} |K_{\sigma,\varepsilon}(u,u',t)| \mathrm{d}u \mathrm{d}u'\right]$$

uniformly for $\sigma, \varepsilon \in \mathbb{Q}_+$. This follows from Kunita-Watanabe's inequality:

$$\begin{aligned} |K_{\sigma,\varepsilon}(u,u',t)| = |\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u')\rangle_t| &\leqslant \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_t^{1/2} \langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_t^{1/2} \\ &\leqslant \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_T^{1/2} \langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_T^{1/2} \end{aligned}$$

and from Cauchy-Schwarz inequality:

$$\begin{aligned} \mathbb{E}\left[\int_0^1 \int_0^1 \sup_{t\leqslant T} |K_{\sigma,\varepsilon}(u,u',t)| \mathrm{d}u \mathrm{d}u'\right] &\leqslant \mathbb{E}\left[\int_0^1 \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_T \mathrm{d}u\right] \\ &= \mathbb{E}\left[\int_0^1 (y_{\sigma,\varepsilon}(u,T) - g(u))^2 \mathrm{d}u\right], \end{aligned}$$

which is bounded uniformly for $\sigma, \varepsilon \in \mathbb{Q}_+$ by Corollary I.26.

We refer to the proof of Lemma I.46 for the second and the third criteria of tightness, and for the rest of the proof, which follows in the same way. It remains to explain why $(K(u,u',t))_{t\in[0,T]}$

is a bounded variation process for almost every $u, u' \in (0,1)$. It follows from Kunita-Watanabe's inequality that:

$$\begin{aligned}\sum_{k=0}^{p-1} |K_{\sigma,\varepsilon}(u,u',t_{k+1}) - K_{\sigma,\varepsilon}(u,u',t_k)| &= \sum_{k=0}^{p-1} |\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u')\rangle_{t_{k+1}} - \langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u')\rangle_{t_k}| \\ &\leqslant \sum_{k=0}^{p-1} \left(\int_{t_k}^{t_{k+1}} \mathrm{d}\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_s\right)^{\frac{1}{2}} \left(\int_{t_k}^{t_{k+1}} \mathrm{d}\langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_s\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\int_{t_0}^{t_p} \mathrm{d}\langle y_{\sigma,\varepsilon}(u), y_{\sigma,\varepsilon}(u)\rangle_s + \frac{1}{2}\int_{t_0}^{t_p} \mathrm{d}\langle y_{\sigma,\varepsilon}(u'), y_{\sigma,\varepsilon}(u')\rangle_s \\ &= \frac{1}{2}\int_{t_0}^{t_p} \frac{\mathrm{d}s}{M_{\sigma,\varepsilon}(u,s)} + \frac{1}{2}\int_{t_0}^{t_p} \frac{\mathrm{d}s}{M_{\sigma,\varepsilon}(u',s)},\end{aligned}$$

Therefore, for every $p \geqslant 1$ and $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_p$, $\sum_{k=0}^{p-1} |K(u,u',t_{k+1}) - K(u,u',t_k)| \leqslant \frac{1}{2}\int_{t_0}^{t_p} \frac{\mathrm{d}s}{m(u,s)} + \frac{1}{2}\int_{t_0}^{t_p} \frac{\mathrm{d}s}{m(u',s)}$. By Proposition I.47, we know that almost surely and for almost every $u \in (0,1)$, $\int_0^T \frac{\mathrm{d}s}{m(u,s)}$ is finite. Thus for almost every u and u' in $(0,1)$, $K(u,u',\cdot)$ is a bounded variation process. This completes the proof of the lemma. □

We use the latter lemma to prove Proposition I.49.

Proof (Proposition I.49). By Lemma I.50 and Skorohod's representation Theorem, we may suppose that $(y_\sigma, K_\sigma)_{\sigma\in\mathbb{Q}_+}$ converges almost surely in $L_2([0,1],\mathcal{C}[0,T]) \times L_1([0,1]\times[0,1],\mathcal{C}[0,T])$ to (y,K). As previously, up to extracting a subsequence, we deduce that for almost every $(\omega,u,u') \in \Omega \times [0,1]\times[0,1]$,

$$\sup_{t\leqslant T} |y_\sigma(u,t) - y(u,t)|(\omega) \underset{\sigma\to 0}{\longrightarrow} 0, \tag{I.56}$$

and

$$\sup_{t\leqslant T} |K_\sigma(u,u',t) - K(u,u',t)|(\omega) \underset{\sigma\to 0}{\longrightarrow} 0. \tag{I.57}$$

Therefore, there exists a (non-random) subset $\mathcal{A}$ of $[0,1]$, such that for every $u,u' \in \mathcal{A}$, (I.56) and (I.57) holds almost surely.

Let $u,u' \in \mathcal{A}$. If $g(u) = g(u')$ then $\tau_{u,u'} = 0$ almost surely, thus (I.55) is clear. Up to exchanging u and u', assume that $g(u) < g(u')$. Let $\delta < 2(g(u') - g(u))$. Almost surely, by (I.56), there exists σ_0 such that for all $\sigma \in (0,\sigma_0)\cap\mathbb{Q}_+$,

$$\begin{aligned}\sup_{t\leqslant T} |y_\sigma(u,t) - y(u,t)| &\leqslant \frac{\delta}{4}, \\ \sup_{t\leqslant T} |y_\sigma(u',t) - y(u',t)| &\leqslant \frac{\delta}{4}.\end{aligned}$$

Define $\tau^\delta_{u,u'} := \inf\{t \geqslant 0 : |y(u,t) - y(u',t)| \leqslant \delta\} \wedge T$. Therefore, for all $t < \tau^\delta_{u,u'}$ and for all $\sigma < \sigma_0$, $|y_\sigma(u,t) - y_\sigma(u',t)| \geqslant \frac{\delta}{2}$. Let $\sigma < \min(\sigma_0, \frac{\delta}{2})$. For all $t < \tau^\delta_{u,u'}$, we have $|y_\sigma(u,t) - y_\sigma(u',t)| \geqslant \sigma$ and thus $m_\sigma(u,u',t) = 0$, hence $K_\sigma(u,u',t) = \int_0^t \frac{m_\sigma(u,u',s)}{m_\sigma(u,s)m_\sigma(u',s)}\mathrm{d}s = 0$ for $t \leqslant \tau^\delta_{u,u'}$. By (I.57), we obtain

$$\sup_{t\leqslant \tau^\delta_{u,u'}} |K(u,u',t)| = 0.$$

Thus for every $\delta > 0$, for every $u,u' \in \mathcal{A}$ and $t \leqslant \tau^\delta_{u,u'}$, $\langle y(u), y(u')\rangle_t = 0$. Since $\tau^\delta_{u,u'} \to \tau_{u,u'}$ when $\delta \to 0$, we have for each $u,u' \in \mathcal{A}$:

$$\langle y(u), y(u')\rangle_{t\wedge\tau_{u,u'}} = 0. \tag{I.58}$$

It remains to show that (I.58) holds with every $(u,u') \in (0,1)^2$. Let $(u,u') \in (0,1)^2$. As previously, we may assume that $g(u) < g(u')$. By continuity of the processes $(y(u,t))_{t\in[0,T]}$ and $(y(u',t))_{t\in[0,T]}$, the first time of coalescence $\tau_{u,u'}$ is almost surely positive. Fix $l \geqslant 1$, $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t$ and a bounded and continuous function $f_l : (L_2(0,1))^l \to \mathbb{R}$. Suppose that $s > 0$. We want to prove that:

$$\mathbb{E}\left[(y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'}) - y(u,s\wedge\tau_{u,u'})y(u',s\wedge\tau_{u,u'}))f_l(Y(s_1),\ldots,Y(s_l))\right] = 0. \quad \text{(I.59)}$$

Let $\varepsilon > 0$. For each $v \in (u,u+\varepsilon)\cap\mathcal{A}$ and $v' \in (u',u'+\varepsilon)\cap\mathcal{A}$ (since $\mathcal{A}$ is of plain measure in $(0,1)$, both sets are non-empty), since we have equality (I.58),

$$0 = \mathbb{E}\big[(y(v,t\wedge\tau_{v,v'})y(v',t\wedge\tau_{v,v'}) - y(v,s\wedge\tau_{v,v'})y(v',s\wedge\tau_{v,v'}))f_l(Y(s_1),\ldots,Y(s_l))\big]. \quad \text{(I.60)}$$

Let $t_0 \in (0,s)$. We define

$$\eta := \sup\{h \geqslant 0 : y(u+h,t_0) = y(u,t_0) \text{ and } y(u'+h,t_0) = y(u',t_0)\}.$$

By the coalescence property given by Proposition I.41, under the event $\{\tau_{u,u'} > t_0\}$, we know that for every $r \geqslant t_0$, for each $v \in (u,u+\eta)$ and $v' \in (u',u'+\eta)$, $y(v,r) = y(u,r)$ and $y(v',r) = y(u',r)$, whence $\tau_{v,v'} = \tau_{u,u'}$. Thus, by equality (I.60), we deduce that for each $v \in (u,u+\varepsilon)\cap\mathcal{A}$ and $v' \in (u',u'+\varepsilon)\cap\mathcal{A}$,

$$\begin{aligned}0 = \mathbb{E}\big[&\mathbb{1}_{\{\eta>\varepsilon\}}\mathbb{1}_{\{\tau_{u,u'}>t_0\}}(y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'})\\ &\qquad - y(u,s\wedge\tau_{u,u'})y(u',s\wedge\tau_{u,u'}))f_l(Y(s_1),\ldots,Y(s_l))\big]\\ &+ \mathbb{E}\big[\mathbb{1}_{\{\eta\leqslant\varepsilon\}\cup\{\tau_{u,u'}\leqslant t_0\}}(y(v,t\wedge\tau_{v,v'})y(v',t\wedge\tau_{v,v'}) - y(v,s\wedge\tau_{v,v'})y(v',s\wedge\tau_{v,v'}))\\ &\qquad\qquad f_l(Y(s_1),\ldots,Y(s_l))\big]. \quad \text{(I.61)}\end{aligned}$$

Let $h > 0$ be such that $(u,u+\varepsilon)$ and $(u',u'+\varepsilon)$ are contained in $(h,1-h)$. Thus for every $v \in (u,u+\varepsilon)\cap\mathcal{A}$, for every $r \in [0,T]$, by inequality (I.42) and by Doob's inequality, we deduce that:

$$\mathbb{E}\left[\sup_{r\leqslant T} y(v,r)^{2\beta}\right] \leqslant \frac{2}{h}\mathbb{E}\left[\int_0^1 \sup_{r\leqslant T} y(x,r)^{2\beta}\mathrm{d}x\right] \leqslant \frac{C_\beta}{h}\mathbb{E}\left[\int_0^1 y(x,T)^{2\beta}\mathrm{d}x\right] \leqslant \frac{\widetilde{C}_\beta}{h},$$

for a β arbitrarily chosen in $(1,\frac{3}{2}-\frac{1}{p})$ (by Corollary I.48). Thus, there exists $\beta > 1$ such that $\mathbb{E}\left[(y(v,t\wedge\tau_{v,v'})y(v',t\wedge\tau_{v,v'})^\beta\right]$ is uniformly bounded for $v \in (u,u+\varepsilon)$ and $v' \in (u',u'+\varepsilon)$. Let $\alpha = 1-\frac{1}{\beta}$. Therefore, we deduce from (I.61) that there is a constant C depending only on u, u' and α such that:

$$\begin{aligned}\mathbb{E}\big[&\mathbb{1}_{\{\eta>\varepsilon\}}\mathbb{1}_{\{\tau_{u,u'}>t_0\}}(y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'}) - y(u,s\wedge\tau_{u,u'})y(u',s\wedge\tau_{u,u'}))\\ &\qquad f_l(Y(s_1),\ldots,Y(s_l))\big] \leqslant C\left(\mathbb{P}\left[\eta\leqslant\varepsilon\right]^\alpha + \mathbb{P}\left[\tau_{u,u'}\leqslant t_0\right]^\alpha\right). \quad \text{(I.62)}\end{aligned}$$

We divide the left hand side of inequality (I.62) into two parts by writing

$$\mathbb{1}_{\{\eta>\varepsilon\}}\mathbb{1}_{\{\tau_{u,u'}>t_0\}} = 1 - \mathbb{1}_{\{\eta\leqslant\varepsilon\}\cup\{\tau_{u,u'}\leqslant t_0\}}$$

and we estimate the second term in the same way as above. We deduce that there is a constant C' such that:

$$\begin{aligned}\mathbb{E}\left[(y(u,t\wedge\tau_{u,u'})y(u',t\wedge\tau_{u,u'}) - y(u,s\wedge\tau_{u,u'})y(u',s\wedge\tau_{u,u'}))f_l(Y(s_1),\ldots,Y(s_l))\right]&\\ \leqslant C'\left(\mathbb{P}\left[\eta\leqslant\varepsilon\right]^\alpha + \mathbb{P}\left[\tau_{u,u'}\leqslant t_0\right]^\alpha\right).&\end{aligned}$$

Let $\delta > 0$. Since $\tau_{u,u'} > 0$ almost surely, we choose $t_0 \in (0, s)$ such that $\mathbb{P}\left[\tau_{u,u'} \leqslant t_0\right]^\alpha \leqslant \delta$. Since $t_0 > 0$, we know by Proposition I.37 that $y(\cdot, t_0)$ is almost surely a step function, so $\eta > 0$ almost surely. Therefore, we can choose $\varepsilon > 0$ so that $\mathbb{P}\left[\eta \leqslant \varepsilon\right]^\alpha \leqslant \delta$. This completes the proof of equality (I.59).

Recall that we suppose that $t \geqslant s > 0$. By continuity of time of $y(u, \cdot)$ and $y(u', \cdot)$, equality (I.59) also holds with $s = 0$. Therefore, $y(u, t \wedge \tau_{u,u'})y(u', t \wedge \tau_{u,u'})$ is a $(\mathcal{F}_t)_{t\in[0,T]}$-martingale and $\langle y(u), y(u')\rangle_{t\wedge\tau_{u,u'}} = 0$. This completes the proof of Proposition I.49. □

I.6 Appendix: Itô's formula for the Wasserstein diffusion

Let $g \in \mathcal{L}^{\uparrow}_{2+}[0, 1]$. We assume, to simplify the notations, that $g(1)$ is finite, but the proof can be easily adapted to functions g with $g(u) \underset{u\to 1}{\longrightarrow} +\infty$. Let y be a process in $\mathcal{D}([0, 1], \mathcal{C}[0, T])$ satisfying $(i) - (iv)$ (see Introduction).

Recall that the process $y(\cdot, t)_{t\in[0,T]}$ can be considered as the quantile function of $(\mu_t)_{t\in[0,T]}$, by setting $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y(\cdot, t)^{-1}$. The latter process has every feature of a Wasserstein diffusion. We describe in this paragraph the dynamics of the process $(\mu_t)_{t\in[0,T]}$, after having introduced a differential calculus on $\mathcal{P}_2(\mathbb{R})$ due to Lions (see [Lio, Car13]). We prove that, for a smooth function $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, the process $(U(\mu_t))_{t\in[0,T]}$ is a semi-martingale with quadratic variation proportional to the square of the gradient of U (see Theorem I.53). This result is a generalization of the formula given by Konarovskyi and von Renesse in [KvR18]. We compare it to a similar result obtained by von Renesse and Sturm [vRS09] for the Wasserstein diffusion on $[0, 1]$ (see Remark I.54).

In order to describe the dynamics of $(\mu_t)_{t\in[0,T]}$, we begin by a discretization in space and by writing the classical Itô formula for that discretized process. Let introduce $\widetilde{\mu}^n_t := \frac{1}{n}\sum_{k\in[n]} \delta_{y(\frac{k}{n},t)}$, where $[n]$ denotes the set $\{1, \dots, n\}$. Fix $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ a continuous function, with respect to the Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$. Let define $U^n(x_1, \dots, x_n) := U(\frac{1}{n}\sum_{j\in[n]} \delta_{x_j})$. Remark that $U(\widetilde{\mu}^n_t) = U^n\left(y(\frac{1}{n}, t), y(\frac{2}{n}, t), \dots, y(1, t)\right)$. Assuming that U^n belongs to $C^2(\mathbb{R}^n)$, and using that $y(\frac{k}{n}, \cdot)$ is a square integrable continuous martingale on $[0, T]$, we have (recall that $g(1)$ is finite):

$$\begin{aligned} U(\widetilde{\mu}^n_t) =& U^n(g(\tfrac{1}{n}), \dots, g(1)) + \sum_{k\in[n]} \int_0^t \partial_k U^n(y(\tfrac{1}{n}, s), \dots, y(1, s))\, \mathrm{d}y(\tfrac{k}{n}, s) \\ &+ \frac{1}{2}\sum_{k,l\in[n]} \int_0^t \partial^2_{k,l} U^n(y(\tfrac{1}{n}, s), \dots, y(1, s))\, \mathrm{d}\langle y(\tfrac{k}{n}, \cdot), y(\tfrac{l}{n}, \cdot)\rangle_s. \end{aligned} \tag{I.63}$$

In order to write the derivatives of U^n in terms of derivatives of U, we should introduce a differential calculus on $\mathcal{P}_2(\mathbb{R})$, well-adapted to the differentiation of empirical measures. P.L. Lions introduces in his lectures at Collège de France (see Section 6.1 of Cardaliaguet's notes [Car13]) a differential calculus on $\mathcal{P}_2(\mathbb{R})$ by using the Hilbertian structure of $L_2(\Omega)$. We set $\widetilde{U}(X) := U(\mathrm{Law}(X))$ for all $X \in L_2(\Omega)$.

A function $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is said to be *L-differentiable* (or differentiable in the sense of Lions) at a point $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ if there is a random variable X_0 with law μ_0 such that $\widetilde{U}$ is Fréchet-differentiable at X_0. The definition does not depend on the choice of the representative X_0 of the law μ_0, and if X_0 and X_1 have the same law, then the laws of $D\widetilde{U}(X_0)$ and $D\widetilde{U}(X_1)$ are equal (see e.g. [Car13]). Furthermore, if $D\widetilde{U} : L_2(\Omega) \to L_2(\Omega)$ is a continuous function, then for all $\mu_0 \in \mathcal{P}_2(\mathbb{R})$, there exists a measurable function $\mathbb{R} \to \mathbb{R}$, denoted by $\partial_\mu U(\mu_0)$, such that for each $X \in L_2(\Omega)$ with law μ_0, we have $D\widetilde{U}(X) = \partial_\mu U(\mu_0)(X)$ almost surely (see [Car13]).

In [CD18a], Carmona and Delarue prove that the L-differentiability of $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$

implies the differentiability of U^n on $\mathbb{R}^n$, and that we have for each $k \in [n]$:

$$\partial_k U^n(x_1, \dots, x_n) = \frac{1}{n} \partial_\mu U(\tfrac{1}{n} \textstyle\sum_{j\in[n]} \delta_{x_j})(x_k).$$

Furthermore, assume that U is L-differentiable and that $(\mu, v) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}$ is continuous. Moreover, we assume that for every $\mu \in \mathcal{P}_2(\mathbb{R})$, the map $v \in \mathbb{R} \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}$ is differentiable on $\mathbb{R}$ in the classical sense and that its derivative is given by a jointly continuous function $(\mu, v) \mapsto \partial_v \partial_\mu U(\mu)(v)$. We also assume that for every $v \in \mathbb{R}$, the map $\mu \mapsto \partial_\mu U(\mu)(v)$ is L-differentiable and its derivative is denoted by $(\mu, v, v') \mapsto \partial_\mu^2 U(\mu)(v, v')$. Then, U^n is $\mathcal{C}^2$ on $\mathbb{R}^n$ and for all $k, l \in [n]$:

$$\partial^2_{k,l} U^n(x_1, \dots, x_n) = \frac{1}{n} \partial_v \partial_\mu U(\tfrac{1}{n} \textstyle\sum_{j\in[n]} \delta_{x_j})(x_k) \mathbb{1}_{\{k=l\}} + \frac{1}{n^2} \partial_\mu^2 U(\tfrac{1}{n} \textstyle\sum_{j\in[n]} \delta_{x_j})(x_k, x_l).$$

Therefore, we obtain from equation (I.63):

$$\begin{aligned} U(\widetilde{\mu}^n_t) = U(\widetilde{\mu}^n_0) + \frac{1}{n} \sum_{k\in[n]} \int_0^t \partial_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n}, s)) \mathrm{d}y(\tfrac{k}{n}, s) + \frac{1}{2n} \sum_{k\in[n]} \int_0^t \partial_v \partial_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n}, s)) \tfrac{\mathrm{d}s}{m(\frac{k}{n}, s)} \\ + \frac{1}{2n^2} \sum_{k,l\in[n]} \int_0^t \partial^2_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n}, s), y(\tfrac{l}{n}, s)) \tfrac{\mathbb{1}_{\{\tau_{\frac{k}{n}, \frac{l}{n}} \leqslant s\}}}{m(\frac{k}{n}, s)} \mathrm{d}s. \end{aligned} \tag{I.64}$$

By property of coalescence, if $\tau_{\frac{k}{n}, \frac{l}{n}} \leqslant s$, we have $y(\frac{k}{n}, s) = y(\frac{l}{n}, s)$, so that the last term in the latter equation is equal to:

$$\frac{1}{2n} \sum_{k\in[n]} \int_0^t \partial^2_\mu U(\widetilde{\mu}^n_s)(y(\tfrac{k}{n}, s), y(\tfrac{k}{n}, s)) \tfrac{\frac{1}{n}\sum_{l\in[n]} \mathbb{1}_{\{\tau_{\frac{k}{n}, \frac{l}{n}} \leqslant s\}}}{m(\frac{k}{n}, s)} \mathrm{d}s.$$

Observe that the difference between $\frac{1}{n} \sum_{l\in[n]} \mathbb{1}_{\{\tau_{\frac{k}{n}, \frac{l}{n}} \leqslant s\}}$ and $m(\frac{k}{n}, s) = \int_0^1 \mathbb{1}_{\{\tau_{\frac{k}{n}, u} \leqslant s\}} \mathrm{d}u$ is bounded by $\frac{2}{n}$, since the set $\{u : \tau_{\frac{k}{n}, u} \leqslant s\}$ is an interval.

We want to let n tend to $+\infty$ in order to obtain an Itô formula for the limit process. We start by proving the convergence of a subsequence of $((\widetilde{\mu}^n_t)_{t\in[0,T]})_{n\geqslant 1}$ to $(\mu_t)_{t\in[0,T]}$ with respect to the L_2-Wasserstein distance.

Proposition I.51. *There exists a subsequence $((\widetilde{\mu}^{\varphi(n)}_t)_{t\in[0,T]})_{n\geqslant 1}$ of $((\widetilde{\mu}^n_t)_{t\in[0,T]})_{n\geqslant 1}$ such that, for almost every $t \in [0, T]$, the sequence $(\widetilde{\mu}^{\varphi(n)}_t)_{n\geqslant 1}$ converges almost surely to μ_t with respect to the Wasserstein distance W_2.*

Remark I.52. We point out that the extraction function φ does not depend on $t \in [0, T]$.

Proof. To obtain the statement of the proposition, it is sufficient to prove that:

$$\mathbb{E}\left[\int_0^T W_2(\widetilde{\mu}^n_t, \mu_t)^2 \mathrm{d}t\right] \to 0.$$

Let V be a uniform random variable on $[0, 1]$, defined on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. Therefore, μ_t is the law of $y(V, t)$ and $\widetilde{\mu}^n_t$ the law of $\sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n} < V \leqslant \frac{k}{n}\}} y(\frac{k}{n}, t)$. Hence we have:

$$\begin{aligned} W_2(\widetilde{\mu}^n_t, \mu_t)^2 &\leqslant \widetilde{\mathbb{E}}\left[\left(\sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n} < V \leqslant \frac{k}{n}\}} y(\tfrac{k}{n}, t) - y(V, t)\right)^2\right] \\ &= \int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n} < u \leqslant \frac{k}{n}\}} |y(\tfrac{k}{n}, t) - y(u, t)|^2 \mathrm{d}u. \end{aligned}$$

Therefore, it is sufficient to show that:

$$\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^2\mathrm{d}u\mathrm{d}t\right] \underset{n\to+\infty}{\longrightarrow} 0. \tag{I.65}$$

Fixing $u\in(0,1), t\in(0,T)$, $\sum_{k\in[n]}\mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\frac{k}{n},t)-y(u,t)|^2$ converges almost surely to 0 by the right-continuity of $y(\cdot,t)$ at point u. To prove (I.65), we have to show a uniform integrability property, *i.e.* that for a certain $\beta>1$,

$$\sup_{n\geqslant 1}\mathbb{E}\left[\left(\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^2\mathrm{d}u\mathrm{d}t\right)^\beta\right]<+\infty.$$

We compute:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^2\mathrm{d}u\mathrm{d}t\right)^\beta\right]^{1/(2\beta)}\\
&\qquad\leqslant T^{\frac{\beta-1}{2\beta}}\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|y(\tfrac{k}{n},t)-y(u,t)|^{2\beta}\mathrm{d}u\mathrm{d}t\right]^{1/(2\beta)}\\
&\qquad\qquad\leqslant T^{\frac{\beta-1}{2\beta}}\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|M_0|^{2\beta}\mathrm{d}u\mathrm{d}t\right]^{1/(2\beta)}\\
&\qquad\qquad\quad+T^{\frac{\beta-1}{2\beta}}\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|M_t-M_0|^{2\beta}\mathrm{d}u\mathrm{d}t\right]^{1/(2\beta)},
\end{aligned}$$

where $M_t=y(\frac{k}{n},t)-y(u,t)$. Recall that by property (i) of the process y, $M_0=g(\frac{k}{n})-g(u)$. We deduce that:

$$\mathbb{E}\left[\int_0^T\int_0^1 \sum_{k\in[n]} \mathbb{1}_{\{\frac{k-1}{n}<u\leqslant\frac{k}{n}\}}|g(\tfrac{k}{n})-g(u)|^{2\beta}\mathrm{d}u\mathrm{d}t\right]\leqslant TC_\beta\mathbb{E}\left[\int_0^1 g(u)^{2\beta}\mathrm{d}u\right].$$

Since g belongs to $\mathcal{L}^{\uparrow}_{2+}[0,1]$, there exists $p>2$ such that $g\in L_p(0,1)$. Therefore, we can choose $\beta>1$ such that $2\beta\leqslant p$. By Burkholder-Davis-Gundy inequality and the martingale property of M, we have:

$$\mathbb{E}\left[(M_t-M_0)^{2\beta}\right]\leqslant C_\beta\mathbb{E}\left[\langle M,M\rangle_t^\beta\right].$$

By property (iv),

$$\begin{aligned}
\langle M,M\rangle_t&=\int_0^t\frac{\mathrm{d}s}{m(\frac{k}{n},s)}+\int_0^t\frac{\mathrm{d}s}{m(u,s)}-2\int_0^t\frac{\mathbb{1}_{\{\tau_{\frac{k}{n},u}\leqslant s\}}}{m(\frac{k}{n},s)^{1/2}m(u,s)^{1/2}}\mathrm{d}s\\
&\leqslant\int_0^t\frac{\mathrm{d}s}{m(\frac{k}{n},s)}+\int_0^t\frac{\mathrm{d}s}{m(u,s)},
\end{aligned}$$

so that there is a constant C_β satisfying:

$$\mathbb{E}\left[\langle M,M\rangle_t^\beta\right]\leqslant C_\beta t^{\beta-1}\mathbb{E}\left[\int_0^t\frac{\mathrm{d}s}{m(\frac{k}{n},s)^\beta}+\int_0^t\frac{\mathrm{d}s}{m(u,s)^\beta}\right].$$

To conclude, we use the following statement: provided $\beta < \frac{3}{2} - \frac{1}{p}$, there is a constant C_β such that for each t and u:

$$\mathbb{E}\left[\int_0^1 \int_0^t \frac{\mathrm{d}s}{m(u,s)^\beta}\mathrm{d}u\right] \leqslant C_\beta \sqrt{t}. \tag{I.66}$$

This statement is Proposition I.47 for the limit process that we constructed in this chapter, or in [Kon17a, Prop. 4.3] for the process constructed by Konarovskyi. This completes the proof. □

By similar arguments of convergence, equation (I.64) leads to the following Itô formula for $(\mu_t)_{t\in[0,T]}$, by letting n tend to ∞. The estimation (I.66) is the key of the proof of those convergences.

Theorem I.53. *Let $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be smooth enough so that U and its derivatives $\partial_\mu U$, $\partial_v \partial_\mu U$ and $\partial_\mu^2 U$ exist, are uniformly continuous and bounded. Almost surely, for each $t \in [0,T]$, we have:*

$$\begin{aligned} U(\mu_t) =& U(\mu_0) + \int_0^1 \int_0^t \partial_\mu U(\mu_s)(y(u,s))\mathrm{d}y(u,s)\mathrm{d}u + \frac{1}{2}\int_0^1 \int_0^t \partial_v \partial_\mu U(\mu_s)(y(u,s))\frac{\mathrm{d}s}{m(u,s)}\mathrm{d}u \\ &+ \frac{1}{2}\int_0^1 \int_0^t \partial_\mu^2 U(\mu_s)(y(u,s), y(u,s))\mathrm{d}s\mathrm{d}u, \end{aligned}$$

where $\displaystyle\int_0^1 \int_0^t \partial_\mu U(\mu_s)(y(u,s))\mathrm{d}y(u,s)\mathrm{d}u$ is a square integrable continuous martingale with a quadratic variation process equal to $t \mapsto \displaystyle\int_0^1 \int_0^t \left(\partial_\mu U(\mu_s)\right)^2 (y(u,s))\mathrm{d}s\mathrm{d}u$.

Remark I.54. Choose in particular $U : \mu \mapsto V\left(\int_{\mathbb{R}} \alpha_1 \mathrm{d}\mu, \ldots, \int_{\mathbb{R}} \alpha_m \mathrm{d}\mu\right) =: V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu\right)$, where $V \in \mathcal{C}^2(\mathbb{R}^m)$ and $\alpha_1, \ldots, \alpha_m$ are bounded $\mathcal{C}^2(\mathbb{R})$-functions, with bounded first and second-order derivatives. In this case, $\partial_\mu U(\mu)(v) = \sum_{i=1}^m \partial_i V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu\right)\alpha_i'(v)$ for all $\mu \in \mathcal{P}_2(\mathbb{R})$ and $v \in \mathbb{R}$. Computing the second-order derivatives, we show that

$$U(\mu_t) - U(\mu_0) - \frac{1}{2}\int_0^t \mathcal{L}_1 U(\mu_s)\mathrm{d}s - \frac{1}{2}\int_0^t \mathcal{L}_2 U(\mu_s)\mathrm{d}s$$

is a martingale with quadratic variation process

$$t \mapsto \int_0^t \int_0^1 \left(\sum_{i=1}^m \partial_i V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu_s\right)\alpha_i'(y(u,s))\right)^2 \mathrm{d}u\mathrm{d}s$$

and an operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ of the form $\mathcal{L}_1 U(\mu_s) := \sum_{i=1}^m \partial_i V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu_s\right)\int_0^1 \frac{\alpha_i''(y(u,s))}{m(u,s)}\mathrm{d}u$ and $\mathcal{L}_2 U(\mu_s) := \sum_{i,j=1}^m \partial_{i,j}^2 V\left(\int \overrightarrow{\alpha}\mathrm{d}\mu_s\right)\int_0^1 \alpha_i'(y(u,s))\alpha_j'(y(u,s))\mathrm{d}u$.

Remark that we have some restrictions on the domain of the generator $\mathcal{L}_1$. We know that for measures with finite support, $\int_0^1 \frac{\mathrm{d}u}{m(u,s)}$ is finite and is equal to the cardinality of the support (see the paragraph preceding Corollary I.39). The fact that the generator of the martingale problem is not defined on the whole Wasserstein space is related to the fact that the process $(\mu_t)_{t\in[0,T]}$ takes values, for every positive time t, on the space of measures with finite support.

We compare this result to Theorem 7.17 in [vRS09]. The generator of the martingale in the case of von Renesse and Sturm's Wasserstein diffusion is $\mathbb{L} = \mathbb{L}_1 + \mathbb{L}_2 + \beta\mathbb{L}_3$, with $\mathbb{L}_1 = \mathcal{L}_2$ and $\mathbb{L}_3$ similar to $\mathcal{L}_1$ up to the lack of the mass function, whereas $\mathbb{L}_2$, which is the part of the generator considering the gaps of the measure μ, does not appear in our model.

Chapter II

Restoration of uniqueness for a Fokker-Planck equation with irregular drift coefficient

Abstract. Restoration of uniqueness for McKean-Vlasov equations with non-smooth coefficients has been intensively studied since the fundamental works of Kac, McKean, Funaki and Sznitman [Kac56, McK66, Fun84, Szn91] among others. Recent studies, motivated by various applications from fluid mechanics to control or game theory, have led to more intricate problems driven by singular coefficients or requiring finer regularity estimates. Among others, this prompted several authors to push their own research towards new results of well-posedness for McKean-Vlasov SDEs under lower regularity conditions. As a typical instance, much effort has been spent on the case where the drift and the diffusion coefficients are bounded and measurable in the space variable and Lipschitz-continuous in the measure variable with respect to the distance in total variation [Jou97, MV, Lac18, CdRF].
In this paper, we study the regularization properties of a diffusion on the L_2-Wasserstein space inspired by [Kon17b, Mar18] and we show that this infinite-dimensional noise restores uniqueness for Fokker-Planck equations with a non-regular drift coefficient b depending on a space and on a measure variable. In particular, we prove well-posedness of SDEs driven by a Wasserstein-type diffusive noise and by a drift coefficient b that may be Hölder-continuous in the measure variable with respect to the distance in total-variation or even just bounded and measurable in the measure variable, provided that a trade-off is respected between the regularity in the finite-dimensional component and the regularity in the measure-argument. In this regard, we show that the higher the regularity of b with respect to its space variable is, the lower regularity we have to assume on b with respect to its measure variable in order to restore uniqueness.

II.1 Introduction

Let $T \in (0, +\infty)$ be a fixed final time.

In this chapter, we will prove uniqueness in law of a solution $(y_t(u))_{t\in[0,T],u\in[0,1]}$ to the

following stochastic differential equation:

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t + \dfrac{1}{\left(\int_0^1 \varphi(y_t(u) - y_t(v))\mathrm{d}v\right)^{1/2}} \displaystyle\int_{\mathbb{R}} f(k)\Re(e^{-iky_t(u)}\mathrm{d}w(k,t)); \\ \quad \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}. \end{cases} \tag{II.1}$$

As in Chapter I, equation (II.1) describes a particle system in interaction; for every $u \in [0,1]$, $(y_t(u))_{t\in[0,T]}$ denotes the trajectory of the particle indexed by u and for every $t \in [0,T]$, μ_t is the distribution of the cloud of particles. There is a mean-field interaction in SDE (II.1), both through a drift term b which takes as an argument the probability measure μ_t, and through the diffusion term, since the denominator $(\int_0^1 \varphi(y_t(u)-y_t(v))\mathrm{d}v)^{1/2}$ also depends on the distribution of the particles on the real line.

Let us start by briefly describing the different terms appearing in equation (II.1).

- as in Chapter I, the unknown $(y_t)_{t\in[0,T]}$ is a time-continuous stochastic process such that for each $t \in [0,T]$, y_t is a random variable with values in the space $L_2^{\uparrow}[0,1]$ of non-decreasing functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 f^2 < +\infty$. Recall that for each $t \in [0,T]$, $u \in [0,1] \mapsto y_t(u)$ can be seen as the quantile function associated to the measure $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}$ belonging to $\mathcal{P}_2(\mathbb{R})$. If μ_t describes the distribution at time t of a cloud of particles indexed by the interval $[0,1]$, $y_t(u)$ is the position at time t of the particle indexed by u.

- the function $b : \mathbb{R}\times L_2[0,1] \to \mathbb{R}$ will be called the *drift function*, or velocity field. In (II.1), the drift is distribution dependent: it depends both on the position $y_t(u)$ of particle u but also on the distribution μ_t of the whole cloud of particle. The type of SDE where the drift and/or the diffusion coefficient depend on the law of the solution is often called mean-field SDE, for a reason that we will explain hereafter.

- the second term appearing in (II.1) is the *diffusion term*. It consists of several parts:

  ▷ we will denote by $m_t(u) := \int_0^1 \varphi(y_t(u)-y_t(v))\mathrm{d}v$ the mass function. As in Chapter I, the quadratic variation of $(y_t(u))_{t\in[0,T]}$ will be proportional to $\int_0^t \frac{\mathrm{d}s}{m_s(u)}$. In order to avoid problems of cancellation of the mass, we will only consider in this chapter functions φ which are positive everywhere on $\mathbb{R}$. A typical example will be the Gaussian density. Our results will also include the case with a constant mass, *i.e.* when $\varphi \equiv 1$.

  ▷ the function f will typically be of the form $f(k) = \frac{1}{(1+k^2)^{\alpha/2}}$. Depending on α, the diffusion term will be more or less smooth with respect to the space variable u. This behaviour recalls the link between regularity of a function and decay at infinity of its Fourier transform. It is not a coincidence since the shape of the diffusion term in (II.1) is constructed in such a way that it is easy to deal with the Fourier transform of f.

  ▷ the complex-valued Brownian sheet w is defined by $w := w^{\Re} + iw^{\Im}$, where $w^{\Re}$ and $w^{\Im}$ are two independent real Brownian sheets on $\mathbb{R}\times[0,T]$. Thus $\Re(e^{-iky_t(u)}\mathrm{d}w(k,t)) = \cos(ky_t(u))\mathrm{d}w^{\Re}(k,t) + \sin(ky_t(u))\mathrm{d}w^{\Im}(k,t)$. By opposition with the diffusion introduced in Chapter I, the Brownian sheet is no more indexed by the other particles, but by the frequency k. We will explain hereafter why we decide to change the model.

One of the main specificity and interest of this diffusion model is that the solution $(y_t)_{t\in[0,T]}$ is a canonical representative of the measure-valued process $(\mu_t)_{t\in[0,T]}$. Indeed, for each time t, y_t is the quantile function, or in other words the increasing rearrangement, of μ_t. In that sense, the diffusion term appearing in equation (II.1) belongs to the same family of diffusions as Konarovskyi's model (see [Kon11, Kon17b, KvR18]) and as the smooth approximation of Konarovskyi's particle system introduced in Chapter I.

By applying Itô's formula to $F(y_t(u))$ for some $F \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ and by using the relation $\int_0^1 F(y_t(u))\mathrm{d}u = \int_{\mathbb{R}} F(x)\mathrm{d}\mu_t(x)$, we get the following SPDE satisfied by the process $(\mu_t)_{t\in[0,T]}$:

$$\mathrm{d}\mu_t = \frac{C_f}{2}\,\partial^2_{xx}\left(\frac{\mu_t}{\varphi * \mu_t}\right)\mathrm{d}t - \partial_x\Big(b(\cdot,\mu_t)\,\mu_t\Big)\mathrm{d}t - \partial_x\left(\frac{1}{(\varphi*\mu_t)^{1/2}}\int_{\mathbb{R}} f(k)\Re\left(e^{-ik\,\cdot}\,\mathrm{d}w(k,t)\right)\,\mu_t\right)$$

where $C_f = \int_{\mathbb{R}} f(k)^2\mathrm{d}k$. The computation is detailled in the introduction of Section II.3.

We are interested in the regularization properties of this family of diffusions; in this chapter, we will wonder how far this Wasserstein-type diffusion may help for restoring uniqueness, in other words for ensuring the well-posedness of the stochastic differential equation (II.1). In this framework, we will divide our preliminary discussion into two parts. First, we will link our problem to a large literature dealing with regularization by noise, in particular for McKean-Vlasov equations, and explain which order of regularity we need on the drift function b. Second, we will focus on the diffusion term, compare it with the diffusion studied in Chapter I and explain why we have to slightly change the model.

II.1.1 Regularization by noise

Let us take the diffusion term equal to zero and consider only the drift part of equation (II.1):

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u),\mu_t)\mathrm{d}t; \\ \quad \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}. \end{cases} \tag{II.2}$$

Then the measure-valued process $(\mu_t)_{t\in[0,T]}$ satisfies the following non-linear Fokker-Planck equation on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$:

$$\partial_t\mu_t + \mathrm{div}(\mu_t\, b(\cdot,\mu_t)) = 0. \tag{II.3}$$

Let us remark that equations (II.2) and (II.3) are deterministic equations.

The problem of regularization by noise arises when we leave the framework of the Cauchy-Lipschitz theory. If $b : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is a Lipschitz-continuous function (with respect to the L_2-Wasserstein distance on $\mathcal{P}_2(\mathbb{R})$), then equation (II.2) is well-posed: the proof is based on a fixed-point method and is similar to the proof of [Szn91, Thm 1.1].

Let us now consider an example where b is not Lipschitz-continuous and where uniqueness fails. This example is derived from the one-dimensional Peano counter-example. Let b be the following function: for each $\mu \in \mathcal{P}_2(\mathbb{R})$, $b(\mu) := 2\,\mathrm{sign}(m)\sqrt{|m|}$, where $m := \int_{\mathbb{R}} x\mu(\mathrm{d}x)$ and $\mathrm{sign}(x) := \mathbb{1}_{\{x\neq 0\}}\frac{x}{|x|}$. In particular, the restriction of b to the set of Dirac measures is, for each $x\in\mathbb{R}$, $b(\delta_x) = 2\,\mathrm{sign}(x)\sqrt{|x|}$. Then the Cauchy problem with initial condition $\mu_0 = \delta_0$ has infinitely many solutions of the form $\mu_t = \delta_{\pm((t-t_0)^+)^2}$, where t_0 can be chosen arbitrarily on $[0,+\infty)$. The function $\mu_t = \delta_0$ for every $t \geqslant 0$ is also solution.

Our aim is to restore uniqueness in law of (II.2) by adding a diffusion term. For ordinary and partial differential equations in finite dimension, this idea of restoring uniqueness by adding a noise comes back to pioneer works of Stroock and Varadhan [SV69], for weak well-posedness, and Zvonkin [Zvo74] for strong well-posedness, which means that the solutions are adapted to the filtration generated by the noise and that they are almost surely indistinguishable. In [SV69, SV79], Stroock and Varadhan have developed a correspondance between parabolic equations in finite dimension and martingale problems. In particular, they have proved weak existence and uniqueness in law for SDEs in finite dimension with bounded and measurable drift and uniformly elliptic and continuous diffusion matrix. Zvonkin [Zvo74] has considered the following SDE in dimension one:

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t,$$

and has proved that strong uniqueness holds if σ is Hölder-continuous, bounded by above and by below (far from 0) and if b is only bounded and measurable. Veretennikov [Ver80] has extended this result to the case of higher dimension d and Krylov and Röckner [KR05] have proved that strong uniqueness still holds if b is only bounded in L^p, for $p > d$.

More recently, restoration of uniqueness by stochastic perturbations has been obtained in infinite-dimension. In [FGP10], Flandoli, Gubinelli and Priola have shown strong existence and pathwise uniqueness of the following transport PDE

$$\mathrm{d}_t u(t,x) = (b(t,x)\cdot Du(t,x))\mathrm{d}t + \sum_{i=1}^{d} e_i \cdot Du(t,x)\circ dW_t^i; \quad u(0,x) = u_0(x),$$

in a case where b is Hölder-continuous. The introduction of a multiplicative noise, where the stochastic integration is regarded in the Stratonovich sense, is necessary in this case to get uniqueness; without noise, some counter-examples to uniqueness exist. Since this breakthrough, several results [DPF10, DPFPR13, DPFPR15] have been obtained on pathwise uniqueness for SDEs on Hilbert spaces of the following form

$$dX_t = (AX_t + B(X_t))\mathrm{d}t + \mathrm{d}W_t,$$

where $A : D(A) \subset H \to H$ is an unbounded operator that satisfies certain properties, $B : H \to H$ is required to be of minimal regularity and W is a cylindrical Wiener process. Pathwise uniqueness is shown to hold for drift functions B that are only measurable and locally bounded. Moreover, similar regularization results obtained for other types of PDEs, like for instance kinetic equations perturbated by noise [FFPV17]. We refer to Flandoli's book [Fla11] for an extensive introduction to restoration of uniqueness by noise in both finite and infinite dimension.

The question of well-posedness has also been intensively studied for McKean-Vlasov equation:

$$\mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X_t, \mu_t)\mathrm{d}W_t, \tag{II.4}$$

where μ_t is the law of X_t. The drift function $b : (t,x,\mu) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \mapsto b(t,x,\mu) \in \mathbb{R}^d$ and the diffusion matrix $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$ are measurable and $(W_t)_{t\geqslant 0}$ is a $\mathbb{R}^m$-valued Brownian motion. This equation is closely linked to the system of particles

$$\mathrm{d}X_t^i = b(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}t + \sigma(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}W_t^i, \quad i = 1, \dots, N, \tag{II.5}$$

where $\overline{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$ and where the Brownian motions $(W_t^i)_{t\geqslant 0,\, 1\leqslant i\leqslant N}$ are independent. Equation (II.5) is often described as a particle system with mean-field interaction, since the coefficients depend on the whole distribution only via $\overline{\mu}_t^N$. The interest in McKean-Vlasov equation dates back to the work of Kac [Kac56] and McKean [McK66, McK67], who have introduced the concept of propagation of chaos. Propagation of chaos means, roughly speaking, that if the particles $(X_0^i)_{i=1,\dots,N}$ are initially independently distributed on $\mathbb{R}^d$, then for any fixed k, the behaviour of k particles satisfying (II.5) is asymptotically close to the behaviour of k independent solutions to (II.4) when N tends to $+\infty$. In [Szn91], Sznitman have proved well-posedness and propagation of chaos of McKean-Vlasov SDE (II.4) for a drift b of the form $b(t,x,\mu) = \int B(t,x,y)\mathrm{d}\mu(y)$. Since, more general particle systems with mean-field interaction have been investigated, including their asymptotic behaviour when the number of particles tends to infinity. For instance, propagation of chaos was also studied for Boltzmann equation (see e.g. [Mé96]), for Burger's equation (see e.g. [BT97]) or for homogeneous Landau equation for hard potentials (see e.g. [FG17]). Let us also mention important results for McKean-Vlasov equations with both common noise and idiosyncratic noise (see e.g. [Vai88, DV95, KX99, KX04]).

Existence and uniqueness of McKean-Vlasov equation (II.4), in a strong or a weak sense, have been studied in very general settings and under various assumptions on the drift and

the diffusion functions. We only give here a short overview of the first works on this topic and of some recent developments that are interesting for our study. Funaki [Fun84] and Gärtner [GÖ88] have proved existence of a solution to the martingale problem associated to (II.4) when b and σ are continuous and satisfy some Lyapunov conditions. Under Lipschitz-continuity assumptions over b and σ, they have also proved unique solvability of the martingale problem. Strong existence and uniqueness of a solution to (II.4) are also studied by Oelschläger [Oel84] under Lipschitz-continuity assumptions over b and σ. More recently, these results have been extended to unbounded coefficients on a domain D of $\mathbb{R}^d$ under a suitable Lyapunov condition (see e.g. [HSS]).

Following Jourdain [Jou97], the question of restoration of uniqueness by noise has led to recent developments in [MV, CdR19, Lac18, CdRF, RZ]. The assumptions on the drift function b require no continuity with respect to t and x and Lipschitz-continuity with respect to μ with respect to the total variation distance $\mathrm{d_{TV}}$, defined as

$$\mathrm{d_{TV}}(\mu,\nu) := \sup\left\{\int_{\mathbb{R}^d} f\mathrm{d}\mu - \int_{\mathbb{R}^d} f\mathrm{d}\nu \,;\, f:\mathbb{R}^d\to\mathbb{R} \text{ measurable}, \|f\|_{L_\infty}\leqslant 1\right\}.$$

In [MV], Mishura and Veretennikov adapted estimates due to Krylov, established in the case of Itô's SDEs, to the case of McKean-Vlasov SDEs. Strong uniqueness is proved when σ is does not depend on μ, is non-degenerate and is Lipschitz-continuous with respect to the space variable and when b grows at most linearly and is Lipschitz-continuous with respect to the measure variable. Lacker [Lac18] gives a simple proof relying on a fixed-point argument, in the case where σ does not depend on μ. The measure is initially frozen and is token as an input in (II.4): therefore, (II.4) becomes an SDE with coefficients depending only on time t and position x and the equation can be solved by Girsanov's Theorem. Then, under the assumption that b is Lipschitz-continuous with respect to $\mathrm{d_{TV}}$, the map associating to every input the law of $(X_t)_{t\in[0,T]}$ under the constructed Girsanov measure is a contraction. Chaudru de Raynal and Frikha [CdRF] proved well-posedness under similar assumptions by a method relying on the study of the transition kernel, which is close to a parametrix method and relies on results by Buckdahn-Li-Peng [BLP09]. Röckner and Zhang [RZ] have recently proved that well-posedness can be extended to the case of coefficients satisfying Krylov-Röckner's conditions; here also, the drift term b should be Lipschitz-continuous with respect to the measure variable.

In this chapter, we give a result of restoration of uniqueness for equation (II.2) by an additive infinite-dimensional noise, as described by equation (II.1). As in Chapter I, the dynamics of the system preserve the monotonicity; more precisely, if the initial condition $u\in[0,1]\mapsto y_0(u)\in\mathbb{R}$ is strictly increasing, then almost surely, $u\mapsto y_t(u)$ is strictly increasing for every $t\in[0,T]$. In other words, if we see the system as a cloud of particles indexed by the interval $[0,1]$, particles do not collide and preserve their respective order. The method of proof for existence and uniqueness of a weak solution relies on Girsanov's Theorem, e.g. as in [SV79].

The fact that the additive noise is of infinite dimension allows us to weaken the assumptions of regularity of the drift function b with respect to the measure variable μ. Roughly speaking, we will assume that $b:\mathbb{R}\times\mathcal{P}_2(\mathbb{R})\to\mathbb{R}$ is twice continuously differentiable with respect to its first variable and bounded and measurable with respect to its second variable. A precise statement of the assumptions on b is given in Definition II.14 and some examples of admissible drift functions b are given in Remark II.15. Let us quote one particular example: let b be of the following type

$$b(x,\mu) = b(\mu) := a\left(\int_{\mathbb{R}} y\mathrm{d}\mu(y)\right),$$

with a a bounded and measurable function, or even Hölder-continuous. Then b is admissible for our problem, whereas it is not necessarily Lipschitz-continuous with respect to the measure

variable and the regularization results for McKean-Vlasov equation mentioned above do not apply in this case.

The drawback of our approach is that we need a $\mathcal{C}^2$-regularity of the velocity field b with respect to its first variable (or, as we will see, at least a H^s-regularity for some $s > \frac{3}{2}$). In Paragraph II.1.2, we explain among others why we need this assumption on the regularity of b with respect to its first variable. It is linked to the inversion of the diffusive part. In Paragraph II.1.3, we will describe a balance between the regularity in the space variable and the regularity in the measure variable.

II.1.2 Fourier shape of the diffusion term

The general idea of the method presented in this chapter starts by proving strong solvability of (II.1) in the case where we only have a diffusion part ($b \equiv 0$). Then we deduce weak solvability of (II.1) for general drift functions by writing the drift term as a perturbation of the noise and by applying Girsanov's Theorem. This method does not need regularity of b with respect to the measure variable μ. The main difficulty in applying this scheme of proof is to inverse the diffusion coefficient and we have to assume some regularity of b with respect to the space variable x. We will first explain the idea of the proof in the case where the diffusion term is the process constructed in Chapter I. We will point out where this idea fails and how we modify the model of Chapter I in order to be able to inverse the diffusion term.

Let us consider the following equation:

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t + \dfrac{1}{\int_0^1 \varphi^2(y_t(u) - y_t(v))\mathrm{d}v} \displaystyle\int_0^1 \varphi(y_t(u) - y_t(u'))\mathrm{d}w(u', t); \\ \quad \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}. \end{cases} \tag{II.6}$$

The diffusion term is similar to the diffusion introduced in (I.3), with the modification that we are now considering a function φ which is positive everywhere on $\mathbb{R}$ instead of a function φ with compact support, in order to avoid that the mass function is zero (and thus, the parameter ε introduced in (I.3) is not needed here). Let us assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is an even Lipschitz-continuous function, non-increasing on $[0, +\infty)$ and such that $\varphi(x) > 0$ for every $x \in \mathbb{R}$. We can consider smooth functions φ, like a Gaussian function $\varphi(x) = e^{-x^2/2}$, or less regular functions φ, like $\varphi(x) = e^{-|x|}$.

As explained previously, the first step consists in proving strong well-posedness of (II.6) in the case where $b \equiv 0$. If φ is smooth and $\varphi \geqslant c > 0$, the proof is a standard fixed-point argument since the diffusion term is bounded and Lipschitz-continuous. In the general setting, we first prove existence and uniqueness up to a stopping time, defined as the first time where the diameter of the cloud of particles becomes larger than a fixed level M. As in Chapter I, the solution $u \mapsto y_t(u)$ owns the property to be increasing for every fixed time t, so that it is the quantile function of the probability measure $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}$. Furthermore, we will show that, under some regularity assumption on the initial condition, for any $t \in [0, T]$, $u \mapsto x_t(u)$ is smooth enough to ensure that μ_t has a continuous density, denoted by p_t. We denote by F_t the c.d.f. associated to μ_t.

Then we want to write the drift term b as a perturbation of the Brownian sheet w in order to apply Girsanov's Theorem. For this purpose, we want to construct an $L_2[0,1]$-process $(h_t)_{t\in[0,T]}$ satisfying for every $x \in \mathbb{R}$

$$b(x, \mu_t) = \frac{1}{\int_0^1 \varphi^2(x - y_t(v))\mathrm{d}v} \int_0^1 \varphi(x - y_t(u'))h_t(u')\mathrm{d}u', \tag{II.7}$$

and then construct a new probability measure under which $\mathrm{d}w(u', t) + h_t(u')\mathrm{d}u'\mathrm{d}t$ is a the infinitesimal increment of a Brownian sheet. Informally, we get by the substitutions $y = y_t(v)$

and $z = y_t(u')$:

$$b(x, \mu_t) = \frac{1}{\int_{\mathbb{R}} \varphi^2(x-y) p_t(y) \mathrm{d}y} \int_{\mathbb{R}} \varphi(x-z) h_t(F_t(z)) p_t(z) \mathrm{d}z.$$

That equation is equivalent, denoting by $\mathcal{F}$ the Fourier transform, to the following equation:

$$\mathcal{F}(b(\cdot, \mu_t) \cdot ((\varphi^2) * p_t)) = \mathcal{F}(\varphi) \cdot \mathcal{F}(h_t \circ F_t \cdot p_t).$$

Thus we want to define $h_t \circ F_t = \frac{1}{p_t} \mathcal{F}^{-1} \left(\frac{\mathcal{F}(b(\cdot, \mu_t) \cdot ((\varphi^2) * p_t))}{\mathcal{F}(\varphi)} \right)$ and prove that h_t is square-integrable.

We see two major obstacles here. The first one is the division by the density p_t, which is equal to zero outside the support of μ_t, since $\frac{\mathcal{F}(b(\cdot, \mu_t) \cdot ((\varphi^2) * p_t))}{\mathcal{F}(\varphi)}$ has no chance to be smooth enough so that $\mathcal{F}^{-1} \left(\frac{\mathcal{F}(b(\cdot, \mu_t) \cdot ((\varphi^2) * p_t))}{\mathcal{F}(\varphi)} \right)$ has a compact support.

The second hindrance is the division by $\mathcal{F}(\varphi)$. In the case where φ is a Gaussian density, $\frac{1}{\mathcal{F}(\varphi)(k)}$ behaves like $e^{k^2/2}$. Even if b is $\mathcal{C}^\infty$ with respect to its first variable, this would not be sufficient to obtain L_2-integrability on h. Let us try to reduce the regularity of φ: if $\varphi(x) = e^{-|x|}$, then $\frac{1}{\mathcal{F}(\varphi)(k)} = 1 + k^2$. Nevertheless, the density p_t cannot be of class $\mathcal{C}^1$ with this choice of function φ (we refer to Paragraph II.2.3 for a justification of this statement). Thus even with smooth functions b, the regularity of $(\varphi^2) * p_t$ is not sufficient to compensate for the term $\frac{1}{\mathcal{F}(\varphi)}$ and to insure that h_t belongs to L_2. Here, we point out that the question of the regularity of φ is crucial. If φ has high regularity, it will be difficult to inverse the interaction kernel. If φ has low regularity, then the solution $u \mapsto y_t(u)$ and consequently the density p_t will not be smooth enough to allow h_t to belong to L_2 and therefore to apply Girsanov's change of measures. The problem is that there is not an intermediate regularity that we could impose on φ so that both problems can be simultaneously solved.

This leads to the introduction of a new model of diffusion, which is used in equation (II.1):

$$\mathrm{d}y_t(u) = \frac{1}{\left(\int_0^1 \varphi(y_t(u) - y_t(v)) \mathrm{d}v \right)^{1/2}} \int_{\mathbb{R}} f(k) \Re(e^{-iky_t(u)} \mathrm{d}w(k,t)), \tag{II.8}$$

The question of inversion of the diffusion coefficient is now the following: by analogy with (II.7), we want to construct an $L_2(\mathbb{R}; \mathbb{C})$-valued process $(h_t)_{t \in [0,T]}$ such that

$$b(x, \mu_t) = \frac{1}{\left(\int_0^1 \varphi(x - y_t(v)) \mathrm{d}v \right)^{1/2}} \int_{\mathbb{R}} f(k) e^{-ikx} h_t(k) \mathrm{d}k. \tag{II.9}$$

A solution to (II.9) will now be of the form

$$h_t(k) = \frac{1}{f(k)} \mathcal{F}^{-1} \left(b(\cdot, \mu_t) \cdot (\varphi * p_t) \right)(k). \tag{II.10}$$

In (II.8), there are two major modifications with respect to the diffusion of (II.6), related to the two obstacles mentioned above. First, in order to avoid the division by p_t, we change the model so that the integral in the right-hand side of (II.9) is written as the Fourier transform of $f \cdot h_t$. In this new setting, the Brownian sheet is now acting on the frequency k instead of acting on the other particles.

The second modification brought to the model is related to the fact that we are now considering two different interaction functions at numerator and denominator, in order to be choose different regularities. At numerator, the regularity will depend on the decay at infinity of f. To fix ideas, let us take f of the form $f(k) = \frac{1}{(1+k^2)^{\alpha/2}}$, for a certain $\alpha > 0$. Then the larger α is, the higher the regularity of $u \mapsto y_t(u)$ is; nevertheless, we will also have to assume some regularity on b with respect to its first variable so that h_t defined by (II.10) belongs to L_2. For

the mass function at denominator, we consider a function φ which is very smooth, typically $\mathcal{C}^\infty$ (actually, we only need that φ has at least the same regularity than $x \mapsto b(x,\mu)$). Let us remark that a possible choice for φ is $\varphi \equiv 1$. In that particular case, the model does not involve a mass. In other words, the quadratic variation is the same for every particle. For a general function φ, $\langle y(u), y(u)\rangle_t = C_f \int_0^t \frac{\mathrm{d}s}{m_s(u)}$, where C_f is a constant depending only on the L_2-norm of f.

Nevertheless, despite that those modifications, the process $(y_t)_{t\in[0,T]}$ solution to (II.8) still has similar properties to the diffusion of Chapter I used in (II.6). It preserves the monotonicity property: we will prove that almost surely, $u \mapsto y_t(u)$ is strictly increasing. Moreover, if two particles u and u' are close at one time t, then their future trajectories will be very correlated with high probability. Moreover, due to the presence of a mass function, the fluctuations of a particle u will be large (resp. small) if there are few (resp. a lot of) particles in the neighbourhood of particle u.

II.1.3 A balance of regularity

Let us recap the assumptions made on the drift function b in the results described above. In [Jou97, Lac18, MV, CdRF, RZ], well-posedness holds for a class of drift coefficients b equal or included in the space of $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ that are measurable, bounded and Lipschitz-continuous in the second variable. The result that we state in Section II.4 applies for functions b that are measurable, bounded and twice continuously differentiable with respect to the first variable. A natural question arises: can we obtain a regularization result for a drift function which has at the same time low regularity in the space variable and in the measure variable?

In Section II.5, we give an example of a diffusion restoring uniqueness for a velocity field b which is neither Lipschitz-continuous in the space variable nor Lipschitz-continuous in the measure variable. That diffusion is inspired by (II.8), but there are some modifications of the model in order to deal with the lack of regularity of b with respect to its first variable. More precisely, we give in Section II.5 a result showing that it is possible to regularize a Fokker-Planck equation of type (II.3) for a certain class of drift functions b such that for every $\mu \in \mathcal{P}_2(\mathbb{R})$, $b(\cdot,\mu)$ belongs to $H^\eta(\mathbb{R})$ and such that for every $x \in \mathbb{R}$, $b(x,\cdot)$ is δ-Hölder continuous with respect to the total variation distance, under the assumption $\eta > \frac{3}{2}(1-\delta)$. In particular, it holds if $\eta = \delta = \frac{2}{3}$, in a case where b is not Lipschitz-continuous in any variable. It establishes a continuum of regularity conditions interpolating the two classes of results described above. If $\delta = 1$, we assume that b is merely bounded and continuous with respect to the first variable, which is close to the assumptions of [Jou97, Lac18, MV, CdRF, RZ] and if $\delta = 0$, we recover the result of Section II.4.

This interpolation result is obtained by writing the drift function b as a sum of a function $\widetilde{b}$ satisfying the conditions needed to apply Lacker's proof and a function $b - \widetilde{b}$ on which we will apply the same proof as in Section II.4. The main modification of the model is the addition of a new Brownian motion $(\beta_t)_{t\in[0,T]}$, independent of the Brownian sheet w, on which we will apply a Girsanov transformation to deal with the drift $\widetilde{b}$. Adding this second noise has a regularizing effect in the direction of the space variable and allows us to prove well-posedness for this larger class of drift functions described above. The form of the SDE, defined on a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$, will be as follows:

$$\begin{cases} \mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t + b(z_t,\mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^{\mu,W}), \quad (\mu,w) \perp\!\!\!\perp (\beta,\xi) \\ z_0 = \xi, \quad \mathcal{L}^{\mathbb{P}}(\xi) = \mu_0, \end{cases} \tag{II.11}$$

where $(\mathcal{G}_t^{\mu,W})_{t\in[0,T]}$ is the filtration generated by the Brownian sheet w and the measure-valued process $(\mu_t)_{t\in[0,T]}$ itself. Recall that previously, in equation (II.1), the measure $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}$ was regarding the law of y_t with respect to the randomness carrying the initial condition, in the

form $y_0(u) = g(u)$ for every $u \in [0,1]$. Implicitly, that allowed to identify μ_t with the conditional law of y_t given the common noise $(w(\cdot, s))_{s \leqslant t}$. In (II.11), μ_t denotes the conditional law of z_t given the realization of $(w(\cdot, s))_{s \leqslant t}$ and μ_t itself, provided that (μ, w) is independent of (β, ξ). It is due to the fact that we will allow for weak solutions, namely for solutions where μ is not necessarily adapted with respect to w. This leads to another difficulty arising due to the lack of adaptness: the observation of z may bias the future realizations of μ, w and β. Thus, we will consider weak solutions of (II.11) satisfying a further assumption, called *compatibility condition*; namely, we assume that for every $t \in [0,T]$, the processes (ξ, w, μ) and β are independent given $\mathcal{G}_t$.

Remark that equation (II.11) does not belong to the general framework of this thesis: until now, we have only studied stochastic differential equations where the solutions can be written as a quantile function, in other words SDEs preserving a monotonicity property. It is due to Girsanov's change of measure involving the noise β and the drift $\widetilde{b}$; this change of measure destrois the monotonicity of the solution. That is why we consider in (II.11) as initial condition a general random variable ξ of distribution μ_0 and not anymore the quantile function $u \mapsto g(u)$ associated with μ_0. We consider here no mean-field interaction in the diffusion coefficient; in other words, the mass is here constant equal to one.

Organisation of the chapter

We will first address in Section II.2 the question of well-posedness, continuity and monotonicity with respect to the variable u of the diffusion process. Then we will obtain higher orders of regularity, depending on the assumptions made on the function f. In Section II.3, we will specify the assumptions needed on the drift term B and we will compute the Fourier inversion of the kernel. Subsequently, in Section II.4, we will prove existence and uniqueness of a weak solution to equation (II.1). We will divide this problem into two steps. First, we will consider the problem of well-posedness up to a stopping time depending on the size of the support of measure μ_t. Then, we will prove well-posedness on the global time interval $[0,T]$. Finally, in Section II.5, we will prove well-posedness of SDE (II.11).

Notations. Throughout the chapter, we will always denote by C_M the constants depending on M, even if they change from one line to the next. We will also denote by $\langle k \rangle := (1+k^2)^{1/2}$.

II.2 Regularity of a diffusion on the Wasserstein space

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. Let $w^{\Re}$ and $w^{\Im}$ be two independent $(\mathcal{G}_t)_{t\in[0,T]}$-adapted Brownian sheets on $\mathbb{R} \times [0,T]$ (see Section I.2 for a definition). As in Section I.3, define $\mathcal{L}^{\uparrow}_{2+}[0,1]$ the set of non-decreasing and càdlàg functions $g : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ such that there is $p > 2$ satisfying $\int_0^1 |g(u)|^p \mathrm{d}u < +\infty$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an even function of class $\mathcal{C}^\infty$. Furthermore, we assume that φ is decreasing on $[0, +\infty)$ and that for every $x \in [0, +\infty)$, $\varphi(x) > 0$. Typical examples of functions φ are the constant function $\varphi \equiv 1$ and the Gaussian density $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Let $f : \mathbb{R} \to \mathbb{R}$ be an even and square integrable function.

Let us define the stochastic differential equation satisfied by a process $(x_t)_{t\in[0,T]}$ with values in $L_2([0,1], \mathbb{C})$:

$$x_t(u) = g(u) + \int_0^t \frac{1}{\left(\int_0^1 \varphi(\Re x_s(u) - \Re x_s(v))\mathrm{d}v\right)^{1/2}} \int_{\mathbb{R}} e^{-ik\Re x_s(u)} f(k) \mathrm{d}w(k,s), \tag{II.12}$$

where w is the complex Brownian sheet defined by $w = w^{\Re} + iw^{\Im}$ and $(\Re x_t)_{t\in[0,T]}$ is the real part of the process $(x_t)_{t\in[0,T]}$.

Actually, we will focus on the real part of equation (II.12). Let $y_t(u) := \Re x_t(u)$ for every $t \in [0,T]$ and $u \in [0,1]$. The process $(y_t)_{t\in[0,T]}$ has to satisfy the following *real* equation:

$$\begin{aligned} y_t(u) = g(u) &+ \int_0^t \frac{1}{m_s(u)^{1/2}} \int_{\mathbb{R}} \cos(ky_s(u)) f(k) \mathrm{d}w^{\Re}(k,s) \\ &+ \int_0^t \frac{1}{m_s(u)^{1/2}} \int_{\mathbb{R}} \sin(ky_s(u)) f(k) \mathrm{d}w^{\Im}(k,s), \end{aligned} \tag{II.13}$$

where $m_s : u \mapsto \int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v$ denotes the mass function at time s and at point u. As in Chapter I, the quadratic variation is proportional to the inverse of the mass. It is given by the following computation

$$\langle y(u), y(u)\rangle_t = \int_0^t \int_{\mathbb{R}} \frac{\cos(ky_s(u))^2 + \sin(ky_s(u))^2}{m_s(u)} f(k)^2 \mathrm{d}k\mathrm{d}s = \|f\|_{L_2}^2 \int_0^t \frac{\mathrm{d}s}{m_s(u)}. \tag{II.14}$$

Moreover, let us compute the co-variation between two particles u and v:

$$\begin{aligned} \langle y(u), y(u')\rangle_t &= \int_0^t \int_{\mathbb{R}} \frac{\cos(ky_s(u))\cos(ky_s(u')) + \sin(ky_s(u))\sin(ky_s(u'))}{m_s(u)^{1/2} m_s(u')^{1/2}} f(k)^2 \mathrm{d}k\mathrm{d}s \\ &= \int_0^t \frac{1}{m_s(u)^{1/2} m_s(u')^{1/2}} \int_{\mathbb{R}} \cos(k(y_s(u) - y_s(u'))) f(k)^2 \mathrm{d}k\mathrm{d}s. \end{aligned} \tag{II.15}$$

The covariation reaches its maximum when $y_s(u) - y_s(u') \in 2\pi\mathbb{Z}$, in particular when $u = u'$.

Remark II.1. Recall that in Chapter I, the covariation between two particles, given by Corollary I.18, was of the form

$$\langle y(u), y(u')\rangle_t = \int_0^t \frac{\int_0^1 \varphi_\sigma(y_s(u) - y_s(v))\ \varphi_\sigma(y_s(u') - y_s(v))\ \mathrm{d}v}{(\varepsilon + \int_0^1 \varphi_\sigma^2(y_s(u) - y_s(v))\mathrm{d}v)\ (\varepsilon + \int_0^1 \varphi_\sigma^2(y_s(u') - y_s(v))\mathrm{d}v)} \mathrm{d}s.$$

Recall that, since φ_σ had a support included in $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$, the covariation satisfied $\mathrm{d}\langle y(u), y(u')\rangle_t = 0$ at times t where the particles u and u' were at a relative distance larger than σ. In the new model presented in this Chapter, particles do not satisfy this property of un-correlated increments when they are far one to each other. Nevertheless, depending on the shape of f, $\langle y(u), y(u')\rangle_t$ may decrease exponentially fast when $|y_t(u) - y_t(u')|$ tends to infinity. Let us illustrate this phenomenon when $f(k) = \frac{1}{(1+k^2)^{1/2}}$; in that case, equality (II.15) reads as follows:

$$\begin{aligned} \langle y(u), y(u')\rangle_t &= \int_0^t \frac{1}{m_s(u)^{1/2} m_s(u')^{1/2}} \int_{\mathbb{R}} \cos(k(y_s(u) - y_s(u'))) \frac{1}{1+k^2} \mathrm{d}k\mathrm{d}s \\ &= \int_0^t \frac{\pi e^{-|y_s(u) - y_s(u')|}}{m_s(u)^{1/2} m_s(u')^{1/2}} \mathrm{d}s. \end{aligned}$$

In Paragraph II.2.1, we will introduce an auxiliary equation where the function φ is replaced by a function φ_M so that the diffusion coefficient is bounded. We will prove strong well-posedness of that equation, continuity and monotonicity with respect to the space variable u of the solution. In Paragraph II.2.2, we will deduce existence and uniqueness of a strong solution to equation (II.13). Finally, in Paragraph II.2.3, we will discuss higher regularity of the process $(y_t)_{t\in[0,T]}$ depending on the decay at infinity of the function f.

II.2.1 Existence, uniqueness and continuity of the diffusion

We will consider in this paragraph an auxiliary equation where the mass function is bounded from below by a positive constant, so that the diffusion coefficient is bounded. For each $M \in \mathbb{N}\backslash\{0\}$,

we define $\varphi_M(x) := \varphi(|x| \wedge M)$. In particular, for every x in $[-M, M]$, $\varphi_M(x) = \varphi(x)$ and for each $x \in \mathbb{R}$, $\varphi_M(x) \geqslant \varphi(M) > 0$. Let us consider the following equation

$$y_t^M(u) = g(u) + \int_0^t \frac{1}{m_s^M(u)^{1/2}} \int_{\mathbb{R}} \cos(k y_s^M(u)) f(k) \mathrm{d}w^{\Re}(k,s) + \int_0^t \frac{1}{m_s^M(u)^{1/2}} \int_{\mathbb{R}} \sin(k y_s^M(u)) f(k) \mathrm{d}w^{\Im}(k,s). \tag{II.16}$$

where $m_s^M(u) = \int_0^1 \varphi_M(y_s^M(u) - y_s^M(v))\mathrm{d}v$. Since the mass function m_s^M is uniformly bounded from below by $\varphi(M)$, this equation is easier to resolve and we expect that the solution also satisfies equation (II.13) up to a certain stopping time.

Let us also recall (see Definition I.3) that a $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process $(M_t)_{t\in[0,T]}$ is said to be an L_2-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale if for each time $t \in [0,T]$, M_t belongs to $L_2([0,1],\mathbb{R})$ and $\mathbb{E}\left[\|M_t\|_{L_2}\right] < +\infty$ and if for each $h \in L_2([0,1],\mathbb{R})$, the scalar product $(M_t, h)_{L_2}$ is a real-valued $(\mathcal{G}_t)_{t\in[0,T]}$-martingale.

Proposition II.2. *Let $g \in \mathcal{L}_{2+}^{\uparrow}[0,1]$. Assume that $k \mapsto \langle k\rangle f(k)$ is square integrable. There exists a unique solution y^M in $\mathcal{C}([0,T], L_2[0,1])$ to equation* (II.16)*. Furthermore, the process $(y_t^M)_{t\in[0,T]}$ is an L_2-valued continuous $(\mathcal{G}_t)_{t\in[0,T]}$-martingale.*

Remark II.3. In this proposition and in every following result, we assume, at least, that $k \mapsto \langle k\rangle f(k)$ is square integrable on $\mathbb{R}$. In the particular case of $f_\alpha(k) = \frac{1}{\langle k\rangle^\alpha} = \frac{1}{(1+k^2)^{\alpha/2}}$, this assumption is equivalent to the condition $\alpha > \frac{3}{2}$. In particular, the assumption holds true for the Cauchy density $f(k) = \frac{1}{1+k^2}$.

Proof. We only give here the main elements of the proof, since it is very similar to the proof of Proposition I.11. This proof is based on a fixed-point argument. Define $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ the space of all $z \in L_2(\Omega, C([0,T], L_2[0,1]))$ such that $(z(\omega)_t)_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process with values in $L_2[0,1]$. The definition of $\|\cdot\|_{\mathcal{M}}$ is given by (I.4). Define

$$\psi(z)_t(u) := g(u) + \int_0^t \frac{1}{m_s^z(u)^{1/2}} \int_{\mathbb{R}} \cos(k z_s(u)) f(k) \mathrm{d}w^{\Re}(k,s) + \int_0^t \frac{1}{m_s^z(u)^{1/2}} \int_{\mathbb{R}} \sin(k z_s(u)) f(k) \mathrm{d}w^{\Im}(k,s),$$

where $m_s^z(u) = \int_0^1 \varphi_M(z_s(u) - z_s(v))\mathrm{d}v$. For each $z \in \mathcal{M}$, $\psi(z)$ belongs to $\mathcal{M}$, since by Burkholder-Davis-Gundy inequality, there is $C > 0$ such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leqslant T} \int_0^1 |\psi(z)_t(u)|^2 \mathrm{d}u\right] &\leqslant 3\|g\|_{L_2}^2 + C\mathbb{E}\left[\int_0^1\int_0^T\int_{\mathbb{R}} \frac{\cos^2(k z_s(u)) f^2(k)}{m_s^z(u)} \mathrm{d}k\mathrm{d}s\mathrm{d}u\right] \\
&\quad + C\mathbb{E}\left[\int_0^1\int_0^T\int_{\mathbb{R}} \frac{\sin^2(k z_s(u)) f^2(k)}{m_s^z(u)} \mathrm{d}k\mathrm{d}s\mathrm{d}u\right] \\
&\leqslant 3\|g\|_{L_2}^2 + C\|f\|_{L_2}^2\, \mathbb{E}\left[\int_0^1\int_0^T \frac{1}{m_s^z(u)} \mathrm{d}s\mathrm{d}u\right] \leqslant 3\|g\|_{L_2}^2 + C_M\|f\|_{L_2}^2,
\end{aligned}$$

because $m_s^z \geqslant \varphi(M) > 0$. Moreover, $(\psi(z)_t)_{t\in[0,T]}$ is an L_2-valued martingale and for each

$t \in [0,T]$

$$\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 |\psi(z^1)_s - \psi(z^2)_s|^2(u)\mathrm{d}u\right]$$
$$\leqslant C\mathbb{E}\left[\int_0^1\int_0^t\int_{\mathbb{R}}\left|\frac{\cos(kz_s^1(u))f(k)}{m_s^{z^1}(u)^{1/2}} - \frac{\cos(kz_s^2(u))f(k)}{m_s^{z^2}(u)^{1/2}}\right|^2 \mathrm{d}k\mathrm{d}s\mathrm{d}u\right]$$
$$+ C\mathbb{E}\left[\int_0^1\int_0^t\int_{\mathbb{R}}\left|\frac{\sin(kz_s^1(u))f(k)}{m_s^{z^1}(u)^{1/2}} - \frac{\sin(kz_s^2(u))f(k)}{m_s^{z^2}(u)^{1/2}}\right|^2 \mathrm{d}k\mathrm{d}s\mathrm{d}u\right].$$

For every $u \in [0,1]$ and every $s \in [0,T]$, $|\cos(kz_s^2(u)) - \cos(kz_s^1(u))| \leqslant k|z_s^2(u) - z_s^1(u)|$ and the same Lipschitz estimate for the sine function. Furthermore, φ_M is bounded from below and Lipschitz, since φ_M is $\mathcal{C}^\infty$ on $(-M,M)$, continuous on $\mathbb{R}$ and constant on $[M,+\infty)$. Thus we have:

$$\left|\frac{1}{\sqrt{m_s^{z^1}(u)}} - \frac{1}{\sqrt{m_s^{z^2}(u)}}\right| = \frac{1}{\sqrt{m_s^{z^1}(u)}\sqrt{m_s^{z^2}(u)}}\frac{1}{\sqrt{m_s^{z^1}(u)}+\sqrt{m_s^{z^2}(u)}}\left|m_s^{z^1}(u) - m_s^{z^2}(u)\right|$$
$$\leqslant \frac{1}{2\varphi(M)^{3/2}}\int_0^1 \left|\varphi_M(z_s^1(u) - z_s^1(v)) - \varphi_M(z_s^2(u) - z_s^2(v))\right|\mathrm{d}v$$
$$\leqslant C_M\left(|z_s^1(u) - z_s^2(u)| + \int_0^1 |z_s^1(v) - z_s^2(v)|\mathrm{d}v\right). \tag{II.17}$$

It follows that:

$$\mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 |\psi(z^1)_s - \psi(z^2)_s|^2(u)\mathrm{d}u\right] \leqslant C_M\mathbb{E}\left[\int_0^1\int_0^t\int_{\mathbb{R}}|z_s^1(u) - z_s^2(u)|^2(1+|k|^2)|f(k)|^2\mathrm{d}k\mathrm{d}s\mathrm{d}u\right]$$
$$\leqslant C_M\int_{\mathbb{R}}\langle k\rangle^2|f(k)|^2\mathrm{d}k\int_0^t\mathbb{E}\left[\sup_{r\leqslant s}\int_0^1|z_r^1(u) - z_r^2(u)|^2\mathrm{d}u\right]\mathrm{d}s.$$

Define $h_n(t) := \mathbb{E}\left[\sup_{s\leqslant t}\int_0^1 |\psi^{\circ n}(z^1)_s - \psi^{\circ n}(z^2)_s|^2(u)\mathrm{d}u\right]$. There is a constant $C_{M,f}$ depending on M and on $\int_{\mathbb{R}}\langle k\rangle^2|f(k)|^2\mathrm{d}k$ such that for all $n \in \mathbb{N}$ and for all $t \in [0,T]$, we have $h_{n+1}(t) \leqslant C_{M,f}\int_0^t h_n(s)\mathrm{d}s$. Therefore, $h_n(t) \leqslant \frac{C_{M,f}^n t^n}{n!}h_0(t)$ and we deduce that $\|\psi^{\circ n}(z^1) - \psi^{\circ n}(z^2)\|_{\mathcal{M}}^2 \leqslant \frac{C_{M,f}T^n}{n!}\|z^1 - z^2\|_{\mathcal{M}}^2$. Let n be large enough so that $\frac{C_{M,f}^n T^n}{n!} < 1$, *i.e.* so that $\psi^{\circ n}$ is a contraction. Then ψ admits a unique fixed point, which we denote by y^M. Since $y^M = \psi(y^M)$, it is an L_2-valued continuous $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. □

In the following two propositions, we prove that the process $(y_t^M)_{t\in[0,T]}$ preserves continuity and monotonicity of the initial condition.

Proposition II.4. *Let $g \in \mathcal{L}_{2+}^{\uparrow}[0,1]$ such that g is α-Hölder for some $\alpha > 0$. Assume that $k \mapsto \langle k\rangle f(k)$ is square integrable. There exists a version of y^M in $\mathcal{C}([0,1]\times[0,T])$.*

Proof. Let $u_1, u_2 \in [0,1]$. Let $p \geqslant 2$ such that $p > \frac{1}{\alpha}$. For every $t \in [0,T]$, by Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\Big[\sup_{s\leqslant t}|y_s^M(u_1) - y_s^M(u_2)|^p\Big]$$
$$\leqslant C_p|g(u_1) - g(u_2)|^p + C_{p,M}\mathbb{E}\Big[\Big(\int_0^t\int_{\mathbb{R}}\langle k\rangle^2 f(k)^2\mathrm{d}k|y_s^M(u_1) - y_s^M(u_2)|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big].$$

It follows that

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\leqslant t}|y_s^M(u_1)-y_s^M(u_2)|^p\Big]&\\ &\leqslant C_p|g(u_1)-g(u_2)|^p+C_{p,M,f}t^{p/2-1}\mathbb{E}\Big[\int_0^t|y_s^M(u_1)-y_s^M(u_2)|^p\mathrm{d}s\Big]\\ &\leqslant C_p|g(u_1)-g(u_2)|^p+C_{p,M,f}t^{p/2-1}\int_0^t\mathbb{E}\left[\sup_{r\leqslant s}|y_r^M(u_1)-y_r^M(u_2)|^p\right]\mathrm{d}s.\end{aligned}$$

By Gronwall's Lemma, and using the α-Hölder regularity of g, we have:

$$\mathbb{E}\left[\sup_{t\leqslant T}|y_t^M(u_1)-y_t^M(u_2)|^p\right]\leqslant C_{M,p,f}|u_1-u_2|^{p\alpha}.$$

Remark that $p\alpha-1>0$. Let us apply Kolmogorov's Lemma (e.g in [RY99, Theorem I.2.1, p.26] with $d=1$, $\gamma=p$ and $\varepsilon=p\alpha-1$). Thus there exists a version $\widetilde{y}^M$ of y^M in $\mathcal{C}([0,1]\times[0,T])$. □

Proposition II.5. *Let $g\in\mathcal{L}^{\uparrow}_{2+}[0,1]$. Let $u_1<u_2\in[0,1]$ be such that $g(u_1)<g(u_2)$. Assume that $k\mapsto\langle k\rangle f(k)$ is square integrable. Let y^M be the solution to equation* (II.16)*. Then almost surely and for every $t\in[0,T]$, $y_t^M(u_1)<y_t^M(u_2)$.*

Proof. Let $u_1<u_2\in[0,1]$ be such that $g(u_1)<g(u_2)$. Thus the process $Y_t=y_t^M(u_2)-y_t^M(u_1)$ satisfies

$$Y_t=g(u_2)-g(u_1)+\int_0^t Y_s\mathrm{d}N_s^M, \tag{II.18}$$

where we denote

$$\begin{aligned}N_t^M=&\int_0^t\int_{\mathbb{R}}\mathbb{1}_{\{y_s^M(u_1)\neq y_s^M(u_2)\}}\frac{\theta_M^{\Re}(y_s^M(u_2),k,s)-\theta_M^{\Re}(y_s^M(u_1),k,s)}{y_s^M(u_2)-y_s^M(u_1)}f(k)\mathrm{d}w^{\Re}(k,s)\\&+\int_0^t\int_{\mathbb{R}}\mathbb{1}_{\{y_s^M(u_1)\neq y_s^M(u_2)\}}\frac{\theta_M^{\Im}(y_s^M(u_2),k,s)-\theta_M^{\Im}(y_s^M(u_1),k,s)}{y_s^M(u_2)-y_s^M(u_1)}f(k)\mathrm{d}w^{\Im}(k,s)\end{aligned}$$

and $\theta_M^{\Re}(x,k,s)=\frac{\cos(kx)}{(\int_0^1\varphi_M(x-y_s^M(v))\mathrm{d}v)^{1/2}}$ and $\theta_M^{\Im}(x,k,s)=\frac{\sin(kx)}{(\int_0^1\varphi_M(x-y_s^M(v))\mathrm{d}v)^{1/2}}$. Thus we have

$$\begin{aligned}\langle N^M,N^M\rangle_t=&\int_0^t\int_{\mathbb{R}}\mathbb{1}_{\{y_s^M(u_1)\neq y_s^M(u_2)\}}\left|\frac{\theta_M^{\Re}(y_s^M(u_2),k,s)-\theta_M^{\Re}(y_s^M(u_1),k,s)}{y_s^M(u_2)-y_s^M(u_1)}\right|^2f(k)^2\mathrm{d}k\mathrm{d}s\\&+\int_0^t\int_{\mathbb{R}}\mathbb{1}_{\{y_s^M(u_1)\neq y_s^M(u_2)\}}\left|\frac{\theta_M^{\Im}(y_s^M(u_2),k,s)-\theta_M^{\Im}(y_s^M(u_1),k,s)}{y_s^M(u_2)-y_s^M(u_1)}\right|^2f(k)^2\mathrm{d}k\mathrm{d}s.\end{aligned}$$

For every x_1, $x_2\in\mathbb{R}$, for every $k\in\mathbb{R}$ and for every $s\in[0,T]$, we have the following two estimates:

$$\begin{aligned}|\cos(kx_2)-\cos(kx_1)|&\leqslant k|x_2-x_1|,\\ \left|\int_0^1\varphi_M(x_2-y_s^M(v))\mathrm{d}v-\int_0^1\varphi_M(x_1-y_s^M(v))\mathrm{d}v\right|&\leqslant\mathrm{Lip}(\varphi_M)|x_2-x_1|.\end{aligned}$$

It follows, by the same computation as (II.17), that

$$\left|\frac{1}{\big(\int_0^1\varphi_M(x_2-y_s^M(v))\mathrm{d}v\big)^{1/2}}-\frac{1}{\big(\int_0^1\varphi_M(x_1-y_s^M(v))\mathrm{d}v\big)^{1/2}}\right|\leqslant C_M|x_2-x_1|.$$

Thus for every x_1, $x_2 \in \mathbb{R}$,

$$\begin{aligned}\left|\theta_M^{\Re}(x_2,k,s)-\theta_M^{\Re}(x_1,k,s)\right| &\\ &\leqslant \left|\frac{\cos(kx_2)-\cos(kx_1)}{(\int_0^1 \varphi_M(x_2-y_s^M(v))\mathrm{d}v)^{1/2}}\right| \\ &\quad + |\cos(kx_1)|\left|\frac{1}{\left(\int_0^1 \varphi_M(x_2-y_s^M(v))\mathrm{d}v\right)^{1/2}}-\frac{1}{\left(\int_0^1 \varphi_M(x_1-y_s^M(v))\mathrm{d}v\right)^{1/2}}\right| \\ &\leqslant C_M\langle k\rangle|x_2-x_1|,\end{aligned}$$

Therefore, for every $s \in [0,T]$,

$$\mathbb{1}_{\{y_s^M(u_1)\neq y_s^M(u_2)\}}\left|\frac{\theta_M^{\Re}(y_s^M(u_2),k,s)-\theta_M^{\Re}(y_s^M(u_1),k,s)}{y_s^M(u_2)-y_s^M(u_1)}\right|^2 \leqslant C_M\langle k\rangle^2.$$

We have a similar bound on $\theta^{\Im}$. We deduce that $\frac{\mathrm{d}\langle N^M,N^M\rangle_s}{\mathrm{d}s} \leqslant C_M\int_{\mathbb{R}}\langle k\rangle^2 f(k)^2\mathrm{d}k$ for each $s \in [0,T]$. Hence the stochastic differential equation (II.18) has a unique solution and it is $Y_t = (g(u_2)-g(u_1))\exp\left(N_t^M-\frac{1}{2}\langle N^M,N^M\rangle_t\right)$. In particular

$$y_t^M(u_2)-y_t^M(u_1) = (g(u_2)-g(u_1))\exp\left(N_t^M-\frac{1}{2}\langle N^M,N^M\rangle_t\right).$$

Since $g(u_1) < g(u_2)$, we deduce that for every $t \in [0,T]$, $Y_t > 0$. Thus for every $t \in [0,T]$, $y_t^M(u_1) < y_t^M(u_2)$. □

Corollary II.6. *Let $g \in \mathcal{L}_{2+}^{\uparrow}[0,1]$ such that g is α-Hölder for some $\alpha > \frac{1}{2}$. Assume that $k \mapsto \langle k\rangle f(k)$ is square integrable. Then there is a version y^M of the solution to equation* (II.16) *in $\mathcal{C}([0,1]\times[0,T])$ such that almost surely, for each $t \in [0,T]$, $u \in [0,1] \mapsto y_t^M(u)$ is strictly increasing.*

Proof. By Proposition II.4, we know that there is a version y^M of the solution to (II.16) jointly continuous in time and space.

Furthermore, by Proposition II.5, there exists an almost sure event $\widetilde{\Omega}$ under which y^M belongs to $\mathcal{C}([0,1]\times[0,T])$ and for every $t \in [0,T]$ and for every $u_1, u_2 \in \mathbb{Q}\cap[0,1]$ such that $u_1 < u_2$, we have $y_t^M(u_1) < y_t^M(u_2)$. Since $u \mapsto y_t^M(u)$ is continuous under the event $\widetilde{\Omega}$, we deduce that $y_t^M(u_1) < y_t^M(u_2)$ holds with every $u_1 < u_2 \in [0,1]$. □

II.2.2 Construction of a non-blowing solution on the global time interval $[0,T]$

Definition II.7. For every $j \in \mathbb{N}$, let $\mathbf{G}^j$ denote the set of $\mathcal{C}^j$-functions $g : [0,1] \to \mathbb{R}$ such that for all $u < v$, $g(u) < g(v)$. Furthermore, for every $j \in \mathbb{N}$ and for every $\theta \in (0,1)$, let $\mathbf{G}^{j+\theta}$ denote the set of functions $g \in \mathbf{G}^j$ such that $\partial_u^{(j)}g$ is θ-Hölder continuous, *i.e.* there is a constant $C > 0$ such that for every $u, v \in [0,1]$, $|\partial_u^{(j)}g(u) - \partial_u^{(j)}g(v)| \leqslant C|u-v|^\theta$.

Remark that every $g \in \mathbf{G}^1$ is the quantile function of a measure μ_0, and μ_0 has a density p_0 with compact support equal to $[g(0),g(1)]$. The set $\mathbf{G}^j$ should be seen as the set of constraints on the initial condition of the SDE.

Let $g \in \mathbf{G}^1$. Let us fix M_0 an integer such that $M_0 > g(1)-g(0)$. We want to construct a solution to equation (II.13) starting at g, well-defined and continuous on the whole interval $[0,T]$. We will construct it on the basis of the family $(y^M)_{M\geqslant M_0}$ of solutions to equation (II.16) for each $M \geqslant M_0$. Since g belongs to $\mathbf{G}^1$, the assumptions made in Propositions II.2 and

II.4 and Corollary II.6 can be applied. Thus for every $u, v \in [0,1]$ and for every $t \in [0,T]$, $|y_t^M(u) - y_t^M(v)| \leqslant y_t^M(1) - y_t^M(0)$. For every $M, M' \geqslant M_0$, define

$$\tau_M(y^{M'}) := \inf\left\{t \geqslant 0 : y_t^{M'}(1) - y_t^{M'}(0) \geqslant M\right\} \wedge T.$$

Since $M > g(1) - g(0)$ and since the process $y_\cdot^{M'}(1) - y_\cdot^{M'}(0)$ is continuous, $\tau_M(y^{M'}) > 0$ almost surely for every $M, M' \geqslant M_0$. Assume that $M \leqslant M'$. Then for every $s \leqslant \tau_M(y^{M'})$, for every $u, v \in [0,1]$, $|y_s^{M'}(u) - y_s^{M'}(v)| \leqslant M \leqslant M'$ and thus

$$\varphi_{M'}(y_s^{M'}(u) - y_s^{M'}(v)) = \varphi(y_s^{M'}(u) - y_s^{M'}(v)) = \varphi_M(y_s^{M'}(u) - y_s^{M'}(v)).$$

Let $\sigma = \tau_M(y^M) \wedge \tau_M(y^{M'})$. We deduce from the latter equality that the processes $(y_{t\wedge\sigma}^M)_{t\in[0,T]}$ and $(y_{t\wedge\sigma}^{M'})_{t\in[0,T]}$ are both solutions to the same stochastic differential equation:

$$z_t(u) = g(u) + \int_0^{t\wedge\sigma} \frac{1}{m_s^z(u)^{1/2}} \left(\int_{\mathbb{R}} \cos(kz_s(u)) f(k) \mathrm{d}w^{\Re}(k,s) + \int_{\mathbb{R}} \sin(kz_s(u)) f(k) \mathrm{d}w^{\Im}(k,s) \right), \tag{II.19}$$

where $m_s^z(u) = \int_0^1 \varphi_M(z_s(u) - z_s(v))\mathrm{d}v$.

Assume that $k \mapsto \langle k \rangle f(k)$ is square integrable. Therefore, by pathwise uniqueness of the solution to equation (II.19), which follows from the same argument as in Proposition II.2, we have for all $u \in [0,1]$, for all $t \in [0,T]$, $y_{t\wedge\sigma}^M(u) = y_{t\wedge\sigma}^{M'}(u)$, whence $\tau_M(y^M) = \tau_M(y^{M'})$. From now on, we will denote that stopping time by τ_M. The sequence of stopping times $(\tau_M)_{M\geqslant 1}$ is non-decreasing.

Setting $\tau_{M_0-1} = 0$, we define $y_t(u) := \mathbb{1}_{\{t=0\}} g(u) + \sum_{M=M_0}^{+\infty} \mathbb{1}_{\{t\in(\tau_{M-1},\tau_M]\}} y_t^M(u)$ for every $t \in [0,T]$ and $u \in [0,1]$. Let $\tau_\infty := \sup_{M\geqslant M_0} \tau_M$. Clearly, $\tau_\infty > 0$ almost surely. Since $\tau_M \leqslant T$ for every $M \geqslant M_0$, we have $\tau_\infty \leqslant T$. Furthermore, for each $M \geqslant M_0$, $y = y^M$ on $[0, \tau_M]$ and on the interval $[0, \tau_\infty)$, $(y_t)_{t\in[0,T]}$ is solution to equation (II.13).

Let us remark that $\mathbb{P}$-almost surely, $u \mapsto y_t(u)$ is strictly increasing for every $t \in [0,T]$. Moreover, the following proposition states that it is the unique solution in $\mathcal{C}([0,1] \times [0,T])$ to equation (II.13).

Proposition II.8. *Let $g \in \mathbf{G}^1$. Assume that $k \mapsto \langle k \rangle f(k)$ is square integrable. There exists a unique solution y in $\mathcal{C}([0,1] \times [0,T])$ to equation* (II.13) *and this solution is defined on $[0,T]$. Furthermore, the process $(y_t)_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-adapted.*

Proof. First, we prove that τ_∞ defined above is almost surely equal to T. Let $M \geqslant M_0$. Let us estimate $\mathbb{P}[\tau_M < T]$. Define $z_t^M := y_t^M(1) - y_t^M(0)$. Then $(z_t^M)_{t\in[0,T]}$ is a continuous and square integrable local martingale on $[0,T]$ and thus there is a standard $\mathbb{P}$-Brownian motion β such that $z_t^M = g(1) - g(0) + \beta_{\langle z^M, z^M\rangle_t}$. Moreover, $\tau_M = \inf\{t : z_t^M \geqslant M\} \wedge T$. Under the event $\{\tau_M < T\}$, there is a random time $t_0 \in [0,T)$ such that $z_{t_0}^M \geqslant M$ whereas for all $t \in [0,T]$, $z_t^M > 0$ by Proposition II.5. Let us define the process $(\gamma_t)_{t\geqslant 0}$ by $\gamma_t := g(1) - g(0) + \beta_t$. Under the measure $\mathbb{P}$, it is a Brownian motion starting at $g(1) - g(0) \in (0, M)$. Moreover, under the event $\{\tau_M < T\}$, $(\gamma_t)_{t\geqslant 0}$ reaches the level M before it reaches the level 0. Therefore, $\mathbb{P}[\tau_M < T] \leqslant \mathbb{P}[(\gamma_t)_{t\geqslant 0} \text{ reaches } M \text{ before } 0] = \frac{g(1)-g(0)}{M}$. Since $\{\tau_M < T\}_{M\geqslant M_0}$ is a non-increasing sequence of events, we deduce that $\mathbb{P}\left[\bigcap_{\{M\geqslant M_0\}} \{\tau_M < T\}\right] = 0$. Thus $\mathbb{P}$-almost surely, there exists $M \geqslant M_0$ such that $\tau_M = T$, whence $y = y^M$. It follows that $\tau_\infty = T$ almost surely. Thus y is a continuous solution to equation (II.13) defined on $[0,T]$.

Let us now prove pathwise uniqueness. Let x^1 and x^2 be two solutions on $(\Omega, \mathcal{G}, \mathbb{P})$ to equation (II.13) in $\mathcal{C}([0,1] \times [0,T])$. Let $\varepsilon > 0$. For every $M \geqslant M_0$, let us define the following event: $A_M^i := \{\omega \in \Omega : \sup_{u\in[0,1], t\in[0,T]} |x_t^i(u)|(\omega) \leqslant \frac{M}{2}\}$, $i = 1, 2$. Let $A_M := A_M^1 \cap A_M^2$. The

sequence of events $(A_M)_{M\geqslant M_0}$ is non-decreasing and it follows from the fact that x^1 and x^2 are continuous that $\mathbb{P}\left[\bigcup_{M\geqslant M_0} A_M\right] = 1$. Thus there is M such that $\mathbb{P}[A_M] > 1-\varepsilon$.

Let M be such that $\mathbb{P}[A_M] > 1-\varepsilon$. Let $\tau_M^i := \inf\{t \geqslant 0 : x_t^i(1) - x_t^i(0) \geqslant M\} \wedge T$ and $\tau_M = \tau_M^1 \wedge \tau_M^2$. For $i = 1, 2$, the same argument as the one given in Corollary II.6 implies that almost surely for each $t \in [0,T]$, $u \mapsto x^i_{t\wedge\tau_M}(u)$ is strictly increasing. Therefore, under the event A_M, the equality $\tau_M = T$ holds. Moreover, the processes $(x^1_{t\wedge\tau_M})_{t\in[0,T]}$ and $(x^2_{t\wedge\tau_M})_{t\in[0,T]}$ satisfy equation (II.16) up to the same stopping time τ_M. By Proposition II.2, pathwise uniqueness holds for equation (II.16), so $\mathbb{P}\left[x^1_{\cdot\wedge\tau_M} \neq x^2_{\cdot\wedge\tau_M}\right] = 0$. In particular, $0 = \mathbb{P}\left[\{x^1_{\cdot\wedge\tau_M} \neq x^2_{\cdot\wedge\tau_M}\} \cap A_M\right] = \mathbb{P}\left[\{x^1 \neq x^2\} \cap A_M\right]$. It follows that $\mathbb{P}\left[x^1 \neq x^2\right] < \varepsilon$ for every $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\mathbb{P}\left[x^1 \neq x^2\right] = 0$ and pathwise uniqueness holds for (II.13). □

II.2.3 Higher regularity depending on the decay at infinity of f

In this paragraph, we discuss the regularity of $u \mapsto y_t(u)$ for a fixed $t \in [0,T]$, depending on the regularity of the initial condition g and the decay at infinity of f. Recall that in Introduction II.1 to this chapter, we announced that under suitable assumptions on f, the process $u \mapsto y_t(u)$ admits a density p_t and that the faster f decays to infinity, the higher regularity we will obtain on p_t.

Of course, if we assume that g belongs to $\mathcal{C}^\infty$ and that f decays faster to infinity than every polynomial, the result stated below insures that $u \mapsto y_t(u)$ and consequently $x \mapsto p_t(x)$ are $\mathcal{C}^\infty$-maps. Nevertheless, we want to have a precise idea of how the order of regularity of p_t grows with the order of regularity of g and the integrability of f, because we want to assume as low integrability as possible on f in order to implement a Fourier inversion in Section II.3.

We have seen in Proposition II.8 that the assumption $g \in \mathbf{G}^1$ implies that y is continuous with respect to u. By differentiating formally y with respect to u, we expect that the derivative of y is a solution to the following linear stochastic differential equation for every $u \in [0,1]$:

$$z_t(u) = g'(u) + \int_0^t z_s(u) \int_{\mathbb{R}} \phi^{\Re}(u,k,s) f(k) \mathrm{d}w^{\Re}(k,s) + \int_0^t z_s(u) \int_{\mathbb{R}} \phi^{\Im}(u,k,s) f(k) \mathrm{d}w^{\Im}(k,s), \quad \text{(II.20)}$$

where

$$\phi^{\Re}(u,k,s) := \frac{-k\sin(ky_s(u))}{(\int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v)^{1/2}} - \frac{\cos(ky_s(u)) \int_0^1 \varphi'(y_s(u) - y_s(v))\mathrm{d}v}{2(\int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v)^{3/2}}; \quad \text{(II.21)}$$

$$\phi^{\Im}(u,k,s) := \frac{k\cos(ky_s(u))}{(\int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v)^{1/2}} - \frac{\sin(ky_s(u)) \int_0^1 \varphi'(y_s(u) - y_s(v))\mathrm{d}v}{2(\int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v)^{3/2}}. \quad \text{(II.22)}$$

Recall definition II.7 of $\mathbf{G}^{j+\theta}$.

Proposition II.9. *Let $\theta \in (0,1)$. Let $g \in \mathbf{G}^{1+\theta}$. Assume that $k \mapsto \langle k\rangle^{1+\theta} f(k)$ is square integrable. Almost surely, for every $t \in [0,T]$, the map $u \mapsto y_t(u)$ belongs to $\mathbf{G}^{1+\theta'}$ for every $0 \leqslant \theta' < \theta$ and $(\partial_u y_t)_{t\in[0,T]}$ satisfies equation* (II.20)*. Moreover, the derivative has the following explicit form:*

$$\partial_u y_t(u) = g'(u) \exp\Bigg(\int_0^t \int_{\mathbb{R}} \phi^{\Re}(u,k,s) f(k) \mathrm{d}w^{\Re}(k,s) + \int_0^t \int_{\mathbb{R}} \phi^{\Im}(u,k,s) f(k) \mathrm{d}w^{\Im}(k,s) - \frac{1}{2} \int_0^t \int_{\mathbb{R}} (\phi^{\Re}(u,k,s)^2 + \phi^{\Im}(u,k,s)^2) f(k)^2 \mathrm{d}k\mathrm{d}s \Bigg). \quad \text{(II.23)}$$

More generally, if for an integer $j \geqslant 1$, g belongs to $\mathbf{G}^{j+\theta}$ and $k \mapsto \langle k\rangle^{j+\theta} f(k)$ is square integrable, then almost surely, for every $t \in [0,T]$, the map $u \mapsto y_t(u)$ belongs to $\mathbf{G}^{j+\theta'}$ for every $0 \leqslant \theta' < \theta$.

Remark II.10. Let us consider the case of $f_\alpha(k) = \frac{1}{\langle k\rangle^\alpha}$. The assumption $\langle k\rangle^{j+\theta} f_\alpha(k) \in L_2(\mathbb{R})$ is equivalent to the condition $\alpha > j + \theta + \frac{1}{2}$. If f is the Cauchy density $f(k) = \frac{1}{1+k^2}$, then the process $u \mapsto y_t(u)$ is differentiable and the derivative is θ'-Hölder continuous for every $\theta' < \frac{1}{2}$.

Remark II.11. By the property of monotonicity of y_t, we deduce that almost surely, for every $t \in [0,T]$, $u \mapsto \partial_u y_t(u) > 0$. Recall that the c.d.f. F_t associated to y_t is equal to $F_t = (y_t)^{-1}$ and that the density of p_t is the first derivative of F_t. Therefore, for every $u \in [0,1]$, $F_t(y_t(u)) = u$ and $p_t(y_t(u))\partial_u y_t(u) = 1$. Thus for every $x \in [y_t(0), y_t(1)]$,

$$p_t(x) = \frac{1}{\partial_u y_t(F_t(x))}.$$

It follows that p_t has the same regularity than $\partial_u y_t$. If $f(k) = \frac{1}{1+k^2}$, then p_t is θ'-Hölder continuous for every $\theta' < \frac{1}{2}$.

As we did to prove well-posedness, we will first consider the auxiliary equation, where φ is replaced by φ_M, so that the coefficient functions $\phi_M^{\Re}$ and $\phi_M^{\Im}$ are bounded.

Recall that M_0 is a fixed integer such that $M_0 > g(1) - g(0)$. Fix $M \geqslant M_0$. We consider the following equation

$$z_t^M(u) = g'(u) + \int_0^t z_s^M(u) \left(\int_{\mathbb{R}} \phi_M^{\Re}(u,k,s) f(k) \mathrm{d}w^{\Re}(k,s) + \int_{\mathbb{R}} \phi_M^{\Im}(u,k,s) f(k) \mathrm{d}w^{\Im}(k,s) \right), \tag{II.24}$$

where $\phi_M^{\Re}$ and $\phi_M^{\Im}$ are defined in analogy with (II.21) and (II.22) by:

$$\phi_M^{\Re}(u,k,s) := \frac{-k\sin(ky_s^M(u))}{(\int_0^1 \varphi_M(y_s^M(u) - y_s^M(v))\mathrm{d}v)^{1/2}} - \frac{\cos(ky_s^M(u)) \int_0^1 \varphi_M'(y_s^M(u) - y_s^M(v))\mathrm{d}v}{2(\int_0^1 \varphi_M(y_s^M(u) - y_s^M(v))\mathrm{d}v)^{3/2}};$$
$$\phi_M^{\Im}(u,k,s) := \frac{k\cos(ky_s^M(u))}{(\int_0^1 \varphi_M(y_s^M(u) - y_s^M(v))\mathrm{d}v)^{1/2}} - \frac{\sin(ky_s^M(u)) \int_0^1 \varphi_M'(y_s^M(u) - y_s^M(v))\mathrm{d}v}{2(\int_0^1 \varphi_M(y_s^M(u) - y_s^M(v))\mathrm{d}v)^{3/2}}.$$

The map φ_M' is well-defined and bounded on $\mathbb{R}\backslash\{-M, M\}$. Because $u \mapsto y_s(u)$ is strictly increasing, the Lebesgue measure of $\{v \in [0,1] : y_s(u) - y_s(v) = -M \text{ or } M\}$ is equal to 0. Furthermore, φ_M is bounded from below by $\varphi(M)$. Therefore, the functions $\frac{1}{k}\phi_M^{\Re}$ and $\frac{1}{k}\phi_M^{\Im}$ are bounded and there is a unique solution to equation (II.24), because $k \mapsto \langle k\rangle f(k)$ is square integrable. The solution is given by an explicit formula, similar to (II.23).

Recall the definition of $\tau_M = \inf\{t \geqslant 0 : y_t(1) - y_t(0) \geqslant M\} \wedge T$ for every $M \geqslant M_0$ and of $\tau_{M_0-1} = 0$. Thus, if we define $z_t(u) := \mathbb{1}_{\{t=0\}} g'(u) + \sum_{M=M_0}^{+\infty} \mathbb{1}_{\{t \in (\tau_{M-1}, \tau_M]\}} z_t^M(u)$, z is solution in $\mathcal{C}([0,T], L_2(0,1))$ to equation (II.20). In order to prove Proposition II.9, we first prove the following two lemmas.

Lemma II.12. *Let $g \in \mathbf{G}^{1+\theta}$. Assume that $k \mapsto \langle k\rangle^{1+\theta} f(k)$ is square integrable. For every $M \geqslant M_0$ and for every $u \in [0,1]$, we have $\frac{y_{\cdot\wedge\tau_M}(u+\varepsilon) - y_{\cdot\wedge\tau_M}(u)}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} z_{\cdot\wedge\tau_M}(u)$ in $L_2(\Omega, \mathcal{C}[0,T])$.*

Proof. Let $u \in [0,1]$. As previously, denote by $m_s(u) := \int_0^1 \varphi(y_s(u) - y_s(v))\mathrm{d}v$. Recall that

$$\begin{aligned}
&z_{t\wedge\tau_M}(u) = g'(u) \\
&+ \int_0^{t\wedge\tau_M} \frac{z_s(u)}{m_s(u)^{1/2}} \int_{\mathbb{R}} kf(k) \Big[-\sin(ky_s(u))\mathrm{d}w^{\Re}(k,s) + \cos(ky_s(u))\mathrm{d}w^{\Im}(k,s) \Big] \\
&- \int_0^{t\wedge\tau_M} \frac{z_s(u) \int_0^1 \varphi'(y_s(u) - y_s(v))\mathrm{d}v}{2m_s(u)^{3/2}} \int_{\mathbb{R}} f(k) \Big[\cos(ky_s(u))\mathrm{d}w^{\Re}(k,s) + \sin(ky_s(u))\mathrm{d}w^{\Im}(k,s) \Big].
\end{aligned}$$

Let $\varepsilon \neq 0$ such that $u+\varepsilon \in (0,1)$. Therefore, we have:

$$\begin{aligned}\frac{y_{t\wedge\tau_M}(u+\varepsilon)-y_{t\wedge\tau_M}(u)}{\varepsilon} &= \frac{g(u+\varepsilon)-g(u)}{\varepsilon}\\ &+\int_0^{t\wedge\tau_M}\frac{1}{m_s(u+\varepsilon)^{1/2}}\int_{\mathbb{R}} f(k)\Bigg[\frac{\cos(ky_s(u+\varepsilon))-\cos(ky_s(u))}{\varepsilon}\mathrm{d}w^{\Re}(k,s)\\ &\qquad\qquad +\frac{\sin(ky_s(u+\varepsilon))-\sin(ky_s(u))}{\varepsilon}\mathrm{d}w^{\Im}(k,s)\Bigg]\\ &+\int_0^{t\wedge\tau_M}\frac{1}{\varepsilon}\left(\frac{1}{m_s(u+\varepsilon)^{\frac12}}-\frac{1}{m_s(u)^{\frac12}}\right)\int_{\mathbb{R}} f(k)\Big[\cos(ky_s(u))\mathrm{d}w^{\Re}(k,s)+\sin(ky_s(u))\mathrm{d}w^{\Im}(k,s)\Big].\end{aligned} \tag{II.25}$$

By Burkholder-Davis-Gundy inequality, there is a constant C such that

$$\mathbb{E}\left[\sup_{s\leqslant t}\left|\frac{y_{s\wedge\tau_M}(u+\varepsilon)-y_{s\wedge\tau_M}(u)}{\varepsilon}-z_{s\wedge\tau_M}(u)\right|^2\right]\leqslant C\sum_{i=1}^4 E_i, \tag{II.26}$$

where

$$\begin{aligned}E_1 &:= \left|\frac{g(u+\varepsilon)-g(u)}{\varepsilon}-g'(u)\right|^2;\\ E_2 &:= \mathbb{E}\left[\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} f(k)^2\left|\frac{\cos(ky_s(u+\varepsilon))-\cos(ky_s(u))}{\varepsilon\, m_s(u+\varepsilon)^{1/2}}+\frac{z_s(u)}{m_s(u)^{1/2}}k\sin(ky_s(u))\right|^2\mathrm{d}k\mathrm{d}s\right];\\ E_3 &:= \mathbb{E}\left[\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} f(k)^2\left|\frac{\sin(ky_s(u+\varepsilon))-\sin(ky_s(u))}{\varepsilon\, m_s(u+\varepsilon)^{1/2}}-\frac{z_s(u)}{m_s(u)^{1/2}}k\cos(ky_s(u))\right|^2\mathrm{d}k\mathrm{d}s\right];\\ E_4 &:= \mathbb{E}\left[\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} f(k)^2\left|\frac{1}{\varepsilon}\left(\frac{1}{m_s(u+\varepsilon)^{\frac12}}-\frac{1}{m_s(u)^{\frac12}}\right)+\frac{z_s(u)}{2m_s(u)^{\frac32}}\int_0^1\varphi'(y_s(u)-y_s(v))\mathrm{d}v\right|^2\mathrm{d}k\mathrm{d}s\right].\end{aligned}$$

To obtain E_4, we used the fact that $\cos(ky_s(u))^2+\sin(ky_s(u))^2=1$.

Study of E_1. Recall that $g\in\mathbf{G}^{1+\theta}$. It follows that

$$E_1=\left|\int_0^1 g'(u+\lambda\varepsilon)\mathrm{d}\lambda-g'(u)\right|^2\leqslant\int_0^1|g'(u+\lambda\varepsilon)-g'(u)|^2\mathrm{d}\lambda\leqslant C\|g\|_{1+\theta}^2|\varepsilon|^{2\theta},$$

where $\|g\|_{1+\theta}=\|g\|_{L_\infty}+\|g'\|_{L_\infty}+\sup_{u\neq v}\frac{|g'(u)-g'(v)|}{|u-v|^\theta}$.

Study of E_2. Furthermore

$$\frac{\cos(ky_s(u+\varepsilon))-\cos(ky_s(u))}{\varepsilon}=-\frac{y_s(u+\varepsilon)-y_s(u)}{\varepsilon}\int_0^1 k\sin(k\lambda y_s(u+\varepsilon)+(1-\lambda)ky_s(u))\mathrm{d}\lambda.$$

Henceforth, denoting $\theta_s(\lambda)=\lambda y_s(u+\varepsilon)+(1-\lambda)y_s(u)$, we have

$$E_2\leqslant 3\mathbb{E}\left[\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} f(k)^2\frac{|\int_0^1 k\sin(k\theta_s(\lambda))\mathrm{d}\lambda|^2}{m_s(u+\varepsilon)}\left|\frac{y_s(u+\varepsilon)-y_s(u)}{\varepsilon}-z_s(u)\right|^2\mathrm{d}k\mathrm{d}s\right] \tag{II.27}$$

$$+3\mathbb{E}\left[\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} f(k)^2\frac{z_s(u)^2}{m_s(u+\varepsilon)}\left|\int_0^1 k\sin(k\theta_s(\lambda))\mathrm{d}\lambda-k\sin(ky_s(u))\right|^2\mathrm{d}k\mathrm{d}s\right] \tag{II.28}$$

$$+3\mathbb{E}\left[\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} f(k)^2\left|z_s(u)k\sin(ky_s(u))\left(\frac{1}{m_s(u+\varepsilon)^{1/2}}-\frac{1}{m_s(u)^{1/2}}\right)\right|^2\mathrm{d}k\mathrm{d}s\right]. \tag{II.29}$$

Let us first remark that for every $s \leqslant \tau_M$, m_s is bounded from below by $\varphi(M)$. Moreover, in order to give an upper bound to (II.28), we note that there is $C > 0$ such that for every $x, y \in \mathbb{R}$, $|\sin(x) - \sin(y)| \leqslant C|x-y|^\theta$. Thus we have,

$$\left| \int_0^1 k \sin(k\theta_s(\lambda))\mathrm{d}\lambda - k\sin(ky_s(u)) \right| \leqslant C|k|^{1+\theta}|y_s(u+\varepsilon) - y_s(u)|^\theta.$$

In order to estimate (II.29), remark that for every $s \in [0,T]$,

$$\begin{aligned}
\mathbb{1}_{\{s\leqslant\tau_M\}} \left| \frac{1}{m_s(u+\varepsilon)^{\frac{1}{2}}} - \frac{1}{m_s(u)^{\frac{1}{2}}} \right| &= \mathbb{1}_{\{s\leqslant\tau_M\}} \frac{|m_s(u+\varepsilon) - m_s(u)|}{m_s(u+\varepsilon)^{\frac{1}{2}} m_s(u)^{\frac{1}{2}} (m_s(u+\varepsilon)^{\frac{1}{2}} + m_s(u)^{\frac{1}{2}})} \\
&\leqslant \frac{\mathbb{1}_{\{s\leqslant\tau_M\}}}{2\varphi(M)^{\frac{3}{2}}} \int_0^1 |\varphi(y_s(u+\varepsilon) - y_s(v)) - \varphi(y_s(u) - y_s(v))|\mathrm{d}v \\
&\leqslant \frac{\sup_{x\in[-M,M]} |\varphi'(x)|}{2\varphi(M)^{\frac{3}{2}}} \mathbb{1}_{\{s\leqslant\tau_M\}} |y_s(u+\varepsilon) - y_s(u)|.
\end{aligned}$$

It follows from (II.27) that

$$\begin{aligned}
E_2 \leqslant C_M \int_{\mathbb{R}} \langle k\rangle^2 f(k)^2 \mathrm{d}k \int_0^t \mathbb{E}\left[\sup_{r\leqslant s} \left| \frac{y_{r\wedge\tau_M}(u+\varepsilon) - y_{r\wedge\tau_M}(u)}{\varepsilon} - z_{r\wedge\tau_M}(u) \right|^2 \right] \mathrm{d}s \\
+ C_M \int_{\mathbb{R}} \langle k\rangle^{2+2\theta} f(k)^2 \mathrm{d}k \; E_{2,1} + C_M \int_{\mathbb{R}} \langle k\rangle^2 f(k)^2 \mathrm{d}k \; E_{2,2},
\end{aligned} \tag{II.30}$$

where

$$\begin{aligned}
E_{2,1} &:= \mathbb{E}\left[\int_0^{t\wedge\tau_M} |z_s(u)|^2 |y_s(u+\varepsilon) - y_s(u)|^{2\theta} \mathrm{d}s \right]; \\
E_{2,2} &:= \mathbb{E}\left[\int_0^{t\wedge\tau_M} |z_s(u)|^2 |y_s(u+\varepsilon) - y_s(u)|^2 \mathrm{d}s \right].
\end{aligned}$$

By assumption on f, $\int_{\mathbb{R}} \langle k\rangle^2 f(k)^2 \mathrm{d}k$ and $\int_{\mathbb{R}} \langle k\rangle^{2+2\theta} f(k)^2 \mathrm{d}k$ are finite. Let $p \geqslant 2$. By Burkholder-Davis-Gundy's inequality,

$$\begin{aligned}
\mathbb{E}\left[|z_{t\wedge\tau_M}(u)|^p\right] &\leqslant C_p |g'(u)|^p + C_M \left| \int_{\mathbb{R}} \langle k\rangle^2 f(k)^2 \mathrm{d}k \right|^{p/2} \mathbb{E}\left[\left| \int_0^{t\wedge\tau_M} |z_s(u)|^2 \mathrm{d}s \right|^{p/2} \right] \\
&\leqslant C_{M,p} \|g'\|_{L_\infty}^p + C_M T^{\frac{p}{2}-1} \int_0^t \mathbb{E}\left[|z_{s\wedge\tau_M}(u)|^p\right] \mathrm{d}s.
\end{aligned}$$

Similarly,

$$\mathbb{E}\left[|y_{t\wedge\tau_M}(u+\varepsilon) - y_{t\wedge\tau_M}(u)|^p\right] \leqslant C_p \|g'\|_{L_\infty}^p |\varepsilon|^p + C_{M,p} T^{\frac{p}{2}-1} \int_0^t \mathbb{E}\left[|y_{s\wedge\tau_M}(u+\varepsilon) - y_{s\wedge\tau_M}(u)|^p\right] \mathrm{d}s.$$

Thus, it follows from Gronwall's inequality that for every $p \geqslant 2$, there is $C_{M,p}$ independent of t such that

$$\mathbb{E}\left[|z_{t\wedge\tau_M}(u)|^p\right] \leqslant C_{M,p}; \tag{II.31}$$

$$\mathbb{E}\left[|y_{t\wedge\tau_M}(u+\varepsilon) - y_{t\wedge\tau_M}(u)|^p\right] \leqslant C_{M,p}|\varepsilon|^p. \tag{II.32}$$

By Hölder's inequality,

$$E_{2,2} \leqslant \mathbb{E}\left[\int_0^{t\wedge\tau_M} |z_s(u)|^{\frac{2}{1-\theta}} \mathrm{d}s \right]^{1-\theta} \mathbb{E}\left[\int_0^{t\wedge\tau_M} |y_s(u+\varepsilon) - y_s(u)|^2 \mathrm{d}s \right]^{\theta} \leqslant C_M |\varepsilon|^{2\theta}.$$

Similarly, $E_{2,1} \leqslant C_{M,p}|\varepsilon|^2$. Finally, it follows from (II.30) that

$$E_2 \leqslant C_M \int_0^t \mathbb{E}\left[\sup_{r\leqslant s}\left|\frac{y_{r\wedge\tau_M}(u+\varepsilon)-y_{r\wedge\tau_M}(u)}{\varepsilon}-z_{r\wedge\tau_M}(u)\right|^2\right]\mathrm{d}s + C_M|\varepsilon|^{2\theta}.$$

Study of E_3. It is analog to E_2.

Study of E_4. Let us give an estimate for E_4. We have

$$\frac{1}{\varepsilon}\left(\frac{1}{m_s(u+\varepsilon)^{1/2}}-\frac{1}{m_s(u)^{1/2}}\right) = -\frac{y_s(u+\varepsilon)-y_s(u)}{\varepsilon}\int_0^1 \frac{\int_0^1 \varphi'(x_\lambda - y_s(v))\mathrm{d}v}{2(\int_0^1\varphi(x_\lambda-y_s(v))\mathrm{d}v)^{3/2}}\mathrm{d}\lambda,$$

where $x_\lambda = \lambda y_s(u+\varepsilon)+(1-\lambda)y_s(u)$. We have to compare the latter term, for each time $s \leqslant \tau_M$, to

$$-\frac{z_s(u)}{2m_s(u)^{3/2}}\int_0^1 \varphi'(y_s(u)-y_s(v))\mathrm{d}v.$$

Using (II.31) and (II.32), we obtain by the same computations as for E_2:

$$\begin{aligned} E_4 &\leqslant C_M \int_0^t \mathbb{E}\left[\sup_{r\leqslant s}\left|\frac{y_{r\wedge\tau_M}(u+\varepsilon)-y_{r\wedge\tau_M}(u)}{\varepsilon}-z_{r\wedge\tau_M}(u)\right|^2\right]\mathrm{d}s + C_M\int_{\mathbb{R}} f(k)^2\mathrm{d}k\; E_{2,1} \\ &\leqslant C_M \int_0^t \mathbb{E}\left[\sup_{r\leqslant s}\left|\frac{y_{r\wedge\tau_M}(u+\varepsilon)-y_{r\wedge\tau_M}(u)}{\varepsilon}-z_{r\wedge\tau_M}(u)\right|^2\right]\mathrm{d}s + C_M|\varepsilon|^2. \end{aligned}$$

Back to (II.26), there is a constant C_M such that:

$$\begin{aligned} \mathbb{E}\left[\sup_{s\leqslant t}\left|\frac{y_{s\wedge\tau_M}(u+\varepsilon)-y_{s\wedge\tau_M}(u)}{\varepsilon}-z_{s\wedge\tau_M}(u)\right|^2\right] &\leqslant C_M|\varepsilon|^{2\theta} \\ &+ C_M\int_0^t \mathbb{E}\left[\sup_{r\leqslant s}\left|\frac{y_{r\wedge\tau_M}(u+\varepsilon)-y_{r\wedge\tau_M}(u)}{\varepsilon}-z_{r\wedge\tau_M}(u)\right|^2\right]\mathrm{d}s. \end{aligned}$$

It follows from Gronwall's Lemma that:

$$\mathbb{E}\left[\sup_{t\leqslant T}\left|\frac{y_{t\wedge\tau_M}(u+\varepsilon)-y_{t\wedge\tau_M}(u)}{\varepsilon}-z_{t\wedge\tau_M}(u)\right|^2\right] \leqslant C_M|\varepsilon|^{2\theta},$$

whence we deduce the lemma. □

Lemma II.13. *Under the same assumptions as Lemma II.12, almost surely, for all $M \geqslant M_0$, for all $u \in [0,1]$, for all $\varepsilon \neq 0$ such that $u+\varepsilon \in (0,1)$ and for all $t \in [0,T]$,*

$$\frac{y_{t\wedge\tau_M}(u+\varepsilon)-y_{t\wedge\tau_M}(u)}{\varepsilon} = \int_0^1 z_{t\wedge\tau_M}(u+\lambda\varepsilon)\mathrm{d}\lambda.$$

Proof. We use the following basic result: if E is a Banach space and if $f, g : [0,1] \to E$ satisfy that

i) for all $v \in [0,1]$, $\frac{f(v+\varepsilon)-f(v)}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} g(v)$;

ii) g is continuous;

then for all $v \in [0,1]$, and for all $\varepsilon \neq 0$ such that $v+\varepsilon \in (0,1)$, $\frac{f(v+\varepsilon)-f(v)}{\varepsilon} = \int_0^1 g(v+\lambda\varepsilon)\mathrm{d}\lambda$.

Let us apply this result to $E = L_2(\Omega, \mathcal{C}[0,T])$, $f : u \mapsto y_{\cdot\wedge\tau_M}(u)$ and $g = z_{\cdot\wedge\tau_M}$. The point $i)$ is a consequence of Lemma II.12. It remains to prove $ii)$, that is the continuity of z^M. We have:

$$\begin{aligned}
z_{t\wedge\tau_M}(u+h) - z_{t\wedge\tau_M}(u) &= g'(u+h) - g'(u) \\
&+ \int_0^{t\wedge\tau_M} (z_s(u+h) - z_s(u)) \int_{\mathbb{R}} f(k)\left[\phi^{\Re}(u+h,k,s)\mathrm{d}w^{\Re}(k,s) + \phi^{\Im}(u+h,k,s)\mathrm{d}w^{\Im}(k,s)\right] \\
&+ \int_0^{t\wedge\tau_M} z_s(u) \int_{\mathbb{R}} f(k)\Big[(\phi^{\Re}(u+h,k,s) - \phi^{\Re}(u,k,s))\mathrm{d}w^{\Re}(k,s) \\
&\qquad + (\phi^{\Im}(u+h,k,s) - \phi^{\Im}(u,k,s))\mathrm{d}w^{\Im}(k,s)\Big].
\end{aligned}$$

First, since $g \in \mathbf{G}^{1+\theta}$, $|g'(u+h) - g'(u)| \leqslant \|g\|_{1+\theta}|h|^\theta$. Moreover, there is C_M such that the following inequalities hold with every u, $u+h \in [0,1]$, for every $k \in \mathbb{R}$ and for every $s \leqslant \tau_M$:

$$\begin{aligned}
\mathbb{1}_{\{s\leqslant\tau_M\}}\, |\phi^{\Re}(u,k,s)| &\leqslant C_M\langle k\rangle; \\
\mathbb{1}_{\{s\leqslant\tau_M\}}\, |\phi^{\Im}(u,k,s)| &\leqslant C_M\langle k\rangle; \\
\mathbb{1}_{\{s\leqslant\tau_M\}}\, |\phi^{\Re}(u+h,k,s) - \phi^{\Re}(u,k,s)| &\leqslant C_M\langle k\rangle^{1+\theta}\, \mathbb{1}_{\{s\leqslant\tau_M\}}\, |y_s(u+h) - y_s(u)|^\theta; \\
\mathbb{1}_{\{s\leqslant\tau_M\}}\, |\phi^{\Im}(u+h,k,s) - \phi^{\Im}(u,k,s)| &\leqslant C_M\langle k\rangle^{1+\theta}\, \mathbb{1}_{\{s\leqslant\tau_M\}}\, |y_s(u+h) - y_s(u)|^\theta.
\end{aligned}$$

Thus by Burkholder-Davis-Gundy inequality, for every $p \geqslant 2$

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\leqslant t} |z_{s\wedge\tau_M}(u+h) - z_{s\wedge\tau_M}(u)|^p\right] &\leqslant C_M|h|^{p\theta} \\
&+ C_M\left|\int_{\mathbb{R}} \langle k\rangle^2 f(k)^2\mathrm{d}k\right|^{p/2} \int_0^t \mathbb{E}\left[\sup_{r\leqslant s} |z_{r\wedge\tau_M}(u+h) - z_{r\wedge\tau_M}(u)|^p\right]\mathrm{d}s \\
&+ C_M\left|\int_{\mathbb{R}} \langle k\rangle^{2+2\theta} f(k)^2\mathrm{d}k\right|^{p/2} \mathbb{E}\left[\int_0^{t\wedge\tau_M} |z_s(u)|^p |y_s(u+h) - y_s(u)|^{p\theta}\mathrm{d}s\right].
\end{aligned} \tag{II.33}$$

By assumption on f, $\int_{\mathbb{R}}\langle k\rangle^2 f(k)^2\mathrm{d}k$ and $\int_{\mathbb{R}}\langle k\rangle^{2+2\theta} f(k)^2\mathrm{d}k$ are finite. Moreover, by Hölder's inequality and by (II.31) and (II.32),

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge\tau_M} |z_s(u)|^p |y_s(u+h) - y_s(u)|^{p\theta}\mathrm{d}s\right] \\
&\qquad \leqslant \mathbb{E}\left[\int_0^{t\wedge\tau_M} |z_s(u)|^{\frac{p}{1-\theta}}\mathrm{d}s\right]^{1-\theta} \mathbb{E}\left[\int_0^{t\wedge\tau_M} |y_s(u+h) - y_s(u)|^p\mathrm{d}s\right]^{\theta} \leqslant C_M|h|^{p\theta}.
\end{aligned}$$

Thus, by Gronwall's Lemma applied to (II.33), for all $u \in [0,1]$ and all h such that $u+h \in (0,1)$,

$$\mathbb{E}\left[\sup_{t\leqslant T} |z_{t\wedge\tau_M}(u+h) - z_{t\wedge\tau_M}(u)|^p\right] \leqslant C_M|h|^{p\theta}. \tag{II.34}$$

Let p be large enough so that $p\theta > 1$. By Kolmogorov's Lemma [RY99, Theorem I.2.1, p.26], we deduce that there is a modification of $z_{t\wedge\tau_M}$ which is γ-Hölder continuous for every $\gamma < \theta - \frac{1}{p}$. Taking p large enough, we deduce that $z_{t\wedge\tau_M}$ is θ'-Hölder continuous for every $\theta' < \theta$. In particular, it is continuous. Hence the point $ii)$ is satisfied.

We deduce that the following equality holds in $L_2(\Omega, \mathcal{C}[0,T])$:

$$\frac{y_{\cdot\wedge\tau_M}(u+\varepsilon) - y_{\cdot\wedge\tau_M}(u)}{\varepsilon} = \int_0^1 z_{\cdot\wedge\tau_M}(u+\lambda\varepsilon)\mathrm{d}\lambda. \tag{II.35}$$

Therefore, almost surely, equality (II.35) holds with every $u \in \mathbb{Q} \cap [0,1]$ and for every $M \geqslant M_0$. Furthermore, $u \in [0,1] \mapsto y_{\cdot \wedge \tau_M}(u) \in \mathcal{C}[0,T]$ is continuous on $[0,1]$ by Proposition II.4 and we have just seen that the same statement is true for $u \mapsto z_{\cdot \wedge \tau_M}(u)$. Hence equality (II.35) holds almost surely for every $u \in [0,1]$, for each $M \in \mathbb{N} \backslash \{0\}$. □

Proof (Proposition II.9). Under the event $\{\tau_M = T\}$, $y_{\cdot \wedge \tau_M} = y$ and $z_{\cdot \wedge \tau_M} = z$. It follows that under $\{\tau_M = T\}$, for all $u \in [0,1]$, for all $\varepsilon \neq 0$ such that $u + \varepsilon \in (0,1)$ and for all $t \in [0,T]$,

$$\frac{y_t(u+\varepsilon) - y_t(u)}{\varepsilon} = \int_0^1 z_t(u + \lambda\varepsilon) \mathrm{d}\lambda. \tag{II.36}$$

Recall that by Proposition II.8, $\mathbb{P}\left[\bigcup_{M \geqslant M_0} \{\tau_M = T\}\right] = 1$. Therefore (II.36) holds almost surely. Since z belongs to $\mathcal{C}([0,1] \times [0,T])$, it is equal to the derivative of y.

The case of higher derivatives can be easily inferred by similar computations. Since no new idea is needed there, the proof will not be detailled. □

II.3 Fourier inversion with respect to the interaction kernel

In this section, we will introduce an important tool to determine the inverse function of the drift term with respect to the interaction kernel previously defined.

More precisely, the aim of this chapter is to prove well-posedness of the following SDE:

$$dz_t(u) = \frac{1}{m_t(u)^{1/2}} \int_{\mathbb{R}} f(k) \left(\cos(kz_t(u)) \mathrm{d}w^{\Re}(k,t) + \sin(kz_t(u)) \mathrm{d}w^{\Im}(k,t) \right) + b(z_t(u), \mu_t) \mathrm{d}t, \tag{II.37}$$

where $m_t(u) = \int_0^1 \varphi(z_t(u) - z_t(v)) \mathrm{d}v$, where f, φ, $w^{\Re}$ and $w^{\Im}$ are defined as in Section II.2, where $\mu_t = \mathrm{Leb}\,|_{[0,1]} \circ z_t^{-1}$ and where $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is a drift as general as possible. If the drift b is equal to zero, this equation is exactly the one which was studied in Section II.2. The question is now: do we keep well-posedness of this diffusion if we add a drift term with low regularity? Typically, we will assume only measurability and boundedness of b with respect to the measure variable, whereas we will assume that b is at least $\mathcal{C}^2$ with respect to its first variable.

Let us write the associated SPDE satisfied by the process $(\mu_t)_{t \in [0,T]}$. Let $F \in \mathcal{C}^2(\mathbb{R}; \mathbb{R})$. Denote by $C_f := \int_{\mathbb{R}} f(k)^2 \mathrm{d}k$. For every $u \in [0,1]$, we have by Itô's formula

$$\begin{aligned} \mathrm{d}F(z_t(u)) = F'(z_t(u)) \, \frac{1}{m_t(u)^{1/2}} \int_{\mathbb{R}} f(k) \left(\cos(kz_t(u)) \mathrm{d}w^{\Re}(k,t) + \sin(kz_t(u)) \mathrm{d}w^{\Im}(k,t) \right) \\ + F'(z_t(u)) \, b(z_t(u), \mu_t) \mathrm{d}t + \frac{1}{2} F''(z_t(u)) \, \frac{C_f}{m_t(u)} \mathrm{d}t. \end{aligned}$$

Moreover,

$$m_t(u) = \int_0^1 \varphi(z_t(u) - z_t(v)) \mathrm{d}v = \int_{\mathbb{R}} \varphi(z_t(u) - x) \mathrm{d}\mu_t(x) = (\varphi * \mu_t)(z_t(u)).$$

Integrating over $u \in [0,1]$, and using the relation $\int_0^1 F(z_t(u)) \mathrm{d}u = \int_{\mathbb{R}} F(x) \mathrm{d}\mu_t(x)$ for F and its derivatives, we deduce that

$$\begin{aligned} \mathrm{d}_t \left(\int_{\mathbb{R}} F(x) \mathrm{d}\mu_t(x) \right) = \int_{\mathbb{R}} \frac{F'(x)}{(\varphi * \mu_t)(x)^{1/2}} \int_{\mathbb{R}} f(k) \left(\cos(kx) \mathrm{d}w^{\Re}(k,t) + \sin(kx) \mathrm{d}w^{\Im}(k,t) \right) \mathrm{d}\mu_t(x) \\ + \int_{\mathbb{R}} F'(x) b(x, \mu_t) \mathrm{d}\mu_t(x) \mathrm{d}t + \frac{1}{2} \int_{\mathbb{R}} F''(x) \frac{C_f}{(\varphi * \mu_t)(x)} \mathrm{d}\mu_t(x) \mathrm{d}t. \end{aligned}$$

By integration by parts, we deduce that $(\mu_t)_{t\in[0,T]}$ satisfies the following SPDE:

$$\mathrm{d}\mu_t = \frac{C_f}{2}\,\partial^2_{xx}\left(\frac{\mu_t}{\varphi * \mu_t}\right)\mathrm{d}t - \partial_x\Big(b(\cdot,\mu_t)\,\mu_t\Big)\mathrm{d}t - \partial_x\left(\frac{1}{(\varphi*\mu_t)^{1/2}}\int_{\mathbb{R}} f(k)\Re\left(e^{-ik\,\cdot}\,\mathrm{d}w(k,t)\right)\,\mu_t\right). \tag{II.38}$$

The main result of this section will be Lemma II.18. We will construct the inverse process $(h_t)_{t\in[0,T]}$ described in (II.9) and we will also specify the relation between the regularity of the drift term B and the integrability of the function f. If B has low regularity, we will only be able to proceed to this inversion for functions f with slow decay at infinity.

II.3.1 Assumptions on the drift function

We will denote by $B : \mathbb{R}\times L_2[0,1]\to\mathbb{R}$ the function defined by $B(x,z) := b(x, \mathrm{Leb}\,|_{[0,1]}\circ z^{-1})$. For every $\mu\in\mathcal{P}_2(\mathbb{R})$ with compact support, we denote by $|\,\mathrm{Supp}\,\mu|$ the Lebesgue measure of the support of μ.

Let us define the assumptions that we will make on b.

Definition II.14. A measurable function $b : \mathbb{R}\times\mathcal{P}_2(\mathbb{R})\to\mathbb{R}$ is said to satisfy the b-hypotheses of order $j\in\mathbb{N}\backslash\{0\}$ if:

- for every $\mu\in\mathcal{P}_2(\mathbb{R})$, $x\mapsto b(x,\mu)$ is continuous and j-times differentiable on $\mathbb{R}$;
- for every $i\in\{0,1,\dots,j\}$, there is a sequence $(C_i(M))_{M\geqslant M_0}$ such that $\left|\partial_1^{(i)}b(x,\mu)\right|\leqslant C_i(M)$ for every $x\in\mathbb{R}$ and for every $\mu\in\mathcal{P}_2(\mathbb{R})$ with compact support satisfying $|\,\mathrm{Supp}\,\mu|\leqslant M$.
- the sequence $(C_0(M))_{M\geqslant M_0}$ satisfies $\frac{C_0(M)}{M}\underset{M\to+\infty}{\longrightarrow}0$.

We say that $B : \mathbb{R}\times L_2[0,1]\to\mathbb{R}$ satisfies the B-hypotheses of order j if the associated b satisfies the b-hypotheses of order j.

Let us remark that, if $z\in L_2[0,1]$ is a continuous and strictly increasing function, the support of $\mu = \mathrm{Leb}\,|_{[0,1]}\circ z^{-1}$ is equal to $z(1)-z(0)$. The assumption $\frac{C_0(M)}{M}\underset{M\to+\infty}{\longrightarrow}0$ will only be necessary in the proof of existence of a weak solution to (II.37) (see Theorem II.33), in order to insure that under the new measure given by Girsanov's Theorem, there exists almost surely a (random) level M so that the diameter of the cloud of particles remains bounded by M for every $t\in[0,T]$.

Remark II.15. Let us give a few types of examples of admissible drift functions:

1. Let $b(x,\mu) := \int_{\mathbb{R}} a(x,y)\mathrm{d}\mu(y)$ or equivalently $B(x,z) := \int_0^1 a(x,z(u))\mathrm{d}u$. If $a:\mathbb{R}^2\to R$ is bounded and $x\mapsto a(x,y)$ is j-times differentiable with bounded derivatives, then b satisfies the b-hypotheses of order j. For instance, $b(x,\mu) = \int_{\mathbb{R}}(x-y)^2\rho(x-y)\mathrm{d}\mu(y)$, where ρ is a $\mathcal{C}^\infty$-function with compact support equal to 1 on $[-1,1]$.
2. Let $b(x,\mu) := a(x)\mathbb{V}\mathrm{ar}\,[Y]^\eta$, where the law of Y is μ and $\eta<\frac{1}{2}$. If a is bounded and j-times differentiable with bounded derivatives, then b satisfies the b-hypotheses of order j. Indeed, if μ has a compact support with $|\,\mathrm{Supp}\,\mu|\leqslant M$, then $\mathbb{V}\mathrm{ar}\,[Y]^\eta\leqslant M^{2\eta}$; thus $\frac{C_0(M)}{M} = \|a_0\|_{L_\infty}\frac{M^{2\eta}}{M}$. For instance, we can take $b(x,\mu) := \frac{\mathbb{V}\mathrm{ar}[Y]^{1//4}}{1+x^2}$ or $b(x,\mu) = e^{-x^2}\mathbb{V}\mathrm{ar}\,[Y]^{1/4}$.
3. Let $b(x,\mu) = a(x,\mathbb{E}\,[\varphi(Y)])$, where the law of Y is μ. If a is bounded and j-times differentiable with bounded derivatives and if $\varphi(Y)$ is bounded and measurable, then b satisfies the b-hypotheses of order j.

4. Let $b(x,\mu) = a\left(\int_{\mathbb{R}}(x-y)\mathrm{d}\mu(y)\right)$. If a is bounded and j-times differentiable with bounded derivatives, then b satisfies the b-hypotheses of order j. Indeed, for every $i \in \mathbb{N}$, we have $\partial_1^{(i)} b(x,\mu) = a^{(i)}\left(\int_{\mathbb{R}}(x-y)\mathrm{d}\mu(y)\right)$. Other similar examples are $b(x,\mu) = a\left(\int_{\mathbb{R}} y\mathrm{d}\mu(y)\right)$ and $b(x,\mu) = a(\sigma_\mu x)$, where $\sigma_\mu = \sqrt{\mathbb{V}\mathrm{ar}\,[Y]}$ with Y of law μ.

We will consider, for a function B satisfying the B-hypotheses, the following equation:

$$dz_t(u) = \frac{1}{m_t(u)^{1/2}}\int_{\mathbb{R}} f(k)\left(\cos(kz_t(u))\mathrm{d}w^{\Re}(k,t) + \sin(kz_t(u))\mathrm{d}w^{\Im}(k,t)\right) + B(z_t(u),z_t)\mathrm{d}t. \tag{II.39}$$

The usual technique to prove well-posedness of equation (II.39), following the method of [SV79], is to use Girsanov's change of measure in order to create a new probability measure on which the process z is a diffusion without drift. For that purpose, we have to write the drift term as a perturbation of the noise $\mathrm{d}w = \mathrm{d}w^{\Re} + i\mathrm{d}w^{\Im}$. The key point is to be able to inverse the kernel, *i.e.* to prove, for a given $L_2([0,1],\mathbb{R})$-valued-process $(x_t)_{t\in[0,T]}$, the existence of a $L_2(\mathbb{R},\mathbb{C})$-valued process $(h_t)_{t\in[0,T]} = (h_t^{\Re} + ih_t^{\Im})_{t\in[0,T]}$ such that

$$B(x_t(u),x_t) = \frac{1}{\left(\int_0^1 \varphi(x_t(u)-x_t(v))\mathrm{d}v\right)^{1/2}}\int_{\mathbb{R}} e^{-ikx_t(u)} f(k)h_t(k)\mathrm{d}k, \tag{II.40}$$

or equivalently, taking the real part of (II.40), to find two $L_2(\mathbb{R},\mathbb{R})$-valued processes $(h_t^{\Re})_{t\in[0,T]}$ and $(h_t^{\Im})_{t\in[0,T]}$ such that

$$B(x_t(u),x_t) = \int_{\mathbb{R}} \frac{\cos(kx_t(u))f(k)}{(\int_0^1 \varphi(x_t(u)-x_t(v))\mathrm{d}v)^{1/2}} h_t^{\Re}(k)\mathrm{d}k + \int_{\mathbb{R}} \frac{\sin(kx_t(u))f(k)}{(\int_0^1 \varphi(x_t(u)-x_t(v))\mathrm{d}v)^{1/2}} h_t^{\Im}(k)\mathrm{d}k.$$

Note that the functions B, φ and f are real-valued.

For every $M \in \mathbb{N}\backslash\{0\}$, we define the following assumptions:

Definition II.16. A process $(x_t)_{t\in[0,T]}$ with values in $\mathcal{C}[0,1]$ is said to satisfy the X_M-hypotheses if:

i) $(x_t)_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-adapted.

ii) almost surely, for every $t \in [0,T]$, $u \mapsto x_t(u)$ is strictly increasing.

iii) almost surely, for every $t \in [0,T]$, $|x_t(1) - x_t(0)| \leqslant M$.

As a consequence of Corollary II.6 and Proposition II.8, the process $(y_{t\wedge\tau_M})_{t\in[0,T]}$ satisfies the assumptions of Definition II.16.

Corollary II.17. *Let $g \in \mathbf{G}^1$. Assume that $k \mapsto \langle k\rangle f(k)$ is square integrable. Let $(y_t)_{t\in[0,T]}$ be the unique solution to equation* (II.13) *given by Proposition II.8. Let $M \geqslant M_0$ and recall that $\tau_M := \inf\{t \geqslant 0 : y_t(1) - y_t(0) \geqslant M\} \wedge T$. Then $(y_{t\wedge\tau_M})_{t\in[0,T]}$ satisfies the X_M-hypotheses.*

II.3.2 Statement of the inversion Lemma

We denote by $\mathcal{F}\phi$ the Fourier transform of a function ϕ; if ϕ belongs to $L_1(\mathbb{R})$, $\mathcal{F}\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy}\phi(y)\mathrm{d}y$. Recall that Plancherel's formula is given by: $\|\phi\|_{L_2} = \|\mathcal{F}\phi\|_{L_2}$. We denote by $\mathcal{F}^{-1}$ the inverse Fourier transform.

Lemma II.18. *Let $M \geqslant M_0$, $j \in \mathbb{N}$ and $\alpha > 0$. Let us assume that*

- *there is $c > 0$ such that for every $k \in \mathbb{R}$, $f(k) \geqslant \frac{c}{\langle k\rangle^\alpha}$;*

- *the function B satisfies the B-hypotheses of order $2j$;*

- *the process $(x_t)_{t\in[0,T]}$ satisfies the X_M-hypotheses.*

If $2j \geqslant \alpha$, then there is a $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process $(h_t)_{t\in[0,T]}$ which is solution, for every $t \in [0,T]$, to equation (II.40) *and such that there exists $C_M > 0$ for which the inequality $\int_0^T \int_{\mathbb{R}} |h_t(k)|^2 \mathrm{d}k\mathrm{d}t \leqslant C_M$ holds almost surely.*

Proof. Let $(x_t)_{t\in[0,T]}$ be a process satisfying the X_M-hypotheses. Therefore, for a fixed $t \in [0,T]$, the map $u \mapsto x_t(u)$ is a continuous strictly increasing function and can be seen as the quantile function of a measure $\mu_t \in \mathcal{P}_2(\mathbb{R})$. Let us denote by F_t and p_t respectively the c.d.f. and the density associated to μ_t. More precisely, $F_t(x_t(u)) = u$ for all $u \in [0,1]$, $F_t(y) = 0$ for all $y \leqslant x_t(0)$ and $F_t(y) = 1$ for all $y \geqslant x_t(1)$. Since almost surely, for every $t \in [0,T]$, $x_t(1) - x_t(0) \leqslant M$, the Lebesgue measure of the support of p_t is bounded by M.

By the substitution $y = x_t(u)$, equation (II.40) is equivalent to

$$B(y, x_t)\left(\int_0^1 \varphi(y - x_t(v))\mathrm{d}v\right)^{1/2} = \int_{\mathbb{R}} e^{-iky} f(k) h_t(k)\mathrm{d}k \tag{II.41}$$

for every $y \in [x_t(0), x_t(1)]$.

Let us fix a $\mathcal{C}^\infty$-function $\Psi : \mathbb{R} \to \mathbb{R}$ that is equal to 0 on $(-\infty, 0]$ and equal to 1 on $[1, +\infty)$. For every $a < b$, we define the cut-off function $\eta_{a,b} : \mathbb{R} \to \mathbb{R}$ by

$$\eta_{a,b}(y) = \eta_t(y)\begin{cases} 1 & \text{on } [a,b], \\ \Psi(y - (a-1)) & \text{on } (a-1, a), \\ \Psi(b + 1 - y) & \text{on } (b, b+1). \\ 0 & \text{elsewhere.} \end{cases} \tag{II.42}$$

Let us denote by $\eta_t := \eta_{x_t(0),x_t(1)}$. For every $y \in [x_t(0), x_t(1)]$, $\eta_t(y) = 1$. Moreover, η_t has a compact support included in $[x_t(0) - 1, x_t(1) + 1]$. Therefore, if a process $(h_t)_{t\in[0,T]}$ satisfies

$$B(y, x_t)\eta_t(y)\left(\int_0^1 \varphi(y - x_t(v))\mathrm{d}v\right)^{1/2} = \int_{\mathbb{R}} e^{-iky} f(k) h_t(k)\mathrm{d}k \tag{II.43}$$

for every $y \in \mathbb{R}$, then it satisfies (II.41) for every $y \in [x_t(0), x_t(1)]$ and thus it satisfies (II.40) for every $u \in [0,1]$.

Thus, equation (II.40) is equivalent to

$$\mathcal{F}(fh_t) = \frac{1}{\sqrt{2\pi}} B(\cdot, x_t)\eta_t\left(\int_0^1 \varphi(\cdot - x_t(v))\mathrm{d}v\right)^{1/2}.$$

Therefore, the process defined by

$$h_t(k) := \frac{1}{\sqrt{2\pi} f(k)} \mathcal{F}^{-1}\left(B(\cdot, x_t)\eta_t\left(\int_0^1 \varphi(\cdot - x_t(v))\mathrm{d}v\right)^{1/2}\right)(k) \tag{II.44}$$

is solution to equation (II.40), provided h_t is square integrable for every $t \in [0,T]$. Define $\Phi_t := B(\cdot, x_t)\eta_t(\int_0^1 \varphi(\cdot - x_t(v))\mathrm{d}v)^{1/2}$. Note that for every $t \in [0,T]$, $y \mapsto \eta_t(y)(\int_0^1 \varphi(y - x_t(v))\mathrm{d}v)^{1/2}$ is a bounded $\mathcal{C}^\infty$-function with compact support and $y \mapsto B(y, x_t)$ is a bounded continuous function. Therefore, Φ_t belongs to $L^1(\mathbb{R}, \mathbb{C})$ and h_t is well-defined. Moreover, since $(\Phi_t)_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-adapted, $(h_t)_{t\in[0,T]}$ is also $(\mathcal{G}_t)_{t\in[0,T]}$-adapted.

Furthermore, we know by assumption that $f(k) \geqslant \frac{c}{\langle k\rangle^\alpha}$ for every $k \in \mathbb{R}$. Therefore, $\frac{1}{f(k)} \leqslant c^{-1}\langle k\rangle^\alpha \leqslant c^{-1}\langle k\rangle^{2j}$. Thus by Plancherel's Theorem (and denoting by Δ the Laplace operator) we have

$$\begin{aligned}\|h_t\|_{L_2}^2 = \int_{\mathbb{R}} |h_t(k)|^2 \mathrm{d}k = \int_{\mathbb{R}} \frac{1}{2\pi}\frac{|\mathcal{F}^{-1}\Phi_t(k)|^2}{|f(k)|^2}\mathrm{d}k &\leqslant C\int_{\mathbb{R}} \left|\langle k\rangle^{2j}\mathcal{F}^{-1}\Phi_t(k)\right|^2 \mathrm{d}k \\ &= C\int_{\mathbb{R}} \left|\mathcal{F}^{-1}((1+\Delta)^j\Phi_t)(k)\right|^2 \mathrm{d}k \\ &= C\left\|(1+\Delta)^j\Phi_t\right\|_{L_2}^2.\end{aligned}$$

Then, on the one hand, we note that B satisfies the B-hypotheses of order $2j$ and, by the X_M-hypotheses, almost surely for every $t \in [0,T]$, $|\operatorname{Supp} p_t| \leqslant M$. Thus almost surely, we have for every $i \in \{0,1,\dots,2j\}$, for every $t \in [0,T]$ and for every $y \in \mathbb{R}$, $|\partial_1^{(i)}B(y,x_t)| \leqslant C_i(M)$. On the other hand, $y \mapsto \eta_t(y)(\int_0^1 \varphi(y - x_t(v))\mathrm{d}v)^{1/2}$ is a $\mathcal{C}^\infty$-function with compact support, thus this function and all its derivatives are bounded on $\mathbb{R}$. We deduce that for every $i \leqslant 2j$, there is a constant C_M such that almost surely, for every $t \in [0,T]$, $\|\partial^i\Phi_t\|_{L_\infty} \leqslant C_M$.

Recall that the support of η_t is included in $[x_t(0)-1, x_t(1)+1]$. Henceforth, almost surely for every $t \in [0,T]$, the Lebesgue measure of the support of Φ_t is bounded by $M+2$. Therefore, for every $i \leqslant 2j$, there is a constant C_M such that almost surely, for every $t \in [0,T]$,

$$\|\partial^i\Phi_t\|_{L_2} \leqslant |\operatorname{Supp}\Phi_t|^{1/2}\|\partial^i\Phi_t\|_{L_\infty} \leqslant C_M.$$

We deduce that there is $C_M > 0$ such that $\int_0^T \|h_t\|_{L_2}^2 \mathrm{d}t \leqslant C_M$ almost surely, which completes the proof. □

Remark II.19. Let us note that, by formula (II.44) defining h_t for each $t \in [0,T]$, there is a measurable map $\mathcal{H} : \mathcal{C}[0,1] \to L_2(\mathbb{R},\mathbb{C})$ such that for each $t \in [0,T]$, $h_t = \mathcal{H}(x_t)$. Indeed, $\mathcal{H}$ is defined for every $\mathbf{x} \in \mathcal{C}[0,1]$ by

$$\mathcal{H}(\mathbf{x}) : k \mapsto \frac{1}{\sqrt{2\pi}f(k)}\mathcal{F}^{-1}\left(B(\cdot,\mathbf{x})\ \eta_{\mathbf{x}(0),\mathbf{x}(1)}\left(\int_0^1 \varphi(\cdot - \mathbf{x}(v))\mathrm{d}v\right)^{1/2}\right)(k). \tag{II.45}$$

Remark II.20. We do not try to give here an optimal assumption on the regularity of B with respect to the space variable. This question will be discussed in Section II.5. Actually, we need to know that $k \mapsto \frac{1}{f(k)}\mathcal{F}^{-1}\Phi_t(k)$ belongs to $L_2(\mathbb{R})$, *i.e.* that Φ_t belongs to the Sobolev space $H^\alpha(\mathbb{R})$. Since $\eta_t(\int_0^1 \varphi(\cdot - x_t(v))\mathrm{d}v)^{1/2}$ is a $\mathcal{C}^\infty$-function with compact support, a sufficient condition to insure that we can apply the Fourier inversion is that there is a constant C_M such that almost surely for every $t \in [0,T]$, $\|B(\cdot,x_t)\|_{H^\alpha} \leqslant C_M$.

II.4 Regularization of an ill-posed Fokker Planck equation

Let $g \in \mathbf{G}^1$. Let $M \geqslant M_0 > g(1) - g(0)$. Let $B : \mathbb{R} \times L_2[0,1] \to \mathbb{R}$ satisfy the B-hypotheses. Let $f : \mathbb{R} \to \mathbb{R}$ be an even and square integrable function.

Definition II.21. We say that $f : \mathbb{R} \to \mathbb{R}$ is of order α if there exist two constants C and $c > 0$ such that $c\frac{1}{\langle k\rangle^\alpha} \leqslant f(k) \leqslant C\frac{1}{\langle k\rangle^\alpha}$ for every $k \in \mathbb{R}$.

In order to insure that $k \mapsto \langle k\rangle f(k)$ is square integrable, we assume that $\alpha > \frac{3}{2}$.

The goal of this section is to prove well-posedness of equation (II.39). We will assume in this section that the order of regularity of B and the order of integrability of f are compatible, *i.e.* that the condition $\alpha \leqslant 2j$ of Lemma II.18 is satisfied. Therefore, this method operates for drift

functions B which have at least $\mathcal{C}^2$-regularity in the space variable (in the light of Remark II.20, the method actually works for functions B which belong to $H^s(\mathbb{R})$ for a certain $s > \frac{3}{2}$).

We will now use the Fourier inversion to apply a Girsanov argument and to be able to prove weak existence and later weak uniqueness of a solution to the drifted SDE (II.39). But we want to stress here the fact that, according to Lemma II.18, the L_2-norm of the inverse process $(h_t)_{t\in[0,T]}$ is bounded by a constant C_M depending on M. This dependence on the size of the support of μ_t can be removed for certain choices of drift functions B and mass kernels φ, but not in general. If we choose $B \equiv 1$ and $\varphi \equiv 1$, we can see in the proof of lemma that we need to multiply the product Bm_t by a cut-off function η_t in order to obtain an L_2-function. Its L_2-norm will therefore depend on the size of the support of η_t, that is on M. The consequence of this remark is that we will need to localize the SDE in order to control the size of the support of μ_t.

We will first prove in Paragraph II.4.1 existence of a weak solution to equation (II.39) up to a stopping time. Then, in Paragraph II.4.2, we will show uniqueness in law for the same equation up to the same stopping time. Finally, in Paragraph II.4.3 we will explain how we can remove the localization in order to prove well-posedness on the global time interval $[0, T]$.

II.4.1 Existence of a weak solution up to a stopping time

Let us consider the following stochastic differential equation:

$$\begin{cases} \mathrm{d}z_t(u) = \mathbb{1}_{\{t\leqslant\zeta_M\}}\Big(\dfrac{1}{m_t(u)^{1/2}}\displaystyle\int_{\mathbb{R}} \cos(kz_t(u))f(k)\ \mathrm{d}w^{\Re}(k,t) \\ \qquad\qquad + \dfrac{1}{m_t(u)^{1/2}}\displaystyle\int_{\mathbb{R}} \sin(kz_t(u))f(k)\ \mathrm{d}w^{\Im}(k,t) + B(z_t(u), z_t)\mathrm{d}t\Big) \\ z_0(u) = g(u), \end{cases} \tag{Eq_M}$$

where $m_t(u) = \int_0^1 \varphi(z_t(u) - z_t(v))\mathrm{d}v$ and $\zeta_M := \inf\{t \geqslant 0 : z_t(1) - z_t(0) \geqslant M\} \wedge T$.

Remark II.22. If $B \equiv 0$, the result of Proposition II.8 states that (Eq_M) is well-posed: in particular, pathwise uniqueness holds for this equation. Actually, if we assume that B is Lipschitz continuous with respect to both variables on $\mathbb{R} \times L_2[0, 1]$, then we can apply the same method based on a fixed-point argument as in Section II.2. Therefore, existence of a strong solution and pathwise uniqueness still hold for equation (II.16) under these regularity assumptions. The aim of what follows is to obtain existence and uniqueness of a weak solution under lower assumptions on the regularity of the drift term B with respect to the measure variable, that is under the B-hypotheses. Nevertheless, we will see that B needs to be at least of class $\mathcal{C}^2$ with respect to its first variable.

In accordance with [KS91], we give the following sense to a weak solution to the SDE (Eq_M).

Definition II.23. A sextuple $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}, z, w)$ is said to be a *weak solution* to equation (Eq_M) if

- $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space satisfying usual conditions,
- $(z_t)_{t\in[0,T]}$ is a continuous $(\mathcal{G}_t)_{t\in[0,T]}$-adapted $\mathcal{C}[0, 1]$-valued process,
- $w = (w^{\Re}, w^{\Im})$ and $(w^{\Re}(k, t), w^{\Im}(k, t))_{k\in\mathbb{R},t\in[0,T]}$ are two independent $(\mathcal{G}_t)_{t\in[0,T]}$-Brownian sheets under $\mathbb{P}$,
- $\mathbb{P}\left[\displaystyle\int_0^1\int_0^{\zeta_M} \frac{1}{m_t(u)}\ \mathrm{d}t\mathrm{d}u < +\infty\right] = 1,$

- $\mathbb{P}$-almost surely, for every $t \in [0,T]$,

$$z_t(u) = g(u) + \int_0^{t\wedge\zeta_M} \frac{1}{m_s(u)^{1/2}} \int_{\mathbb{R}} \cos(kz_s(u)) f(k) \mathrm{d}w^{\Re}(k,s) + \int_0^{t\wedge\zeta_M} \frac{1}{m_s(u)^{1/2}} \int_{\mathbb{R}} \sin(kz_s(u)) f(k) \mathrm{d}w^{\Im}(k,s) + \int_0^{t\wedge\zeta_M} B(z_s(u), z_s) \mathrm{d}s. \quad \text{(II.46)}$$

Proposition II.24. *Let $g \in \mathbf{G}^1$, $M \geqslant M_0$ and $j \in \mathbb{N}\backslash\{0\}$. Let f be a function of order $\alpha > \frac{3}{2}$. Let $B : \mathbb{R} \times L_2[0,1] \to \mathbb{R}$ satisfy the B-hypotheses of order $2j$. If $2j \geqslant \alpha$, there exists a weak solution to equation* (Eq_M).

Proof. Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and $(w(u,t))_{u\in[0,1],t\in[0,T]}$ be a $(\mathcal{G}_t)_{t\in[0,T]}$-Brownian sheet. Since $\alpha > \frac{3}{2}$, the map $k \mapsto \langle k\rangle f(k)$ is square integrable. Thus, by Proposition II.8, there is a unique process $(y_t)_{t\in[0,T]}$ satisfying equation (II.13) for every $u \in [0,1]$. Furthermore, by Corollary II.17, the process $(y_{t\wedge\tau_M})_{t\in[0,T]}$ satisfies the X_M-hypotheses.

Therefore, by Lemma II.18, there is a process $(h_t)_{t\in[0,T]} = (h_t^{\Re} + ih_t^{\Im})_{t\in[0,T]}$ with values in $L_2(\mathbb{R},\mathbb{C})$ satisfying for every $t \in [0,T]$ and $u \in [0,1]$:

$$B(y_{t\wedge\tau_M}(u), y_{t\wedge\tau_M}) = \int_{\mathbb{R}} \frac{\cos(ky_{t\wedge\tau_M}(u)) f(k)}{(\int_0^1 \varphi(y_{t\wedge\tau_M}(u) - y_{t\wedge\tau_M}(v))\mathrm{d}v)^{1/2}} h_t^{\Re}(k)\mathrm{d}k + \int_{\mathbb{R}} \frac{\sin(ky_{t\wedge\tau_M}(u)) f(k)}{(\int_0^1 \varphi(y_{t\wedge\tau_M}(u) - y_{t\wedge\tau_M}(v))\mathrm{d}v)^{1/2}} h_t^{\Im}(k)\mathrm{d}k$$

and such that there exists a constant C_M such that almost surely,

$$\int_0^T \int_{\mathbb{R}} |h_t(k)|^2 \mathrm{d}k\mathrm{d}t \leqslant C_M. \quad \text{(II.47)}$$

Therefore, we can rewrite equation (II.13) as follows:

$$\begin{aligned} y_{t\wedge\tau_M}(u) &= g(u) + \int_0^{t\wedge\tau_M} \int_{\mathbb{R}} \frac{\cos(ky_s(u)) f(k)}{m_s(u)^{1/2}} \mathrm{d}w^{\Re}(k,s) + \int_0^{t\wedge\tau_M} \int_{\mathbb{R}} \frac{\sin(ky_s(u)) f(k)}{m_s(u)^{1/2}} \mathrm{d}w^{\Im}(k,s) \\ &\quad + \int_0^{t\wedge\tau_M} B(y_s(u), y_s)\mathrm{d}s \\ &\quad - \int_0^{t\wedge\tau_M} \int_{\mathbb{R}} \frac{\cos(ky_s(u)) f(k)}{m_s(u)^{1/2}} h_s^{\Re}(k)\mathrm{d}k - \int_0^{t\wedge\tau_M} \int_{\mathbb{R}} \frac{\sin(ky_s(u)) f(k)}{m_s(u)^{1/2}} h_t^{\Im}(k)\mathrm{d}k \\ &= g(u) + \int_0^{t\wedge\tau_M} \int_{\mathbb{R}} \frac{\cos(ky_s(u)) f(k)}{m_s(u)^{1/2}} \mathrm{d}\widetilde{w}^{\Re}(k,s) + \int_0^{t\wedge\tau_M} \int_{\mathbb{R}} \frac{\sin(ky_s(u)) f(k)}{m_s(u)^{1/2}} \mathrm{d}\widetilde{w}^{\Im}(k,s) \\ &\quad + \int_0^{t\wedge\tau_M} B(y_s(u), y_s)\mathrm{d}s, \end{aligned} \quad \text{(II.48)}$$

where we define for every $k \in \mathbb{R}$ and every $s \in [0,T]$

$$\mathrm{d}\widetilde{w}^{\Re}(k,s) := \mathrm{d}w^{\Re}(k,s) - \mathbb{1}_{\{s\leqslant\tau_M\}} h_s^{\Re}(k)\mathrm{d}k\mathrm{d}s,$$
$$\mathrm{d}\widetilde{w}^{\Im}(k,s) := \mathrm{d}w^{\Im}(k,s) - \mathbb{1}_{\{s\leqslant\tau_M\}} h_s^{\Im}(k)\mathrm{d}k\mathrm{d}s.$$

Let us consider the process $(G_t)_{t\in[0,T]}$ defined by:

$$G_t := \exp\left(\int_0^{t\wedge\tau_M} \int_{\mathbb{R}} h_s^{\Re}(k)\mathrm{d}w^{\Re}(k,s) + \int_0^{t\wedge\tau_M} \int_{\mathbb{R}} h_s^{\Im}(k)\mathrm{d}w^{\Im}(k,s) - \frac{1}{2}\int_0^{t\wedge\tau_M} \int_{\mathbb{R}} |h_s(k)|^2 \mathrm{d}k\mathrm{d}s \right).$$

By (II.47), there is $C_M > 0$ such that $\exp(\frac{1}{2}\int_0^{\tau_M}\int_{\mathbb{R}} |h_s(k)|^2 \mathrm{d}k\mathrm{d}s) \leqslant C_M$ almost surely. Thus Novikov's condition holds and the process $(G_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale. Let us define the probability measure $\mathbb{Q}$ by the absolutely continuous measure with respect to $\mathbb{P}$ with density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = G_T$.

Remark that the process $(G_t)_{t\in[0,T]}$ and the measure $\mathbb{Q}$ depend on M. For simplicity, we do not mention this dependence in the notation.

By Girsanov's Theorem, under the probability measure $\mathbb{Q}$, $(\widetilde{w}^{\Re}(k,t),\widetilde{w}^{\Im}(k,t))_{k\in\mathbb{R},t\in[0,T]}$ are two independent Brownian sheets on $\mathbb{R}\times[0,T]$ and the couple $(y_{\cdot\wedge\tau_M},\widetilde{w})$ satisfies equation (Eq_M). Furthermore, $\mathbb{Q}$-almost surely, $\int_0^1\int_0^{\tau_M}\frac{1}{m_t(u)}\mathrm{d}t\mathrm{d}u\leqslant C_M<+\infty$.

Thus $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\in[0,T]},\mathbb{Q},y_{\cdot\wedge\tau_M},\widetilde{w})$ is a weak solution of equation (Eq_M). □

II.4.2 Uniqueness in law of a solution up to a stopping time

Definition II.25. We say that *uniqueness in law* holds for equation (Eq_M) if for any two weak solutions $(\Omega^i,\mathcal{G}^i,(\mathcal{G}^i_t)_{t\in[0,T]},\mathbb{P}^i,z^i,w^i)$, $i=1,2$, the two processes $(z^1_t)_{t\in[0,T]}$ and $(z^2_t)_{t\in[0,T]}$ have the same law on $\mathcal{C}([0,1]\times[0,T])$.

Lemma II.26. *Let us assume that $(\Omega^i,\mathcal{G}^i,(\mathcal{G}^i_t)_{t\in[0,T]},\mathbb{P}^i,z^i,w^i)$, $i=1,2$, are two weak solutions to equation (Eq_M) in the case where $B\equiv 0$. Then (z^1,w^1) and (z^2,w^2) have same law in $\mathcal{C}([0,1]\times[0,T])\times\mathcal{C}(\mathbb{R}\times[0,T],\mathbb{R}^2)$.*

Let us remark that the Brownian sheets w^1 and w^2 are seen here as taking values in $\mathbb{R}^2$, by an identification of $\mathbb{R}^2$ with $\mathbb{C}$.

Proof. In the case where $B\equiv 0$, equation (Eq_M) has a unique pathwise solution by Proposition II.8. By an infinite-dimensional version of Yamada-Watanabe result (see [KS91, Prop 5.3.20]), it implies that the law of (z^1,w^1) under $\mathbb{P}^1$ is equal to the law of (z^2,w^2) under $\mathbb{P}^2$. □

The proof of uniqueness in law for equation (Eq_M) is based on Girsanov's Theorem. As in the proof of Proposition II.24, we will apply Lemma II.18 to the drift function B and to a weak solution to equation (Eq_M). In order to apply it, we need to know that any weak solution to (Eq_M) satisfies the X_M-hypotheses (recall Definition II.16 of these assumptions). It is the statement of the next lemma.

Lemma II.27. *Let $B:\mathbb{R}\times L_2[0,1]\to\mathbb{R}$ satisfy the B-hypotheses of order 1. Let $(z_t)_{t\in[0,T]}$ be a solution to equation (II.46). Then it satisfies the X_M-hypotheses.*

Proof. The process $(z_t)_{t\in[0,T]}$ is $(\mathcal{G}_t)_{t\in[0,T]}$-adapted. Moreover, by definition of equation (Eq_M), $(z_t)_{t\in[0,T]}=(z_{t\wedge\tau_M})_{t\in[0,T]}$. Therefore, we deduce that almost surely, for every $t\in[0,T]$, $z_t(1)-z_t(0)\leqslant M$. It remains to check assumption $ii)$ of the X_M-hypotheses. We adapt the argument given in Proposition II.5. Let $u_1,u_2\in[0,1]$ such that $u_1<u_2$. The process $Z_t=z_t(u_2)-z_t(u_1)$ is solution to

$$Z_t=g(u_2)-g(u_1)+\int_0^t Z_s\mathrm{d}N^M_s,$$

where N^M_t is the following semi-martingale:

$$\begin{aligned}N^M_t:=&\int_0^{t\wedge\zeta_M}\int_{\mathbb{R}}\mathbb{1}_{\{z_s(u_1)\neq z_s(u_2)\}}\frac{\theta^{\Re}_M(z_s(u_2),k,s)-\theta^{\Re}_M(z_s(u_1),k,s)}{z_s(u_2)-z_s(u_1)}f(k)\mathrm{d}w^{\Re}(k,s)\\&+\int_0^{t\wedge\zeta_M}\int_{\mathbb{R}}\mathbb{1}_{\{z_s(u_1)\neq z_s(u_2)\}}\frac{\theta^{\Im}_M(z_s(u_2),k,s)-\theta^{\Im}_M(z_s(u_1),k,s)}{z_s(u_2)-z_s(u_1)}f(k)\mathrm{d}w^{\Im}(k,s)\\&+\int_0^{t\wedge\zeta_M}\mathbb{1}_{\{z_s(u_1)\neq z_s(u_2)\}}\frac{B(z_s(u_2),z_s)-B(z_s(u_1),z_s)}{z_s(u_2)-z_s(u_1)}\mathrm{d}s,\end{aligned}$$

where $\theta^{\Re}_M(x,k,s)=\frac{\cos(kx)}{(\int_0^1\varphi(x-z_s(v))\mathrm{d}v)^{1/2}}$ and $\theta^{\Im}_M(x,k,s)=\frac{\sin(kx)}{(\int_0^1\varphi(x-z_s(v))\mathrm{d}v)^{1/2}}$. As in the proof of Proposition II.5, we show that $(Z_t)_{t\in[0,T]}$ admits the following exponential form:

$$Z_t=(g(u_2)-g(u_1))\exp\left(N^M_t-\langle N^M,N^M\rangle_t\right). \tag{II.49}$$

As in Proposition II.5, equation (II.49) relies on the fact that almost surely, for every $s \in [0,T]$, $\frac{\mathrm{d}\langle N^M, N^M\rangle_s}{\mathrm{d}s} \leqslant C_{M,T}$. Indeed, by assumption on B, for every $s \in [0,T]$,

$$\mathbb{1}_{\{s\leqslant\zeta_M\}}\mathbb{1}_{\{z_s(u_1)\neq z_s(u_2)\}}\left|\frac{B(z_s(u_2),z_s)-B(z_s(u_1),z_s)}{z_s(u_2)-z_s(u_1)}\right| \leqslant C_1(M).$$

We deduce from (II.49) that almost surely, for every $t \leqslant T$, $u \mapsto z_t(u)$ is strictly increasing. Thus, $(z_t)_{t\in[0,T]}$ satisfies the X_M-hypotheses. □

Theorem II.28. *Let $g \in \mathbf{G}^1$, $M \geqslant M_0$ and $j \in \mathbb{N}\backslash\{0\}$. Let f be a function of order $\alpha > \frac{3}{2}$. Let $B : \mathbb{R}\times L_2[0,1] \to \mathbb{R}$ satisfy the B-hypotheses of order $2j$. If $2j \geqslant \alpha$, uniqueness in law holds for equation* (Eq_M).

Moreover, if $(\Omega^i, \mathcal{G}^i, (\mathcal{G}^i_t)_{t\in[0,T]}, \mathbb{P}^i, z^i, w^i)$, $i = 1,2$, are two weak solutions of equation (Eq_M), *then the laws of (z^1,w^1) and (z^2,w^2) are the same in $\mathcal{C}([0,1]\times[0,T])\times\mathcal{C}(\mathbb{R}\times[0,T],\mathbb{R}^2)$.*

Proof. Let us consider two weak solutions $(\Omega^i, \mathcal{G}^i, (\mathcal{G}^i_t)_{t\in[0,T]}, \mathbb{P}^i, z^i, w^i)$, for $i = 1,2$, to equation (Eq_M). Let $i = 1$ or 2. Let $\zeta^i_M := \inf\{t \geqslant 0 : z^i_t(1) - z^i_t(0) \geqslant M\} \wedge T$.

By Lemma II.27, the process $(z^i_t)_{t\in[0,T]}$ satisfies the X_M-hypotheses. Thus by Lemma II.18, there are for $i = 1,2$ $(\mathcal{G}^i_t)_{t\in[0,T]}$-adapted processes $(h^i_t)_{t\in[0,T]}$ such that $\int_0^T\int_{\mathbb{R}}|h^i_t(k)|^2\mathrm{d}k\mathrm{d}s \leqslant C_M$ almost surely and for all $t \in [0,T]$ and $u \in [0,1]$,

$$\begin{aligned}B(z^i_t(u),z^i_t) = &\int_{\mathbb{R}}\frac{\cos(kz^i_t(u))f(k)}{(\int_0^1\varphi(z^i_t(u)-z^i_t(v))\mathrm{d}v)^{1/2}}h^{i,\Re}_t(k)\mathrm{d}k\\ &+\int_{\mathbb{R}}\frac{\sin(kz^i_t(u))f(k)}{(\int_0^1\varphi(z^i_t(u)-z^i_t(v))\mathrm{d}v)^{1/2}}h^{i,\Im}_t(k)\mathrm{d}k\end{aligned}$$

Furthermore, by formula (II.45), there is a measurable map $\mathcal{H} : \mathcal{C}[0,1] \to L_2(\mathbb{R},\mathbb{C})$ such that $h^i_t = \mathcal{H}(z^i_t)$ for every $t \in [0,T]$ and for $i = 1,2$.

Since $(z^i_t)_{t\in[0,T]}$ is solution to equation (II.46), we have $\mathbb{P}^i$-almost surely for every $t \in [0,T]$ and for every $u \in [0,1]$:

$$\begin{aligned}z^i_t(u) = g(u) &+ \int_0^{t\wedge\zeta^i_M}\frac{1}{m^i_s(u)^{1/2}}\int_{\mathbb{R}}\cos(kz^i_s(u))f(k)(\mathrm{d}w^{i,\Re}(k,s)+h^{i,\Re}_s(k)\mathrm{d}k\mathrm{d}s)\\ &+\int_0^{t\wedge\zeta^i_M}\frac{1}{m^i_s(u)^{1/2}}\int_{\mathbb{R}}\sin(kz^i_s(u))f(k)(\mathrm{d}w^{i,\Im}(k,s)+h^{i,\Im}_s(k)\mathrm{d}k\mathrm{d}s),\end{aligned}$$

where $m^i_s(u) = \int_0^1\varphi(z^i_s(u)-z^i_s(v))\mathrm{d}v$. Let us define for every $k \in \mathbb{R}$ and every $s \in [0,T]$

$$\begin{aligned}\mathrm{d}\widetilde{w}^{i,\Re}(k,s) &:= \mathrm{d}w^{i,\Re}(k,s) + \mathbb{1}_{\{s\leqslant\zeta^i_M\}}h^{i,\Re}_s(k)\mathrm{d}k\mathrm{d}s,\\ \mathrm{d}\widetilde{w}^{i,\Im}(k,s) &:= \mathrm{d}w^{i,\Im}(k,s) + \mathbb{1}_{\{s\leqslant\zeta^i_M\}}h^{i,\Im}_s(k)\mathrm{d}k\mathrm{d}s.\end{aligned}$$

Let us consider the process $(G^i_t)_{t\in[0,T]}$ defined by:

$$G^i_t := \exp\left(-\int_0^{t\wedge\zeta^i_M}\int_{\mathbb{R}}h^{i,\Re}_s(k)\mathrm{d}w^{i,\Re}(k,s)-\int_0^{t\wedge\zeta^i_M}\int_{\mathbb{R}}h^{i,\Im}_s(k)\mathrm{d}w^{i,\Im}(k,s)-\frac{1}{2}\int_0^{t\wedge\zeta^i_M}\int_{\mathbb{R}}|h^i_s(k)|^2\mathrm{d}k\mathrm{d}s\right).$$

Novikov's condition applies because $\int_0^T\int_{\mathbb{R}}|h^i_t(k)|^2\mathrm{d}k\mathrm{d}s \leqslant C_M$ almost surely, so the process $(G^i_t)_{t\in[0,T]}$ is a $\mathbb{P}^i$-martingale. We define the probability measure $\mathbb{Q}^i$ by the absolutely continuous measure with respect to $\mathbb{P}^i$ with density $\frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{P}^i} = G^i_T$. By Girsanov's Theorem, under $\mathbb{Q}^i$, $\widetilde{w}^i =$

$(\widetilde{w}^{i,\Re}, \widetilde{w}^{i,\Im})$ is a couple of two independent Brownian sheets and $\mathbb{Q}^i$-almost surely, for every $t \in [0,T]$,

$$z_t^i(u) = g(u) + \int_0^{t\wedge\zeta_M^i} \frac{1}{m_s^i(u)^{1/2}} \int_{\mathbb{R}} f(k) \left(\cos(kz_s^i(u)) \mathrm{d}\widetilde{w}^{i,\Re}(k,s) + \sin(kz_s^i(u)) \mathrm{d}\widetilde{w}^{i,\Im}(k,s) \right).$$

Thus $(\Omega^i, \mathcal{G}^i, (\mathcal{G}_t^i)_{t\in[0,T]}, \mathbb{Q}^i, z^i, \widetilde{w}^i)$, for $i = 1, 2$, are two weak solutions to equation (Eq_M) in the case where $B \equiv 0$. By Lemma II.26, it follows that for every measurable function $\psi : \mathcal{C}([0,1]\times[0,T]) \times \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2) \to \mathbb{R}$ such that $\mathbb{E}^{\mathbb{Q}^i}\left[|\psi(z^i, \widetilde{w}^i)|\right] < +\infty$ for $i = 1, 2$, we have

$$\mathbb{E}^{\mathbb{Q}^1}\left[\psi(z^1, \widetilde{w}^1)\right] = \mathbb{E}^{\mathbb{Q}^2}\left[\psi(z^2, \widetilde{w}^2)\right]. \tag{II.50}$$

Let $\phi : \mathcal{C}([0,1]\times[0,T]) \times \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2) \to \mathbb{R}$ be a bounded and measurable function. We have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^i}\left[\phi(z^i, w^i)\right] &= \mathbb{E}^{\mathbb{Q}^i}\left[\phi(z^i, w^i)(G_T^i)^{-1}\right] \\
&= \mathbb{E}^{\mathbb{Q}^i}\Bigg[\phi(z^i, w^i) \exp\Bigg(\int_0^{\zeta_M^i}\int_{\mathbb{R}} h_s^{i,\Re}(k)\mathrm{d}w^{i,\Re}(k,s) \\
&\qquad + \int_0^{\zeta_M^i}\int_{\mathbb{R}} h_s^{i,\Im}(k)\mathrm{d}w^{i,\Im}(k,s) + \frac{1}{2}\int_0^{\zeta_M^i}\int_{\mathbb{R}} |h_s^i(k)|^2 \mathrm{d}k\mathrm{d}s \Bigg)\Bigg] \\
&= \mathbb{E}^{\mathbb{Q}^i}\Bigg[\phi(z^i, \widetilde{w}^i + \textstyle\int_0^{\cdot}\int_0^{\cdot} h_s^i(k)\mathrm{d}k\mathrm{d}s) \exp\Bigg(\displaystyle\int_0^{\zeta_M^i}\int_{\mathbb{R}} h_s^{i,\Re}(k)\mathrm{d}\widetilde{w}^{i,\Re}(k,s) \\
&\qquad + \int_0^{\zeta_M^i}\int_{\mathbb{R}} h_s^{i,\Im}(k)\mathrm{d}\widetilde{w}^{i,\Im}(k,s) - \frac{1}{2}\int_0^{\zeta_M^i}\int_{\mathbb{R}} |h_s^i(k)|^2 \mathrm{d}k\mathrm{d}s \Bigg)\Bigg] \\
&= \mathbb{E}^{\mathbb{Q}^i}\left[\psi(z^i, \widetilde{w}^i)\right]
\end{aligned} \tag{II.51}$$

where $\psi : \mathcal{C}([0,1]\times[0,T]) \times \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2) \to \mathbb{R}$ is a measurable function, because for each $t \in [0,T]$, $h_t^i = \mathcal{H}(z_t^i)$ with $\mathcal{H} : \mathcal{C}[0,1] \to L_2(\mathbb{R}, \mathbb{C})$ a measurable function. By equality (II.50), we deduce that $\mathbb{E}^{\mathbb{P}^1}\left[\phi(z^1, w^1)\right] = \mathbb{E}^{\mathbb{P}^2}\left[\phi(z^2, w^2)\right]$. Thus (z^1, w^1) and (z^2, w^2) have the same law and this completes the proof of the Theorem. □

II.4.3 Existence and uniqueness on the interval $[0,T]$

We consider the following stochastic differential equation:

$$\begin{cases} \mathrm{d}z_t(u) = \dfrac{1}{m_t(u)^{1/2}} \displaystyle\int_{\mathbb{R}} f(k)(\cos(kz_t(u))\mathrm{d}w^{\Re}(k,t) + \sin(kz_t(u))\mathrm{d}w^{\Im}(k,t)) + B(z_t(u), z_t)\mathrm{d}t \\ z_0(u) = g(u), \end{cases} \tag{Eq_∞}$$

where $m_t(u) = \int_0^1 \varphi(z_t(u) - z_t(v))\mathrm{d}v$.

Definition II.29. A sextuple $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}, z, w)$ is said to be a *weak solution* to equation (Eq_∞) if the conditions of Definition II.23 are satisfied, except the last condition replaced by: $\mathbb{P}$-almost surely, for every $t \in [0,T]$,

$$\begin{aligned}
z_t(u) = g(u) &+ \int_0^t \frac{1}{m_s(u)^{1/2}} \int_{\mathbb{R}} \cos(kz_s(u)) f(k)\mathrm{d}w^{\Re}(k,s) \\
&+ \int_0^t \frac{1}{m_s(u)^{1/2}} \int_{\mathbb{R}} \sin(kz_s(u)) f(k)\mathrm{d}w^{\Im}(k,s) + \int_0^t B(z_s(u), z_s)\mathrm{d}s.
\end{aligned}$$

Let $M_0 > g(1) - g(0)$. For each $M \geqslant M_0$, we consider the solution to equation (Eq_M) on the canonical probability space. Let $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 := \mathcal{C}([0,1] \times [0,T])$ and $\Omega_2 := \mathcal{C}(\mathbb{R} \times [0,T], \mathbb{R}^2)$, equipped with the class $\mathcal{B}(\Omega)$ of Borel subsets of Ω. To every $z \in \Omega_1$, we associate $\zeta_z^M = \inf\{t \geqslant 0 : z_t(1) - z_t(0) \geqslant M\} \wedge T$. If no confusion is possible, we will also write $\zeta^M = \zeta_z^M$.

Let $\mathcal{G}^M$ be the σ-algebra generated by the map $\pi^M : z \in \Omega_1 \mapsto z_{\cdot \wedge \zeta_z^M} \in \Omega_1$. Recall that by Proposition II.24, there is a weak solution to equation (Eq_M). Considering the law $\mathbb{Q}^M$ of that solution on the space $(\Omega, \mathcal{G}^M \otimes \mathcal{B}(\Omega_2))$ constructed above, we deduce that for each $M \geqslant M_0$, there is a weak solution $(\Omega, \mathcal{G}^M \otimes \mathcal{B}(\Omega_2), (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q}^M, z_{\cdot\wedge\zeta_z^M}, w)$ to equation (Eq_M), where for each $\omega = (\omega_1, \omega_2) \in \Omega$, $(z, w)(\omega) = (\omega_1, \omega_2)$.

Let us prove that the family $(\mathbb{Q}^M)_{M\geqslant M_0}$ is consistent, in the following sense:

Proposition II.30. *The family $(\mathbb{Q}^M)_{M\geqslant M_0}$ is consistent, i.e. for each $M' \geqslant M \geqslant M_0$, for each $A \in \mathcal{B}(\Omega_1)$, for each $B \in \mathcal{B}(\Omega_2)$*

$$\mathbb{Q}^M \left[z_{\cdot\wedge\zeta^M} \in A, w \in B \right] = \mathbb{Q}^{M'} \left[z_{\cdot\wedge\zeta^M} \in A, w \in B \right]. \tag{II.52}$$

In other words, for each $A^M \in \mathcal{G}^M$, for each $B \in \mathcal{B}(\Omega_2)$

$$\mathbb{Q}^M \left[z \in A^M, w \in B \right] = \mathbb{Q}^{M'} \left[z \in A^M, w \in B \right]. \tag{II.53}$$

Proof. Let $M' \geqslant M \geqslant M_0$. Then $(\Omega, \mathcal{G}^M \otimes \mathcal{B}(\Omega_2), (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q}^M, z_{\cdot\wedge\zeta^M}, w)$ and $(\Omega, \mathcal{G}^{M'} \otimes \mathcal{B}(\Omega_2), (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q}^{M'}, z_{\cdot\wedge\zeta^M}, w)$ are two weak solutions to equation (Eq_M). By Theorem II.28, the laws of both processes are the same, whence we deduce (II.52). □

We want to construct a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{B}(\Omega))$ which is equal for every $M \geqslant M_0$ to $\mathbb{Q}^M$ on $\mathcal{G}^M \times \mathcal{B}(\Omega_2)$. For this purpose, we check in the following proposition the assumptions of Theorem V.4.2 in [Par67, p.143].

Remark II.31. We recall the definitions given in [Par67]. An atom of a Borel space $(X, \mathcal{B})$ is a set $A_0 \in \mathcal{B}$ such that the relations $A \subseteq A_0$, $A \in \mathcal{B}$ imply that $A = A_0$ or $A = \emptyset$. If Y is a metric space, $\mathcal{B}_Y$ denotes the Borel σ-algebra of Y. A countably generated Borel space $(X, \mathcal{B})$ is called *standard* if there exists a complete separable metric space Y such that the σ-algebras $\mathcal{B}$ and $\mathcal{B}_Y$ are σ-isomorphic, that is if there is a one-to-one map between $\mathcal{B}$ and $\mathcal{B}_Y$ preserving countable sets operations.

Proposition II.32. *The family $(\Omega, \mathcal{G}^M \otimes \mathcal{B}(\Omega_2))_{M\geqslant M_0}$ satisfies the assumptions of Theorem V.4.2 in [Par67, p.143], that is*

i) $\mathcal{G}^M \otimes \mathcal{B}(\Omega_2) \subseteq \mathcal{G}^{M'} \otimes \mathcal{B}(\Omega_2)$ *for each* $M \leqslant M'$;

ii) $(\Omega, \mathcal{G}^M \otimes \mathcal{B}(\Omega_2))$ *is a standard Borel space for all* M;

iii) $\bigcup_{M\geqslant M_0}(\mathcal{G}^M \otimes \mathcal{B}(\Omega_2))$ *generates* $\mathcal{B}(\Omega)$;

iv) for any sequence $(A_n)_{n\geqslant 1}$ of sets such that $A_0 \supseteq A_1 \supseteq \ldots$ and A_n is an atom of $\mathcal{G}^{M_n} \otimes \mathcal{B}(\Omega_2)$, where $(M_n)_{n\geqslant 1}$ is an increasing sequence, one has $\bigcap_n A_n \neq \varnothing$.

Proof. Statements *i)* and *iii)* are clear.

Let us check statement *ii)*. Let $\chi^M := \{x \in \Omega_1 : x = x_{\cdot\wedge\zeta_x^M}\}$, i.e. the set of functions $x \in \Omega_1 = \mathcal{C}([0,1] \times [0,T])$ which remain constant as soon as $x_t(1) - x_t(0)$ reaches the level M. Let $\mathcal{F}^M$ be the class of Borel subsets of χ^M. The set χ^M is a Borel set of Ω_1 (nevertheless, it is not closed in Ω_1). Since Ω_1 is a Polish space, it follows from [Par67, Theorem V.2.2] that $(\chi^M, \mathcal{F}^M)$ is standard if and only if it is separable.

Let us check that $(\chi^M, \mathcal{F}^M)$ is a separable metric space, with respect to the L_∞-distance on $[0,1]\times[0,T]$. Let $(f^n)_{n\in\mathbb{N}}$ be a countable dense subset of Ω_1. Thus

$$\left(\{f^n\,;n\in\mathbb{N}\}\cup\left\{\left(1-\frac{1}{m}\right)f^n\,;n\in\mathbb{N},m\in\mathbb{N}\backslash\{0\}\right\}\right)\cap\chi^M$$

is a countable and dense subset of χ^M. Indeed, let $x\in\chi^M$. Remark that $U^M:=\{x\in\Omega_1:\sup_{t\leqslant T}|x_t(1)-x_t(0)|<M\}$ is an open set of Ω_1 contained in χ^M. Thus every $x\in U^M$ is limit of elements of $\{f^n\,;n\in\mathbb{N}\}\cap\chi^M$. If $x\in\chi^M\backslash U^M$, then $\sup_{t\leqslant T}|x_t(1)-x_t(0)|=M$. Let $\varepsilon\in(0,1)$. There is $n\in\mathbb{N}$ such that $\|f^n-x\|_{L_\infty}<\frac{\varepsilon}{2}$. Then $\sup_{t\leqslant T}|f^n_t(1)-f^n_t(0)|<M+\varepsilon$. Let m be the integer satisfying $\frac{M}{\varepsilon}\leqslant m<1+\frac{M}{\varepsilon}$. Thus we have

$$\begin{aligned}\sup_{t\leqslant T}\left|\left(1-\frac{1}{m}\right)f^n_t(1)-\left(1-\frac{1}{m}\right)f^n_t(0)\right|&\leqslant\left(1-\frac{1}{m}\right)\sup_{t\leqslant T}|f^n_t(1)-f^n_t(0)|\\&<\left(1-\frac{1}{m}\right)(M+\varepsilon)<M.\end{aligned}$$

Thus $\left(1-\frac{1}{m}\right)f^n$ belongs to U^M and therefore belongs to χ^M. Furthermore,

$$\left\|\left(1-\frac{1}{m}\right)f^n-x\right\|_{L_\infty}\leqslant\left\|\left(1-\frac{1}{m}\right)x-x\right\|_{L_\infty}+\frac{\varepsilon}{2}\leqslant\frac{1}{m}\|x\|_{L_\infty}+\frac{\varepsilon}{2}\leqslant\frac{\varepsilon}{M}\|x\|_{L_\infty}+\frac{\varepsilon}{2}.$$

Thus $(\chi^M,\mathcal{F}^M)$ is a separable metric space.

We deduce that $(\Omega_1,\mathcal{G}^M)$ is also standard, since $\mathcal{F}^M$ and $\mathcal{G}^M$ are σ-isomorphic. Indeed, let $p:x\in\Omega_1\mapsto x_{.\wedge\zeta^M_x}\in\chi^M$. Let $p^{-1}:\mathcal{F}^M\to\mathcal{G}^M$ the map associating to every element of $\mathcal{F}^M$ its pre-image by p. Then p^{-1} is a σ-isomorphism, since $\mathcal{G}^M=\left\{(\pi^M)^{-1}(A):A\in\mathcal{B}(\Omega_1)\right\}=\left\{p^{-1}(A):A\in\mathcal{F}^M\right\}$. Therefore, $(\Omega,\mathcal{G}^M\otimes\mathcal{B}(\Omega_2))$ is also a standard Borel space.

Let us check statement (iv). Let A_n be an atom of $\mathcal{G}^{M_n}\otimes\mathcal{B}(\Omega_2)$. There exist $w_1^n\in\chi^{M_n}$ and $w_2^n\in\Omega_2$ such that $A_n=\pi_n^{-1}(\{w_1^n\})\times\{w_2^n\}$, where $\pi_n=\pi^{M_n}$. If $n\leqslant n'$, we have $A_n\supseteq A_{n'}$, thus $w_2^n=w_2^{n'}$. If $y\in\pi_n^{-1}(\{w_1^n\})\cap\pi_{n'}^{-1}(\{w_1^{n'}\})$ then

$$\begin{aligned}&\text{for every } t\leqslant\zeta_y^{M_n}, y_t=(w_1^n)_t;\\&\text{for every } t\leqslant\zeta_y^{M_{n'}}, y_t=(w_1^{n'})_t.\end{aligned}$$

In particular, $(w_1^n)_t=(w_1^{n'})_t$ for every $t\leqslant\zeta^{M_n}_{w_1^n}$. By recursion, we construct a couple $(y,z)\in\Omega_1\times\Omega_2$ such that for every n, $y_t=(w_1^n)_t$ for every $t\leqslant\zeta_y^{M_n}$ and $z=w_2^n$. Thus $(y,z)\in\bigcap_n A_n$ and $\bigcap_n A_n$ is non empty. □

We deduce the existence and the uniqueness of a weak solution to equation (Eq_∞):

Theorem II.33. *Let $g\in\mathbf{G}^1$ and $j\in\mathbb{N}\backslash\{0\}$. Let f be a function of order $\alpha>\frac{3}{2}$. Let $B:\mathbb{R}\times L_2[0,1]\to\mathbb{R}$ satisfy the B-hypotheses of order $2j$. If $2j\geqslant\alpha$, there exists a weak solution to equation (Eq_∞) and uniqueness in law holds for this equation.*

Proof. By Proposition II.30, the family of probability measures $(\mathbb{Q}^M)_{M\geqslant M_0}$ is consistent on $\mathcal{G}^M\otimes\mathcal{B}(\Omega_2)$. By Proposition II.32 and by Theorem *V*.4.2 in [Par67, p.143], there is a probability measure $\mathbb{Q}$ defined on $(\Omega,\mathcal{B}(\Omega))$ such that for each $M\geqslant M_0$, for each $A\in\mathcal{G}^M$ and for each $B\in\mathcal{B}(\Omega_2)$, $\mathbb{Q}[A\times B]=\mathbb{Q}^M[A\times B]$.

In particular, since the event $\{z\in\Omega_1:\sup_{t\leqslant T}|z_t(1)-z_t(0)|<M\}$ is equal to the event $\{z\in\Omega_1:\sup_{t\leqslant T}|z_{t\wedge\zeta^M}(1)-z_{t\wedge\zeta^M}(0)|<M\}$ and belongs to $\mathcal{G}^M$, we have

$$\begin{aligned}\mathbb{Q}\left[\zeta^M=T\right]\geqslant\mathbb{Q}\left[\sup_{t\leqslant T}|z_t(1)-z_t(0)|<M\right]&=\mathbb{Q}\left[\sup_{t\leqslant T}|z_{t\wedge\zeta^M}(1)-z_{t\wedge\zeta^M}(0)|<M\right]\\&=\mathbb{Q}^M\left[\sup_{t\leqslant T}|z_{t\wedge\zeta^M}(1)-z_{t\wedge\zeta^M}(0)|<M\right].\end{aligned}\quad(\text{II.54})$$

We want to prove that $\mathbb{Q}^M\left[\sup_{t\leqslant T}|z_{t\wedge\zeta^M}(1)-z_{t\wedge\zeta^M}(0)|<M\right]$ tends to 1 when $M\to+\infty$.

Let us define $\widetilde{z}_t := z_t - \int_0^t B(z_s(\cdot),z_s)\mathrm{d}s$ and $N_t := \widetilde{z}_{t\wedge\zeta^M}(1)-\widetilde{z}_{t\wedge\zeta^M}(0)$. We have

$$N_t = g(1)-g(0)+\int_0^{t\wedge\zeta^M}\int_{\mathbb{R}} f(k)\left(\frac{\cos(kz_s(1))}{m_s(1)^{1/2}}-\frac{\cos(kz_s(0))}{m_s(0)^{1/2}}\right)\mathrm{d}w^{\Re}(k,s)$$
$$+\int_0^{t\wedge\zeta^M}\int_{\mathbb{R}} f(k)\left(\frac{\sin(kz_s(1))}{m_s(1)^{1/2}}-\frac{\sin(kz_s(0))}{m_s(0)^{1/2}}\right)\mathrm{d}w^{\Im}(k,s).$$

Therefore, $(N_t)_{t\in[0,T]}$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale under the measure $\mathbb{Q}^M$. Furthermore, for each $t\in[0,T]$,

$$N_t = g(1)-g(0)+\int_0^t N_s\mathrm{d}R_s^{\Re}+\int_0^t N_s\mathrm{d}R_s^{\Im},$$

where

$$R_t^{\Re}=\int_0^{t\wedge\zeta^M}\int_{\mathbb{R}}\frac{\mathbb{1}_{\{\widetilde{z}_s(1)\neq\widetilde{z}_s(0)\}}}{\widetilde{z}_s(1)-\widetilde{z}_s(0)}f(k)\left(\frac{\cos(kz_s(1))}{m_s(1)^{1/2}}-\frac{\cos(kz_s(0))}{m_s(0)^{1/2}}\right)\mathrm{d}w^{\Re}(k,s),$$
$$R_t^{\Im}=\int_0^{t\wedge\zeta^M}\int_{\mathbb{R}}\frac{\mathbb{1}_{\{\widetilde{z}_s(1)\neq\widetilde{z}_s(0)\}}}{\widetilde{z}_s(1)-\widetilde{z}_s(0)}f(k)\left(\frac{\sin(kz_s(1))}{m_s(1)^{1/2}}-\frac{\sin(kz_s(0))}{m_s(0)^{1/2}}\right)\mathrm{d}w^{\Im}(k,s).$$

We want to check that

$$\mathbb{1}_{s\leqslant\zeta^M}\int_{\mathbb{R}}\left|\frac{\mathbb{1}_{\{\widetilde{z}_s(1)\neq\widetilde{z}_s(0)\}}}{\widetilde{z}_s(1)-\widetilde{z}_s(0)}f(k)\left(\frac{\cos(kz_s(1))}{m_s(1)^{1/2}}-\frac{\cos(kz_s(0))}{m_s(0)^{1/2}}\right)\right|^2\mathrm{d}k\leqslant C_M. \tag{II.55}$$

As in the proof of Proposition II.5, we can show that

$$\left|\frac{\cos(kz_s(1))}{m_s(1)^{1/2}}-\frac{\cos(kz_s(0))}{m_s(0)^{1/2}}\right|^2\leqslant C_M\langle k\rangle^2|z_s(1)-z_s(0)|^2.$$

On the one hand, $\int_{\mathbb{R}}\langle k\rangle^2 f(k)^2\mathrm{d}k<\infty$ by assumption on f. On the other hand, we have

$$z_{t\wedge\zeta^M}(1)-z_{t\wedge\zeta^M}(0)=\widetilde{z}_{t\wedge\zeta^M}(1)-\widetilde{z}_{t\wedge\zeta^M}(0)+\int_0^{t\wedge\zeta^M}(B(z_s(1),z_s)-B(z_s(0),z_s))\mathrm{d}s.$$

By assumption on B (recall Definition II.14), for every $s\leqslant\zeta_M$, $|B(z_s(1),z_s)-B(z_s(0),z_s)|\leqslant C_1(M)|z_s(1)-z_s(0)|$. Thus almost surely, for every $t\in[0,T]$, we have

$$|z_{t\wedge\zeta^M}(1)-z_{t\wedge\zeta^M}(0)|\leqslant|\widetilde{z}_{t\wedge\zeta^M}(1)-\widetilde{z}_{t\wedge\zeta^M}(0)|+C_1(M)\int_0^t|z_{s\wedge\zeta^M}(1)-z_{s\wedge\zeta^M}(0)|\mathrm{d}s.$$

By Gronwall's Lemma, we deduce that almost surely, for every $t\in[0,T]$,

$$|z_{t\wedge\zeta^M}(1)-z_{t\wedge\zeta^M}(0)|\leqslant C_M|\widetilde{z}_{t\wedge\zeta^M}(1)-\widetilde{z}_{t\wedge\zeta^M}(0)|.$$

Therefore inequality (II.55) holds true. We deduce as in the proof of Proposition II.5 that N_t has an exponential form, and then that $N_t>0$ almost surely for every $t\in[0,T]$.

Therefore, under $\mathbb{Q}^M$, $(N_t)_{t\in[0,T]}$ is a strictly positive $(\mathcal{G}_t)_{t\in[0,T]}$-martingale; hence there is a Brownian motion $(\beta_t)_{t\geqslant0}$ under $\mathbb{Q}^M$ such that $N_t=g(1)-g(0)+\beta_{\langle N,N\rangle_t}$. We denote by $(\gamma_t)_{t\geqslant0}$ the Brownian motion $(g(1)-g(0)+\beta_t)_{t\geqslant0}$. We have $N_t=\gamma_{\langle N,N\rangle_t}$ for each $t\in[0,T]$ and we deduce that

$$\mathbb{Q}^M\left[\sup_{t\leqslant T}|N_t|\geqslant\frac{M}{2}\right]\leqslant\mathbb{Q}^M\left[(\gamma_t)_{t\geqslant0}\text{ reaches }\frac{M}{2}\text{ before }0\right]\leqslant\frac{g(1)-g(0)}{M/2}\underset{M\to+\infty}{\longrightarrow}0. \tag{II.56}$$

Moreover, by assumption on B, for every $s \leqslant \zeta_M$, $|B(z_s(1), z_s) - B(z_s(0), z_s)| \leqslant 2C_0(M)$. Thus by Markov's inequality

$$\mathbb{Q}^M \left[\sup_{t \leqslant T} \left| \int_0^{t \wedge \zeta^M} (B(z_s(1), z_s) - B(z_s(0), z_s)) \mathrm{d}s \right| \geqslant \frac{M}{2} \right] \leqslant \frac{2}{M} 2TC_0(M) \underset{M \to +\infty}{\longrightarrow} 0, \qquad \text{(II.57)}$$

since by assumption on B, $\frac{C_0(M)}{M} \underset{M \to +\infty}{\longrightarrow} 0$.

By (II.56) and (II.57), it follows that

$$\begin{aligned} &\mathbb{Q}^M \left[\sup_{t \leqslant T} |z_{t \wedge \zeta^M}(1) - z_{t \wedge \zeta^M}(0)| \geqslant M \right] \\ &\leqslant \mathbb{Q}^M \left[\sup_{t \leqslant T} |N_t| \geqslant \frac{M}{2} \right] + \mathbb{Q}^M \left[\sup_{t \leqslant T} \left| \int_0^{t \wedge \zeta^M} (B(z_s(1), z_s) - B(z_s(0), z_s)) \mathrm{d}s \right| \geqslant \frac{M}{2} \right] \underset{M \to +\infty}{\longrightarrow} 0. \end{aligned}$$

Recalling inequality (II.54), we conclude that $\mathbb{Q}\left[\zeta^M = T\right] \underset{M \to +\infty}{\longrightarrow} 1$.

It follows that $\mathbb{Q}\left[\bigcup_{\{M \geqslant M_0\}} \{\zeta^M = T\}\right] = 1$. We deduce that $\mathbb{Q}$-almost surely, there is $M \geqslant M_0$ such that $\zeta^M = T$. Furthermore, let us define for each $M \geqslant M_0$

$$\begin{aligned} E^M := \Big\{ & z \in \Omega_1 : \forall t \in [0,T], \forall u \in [0,1], z_{t \wedge \zeta^M}(u) = g(u) \\ & + \int_0^{t \wedge \zeta_M} \int_{\mathbb{R}} \frac{f(k)}{m_s(u)^{1/2}} \Big(\cos(k z_s(u)) \mathrm{d}w^{\Re}(k,s) + \sin(k z_s(u)) \mathrm{d}w^{\Im}(k,s) \Big) + \int_0^{t \wedge \zeta^M} B(z_s(u), z_s) \mathrm{d}s \Big\}. \end{aligned}$$

The event E^M is $\mathcal{G}^M$-measurable and $\mathbb{Q}^M\left[E^M\right] = 1$. It follows that $\mathbb{Q}\left[E^M\right] = 1$, whence we obtain $\mathbb{Q}\left[\bigcap_{\{M \geqslant M_0\}} E^M\right] = 1$. Therefore, $\mathbb{Q}$-almost surely, for every $t \in [0,T]$, equation (Eq_∞) holds true. Thus $(\Omega, \mathcal{B}(\Omega), (\mathcal{G}_t)_{t \in [0,T]}, \mathbb{Q}, z, w)$ is a weak solution to equation (Eq_∞).

Furthermore, let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0,T]}, \mathbb{P}, z, w)$ be a weak solution to equation (Eq_∞). Thus for each $M \geqslant M_0$, $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0,T]}, \mathbb{P}, z_{\cdot \wedge \zeta^M}, w)$ is a weak solution to equation (Eq_M). By Theorem II.28, we deduce that for each $M \geqslant M_0$, the law of $(z_{t \wedge \zeta^M})_{t \in [0,T]}$ is uniquely determined. Furthermore, by the previous martingale argument, we have $\mathbb{P}\left[\bigcup_{\{M \geqslant M_0\}} \left\{\zeta^M = T\right\}\right] = 1$. We deduce that the law of $(z_t)_{t \in [0,T]}$ is uniquely determined. □

II.5 A continuum of admissible drift functions

In this section, we make the connection between the result of restoration of uniqueness obtained in the previous section and earlier results of existence and uniqueness for standard McKean-Vlasov equations driven by a velocity field that is merely measurable in the space variable. The connection reads in the form of a new existence and uniqueness result but for a suitable notion of weak solution and for a class of admissible drifts. We address both in the next two subsections.

II.5.1 Description of the class of admissible drift functions

Recall the definition of the distance in total variation between two probability measures. For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$,

$$\mathrm{d}_{\mathrm{TV}}(\mu, \nu) = 2 \inf_{\substack{\mathcal{L}(X) = \mu \\ \mathcal{L}(Y) = \nu}} \mathbb{P}\left[X \neq Y\right]. \qquad \text{(II.58)}$$

The infimum is taken over every coupling (X, Y) of random variables X and Y in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ with respective distributions μ and ν, where $(\Omega, \mathcal{F}, \mathbb{P})$ is any fixed Polish and atomless probability space.

Let us define the following space on which we will consider the drift function:

Definition II.34. Let $\eta > 0$ and $\delta \in [0, 1]$. We say that $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is of class $(H^\eta, \mathcal{C}^\delta)$ if there are measurable functions $\lambda^{\Re}, \lambda^{\Im} : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ and $\Lambda : \mathbb{R} \to \mathbb{R}_+$ such that for every $x \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$b(x, \mu) = \int_{\mathbb{R}} \langle k \rangle^{-\eta} \Big(\cos(kx)\lambda^{\Re}(k, \mu) + \sin(kx)\lambda^{\Im}(k, \mu) \Big) \mathrm{d}k,$$

where

- $\lambda := \lambda^{\Re} + i\lambda^{\Im}$ is bounded in the measure variable: for every $k \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$, $|\lambda(k, \mu)| \leqslant \Lambda(k)$;
- λ is δ-Hölder continuous in the measure variable: for every k and for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $|\lambda(k, \mu) - \lambda(k, \nu)| \leqslant \Lambda(k) \mathrm{d}_{\mathrm{TV}}(\mu, \nu)^\delta$;
- $\Lambda \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

In particular, if b is of class $(H^\eta, \mathcal{C}^\delta)$, then for every $\mu \in \mathcal{P}_2(\mathbb{R})$, the map $x \mapsto b(x, \mu)$ belongs to the Sobolev space $H^\eta(\mathbb{R})$. Indeed, denoting by $\mathcal{F}(b(\cdot, \mu))$ the Fourier transform of $b(\cdot, \mu)$, we have

$$\int_{\mathbb{R}} |\langle k \rangle^\eta \mathcal{F}(b(\cdot, \mu))(k)|^2 \mathrm{d}k \leqslant C \int_{\mathbb{R}} |\lambda(-k, \mu)|^2 \mathrm{d}k \leqslant C \int_{\mathbb{R}} \Lambda(-k)^2 \mathrm{d}k < +\infty.$$

Moreover, if b is of class $(H^\eta, \mathcal{C}^\delta)$, then for every $x \in \mathbb{R}$, $\mu \mapsto b(x, \mu)$ is δ-Hölder continuous in total variation distance:

$$|b(x, \mu) - b(x, \nu)| \leqslant \int_{\mathbb{R}} \langle k \rangle^{-\eta} |\lambda(k, \mu) - \lambda(k, \nu)| \mathrm{d}k \leqslant \int_{\mathbb{R}} \langle k \rangle^{-\eta} \Lambda(k) \mathrm{d}k \; \mathrm{d}_{\mathrm{TV}}(\mu, \nu)^\delta.$$

Since $\eta \geqslant 0$ and $\Lambda \in L_1(\mathbb{R})$, $\int_{\mathbb{R}} \langle k \rangle^{-\eta} \Lambda(k) \mathrm{d}k$ is finite.

In order to apply our strategy, we need to assume the following minimal regularity assumption on the drift b:

$$\eta > \frac{3}{2}(1 - \delta).$$

It describes a continuum of admissible drift functions b between the following two extremal classes:

- if $\delta = 0$: the drift is only bounded in the measure variable. We have seen in Section II.4 that we were able to restore uniqueness if the regularity in the space variable is $\mathcal{C}^2(\mathbb{R})$. By Remark II.20, it actually works for drift functions b which are at least $H^\eta(\mathbb{R})$ in the space variable, with $\eta > \frac{3}{2}$. It is consistent with $\eta > \frac{3}{2}(1 - \delta)$.
- if $\delta = 1$: the drift is Lipschitz-continuous in total variation with respect to the measure distance. Jourdain [Jou97], Mishura-Veretennikov [MV], Lacker [Lac18], Chaudru de Raynal-Frikha [CdRF] among others have proved results under this assumption if b is only measurable and bounded in the space variable. Our result applies if b belongs to $H^\eta(\mathbb{R})$ for some $\eta > 0$ and if the Fourier transform of b belongs to $L_1(\mathbb{R})$; it is a subset of the space $\mathcal{C}_0(\mathbb{R})$ of continuous functions vanishing at infinity.

II.5.2 Definition of the problem

Let us consider a new model, with the purpose to make a link between the results obtained in this chapter and recent regularization by noise results for McKean-Vlasov equations obtained among others by [Jou97, MV, Lac18, CdRF]. There are some important changes with respect to the model (II.37) studied in this chapter. The main modification consists in adding a Brownian motion β, independent of w, in order to take benefit from some additional regularizing effect. In short, the role of β in the model below is to smooth out the (finite dimensional) space variable in the drift coefficient. Obviously, this comes in contrast with the role of the Brownian sheet w, the action of which is to mollify the velocity field in the measure argument, as made clear by Theorem II.33. Of course, we know from the standard diffusive case (i.e. $w \equiv 0$ and $b(x,\mu) \equiv b(x)$) that, in order to fully benefit from the action of β onto the space variable, we should average out over all the possible realizations of β (for instance, we may consider the semi-group generated by the diffusion process). In the present context, this prompts us to disentangle the roles of the two noises β and w in the mean-field interaction. Similarly to the standard McKean-Vlasov model, we shall compute the law of the particle (i.e. the mean-field component) with respect to the noise carrying β and the initial condition, but, similarly to the model addressed in the previous section, we shall freeze the realization of w. According to the terminology that has been used in the literature (see in particular the mean-field game literature [GLL11, CD18b], see also the earlier references [Vai88, DV95, KX99, KX04, CF16]), β will be regarded as an idiosyncratic noise acting independently on each particle and w as a common (or systemic) noise. To sum-up, in the previous sections, we defined μ_t as $\mu_t = \mathrm{Leb}_{[0,1]} \circ (z_t)^{-1}$, the space $[0,1]$ therein carrying the initial condition in the form $z_0(u) = g(u)$ for $u \in [0,1]$. Implicitly, this allowed us to identify μ_t with the conditional law of z_t given $(w(k,s))_{k\in\mathbb{R}, s\leqslant t}$. Now, μ_t will be understood as the law of the particle over the randomness carrying both β and the initial condition. This remark is made more precise in Remark II.36. Importantly, we will address in Chapter III the regularization properties of the semi-group associated to a similar model featuring both an infinite-dimensional common noise and a one-dimensional idiosyncratic one.

There are two other modifications of the model introduced in this section. In the Girsanov's arguments that we will use in the following proofs, we will not be able to preserve the monotonicity of the solution with respect to the variable u, which is the common point of the processes studied in this thesis. So we decide to use the same framework as usual in the literature on McKean-Vlasov SDEs, namely we take as initial condition a random variable ξ of prescribed law, independent from β and w. Furthermore, we decide to consider the easiest possible assumption on the mass, namely that it is constant equal to one.

Let $\eta > 0$ and $\delta \in [0,1]$ be such that $\eta > \frac{3}{2}(1-\delta)$. Let $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be of class $(H^\eta, \mathcal{C}^\delta)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of order α, such that

$$\frac{3}{2} < \alpha \leqslant \frac{\eta}{1-\delta}, \tag{II.59}$$

(if $\delta = 1$, we just require that $\alpha > \frac{3}{2}$). The condition $\eta > \frac{3}{2}(1-\delta)$ insures that this choice of α is possible. Let μ_0 be any given initial condition in $\mathcal{P}_2(\mathbb{R})$. Let us consider the following SDE:

$$\begin{cases} \mathrm{d}z_t = \displaystyle\int_{\mathbb{R}} f(k)\Re(e^{-ikz_t}\mathrm{d}w(k,t)) + \mathrm{d}\beta_t + b(z_t,\mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^{\mu,W}) \text{ a.s.} \\ z_0 = \xi, \quad \mathcal{L}^{\mathbb{P}}(\xi) = \mu_0, \end{cases} \tag{II.60}$$

where the filtration $(\mathcal{G}_t^{\mu,W})_{t\in[0,T]}$ is defined by $\mathcal{G}_t^{\mu,W} := \sigma\{w(\cdot,s), \mu_s\,; s \leqslant t\}$ and where (μ, w) is independent of (β, ξ). *Note:* In that equation and in all this section, $\mathcal{L}^{\mathbb{P}}(X)$ denotes the law of the random variable X under the probability measure $\mathbb{P}$, that is the distribution $\mathbb{P} \circ X^{-1}$.

Let us define the notion of weak solution to (II.60):

Definition II.35. An element $\mathbf{\Omega} = (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}, z, w, \beta, \xi)$ is said to be a *weak solution* to equation (II.60) if

- $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space satisfying usual conditions,
- (w, β, ξ) are independent random variables on $(\Omega, \mathcal{G})$, where

  ▷ $w := (w^{\Re}, w^{\Im})$, with $(w^{\Re}(k,t))_{k\in\mathbb{R},t\in[0,T]}$ and $(w^{\Im}(k,t))_{k\in\mathbb{R},t\in[0,T]}$ two independent $(\mathcal{G}_t)_{t\in[0,T]}$-Brownian sheets under $\mathbb{P}$,

  ▷ $(\beta_t)_{t\in[0,T]}$ is a standard $(\mathcal{G}_t)_{t\in[0,T]}$-Brownian motion under $\mathbb{P}$,

  ▷ for any $t \in [0,T]$, the σ-field $\sigma\{w^{\Re}(k,t') - w^{\Re}(k,t),\ w^{\Im}(k,t') - w^{\Im}(k,t),\ \beta_{t'} - \beta_t;$ $k \in \mathbb{R},\ t' \in [t,T]\}$ is independent of $\mathcal{G}_t$ under $\mathbb{P}$,

  ▷ ξ has distribution μ_0 under $\mathbb{P}$;
- $(z_t)_{t\in[0,T]}$ is a continuous $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process satisfying $\mathbb{P}$-almost surely, for every $t \in [0,T]$,

$$z_t = \xi + \int_0^t \int_{\mathbb{R}} f(k)\Re(e^{-ikz_s}\mathrm{d}w(k,s)) + \beta_t + \int_0^t b(z_s, \mu_s)\mathrm{d}s.$$

- $(\mu_t)_{t\in[0,T]}$ is a $\mathcal{P}_2(\mathbb{R})$-valued continuous $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process such that, for every $t \in [0,T]$, $\mathbb{P}$-almost surely, $\mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^{\mu,W})$, where $\mathcal{G}_t^{\mu,W} := \sigma\{w(k,s), \mu_s\ ; k \in \mathbb{R}, s \leqslant t\}$,
- (μ, w) is independent of (β, ξ) under $\mathbb{P}$ (and thus (μ, w), β and ξ are independent) and, more generally, for every $t \in [0,T]$, the processes (ξ, w, μ) and β are conditionally independent given $\mathcal{G}_t$.

Remark II.36. The last two conditions are certainly the most difficult ones to understand. In fact, both are dictated by the fact that we are looking for weak solutions only: *a priori*, nothing is said on the measurability of z and μ with respect to the inputs ξ, w and β. In particular, at this stage, μ may not be measurable with respect to w (which comes in contrast with the intuitive explanations we gave in introduction of the section). This is the rationale for defining the McKean-Vlasov constraint in terms of the conditional law of z_t given the σ-field generated (up to time t) not only by w but also by μ itself. Similarly, the last constraint is known as a compatibility condition and has been widely used (in a slightly stronger manner) in the analysis of weak solutions to stochastic equations, see for instance [Kur07, Kur14]. In short, it says that the observation of z does not corrupt the independence property of (ξ, μ, w) and β. Quite obviously, see for instance [CD18b, Remark I.11], compatibility is automatically satisfied if μ is adapted with respect to the completion of $\mathcal{G}^W$, in which case the solution should be called semi-strong.

We will prove weak well-posedness for the SDE (II.60) in three steps: *i*) when the drift b is equal to zero; *ii*) when the drift b is bounded and Lipschitz-continuous in total variation distance with respect to the measure variable; *iii*) in the general case, when the drift b belongs to the class $(H^\eta, \mathcal{C}^\delta)$.

Let us first consider the case where the drift is zero:

$$\begin{cases} \mathrm{d}z_t = \displaystyle\int_{\mathbb{R}} f(k)\Re(e^{-ikz_t}\mathrm{d}w(k,t)) + \mathrm{d}\beta_t, \\ z_0 = \xi, \quad \mathcal{L}^{\mathbb{P}}(\xi) = \mu_0. \end{cases} \tag{II.61}$$

In this case, well-posedness holds even in a strong sense.

Proposition II.37. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of order $\alpha > \frac{3}{2}$. Then there is a unique strong solution to equation* (II.61)*. Moreover, if $\mathbf{\Omega^1}$ and $\mathbf{\Omega^2}$ are two solutions to* (II.61)*, then $\mathcal{L}^{\mathbb{P}^1}(z^1, w^1, \beta^1) = \mathcal{L}^{\mathbb{P}^2}(z^2, w^2, \beta^2)$.*

Proof. Strong well-posedness can be proved by a classical fixed-point argument, as in the proof of Proposition II.2 for example (but the proof is now easier since the mass is equal to 1 everywhere). The additional noise β does not change anything to this proof. Moreover, the assumption $\alpha > \frac{3}{2}$ insures that the assumption of square integrability of $k \mapsto \langle k \rangle f(k)$ is satisfied (see Proposition II.2); in other words, it insures that the diffusive coefficient in front of the noise w is Lipschitz-continuous.

Furthermore, by Yamada-Watanabe Theorem, the law of (z, w, β) solution to (II.61) is uniquely determined. That result is stated and proved in [KS91, Prop 5.3.20, p.309] in a finite-dimensional case, but the proof is the same for an infinite dimensional noise. Moreover, a corollary to Yamada-Watanabe Theorem [KS91, Cor 5.3.23, p.310] states the following result: if $\mathbf{\Omega} = (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}, z, w, \beta, \xi)$ is a solution to (II.61), then $\mathbb{P}$-almost surely, for every $t \in [0, T]$,

$$z_t = \mathcal{Z}_t(\xi, w, \beta),$$

where $\mathcal{Z}$ is a function defined on the canonical space

$$\begin{aligned} \mathcal{Z} : \mathbb{R} \times \mathcal{C}(\mathbb{R} \times [0,T], \mathbb{R}^2) \times \mathcal{C}([0,T], \mathbb{R}) &\to \mathcal{C}([0,T], \mathbb{R}) \\ (x, \omega^W, \omega^\beta) &\mapsto \mathcal{Z}(x, \omega^W, \omega^\beta) \end{aligned} \tag{II.62}$$

which is progressively measurable with respect to the canonical filtration on $\mathcal{C}(\mathbb{R} \times [0,T], \mathbb{R}^2) \times \mathcal{C}([0,T], \mathbb{R})$. Remark that $\mathcal{C}([0,T], \mathbb{R})$ represents here the canonical space on which we define the Wiener measure of a standard Wiener process on $[0,T]$, and $\mathcal{C}(\mathbb{R} \times [0,T], \mathbb{R}^2)$ represents the Wiener space associated to the measure of a $\mathbb{R}^2$-valued Brownian sheet $(w^{\Re}, w^{\Im})$ on $\mathbb{R} \times [0,T]$. □

II.5.3 Resolution of the SDE when the drift is Lipschitz continuous

Let us assume that $\widetilde{b} : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is uniformly bounded and uniformly Lipschitz-continuous in the measure variable. We consider the following SDE with a drift $\widetilde{b}$:

$$\begin{cases} \mathrm{d}z_t = \displaystyle\int_{\mathbb{R}} f(k) \Re(e^{-ikz_t} \mathrm{d}w(k,t)) + \mathrm{d}\beta_t + \widetilde{b}(z_t, \mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu, W}), \\ z_0 = \xi, \quad \mathcal{L}^{\mathbb{P}}(\xi) = \mu_0, \end{cases} \tag{II.63}$$

with the same assumptions and the same interpretation as in Definition II.35. Let us prove existence and uniqueness of a weak solution.

Proposition II.38. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of order $\alpha > \frac{3}{2}$. Let $\widetilde{b} : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a function such that there exists $C > 0$ satisfying for every $x \in \mathbb{R}$ and for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$*

- $|\widetilde{b}(x, \mu)| \leqslant C$;
- $|\widetilde{b}(x, \mu) - \widetilde{b}(x, \nu)| \leqslant C \mathrm{d}_{\mathrm{TV}}(\mu, \nu)$.

Then there exists a weak solution to (II.63)*.*

Proposition II.39. *Under the same assumptions as Proposition II.38, if $\mathbf{\Omega^1}$ and $\mathbf{\Omega^2}$ are two weak solutions to* (II.63)*, then $\mathcal{L}^{\mathbb{P}^1}(z^1, w^1) = \mathcal{L}^{\mathbb{P}^2}(z^2, w^2)$. In particular, uniqueness in law holds for the SDE* (II.63)*. Moreover, for any weak solution $\mathbf{\Omega}$, $(\mu_t)_{t\in[0,T]}$ is adapted to the completion of $(\mathcal{G}_t^W = \sigma\{w(\cdot, s)\ ; s \leqslant t\})_{t\in[0,T]}$.*

Note that the statement of Proposition II.39 shows that the weak solution of (II.63) is adapted to the filtration generated by the noise w.

Remark II.40. The question of the filtration under which the measure-valued process $(\mu_t)_{t\in[0,T]}$ is adapted is important here. Actually, we will see in the proof of existence that the weak solution that we will construct is automatically adapted with respect to the filtration generated by w. Nevertheless, we want to give a more general statement for uniqueness, *i.e.* we want to be able to compare two weak solutions where $(\mu_t)_{t\in[0,T]}$ is adapted with respect to a filtration generated by w and possibly another source of randomness, provided (μ, W) remains independent of (β, ξ). This will be useful in the proof of Theorem II.43, which states well-posedness for the SDE with $(H^\eta, \mathcal{C}^\delta)$-drift b, since for this general case, our proof based on Girsanov's Theorem does not imply that $(\mu_t)_{t\in[0,T]}$ is adapted with respect to the filtration generated by w (see Remark II.45).

The assumptions on $\widetilde{b}$ are the same as in [Lac18]. We will essentially apply the same proof, which we will recall hereafter.

II.5.3.1 Existence of a weak solution to the intermediate SDE

Let us prove in this paragraph Proposition II.38. We begin by constructing a weak solution on the canonical space.

Proof (Proposition II.38). Let us consider the filtered canonical probability space, denoted by $(\Omega^W, \mathcal{G}^W, (\mathcal{G}_t^W)_{t\in[0,T]}, \mathbb{P}^W)$, where $\Omega^W := \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2)$, $\mathcal{G}^W$ is the Borel σ-algebra on Ω^W, $(\mathcal{G}_t^W)_{t\in[0,T]}$ is the canonical filtration on $(\Omega^W, \mathcal{G}^W)$ and $\mathbb{P}^W$ is the probability measure on $(\Omega^W, \mathcal{G}^W)$ such that the distribution of the random variable $w^W \mapsto w^W$ is the law of two independent (real-valued) Brownian sheets on $\mathbb{R}\times[0,T]$.

Let $(\Omega^{\beta,\xi}, \mathcal{G}^{\beta,\xi}, (\mathcal{G}_t^{\beta,\xi})_{t\in[0,T]}, \mathbb{P}^{\beta,\xi})$ be another filtered probability space on which we define two independent random variables ξ and $(\beta_t)_{t\in[0,T]}$ such that $(\beta_t)_{t\in[0,T]}$ is a $(\mathcal{G}_t^{\beta,\xi})_{t\in[0,T]}$-adapted Brownian motion and such that the law of ξ is μ_0.

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ be the product space: $\Omega = \Omega^W \times \Omega^{\beta,\xi}$, $\mathcal{G} = \mathcal{G}^W \otimes \mathcal{G}^{W,\beta}$, $\mathcal{G}_t = \sigma(\mathcal{G}_t^W, \mathcal{G}_t^{\beta,\xi})$ and $\mathbb{P} = \mathbb{P}^W \otimes \mathbb{P}^{\beta,\xi}$. In particular, w is independent of (β, ξ) under $\mathbb{P}$. Up to adding negligible subsets, we assume that the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ is complete. Let $(z_t)_{t\in[0,T]}$ be the unique solution on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ of the SDE:

$$\begin{cases} \mathrm{d}z_t = \displaystyle\int_{\mathbb{R}} f(k)\Re(e^{-ikz_t}\mathrm{d}w(k,t)) + \mathrm{d}\beta_t, \\ z_0 = \xi, \quad \mathcal{L}^{\mathbb{P}}(\xi) = \mu_0. \end{cases} \tag{II.64}$$

Existence and uniqueness of a strong solution to (II.64) is given by Proposition II.37. Furthermore, by Yamada-Watanabe Theorem, there is a $(\mathcal{G}_t)_{t\in[0,T]}$-progressively measurable map $\mathcal{Z}_t$ as defined in (II.62) such that $\mathbb{P}$-almost surely, $z_t = \mathcal{Z}_t(\xi, w, \beta)$.

Let us denote by $\mathcal{C}$ the space $\mathcal{C}([0,T], \mathbb{R})$ and by $\mathcal{P}(\mathcal{C})$ the space of probability measures on $\mathcal{C}$. For each time $t \in [0,T]$, let us denote by $\pi_t : \mu^T \in \mathcal{P}(\mathcal{C}) \mapsto \mu^t \in \mathcal{P}(\mathcal{C})$ the map associating to μ^T the push-forward measure of μ^T by the map $x \in \mathcal{C} \mapsto x_{\cdot\wedge t} \in \mathcal{C}$. Let $(\mathcal{X}, d)$ be the complete metric space of functions

$$\begin{aligned} \mu : \Omega^W = \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2) &\to \mathcal{P}(\mathcal{C}) \\ w &\mapsto \mu^T(w), \end{aligned}$$

such that, for each $t \in [0,T]$, $(\mu^t = \pi^t(\mu^T))_{t\in[0,T]}$ is $(\mathcal{G}_t^W)_{t\in[0,T]}$-progressively measurable. The distance d is defined by $d(\mu,\nu) := \mathbb{E}^W\left[\mathrm{d_{TV}}(\mu^T, \nu^T)^2\right]^{1/2}$, where $\mathrm{d_{TV}}$ is here understood as the

total variation distance on $\mathcal{P}(\mathcal{C})$ (while we defined it before on $\mathcal{P}(\mathbb{R})$). Furthermore, for $\mu \in \mathcal{X}$ and for $t \in [0,T]$, we call μ_t the image of μ by the mapping $x \in \mathcal{C} \mapsto x_t \in \mathbb{R}$.

Let $\nu \in \mathcal{X}$. Recall that $\widetilde{b} : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is uniformly bounded. Therefore

$$\mathcal{E}_t^\nu := \exp\left(\int_0^t \widetilde{b}(z_s, \nu_s)\mathrm{d}\beta_s - \frac{1}{2}\int_0^t |\widetilde{b}(z_s,\nu_s)|^2 \mathrm{d}s\right)$$

is a $(\mathcal{G}_t)_{t\in[0,T]}$-martingale. Let $\mathbb{P}^\nu$ be the probability measure on $(\Omega, \mathcal{G})$ absolutely continuous with respect to $\mathbb{P} = \mathbb{P}^W \otimes \mathbb{P}^{\beta,\xi}$, with density:

$$\frac{\mathrm{d}\mathbb{P}^\nu}{\mathrm{d}\mathbb{P}} = \mathcal{E}_T^\nu. \tag{II.65}$$

For every $w \in \Omega^W$, let us denote by $\mathbb{P}^{\nu,\beta,\xi}(w)$ the probability measure on $(\Omega^{\beta,\xi}, \mathcal{G}^{\beta,\xi})$ with the following density with respect to $\mathbb{P}^{\beta,\xi}$:

$$\frac{\mathrm{d}\mathbb{P}^{\nu,\beta,\xi}(w)}{\mathrm{d}\mathbb{P}^{\beta,\xi}} = \mathcal{E}_T^\nu(w).$$

Equivalently, $\mathbb{P}^{\nu,\beta,\xi} : \Omega^W \times \mathcal{G}^{\beta,\xi} \to \mathbb{R}_+$ is also defined as the conditional probability satisfying for every $(A^W, A^{\beta,\xi}) \in \mathcal{G}^W \times \mathcal{G}^{\beta,\xi}$

$$\mathbb{P}^\nu(A^W \times A^{\beta,\xi}) = \int_{A^W} \mathbb{P}^{\nu,\beta,\xi}(w, A^{\beta,\xi})\mathrm{d}\mathbb{P}^W(w).$$

Let us define $\mathrm{d}\widetilde{\beta}_t^\nu := \mathrm{d}\beta_t - \widetilde{b}(z_t,\nu_t)\mathrm{d}t$. By Girsanov's Theorem, $(\widetilde{\beta}_t^\nu)_{t\in[0,T]}$ is a Brownian motion under the measure $\mathbb{P}^\nu$, $(\widetilde{\beta}^\nu, \xi, w)$ are independent under $\mathbb{P}^\nu$ and, for any $t \in [0,T]$, the σ-field $\sigma\{w(k,t') - w(k,t), \widetilde{\beta}_{t'}^\nu - \widetilde{\beta}_t^\nu;\ k \in \mathbb{R},\ t' \in [t,T]\}$ is independent of $\mathcal{G}_t$ under $\mathbb{P}^\nu$. Moreover the process $(z_t)_{t\in[0,T]}$ satisfies:

$$\mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t}\mathrm{d}w(k,t)) + \mathrm{d}\widetilde{\beta}_t^\nu + \widetilde{b}(z_t,\nu_t)\mathrm{d}t.$$

If ν satisfies for every $t \in [0,T]$, $\mathbb{P}^W$-almost surely,

$$\nu_t = \mathcal{L}^{\mathbb{P}^\nu}(z_t | \mathcal{G}_t^W), \tag{II.66}$$

then it also satisfies for every $t \in [0,T]$, $\mathbb{P}^W$-almost surely, $\nu_t = \mathcal{L}^{\mathbb{P}^\nu}(z_t|\mathcal{G}_t^{\nu,W})$, where $\mathcal{G}_t^{\nu,W} = \sigma\{w(\cdot,s), \nu_s\ ; s \leqslant t\}$. Furthermore, (ν, w) is adapted to the completion of $\mathcal{G}^W$; hence under $\mathbb{P}^\nu$, (ν, w) is independent of $(\tilde{\beta}^\nu, \xi)$, and by Remark II.36 the compatibility condition is automatically satisfied. Thus if (II.66) is satisfied for any $t \in [0,T]$ $\mathbb{P}^W$-almost surely, then $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}^\nu, z, w, \widetilde{\beta}^\nu, \xi)$ is a weak solution to (II.63). Equivalently, it is solution if for $\mathbb{P}^W$-almost every $w \in \Omega^W$, for every $t \in [0,T]$, $\nu_t(w) = \mathcal{L}^{\mathbb{P}^{\nu,\beta,\xi}(w)}(\mathcal{Z}_t(\cdot,w,\cdot))$ (the latter obviously implying (II.66) and the converse following from the fact that, in (II.66), $\mathcal{G}_t^W$ can be replaced by $\mathcal{G}_T^W$, which implies not only that, for any $t \in [0,T]$, for $\mathbb{P}^W$-almost every $w \in \Omega^W$, $\nu_t(w) = \mathcal{L}^{\mathbb{P}^{\nu,\beta,\xi}(w)}(\mathcal{Z}_t(\cdot,w,\cdot))$ but also that the quantifiers *for all* t and *for* $\mathbb{P}^W$*-almost every* can be exchanged by a standard continuity argument). Notice in particular, that by Fubini's Theorem, $w \mapsto \mathcal{L}^{\mathbb{P}^{\nu,\beta,\xi}(w)}(\mathcal{Z}_t(\cdot,w,\cdot))$ is $\mathcal{G}_t^W$-measurable.

Let us prove that there is a process $\nu \in \mathcal{X}$ satisfying (II.66). For every $\nu \in \mathcal{X}$, let us define $\phi(\nu)_t := w \mapsto \mathcal{L}^{\mathbb{P}^{\nu,\beta,\xi}(w)}(\mathcal{Z}_t(\cdot,w,\cdot))$. By construction, $\phi(\nu)$ also belongs to $\mathcal{X}$. For every $\mu^T \in \mathcal{X}$ and for every $t \in [0,T]$, let us denote by $\mu^t \in \mathcal{P}(\mathcal{C})$ the push-forward measure of μ^T through the map $x \in \mathcal{C} \mapsto x_{\cdot\wedge t} \in \mathcal{C}$. In particular, for every $t \in [0,T]$, $\phi(\nu)^t = \mathcal{L}^{\mathbb{P}^{\nu,\beta,\xi}(w)}(\mathcal{Z}_{\cdot\wedge t}(\cdot,w,\cdot))$. For $\mu, \nu \in \mathcal{P}_2(\mathcal{C})$, let us denote by $H(\mu|\nu)$ the relative entropy

$$H(\mu|\nu) = \int_{\mathcal{C}} \ln\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\mathrm{d}\mu \quad \text{if } \mu \ll \nu, \qquad H(\mu|\nu) = +\infty \quad \text{otherwise.}$$

Let us apply the same proof as [Lac18, Thm 2.4]. Let us state the following lemma, which is shown at the end of the current proof.

Lemma II.41. *For every $\mu, \nu \in \mathcal{X}$ and for every $t \in [0,T]$,*

$$H(\phi(\mu)^t|\phi(\nu)^t) = \frac{1}{2}\mathbb{E}^{\mathbb{P}^\mu}\left[\int_0^t |\widetilde{b}(z_s,\nu_s) - \widetilde{b}(z_s,\mu_s)|^2 \mathrm{d}s \Big| \mathcal{G}_t^W\right].$$

By Lipschitz-continuity of $\widetilde{b}$, there is $C > 0$ such that

$$\begin{aligned} H(\phi(\mu)^t|\phi(\nu)^t) &\leqslant C\int_0^t \mathbb{E}^{\mathbb{P}^\mu}\left[\mathrm{d}_{\mathrm{TV}}(\mu_s,\nu_s)^2\Big|\mathcal{G}_t^W\right]\mathrm{d}s = C\int_0^t \mathrm{d}_{\mathrm{TV}}(\mu_s,\nu_s)^2\mathrm{d}s \\ &\leqslant C\int_0^t \mathrm{d}_{\mathrm{TV}}(\mu^s,\nu^s)^2\mathrm{d}s. \end{aligned}$$

By Pinsker's inequality, $\mathrm{d}_{\mathrm{TV}}(\phi(\mu)^t,\phi(\nu)^t)^2 \leqslant 2H(\phi(\mu)^t|\phi(\nu)^t)$. Therefore, there is C such that for every $t \in [0,T]$,

$$\mathbb{E}^W\left[\mathrm{d}_{\mathrm{TV}}(\phi(\mu)^t,\phi(\nu)^t)^2\right] \leqslant C\int_0^t \mathbb{E}^W\left[\mathrm{d}_{\mathrm{TV}}(\mu^s,\nu^s)^2\right]\mathrm{d}s. \tag{II.67}$$

For every $n \in \mathbb{N}\backslash\{0\}$, let us write $\phi^{\circ n}$ for $\underbrace{\phi\circ\cdots\circ\phi}_{n \text{ times}}$. It follows from a simple recursion and from (II.67) that for every $t \in [0,T]$ and $n \in \mathbb{N}$

$$\mathbb{E}^W\left[\mathrm{d}_{\mathrm{TV}}(\phi^{\circ n}(\mu)^t,\phi^{\circ n}(\nu)^t)^2\right] \leqslant \frac{C^n t^n}{n!}\mathbb{E}^W\left[\mathrm{d}_{\mathrm{TV}}(\mu^t,\nu^t)^2\right].$$

Recall that the distance d on $\mathcal{X}$ is defined by $d(\mu,\nu) = \mathbb{E}^W\left[\mathrm{d}_{\mathrm{TV}}(\mu^T,\nu^T)^2\right]^{1/2}$. Thus for every $n \geqslant 1$ and for every $\mu,\nu \in \mathcal{X}$,

$$d(\phi^{\circ n}(\mu),\phi^{\circ n}(\nu))^2 \leqslant \frac{C^n T^n}{n!}d(\mu,\nu)^2.$$

Therefore, for n large enough so that $\frac{C^n T^n}{n!} < 1$, $\phi^{\circ n}$ is a contraction. Therefore, by Picard's fixed-point Theorem, there is a unique solution, called $\mu^\phi \in \mathcal{X}$, of $\mu^\phi = \phi(\mu^\phi)$. In particular, there exists a weak solution to equation (II.63). This completes the proof of Proposition II.38. □

Proof (Lemma II.41). Let us first compute for every $t \in [0,T]$,

$$H(\phi(\mu)^t|\phi(\nu)^t) = \int_{\mathcal{C}} \ln\frac{\mathrm{d}\phi(\mu)^t}{\mathrm{d}\phi(\nu)^t}\mathrm{d}\phi(\mu)^t = \mathbb{E}^{\mathbb{P}^\mu}\left[\ln\frac{\mathrm{d}\phi(\mu)^t}{\mathrm{d}\phi(\nu)^t}(z_{\cdot\wedge t})\Big|\mathcal{G}_t^W\right]. \tag{II.68}$$

Let us prove that

$$\frac{\mathrm{d}\phi(\mu)^t}{\mathrm{d}\phi(\nu)^t}(z_{\cdot\wedge t}) = \mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}\Big|\mathcal{G}_t^{z,W}\right], \tag{II.69}$$

where $\mathcal{G}_t^{z,W} = \sigma\{z_s, w(\cdot,s)\ ; s \leqslant t\}$. Indeed, for every measurable and bounded functions $f : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ and $g : \mathcal{C}(\mathbb{R}\times[0,T],\mathbb{R}^2) \to \mathbb{R}$ (recall that we denote by $\mathcal{C}$ the space $\mathcal{C}([0,T],\mathbb{R})$):

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}f(z_{\cdot\wedge t})g(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{P}^\mu}\left[f(z_{\cdot\wedge t})g(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{P}^\mu}\left[\int_{\mathcal{C}} f(x)\mathrm{d}\phi(\mu)^t(x)\ g(w_{\cdot\wedge t})\right] \\ &= \mathbb{E}^{\mathbb{P}^\mu}\left[\int_{\mathcal{C}} f(x)\frac{\mathrm{d}\phi(\mu)^t}{\mathrm{d}\phi(\nu)^t}(x)\mathrm{d}\phi(\nu)^t(x)\ g(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}\int_{\mathcal{C}} f(x)\frac{\mathrm{d}\phi(\mu)^t}{\mathrm{d}\phi(\nu)^t}(x)\mathrm{d}\phi(\nu)^t(x)\ g(w_{\cdot\wedge t})\right] \\ &= \mathbb{E}^{\mathbb{P}^\nu}\left[\mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}\Big|\mathcal{G}_t^W\right]\int_{\mathcal{C}} f(x)\frac{\mathrm{d}\phi(\mu)^t}{\mathrm{d}\phi(\nu)^t}(x)\mathrm{d}\phi(\nu)^t(x)\ g(w_{\cdot\wedge t})\right] \\ &= \mathbb{E}^{\mathbb{P}^\nu}\left[\mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}\Big|\mathcal{G}_t^W\right]f(z_{\cdot\wedge t})\frac{\mathrm{d}\phi(\mu)^t}{\mathrm{d}\phi(\nu)^t}(z_{\cdot\wedge t})\ g(w_{\cdot\wedge t})\right]. \end{aligned} \tag{II.70}$$

Moreover, recalling the relation $\mathrm{d}\beta_s = \mathrm{d}\widetilde{\beta}^\nu_s + \widetilde{b}(z_s,\nu_s)\mathrm{d}s$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}\Big|\mathcal{G}^W_t\right] = \mathbb{E}^{\mathbb{P}^\nu}\left[\mathcal{E}^\mu_T(\mathcal{E}^\nu_T)^{-1}\Big|\mathcal{G}^W_t\right]\\
&= \mathbb{E}^{\mathbb{P}^\nu}\left[\exp\left(\int_0^T(\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}\beta_s - \frac{1}{2}\int_0^T|\widetilde{b}(z_s,\mu_s)|^2\mathrm{d}s + \frac{1}{2}\int_0^T|\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\right)\Big|\mathcal{G}^W_t\right]\\
&= \mathbb{E}^{\mathbb{P}^\nu}\left[\exp\left(\int_0^T(\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}\widetilde{\beta}^\nu_s - \frac{1}{2}\int_0^T|\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\right)\Big|\mathcal{G}^W_t\right].
\end{aligned}$$

For every bounded and measurable $g : \mathcal{C}(\mathbb{R}\times[0,T],\mathbb{R}^2)\to\mathbb{R}$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\nu}\Big[\exp\Big(\int_0^T(\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}\widetilde{\beta}^\nu_s - \frac{1}{2}\int_0^T|\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\Big)g(w_{\cdot\wedge t})\Big]\\
&= \mathbb{E}^{\mathbb{P}^\nu}\Big[\mathbb{E}^{\mathbb{P}^{\nu,\beta,\xi}(w)}\Big[\exp\Big(\int_0^T(\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}\widetilde{\beta}^\nu_s - \frac{1}{2}\int_0^T|\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\Big)\Big]g(w_{\cdot\wedge t})\Big]\\
&= \mathbb{E}^{\mathbb{P}^\nu}\left[g(w_{\cdot\wedge t})\right],
\end{aligned}$$

since under $\mathbb{P}^\nu$, $\widetilde{\beta}^\nu$ and w are independent and since the exponential is a $\mathbb{P}^{\nu,\beta,\xi}(w)$-martingale by Novikov's condition (recalling that $\widetilde{b}$ is uniformly bounded). Thus

$$\mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}\Big|\mathcal{G}^W_t\right] = 1. \tag{II.71}$$

Using equalities (II.70) and (II.71), we get equality (II.69).

Therefore, back to equality (II.68), we obtain

$$H(\phi(\mu)^t|\phi(\nu)^t) = \mathbb{E}^{\mathbb{P}^\mu}\left[\ln\mathbb{E}^{\mathbb{P}^\nu}\left[\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}^\nu}\Big|\mathcal{G}^{z,W}_t\right]\Big|\mathcal{G}^W_t\right].$$

Recall that $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\in[0,T]},\mathbb{P}^\nu,z,w,\widetilde{\beta}^\nu,\xi)$ is a weak solution to (II.63). Thus $\mathbb{P}^\nu$-almost surely, $\widetilde{\beta}^\nu$ satisfies for every $t\in[0,T]$:

$$\widetilde{\beta}^\nu_t = z_t - z_0 - \int_0^t\int_{\mathbb{R}} f(k)\Re(e^{-ikz_s}\mathrm{d}w(k,s)) - \int_0^t\widetilde{b}(z_s,\mu_s)\mathrm{d}s.$$

Thus $(\widetilde{\beta}^\nu_t)_{t\in[0,T]}$ is $(\mathcal{G}^{z,W}_t)_{t\in[0,T]}$-adapted and we deduce that

$$\begin{aligned}
&H(\phi(\mu)^t|\phi(\nu)^t)\\
&\qquad= \mathbb{E}^{\mathbb{P}^\mu}\Big[\ln\exp\Big(\int_0^t(\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}\widetilde{\beta}^\nu_s - \frac{1}{2}\int_0^t|\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\Big)\Big|\mathcal{G}^W_t\Big]\\
&\qquad= \mathbb{E}^{\mathbb{P}^\mu}\Big[\int_0^t(\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}\widetilde{\beta}^\nu_s - \frac{1}{2}\int_0^t|\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\Big|\mathcal{G}^W_t\Big]\\
&\qquad= \mathbb{E}^{\mathbb{P}^\mu}\Big[\int_0^t(\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}\widetilde{\beta}^\mu_s + \frac{1}{2}\int_0^t|\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\Big|\mathcal{G}^W_t\Big]\\
&\qquad= \mathbb{E}^{\mathbb{P}^\mu}\Big[\frac{1}{2}\int_0^t|\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\Big|\mathcal{G}^W_t\Big],
\end{aligned}$$

because $\mathrm{d}\widetilde{\beta}^\nu_s - \mathrm{d}\widetilde{\beta}^\mu_s = (\widetilde{b}(z_s,\mu_s)-\widetilde{b}(z_s,\nu_s))\mathrm{d}s$. This completes the proof of the Lemma. □

II.5.3.2 Uniqueness in law for the intermediate SDE

Let us prove in this paragraph Proposition II.39.

Proof (Proposition II.39). Let $\mathbf{\Omega^1}$ and $\mathbf{\Omega^2}$ be two weak solutions to (II.63), often denoted by $\mathbf{\Omega^n}$, $n = 1, 2$. In particular, the process $(z_t^n)_{t\in[0,T]}$ satisfies $\mathbb{P}^n$-almost surely,

$$z_t^n = \xi^n + \int_0^t \int_{\mathbb{R}} f(k)\Re(e^{-ikz_s^n} \mathrm{d}w^n(k,s)) + \beta_t^n + \int_0^t \widetilde{b}(z_s^n, \mu_s^n)\mathrm{d}s,$$

where for every $t \in [0,T]$, $\mathbb{P}^n$-almost surely $\mu_t^n = \mathcal{L}^{\mathbb{P}^n}(z_t^n | \mathcal{G}_t^{\mu^n, W^n})$ and where (μ^n, w^n) is independent of (β^n, ξ^n).

Let $\mathbb{Q}^n$ be the probability measure on $(\Omega^n, \mathcal{G}^n)$ with the following density with respect to $\mathbb{P}^n$,

$$\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}^n} = \exp\left(- \int_0^T \widetilde{b}(z_s^n, \mu_s^n)\mathrm{d}\beta_s^n - \frac{1}{2}\int_0^T |\widetilde{b}(z_s^n, \mu_s^n)|^2 \mathrm{d}s \right). \tag{II.72}$$

Let $\widetilde{\beta}_t^n = \beta_t^n + \int_0^t \widetilde{b}(z_s^n, \mu_s^n)\mathrm{d}s$. By Girsanov's Theorem, $\mathcal{L}^{\mathbb{Q}^n}(w^n, \widetilde{\beta}^n, \xi^n) = \mathcal{L}^{\mathbb{P}^n}(w^n, \beta^n, \xi^n)$ and for any $t \in [0,T]$, the σ-field $\sigma\{w^n(k,t') - w^n(k,t), \widetilde{\beta}_{t'}^n - \widetilde{\beta}_t^n,\ k \in \mathbb{R},\ t' \in [t,T]\}$ is independent of $\mathcal{G}_t^n$ under $\mathbb{Q}^n$. It follows that $\widetilde{\mathbf{\Omega^n}} = (\Omega^n, \mathcal{G}^n, (\mathcal{G}_t^n)_{t\in[0,T]}, \mathbb{Q}^n, z^n, w^n, \widetilde{\beta}^n, \xi^n)$ is a weak solution to the SDE (II.61) with zero drift. By Proposition II.37, $\mathcal{L}^{\mathbb{Q}^1}(z^1, w^1, \widetilde{\beta}^1) = \mathcal{L}^{\mathbb{Q}^2}(z^2, w^2, \widetilde{\beta}^2)$ and $\mathbb{Q}^n$-almost surely, $z^n = \mathcal{Z}(\xi^n, w^n, \widetilde{\beta}^n)$, where $\mathcal{Z}$ is of the form (II.62).

Moreover, recall that for $n = 1, 2$, $\mu_t^n = \mathcal{L}^{\mathbb{P}^n}(z_t^n | \mathcal{G}_t^{\mu^n, W^n})$. Recall also that μ^ϕ is defined as being the unique fixed-point of ϕ in $\mathcal{X}$ (see proof of Proposition II.38). Let us state the following lemma, which will be shown at the end of the current proof.

Lemma II.42. *Let $n = 1, 2$. Then $\mathbb{Q}^n$-almost surely, for every $t \in [0,T]$, $\mu_t^n = \mu^\phi(w^n)_t$. In particular, $(\mu_t^n)_{t\in[0,T]}$ is adapted to the completion of $(\mathcal{G}_t^{W^n})_{t\in[0,T]}$, where $\mathcal{G}_t^{W^n} := \sigma\{w^n(k,s)\,;\, k \in \mathbb{R}, s \leqslant t\}$.*

Let us consider a measurable function $\psi : \mathcal{C}([0,T], \mathbb{R}) \times \mathcal{C}(\mathbb{R} \times [0,T], \mathbb{R}^2) \to \mathbb{R}$ such that $\mathbb{E}^{\mathbb{P}^n}[|\psi(z^n, w^n)|] < +\infty$ for $n = 1, 2$. It follows from (II.72) and from Lemma II.42 that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^n}[\psi(z^n, w^n)] &= \mathbb{E}^{\mathbb{Q}^n}\left[\psi(z^n, w^n) \exp\left(\int_0^T \widetilde{b}(z_s^n, \mu_s^n)\mathrm{d}\beta_s^n + \frac{1}{2}\int_0^T |\widetilde{b}(z_s^n, \mu_s^n)|^2 \mathrm{d}s \right)\right] \\
&= \mathbb{E}^{\mathbb{Q}^n}\left[\psi(z^n, w^n) \exp\left(\int_0^T \widetilde{b}(z_s^n, \mu_s^n)\mathrm{d}\widetilde{\beta}_s^n - \frac{1}{2}\int_0^T |\widetilde{b}(z_s^n, \mu_s^n)|^2 \mathrm{d}s \right)\right] \\
&= \mathbb{E}^{\mathbb{Q}^n}\left[\psi(z^n, w^n) \exp\left(\int_0^T \widetilde{b}(z_s^n, \mu^\phi(w^n)_s)\mathrm{d}\widetilde{\beta}_s^n - \frac{1}{2}\int_0^T |\widetilde{b}(z_s^n, \mu^\phi(w^n)_s)|^2 \mathrm{d}s \right)\right] \\
&= \mathbb{E}^{\mathbb{Q}^n}\left[\widetilde{\psi}(z^n, w^n, \widetilde{\beta}^n)\right]
\end{aligned}$$

where $\widetilde{\psi}$ is a measurable map such that $\mathbb{E}^{\mathbb{Q}^n}\left[|\widetilde{\psi}(z^n, w^n, \widetilde{\beta}^n)|\right] < +\infty$; the measurability of $\widetilde{\psi}$ follows from the fact that μ^ϕ belongs to $\mathcal{X}$. Furthermore, μ^ϕ does not depend on $n = 1, 2$, since it is the unique fixed-point of ϕ. Recalling the equality $\mathcal{L}^{\mathbb{Q}^1}(z^1, w^1, \widetilde{\beta}^1) = \mathcal{L}^{\mathbb{Q}^2}(z^2, w^2, \widetilde{\beta}^2)$, we conclude that $\mathbb{E}^{\mathbb{P}^1}[\psi(z^1, w^1)] = \mathbb{E}^{\mathbb{P}^2}[\psi(z^2, w^2)]$. Moreover, by Lemma II.42, $(\mu_t^n)_{t\in[0,T]}$ is $(\mathcal{G}_t^{W^n})_{t\in[0,T]}$-measurable. This completes the proof of Proposition II.39. □

Proof (Lemma II.42). Let us forget about the exponent n in this proof. On the one hand, the process $(\mu_t)_{t\in[0,T]}$ satisfies for every $t \in [0,T]$, $\mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu, W})$ and (μ, w) is independent of

(β, ξ). Moreover, it follows from equality (II.72) that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ with a density given by

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} &= \exp\left(\int_0^T \widetilde{b}(z_s,\mu_s)\mathrm{d}\beta_s + \frac{1}{2}\int_0^T |\widetilde{b}(z_s,\mu_s)|^2\mathrm{d}s\right)\\ &= \exp\left(\int_0^T \widetilde{b}(z_s,\mu_s)\mathrm{d}\widetilde{\beta}_s - \frac{1}{2}\int_0^T |\widetilde{b}(z_s,\mu_s)|^2\mathrm{d}s\right). \end{aligned} \tag{II.73}$$

On the other hand, since μ^ϕ is the fixed point of ϕ, the process $(\mu^\phi(w)_t)_{t\in[0,T]}$ satisfies $\mu^\phi(w) = \phi(\mu^\phi)(w) = \mathcal{L}^{\mathbb{P}^{\mu^\phi,\beta,\xi}(w)}(\mathcal{Z}(\cdot,w,\cdot))$. Since under $\mathbb{Q}$, $z = \mathcal{Z}(\xi, w, \widetilde{\beta})$, we deduce that for every $t \in [0,T]$, $\mu^\phi(w)_t = \mathcal{L}^{\mathbb{R}}(z_t|\mathcal{G}_t^W)$, where $\mathcal{G}_t^W := \sigma\{w(k,s)\ ; k \in \mathbb{R}, s \leqslant t\}$ and $\mathbb{R}$ is defined by

$$\frac{\mathrm{d}\mathbb{R}}{\mathrm{d}\mathbb{Q}} = \exp\left(\int_0^T \widetilde{b}(z_s,\mu^\phi(w)_s)\mathrm{d}\widetilde{\beta}_s - \frac{1}{2}\int_0^T |\widetilde{b}(z_s,\mu^\phi(w)_s)|^2\mathrm{d}s\right). \tag{II.74}$$

Let us prove that

(1) under the probability measure $\mathbb{Q}$, (μ, w) is independent of $(\widetilde{\beta}, \xi)$;

(2) for every $t \in [0,T]$, $\mathbb{R}$-almost surely, $\mu^\phi(w)_t = \mathcal{L}^{\mathbb{R}}(z_t|\mathcal{G}_t^{\mu,W})$;

(3) conclude the proof of the lemma by comparing, for every $t \in [0,T]$, $\mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^{\mu,W})$ with $\mu^\phi(w)_t = \mathcal{L}^{\mathbb{R}}(z_t|\mathcal{G}_t^{\mu,W})$.

Proof of (1)**.** By definition of a weak solution, under probability measure $\mathbb{P}$, w, β and ξ are independent random variables and (μ, w) is independent of (β, ξ). Let us consider bounded and measurable functions $f : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R})) \times \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2) \to \mathbb{R}$ and let $g : [0,T] \to \mathbb{R}$ be a deterministic square integrable function. Recalling that $\mathrm{d}\widetilde{\beta}_t = \mathrm{d}\beta_t + \widetilde{b}(z_t,\mu_t)\mathrm{d}t$, let us compute

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\psi(\mu,w)f(\xi)\exp\Big(\textstyle\int_0^T g_s\mathrm{d}\widetilde{\beta}_s - \frac{1}{2}\int_0^T g_s^2\mathrm{d}s\Big)\right]\\ &= \mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)f(\xi)\exp\Big(\textstyle\int_0^T g_s\mathrm{d}\widetilde{\beta}_s - \frac{1}{2}\int_0^T g_s^2\mathrm{d}s\Big)\exp\Big(-\int_0^T \widetilde{b}(z_s,\mu_s)\mathrm{d}\beta_s - \frac{1}{2}\int_0^T |\widetilde{b}(z_s,\mu_s)|^2\mathrm{d}s\Big)\right]\\ &= \mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)f(\xi)\exp\Big(\textstyle\int_0^T (g_s - \widetilde{b}(z_s,\mu_s))\mathrm{d}\beta_s - \frac{1}{2}\int_0^T |g_s - \widetilde{b}(z_s,\mu_s)|^2\mathrm{d}s\Big)\right].\end{aligned}$$

We now show that the last line is in fact equal to $\mathbb{E}^{\mathbb{P}}[\psi(\mu,w)f(\xi)]$. By expanding the exponential martingale by Itô's formula, it is in fact sufficient to prove that, for any $(\mathcal{G}_t)_{t\in[0,T]}$ progressively-measurable and square integrable process $(H_t)_{t\in[0,T]}$, the stochastic integral $\int_0^T H_s d\beta_s$ is orthogonal to $\psi(\mu,w)f(\xi)$ under $\mathbb{P}$. By a standard approximation, it is even sufficient to do so for simple processes $(H_t)_{t\in[0,T]}$. In other words, it suffices to prove that, for any $0 \leqslant t \leqslant t' \leqslant T$, for any $\mathcal{G}_t$-measurable square-integrable random variable H_t,

$$\mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)f(\xi)H_t(\beta_{t'} - \beta_t)\right] = 0.$$

By taking the conditional expectation given $\mathcal{G}_t$ in the expectation appearing in the left-hand side, it is sufficient to prove that, for any $0 \leqslant t \leqslant t' \leqslant T$,

$$\mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)f(\xi)(\beta_{t'} - \beta_t)\Big|\mathcal{G}_t\right] = 0.$$

Thanks to the compatibility condition in Definition II.35,

$$\mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)f(\xi)(\beta_{t'} - \beta_t)\Big|\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)f(\xi)\Big|\mathcal{G}_t\right]\mathbb{E}^{\mathbb{P}}\left[(\beta_{t'} - \beta_t)\Big|\mathcal{G}_t\right] = 0$$

because $\beta_{t'} - \beta_t$ is independent of $\mathcal{G}_t$. Therefore,

$$\mathbb{E}^{\mathbb{Q}}\left[\psi(\mu,w)f(\xi)\exp\left(\int_0^T g_s \mathrm{d}\widetilde{\beta}_s - \tfrac{1}{2}\int_0^T g_s^2 \mathrm{d}s\right)\right] = \mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)f(\xi)\right].$$

It follows that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\psi(\mu,w)f(\xi)\exp\left(\int_0^T g_s \mathrm{d}\widetilde{\beta}_s - \tfrac{1}{2}\int_0^T g_s^2 \mathrm{d}s\right)\right] &= \mathbb{E}^{\mathbb{P}}\left[\psi(\mu,w)\right]\cdot\mathbb{E}^{\mathbb{P}}\left[f(\xi)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\psi(\mu,w)\right]\cdot\mathbb{E}^{\mathbb{Q}}\left[f(\xi)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\psi(\mu,w)\right]\cdot\mathbb{E}^{\mathbb{Q}}\left[f(\xi)\right]\cdot\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^T g_s \mathrm{d}\widetilde{\beta}_s - \tfrac{1}{2}\int_0^T g_s^2 \mathrm{d}s\right)\right],\end{aligned}$$

since $\left(\exp\left(\int_0^T g_s \mathrm{d}\widetilde{\beta}_s - \frac{1}{2}\int_0^T g_s^2 \mathrm{d}s\right)\right)_{t\in[0,T]}$ is a martingale under the measure $\mathbb{Q}$. Moreover, the linear span of $\{\exp(\int_0^T g_s \mathrm{d}\widetilde{\beta}_s), g \in L_2([0,T],\mathbb{R})\}$ is dense in $L_2(\Omega, \mathcal{G}^{\widetilde{\beta}}, \mathbb{Q})$, where $\mathcal{G}^{\widetilde{\beta}}$ is the σ-algebra generated by $(\widetilde{\beta}_t)_{t\in[0,T]}$. Therefore, $(f(\xi), \exp(\int_0^T g_s \mathrm{d}\widetilde{\beta}_s))$ generates the σ-algebra $\mathcal{G}^{\xi,\widetilde{\beta}}$, and thus (μ, w) and $(\xi, \widetilde{\beta})$ are independent under the probability measure $\mathbb{Q}$.

Proof of (2)**.** Recall that for every $t \in [0,T]$, $\mu^\phi(w)_t = \mathcal{L}^{\mathbb{R}}(z_t|\mathcal{G}_t^W)$, and let us prove that for every $t \in [0,T]$, $\mathbb{R}$-almost surely, $\mu^\phi(w)_t = \mathcal{L}^{\mathbb{R}}(z_t|\mathcal{G}_t^{\mu,W})$. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R})) \to \mathbb{R}$ and $h : \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2) \to \mathbb{R}$ be bounded and measurable functions. Fix $t \in [0,T]$. By (II.74), we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{R}}\left[f(z_t)g(\mu_{\cdot\wedge t})h(w_{\cdot\wedge t})\right] \\ &\quad= \mathbb{E}^{\mathbb{Q}}\left[f(z_t)\exp\left(\int_0^t \widetilde{b}(z_s,\mu^\phi(w)_s)\mathrm{d}\widetilde{\beta}_s - \tfrac{1}{2}\int_0^t |\widetilde{b}(z_s,\mu^\phi(w)_s)|^2\mathrm{d}s\right)g(\mu_{\cdot\wedge t})h(w_{\cdot\wedge t})\right] \\ &\quad= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[F_t \,|\, \mathcal{G}_t^{\mu,W}\right]\; g(\mu_{\cdot\wedge t})\; h(w_{\cdot\wedge t})\right],\end{aligned} \tag{II.75}$$

where (recall that z has the form $z = \mathcal{Z}(\xi, w, \widetilde{\beta})$)

$$\begin{aligned}F_t &:= f(z_t)\exp\left(\int_0^t \widetilde{b}(z_s,\mu^\phi(w)_s)\mathrm{d}\widetilde{\beta}_s - \tfrac{1}{2}\int_0^t |\widetilde{b}(z_s,\mu^\phi(w)_s)|^2\mathrm{d}s\right) \\ &= f(\mathcal{Z}_t(\xi,w,\widetilde{\beta}))\exp\left(\int_0^t \widetilde{b}(\mathcal{Z}_s(\xi,w,\widetilde{\beta}),\mu^\phi(w)_s)\mathrm{d}\widetilde{\beta}_s - \tfrac{1}{2}\int_0^t |\widetilde{b}(\mathcal{Z}_s(\xi,w,\widetilde{\beta}),\mu^\phi(w)_s)|^2\mathrm{d}s\right).\end{aligned}$$

Note that F_t is $\mathcal{G}_t^{W,\beta,\xi}$-measurable. By statement (1), under probability measure $\mathbb{Q}$, (μ, w) is independent of $(\widetilde{\beta}, \xi)$. Hence $\mathbb{E}^{\mathbb{Q}}\left[F_t \,|\, \mathcal{G}_t^{\mu,W}\right]$ is $\mathcal{G}_t^W$-measurable. Thus it follows from (II.75) that

$$\mathbb{E}^{\mathbb{R}}\left[f(z_t)g(\mu_{\cdot\wedge t})h(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[F_t \,|\, \mathcal{G}_t^{\mu,W}\right]\; \mathbb{E}^{\mathbb{Q}}\left[g(\mu_{\cdot\wedge t})|\mathcal{G}_t^W\right] h(w_{\cdot\wedge t})\right].$$

Since $\mathbb{E}^{\mathbb{Q}}\left[g(\mu_{\cdot\wedge t})|\mathcal{G}_t^W\right] h(w_{\cdot\wedge t})$ is $\mathcal{G}_t^W$-measurable and bounded, there is a bounded and measurable function $k : \mathcal{C}(\mathbb{R}\times[0,T],\mathbb{R}^2) \to \mathbb{R}$ such that $\mathbb{E}^{\mathbb{Q}}\left[g(\mu_{\cdot\wedge t})|\mathcal{G}_t^W\right] h(w_{\cdot\wedge t}) = k(w_{\cdot\wedge t})$. Thus, redoing the same computations in reverse, we obtain:

$$\begin{aligned}&\mathbb{E}^{\mathbb{R}}\left[f(z_t)g(\mu_{\cdot\wedge t})h(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[F_t \,|\, \mathcal{G}_t^{\mu,W}\right]\; k(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{Q}}\left[F_t\; k(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{R}}\left[f(z_t)\; k(w_{\cdot\wedge t})\right] \\ &= \mathbb{E}^{\mathbb{R}}\left[\int_{\mathbb{R}} f(x)\mathrm{d}\mu^\phi(w)_t(x)\; k(w_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{R}}\left[\int_{\mathbb{R}} f(x)\mathrm{d}\mu^\phi(w)_t(x)\; \mathbb{E}^{\mathbb{Q}}\left[g(\mu_{\cdot\wedge t})|\mathcal{G}_t^W\right] h(w_{\cdot\wedge t})\right] \\ &= \mathbb{E}^{\mathbb{R}}\left[\int_{\mathbb{R}} f(x)\mathrm{d}\mu^\phi(w)_t(x)\; g(\mu_{\cdot\wedge t})\; h(w_{\cdot\wedge t})\right].\end{aligned} \tag{II.76}$$

Since the process $(\mu^\phi(w)_t)_{t\in[0,T]}$ is $(\mathcal{G}_t^W)_{t\in[0,T]}$-adapted, it is in particular $(\mathcal{G}_t^{\mu,W})_{t\in[0,T]}$-adapted, thus equality (II.76) implies that $\mu^\phi(w)_t = \mathcal{L}^{\mathbb{R}}(z_t|\mathcal{G}_t^{\mu,W})$. It completes the proof of (2).

Proof of (3)**.** Let us denote for every $t \in [0,T]$, $\nu_t = \mu^\phi(w)_t$. We want to prove that $(\mu_t)_{t\in[0,T]} = (\nu_t)_{t\in[0,T]}$. Recall that for every $t \in [0,T]$, $\mathbb{Q}$-almost surely, $\mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^{\mu,W})$ and, by point (2), $\nu_t = \mathcal{L}^{\mathbb{R}}(z_t|\mathcal{G}_t^{\mu,W})$. By (II.73) and (II.74),

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = \exp\left(\int_0^T \widetilde{b}(z_s,\mu_s)\mathrm{d}\widetilde{\beta}_s - \frac{1}{2}\int_0^T |\widetilde{b}(z_s,\mu_s)|^2\mathrm{d}s\right);$$
$$\frac{\mathrm{d}\mathbb{R}}{\mathrm{d}\mathbb{Q}} = \exp\left(\int_0^T \widetilde{b}(z_s,\nu_s)\mathrm{d}\widetilde{\beta}_s - \frac{1}{2}\int_0^T |\widetilde{b}(z_s,\nu_s)|^2\mathrm{d}s\right).$$

Let us apply the same computation as in the proof of Lemma II.41. Recall that for every $t \in [0,T]$, μ^t denotes $\mathcal{L}^{\mathbb{P}}(z_{\cdot\wedge t}|\mathcal{G}_t^{\mu,W})$ and $\nu^t := \mathcal{L}^{\mathbb{R}}(z_{\cdot\wedge t}|\mathcal{G}_t^{\mu,W})$. For every $t \in [0,T]$,

$$H(\nu^t|\mu^t) = \int_{\mathbb{R}} \ln\frac{\mathrm{d}\nu^t}{\mathrm{d}\mu^t}\mathrm{d}\nu^t = \mathbb{E}^{\mathbb{R}}\left[\ln\frac{\mathrm{d}\nu^t}{\mathrm{d}\mu^t}(z_{\cdot\wedge t})\Big|\mathcal{G}_t^{\mu,W}\right].$$

We use the fact that under $\mathbb{P}$, β is independent of (μ,w) in order to prove, exactly as in the proof of Lemma II.41, that for every $t \in [0,T]$, $\mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{R}}{\mathrm{d}\mathbb{P}}\Big|\mathcal{G}_t^{\mu,W}\right] = 1$. Again by mimicking the proof of (II.69), this leads to

$$\frac{\mathrm{d}\nu^t}{\mathrm{d}\mu^t}(z_{\cdot\wedge t}) = \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{R}}{\mathrm{d}\mathbb{P}}\Big|\mathcal{G}_t^{z,\mu,W}\right].$$

Therefore, we finally obtain

$$H(\nu^t|\mu^t) = \frac{1}{2}\mathbb{E}^{\mathbb{R}}\left[\int_0^t |\widetilde{b}(z_s,\nu_s) - \widetilde{b}(z_s,\mu_s)|^2\mathrm{d}s\Big|\mathcal{G}_t^{\mu,W}\right].$$

Applying Pinsker's inequality and using the fact that $\widetilde{b}$ is Lipschitz-continuous with respect to the measure variable, we finally obtain for every $t \in [0,T]$,

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}_{\mathrm{TV}}(\nu_t,\mu_t)^2\right] \leqslant \mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}_{\mathrm{TV}}(\nu^t,\mu^t)^2\right] \leqslant C\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}_{\mathrm{TV}}(\nu_s,\mu_s)^2\right]\mathrm{d}s.$$

Thus by Gronwall's inequality, we obtain that for every $t \in [0,T]$, $\mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}_{\mathrm{TV}}(\nu_t,\mu_t)^2\right] = 0$. In particular, $\mathbb{Q}$-almost surely, the two continuous processes $(\mu_t)_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$ are equal. This completes the proof of the lemma. □

II.5.4 Resolution of the SDE when the drift is general

Let us state the well-posedness result for the general case:

Theorem II.43. *Let $\eta > 0$ and $\delta \in [0,1]$ satisfy the inequality $\eta > \frac{3}{2}(1-\delta)$. Let $b : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be of class $(H^\eta, \mathcal{C}^\delta)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of order $\alpha \in (\frac{3}{2}, \frac{\eta}{1-\delta}]$.*

Then existence of a weak solution and uniqueness in law hold for equation (II.60)*.*

Note: The assumption on α is the same as the one given by inequality (II.59).

As a first step, let us prove that a drift function b satisfying the assumptions of Theorem II.43 can be written as a sum $\widetilde{b} + (b - \widetilde{b})$, where $\widetilde{b}$ satisfies the assumptions of Proposition II.38 and where $b - \widetilde{b}$ satisfies assumptions similar to Definition II.14, and apply on $b - \widetilde{b}$ the same Fourier inversion as in Section II.3.

Recall that by Definition II.34, b can be written as

$$b(x,\mu) = \int_{\mathbb{R}} \langle k\rangle^{-\eta}\Big(\cos(kx)\lambda^{\Re}(k,\mu) + \sin(kx)\lambda^{\Im}(k,\mu)\Big)\mathrm{d}k, \tag{II.77}$$

where $\lambda = \lambda^{\Re} + i\lambda^{\Im}$ satisfies for every $k \in \mathbb{R}$ and for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$,

$$|\lambda(k,\mu)| \leqslant \Lambda(k); \tag{II.78}$$

$$|\lambda(k,\mu) - \lambda(k,\nu)| \leqslant \Lambda(k) \mathrm{d}_{\mathrm{TV}}(\mu,\nu)^{\delta}, \tag{II.79}$$

and Λ belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

Lemma II.44. *Let $\theta := \frac{\alpha-\eta}{\delta}$. There exists $\widetilde{\lambda} = \widetilde{\lambda}^{\Re} + i\widetilde{\lambda}^{\Im}$, where $\widetilde{\lambda}^{\Re}, \widetilde{\lambda}^{\Im} : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, such that for each $k \in \mathbb{R}$ and for each $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$,*

$$|\lambda(k,\mu) - \widetilde{\lambda}(k,\mu)| \leqslant \frac{C}{\langle k \rangle^{\theta\delta}} \Lambda(k); \tag{II.80}$$

$$|\widetilde{\lambda}(k,\mu) - \widetilde{\lambda}(k,\nu)| \leqslant C \langle k \rangle^{\theta(1-\delta)} \Lambda(k) \mathrm{d}_{\mathrm{TV}}(\mu,\nu), \tag{II.81}$$

where C is independent of k, μ, ν and θ.

Proof. Let us fix $k \in \mathbb{R}$. We will focus on the proof for the real part; the case of the imaginary part is identical.

Let us define $u : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ by $u(\mu) := \frac{\lambda^{\Re}(k,\mu)}{\Lambda(k)}$. By (II.78) and (II.79), for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $|u(\mu)| \leqslant 1$ and $|u(\mu) - u(\nu)| \leqslant \mathrm{d}_{\mathrm{TV}}(\mu,\nu)^{\delta}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Polish and atomless probability space. Let us define $v : L_2(\Omega) \to \mathbb{R}$ by $v(X) := u(\mathcal{L}(X))$.

The following approximation method is inspired by the inf-convolution techniques. Let $\varepsilon > 0$. Let us define $v^{\varepsilon} : L_2(\Omega) \to \mathbb{R}$ by

$$v^{\varepsilon}(X) := \inf_{Y \in L_2(\Omega \times [0,1])} \left\{ v(Y) + \frac{1}{2\varepsilon} (\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X \neq Y \right]^2 \right\}. \tag{II.82}$$

We consider here the infimum over random variables in a larger probability space in order to be enseure the existence of a random variable Y independent of X. In (II.82), the map v is extended to $L_2(\Omega \in [0,1]) \times \mathbb{R}$ by $v(Y) := u(\mathcal{L}(Y))$. Let us prove that

(*i*) $v^{\varepsilon}(X)$ depends only on the law of X; thus we can define $u^{\varepsilon}(\mu)$ by letting $u^{\varepsilon}(\mu) := v^{\varepsilon}(X)$, whatever the choice of the random variable X with distribution μ.

(*ii*) for every $\mu \in \mathcal{P}_2(\mathbb{R})$, $|u^{\varepsilon}(\mu) - u(\mu)| \leqslant C\varepsilon^{\frac{\delta}{2-\delta}}$.

(*iii*) for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $|u^{\varepsilon}(\mu) - u^{\varepsilon}(\nu)| \leqslant C\varepsilon^{\frac{\delta-1}{2-\delta}} \mathrm{d}_{\mathrm{TV}}(\mu,\nu)$.

Proof of (*i*)**.** Let $X, X' \in L_2(\Omega)$ with same law. We want to prove that $v^{\varepsilon}(X) = v^{\varepsilon}(X')$. Remark that by definition of v, $v(X)$ depends only on the law of X. Fix $\eta > 0$. There is $Y^{\eta} \in L_2(\Omega \times [0,1])$ such that

$$v(Y^{\eta}) + \frac{1}{2\varepsilon} (\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X \neq Y^{\eta} \right]^2 \leqslant v^{\varepsilon}(X) + \eta. \tag{II.83}$$

Let $\nu : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ be the conditional law of Y^{η} given X; in other words, for every fixed $x \in \mathbb{R}$, $\nu(x,\cdot)$ belongs to $\mathcal{P}_2(\mathbb{R})$, for every fixed $A \in \mathcal{B}(\mathbb{R})$, $x \mapsto \nu(x,A)$ is measurable and for every $f : \mathbb{R}^2 \to \mathbb{R}$ bounded and measurable, $(\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) [f(X, Y^{\eta})] = (\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) [\int_{\mathbb{R}} f(X,y) \nu(X, \mathrm{d}y)]$.

Furthermore, for every fixed $x \in \mathbb{R}$, let us denote by $u \in [0,1] \mapsto g(x,u)$ the quantile function associated to the probability measure $\nu(x,\cdot)$. For every $t \in \mathbb{R}$ and for every $u \in [0,1]$, $\{x : g(x,u) \leqslant t\} = \{x : \nu(x,(-\infty,t]) \leqslant u\} \in \mathcal{B}(\mathbb{R})$, so we deduce that for every $u \in [0,1]$, $x \mapsto g(x,u)$ is measurable. Moreover, $u \mapsto g(x,u)$ is a càdlàg function. It follows from [KS91, Proposition 1.13] that $(x,u) \mapsto g(x,u)$ is measurable.

Let $U \in L_2([0,1])$ be a random variable with uniform law on $[0,1]$; in particular, it is independent of X' (remark that we have considered a larger probability space in order to ensure

the existence of U independent of X'). Let $Y' := g(X', U)$. Then for every $f : \mathbb{R}^2 \to \mathbb{R}$ bounded and measurable

$$\begin{aligned}(\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) \left[f(X', Y')\right] &= (\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) \left[f(X', g(X', U))\right] \\ &= (\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) \left[\int_0^1 f(X', g(X', u))\mathrm{d}u\right] \\ &= (\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) \left[\int_{\mathbb{R}} f(X', y)\nu(X', \mathrm{d}y)\right].\end{aligned}$$

Since X and X' have same law, we deduce that

$$(\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) \left[f(X', Y')\right] = (\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) \left[\int_{\mathbb{R}} f(X, y)\nu(X, \mathrm{d}y)\right] = (\mathbb{E} \otimes \mathrm{Leb}_{[0,1]}) \left[f(X, Y^\eta)\right].$$

Therefore, the pair (X', Y') has same distribution as (X, Y^η). It follows that

$$(\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X \neq Y^\eta\right] = (\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X' \neq Y'\right]$$

and $v(Y^\eta) = v(Y')$ since v depends only on the law of the random variable. Thus by inequality (II.83),

$$v(Y') + \frac{1}{2\varepsilon}(\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X' \neq Y'\right]^2 \leqslant v^\varepsilon(X) + \eta.$$

By definition (II.82) of v^ε, $v^\varepsilon(X') \leqslant v(Y') + \frac{1}{2\varepsilon}(\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X' \neq Y'\right]^2$, thus $v^\varepsilon(X') \leqslant v^\varepsilon(X) + \eta$. We proved that the inequality holds with every $\eta > 0$, thus $v^\varepsilon(X') \leqslant v^\varepsilon(X)$. By symmetry, $v^\varepsilon(X) \leqslant v^\varepsilon(X')$, hence the equality holds true.

Proof of (ii)**.** Let us prove that for every $X \in L_2(\Omega)$, $|v^\varepsilon(X) - v(X)| \leqslant C\varepsilon^{\frac{\delta}{2-\delta}}$. Fix $X \in L_2(\Omega)$. By definition (II.82), it is obvious that $v^\varepsilon(X) \leqslant v(X)$. Thus it is sufficient to prove that $v(X) - v^\varepsilon(X) \leqslant C\varepsilon^{\frac{\delta}{2-\delta}}$.

Fix $\eta > 0$. There exists Y^η such that (II.83). It follows that

$$v(Y^\eta) + \frac{1}{2\varepsilon}(\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X \neq Y^\eta\right]^2 \leqslant v(X) + \eta.$$

By definition of v, $|v(X) - v(Y^\eta)| = |u(\mathcal{L}(X)) - u(\mathcal{L}(Y^\eta))| \leqslant \mathrm{d_{TV}}(\mathcal{L}(X), \mathcal{L}(Y^\eta))^\delta$. Therefore, by (II.58),

$$\mathrm{d_{TV}}(\mathcal{L}(X), \mathcal{L}(Y^\eta))^2 \leqslant 4(\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X \neq Y^\eta\right]^2 \leqslant 8\varepsilon \left[\mathrm{d_{TV}}(\mathcal{L}(X), \mathcal{L}(Y^\eta))^\delta + \eta\right]. \tag{II.84}$$

Let $l := \limsup_{\eta \searrow 0} \mathrm{d_{TV}}(\mathcal{L}(X), \mathcal{L}(Y^\eta))$. Thus $l^2 \leqslant 8\varepsilon l^\delta$, hence we get $l^{2-\delta} \leqslant 8\varepsilon$. It follows that

$$\limsup_{\eta \searrow 0} \mathrm{d_{TV}}(\mathcal{L}(X), \mathcal{L}(Y^\eta)) \leqslant 2^{\frac{3}{2-\delta}} \varepsilon^{\frac{1}{2-\delta}}. \tag{II.85}$$

By inequality (II.83),

$$\begin{aligned}v(X) - v^\varepsilon(X) &\leqslant v(X) - v(Y^\eta) - \frac{1}{2\varepsilon}(\mathbb{P} \otimes \mathrm{Leb}_{[0,1]}) \left[X \neq Y^\eta\right]^2 + \eta \\ &\leqslant |v(X) - v(Y^\eta)| + \eta \leqslant \mathrm{d_{TV}}(\mathcal{L}(X), \mathcal{L}(Y^\eta))^\delta + \eta.\end{aligned}$$

By passing to the limit $\eta \searrow 0$, we obtain $v(X) - v^\varepsilon(X) \leqslant C\varepsilon^{\frac{\delta}{2-\delta}}$, which completes the proof of (ii).

Proof of *(iii)***.** Let us first prove that u^ε is also δ-Hölder continuous. Let $\mu,\nu\in\mathcal{P}_2(\mathbb{R})$. Let X and $X'\in L_2(\Omega)$ with respective distributions μ and ν. Fix $\eta>0$. Let Y^η satisfying (II.83). Then

$$\begin{aligned}v^\varepsilon(X')-v^\varepsilon(X)&\leqslant v^\varepsilon(X')-v(Y^\eta)-\frac{1}{2\varepsilon}(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y^\eta\right]^2+\eta\\&\leqslant v(Y^\eta+X'-X)+\frac{1}{2\varepsilon}(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y^\eta+X'-X\right]^2\\&\quad-v(Y^\eta)-\frac{1}{2\varepsilon}(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y^\eta\right]^2+\eta\\&\leqslant|v(Y^\eta+X'-X)-v(Y^\eta)|+\eta\leqslant\mathrm{d_{TV}}(\mathcal{L}(Y^\eta+X'-X),\mathcal{L}(Y^\eta))^\delta+\eta.\end{aligned}$$

By definition of the distance in total variation, $\mathrm{d_{TV}}(\mathcal{L}(Y^\eta+X'-X),\mathcal{L}(Y^\eta))\leqslant 2\mathbb{P}\left[X'\neq X\right]$. Thus for every $\eta>0$, $v^\varepsilon(X')-v^\varepsilon(X)\leqslant 2^\delta\mathbb{P}\left[X'\neq X\right]^\delta+\eta$. By letting η tend to zero and by symmetry, we deduce that there is $C>0$ depending only on δ such that

$$|u^\varepsilon(\mu)-u^\varepsilon(\nu)|=|v^\varepsilon(X)-v^\varepsilon(X')|\leqslant C\mathbb{P}\left[X'\neq X\right]^\delta.$$

By taking the infimum over every coupling (X,X') of (μ,ν), we finally get

$$|u^\varepsilon(\mu)-u^\varepsilon(\nu)|\leqslant C\mathrm{d_{TV}}(\mu,\nu)^\delta. \tag{II.86}$$

Therefore, u^ε is also δ-Hölder continuous.

Keep $X,X'\in L_2(\Omega)$ two random variables with laws μ and ν. Let $(Y^\eta)_{\eta>0}$ satisfy (II.83). It follows from (II.84) and (II.85) that

$$\limsup_{\eta\searrow 0}(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y^\eta\right]\leqslant\sqrt{2\varepsilon 2^{\frac{3\delta}{2-\delta}}\varepsilon^{\frac{\delta}{2-\delta}}}=2^{\frac{1+\delta}{2-\delta}}\varepsilon^{\frac{1}{2-\delta}}.$$

For every $\eta>0$, let us define

$$\begin{aligned}S^\eta:=\Big\{Y\in L_2(\Omega\times[0,1]):(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y\right]&\leqslant 2^{\frac{1+\delta}{2-\delta}}\varepsilon^{\frac{1}{2-\delta}}+\eta\\&\text{or}\quad(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y\right]\leqslant 2^{\frac{1+\delta}{2-\delta}}\varepsilon^{\frac{1}{2-\delta}}+\eta\Big\}.\end{aligned}$$

Fix $\eta>0$. Thus there is $Y^\eta\in S^\eta$ such that (II.83) holds true. We deduce that

$$\begin{aligned}v^\varepsilon(X')-v^\varepsilon(X)&\leqslant v(Y^\eta)+\frac{1}{2\varepsilon}(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y^\eta\right]^2\\&\quad-v(Y^\eta)-\frac{1}{2\varepsilon}(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y^\eta\right]^2+\eta\\&\leqslant\frac{1}{2\varepsilon}\Big((\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y^\eta\right]^2-(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y^\eta\right]^2\Big)+\eta.\end{aligned}$$

By symmetry, we deduce that

$$|v^\varepsilon(X')-v^\varepsilon(X)|\leqslant\frac{1}{2\varepsilon}\sup_{Y\in S^\eta}\Big|(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y\right]^2-(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y\right]^2\Big|+\eta.$$

Moreover,

$$\begin{aligned}&\Big|(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y\right]^2-(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y\right]^2\Big|\\&\qquad\leqslant\Big|(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y\right]-(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y\right]\Big|\\&\qquad\quad\cdot\Big|(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y\right]+(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y\right]\Big|\\&\qquad\leqslant\mathbb{P}\left[X'\neq X\right]\cdot\Big|(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X'\neq Y\right]+(\mathbb{P}\otimes\mathrm{Leb}_{[0,1]})\left[X\neq Y\right]\Big|.\end{aligned}$$

For every $Y \in S^\eta$, we have

$$\Big|(\mathbb{P}\otimes \mathrm{Leb}_{[0,1]})\left[X' \neq Y\right] + (\mathbb{P}\otimes \mathrm{Leb}_{[0,1]})\left[X \neq Y\right]\Big| \leqslant \mathbb{P}\left[X \neq X'\right] + 2\left(2^{\frac{1+\delta}{2-\delta}}\varepsilon^{\frac{1}{2-\delta}} + \eta\right).$$

By passing to the limit $\eta \searrow 0$ it follows that there exists $C > 0$ depending on δ such that for every $X, X' \in L_2(\Omega)$ with respective distributions μ and ν,

$$|u^\varepsilon(\mu) - u^\varepsilon(\nu)| = |v^\varepsilon(X) - v^\varepsilon(X')| \leqslant \frac{1}{2\varepsilon}\mathbb{P}\left[X' \neq X\right]\left(\mathbb{P}\left[X' \neq X\right] + C\varepsilon^{\frac{1}{2-\delta}}\right).$$

Let us distinguish two cases:

- if $\mathrm{d_{TV}}(\mu,\nu) < \varepsilon^{\frac{1}{2-\delta}}$: by definition (II.58), there exists a coupling (X, X') of law (μ,ν) such that $\mathbb{P}\left[X \neq X'\right] < \varepsilon^{\frac{1}{2-\delta}}$. Thus

$$|u^\varepsilon(\mu) - u^\varepsilon(\nu)| \leqslant C\frac{\varepsilon^{\frac{1}{2-\delta}}}{\varepsilon}\mathbb{P}\left[X' \neq X\right] \leqslant C\varepsilon^{\frac{\delta-1}{2-\delta}}\mathrm{d_{TV}}(\mu,\nu).$$

- if $\mathrm{d_{TV}}(\mu,\nu) \geqslant \varepsilon^{\frac{1}{2-\delta}}$: recall that u^ε is δ-Hölder continuous (see (II.86)). Thus

$$|u^\varepsilon(\mu) - u^\varepsilon(\nu)| \leqslant C\mathrm{d_{TV}}(\mu,\nu)^\delta \leqslant C\frac{\mathrm{d_{TV}}(\mu,\nu)}{\mathrm{d_{TV}}(\mu,\nu)^{1-\delta}} \leqslant C\varepsilon^{\frac{\delta-1}{2-\delta}}\mathrm{d_{TV}}(\mu,\nu).$$

This completes the proof of *(iii)*.

Let us conclude the proof of Lemma II.44. Let us define $\widetilde{\lambda}^{\Re}(k,\mu) := \Lambda(k)u^\varepsilon(\mu)$, with $\varepsilon = \frac{1}{\langle k\rangle^{\theta(2-\delta)}}$. For every $\mu,\nu \in \mathcal{P}_2(\mathbb{R})$, we have

$$|\widetilde{\lambda}^{\Re}(k,\mu) - \lambda^{\Re}(k,\mu)| \leqslant \Lambda(k)|u^\varepsilon(\mu) - u(\mu)| \leqslant C\Lambda(k)\varepsilon^{\frac{\delta}{2-\delta}} \leqslant C\Lambda(k)\frac{1}{\langle k\rangle^{\theta\delta}};$$

$$|\widetilde{\lambda}^{\Re}(k,\mu) - \widetilde{\lambda}^{\Re}(k,\nu)| \leqslant \Lambda(k)|u^\varepsilon(\mu) - u^\varepsilon(\nu)| \leqslant C\Lambda(k)\varepsilon^{\frac{\delta-1}{2-\delta}}\mathrm{d_{TV}}(\mu,\nu) \leqslant C\Lambda(k)\langle k\rangle^{\theta(1-\delta)}\mathrm{d_{TV}}(\mu,\nu).$$

It completes the proof of (II.80) and (II.81) for the case of the real part. The proof for the imaginary part is the same. □

In particular, it follows from (II.78) and from (II.80) that there is $C > 0$ such that for each $k \in \mathbb{R}$, $|\widetilde{\lambda}(k,\cdot)| \leqslant C\Lambda(k)$.

Let us define

$$\widetilde{b}(x,\mu) := \int_{\mathbb{R}} \langle k\rangle^{-\eta}\Big(\cos(kx)\widetilde{\lambda}^{\Re}(k,\mu) + \sin(kx)\widetilde{\lambda}^{\Im}(k,\mu)\Big)\mathrm{d}k. \tag{II.87}$$

For every $x \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$|\widetilde{b}(x,\mu)| \leqslant C\int_{\mathbb{R}} \langle k\rangle^{-\eta}(|\widetilde{\lambda}^{\Re}(k,\mu)| + |\widetilde{\lambda}^{\Im}(k,\mu)|)\mathrm{d}k \leqslant C\int_{\mathbb{R}} \langle k\rangle^{-\eta}\Lambda(k)\mathrm{d}k \leqslant C,$$

since $\eta > 0$ and $\Lambda \in L_1(\mathbb{R})$. Furthermore, by (II.81), for every $x \in \mathbb{R}$ and for every $\mu,\nu \in \mathcal{P}_2(\mathbb{R})$,

$$\begin{aligned}|\widetilde{b}(x,\mu) - \widetilde{b}(x,\nu)| &\leqslant \int_{\mathbb{R}} \langle k\rangle^{-\eta}\Big(|\widetilde{\lambda}^{\Re}(k,\mu) - \widetilde{\lambda}^{\Re}(k,\nu)| + |\widetilde{\lambda}^{\Im}(k,\mu) - \widetilde{\lambda}^{\Im}(k,\nu)|\Big)\mathrm{d}k \\ &\leqslant C\int_{\mathbb{R}} \langle k\rangle^{-\eta}\langle k\rangle^{\theta(1-\delta)}\Lambda(k)\mathrm{d}k\ \mathrm{d_{TV}}(\mu,\nu).\end{aligned}$$

Moreover, $\eta - \theta(1-\delta) \geqslant 0$. Indeed, $\eta - \theta(1-\delta) = \eta + \theta\delta - \theta = \alpha - \frac{\alpha-\eta}{\delta} = \frac{\eta-\alpha(1-\delta)}{\delta} \geqslant 0$ by inequality (II.59). Since Λ belongs to $L_1(\mathbb{R})$, it implies that $\int_{\mathbb{R}} \langle k\rangle^{-\eta}\langle k\rangle^{\theta(1-\delta)}\Lambda(k)\mathrm{d}k < +\infty$. Therefore, the drift function $\widetilde{b}$ is uniformly bounded and uniformly Lipschitz-continuous in the measure variable.

II.5.4.1 Existence of a weak solution to the SDE with drift function b.

Let us prove existence of a weak solution to equation (II.60). We follow the same idea as in Paragraph II.4.1.

Proof (Theorem II.43, existence part). Let $\mathbf{\Omega}$ be a weak solution to the SDE (II.63) with drift $\widetilde{b}$ given by (II.87). In particular, $\mathbb{P}$-almost surely and for every $t \in [0,T]$,

$$z_t = \xi + \int_0^t \int_{\mathbb{R}} f(k)\Re(e^{-ikz_s}\mathrm{d}w(k,s)) + \beta_t + \int_0^t \widetilde{b}(z_s,\mu_s)\mathrm{d}s,$$

where for every $t \in [0,T]$, $\mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^W)$ and $\mathcal{G}_t^W := \sigma\{w(k,s), k \in \mathbb{R}, s \leqslant t\}$. Recall that Proposition II.39 states that every weak solution has this form, *i.e.* $(\mu_t)_{t\in[0,T]}$ is adapted to (the completion of) $(\mathcal{G}_t^W)_{t\in[0,T]}$.

Let $(h_t)_{t\in[0,T]} = (h_t^{\Re} + ih_t^{\Im})_{t\in[0,T]}$ be a process with values in $L_2(\mathbb{R},\mathbb{C})$ satisfying for every $t \in [0,T]$ and for every $x \in \mathbb{R}$,

$$(b-\widetilde{b})(x,\mu_t) = \int_{\mathbb{R}} f(k)\Big(\cos(kx)h_t^{\Re}(k) + \sin(kx)h_t^{\Im}(k)\Big)\mathrm{d}k. \tag{II.88}$$

By (II.77) and (II.87), the unique solution to (II.88) is given, for every $k \in \mathbb{R}$ and for every $t \in [0,T]$, by

$$h_t(k) = \frac{1}{f(k)}\langle k\rangle^{-\eta}(\lambda(k,\mu_t) - \widetilde{\lambda}(k,\mu_t)).$$

Since μ_t is a $(\mathcal{G}_t^W)_{t\in[0,T]}$-adapted process, the process $(h_t)_{t\in[0,T]}$ is also $(\mathcal{G}_t^W)_{t\in[0,T]}$-adapted. Furthermore, by (II.80) and since f is of order α,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} |h_t(k)|^2\mathrm{d}k\mathrm{d}t &\leqslant C\int_0^T\int_{\mathbb{R}} \langle k\rangle^{2\alpha-2\eta}|\lambda(k,\mu_t) - \widetilde{\lambda}(k,\mu_t)|^2\mathrm{d}k\mathrm{d}t\\ &\leqslant C\int_0^T\int_{\mathbb{R}} \langle k\rangle^{2\alpha-2\eta-2\theta\delta}\Lambda(k)^2\mathrm{d}k\mathrm{d}t.\end{aligned}$$

Since $\alpha = \eta+\theta\delta$ and $\Lambda \in L_2(\mathbb{R})$, we deduce that $\int_0^T\int_{\mathbb{R}} |h_t(k)|^2\mathrm{d}k\mathrm{d}t$ is bounded by a deterministic constant. Therefore, the measure $\mathbb{Q}$ on $(\Omega,\mathcal{G})$ with the following density with respect to $\mathbb{P}$:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^T\int_{\mathbb{R}} h_t^{\Re}(k)\mathrm{d}w^{\Re}(k,t) + \int_0^T\int_{\mathbb{R}} h_t^{\Im}(k)\mathrm{d}w^{\Im}(k,t) - \frac{1}{2}\int_0^T\int_{\mathbb{R}} |h_t(k)|^2\mathrm{d}k\mathrm{d}t\right)$$

is a probability measure. Let us define $\widetilde{w}(k,t) = \widetilde{w}^{\Re}(k,t) + i\widetilde{w}^{\Im}(k,t)$, where

$$\widetilde{w}^{\Re}(k,t) := w^{\Re}(k,t) - \int_0^t\int_0^k h_s^{\Re}(l)\ \mathrm{d}l\mathrm{d}s, \tag{II.89}$$

$$\widetilde{w}^{\Im}(k,t) := w^{\Im}(k,t) - \int_0^t\int_0^k h_s^{\Im}(l)\ \mathrm{d}l\mathrm{d}s. \tag{II.90}$$

By Girsanov's Theorem, $\mathcal{L}^{\mathbb{Q}}(\widetilde{w},\beta,\xi) = \mathcal{L}^{\mathbb{P}}(w,\beta,\xi)$ and for any $t \in [0,T]$, the σ-field $\sigma\{\widetilde{w}(k,t') - \widetilde{w}(k,t), \beta_{t'} - \beta_t,\ k \in \mathbb{R},\ t' \in [t,T]\}$ is independent of $\mathcal{G}_t$ under $\mathbb{Q}$. Moreover, $\mathbb{Q}$-almost surely, the process $(z_t)_{t\in[0,T]}$ satisfies

$$\begin{aligned}z_t &= \xi + \int_0^t\int_{\mathbb{R}} f(k)\Re(e^{-ikz_s}\mathrm{d}\widetilde{w}(k,s)) + \int_0^t\int_{\mathbb{R}} f(k)\Big(\cos(kz_s)h_t^{\Re}(k) + \sin(kz_s)h_t^{\Im}(k)\Big)\mathrm{d}k\mathrm{d}s\\ &\quad + \beta_t + \int_0^t \widetilde{b}(z_s,\mu_s)\mathrm{d}s\\ &= \xi + \int_0^t\int_{\mathbb{R}} f(k)\Re(e^{-ikz_s}\mathrm{d}\widetilde{w}(k,s)) + \int_0^t (b-\widetilde{b})(z_s,\mu_s)\mathrm{d}s + \beta_t + \int_0^t \widetilde{b}(z_s,\mu_s)\mathrm{d}s\\ &= \xi + \int_0^t\int_{\mathbb{R}} f(k)\Re(e^{-ikz_s}\mathrm{d}\widetilde{w}(k,s)) + \beta_t + \int_0^t b(z_s,\mu_s)\mathrm{d}s.\end{aligned}$$

Furthermore, recall that for every $t \in [0,T]$ $\mathbb{P}$-almost surely, $\mu_t = \mathcal{L}^{\mathbb{P}}(z_t|\mathcal{G}_t^W)$. We want to prove that for every $t \in [0,T]$ $\mathbb{Q}$-almost surely, $\mu_t = \mathcal{L}^{\mathbb{Q}}(z_t|\mathcal{G}_t^{\mu,\widetilde{W}})$, where the filtration $(\mathcal{G}_t^{\mu,\widetilde{W}})_{t\in[0,T]}$ is defined by $\mathcal{G}_t^{\mu,\widetilde{W}} = \sigma\{\widetilde{w}(k,s), \mu_s \ ; k \in \mathbb{R}, s \leqslant t\}$. Let $\psi : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R})) \times \mathcal{C}(\mathbb{R} \times [0,T], \mathbb{R}^2) \to \mathbb{R}$ be bounded and measurable functions. Fix $t \in [0,T]$. Then

$$\mathbb{E}^{\mathbb{Q}}\left[\psi(z_t)\ \varphi(\mu_{\cdot\wedge t}, \widetilde{w}_{\cdot\wedge t})\right] = \mathbb{E}^{\mathbb{P}}\left[\psi(z_t)\ \varphi(\mu_{\cdot\wedge t}, \widetilde{w}_{\cdot\wedge t}) \exp\Big(\int_0^t\int_{\mathbb{R}} \Re(\overline{h_t(k)}\mathrm{d}w(k,t)) - \frac{1}{2}\int_0^t\int_{\mathbb{R}} |h_t(k)|^2 \mathrm{d}k\mathrm{d}t\Big)\right].$$

Recall that the process $(h_t)_{t\in[0,T]}$ is $(\mathcal{G}_t^W)_{t\in[0,T]}$-adapted. It follows that the process $(\widetilde{w}_{\cdot\wedge t})_{t\in[0,T]}$ is also $(\mathcal{G}_t^W)_{t\in[0,T]}$-adapted, since $\widetilde{w}_{\cdot\wedge t} = w_{\cdot\wedge t} - \int_0^{\cdot\wedge t}\int_0^{\cdot} h_s(l)\mathrm{d}l\mathrm{d}s$. Thus

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\psi(z_t)\ \varphi(\mu_{\cdot\wedge t}, \widetilde{w}_{\cdot\wedge t})\right] \\
&\quad= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\psi(z_t)|\mathcal{G}_t^W\right]\varphi(\mu_{\cdot\wedge t}, \widetilde{w}_{\cdot\wedge t}) \exp\Big(\int_0^t\int_{\mathbb{R}} \Re(\overline{h_t(k)}\mathrm{d}w(k,t)) - \frac{1}{2}\int_0^t\int_{\mathbb{R}} |h_t(k)|^2 \mathrm{d}k\mathrm{d}t\Big)\right] \\
&\quad= \mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}} \psi(x)\mathrm{d}\mu_t(x)\ \varphi(\mu_{\cdot\wedge t}, \widetilde{w}_{\cdot\wedge t}) \exp\Big(\int_0^t\int_{\mathbb{R}} \Re(\overline{h_t(k)}\mathrm{d}w(k,t)) - \frac{1}{2}\int_0^t\int_{\mathbb{R}} |h_t(k)|^2 \mathrm{d}k\mathrm{d}t\Big)\right] \\
&\quad= \mathbb{E}^{\mathbb{Q}}\left[\int_{\mathbb{R}} \psi(x)\mathrm{d}\mu_t(x)\ \varphi(\mu_{\cdot\wedge t}, \widetilde{w}_{\cdot\wedge t})\right].
\end{aligned} \tag{II.91}$$

Therefore, for every $t \in [0,T]$, $\mathbb{E}^{\mathbb{Q}}\left[\psi(z_t)|\mathcal{G}_t^{\mu,\widetilde{W}}\right] = \int_{\mathbb{R}} \psi(x)\mathrm{d}\mu_t(x)$. We deduce that for every $t \in [0,T]$, $\mathbb{Q}$-almost surely $\mu_t = \mathcal{L}^{\mathbb{Q}}(z_t|\mathcal{G}_t^{\mu,\widetilde{W}})$.

Furthermore, the pair $(\mu, \widetilde{w})$ is $\mathcal{G}^W$-measurable and, subsequently, $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is also $\mathcal{G}^W$-measurable. By independence of (ξ, w, β) under $\mathbb{P}$, we deduce that $(\mu, \widetilde{w})$ and (β, ξ) are independent under $\mathbb{Q}$. By the same argument and by the compatibility property under $\mathbb{P}$, we deduce that, under $\mathbb{Q}$, for any $t \in [0,T]$, $(\xi, \widetilde{w}, \mu)$ and β are conditionally independent given $\mathcal{G}_t$, which is the required compatibility condition.

Therefore $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q}, z, \widetilde{w}, \beta, \xi)$ is a weak solution to (II.60). This proves the first statement of Theorem II.43. □

Remark II.45. In Remark II.40, we emphasized the importance of the filtration under which $(\mu_t)_{t\in[0,T]}$ is adapted. It makes sense in (II.91), because in order to identify $\int_{\mathbb{R}} \psi(x)\mathrm{d}\mu_t(x)$ with $\mathbb{E}^{\mathbb{Q}}\left[\psi(z_t)|\mathcal{G}_t^{\mu,\widetilde{W}}\right]$, we need to know that μ_t is $\mathcal{G}_t^{\mu,\widetilde{W}}$-measurable. This is obviously true, but it is not necessarily true with $\mathcal{G}_t^{\widetilde{W}}$ instead of $\mathcal{G}_t^{\mu,\widetilde{W}}$.

II.5.4.2 Uniqueness in law for the SDE with drift function b.

Let us conclude the proof of Theorem II.43 by showing uniqueness in law for equation (II.60). We follow the same idea as in Paragraph II.4.2.

Proof (Theorem II.43, uniqueness part). Let $\mathbf{\Omega^1}$ and $\mathbf{\Omega^2}$ be two weak solutions to (II.60). We want to prove that $\mathcal{L}^{\mathbb{P}^1}(z^1) = \mathcal{L}^{\mathbb{P}^2}(z^2)$. In particular, for $n = 1, 2$, $\mathbb{P}^n$-almost surely, the process $(z_t^n)_{t\in[0,T]}$ satisfies

$$z_t^n = \xi^n + \int_0^t\int_{\mathbb{R}} f(k)\Re(e^{-ikz_s^n}\mathrm{d}w^n(k,s)) + \beta_t^n + \int_0^t b(z_s^n, \mu_s^n)\mathrm{d}s,$$

where for every $t \in [0,T]$, $\mu_t^n = \mathcal{L}^{\mathbb{P}^n}(z_t^n | \mathcal{G}_t^{\mu^n, W^n})$, $\mathcal{G}_t^{\mu^n,W^n} := \sigma\{w^n(k,s), \mu_s^n \, ; k \in \mathbb{R}, s \leqslant t\}$ and (μ^n, w^n) is independent of (β^n, ξ^n) under $\mathbb{P}^n$.

For $n = 1, 2$, define the process $(h_t^n)_{t\in[0,T]}$ by $h_t^n(k) := \frac{1}{f(k)}\langle k\rangle^{-\eta}(\lambda(k,\mu_t^n) - \widetilde{\lambda}(k,\mu_t^n))$ for every $k \in \mathbb{R}$ and for every $t \in [0,T]$. It is $(\mathcal{G}_t^{\mu^n})_{t\in[0,T]}$-adapted, $\int_0^T \int_{\mathbb{R}} |h_t^n(k)|^2 \mathrm{d}k\mathrm{d}t$ is bounded and $(h_t^n)_{t\in[0,T]}$ satisfies for every $x \in \mathbb{R}$ and for every $t \in [0,T]$

$$(b - \widetilde{b})(x, \mu_t^n) = \int_{\mathbb{R}} f(k)\Big(\cos(kx) h_t^{\Re,i}(k) + \sin(kx) h_t^{\Im,i}(k)\Big)\mathrm{d}k.$$

Let us define $\mathbb{Q}^n$ as the absolutely continuous probability measure with respect to $\mathbb{P}^n$ with density

$$\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}^n} = \exp\left(- \int_0^T \int_{\mathbb{R}} \Re(\overline{h_t^n(k)} \mathrm{d}w^n(k,t)) - \frac{1}{2}\int_0^T \int_{\mathbb{R}} |h_t^n(k)|^2 \mathrm{d}k\mathrm{d}t \right).$$

Let us denote $\mathrm{d}\widetilde{w}^n(k,t) = \mathrm{d}w^n(k,t) + h_t^n(k)\mathrm{d}k\mathrm{d}t$. It follows from Girsanov's Theorem that $\mathcal{L}^{\mathbb{Q}^n}(\widetilde{w}^n, \beta^n, \xi^n) = \mathcal{L}^{\mathbb{P}^n}(w^n, \beta^n, \xi^n)$ and that, for any $t \in [0,T]$, the σ-field $\sigma\{\widetilde{w}^n(k,t') - \widetilde{w}^n(k,t), \beta_{t'}^n - \beta_t^n, \ k \in \mathbb{R}, \ t' \in [t,T]\}$ is independent of $\mathcal{G}_t^n$ under $\mathbb{Q}^n$. Moreover, $(z_t^n)_{t\in[0,T]}$ satisfies

$$z_t^n = \xi^n + \int_0^t \int_{\mathbb{R}} f(k) \Re(e^{-ikz_s^n} \mathrm{d}\widetilde{w}^n(k,s)) + \beta_t^n + \int_0^t \widetilde{b}(z_s^n, \mu_s^n)\mathrm{d}s.$$

Let us remark that $\Big(\exp(-\int_0^t \int_{\mathbb{R}} \Re(\overline{h_s^n(k)}\mathrm{d}w^n(k,s)) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} |h_s^n(k)|^2 \mathrm{d}k\mathrm{d}s)\Big)_{t\in[0,T]}$ and $\widetilde{w}_{\cdot\wedge t}^n = w_{\cdot\wedge t}^n - \int_0^{\cdot\wedge t}\int_0^{\cdot} h_s^n(l)\mathrm{d}l\mathrm{d}s$ are $(\mathcal{G}_t^{\mu^n,W^n})_{t\in[0,T]}$-adapted. Let us consider the same function φ and ψ as in equality (II.91). We obtain by a similar computation:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^n}\left[\psi(z_t^n)\ \varphi(\mu_{\cdot\wedge t}^n, \widetilde{w}_{\cdot\wedge t}^n)\right] \\
&= \mathbb{E}^{\mathbb{P}^n}\left[\psi(z_t^n)\ \varphi(\mu_{\cdot\wedge t}^n, \widetilde{w}_{\cdot\wedge t}^n) \exp\Big(- \int_0^t \int_{\mathbb{R}} \Re(\overline{h_s^n(k)}\mathrm{d}w^n(k,s)) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} |h_s^n(k)|^2 \mathrm{d}k\mathrm{d}s\Big)\right] \\
&= \mathbb{E}^{\mathbb{P}^n}\Bigg[\mathbb{E}^{\mathbb{P}^n}\left[\psi(z_t^n) | \mathcal{G}_t^{\mu^n,W^n}\right] \varphi(\mu_{\cdot\wedge t}^n, \widetilde{w}_{\cdot\wedge t}^n) \\
&\qquad \cdot \exp\Big(-\int_0^t \int_{\mathbb{R}} \Re(\overline{h_s^n(k)}\mathrm{d}w^n(k,s)) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} |h_s^n(k)|^2 \mathrm{d}k\mathrm{d}s\Big)\Bigg] \\
&= \mathbb{E}^{\mathbb{P}^n}\left[\int_{\mathbb{R}} \psi(x)\mathrm{d}\mu_t^n(x)\ \varphi(\mu_{\cdot\wedge t}^n, \widetilde{w}_{\cdot\wedge t}^n) \exp\Big(- \int_0^t \int_{\mathbb{R}} \Re(\overline{h_s^n(k)}\mathrm{d}w^n(k,s)) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} |h_s^n(k)|^2 \mathrm{d}k\mathrm{d}s\Big)\right] \\
&= \mathbb{E}^{\mathbb{Q}^n}\left[\int_{\mathbb{R}} \psi(x)\mathrm{d}\mu_t^n(x)\ \varphi(\mu_{\cdot\wedge t}^n, \widetilde{w}_{\cdot\wedge t}^n)\right],
\end{aligned}$$

and thus for every $t \in [0,T]$, $\mathbb{Q}^n$-almost surely, $\mu_t^n = \mathcal{L}^{\mathbb{Q}^n}(z_t^n | \mathcal{G}_t^{\mu^n, \widetilde{W}^n})$.

Moreover $(\mu^n, \widetilde{w}^n)$ and $\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}^n}$ are $\mathcal{G}^{\mu^n,W^n}$-measurable and under $\mathbb{P}^n$, (μ^n, w^n) is independent of (β^n, ξ^n). Thus for any bounded and measurable functions $g : \mathcal{C}([0,T],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $f : \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R})) \times \mathcal{C}(\mathbb{R} \times [0,T], \mathbb{R}^2) \to \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^n}\left[f(\mu^n, \widetilde{w}^n)\ g(\beta^n, \xi^n)\right] &= \mathbb{E}^{\mathbb{P}^n}\left[f(\mu^n, \widetilde{w}^n)\ \frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}^n}\ g(\beta^n, \xi^n)\right] \\
&= \mathbb{E}^{\mathbb{P}^n}\left[f(\mu^n, \widetilde{w}^n)\ \frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}^n}\right] \cdot \mathbb{E}^{\mathbb{P}^n}\left[g(\beta^n, \xi^n)\right] \\
&= \mathbb{E}^{\mathbb{Q}^n}\left[f(\mu^n, \widetilde{w}^n)\right] \cdot \mathbb{E}^{\mathbb{Q}^n}\left[g(\beta^n, \xi^n)\right].
\end{aligned}$$

Thus under $\mathbb{Q}^n$, $(\mu^n, \widetilde{w}^n)$ is independent of (β^n, ξ^n). By the same argument and by the compatibility property under $\mathbb{P}^n$, we get that, under $\mathbb{Q}^n$, for any $t \in [0,T]$, $(\xi^n, \widetilde{w}^n, \mu^n)$ and β^n are conditionally independent given $\mathcal{G}_t^n$, which proves compatibility under $\mathbb{Q}^n$.

Thus we deduce that for $n = 1, 2$, $(\Omega^n, \mathcal{G}^n, (\mathcal{G}^n_t)_{t\in[0,T]}, \mathbb{Q}^n, z^n, \widetilde{w}^n, \beta^n, \xi^n)$ are weak solutions to the SDE (II.63) with drift $\widetilde{b}$. By Proposition II.39, it follows that $\mathcal{L}^{\mathbb{Q}^1}(z^1, \widetilde{w}^1) = \mathcal{L}^{\mathbb{Q}^2}(z^2, \widetilde{w}^2)$ and that for every $t \in [0,T]$, $\mu^n_t = \mathcal{L}^{\mathbb{Q}^n}(z^n_t|\mathcal{G}^{\widetilde{W}^n}_t)$. Then, we apply the same computation as (II.51): for each bounded and measurable $\phi : \mathcal{C}([0,T], \mathbb{R}) \to \mathbb{R}$,

$$\mathbb{E}^{\mathbb{P}^n}\left[\phi(z^n)\right] = \mathbb{E}^{\mathbb{Q}^n}\left[\phi(z^n)\exp\Big(\int_0^T\int_{\mathbb{R}} \Re\big(\overline{h^n_t(k)}\mathrm{d}\widetilde{w}^n(k,t)\big) - \frac{1}{2}\int_0^T\int_{\mathbb{R}} |h^n_t(k)|^2\mathrm{d}k\mathrm{d}t\Big)\right].$$

Recall that $h^n_t(k) = \frac{1}{f(k)}\langle k\rangle^{-\eta}(\lambda(k,\mu^n_t) - \widetilde{\lambda}(k,\mu^n_t))$ and that $\mu^n_t = \mathcal{L}^{\mathbb{Q}^n}(z^n_t|\mathcal{G}^{\widetilde{W}^n}_t)$. Hence the process $(h^n_t)_{t\in[0,T]}$ is $(\mathcal{G}^{\widetilde{W}^n}_t)_{t\in[0,T]}$-progressively measurable. It follows that there is a measurable map $\psi : \mathcal{C}(\mathbb{R}\times[0,T], \mathbb{R}^2) \to \mathbb{R}$, independent of n, such that $\mathbb{E}^{\mathbb{Q}^n}\left[|\psi(\widetilde{w}^n)|\right] < +\infty$ and

$$\mathbb{E}^{\mathbb{P}^1}\Big[\phi(z^1)\Big] = \mathbb{E}^{\mathbb{Q}^1}\Big[\phi(z^1)\psi(\widetilde{w}^1)\Big] = \mathbb{E}^{\mathbb{Q}^2}\Big[\phi(z^2)\psi(\widetilde{w}^2)\Big] = \mathbb{E}^{\mathbb{P}^2}\Big[\phi(z^2)\Big].$$

We conclude that $\mathcal{L}^{\mathbb{P}^1}(z^1) = \mathcal{L}^{\mathbb{P}^2}(z^2)$. This completes the proof of Theorem II.43. □

Remark II.46. As the last computation right above shows it, we have in fact that $\mathcal{L}^{\mathbb{P}^1}(z^1, \widetilde{w}^1) = \mathcal{L}^{\mathbb{P}^2}(z^2, \widetilde{w}^2)$ and then $\mathcal{L}^{\mathbb{P}^1}(z^1, \mu^1) = \mathcal{L}^{\mathbb{P}^2}(z^2, \mu^2)$.

Chapter III

Bismut-Elworthy inequality for a Wasserstein diffusion on the torus

Abstract. In the previous chapters, we introduced a family of diffusions on the Wasserstein space with smooth volatility coefficient. Moreover, adopting the particle point of view, those measure-valued processes $(\mu_t)_{t\in[0,T]}$ shared the property of being associated with a canonical lift $(x_t)_{t\in[0,T]}$, where for each t, x_t is *the* quantile function of μ_t and where $(x_t(u))_{t\in[0,T]}$ is a continuous real-valued process representing, for a given $u \in [0,1]$, the path of the particle with label u. In all those dynamics, the interactions between particles were ruled by a common infinite-dimensional noise. In Chapter II, we were interested in restoration of uniqueness for a certain diffusion with the above-mentioned features. In the current chapter, we prove another regularization property for a process of that family, namely a gradient estimate often called Bismut-Elworthy or Bismut-Elworthy-Li inequality after [Bis81, Elw92, EL94]. There are two main modifications with respect to the previously studied diffusions: on the one hand, the processes of this chapter are defined on the probability space of the one-dimensional torus, instead of $\mathcal{P}_2(\mathbb{R})$, and on the other hand, we add to the dynamics an idiosyncratic noise β and the law μ_t is now considered with respect to both the noise carrying the initial condition and the Brownian motion β, as it is standard in McKean-Vlasov theory and as it has also been done in Section II.5.

III.1 Introduction

In Chapter I [Mar18], we introduced a smooth version of Konarovskyi's diffusion on the Wasserstein space (see [Kon11, Kon17b]) and we have shown in Chapter II that this kind of diffusion has a regularization effect, namely that it restores uniqueness of Fokker-Planck equation with non-smooth drift. Recall that we had to slightly modify the shape of the diffusion in order to invert the interaction kernel. In this chapter, we are interested in a smoothing property of a Wasserstein diffusion on the torus; more precisely we will prove a Bismut-Elworthy-like inequality. Let us first introduce in Paragraph III.1.1 the diffusion studied in this chapter; we will describe the two main differences with the diffusion studied in Chapter II, *i.e.* the fact that we are now considering a particle system evolving on the one-dimensional torus, that is on the circle, instead of the real line, and the fact that we add an idiosyncratic source of randomness β as we did in Section II.5. Then in Paragraph III.1.2 we will link our result to the original works of Bismut, Elworthy and Li on gradient estimates and to recent developments around gradient

estimates for McKean-Vlasov SDEs. In Paragraph III.1.3, we will explain the general strategy of proof followed in this chapter and highlight the reasons that have led us to modify the structure of the diffusion with respect to the previous chapter.

III.1.1 A diffusion on the Wasserstein space of the torus

Recall SDE (II.12), studied in Chapter II:

$$\mathrm{d}y_t(u) = \frac{1}{\left(\int_0^1 \varphi(y_t(u) - y_t(v))\mathrm{d}v\right)^{1/2}} \int_{\mathbb{R}} f(k)\Re(e^{-iky_t(u)}\mathrm{d}w(k,t)) \tag{III.1}$$

In this chapter, we study the regularization properties of a diffusion process close to the solution to (III.1). First, we consider processes with constant mass, *i.e.* we take $\varphi \equiv 1$ in (III.1). Moreover, we introduce two major modifications with respect to the previous chapter. First, we consider a system of particles evolving on the one-dimensional torus, instead of the real line. It allows us to construct a process $(\mu_t)_{t\in[0,T]}$ having a density vanishing nowhere on the state space. The second change is the introduction of an idiosyncratic noise, with an additive Brownian motion $(\beta_t)_{t\in[0,T]}$, which allows us to obtain a regularizing effect of the diffusion. This effect has already been noted in Section II.5, where we have also added a noise $(\beta_t)_{t\in[0,T]}$ in order to restore uniqueness of a Fokker-Planck equation with a non-regular drift function b both in the space and in the measure variable. It allowed us to regularize the space component of the drift: a similar phenomenon occurs in this chapter.

Let us introduce the diffusion process $(x_t(u))_{t\in[0,T],u\in[0,1]}$ with constant mass on the one-dimensional torus $\mathbb{T}$, which we will identify with the compact interval $[0, 2\pi]$. This diffusion process describes a system of infinitely many particles covering the torus and evolving according to a common infinite-dimensional noise. That diffusion process on the torus is the counterpart of (III.1) on the real line. It will be defined as the unique solution $(x_t^g(u))_{t\in[0,T],u\in[0,1]}$ to the following equation

$$x_t^g(u) = g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re(e^{-ikx_s^g(u)}\mathrm{d}W_s^k). \tag{III.2}$$

Let us briefly describe equation (III.2). As in the first two chapters, $g : [0,1] \to \mathbb{T}$ is an increasing initial condition, viewed as the quantile function of a probability measure μ_0 on the torus. We should give a precise meaning to the notion of monotonicity on the torus. The notion of quantile function on the torus will be defined in Paragraph III.2.1; at this stage, the reader can simply see g as the restriction to $[0,1]$ of a strictly increasing $\mathcal{C}^1$-function $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ satisfying $\widetilde{g}(u+1) = \widetilde{g}(u) + 2\pi$ for each $u \in \mathbb{R}$.

Remember that the flow of the diffusion (III.1) introduced in Chapter II preserves the monotonicity of the solution (see Proposition II.5). The flow of SDE (III.2) shares this property. It implies that the solution to equation (III.2) can be seen as a process $(\mu_t)_{t\in[0,T]}$ with values in the Wasserstein space $\mathcal{P}(\mathbb{T})$ of the one-dimensional torus and that for each $t \in [0,T]$, the map $u \mapsto x_t(u)$ is a canonical representative of the probability measure μ_t, since it is the quantile function associated with μ_t. Therefore, it makes sense to see equation (III.2) as the counterpart on the one-dimensional torus of the diffusion (III.1) on $\mathbb{R}$.

Moreover, $(f_k)_{k\in\mathbb{Z}}$ is a given sequence of real and deterministic numbers. It plays the same role as the function f appearing in (III.1): the regularity in the variable u of the solution $(x_t(u))_{t\in[0,T],u\in[0,1]}$ is linked to the decay at $+\infty$ and $-\infty$ of $(f_k)_{k\in\mathbb{Z}}$. In (III.2), $(W^k)_{k\in\mathbb{Z}} = (W^{\Re,k} + iW^{\Im,k})_{k\in\mathbb{Z}}$ is a sequence of complex-valued independent Brownian motions: equation (III.2) may be rewritten as

$$x_t^g(u) = g(u) + f_0 W_t^{\Re,0} + \sum_{k\neq 0} f_k \left(\int_0^t \cos(kx_s^g(u))\mathrm{d}W_s^{\Re,k} + \int_0^t \sin(kx_s^g(u))\mathrm{d}W_s^{\Im,k}\right). \tag{III.3}$$

In that equation, we remark that the term $f_0 W_t^{\Re,0}$ is common for every particle; it does not depend on the variable u. Let us rewrite it as $f_0 W_t^{\Re,0} + \beta_t$, where $(\beta_t)_{t\in[0,T]}$ is a Brownian motion independent of the collection $(W_t^k)_{t\in[0,T],k\in\mathbb{Z}}$:

$$x_t^g(u) = g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re(e^{-ikx_s^g(u)} \mathrm{d}W_s^k) + \beta_t. \tag{III.4}$$

The addition of a Brownian motion $(\beta_t)_{t\in[0,T]}$ is equivalent to change the diffusion rate of $W_t^{\Re,0}$, since we can rewrite the equation with $f_0 W_t^{\Re,0} + \beta_t = \widetilde{f}_0 \widetilde{W}_t^{\Re,0}$.

The interest of dissociating W and β is that we will see W as a common noise with respect to the diffusion and that β will be seen as an idiosyncratic source of randomness. More precisely, we consider two distinct probability spaces: $(\Omega^W, \mathcal{G}^W, \mathbb{P}^W)$ carrying the collection of Brownian motions $(W^k)_{k\in\mathbb{Z}}$ and $(\Omega^\beta, \mathcal{G}^\beta, \mathbb{P}^\beta)$ carrying the Brownian motion β. Then we define equation (III.4) on the product space $(\Omega, \mathcal{G}, \mathbb{P})$, where $\Omega = \Omega^W \times \Omega^\beta$, $\mathbb{P} = \mathbb{P}^W \otimes \mathbb{P}^\beta$ and $\mathcal{G}$ is the product of the σ-algebras $\mathcal{G}^W$ and $\mathcal{G}^\beta$ completed with the null sets of $\mathbb{P}$. We associate to the solution $(x_t^g)_{t\in[0,T]}$ to SDE (III.4) a measure-valued process $(\mu_t^g)_{t\in[0,T]}$ defined for every $t \in [0,T]$ as

$$\mu_t^g := (\mathrm{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ (x_t^g)^{-1}. \tag{III.5}$$

In other terms, μ_t^g is the conditional law of x_t^g given $(W^k)_{k\in\mathbb{Z}}$. The interest of adding β is to add a level of regularization to that equation: this phenomenon will be explained in more details in Paragraph III.1.3.

III.1.2 Bismut-Elworthy inequality: a regularization result

The aim of this chapter is to prove a regularization property satisfied by the measure-valued process $(\mu_t^g)_{t\in[0,T]}$ defined by (III.5). In Chapter II, we have already proved that this type of diffusion processes on the Wasserstein space allows to restore uniqueness of PDEs defined on spaces of infinite dimension, e.g. Fokker-Planck equation. Bismut-Elworthy inequality focuses on the smoothing effect of the semi-group and also provides a rate describing how fast the regularization happens.

Roughly speaking, we will define a class of bounded functions $\phi : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ that are continuous with respect to the Wasserstein distance and we will construct a semi-group $(P_t)_{t\in[0,T]}$ on that space by letting $P_t\phi(\mu_0^g) := \mathbb{E}^W[\phi(\mu_t^g)]$, where μ_0^g is the probability measure $\mu_0^g = \mathrm{Leb}_{[0,1]} \circ g^{-1}$ associated with the initial condition g. If g is sufficiently smooth with respect to u (typically, we will need a regularity slightly better than $\mathcal{C}^3$), then we will prove that we can differentiate $P_t\phi$ in certain smooth directions: Theorem III.45 states that for each ρ sufficiently close to 0 and for every $\theta \in (0,1)$,

$$\left| P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g) \right| \leqslant |\rho| C_{g,h} \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}}, \tag{III.6}$$

where $C_{g,h}$ is a constant depending on g and on the direction of perturbation h. Of course, the locally Lipschitz bound tends to $+\infty$ when t becomes close to zero, since ϕ is not necessarily Lipschitz-continuous. Before giving more comments on the rate $t^{-(2+\theta)}$ appearing in that formula, let us come back to historical results around Bismut-Elworthy inequality and comment on its usual applications.

Bismut-Elworthy inequality derives from an integration by parts formula for the heat semi-group on $\mathbb{R}^n$ or on a compact manifold, obtained by Bismut [Bis81], Elworthy [Elw92] and Elworthy-Li [EL94]. Roughly speaking, for a stochastic differential equation on $\mathbb{R}^n$ of type

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$$

with initial condition $X_0 = x_0$, Bismut's integration by parts formula has the following form:

$$d(P_t\phi)_{x_0}(v_0) = \frac{1}{t}\mathbb{E}\left[\phi(X_t)\int_0^t \langle V_s, \sigma(X_s)\mathrm{d}W_s\rangle\right], \tag{III.7}$$

where V_s is a certain stochastic process starting at v_0 (see [EL94]). Bismut-Elworthy inequality is a corollary to that integration by parts formula and consists in estimating the right-hand-side of (III.7). It provides a gradient estimate for the heat semi-group. Let us note that for the heat equation on $\mathbb{R}^n$, the upper bound for the gradient estimate is of order $t^{-1/2}$. Bismut-Elworthy inequality is part of the general framework of Malliavin calculus (see [Nua06, Nor86] for an introduction to Malliavin calculus), but it can be shown independently.

An important domain of application of Bismut-Elworthy inequality is geometry. The techniques inspired by the works of Bismut, Elworthy and Li have led to the obtention of gradient estimates for PDEs on manifolds, among others by Thalmaier, Wang and Arnaudon [Tha97, TW98, ATW06]. It has improved gradient estimates obtained by a coupling method of two Brownian motions (see Cranston [Cra91]). Furthermore, Bismut-Elworthy inequality has led to get gradient estimates for non-linear PDEs, for instance Kolmogorov backward equation or Forward-Backward equations, see [Del03, Zha05]. The integration by parts formula is useful to compute price sensitivities, so-called Greeks, in finance; e.g. in [FLL$^+$99] the authors used Malliavin calculus techniques to obtain an integration by parts formula and obtain better convergence rates than Monte-Carlo methods. Bismut-Elworthy-Li integration by parts formula is also used in [MT06] to compute Greeks. The work of Bismut, Elworthy and Li allows to improve numerical methods of resolution of SDEs, since it provides a convergence rate for finite-dimensional approximations of a solution.

The pioneer results of Bismut, Elworthy and Li have been extended to the case of SPDEs in infinite-dimension. In a large survey [Cer01], Cerrai has recalled smoothing results for Kolmogorov equations in finite dimension and has extended Bismut-Elworthy formula to the case of reaction-diffusion systems in bounded domains of $\mathbb{R}^n$ and of Kolmorogov equations on Hilbert spaces, following results previously obtained in [DPEZ95]. For the reader interested in smoothing properties of Kolmogorov equations on infinite dimension, let us mention the Da Prato's textbook [DP04].

More recently, smoothing properties have been obtained for McKean-Vlasov equation:

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X_t, \mu_t)\mathrm{d}W_t, \\ \mu_t = \mathcal{L}(X_t). \end{cases} \tag{III.8}$$

Let us introduce the decoupled equation associated to (III.8): it is called decoupled since we first fix X_t^θ the solution to (III.8) starting at θ and then solve the SDE given by

$$\begin{cases} \mathrm{d}X_t^{x,[\theta]} = b(t, X_t^{x,[\theta]}, \mathcal{L}(X_t^\theta))\mathrm{d}t + \sigma(t, X_t^{x,[\theta]}, \mathcal{L}(X_t^\theta))\mathrm{d}W_t, \\ X_0^{x,[\theta]} = x. \end{cases} \tag{III.9}$$

In [BLPR17], Buckdahn, Li, Peng and Rainer have proved that the function $V(t, x, \mathcal{L}(\xi)) := \mathbb{E}\left[\phi(X_T^{t,x,[\xi]}, \mathcal{L}(X_T^{t,[\xi]}))\right]$ has some regularity and satisfies a classical second order PDE, involving the first and second derivatives with respect to the space and the measure variable. Chassagneux, Crisan and Delarue [CCD19] have studied similar smoothing properties of the stochastic flow generated by a forward-backward stochastic system of McKean-Vlasov type; it is linked to the theory of Mean-Field Games, where these equations arise and describe the asymptotic behaviour of a large population of players with mean-field interaction. The equation followed by the flow of the decoupled equation associated with the forward-backward system is called the master equation. In [CM18], Crisan and McMurray have shown some integration by parts formulae,

using Malliavin calculus, for the decoupled equation (III.9), for derivatives in the directions of both x and the measure variable. Using those integration by parts formulae, Crisan and McMurray have obtained estimates on the derivatives of the density associated to the solution of the decoupled equation. In the case where the mean-field interaction is scalar, *i.e.* when $b(t, X_t, \mu_t) = b(t, X_t, \mathbb{E}[f(X_t)])$ and $\sigma(t, X_t, \mu_t) = \sigma(t, X_t, \mathbb{E}[g(X_t)])$, Baños [Bn18] proved a Bismut-Elworthy-Li integration by parts formula, in the case where both b and σ are continuously differentiable with bounded Lipschitz derivatives in both variables. Let us also cite a result obtained by Chaudru de Raynal [CdR19] using parametrix techniques under the same kind of assumptions. In [CdRF], Chaudru de Raynal and Frikha have proved smoothing properties of the densities of the solution to equation (III.8).

III.1.3 Integration by parts formula and addition of an idiosyncratic noise

In this chapter, we prove an inequality of Bismut-Elworthy-Li type, namely inequality (III.6). In this paragraph, we briefly explain the strategy of proof followed in this chapter by highlighting the reasons why we are considering a diffusion on the torus and why we add an idiosyncratic source of randomness β with respect to the diffusion studied in Chapter II.

Without justifying the computations done hereafter, we want to give an idea of how we obtain an integration by parts formula for the semi-group associated to the measure-valued process $(\mu_t^g)_{t\in[0,T]}$ defined by (III.5). The strategy is based on Kunita expansion and on Girsanov's Theorem. Recall that $P_t\phi(\mu_0^g) = \mathbb{E}^W[\phi(\mu_t^g)]$. Using Lions' differential calculus, the computation of the derivative of $P_t\phi$ at point g and in direction $g'h$ (where g' denotes the derivative of g) is:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho g'h}) = \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\,\partial_u x_t^g(u)\,h(u)\mathrm{d}u\right].$$

Note that we choose here to differentiate in the direction $g'h$ instead of h in order to be coherent with the computations made later in the chapter. Of course, replacing h by $\frac{h}{g'}$, it is straightforward to obtain the results corresponding to a differentiation in direction h.

Let us denote, for every $t \in [0,T]$, $A_t^g := (\partial_u x_t^g \cdot h) \circ F_t^g$, where F_t^g denotes the inverse map of x_t^g, that is the c.d.f. associated with the measure μ_t^g. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho g'h}) &= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\,\partial_u x_t^g(u)\int_0^t \frac{A_s^g(x_s^g(u))}{\partial_u x_s^g(u)}\mathrm{d}s\mathrm{d}u\right]\\ &= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\,\frac{\partial_u x_t^g(u)}{g'(u)}a_t^g(u)\mathrm{d}u\right],\end{aligned}$$

where $a_t^g(u) = \int_0^t \frac{g'(u)}{\partial_u x_s^g(u)} A_s^g(x_s^g(u))\mathrm{d}s$. Following the method of proof of [Tha97], let us expand the equation satisfied by the trajectory $(x_t^{g+\rho a_t^g})_{t\in[0,T]}$, where the initial condition is also evolving with time:

$$x_t^{g+\rho a_t^g}(u) = g(u) + \sum_{k\in\mathbb{Z}}\int_0^t f_k\Re\left(e^{-ikx_s^{g+\rho a_s^g}(u)}\mathrm{d}W_s^k\right) + \beta_t + \rho\int_0^t A_s^g(x_s^{g+\rho a_s^g}(u))\mathrm{d}s + \mathcal{O}(\rho^2). \tag{III.10}$$

That expansion is due to Kunita [Kun90]. As we did in Chapter II, we want to define a collection of square-integrable processes $((\lambda_t^k)_{t\in[0,T]})_{k\in\mathbb{Z}}$ such that for every $s \in [0,T]$ and for every $x \in \mathbb{R}$

$$A_s^g(x) = \sum_{k\in\mathbb{Z}} f_k\Re(e^{-ikx}\lambda_s^k). \tag{III.11}$$

Using (III.11), we can rewrite (III.10) into:

$$x_t^{g+\rho a_t^g}(u) = g(u) + \sum_{k\in\mathbb{Z}} \int_0^t f_k \Re\left(e^{-ikx_s^{g+\rho a_s^g}(u)}(\mathrm{d}W_s^k + \rho\lambda_s^k \mathrm{d}s)\right) + \beta_t + \mathcal{O}(\rho^2),$$

and by Girsanov's Theorem, $\mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^{g+\rho a_t^g})\mathcal{E}_t^g\right] = \mathbb{E}^W\left[\phi(\mu_t^g)\right]$, where $(\mathcal{E}_t^\rho)_{t\in[0,T]}$ is an exponential martingale. Thus the left-hand-side of the latter equality does not depend on ρ and we obtain the following equality by differentiating with respect to ρ:

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\,\frac{\partial_u x_t^g(u)}{g'(u)} a_t^g(u)\mathrm{d}u\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right],$$

where $\overline{\lambda_s^k}$ is the complex conjugate of λ_s^k. Finally, we get the following Bismut-Elworthy integration by parts formula:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho g'h}) = \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right].$$

Unfortunately, the strategy detailled below cannot be applied directly. The key point of the proof is, similar to Chapter II, the Fourier inversion given by (III.11). Let us recap: given a direction of perturbation h, does there exist a collection of $\mathbb{C}$-valued adapted processes $((\lambda_t^k)_{t\in[0,T]})_{k\in\mathbb{Z}}$ such that for each $s\in[0,T]$ and each $x\in\mathbb{R}$,

$$(\partial_u x_s^g \cdot h)\circ F_s^g(x) = \sum_{k\in\mathbb{Z}} f_k \Re(e^{-ikx}\lambda_s^k), \tag{III.12}$$

and $\sum_{k\in\mathbb{Z}}\int_0^t |\lambda_s^k|^2\mathrm{d}s$ is almost surely finite?

A first problem to get (III.12) is that we have to make sure that the derivative $\partial_u x_s^g$ of the quantile function does not tend to $+\infty$. Recall that it tends to $+\infty$ if and only if the density of the measure μ_t^g tends to zero. That explains why we decide to consider a diffusion on the torus in this chapter. Indeed, if the initial condition μ_0^g is a probability measure on the torus which is absolutely continuous with respect to the Lebesgue measure and with a density positive everywhere on the torus, then we will prove that the flow of the SDE (III.4) preserves this property and we will be able to obtain controls on the integrability of the process $(\partial_u x_t^g)_{t\in[0,T]}$. The fact that we consider the diffusion (III.4) on the torus allows us also to take advantage of the compactness of the torus.

Nevertheless, working on the torus is not sufficient to resolve (III.4). Actually, we get to the same problem as in Chapter II, namely we will have a balance between the order of regularity of $(x_t^g)_{t\in[0,T]}$ and the decay at infinity of $(f_k)_{k\in\mathbb{Z}}$. Let us assume to simplify the discussion that $f_k = \frac{1}{(1+k^2)^{\alpha/2}}$. In order to insure some L_2-integrability for $((\lambda_t^k)_{t\in[0,T]})_{k\in\mathbb{Z}}$, α should be as small as possible. But the regularity of $u\mapsto x_t^g(u)$ for any t increases when α becomes larger. Unfortunately, as it will become clear with the computations of Lemma III.30, there is not an exponent α for which $((\lambda_t^k)_{t\in[0,T]})_{k\in\mathbb{Z}}$ is square-integrable.

In order to avoid that problem, we will add a step of regularization. We will consider a sequence $(A_s^{g,\varepsilon})_{\varepsilon>0}$ of smooth approximations of A_s^g. On the one side, we will apply to $A_s^{g,\varepsilon}$ the previous method. The bound in Bismut-Elworthy inequality will depend on ε and explode when $\varepsilon\to 0^+$. On the other side, we deal with $A_s^g - A_s^{g,\varepsilon}$; we will apply once more Girsanov's Theorem, but this time with respect to the Brownian motion $(\beta_t)_{t\in[0,T]}$. This emphasizes the regularizing effect of β.

Of course, we may wonder if adding β is necessary to get a regularizing effect. The following computations may give an idea why it seems reasonable to add a new source of randomness. Let

us denote by p_t^g the density of the measure associated with the solution x_t^g of the SDE (III.2) without β-term. For every 2π-periodic function $\psi : \mathbb{R} \to \mathbb{R}$, we have $\int_0^1 \psi(x_t^g(u))\mathrm{d}u = \int_{\mathbb{T}} \psi(v)p_t^g(v)\mathrm{d}v$. Formally, we get that $(p_t^g)_{t\in[0,T]}$ satisfies the following SDE: for every $v \in \mathbb{T}$,

$$\mathrm{d}p_t^g(v) = -\partial_v\left(p_t(v)\sum_{k\in\mathbb{Z}} f_k\Re(e^{-ikv}\mathrm{d}W_t^k)\right) + \lambda p_t''(v)\mathrm{d}t.$$

If we assume that $\sum_{k\in\mathbb{Z}} f_k^2 = 1$, then $\lambda = \frac{1}{2}$. Following the work of Denis, Matoussi, Stoica [DS04, DMS05], there is, at least for noises of finite dimension, a critical threshold at $\lambda > \frac{1}{2}$ to obtain a regularizing effect for nonlinear SPDEs where the coefficients depend on the derivatives of the solution. Adding the noise β, the equation followed by the density process $(p_t^g)_{t\in[0,T]}$ becomes:

$$\mathrm{d}p_t^g(v) = -\partial_v\left(p_t(v)\sum_{k\in\mathbb{Z}} f_k\Re(e^{-ikv}\mathrm{d}W_t^k)\right) + p_t''(v)\mathrm{d}t,$$

and thus the condition of Denis-Stoica is satisfied.

That idea of adding an idiosyncratic noise β will allow us to prove a Bismut-Elworthy-like inequality. However, due to the interpolation argument, more precisely due to the choice of ε which will depend on t, we cannot expect to obtain a rate of order $t^{-1/2}$. As previously announced, the best rate that we are able to get with that strategy is of order $t^{-2-\theta}$, for θ as close to zero as desired. We will devote Chapter IV to the optimization of that rate and we will give an application of this result to a gradient estimate of a PDE on the Wasserstein space with source term.

Organisation of the chapter

Before introducing the diffusion model, we will devote Section III.2 to the introduction of some useful tools needed in this chapter, namely the notion of quantile functions of probability measures on the torus and a differential calculus for functions of probability measures on the torus. Then, in Section III.3, we will construct the diffusion on the torus and prove its properties of differentiability and integrability. The core of the chapter is Section III.4, in which we will introduce the semi-group associated to the diffusion process and prove a Bismut-Elworthy-like inequality. The proof will be based on the scheme given in this introduction, taking into account the difficulties raised above. We will first assume that ϕ is smooth and finally get the result for every bounded and continuous $\phi : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ by a regularization argument.

III.2 Probability measures on the torus

Let $\mathbb{T}$ be the one-dimensional torus, that we will identify with the interval $[0, 2\pi]$. Let $\mathcal{P}(\mathbb{T})$ be the space of probability measures on the torus. Let us consider the L_2-Wasserstein metric $W_2^{\mathbb{T}}$ on $\mathcal{P}(\mathbb{T})$:

$$W_2^{\mathbb{T}}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)}\left(\int_{\mathbb{T}^2} d^{\mathbb{T}}(x,y)^2\mathrm{d}\pi(x,y)\right)^{1/2},$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathbb{T}^2$ with first marginal μ and second marginal ν and where $d^{\mathbb{T}}$ is the distance on the torus: for every $x, y \in \mathbb{R}$, $d^{\mathbb{T}}(x,y) := \inf_{k\in\mathbb{Z}} |x - y - 2k\pi|$.

Before defining in the next section a diffusion on the torus, we will focus in this section on a few properties of the topology of $\mathbb{T}$ that are useful for our purposes. Recall that the diffusions constructed in the previous chapters shared the property of preserving the relative order of

the particles, *i.e.* particles do not collide. In other words, the dynamics of these diffusions on the space of probability measures $\mathcal{P}_2(\mathbb{R})$ can be described as dynamics of the associated quantile functions. We want to construct in this chapter a diffusion on $\mathcal{P}(\mathbb{T})$ that preserves also the structure of quantile functions. In Paragraph III.2.1, we will introduce a notion of quantile function on the torus. We will show that there is a canonical way to associate to every probability measure $\mu \in \mathcal{P}_2(\mathbb{R})$ with positive density on $\mathbb{T}$ an equivalence class of increasing and pseudo-periodic functions.

In this chapter, we are interested in regularizing properties of that diffusion process on $\mathcal{P}(\mathbb{T})$. More precisely, we will study the regularity of the semi-group associated to this diffusion. To that purpose, we will define in Paragraph III.2.2 a particular class of functions $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfying both a periodicity and a regularity assumption. First, ϕ should be constant on every equivalence class of measures on $\mathbb{R}$ having same traces on $\mathbb{T}$. Second, ϕ should be differentiable with bounded and Lipschitz-continuous derivative, in the sense of Lions' differential calculus. We will show a periodicity property of Lions' derivatives of functions ϕ belonging to that class.

III.2.1 Density function and quantile function on the torus

We will focus on a class of probability measures on $\mathbb{T}$ having a density with respect to the Lebesgue measure on $\mathbb{T}$, and more precisely densities that are continuous and positive everywhere on the torus. The following definition makes it precise.

Definition III.1. Let $\mathcal{P}^+$ be the set of continuous functions $p : \mathbb{T} \to \mathbb{R}$ such that for every $x \in \mathbb{T}$, $p(x) > 0$ and $\int_{\mathbb{T}} p = 1$. In other words, $\mathcal{P}^+$ is the set of 2π-periodic and continuous functions $p : \mathbb{R} \to (0, +\infty)$ such that $\int_0^{2\pi} p(x)\mathrm{d}x = 1$.

As in the previous chapters, we want to associate a quantile function to each measure on the torus. Since we work here on a periodic domain, we will carefully define this notion in what follows.

Let $p \in \mathcal{P}^+$. Let $x_0 \in \mathbb{T}$ be an arbitrary point on the torus. Let us define a cumulative distribution function (c.d.f.) $F_0 : \mathbb{R} \to \mathbb{R}$ as follows: for each $x \in \mathbb{R}$,

$$F_0(x) = \int_{x_0}^{x} p(y)\mathrm{d}y. \tag{III.13}$$

Since p is 2π-periodic and $\int_0^{2\pi} p = 1$, F_0 satisfies $F_0(x + 2\pi) = F_0(x) + 1$ for each $x \in \mathbb{R}$. It follows from the continuity and the positivity of p that F_0 is a $\mathcal{C}^1$-function and for every $x \in \mathbb{R}$, $F_0'(x) = p(x) > 0$, so that F_0 is strictly increasing. Therefore, we can define an inverse function $g_0 := F_0^{-1} : \mathbb{R} \to \mathbb{R}$. The following properties of g_0 are straightforward:

- for every $x \in \mathbb{R}$, $g_0 \circ F_0(x) = x$ and for every $u \in \mathbb{R}$, $F_0 \circ g_0(u) = u$;
- g_0 is a strictly increasing $\mathcal{C}^1$-function and for each $u \in \mathbb{R}$, $g_0'(u) = \frac{1}{p(g_0(u))}$;
- $g_0(0) = x_0$ and for every $u \in \mathbb{R}$, $g_0(u + 1) = g_0(u) + 2\pi$ (we sometimes say that g_0 is pseudo-periodic);
- $g_0' : \mathbb{R} \to \mathbb{R}$ is positive everywhere and is a 1-periodic function.

Moreover, let us choose another point x_1 on the torus and let us associate to x_1 the c.d.f. F_1 defined by $F_1(x) = \int_{x_1}^{x} p$ for every $x \in \mathbb{R}$. Then

$$F_1(x) = \int_{x_1}^{x} p = F_0(x) + c,$$

where $c := \int_{x_1}^{x_0} p$. Therefore, if $g_1 = F_1^{-1}$, we get for every $u \in \mathbb{R}$,

$$g_0(u-c) = g_0(F_1 \circ g_1(u) - c) = g_0(F_0(g_1(u)) + c - c) = g_0 \circ F_0(g_1(u)) = g_1(u).$$

It becomes natural to define the following equivalence relation.

Definition III.2. Let $j \in \mathbb{N}\backslash\{0\}$ and $\theta \in [0,1]$. Let $\mathfrak{G}^{j+\theta}$ be the set of $\mathcal{C}^j$-functions $g : \mathbb{R} \to \mathbb{R}$ such that $\partial_u^{(j)} g$ is θ-Hölder continuous and such that for every $u \in \mathbb{R}$, $g'(u) > 0$ and $g(u+1) = g(u) + 2\pi$. Let $\sim$ be the following equivalence relation on $\mathfrak{G}^{j+\theta}$: $g_1 \sim g_2$ if and only if there exists $c \in \mathbb{R}$ such that $g_2(\cdot) = g_1(\cdot + c)$.

We denote by $\mathbf{G}^{j+\theta}$ the set of equivalence classes $\mathfrak{G}^{j+\theta}/\sim$ and by $\overline{g}$ the equivalence class of an element g of $\mathfrak{G}^{j+\theta}$.

Proposition III.3. *There is a one-to-one correspondence between the set $\mathcal{P}^+$ and the set $\mathbf{G}^1$.*

Proof. Let $\iota : \mathcal{P}^+ \to \mathbf{G}^1$ be the map such that for every $p \in \mathcal{P}^+$, $\overline{g} = \iota(p)$ is the equivalence class given by the above construction. Let us show that ι is one-to-one.

First, ι is injective. Indeed, let $p_1, p_2 \in \mathcal{P}^+$ such that $\iota(p_1) = \iota(p_2)$. Let $x_0 \in \mathbb{T}$ and define, for $i = 1, 2$, $F_i(x) = \int_{x_0}^x p_i(y)\mathrm{d}y$ and $g_i = F_i^{-1}$. Then by construction $\overline{g_1} = \iota(p_1) = \iota(p_2) = \overline{g_2}$. Therefore, there is $c \in \mathbb{R}$ such that $g_2(\cdot) = g_1(\cdot + c)$. Thus for every $x \in \mathbb{R}$,

$$F_1(x) = F_1(g_2 \circ F_2(x)) = F_1(g_1(F_2(x) + c)) = F_2(x) + c.$$

Thus F_1 and F_2 share the same derivative: $p_1 = p_2$.

Second, ι is surjective. Let $\overline{g} \in \mathbf{G}^1$. Let g be a representative of the class $\overline{g}$. It is a $\mathcal{C}^1$-function such that $g'(u) > 0$ for every $u \in \mathbb{R}$ and, since $g(u+1) = g(u) + 2\pi$ for every $u \in \mathbb{R}$, g' is 1-periodic.

Let $F := g^{-1} : \mathbb{R} \to \mathbb{R}$. In particular, F is a $\mathcal{C}^1$-function such that $F' > 0$ and for every $x \in \mathbb{R}$, $F(x + 2\pi) = F(x) + 1$. Thus $p := F'$ is a continuous function with values in $(0, +\infty)$ and for every $x \in \mathbb{R}$, $p(x) = \frac{1}{g'(F(x))}$. Thus for every $x \in \mathbb{R}$,

$$p(x + 2\pi) = \frac{1}{g'(F(x + 2\pi))} = \frac{1}{g'(F(x) + 1)} = \frac{1}{g'(F(x))} = p(x),$$

and

$$\int_0^{2\pi} p(x)\mathrm{d}x = \int_0^{2\pi} F'(x)\mathrm{d}x = F(2\pi) - F(0) = 1.$$

Therefore p belongs to $\mathcal{P}^+$. Let us check that $\overline{g} = \iota(p)$. Let x_0 be an arbitrary point in $\mathbb{T}$ and let F_0 be defined by (III.13) and $g_0 := F_0^{-1}$. Since $F_0' = p = F'$, there is $c \in \mathbb{R}$ such that $F_0(\cdot) = F(\cdot) + c$. Therefore, $g_0(\cdot) = g(\cdot + c)$, whence $g_0 \sim g$. This completes the proof. □

Remark III.4. The map ι induces naturally a bijection between $\mathbf{G}^{j+\theta}$ and $\mathcal{P}^+ \cap \mathcal{C}^{j+\theta}(\mathbb{R})$.

III.2.2 Functions of probability measures on the torus

In order to study the flow of an SDE defined on the torus, we define in this paragraph an appropriate class of functions from $\mathcal{P}(\mathbb{T})$ to $\mathbb{R}$. Nevertheless, since we need to differentiate these functions, it is be more convenient to define a class of functions from $\mathcal{P}_2(\mathbb{R})$ to $\mathbb{R}$ with a certain periodicity condition.

Definition III.5. Let us define the following equivalence class on $\mathcal{P}_2(\mathbb{R})$. We say that $\mu \sim \nu$ if for every $A \in \mathcal{B}([0,2\pi])$, $\mu(A+2\pi\mathbb{Z}) = \nu(A+2\pi\mathbb{Z})$; in other words if the traces of μ and ν on the torus $\mathbb{T}$ are the same.

We say that a function $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is $\mathbb{T}$-*stable* if $\phi(\mu) = \phi(\nu)$ for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ such that $\mu \sim \nu$. For every $\mu \in \mathcal{P}_2(\mathbb{R})$, we define $\widehat{\mu}$ the measure satisfying $\widehat{\mu}(A) = \mu(A+2\pi\mathbb{Z})$ for every $A \in \mathcal{B}([0,2\pi])$. Clearly, $\widehat{\mu}$ belongs to $\mathcal{P}(\mathbb{T})$ and $\widehat{\mu} = \widehat{\nu}$ for every $\mu \sim \nu$. Thus ϕ induces a map from $\mathcal{P}(\mathbb{T})$ to $\mathbb{R}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space rich enough so that for any probability measure μ on any Polish space, we can construct on $(\Omega, \mathcal{F}, \mathbb{P})$ a random variable with distribution μ; a sufficient condition is that $(\Omega, \mathcal{F}, \mathbb{P})$ is Polish and atomless. Let $L_2(\Omega)$ be the set of square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, modulo the equivalence relation of almost sure equality. Let us define $\widehat{\phi} : L_2(\Omega) \to \mathbb{R}$ by $\widehat{\phi}(X) = \phi(\mathcal{L}(X))$, where $\mathcal{L}(X)$ denotes the law of X. If $\widehat{\phi} : L_2(\Omega) \to \mathbb{R}$ is Fréchet-differentiable, we will denote by $D\widehat{\phi}(X)$ its derivative at X and we will identify it with an element of $L_2(\Omega)$. Recall that $D\widehat{\phi}$ is independent of the choice of the probability space Ω (provided that Ω satisfies the condition mentioned above): there is a measurable function $\partial_\mu \phi(\mu) : \mathbb{R} \to \mathbb{R}$ such that for every X with distribution μ, $D\widehat{\phi}(X) = \partial_\mu \phi(\mu)(X)$ (see Section 0.4).

For every $x \in \mathbb{R}$, let $\{x\}$ be the unique number in $[0, 2\pi)$ such that $x - \{x\} \in 2\pi\mathbb{Z}$. For each $X \in L_2(\Omega)$, we construct $\{X\} \in L_2(\Omega)$, defined by $\{X\}(\omega) := \{X(\omega)\}$. For every $A \in \mathcal{B}([0,2\pi])$, $\mathbb{P}[X \in A + 2\pi\mathbb{Z}] = \mathbb{P}[\{X\} \in A + 2\pi\mathbb{Z}]$, whence we deduce that the laws $\mathcal{L}(X)$ and $\mathcal{L}(\{X\})$ are equivalent in the sense of Definition III.5. In particular, for every $X \in L_2(\Omega)$, for every $\mathbb{T}$-stable function ϕ,

$$\widehat{\phi}(X) = \widehat{\phi}(\{X\}). \tag{III.14}$$

Let us quote two classes of examples of $\mathbb{T}$-stable functions:

- if $h : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function, the map $\phi : \mu \in \mathcal{P}_2(\mathbb{R}) \mapsto \int_{\mathbb{R}} h(x)\mathrm{d}\mu(x)$ is $\mathbb{T}$-stable. The 2π-periodicity condition ensures that $\widehat{\phi}(X) = \mathbb{E}[h(X)] = \mathbb{E}[h(\{X\})] = \widehat{\phi}(\{X\})$.
- if $h : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function, the map $\phi : \mu \in \mathcal{P}_2(\mathbb{R}) \mapsto \int_{\mathbb{R}^2} h(x-y)\mathrm{d}(\mu\otimes\mu)(x,y)$ is also $\mathbb{T}$-stable. We have $\widehat{\phi}(X) = \mathbb{E}\mathbb{E}'[h(X-X')]$, where X' is a copy of X defined on $(\Omega', \mathcal{F}', \mathbb{P}')$, copy of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In this chapter, we will first get results for a class of regular functions ϕ and then extend those results by a regularization argument to a more general class. The class of smooth functions that we will consider in the first instance is the set of functions ϕ such that $\widehat{\phi}$ belongs to $C^{1,1}_{\mathrm{b}}(L_2(\Omega))$. More precisely, ϕ should satisfy the following assumptions:

Definition III.6. A function $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is said to satisfy the ϕ-*assumptions* if the following three conditions hold:

(ϕ1) ϕ is $\mathbb{T}$-stable, bounded and continuous on $\mathcal{P}_2(\mathbb{R})$.

(ϕ2) ϕ is L-differentiable (differentiable in the sense of Lions, see Section 0.4) and

$$\sup_{\mu\in\mathcal{P}_2(\mathbb{R})} \int_{\mathbb{R}} |\partial_\mu \phi(\mu)(x)|^2 \mathrm{d}\mu(x) < +\infty.$$

(ϕ3) $D\widehat{\phi}$ is Lipschitz-continuous: there is $C > 0$ such that for every $X, Y \in L_2(\Omega)$,

$$\mathbb{E}\left[|D\widehat{\phi}(X) - D\widehat{\phi}(Y)|^2\right] \leqslant C\mathbb{E}\left[|X-Y|^2\right].$$

Remark III.7. It follows from ($\phi 2$) that $\widehat{\phi}$ is a Lipschitz-continuous function, by the following computation:

$$\begin{aligned}|\widehat{\phi}(X)-\widehat{\phi}(Y)| &= \left|\mathbb{E}\left[\int_0^1 D\widehat{\phi}(\lambda X+(1-\lambda)Y)\,(X-Y)\mathrm{d}\lambda\right]\right| \\ &= \left|\int_0^1 \mathbb{E}\left[\partial_\mu\phi(\mathcal{L}(\lambda X+(1-\lambda)Y))(\lambda X+(1-\lambda)Y)\,(X-Y)\right]\mathrm{d}\lambda\right| \\ &\leqslant \left|\sup_{\mu\in\mathcal{P}_2(\mathbb{R})}\int_{\mathbb{R}}|\partial_\mu\phi(\mu)(x)|^2\mathrm{d}\mu(x)\right|^{1/2}\|X-Y\|_{L_2(\Omega)}.\end{aligned}$$

We want to show that every function ϕ satisfying the ϕ-assumptions has a 2π-periodic L-derivative. Let us first recall the following result, which applies more generally to every function $\phi:\mathcal{P}_2(\mathbb{R})\to\mathbb{R}$ with Lipschitz-continuous L-derivative.

Lemma III.8. *Let $\phi:\mathcal{P}_2(\mathbb{R})\to\mathbb{R}$ be a function satisfying the ϕ-assumptions. Then there is a constant $C>0$ such that for every $\mu\in\mathcal{P}_2(\mathbb{R})$, we can redefine $\partial_\mu\phi(\mu)(\cdot)$ on a μ-negligible set in such a way that for every $v,v'\in\mathbb{R}$,*

$$|\partial_\mu\phi(\mu)(v)-\partial_\mu\phi(\mu)(v')|\leqslant C|v-v'|, \tag{III.15}$$

and $(\mu,v)\mapsto\partial_\mu\phi(\mu)(v)$ is continuous at any point (μ,v) such that v belongs to the support of μ. Furthermore, there is $C>0$ such that for every $\mu\in\mathcal{P}_2(\mathbb{R})$, for every $v\in\mathbb{R}$,

$$|\partial_\mu\phi(\mu)(v)|\leqslant C(1+|v|)+C\int|x|\mathrm{d}\mu(x). \tag{III.16}$$

Proof. By [CD18a, Proposition 5.36], inequality (III.15) is a consequence of assumption ($\phi 3$). The proof of the continuity of $(\mu,v)\mapsto\partial_\mu\phi(\mu)(v)$ at any point where v belongs to the support of μ is given in [CD18a, Corollary 5.38]. Integrating (III.15) over v' under the measure μ, we get

$$\begin{aligned}\left|\partial_\mu\phi(\mu)(v)-\int_{\mathbb{R}}\partial_\mu\phi(\mu)(v')\mathrm{d}\mu(v')\right| &\leqslant \int_{\mathbb{R}}|\partial_\mu\phi(\mu)(v)-\partial_\mu\phi(\mu)(v')|\,\mathrm{d}\mu(v') \\ &\leqslant C\int_{\mathbb{R}}|v-v'|\,\mathrm{d}\mu(v')\leqslant C|v|+C\int_{\mathbb{R}}|v'|\,\mathrm{d}\mu(v').\end{aligned}$$

By assumption ($\phi 2$), $|\int_{\mathbb{R}}\partial_\mu\phi(\mu)(v')\mathrm{d}\mu(v')|\leqslant(\int_{\mathbb{R}}|\partial_\mu\phi(\mu)(v')|^2\mathrm{d}\mu(v'))^{1/2}\leqslant C$. Thus we deduce inequality (III.16). □

Proposition III.9. *Let $\phi:\mathcal{P}_2(\mathbb{R})\to\mathbb{R}$ be a function satisfying the ϕ-assumptions. Let $\mu\in\mathcal{P}_2(\mathbb{R})$. Then, up to redefining $\partial_\mu\phi(\mu)(\cdot)$ on a μ-negligible set, the map $v\mapsto\partial_\mu\phi(\mu)(v)$ is 2π-periodic.*

Proof. Let $X\in L_2(\Omega)$ be a random variable with distribution μ.

For any $Y\in L_2(\Omega)$ and for any random variable K with values in $\mathbb{Z}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\widehat{\phi}(X+2K\pi+\varepsilon Y)=\mathbb{E}\left[D\widehat{\phi}(X+2K\pi)\,Y\right]=\mathbb{E}\left[\partial_\mu\phi(\mathcal{L}(X+2K\pi))(X+2K\pi)\,Y\right].$$

Moreover, for any ε, $\widehat{\phi}(X+2K\pi+\varepsilon Y)=\widehat{\phi}(X+\varepsilon Y)$ because $\mathcal{L}(X+2K\pi+\varepsilon Y)\sim\mathcal{L}(X+\varepsilon Y)$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\widehat{\phi}(X+2K\pi+\varepsilon Y)=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0}\widehat{\phi}(X+\varepsilon Y)=\mathbb{E}\left[\partial_\mu\phi(\mathcal{L}(X))(X)\,Y\right].$$

Thus for any $Y \in L_2(\Omega)$, $\mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X+2K\pi))(X+2K\pi)Y\right] = \mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X))(X)Y\right]$. We deduce that for any $\mathbb{Z}$-valued random variable K, almost surely,

$$\partial_\mu \phi(\mathcal{L}(X+2K\pi))(X+2K\pi) = \partial_\mu \phi(\mathcal{L}(X))(X). \tag{III.17}$$

First, assume that the support of $\mu = \mathcal{L}(X)$ is equal to $\mathbb{R}$. For every $\delta \in (0,1)$, let K^δ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X with a Bernoulli distribution of parameter δ. Thus it follows from (III.17) that

$$\begin{aligned} 1 = (1-\delta)\,\mathbb{P}\left[\partial_\mu \phi(\mathcal{L}(X+2K^\delta\pi))(X) = \partial_\mu \phi(\mathcal{L}(X))(X)\right] \\ + \delta\,\mathbb{P}\left[\partial_\mu \phi(\mathcal{L}(X+2K^\delta\pi))(X+2\pi) = \partial_\mu \phi(\mathcal{L}(X))(X)\right]. \end{aligned}$$

We deduce that $\mathbb{P}\left[\partial_\mu \phi(\mathcal{L}(X+2K^\delta\pi))(X+2\pi) = \partial_\mu \phi(\mathcal{L}(X))(X)\right] = 1$ for any $\delta \in (0,1)$. Since the support of $\mathcal{L}(X)$ is equal to $\mathbb{R}$, it follows from Lemma III.8 that $(\nu, x) \mapsto \partial_\mu \phi(\nu)(x)$ is continuous at (μ, x) for every $x \in \mathbb{R}$. Moreover $\mathcal{L}(X+2K^\delta\pi)$ tends in L_2-Wasserstein distance to $\mathcal{L}(X) = \mu$ when $\delta \to 0$. So, there exists an event $\widetilde{\Omega}$ of probability one such that for every $\omega \in \widetilde{\Omega}$ and every $\delta \in (0,1) \cap \mathbb{Q}$, $\partial_\mu \phi(\mathcal{L}(X+2K^\delta\pi))(X(\omega)+2\pi) = \partial_\mu \phi(\mathcal{L}(X))(X(\omega))$. Thus for every $\omega \in \widetilde{\Omega}$, $\partial_\mu \phi(\mu)(X(\omega)+2\pi) = \partial_\mu \phi(\mu)(X(\omega))$. Since $\mathbb{P}\left[\widetilde{\Omega}\right] = 1$ and the support of μ is $\mathbb{R}$, we deduce that $\partial_\mu \phi(\mu)(x+2\pi) = \partial_\mu \phi(\mu)(x)$ holds with every x in a dense subset of $\mathbb{R}$. By continuity, of $\partial_\mu \phi(\mu)(\cdot)$, the last inequality holds with every $x \in \mathbb{R}$. We deduce that $\partial_\mu \phi(\mu)(\cdot)$ is 2π-periodic.

Then, let us consider a general $\mu \in \mathcal{P}_2(\mathbb{R})$. Let Z be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X with normal distribution $\mathcal{N}(0,1)$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that for all $n \in \mathbb{N}$, $a_n \in (0,1)$ and $a_n \to_{n\to+\infty} 0$. For every $n \in \mathbb{N}$, the support of the distribution $\mathcal{L}(X+a_nZ)$ is equal to $\mathbb{R}$. Thus for every $n \in \mathbb{N}$, $v \mapsto \partial_\mu \phi(\mathcal{L}(X+a_nZ))(v)$ is 2π-periodic.

By (III.15), the sequence of continuous functions $(\partial_\mu \phi(\mathcal{L}(X+a_nZ)))_{n\in\mathbb{N}}$ is equicontinuous. Furthermore, (III.16) implies that for every $v \in [0, 2\pi]$,

$$\left|\partial_\mu \phi(\mathcal{L}(X+a_nZ))(v)\right| \leqslant C(1+|v|) + C\mathbb{E}\left[|X+a_nZ|\right].$$

Since (a_n) is bounded by 1 and $X \in L_2(\Omega)$, there exists $C > 0$ such that for every $n \in \mathbb{N}$ and for every $v \in [0, 2\pi]$

$$\left|\partial_\mu \phi(\mathcal{L}(X+a_nZ))(v)\right| \leqslant C(1+|v|) \leqslant C(1+2\pi). \tag{III.18}$$

Recall that $v \mapsto \partial_\mu \phi(\mathcal{L}(X+a_nZ))(v)$ is 2π-periodic for every $n \in \mathbb{N}$. Thus the sequence $(\partial_\mu \phi(\mathcal{L}(X+a_nZ)))_{n\in\mathbb{N}}$ is uniformly bounded on $\mathbb{R}$. By Arzela-Ascoli's Theorem, up to extracting a subsequence, $(\partial_\mu \phi(\mathcal{L}(X+a_nZ)))_{n\in\mathbb{N}}$ converges uniformly to a limit $u : \mathbb{R} \to \mathbb{R}$. In particular, u is a 2π-periodic function.

Moreover, let us prove that the following quantity tends to zero. Let $Y \in L_2(\Omega)$.

$$\begin{aligned} &\left|\mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X+a_nZ))(X+a_nZ)Y\right] - \mathbb{E}\left[u(X)Y\right]\right| \\ &\qquad \leqslant \left|\mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X+a_nZ))(X+a_nZ)Y\right] - \mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X+a_nZ))(X)Y\right]\right| \\ &\qquad\quad + \left|\mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X+a_nZ))(X)Y\right] - \mathbb{E}\left[u(X)Y\right]\right| \\ &\qquad \leqslant Ca_n\mathbb{E}\left[|ZY|\right] + \left|\mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X+a_nZ))(X)Y\right] - \mathbb{E}\left[u(X)Y\right]\right|, \end{aligned}$$

by inequality (III.15). Since ZY is integrable, $Ca_n\mathbb{E}\left[|ZY|\right] \to_{n\to+\infty} 0$. Moreover, let us show that $\left|\mathbb{E}\left[\partial_\mu \phi(\mathcal{L}(X+a_nZ))(X)Y\right] - \mathbb{E}\left[u(X)Y\right]\right| \to_{n\to+\infty} 0$. Remark that $(\partial_\mu \phi(\mathcal{L}(X+a_nZ)))_{n\in\mathbb{N}}$

converges uniformly to u, hence it converges pointwise to u. Moreover, we have a uniform integrability property. Indeed, by 2π-periodicity of $\partial_\mu\phi(\mathcal{L}(X+a_nZ))$ and by (III.18)

$$\begin{aligned}\mathbb{E}\left[|\partial_\mu\phi(\mathcal{L}(X+a_nZ))(X)Y|^{3/2}\right] &\leqslant \mathbb{E}\left[|\partial_\mu\phi(\mathcal{L}(X+a_nZ))(X)|^6\right]^{1/4}\mathbb{E}\left[|Y|^2\right]^{3/4}\\ &= \mathbb{E}\left[|\partial_\mu\phi(\mathcal{L}(X+a_nZ))(\{X\})|^6\right]^{1/4}\mathbb{E}\left[|Y|^2\right]^{3/4}\\ &\leqslant C\mathbb{E}\left[|Y|^2\right]^{3/4}.\end{aligned}$$

Since Y is square integrable, $\sup_{n\in\mathbb{N}}\mathbb{E}\left[|\partial_\mu\phi(\mathcal{L}(X+a_nZ))(X)Y|^{3/2}\right] < +\infty$. Thus the sequence $(\partial_\mu\phi(\mathcal{L}(X+a_nZ))(X)Y)_{n\in\mathbb{N}}$ is uniformly integrable. By Fatou's Lemma, if follows that $\mathbb{E}\left[|u(X)Y|^{3/2}\right] < +\infty$, thus we conclude that $\mathbb{E}\left[\partial_\mu\phi(\mathcal{L}(X+a_nZ))(X)Y\right] \to_{n\to+\infty} \mathbb{E}\left[u(X)Y\right]$. Therefore,

$$\mathbb{E}\left[\partial_\mu\phi(\mathcal{L}(X+a_nZ))(X+a_nZ)Y\right] \to_{n\to+\infty} \mathbb{E}\left[u(X)Y\right].$$

On the other hand,

$$\mathbb{E}\left[\partial_\mu\phi(\mathcal{L}(X+a_nZ))(X+a_nZ)Y\right] = \mathbb{E}\left[D\widehat{\phi}(X+a_nZ)Y\right] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left[D\widehat{\phi}(X)Y\right]$$

because $D\widehat{\phi}$ is Lipschitz by assumption ($\phi 3$). We deduce that $\mathbb{E}\left[u(X)Y\right] = \mathbb{E}\left[D\widehat{\phi}(X)Y\right]$ for every $Y \in L_2(\Omega)$, hence almost surely, $u(X) = \partial_\mu\phi(\mu)(X)$. Recall that u is continuous and 2π-periodic. Therefore, up to redefining $\partial_\mu\phi(\mu)(\cdot)$ on a μ-negligible set, $v \mapsto \partial_\mu\phi(\mu)(v)$ is continuous and 2π-periodic. □

Recall that, as explained in Definition III.5, we can associate to every $\mu \in \mathcal{P}_2(\mathbb{R})$ the measure $\widehat{\mu} \in \mathcal{P}(\mathbb{T})$, defined by $\widehat{\mu}(A) = \mu(A + 2\pi\mathbb{Z})$ for every $A \in \mathcal{B}([0,2\pi])$.

Corollary III.10. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a function satisfying the ϕ-assumptions. Let $\mu \in \mathcal{P}_2(\mathbb{R})$ be such that $\widehat{\mu}$ has a density belonging to $\mathcal{P}^+$. Then there is a unique 2π-periodic and continuous version of $\partial_\mu\phi(\mu)(\cdot)$. Furthermore, for every $v \in [0,2\pi]$, $\partial_\mu\phi(\mu)(v) = \partial_\mu\phi(\widehat{\mu})(v)$.*

Proof. Let $X \in L_2(\Omega)$ with distribution μ. Then the law of $\{X\}$ is $\widehat{\mu}$, seen as an element of $\mathcal{P}_2(\mathbb{R})$ with support included in $[0,2\pi]$. Since $\widehat{\mu} \in \mathcal{P}^+$, the support of $\widehat{\mu}$ is equal to $[0,2\pi]$, because the density of $\widehat{\mu}$ is positive everywhere on $[0,2\pi]$.

Furthermore, by equality (III.17) applied to $K = \frac{1}{2\pi}(\{X\} - X)$, *i.e* $X + 2K\pi = \{X\}$, the following equality holds almost surely: $\partial_\mu\phi(\mathcal{L}(\{X\}))(\{X\}) = \partial_\mu\phi(\mathcal{L}(X))(X)$. By Proposition III.9, $\partial_\mu\phi(\mathcal{L}(X))(\cdot)$ is 2π-periodic, so $\partial_\mu\phi(\mathcal{L}(X))(X) = \partial_\mu\phi(\mathcal{L}(X))(\{X\})$. Since the support of $\mathcal{L}(\{X\})$ is equal to $[0,2\pi]$, we deduce that for every $v \in [0,2\pi]$, $\partial_\mu\phi(\mathcal{L}(X))(v) = \partial_\mu\phi(\mathcal{L}(\{X\}))(v)$. This shows that there is a unique 2π-periodic and continuous version $\partial_\mu\phi(\mathcal{L}(X))(\cdot)$. □

In Section 0.4, we introduced another notion of differentiability on the Wasserstein space, the linear functional derivative $\frac{\delta\phi}{\delta m}$. Moreover, for any ϕ satisfying the ϕ-assumptions, there is a simple relation between the linear functional derivative and Lions' derivative. Indeed, for every $\mu \in \mathcal{P}_2(\mathbb{R})$, $\partial_\mu\phi(\mu)(\cdot) = \partial_v\frac{\delta\phi}{\delta m}(\mu)(\cdot)$. Let us prove that the linear functional derivative is also 2π-periodic.

Proposition III.11. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a function satisfying the ϕ-assumptions. Let $\mu \in \mathcal{P}_2(\mathbb{R})$ be such that $\widehat{\mu}$ has a density belonging to $\mathcal{P}^+$. Then $\int_{\mathbb{T}}\partial_\mu\phi(\mu)(v)\mathrm{d}v = 0$. In other words, $v \mapsto \frac{\delta\phi}{\delta m}(\mu)(v)$ is 2π-periodic.*

Proof. By Corollary III.10, it is sufficient to prove that $\int_{\mathbb{T}} \partial_\mu \phi(\widehat{\mu})(v)\mathrm{d}v = 0$. Let Y_0 be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution equal to $\widehat{\mu}$. Let $p : \mathbb{R} \to \mathbb{R}$ denote its density, extended by 2π-continuity. By assumption, $p(v) > 0$ for every $v \in [0, 2\pi]$, hence for every $v \in \mathbb{R}$.

Let us define the following ordinary differential equation:

$$\dot{Y}_t = \frac{1}{p(Y_t)},$$

with initial condition Y_0. Denoting by $F := x \mapsto \int_0^x p(v)\mathrm{d}v$ and $g = F^{-1}$ respectively the c.d.f. and the quantile function associated to p, we have $\frac{\mathrm{d}}{\mathrm{d}t}F(Y_t) = 1$. Thus for every $t \geqslant 0$, $F(Y_t) = F(Y_0) + t$ and $Y_t = g(F(Y_0) + t) = g_t(F(Y_0))$, where $g_t(\cdot) = g(\cdot + t)$.

Fix $t \geqslant 0$. Since $F(Y_0)$ has a uniform distribution on $[0, 1]$, $Y_t = g_t(F(Y_0))$ implies that g_t is the quantile function of the random variable Y_t. By Definition III.2, $g_t \sim g$, thus we deduce that the law of $\{Y_t\}$ is $\widehat{\mu}$. Since ϕ is $\mathbb{T}$-stable, $\widehat{\phi}(Y_t) = \widehat{\phi}(\{Y_t\}) = \widehat{\phi}(Y_0)$ for every $t \geqslant 0$. Thus $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\widehat{\phi}(Y_t) = 0$. Thus

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\widehat{\phi}(Y_t) &= \mathbb{E}\left[D\widehat{\phi}(Y_0)\dot{Y}_0\right] = \mathbb{E}\left[D\widehat{\phi}(Y_0)\frac{1}{p(Y_0)}\right] \\ &= \int_{\mathbb{R}} \partial_\mu \phi(\widehat{\mu})(v)\frac{1}{p(v)}\mathrm{d}\widehat{\mu}(v) = \int_0^{2\pi} \partial_\mu \phi(\widehat{\mu})(v)\frac{p(v)}{p(v)}\mathrm{d}v = \int_0^{2\pi} \partial_\mu \phi(\widehat{\mu})(v)\mathrm{d}v, \end{aligned}$$

since p is the density of the measure $\widehat{\mu}$. The statement of the proposition follows. □

III.3 A diffusion model on the torus

Let us define the probability spaces and the noises that we will use in this section.

Let $T > 0$ be a fixed final time. Let $(\Omega^W, \mathcal{G}^W, (\mathcal{G}_t^W)_{t\in[0,T]}, \mathbb{P}^W)$ be a filtered probability space satisfying usual conditions. Let us define on this space a $(\mathcal{G}_t^W)_{t\in[0,T]}$-adapted collection $((W_t^{\Re,k})_{t\in[0,T]}, (W_t^{\Im,k})_{t\in[0,T]})_{k\in\mathbb{Z}}$ of independent real standard Brownian motions on $[0, T]$. We denote by $W_\cdot^k$ the $\mathbb{C}$-valued Brownian motion $W_\cdot^{\Re,k} + iW_\cdot^{\Im,k}$.

Let $(\Omega^\beta, \mathcal{G}^\beta, (\mathcal{G}_t^\beta)_{t\in[0,T]}, \mathbb{P}^\beta)$ be another filtered probability space. Let us define on this space a $(\mathcal{G}_t^\beta)_{t\in[0,T]}$-adapted standard Brownian motion $(\beta_t)_{t\in[0,T]}$. Let $(\Omega^0, \mathcal{G}^0, \mathbb{P}^0)$ be another probability space rich enough to support $\mathbf{G}^1$-valued random variables with any distribution that could be introduced later in this chapter. We will denote by $\mathbb{E}^B$, $\mathbb{E}^W$ and $\mathbb{E}^0$ the expectations respectively associated to $\mathbb{P}^\beta$, $\mathbb{P}^W$ and $\mathbb{P}^0$.

Let us now define $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ the filtered probability space defined by $\Omega := \Omega^W \times \Omega^\beta \times \Omega^0$, $\mathcal{G} := \mathcal{G}^W \otimes \mathcal{G}^\beta \otimes \mathcal{G}^0$, $\mathcal{G}_t := \sigma((\mathcal{G}_s^W)_{s\leqslant t}, (\mathcal{G}_s^\beta)_{s\leqslant t}, \mathcal{G}^0)$ and $\mathbb{P} := \mathbb{P}^W \otimes \mathbb{P}^\beta \otimes \mathbb{P}^0$. Without loss of generality, we assume the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ to be complete and, up to adding negligible subsets to $\mathcal{G}^0$, we assume that $\mathcal{G}_0 = \mathcal{G}^0$.

III.3.1 Construction of a diffusion on the torus

The diffusion on the torus is very close to the diffusion on the real line constructed in Section II.2. In this paragraph, we define the SDE on the torus, we show its well-posedness and we justify that the main properties of the process $(y_t^g)_{t\in[0,T]}$ constructed in Section II.2 are preserved by this diffusion on the torus.

Let $f := (f_k)_{k\in\mathbb{Z}}$ be a real sequence (indexed by $\mathbb{Z}$). We say that f is of order $\alpha > 0$ if there is $c > 0$ and $C > 0$ such that $\frac{c}{\langle k\rangle^\alpha} \leqslant |f_k| \leqslant \frac{C}{\langle k\rangle^\alpha}$ for every $k \in \mathbb{Z}$, where $\langle k\rangle := (1 + |k|^2)^{1/2}$.

Let g be an element of $\mathfrak{G}^1$, or equivalently a representative of a class $\overline{g} \in \mathbf{G}^1$. Recall that g, f and the Brownian motions β, $(W^{\Re,k})_{k\in\mathbb{Z}}$ and $(W^{\Im,k})_{k\in\mathbb{Z}}$ are real-valued. Let us define the

following SDE satisfied by the real-valued process $(x_t^g)_{t\in[0,T]}$: for everu $u \in \mathbb{R}$,

$$dx_t^g(u) = \sum_{k\in\mathbb{Z}} f_k \cos(kx_t^g(u))\mathrm{d}W_t^{\Re,k} + \sum_{k\in\mathbb{Z}} f_k \sin(kx_t^g(u))\mathrm{d}W_t^{\Im,k} + \mathrm{d}\beta_t, \tag{III.19}$$

with initial condition $x_0^g = g$. Recalling the notation $W_\cdot^k = W_\cdot^{\Re,k} + iW_\cdot^{\Im,k}$, the SDE may be rewritten:

$$dx_t^g(u) = \sum_{k\in\mathbb{Z}} f_k \Re\left(e^{-ikx_t^g(u)}\mathrm{d}W_t^k\right) + \mathrm{d}\beta_t, \tag{III.20}$$

with initial condition $x_0^g = g$.

If f is of order $\alpha > \frac{1}{2}$, then $\sum_{k\in\mathbb{Z}} f_k^2 < \infty$ and we can compute the quadratic variation of the particle $(x_t^g(u))_{t\in[0,T]}$, for a fixed $u \in \mathbb{R}$:

$$d\langle x^g(u), x^g(u)\rangle_t = \sum_{k\in\mathbb{Z}} f_k^2(\cos^2(kx_t^g(u)) + \sin^2(kx_t^g(u)))\mathrm{d}t + \mathrm{d}t = \Big(1 + \sum_{k\in\mathbb{Z}} f_k^2\Big)\mathrm{d}t.$$

Since the mass function is equal to one in this chapter, the quadratic variation is constant for each variable; in other words it does not depend on the variable u. Let us also compute the covariation between two particles: for every $u, v \in \mathbb{R}$,

$$d\langle x^g(u), x^g(v)\rangle_t = \Big(1 + \sum_{k\in\mathbb{Z}} f_k^2 \cos\left(k(x_t^g(u) - x_t^g(v))\right)\Big)\mathrm{d}t.$$

In the following two propositions, we will show that the SDE (III.20) is well-posed, that its unique solution is continuous, strictly increasing in u and that its regularity depends on the regularity of the initial condition g and of α. These statements are not new: they are the adaptations of the results of Section II.2 to the case of a diffusion on the torus.

Proposition III.12. *Let $g \in \mathfrak{S}^1$. Let f be of order $\alpha > \frac{3}{2}$. Then for each $u \in \mathbb{R}$, strong existence and pathwise uniqueness hold for the stochastic differential equation* (III.20)*. Furthermore, there is a unique version of the solution in $\mathcal{C}(\mathbb{R}\times[0,T])$. Also, $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $u \in [0,1]$, $(x_t^g(u))_{t\in[0,T]}$ satisfies SDE* (III.20) *and for every $t \in [0,T]$, $u \mapsto x_t^g(u)$ is strictly increasing.*

Proof. We do not give details of this proof, since it is very close to the proofs of Proposition II.2, II.4 and II.5 and Corollary II.6. We just need to adapt the proofs to the case of the torus. The additional Brownian motion $(\beta_t)_{t\in[0,T]}$ does not add any difficulty, since it does depend neither on the initial condition g nor on the variable u.

We only give the analogous formula for (II.18): if $u_1, u_2 \in \mathbb{R}$ satisfy $u_1 < u_2$, then the process $Y_t = x_t^g(u_2) - x_t^g(u_1)$ satisfies the SDE: $Y_t = g(u_2) - g(u_1) + \int_0^t Y_s \mathrm{d}N_s$, where

$$N_t = \sum_{k\in\mathbb{Z}} f_k \int_0^t \mathbb{1}_{\{x_s^g(u_2)\neq x_s^g(u_1)\}} \Re\Big(\frac{e^{-ikx_s^g(u_2)} - e^{-ikx_s^g(u_1)}}{x_s^g(u_2) - x_s^g(u_1)}\mathrm{d}W_s^k\Big).$$

We show that $x_t^g(u_2) - x_t^g(u_1) = (g(u_2) - g(u_1))\exp(N_t - \frac{1}{2}\langle N, N\rangle_t)$ as in the proof of Proposition II.5. □

For every $j \in \mathbb{N}$ and $\theta \in [0,1)$, $\mathcal{C}^{j+\theta}$ denotes the set of $\mathcal{C}^j$-functions whose derivative of order j is θ-Hölder continuous.

Proposition III.13. *Let $j \in \mathbb{N}\backslash\{0\}$ and $\theta \in (0,1)$. Let $g \in \mathfrak{G}^{j+\theta}$ and f be of order $\alpha > j+\frac{1}{2}+\theta$. Then $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $t \in [0,T]$, the map $u \mapsto x_t^g(u)$ is a $\mathcal{C}^{j+\theta'}$-function for every $\theta' < \theta$. Moreover, its first derivative has the following explicit form: $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $u \in [0,1]$ and for every $t \in [0,T]$,*

$$\partial_u x_t^g(u) = g'(u) \exp\Big(\sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(-ike^{-ikx_s^g(u)} \mathrm{d}W_s^k\right) - \frac{t}{2} \sum_{k\in\mathbb{Z}} f_k^2 k^2 \Big). \tag{III.21}$$

Proof. We refer to Proposition II.9 for a complete proof of an analogous statement. Let us just write the SDE satisfied by $\partial_u x_t^g$:

$$\begin{aligned}\partial_u x_t^g(u) &= g'(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \partial_u x_s^g(u) \left\{ -k \sin(kx_s^g(u)) \mathrm{d}W_s^{\Re,k} + k\cos(kx_s^g(u)) \mathrm{d}W_s^{\Im,k} \right\} \\ &= g'(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \partial_u x_s^g(u) \Re\left(-ike^{-ikx_s^g(u)} \mathrm{d}W_s^k\right). \end{aligned} \tag{III.22}$$

We also remark that this equation does not involve the Brownian motion $(\beta_t)_{t\in[0,T]}$. □

In the next proposition, we show that the dynamics of the SDE preserve the structure of quantile functions.

Proposition III.14. *Let $\theta \in (0,1)$, $g \in \mathfrak{G}^{1+\theta}$ and f be of order $\alpha > \frac{3}{2}+\theta$. Then $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $t \in [0,T]$, the map $u \mapsto x_t^g(u)$ belongs to $\mathfrak{G}^1$.*

Moreover, if $g_1 \sim g_2$, then $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, $x_t^{g_1} \sim x_t^{g_2}$ for every $t \in [0,T]$.

Proof. By Propositions III.12 and III.13, it is clear that $u \mapsto x_t^g(u)$ belongs to $\mathcal{C}^1$ and that $\partial_u x_t^g(u) > 0$ for every $u \in \mathbb{R}$. Furthermore, let $(y_t^g)_{t\in[0,T]}$ be the process defined by $y_t^g(u) := x_t^g(u+1) - 2\pi$. Then for every $t \in [0,T]$, for every $u \in \mathbb{R}$,

$$y_t^g(u) = g(u+1) - 2\pi + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-ikx_s^g(u+1)} \mathrm{d}W_s^k\right) + \beta_t.$$

Let us recall that by definition of $\mathfrak{G}^{1+\theta}$, $g(u+1) - 2\pi = g(u)$ and remark that $e^{-ikx_s^g(u+1)} = e^{-iky_s^g(u)}$ for every $k \in \mathbb{Z}$. Thus

$$y_t^g(u) = g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-iky_s^g(u)} \mathrm{d}W_s^k\right) + \beta_t.$$

Therefore $(y_t^g(u))_{t\in[0,T],u\in\mathbb{R}}$ and $(x_t^g(u))_{t\in[0,T],u\in\mathbb{R}}$ satisfy the same equation and belong to $\mathcal{C}(\mathbb{R}\times[0,T])$. By Proposition III.12, there is a unique solution in this space. Thus for every $t \in [0,T]$ and every $u \in \mathbb{R}$, $x_t^g(u+1) - 2\pi = x_t^g(u)$. We deduce that x_t^g belongs to $\mathfrak{G}^1$ for every $t \in [0,T]$.

The proof of the second statement is similar; if there is $c \in \mathbb{R}$ such that $g_2(u) = g_1(u+c)$ for every $u \in \mathbb{R}$, then the processes $(x_t^{g_2}(u))_{t\in[0,T],u\in\mathbb{R}}$ and $(x_t^{g_1}(u+c))_{t\in[0,T],u\in\mathbb{R}}$ satisfy the same equation and are equal. □

By Proposition III.14, we are able to give a meaning to the equation (III.20) with initial value $g \in \mathbf{G}^{1+\theta}$. Indeed, for each $t \in [0,T]$, the solution x_t^g will take its values in $\mathbf{G}^{1+\theta'}$ for every $\theta' < \theta$. More generally, by Proposition III.13, if the initial condition g belongs to $\mathbf{G}^{j+\theta}$ for $j \in \mathbb{N}\backslash\{0\}$ and $\theta \in (0,1)$, then for each $t \in [0,T]$, x_t^g belongs to $\mathbf{G}^{j+\theta'}$ for every $\theta' < \theta$.

Recall that by Proposition III.3, there is a one-to-one correspondence betwenn $\mathbf{G}^1$ and the set $\mathcal{P}^+$. For every $t \in [0,T]$, let us denote by q_t^g the element of $\mathcal{P}^+$ associated to $x_t^g \in \mathbf{G}^1$. In other words, $q_t^g = \iota^{-1}(x_t^g)$, where ι is defined in the proof of Proposition III.3. Moreover, denoting by F_t^g the c.d.f. associated to x_t^g, we have by definition that for every $x \in \mathbb{R}$, $x_t^g \circ F_t^g(x) = x$,

whence $q_t^g(x) = \frac{1}{\partial_u x_t^g(F_t^g(x))}$. Then, for every bounded measurable function Υ on the torus $\mathbb{T}$ (or equivalently for every bounded measurable 2π-periodic function $\Upsilon : \mathbb{R} \to \mathbb{R}$), we have by the substitution $u = F_t^g(v)$:

$$\int_0^1 \Upsilon(x_t^g(u))\mathrm{d}u = \int_{x_t^g(0)}^{x_t^g(1)} \Upsilon(v) q_t^g(v)\mathrm{d}v = \int_{\mathbb{T}} \Upsilon(x) q_t^g(x)\mathrm{d}x. \tag{III.23}$$

III.3.2 Integrability of the solution and of its derivatives

In this paragraph, we are interested in controlling the integrability of the derivative $(\partial_u x_t^g)_{t\in[0,T]}$, of the inverse of the derivative $\left(\frac{1}{\partial_u x_t^g}\right)_{t\in[0,T]}$ and of the higher derivatives $(\partial_u^{(j)} x_t^g)_{t\in[0,T]}$, for $j \geqslant 2$. We will give upper bounds for the L_p-norms in space and L_∞-norms in time of those different processes and we want to obtain a precise dependence of these bounds with respect to the initial condition g.

Let us start with the process $(\partial_u x_t^g)_{t\in[0,T]}$. If $g \in \mathbf{G}^{1+\theta}$ and if f is of order $\alpha > \frac{3}{2} + \theta$ for some $\theta \in (0,1)$, recall that the derivative process $(\partial_u x_t^g)_{t\in[0,T]}$ satisfies for every $t \in [0,T]$ and for every $u \in \mathbb{R}$ the SDE (III.22):

$$\partial_u x_t^g(u) = g'(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \partial_u x_s^g(u) \Re\left(-ike^{-ikx_s^g(u)}\mathrm{d}W_s^k\right).$$

In Proposition III.14, we proved that $u \mapsto x_t^g(u)$ belongs almost surely to $\mathbf{G}^{1+\theta'}$. In particular, the map $u \mapsto \partial_u x_t^g(u)$ is a 1-periodic function. In the next proposition, we give a control on the $L_p(\Omega^W \times \Omega^\beta, \mathcal{C}([0,T], L_p[0,1]))$-norm of $\partial_u x_t^g$ with respect to the $L_p[0,1]$-norm of $\partial_u x_0^g = g'$.

Remark III.15. In the following proposition and in the next statements of this paragraph, we will consider an initial condition g that is random, the randomness being carried out by the probability space $(\Omega^0, \mathcal{G}^0, \mathbb{P}^0)$. Recall that in this framework, g is independent of $((W^k)_{k\in\mathbb{Z}}, \beta)$.

Proposition III.16. *Let $\theta \in (0,1)$. Let g be a $\mathcal{G}_0$-measurable random variable with values in $\mathbf{G}^{1+\theta}$. Assume that f is of order $\alpha > \frac{3}{2} + \theta$. Then for every $p \geqslant 2$, there exists a constant C_p independent of θ and g such that $\mathbb{P}^0$-almost surely,*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1 |\partial_u x_t^g(v)|^p \,\mathrm{d}v\right] \leqslant C_p \|g'\|_{L_p[0,1]}^p. \tag{III.24}$$

Moreover, there exists a constant C_p independent of θ and g such that $\mathbb{P}^0$-almost surely,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T} |\partial_u x_t^g(0)|^p\right] \leqslant C_p g'(0)^p. \tag{III.25}$$

Proof. Fix $p \in [2, +\infty)$.

Note: in this proof and in the following proofs of this paragraph, we always denote by C_p a constant depending on p (and possible on f and on T), even if it changes from a line to the next.

Let $M > 0$ be strictly larger than $\int_0^1 |g'(v)|^p \mathrm{d}v$. Let us define the stopping time $\sigma^M := \inf\{t \geqslant 0 : \int_0^1 |\partial_u x_t^g(v)|^p \mathrm{d}v \geqslant M\}$. By equation (III.22), the following inequality holds with

every $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\Big[\sup_{s\leqslant t\wedge\sigma^M}\int_0^1|\partial_u x_s^g(v)|^p\,\mathrm{d}v\Big]\\
&\quad\leqslant C_p\int_0^1|g'(v)|^p\,\mathrm{d}v + C_p\mathbb{E}^W\mathbb{E}^\beta\Big[\sup_{s\leqslant t\wedge\sigma^M}\int_0^1\Big|\sum_{k\in\mathbb{Z}}f_k\int_0^s\partial_u x_r^g(v)\Re\left(-ike^{-ikx_r^g(v)}\mathrm{d}W_r^k\right)\Big|^p\mathrm{d}v\Big]\\
&\quad\leqslant C_p\|g'\|^p_{L_p[0,1]} + C_p\int_0^1\mathbb{E}^W\mathbb{E}^\beta\Big[\sup_{s\leqslant t\wedge\sigma^M}\Big|\sum_{k\in\mathbb{Z}}f_k\int_0^s\partial_u x_r^g(v)\Re\left(-ike^{-ikx_r^g(v)}\mathrm{d}W_r^k\right)\Big|^p\Big]\mathrm{d}v.
\end{aligned}\tag{III.26}$$

Thus by Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\Big[\sup_{s\leqslant t\wedge\sigma^M}\int_0^1|\partial_u x_s^g(v)|^p\,\mathrm{d}v\Big]\\
&\quad\leqslant C_p\|g'\|^p_{L_p[0,1]} + C_p\int_0^1\mathbb{E}^W\mathbb{E}^\beta\Big[\Big|\sum_{k\in\mathbb{Z}}f_k^2\int_0^{t\wedge\sigma^M}|\partial_u x_s^g(v)|^2\mid -ike^{-ikx_s^g(v)}|^2\mathrm{d}s\Big|^{p/2}\Big]\mathrm{d}v\\
&\quad\leqslant C_p\|g'\|^p_{L_p[0,1]} + C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^{p/2}T^{p/2-1}\int_0^1\mathbb{E}^W\mathbb{E}^\beta\Big[\int_0^{t\wedge\sigma^M}|\partial_u x_s^g(v)|^p\mathrm{d}s\Big]\mathrm{d}v\\
&\quad\leqslant C_p\|g'\|^p_{L_p[0,1]} + C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^{p/2}T^{p/2-1}\int_0^t\mathbb{E}^W\mathbb{E}^\beta\Big[\sup_{r\leqslant s\wedge\sigma^M}\int_0^1|\partial_u x_r^g(v)|^p\mathrm{d}v\Big]\mathrm{d}s.
\end{aligned}\tag{III.27}$$

Since f is of order $\alpha > \frac{3}{2}$, $\sum_{k\in\mathbb{Z}}f_k^2k^2$ converges. By Gronwall's Lemma, we deduce that there is a constant C_p such that for every M large enough,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant\sigma^M}\int_0^1|\partial_u x_t^g(v)|^p\,\mathrm{d}v\right]\leqslant C_p\|g'\|^p_{L_p[0,1]}\tag{III.28}$$

Moreover,

$$\mathbb{P}^W\otimes\mathbb{P}^\beta\left[\sigma^M<T\right]\leqslant\mathbb{P}^W\otimes\mathbb{P}^\beta\left[\sup_{t\leqslant\sigma^M}\int_0^1|\partial_u x_t^g(v)|^p\,\mathrm{d}v\geqslant M\right]\leqslant\frac{C_p}{M}\|g'\|^p_{L_p[0,1]},$$

hence we deduce $\mathbb{P}^W\otimes\mathbb{P}^\beta\left[\bigcup_M\{\sigma^M=T\}\right]=1$. Thus, we let M tend to $+\infty$ in (III.28) and inequality (III.24) follows.

Similarly, we prove that for every $t\in[0,T]$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}|\partial_u x_s^g(0)|^p\right]\leqslant C_pg'(0)^p + C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^{p/2}T^{p/2-1}\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s}|\partial_u x_r^g(0)|^p\right]\mathrm{d}s$$

and we deduce (III.25) by Gronwall's Lemma. Note that rigorously, we should localize as we did with σ^M since the right-hand side can be equal to $+\infty$. We omit this part of the proof since there is no difference with the previous case. □

In the following proposition, we will control the L_p-norm of the density. Recall that we associate to the map $u\mapsto x_t^g(u)$ a density function q_t^g on the torus $\mathbb{T}$. For every $t\in[0,T]$ and for every $x\in\mathbb{R}$, we have $x_t^g\circ F_t^g(x)=x$, whence $q_t^g(x)\partial_u x_t^g(F_t^g(x))=1$. Therefore, the substitution $v=F_t^g(x)$ leads to the following equality

$$\int_0^1\frac{1}{|\partial_u x_t^g(v)|^p}\mathrm{d}v=\int_{x_t^g(0)}^{x_t^g(1)}\frac{q_t^g(x)}{|\partial_u x_t^g(F_t^g(x))|^p}\mathrm{d}x=\int_{x_t^g(0)}^{x_t^g(0)+2\pi}q_t^g(x)^{p+1}\mathrm{d}x=\int_{\mathbb{T}}q_t^g(x)^{p+1}\mathrm{d}x.$$

Proposition III.17. *Let $\theta \in (0,1)$. Let g be a $\mathcal{G}_0$-measurable random variable with values in $\mathbf{G}^{1+\theta}$. Assume that f is of order $\alpha > \frac{3}{2}+\theta$. Then for every $p \geqslant 2$, there exists a constant C_p independent of θ and g such that $\mathbb{P}^0$-almost surely,*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1 \frac{1}{|\partial_u x_t^g(v)|^p}\mathrm{d}v\right] \leqslant C_p\left\|\frac{1}{g'}\right\|^p_{L_p[0,1]}. \tag{III.29}$$

Moreover, there also exists a constant C_p independent of θ and g such that $\mathbb{P}^0$-almost surely,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\frac{1}{|\partial_u x_t^g(0)|^p}\right] \leqslant C_p\frac{1}{g'(0)^p}. \tag{III.30}$$

Proof. Recall that $\mathbb{P}^W\otimes\mathbb{P}^\beta$-almost surely, for every $t\in[0,T]$ and for every $v\in[0,1]$, $\partial_u x_t^g(v)>0$. By Itô's formula applied to $x\in(0,+\infty)\mapsto\frac{1}{x}$, it follows from equation (III.22) that

$$\begin{aligned}\frac{1}{\partial_u x_t^g(v)} &= \frac{1}{g'(v)} - \sum_{k\in\mathbb{Z}} f_k\int_0^t \frac{1}{\partial_u x_s^g(v)^2}\partial_u x_s^g(v)\Re\left(-ike^{-ikx_s^g(v)}\mathrm{d}W_s^k\right)\\ &\quad+\frac{1}{2}\sum_{k\in\mathbb{Z}} f_k^2\int_0^t\frac{2}{\partial_u x_s^g(v)^3}\partial_u x_s^g(v)^2k^2\mathrm{d}s\\ &= \frac{1}{g'(v)} - \sum_{k\in\mathbb{Z}} f_k\int_0^t \frac{1}{\partial_u x_s^g(v)}\Re\left(-ike^{-ikx_s^g(v)}\mathrm{d}W_s^k\right) + \sum_{k\in\mathbb{Z}} f_k^2k^2\int_0^t\frac{1}{\partial_u x_s^g(v)}\mathrm{d}s.\end{aligned}$$

Fix $p\in[2,+\infty)$. Therefore, by the same computation as (III.26)-(III.27), we obtain

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\int_0^1\frac{1}{|\partial_u x_s^g(v)|^p}\mathrm{d}v\right] &\leqslant C_p\left\|\frac{1}{g'}\right\|^p_{L_p[0,1]}\\ &+C_p\left(\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^{p/2}T^{p/2-1}+\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^pT^{p-1}\right)\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s}\int_0^1\frac{1}{|\partial_u x_r^g(v)|^p}\mathrm{d}v\right]\mathrm{d}s.\end{aligned}$$

The sum $\sum_{k\in\mathbb{Z}}f_k^2k^2$ converges because $\alpha>\frac{3}{2}$. By Gronwall's Lemma, we deduce (III.29). A very close argument shows that (III.30) holds true too. □

We deduce estimates for higher derivatives of $(x_t^g)_{t\in[0,T]}$.

Proposition III.18. *Let $j\geqslant 2$ be an integer and let $\theta\in(0,1)$. Let g be a $\mathcal{G}_0$-measurable random variable with values in $\mathbf{G}^{j+\theta}$. Assume that f is of order $\alpha>j+\frac{1}{2}+\theta$. Then for every $p\geqslant 2$, there exists a constant $C_{p,j}$ independent of θ and g such that $\mathbb{P}^0$-almost surely,*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1\left|\partial_u^{(j)}x_t^g(v)\right|^p\mathrm{d}v\right]\leqslant C_{p,j}\Big\{1+\|\partial_u^{(j)}g\|^p_{L_p[0,1]}+\sum_{k=1}^{j-1}\|\partial_u^{(k)}g\|^{jp}_{L_\infty[0,1]}\Big\}. \tag{III.31}$$

Moreover, there exists a constant $C_{p,j}$ independent of θ and g such that $\mathbb{P}^0$-almost surely,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\left|\partial_u^{(j)}x_t^g(0)\right|^p\right]\leqslant C_{p,j}\Big\{1+|\partial_u^{(j)}g(0)|^p+\sum_{k=1}^{j-1}\|\partial_u^{(k)}g\|^{jp}_{L_\infty[0,1]}\Big\}. \tag{III.32}$$

Proof. Let us start by proving (III.31) for $j=2$. In Proposition III.13, we have seen that if g belongs to $\mathbf{G}^{2+\theta}$ and $\alpha>\frac{5}{2}+\theta$, then the second derivative $(\partial_u^{(2)}x_t^g)_{t\in[0,T]}$ exists. It satisfies the following equation:

$$\begin{aligned}\partial_u^{(2)}x_t^g(u) &= g''(u)+\sum_{k\in\mathbb{Z}}f_k\int_0^t\partial_u^{(2)}x_s^g(u)\Re\left(-ike^{-ikx_s^g(u)}\mathrm{d}W_s^k\right)\\ &\quad+\sum_{k\in\mathbb{Z}}f_k\int_0^t\partial_u x_s^g(u)^2\Re\left(-k^2e^{-ikx_s^g(u)}\mathrm{d}W_s^k\right).\end{aligned}$$

For every $p \geqslant 2$, for every $t \in [0,T]$, we have

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\int_0^1\left|\partial_u^{(2)}x_s^g(v)\right|^p\mathrm{d}v\right]\\
&\quad\leqslant C_p\|g''\|^p_{L_p[0,1]}+C_p\int_0^1\mathbb{E}^W\mathbb{E}^\beta\left[\left|\sum_{k\in\mathbb{Z}}f_k^2\int_0^t|\partial_u^{(2)}x_s^g(v)|^2\mid -ike^{-ikx_s^g(v)}|^2\mathrm{d}s\right|^{p/2}\right]\mathrm{d}v\\
&\qquad+C_p\int_0^1\mathbb{E}^W\mathbb{E}^\beta\left[\left|\sum_{k\in\mathbb{Z}}f_k^2\int_0^t|\partial_u x_s^g(v)|^4\mid -k^2e^{-ikx_s^g(v)}|^2\mathrm{d}s\right|^{p/2}\right]\mathrm{d}v\\
&\quad\leqslant C_p\|g''\|^p_{L_p[0,1]}+C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^{p/2}T^{p/2-1}\int_0^1\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^t|\partial_u^{(2)}x_s^g(v)|^p\mathrm{d}s\right]\mathrm{d}v\\
&\qquad+C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2k^4\Big)^{p/2}T^{p/2-1}\int_0^1\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^t|\partial_u x_s^g(v)|^{2p}\mathrm{d}s\right]\mathrm{d}v.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\int_0^1\left|\partial_u^{(2)}x_s^g(v)\right|^p\mathrm{d}v\right]\\
&\qquad\leqslant C_p\|g''\|^p_{L_p[0,1]}+C_p\|g'\|^{2p}_{L_{2p}[0,1]}+C_p\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s}\int_0^1|\partial_u^{(2)}x_r^g(v)|^p\mathrm{d}v\right]\mathrm{d}s.
\end{aligned}$$

Note that we use inequality (III.24) applied with $2p$ instead of p to obtain the last term. We also use the fact that f is of order $\alpha > \frac{5}{2}$ so that $\sum_{k\in\mathbb{Z}} f_k^2k^4$ converges. Remark that $\|g'\|_{L_{2p}[0,1]} \leqslant \|g'\|_{L_\infty[0,1]}$. Thus (III.31) follows by Gronwall's Lemma.

Let us now prove (III.31) for $j=3$. We obtain by similar computations

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\int_0^1\left|\partial_u^{(3)}x_s^g(v)\right|^p\mathrm{d}v\right]\\
&\quad\leqslant C_p\|g'''\|^p_{L_p[0,1]}+C_p\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s}\int_0^1|\partial_u^{(3)}x_r^g(v)|^p\mathrm{d}v\right]\mathrm{d}s\\
&\qquad+C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\int_0^1|\partial_u x_s^g(v)|^{3p}\,\mathrm{d}v\right]+C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\int_0^1\left|\partial_u^{(2)}x_s^g(v)\right|^p|\partial_u x_s^g(v)|^p\,\mathrm{d}v\right].
\end{aligned}$$

Since $|\partial_u^{(2)}x_s^g(v)|^p|\partial_u x_s^g(v)|^p \leqslant \frac{2}{3}|\partial_u^{(2)}x_s^g(v)|^{3p/2}+\frac{1}{3}|\partial_u x_s^g(v)|^{3p}$, we deduce that

$$\begin{aligned}
\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\int_0^1\left|\partial_u^{(3)}x_s^g(v)\right|^p\mathrm{d}v\right]&\leqslant C_p\left(\|g'''\|^p_{L_p[0,1]}+\|g'\|^{3p}_{L_\infty[0,1]}+\|g''\|^{3p/2}_{L_\infty[0,1]}\right)\\
&\quad+C_p\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s}\int_0^1|\partial_u^{(3)}x_r^g(v)|^p\mathrm{d}v\right]\mathrm{d}s.
\end{aligned}$$

Moreover, $\|g''\|^{3p/2}_{L_\infty[0,1]} \leqslant 1+\|g''\|^{3p}_{L_\infty[0,1]}$, thus (III.31) follows from Gronwall's Lemma. The proof for higher orders of differentiation ($j \geqslant 4$) follows recursively. As previously, the proofs for (III.32) are similar. □

Corollary III.19. *Let $j \geqslant 1$ be an integer and let $\theta \in (0,1)$. Let g be a $\mathcal{G}_0$-measurable random variable with values in $\mathbf{G}^{j+1+\theta}$. Assume that f is of order $\alpha > j+\frac{3}{2}+\theta$. Then for every $p \geqslant 2$,*

there exists a constant $C_{p,j}$ independent of θ and g such that $\mathbb{P}^0$-almost surely,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\left\|\partial_u^{(j)}x_t^g\right\|_{L_\infty}^p\right]\leqslant C_{p,j}\left\{1+\|\partial_u^{(j+1)}g\|_{L_p}^p+\sum_{k=1}^j\|\partial_u^{(k)}g\|_{L_\infty}^{(j+1)p}\right\}. \tag{III.33}$$

Moreover, if g belongs to $\mathbf{G}^{2+\theta}$ and $\alpha>\frac{5}{2}+\theta$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\left\|\frac{1}{\partial_u x_t^g}\right\|_{L_\infty}^p\right]\leqslant C_p\Big\{1+\frac{1}{g'(0)^p}+\left\|\frac{1}{g'}\right\|_{L_{4p}}^{4p}+\|g''\|_{L_{2p}}^{2p}+\|g'\|_{L_\infty}^{4p}\Big\}. \tag{III.34}$$

Proof. Note that for every 1-periodic function f, $\|f\|_{L_\infty}=\sup_{u\in[0,1]}|f(u)|$, and for every $u\in[0,1]$, $f(u)=f(0)+\int_0^u\partial_u f(v)\mathrm{d}v$. We deduce that $\|f\|_{L_\infty}\leqslant|f(0)|+\int_0^1|\partial_u f(v)|\mathrm{d}v$.

Applying this remark, we obtain for every $p\geqslant 2$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\left\|\partial_u^{(j)}x_t^g\right\|_{L_\infty}^p\right]\leqslant C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}|\partial_u^{(j)}x_t^g(0)|^p\right]+C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1\left|\partial_u^{(j+1)}x_t^g(v)\right|^p\mathrm{d}v\right].$$

If g belongs to $\mathbf{G}^{j+1+\theta}$ and $\alpha>j+\frac{3}{2}+\theta$, we have by (III.31) and (III.32)

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}|\partial_u^{(j)}x_t^g(0)|^p\right]&\leqslant C_{p,j}\Big\{1+|\partial_u^{(j)}g(0)|^p+\sum_{k=1}^{j-1}\|\partial_u^{(k)}g\|_{L_\infty[0,1]}^{jp}\Big\}\\&\leqslant C_{p,j}\Big\{1+\sum_{k=1}^{j}\|\partial_u^{(k)}g\|_{L_\infty[0,1]}^{jp}\Big\};\\\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1\left|\partial_u^{(j+1)}x_t^g(v)\right|^p\mathrm{d}v\right]&\leqslant C_{p,j}\Big\{1+\|\partial_u^{(j+1)}g\|_{L_p[0,1]}^p+\sum_{k=1}^{j}\|\partial_u^{(k)}g\|_{L_\infty[0,1]}^{(j+1)p}\Big\}.\end{aligned}$$

We deduce (III.33).

Moreover, by the same observation, we have

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\left\|\frac{1}{\partial_u x_t^g}\right\|_{L_\infty}^p\right]\leqslant C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\frac{1}{|\partial_u x_t^g(0)|^p}\right]+C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1\left|\frac{\partial_u^{(2)}x_t^g(v)}{\partial_u x_t^g(v)^2}\right|^p\mathrm{d}v\right].$$

By (III.30), $\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\frac{1}{|\partial_u x_t^g(0)|^p}\right]\leqslant C_p\frac{1}{g'(0)^p}$. By Cauchy-Schwarz inequality and by (III.29) and (III.31), we have

$$\begin{aligned}&\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1\left|\frac{\partial_u^{(2)}x_t^g(v)}{\partial_u x_t^g(v)^2}\right|^p\mathrm{d}v\right]\\&\quad\leqslant\mathbb{E}^W\mathbb{E}^\beta\left[\Big|\sup_{t\leqslant T}\int_0^1\left|\partial_u^{(2)}x_t^g(v)\right|^{2p}\mathrm{d}v\Big|^{1/2}\Big|\sup_{t\leqslant T}\int_0^1\frac{1}{|\partial_u x_t^g(v)|^{4p}}\mathrm{d}v\Big|^{1/2}\right]\\&\quad\leqslant\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1\left|\partial_u^{(2)}x_t^g(v)\right|^{2p}\mathrm{d}v\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\int_0^1\frac{1}{|\partial_u x_t^g(v)|^{4p}}\mathrm{d}v\right]^{1/2}\\&\quad\leqslant C_p(1+\|g''\|_{L_{2p}}^p+\|g'\|_{L_\infty}^{2p})\left\|\tfrac{1}{g'}\right\|_{L_{4p}}^{2p}\leqslant C_p\Big\{1+\|g''\|_{L_{2p}}^{2p}+\|g'\|_{L_\infty}^{4p}+\left\|\tfrac{1}{g'}\right\|_{L_{4p}}^{4p}\Big\}.\end{aligned}$$

We deduce (III.34). □

III.3.3 Parametric SDE and Kunita expansion

The aim of this paragraph is to study the evolution of the process $(x_t^{g+\varepsilon a_t})_{t\in[0,T]}$, where for each time $t \in [0,T]$, we are considering the value at time t of the solution to the equation starting at a point $g + \varepsilon a_t$ depending on t. The small perturbation $a_t : \mathbb{R} \to \mathbb{R}$ is a 1-periodic function. In order to apply Kunita's expansion, which is stated in [Kun90] for initial conditions in $\mathbb{R}^d$, we will introduce the following parametric SDE. For every $x \in \mathbb{R}$, let us consider

$$Z_t^x = x + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-ikZ_s^x} \mathrm{d}W_s^k\right) + \beta_t. \tag{III.35}$$

Formally, we see that for each $g \in \mathbf{G}^1$, $Z_t^{g(u)} = x_t^g(u)$. We will prove this equality in (III.40).

Let us start by giving some controls on the regularity and on the integrability of the process $(Z_t^x)_{t\in[0,T]}$ and of its first derivative $(\partial_x Z_t^x)_{t\in[0,T]}$. The next two propositions are very similar to the statements obtained in the previous paragraph. In this prospect, it would have been possible to prove first the former two propositions and then to derive the latter inequalities of Paragraph III.3.2, but we felt better to show that the inequalities of Paragraph III.3.2 could be obtained directly with explicit bounds depending on g.

Proposition III.20. *Let f be of order $\alpha > \frac{3}{2}$. Then there is a collection $(Z_t^x)_{t\in[0,T],x\in\mathbb{R}}$ such that for each $x \in \mathbb{R}$, $(Z_t^x)_{t\in[0,T]}$ is the unique solution to* (III.35) *and $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, the flow $(t,x) \in [0,T] \times \mathbb{R} \mapsto Z_t^x$ is continuous. Furthermore, for every $p \geqslant 2$, there is a constant $C_p > 0$ such that for every $x, y \in \mathbb{R}$,*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T} |Z_t^x - Z_t^y|^p\right] \leqslant C_p|x-y|^p. \tag{III.36}$$

Proof. Well-posedness of (III.35) for a fixed $x \in \mathbb{R}$ under the assumption $\alpha > \frac{3}{2}$ is a consequence of well-posedness of (III.20) for a fixed $u \in \mathbb{R}$.

Moreover, fix $p \geqslant 2$. For every $t \in [0,T]$ and for every $x, y \in \mathbb{R}$, by Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t} |Z_s^x - Z_s^y|^p\right] &\leqslant C|x-y|^p + C\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\left|\sum_{k\in\mathbb{Z}} f_k \int_0^s \Re\left((e^{-ikZ_r^x} - e^{-ikZ_r^y})\mathrm{d}W_r^k\right)\right|^p\right] \\ &\leqslant C|x-y|^p + C\mathbb{E}^W\mathbb{E}^\beta\left[\left(\sum_{k\in\mathbb{Z}} f_k^2 \int_0^t \left|e^{-ikZ_s^x} - e^{-ikZ_s^y}\right|^2 \mathrm{d}s\right)^{p/2}\right].\end{aligned}$$

Thus

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t} |Z_s^x - Z_s^y|^p\right] \leqslant C|x-y|^p + C\Big(\sum_{k\in\mathbb{Z}} f_k^2 k^2\Big)^{p/2} T^{p/2-1} \int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s} |Z_r^x - Z_r^y|^p\right] \mathrm{d}s.$$

Since f is of order $\alpha > \frac{3}{2}$, the sum $\sum_{k\in\mathbb{Z}} f_k^2 k^2$ is finite. Therefore, by Gronwall's Lemma, we deduce (III.36). By Kolmogorov's Lemma, it follows from (III.36) that $(t,x) \mapsto Z_t^x$ has a continuous version. □

Proposition III.21. *Let f be of order $\alpha > \frac{3}{2} + \theta$ for some $\theta \in (0,1)$. Then $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, the map $x \mapsto Z_t^x$ is differentiable on $\mathbb{R}$ for every $t \in [0,T]$ and the partial derivative $(\partial_x Z_t^x)_{t\in[0,T]}$ satisfies the following equation:*

$$\partial_x Z_t^x = 1 + \sum_{k\in\mathbb{Z}} f_k \int_0^t \partial_x Z_s^x \Re\left(-ike^{-ikZ_s^x} \mathrm{d}W_s^k\right).$$

Moreover, there is a continuous version of the map $(t,x)\in[0,T]\times\mathbb{R}\mapsto\partial_x Z_t^x$ and the following exponential form holds true $\mathbb{P}^W\otimes\mathbb{P}^\beta$-almost surely for every $t\in[0,T]$ and for every $x\in\mathbb{R}$:

$$\partial_x Z_t^x=\exp\left(\sum_{k\in\mathbb{Z}}f_k\int_0^t\Re\left(-ike^{-ikZ_s^x}\mathrm{d}W_s^k\right)-\frac{t}{2}\sum_{k\in\mathbb{Z}}f_k^2k^2\right). \tag{III.37}$$

For every $p\geqslant 2$, there is C_p such that for every $x,y\in\mathbb{R}$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}|\partial_x Z_t^x|^p\right]\leqslant C_p, \tag{III.38}$$

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}|\partial_x Z_t^x-\partial_x Z_t^y|^p\right]\leqslant C_p|x-y|^{p\theta}. \tag{III.39}$$

If $\alpha>\frac{5}{2}$, then (III.39) *holds with θ equal to* 1.

Proof. The proof of the differentiability of the flow with respect to the variable $x\in\mathbb{R}$ and the proof of (III.37) are similar to the proofs of Propositions II.9 and III.13.

Moreover, for every $p\geqslant 2$, for every $t\in[0,T]$ and for every $x\in\mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}|\partial_x Z_s^x|^p\right]&\leqslant C_p+C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\left|\sum_{k\in\mathbb{Z}}f_k\int_0^s\partial_x Z_r^x\Re\left(-ike^{-ikZ_r^x}\mathrm{d}W_r^k\right)\right|^p\right]\\&\leqslant C_p+C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^{p/2}\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s}|\partial_x Z_r^x|^p\right]\mathrm{d}s.\end{aligned}$$

Thus by Gronwall's Lemma, inequality (III.38) follows. For every $x,y\in\mathbb{R}$, we also have

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}|\partial_x Z_s^x-\partial_x Z_s^y|^p\right]&\leqslant C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\left|\sum_{k\in\mathbb{Z}}f_k\int_0^s(\partial_x Z_r^x-\partial_x Z_r^y)\Re\left(-ike^{-ikZ_r^x}\mathrm{d}W_r^k\right)\right|^p\right]\\&\quad+C_p\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}\left|\sum_{k\in\mathbb{Z}}f_k\int_0^s\partial_x Z_r^y\Re\left(-ik(e^{-ikZ_r^x}-e^{-ikZ_r^y})\mathrm{d}W_r^k\right)\right|^p\right].\end{aligned}$$

By Burkholder-Davis-Gundy inequality, we deduce that

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{s\leqslant t}|\partial_x Z_s^x-\partial_x Z_s^y|^p\right]&\leqslant C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2k^2\Big)^{p/2}\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant s}|\partial_x Z_r^x-\partial_x Z_r^y|^p\right]\mathrm{d}r\\&\quad+C_p\Big(\sum_{k\in\mathbb{Z}}f_k^2|k|^{2+2\theta}\Big)^{p/2}\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[|\partial_x Z_s^y|^p\,|Z_s^x-Z_s^y|^{p\theta}\right]\mathrm{d}s.\end{aligned}$$

Furthermore, by (III.36) and (III.38), we obtain for every $s\in[0,T]$ and for every x and y,

$$\mathbb{E}^W\mathbb{E}^\beta\left[|\partial_x Z_s^y|^p\,|Z_s^x-Z_s^y|^{p\theta}\right]\leqslant\mathbb{E}^W\mathbb{E}^\beta\left[|\partial_x Z_s^y|^{\frac{p}{1-\theta}}\right]^{1-\theta}\mathbb{E}^W\mathbb{E}^\beta\left[|Z_s^x-Z_s^y|^p\right]^\theta\leqslant C_{p,\theta}|x-y|^{p\theta}.$$

Recall that f is of order $\alpha>\frac{3}{2}+\theta$, so the sum $\sum_{k\in\mathbb{Z}}f_k^2|k|^{2+2\theta}$ is finite. By Gronwall's Lemma, we deduce (III.39). By Kolmogorov's Lemma, it follows from (III.39) (with p larger than $\frac{1}{\theta}$) that there is a continuous version of $(t,x)\mapsto\partial_x Z_t^x$. □

Proposition III.22. *Let $g\in\mathbf{G}^1$ and f be of order $\alpha>\frac{3}{2}$. Then $\mathbb{P}^W\otimes\mathbb{P}^\beta$-almost surely, for every $t\in[0,T]$ and for every $u\in[0,1]$,*

$$Z_t^{g(u)}=x_t^g(u). \tag{III.40}$$

If we assume moreover that $g \in \mathbf{G}^{1+\theta}$ and $\alpha > \frac{3}{2} + \theta$ for some $\theta \in (0,1)$, then $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $t \in [0,T]$ and for every $u \in [0,1]$,

$$\partial_x Z_t^{g(u)} = \frac{1}{g'(u)} \partial_u x_t^g(u). \tag{III.41}$$

Proof. Let us fix $g \in \mathbf{G}^1$ and $u \in \mathbb{R}$. Then the processes $(Z_t^{g(u)})_{t\in[0,T]}$ and $(x_t^g(u))_{t\in[0,T]}$ both satisfy the same SDE (III.20) with same initial condition. Since f is of order $\alpha > \frac{3}{2}$, pathwise uniqueness holds for this equation. Hence for every $u \in \mathbb{R}$, $Z_t^{g(u)} = x_t^g(u)$ holds almost surely. Moreover, since $u \in \mathbb{R} \mapsto x_t^g(u)$ and $x \in \mathbb{R} \mapsto Z_t^x$ are $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely continuous, and g is continuous, we deduce that (III.40) holds almost surely for every $u \in \mathbb{R}$.

By (III.21) and (III.37), we have $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $u \in [0,1]$ and for every $t \in [0,T]$,

$$\partial_u x_t^g(u) = g'(u) \exp\left(\sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(-ike^{-ikx_s^g(u)} \mathrm{d}W_s^k\right) - \frac{t}{2} \sum_{k\in\mathbb{Z}} f_k^2 k^2 \right);$$

$$\partial_x Z_t^x = \exp\left(\sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(-ike^{-ikZ_s^x} \mathrm{d}W_s^k\right) - \frac{t}{2} \sum_{k\in\mathbb{Z}} f_k^2 k^2 \right).$$

Thus equality (III.41) follows from (III.40). □

Finally, we give the following expansion due to Kunita [Kun90, Chapter III, Theorem 3.3.1].

Proposition III.23. *Let f be of order $\alpha > \frac{3}{2} + \theta$ for some $\theta \in (0,1)$. Let $(\zeta_t)_{t\in[0,T]}$ be a $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process such that $t \in [0,T] \mapsto \zeta_t$ is absolutely continuous, $\zeta_0 = 0$ almost surely and $\mathbb{E}\left[\int_0^T |\zeta_t| \mathrm{d}t\right]$ is finite.*

Then almost surely, for every $x \in \mathbb{R}$, for every $t \in [0,T]$ and for every $\varepsilon \in R$,

$$Z_t^{x+\varepsilon\zeta_t} = Z_0^x + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-ikZ_s^{x+\varepsilon\zeta_s}} \mathrm{d}W_s^k\right) + \beta_t + \varepsilon \int_0^t \partial_x Z_s^{x+\varepsilon\zeta_s} \, \dot{\zeta}_s \, \mathrm{d}s. \tag{III.42}$$

III.4 Smoothing property of the diffusion on the torus

Let $g \in \mathbf{G}^1$. Recall that for every time $t \in [0,T]$, q_t^g denotes the density function in $\mathcal{P}^+$ associated to the equivalence class of $u \mapsto x_t^g(u)$ by the one-to-one correspondence given by Proposition III.3. Let us define for every $t \in [0,T]$ and for every $x \in \mathbb{R}$, the following density

$$p_t^g(x) := \mathbb{E}^\beta\left[q_t^g(x)\right] = \mathbb{E}\left[q_t^g(x) \middle| \mathcal{G}^0 \vee \mathcal{G}^W\right].$$

By Fubini's Theorem and by (III.23), the following equality holds with every bounded measurable 2π-periodic function $\Upsilon : \mathbb{R} \to \mathbb{R}$:

$$\mathbb{E}^\beta\left[\int_0^1 \Upsilon(x_t^g(u)) \mathrm{d}u\right] = \int_{\mathbb{T}} \Upsilon(x) p_t^g(x) \mathrm{d}x. \tag{III.43}$$

Definition III.24. Fix $t \in [0,T]$ and $\omega \in \Omega^W \times \Omega^0$. We denote by $\mu_t^g(\omega)$ the law of the random variable $x_t^g(\omega) : (u, \omega^\beta) \in [0,1] \times \Omega^\beta \mapsto x_t^g(\omega)(u, \omega^\beta)$:

$$\mu_t^g(\omega) := \left(\mathrm{Leb}_{[0,1]} \otimes \mathbb{P}^\beta\right) \circ \left(x_t^g(\omega)\right)^{-1}.$$

Remark III.25. For every $t \in [0,T]$, μ_t^g is a random variable defined on $\Omega^W \times \Omega^0$ with values in $\mathcal{P}_2(\mathbb{R})$.

In this section, we will show that the diffusion process $(\mu_t)_{t\in[0,T]}$ has a regularizing effect. In this regard, the role of the idiosyncratic noise β is similar to its role in Section II.5, where it allowed to regularize the spatial component of the velocity field b. Here, the averaging over the noise carrying β regularizes the spatial component of the derivative of ϕ: it will be shown in Paragraph III.4.3 and in particular in the proof of Lemma III.38.

III.4.1 Definition and properties of the semi-group $(P_t)_{t\in[0,T]}$

In this paragraph, we will first define the semi-group $(P_t)_{t\in[0,T]}$ associated to the diffusion process $(\mu_t)_{t\in[0,T]}$. Then, we will show in Proposition III.26 that the class of functions $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfying the ϕ-assumptions (recall Definition III.6) is stable by the action of $(P_t)_{t\in[0,T]}$. Finally, we will give in Lemma III.28 a useful formula expressing the derivative of the semi-group in terms of Lions' derivative of ϕ or in terms of the linear functional derivative of $P_t\phi$.

Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a function satisfying the ϕ-assumptions. Let us define $\widehat{\phi}(X) := \phi(\mathcal{L}_{[0,1]\times\Omega^\beta}(X))$ for every $X \in L_2([0,1]\times\Omega^\beta)$, where $\mathcal{L}_{[0,1]\times\Omega^\beta}(X) := (\mathrm{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ X^{-1}$ denotes the law of the random variable X defined on $[0,1]\times\Omega^\beta$. In particular, for every $g \in \mathbf{G}^1$ and for every $t \in [0,T]$, we have $\mathbb{P}^W$-almost surely

$$\widehat{\phi}(x_t^g) = \phi(\mathcal{L}_{[0,1]\times\Omega^\beta}(x_t^g)) = \phi(\mu_t^g).$$

For every $t \in [0,T]$ and for every $X \in L_2[0,1]$, define

$$\widehat{P_t\phi}(X) := \mathbb{E}^W\left[\widehat{\phi}(Z_t^X)\right],$$

and for every $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$P_t\phi(\mu) := \widehat{P_t\phi}(X), \quad \mathcal{L}_{[0,1]}(X) = \mu, \tag{III.44}$$

where $\mathcal{L}_{[0,1]}(X) = \mathrm{Leb}_{[0,1]} \circ X^{-1}$.

Let us check that $P_t\phi$ is well-defined. Recall that the parametric SDE (III.35) satisfied by $(Z_t^x)_{t\in[0,T],x\in\mathbb{R}}$ is strongly well-posed. Thus, if $X, X' \in L_2[0,1]$ satisfy the relation $\mathcal{L}_{[0,1]}(X) = \mathcal{L}_{[0,1]}(X')$, then $\mathbb{P}^W$-almost surely for every $t \in [0,T]$, $\mathcal{L}_{[0,1]\times\Omega^\beta}(Z_t^X) = \mathcal{L}_{[0,1]\times\Omega^\beta}(Z_t^{X'})$. It follows that $\mathbb{E}^W\left[\widehat{\phi}(Z_t^X)\right] = \mathbb{E}^W\left[\widehat{\phi}(Z_t^{X'})\right]$, thus $\widehat{P_t\phi}(X) = \widehat{P_t\phi}(X')$.

Moreover, for every deterministic $g \in \mathbf{G}^1$, we know by Proposition III.22 that $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $t \in [0,T]$ and for every $u \in [0,1]$, $Z_t^{g(u)} = x_t^g(u)$. In particular, $\mathbb{P}^W$-almost surely and for every $t \in [0,T]$, $Z_t^{g(u)} = x_t^g(u)$ holds true $\mathrm{Leb}|_{[0,1]} \otimes \mathbb{P}^\beta$-almost surely. Therefore,

$$P_t\phi(\mu_0^g) = \widehat{P_t\phi}(g) = \mathbb{E}^W\left[\widehat{\phi}\left(Z_t^g\right)\right] = \mathbb{E}^W\left[\widehat{\phi}\left(x_t^g\right)\right] = \mathbb{E}^W\left[\phi(\mu_t^g)\right]. \tag{III.45}$$

Proposition III.26. *Assume that f is of order $\alpha > \frac{5}{2}$. Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a function satisfying the ϕ-assumptions. Then for every $t \in [0,T]$, $P_t\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ also satisfies the ϕ-assumptions.*

Proof. Let us fix $t \in [0,T]$.

Assumption (ϕ1)**:** Let us start by proving that $P_t\phi$ is $\mathbb{T}$-stable. Let $\mu \sim \nu$ in the sense of Definition III.5. Let $X, Y \in L_2[0,1]$ satisfy $\mathcal{L}_{[0,1]}(X) = \mu$ and $\mathcal{L}_{[0,1]}(Y) = \nu$. Recall that $\{x\}$ denotes the unique number in $[0,2\pi)$ such that $x - \{x\} \in 2\pi\mathbb{Z}$. Since $\mathcal{L}_{[0,1]}(X) \sim \mathcal{L}_{[0,1]}(Y)$, we have $\mathcal{L}_{[0,1]}(\{X\}) = \mathcal{L}_{[0,1]}(\{Y\})$. By Proposition III.14, it follows that $\mathbb{P}^W$-almost surely, the laws of $\{Z_t^X\}$ and of $\{Z_t^Y\}$ under $[0,1]\times\Omega^\beta$ are equal. Recall that for any $Z \in L_2([0,1]\times\Omega^\beta)$,

$\widehat{\phi}(Z)$ only depends on the law of Z under $[0,1]\times\Omega^\beta$ and $\widehat{\phi}(\{Z\}) = \widehat{\phi}(Z)$ because ϕ is $\mathbb{T}$-stable. Thus $\mathbb{P}^W$-almost surely, $\widehat{\phi}(Z_t^X) = \widehat{\phi}(\{Z_t^X\}) = \widehat{\phi}(\{Z_t^Y\}) = \widehat{\phi}(Z_t^Y)$. By Definition (III.44), $P_t\phi(\mu) = \widehat{P_t\phi}(X) = \mathbb{E}^W[\widehat{\phi}(Z_t^X)] = \mathbb{E}^W[\widehat{\phi}(Z_t^Y)] = \widehat{P_t\phi}(Y) = P_t\phi(\nu)$. Thus $P_t\phi$ is $\mathbb{T}$-stable.

By (III.44), it is clear that $P_t\phi$ is bounded on $\mathcal{P}_2(\mathbb{R})$, because ϕ is bounded. Furthermore, $P_t\phi$ is continuous on $\mathcal{P}_2(\mathbb{R})$, and even Lipschitz-continuous. Indeed, let $\mu,\nu\in\mathcal{P}_2(\mathbb{R})$. Let $X,Y\in L_2[0,1]$ be the quantile functions respectively associated with μ and ν: $\mu = \mathcal{L}_{[0,1]}(X)$ and $\nu = \mathcal{L}_{[0,1]}(Y)$; in other words, X (resp. Y) is the increasing rearrangement of μ (resp. ν). A classical result in optimal transportation (see e.g. [Vil03, Theorem 2.18]) states that (X,Y) realises the optimal coupling in the definition of the L_2-Wasserstein distance: $W_2(\mu,\nu)^2 = \int_0^1 |X(u)-Y(u)|^2\mathrm{d}u$.

By Remark III.7, $\widehat{\phi}$ is Lipschitz-continuous, thus:

$$\begin{aligned}|P_t\phi(\mu) - P_t\phi(\nu)| = |\widehat{P_t\phi}(X) - \widehat{P_t\phi}(Y)| &\leqslant \mathbb{E}^W\left[|\widehat{\phi}(Z_t^X) - \widehat{\phi}(Z_t^Y)|\right]\\ &\leqslant \|\widehat{\phi}\|_{\mathrm{Lip}}\,\mathbb{E}^W\left[\|Z_t^X - Z_t^Y\|_{L_2([0,1]\times\Omega^\beta)}\right]\\ &\leqslant \|\widehat{\phi}\|_{\mathrm{Lip}}\,\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |Z_t^{X(u)} - Z_t^{Y(u)}|^2\mathrm{d}u\right]^{1/2}.\end{aligned}$$

By (III.36), we have

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |Z_t^{X(u)} - Z_t^{Y(u)}|^2\mathrm{d}u\right] \leqslant C_2\int_0^1 |X(u)-Y(u)|^2\mathrm{d}u = W_2(\mu,\nu)^2.$$

Thus $|P_t\phi(\mu) - P_t\phi(\nu)| \leqslant CW_2(\mu,\nu)$; in particular, $P_t\phi$ is continuous.

Assumption (ϕ2)**:** Let us prove that $P_t\phi$ is L-differentiable. Let $\mu,\nu\in\mathcal{P}_2(\mathbb{R})$ and let $X,Y\in L_2[0,1]$ such that $\mathcal{L}_{[0,1]}(X) = \mu$ and $\mathcal{L}_{[0,1]}(Y) = \nu$. Let us prove that the Fréchet derivative of $\widehat{P_t\phi}$ at point X is given by

$$D\widehat{P_t\phi}(X)\cdot Y = \int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[D\widehat{\phi}(Z_t^X)_u\,\partial_x Z_t^{X(u)}\right]Y(u)\mathrm{d}u. \tag{III.46}$$

Remark that $D\widehat{\phi}(Z_t^X)$ is an element of the dual of $L_2([0,1]\times\Omega^\beta)$ and that we identify it with an element of $L_2([0,1]\times\Omega^\beta)$. Thus $D\widehat{\phi}(Z_t^X)_u$ denotes the value at point $u\in[0,1]$ of the $L_2([0,1]\times\Omega^\beta)$-function $D\widehat{\phi}(Z_t^X)$.

Let us assume that $\|Y\|_{L_2}\leqslant 1$. Let us compute

$$\begin{aligned}\widehat{P_t\phi}(X+Y) - \widehat{P_t\phi}(X) &= \mathbb{E}^W\left[\widehat{\phi}(Z_t^{X+Y}) - \widehat{\phi}(Z_t^X)\right] = \mathbb{E}^W\left[\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{\phi}(Z_t^{X+\lambda Y})\mathrm{d}\lambda\right]\\ &= \int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(Z_t^{X+\lambda Y})_u\,\partial_x Z_t^{(X+\lambda Y)(u)}Y(u)\mathrm{d}u\right]\mathrm{d}\lambda,\end{aligned}$$

since $\widehat{\phi}$ is Fréchet-differentiable. Therefore

$$\left|\widehat{P_t\phi}(X+Y) - \widehat{P_t\phi}(X) - \int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[D\widehat{\phi}(Z_t^X)_u\,\partial_x Z_t^{X(u)}\right]Y(u)\mathrm{d}u\right| \leqslant |D_1| + |D_2|,$$

where

$$\begin{aligned}D_1 &:= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^1 \left(D\widehat{\phi}(Z_t^{X+\lambda Y})_u - D\widehat{\phi}(Z_t^X)_u\right)\partial_x Z_t^{X(u)}Y(u)\,\mathrm{d}\lambda\mathrm{d}u\right];\\ D_2 &:= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^1 D\widehat{\phi}(Z_t^{X+\lambda Y})_u\left(\partial_x Z_t^{(X+\lambda Y)(u)} - \partial_x Z_t^{X(u)}\right)Y(u)\,\mathrm{d}\lambda\mathrm{d}u\right].\end{aligned}$$

Let us start by estimating D_1. By Cauchy-Schwarz inequality,

$$|D_1| \leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^1 \left|D\widehat{\phi}(Z_t^{X+\lambda Y})_u - D\widehat{\phi}(Z_t^X)_u\right|^2 \mathrm{d}\lambda\mathrm{d}u\right]^{\frac{1}{2}} \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |\partial_x Z_t^{X(u)}Y(u)|^2\mathrm{d}u\right]^{\frac{1}{2}}.$$

On the one hand, by assumption (ϕ3) and by (III.36), we have (with constants modified from a line to the next):

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^1 \left|D\widehat{\phi}(Z_t^{X+\lambda Y})_u - D\widehat{\phi}(Z_t^X)_u\right|^2 \mathrm{d}\lambda\mathrm{d}u\right]&\\ \leqslant C\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^1 \left|Z_t^{(X+\lambda Y)(u)} - Z_t^{X(u)}\right|^2 \mathrm{d}\lambda\mathrm{d}u\right]&\\ \leqslant C\int_0^1\int_0^1 \lambda^2|Y(u)|^2\mathrm{d}\lambda\mathrm{d}u \leqslant C\|Y\|_{L_2}^2.&\end{aligned}$$

On the other hand

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |\partial_x Z_t^{X(u)}Y(u)|^2\mathrm{d}u\right] = \int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[|\partial_x Z_t^{X(u)}|^2\right]|Y(u)|^2\mathrm{d}u.$$

By (III.38), there is $C>0$ such that for every $u\in[0,1]$, $\mathbb{E}^W\mathbb{E}^\beta\left[|\partial_x Z_t^{X(u)}|^2\right]\leqslant C$. Thus

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |\partial_x Z_t^{X(u)}Y(u)|^2\mathrm{d}u\right] \leqslant C\|Y\|_{L_2}^2.$$

Finally, we get

$$|D_1| \leqslant C\|Y\|_{L_2}^2. \tag{III.47}$$

Moreover, let us compute D_2. By Cauchy-Schwarz inequality, $|D_2| \leqslant |D_{2,1}|^{1/2}\cdot|D_{2,2}|^{1/2}$ where

$$\begin{aligned}D_{2,1} &:= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^1 \left|D\widehat{\phi}(Z_t^{X+\lambda Y})_u\, Y(u)\right|^2 \mathrm{d}\lambda\mathrm{d}u\right];\\ D_{2,2} &:= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^1 \left|\partial_x Z_t^{(X+\lambda Y)(u)} - \partial_x Z_t^{X(u)}\right|^2 \mathrm{d}\lambda\mathrm{d}u\right].\end{aligned}$$

On the one hand,

$$D_{2,1} = \int_0^1\int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[\left|D\widehat{\phi}(Z_t^{X+\lambda Y})_u\right|^2\right]|Y(u)|^2\mathrm{d}\lambda\mathrm{d}u.$$

Recall that ϕ is $\mathbb{T}$-stable. It follows that for any random variables $U,V\in L_2([0,1]\times\Omega^\beta)$, $D\widehat{\phi}(U)\cdot V = \lim_{\varepsilon\to0}\frac{\widehat{\phi}(U+\varepsilon V)-\widehat{\phi}(U)}{\varepsilon} = \lim_{\varepsilon\to0}\frac{\widehat{\phi}(\{U\}+\varepsilon V)-\widehat{\phi}(\{U\})}{\varepsilon} = D\widehat{\phi}(\{U\})\cdot V$. Hence for every $U\in L_2([0,1]\times\Omega^\beta)$, $D\widehat{\phi}(U) = D\widehat{\phi}(\{U\})$.

Let us denote by $\xi := \mathcal{L}_{[0,1]\times\Omega^\beta}(\{Z_t^{X+\lambda Y}\})$. Then

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\left|D\widehat{\phi}(Z_t^{X+\lambda Y})_u\right|^2\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\left|D\widehat{\phi}(\{Z_t^{X+\lambda Y}\})_u\right|^2\right] &= \mathbb{E}^W\mathbb{E}^\beta\left[\left|\partial_\mu\phi(\xi)(\{Z_t^{(X+\lambda Y)(u)}\})\right|^2\right]\\ &\leqslant \mathbb{E}^W\left[\sup_{v\in\mathbb{R}}|\partial_\mu\phi(\xi)(v)|^2\right].\end{aligned}$$

By Proposition III.9, $v \mapsto \partial_\mu \phi(\xi)(v)$ is 2π-periodic. By (III.16), it follows that

$$\begin{aligned}\sup_{v\in\mathbb{R}} |\partial_\mu \phi(\xi)(v)| = \sup_{v\in[0,2\pi]} |\partial_\mu \phi(\xi)(v)| &\leqslant C(1+2\pi) + C\int_{\mathbb{R}} |x| \mathrm{d}\xi(x) \\ &= C(1+2\pi) + C\mathbb{E}^\beta\left[\int_0^1 |\{Z_t^{(X+\lambda Y)(u)}\}| \mathrm{d}u\right] \\ &\leqslant C(1+2\pi) + C2\pi,\end{aligned}$$

since $\{Z_t^{(X+\lambda Y)(u)}\}$ takes values in $[0,2\pi)$. Thus $\mathbb{E}^W\left[\sup_{v\in\mathbb{R}} |\partial_\mu \phi(\xi)(v)|^2\right] \leqslant C$, where C is independent of X, λ and Y. We deduce that $D_{2,1} \leqslant \int_0^1\int_0^1 C|Y(u)|^2 \mathrm{d}\lambda \mathrm{d}u \leqslant C\|Y\|_{L_2}^2$.

On the other hand, since f is of order $\alpha > \frac{5}{2}$, inequality (III.39) holds with $\theta = 1$. Thus

$$D_{2,2} \leqslant C\int_0^1\int_0^1 \lambda^2 |Y(u)|^2 \mathrm{d}\lambda \mathrm{d}u \leqslant C\|Y\|_{L_2}^2.$$

We finally obtain

$$|D_2| \leqslant C\|Y\|_{L_2}^2. \tag{III.48}$$

It follows from (III.47) and (III.48) that for every $\|Y\|_{L_2} \leqslant 1$,

$$\left|\widehat{P_t\phi}(X+Y) - \widehat{P_t\phi}(X) - \int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[D\widehat{\phi}(Z_t^X)_u\, \partial_x Z_t^{X(u)}\right] Y(u)\mathrm{d}u\right| \leqslant C\|Y\|_{L_2}^2.$$

Thus $\widehat{P_t\phi}$ is Fréchet-differentiable at point X and the derivative is given by (III.46). Thus $P_t\phi$ is L-differentiable and its derivative is given by

$$\partial_\mu P_t\phi(\mu)(X)\cdot Y = D\widehat{P_t\phi}(X)\cdot Y = \int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[D\widehat{\phi}(Z_t^X)_u\, \partial_x Z_t^{X(u)}\right] Y(u)\mathrm{d}u.$$

Moreover, let us prove that

$$\sup_{\mu\in\mathcal{P}_2(\mathbb{R})} \int_{\mathbb{R}} |\partial_\mu P_t\phi(\mu)(x)|^2 \mathrm{d}\mu(x) = \sup_{X\in L_2[0,1]} \int_0^1 |D\widehat{P_t\phi}(X)_u|^2 \mathrm{d}u < +\infty.$$

Let us apply (III.46) with $Y = D\widehat{P_t\phi}(X)$. We obtain

$$\begin{aligned}\int_0^1 |D\widehat{P_t\phi}(X)_u|^2 \mathrm{d}u &= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(Z_t^X)_u\, \partial_x Z_t^{X(u)} D\widehat{P_t\phi}(X)_u \mathrm{d}u\right] \\ &\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |D\widehat{\phi}(Z_t^X)_u|^2 \mathrm{d}u\right]^{\frac{1}{2}} \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |\partial_x Z_t^{X(u)} D\widehat{P_t\phi}(X)_u|^2 \mathrm{d}u\right]^{\frac{1}{2}}.\end{aligned} \tag{III.49}$$

By (III.38),

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |\partial_x Z_t^{X(u)} D\widehat{P_t\phi}(X)_u|^2 \mathrm{d}u\right] &= \int_0^1 \mathbb{E}^W\mathbb{E}^\beta\left[|\partial_x Z_t^{X(u)}|^2\right] |D\widehat{P_t\phi}(X)_u|^2 \mathrm{d}u \\ &\leqslant C\int_0^1 |D\widehat{P_t\phi}(X)_u|^2 \mathrm{d}u.\end{aligned} \tag{III.50}$$

It follows from (III.49) and (III.50) that

$$\int_0^1 |D\widehat{P_t\phi}(X)_u|^2 \mathrm{d}u \leqslant C\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |D\widehat{\phi}(Z_t^X)_u|^2 \mathrm{d}u\right] = C\mathbb{E}^W\left[\int_{\mathbb{R}} |\partial_\mu \phi(\xi)(x)|^2 \mathrm{d}\xi(x)\right],$$

where $\xi = \mathcal{L}_{[0,1]\times\Omega^\beta}(Z_t^{X(u)})$. The last term is bounded by a constant independent of ξ because by assumption ($\phi 2$), $\sup_{\mu\in\mathcal{P}_2(\mathbb{R})}\int_{\mathbb{R}}|\partial_\mu\phi(\mu)(x)|^2\mathrm{d}\mu(x) < +\infty$.

Assumption ($\phi 3$)**:** Let us prove that for every $X_1, X_2, Y \in L_2[0,1]$,

$$|D\widehat{P_t\phi}(X_1)\cdot Y - D\widehat{P_t\phi}(X_2)\cdot Y| \leqslant C\|X_1 - X_2\|_{L_2[0,1]}\|Y\|_{L_2[0,1]}. \tag{III.51}$$

By formula (III.46), $|D\widehat{P_t\phi}(X_1)\cdot Y - D\widehat{P_t\phi}(X_2)\cdot Y| \leqslant |D_3| + |D_4|$, where

$$D_3 := \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 (D\widehat{\phi}(Z_t^{X_1})_u - D\widehat{\phi}(Z_t^{X_2})_u)\partial_x Z_t^{X_1(u)}Y(u)\mathrm{d}u\right];$$
$$D_4 := \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(Z_t^{X_2})_u(\partial_x Z_t^{X_1(u)} - \partial_x Z_t^{X_2(u)})Y(u)\mathrm{d}u\right].$$

Up to replacing X and $X+\lambda Y$ by X_1 and X_2, D_3 and D_4 are equivalent to D_1 and D_2. Thus we get by the same computations as for D_1 and D_2:

$$|D_3| \leqslant C\|X_1 - X_2\|_{L_2[0,1]}\|Y\|_{L_2[0,1]}$$
$$|D_4| \leqslant C\|X_1 - X_2\|_{L_2[0,1]}\|Y\|_{L_2[0,1]}.$$

This completes the proofs of (III.51) and of the proposition. □

Let us consider $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfying the ϕ-assumptions. We are interested in this chapter in regularization properties of the semi-group $(P_t)_{t\in[0,T]}$ and more precisely, we will study the behaviour of $\frac{\widehat{P_t\phi}(g+\rho h)-\widehat{P_t\phi}(g)}{\rho}$, or equivalently $\frac{P_t\phi(\mu_0^{g+\rho h})-P_t\phi(\mu_0^g)}{\rho}$, when ρ tends to zero. In other words, we will fix an initial condition, given by a quantile function $g \in \mathbf{G}^1$, or equivalently by the associated measure μ_0^g and compare the law of the system at final time T with the law of a system which started at a perturbed initial condition, of the form $g + \rho h$ for ρ close to zero. Let us remark that the point of view is, as in the previous chapters, to study the system in terms of the process of quantile functions more than in terms of the process of probability measures, which explains that we consider linear perturbations of the initial quantile function. If we see the process as a system of particles, then for each $u \in \mathbb{R}$, $\rho h(u)$ describes the direction and the intensity of the perturbation of the initial position of the particle indexed by u.

For now, we consider an initial condition $g \in \mathbf{G}^1$, but we will need more regularity in order to prove a Bismut-Elworthy-like formula. In Theorem III.29, we will assume that g belongs to $\mathbf{G}^{3+\theta}$, for some positive θ. Concerning the direction of perturbation h, we assume that it belongs to Δ^1 defined below:

Definition III.27. We denote by Δ^1 the set of 1-periodic $\mathcal{C}^1$-functions $h : \mathbb{R} \to \mathbb{R}$. We define the following norm on Δ^1:

$$\|h\|_{\mathcal{C}^1} := \sup_{u\in[0,1]}|h(u)| + \sup_{u\in[0,1]}|\partial_u h(u)|.$$

This ensures that for ρ close to zero, $g + \rho h$ still belongs to $\mathbf{G}^1$.Indeed, the periodicity assumption insures that $g + \rho h$ still satisfies the pseudo-periodic condition. Moreover, let $c := \inf_{u\in\mathbb{R}} g'(u)$. Remark that c does not depend on the representative g of the equivalence class $\overline{g}$, and that $c > 0$, since g' is continuous, positive everywhere, and 1-periodic. Thus for every $\rho \in (-\rho_0, \rho_0)$, where $\rho_0 = \frac{c}{\|h'\|_{L_\infty}}$, we have $g'(u) + \rho h'(u) > 0$ for every $u \in \mathbb{R}$.

Let us compute $\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0}P_t\phi(\mu_0^{g+\rho h})$. We will give in the following lemma the expression of the derivative in terms of Lions' derivative of ϕ and also in terms of the linear functional derivative of $P_t\phi$. For later purposes, we will state the result considering random initial functions g and h, with a randomness which is $\mathcal{G}_0$-measurable. Recall that within this framework, g and h are independent of $((W^k)_{k\in\mathbb{Z}}, \beta)$.

Lemma III.28. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfy the ϕ-assumptions. Let $\theta \in (0,1)$ and assume that f is of order $\alpha > \frac{3}{2} + \theta$. Let g and h be $\mathcal{G}_0$-measurable random variables with values respectively in $\mathbf{G}^{1+\theta}$ and Δ^1. For every $t \in [0,T]$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) &= \int_0^1 \partial_\mu(P_t\phi)(\mu_0^g)(g(u))\, h(u)\mathrm{d}u \\ &= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\, \partial_u x_t^g(u)\frac{h(u)}{g'(u)}\mathrm{d}u\right] \qquad \text{(III.52)} \\ &= -\int_0^1 \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u))\, \partial_u\left(\frac{h}{g'}\right)(u)\mathrm{d}u.\end{aligned}$$

Proof. Let us fix ω^0 in an almost-sure event of Ω^0 such that $g = g(\omega^0)$ belongs to $\mathbf{G}^{1+\theta}$ and $h = h(\omega^0)$ belongs to Δ^1. Let $\rho \in (-\rho_0, \rho_0)$, where $\rho_0 = \frac{1}{\|h'\|_{L^\infty}} \inf g'$. Thus $g + \rho h$ belongs to $\mathbf{G}^1$. Let us consider the process $(\mu_t^{g+\rho h})_{t\in[0,T]}$. The first equality in (III.52) follows from the definition of Lions' differential calculus.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) = \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} \widehat{P_t\phi}(g+\rho h) = D\widehat{P_t\phi}(g)\cdot h &= \int_0^1 D\widehat{P_t\phi}(g)_u\, h(u)\mathrm{d}u \qquad \text{(III.53)} \\ &= \int_0^1 \partial_\mu(P_t\phi)(\mu_0^g)(g(u))\, h(u)\mathrm{d}u.\end{aligned}$$

It follows from (III.45) that for every $t \in [0,T]$,

$$P_t\phi(\mu_0^{g+\rho h}) = \mathbb{E}^W\left[\phi\left(\mu_t^{g+\rho h}\right)\right] = \mathbb{E}^W\left[\widehat{\phi}\left(x_t^{g+\rho h}\right)\right].$$

The second equality in (III.52) is a consequence of formula (III.46):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) &= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(Z_t^g)_u\, \partial_x Z_t^{g(u)} h(u)\mathrm{d}u\right] \\ &= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(x_t^g)_u\, \frac{\partial_u x_t^g(u)}{g'(u)} h(u)\mathrm{d}u\right] \\ &= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\partial_u x_t^g(u)\frac{h(u)}{g'(u)}\mathrm{d}u\right],\end{aligned}$$

by equalities (III.40) and (III.41).

Let us compute the third equality in (III.52). By Proposition 5.48 of [CD18a], Lions' derivative is equal to the gradient of the linear functional derivative (see Section 0.4): for every ϕ and μ, $\partial_\mu\phi(\mu)(\cdot) = \partial_v\frac{\delta\phi}{\delta m}(\mu)(\cdot)$. Therefore,

$$\begin{aligned}\int_0^1 \partial_\mu(P_t\phi)(\mu_0^g)(g(u))h(u)\mathrm{d}u &= \int_0^1 \partial_v\left\{\frac{\delta P_t\phi}{\delta m}(\mu_0^g)\right\}(g(u))h(u)\mathrm{d}u \\ &= \int_0^1 \partial_u\left\{\frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(\cdot))\right\}(u)\frac{h(u)}{g'(u)}\mathrm{d}u \\ &= \left[\frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u))\frac{h(u)}{g'(u)}\right]_0^1 - \int_0^1 \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u))\partial_u\left(\frac{h}{g'}\right)(u)\mathrm{d}u,\end{aligned}$$

by an integration by parts formula. Furthermore, by Proposition III.26, $P_t\phi$ satisfies the ϕ-assumptions, and the probability measure μ_0^g satisfies the assumption of Proposition III.11 since $g'(u) > 0$ for every $u \in \mathbb{R}$. Therefore, it follows from Proposition III.9 that $v \mapsto \partial_\mu P_t\phi(\mu_0^g)(v)$ is 2π-periodic and from Proposition III.11 that $v \mapsto \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(v)$ is 2π-periodic. It follows that $\frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(1)) = \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(0)+2\pi) = \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(0))$. Since $\frac{h}{g'}$ is 1-periodic, we conclude that $\left[\frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u))\frac{h(u)}{g'(u)}\right]_0^1 = 0$. □

For every ϕ, the linear functional derivative $\frac{\delta\phi}{\delta m}(\mu)$ is defined up to an additive constant. For every 1-periodic $\mathcal{C}^1$-function h, we easily see that

$$\int_0^1 \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u))\partial_u\left(\frac{h}{g'}\right)(u)\mathrm{d}u \\ = \int_0^1 \left[\frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u)) - \int_0^1 \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u'))\mathrm{d}u'\right]\partial_u\left(\frac{h}{g'}\right)(u)\mathrm{d}u.$$

Let us introduce the following notation. $\left[\frac{\delta\phi}{\delta m}\right]$ denotes the zero-average linear functional derivative; for every $\mu\in\mathcal{P}_2(\mathbb{R})$ and $v\in\mathbb{R}$,

$$\left[\frac{\delta\phi}{\delta m}\right](\mu)(v) := \frac{\delta\phi}{\delta m}(\mu)(v) - \int_{\mathbb{R}}\frac{\delta\phi}{\delta m}(\mu)(v')\mathrm{d}\mu(v'). \tag{III.54}$$

In particular, for every $v\in\mathbb{R}$,

$$\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(v) = \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(v) - \mathbb{E}^\beta\left[\int_0^1\frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u))\mathrm{d}u\right] \\ = \frac{\delta P_t\phi}{\delta m}(\mu_0^g)(v) - \int_0^1\frac{\delta P_t\phi}{\delta m}(\mu_0^g)(g(u))\mathrm{d}u.$$

Thus we also have the following writing for equation (III.52):

$$\int_0^1 \partial_\mu(P_t\phi)(\mu_0^g)(g(u))\,h(u)\mathrm{d}u = -\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(g(u))\,\partial_u\left(\frac{h}{g'}\right)(u)\mathrm{d}u. \tag{III.55}$$

III.4.2 Fourier inversion on the torus

The aim of the following three paragraphs is to prove this theorem. For every $X\in L_2(\Omega^W)$, we denote by $\mathbb{V}^W[X]$ the variance of the random variable X, *i.e.* $\mathbb{V}^W[X] := \mathbb{E}^W\left[(X-\mathbb{E}^W[X])^2\right]$.

Theorem III.29. *Let $\phi:\mathcal{P}_2(\mathbb{R})\to\mathbb{R}$ satisfy the ϕ-assumptions. Let $\theta\in(0,1)$ and f be of order $\alpha=\frac{7}{2}+\theta$. Let $g\in\mathbf{G}^{3+\theta}$ and $h\in\Delta^1$ be two deterministic functions. Then there is C_g independent of h such that for every $t\in(0,T]$*

$$\left|\frac{\mathrm{d}}{\mathrm{d}\rho_{|\rho=0}}P_t\phi(\mu_0^{g+\rho h})\right| = \left|\int_0^1\partial_\mu(P_t\phi)(\mu_0^g)(g(u))\,h(u)\mathrm{d}u\right| \leqslant C_g\frac{\mathbb{V}^W[\phi(\mu_t^g)]^{1/2}}{t^{2+\theta}}\|h\|_{\mathcal{C}^1}, \tag{III.56}$$

where C_g is bounded when $\|g'''\|_{L_\infty}+\|g''\|_{L_\infty}+\|g'\|_{L_\infty}+\|\frac{1}{g'}\|_{L_\infty}$ is bounded.

Let us first explain the general strategy.

Let g and h be $\mathcal{G}_0$-measurable random variables with values respectively in $\mathbf{G}^{3+\theta}$ and Δ^1, for some $\theta\in(0,1)$. Then $g'h$ also belongs to Δ^1, since g' is a 1-periodic function with at least $\mathcal{C}^1$-regularity. By formulas (III.52) and (III.55) applied to $g'h$ instead of h:

$$\left|\int_0^1\partial_\mu(P_t\phi)(\mu_0^g)(g(u))\,g'(u)h(u)\mathrm{d}u\right| = \left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(g(u))\,\partial_u h(u)\mathrm{d}u\right| \\ = \left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu\phi(\mu_t^g)(x_t^g(u))\,\partial_u x_t^g(u)\,h(u)\mathrm{d}u\right]\right|. \tag{III.57}$$

We have the following writing for h:

$$h(u) = \frac{1}{t}\int_0^t h(u)\mathrm{d}s = \frac{1}{t}\int_0^t\frac{1}{\partial_u x_s^g(u)}\partial_u x_s^g(F_s^g(x_s^g(u)))h(F_s^g(x_s^g(u)))\mathrm{d}s \\ = \frac{1}{t}\int_0^t\frac{1}{\partial_u x_s^g(u)}A_s^g(x_s^g(u))\mathrm{d}s, \tag{III.58}$$

where $(A_t^g)_{t\in[0,T]}$ is defined by

$$A_t^g := \partial_u x_t^g(F_t^g(\cdot))h(F_t^g(\cdot)). \tag{III.59}$$

Recall that $F_t^g = (x_t^g)^{-1}$ and that for each $x \in \mathbb{R}$, $F_t^g(x+2\pi) = F_t^g(x)+1$. Using the fact that h and $\partial_u x_t^g$ are 1-periodic, we deduce that A_t^g is 2π-periodic:

$$A_t^g(x+2\pi) = \partial_u x_t^g(F_t^g(x+2\pi))h(F_t^g(x+2\pi)) = \partial_u x_t^g(F_t^g(x)+1)h(F_t^g(x)+1) = A_t^g(x).$$

Let us note that for each $t \in [0,T]$, it follows from Proposition III.13 and from the assumptions on g and h that A_t^g is a $\mathcal{C}^1$-function. This will not be sufficient in order to apply a Fourier inversion of A_t^g. Of course, one could possibly assume that h is $\mathcal{C}^3$ in order to increase the regularity of A_t^g: in that case, it would be of class $C^{2+\theta'}$, due to the derivative $\partial_u x_t^g$. Even in that case, this will not be sufficient for our purpose. So we will regularize the process $(A_t^g)_{t\in[0,T]}$.

Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a collection of mollifiers, defined by $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, where $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. For every $t \in [0,T]$, let us define

$$A_t^{g,\varepsilon} := A_t^g * \varphi_\varepsilon = \int_{\mathbb{R}} A_t^g(\cdot - y)\varphi_\varepsilon(y)\mathrm{d}y. \tag{III.60}$$

Observe that $A_t^{g,\varepsilon}$ is also 2π-periodic.

It follows from (III.58) that

$$\begin{aligned} I &:= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\partial_u x_t^g(u)h(u)\mathrm{d}u\right] \\ &= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\partial_u x_t^g(u)\int_0^t \frac{A_s^g(x_s^g(u))}{\partial_u x_s^g(u)}\mathrm{d}s\mathrm{d}u\right] = I_1 + I_2, \end{aligned} \tag{III.61}$$

where

$$I_1 := \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}s\mathrm{d}u\right]; \tag{III.62}$$

$$I_2 := \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}(A_s^g - A_s^{g,\varepsilon})(x_s^g(u))\mathrm{d}s\mathrm{d}u\right]. \tag{III.63}$$

In this paragraph, we will focus on I_1. In Paragraph III.4.3, we will focus on I_2. These two studies will lead to Proposition III.39, which provides a first estimate of I. In Paragraph III.4.4, we will complete the proof of Theorem III.29 by interpolating the two results in ε.

The aim of this paragraph is to obtain an upper bound for I_1, depending on ε. It will be given in Proposition III.33. Of course, that bound explodes when $\varepsilon \to 0$, but it will be caught up by the estimate of I_2. The scheme of the proof is close to the strategy of Chapter II. In the next two lemmas, we will define a Fourier inverse $(\lambda_t^k)_{t\in[0,T],k\in\mathbb{Z}}$ of the process $(A_t^{g,\varepsilon})_{t\in[0,T]}$, by analogy with Lemma II.18 but in terms of Fourier series instead of Fourier transform. Then, we will see, for each $k \in \mathbb{Z}$, $(\lambda_t^k)_{t\in[0,T]}$ as a perturbation of the noise $(W_t^k)_{t\in[0,T]}$ and apply a Fourier inversion.

In Lemma III.30, we give a first estimate on the L_2-norm of $(\lambda_t^k)_{t\in[0,T],k\in\mathbb{Z}}$ with respect to ε and to $\|A_t^g\|_{\mathcal{C}^1}$.

Lemma III.30. *Let $\theta \in (0,1)$, $g \in \mathbf{G}^{2+\theta}$ and f be of order $\alpha = \frac{7}{2}+\theta$. Let us fix $\varepsilon \in (0,1)$. Then there is a collection of $(\mathcal{G}_t)_{t\in[0,T]}$-adapted $\mathbb{C}$-valued processes $((\lambda_t^k)_{t\in[0,T]})_{k\in\mathbb{Z}}$ such that for every $t \in [0,T]$, the following equality holds*

$$A_t^{g,\varepsilon}(y) = \sum_{k\in\mathbb{Z}} f_k e^{-iky}\lambda_t^k, \tag{III.64}$$

and such that there is a constant $C > 0$ independent of ε satisfying the following inequality $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely for every $t \in [0,T]$

$$\left(\sum_{k\in\mathbb{Z}} \int_0^t |\lambda_s^k|^2 \mathrm{d}s \right)^{1/2} \leqslant \frac{C}{\varepsilon^{3+2\theta}} \left(\int_0^t \|A_s^g\|_{\mathcal{C}^1}^2 \mathrm{d}s \right)^{1/2}. \tag{III.65}$$

Proof. Let us fix $t \in (0,T]$. The map $y \mapsto A_t^{g,\varepsilon}(y)$ is a 2π-periodic $\mathcal{C}^1$-function. Therefore, by Dirichlet's Theorem, that map is equal to the sum of its Fourier series; we have

$$A_t^{g,\varepsilon}(y) = \sum_{k\in\mathbb{Z}} c_k(A_t^{g,\varepsilon}) e^{-iky},$$

where for every 2π-periodic function A and for every $k \in \mathbb{Z}$, $c_k(A) := \frac{1}{2\pi}\int_0^{2\pi} A(y)e^{iky}\mathrm{d}y$.

Let us define $\lambda_t^k := \frac{c_k(A_t^{g,\varepsilon})}{f_k}$. Since $(A_t^{g,\varepsilon})_{t\in[0,T]}$ is $(\mathcal{G}_t)$-adapted, it is clear that for each $k \in \mathbb{Z}$, $(\lambda_t^k)_{t\in[0,T]}$ is also $(\mathcal{G}_t)$-adapted. Equality (III.64) clearly holds true. Moreover,

$$\sum_{k\in\mathbb{Z}} \int_0^t |\lambda_s^k|^2 \mathrm{d}s = \sum_{k\in\mathbb{Z}} \int_0^t \left| \frac{c_k(A_s^{g,\varepsilon})}{f_k} \right|^2 \mathrm{d}s.$$

Let us compute the Fourier coefficient $c_k(A_s^{g,\varepsilon})$:

$$\begin{aligned} c_k(A_s^{g,\varepsilon}) = c_k(A_s^g * \varphi_\varepsilon) &= \frac{1}{2\pi}\int_0^{2\pi} \left(\int_{\mathbb{R}} A_s^g(y-x)\varphi_\varepsilon(x)\mathrm{d}x \right) e^{iky}\mathrm{d}y \\ &= \int_{\mathbb{R}} \varphi_\varepsilon(x) e^{ikx} \left(\frac{1}{2\pi}\int_0^{2\pi} A_s^g(y-x)e^{ik(y-x)}\mathrm{d}y \right) \mathrm{d}x \\ &= c_k(A_s^g) \int_{\mathbb{R}} \varphi(x) e^{ik\varepsilon x}\mathrm{d}x. \end{aligned} \tag{III.66}$$

Since $\int_{\mathbb{R}} \varphi(x)e^{ik\varepsilon x}\mathrm{d}x$ is equal to the value of the Fourier transform of φ (that is φ itself) at point $k\varepsilon$, we know in particular that there is $C > 0$ such that for every $k \in \mathbb{Z}\backslash\{0\}$ and for every $\varepsilon > 0$, $\left| \int_{\mathbb{R}} \varphi(x)e^{ik\varepsilon x}\mathrm{d}x \right| \leqslant \frac{C}{|k\varepsilon|^{3+2\theta}}$.

Moreover, since g belongs to $\mathbf{G}^{2+\theta}$, A_s^g is a $\mathcal{C}^1$-function. Indeed, by Proposition III.13, it follows from the fact that g belongs to $\mathbf{G}^{2+\theta}$ and $\alpha = \frac{7}{2}+\theta$ that x_t^g (and thus F_t^g) is a $\mathcal{C}^2$-function, so we deduce that $u \mapsto \partial_u x_t^g(u)$ is a $\mathcal{C}^1$-function. Since h is also $\mathcal{C}^1$, A_t^g is $\mathcal{C}^1$ by product and composition. Thus there is $C > 0$ independent of k and s such that for every $k \in \mathbb{Z}\backslash\{0\}$, $|c_k(A_s^g)| \leqslant \frac{C}{|k|}\|\partial_x A_s^g\|_{L_\infty}$. Moreover, $|c_0(A_s^{g,\varepsilon})| = |c_0(A_s^g)| \leqslant \|A_s^g\|_{L_\infty}$.

Since f is of order $\alpha = \frac{7}{2}+\theta$, there is C such that for every $k \in \mathbb{Z}\backslash\{0\}$, $\frac{1}{|f_k|} \leqslant C|k|^{\frac{7}{2}+\theta}$. Thus we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \int_0^t \frac{|c_k(A_s^{g,\varepsilon})|^2}{f_k^2}\mathrm{d}s &= \int_0^t \frac{|c_0(A_s^{g,\varepsilon})|^2}{f_0^2}\mathrm{d}s + \sum_{k\neq 0}\int_0^t \frac{|c_k(A_s^{g,\varepsilon})|^2}{f_k^2}\mathrm{d}s \\ &\leqslant C\int_0^t \|A_s^g\|_{L_\infty}^2\mathrm{d}s + C\sum_{k\neq 0}\int_0^t |k|^{7+2\theta}\frac{1}{|k\varepsilon|^{6+4\theta}}\frac{1}{|k|^2}\|\partial_x A_s^g\|_{L_\infty}^2\mathrm{d}s \\ &\leqslant \frac{C}{\varepsilon^{6+4\theta}}\int_0^t \|A_s^g\|_{\mathcal{C}^1}^2\mathrm{d}s, \end{aligned}$$

because $1 \leqslant \frac{1}{\varepsilon}$ and because the sum $\sum_{k\neq 0}\frac{1}{|k|^{1+2\theta}}$ converges. This completes the proof of the lemma. □

In Lemma III.30, we have obtained a control in terms of the L_2-norm of $t \mapsto \|A_t^g\|_{\mathcal{C}^1}$. That quantity is almost surely bounded but not uniformly in $(\omega^W, \omega^\beta) \in \Omega^W \times \Omega^\beta$. In Lemma III.31, we will obtain a uniform almost sure bound, after localization of the process $(\lambda_t^k)_{t\in[0,T],k\in\mathbb{Z}}$. Here, we are not interested in the precise dependence of the bound with respect to ε; thus, we will get a bound in terms of the L_2-norm of A_t^g instead of the $\mathcal{C}^1$-norm and we only have to bound uniformly the L_2-norm of $t \mapsto \|A_t^g\|_{L_2}$.

Let M_0 be an integer large enough so that for every $u \in \mathbb{R}$, $\frac{1}{M_0} < g'(u) < M_0$. For every $M \geqslant M_0$, let us define the following stopping times:

$$\begin{aligned}
\tau_M^1 &:= \inf\{t \geqslant 0 : \|\partial_u x_t^g\|_{L_\infty} \geqslant M\} \wedge T;\\
\tau_M^2 &:= \inf\{t \geqslant 0 : \left\|\tfrac{1}{\partial_u x_t^g(\cdot)}\right\|_{L_\infty} \geqslant M\} \wedge T;\\
\tau_M &:= \tau_M^1 \wedge \tau_M^2.
\end{aligned} \tag{III.67}$$

Remark that

$$\begin{aligned}
\mathbb{P}^W \otimes \mathbb{P}^\beta [\tau_M < T] &\leqslant \mathbb{P}^W \otimes \mathbb{P}^\beta \left[\tau_M^1 < T\right] + \mathbb{P}^W \otimes \mathbb{P}^\beta \left[\tau_M^2 < T\right]\\
&\leqslant \mathbb{P}^W \otimes \mathbb{P}^\beta \left[\sup_{t\leqslant T} \|\partial_u x_t^g\|_{L_\infty} \geqslant M\right] + \mathbb{P}^W \otimes \mathbb{P}^\beta \left[\sup_{t\leqslant T} \left\|\tfrac{1}{\partial_u x_t^g(\cdot)}\right\|_{L_\infty} \geqslant M\right]\\
&\leqslant \frac{1}{M}\left(\mathbb{E}^W\mathbb{E}^\beta \left[\sup_{t\leqslant T} \|\partial_u x_t^g\|_{L_\infty}\right] + \mathbb{E}^W\mathbb{E}^\beta \left[\sup_{t\leqslant T} \left\|\tfrac{1}{\partial_u x_t^g(\cdot)}\right\|_{L_\infty}\right]\right) \leqslant \frac{C}{M}.
\end{aligned}$$

The last inequality holds by (III.33) and (III.34) because g belongs to $\mathbf{G}^{2+\theta}$ and f is of order $\alpha > \frac{7}{2}$. We deduce that

$$\mathbb{P}^W \otimes \mathbb{P}^\beta [\tau_M < T] \underset{M\to+\infty}{\longrightarrow} 0. \tag{III.68}$$

Lemma III.31. *Let $M \geqslant M_0$. Let $\theta \in (0,1)$, $g \in \mathbf{G}^{2+\theta}$ and f be of order $\alpha = \frac{7}{2} + \theta$. Let us fix $\varepsilon \in (0,1)$. Then there is a collection of $(\mathcal{G}_t)_{t\in[0,T]}$-adapted $\mathbb{C}$-valued processes $((\lambda_t^{k,M})_{t\in[0,T]})_{k\in\mathbb{Z}}$ such that for every $t \in [0,T]$, the following equality holds*

$$\mathbb{1}_{\{t\leqslant\tau_M\}} A_t^{g,\varepsilon}(y) = \sum_{k\in\mathbb{Z}} f_k e^{-iky} \lambda_t^{k,M}, \tag{III.69}$$

and there is a constant $C_M > 0$ depending on M and ε such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, $\sum_{k\in\mathbb{Z}} \int_0^T |\lambda_t^{k,M}|^2 \mathrm{d}t \leqslant C_M$.

Proof. Define for every $t \in [0,T]$, $\lambda_t^{k,M} := \mathbb{1}_{\{t\leqslant\tau_M\}} \frac{c_k(A_t^{g,\varepsilon})}{f_k} = \mathbb{1}_{\{t\leqslant\tau_M\}} \lambda_t^k$.

The aim of this proof is to obtain an almost-sure upper bound for $\sum_{k\in\mathbb{Z}} \int_0^T |\lambda_t^{k,M}|^2 \mathrm{d}t$, which can depend on M and on ε. Since we are not interested in the dependance on ε here, the constant can be worse than in the proof of Lemma III.30. Similarly as in that proof, there is a constant $C > 0$ such that for every $k \in \mathbb{Z}\backslash\{0\}$ and for every $\varepsilon > 0$, $\left|\int_{\mathbb{R}} \varphi(x) e^{ik\varepsilon x} \mathrm{d}x\right| \leqslant \frac{C}{|k\varepsilon|^{4+2\theta}}$. Furthermore, for every $k \in \mathbb{Z}$, $|c_k(A_s^g)| \leqslant \|A_s^g\|_{L_2(\mathbb{T})}$. Thus we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \int_0^T |\lambda_t^{k,M}|^2 \mathrm{d}t &= \sum_{k\in\mathbb{Z}} \int_0^T \mathbb{1}_{\{t\leqslant\tau_M\}} \frac{|c_k(A_t^{g,\varepsilon})|^2}{f_k^2} \mathrm{d}t\\
&\leqslant C \int_0^T \mathbb{1}_{\{t\leqslant\tau_M\}} |c_0(A_t^g)|^2 \mathrm{d}t + \sum_{k\neq 0} \int_0^T \mathbb{1}_{\{t\leqslant\tau_M\}} |k|^{7+2\theta} \frac{1}{|k\varepsilon|^{8+4\theta}} |c_k(A_t^g)|^2 \mathrm{d}t\\
&\leqslant \frac{C}{\varepsilon^{8+4\theta}} \int_0^T \mathbb{1}_{\{t\leqslant\tau_M\}} \|A_t^g\|_{L_2(\mathbb{T})}^2 \mathrm{d}t.
\end{aligned}$$

By Definition (III.59), for every $s \in [0,T]$,

$$\mathbb{1}_{\{t\leqslant\tau_M\}} \|A_t^g\|_{L_2(\mathbb{T})} \leqslant C\mathbb{1}_{\{t\leqslant\tau_M\}} \|A_t^g\|_{L_\infty(\mathbb{T})} \leqslant C\mathbb{1}_{\{t\leqslant\tau_M^1\}} \|h\|_{L_\infty} \|\partial_u x_t^g\|_{L_\infty} \leqslant CM \|h\|_{L_\infty}.$$

Since the constant does not depend on t, we deduce the statement of the lemma. □

Before proving the bound for I_1, we give in the following lemma two inequalities which follows from the definition of the stopping time τ_M. Let us define the $(\mathcal{G}_t)$-adapted process $(a_t^M)_{t\in[0,T]}$ by

$$a_t^M(u) := \int_0^t \mathbb{1}_{\{s\leqslant\tau_M\}} \frac{g'(u)}{\partial_u x_s^g(u)} A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}s. \tag{III.70}$$

We easily check that for every $u \in \mathbb{R}$, $a_0^M(u) = 0$ and that $\dot{a}_t^M(u) = \frac{g'(u)}{\partial_u x_t^g(u)} \mathbb{1}_{\{t\leqslant\tau_M\}} A_t^{g,\varepsilon}(x_t^g(u))$ is a 1-periodic and continuous function of $u \in \mathbb{R}$.

Lemma III.32. *Let $\varepsilon \in (0,1)$. For every $M \geqslant M_0$, there are two constants C_M^a (depending on T, M, g' and h) and $C_{M,\varepsilon}$ (depending on T, M, ε and h) such that for every $t \in [0,T]$*

$$\|a_t^M\|_{L_\infty} \leqslant C_M^a; \tag{III.71}$$

$$\int_0^{t\wedge\tau_M} \|A_s^{g,\varepsilon}\|_{\mathcal{C}^1}^2 \mathrm{d}s \leqslant C_{M,\varepsilon}. \tag{III.72}$$

Proof. By definition of τ_M, for every $t \in [0,T]$ and every $u \in \mathbb{R}$,

$$|a_t^M(u)| \leqslant T\|g'\|_{L_\infty} M \sup_{s\leqslant\tau_M} \|A_s^{g,\varepsilon}\|_{L_\infty}$$

Since $A_s^{g,\varepsilon}$ is 2π-periodic and $A_s^{g,\varepsilon} = A_s^g * \varphi_\varepsilon$, with $\|\varphi_\varepsilon\|_{L_1(\mathbb{R})} = 1$, we have $\|A_s^{g,\varepsilon}\|_{L_\infty} \leqslant \|A_s^g\|_{L_\infty(\mathbb{T})}$. Recall that by definition (III.59), $\|A_s^g\|_{L_\infty(\mathbb{T})} \leqslant \|h\|_{L_\infty} \|\partial_u x_s^g\|_{L_\infty}$. We deduce that

$$\mathbb{1}_{\{s\leqslant\tau_M\}} \|A_s^{g,\varepsilon}\|_{L_\infty(\mathbb{T})} \leqslant \|h\|_{L_\infty} \mathbb{1}_{\{s\leqslant\tau_M^1\}} \|\partial_u x_s^g\|_{L_\infty} \leqslant M \|h\|_{L_\infty}. \tag{III.73}$$

Thus for every $t \in [0,T]$ and for every $u \in \mathbb{R}$, $|a_t^M(u)| \leqslant T\|g'\|_{L_\infty} M^2 \|h\|_{L_\infty}$. Therefore, inequality (III.71) holds with $C_M^a := T\|g'\|_{L_\infty} M^2 \|h\|_{L_\infty}$.

For every $t \in [0,T]$, $\partial_x A_t^{g,\varepsilon} = A_t^g * \partial_x \varphi_\varepsilon$. Since $\|\partial_x \varphi_\varepsilon\|_{L_1(\mathbb{R})} \leqslant \frac{C}{\varepsilon}$, we obtain

$$\mathbb{1}_{\{t\leqslant\tau_M\}} \|\partial_x A_t^{g,\varepsilon}\|_{L_\infty(\mathbb{T})} \leqslant \frac{C}{\varepsilon} \|h\|_{L_\infty} \mathbb{1}_{\{t\leqslant\tau_M^1\}} \|\partial_u x_t^g\|_{L_\infty} \leqslant \frac{CM}{\varepsilon} \|h\|_{L_\infty}. \tag{III.74}$$

It follows from (III.73) and (III.74) that $\int_0^{t\wedge\tau_M} \|A_s^{g,\varepsilon}\|_{\mathcal{C}^1}^2 \mathrm{d}s \leqslant T\frac{C}{\varepsilon^2} M^2 \|h\|_{L_\infty}^2$ for every $t \in [0,T]$, whence we obtain (III.72). □

Let us now state the main result of this paragraph.

Proposition III.33. *Let $\theta \in (0,1)$, $g \in \mathbf{G}^{3+\theta}$ and f be of order $\alpha = \frac{7}{2} + \theta$. Then there is $C > 0$ independent of g, h and θ such that for every $t \in [0,T]$, for every $\varepsilon \in (0,1)$,*

$$\left| \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \partial_\mu \phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)} A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}u \right] \right| \leqslant C \frac{\sqrt{t}}{\varepsilon^{3+2\theta}} \mathbb{V}^W \left[\phi(\mu_t^g)\right]^{1/2} C_1(g) \|h\|_{\mathcal{C}^1},$$

where $C_1(g) = 1 + \|g'''\|_{L_4}^2 + \|g''\|_{L_\infty}^6 + \|g'\|_{L_\infty}^6 + \left\|\frac{1}{g'}\right\|_{L_\infty}^8$.

In particular, by definition (III.62) of I_1, we have

$$|I_1| \leqslant \frac{C}{\sqrt{t}\,\varepsilon^{3+2\theta}} \mathbb{V}^W \left[\phi(\mu_t^g)\right]^{1/2} C_1(g) \|h\|_{\mathcal{C}^1}.$$

Proof. Fix $\varepsilon \in (0,1)$. Let $u \in \mathbb{R}$.

By inequality (III.71), $(\zeta_t)_{t\in[0,T]} := (a_t^M(u))_{t\in[0,T]}$ satisfies $\mathbb{E}\left[\int_0^T |\zeta_t|\mathrm{d}t\right] \leqslant C_M$. Thus the process $(\zeta_t)_{t\in[0,T]}$ satisfies the assumptions of Proposition III.23. Let us apply Kunita's expansion (III.42) to $x = g(u)$ and $\zeta_t = a_t^M(u)$. Using equality (III.40), $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, the following equation holds true for every $t \in [0,T]$:

$$\begin{aligned} Z_t^{g(u)+\rho a_t^M(u)} &= Z_0^{g(u)} + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-ikZ_s^{g(u)+\rho a_s^M(u)}}\mathrm{d}W_s^k\right) + \beta_t + \rho\int_0^t \partial_x Z_s^{g(u)+\rho a_s^M(u)}\dot{a}_s(u)\mathrm{d}s \\ &= g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-ikZ_s^{g(u)+\rho a_s^M(u)}}\mathrm{d}W_s^k\right) + \beta_t \\ &\quad + \rho\int_0^t \partial_x Z_s^{g(u)+\rho a_s^M(u)} \frac{g'(u)}{\partial_u x_s^g(u)} \mathbb{1}_{\{s\leqslant\tau_M\}} A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}s. \end{aligned}$$

Since both sides of the latter equation are almost surely continuous with respect to u, we deduce that the latter equality holds almost surely for every $u \in \mathbb{R}$. Recall that by (III.40) and by (III.41), $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely for every $s \in [0,T]$ and for every $u \in \mathbb{R}$, $x_s^g(u) = Z_s^{g(u)}$ and $\partial_x Z_s^{g(u)} = \frac{1}{g'(u)}\partial_u x_s^g(u)$. Thus we have

$$\begin{aligned} Z_t^{g(u)+\rho a_t^M(u)} &= g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-ikZ_s^{g(u)+\rho a_s^M(u)}}\mathrm{d}W_s^k\right) + \beta_t + \rho\int_0^t \mathbb{1}_{\{s\leqslant\tau_M\}} A_s^{g,\varepsilon}(Z_s^{g(u)})\mathrm{d}s \\ &\quad + \rho\int_0^t (\partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)}) \frac{g'(u)}{\partial_u x_s^g(u)} \mathbb{1}_{\{s\leqslant\tau_M\}} A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}s. \end{aligned} \tag{III.75}$$

Let us define an auxiliary process $(Y_t^{\rho,M})_{t\in[0,T]}$ solution to

$$Y_t^{\rho,M}(u) = g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(e^{-ikY_s^{\rho,M}(u)}\mathrm{d}W_s^k\right) + \beta_t + \rho\int_0^t \mathbb{1}_{\{s\leqslant\tau_M\}} A_s^{g,\varepsilon}(Y_s^{\rho,M}(u))\mathrm{d}s. \tag{III.76}$$

Recall the constant C_M^a from inequality (III.71) and define $\rho_0 := \rho_0(M) = \frac{1}{2C_M^a}$.

Let us state the following three lemmas (we prove them at the end of this proof).

Lemma III.34. *Assume that f is of order $\alpha > \frac{5}{2}$. There exists C depending on M, f, g', h, T and ε such that for every $\rho \in (-\rho_0, \rho_0)$ and for every $t \in [0,T]$,*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |Z_t^{g(u)+\rho a_t^M(u)} - Y_t^{\rho,M}(u)|^2\mathrm{d}u\right]^{1/2} \leqslant C|\rho|^{5/4}. \tag{III.77}$$

Lemma III.35. *Let $\theta \in (0,1)$, $g \in \mathbf{G}^{2+\theta}$ and f be of order $\alpha = \frac{7}{2}+\theta$. For every $M \geqslant M_0$ and for every $t \in [0,T]$,*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{g'(u)} a_t^M(u)\mathrm{d}u\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k)\right], \tag{III.78}$$

where $\overline{\lambda_s^{k,M}}$ denotes the complex conjugate of $\lambda_s^{k,M}$.

Lemma III.36. *Let $\theta \in (0,1)$, $g \in \mathbf{G}^{3+\theta}$ and f be of order $\alpha = \frac{7}{2}+\theta$. For every $t \in [0,T]$,*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)} A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}s\mathrm{d}u\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right]. \tag{III.79}$$

Moreover,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t|\lambda_s^k|^2\mathrm{d}s\right]\leqslant\frac{Ct}{\varepsilon^{6+4\theta}}\|h\|_{\mathcal{C}^1}^2C_1(g)^2, \tag{III.80}$$

where $C_1(g)=1+\|g'''\|_{L_4}^2+\|g''\|_{L_\infty}^6+\|g'\|_{L_\infty}^6+\left\|\frac{1}{g'}\right\|_{L_\infty}^8$.

Let us finalize the proof of Proposition III.33. It follows from (III.79) that

$$\begin{aligned}&\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t\partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}u\mathrm{d}s\right]\\&=\mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t\Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right]=\mathbb{E}^W\mathbb{E}^\beta\left[\Big(\phi(\mu_t^g)-\mathbb{E}^W\left[\phi(\mu_t^g)\right]\Big)\sum_{k\in\mathbb{Z}}\int_0^t\Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right]\end{aligned} \tag{III.81}$$

because $\mathbb{E}^W\mathbb{E}^\beta\left[\mathbb{E}^W\left[\phi(\mu_t^g)\right]\sum_{k\in\mathbb{Z}}\int_0^t\Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right]=\mathbb{E}^W\left[\phi(\mu_t^g)\right]\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t\Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right]=0$. By Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}&\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t\partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}u\mathrm{d}s\right]\right|\\&\qquad\leqslant\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t|\lambda_s^k|^2\mathrm{d}s\right]^{1/2}\\&\qquad\leqslant C\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}\frac{\sqrt{t}}{\varepsilon^{3+2\theta}}C_1(g)\|h\|_{\mathcal{C}^1},\end{aligned}$$

by inequality (III.80). It completes the proof of Proposition III.33. □

It remains to prove Lemmas III.34, III.35 and III.36.

Proof (Lemma III.34). Throughout the proof, C is a constant depending on M, f, g', h, T and ε. Its value can change from a line to the next. Recall that we want to prove that for each $\rho\in(-\rho_0,\rho_0)$ and for each $t\in[0,T]$, $\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1|Z_t^{g(u)+\rho a_t^M(u)}-Y_t^{\rho,M}(u)|^2\mathrm{d}u\right]^{1/2}\leqslant C|\rho|^{5/4}$.

The processes $(Z_t^{g(u)+\rho a_t^M(u)})_{t\in[0,T]}$ and $(Y_t^{\rho,M}(u))_{t\in[0,T]}$ satisfy equation (III.75) and equation (III.76) respectively. Thus for every $t\in[0,T]$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1|Z_t^{g(u)+\rho a_t^M(u)}-Y_t^{\rho,M}(u)|^2\mathrm{d}u\right]\leqslant 3(E_1+\rho^2E_2+\rho^2E_3), \tag{III.82}$$

where

$$\begin{aligned}E_1&:=\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left|\sum_{k\in\mathbb{Z}}\int_0^t f_k\Re\left((e^{-ikZ_s^{g(u)+\rho a_s^M(u)}}-e^{-ikY_s^{\rho,M}(u)})\mathrm{d}W_s^k\right)\right|^2\mathrm{d}u\right]\\E_2&:=\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left|\int_0^t\mathbb{1}_{\{s\leqslant\tau_M\}}(A_s^{g,\varepsilon}(Z_s^{g(u)})-A_s^{g,\varepsilon}(Y_s^{\rho,M}(u)))\mathrm{d}s\right|^2\mathrm{d}u\right]\\E_3&:=\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left|\int_0^t(\partial_xZ_s^{g(u)+\rho a_s^M(u)}-\partial_xZ_s^{g(u)})\frac{g'(u)}{\partial_u x_s^g(u)}\mathbb{1}_{\{s\leqslant\tau_M\}}A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}s\right|^2\mathrm{d}u\right].\end{aligned}$$

By Itô's isometry,

$$\begin{aligned}
E_1 &= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \sum_{k\in\mathbb{Z}} f_k^2 \left|e^{-ikZ_s^{g(u)+\rho a_s^M(u)}} - e^{-ikY_s^{\rho,M}(u)}\right|^2 \mathrm{d}s\mathrm{d}u\right]\\
&\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \sum_{k\in\mathbb{Z}} f_k^2k^2 \left|Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho,M}(u)\right|^2 \mathrm{d}s\mathrm{d}u\right]\\
&\leqslant C\int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \left|Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho,M}(u)\right|^2 \mathrm{d}u\right]\mathrm{d}s, \qquad \text{(III.83)}
\end{aligned}$$

because $\sum_{k\in\mathbb{Z}} f_k^2k^2 < +\infty$, since $\alpha > \frac{3}{2}$.

Let us estimate E_2. Recall that there is $C > 0$ such that $\int_0^{t\wedge\tau_M} \|\partial_x A_s^{g,\varepsilon}\|_{L_\infty(\mathbb{T})}^2 \,\mathrm{d}s \leqslant C$ by (III.72). It follows from Cauchy-Schwarz inequality that

$$\begin{aligned}
E_2 &\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \left|\int_0^{t\wedge\tau_M} \|\partial_x A_s^{g,\varepsilon}\|_{L_\infty(\mathbb{T})} |Z_s^{g(u)} - Y_s^{\rho,M}(u)|\mathrm{d}s\right|^2 \mathrm{d}u\right]\\
&\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \int_0^{t\wedge\tau_M} \|\partial_x A_s^{g,\varepsilon}\|_{L_\infty(\mathbb{T})}^2 \,\mathrm{d}s \int_0^t |Z_s^{g(u)} - Y_s^{\rho,M}(u)|^2\mathrm{d}s\mathrm{d}u\right]\\
&\leqslant C\int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |Z_s^{g(u)} - Y_s^{\rho,M}(u)|^2\mathrm{d}u\right]\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_2 &\leqslant C\int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \left|Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho,M}(u)\right|^2 \mathrm{d}u\right]\mathrm{d}s\\
&\quad + C\int_0^1\int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\left|Z_s^{g(u)} - Z_s^{g(u)+\rho a_s^M(u)}\right|^2\right]\mathrm{d}s\mathrm{d}u.
\end{aligned}$$

Moreover, by inequality (III.71), $|\rho a_t^M(u)| \leqslant \rho_0 C_M^a = \frac{1}{2}$ for every $t \in [0,T]$, $u \in \mathbb{R}$ and $\rho \in (-\rho_0, \rho_0)$. Fix $u \in [0,1]$. Let J_u be the interval $[g(u) - \frac{1}{2}, g(u) + \frac{1}{2}]$. By inequality (III.36), there is C independent of u such that for every $x, y \in J_u$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T} |Z_t^x - Z_t^y|^2\right] \leqslant C|x-y|^2.$$

It follows from Kolmogorov's Lemma (see [RY99, p.26, Thm I.2.1] and take therein $\gamma = 2$, $d = 1$ and $\varepsilon = 1$) that, up to considering a modification of the process $(Z_t^x)_{x\in J_u}$, there is a constant C_{Kol} independent of u such that

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{x,y\in J_u, x\neq y} \frac{\sup_{t\leqslant T}|Z_t^x - Z_t^y|^2}{|x-y|^{1/2}}\right] \leqslant C_{\mathrm{Kol}}. \qquad \text{(III.84)}$$

Thus, by (III.84), we deduce that for every $\rho \in (-\rho_0, \rho_0)$,

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\left[\left|Z_s^{g(u)} - Z_s^{g(u)+\rho a_s^M(u)}\right|^2\right]\\
&\qquad = \mathbb{E}^W\mathbb{E}^\beta\left[\mathbb{1}_{\{\rho a_t^M(u)\neq 0\}} \frac{\sup_{t\leqslant T}|Z_t^{g(u)} - Z_t^{g(u)+\rho a_t^M(u)}|^2}{|\rho a_t^M(u)|^{1/2}} |\rho a_t^M(u)|^{1/2}\right]\\
&\qquad \leqslant |\rho|^{1/2}(C_M^a)^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{x,y\in J_u, x\neq y} \frac{\sup_{t\leqslant T}|Z_t^x - Z_t^y|^2}{|x-y|^{1/2}}\right]\\
&\qquad \leqslant CC_{\mathrm{Kol}}|\rho|^{1/2},
\end{aligned}$$

where the constants are independent of s and u. We conclude that for every $\rho \in (-\rho_0, \rho_0)$

$$E_2 \leqslant C \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho,M}(u) \right|^2 \mathrm{d}u \right] \mathrm{d}s + C|\rho|^{1/2}. \tag{III.85}$$

Let us consider now E_3. By Definition (III.67) of τ_M^2, for every $s \leqslant \tau_M$, $\left\| \frac{1}{\partial_u x_s^g(\cdot)} \right\|_{L_\infty} \leqslant M$. Therefore, by (III.72),

$$\begin{aligned}
E_3 &\leqslant \|g'\|_{L_\infty} M \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| \int_0^{t\wedge\tau_M} \left| \partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)} \right| \, \|A_s^{g,\varepsilon}\|_{L_\infty} \mathrm{d}s \right|^2 \mathrm{d}u \right] \\
&\leqslant \|g'\|_{L_\infty} M \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \left| \partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)} \right|^2 \mathrm{d}s \int_0^{t\wedge\tau_M} \|A_s^{g,\varepsilon}\|_{L_\infty}^2 \mathrm{d}s \mathrm{d}u \right] \\
&\leqslant C \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^t \left| \partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)} \right|^2 \mathrm{d}s \mathrm{d}u \right]
\end{aligned}$$

By inequality (III.39) and the fact that f is of order $\alpha > \frac{5}{2}$, we can apply similarly Kolmogorov's Lemma, replacing Z by $\partial_x Z$. And we get for every $\rho \in (-\rho_0, \rho_0)$,

$$\mathbb{E}^W \mathbb{E}^\beta \left[\left| \partial_x Z_s^{g(u)+\rho a_s^M(u)} - \partial_x Z_s^{g(u)} \right|^2 \right] \leqslant C C_{\mathrm{Kol}} |\rho|^{1/2}.$$

Therefore, for every $\rho \in (-\rho_0, \rho_0)$, $E_3 \leqslant C|\rho|^{1/2}$.

Putting together the last inequality with (III.82), (III.83) and (III.85), we obtain for every $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 |Z_t^{g(u)+\rho a_t^M(u)} - Y_t^{\rho,M}(u)|^2 \mathrm{d}u \right] \\
&\qquad \leqslant C \int_0^t \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \left| Z_s^{g(u)+\rho a_s^M(u)} - Y_s^{\rho,M}(u) \right|^2 \mathrm{d}u \right] \mathrm{d}s + C|\rho|^{2+1/2}.
\end{aligned}$$

By Gronwall's inequality, the proof of Lemma III.34 is complete. □

Proof (Lemma III.35). Let us take the real part of equality (III.69) and apply this equality with $y = Y_s^{\rho,M}(u)$. Recall that $A_s^{g,\varepsilon}$ and f_k are real-valued. We obtain for every $M \geqslant M_0$, for every $u \in \mathbb{R}$, for every $\rho \in (-\rho_0(M), \rho_0(M))$ and for every $s \in [0, T]$,

$$\mathbb{1}_{\{s \leqslant \tau_M\}} A_s^{g,\varepsilon}(Y_s^{\rho,M}(u)) = \sum_{k\in\mathbb{Z}} f_k \Re\left(e^{-ikY_s^{\rho,M}(u)} \lambda_s^{k,M} \right).$$

It follows from the latter equality and from (III.76) that for every $t \in [0, T]$

$$Y_t^{\rho,M}(u) = g(u) + \sum_{k\in\mathbb{Z}} \int_0^t f_k \Re\left(e^{-ikY_s^{\rho,M}(u)} (\mathrm{d}W_s^k + \rho \lambda_s^{k,M} \mathrm{d}s) \right) + \beta_t.$$

Recall that for each $k \in \mathbb{Z}$ and each $s \in [0, T]$, $\lambda_s^{k,M}$ is complex-valued. We denote $\lambda_s^{k,M} = \lambda_s^{\Re,k,M} + i\lambda_s^{\Im,k,M}$. Let us define, for every $t \in [0, T]$,

$$\begin{aligned}
\mathcal{E}_t^\rho &= \exp\left(-\rho \sum_{k\in\mathbb{Z}} \int_0^t (\lambda_s^{\Re,k,M} \mathrm{d}W_s^{\Re,k} + \lambda_s^{\Im,k,M} \mathrm{d}W_s^{\Im,k}) - \frac{\rho^2}{2} \sum_{k\in\mathbb{Z}} \int_0^t |\lambda_s^{k,M}|^2 \mathrm{d}s \right) \\
&= \exp\left(-\rho \sum_{k\in\mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^{k,M}} \mathrm{d}W_s^k) - \frac{\rho^2}{2} \sum_{k\in\mathbb{Z}} \int_0^t |\lambda_s^{k,M}|^2 \mathrm{d}s \right).
\end{aligned}$$

Recall that by Lemma III.31, there is a constant $C_M > 0$ such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, $\sum_{k\in\mathbb{Z}} \int_0^T |\lambda_s^{k,M}|^2 \mathrm{d}s \leqslant C_M$. It follows from Novikov's condition that the process $(\mathcal{E}_t^\rho)_{t\in[0,T]}$ is a $\mathbb{P}^W \otimes \mathbb{P}^\beta$-martingale. Let $\mathbb{P}^\rho$ be the probability measure on $\Omega^W \times \Omega^\beta$ such that $\mathbb{P}^\rho$ is absolutely continuous with respect to $\mathbb{P}^W \otimes \mathbb{P}^\beta$ with the following density

$$\frac{\mathrm{d}\mathbb{P}^\rho}{\mathrm{d}(\mathbb{P}^W \otimes \mathbb{P}^\beta)} = \mathcal{E}_T^\rho.$$

By Girsanov's Theorem, $((W_t^k + \rho\lambda_t^{k,M})_{t\in[0,T]})_{k\in\mathbb{Z}}$ is a collection of independent Brownian motions under $\mathbb{P}^\rho$, independent of $(\beta, \mathcal{G}_0)$. By uniqueness in law of equation (III.20), the law of $(Y_t^{\rho,M})_{t\in[0,T]}$ under $\mathbb{P}^\rho$ is equal to the law of $(x_t^g)_{t\in[0,T]}$ under $\mathbb{P}^W \otimes \mathbb{P}^\beta$.

Fix $t \in [0,T]$. We deduce that

$$\mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Y_t^{\rho,M})\mathcal{E}_t^\rho\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Y_t^{\rho,M})\frac{\mathrm{d}\mathbb{P}^\rho}{\mathrm{d}(\mathbb{P}^W \otimes \mathbb{P}^\beta)}\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(x_t^g)\right].$$

Since $\mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(x_t^g)\right]$ does not depend on ρ, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Y_t^{\rho,M})\mathcal{E}_t^\rho\right] = 0. \tag{III.86}$$

Recall that by assumption $(\phi 2)$ and Remark III.7, $\widehat{\phi}$ is a Lipschitz-continuous function. By Lemma III.34, we have for every $\rho \in (-\rho_0, \rho_0)$

$$\begin{aligned}
&\left|\mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^{g+\rho a_t^M})\mathcal{E}_t^\rho\right] - \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Y_t^{\rho,M})\mathcal{E}_t^\rho\right]\right| \\
&\qquad\leqslant \mathbb{E}^W\left[|\widehat{\phi}(Z_t^{g+\rho a_t^M}) - \widehat{\phi}(Y_t^{\rho,M})|^2\right]^{1/2} \mathbb{E}^W\mathbb{E}^\beta\left[|\mathcal{E}_t^\rho|^2\right]^{1/2} \\
&\qquad\leqslant \|\widehat{\phi}\|_{\mathrm{Lip}}\mathbb{E}^W\left[\left\|Z_t^{g+\rho a_t^M} - Y_t^{\rho,M}\right\|^2_{L_2([0,1]\times\Omega^\beta)}\right]^{1/2} \mathbb{E}^W\mathbb{E}^\beta\left[|\mathcal{E}_t^\rho|^2\right]^{1/2} \\
&\qquad\leqslant C_M|\rho|^{5/4}\mathbb{E}^W\mathbb{E}^\beta\left[|\mathcal{E}_t^\rho|^2\right]^{1/2}.
\end{aligned}$$

Moreover, recalling that $\sum_{k\in\mathbb{Z}} \int_0^T |\lambda_s^{k,M}|^2 \mathrm{d}s \leqslant C_M$

$$\begin{aligned}
\mathbb{E}^W\mathbb{E}^\beta\left[|\mathcal{E}_t^\rho|^2\right] &= \mathbb{E}^W\mathbb{E}^\beta\left[\exp\left(-2\rho\textstyle\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k) - \rho^2\sum_{k\in\mathbb{Z}}\int_0^t |\lambda_s^{k,M}|^2\mathrm{d}s\right)\right] \\
&= \mathbb{E}^W\mathbb{E}^\beta\Big[\exp\left(-2\rho\textstyle\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k) - \frac{(2\rho)^2}{2}\sum_{k\in\mathbb{Z}}\int_0^t |\lambda_s^{k,M}|^2\mathrm{d}s\right) \\
&\qquad\qquad \cdot \exp\left(\rho^2\textstyle\sum_{k\in\mathbb{Z}}\int_0^t |\lambda_s^{k,M}|^2\mathrm{d}s\right)\Big] \\
&\leqslant e^{\rho_0^2 C_M}\ \mathbb{E}^W\mathbb{E}^\beta\left[\exp\left(-2\rho\textstyle\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k) - \frac{(2\rho)^2}{2}\sum_{k\in\mathbb{Z}}\int_0^t |\lambda_s^{k,M}|^2\mathrm{d}s\right)\right] \\
&= e^{\rho_0^2 C_M},
\end{aligned}$$

since the exponential on the last line is a $\mathbb{P}^W \otimes \mathbb{P}^\beta$-martingale. Therefore,

$$\left|\mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^{g+\rho a_t^M})\mathcal{E}_t^\rho\right] - \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Y_t^{\rho,M})\mathcal{E}_t^\rho\right]\right| \leqslant C_M|\rho|^{5/4}. \tag{III.87}$$

It follows from (III.86) and (III.87) that $\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0}\mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^{g+\rho a_t^M})\mathcal{E}_t^\rho\right] = 0$. Moreover, let us compute that derivative:

$$\frac{1}{\rho}\left(\mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^{g+\rho a_t^M})\mathcal{E}_t^\rho\right] - \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^g)\right]\right) = \mathbb{E}^W\mathbb{E}^\beta\left[\frac{\widehat{\phi}(Z_t^{g+\rho a_t^M}) - \widehat{\phi}(Z_t^g)}{\rho}\right] + \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^{g+\rho a_t^M})\frac{\mathcal{E}_t^\rho - 1}{\rho}\right].$$

It follows from (III.46) that

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0}\mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^{g+\rho a_t^M})\mathcal{E}_t^\rho\right] \\
&= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(Z_t^g)_u\,\partial_x Z_t^{g(u)} a_t^M(u)\mathrm{d}u\right] - \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{\phi}(Z_t^g)\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k)\right] \\
&= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{g'(u)}a_t^M(u)\mathrm{d}u\right] - \mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k)\right].
\end{aligned}$$

Therefore, equality (III.78) holds true. □

Proof (Lemma III.36). Recall that for every $t\in[0,T]$, $\lambda_t^{k,M} = \mathbb{1}_{\{t\leqslant\tau_M\}}\lambda_t^k$. In order to obtain (III.79), we pass to the limit when $M\to+\infty$ in (III.78). Let us prove this convergence.

By (III.68), $\mathbb{P}^W\otimes\mathbb{P}^\beta[\tau_M < T] \to_{M\to+\infty} 0$. Since $\{\tau_M < T\}_{M\geqslant M_0}$ is a non-increasing sequence of events, it follows that $\mathbb{P}^W\otimes\mathbb{P}^\beta$-almost surely, there exists M_1 such that $\tau_M = T$ for every $M\geqslant M_1$. In order to obtain (III.79), it is sufficient to prove uniform integrability of both terms. Hence it remains to prove the following two statements

$$\sup_{M\geqslant M_0}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left(\partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{g'(u)}a_t^M(u)\right)^{3/2}\mathrm{d}u\right] < +\infty; \tag{III.88}$$

$$\sup_{M\geqslant M_0}\mathbb{E}^W\mathbb{E}^\beta\left[\left(\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t\Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k)\right)^2\right] < +\infty. \tag{III.89}$$

Let us first prove (III.88). For every $M\geqslant M_0$, by Hölder's inequality

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left(\partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{g'(u)}a_t^M(u)\right)^{3/2}\mathrm{d}u\right] \\
&\qquad\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left(\partial_\mu\phi(\mu_t^g)(x_t^g(u))\right)^2\mathrm{d}u\right]^{3/4}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left|\frac{\partial_u x_t^g(u)}{g'(u)}a_t^M(u)\right|^6\mathrm{d}u\right]^{1/4}.
\end{aligned}$$

By assumption $(\phi 2)$, $\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left(\partial_\mu\phi(\mu_t^g)(x_t^g(u))\right)^2\mathrm{d}u\right]$ is bounded. Moreover

$$\begin{aligned}
&\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left|\frac{\partial_u x_t^g(u)}{g'(u)}a_t^M(u)\right|^6\mathrm{d}u\right] \\
&\qquad\leqslant \left\|\frac{1}{g'}\right\|_{L_\infty}^6\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1|\partial_u x_t^g(u)|^{12}\mathrm{d}u\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left|a_t^M(u)\right|^{12}\mathrm{d}u\right]^{1/2} \\
&\qquad\leqslant C\left\|\frac{1}{g'}\right\|_{L_\infty}^6\|g'\|_{L_{12}}^6\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left|a_t^M(u)\right|^{12}\mathrm{d}u\right]^{1/2},
\end{aligned}$$

by inequality (III.24). By definition (III.70) of a_t^M, we have

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |a_t^M(u)|^{12}\mathrm{d}u\right] \leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 T^{11}\int_0^t \left|\frac{g'(u)}{\partial_u x_s^g(u)}A_s^{g,\varepsilon}(x_s^g(u))\right|^{12}\mathrm{d}s\mathrm{d}u\right].$$

Remark that for every $s\in[0,T]$, $\|A_s^{g,\varepsilon}\|_{L_\infty}\leqslant\|A_s^g\|_{L_\infty}\leqslant\|\partial_u x_s^g\|_{L_\infty}\|h\|_{L_\infty}$. Thus

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |a_t^M(u)|^{12}\mathrm{d}u\right] &\leqslant C\|g'\|_{L_\infty}^{12}\|h\|_{L_\infty}^{12}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\|\partial_u x_t^g\|_{L_\infty}^{12}\sup_{t\leqslant T}\left\|\tfrac{1}{\partial_u x_t^g(\cdot)}\right\|_{L_\infty}^{12}\right]\\ &\leqslant C\|g'\|_{L_\infty}^{12}\|h\|_{L_\infty}^{12}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\|\partial_u x_t^g\|_{L_\infty}^{24}\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\left\|\tfrac{1}{\partial_u x_t^g(\cdot)}\right\|_{L_\infty}^{24}\right]^{1/2}\leqslant C,\end{aligned}$$

where the constant C does not depend on M. The last inequality is obtained by inequalities (III.33) and (III.34), because $g\in\mathbf{G}^{2+\theta}$ and $\alpha>\frac{5}{2}+\theta$. We deduce (III.88).

Furthermore, for every $M\geqslant M_0$, for every $t\in[0,T]$, we have

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\left(\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^t \Re(\overline{\lambda_s^{k,M}}\mathrm{d}W_s^k)\right)^2\right] &\leqslant \|\phi\|_{L_\infty}^2\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t|\lambda_s^{k,M}|^2\mathrm{d}s\right]\\ &\leqslant \|\phi\|_{L_\infty}^2\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t|\lambda_s^{k}|^2\mathrm{d}s\right],\end{aligned}\tag{III.90}$$

since $\lambda_s^{k,M}=\mathbb{1}_{\{s\leqslant\tau_M\}}\lambda_s^k$. By inequality (III.65), we have

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t|\lambda_s^k|^2\mathrm{d}s\right]\leqslant\frac{C}{\varepsilon^{6+4\theta}}\int_0^t\mathbb{E}^W\mathbb{E}^\beta\left[\|A_s^g\|_{\mathcal{C}^1}^2\right]\mathrm{d}s.$$

Recall that $\|A_s^g\|_{L_\infty}\leqslant\|\partial_u x_s^g\|_{L_\infty}\|h\|_{L_\infty}$. Thus for every $s\in[0,T]$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|A_s^g\|_{L_\infty}^2\right]\leqslant\|h\|_{L_\infty}^2\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\|\partial_u x_t^g\|_{L_\infty}^2\right]\leqslant C\|h\|_{L_\infty}^2(1+\|g''\|_{L_2}^2+\|g'\|_{L_\infty}^4),$$

by inequality (III.33). Moreover, let us compute the derivative of A_s^g:

$$\partial_x A_s^g(x)=\left(\partial_u h\partial_u x_s^g+h\partial_u^{(2)}x_s^g\right)(F_s^g(x))\frac{1}{\partial_u x_s^g(F_s^g(x))}=\partial_u h(F_s^g(x))+h(F_s^g(x))\frac{\partial_u^{(2)}x_s^g(F_s^g(x))}{\partial_u x_s^g(F_s^g(x))}.$$

We deduce that

$$\|\partial_x A_s^g\|_{L_\infty}\leqslant C\|h\|_{\mathcal{C}^1}\left(1+\|\partial_u^{(2)}x_s^g\|_{L_\infty}\left\|\frac{1}{\partial_u x_s^g}\right\|_{L_\infty}\right).\tag{III.91}$$

Therefore, for every $s\leqslant T$,

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[\|\partial_x A_s^g\|_{L_\infty}^2\right] &\leqslant C\|h\|_{\mathcal{C}^1}^2\left(1+\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\|\partial_u^{(2)}x_t^g\|_{L_\infty}^4\right]+\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{t\leqslant T}\left\|\frac{1}{\partial_u x_t^g}\right\|_{L_\infty}^4\right]\right)\\ &\leqslant C\|h\|_{\mathcal{C}^1}^2\left(1+\|g'''\|_{L_4}^4+\|g''\|_{L_\infty}^{12}+\|g'\|_{L_\infty}^{16}+\left\|\frac{1}{g'}\right\|_{L_\infty}^{16}\right),\end{aligned}\tag{III.92}$$

by (III.33) and (III.34) and because because g belongs to $\mathbf{G}^{3+\theta}$ and $\alpha=\frac{7}{2}+\theta$. We deduce that

$$\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t|\lambda_s^k|^2\mathrm{d}s\right]\leqslant\frac{Ct}{\varepsilon^{6+4\theta}}\|h\|_{\mathcal{C}^1}^2\left(1+\|g'''\|_{L_4}^4+\|g''\|_{L_\infty}^{12}+\|g'\|_{L_\infty}^{16}+\left\|\frac{1}{g'}\right\|_{L_\infty}^{16}\right),$$

which is inequality (III.80). It follows from (III.90) and (III.80) that inequality (III.89) holds. It completes the proof. □

III.4.3 Behaviour of the remainder term for small ε

Let $\theta \in (0,1)$, $g \in \mathbf{G}^{1+\theta}$ and f be of order $\alpha > \frac{3}{2} + \theta$. Recall that, on $(\Omega^W \times \Omega^\beta, \mathcal{G}^W \otimes \mathcal{G}^\beta)$, $(x_t^g(u))_{t\in[0,T],u\in\mathbb{R}}$ is the unique solution to:

$$x_t^g(u) = g(u) + \sum_{k\in\mathbb{Z}} \int_0^t f_k \Re\left(e^{-ikx_s^g(u)} \mathrm{d}W_s^k\right) + \beta_t. \tag{III.93}$$

Recall that for every $t \in [0,T]$, $\mu_t^g = \mathcal{L}_{[0,1]\times\Omega^\beta}(x_t^g)$ is a random measure. Let $h \in \Delta^1$ and recall the definitions of A_t^g and $A_t^{g,\varepsilon}$:

$$\begin{aligned} A_t^g &:= \partial_u x_t^g(F_t^g(\cdot))h(F_t^g(\cdot)); \\ A_t^{g,\varepsilon} &:= A_t^g * \varphi_\varepsilon. \end{aligned}$$

Let us fix $\varepsilon > 0$ and let us define

$$H_t^{g,\varepsilon}(u) := \frac{1}{\partial_u x_t^g(u)}(A_t^g - A_t^{g,\varepsilon})(x_t^g(u)). \tag{III.94}$$

Let us define the canonical space $(\Theta, \mathcal{B}(\Theta))$ by $\Theta = \mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}} \times \mathcal{C}([0,T],\mathbb{R})$ and $\mathcal{B}(\Theta) = \mathcal{B}(\mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}}) \otimes \mathcal{B}(\mathcal{C}([0,T],\mathbb{R}))$. Let $\mathbf{P}$ be the probability measure on $(\Theta, \mathcal{B}(\Theta))$ defined as being the distribution of $((W^k)_{k\in\mathbb{Z}}, \beta)$ on $\Omega^W \times \Omega^\beta$. Let $\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{R})) := \sigma(\mathbf{x}(s); 0 \leqslant s \leqslant t)$; in other words $(\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{R})))_{t\in[0,T]}$ is the canonical filtration on $(\mathcal{C}([0,T],\mathbb{R}), \mathcal{B}(\mathcal{C}([0,T],\mathbb{R})))$. Similarly, let $(\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}}))_{t\in[0,T]}$ be the canonical filtration on $(\mathcal{C}([0,T],\mathbb{C})^Z, \mathcal{B}(\mathcal{C}([0,T],\mathbb{C})^Z))$. Let $(\widehat{\mathcal{B}}_t(\Theta))_{t\in[0,T]}$ be the augmentation of the filtration $(\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}}) \otimes \mathcal{B}_t(\mathcal{C}([0,T],\mathbb{R})))_{t\in[0,T]}$ by the null sets of $\mathbf{P}$. We used here almost the same notations as Karatzas-Shreve [KS91, pp.308-311].

Notation: we denote in bold elements of Θ, e.g. $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in \Theta$.

We want to prove that

(*a*) There is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\Theta)/\mathcal{B}(\mathcal{C}([0,T],\mathbb{R}))$-measurable function

$$\begin{aligned} \mathcal{X} : \mathbb{R} \times \Theta &\to \mathcal{C}([0,T],\mathbb{R}) \\ (u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \end{aligned}$$

which is, for every fixed $t \in [0,T]$, $\mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{R}))$-measurable, such that $\mathcal{X}$ is continuous in u for $\mathbf{P}$-almost every fixed $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in \Theta$ and such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0,T]$,

$$x_t^g(u) = \mathcal{X}_t(u, (W^k)_{k\in\mathbb{Z}}, \beta). \tag{III.95}$$

(*b*) There is a $\mathcal{B}(\mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}})/\mathcal{B}(\mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R})))$-measurable function

$$\begin{aligned} \mathcal{P} : \mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}} &\to \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R})) \\ (\mathbf{w}^k)_{k\in\mathbb{Z}} &\mapsto \mathcal{P}((\mathbf{w}^k)_{k\in\mathbb{Z}}) \end{aligned}$$

which is, for every fixed $t \in [0,T]$, $\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}})/\mathcal{B}_t(\mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R})))$-measurable, such that $\mathbb{P}^W$-almost surely, for every $t \in [0,T]$,

$$\mu_t^g = \mathcal{P}_t((W^k)_{k\in\mathbb{Z}}). \tag{III.96}$$

(*c*) There is a progressively-measurable function $\mathcal{H} : [0,T] \times \mathbb{R} \times \Theta \to \mathbb{R}$, *i.e.* for every $t \in [0,T]$,

$$\begin{aligned} [0,t] \times \mathbb{R} \times \Theta &\to \mathbb{R} \\ (s, u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{H}_s(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \end{aligned}$$

is $\mathcal{B}[0,t] \otimes \mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$-measurable, such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0,T]$,

$$H_t^{g,\varepsilon}(u) = \mathcal{H}_t(u, (W^k)_{k\in\mathbb{Z}}, \beta). \tag{III.97}$$

Proposition III.37. *Let $\theta \in (0,1)$, $g \in \mathbf{G}^{1+\theta}$ and f be of order $\alpha > \frac{3}{2} + \theta$. Let $h \in \Delta^1$ and $\varepsilon > 0$. Then there are three functions $\mathcal{X}$, $\mathcal{P}$ and $\mathcal{H}$ such that statements (a), (b) and (c) defined above are satisfied.*

Proof. Let us consider the canonical space $(\Theta, \mathcal{B}(\Theta), (\widehat{\mathcal{B}}_t(\Theta))_{t\in[0,T]}, \mathbf{P})$. By Proposition III.12, there is a strong solution to (III.93) and pathwise uniqueness holds for that equation. Therefore, for every fixed $u \in \mathbb{R}$, there is a unique solution $(\mathbf{x}_t^g(u))_{t\in[0,T]}$ to

$$\mathbf{x}_t^g(u) = g(u) + \sum_{k\in\mathbb{Z}} \int_0^t f_k \Re\left(e^{-ik\mathbf{x}_s^g(u)} \mathrm{d}\mathbf{w}_s^k\right) + \mathbf{b}_t.$$

Proof of (*a*)**.** By Yamada-Watanabe Theorem, the law of $(x^g, (W^k)_{k\in\mathbb{Z}}, \beta)$ under $\mathbb{P}^W \otimes \mathbb{P}^\beta$ is equal to the law of $(\mathbf{x}^g, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b})$ under $\mathbf{P}$. This result is proved in [KS91, Proposition 5.3.20, p.309] for a finite-dimensional noise, but the proof is the same for the infinite-dimensional noise $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in \Theta$.

By a corollary to this theorem [KS91, Corollary 5.3.23, p.310], it follows that for every $u \in \mathbb{Q}$, there is a $\mathcal{B}(\Theta)/\mathcal{B}(\mathcal{C}([0,T],\mathbb{R}))$-measurable function

$$\begin{aligned} \mathcal{X}^u : \Theta &\to \mathcal{C}([0,T],\mathbb{R}) \\ ((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}^u((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \end{aligned}$$

which is, for every fixed $t \in [0,T]$, $\widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{R}))$-measurable, such that $\mathbf{P}$-almost surely, for every $t \in [0,T]$,

$$\mathbf{x}_t^g(u) = \mathcal{X}_t^u((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}). \tag{III.98}$$

Moreover, again by Proposition III.12, there is an event $A \in \mathcal{B}(\Theta)$ of probability $\mathbf{P}[A] = 1$ such that for every $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in A$, the function $(t,u) \mapsto \mathbf{x}_t^g(u)$ is continuous on $[0,T] \times \mathbb{R}$. Up to modifying the almost-sure event A, we may assume that for every $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in A$ and for every $u \in \mathbb{Q}$, equality (III.98) holds. Therefore, we can define a continuous function in the variable $u \in \mathbb{R}$ by extending $u \in \mathbb{Q} \mapsto \mathcal{X}^u$. More precisely, define for every $u \in \mathbb{R}$, $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in \Theta$,

$$\mathcal{X}(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) = \begin{cases} \lim\limits_{\substack{u_n \to u \\ (u_n)_n \in \mathbb{Q}^{\mathbb{N}}}} \mathcal{X}^{u_n}((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) & \text{if } ((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

In the latter definition, the limit exists and for every $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in A$, $\mathcal{X}_t(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) = \mathbf{x}_t^g(u)$ for every $t \in [0,T]$ and for every $u \in \mathbb{R}$. By construction, for every $((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in \Theta$, $u \in$

$\mathbb{R} \mapsto \mathcal{X}(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) \in \mathcal{C}([0,T], \mathbb{R})$ is continuous. It remains to show that $\mathcal{X}$ is progressively-measurable. Let us fix $t \in [0,T]$. By construction of $\mathcal{X}^u$, we know that for every $u \in \mathbb{Q}$,

$$\begin{aligned}[0,t] \times \Theta &\to \mathbb{R}\\ (s, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}_s^u((\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b})\end{aligned}$$

is $\mathcal{B}[0,t] \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$-measurable. Since $\mathcal{X}$ is the limit of $\mathcal{X}_n := \sum_{k\in\mathbb{Z}} \mathcal{X}^{k/n} \mathbb{1}_{\{u\in[\frac{k}{n}, \frac{k+1}{n})\}}$, we deduce that for every $t \in [0,T]$,

$$\begin{aligned}[0,t] \times \mathbb{R} \times \Theta &\to \mathbb{R}\\ (s, u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) &\mapsto \mathcal{X}_s(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b})\end{aligned}$$

is $\mathcal{B}[0,t] \otimes \mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$-measurable.

Recall that $\mathcal{L}^{\mathbb{P}^W \otimes \mathbb{P}^\beta}(x^g, (W^k)_{k\in\mathbb{Z}}, \beta) = \mathcal{L}^{\mathbf{P}}(\mathbf{x}^g, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b})$. Since $\mathbb{P}$-almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0,T]$, $\mathbf{x}_t^g(u) = \mathcal{X}_t(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b})$, we deduce that $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0,T]$, equality (III.95) holds. It completes the proof of (a).

Proof of (b)**.** Proving (b) is equivalent to find $\mathcal{P} : \mathcal{C}([0,T], \mathbb{C})^{\mathbb{Z}} \to \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}))$ such that for every bounded measurable function $\Upsilon : \mathbb{R} \to \mathbb{R}$, the function

$$\begin{aligned}\langle \Upsilon, \mathcal{P}\rangle : \mathcal{C}([0,T], \mathbb{C})^{\mathbb{Z}} &\to \mathcal{C}([0,T], \mathbb{R})\\ (\mathbf{w}^k)_{k\in\mathbb{Z}} &\mapsto \langle \Upsilon, \mathcal{P}((\mathbf{w}^k)_{k\in\mathbb{Z}})\rangle = \int_{\mathbb{R}} \Upsilon(x) \mathrm{d}\mathcal{P}((\mathbf{w}^k)_{k\in\mathbb{Z}})(x)\end{aligned}$$

is, for every fixed $t \in [0,T]$, $\mathcal{B}_t(\mathcal{C}([0,T], \mathbb{C})^{\mathbb{Z}})/\mathcal{B}_t(\mathcal{C}([0,T], \mathbb{R}))$-measurable.

Let us fix a bounded measurable function $\Upsilon : \mathbb{R} \to \mathbb{R}$. Then, by Definition III.24, $\mathbb{P}^W$-almost surely, for every $t \in [0,T]$,

$$\langle \Upsilon, \mu_t^g\rangle := \int_{\mathbb{R}} \Upsilon(x) \mu_t^g(\mathrm{d}x) = \mathbb{E}^\beta \left[\int_0^1 \Upsilon(x_t^g(v)) \mathrm{d}v\right] = \mathbb{E}^\beta \left[\int_0^1 \Upsilon(\mathcal{X}_t(v, (W^k)_{k\in\mathbb{Z}}, \beta)) \mathrm{d}v\right].$$

Let us define $\mathcal{P}$ by duality: for every $\Upsilon : \mathbb{R} \to \mathbb{R}$ bounded and measurable,

$$\langle \Upsilon, \mathcal{P}((\mathbf{w}^k)_{k\in\mathbb{Z}})\rangle := \int_{\mathcal{C}([0,T],\mathbb{R})} \int_0^1 \Upsilon(\mathcal{X}(v, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}))\, \mathrm{d}v\, \mathrm{d}\mu_{\text{Wiener}}(\mathbf{b}).$$

Thus $\mathbb{P}^W$-almost surely, for every $t \in [0,T]$,

$$\langle \Upsilon, \mathcal{P}((W^k)_{k\in\mathbb{Z}})\rangle = \mathbb{E}^\beta \left[\int_0^1 \Upsilon(\mathcal{X}(v, (W^k)_{k\in\mathbb{Z}}, \beta)) \mathrm{d}v\right] = \langle \Upsilon, \mu_t^g\rangle.$$

Equality (III.96) follows.

Moreover, for every $t \in [0,T]$, by composition of two measurable functions,

$$\begin{aligned}[0,t] \times \mathbb{R} \times \Theta &\to \mathbb{R}\\ (s, u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}) &\mapsto \Upsilon(\mathcal{X}_s(u, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}))\end{aligned}$$

is $\mathcal{B}[0,t] \otimes \mathcal{B}(\mathbb{R}) \otimes \widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$-measurable. By Fubini's Theorem, it follows that for every $t \in [0,T]$,

$$\begin{aligned}[0,t] \times \mathcal{C}([0,T], \mathbb{C})^{\mathbb{Z}} &\to \mathbb{R}\\ (s, (\mathbf{w}^k)_{k\in\mathbb{Z}}) &\mapsto \int_{\mathcal{C}([0,T],\mathbb{R})} \int_0^1 \Upsilon(\mathcal{X}_s(v, (\mathbf{w}^k)_{k\in\mathbb{Z}}, \mathbf{b}))\, \mathrm{d}v\, \mathrm{d}\mu_{\text{Wiener}}(\mathbf{b}),\end{aligned}$$

is $\mathcal{B}[0,t]\otimes\mathcal{B}_t(\mathcal{C}([0,T],\mathbb{C})^{\mathbb{Z}})/\mathcal{B}(\mathbb{R})$-measurable, where μ_{Wiener} denotes the Wiener measure on $\mathcal{C}([0,T],\mathbb{R})$. This completes the proof of (b).

Proof of (c)**.** Let us define, on the canonical space $(\Theta,\mathcal{B}(\Theta))$, $\mathbf{F}^g_t=(\mathbf{x}^g_t)^{-1}$ and

$$\begin{aligned}\mathbf{A}^g_t &:= \partial_u\mathbf{x}^g_t(\mathbf{F}^g_t(\cdot))h(\mathbf{F}^g_t(\cdot));\\ \mathbf{A}^{g,\varepsilon}_t &:= \int_{\mathbb{R}}\mathbf{A}^g_t(\cdot-y)\varphi_\varepsilon(y)\mathrm{d}y;\\ \mathbf{H}^{g,\varepsilon}_t(u) &:= \frac{1}{\partial_u\mathbf{x}^g_t(u)}(\mathbf{A}^g_t-\mathbf{A}^{g,\varepsilon}_t)(\mathbf{x}^g_t(u)).\end{aligned}$$

In order to prove that $\mathbf{H}^{g,\varepsilon}$ can be written as a progressively measurable function of u and $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$, we will prove successively that this property holds for $\partial_u\mathbf{x}^g$, $\mathbf{F}^g$, $\mathbf{A}^g$ and $\mathbf{A}^{g,\varepsilon}$ and we will deduce the result for $\mathbf{H}^{g,\varepsilon}$ by composition of progressively measurable functions.

Let us start with $\partial_u\mathbf{x}^g$. By Proposition III.13, since $g\in\mathbf{G}^{1+\theta}$ and $\alpha>\frac{3}{2}+\theta$, $\mathbf{P}$-almost surely, for every $t\in[0,T]$, the map $u\mapsto\mathbf{x}^g_t(u)$ is of class $\mathcal{C}^1$. Thus there exists a $\mathbf{P}$-almost-sure event $A\in\mathcal{B}(\theta)$ such that for every $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\in A$, $\mathbf{x}^g_t(u)=\mathcal{X}_t(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$ holds with every $(t,u)\in[0,T]\times\mathbb{R}$ and $u\mapsto x^g_t(u)$ belongs to $\mathcal{C}^1$. Let us define for every $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\in A$, for every $(t,u)\in[0,T]\times\mathbb{R}$,

$$\partial_u\mathcal{X}_t(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b}) := \limsup_{\eta\searrow0}\frac{\mathcal{X}_t(u+\eta,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})-\mathcal{X}_t(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})}{\eta}$$

Thus for every $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\in A$ and for every $(t,u)\in[0,T]\times\mathbb{R}$, $\partial_u\mathbf{x}^g_t(u)=\partial_u\mathcal{X}_t(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$. If $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\notin A$, then define $\partial_u\mathcal{X}_t(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})=0$ for every $(t,u)\in[0,T]\times\mathbb{R}$. Moreover, by progressively-measurability of $\mathcal{X}$, it follows from the definition of $\partial_u\mathcal{X}$ is also progressively measurable; more precisely, for every $t\in[0,T]$,

$$\begin{aligned}[0,t]\times\mathbb{R}\times\Theta&\to\mathbb{R}\\ (s,u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})&\mapsto\partial_u\mathcal{X}_s(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\end{aligned}$$

is $\mathcal{B}[0,t]\otimes\mathcal{B}(\mathbb{R})\otimes\widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$-measurable.

Let us proceed with $\mathbf{F}^g$. Let us define for every $x\in[0,2\pi]$

$$\widetilde{\mathbf{F}}^g_t(x) := \int_0^1\mathbb{1}_{\{\mathbf{x}^g_t(v)-\mathbf{x}^g_t(0)\leqslant x\}}\mathrm{d}v. \tag{III.99}$$

Thus we have for every $x\in[\mathbf{x}_t(0),\mathbf{x}_t(0)+2\pi]$

$$\widetilde{\mathbf{F}}^g_t(x-\mathbf{x}^g_t(0)) = \int_0^1\mathbb{1}_{\{\mathbf{x}^g_t(v)\leqslant x\}}\mathrm{d}v = \int_0^1\mathbb{1}_{\{v\leqslant\mathbf{F}^g_t(x)\}}\mathrm{d}v = \mathbf{F}^g_t(x).$$

Therefore, since for every $x\in\mathbb{R}$, $\mathbf{F}^g_t(x+2\pi)=\mathbf{F}^g_t(x)+1$, we have

$$\mathbf{F}^g_t(x)=\sum_{k\in\mathbb{Z}}\mathbb{1}_{\{x-2\pi k\in[\mathbf{x}_t(0),\mathbf{x}_t(0)+2\pi)\}}\left(\widetilde{\mathbf{F}}^g_t(x-2\pi k-\mathbf{x}^g_t(0))+k\right).$$

Hence it is sufficient to prove that we can write $\widetilde{\mathbf{F}}^g_t$ as a progressively measurable function of x and $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$. Recall that $\mathbf{P}$-almost surely, $u\mapsto\mathbf{x}^g(u)=\mathcal{X}(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$ is continuous. Thus there is $\mathcal{I}$ such that $\mathbf{P}$-almost surely, for every $v\in[0,1]$, for every $x\in[0,2\pi]$, $\mathbb{1}_{\{\mathbf{x}^g(v)-\mathbf{x}^g(0)\leqslant x\}}=\mathcal{I}_\cdot(v,x,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$ and such that for every $t\in[0,T]$,

$$\begin{aligned}[0,t]\times[0,1]\times[0,2\pi]\times\Theta&\to\mathbb{R}\\ (s,v,x,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})&\mapsto\mathcal{I}_s(v,x,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\end{aligned}$$

is $\mathcal{B}[0,t]\otimes\mathcal{B}([0,1]\times[0,2\pi])\otimes\widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$-measurable. It follows from Fubini's Theorem and from (III.99) that for every $t\in[0,T]$,

$$[0,t]\times[0,2\pi]\times\Theta\to\mathbb{R}$$
$$(s,x,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\mapsto\int_0^1\mathcal{I}_s(v,x,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})\mathrm{d}v=\widetilde{\mathbf{F}}^g_s(x)$$

is $\mathcal{B}[0,t]\otimes\mathcal{B}([0,2\pi])\otimes\widehat{\mathcal{B}}_t(\Theta)/\mathcal{B}(\mathbb{R})$-measurable.

Let us conclude with $\mathbf{A}^g$, $\mathbf{A}^{g,\varepsilon}$ and $\mathbf{H}^{g,\varepsilon}$. First, remark that $\mathbf{A}^g$ is obtained by product and compositions of $\partial_u\mathbf{x}^g$, $\mathbf{F}^g$ and h, where h is a $\mathcal{C}^1$-function. Thus $x\mapsto\mathbf{A}^g(x)$ is a progressively measurable function of x and $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$. It follows also that $(x,y)\mapsto\mathbf{A}^g(x-y)\varphi_\varepsilon(y)$ is a progressively measurable function of x, y and $(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b}$. By Fubini's Theorem, it follows that $x\mapsto\mathbf{A}^{g,\varepsilon}(x)$ is a progressively measurable function of x and $((\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b})$. Again by products and compositions, it follows that there is a progressively measurable function $\mathcal{H}$ such that $\mathbf{P}$-almost surely, for every $u\in\mathbb{R}$ and for every $t\in[0,T]$,

$$\mathbf{H}^{g,\varepsilon}_t(u)=\mathcal{H}_t(u,(\mathbf{w}^k)_{k\in\mathbb{Z}},\mathbf{b}).$$

It follows that $\mathbf{P}^W\otimes\mathbf{P}^\beta$-almost surely, equality (III.97) holds. It completes the proof of (c) and of the proposition. □

By definition (III.54) of the notation $\left[\frac{\delta\phi}{\delta m}\right]$, we have for every $v\in\mathbb{R}$:

$$\left[\frac{\delta\phi}{\delta m}\right](\mu^g_t)(v)=\frac{\delta\phi}{\delta m}(\mu^g_t)(v)-\mathbb{E}^\beta\left[\int_0^1\frac{\delta\phi}{\delta m}(\mu^g_t)(x^g_t(u))\mathrm{d}u\right].$$

Let us prove that for every $t\in[0,T]$, $g\in\mathbf{G}^1$ and $p\geqslant1$, there is $C_p>0$ such that for every $v\in\mathbb{R}$,

$$\mathbb{E}^W\left[\left|\left[\frac{\delta\phi}{\delta m}\right](\mu^g_t)(v)\right|^p\right]\leqslant C_p; \tag{III.100}$$
$$\mathbb{E}^W\left[\left|\partial_v\left\{\left[\frac{\delta\phi}{\delta m}\right](\mu^g_t)\right\}(v)\right|^p\right]\leqslant C_p. \tag{III.101}$$

By inequality (III.16), there is a $C>0$ such that $\mathbb{P}^W$-almost surely for every $x\in[0,2\pi]$,

$$\left|\partial_v\left\{\left[\frac{\delta\phi}{\delta m}\right](\mu^g_t)\right\}(x)\right|=\left|\partial_v\left\{\frac{\delta\phi}{\delta m}(\mu^g_t)\right\}(x)\right|=|\partial_\mu\phi(\mu^g_t)(x)|$$
$$\leqslant C(1+2\pi)+C\mathbb{E}^\beta\left[\int_0^1|x^g_t(u')|\mathrm{d}u'\right]. \tag{III.102}$$

By Proposition III.9, $x\mapsto\partial_\mu\phi(\mu^g_t)(x)$ is 2π-periodic, thus inequality (III.102) holds for every $x\in\mathbb{R}$. Moreover, for every $p\geqslant1$, $\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1|x^g_t(u')|^p\mathrm{d}u'\right]$ is bounded by a constant depending on g, t and p. This implies inequality (III.101).

By integrating inequality (III.102) for x between v and v', we deduce that there is $C>0$ such that $\mathbb{P}^W$-almost surely, for every $v,v'\in[0,2\pi]$,

$$\left|\frac{\delta\phi}{\delta m}(\mu^g_t)(v)-\frac{\delta\phi}{\delta m}(\mu^g_t)(v')\right|\leqslant C+C\mathbb{E}^\beta\left[\int_0^1|x^g_t(u')|\mathrm{d}u'\right].$$

By Proposition III.11, $v\mapsto\frac{\delta\phi}{\delta m}(\mu^g_t)(v)$ is 2π-periodic, thus the latter inequality holds with every $v,v'\in\mathbb{R}$. We deduce that there is $C>0$ such that $\mathbb{P}^W$-almost surely for every $v\in\mathbb{R}$,

$$\left|\left[\frac{\delta\phi}{\delta m}\right](\mu^g_t)(v)\right|\leqslant C+C\mathbb{E}^\beta\left[\int_0^1|x^g_t(u')|\mathrm{d}u'\right].$$

Inequality (III.100) follows as previously.

Let us state the following Lemma.

Lemma III.38. *Let $\theta \in (0,1)$, $g \in \mathbf{G}^{1+\theta}$ and f be of order $\alpha > \frac{3}{2} + \theta$. Let $h \in \Delta^1$ and $\varepsilon > 0$. Fix $u \in [0,1]$ and $s < t \in [0,T]$. Thus the following equality holds true*

$$\mathbb{E}^W \mathbb{E}^\beta \left[\partial_u \left\{ \left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(\cdot)) \right\} (u)\ H_s^{g,\varepsilon}(u) \right] = \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(x_t^g(u))\ H_s^{g,\varepsilon}(u)\ \frac{1}{t-s} \int_s^t \partial_u x_r^g(u) \mathrm{d}\beta_r \right]. \quad \text{(III.103)}$$

Proof. Let $u \in [0,1]$ and $s < t \in [0,T]$. Let us define, for every $r \in [0,T]$, $\xi_r := \frac{1}{t-s} \int_0^r \mathbb{1}_{\{z \in [s,t]\}} \mathrm{d}z$.

For every $\nu \in [-1,1]$, denote by $(x_r^\nu)_{r \in [0,T]}$ the process $(x_r^{g+\nu\xi_r})_{r \in [0,T]}$. By (III.40), $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, $x_r^\nu(u) = x_r^{g+\nu\xi_r}(u) = Z_r^{g(u)+\nu\xi_r}$, where ξ_r is independent of u. Let us apply (III.42) to $x = g(u)$ and $\zeta_t = \xi_t$. We obtain for every $u \in \mathbb{R}$, $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely for every $r \in [0,T]$ and every $\nu \in [-1,1]$:

$$x_r^\nu(u) = g(u) + \sum_{k \in \mathbb{Z}} f_k \int_0^r \Re \left(e^{-ikx_z^\nu(u)} \mathrm{d}W_z^k \right) + \beta_r + \nu \int_0^r \partial_x Z_z^{g(u)+\nu\xi_z} \dot{\xi}_z \mathrm{d}z.$$

Since both terms of the last equality are almost surely continuous with respect to $u \in \mathbb{R}$, that equality holds almost surely for every $u \in \mathbb{R}$.

For every $\nu \in [-1,1]$, us define the following stopping time

$$\sigma^\nu := \inf \left\{ r \geqslant 0 : \left| \nu \int_0^r \partial_x Z_z^{g(u)+\nu\xi_z} \dot{\xi}_z \mathrm{d}\beta_z \right| \geqslant 1 \right\} \wedge T.$$

Let us define the process $(y_r^\nu)_{r \in [0,T]}$ solution to the SDE:

$$\mathrm{d}y_r^\nu(u) = \sum_{k \in \mathbb{Z}} f_k \Re \left(e^{-iky_r^\nu(u)} \mathrm{d}W_r^k \right) + \mathrm{d}\beta_r + \nu \mathbb{1}_{\{r \leqslant \sigma^\nu\}} \partial_x Z_r^{g(u)+\nu\xi_r} \dot{\xi}_r \mathrm{d}r.$$

Remark that the processes $(x_r^\nu)_{r \in [0,T]}$ and $(y_r^\nu)_{r \in [0,T]}$ are equal until σ^ν. In particular, they are equal under the event $\{\sigma^\nu = T\}$. Let us define for every $r \in [0,T]$

$$\mathcal{E}_r^\nu = \exp \left(-\nu \int_0^{r \wedge \sigma^\nu} \partial_x Z_z^{g(u)+\nu\xi_z} \dot{\xi}_z \mathrm{d}\beta_z - \frac{\nu^2}{2} \int_0^{r \wedge \sigma^\nu} \left| \partial_x Z_z^{g(u)+\nu\xi_z} \dot{\xi}_z \right|^2 \mathrm{d}z \right).$$

By definition of σ^ν, we have $\mathcal{E}_r^\nu \leqslant \exp\left(-\nu \int_0^{r \wedge \sigma^\nu} \partial_x Z_z^{g(u)+\nu\xi_z} \dot{\xi}_z \mathrm{d}\beta_z \right) \leqslant \exp(1)$. In particular, $(\mathcal{E}_r^\nu)_{r \in [0,T]}$ is a $\mathbb{P}^W \otimes \mathbb{P}^\beta$-martingale. Let us define $\mathbb{P}^\nu$ as the absolutely continuous probability measure with respect to $\mathbb{P}^W \otimes \mathbb{P}^\beta$ with density $\frac{\mathrm{d}\mathbb{P}^\nu}{\mathrm{d}(\mathbb{P}^W \otimes \mathbb{P}^\beta)} = \mathcal{E}_T^\nu$. Thus by Girsanov's Theorem, the law under $\mathbb{P}^\nu$ of $((W^k)_{k \in \mathbb{Z}}, \beta^\nu)$ is equal to the law under $\mathbb{P}^W \otimes \mathbb{P}^\beta$ of $((W^k)_{k \in \mathbb{Z}}, \beta)$. It follows that $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0,T]}, \mathbb{P}^\nu, y^\nu, (W^k)_{k \in \mathbb{Z}}, \beta^\nu)$ is a weak solution to (III.93).

By Yamada-Watanabe Theorem (see the proof of (a) in Proposition III.37),

$$\mathcal{L}^{\mathbb{P}^\nu}(y^\nu, (W^k)_{k \in \mathbb{Z}}, \beta^\nu) = \mathcal{L}^{\mathbb{P}^W \otimes \mathbb{P}^\beta}(x^g, (W^k)_{k \in \mathbb{Z}}, \beta) = \mathcal{L}^{\mathbf{P}}(\mathbf{x}^g, (\mathbf{w}^k)_{k \in \mathbb{Z}}, \mathbf{b}),$$

and $\mathbb{P}^\nu$-almost surely, for every $u \in \mathbb{R}$ and for every $t \in [0,T]$,

$$y_t^\nu(u) = \mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^\nu). \quad \text{(III.104)}$$

Thus it follows from (III.104) that

$$\mathbb{E}^\nu \left[\left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(y_t^\nu(u)) H_s^{g,\varepsilon}(u) \right] = \mathbb{E}^\nu \left[\left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^\nu)) H_s^{g,\varepsilon}(u) \right] = \mathbb{E}^W \mathbb{E}^\beta \left[\left[\frac{\delta \phi}{\delta m} \right] (\mu_t^g)(\mathcal{X}_t(u, (W^k)_{k \in \mathbb{Z}}, \beta^\nu)) H_s^{g,\varepsilon}(u) \frac{\mathrm{d}\mathbb{P}^\nu}{\mathrm{d}(\mathbb{P}^W \otimes \mathbb{P}^\beta)} \right].$$

Furthermore, by Proposition III.37,

$$\begin{aligned}
&\mathbb{E}^\nu\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(y_t^\nu(u))H_s^{g,\varepsilon}(u)\right]\\
&\quad=\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right]\left(\mathcal{P}_t((W^k)_{k\in\mathbb{Z}})\right)\left(\mathcal{X}_t(u,(W^k)_{k\in\mathbb{Z}},\beta^\nu)\right)\mathcal{H}_s(u,(W^k)_{k\in\mathbb{Z}},\beta)\,\frac{\mathrm{d}\mathbb{P}^\nu}{\mathrm{d}(\mathbb{P}^W\otimes\mathbb{P}^\beta)}\right]\\
&\quad=\mathbb{E}^\nu\left[\left[\frac{\delta\phi}{\delta m}\right]\left(\mathcal{P}_t((W^k)_{k\in\mathbb{Z}})\right)\left(\mathcal{X}_t(u,(W^k)_{k\in\mathbb{Z}},\beta^\nu)\right)\mathcal{H}_s(u,(W^k)_{k\in\mathbb{Z}},\beta)\right].
\end{aligned}$$

Moreover, remark that the processes $(\beta_r)_{r\in[0,s]}$ and $(\beta_r^\nu)_{r\in[0,s]}$ are equal, because $\xi_r\equiv 0$ on $[0,s]$. Since $(\mathcal{H}_s)_{s\in[0,T]}$ is progressively measurable, $\mathbb{P}^\nu$-almost surely, $\mathcal{H}_s(u,(W^k)_{k\in\mathbb{Z}},\beta)=\mathcal{H}_s(u,(W^k)_{k\in\mathbb{Z}},\beta^\nu)$. Therefore,

$$\begin{aligned}
&\mathbb{E}^\nu\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(y_t^\nu(u))H_s^{g,\varepsilon}(u)\right]\\
&\qquad=\mathbb{E}^\nu\left[\left[\frac{\delta\phi}{\delta m}\right]\left(\mathcal{P}_t((W^k)_{k\in\mathbb{Z}})\right)\left(\mathcal{X}_t(u,(W^k)_{k\in\mathbb{Z}},\beta^\nu)\right)\mathcal{H}_s(u,(W^k)_{k\in\mathbb{Z}},\beta^\nu)\right]\\
&\qquad=\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right]\left(\mathcal{P}_t((W^k)_{k\in\mathbb{Z}})\right)\left(\mathcal{X}_t(u,(W^k)_{k\in\mathbb{Z}},\beta)\right)\mathcal{H}_s(u,(W^k)_{k\in\mathbb{Z}},\beta)\right],
\end{aligned}$$

since the law of $((W^k)_{k\in\mathbb{Z}},\beta^\nu)$ under $\mathbb{P}^\nu$ is equal to the law of $((W^k)_{k\in\mathbb{Z}},\beta)$ under $\mathbb{P}^W\otimes\mathbb{P}^\beta$. The last term of that equality does not depend on ν, so

$$\frac{\mathrm{d}}{\mathrm{d}\nu}_{|\nu=0}\mathbb{E}^\nu\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(y_t^\nu(u))H_s^{g,\varepsilon}(u)\right]=0. \tag{III.105}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}^\nu\Big[\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(y_t^\nu(u))H_s^{g,\varepsilon}(u)\Big]&=\mathbb{E}^W\mathbb{E}^\beta\left[\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(y_t^\nu(u))H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right]\\
&=\mathbb{E}^W\mathbb{E}^\beta\left[\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(x_t^\nu(u))H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right]\\
&\quad+\mathbb{E}^W\mathbb{E}^\beta\left[\mathbb{1}_{\{\sigma^\nu<T\}}\left(\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(y_t^\nu(u))-\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(x_t^\nu(u))\right)H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right]
\end{aligned} \tag{III.106}$$

because $\mathbb{1}_{\{\sigma^\nu=T\}}(x_t^\nu(u)-y_t^\nu(u))=0$. By Hölder's inequality and by the fact that $\mathcal{E}_t^\nu\leqslant\exp(1)$, we have

$$\begin{aligned}
&\Big|\mathbb{E}^W\mathbb{E}^\beta\left[\mathbb{1}_{\{\sigma^\nu<T\}}\Big(\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(y_t^\nu(u))-\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(x_t^\nu(u))\Big)H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right]\Big|\\
&\qquad\leqslant\exp(1)\,\mathbb{E}^W\mathbb{E}^\beta\left[H_s^{g,\varepsilon}(u)^2\right]^{1/2}\,(\mathbb{P}^W\otimes\mathbb{P}^\beta)\Big[\sigma^\nu<T\Big]^{1/4}\\
&\qquad\quad\cdot\Big(\mathbb{E}^W\mathbb{E}^\beta\left[\Big|\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(y_t^\nu(u))\Big|^4\right]^{1/4}+\mathbb{E}^W\mathbb{E}^\beta\left[\Big|\Big[\frac{\delta\phi}{\delta m}\Big](\mu_t^g)(x_t^\nu(u))\Big|^4\right]^{1/4}\Big).
\end{aligned} \tag{III.107}$$

By Burkholder-Davis-Gundy inequality, and by inequality (III.38), for every $p\geqslant 2$,

$$\begin{aligned}
(\mathbb{P}^W\otimes\mathbb{P}^\beta)\Big[\sigma^\nu<T\Big]&=(\mathbb{P}^W\otimes\mathbb{P}^\beta)\left[\sup_{r\leqslant T}\Big|\nu\int_0^r\partial_x Z_z^{g(u)+\nu\xi_z}\dot\xi_z\mathrm{d}\beta_z\Big|\geqslant 1\right]\\
&\leqslant\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant T}\Big|\nu\int_0^r\partial_x Z_z^{g(u)+\nu\xi_z}\frac{\mathbb{1}_{\{z\in[s,t]\}}}{t-s}\mathrm{d}\beta_z\Big|^p\right]\\
&\leqslant C_p|\nu|^p\,\mathbb{E}^W\mathbb{E}^\beta\left[\Big|\int_s^t\Big|\partial_x Z_r^{g(u)+\nu\xi_r}\Big|^2\frac{1}{(t-s)^2}\mathrm{d}r\Big|^{p/2}\right]\\
&\leqslant\frac{C_p}{(t-s)^{\frac{p}{2}+1}}|\nu|^p\,\mathbb{E}^W\mathbb{E}^\beta\left[\int_s^t\Big|\partial_x Z_r^{g(u)+\nu\xi_r}\Big|^p\mathrm{d}r\right]\leqslant C_p|\nu|^p,
\end{aligned}$$

where the constant C_p changes from a line to the next and depends on s and t. In particular, there is C such that for every $\nu \in [-1,1]$, $(\mathbb{P}^W \otimes \mathbb{P}^\beta)\Big[\sigma^\nu < T\Big] \leqslant C|\nu|^8$. Moreover, by (III.100), for every $\nu \in [-1,1]$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\left|\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(y_t^\nu(u))\right|^4\right]^{1/4} \leqslant C,$$

where C depends only on g and t and is independent of ν. Similarly, there is also $C>0$ for every $\nu \in [-1,1]$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\left|\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^\nu(u))\right|^4\right]^{1/4} \leqslant C. \tag{III.108}$$

Furthermore, let us compute

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta\left[H_s^{g,\varepsilon}(u)^2\right]^{1/2} &= \mathbb{E}^W\mathbb{E}^\beta\left[\frac{1}{|\partial_u x_s^g(u)|^2}|A_s^g - A_s^{g,\varepsilon}|^2(x_s^g(u))\right]^{1/2}\\ &\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\left\|\frac{1}{\partial_u x_s^g}\right\|_{L_\infty}^4\right]^{1/4}\mathbb{E}^W\mathbb{E}^\beta\left[\|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty}^4\right]^{1/4}.\end{aligned}$$

By (III.34), $\mathbb{E}^W\mathbb{E}^\beta\left[\left\|\frac{1}{\partial_u x_s^g}\right\|_{L_\infty}^4\right]$ is finite. Moreover, recall that $A_s^{g,\varepsilon} = A_s^g * \varphi_\varepsilon$. Therefore, for every $x \in \mathbb{R}$,

$$\begin{aligned}|A_s^g - A_s^{g,\varepsilon}|(x) = \left|\int_{\mathbb{R}}\left(A_s^g(x) - A_s^g(x-y)\right)\frac{1}{\varepsilon}\varphi\left(\frac{y}{\varepsilon}\right)\mathrm{d}y\right| &= \left|\int_{\mathbb{R}}\left(A_s^g(x) - A_s^g(x-\varepsilon y)\right)\varphi(y)\mathrm{d}y\right|\\ &\leqslant \int_{\mathbb{R}}\left|\int_0^1 \partial_x A_s^g(x-\varepsilon\lambda y)\varepsilon y\mathrm{d}\lambda\right|\varphi(y)\mathrm{d}y\\ &\leqslant C\varepsilon\|\partial_x A_s^g\|_{L_\infty},\end{aligned} \tag{III.109}$$

where $C = \int_{\mathbb{R}}|y|\varphi(y)\mathrm{d}y$. Thus $\mathbb{E}^W\mathbb{E}^\beta\left[\|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty}^4\right] \leqslant C\varepsilon^4\mathbb{E}^W\mathbb{E}^\beta\left[\|\partial_x A_s^g\|_{L_\infty}^4\right]$, which is finite. To show that last statement, we use (III.91) and an analog to (III.92), where we replace the exponent 2 by an exponent 4. We deduce that $\mathbb{E}^W\mathbb{E}^\beta\left[H_s^{g,\varepsilon}(u)^2\right]^{1/2}$ is finite.

Thus it follows from (III.107) that for every $\nu \in [-1,1]$,

$$\left|\mathbb{E}^W\mathbb{E}^\beta\left[\mathbb{1}_{\{\sigma^\nu<T\}}\left(\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(y_t^\nu(u)) - \left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^\nu(u))\right)H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right]\right| \leqslant C|\nu|^2. \tag{III.110}$$

Combining (III.106) and (III.110), this proves that there is $C>0$ such that for every $\nu \in [-1,1]$,

$$\left|\mathbb{E}^\nu\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(y_t^\nu(u))H_s^{g,\varepsilon}(u)\right] - \mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^\nu(u))H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right]\right| \leqslant C|\nu|^2.$$

Recalling (III.105), we deduce that

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}\nu}_{|\nu=0}\mathbb{E}^\nu\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(y_t^\nu(u))H_s^{g,\varepsilon}(u)\right] &= \frac{\mathrm{d}}{\mathrm{d}\nu}_{|\nu=0}\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^\nu(u))H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}\nu}_{|\nu=0}\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(Z_t^{g(u)+\nu\xi_t})H_s^{g,\varepsilon}(u)\mathcal{E}_t^\nu\right].\end{aligned}$$

It follows from (III.108) that $\left(\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(Z_t^{g(u)+\nu\xi_t})\right)_{\nu\in[-1,1]}$ is uniformly integrable. By inequality (III.101), we prove in the same way that $\left(\partial_v\left\{\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)\right\}(Z_t^{g(u)+\nu\xi_t})\right)_{\nu\in[-1,1]}$ is also uniformly

integrable. Recall that, by inequality (III.38), $(\partial_x Z_t^{g(u)+\nu\xi_t})_{\nu\in[-1,1]}$ is uniformly integrable. Thus we get by differentiation:

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}\nu}_{|\nu=0}\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(Z_t^{g(u)+\nu\xi_t})\ H_s^{g,\varepsilon}(u)\ \mathcal{E}_t^\nu\right]\\ &= \mathbb{E}^W\mathbb{E}^\beta\left[\partial_v\Big\{\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)\Big\}(Z_t^{g(u)})\partial_x Z_t^{g(u)}\xi_t\ H_s^{g,\varepsilon}(u)\right]\\ &\quad-\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(Z_t^{g(u)})\ H_s^{g,\varepsilon}(u)\int_0^t\partial_x Z_r^{g(u)}\dot\xi_r\mathrm{d}\beta_r\right]\end{aligned}\tag{III.111}$$

On the one hand, using equalities (III.40) and (III.41), we have

$$\begin{aligned}&\mathbb{E}^W\mathbb{E}^\beta\left[\partial_v\Big\{\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)\Big\}(Z_t^{g(u)})\partial_x Z_t^{g(u)}\xi_t\ H_s^{g,\varepsilon}(u)\right]\\ &\qquad=\mathbb{E}^W\mathbb{E}^\beta\left[\partial_v\Big\{\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)\Big\}(x_t^g(u))\frac{\partial_u x_t^g(u)}{g'(u)}\xi_t\ H_s^{g,\varepsilon}(u)\right]\\ &\qquad=\mathbb{E}^W\mathbb{E}^\beta\left[\partial_u\Big\{\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(\cdot))\Big\}(u)\ \frac{1}{g'(u)}\ H_s^{g,\varepsilon}(u)\right],\end{aligned}\tag{III.112}$$

because $\xi_t = 1$. On the other hand, by definition of ξ,

$$\begin{aligned}&\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(Z_t^{g(u)})\ H_s^{g,\varepsilon}(u)\int_0^t\partial_x Z_r^{g(u)}\dot\xi_r\mathrm{d}\beta_r\right]\\ &\qquad=\mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))H_s^{g,\varepsilon}(u)\frac{1}{t-s}\int_s^t\frac{\partial_u x_r^g(u)}{g'(u)}\mathrm{d}\beta_r\right],\end{aligned}\tag{III.113}$$

Putting together equalities (III.111), (III.112) and (III.113), we obtain equality (III.103). □

Recall that h is a $\mathcal{G}_0$-measurable random variable with values in Δ^1 and the definitions (III.59) and (III.60) of the processes $(A_t^g)_{t\in[0,T]}$ and $(A_t^{g,\varepsilon})_{t\in[0,T]}$. Let us define the process $(K_t^{g,\varepsilon})_{t\in[0,T]}$, also depending on h, by

$$K_t^{g,\varepsilon}(u) := \int_0^t\frac{H_s^{g,\varepsilon}(u)}{t-s}\int_s^t\partial_u x_r^g(u)\mathrm{d}\beta_r\mathrm{d}s = \int_0^t\frac{(A_s^g - A_s^{g,\varepsilon})(x_s^g(u))}{\partial_u x_s^g(u)}\frac{1}{t-s}\int_s^t\partial_u x_r^g(u)\mathrm{d}\beta_r\mathrm{d}s.\tag{III.114}$$

Thus we state the following proposition. Recall the notation $\left[\frac{\delta P_t\phi}{\delta m}\right]$ and $\left[\frac{\delta\phi}{\delta m}\right]$ defined by (III.54).

Proposition III.39. *Let $\theta\in(0,1)$ and f be of order $\alpha = \frac{7}{2}+\theta$. Let g and h be $\mathcal{G}_0$-measurable random variables with values respectively in $\mathbf{G}^{3+\theta}$ and Δ^1. Let $(K_t^{g,\varepsilon})_{t\in[0,T]}$ be the process defined by* (III.114)*. Then there is $C>0$ independent of ε, g, h and θ such that $\mathbb{P}^0$-almost surely, for every $t\in(0,T]$,*

$$\begin{aligned}\left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(g(u))\partial_u h(u)\mathrm{d}u\right| &\leqslant C\frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{\varepsilon^{3+2\theta}\sqrt{t}}C_1(g)\|h\|_{\mathcal{C}^1}\\ &\quad+\frac{C}{\sqrt{t}}\varepsilon\|h\|_{\mathcal{C}^1}C_2(g)\mathbb{E}^W\mathbb{E}^\beta\left[\left|\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))\frac{K_t^{g,\varepsilon}(u)}{\|K_t^{g,\varepsilon}\|_{L_\infty}}\mathrm{d}u\right|^2\right]^{1/2},\end{aligned}\tag{III.115}$$

where $C_1(g) = 1+\|g'''\|_{L_4}^2+\|g''\|_{L_\infty}^6+\|g'\|_{L_\infty}^8+\left\|\frac{1}{g'}\right\|_{L_\infty}^8$ and $C_2(g) = 1+\|g'''\|_{L_8}^3+\|g''\|_{L_\infty}^{12}+\|g'\|_{L_\infty}^{12}+\left\|\frac{1}{g'}\right\|_{L_\infty}^{24}$.

Proof. By (III.57), $\left|\int_0^1 \left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(g(u))\,\partial_u h(u)\mathrm{d}u\right| = |I|$, where I is given by (III.61):

$$I := \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\partial_u x_t^g(u)h(u)\mathrm{d}u\right] = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &:= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}A_s^{g,\varepsilon}(x_s^g(u))\mathrm{d}s\mathrm{d}u\right];\\
I_2 &:= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}(A_s^g - A_s^{g,\varepsilon})(x_s^g(u))\mathrm{d}s\mathrm{d}u\right]\\
&= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \partial_\mu\phi(\mu_t^g)(x_t^g(u))\partial_u x_t^g(u)H_s^{g,\varepsilon}(u)\mathrm{d}s\mathrm{d}u\right],
\end{aligned}$$

where $H_s^{g,\varepsilon}$ is a 1-periodic function given by (III.94). By Proposition III.33,

$$|I_1| \leqslant C\frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{\varepsilon^{3+2\theta}\sqrt{t}}C_1(g)\|h\|_{\mathcal{C}^1}.$$

Furthermore, $\partial_\mu\phi(\mu_t^g) = \partial_v\left\{\frac{\delta\phi}{\delta m}(\mu_t^g)\right\}$, thus

$$\begin{aligned}
I_2 &= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^t \partial_v\left\{\frac{\delta\phi}{\delta m}(\mu_t^g)\right\}(x_t^g(u))\partial_u x_t^g(u)H_s^{g,\varepsilon}(u)\mathrm{d}s\mathrm{d}u\right]\\
&= \frac{1}{t}\int_0^1\int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\partial_u\left\{\frac{\delta\phi}{\delta m}(\mu_t^g)(x_t^g(\cdot))\right\}(u)H_s^{g,\varepsilon}(u)\right]\mathrm{d}s\mathrm{d}u.
\end{aligned}$$

Recall Definition (III.54) of $\left[\frac{\delta\phi}{\delta m}\right]$. We can remove the average of $\frac{\delta\phi}{\delta m}(\mu_t^g)(x_t^g(\cdot))$ since we are considering the derivative with respect to u. Therefore

$$I_2 = \frac{1}{t}\int_0^1\int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\partial_u\left\{\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(\cdot))\right\}(u)H_s^{g,\varepsilon}(u)\right]\mathrm{d}s\mathrm{d}u.$$

By equality (III.103) proved in Lemma (III.38), we deduce that

$$\begin{aligned}
I_2 &= \frac{1}{t}\int_0^1\int_0^t \mathbb{E}^W\mathbb{E}^\beta\left[\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))H_s^{g,\varepsilon}(u)\frac{1}{t-s}\int_s^t \partial_u x_r^g(u)\mathrm{d}\beta_r\right]\mathrm{d}s\mathrm{d}u\\
&= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))\int_0^t H_s^{g,\varepsilon}(u)\frac{1}{t-s}\int_s^t \partial_u x_r^g(u)\mathrm{d}\beta_r\mathrm{d}s\mathrm{d}u\right]\\
&= \frac{1}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))K_t^{g,\varepsilon}(u)\mathrm{d}u\right],
\end{aligned}$$

where we recall that $K_t^{g,\varepsilon}(u) = \int_0^t H_s^{g,\varepsilon}(u)\frac{1}{t-s}\int_s^t \partial_u x_r^g(u)\mathrm{d}\beta_r\mathrm{d}s$, which is a 1-periodic random function. By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))K_t^{g,\varepsilon}(u)\mathrm{d}u\right]\right|\\
&\qquad\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\left|\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))\frac{K_t^{g,\varepsilon}(u)}{\|K_t^{g,\varepsilon}\|_{L_\infty}}\mathrm{d}u\right|^2\right]^{1/2}.
\end{aligned}$$

Let us estimate $\mathbb{E}^W\mathbb{E}^\beta\left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2\right]$. By definition of $K_t^{g,\varepsilon}$ given by (III.114), we have for every $u\in[0,1]$,

$$|K_t^{g,\varepsilon}(u)| \leqslant \int_0^t \|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty}\frac{1}{|\partial_u x_s^g(u)|}\frac{1}{t-s}\left|\int_s^t \partial_u x_r^g(u)\mathrm{d}\beta_r\right|\mathrm{d}s$$

By (III.109) and by (III.91),

$$\|A_s^g - A_s^{g,\varepsilon}\|_{L_\infty} \leqslant C\varepsilon \, \|\partial_x A_s^g\|_{L_\infty} \leqslant C\varepsilon \|h\|_{\mathcal{C}^1} \left(1 + \|\partial_u^{(2)} x_s^g\|_{L_\infty} \left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty}\right).$$

Thus we obtain

$$\begin{aligned}
\|K_t^{g,\varepsilon}\|_{L_\infty} &\leqslant C\varepsilon\|h\|_{\mathcal{C}^1} \int_0^t \frac{1}{t-s} \left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty} \left\| \int_s^t \partial_u x_r^g(\cdot) \mathrm{d}\beta_r \right\|_{L_\infty} \mathrm{d}s \\
&\quad + C\varepsilon\|h\|_{\mathcal{C}^1} \int_0^t \frac{1}{t-s} \left\| \partial_u^{(2)} x_s^g \right\|_{L_\infty} \left\| \frac{1}{\partial_u x_s^g} \right\|_{L_\infty}^2 \left\| \int_s^t \partial_u x_r^g(\cdot) \mathrm{d}\beta_r \right\|_{L_\infty} \mathrm{d}s \\
&\leqslant C\varepsilon\|h\|_{\mathcal{C}^1} \left\{ 1 + \sup_{r\leqslant T} \left\| \partial_u^{(2)} x_r^g \right\|_{L_\infty} \right\} \left\{ \sup_{r\leqslant T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty} + \sup_{r\leqslant T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty}^2 \right\} \\
&\quad \cdot \int_0^t \frac{1}{t-s} \left\| \int_s^t \partial_u x_r^g(\cdot) \mathrm{d}\beta_r \right\|_{L_\infty} \mathrm{d}s.
\end{aligned}$$

Recall that h is a random variable on Ω^0. Thus, by Hölder's equality, we obtain

$$\mathbb{E}^W \mathbb{E}^\beta \left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2 \right]^{1/2} \leqslant C\varepsilon \|h\|_{\mathcal{C}^1} E_1 E_2 E_3. \tag{III.116}$$

where

$$\begin{aligned}
E_1 &:= 1 + \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r\leqslant T} \left\| \partial_u^{(2)} x_r^g \right\|_{L_\infty}^8 \right]^{1/8}; \\
E_2 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r\leqslant T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty}^8 \right]^{1/8} + \mathbb{E}^W \mathbb{E}^\beta \left[\sup_{r\leqslant T} \left\| \frac{1}{\partial_u x_r^g} \right\|_{L_\infty}^{16} \right]^{1/8}; \\
E_3 &:= \mathbb{E}^W \mathbb{E}^\beta \left[\left(\int_0^t \frac{1}{t-s} \left\| \int_s^t \partial_u x_r^g(\cdot) \mathrm{d}\beta_r \right\|_{L_\infty} \mathrm{d}s \right)^4 \right]^{1/4}.
\end{aligned}$$

Recall that by assumptions, g belongs to $\mathbf{G}^{3+\theta}$ and f is of order $\alpha > \frac{7}{2} + \theta$. By (III.33) and by (III.34)

$$\begin{aligned}
E_1 &\leqslant C(1 + \|g'''\|_{L_8} + \|g''\|_{L_\infty}^3 + \|g'\|_{L_\infty}^3); \\
E_2 &\leqslant C(1 + \|g''\|_{L_\infty}^4 + \|g'\|_{L_\infty}^4 + \|\tfrac{1}{g'}\|_{L_\infty}^8).
\end{aligned}$$

Furthermore, $E_3 \leqslant E_{3,1} + E_{3,2}$, where

$$\begin{aligned}
E_{3,1} &:= \mathbb{E}^W \mathbb{E}^\beta \left[\Big(\int_0^t \frac{1}{t-s} \Big| \int_s^t \partial_u x_r^g(0) \mathrm{d}\beta_r \Big| \mathrm{d}s \Big)^4 \right]^{1/4}; \\
E_{3,2} &:= \mathbb{E}^W \mathbb{E}^\beta \left[\Big(\int_0^t \frac{1}{t-s} \int_0^1 \Big| \int_s^t \partial_u^{(2)} x_r^g(v) \mathrm{d}\beta_r \Big| \mathrm{d}s \mathrm{d}v \Big)^4 \right]^{1/4}.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
(E_{3,1})^4 &= \mathbb{E}^W \mathbb{E}^\beta \left[\Big(\int_0^t \frac{1}{|t-s|^{3/8}} \, \frac{1}{|t-s|^{5/8}} \Big| \int_s^t \partial_u x_r^g(0) \mathrm{d}\beta_r \Big| \mathrm{d}s \Big)^4 \right] \\
&\leqslant \mathbb{E}^W \mathbb{E}^\beta \left[\Big(\int_0^t \frac{1}{|t-s|^{1/2}} \mathrm{d}s \Big)^3 \int_0^t \frac{1}{|t-s|^{5/2}} \Big| \int_s^t \partial_u x_r^g(0) \mathrm{d}\beta_r \Big|^4 \mathrm{d}s \right] \\
&\leqslant C\sqrt{t}^3 \int_0^t \frac{1}{|t-s|^{5/2}} \mathbb{E}^W \mathbb{E}^\beta \left[\Big| \int_s^t \partial_u x_r^g(0) \mathrm{d}\beta_r \Big|^4 \right] \mathrm{d}s.
\end{aligned}$$

By Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}^W\mathbb{E}^\beta\left[\Big|\int_s^t \partial_u x_r^g(0)\mathrm{d}\beta_r\Big|^4\right] \leqslant C\mathbb{E}^W\mathbb{E}^\beta\left[\Big|\int_s^t |\partial_u x_r^g(0)|^2\mathrm{d}r\Big|^2\right] \leqslant C(t-s)^2\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant T}|\partial_u x_r^g(0)|^4\right].$$

Thus

$$E_{3,1} \leqslant C\Big(\sqrt{t}^3\int_0^t \frac{|t-s|^2}{|t-s|^{5/2}}\mathrm{d}s\Big)^{1/4}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant T}|\partial_u x_r^g(0)|^4\right]^{1/4} \leqslant C\sqrt{t}\|g'\|_{L_\infty},$$

where the last inequality holds by (III.25). By the same computation,

$$E_{3,2} \leqslant C\sqrt{t}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant T}\int_0^1 |\partial_u^{(2)} x_r^g(v)|^4\mathrm{d}v\right]^{1/4} \leqslant C\sqrt{t}\,(1+\|g''\|_{L_\infty}+\|g'\|_{L_\infty}^2),$$

where the last inequality holds by (III.31). We deduce that $E_3 \leqslant C\sqrt{t}\,(1+\|g''\|_{L_\infty}+\|g'\|_{L_\infty}^2)$. By inequality (III.116) and the estimations on E_i, for $i=1,2,3$, we finally get:

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|K_t^{g,\varepsilon}\|_{L_\infty}^2\right]^{1/2} \leqslant C\sqrt{t}\varepsilon\|h\|_{\mathcal{C}^1}C_2(g),$$

where $C_2(g) = 1+\|g'''\|_{L_8}^3+\|g''\|_{L_\infty}^{12}+\|g'\|_{L_\infty}^{12}+\|\frac{1}{g'}\|_{L_\infty}^{24}$. □

III.4.4 Bismut-Elworthy inequality for the semi-group $(P_t)_{t\in[0,T]}$

The aim of this paragraph is to complete the proof of Theorem III.29. We start from the result given in Proposition III.39. In Proposition III.41, we will choose a suitable value of ε. Those results give an estimation of the linear functional derivative of $P_t\phi$ in terms of the linear functional derivative of ϕ; in Proposition III.42 and in Proposition III.43, we compare $P_{2t}\phi$ to $P_t\phi$. Finally, we will take a supremum over every t in order to get rid of the second term on the right-hand side. This will achieve the proof of Theorem III.29.

Definition III.40. Let $\mathcal{K}_t$ be the set of $\mathcal{G}_t$-measurable random variables taking $\mathbb{P}$-almost surely their values in the set of 1-periodic functions $k:\mathbb{R}\to\mathbb{R}$ satisfying $\|k\|_{L_\infty}=1$.

We denote by $\mathbb{E}[\cdot|\mathcal{G}_0]$ the conditional expectation with respect to the σ-algebra $\mathcal{G}_0$. Note that for each random variable X on Ω, we have $\mathbb{E}^W\mathbb{E}^\beta[X] = \mathbb{E}[X|\mathcal{G}_0]$. Indeed, for each $\mathcal{G}_0$-measurable random variable Y, $\mathbb{E}[XY] = \mathbb{E}^0\mathbb{E}^W\mathbb{E}^\beta[XY] = \mathbb{E}^0\left[\mathbb{E}^W\mathbb{E}^\beta[X]\,Y\right]$.

Proposition III.41. *Let $\theta\in(0,1)$ and f be of order $\alpha=\frac{7}{2}+\theta$. Let g and h be $\mathcal{G}_0$-measurable random variables with values respectively in $\mathbf{G}^{3+\theta}$ and Δ^1. Let $(K_t^{g,\varepsilon})_{t\in[0,T]}$ be the process defined on the basis of h by* (III.114)*. Then $\mathbb{P}^0$-almost surely, there is ε_0 such that for every $t\in(0,T]$,*

$$\mathbb{E}\left[\Big|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(g(u))\partial_u h(u)\mathrm{d}u\Big|^2\Big|\mathcal{G}_0\right]^{1/2} \leqslant C\frac{\mathbb{V}^W[\phi(\mu_t^g)]^{1/2}}{t^{2+\theta}}C_3(g)\|h\|_{\mathcal{C}^1}^{4+2\theta} + \frac{1}{2^{3+\theta}}\mathbb{E}\left[\Big|\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))\frac{K_t^{g,\varepsilon_0}(u)}{\|K_t^{g,\varepsilon_0}\|_{L_\infty}}\mathrm{d}u\Big|^2\Big|\mathcal{G}_0\right]^{1/2}, \quad \text{(III.117)}$$

where $C_3(g) = C_1(g)C_2(g)^{3+2\theta}$ and C is independent of g, h and t. Moreover the random variable $u\mapsto\frac{K_t^{g,\varepsilon_0}(u)}{\|K_t^{g,\varepsilon_0}\|_{L_\infty}}$ belongs to $\mathcal{K}_t$.

Proof. By (III.115), we obtain

$$\left|\int_0^1 \left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(g(u))\partial_u h(u)\mathrm{d}u\right| \leqslant C_a\frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{\varepsilon^{3+2\theta}\sqrt{t}}C_1(g)\|h\|_{\mathcal{C}^1} + C_b\frac{\varepsilon\|h\|_{\mathcal{C}^1}}{\sqrt{t}}C_2(g)\mathbb{E}^W\mathbb{E}^\beta\left[\left|\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))\frac{K_t^{g,\varepsilon}(u)}{\|K_t^{g,\varepsilon}\|_{L_\infty}}\mathrm{d}u\right|^2\right]^{1/2},$$

where $C_1(g)$ and $C_2(g)$ are given by Proposition III.39 and C_a, C_b are two constants.

Let us choose $\varepsilon_0 = \frac{\sqrt{t}}{C_b\|h\|_{\mathcal{C}^1}C_2(g)}\frac{1}{2^{3+\theta}}$. We obtain

$$\left|\int_0^1 \left[\frac{\delta P_t\phi}{\delta m}\right](\mu_0^g)(g(u))\partial_u h(u)\mathrm{d}u\right| \leqslant C\frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{\sqrt{t}^{4+2\theta}}C_3(g)\|h\|_{\mathcal{C}^1}^{4+2\theta} + \frac{1}{2^{3+\theta}}\mathbb{E}^W\mathbb{E}^\beta\left[\left|\int_0^1\left[\frac{\delta\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))\frac{K_t^{g,\varepsilon_0}(u)}{\|K_t^{g,\varepsilon_0}\|_{L_\infty}}\mathrm{d}u\right|^2\right]^{1/2},$$

where $C_3(g) = C_1(g)C_2(g)^{3+2\theta}$. By the fact that g and h are $\mathcal{G}_0$-measurable and by the property $\mathbb{E}^W\mathbb{E}^\beta[\cdot] = \mathbb{E}[\cdot|\mathcal{G}_0]$, we deduce the result of the proposition. □

In the following proposition, we use the flow property to compare the derivative of $P_{2t}\phi$ to the derivative of $P_t\phi$.

Proposition III.42. *Let $\theta \in (0,1)$ and f be of order $\alpha = \frac{7}{2}+\theta$. Let $g : \mathbb{R}\to\mathbb{R}$ be a deterministic initial condition belonging to $\mathbf{G}^{3+\theta}$. Fix $t_0 \in (0,T]$. Let $t \in (0, \frac{t_0}{2}]$. Let $h : \mathbb{R}\to\mathbb{R}$ be a $\mathcal{G}_{t_0-2t}$-measurable random variable with values in Δ^1 satisfying $\mathbb{P}$-almost surely $\|h\|_{\mathcal{C}^1} \leqslant 4$. Then there is a constant $C_g > 0$ depending on the initial condition g and independent of t_0, t and h such that*

$$\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu_{t_0-2t}^g)(x_{t_0-2t}^g(u))\partial_u h(u)\mathrm{d}u\right|^2\right]^{1/2} \leqslant C_g\frac{\mathbb{V}^W\left[\phi(\mu_{t_0}^g)\right]^{1/2}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}}\sup_{k\in\mathcal{K}_{t_0-t}}\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_{t_0-t}^g)(x_{t_0-t}^g(u))k(u)\mathrm{d}u\right|^2\right]^{1/2}. \quad \text{(III.118)}$$

Proof. Let $t \in (0, \frac{t_0}{2}]$. Apply inequality (III.117) to $P_t\phi$ instead of ϕ. Remark that $\|P_t\phi\|_{L_\infty} \leqslant \|\phi\|_{L_\infty}$. We obtain that

$$\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu_0^g)(g(u))\partial_u h(u)\mathrm{d}u\right|^2\Big|\mathcal{G}_0\right]^{1/2} \leqslant C\frac{\mathbb{V}^W\left[\phi(\mu_{2t}^g)\right]^{1/2}}{t^{2+\theta}}C_3(g)\|h\|_{\mathcal{C}^1}^{4+2\theta} + \frac{1}{2^{3+\theta}}\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_t^g)(x_t^g(u))\frac{K_t^{g,\varepsilon_0}(u)}{\|K_t^{g,\varepsilon_0}\|_{L_\infty}}\mathrm{d}u\right|^2\Big|\mathcal{G}_0\right]^{1/2}.$$

Fix $s = t_0 - 2t$. Let us consider h a $\mathcal{G}_s$-measurable random variable, taking almost surely values in the set of 1-periodic $\mathcal{C}^1$-functions such that $\|h\|_{\mathcal{C}^1} \leqslant 4$. We start again the whole argument with this new h and after replacing the initial condition g by x_s^g, which is also $\mathcal{G}_s$-

measurable. The computations remains the same and we finally get

$$\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu_s^{s,x_s^g})(x_s^g(u))\partial_u h(u)\mathrm{d}u\right|^2\Big|\mathcal{G}_s\right]^{1/2}\leqslant C\frac{\mathbb{V}\left[\phi(\mu_{2t+s}^{s,x_s^g})|\mathcal{G}_s\right]^{1/2}}{t^{2+\theta}}C_3(x_s^g)\|h\|_{\mathcal{C}^1}^{4+2\theta}$$
$$+\frac{1}{2^{3+\theta}}\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_{t+s}^{s,x_s^g})(x_{t+s}^{s,x_s^g}(u))\frac{K_{t+s}^{s,x_s^g,\varepsilon_0}(u)}{\left\|K_{t+s}^{s,x_s^g,\varepsilon_0}\right\|_{L_\infty}}\mathrm{d}u\right|^2\Big|\mathcal{G}_s\right]^{1/2}.$$

In the latter inequality, $x_{t+s}^{s,x_s^g}(u)$ denotes the value at time $t+s$ and at point u of the unique solution to (III.20) which is equal to x_s^g at time s. By strong uniqueness of (III.20), we have the following flow property: $x_{t+s}^{s,x_s^g}(u)=x_{t+s}^g(u)$. Similarly, $\mu_s^{s,x_s^g}=\mu_s^g$, $\mu_{2t+s}^{s,x_s^g}=\mu_{2t+s}^g$ and $\mu_{t+s}^{s,x_s^g}=\mu_{t+s}^g$. Therefore,

$$\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu_s^g)(x_s^g(u))\partial_u h(u)\mathrm{d}u\right|^2\Big|\mathcal{G}_s\right]^{1/2}\leqslant C\frac{\mathbb{V}\left[\phi(\mu_{2t+s}^g)|\mathcal{G}_s\right]^{1/2}}{t^{2+\theta}}C_3(x_s^g)\|h\|_{\mathcal{C}^1}^{4+2\theta}$$
$$+\frac{1}{2^{3+\theta}}\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_{t+s}^g)(x_{t+s}^g(u))\frac{K_{t+s}^{s,x_s^g,\varepsilon_0}(u)}{\left\|K_{t+s}^{s,x_s^g,\varepsilon_0}\right\|_{L_\infty}}\mathrm{d}u\right|^2\Big|\mathcal{G}_s\right]^{1/2}. \quad \text{(III.119)}$$

Here, the random variable $u\mapsto\frac{K_{t+s}^{s,x_s^g,\varepsilon_0}(u)}{\left\|K_{t+s}^{s,x_s^g,\varepsilon_0}\right\|_{L_\infty}}$ belongs to $\mathcal{K}_{t+s}$. Taking the expectation of the left hand side of (III.119), we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu_s^g)(x_s^g(u))\partial_u h(u)\mathrm{d}u\right|^2\right]^{1/2}&\leqslant C\frac{1}{t^{2+\theta}}\mathbb{E}\left[\mathbb{V}\left[\phi(\mu_{2t+s}^g)|\mathcal{G}_s\right]^{1/2}C_3(x_s^g)\|h\|_{\mathcal{C}^1}^{4+2\theta}\right]\\&\quad+\frac{1}{2^{3+\theta}}\sup_{k\in\mathcal{K}_{t+s}}\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_{t+s}^g)(x_{t+s}^g(u))k(u)\mathrm{d}u\right|^2\right]^{1/2}\\&\leqslant C\frac{1}{t^{2+\theta}}\mathbb{V}\left[\phi(\mu_{2t+s}^g)\right]^{1/2}\mathbb{E}\left[C_3(x_s^g)^2\|h\|_{\mathcal{C}^1}^{8+4\theta}\right]^{1/2}\\&\quad+\frac{1}{2^{3+\theta}}\sup_{k\in\mathcal{K}_{t+s}}\mathbb{E}\left[\left|\int_0^1\left[\frac{\delta P_t\phi}{\delta m}\right](\mu_{t+s}^g)(x_{t+s}^g(u))k(u)\mathrm{d}u\right|^2\right]^{1/2}.\end{aligned} \quad \text{(III.120)}$$

Recall that almost surely $\|h\|_{\mathcal{C}^1}\leqslant 4$. Recall that the definition of $C_3(\cdot)$ is given by Proposition III.41.

$$\begin{aligned}\mathbb{E}\left[C_3(x_s^g)^2\right]&=\mathbb{E}\left[C_1(x_s^g)^2C_2(x_s^g)^{6+4\theta}\right]\\&=\mathbb{E}\Bigg[\left(1+\|\partial_u^{(3)}x_s^g\|_{L_4}^2+\|\partial_u^{(2)}x_s^g\|_{L_\infty}^6+\|\partial_u x_s^g\|_{L_\infty}^8+\left\|\frac{1}{\partial_u x_s^g}\right\|_{L_\infty}^8\right)^2\\&\qquad\cdot\left(1+\|\partial_u^{(3)}x_s^g\|_{L_8}^3+\|\partial_u^{(2)}x_s^g\|_{L_\infty}^{12}+\|\partial_u x_s^g\|_{L_\infty}^{12}+\|\tfrac{1}{\partial_u x_s^g}\|_{L_\infty}^{24}\right)^{6+4\theta}\Bigg].\end{aligned} \quad \text{(III.121)}$$

We refer to (III.31), (III.33) and (III.34) to argue that every term occurring in the right hand side of (III.121) are bounded by a constant depending polynomially on $\|g'''\|_{L_\infty}$, $\|g''\|_{L_\infty}$, $\|g'\|_{L_\infty}$ and $\|\frac{1}{g'}\|_{L_\infty}$. The constant is finite since g belongs to $\mathbf{G}^{3+\theta}$. It depends also on T but is independent of s. Therefore, there is a constant C_g depending on the initial condition g and not on s such that $\mathbb{E}\left[C_3(x_s^g)\right]\leqslant C_g$.

We conclude the proof of (III.118) by replacing in (III.120) s by its value $s=t_0-2t$. □

For every $t \in (0, t_0]$, let us define

$$S_t := \sup_{k \in \mathcal{K}_{t_0-t}} \mathbb{E}\left[\left|\int_0^1 \left[\frac{\delta P_t \phi}{\delta m}\right](\mu^g_{t_0-t})(x^g_{t_0-t}(u))k(u)\mathrm{d}u\right|^2\right]^{1/2},$$

where $\mathcal{K}_{t_0-t}$ is defined by Definition III.40.

Proposition III.43. *Let $\theta \in (0,1)$ and f be of order $\alpha = \frac{7}{2}+\theta$. Let $g : \mathbb{R} \to \mathbb{R}$ be a deterministic initial condition belonging to $\mathbf{G}^{3+\theta}$. Fix $t_0 \in (0, T]$. For every $t \in (0, \frac{t_0}{2}]$, we have:*

$$S_{2t} \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu^g_{t_0})\right]^{1/2}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t. \tag{III.122}$$

Proof. Let $t \in (0, \frac{t_0}{2}]$. Let $k \in \mathcal{K}_{t_0-2t}$. By Definition III.40, $k : \mathbb{R} \to \mathbb{R}$ is a 1-periodic function and a $\mathcal{G}_{t_0-2t}$-measurable random variable so that $\mathbb{P}$-almost surely, $\|k\|_{L_\infty} = \sup_{u\in[0,1]} |k(u)| = 1$.

Let us denote by h the map defined for every $u \in \mathbb{R}$ by $h(u) := \int_0^u (k(v) - m)\mathrm{d}v$, where $m = \int_0^1 k(v)\mathrm{d}v$. Clearly, h is $\mathcal{G}_{t_0-2t}$-measurable. Furthermore, since k is 1-periodic, we obtain for every $u \in [0,1]$:

$$h(u+1) - h(u) = \int_u^{u+1} (k(v) - m)\mathrm{d}v = \int_0^1 (k(v) - m)\mathrm{d}v = m - m = 0.$$

Thus h is also 1-periodic. Moreover

$$\|h\|_{L_\infty} = \sup_{u\in[0,1]} |h(u)| \leqslant \int_0^1 |k(v) - m|\mathrm{d}v \leqslant \int_0^1\int_0^1 |k(v) - k(u)|\mathrm{d}u\mathrm{d}v \leqslant 2\|k\|_{L_\infty} \leqslant 2,$$

$$\|\partial_u h\|_{L_\infty} = \sup_{u\in[0,1]} |\partial_u h(u)| \leqslant \sup_{u\in[0,1]} |k(u) - m| \leqslant \sup_{u\in[0,1]} \int_0^1 |k(u) - k(v)|\mathrm{d}v \leqslant 2\|k\|_{L_\infty} \leqslant 2.$$

Therefore, $\|h\|_{\mathcal{C}^1} \leqslant 4$. Thus the assumptions of Proposition III.42 are satisfied, hence we have by inequality (III.118):

$$\mathbb{E}\left[\left|\int_0^1 \left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu^g_{t_0-2t})(x^g_{t_0-2t}(u))\partial_u h(u)\mathrm{d}u\right|^2\right]^{1/2} \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu^g_{t_0})\right]^{1/2}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t.$$

Moreover, $\partial_u h(u) = k(u) - m$ and by definition of $\left[\frac{\delta P_{2t}\phi}{\delta m}\right]$, $\int_0^1 \left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu^g_{t_0-2t})(x^g_{t_0-2t}(u))\, m\mathrm{d}u = 0$. Thus we obtain

$$\mathbb{E}\left[\left|\int_0^1 \left[\frac{\delta P_{2t}\phi}{\delta m}\right](\mu^g_{t_0-2t})(x^g_{t_0-2t}(u))k(u)\mathrm{d}u\right|^2\right]^{1/2} \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu^g_{t_0})\right]^{1/2}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t,$$

and by taking the supremum over all k's in $\mathcal{K}_{t_0-2t}$, we get $S_{2t} \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu^g_{t_0})\right]^{1/2}}{t^{2+\theta}} + \frac{1}{2^{3+\theta}} S_t$. □

Let us complete the proof of Theorem III.29.

Proof (Theorem III.29). By (III.122), we deduce that for every $t \in (0, \frac{t_0}{2}]$,

$$(2t)^{2+\theta} S_{2t} \leqslant 2^{2+\theta} C_g \mathbb{V}^W\left[\phi(\mu^g_{t_0})\right]^{1/2} + \frac{1}{2} t^{2+\theta} S_t.$$

Therefore, denoting by $\mathbf{S} := \sup_{t\in(0,t_0]} t^{2+\theta} S_t$, we have $\mathbf{S} \leqslant 2^{2+\theta} C_g \mathbb{V}^W\left[\phi(\mu^g_{t_0})\right]^{1/2} + \frac{1}{2}\mathbf{S}$. Since $\mathbf{S} < +\infty$, we obtain $\mathbf{S} \leqslant 2^{3+\theta} C_g \mathbb{V}^W\left[\phi(\mu^g_{t_0})\right]^{1/2}$. Applying this for $t = t_0$, we proved that for

every $t_0 \in (0,T]$, $S_{t_0} \leqslant 2^{3+\theta} C_g \frac{1}{t_0^{2+\theta}} \mathbb{V}^W \left[\phi(\mu_{t_0}^g)\right]^{1/2}$. We deduce that there is $C_g > 0$ such that for every deterministic 1-periodic function $k : \mathbb{R} \to \mathbb{R}$ and for every $t \in (0,T]$, we have

$$\mathbb{E}\left[\left|\int_0^1 \left[\frac{\delta P_t \phi}{\delta m}\right](\mu_0^g)(g(u))k(u)\mathrm{d}u\right|^2\right]^{1/2} \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{t^{2+\theta}} \|k\|_{L_\infty}.$$

Since g and k are deterministic functions, we have

$$\left|\int_0^1 \left[\frac{\delta P_t \phi}{\delta m}\right](\mu_0^g)(g(u))k(u)\mathrm{d}u\right| \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{t^{2+\theta}} \|k\|_{L_\infty}.$$

Let $h \in \Delta^1$. Thus $k = \partial_u\left(\frac{h}{g'}\right)$ is a 1-periodic function and we deduce that

$$\begin{aligned}\left|\int_0^1 \left[\frac{\delta P_t \phi}{\delta m}\right](\mu_0^g)(g(u))\partial_u\left(\frac{h}{g'}\right)(u)\mathrm{d}u\right| &\leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{t^{2+\theta}} \|\partial_u \tfrac{h}{g'}\|_{L_\infty} \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{\frac{1}{2}}}{t^{2+\theta}} \left\|\frac{h}{g'}\right\|_{\mathcal{C}^1} \\ &\leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{\frac{1}{2}}}{t^{2+\theta}} \|h\|_{\mathcal{C}^1},\end{aligned}$$

for a modified constant C_g. Let us apply equality (III.55):

$$\left|\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h})\right| \leqslant C_g \frac{\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}}{t^{2+\theta}} \|h\|_{\mathcal{C}^1},$$

which is the result of the theorem. □

III.4.5 Regularization of the function ϕ

In Theorem III.29, we have assumed that ϕ satisfies the ϕ-assumptions, *i.e.* that $\widehat{\phi}$ belongs to $C_{\mathrm{b}}^{1,1}(L_2(\Omega))$. However, Bismut-Elworthy-like inequality (III.56) does not involve any derivative of ϕ. In this paragraph, we show that a locally Lipschitz bound holds true for a larger class of $\mathbb{T}$-stable functions $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, namely for every bounded and uniformly continuous function ϕ. We start by showing that this class of functions is linked with the set of bounded and continuous functions $\widehat{\phi}$ on $\mathcal{P}(\mathbb{T})$.

Recall that $W_2^{\mathbb{T}}$ is the L_2-Wasserstein distance on $\mathcal{P}(\mathbb{T})$. Let $\widetilde{\phi} : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ be a bounded and continuous function with respect to $W_2^{\mathbb{T}}$. For every $\mu \in \mathcal{P}_2(\mathbb{R})$, we associate the probability measure $\widetilde{\mu} \in \mathcal{P}(\mathbb{T})$ defined by $\widetilde{\mu}(A) = \mu(A + 2\pi\mathbb{Z})$ for every $A \in \mathcal{B}([0,2\pi])$. If X is a random variable with distribution μ, then the law of $\{X\}$ is $\widetilde{\mu}$. Let us define $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ by $\phi(\mu) := \widetilde{\phi}(\widetilde{\mu})$.

Proposition III.44. *If $\widetilde{\phi} : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ is a bounded and continuous function with respect to the Wasserstein distance $W_2^{\mathbb{T}}$, then $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ defined as above is $\mathbb{T}$-stable, bounded and uniformly continuous with respect to the usual L_2-Wasserstein distance on $\mathbb{R}$.*

Proof. The properties of $\mathbb{T}$-stability and of boundedness are clear. Let us prove uniform continuity.

Let $\varepsilon > 0$. Since $\mathbb{T}$ is compact, it follows that $\mathcal{P}(\mathbb{T})$ is compact for the weak topology. Moreover, again by compactness of $\mathbb{T}$, the weak topology on $\mathcal{P}(\mathbb{T})$ is equal to the topology induced by the distance $W_2^{\mathbb{T}}$. Thus the continuity of the map $\widetilde{\phi} : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ with respect to $W_2^{\mathbb{T}}$ implies its uniform continuity: there is $\delta > 0$ such that for every $\widehat{\mu}, \widehat{\nu}$ in $\mathcal{P}(\mathbb{T})$ such that $W_2^{\mathbb{T}}(\widehat{\mu}, \widehat{\nu}) < \delta$, we have $|\widetilde{\phi}(\widehat{\mu}) - \widetilde{\phi}(\widehat{\nu})| < \varepsilon$.

Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ such that $W_2(\mu,\nu) < \delta$. Then there is a couple (X,Y) of random variables, X with law μ, Y with law ν, such that $\mathbb{E}\left[|X-Y|^2\right]^{1/2} < \delta$. Thus the law of $\{X\}$ is $\widetilde{\mu}$ and the law of $\{Y\}$ is $\widetilde{\nu}$. Moreover, $d^{\mathbb{T}}(\{X\},\{Y\}) \leqslant |X-Y|$, thus we have

$$W_2^{\mathbb{T}}(\widetilde{\mu},\widetilde{\nu}) \leqslant \mathbb{E}\left[d^{\mathbb{T}}(\{X\},\{Y\})^2\right]^{1/2} \leqslant \mathbb{E}\left[|X-Y|^2\right]^{1/2} < \delta.$$

Hence $|\widetilde{\phi}(\widetilde{\mu}) - \widetilde{\phi}(\widetilde{\nu})| < \varepsilon$ and it follows by definition of ϕ that $|\phi(\mu)-\phi(\nu)| < \varepsilon$. This proves that ϕ is uniformly continuous. □

Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a $\mathbb{T}$-stable function, which is bounded and uniformly continuous on $\mathcal{P}_2(\mathbb{R})$. Recall that by (III.45), for every $t \in [0,T]$, for every $g \in \mathbf{G}^1$,

$$P_t\phi(\mu_0^g) = \mathbb{E}^W\left[\phi(\mu_t^g)\right] = \mathbb{E}^W\left[\widehat{\phi}(x_t^g)\right],$$

where $\widehat{\phi}(X) := \phi(\mathcal{L}_{[0,1]\times\Omega^\beta}(X))$ for every $X \in L_2([0,1]\times\Omega^\beta)$. Moreover, $\widehat{\phi}$ is bounded and uniformly continuous on $L_2([0,1]\times\Omega^\beta)$.

Let $\Delta^{3+\theta}$ be the set of 1-periodic $\mathcal{C}^{3+\theta}$-functions $h : \mathbb{R} \to \mathbb{R}$.

Theorem III.45. *Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a $\mathbb{T}$-stable function, which is bounded and uniformly continuous on $\mathcal{P}_2(\mathbb{R})$. Let $\theta \in (0,1)$ and f be of order $\alpha = \frac{7}{2} + \theta$. Let $g \in \mathbf{G}^{3+\theta}$ and $h \in \Delta^{3+\theta}$. Then there is $\rho_0 > 0$ and a constant $C_{g,h}$ both depending on g and h such that for every $t \in (0,T]$ and for every $\rho \in [-\rho_0,\rho_0]$,*

$$\left|P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g)\right| \leqslant |\rho| C_{g,h} \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}},$$

where $C_{g,h}$ is bounded when $\|g'''\|_{L_\infty} + \|g''\|_{L_\infty} + \|g'\|_{L_\infty} + \|\frac{1}{g'}\|_{L_\infty} + \|h\|_{\mathcal{C}^3}$ is bounded.

Proof. **Step 1:** Let us assume that ϕ satisfies the ϕ-assumptions. Let $\rho_0 = \frac{1}{2\|h'\|_{L_\infty}\|\frac{1}{g'}\|_{L_\infty}}$. Then for every $|\rho| \leqslant \rho_0$ and for every $u \in [0,1]$, $g'(u) + \rho h'(u) \geqslant \frac{g'(u)}{2}$. We have for every $|\rho| \leqslant \rho_0$:

$$P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g) = \int_0^\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\left\{P_t\phi(\mu_0^{g+rh})\right\}\mathrm{d}r.$$

Let us apply, for every r between 0 and ρ, the result of Theorem III.29 to μ_0^{g+rh}:

$$\left|P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g)\right| \leqslant \left|\int_0^\rho C_{g+rh}\frac{\|\phi\|_{L_\infty}}{t^{2+\theta}}\|h\|_{\mathcal{C}^1}\mathrm{d}r\right|,$$

where C_{g+rh} is bounded when $\|g''' + rh'''\|_{L_\infty} + \|g'' + rh''\|_{L_\infty} + \|g' + rh'\|_{L_\infty} + \|\frac{1}{g'+rh'}\|_{L_\infty}$ is bounded. Note that for every $|r| \leqslant \rho_0$,

$$\left\|\frac{1}{g'+rh'}\right\|_{L_\infty} \leqslant 2\left\|\frac{1}{g'}\right\|_{L_\infty};$$
$$\|g^{(k)} + rh^{(k)}\|_{L_\infty} \leqslant \|g^{(k)}\|_{L_\infty} + \rho_0\|h^{(k)}\|_{L_\infty}.$$

Thus, if we choose a constant $C_{g,h}$ being equal to $\|h\|_{\mathcal{C}^1}\sup_{r\in[-\rho_0,\rho_0]} C_{g+rh}$, we have

$$\left|P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g)\right| \leqslant |\rho| C_{g,h}\frac{\|\phi\|_{L_\infty}}{t^{2+\theta}},$$

and $C_{g,h}$ is bounded when $\|g'''\|_{L_\infty} + \|g''\|_{L_\infty} + \|g'\|_{L_\infty} + \|\frac{1}{g'}\|_{L_\infty} + \|h\|_{\mathcal{C}^3}$ is bounded. Hence the statement of the theorem is true in the case where ϕ satisfies the ϕ-assumptions.

Step 2: Let us assume now that ϕ is $\mathbb{T}$-stable, bounded and continuons on $\mathcal{P}_2(\mathbb{R})$. Recall that $\widehat{\phi}$ is bounded and uniformly continuous on $H := L_2([0,1]\times\Omega^\beta)$. Furthermore, $\widehat{\phi}(X) = \widehat{\phi}(X')$ if $\mathcal{L}(X) \sim \mathcal{L}(X')$ in the sense of Definition III.5. Therefore, let us apply the inf-sup-convolution method introduced by Lasry and Lions [LL86]. Let us define for every $0 < \delta < \varepsilon$, for every $X \in H$:

$$\widehat{\phi}^\varepsilon(X) := \sup_{Y\in H}\left\{\widehat{\phi}(Y) - \frac{1}{2\varepsilon}\|X-Y\|_H\right\};$$
$$\widehat{\phi}^{\varepsilon,\delta}(X') := \inf_{Y\in H}\left\{\widehat{\phi}^\varepsilon(Y) + \frac{1}{2\delta}\|X'-Y\|_H\right\},$$

where $\|\cdot\|_H$ denotes the L_2-norm on $[0,1]\times\Omega^\beta$. Remark that for the sup-convolution we should actually extend $\widehat{\phi}$ to $H_2 = L_2([0,1]\times\Omega^\beta\times[0,1])$ and take a supremum over $Y \in H_2$ of $\widehat{\phi}(Y) - \frac{1}{2\varepsilon}\|X-Y\|_{H_2}$, in order to ensure, for any $X \in H$, the existence of a random variable on H_2 independent of X (and do the same for the inf-convolution). We refer to the proof of Lemma II.44 for details about this strategy; for simplicity, we will assume that for any $X \in H$, we can construct on H a random variable independent of X. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 and $\psi^n := \widehat{\phi}^{\varepsilon_n,\varepsilon_n/2}$. By [LL86], for every $n \in \mathbb{N}$, ψ^n belongs to $C_{\mathrm{b}}^{1,1}(H)$, the set of bounded $\mathcal{C}^1$-functions defined on H such that the gradient is bounded and Lipschitz on H. Furthermore, the sequence $(\psi^n)_{n\in\mathbb{N}}$ converges to $\widehat{\phi}$ uniformly on H.

Let us check that for each $n \in \mathbb{N}$, there is $\phi^n : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfying the ϕ-assumptions such that $\widehat{\phi^n} = \psi^n$. First, we have to prove that if $X, X' \in H$ are two random variables such that $\mathcal{L}(X) \sim \mathcal{L}(X')$, then $\psi^n(X) = \psi^n(X')$. Recall that $\psi^n := \widehat{\phi}^{\varepsilon_n,\varepsilon_n/2}$; thus we only prove that property for $\widehat{\phi}^\varepsilon$ since the definition of $\widehat{\phi}^{\varepsilon,\delta}$ is symmetric.

Let $\varepsilon > 0$. First of all, recall that there is a $\mathbb{Z}$-valued random variable $k(X)$ such that $X = \{X\} + 2k(X)\pi$. Since X and $\{X\}$ belong to $H = L_2([0,1]\times\Omega^\beta)$, $2k(X)\pi$ also belongs to H. Thus

$$\begin{aligned}\widehat{\phi}^\varepsilon(\{X\}) = \sup_{Y\in H}\left\{\widehat{\phi}(Y) - \frac{1}{2\varepsilon}\|\{X\}-Y\|_H\right\} &= \sup_{Y\in H}\left\{\widehat{\phi}(Y) - \frac{1}{2\varepsilon}\|X-2k(X)\pi-Y\|_H\right\}\\ &= \sup_{Y\in H}\left\{\widehat{\phi}(Y+2k(X)\pi) - \frac{1}{2\varepsilon}\|X-Y\|_H\right\}\\ &= \sup_{Y\in H}\left\{\widehat{\phi}(Y) - \frac{1}{2\varepsilon}\|X-Y\|_H\right\} = \widehat{\phi}^\varepsilon(X)\end{aligned}$$

because $\widehat{\phi}(Y+2k(X)\pi) = \widehat{\phi}(Y)$.

Let X and X' be two random variables such that $\mathcal{L}(X) \sim \mathcal{L}(X')$. Then $\mathcal{L}(\{X\}) = \mathcal{L}(\{X'\})$. Let us prove that $\widehat{\phi}^\varepsilon(\{X\}) = \widehat{\phi}^\varepsilon(\{X'\})$. Let $\delta > 0$. By definition of $\widehat{\phi}^\varepsilon(\{X\})$, there is $Y^\delta \in H$ such that

$$\widehat{\phi}^\varepsilon(\{X\}) - \delta \leqslant \widehat{\phi}(Y^\delta) - \frac{1}{2\varepsilon}\|\{X\}-Y^\delta\|_H. \tag{III.123}$$

Let us denote by η the conditional distribution of Y^δ given $\{X\}$. More precisely, $\eta : \mathbb{R}\times\mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is a function such that for every $x \in \mathbb{R}$, $\eta(x,\cdot)$ belongs to $\mathcal{P}(\mathbb{R})$, for every $A \in \mathcal{B}(\mathbb{R})$, $x \mapsto \eta(x,A)$ is measurable, and for every $f : \mathbb{R}^2 \to \mathbb{R}$ bounded and measurable, $\mathbb{E}\left[f(\{X\},Y^\delta)\right] = \mathbb{E}\left[\int_{\mathbb{R}} f(\{X\},y)\eta(\{X\},\mathrm{d}y)\right]$, where $\mathbb{E}[\cdot] := \mathbb{E}^\beta\left[\int_0^1 \cdot\,\mathrm{d}u\right]$.

Moreover, for every $x \in \mathbb{R}$, let us denote by $u \in [0,1] \mapsto g(x,u)$ the quantile function associated to the probability measure $\eta(x,\cdot)$. For every $t \in \mathbb{R}$ and for every $u \in [0,1]$, $\{x : g(x,u) \leqslant t\} = \{x : \eta(x,(-\infty,t]) \leqslant u\} \in \mathcal{B}(\mathbb{R})$, so we deduce that for every $u \in [0,1]$, $x \mapsto g(x,u)$ is measurable. Moreover, $u \mapsto g(x,u)$ is a càdlàg function. It follows from [KS91, Proposition 1.13] that $(x,u) \mapsto g(x,u)$ is measurable.

Let us define U be a random variable independent of $\{X'\}$ and with a uniform distribution on $[0,1]$. Let $Y' := g(\{X'\}, U)$. Then the following computation shows that the law of $(\{X'\}, Y')$ is equal to the law of $(\{X\}, Y^\delta)$. Indeed, for every $f : \mathbb{R}^2 \to \mathbb{R}$ bounded and measurable

$$\begin{aligned}\mathbb{E}\left[f(\{X'\}, Y')\right] = \mathbb{E}\left[f(\{X'\}, g(\{X'\}, U))\right] &= \mathbb{E}\left[\int_0^1 f(\{X'\}, g(\{X'\}, u))\mathrm{d}u\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} f(\{X'\}, y)\eta(\{X'\}, \mathrm{d}y)\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} f(\{X\}, y)\eta(\{X\}, \mathrm{d}y)\right] = \mathbb{E}\left[f(\{X\}, Y^\delta)\right],\end{aligned}$$

because $\mathcal{L}(\{X\}) = \mathcal{L}(\{X'\})$. Thus by (III.123), for every $\delta > 0$,

$$\widehat{\phi}^\varepsilon(\{X\}) - \delta \leqslant \widehat{\phi}(Y^\delta) - \frac{1}{2\varepsilon}\|\{X\} - Y^\delta\|_H = \widehat{\phi}(Y') - \frac{1}{2\varepsilon}\|\{X'\} - Y'\|_H \leqslant \widehat{\phi}^\varepsilon(\{X'\}).$$

We deduce that $\widehat{\phi}^\varepsilon(\{X\}) \leqslant \widehat{\phi}^\varepsilon(\{X'\})$. By symmetry, it follows that $\widehat{\phi}^\varepsilon(\{X\}) = \widehat{\phi}^\varepsilon(\{X'\})$. Thus $\widehat{\phi}^\varepsilon(X) = \widehat{\phi}^\varepsilon(X')$.

Therefore, for every $n \in \mathbb{N}$, $\psi^n(X) = \psi^n(X')$ if $\mathcal{L}(X) \sim \mathcal{L}(X')$. Let $\phi^n : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be the unique function satisfying $\phi^n(\mathcal{L}_{[0,1]\times\Omega^\beta}(X)) = \psi^n(X)$ for every $X \in L_2([0,1] \times \Omega^\beta)$. Therefore, ϕ^n is $\mathbb{T}$-stable. Furthermore, the fact that ψ^n belongs to $C^{1,1}_{\mathrm{b}}(H)$ implies that ϕ^n satisfy the ϕ-assumptions.

By Step 1, it follows that for every $n \in \mathbb{N}$

$$\left|P_t\phi^n(\mu_0^{g+\rho h}) - P_t\phi^n(\mu_0^g)\right| \leqslant |\rho| C_{g,h}\frac{\|\phi^n\|_{L_\infty}}{t^{2+\theta}},$$

Recall that $(\psi^n)_{n\in\mathbb{N}}$ converges to $\widehat{\phi}$ uniformly on H and that $\psi^n = \widehat{\phi^n}$. Thus for n large enough, $\|\phi^n\|_{L_\infty} \leqslant 2\|\phi\|_{L_\infty}$. Moreover, $P_t\phi^n(\mu_0^g) = \mathbb{E}^W\left[\psi^n(x_t^g)\right]$ for every $g \in \mathbf{G}^1$. We deduce that

$$\begin{aligned}\left|P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g)\right| &= \left|\mathbb{E}^W\left[\widehat{\phi}(x_t^{g+\rho h})\right] - \mathbb{E}^W\left[\widehat{\phi}(x_t^g)\right]\right| \\ &\leqslant \left|\mathbb{E}^W\left[\psi^n(x_t^{g+\rho h})\right] - \mathbb{E}^W\left[\psi^n(x_t^g)\right]\right| + 2\sup_{X\in H}|\psi^n(X) - \widehat{\phi}(X)| \\ &= \left|P_t\phi^n(\mu_0^{g+\rho h}) - P_t\phi^n(\mu_0^g)\right| + 2\sup_{X\in H}|\psi^n(X) - \widehat{\phi}(X)| \\ &\leqslant |\rho| C_{g,h}\frac{\|\phi\|_{L_\infty}}{t^{2+\theta}} + 2\sup_{X\in H}|\psi^n(X) - \widehat{\phi}(X)|.\end{aligned}$$

The statement of the theorem follows since $\sup_{X\in H}|\psi^n(X) - \widehat{\phi}(X)| \to_{n\to+\infty} 0$. □

Chapter IV

Gradient estimate for an inhomogeneous SPDE

Abstract. This chapter consists in improving the explosion rate in the gradient estimate obtained in Chapter III for a diffusion on the Wasserstein space on the torus. That rate is approximatively of order t^{-2}. Assuming more regularity on the points at which we want to obtain a gradient estimate and more regularity on the directions of perturbation, we obtain a rate approximatively of order t^{-1}. This is achieved by means of an interpolation method, which allows us to overcome the lack of regularity of the derivative of the diffusion, which was a hindrance in the previous chapter. As an application of this result, we give a gradient estimate for a related inhomogeneous SPDE featuring a non-trivial source term.

IV.1 Introduction

In the previous chapter, we obtained a gradient estimate of Bismut-Elworthy-type for the semi-group associated with a diffusion on the Wasserstein space on the torus. Recall that the explosion rate of that estimate is of order $t^{-(2+\theta)}$ (see Theorems III.29 and III.45), which remains quite far from the explosion rate of order $t^{-1/2}$ of the heat semi-group and which is closer to the rate observed for hypoelliptic models in finite dimension, see for instance [KS84, KS85, KS87]. In this chapter, our aim is to improve the rate by assuming more regularity on the admissible initial condition g and on the admissible direction of perturbation h (see the aforementioned two statements for the precise meaning of g and h). We obtain a new rate of order $t^{-(1+\theta)}$. This is enough, as we will see, to get a gradient estimate for the SPDE driven by the generator of the diffusion and with a non-trivial source term.

Recall the framework of Chapter III. In Section III.3, we introduced the probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$, defined as the product space of $(\Omega^W, \mathcal{G}^W, \mathbb{P}^W)$, $(\Omega^\beta, \mathcal{G}^\beta, \mathbb{P}^\beta)$ and $(\Omega^0, \mathcal{G}^0, \mathbb{P}^0)$. Let $g \in \mathbf{G}^1$ (recall definition III.2). We consider, as in (III.20), the following SDE on the space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$:

$$\begin{cases} dx_t^g(u) = \displaystyle\sum_{k\in\mathbb{Z}} f_k \Re\left(e^{-ikx_t^g(u)} \mathrm{d}W_t^k\right) + \mathrm{d}\beta_t, \\ x_0^g = g, \end{cases} \tag{IV.1}$$

where the noises $(W^k)_{k\in\mathbb{Z}}$ and β are respectively carried out by $(\Omega^W, \mathcal{G}^W, \mathbb{P}^W)$ and $(\Omega^\beta, \mathcal{G}^\beta, \mathbb{P}^\beta)$ and hence are independent. Recall the definition of the process $(\mu_t^g)_{t\in[0,T]}$, given in Defini-

tion III.24:

$$\mu_t^g := \left(\mathrm{Leb}_{[0,1]} \otimes \mathbb{P}^\beta\right) \circ (x_t^g)^{-1}.$$

In other words, μ_t^g is the conditional law of x_t^g given $(W^k)_{k\in\mathbb{Z}}$. Let us write the stochastic partial differential equation satisfied by the process $(\mu_t^g)_{t\in[0,T]}$, by the same method as the one used to get (II.38). Let $\Upsilon : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic $\mathcal{C}^2$-function. By Itô's formula, for each $u \in [0,1]$,

$$\mathrm{d}\Upsilon(x_t^g(u)) = \sum_{k\in\mathbb{Z}} \Upsilon'(x_t^g(u)) f_k \Re\left(e^{-ikx_t^g(u)}\mathrm{d}W_t^k\right) + \Upsilon'(x_t^g(u))\mathrm{d}\beta_t + \frac{1}{2}\Upsilon''(x_t^g(u))(C_f+1)\mathrm{d}t, \tag{IV.2}$$

where $C_f = \sum_{k\in\mathbb{Z}} f_k^2$. Recall equality (III.43): $\mathbb{E}^\beta\left[\int_0^1 \Upsilon(x_t^g(u))\mathrm{d}u\right] = \int_{\mathbb{T}} \Upsilon(x)\mathrm{d}\mu_t^g(x)$, which also holds with Υ' and Υ''. Therefore, integrating equation (IV.2) with respect to $\mathbb{P}^\beta \otimes \mathrm{Leb}_{[0,1]}$, we get:

$$\mathrm{d}_t \int_{\mathbb{T}} \Upsilon(x)\mathrm{d}\mu_t^g(x) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} \Upsilon'(x) f_k \Re(e^{-ikx}\mathrm{d}W_t^k)\mathrm{d}\mu_t^g(x) + \frac{C_f+1}{2}\int_{\mathbb{T}} \Upsilon''(x)\mathrm{d}\mu_t^g(x)\mathrm{d}t.$$

By integration by parts, using the periodicity assumption on Υ, it follows that the process $(\mu_t^g)_{t\in[0,T]}$ satisfies the following SPDE:

$$\begin{cases} \mathrm{d}\mu_t^g - \mathbf{L}(\mu_t^g) = 0, \\ \qquad \mu_t^g\big|_{t=0} = \mu_0^g, \end{cases} \tag{IV.3}$$

where

$$\mathbf{L}(\mu_t^g) := \frac{C_f+1}{2}\,\partial_{xx}^2(\mu_t^g)\mathrm{d}t - \partial_x\left(\sum_{k\in\mathbb{Z}} f_k \Re(e^{-ik\cdot}\,\mathrm{d}W_t^k)\;\mu_t^g\right). \tag{IV.4}$$

Recall that the semi-group $(P_t)_{t\in[0,T]}$ associated to the SPDE (IV.3) is defined by

$$P_t\phi(\mu_0^g) = \mathbb{E}^W\left[\phi(\mu_t^g)\right].$$

Let us introduce the main result of this chapter, which consists in an improvement of the rate obtained in Theorem III.29. The assumptions on the initial condition g and the direction of perturbation h are the following: we assume g to belong to $\mathbf{G}^{4+\theta}$ (recall definition III.2) and h to belong to $\Delta^{4+\theta}$, the latter standing for the set of 1-periodic $\mathcal{C}^{4+\theta}$-functions. The definition of $\mathcal{C}^{4+\theta}$ and of the norm $\|\cdot\|_{\mathcal{C}^{4+\theta}}$ are given by:

Definition IV.1. For every $p \in \mathbb{N}$ and for every $\alpha \in (0,1)$, we denote by $\mathcal{C}^{p+\alpha}$ the set of $\mathcal{C}^p$-functions $f : [0,1] \to \mathbb{R}$ with an α-Hölder-continuous derivative of order p. We define the following norm on $\mathcal{C}^{p+\alpha}$:

$$\|f\|_{\mathcal{C}^{p+\alpha}} := \sum_{k=0}^{p} |\partial_u^{(k)} f(0)| + \sup_{\substack{u,v\in[0,1]\\ u\neq v}} \frac{|\partial_u^{(p)} f(u) - \partial_u^{(p)} f(v)|}{|u-v|^\alpha}.$$

For every $p \in \mathbb{N}$, we will denote by $\|\cdot\|_{C^p}$ the usual norm: $\|f\|_{\mathcal{C}^p} := \sum_{k=0}^p \|\partial_u^{(k)} f\|_{L_\infty[0,1]}$.

Under these assumptions on g and h, we obtain a rate of order $t^{-(1+3\theta)}$:

Theorem IV.2. *Let $\theta \in (0, \frac{1}{2})$. Let f be of order $\alpha = \frac{7}{2} + \theta$. Let $g \in \mathbf{G}^{4+\theta}$ and $h \in \Delta^{4+\theta}$. Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfy the ϕ-assumptions given by definition III.6. Then, there is a constant C depending polynomially on $\|g\|_{\mathcal{C}^{4+\theta}}$ and independent of h, ϕ, θ and t such that for every $t \in (0, T]$,*

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t \phi(\mu_0^{g+\rho h}) \right| \leqslant \frac{C}{t^{1+3\theta}} \mathbb{V}^W \left[\phi(\mu_t^g)\right]^{1/2} \|h\|_{\mathcal{C}^{4+\theta}}. \tag{IV.5}$$

Recall that the hindrance in Theorem III.29, explaining why the rate therein cannot be better than $t^{-(2+\theta)}$, follows from the lack of regularity of $A_t^g = (\partial_u x_t^g \cdot h) \circ F_t^g$ for any $t > 0$, the latter matching the left-hand side of (III.12). In order to obtain in (III.11) a square-integrable inverse process $(\lambda_t^k)_{t\in[0,T]}$, we need $\sum_{k\in\mathbb{Z}} \frac{|c_k(A_t^g)|^2}{f_k^2}$ to be finite, where c_k is a generic notation for the kth Fourier coefficient of a periodic function. Thus, if f is of order $\alpha = \frac{7}{2} + \theta$, we need A_t^g to be at least of class $\mathcal{C}^{4+\theta}$ for every $t \in [0, T]$. Here, we assume that $h \in \mathcal{C}^{4+\theta}$; nevertheless, it is not sufficient to ensure the desired regularity on A_t^g. Indeed, f being of order $\frac{7}{2} + \theta$, it follows from Proposition III.13 that $u \mapsto \partial_u x_t^g(u)$ only belongs to $\mathcal{C}^{2+\theta'}$, for some $\theta' < \theta$. Again from Proposition III.13, recall that $u \mapsto \partial_u x_t^g(u)$ may be explicitly represented in the form:

$$\partial_u x_t^g(u) = g'(u) \exp\Big(\sum_{k\in\mathbb{Z}} f_k \int_0^t \Re\left(-ike^{-ikx_s^g(u)} \mathrm{d}W_s^k\right) - \frac{t}{2} \sum_{k\in\mathbb{Z}} f_k^2 k^2 \Big). \tag{IV.6}$$

The main idea of the proof of Theorem IV.2 consists in replacing $\partial_u x_t^g(u)$ by

$$u \mapsto g'(u) \exp\Big(\sum_{|k|\leqslant N_t} f_k \int_0^t \Re\left(-ike^{-ikg(u)} \mathrm{d}W_s^k\right) - \frac{t}{2} \sum_{k\in\mathbb{Z}} f_k^2 k^2 \Big). \tag{IV.7}$$

Both modifications (replacing x_s^g by g and truncating the sum at a finite range N_t depending on t) allows (IV.7) to belong to $\mathcal{C}^{4+\theta'}$ for a certain θ', as we will show in this chapter. Then, we have to deal with the remainder term, consisting in the difference between (IV.6) and (IV.7), and to interpolate both results to get (IV.5).

Let us now give an application of Theorem IV.2. We can easily deduce a gradient estimate for the following SPDE, which is an extension of (IV.3) to the case of a non-trivial source term F:

$$\begin{cases} \mathrm{d}\mu_t^g - \mathbf{L}(\mu_t^g) = F(t, \mu_t^g), \\ \qquad \mu_t^g\big|_{t=0} = \mu_0^g, \end{cases} \tag{IV.8}$$

where $F : [0, T] \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$. Let $(Q_t)_{t\in[0,T]}$ be the semi-group associated to the solution $(\mu_t^g)_{t\in[0,T]}$ to equation (IV.8). Then, under a uniform Hölder-regularity assumption on F, we prove a gradient estimate for $(Q_t)_{t\in[0,T]}$ in the form of a Bismut-Elworthy-like inequality:

Theorem IV.3. *Let $\theta \in (0, \frac{1}{8})$. Let f be of order $\alpha = \frac{7}{2} + \theta$. Let $g \in \mathbf{G}^{4+\theta}$ and $h \in \Delta^{4+\theta}$. Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfy the ϕ-assumptions given by definition III.6. Let the source term F satisfy the following Hölder-assumption: there is $C > 0$ such that for every $t \in [0, T]$ and for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $|F(t, \mu) - F(t, \nu)| \leqslant C W_2(\mu, \nu)^{8\theta}$.*

Then, there is a constant C depending on g, ϕ and F and independent of h, θ and t such that for every $t \in (0, T]$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} Q_t \phi(\mu_0^{g+\rho h}) \right| \leqslant \frac{C}{t^{1+3\theta}} \|h\|_{\mathcal{C}^{4+\theta}}.$$

The Hölder-regularity assumption on F follows from the fact that the gradient of the semi-group is nearly, but is not, integrable in small time. Somehow, the additional gap to get integrability is here filled by the extra regularity of F. To make it clear, we need $s \mapsto \frac{1}{(t-s)^{1+3\theta}} \mathbb{V}^W \left[F(s, \mu^g_{t-s})\right]^{1/2}$ to be integrable on $[0,t]$. In other words, the variance term has to compensate the fact that $t \mapsto t^{-(1+3\theta)}$ is not integrable on $[0,T]$. The assumption on F is tailored-made to force $\mathbb{V}^W \left[F(s, \mu^g_{t-s})\right]^{1/2}$ to be of order $(t-s)^{4\theta}$ and hence to obtain the desired result.

Organisation of the chapter

Sections IV.2 and IV.3 are devoted to the proof of Theorem IV.2. In Section IV.2, we will explain and prove the interpolation method, assuming that we can split $A^g_t = (\partial_u x^g_t \cdot h) \circ F^g_t$ into two terms, the dominant one having sufficient regularity and the remainder term being small with t when t tends to 0^+. Then in Section IV.3, we will prove that the aforementioned splitting holds true, with suitable controls on the decomposition (for a relevant choice of the underlying norms). Essentially, such a splitting consists in replacing the derivative (IV.6) by (IV.7) and applying a similar decomposition for the c.d.f F^g_t. Finally, in Section IV.4, we will prove Theorem IV.3 as an application of the previous result.

IV.2 Interpolation argument

Throughout this chapter, θ is a fixed real number in $(0, \frac{1}{2})$ and f is given and of order $\alpha = \frac{7}{2} + \theta$. Every theorem, proposition, corollary, lemma hold under this assumption, that will not be necessarily recalled. Moreover, for every $k \in \mathbb{Z}$, $\langle k \rangle$ denotes the quantity $(1+|k|^2)^{1/2}$.

We want to prove Theorem IV.2, using an interpolation method. Let us start by explaining it. Let $g \in \mathbf{G}^{4+\theta}$. Let $h \in \Delta^{4+\theta}$. Let $t \in (0,T]$. Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfy the ϕ-assumptions (recall definition III.6). By (III.52), we have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) = \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \partial_\mu \phi(\mu^g_t)(x^g_t(u)) \partial_u x^g_t(u) \frac{h(u)}{g'(u)} \mathrm{d}u \right].$$

Let us write h as follows:

$$h(u) = \frac{2}{t} \int_0^{t/2} h(u) \mathrm{d}s = \frac{2}{t} \int_0^{t/2} \frac{\partial_u x^g_s(u)}{\partial_u x^g_s(u)} h(u) \mathrm{d}s.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) = \frac{2}{t} \mathbb{E}^W \mathbb{E}^\beta \left[\int_0^1 \int_0^{t/2} \partial_\mu \phi(\mu^g_t)(x^g_t(u)) \frac{\partial_u x^g_t(u)}{\partial_u x^g_s(u)} \mathcal{A}_s(u) \mathrm{d}s \mathrm{d}u \right],$$

where $\mathcal{A}_s(u) := \frac{\partial_u x^g_s(u)}{g'(u)} h(u)$ for every $s \in [0, t/2]$ and every $u \in [0,1]$. Let us remark that $\mathcal{A}_s$ looks like $A^g_s = (\partial_u x^g_s \cdot h) \circ F^g_s$ introduced in the previous chapter, but is not exactly equal.

Assumption: all along this section, we will assume that the following splitting holds. We will then prove it in Section IV.3. Let us assume that $(\mathcal{A}_s)_{s\in[0,t/2]}$ can be written in the following way:

$$\mathcal{A}_s(u) = \mathcal{A}^0_s(x^g_s(u)) + \mathcal{A}^1_s(u), \tag{IV.9}$$

where $(\mathcal{A}^0_s)_{s\in[0,t/2]}$ and $(\mathcal{A}^1_s)_{s\in[0,t/2]}$ are $(\mathcal{G}_t)_{t\in[0,T]}$-adapted processes, where for every $s \in [0,t/2]$, $\mathcal{A}^0_s : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic $\mathcal{C}^{4+\theta/16}$-function and $\mathcal{A}^1_s : \mathbb{R} \to \mathbb{R}$ is a 1-periodic $\mathcal{C}^1$-function such

that for every $p \geqslant 2$, there exists $C_p > 0$ satisfying for every $s \in (0, t/2]$:

$$\mathbb{E}^W \mathbb{E}^\beta \left[\|\mathcal{A}_s^0\|_{\mathcal{C}^{4+\theta/16}}^p \right]^{1/p} \leqslant \frac{C_p}{s^{\frac{1}{2}-2\theta}} \|h\|_{\mathcal{C}^{4+\theta}}; \tag{IV.10}$$

$$\mathbb{E}^W \mathbb{E}^\beta \left[\|\mathcal{A}_s^0\|_{\mathcal{C}^3}^p \right]^{1/p} \leqslant C_p \|h\|_{\mathcal{C}^{3+\theta}}; \tag{IV.11}$$

$$\mathbb{E}^W \mathbb{E}^\beta \left[\|\mathcal{A}_s^1\|_{\mathcal{C}^1}^p \right]^{1/p} \leqslant C_p s^{1-2\theta} \|h\|_{\mathcal{C}^2}. \tag{IV.12}$$

Under that assumption, let us now prove Theorem IV.2. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a collection of mollifiers, defined by $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. For every $s \in [0, t/2]$ and $u \in [0,1]$, we have $\mathcal{A}_s(u) = (\mathcal{A}_s^0 * \varphi_s)(x_s^g(u)) + (\mathcal{A}_s^0 - \mathcal{A}_s^0 * \varphi_s)(x_s^g(u)) + \mathcal{A}_s^1(u)$. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) &= \frac{2}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2} \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}(\mathcal{A}_s^0 * \varphi_s)(x_s^g(u))\mathrm{d}s\mathrm{d}u\right] \\ &+ \frac{2}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2} \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}\left\{(\mathcal{A}_s^0 - \mathcal{A}_s^0 * \varphi_s)(x_s^g(u)) + \mathcal{A}_s^1(u)\right\}\mathrm{d}s\mathrm{d}u\right]. \end{aligned} \tag{IV.13}$$

Let us first focus on the first term on the right-hand side of (IV.13).

Proposition IV.4. *Let $t \in (0,T]$ and let $(\mathcal{A}_s^0)_{s\in[0,t/2]}$ be a $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process such that for every $s \in [0,t/2]$, $\mathcal{A}_s^0$ is a $\mathcal{C}^{4+\theta/2}$-function and* (IV.10) *and* (IV.11) *hold true. Then for every $t \in [0,T]$*

$$\begin{aligned}\frac{2}{t}\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2} \partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}(\mathcal{A}_s^0 * \varphi_s)(x_s^g(u))\mathrm{d}s\mathrm{d}u\right]\right| \\ \leqslant \frac{C}{t^{1-\theta}}\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}\|h\|_{\mathcal{C}^{4+\theta}}.\end{aligned} \tag{IV.14}$$

Proof. The statement of this proposition is very similar to Proposition III.33. Thus we will follow the same strategy to obtain (IV.14). The process $(\mathbb{1}_{\{s\leqslant t/2\}}(\mathcal{A}_s^0 * \varphi_s))_{s\in[0,T]}$ will play the role of $(\mathcal{A}_s^{g,\varepsilon})_{s\in[0,T]}$. We will first define processes $(\lambda_s^k)_{s\in[0,T]}$ and $(\lambda_s^{k,M})_{s\in[0,T]}$ and stopping times $(\tau_M)_{M\geqslant 1}$ such that the statements of Lemmas III.30, III.31, III.32 and III.36 still hold with $(\mathbb{1}_{\{s\leqslant t/2\}}(\mathcal{A}_s^0 * \varphi_s))_{s\in[0,T]}$ instead of $(\mathcal{A}_s^{g,\varepsilon})_{s\in[0,T]}$. Then we will get for free the statement of Proposition III.33.

Definition of λ_s^k and statement of Lemma III.30. For every $k \in \mathbb{Z}$ and for every $s \in [0,T]$, define $\lambda_s^k := \mathbb{1}_{\{s\leqslant t/2\}}\frac{c_k(\mathcal{A}_s^0 * \varphi_s)}{f_k}$, where c_k is the kth Fourier coefficient. Following the proof of Lemma III.30, the collection of $(\mathcal{G}_s)_{s\in[0,T]}$-adapted processes $((\lambda_s^k)_{s\in[0,T]})_{k\in\mathbb{Z}}$ satisfies the following equality for every $s \in [0,T]$

$$\mathbb{1}_{\{s\leqslant t/2\}}(\mathcal{A}_s^0 * \varphi_s)(y) = \sum_{k\in\mathbb{Z}} f_k e^{-iky}\lambda_s^k.$$

Furthermore

$$\sum_{k\in\mathbb{Z}}\int_0^T |\lambda_s^k|^2\mathrm{d}s = \sum_{k\in\mathbb{Z}}\int_0^{t/2}\left|\frac{c_k(\mathcal{A}_s^0 * \varphi_s)}{f_k}\right|^2\mathrm{d}s.$$

Moreover, by the calculation made in (III.66), there is $C > 0$ depending on T such that for every $k \in \mathbb{Z}$ and for every $s \in (0,T]$,

$$|c_k(\mathcal{A}_s^0 * \varphi_s)| \leqslant \frac{C}{\langle k\rangle^\theta s^\theta}|c_k(\mathcal{A}_s^0)|.$$

Recall that for every $s \in [0, t/2]$, $\mathcal{A}^0_s$ is a $\mathcal{C}^{4+\theta/16}$-function. Recall the following property of Fourier coefficients: for every $j \in \mathbb{N}$ and $\theta \in [0,1)$, there is a constant C depending only on j and θ such that for every 2π-periodic $\mathcal{C}^{j+\theta}$-function f and for every $k \in \mathbb{Z}$, $|c_k(f)| \leqslant \frac{C}{\langle k\rangle^{j+\theta}}$. Thus for every $k \in \mathbb{Z}$, $|c_k(\mathcal{A}^0_s)| \leqslant \frac{C}{\langle k\rangle^{4+\theta/16}} \|\mathcal{A}^0_s\|_{\mathcal{C}^{4+\theta/16}}$. We deduce that

$$\sum_{k\in\mathbb{Z}} \int_0^{t/2} \left| \frac{c_k(\mathcal{A}^0_s * \varphi_s)}{f_k} \right|^2 \mathrm{d}s \leqslant C \sum_{k\in\mathbb{Z}} \int_0^{t/2} \langle k\rangle^{7+2\theta} \frac{1}{\langle k\rangle^{2\theta} s^{2\theta}} \frac{1}{\langle k\rangle^{8+\theta/8}} \|\mathcal{A}^0_s\|^2_{\mathcal{C}^{4+\theta/16}} \mathrm{d}s$$
$$\leqslant C \sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^{1+\theta/8}} \int_0^{t/2} \frac{\|\mathcal{A}^0_s\|^2_{\mathcal{C}^{4+\theta/16}}}{s^{2\theta}} \mathrm{d}s.$$

The sum $\sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^{1+\theta/8}}$ is finite. Therefore, it follows from (IV.10) that

$$\mathbb{E}^W\mathbb{E}^\beta \left[\sum_{k\in\mathbb{Z}} \int_0^T |\lambda^k_s|^2 \mathrm{d}s \right] \leqslant C \int_0^{t/2} \frac{\mathbb{E}^W\mathbb{E}^\beta \left[\|\mathcal{A}^0_s\|^2_{\mathcal{C}^{4+\theta/16}} \right]}{s^{2\theta}} \mathrm{d}s \leqslant C \int_0^{t/2} \frac{1}{s^{2\theta}} \frac{\|h\|^2_{\mathcal{C}^{4+\theta}}}{s^{1-4\theta}} \mathrm{d}s$$
$$\leqslant C\|h\|^2_{\mathcal{C}^{4+\theta}} t^{2\theta}. \tag{IV.15}$$

Definition of τ_M and $\lambda^{k,M}_s$ and statement of Lemma III.31. For every $M \geqslant 1$, let us define $\tau_M = \tau^1_M \wedge \tau^2_M$, where

$$\tau^1_M := \inf \left\{ s \geqslant 0 : \int_0^s \sum_{k\in\mathbb{Z}} |\lambda^k_r|^2 \mathrm{d}r \geqslant M \right\} \wedge T;$$
$$\tau^2_M := \inf \left\{ s \geqslant 0 : \left\| \tfrac{1}{\partial_u x^g_s(\cdot)} \right\|_{L_\infty} \geqslant M \right\} \wedge T;$$
$$\tau^3_M := \inf \left\{ s \geqslant 0 : \int_0^{s\wedge(t/2)} \left\| \mathcal{A}^0_r \right\|^2_{\mathcal{C}^1} \mathrm{d}r \geqslant M \right\} \wedge T.$$

The definition of τ^2_M is the same as in (III.67). Thus $\mathbb{P}^W \otimes \mathbb{P}^\beta \left[\tau^2_M < T\right] \to_{M\to+\infty} 0$. Moreover,

$$\mathbb{P}^W \otimes \mathbb{P}^\beta \left[\tau^1_M < T\right] \leqslant \frac{1}{M} \mathbb{E}^W\mathbb{E}^\beta \left[\sum_{k\in\mathbb{Z}} \int_0^T |\lambda^k_s|^2 \mathrm{d}s \right] \to_{M\to+\infty} 0,$$

by inequality (IV.15). By inequality (IV.11),

$$\mathbb{P}^W \otimes \mathbb{P}^\beta \left[\tau^3_M < T\right] \leqslant \frac{1}{M} \int_0^{t/2} \mathbb{E}^W\mathbb{E}^\beta \left[\|\mathcal{A}^0_s\|^2_{\mathcal{C}^1} \right] \mathrm{d}r \leqslant \frac{C}{M} \|h\|_{\mathcal{C}^{3+\theta}} \to_{M\to+\infty} 0.$$

Therefore $\mathbb{P}^W \otimes \mathbb{P}^\beta \left[\tau_M < T\right] \to_{M\to+\infty} 0$.

For every $k \in \mathbb{Z}$ and $s \in [0,T]$, let us denote by $\lambda^{k,M}_s := \mathbb{1}_{\{s\leqslant\tau_M\}} \lambda^k_s$. For every $s \in [0,T]$,

$$\mathbb{1}_{\{s\leqslant\tau_M\wedge\frac{t}{2}\}} (\mathcal{A}^0_s * \varphi_s)(y) = \sum_{k\in\mathbb{Z}} f_k e^{-iky} \lambda^{k,M}_s.$$

Thus $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely,

$$\int_0^T \sum_{k\in\mathbb{Z}} |\lambda^{k,M}_s|^2 \mathrm{d}s = \int_0^{\tau_M} \sum_{k\in\mathbb{Z}} |\lambda^k_s|^2 \mathrm{d}s \leqslant \int_0^{\tau^1_M} \sum_{k\in\mathbb{Z}} |\lambda^k_s|^2 \mathrm{d}s \leqslant M.$$

Hence the statement of Lemma III.31 holds with the process $(\mathbb{1}_{\{s\leqslant t/2\}} (\mathcal{A}^0_s * \varphi_s))_{s\in[0,T]}$ instead of $(\mathcal{A}^{g,\varepsilon}_s)_{s\in[0,T]}$.

Definition of a_s^M and statement of Lemma III.32. As in (III.70), we define the $(\mathcal{G}_s)_{s\in[0,T]}$-adapted process $(a_s^M)_{s\in[0,T]}$ by

$$a_s^M(u) := \int_0^s \mathbb{1}_{\{r\leqslant\tau_M\}} \frac{g'(u)}{\partial_u x_r^g(u)} \mathbb{1}_{\{r\leqslant t/2\}} (\mathcal{A}_r^0 * \varphi_r)(x_r^g(u)) \mathrm{d}r. \tag{IV.16}$$

It follows from the definition of τ_M^2 that for every $s\in[0,T]$,

$$\|a_s^M\|_{L_\infty} \leqslant \|g'\|_{L_\infty} M \int_0^s \mathbb{1}_{\{r\leqslant\tau_M\wedge(t/2)\}} \|\mathcal{A}_r^0 * \varphi_r\|_{L_\infty} \mathrm{d}r$$

Recall that $\|\varphi_r\|_{L_1} = 1$ for every $r\in[0,T]$. Thus for every $r\in[0,T]$, $\|\mathcal{A}_r^0 * \varphi_r\|_{L_\infty} \leqslant \|\mathcal{A}_r^0\|_{L_\infty}$. Hence,

$$\begin{aligned}\|a_s^M\|_{L_\infty} &\leqslant \|g'\|_{L_\infty} M \int_0^{s\wedge\tau_M^3\wedge(t/2)} \|\mathcal{A}_r^0\|_{L_\infty} \mathrm{d}r \\ &\leqslant \|g'\|_{L_\infty} M T^{1/2} \left(\int_0^{s\wedge\tau_M^3\wedge(t/2)} \|\mathcal{A}_r^0\|_{L_\infty}^2 \mathrm{d}r \right)^{1/2} \leqslant \|g'\|_{L_\infty} M T^{1/2} M^{1/2}\end{aligned}$$

Therefore, for every $s\in[0,T]$, $\|a_s^M\|_{L_\infty} \leqslant \|g'\|_{L_\infty} M^{3/2} T^{1/2}$. Furthermore, by definition of τ_M^3,

$$\int_0^{s\wedge\tau_M\wedge(t/2)} \left\| \mathcal{A}_r^0 * \varphi_r \right\|_{\mathcal{C}^1}^2 \mathrm{d}r \leqslant \int_0^{s\wedge\tau_M^3\wedge(t/2)} \left\| \mathcal{A}_r^0 \right\|_{\mathcal{C}^1}^2 \mathrm{d}r \leqslant M.$$

Therefore, inequalities (III.71) and (III.72) of Lemma III.32 are still satisfied with the process $(\mathbb{1}_{\{s\leqslant t/2\}}(\mathcal{A}_s^0 * \varphi_s))_{s\in[0,T]}$ instead of $(\mathcal{A}_s^{g,\varepsilon})_{s\in[0,T]}$.

Statement of Lemma III.36. Since we checked that the statements of Lemmas III.30, III.31 and III.32 were true up to the aforementioned modifications, we can apply the same proofs to obtain Lemma III.34 and Lemma III.35. Therefore we obtain the following equality for every $M\geqslant 1$:

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta \left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u)) \partial_u x_t^g(u) \int_0^{t/2} \mathbb{1}_{\{s\leqslant\tau_M\}} \frac{(\mathcal{A}_s^0 * \varphi_s)(x_s^g(u))}{\partial_u x_s^g(u)} \mathrm{d}s\mathrm{d}u \right] \\ = \mathbb{E}^W\mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k\in\mathbb{Z}} \int_0^{t/2} \Re(\overline{\lambda_s^{k,M}} \mathrm{d}W_s^k) \right].\end{aligned}$$

As in Lemma III.36, we want to prove that we can pass to the limit when M tends to $+\infty$. Recall that $\mathbb{P}^W\otimes\mathbb{P}^\beta[\tau_M < T] \to_{M\to+\infty} 0$. Furthermore, let us check that

$$\sup_{M\geqslant M_0} \mathbb{E}^W\mathbb{E}^\beta \left[\int_0^1 \left(\partial_\mu\phi(\mu_t^g)(x_t^g(u)) \frac{\partial_u x_t^g(u)}{g'(u)} a_{t/2}^M(u) \right)^{3/2} \mathrm{d}u \right] < +\infty; \tag{IV.17}$$

$$\sup_{M\geqslant M_0} \mathbb{E}^W\mathbb{E}^\beta \left[\left(\phi(\mu_t^g) \sum_{k\in\mathbb{Z}} \int_0^{t/2} \Re(\overline{\lambda_s^{k,M}} \mathrm{d}W_s^k) \right)^2 \right] < +\infty. \tag{IV.18}$$

In order to obtain (IV.17), we apply the same strategy as for proving inequality (III.88). We only have to check that $\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 |a_{t/2}^M(u)|^{12}\mathrm{d}u\right]$ is uniformly bounded in M.

$$\begin{aligned}\mathbb{E}^W\mathbb{E}^\beta \left[\int_0^1 |a_{t/2}^M(u)|^{12} \mathrm{d}u \right] &\leqslant T^{11} \|g'\|_{L_\infty}^{12} \mathbb{E}^W\mathbb{E}^\beta \left[\sup_{s\leqslant T} \left\| \tfrac{1}{\partial_u x_s^g(\cdot)} \right\|_{L_\infty}^{12} \int_0^1 \int_0^{t/2} \|\mathcal{A}_s^0\|_{L_\infty}^{12} \mathrm{d}s\mathrm{d}u \right] \\ &\leqslant T^{11} \|g'\|_{L_\infty}^{12} \mathbb{E}^W\mathbb{E}^\beta \left[\sup_{s\leqslant T} \left\| \tfrac{1}{\partial_u x_s^g(\cdot)} \right\|_{L_\infty}^{24} \right]^{\frac{1}{2}} \left(\int_0^{t/2} \mathbb{E}^W\mathbb{E}^\beta \left[\|\mathcal{A}_s^0\|_{L_\infty}^{24} \right] \mathrm{d}s \right)^{\frac{1}{2}},\end{aligned}$$

which is bounded and independent of M by inequality (III.34) and by inequality (IV.11). Moreover, inequality (IV.18) follows from inequalities (III.90) and (IV.15). Thus we obtain

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 \partial_\mu\phi(\mu_t^g)(x_t^g(u))\partial_u x_t^g(u)\int_0^{t/2}\frac{(\mathcal{A}_s^0 * \varphi_s)(x_s^g(u))}{\partial_u x_s^g(u)}\mathrm{d}s\mathrm{d}u\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\phi(\mu_t^g)\sum_{k\in\mathbb{Z}}\int_0^{t/2}\Re(\overline{\lambda_s^k}\mathrm{d}W_s^k)\right].$$

Therefore, substracting $\mathbb{E}^W[\phi(\mu_t^g)]$ in the right-hand-side as we did in (III.81), we obtain

$$\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2}\partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}(\mathcal{A}_s^0 * \varphi_s)(x_s^g(u))\mathrm{d}s\mathrm{d}u\right]\right| \leqslant \mathbb{V}^W[\phi(\mu_t^g)]^{1/2}\,\mathbb{E}^W\mathbb{E}^\beta\left[\sum_{k\in\mathbb{Z}}\int_0^t|\lambda_s^k|^2\mathrm{d}s\right]^{1/2}. \quad \text{(IV.19)}$$

Conclusion of the proof. Putting together inequalities (IV.19) and (IV.15), we obtain

$$\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2}\partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}(\mathcal{A}_s^0 * \varphi_s)(x_s^g(u))\mathrm{d}s\mathrm{d}u\right]\right| \leqslant C\mathbb{V}^W[\phi(\mu_t^g)]^{1/2}\,t^\theta\|h\|_{\mathcal{C}^{4+\theta}}.$$

This completes the proof of (IV.14). □

Let us now focus on the second term on the right-hand side of (IV.13).

Proposition IV.5. *Let $t \in (0,T]$ and let $(\mathcal{A}_s^0)_{s\in[0,t/2]}$ and $(\mathcal{A}_s^1)_{s\in[0,t/2]}$ be $(\mathcal{G}_t)_{t\in[0,T]}$-adapted processes such that for every $s \in [0,t/2]$, $\mathcal{A}_s^0$ is a $\mathcal{C}^{4+\theta/2}$-function and $\mathcal{A}_s^1$ is a $\mathcal{C}^1$-function. Assume that inequalities* (IV.11) *and* (IV.12) *hold true. Then for every $t \in [0,T]$*

$$\frac{2}{t}\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2}\partial_\mu\phi(\mu_t^g)(x_t^g(u))\frac{\partial_u x_t^g(u)}{\partial_u x_s^g(u)}\left\{(\mathcal{A}_s^0 - \mathcal{A}_s^0 * \varphi_s)(x_s^g(u)) + \mathcal{A}_s^1(u)\right\}\mathrm{d}s\mathrm{d}u\right]\right| \leqslant \frac{C_g}{t^{1+3\theta}}\mathbb{V}^W[\phi(\mu_t^g)]^{1/2}\,\|h\|_{\mathcal{C}^{3+\theta}}. \quad \text{(IV.20)}$$

Before we prove Proposition IV.5, let us show the following lemma.

Lemma IV.6. *Let $t \in [0,T]$. Let Θ be a $\mathcal{G}_{t/2}$-measurable random variable with values in $L_2([0,1]\times\Omega^W\times\Omega^\beta)$. Then*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu\phi(\mu_t^g)(x_t^g(u))\partial_u x_t^g(u)\Theta(u)\mathrm{d}u\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu(P_{t/2}\phi)(\mu_{t/2}^g)(x_{t/2}^g(u))\partial_u x_{t/2}^g(u)\Theta(u)\mathrm{d}u\right]. \quad \text{(IV.21)}$$

Proof. Recall (III.46): for every $g \in \mathbf{G}^{1+\theta}$ and for every $h \in L_2([0,1]\times\Omega^\beta)$, the Fréchet derivative of $\widehat{P_{t/2}\phi}$ at point g in the direction h is given by:

$$D\widehat{P_{t/2}\phi}(g)\cdot h = \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(x_{t/2}^g)_u\,\partial_x Z_{t/2}^{g(u)}h(u)\mathrm{d}u\right] = \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(x_{t/2}^g)_u\frac{\partial_u x_{t/2}^g(u)}{g'(u)}h(u)\mathrm{d}u\right].$$

Thus for every $g \in \mathbf{G}^{1+\theta}$ and for every $\mathcal{G}_{t/2}$-measurable random variable h in $L_2([0,1]\times\Omega^\beta)$, we have

$$\widehat{DP_{t/2}\phi}(x^g_{t/2})\cdot h = \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(x^g_t)_u\,\frac{\partial_u x^g_t(u)}{\partial_u x^g_{t/2}(u)}h(u)\mathrm{d}u\,\middle|\mathcal{G}_{t/2}\right].$$

Therefore, for every $g\in\mathbf{G}^{1+\theta}$ and for every $\mathcal{G}_{t/2}$-measurable random variable Θ with values in $L_2([0,1]\times\Omega^W\times\Omega^\beta)$,

$$\begin{aligned}
\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu(P_{t/2}\phi)(\mu^g_{t/2})(x^g_{t/2}(u))\partial_u x^g_{t/2}(u)\Theta(u)\mathrm{d}u\right] &= \mathbb{E}^W\mathbb{E}^\beta\left[\widehat{DP_{t/2}\phi}(x^g_{t/2})\cdot\partial_u x^g_{t/2}\,\Theta\right]\\
&= \mathbb{E}^W\mathbb{E}^\beta\left[\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1 D\widehat{\phi}(x^g_t)_u\,\frac{\partial_u x^g_t(u)}{\partial_u x^g_{t/2}(u)}\partial_u x^g_{t/2}(u)\,\Theta(u)\mathrm{d}u\,\middle|\mathcal{G}_{t/2}\right]\right]\\
&= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu\phi(\mu^g_t)(x^g_t(u))\frac{\partial_u x^g_t(u)}{\partial_u x^g_{t/2}(u)}\partial_u x^g_{t/2}(u)\,\Theta(u)\mathrm{d}u\right]\\
&= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu\phi(\mu^g_t)(x^g_t(u))\partial_u x^g_t(u)\,\Theta(u)\mathrm{d}u\right],
\end{aligned}$$

which completes the proof of the lemma. □

Let us apply this lemma to the left-hand side of (IV.20).

Proof (Proposition IV.5). Let $\Theta(u) := \int_0^{t/2}\frac{1}{\partial_u x^g_s(u)}((\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)(x^g_s(u))+\mathcal{A}^1_s(u))\mathrm{d}s$. Clearly, Θ is $\mathcal{G}_{t/2}$-measurable. Let $\widetilde{g} := x^g_{t/2}$. By (IV.21), we have

$$\begin{aligned}
\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2}\partial_\mu\phi(\mu^g_t)(x^g_t(u))\frac{\partial_u x^g_t(u)}{\partial_u x^g_s(u)}\left\{(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)(x^g_s(u))+\mathcal{A}^1_s(u)\right\}\mathrm{d}s\mathrm{d}u\right]&\\
= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu\phi(\mu^g_t)(x^g_t(u))\partial_u x^g_t(u)\,\Theta(u)\mathrm{d}u\right]&\\
= \mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu(P_{t/2}\phi)(\mu^{\widetilde{g}}_0)(\widetilde{g}(u))\widetilde{g}'(u)\,\Theta(u)\mathrm{d}u\right].&
\end{aligned}$$

By Bismut-Elworthy inequality (III.56), we have

$$\left|\int_0^1\partial_\mu(P_{t/2}\phi)(\mu^{\widetilde{g}}_0)(\widetilde{g}(u))\widetilde{g}'(u)\,\Theta(u)\mathrm{d}u\right| \leqslant C_{\widetilde{g}}\,\frac{\mathbb{E}^W\left[\left(\phi(\mu^g_t)-\mathbb{E}^W[\phi(\mu^g_t)]\right)^2|\mathcal{G}^W_{t/2}\right]^{1/2}}{(t/2)^{2+\theta}}\|\partial_u\widetilde{g}\,\Theta\|_{\mathcal{C}^1}.$$

We deduce that

$$\begin{aligned}
\frac{2}{t}\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2}\partial_\mu\phi(\mu^g_t)(x^g_t(u))\frac{\partial_u x^g_t(u)}{\partial_u x^g_s(u)}\left\{(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)(x^g_s(u))+\mathcal{A}^1_s(u)\right\}\mathrm{d}s\mathrm{d}u\right]\right|&\\
= \frac{2}{t}\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\partial_\mu(P_{t/2}\phi)(\mu^{\widetilde{g}}_0)(\widetilde{g}(u))\widetilde{g}'(u)\,\Theta(u)\mathrm{d}u\right]\right|&\\
\leqslant \frac{2}{t}\mathbb{E}^W\mathbb{E}^\beta\left[\frac{C_{\widetilde{g}}}{(t/2)^{2+\theta}}\,\mathbb{E}^W\left[\left(\phi(\mu^g_t)-\mathbb{E}^W[\phi(\mu^g_t)]\right)^2|\mathcal{G}^W_{t/2}\right]^{1/2}\|\partial_u\widetilde{g}\,\Theta\|_{\mathcal{C}^1}\right] &\quad\text{(IV.22)}\\
\leqslant \frac{C}{t^{3+\theta}}\mathbb{V}^W\left[\phi(\mu^g_t)\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[(C_{\widetilde{g}})^2\|\partial_u\widetilde{g}\,\Theta\|^2_{\mathcal{C}^1}\right]^{1/2}.&
\end{aligned}$$

Furthermore,

$$\mathbb{E}^W\mathbb{E}^\beta\left[(C_{\widetilde{g}})^2\|\partial_u\widetilde{g}\,\Theta\|_{\mathcal{C}^1}^2\right]^{1/2} \leqslant \mathbb{E}^W\mathbb{E}^\beta\left[(C_{\widetilde{g}})^2\|\partial_u\widetilde{g}\|_{\mathcal{C}^1}^2\|\Theta\|_{\mathcal{C}^1}^2\right]^{1/2}$$
$$\leqslant \mathbb{E}^W\mathbb{E}^\beta\left[(C_{x^g_{t/2}})^4\|\partial_u x^g_{t/2}\|_{\mathcal{C}^1}^4\right]^{1/4}\mathbb{E}^W\mathbb{E}^\beta\left[\|\Theta\|_{\mathcal{C}^1}^4\right]^{1/4}.$$

We use the estimates given in Paragraph III.3.2, similarly as in (III.121), to argue that there is C_g depending on $\|g'''\|_{L_\infty}+\|g''\|_{L_\infty}+\|g'\|_{L_\infty}+\|\frac{1}{g'}\|_{L_\infty}$ such that $\mathbb{E}^W\mathbb{E}^\beta\left[(C_{x^g_{t/2}})^4\|\partial_u x^g_{t/2}\|_{\mathcal{C}^1}^4\right]^{1/4}\leqslant C_g$. Moreover

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\Theta\|_{\mathcal{C}^1}^4\right] \leqslant \mathbb{E}^W\mathbb{E}^\beta\left[\left|\int_0^{t/2}\left\|\frac{1}{\partial_u x^g_s(\cdot)}\right\|_{\mathcal{C}^1}\left\{\|(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)\circ x^g_s\|_{\mathcal{C}^1}+\|\mathcal{A}^1_s\|_{\mathcal{C}^1}\right\}\mathrm{d}s\right|^4\right]$$
$$\leqslant Ct^3\int_0^{t/2}\mathbb{E}^W\mathbb{E}^\beta\left[\left\|\frac{1}{\partial_u x^g_s(\cdot)}\right\|_{\mathcal{C}^1}^4\left\{\|(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)\circ x^g_s\|_{\mathcal{C}^1}+\|\mathcal{A}^1_s\|_{\mathcal{C}^1}\right\}^4\right]\mathrm{d}s$$
$$\leqslant Ct^3\int_0^{t/2}\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant T}\left\|\frac{1}{\partial_u x^g_r(\cdot)}\right\|_{\mathcal{C}^1}^8\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\|(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)\circ x^g_s\|_{\mathcal{C}^1}^8+\|\mathcal{A}^1_s\|_{\mathcal{C}^1}^8\right]^{1/2}\mathrm{d}s. \tag{IV.23}$$

Remark that $\left\|\frac{1}{\partial_u x^g_r(\cdot)}\right\|_{\mathcal{C}^1}\leqslant\left\|\frac{1}{\partial_u x^g_r(\cdot)}\right\|_{L_\infty}+\left\|\partial_u^{(2)}x^g_r\right\|_{L_\infty}\left\|\frac{1}{\partial_u x^g_r(\cdot)}\right\|_{L_\infty}^2$. By inequality (III.33) and by inequality (III.34), we deduce that $\mathbb{E}^W\mathbb{E}^\beta\left[\sup_{r\leqslant T}\left\|\frac{1}{\partial_u x^g_r(\cdot)}\right\|_{\mathcal{C}^1}^8\right]^{1/2}\leqslant C_g$. Moreover, by assumption (IV.12) on the integrability of $\mathcal{A}^1_s$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\mathcal{A}^1_s\|_{\mathcal{C}^1}^8\right]^{1/2}\leqslant Cs^{4-8\theta}\|h\|_{\mathcal{C}^2}^4. \tag{IV.24}$$

Furthermore

$$\|(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)\circ x^g_s\|_{\mathcal{C}^1}\leqslant\|\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s\|_{L_\infty}+\|\partial_u\{(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)\circ x^g_s\}\|_{L_\infty}$$
$$\leqslant\|\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s\|_{L_\infty}+\|\partial_u x^g_s\|_{L_\infty}\|(\partial_x\mathcal{A}^0_s)-(\partial_x\mathcal{A}^0_s)*\varphi_s\|_{L_\infty}$$
$$\leqslant Cs\|\partial_x\mathcal{A}^0_s\|_{L_\infty}+Cs\|\partial_u x^g_s\|_{L_\infty}\|\partial_x^{(2)}\mathcal{A}^0_s\|_{L_\infty}$$
$$\leqslant Cs(1+\sup_{r\leqslant T}\|\partial_u x^g_r\|_{L_\infty})\|\mathcal{A}^0_s\|_{\mathcal{C}^2}.$$

By inequality (III.33) and by assumption (IV.11) on the integrability of $\mathcal{A}^0_s$,

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)\circ x^g_s\|_{\mathcal{C}^1}^8\right]^{1/2}\leqslant Cs^4\mathbb{E}^W\mathbb{E}^\beta\left[(1+\sup_{r\leqslant T}\|\partial_u x^g_r\|_{L_\infty})^{16}\right]^{1/4}\mathbb{E}^W\mathbb{E}^\beta\left[\|\mathcal{A}^0_s\|_{\mathcal{C}^2}^{16}\right]^{1/4}$$
$$\leqslant C_g s^4\|h\|_{\mathcal{C}^{3+\theta}}^4. \tag{IV.25}$$

Putting together (IV.23), (IV.24) and (IV.25), we obtain

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\Theta\|_{\mathcal{C}^1}^4\right]\leqslant Ct^3\int_0^{t/2}C_g s^{4-8\theta}\|h\|_{\mathcal{C}^{3+\theta}}^4\mathrm{d}s\leqslant C_g t^{8-8\theta}\|h\|_{\mathcal{C}^{3+\theta}}^4.$$

Back to (IV.22), we obtain

$$\frac{2}{t}\left|\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1\int_0^{t/2}\partial_\mu\phi(\mu^g_t)(x^g_t(u))\frac{\partial_u x^g_t(u)}{\partial_u x^g_s(u)}\left\{(\mathcal{A}^0_s-\mathcal{A}^0_s*\varphi_s)(x^g_s(u))+\mathcal{A}^1_s(u)\right\}\mathrm{d}s\mathrm{d}u\right]\right|$$
$$\leqslant\frac{C_g}{t^{3+\theta}}\mathbb{V}^W\left[\phi(\mu^g_t)\right]^{1/2}\mathbb{E}^W\mathbb{E}^\beta\left[\|\Theta\|_{\mathcal{C}^1}^4\right]^{1/4}$$
$$\leqslant\frac{C_g}{t^{3+\theta}}\mathbb{V}^W\left[\phi(\mu^g_t)\right]^{1/2}t^{2-2\theta}\|h\|_{\mathcal{C}^{3+\theta}}.$$

Therefore, inequality (IV.20) follows. □

It follows from the previous two propositions that:

Proposition IV.7. *Assume that the process* $(\mathcal{A}_s)_{s\in[0,t/2]} := (\frac{\partial_u x_s^g(\cdot)}{g'}h)_{s\in[0,t/2]}$ *can be written as* (IV.9)*, with* $(\mathcal{A}_s^0)_{s\in[0,t/2]}$ *and* $(\mathcal{A}_s^1)_{s\in[0,t/2]}$ *satisfying inequalities* (IV.10)*,* (IV.11) *and* (IV.12)*. Then for every* $t \in (0,T]$*,*

$$\left|\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h})\right| \leqslant \frac{C}{t^{1+3\theta}}\mathbb{V}^W\left[\phi(\mu_t^g)\right]^{1/2}\|h\|_{\mathcal{C}^{4+\theta}}. \tag{IV.26}$$

Proof. Inequality (IV.26) follows immediately from Proposition IV.4 and from Proposition IV.5 applied to equality (IV.13). □

In other words, we have proved that Theorem IV.2 holds true, provided the assumption on the splitting of $\mathcal{A}_s$ is satisfied. In the next section, we will prove it.

IV.3 Integrability of the dominant term and of the remainder

Let us show that the assumption made in the previous section holds true. We first construct the splitting in Paragraph IV.3.1 and then prove that inequalities (IV.10), (IV.11) and (IV.12) are satisfied for that splitting in Paragraph IV.3.2.

IV.3.1 Construction of the splitting

In this paragraph, we fix the initial condition $g \in \mathbf{G}^{4+\theta}$ and we denote by $(x_t)_{t\in(0,T]}$ the solution to (III.20) starting from g. Let $h \in \Delta^{4+\theta}$.

For every $t \in (0,T]$ and for every $u \in [0,1]$, let us denote by $\mathcal{A}_t(u) := \frac{\partial_u x_t(u)}{g'(u)}h(u)$. The aim of this paragraph is to find two processes $(\mathcal{A}_t^0)_{t\in(0,T]}$ and $(\mathcal{A}_t^1)_{t\in(0,T]}$ such that for every $u \in [0,1]$ and for every $u \in [0,1]$,

$$\mathcal{A}_t(u) = \frac{\partial_u x_t(u)}{g'(u)}h(u) = \mathcal{A}_t^0(x_t(u)) + \mathcal{A}_t^1(u), \tag{IV.27}$$

such that $\mathbb{P}^W \otimes \mathbb{P}^\beta$-almost surely, for every $t \in (0,T]$, $\mathcal{A}_t^0$ is a 2π-periodic $\mathcal{C}^{4+\theta/2}$-function and $\mathcal{A}_t^1$ is a 1-periodic $\mathcal{C}^1$-function satisfying for every $s \in (0,T]$ the inequalities (IV.10), (IV.11) and (IV.12).

Splitting of the derivative. Let us start by writing $\partial_u x_t$ as the sum of a main part, that will belong to $\mathcal{C}^{4+\theta/2}$, and a remainder term that is small in $\mathcal{C}^1$-norm when t is small. Recall that by (III.21), we have for every $u \in \mathbb{R}$,

$$\partial_u x_t(u) = g'(u)\exp\left(\sum_{k\in\mathbb{Z}} f_k k \int_0^t \Re\left(-ie^{-ikx_s(u)}\mathrm{d}W_s^k\right) - \frac{C_f t}{2}\right),$$

where $C_f := \sum_{k\in\mathbb{Z}} f_k^2 k^2$. For every $t \in (0,T]$, let $N_t := \lfloor \frac{t^\theta}{t^{1/2}} \rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor. Recall that $\theta \in (0, \frac{1}{2})$, thus N_t tends to $+\infty$ when $t \to 0^+$. Let us define $a_t(u) := a_t^0(u) + a_t^1(u)$, where

$$a_t^0(u) := \sum_{|k|\leqslant N_t} f_k k \int_0^t \Re\left(-ie^{-ikg(u)}\mathrm{d}W_s^k\right); \tag{IV.28}$$

$$a_t^1(u) := \sum_{|k|> N_t} f_k k \int_0^t \Re\left(-ie^{-ikg(u)}\mathrm{d}W_s^k\right) + \sum_{k\in\mathbb{Z}} f_k k \int_0^t \Re\left(-i(e^{-ikx_s(u)} - e^{-ikg(u)})\mathrm{d}W_s^k\right). \tag{IV.29}$$

Clearly, we have $\partial_u x_t(u) = g'(u)\exp\left(a_t(u) - \frac{C_f t}{2}\right)$. Moreover, the splitting preserves periodicity: for every $t \in (0,T]$, $u \mapsto a_t^0(u)$ and $u \mapsto a_t^1(u)$ are 1-periodic functions. Furthermore, a simple computation gives the splitting

$$\partial_u x_t(u) = g'(u)\left(b_t^0(u) + b_t^1(u)\right), \tag{IV.30}$$

where

$$b_t^0(u) := \gamma_t \exp\left(a_t^0(u) - \frac{C_f t}{2}\right); \tag{IV.31}$$

$$b_t^1(u) := \exp\left(a_t^0(u) - \frac{C_f t}{2}\right)\left(\exp(a_t^1(u)) - \gamma_t\right); \tag{IV.32}$$

$$\gamma_t := \frac{2\pi}{\int_0^1 g'(v)\exp\left(a_t^0(v) - \frac{C_f t}{2}\right)\mathrm{d}v}. \tag{IV.33}$$

Remark that b_t^0 and b_t^1 are also 1-periodic functions almost surely for every $t \in (0,T]$. The multiplication by γ_t insures that the primitives of $u \mapsto g'(u)b_t^0(u)$ belong to $\mathbf{G}^1$. Indeed, it follows from 1-periodicity that for every $u \in \mathbb{R}$

$$\int_u^{u+1} g'(v)b_t^0(v)\mathrm{d}v = \int_0^1 g'(v)b_t^0(v)\mathrm{d}v = \gamma_t \int_0^1 g'(v)\exp\left(a_t^0(v) - \frac{C_f t}{2}\right)\mathrm{d}v = 2\pi. \tag{IV.34}$$

Splitting of the cumulative distribution function. The splitting of the derivative induces the following decompositon for the process $(x_t)_{t\in(0,T]}$ into a dominant process $(x_t^0)_{t\in(0,T]}$ that is of $\mathcal{C}^{4+\theta/2}$-regularity and a remainder process $(x_t^1)_{t\in(0,T]}$ that is small in $\mathcal{C}^1$-norm when t is small. Let us define for every $t \in (0,T]$ and $u \in \mathbb{R}$

$$x_t^0(u) := x_t(0) + \int_0^u g'(v)b_t^0(v)\mathrm{d}v; \tag{IV.35}$$

$$x_t^1(u) := \int_0^u g'(v)b_t^1(v)\mathrm{d}v. \tag{IV.36}$$

For every $t \in (0,T]$ and for every $u \in [0,1]$, $x_t(u) = x_t^0(u) + x_t^1(u)$, because $u \mapsto x_t(u)$ and $u \mapsto x_t^0(u) + x_t^1(u)$ share the same value at $t = 0$ and the same derivative. Remark that almost surely, for every $t \in (0,T]$, for every $u \in [0,1]$, $\partial_u x_t^0(u) = \gamma_t g'(u)\exp(a_t^0(u) - \frac{C_f t}{2}) > 0$. Moreover, by equality (IV.34), the equality $x_t^0(u+1) - x_t^0(u) = 2\pi$ holds with every $u \in \mathbb{R}$. Thus x_t^0 belongs almost surely to $\mathbf{G}^1$. Furthermore, for every $u \in [0,1]$,

$$x_t^1(u+1) - x_t^1(u) = (x_t(u+1) - x_t(u)) - (x_t^0(u+1) - x_t^0(u)) = 2\pi - 2\pi = 0,$$

thus x_t^1 is a 1-periodic function.

As we did in Paragraph III.2.1, we denote by $F_t : \mathbb{R} \to \mathbb{R}$ and $F_t^0 : \mathbb{R} \to \mathbb{R}$ the inverse functions associated respectively to x_t and x_t^0; that is for every $u \in \mathbb{R}$, $F_t \circ x_t(u) = F_t^0 \circ x_t^0(u) = u$ and for every $v \in \mathbb{R}$, $x_t \circ F_t(v) = x_t^0 \circ F_t^0(v) = v$. Let us remark that almost surely, for every $t \in (0,T]$, F_t and F_t^0 satisfy the relation $F(x + 2\pi) = F(x) + 1$ for every $x \in \mathbb{R}$.

Definition of $\mathcal{A}_t^0$ and $\mathcal{A}_t^1$. Let us define for every $u, v \in \mathbb{R}$

$$\mathcal{A}_t^0(v) := (b_t^0 \cdot h) \circ F_t^0(v); \tag{IV.37}$$

$$\mathcal{A}_t^1(u) := b_t^1(u)h(u) + \left[(b_t^0 \cdot h)(u) - (b_t^0 \cdot h) \circ F_t^0(x_t(u))\right]. \tag{IV.38}$$

Periodicity holds: almost surely for every $t \in (0,T]$, $\mathcal{A}_t^0$ is 2π-periodic and $\mathcal{A}_t^1$ is 1-periodic. Moreover, equality (IV.27) holds: for every $u \in \mathbb{R}$,

$$\mathcal{A}_t^0(x_t(u)) + \mathcal{A}_t^1(u) = b_t^1(u)h(u) + b_t^0(u)h(u) = \frac{\partial_u x_t(u)}{g'(u)}h(u),$$

where the last equality follows from (IV.30).

IV.3.2 Integrability of $\mathcal{A}^0$ and of $\mathcal{A}^1$

Recall the definition of a_t^0 and a_t^1, given by (IV.28) and by (IV.29):

$$a_t^0(u) := \sum_{|k|\leqslant N_t} f_k k \int_0^t \Re\left(-ie^{-ikg(u)}\mathrm{d}W_s^k\right);$$
$$a_t^1(u) := \sum_{|k|> N_t} f_k k \int_0^t \Re\left(-ie^{-ikg(u)}\mathrm{d}W_s^k\right) + \sum_{k\in\mathbb{Z}} f_k k \int_0^t \Re\left(-i(e^{-ikx_s(u)} - e^{-ikg(u)})\mathrm{d}W_s^k\right),$$

where N_t is the floor of $\frac{1}{t^{1/2-\theta}}$.

Lemma IV.8. *Let $g \in \mathbf{G}^{4+\theta}$. Then almost surely, $u \mapsto a_t^0(u)$ belongs to $\mathcal{C}^{4+\theta/2}$ and $u \mapsto a_t^1(u)$ belongs almost surely to $\mathcal{C}^{1+\theta/2}$ for every $t \in (0,T]$. Moreover, for every $p \geqslant 2$, there is $C_p > 0$ such that for every $t \in (0,T]$*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|a_t^0\|^p_{\mathcal{C}^{4+\theta/2}}\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}; \tag{IV.39}$$

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|a_t^0\|^p_{\mathcal{C}^{3+\theta/2}}\right]^{1/p} \leqslant C_p; \tag{IV.40}$$

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|a_t^1\|^p_{\mathcal{C}^{1+\theta/2}}\right]^{1/p} \leqslant C_p t^{1-\theta}. \tag{IV.41}$$

Remark IV.9. Informally speaking, Lemma IV.8 establishes that the cost of every further order of derivation is of order $t^{-1/2}$. In other words, the L_p-norm of $\|a_t^0\|_{\mathcal{C}^j}$ is controlled by $t^{\frac{3-j}{2}+}$ for every $j = 0, 1, \ldots, 4$. Similarly, the L_p-norm of $\|a_t^1\|_{\mathcal{C}^j}$ is controlled by $t^{\frac{3-j}{2}-}$ for $j = 0$ or $j = 1$. The signs $+$ (resp. $-$) are indicating that the control is a little bit better (resp. worse) than $t^{\frac{3-j}{2}}$.

Proof. All along this proof, we will write $\mathbb{E}[\cdot]$ instead of $\mathbb{E}^W\mathbb{E}^\beta[\cdot]$.

Regularity and integrability of a_t^0. Recall that by Definition IV.1,

$$\|a_t^0\|_{\mathcal{C}^{4+\theta/2}} = \sum_{j=0}^4 |\partial_u^{(j)} a_t^0(0)| + \sup_{u\neq v} \frac{|\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)|}{|u-v|^{\theta/2}}.$$

For every $u \in [0,1]$,

$$\partial_u^{(4)} a_t^0(u) = \sum_{|k|\leqslant N_t} f_k k \int_0^t \Re\left(-i\partial_u^{(4)}\{e^{-ikg(u)}\}\mathrm{d}W_s^k\right).$$

By a simple computation on the derivatives of exp, there exists a constant $C > 0$ such that $|\partial_u^{(4)}\{e^{-ikg(\cdot)}\}(0)| \leqslant C\langle k\rangle^4 \|g\|^4_{\mathcal{C}^4}$.

Remark IV.10. Recall that $\|g\|_{\mathcal{C}^4}$ denotes, by Definition IV.1, the C^4-norm of the restriction of g to $[0,1]$, which is finite. Of course, since $g(u+1) = g(u) + 2\pi$ for every $u \in \mathbb{R}$, g is not bounded on $\mathbb{R}$.

Moreover, f is of order $\alpha = \frac{7}{2} + \theta$. Thus $|f_k|^2 \leqslant C\frac{1}{\langle k\rangle^{7+2\theta}}$. Therefore, for every $p \geqslant 2$, by Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[|\partial_u^{(4)} a_t^0(0)|^p\right]^{1/p} \leqslant C_p\mathbb{E}\left[\Big|\sum_{|k|\leqslant N_t}\int_0^t |f_k|^2|k|^2|\partial_u^{(4)}\{e^{-ikg(\cdot)}\}(0)|^2\mathrm{d}s\Big|^{p/2}\right]^{1/p}$$
$$\leqslant C_p\left|t\sum_{|k|\leqslant N_t}\frac{1}{\langle k\rangle^{7+2\theta}}\langle k\rangle^2\langle k\rangle^8\right|^{1/2} \leqslant C_p\left|t\sum_{|k|\leqslant N_t}\langle k\rangle^{3-2\theta}\right|^{1/2}.$$

Recall that $N_t \to +\infty$ when $t \to 0^+$, thus $\sum_{|k|\leqslant N_t}\langle k\rangle^{3-2\theta} \leqslant CN_t^{4-2\theta} \leqslant Ct^{(-\frac{1}{2}+\theta)(4-2\theta)}$. Since θ belongs to $(0,\frac{1}{2})$, $(-\frac{1}{2}+\theta)(4-2\theta) = -2+5\theta-2\theta^2 \geqslant -2+5\theta-\theta = -2+4\theta$. Thus for every $t \leqslant T$, $t^{(-\frac{1}{2}+\theta)(4-2\theta)} \leqslant C_T t^{-2+4\theta}$. It follows that

$$\mathbb{E}\left[|\partial_u^{(4)} a_t^0(0)|^p\right]^{1/p} \leqslant C_p\Big|t\cdot t^{-2+4\theta}\Big|^{1/2} \leqslant C_p t^{-\frac{1}{2}+2\theta}.$$

Similarly, we also have

$$\begin{aligned}\mathbb{E}\left[|\partial_u^{(3)} a_t^0(0)|^p\right]^{1/p} &\leqslant C_p\mathbb{E}\left[\Big|\sum_{|k|\leqslant N_t}\int_0^t |f_k|^2|k|^2|\partial_u^{(3)}\{e^{-ikg(\cdot)}\}(0)|^2\mathrm{d}s\Big|^{p/2}\right]^{1/p}\\ &\leqslant C_p\mathbb{E}\left[\Big|t\sum_{|k|\leqslant N_t}\frac{1}{\langle k\rangle^{7+2\theta}}\langle k\rangle^2\langle k\rangle^6\Big|^{p/2}\right]^{1/p}\\ &\leqslant C_p\Big|t\sum_{|k|\leqslant N_t}\langle k\rangle^{1-2\theta}\Big|^{1/2} \leqslant C_p\Big|tN_t^{2-2\theta}\Big|^{1/2} \leqslant C_p t^{\theta} \leqslant C_p.\end{aligned}$$

Actually, we can prove that for every j, $\mathbb{E}\left[|\partial_u^{(j)} a_t^0(0)|^p\right]^{1/p} \leqslant C_p t^{\frac{3-j}{2}}$. Thus

$$\mathbb{E}\left[\textstyle\sum_{j=0}^4 |\partial_u^{(j)} a_t^0(0)|^p\right]^{1/p} \leqslant C_p t^{-\frac{1}{2}+\frac{3}{2}\theta}; \tag{IV.42}$$

$$\mathbb{E}\left[\textstyle\sum_{j=0}^3 |\partial_u^{(j)} a_t^0(0)|^p\right]^{1/p} \leqslant C_p. \tag{IV.43}$$

Furthermore, let $u, v \in [0,1]$. For every $p \geqslant 2$,

$$\begin{aligned}&\mathbb{E}\left[|\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)|^p\right]^{1/p}\\ &\qquad\leqslant C_p\mathbb{E}\left[\Big|\sum_{|k|\leqslant N_t}\int_0^t |f_k|^2|k|^2|\partial_u^{(4)}\{e^{-ikg(\cdot)}\}(u) - \partial_u^{(4)}\{e^{-ikg(\cdot)}\}(v)|^2\mathrm{d}s\Big|^{p/2}\right]^{1/p}.\end{aligned}$$

For every $\theta' < \theta$, $\left|\partial_u^{(4)}\{e^{-ikg(\cdot)}\}(u) - \partial_u^{(4)}\{e^{-ikg(\cdot)}\}(v)\right| \leqslant C\|g\|^2_{\mathcal{C}^{4+\theta}}\langle k\rangle^{4+\theta'}|u-v|^{\theta'}$. Indeed, the dominant term appearing in the computation of $\partial_u^{(4)}\{e^{-ikg(\cdot)}\}(u) - \partial_u^{(4)}\{e^{-ikg(\cdot)}\}(v)$ can be bounded by:

$$\begin{aligned}\left|k^4\partial_u^{(4)} g(u)e^{-ikg(u)} - k^4\partial_u^{(4)} g(v)e^{-ikg(v)}\right| &\leqslant k^4\left|\partial_u^{(4)} g(u) - \partial_u^{(4)} g(v)\right| + k^4\|g\|_{\mathcal{C}^4}\left|e^{-ikg(u)} - e^{-ikg(v)}\right|\\ &\leqslant k^4\|g\|_{\mathcal{C}^{4+\theta}}|u-v|^{\theta} + k^4\|g\|_{\mathcal{C}^4}|k|^{\theta'}|g(u)-g(v)|^{\theta'}\\ &\leqslant k^4\|g\|_{\mathcal{C}^{4+\theta}}|u-v|^{\theta} + |k|^{4+\theta'}\|g\|_{\mathcal{C}^4}\|g\|^{\theta'}_{\mathcal{C}^1}|u-v|^{\theta'}.\end{aligned}$$

It follows that for every $\theta' < \theta$,

$$\begin{aligned}\mathbb{E}\left[|\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)|^p\right]^{1/p} &\leqslant C_p\mathbb{E}\left[\Big|\sum_{|k|\leqslant N_t} t\frac{1}{\langle k\rangle^{7+2\theta}}\langle k\rangle^2\langle k\rangle^{8+2\theta'}|u-v|^{2\theta'}\Big|^{p/2}\right]^{1/p}\\ &\leqslant C_p\Big|t\sum_{|k|\leqslant N_t}\langle k\rangle^{3+2\theta'-2\theta}|u-v|^{2\theta'}\Big|^{1/2}.\end{aligned}$$

Let us choose $\theta' = \frac{3}{4}\theta$. Then

$$\sum_{|k|\leqslant N_t}\langle k\rangle^{3+2\theta'-2\theta} = \sum_{|k|\leqslant N_t}\langle k\rangle^{3-\theta/2} \leqslant CN_t^{4-\theta/2} \leqslant Ct^{(-1/2+\theta)(4-\theta/2)} \leqslant C_T t^{-2+4\theta}.$$

We obtain

$$\mathbb{E}\left[|\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)|^p\right]^{1/p} \leqslant C_p t^{-\frac{1}{2}+2\theta} |u-v|^{\frac{3}{4}\theta}.$$

Therefore,

$$\mathbb{E}\left[\left|\frac{\partial_u^{(4)} a_t^0(u)}{t^{-\frac{1}{2}+2\theta}} - \frac{\partial_u^{(4)} a_t^0(v)}{t^{-\frac{1}{2}+2\theta}}\right|^p\right] \leqslant C_p |u-v|^{p\frac{3\theta}{4}} = C_p |u-v|^{1+(p\frac{3\theta}{4}-1)}. \tag{IV.44}$$

Let q be chosen such that $q > \frac{4}{\theta}$. We have $\frac{3\theta}{4} - \frac{1}{q} > \frac{\theta}{2}$. Therefore, let us apply Kolmogorov's Lemma [RY99, p.26, Thm I.2.1] to inequality (IV.44), with q instead of p. We deduce that $u \mapsto a_t^0$ has a $\mathcal{C}^{4+\theta/2}$-version and that there is C_{Kol} such that

$$\mathbb{E}\left[\sup_{u\neq v} \frac{\left|\frac{\partial_u^{(4)} a_t^0(u)}{t^{-\frac{1}{2}+2\theta}} - \frac{\partial_u^{(4)} a_t^0(v)}{t^{-\frac{1}{2}+2\theta}}\right|^q}{|u-v|^{q\frac{\theta}{2}}}\right] \leqslant C_{\text{Kol}}.$$

Thus for every $q > \frac{4}{\theta}$,

$$\mathbb{E}\left[\left(\sup_{u\neq v} \frac{\left|\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)\right|}{|u-v|^{\frac{\theta}{2}}}\right)^q\right]^{1/q} \leqslant C_q t^{-\frac{1}{2}+2\theta}.$$

By Hölder's inequality, we deduce that for every $p \geqslant 2$

$$\mathbb{E}\left[\left(\sup_{u\neq v} \frac{\left|\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)\right|}{|u-v|^{\frac{\theta}{2}}}\right)^p\right]^{1/p} \leqslant C_p t^{-\frac{1}{2}+2\theta}. \tag{IV.45}$$

Putting together (IV.42) and (IV.45), we obtain inequality (IV.39). Similarly, we prove that for every $p \geqslant 2$

$$\mathbb{E}\left[\left(\sup_{u\neq v} \frac{\left|\partial_u^{(3)} a_t^0(u) - \partial_u^{(3)} a_t^0(v)\right|}{|u-v|^{\frac{\theta}{2}}}\right)^p\right]^{1/p} \leqslant C_p. \tag{IV.46}$$

It follows from (IV.43) and (IV.46) that $\mathbb{E}\left[\|a_t^0\|^p_{\mathcal{C}^{3+\theta/2}}\right]^{1/p} \leqslant C_p$. This is inequality (IV.40).

Regularity and integrability of a_t^1. Let us divide a_t^1 into two terms: $a_t^1 = a_t^{1,1} + a_t^{1,2}$, where

$$a_t^{1,1}(u) := \sum_{|k|>N_t} f_k k \int_0^t \Re\left(-ie^{-ikg(u)} \mathrm{d}W_s^k\right);$$
$$a_t^{1,2}(u) := \sum_{k\in\mathbb{Z}} f_k k \int_0^t \Re\left(-i(e^{-ikx_s(u)} - e^{-ikg(u)})\mathrm{d}W_s^k\right).$$

Let us first consider the case of $a_t^{1,1}$. For every $p \geqslant 2$,

$$\mathbb{E}\left[|\partial_u a_t^{1,1}(0)|^p\right]^{1/p} \leqslant C_p \mathbb{E}\left[\left|\sum_{|k|>N_t} \int_0^t |f_k|^2 |k|^2 |\partial_u\{e^{-ikg(\cdot)}\}(0)|^2 \mathrm{d}s\right|^{p/2}\right]^{1/p}$$
$$\leqslant C_p \left|t \sum_{|k|>N_t} \frac{1}{\langle k\rangle^{7+2\theta}} \langle k\rangle^2 \langle k\rangle^2\right|^{1/2} \leqslant C_p \left|t \sum_{|k|>N_t} \frac{1}{\langle k\rangle^{3+2\theta}}\right|^{1/2}.$$

Moreover, $\sum_{|k|>N_t} \frac{1}{\langle k\rangle^{3+2\theta}} \leqslant C\frac{1}{N_t^{2+2\theta}} \leqslant Ct^{(\frac{1}{2}-\theta)(2+2\theta)} \leqslant Ct^{1-\theta-2\theta^2} \leqslant Ct^{1-2\theta}$ because $\theta \in (0,\frac{1}{2})$. Therefore,

$$\mathbb{E}\left[|\partial_u a_t^{1,1}(0)|^p\right]^{1/p} \leqslant C_p t^{1-\theta}.$$

Moreover, for every $u,v\in[0,1]$, for every $p\geqslant 2$ and for every $\theta'<\theta$,

$$\begin{aligned}\mathbb{E}\left[|\partial_u a_t^{1,1}(u)-\partial_u a_t^{1,1}(v)|^p\right]^{\frac{1}{p}} &\leqslant C_p\mathbb{E}\left[\left(\sum_{|k|>N_t}\int_0^t |f_k|^2|k|^2|g'(u)e^{-ikg(u)}-g'(v)e^{-ikg(v)}|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}\\ &\leqslant C_p\left(\sum_{|k|>N_t}\int_0^t |f_k|^2|k|^2\left|\langle k\rangle^{1+\theta'}|u-v|^{\theta'}\right|^2\mathrm{d}s\right)^{1/2}\\ &\leqslant C_p\left(t\sum_{|k|>N_t}\frac{1}{\langle k\rangle^{3+2\theta-2\theta'}}|u-v|^{2\theta'}\right)^{1/2}.\end{aligned}$$

Choosing again $\theta'=\frac{3\theta}{4}$, we prove that

$$\mathbb{E}\left[|\partial_u a_t^{1,1}(u)-\partial_u a_t^{1,1}(v)|^p\right]^{1/p} \leqslant C_p t^{1-\theta}|u-v|^{\frac{3}{4}\theta}.$$

By Kolmogorov's Lemma (applied as previously for p large enough), we deduce that $u\mapsto a_t^{1,1}$ is a $\mathcal{C}^{1+\theta/2}$-function and there is C_{Kol} such that for p large enough

$$\mathbb{E}\left[\sup_{u\neq v}\frac{\left|\partial_u a_t^{1,1}(u)-\partial_u a_t^{1,1}(v)\right|^p}{|u-v|^{p\frac{\theta}{2}}}\right]^{1/p} \leqslant C_p t^{1-\theta}.$$

We deduce that

$$\mathbb{E}\left[\|a_t^{1,1}\|^p_{\mathcal{C}^{1+\theta/2}}\right]^{1/p} \leqslant C_p t^{1-\theta}. \tag{IV.47}$$

Let us now consider the case of $a_t^{1,2}$. For every $u\in[0,1]$,

$$\partial_u a_t^{1,2}(u) = \sum_{|k|\leqslant N_t} f_k k\int_0^t \Re\left(-k(\partial_u x_s(u)e^{-ikx_s(u)}-g'(u)e^{-ikg(u)})\mathrm{d}W_s^k\right).$$

For every $p\geqslant 2$,

$$\begin{aligned}\mathbb{E}\left[|\partial_u a_t^{1,2}(0)|^p\right]^{1/p} &\leqslant C_p\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}}\int_0^t |f_k|^2\langle k\rangle^4\left|\partial_u x_s(0)e^{-ikx_s(0)}-g'(0)e^{-ikg(0)}\right|^2\mathrm{d}s\right)^{p/2}\right]^{1/p}\\ &\leqslant C_p\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}}\int_0^t |f_k|^2\langle k\rangle^4\left|\partial_u x_s(0)-g'(0)\right|^2\mathrm{d}s\right)^{p/2}\right]^{1/p}\\ &\quad + C_p\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}}\int_0^t |f_k|^2\langle k\rangle^6\left|x_s(0)-g(0)\right|^2\mathrm{d}s\right)^{p/2}\right]^{1/p}.\end{aligned}$$

Since $\sum_{k\in\mathbb{Z}} |f_k|^2\langle k\rangle^6 \leqslant C\sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^{1+2\theta}}$ is finite, we obtain

$$\begin{aligned}\mathbb{E}\left[|\partial_u a_t^{1,2}(0)|^p\right]^{\frac{1}{p}} &\leqslant C_p\mathbb{E}\left[\Big(\int_0^t |\partial_u x_s(0)-g'(0)|^2\,\mathrm{d}s\Big)^{p/2}\right]^{\frac{1}{p}} + C_p\mathbb{E}\left[\Big(\int_0^t |x_s(0)-g(0)|^2\,\mathrm{d}s\Big)^{p/2}\right]^{\frac{1}{p}}\\ &\leqslant C_p\left(t^{\frac{p}{2}-1}\int_0^t \mathbb{E}\left[|\partial_u x_s(0)-g'(0)|^p\right]\mathrm{d}s + t^{\frac{p}{2}-1}\int_0^t \mathbb{E}\left[|x_s(0)-g(0)|^p\right]\mathrm{d}s\right)^{1/p}\end{aligned}$$

Recall that $(\partial_u x_s(0))_{s\in[0,T]}$ satisfies (III.22) and that $\partial_u x_0(0) = g'(0)$. Therefore, there is C_p such that $\mathbb{E}\left[|\partial_u x_s(0)-g'(0)|^p\right] \leqslant C_p s^{p/2}$. There is also C_p such that $\mathbb{E}\left[|x_s(0)-g(0)|^p\right] \leqslant C_p s^{p/2}$. It follows that

$$\mathbb{E}\left[|\partial_u a_t^{1,2}(0)|^p\right]^{\frac{1}{p}} \leqslant C_p\Big|t^{\frac{p}{2}-1}\int_0^t s^{p/2}\mathrm{d}s\Big|^{1/p} \leqslant C_p t.$$

Moreover, for every $u, v \in [0,1]$ and for every $p \geqslant 2$,

$$\begin{aligned}\mathbb{E}\left[|\partial_u a_t^{1,2}(u)-\partial_u a_t^{1,2}(v)|^p\right]^{1/p} \leqslant C_p\mathbb{E}\Bigg[\Big|\sum_{k\in\mathbb{Z}}\int_0^t |f_k|^2\langle k\rangle^4 \Big|\partial_u x_s(u)e^{-ikx_s(u)} - g'(u)e^{-ikg(u)}\\ - \partial_u x_s(v)e^{-ikx_s(v)} + g'(v)e^{-ikg(v)}\Big|^2\mathrm{d}s\Big|^{p/2}\Bigg]^{1/p}. \quad \text{(IV.48)}\end{aligned}$$

On the one hand,

$$\begin{aligned}&\Big|\partial_u x_s(u)e^{-ikx_s(u)} - g'(u)e^{-ikg(u)} - \partial_u x_s(v)e^{-ikx_s(v)} + g'(v)e^{-ikg(v)}\Big|\\ &\leqslant \Big|\partial_u x_s(u)e^{-ikx_s(u)} - g'(u)e^{-ikg(u)}\Big| + \Big|\partial_u x_s(v)e^{-ikx_s(v)} - g'(v)e^{-ikg(v)}\Big|\\ &\leqslant C(1+\|g'\|_{L_\infty})\langle k\rangle\Big(|\partial_u x_s(u)-g'(u)| + |x_s(u)-g(u)| + |\partial_u x_s(v)-g'(v)| + |x_s(v)-g(v)|\Big).\end{aligned} \quad \text{(IV.49)}$$

On the other hand,

$$\begin{aligned}&\Big|\partial_u x_s(u)e^{-ikx_s(u)} - g'(u)e^{-ikg(u)} - \partial_u x_s(v)e^{-ikx_s(v)} + g'(v)e^{-ikg(v)}\Big|\\ &\qquad\leqslant \Big|\partial_u x_s(u)e^{-ikx_s(u)} - \partial_u x_s(v)e^{-ikx_s(v)}\Big| + \Big|g'(u)e^{-ikg(u)} - g'(v)e^{-ikg(v)}\Big|\\ &\qquad\leqslant C(\|\partial_u^{(2)}x_s\|_{L_\infty} + \|\partial_u x_s\|_{L_\infty}^2 + \|g''\|_{L_\infty} + \|g'\|_{L_\infty}^2)\langle k\rangle|u-v|.\end{aligned} \quad \text{(IV.50)}$$

Let $\theta' \in (0,1)$. By multiplying $\text{(IV.49)}^{1-\theta'}$ with $\text{(IV.50)}^{\theta'}$, there is a constant C_g depending on g such that

$$\begin{aligned}&\Big|\partial_u x_s(u)e^{-ikx_s(u)} - g'(u)e^{-ikg(u)} - \partial_u x_s(v)e^{-ikx_s(v)} + g'(v)e^{-ikg(v)}\Big|\\ &\leqslant C_g(1+\|\partial_u^{(2)}x_s\|_{L_\infty}^{\theta'} + \|\partial_u x_s\|_{L_\infty}^{2\theta'})\langle k\rangle|u-v|^{\theta'}\\ &\quad\cdot\Big(|\partial_u x_s(u)-g'(u)|^{1-\theta'} + |x_s(u)-g(u)|^{1-\theta'} + |\partial_u x_s(v)-g'(v)|^{1-\theta'} + |x_s(v)-g(v)|^{1-\theta'}\Big).\end{aligned} \quad \text{(IV.51)}$$

We insert the different terms of the right-hand side of inequality (IV.51) into the right-hand side

of inequality (IV.48). For instance, let us compute

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{k\in\mathbb{Z}}\int_0^t |f_k|^2\langle k\rangle^6|u-v|^{2\theta'}\|\partial_u^{(2)}x_s\|_{L_\infty}^{2\theta'}|\partial_u x_s(u)-g'(u)|^{2-2\theta'}\mathrm{d}s\Big|^{p/2}\Big]&\\ \leqslant Ct^{\frac{p}{2}-1}|u-v|^{p\theta'}\int_0^t \mathbb{E}\left[\|\partial_u^{(2)}x_s\|_{L_\infty}^{p\theta'}|\partial_u x_s(u)-g'(u)|^{p(1-\theta')}\right]\mathrm{d}s&\\ \leqslant Ct^{\frac{p}{2}-1}|u-v|^{p\theta'}\int_0^t \mathbb{E}\left[\|\partial_u^{(2)}x_s\|_{L_\infty}^{2p\theta'}\right]^{1/2}\mathbb{E}\left[|\partial_u x_s(u)-g'(u)|^{2p(1-\theta')}\right]^{1/2}\mathrm{d}s&\\ \leqslant Ct^{\frac{p}{2}-1}|u-v|^{p\theta'}\int_0^t s^{\frac{p}{2}(1-\theta')}\mathrm{d}s \leqslant Ct^{p-\frac{p}{2}\theta'}|u-v|^{p\theta'}.&\end{aligned}$$

We estimate the other terms of the right-hand side of (IV.51) in the same way. Back to (IV.48) and choosing $\theta' = \frac{3}{4}\theta$, we get

$$\mathbb{E}\left[|\partial_u a_t^{1,2}(u)-\partial_u a_t^{1,2}(v)|^p\right]^{1/p} \leqslant Ct^{1-\frac{3}{8}\theta}|u-v|^{\frac{3}{4}\theta}.$$

Applying as previously Kolmogorov's Lemma, we deduce that

$$\mathbb{E}\left[\|a_t^{1,2}\|_{\mathcal{C}^{1+\theta/2}}^p\right]^{1/p} \leqslant C_p t^{1-\frac{3}{8}\theta}. \tag{IV.52}$$

Finally, putting together (IV.52) and (IV.47), we obtain inequality (IV.41). □

Let us recall Definitions (IV.31), (IV.32) and (IV.33) of b_t^0, b_t^1 and γ_t:

$$\begin{aligned} b_t^0(u) &:= \gamma_t \exp\left(a_t^0(u) - \frac{C_f t}{2}\right);\\ b_t^1(u) &:= \exp\left(a_t^0(u) - \frac{C_f t}{2}\right)\left(\exp(a_t^1(u)) - \gamma_t\right);\\ \gamma_t &:= \frac{2\pi}{\int_0^1 g'(v)\exp\left(a_t^0(v) - \frac{C_f t}{2}\right)\mathrm{d}v}.\end{aligned}$$

Lemma IV.11. *Let $g \in \mathbf{G}^{4+\theta}$. Then almost surely, $u \mapsto b_t^0(u)$ belongs to $\mathcal{C}^{4+\theta/4}$ and $u \mapsto b_t^1(u)$ belongs almost surely to $\mathcal{C}^{1+\theta/4}$ for every $t \in (0,T]$. Moreover, for every $p \geqslant 2$, there is $C_p > 0$ such that for every $t \in (0,T]$*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|b_t^0\|_{\mathcal{C}^{4+\theta/4}}^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}; \tag{IV.53}$$

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|b_t^0\|_{\mathcal{C}^{3+\theta/4}}^p\right]^{1/p} \leqslant C_p; \tag{IV.54}$$

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|b_t^1\|_{\mathcal{C}^{1+\theta/4}}^p\right]^{1/p} \leqslant C_p t^{1-\theta}. \tag{IV.55}$$

Proof. All along this proof, we will write $\mathbb{E}[\cdot]$ instead of $\mathbb{E}^W\mathbb{E}^\beta[\cdot]$.

Regularity and integrability of b_t^0. Since $u \mapsto a_t^0(u)$ belongs to $\mathcal{C}^{4+\theta/2}$, the map $u \mapsto b_t^0(u)$ also belongs to the same space. Moreover, for every $u \in [0,1]$,

$$\begin{aligned}\partial_u^{(4)}b_t^0(u) = \gamma_t\Big[&\partial_u^{(4)}a_t^0(u) + 4\partial_u^{(3)}a_t^0(u)\cdot\partial_u a_t^0(u) + 3(\partial_u^{(2)}a_t^0(u))^2\\ &+ 6\partial_u^{(2)}a_t^0(u)\cdot(\partial_u a_t^0(u))^2 + (\partial_u a_t^0(u))^4\Big]\exp\left(a_t^0(u) - \frac{C_f t}{2}\right).\end{aligned} \tag{IV.56}$$

We will focus on the term $b_t^{0,4}(u) := \gamma_t \partial_u^{(4)} a_t^0(u) \exp\left(a_t^0(u) - \frac{C_f t}{2}\right)$, which is the dominant term: it is the only one including the derivative of order 4 of a_t^0. We will not estimate the other terms but the proof is very close.

Let us consider $b_t^{0,4}$. For every $p \geqslant 2$,

$$\mathbb{E}\left[|b_t^{0,4}(0)|^p\right]^{1/p} \leqslant \mathbb{E}\left[|\gamma_t|^{3p}\right]^{1/(3p)} \mathbb{E}\left[|\partial_u^{(4)} a_t^0(0)|^{3p}\right]^{1/(3p)} \mathbb{E}\left[\left|\exp\left(a_t^0(0) - \tfrac{C_f t}{2}\right)\right|^{3p}\right]^{1/(3p)}.$$

Let $q = 3p$. By (IV.39), we have $\mathbb{E}\left[|\partial_u^{(4)} a_t^0(0)|^q\right]^{1/q} \leqslant C_q \frac{1}{t^{\frac{1}{2}-2\theta}}$. Moreover, for every $u \in [0,1]$, let us denote by $(M_s^t)_{s\in[0,T]}$ the following process:

$$M_s^t := \exp\left(q \sum_{|k|\leqslant N_t} f_k k \int_0^s \Re\left(-ie^{-ikg(u)} \mathrm{d}W_r^k\right) - \frac{q^2}{2} C_{f,t} s\right),$$

where $C_{f,t} := \sum_{|k|\leqslant N_t} |f_k|^2 |k|^2$. Recall that $C_f = \sum_{k\in\mathbb{Z}} |f_k|^2 |k|^2$. Thus $C_{f,t} \leqslant C_f$. By Novikov's condition, $(M_s^t)_{s\in[0,T]}$ is a martingale. Therefore,

$$1 = \mathbb{E}\left[M_0^t\right] = \mathbb{E}\left[M_t^t\right] = \mathbb{E}\left[\exp\left(q a_t^0(u) - \frac{q^2 C_{f,t} t}{2}\right)\right].$$

We deduce that for every $u \in [0,1]$,

$$\begin{aligned}\mathbb{E}\left[\left|\exp\left(a_t^0(u) - \tfrac{C_f t}{2}\right)\right|^q\right]^{1/q} &= \mathbb{E}\left[\exp\left(q a_t^0(u) - \frac{q^2 C_{f,t} t}{2}\right)\right]^{1/q} \exp\left(\frac{q C_{f,t} t}{2} - \frac{C_f t}{2}\right) \\ &= \exp\left(\frac{q C_{f,t} t}{2} - \frac{C_f t}{2}\right) \leqslant \exp\left(\frac{(q-1) C_f T}{2}\right). \qquad \text{(IV.57)}\end{aligned}$$

Furthermore, since $g' > 0$, we have:

$$\int_0^1 g'(v) \exp\left(a_t^0(v) - \tfrac{C_f t}{2}\right) \mathrm{d}v \geqslant \inf_{v\in[0,1]} g'(v) \cdot \inf_{v\in[0,1]} \exp\left(a_t^0(v) - \tfrac{C_f t}{2}\right).$$

It follows that

$$\begin{aligned}|\gamma_t| &\leqslant 2\pi \sup_{v\in[0,1]} \frac{1}{g'(v)} \sup_{v\in[0,1]} \exp\left(-a_t^0(v) + \tfrac{C_f t}{2}\right) \\ &\leqslant 2\pi \sup_{v\in[0,1]} \frac{1}{g'(v)} \left[\exp\left(-a_t^0(0) + \tfrac{C_f t}{2}\right) + \int_0^1 |\partial_u a_t^0(v)| \exp\left(-a_t^0(v) + \tfrac{C_f t}{2}\right) \mathrm{d}v\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[|\gamma_t|^q\right]^{1/q} \leqslant\ & C \left\|\frac{1}{g'}\right\|_{L_\infty} \mathbb{E}\left[\left|\exp\left(-a_t^0(0) + \tfrac{C_f t}{2}\right)\right|^q\right]^{1/q} \\ &+ C \left\|\frac{1}{g'}\right\|_{L_\infty} \mathbb{E}\left[\int_0^1 |\partial_u a_t^0(v)|^{2q} \mathrm{d}v\right]^{1/(2q)} \mathbb{E}\left[\int_0^1 \left|\exp\left(-a_t^0(v) + \tfrac{C_f t}{2}\right)\right|^{2q} \mathrm{d}v\right]^{1/(2q)}.\end{aligned}$$

By (IV.40), there is $C_q > 0$ such that $\mathbb{E}\left[\int_0^1 |\partial_u a_t^0(v)|^{2q} \mathrm{d}v\right]^{1/(2q)} \leqslant C_q$. Moreover, there is a constant $C_q > 0$ depending on f, T and q such that $\mathbb{E}\left[\left|\exp\left(-a_t^0(0) + \frac{C_f t}{2}\right)\right|^q\right]^{1/q} \leqslant C_q$ and $\mathbb{E}\left[\int_0^1 \left|\exp\left(-a_t^0(v) + \frac{C_f t}{2}\right)\right|^{2q} \mathrm{d}v\right]^{1/(2q)} \leqslant C_q$. Thus

$$\mathbb{E}\left[|\gamma_t|^q\right]^{1/q} \leqslant C_q. \qquad \text{(IV.58)}$$

Finally, we obtain

$$\mathbb{E}\left[|b_t^{0,4}(0)|^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}. \tag{IV.59}$$

Moreover, let us compute the Hölder norm of $b_t^{0,4}$. For every $u, v \in [0,1]$, for every $p \geqslant 2$, we have

$$\begin{aligned}
&\mathbb{E}\left[|b_t^{0,4}(u) - b_t^{0,4}(v)|^p\right]^{1/p} \\
&\quad = \mathbb{E}\left[|\gamma_t|^p \left|\partial_u^{(4)} a_t^0(u) \exp\left(a_t^0(u) - \tfrac{C_f t}{2}\right) - \partial_u^{(4)} a_t^0(v) \exp\left(a_t^0(v) - \tfrac{C_f t}{2}\right)\right|^p\right]^{1/p} \leqslant E_1 + E_2,
\end{aligned}$$

where

$$\begin{aligned}
E_1 &:= \mathbb{E}\left[|\gamma_t|^p \, |\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)|^p \left| \exp\left(a_t^0(u) - \tfrac{C_f t}{2}\right)\right|^p\right]^{1/p}; \\
E_2 &:= \mathbb{E}\left[|\gamma_t|^p \, |\partial_u^{(4)} a_t^0(v)|^p \left| \exp\left(a_t^0(u) - \tfrac{C_f t}{2}\right) - \exp\left(a_t^0(v) - \tfrac{C_f t}{2}\right)\right|^p\right]^{1/p}.
\end{aligned}$$

To get an upper bound for E_1, we use inequalities (IV.57), (IV.58) and (IV.39):

$$\begin{aligned}
E_1 &\leqslant \mathbb{E}\left[|\gamma_t|^{3p}\right]^{1/(3p)} \mathbb{E}\left[|\partial_u^{(4)} a_t^0(u) - \partial_u^{(4)} a_t^0(v)|^{3p}\right]^{1/(3p)} \mathbb{E}\left[\left| \exp\left(a_t^0(u) - \tfrac{C_f t}{2}\right)\right|^{3p}\right]^{1/(3p)} \\
&\leqslant C_p |u-v|^{\theta/2} \mathbb{E}\left[\|a_t^0\|_{\mathcal{C}^{4+\theta/2}}^{3p}\right]^{1/(3p)} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}} |u-v|^{\theta/2}.
\end{aligned}$$

Moreover, we use the fact that for every $x, y \in \mathbb{R}$, $|e^x - e^y| \leqslant |x-y|(e^x + e^y)$ and the same inequalities to obtain that

$$\begin{aligned}
E_2 &\leqslant \mathbb{E}\left[|\gamma_t|^{3p}\right]^{1/(3p)} \mathbb{E}\left[|\partial_u^{(4)} a_t^0(v)|^{3p}\right]^{1/(3p)} \mathbb{E}\left[\left| \exp\left(a_t^0(u) - \tfrac{C_f t}{2}\right) - \exp\left(a_t^0(v) - \tfrac{C_f t}{2}\right)\right|^{3p}\right]^{1/(3p)} \\
&\leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}} \mathbb{E}\left[|a_t^0(u) - a_t^0(v)|^{6p}\right]^{1/(6p)} \left(\mathbb{E}\left[\left| \exp\left(a_t^0(u) - \tfrac{C_f t}{2}\right)\right|^{6p}\right]^{1/(6p)} \right. \\
&\qquad\qquad \left. + \mathbb{E}\left[\left| \exp\left(a_t^0(v) - \tfrac{C_f t}{2}\right)\right|^{6p}\right]^{1/(6p)}\right) \\
&\leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}} \mathbb{E}\left[\|a_t^0\|_{\mathcal{C}^1}^{6p}\right]^{1/(6p)} |u-v| \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}} |u-v|.
\end{aligned}$$

Remark that the last inequality is a consequence of (IV.40). Therefore, we proved that for every $u, v \in [0,1]$ and for every $p \geqslant 2$,

$$\mathbb{E}\left[|b_t^{0,4}(u) - b_t^{0,4}(v)|^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}} |u-v|^{\theta/2}.$$

By Kolmogorov's Lemma, it follows that for p large enough,

$$\mathbb{E}\left[\left|\sup_{u \neq v} \frac{|b_t^{0,4}(u) - b_t^{0,4}(v)|}{|u-v|^{\frac{\theta}{4}}}\right|^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}. \tag{IV.60}$$

Inequality (IV.60) also holds with every $p \geqslant 2$ by Hölder's inequality. By (IV.59) and (IV.60), we deduce that for every $p \geqslant 2$, $\mathbb{E}\left[\|b_t^{0,4}\|_{\mathcal{C}^{\frac{\theta}{4}}}^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}$. We obtain the same control on the other terms appearing in (IV.56). Thus for every $p \geqslant 2$, $\mathbb{E}\left[\|\partial_u^{(4)} b_t^0\|_{\mathcal{C}^{\frac{\theta}{4}}}^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}$.

In the expansion of $\partial_u^{(3)} b_t^0$, only the first three derivatives of a_t^0 appear. Since the control over the $\mathcal{C}^{3+\theta/2}$-norm of a_t^0, given by inequality (IV.40), is independent of t, we deduce that for every $p \geqslant 2$, $\mathbb{E}\left[\|\partial_u^{(j)} b_t^0\|^p_{\mathcal{C}^{\frac{\theta}{4}}}\right]^{1/p} \leqslant C_p t^{\frac{3-j}{2}}$. Thus we deduce (IV.53) and (IV.54).

Regularity and integrability of b_t^1. Since $u \mapsto a_t^1(u)$ belongs to $\mathcal{C}^{1+\theta/2}$, the map $u \mapsto b_t^1(u)$ also belongs to the same space. For every $u \in [0,1]$,

$$\partial_u b_t^1(u) = \partial_u a_t^0(u) \exp\left(a_t^0(u) - \frac{C_f t}{2}\right)\left(\exp(a_t^1(u)) - \gamma_t\right) + \partial_u a_t^1(u) \exp\left(a_t^0(u) - \frac{C_f t}{2}\right)\exp(a_t^1(u)).$$

Thus for every $p \geqslant 2$,

$$\begin{aligned}\mathbb{E}\left[|\partial_u b_t^1(0)|^p\right]^{1/p} \leqslant{}& \mathbb{E}\left[|\partial_u a_t^0(0)|^{3p}\right]^{\frac{1}{3p}} \mathbb{E}\left[\left|\exp\left(a_t^0(0) - \tfrac{C_f t}{2}\right)\right|^{3p}\right]^{\frac{1}{3p}} \mathbb{E}\left[\left|\exp(a_t^1(0)) - \gamma_t\right|^{3p}\right]^{\frac{1}{3p}} \\ &+ \mathbb{E}\left[|\partial_u a_t^1(0)|^{3p}\right]^{\frac{1}{3p}} \mathbb{E}\left[\left|\exp\left(a_t^0(0) - \tfrac{C_f t}{2}\right)\right|^{3p}\right]^{\frac{1}{3p}} \mathbb{E}\left[\left|\exp(a_t^1(0))\right|^{3p}\right]^{\frac{1}{3p}}.\end{aligned} \tag{IV.61}$$

By (IV.40) and (IV.41), we have $\mathbb{E}\left[|\partial_u a_t^0(0)|^{3p}\right]^{\frac{1}{3p}} \leqslant C_p$ and $\mathbb{E}\left[|\partial_u a_t^1(0)|^{3p}\right]^{\frac{1}{3p}} \leqslant C_p t^{1-\theta}$. Furthermore, by (IV.57), $\mathbb{E}\left[\left|\exp\left(a_t^0(0) - \frac{C_f t}{2}\right)\right|^{3p}\right]^{\frac{1}{3p}} \leqslant C_p$. Furthermore, recall that by construction, $a_t^1 = a_t - a_t^0$. Therefore, for every $q \geqslant 2$,

$$\begin{aligned}\mathbb{E}\left[\left|\exp(a_t^1(0))\right|^q\right]^{1/q} &= \mathbb{E}\left[\left|\exp\left(a_t(0) - \tfrac{C_f t}{2}\right)\right|^q \left|\exp\left(-a_t^0(0) + \tfrac{C_f t}{2}\right)\right|^q\right]^{1/q} \\ &\leqslant \mathbb{E}\left[\left|\exp\left(a_t(0) - \tfrac{C_f t}{2}\right)\right|^{2q}\right]^{\frac{1}{2q}} \mathbb{E}\left[\left|\exp\left(-a_t^0(0) + \tfrac{C_f t}{2}\right)\right|^q\right]^{\frac{1}{2q}}\end{aligned}$$

We have already seen that $\mathbb{E}\left[\left|\exp\left(-a_t^0(0) + \frac{C_f t}{2}\right)\right|^q\right]^{\frac{1}{2q}} \leqslant C_q$. Moreover, recall that $(a_s)_{s\in[0,T]}$ is a martingale and that its quadratic variation is equal to $\langle a, a\rangle_s = C_f s$. Therefore, for every $r \geqslant 2$, $\left(\exp\left(r a_s(0) - \frac{r^2 C_f s}{2}\right)\right)_{s\in[0,T]}$ is a martingale. We deduce that there is a constant C_q depending on q, f, T such that

$$\mathbb{E}\left[\left|\exp\left(a_t(0) - \tfrac{C_f t}{2}\right)\right|^{2q}\right]^{\frac{1}{2q}} \leqslant C_q. \tag{IV.62}$$

Thus for every $q \geqslant 2$, there is $C_q > 0$ such that

$$\mathbb{E}\left[\left|\exp(a_t^1(0))\right|^q\right]^{1/q} \leqslant C_q. \tag{IV.63}$$

Furthermore, for every $q \geqslant 2$,

$$\mathbb{E}\left[\left|\exp(a_t^1(0)) - \gamma_t\right|^q\right]^{1/q} \leqslant \mathbb{E}\left[\left|\exp(a_t^1(0)) - 1\right|^q\right]^{1/q} + \mathbb{E}\left[|\gamma_t - 1|^q\right]^{1/q}.$$

On the one hand, we use the fact that for every $x \in \mathbb{R}$, $|e^x - 1| \leqslant |x|(e^x + 1)$

$$\mathbb{E}\left[\left|\exp(a_t^1(0)) - 1\right|^q\right]^{1/q} \leqslant \mathbb{E}\left[|a_t^1(0)|^{2q}\right]^{\frac{1}{2q}} \left(1 + \mathbb{E}\left[\left|\exp(a_t^1(0))\right|^{2q}\right]^{\frac{1}{2q}}\right) \leqslant C_q t^{1-\theta},$$

by inequalities (IV.41) and (IV.63). On the other hand, we have $\int_0^1 g'(v)\exp\left(a_t(v)-\frac{C_f t}{2}\right)\mathrm{d}v = \int_0^1 \partial_u x_t(v)\mathrm{d}v = x_t(1)-x_t(0) = 2\pi$. Therefore

$$\begin{aligned}\gamma_t - 1 &= \frac{2\pi}{\int_0^1 g'(v)\exp\left(a_t^0(v)-\frac{C_f t}{2}\right)\mathrm{d}v} - 1\\ &= \frac{\int_0^1 g'(v)\left(\exp\left(a_t(v)-\frac{C_f t}{2}\right)-\exp\left(a_t^0(v)-\frac{C_f t}{2}\right)\right)\mathrm{d}v}{\int_0^1 g'(v)\exp\left(a_t^0(v)-\frac{C_f t}{2}\right)\mathrm{d}v}\\ &= \frac{\gamma_t}{2\pi}\int_0^1 g'(v)\left(\exp\left(a_t(v)-\frac{C_f t}{2}\right)-\exp\left(a_t^0(v)-\frac{C_f t}{2}\right)\right)\mathrm{d}v.\end{aligned}$$

Thus, using again that for every $x, y \in \mathbb{R}$, $|e^x - e^y| \leqslant |x-y|(e^x+e^y)$, we obtain

$$\begin{aligned}\mathbb{E}\left[|\gamma_t - 1|^q\right]^{1/q} &\leqslant C\|g'\|_{L_\infty}\mathbb{E}\left[|\gamma_t|^{2q}\right]^{\frac{1}{2q}}\mathbb{E}\left[\int_0^1\left|\exp\left(a_t(v)-\tfrac{C_f t}{2}\right)-\exp\left(a_t^0(v)-\tfrac{C_f t}{2}\right)\right|^{2q}\mathrm{d}v\right]^{\frac{1}{2q}}\\ &\leqslant C_q\mathbb{E}\left[\int_0^1 |a_t^1(v)|^{4q}\mathrm{d}v\right]^{\frac{1}{4q}}\end{aligned}$$

by inequalities (IV.57), (IV.58) and (IV.62). By the control (IV.41) on the integrability of a_t^1, we deduce that $\mathbb{E}\left[|\gamma_t - 1|^q\right]^{1/q} \leqslant C_q t^{1-\theta}$. Back to (IV.61), we finally obtain that for every $p \geqslant 2$, there is $C_p > 0$ such that

$$\mathbb{E}\left[|\partial_u b_t^1(0)|^p\right]^{1/p} \leqslant C_p t^{1-\theta}. \tag{IV.64}$$

Moreover, for every $u, v \in [0,1]$, we have

$$\begin{aligned}|\partial_u b_t^1(u) - \partial_u b_t^1(v)| \leqslant &\left|\partial_u a_t^0(u)\exp\left(a_t^0(u)-\tfrac{C_f t}{2}\right)\left(\exp(a_t^1(u))-\gamma_t\right)\right.\\ &\left.-\partial_u a_t^0(v)\exp\left(a_t^0(v)-\tfrac{C_f t}{2}\right)\left(\exp(a_t^1(v))-\gamma_t\right)\right|\\ &+\left|\partial_u a_t^1(u)\exp\left(a_t^0(u)-\tfrac{C_f t}{2}\right)\exp(a_t^1(u))\right.\\ &\left.-\partial_u a_t^1(v)\exp\left(a_t^0(v)-\tfrac{C_f t}{2}\right)\exp(a_t^1(v))\right|.\end{aligned} \tag{IV.65}$$

Let us give the details of the estimation of $|\partial_u a_t^1(u) - \partial_u a_t^1(v)|\exp\left(a_t^0(v)-\frac{C_f t}{2}\right)\exp(a_t^1(v))$. For every $p \geqslant 2$,

$$\mathbb{E}\left[\left|(\partial_u a_t^1(u)-\partial_u a_t^1(v))\exp\left(a_t^0(v)-\tfrac{C_f t}{2}\right)\exp(a_t^1(v))\right|^p\right]^{1/p} \leqslant C_p\mathbb{E}\left[\left|\partial_u a_t^1(u)-\partial_u a_t^1(v)\right|^{3p}\right]^{\frac{1}{3p}}$$

by inequalities (IV.57) and (IV.63). Furthermore, since we have the control (IV.41) on the $\frac{\theta}{2}$-Hölder norm of $\partial_u a_t^1$, we deduce that

$$\mathbb{E}\left[\left|(\partial_u a_t^1(u)-\partial_u a_t^1(v))\exp\left(a_t^0(v)-\tfrac{C_f t}{2}\right)\exp(a_t^1(v))\right|^p\right]^{1/p} \leqslant C_p t^{1-\theta}|u-v|^{\theta/2}.$$

We obtain similar bounds for the other terms appearing in the right-hand side of (IV.65). Finally, we get for every $p \geqslant 2$,

$$\mathbb{E}\left[|\partial_u b_t^1(u) - \partial_u b_t^1(v)|^p\right]^{1/p} \leqslant C_p t^{1-\theta}|u-v|^{\theta/2}.$$

By Kolmogorov's Lemma, we deduce that

$$\mathbb{E}\left[\left|\sup_{u\neq v}\frac{|\partial_u b_t^1(u)-\partial_u b_t^1(v)|}{|u-v|^{\frac{\theta}{4}}}\right|^p\right]^{1/p}\leqslant C_p t^{1-\theta}. \tag{IV.66}$$

Putting together (IV.64) and (IV.66), we finally obtain inequality (IV.55). □

Recall the Definitions (IV.35) and (IV.36) of x_t^0 and x_t^1:

$$x_t^0(u) := x_t(0)+\int_0^u g'(v)b_t^0(v)\mathrm{d}v;$$
$$x_t^1(u) := \int_0^u g'(v)b_t^1(v)\mathrm{d}v.$$

Moreover, F_t^0 was defined as the inverse map of x_t^0.

Lemma IV.12. *Let $g\in\mathbf{G}^{4+\theta}$. Then almost surely, the maps $u\mapsto x_t^0(u)$ and $v\mapsto F_t^0(v)$ belong to $\mathcal{C}^{4+\theta/4}$ and the map $u\mapsto x_t^1(u)$ belongs to $\mathcal{C}^{2+\theta/4}$. Moreover, for every $p\geqslant 2$, there is $C_p>0$ such that for every $t\in(0,T]$*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\partial_u x_t^0\|_{\mathcal{C}^{3+\theta/4}}^p\right]^{1/p}\leqslant C_p; \tag{IV.67}$$
$$\mathbb{E}^W\mathbb{E}^\beta\left[\|x_t^1\|_{\mathcal{C}^{2+\theta/4}}^p\right]^{1/p}\leqslant C_p t^{1-\theta}; \tag{IV.68}$$
$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\partial_v F_t^0\|_{\mathcal{C}^{3+\theta/8}}^p\right]^{1/p}\leqslant C_p. \tag{IV.69}$$

Proof. All along this proof, we will write $\mathbb{E}[\cdot]$ instead of $\mathbb{E}^W\mathbb{E}^\beta[\cdot]$.

Regularity and integrability of x_t^0 and x_t^1. Note that $\partial_u x_t^0 = g'\cdot b_t^0$. Both g' and b_t^0 are $\mathcal{C}^{3+\theta/4}$-functions. We deduce that x_t^0 is a $\mathcal{C}^{4+\theta/4}$-function. Moreover, inequality (IV.67) follow immediately from inequality (IV.54) on b_t^0.

Moreover, recall that x_t^1 is a 1-periodic function. Thus it follows from Definition (IV.36) of x_t^1 that $\|x_t^1\|_{L_\infty}\leqslant \sup_{u\in[0,1]}|\int_0^u g'(v)b_t^1(v)\mathrm{d}v|\leqslant \|g'\|_{L_\infty}\|b_t^1\|_{L_\infty}$. Therefore, by inequality (IV.55) on b_t^1, we have for every $p\geqslant 2$, $\mathbb{E}\left[\|x_t^1\|_{L_\infty}^p\right]^{1/p}\leqslant C_p t^{1-\theta}$. Furthermore, $\partial_u x_t^1 = g'\cdot b_t^1$, thus it follows that $\partial_u x_t^1$ is a $\mathcal{C}^{1+\theta/4}$-function and by inequality (IV.55), $\mathbb{E}^W\mathbb{E}^\beta\left[\|\partial_u x_t^1\|_{\mathcal{C}^{1+\theta/4}}^p\right]^{1/p}\leqslant C_p t^{1-\theta}$. Thus inequality (IV.68) follows.

Regularity and integrability of F_t^0. Recall that $F_t^0=(x_t^0)^{-1}$ exists and satisfies the relation $F_t^0(v+2\pi)-F(v)=1$ for every $v\in\mathbb{R}$. Thus $\partial_v F_t^0$ is 2π-periodic. Furthermore,

$$\partial_v F_t^0 = \frac{1}{(\partial_u x_t^0)\circ F_t^0};$$
$$\partial_v^{(2)} F_t^0 = -\frac{(\partial_u^{(2)}x_t^0)\circ F_t^0}{(\partial_u x_t^0)^2\circ F_t^0}\,\partial_v F_t^0 = -\frac{(\partial_u^{(2)}x_t^0)\circ F_t^0}{(\partial_u x_t^0)^3\circ F_t^0}.$$

By further computations, $\partial_v^{(4)}F_t^0 = -\frac{(\partial_u^{(4)}x_t^0)\circ F_t^0}{(\partial_u x_t^0)^5\circ F_t^0}+F_t^{0,2}$, where the terms appearing in $F_t^{0,2}$ are positive powers of $(\partial_u^{(2)}x_t^0)\circ F_t^0$, of $(\partial_u^{(3)}x_t^0)\circ F_t^0$ and of $\frac{1}{(\partial_u x_t^0)\circ F_t^0}$. We will focus on $F_t^{0,1} = -\frac{(\partial_u^{(4)}x_t^0)\circ F_t^0}{(\partial_u x_t^0)^5\circ F_t^0}$. For every $p\geqslant 2$,

$$\begin{aligned}\mathbb{E}\left[|F_t^{0,1}(0)|^p\right]^{1/p} &\leqslant \mathbb{E}\left[\|\partial_u^{(4)}x_t^0\|_{L_\infty}^p\left\|\tfrac{1}{\partial_u x_t^0}\right\|_{L_\infty}^{5p}\right]^{1/p}\\ &\leqslant \mathbb{E}\left[\|\partial_u^{(4)}x_t^0\|_{L_\infty}^{2p}\right]^{\frac{1}{2p}}\mathbb{E}\left[\left\|\tfrac{1}{\partial_u x_t^0}\right\|_{L_\infty}^{10p}\right]^{\frac{5}{10p}}.\end{aligned}$$

By (IV.67), $\mathbb{E}\left[\|\partial_u^{(4)} x_t^0\|_{L_\infty}^{2p}\right]^{\frac{1}{2p}} \leqslant C_p$. Moreover, $\left\|\frac{1}{\partial_u x_t^0}\right\|_{L_\infty} \leqslant \left\|\frac{1}{g'}\right\|_{L_\infty} \left\|\frac{1}{b_t^0}\right\|_{L_\infty}$. For every $u \in [0,1]$, $(b_t^0)^{-1}(u) = \gamma_t^{-1} \exp\left(-a_t^0(u) + \frac{C_f t}{2}\right)$. We prove the inequality $\mathbb{E}\left[|\gamma_t^{-1}|^q\right]^{1/q} \leqslant C_q$ in the same way as we proved inequality (IV.58). Therefore, we obtain the same bound for $(b_t^0)^{-1}$ as the bound obtained for b_t^0. Thus for every $q \geqslant 2$, we have $\mathbb{E}\left[\left\|\frac{1}{b_t^0}\right\|_{L_\infty}^q\right]^{1/q} \leqslant C_q$. It follows that

$$\mathbb{E}\left[|F_t^{0,1}(0)|^p\right]^{1/p} \leqslant C_p.$$

Furthermore, for every $x, y \in [0, 2\pi]$,

$$\begin{aligned}
|F_t^{0,1}(x) - F_t^{0,1}(y)| &= \left|\frac{(\partial_u^{(4)} x_t^0) \circ F_t^0(x)}{(\partial_u x_t^0)^5 \circ F_t^0(x)} - \frac{(\partial_u^{(4)} x_t^0) \circ F_t^0(y)}{(\partial_u x_t^0)^5 \circ F_t^0(y)}\right| \\
&\leqslant \left|\frac{(\partial_u^{(4)} x_t^0) \circ F_t^0(x) - (\partial_u^{(4)} x_t^0) \circ F_t^0(y)}{(\partial_u x_t^0)^5 \circ F_t^0(x)}\right| \\
&\quad + |(\partial_u^{(4)} x_t^0) \circ F_t^0(y)| \left|\frac{1}{(\partial_u x_t^0)^5 \circ F_t^0(x)} - \frac{1}{(\partial_u x_t^0)^5 \circ F_t^0(y)}\right|.
\end{aligned}$$

Let us remark that

$$\begin{aligned}
\left|\frac{1}{(\partial_u x_t^0)^5 \circ F_t^0(x)} - \frac{1}{(\partial_u x_t^0)^5 \circ F_t^0(y)}\right| &\leqslant 5 \left\|\frac{(\partial_u^{(2)} x_t^0) \circ F_t^0}{(\partial_u x_t^0)^7 \circ F_t^0}\right\|_{L_\infty} |x - y| \\
&\leqslant 5 \|\partial_u^{(2)} x_t^0\|_{L_\infty} \left\|\frac{1}{\partial_u x_t^0}\right\|_{L_\infty}^7 |x - y|.
\end{aligned}$$

It follows that for every $p \geqslant 2$,

$$\begin{aligned}
\mathbb{E}\left[|F_t^{0,1}(x) - F_t^{0,1}(y)|^p\right]^{1/p} &\leqslant \mathbb{E}\left[|(\partial_u^{(4)} x_t^0) \circ F_t^0(x) - (\partial_u^{(4)} x_t^0) \circ F_t^0(y)|^{2p}\right]^{\frac{1}{2p}} \mathbb{E}\left[\left\|\frac{1}{\partial_u x_t^0}\right\|_{L_\infty}^{10p}\right]^{\frac{5}{10p}} \\
&\quad + 5\mathbb{E}\left[\|\partial_u^{(4)} x_t^0\|_{L_\infty}^{2p}\right]^{\frac{1}{2p}} \mathbb{E}\left[\|\partial_u^{(2)} x_t^0\|_{L_\infty}^{2p} \left\|\frac{1}{\partial_u x_t^0}\right\|_{L_\infty}^{14p}\right]^{\frac{1}{2p}} |x - y|.
\end{aligned}$$

Moreover, for every $q \geqslant 2$,

$$\begin{aligned}
\mathbb{E}\left[|(\partial_u^{(4)} x_t^0) \circ F_t^0(x) - (\partial_u^{(4)} x_t^0) \circ F_t^0(y)|^q\right]^{1/q} &\leqslant \mathbb{E}\left[\|\partial_u^{(4)} x_t^0\|_{\mathcal{C}^{\theta/4}}^q |F_t^0(x) - F_t^0(y)|^{\frac{q}{4}\theta}\right]^{1/q} \\
&\leqslant \mathbb{E}\left[\|\partial_u^{(4)} x_t^0\|_{\mathcal{C}^{\theta/4}}^{2q}\right]^{\frac{1}{2q}} \mathbb{E}\left[\|\partial_v F_t^0\|_{L_\infty}^{\frac{q}{2}\theta}\right]^{\frac{1}{2q}} |x - y|^{\theta/4}.
\end{aligned}$$

By inequality (IV.67), it follows that for every $x, y \in [0, 2\pi]$ and for every $p \geqslant 2$,

$$\mathbb{E}\left[|F_t^{0,1}(x) - F_t^{0,1}(y)|^p\right]^{1/p} \leqslant C_p |x - y|^{\theta/4}.$$

By Kolmogorov's Lemma, we deduce that $\mathbb{E}\left[\|F_t^{0,1}\|_{\mathcal{C}^{\theta/8}}^p\right]^{1/p} \leqslant C_p$. We obtain the same control for $F_t^{0,2}$. Thus inequality (IV.69) follows. □

Recall the Definitions (IV.37) and (IV.38) of $\mathcal{A}_t^0$ and $\mathcal{A}_t^1$:

$$\begin{aligned}
\mathcal{A}_t^0(v) &:= (b_t^0 \cdot h) \circ F_t^0(v); \\
\mathcal{A}_t^1(u) &:= b_t^1(u) h(u) + \left[(b_t^0 \cdot h)(u) - (b_t^0 \cdot h) \circ F_t^0(x_t(u))\right].
\end{aligned}$$

Proposition IV.13. *Let $g \in \mathbf{G}^{4+\theta}$ and $h \in \Delta^{4+\theta}$. Then almost surely, $v \mapsto \mathcal{A}_t^0(v)$ belongs to $\mathcal{C}^{4+\theta/8}$ and $u \mapsto \mathcal{A}_t^1(u)$ belongs almost surely to $\mathcal{C}^{1+\theta/4}$ for every $t \in (0,T]$. Moreover, for every $p \geqslant 2$, there is $C_p > 0$ such that for every $t \in (0,T]$*

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\mathcal{A}_t^0\|^p_{\mathcal{C}^{4+\theta/16}}\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}\|h\|_{\mathcal{C}^{4+\theta}}; \tag{IV.70}$$

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\mathcal{A}_t^0\|^p_{\mathcal{C}^{3+\theta/16}}\right]^{1/p} \leqslant C_p\|h\|_{\mathcal{C}^{3+\theta}}; \tag{IV.71}$$

$$\mathbb{E}^W\mathbb{E}^\beta\left[\|\mathcal{A}_t^1\|^p_{\mathcal{C}^{1+\theta/8}}\right]^{1/p} \leqslant C_p t^{1-2\theta}\|h\|_{\mathcal{C}^2}. \tag{IV.72}$$

Proof. All along this proof, we will write $\mathbb{E}[\cdot]$ instead of $\mathbb{E}^W\mathbb{E}^\beta[\cdot]$.

Regularity and integrability of $\mathcal{A}_t^0$. Note that b_t^0, h and F_t^0 belong to $\mathcal{C}^{4+\theta/8}$. By inequalities (IV.53) and (IV.54), we deduce that $\mathbb{E}\left[\|b_t^0 \cdot h\|^p_{\mathcal{C}^{4+\theta/4}}\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}\|h\|_{\mathcal{C}^{4+\theta/4}}$ and that $\mathbb{E}\left[\|b_t^0 \cdot h\|^p_{\mathcal{C}^{3+\theta/4}}\right]^{1/p} \leqslant C_p\|h\|_{\mathcal{C}^{3+\theta/4}}$. Furthermore, for every $v \in [0,2\pi]$,

$$\partial_v^{(4)}\mathcal{A}_t^0(v) = (\partial_v F_t^0(v))^4\left[(\partial_u^{(4)}(b_t^0 \cdot h)) \circ F_t^0\right](v) + \partial_v^{(4)}F_t^0(v)\left[(b_t^0 \cdot h) \circ F_t^0\right](v) + \mathcal{A}_t^{0,3}(v), \tag{IV.73}$$

where $\mathcal{A}_t^{0,3}$ involves the derivatives $\partial_v^{(j)}F_t^0$ and $(\partial_u^{(j)}(b_t^0 \cdot h)) \circ F_t^0$ for $j = 1,2,3$. We will focus on the first two terms of the right-hand side of (IV.73).

Let $\mathcal{A}_t^{0,1}(v) := (\partial_v F_t^0(v))^4\left[(\partial_u^{(4)}(b_t^0 \cdot h)) \circ F_t^0\right](v)$. For every $p \geqslant 2$, by inequality (IV.69) on F_t^0,

$$\mathbb{E}\left[|\mathcal{A}_t^{0,1}(0)|^p\right]^{1/p} \leqslant \mathbb{E}\left[|\partial_v F_t^0(0)|^{8p}\right]^{\frac{1}{2p}}\mathbb{E}\left[\|\partial_u^{(4)}(b_t^0 \cdot h)\|^{2p}_{L_\infty}\right]^{\frac{1}{2p}} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}\|h\|_{\mathcal{C}^{4+\theta/4}}.$$

Moreover, for every $x, y \in [0,2\pi]$,

$$\begin{aligned}\mathbb{E}\left[|\mathcal{A}_t^{0,1}(x) - \mathcal{A}_t^{0,1}(y)|^p\right]^{1/p} &\leqslant \mathbb{E}\left[|(\partial_v F_t^0(x))^4 - (\partial_v F_t^0(y))^4|^{2p}\right]^{\frac{1}{2p}}\mathbb{E}\left[\|\partial_u^{(4)}(b_t^0 \cdot h)\|^{2p}_{L_\infty}\right]^{\frac{1}{2p}} \\ &\quad + \mathbb{E}\left[|\partial_v F_t^0(x)|^{8p}\right]^{\frac{1}{2p}}\mathbb{E}\left[\|b_t^0 \cdot h\|^{2p}_{\mathcal{C}^{4+\theta/4}}|F_t^0(x) - F_t^0(y)|^{\frac{p}{2}\theta}\right]^{\frac{1}{2p}} \\ &\leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}\|h\|_{\mathcal{C}^{4+\theta/4}}|x-y|^{\theta/4}.\end{aligned}$$

Let $\mathcal{A}_t^{0,2}(v) := \partial_v^{(4)}F_t^0(v)\left[(b_t^0 \cdot h) \circ F_t^0\right](v)$. For every $p \geqslant 2$, by inequality (IV.69) on F_t^0,

$$\mathbb{E}\left[|\mathcal{A}_t^{0,2}(0)|^p\right]^{1/p} \leqslant \mathbb{E}\left[|\partial_v^{(4)}F_t^0(0)|^{2p}\right]^{\frac{1}{2p}}\mathbb{E}\left[\|b_t^0 \cdot h\|^{2p}_{L_\infty}\right]^{\frac{1}{2p}} \leqslant C_p\|h\|_{L_\infty}.$$

Moreover, for every $x, y \in [0,2\pi]$,

$$\begin{aligned}\mathbb{E}\left[|\mathcal{A}_t^{0,2}(x) - \mathcal{A}_t^{0,2}(y)|^p\right]^{1/p} &\leqslant \mathbb{E}\left[\|\partial_v^{(4)}F_t^0\|^{2p}_{\mathcal{C}^{\theta/8}}\right]^{\frac{1}{2p}}|x-y|^{\theta/8}\mathbb{E}\left[\|b_t^0 \cdot h\|^{2p}_{L_\infty}\right]^{\frac{1}{2p}} \\ &\quad + \mathbb{E}\left[|\partial_v^{(4)}F_t^0(x)|^{2p}\right]^{\frac{1}{2p}}\mathbb{E}\left[\|b_t^0 \cdot h\|^{2p}_{\mathcal{C}^1}|F_t^0(x) - F_t^0(y)|^{2p}\right]^{\frac{1}{2p}} \\ &\leqslant C_p\|h\|_{\mathcal{C}^1}|x-y|^{\theta/8}.\end{aligned}$$

The computations on $\mathcal{A}_t^{0,3}$ are similar. Therefore, we obtain for every $p \geqslant 2$, for every $x, y \in [0,2\pi]$,

$$\mathbb{E}\left[|\partial_v^{(4)}\mathcal{A}_t^0(0)|^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}\|h\|_{\mathcal{C}^{4+\theta/4}}$$

$$\mathbb{E}\left[|\partial_v^{(4)}\mathcal{A}_t^0(x) - \partial_v^{(4)}\mathcal{A}_t^0(y)|^p\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}\|h\|_{\mathcal{C}^{4+\theta/4}}|x-y|^{\theta/8}.$$

It follows from Kolmogorov's Lemma that $\mathbb{E}\left[\|\partial_v^{(4)}\mathcal{A}_t^0\|^p_{\mathcal{C}^{\theta/16}}\right]^{1/p} \leqslant \frac{C_p}{t^{\frac{1}{2}-2\theta}}\|h\|_{\mathcal{C}^{4+\theta/4}}$. In order to bound $\partial_v^{(j)}\mathcal{A}_t^0$ for $j = 0,1,2,3$, we only need to diffentiate 3 times $b_t^0 \cdot h$, thus we only need inequality (IV.54) and we will obtain a control independent of t. Therefore, we obtain in the same way that $\mathbb{E}\left[\|\partial_v^{(j)}\mathcal{A}_t^0\|^p_{\mathcal{C}^{\theta/16}}\right]^{1/p} \leqslant C_p\|h\|_{\mathcal{C}^{4+\theta/4}}$. Henceforth, inequalities (IV.70) and (IV.71) hold true.

Regularity and integrability of $\mathcal{A}_t^1$. Every term appearing in Definition (IV.38) of $\mathcal{A}_t^1$ belongs to $\mathcal{C}^{1+\theta/4}$, so $\mathcal{A}_t^1$ also belongs to $\mathcal{C}^{1+\theta/4}$. It follows directly from inequality (IV.55) on b_t^1 that for every $p \geqslant 2$, $\mathbb{E}\left[\|b_t^1 \cdot h\|^p_{\mathcal{C}^{1+\theta/4}}\right]^{1/p} \leqslant C_p t^{1-\theta}\|h\|_{\mathcal{C}^{1+\theta}}$.

Let $\mathcal{A}_t^{1,2}(u) := (b_t^0 \cdot h)(u) - (b_t^0 \cdot h) \circ F_t^0(x_t(u))$. For every $u \in [0,1]$,

$$\begin{aligned}\partial_u\mathcal{A}_t^{1,2}(u) &= \partial_u(b_t^0 \cdot h)(u) - \partial_u x_t(u)\partial_v F_t^0(x_t(u))(\partial_u(b_t^0 \cdot h)) \circ F_t^0(x_t(u)) \\ &= \partial_u x_t^0(u)\partial_v F_t^0(x_t^0(u))(\partial_u(b_t^0 \cdot h)) \circ F_t^0(x_t^0(u)) \\ &\quad - \partial_u x_t(u)\partial_v F_t^0(x_t(u))(\partial_u(b_t^0 \cdot h)) \circ F_t^0(x_t(u)), \end{aligned} \tag{IV.74}$$

because for every $u \in [0,1]$, $F_t^0(x_t^0(u)) = u$ and $\partial_u x_t^0(u)\partial_v F_t^0(x_t^0(u)) = 1$. Recall that $x_t - x_t^0 = x_t^1$. Therefore, for every $p \geqslant 2$, $\mathbb{E}\left[|\partial_u\mathcal{A}_t^{1,2}(0)|^p\right]^{1/p} \leqslant E_1 + E_2 + E_3$, where

$$\begin{aligned} E_1 &:= \mathbb{E}\left[|\partial_u x_t^1(0)|^p\,|\partial_v F_t^0(x_t^0(0))|^p\,|\partial_u(b_t^0 \cdot h)(0)|^p\right]^{1/p}; \\ E_2 &:= \mathbb{E}\left[|\partial_u x_t(0)|^p\,|\partial_v F_t^0(x_t^0(0)) - \partial_v F_t^0(x_t(0))|^p\,|\partial_u(b_t^0 \cdot h)(0)|^p\right]^{1/p}; \\ E_3 &:= \mathbb{E}\left[|\partial_u x_t(0)|^p\,|\partial_v F_t^0(x_t(0))|^p\,|(\partial_u(b_t^0 \cdot h)) \circ F_t^0(x_t^0(0)) - (\partial_u(b_t^0 \cdot h)) \circ F_t^0(x_t(0))|^p\right]^{1/p}. \end{aligned}$$

The following three inequalities follow from (III.33), (IV.54), (IV.68) and (IV.69):

$$E_1 \leqslant \mathbb{E}\left[\|x_t^1\|^{3p}_{\mathcal{C}^1}\right]^{\frac{1}{3p}}\mathbb{E}\left[\|\partial_v F_t^0\|^{3p}_{L_\infty}\right]^{\frac{1}{3p}}\mathbb{E}\left[\|b_t^0 \cdot h\|^{3p}_{\mathcal{C}^1}\right]^{\frac{1}{3p}} \leqslant C_p t^{1-\theta}\|h\|_{\mathcal{C}^1}.$$

Moreover,

$$E_2 \leqslant \mathbb{E}\left[\|\partial_u x_t\|^{3p}_{L_\infty}\right]^{\frac{1}{3p}}\mathbb{E}\left[\|\partial_v F_t^0\|^{6p}_{\mathcal{C}^1}\right]^{\frac{1}{6p}}\mathbb{E}\left[\|x_t^1\|^{6p}_{L_\infty}\right]^{\frac{1}{6p}}\mathbb{E}\left[\|b_t^0 \cdot h\|^{3p}_{\mathcal{C}^1}\right]^{\frac{1}{3p}} \leqslant C_p t^{1-\theta}\|h\|_{\mathcal{C}^1}.$$

Furthermore,

$$\begin{aligned} E_3 &\leqslant \mathbb{E}\left[\|\partial_u x_t\|^{3p}_{L_\infty}\right]^{\frac{1}{3p}}\mathbb{E}\left[\|\partial_v F_t^0\|^{3p}_{L_\infty}\right]^{\frac{1}{3p}}\mathbb{E}\left[\|(\partial_u(b_t^0 \cdot h)) \circ F_t^0\|^{6p}_{\mathcal{C}^1}\right]^{\frac{1}{6p}}\mathbb{E}\left[\|x_t^1\|^{6p}_{L_\infty}\right]^{\frac{1}{6p}} \\ &\leqslant \mathbb{E}\left[\|\partial_u x_t\|^{3p}_{L_\infty}\right]^{\frac{1}{3p}}\mathbb{E}\left[\|\partial_v F_t^0\|^{3p}_{L_\infty}\right]^{\frac{1}{3p}}\mathbb{E}\left[\|b_t^0 \cdot h\|^{12p}_{\mathcal{C}^2}\right]^{\frac{1}{12p}}\mathbb{E}\left[\|\partial_v F_t^0\|^{12p}_{L_\infty}\right]^{\frac{1}{12p}}\mathbb{E}\left[\|x_t^1\|^{6p}_{L_\infty}\right]^{\frac{1}{6p}} \\ &\leqslant C_p t^{1-\theta}\|h\|_{\mathcal{C}^2}. \end{aligned}$$

Thus for every $p \geqslant 2$,

$$\mathbb{E}\left[|\partial_u\mathcal{A}_t^{1,2}(0)|^p\right]^{1/p} \leqslant C_p t^{1-\theta}\|h\|_{\mathcal{C}^2}. \tag{IV.75}$$

It remains to control $\mathbb{E}\left[|\partial_u\mathcal{A}_t^{1,2}(u) - \partial_u\mathcal{A}_t^{1,2}(v)|^p\right]^{1/p}$ for every $u, v \in [0,1]$. We use (IV.74) to write the difference $\partial_u\mathcal{A}_t^{1,2}(u) - \partial_u\mathcal{A}_t^{1,2}(v)$. Since the proofs are very similar for the different terms arising in the expansion of that difference, we will only give the details for the following term:

$$E_4 := \mathbb{E}\left[\left|\partial_v F_t^0(x_t^0(u)) - \partial_v F_t^0(x_t(u)) - \partial_v F_t^0(x_t^0(v)) + \partial_v F_t^0(x_t(v))\right|^p\right]^{1/p}.$$

On the one hand, it follows from inequalities (IV.68) and (IV.69) that

$$E_4 \leqslant \mathbb{E}\left[\|\partial_v F_t^0\|_{\mathcal{C}^1}^{2p}\right]^{\frac{1}{2p}} \left(\mathbb{E}\left[|x_t^1(u)|^{2p}\right]^{\frac{1}{2p}} + \mathbb{E}\left[|x_t^1(v)|^{2p}\right]^{\frac{1}{2p}}\right) \leqslant C_p t^{1-\theta}. \tag{IV.76}$$

On the other hand, it follows from inequalities (III.33), (IV.67) and (IV.69) that

$$\begin{aligned} E_4 &\leqslant \mathbb{E}\left[\|\partial_v F_t^0\|_{\mathcal{C}^1}^{2p}\right]^{\frac{1}{2p}} \left(\mathbb{E}\left[|x_t^0(u) - x_t^0(v)|^{2p}\right]^{\frac{1}{2p}} + \mathbb{E}\left[|x_t(u) - x_t(v)|^{2p}\right]^{\frac{1}{2p}}\right) \\ &\leqslant \mathbb{E}\left[\|\partial_v F_t^0\|_{\mathcal{C}^1}^{2p}\right]^{\frac{1}{2p}} \left(\mathbb{E}\left[\|\partial_u x_t^0\|_{L_\infty}^{2p}\right]^{\frac{1}{2p}} + \mathbb{E}\left[\|\partial_u x_t\|_{L_\infty}^{2p}\right]^{\frac{1}{2p}}\right) |u-v| \leqslant C_p|u-v|. \end{aligned} \tag{IV.77}$$

Let us multiply (IV.76)$^{1-\theta/4}$ by (IV.77)$^{\theta/4}$:

$$E_4 \leqslant C_p t^{(1-\theta)(1-\frac{1}{4}\theta)}|u-v|^{\theta/4} \leqslant C_p t^{1-\frac{5}{4}\theta}|u-v|^{\theta/4} \leqslant C_p t^{1-2\theta}|u-v|^{\theta/4}.$$

Estimating the other terms in the same way, we finally obtain for every $u, v \in [0,1]$ and for every $p \geqslant 2$

$$\mathbb{E}\left[|\partial_u \mathcal{A}_t^{1,2}(u) - \partial_u \mathcal{A}_t^{1,2}(v)|^p\right]^{1/p} \leqslant C_p t^{1-2\theta}|u-v|^{\theta/4}\|h\|_{\mathcal{C}^2}. \tag{IV.78}$$

It follows from (IV.75) and (IV.78) that

$$\mathbb{E}\left[\|\mathcal{A}_t^{1,2}\|_{\mathcal{C}^{1+\theta/8}}^p\right]^{1/p} \leqslant C_p t^{1-2\theta}\|h\|_{\mathcal{C}^2}.$$

This completes the proof of (IV.72). □

IV.4 Application to an inhomogeneous SPDE

Let us prove Theorem IV.3.

Proof (Theorem IV.3). Let $\theta \in (0, \frac{1}{8})$. Let us consider the inhomogeneous SPDE (IV.8) with source term $F : [0,T] \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$. Let $(Q_t)_{t\in[0,T]}$ be the semi-group associated to that equation. Let us assume that there is $C > 0$ such that for every $t \in [0,T]$, for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $|F(t,\mu) - F(t,\nu)| \leqslant CW_2(\mu,\nu)^{8\theta}$. By Duhamel's formula, for every $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfying the ϕ-assumptions,

$$Q_t\phi(\mu_0^g) = P_t\phi(\mu_0^g) + \int_0^t P_{t-s}(F(s,\cdot))(\mu_0^g)\mathrm{d}s.$$

Recall that $P_t\phi(\mu_0^{g_0}) = \widehat{P_t\phi}(g_0)$ for every $g_0 \in \mathbb{G}^1$. Let us also define $\widehat{Q_t\phi}(g_0) := Q_t\phi(\mu_0^{g_0})$ for every $g_0 \in \mathbb{G}^1$. Thus we have

$$\widehat{Q_t\phi}(g) = \widehat{P_t\phi}(g) + \int_0^t \{P_{t-s}\widehat{(F(s,\cdot))}\}(g)\mathrm{d}s.$$

As in (III.53), we have:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} Q_t\phi(\mu_0^{g+\rho h}) = D\widehat{Q_t\phi}(g)\cdot h,$$

where $D\widehat{Q_t\phi}(g)\cdot h$ denotes the Fréchet derivative of $\widehat{Q_t\phi}$ at point g in the direction h. For every $g \in \mathbf{G}^{4+\theta}$, for every $h \in \Delta^{4+\theta}$,

$$D\widehat{Q_t\phi}(g)\cdot h = D\widehat{P_t\phi}(g)\cdot h + \int_0^t D\{P_{t-s}\widehat{(F(s,\cdot))}\}(g)\cdot h\mathrm{d}s.$$

By Proposition IV.7,

$$\left|D\widehat{Q_t\phi}(g)\cdot h\right| \leqslant \frac{C}{t^{1+3\theta}}\|\phi\|_{L_\infty}\|h\|_{\mathcal{C}^{4+\theta}} + \int_0^t \frac{C}{(t-s)^{1+3\theta}}\mathbb{V}^W\left[F(s,\mu^g_{t-s})\right]^{1/2}\|h\|_{\mathcal{C}^{4+\theta}}\mathrm{d}s. \quad \text{(IV.79)}$$

For every $c\in\mathbb{R}$, $\mathbb{V}^W\left[F(s,\mu^g_{t-s})\right] \leqslant \mathbb{E}^W\left[(F(s,\mu^g_{t-s})-c)^2\right]$. In particular,

$$\begin{aligned}\mathbb{V}^W\left[F(s,\mu^g_{t-s})\right] \leqslant \mathbb{E}^W\left[(F(s,\mu^g_{t-s})-F(s,\mu^g_0))^2\right] &\leqslant C\mathbb{E}^W\left[W_2(\mu^g_{t-s},\mu^g_0)^{16\theta}\right]\\ &\leqslant C\mathbb{E}^W\mathbb{E}^\beta\left[\int_0^1|x^g_{t-s}(u)-x^g_0(u)|^2\mathrm{d}u\right]^{8\theta}\\ &\leqslant C(t-s)^{8\theta}.\end{aligned}$$

Back to (IV.79),

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0}Q_t\phi(\mu_0^{g+\rho h})\right| = \left|D\widehat{Q_t\phi}(g)\cdot h\right| &\leqslant \frac{C}{t^{1+3\theta}}\|\phi\|_{L_\infty}\|h\|_{\mathcal{C}^{4+\theta}} + \int_0^t\frac{C}{(t-s)^{1+3\theta}}(t-s)^{4\theta}\|h\|_{\mathcal{C}^{4+\theta}}\mathrm{d}s\\ &\leqslant \frac{C}{t^{1+3\theta}}\|\phi\|_{L_\infty}\|h\|_{\mathcal{C}^{4+\theta}} + Ct^\theta\|h\|_{\mathcal{C}^{4+\theta}}\\ &\leqslant \frac{C}{t^{1+3\theta}}\|h\|_{\mathcal{C}^{4+\theta}},\end{aligned}$$

where the constant C depends on g, ϕ and F. This completes the proof of the theorem. □

Bibliography

[AGS05] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[Arr79] R. A. Arratia. *Coalescing Brownian motions on the line*. ProQuest LLC, Ann Arbor, MI, 1979. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[ATW06] M. Arnaudon, A. Thalmaier, and F.-Y. Wang. Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below. *Bull. Sci. Math.*, 130(3):223–233, 2006.

[AvR10] S. Andres and M.-K. von Renesse. Particle approximation of the Wasserstein diffusion. *J. Funct. Anal.*, 258(11):3879–3905, 2010.

[BB00] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.*, 84(3):375–393, 2000.

[Bil99] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[Bis81] J.-M. Bismut. Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. *Z. Wahrsch. Verw. Gebiete*, 56(4):469–505, 1981.

[BLP09] R. Buckdahn, J. Li, and S. Peng. Mean-field backward stochastic differential equations and related partial differential equations. *Stochastic Process. Appl.*, 119(10):3133–3154, 2009.

[BLPR17] R. Buckdahn, J. Li, S. Peng, and C. Rainer. Mean-field stochastic differential equations and associated PDEs. *Ann. Probab.*, 45(2):824–878, 2017.

[Bn18] D. Baños. The Bismut-Elworthy-Li formula for mean-field stochastic differential equations. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(1):220–233, 2018.

[BT97] M. Bossy and D. Talay. A stochastic particle method for the McKean-Vlasov and the Burgers equation. *Math. Comp.*, 66(217):157–192, 1997.

[Car13] P. Cardaliaguet. Notes on Mean Field Games (from P.-L. Lions' lectures at collège de France). www.ceremade.dauphine.fr/∼cardalia/, 2013, 2013.

[CCD19] J.-F. Chassagneux, D. Crisan, and F. Delarue. Numerical method for FBSDEs of McKean-Vlasov type. *Ann. Appl. Probab.*, 29(3):1640–1684, 2019.

[CD18a] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I*, volume 83 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean field FBSDEs, control, and games.

[CD18b] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. II*, volume 84 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean field games with common noise and master equations.

[CdR19] P.-E. Chaudru de Raynal. Strong well posedness of McKean-Vlasov stochastic differential equation with Hölder drift. *Stochastic Processes and their Applications*, 2019.

[CdRF] P.-E. Chaudru de Raynal and N. Frikha. Well-posedness for some non-linear diffusion processes and related PDE on the Wasserstein space. arXiv preprint arXiv:1811.06904, 2018.

[Cer01] S. Cerrai. *Second order PDE's in finite and infinite dimension*, volume 1762 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001. A probabilistic approach.

[CF16] M. Coghi and F. Flandoli. Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.*, 26(3):1407–1442, 2016.

[Cho12] O. Chodosh. A lack of Ricci bounds for the entropic measure on Wasserstein space over the interval. *J. Funct. Anal.*, 262(10):4570–4581, 2012.

[CM18] D. Crisan and E. McMurray. Smoothing properties of McKean-Vlasov SDEs. *Probab. Theory Related Fields*, 171(1-2):97–148, 2018.

[Cra91] M. Cranston. Gradient estimates on manifolds using coupling. *J. Funct. Anal.*, 99(1):110–124, 1991.

[Del03] F. Delarue. Estimates of the solutions of a system of quasi-linear PDEs. A probabilistic scheme. In *Séminaire de Probabilités XXXVII*, volume 1832 of *Lecture Notes in Math.*, pages 290–332. Springer, Berlin, 2003.

[Del19] F. Delarue. Restoring uniqueness to mean-field games by randomizing the equilibria. *Stochastics and Partial Differential Equations: Analysis and Computations*, Mar 2019.

[DG87] D. A. Dawson and J. Gärtner. Large deviations from the McKean-Vlasov limit for weakly interacting diffusions. *Stochastics*, 20(4):247–308, 1987.

[DMS05] L. Denis, A. Matoussi, and L. Stoica. L^p estimates for the uniform norm of solutions of quasilinear SPDE's. *Probab. Theory Related Fields*, 133(4):437–463, 2005.

[Dor04] A. A. Dorogovtsev. One Brownian stochastic flow. *Theory Stoch. Process.*, 10(3-4):21–25, 2004.

[DP04] G. Da Prato. *Kolmogorov equations for stochastic PDEs*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2004.

[DPEZ95] G. Da Prato, K. D. Elworthy, and J. Zabczyk. Strong Feller property for stochastic semilinear equations. *Stochastic Anal. Appl.*, 13(1):35–45, 1995.

[DPF10] G. Da Prato and F. Flandoli. Pathwise uniqueness for a class of SDE in Hilbert spaces and applications. *J. Funct. Anal.*, 259(1):243–267, 2010.

[DPFPR13] G. Da Prato, F. Flandoli, E. Priola, and M. Röckner. Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.*, 41(5):3306–3344, 2013.

[DPFPR15] G. Da Prato, F. Flandoli, E. Priola, and M. Röckner. Strong uniqueness for stochastic evolution equations with unbounded measurable drift term. *J. Theoret. Probab.*, 28(4):1571–1600, 2015.

[DS04] L. Denis and L. Stoica. A general analytical result for non-linear SPDE's and applications. *Electron. J. Probab.*, 9:no. 23, 674–709, 2004.

[DS09] M. Döring and W. Stannat. The logarithmic Sobolev inequality for the Wasserstein diffusion. *Probab. Theory Related Fields*, 145(1-2):189–209, 2009.

[DV95] D. Dawson and J. Vaillancourt. Stochastic McKean-Vlasov equations. *NoDEA Nonlinear Differential Equations Appl.*, 2(2):199–229, 1995.

[EL94] K. D. Elworthy and X.-M. Li. Formulae for the derivatives of heat semigroups. *J. Funct. Anal.*, 125(1):252–286, 1994.

[Elw92] K. D. Elworthy. Stochastic flows on Riemannian manifolds. In *Diffusion processes and related problems in analysis, Vol. II (Charlotte, NC, 1990)*, volume 27 of *Progr. Probab.*, pages 37–72. Birkhäuser Boston, Boston, MA, 1992.

[FFPV17] E. Fedrizzi, F. Flandoli, E. Priola, and J. Vovelle. Regularity of stochastic kinetic equations. *Electron. J. Probab.*, 22:Paper No. 48, 42, 2017.

[FG17] N. Fournier and A. Guillin. From a Kac-like particle system to the Landau equation for hard potentials and Maxwell molecules. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(1):157–199, 2017.

[FGP10] F. Flandoli, M. Gubinelli, and E. Priola. Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.*, 180(1):1–53, 2010.

[Fis14] M. Fischer. On the form of the large deviation rate function for the empirical measures of weakly interacting systems. *Bernoulli*, 20(4):1765–1801, 2014.

[Fla11] F. Flandoli. *Random perturbation of PDEs and fluid dynamic models*, volume 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[FLL+99] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance. *Finance Stoch.*, 3(4):391–412, 1999.

[FOT11] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.

[Fun84] T. Funaki. A certain class of diffusion processes associated with nonlinear parabolic equations. *Z. Wahrsch. Verw. Gebiete*, 67(3):331–348, 1984.

[Gär88] J. Gärtner. On the McKean-Vlasov limit for interacting diffusions. *Math. Nachr.*, 137:197–248, 1988.

[GLL11] O. Guéant, J.-M. Lasry, and P.-L. Lions. Mean field games and applications. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 205–266. Springer, Berlin, 2011.

[GM11] L. Gawarecki and V. Mandrekar. *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[HM86] M. Hitsuda and I. Mitoma. Tightness problem and stochastic evolution equation arising from fluctuation phenomena for interacting diffusions. *J. Multivariate Anal.*, 19(2):311–328, 1986.

[HMC06] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.

[HSS] W. Hammersley, D. Siška, and L. Szpruch. McKean-Vlasov SDEs under measure dependent Lyapunov conditions. arXiv preprint arXiv:1802.03974, 2018.

[JKO98] R. Jordan, D. Kinderlehrer, and F. Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Anal.*, 29(1):1–17, 1998.

[JM98] B. Jourdain and S. Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(6):727–766, 1998.

[Jou97] B. Jourdain. Diffusions with a nonlinear irregular drift coefficient and probabilistic interpretation of generalized Burgers' equations. *ESAIM Probab. Statist.*, 1:339–355, 1995/97.

[Kac56] M. Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III*, pages 171–197. University of California Press, Berkeley and Los Angeles, 1956.

[KLvR] V. Konarovskyi, T. Lehmann, and M. von Renesse. On Dean-Kawasaki Dynamics with Smooth Drift Potential. arXiv preprint arXiv:1812.11068, 2018.

[KLvR19] V. Konarovskyi, T. Lehmann, and M.-K. von Renesse. Dean-Kawasaki dynamics: ill-posedness vs. triviality. *Electronic Communications in Probability*, 24, 2019.

[Kon] V. Konarovskyi. Coalescing-fragmentating Wasserstein dynamics: particle approach. arXiv preprint arXiv:1711.03011, 2017.

[Kon11] V. V. Konarovskii. On an infinite system of diffusing particles with coalescing. *Theory Probab. Appl.*, 55(1):134–144, 2011.

[Kon17a] V. Konarovskyi. On asymptotic behavior of the modified Arratia flow. *Electron. J. Probab.*, 22:Paper No. 19, 31, 2017.

[Kon17b] V. Konarovskyi. A system of coalescing heavy diffusion particles on the real line. *Ann. Probab.*, 45(5):3293–3335, 2017.

[KR05] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.

[KS79] N. V. Krylov and M. V. Safonov. An estimate for the probability of a diffusion process hitting a set of positive measure. *Dokl. Akad. Nauk SSSR*, 245(1):18–20, 1979.

[KS84] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. I. In *Stochastic analysis (Katata/Kyoto, 1982)*, volume 32 of *North-Holland Math. Library*, pages 271–306. North-Holland, Amsterdam, 1984.

[KS85] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 32(1):1–76, 1985.

[KS87] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):391–442, 1987.

[KS91] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[Kun90] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[Kur07] T. G. Kurtz. The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. *Electron. J. Probab.*, 12:951–965, 2007.

[Kur14] T. G. Kurtz. Weak and strong solutions of general stochastic models. *Electron. Commun. Probab.*, 19:no. 58, 16, 2014.

[KvR] V. Konarovskyi and M. von Renesse. Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line. arXiv preprint arXiv:1709.02839, 2017.

[KvR18] V. Konarovskyi and M.-K. von Renesse. Modified Massive Arratia Flow and Wasserstein Diffusion. *Communications on Pure and Applied Mathematics*, 2018. doi:10.1002/cpa.21758.

[KX99] T. G. Kurtz and J. Xiong. Particle representations for a class of nonlinear SPDEs. *Stochastic Process. Appl.*, 83(1):103–126, 1999.

[KX04] T. G. Kurtz and J. Xiong. A stochastic evolution equation arising from the fluctuations of a class of interacting particle systems. *Commun. Math. Sci.*, 2(3):325–358, 2004.

[Lac18] D. Lacker. On a strong form of propagation of chaos for McKean-Vlasov equations. *Electron. Commun. Probab.*, 23:Paper No. 45, 11, 2018.

[Lio] P.-L. Lions. Cours au Collège de France. www.college-de-france.fr.

[LJR04] Y. Le Jan and O. Raimond. Flows, coalescence and noise. *Ann. Probab.*, 32(2):1247–1315, 2004.

[LL86] J.-M. Lasry and P.-L. Lions. A remark on regularization in Hilbert spaces. *Israel J. Math.*, 55(3):257–266, 1986.

[LL06a] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.

[LL06b] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.

[LL07] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.

[Mar18] V. Marx. A new approach for the construction of a Wasserstein diffusion. *Electron. J. Probab.*, 23:Paper No. 124, 54, 2018.

[McK66] H. P. McKean, Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, 56:1907–1911, 1966.

[McK67] H. P. McKean, Jr. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57. Air Force Office Sci. Res., Arlington, Va., 1967.

[MT06] P. Malliavin and A. Thalmaier. *Stochastic calculus of variations in mathematical finance*. Springer Finance. Springer-Verlag, Berlin, 2006.

[MV] Y. S. Mishura and A. Y. Veretennikov. Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations. arXiv preprint arXiv:1603.02212, 2016.

[Mé96] S. Méléard. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.

[Nor86] J. Norris. Simplified Malliavin calculus. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 101–130. Springer, Berlin, 1986.

[NT15] J. Norris and A. Turner. Weak convergence of the localized disturbance flow to the coalescing Brownian flow. *Ann. Probab.*, 43(3):935–970, 2015.

[Nua06] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[Oel84] K. Oelschläger. A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.*, 12(2):458–479, 1984.

[Ott01] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[Par67] K. R. Parthasarathy. *Probability measures on metric spaces*. Probability and Mathematical Statistics, No. 3. Academic Press, Inc., New York-London, 1967.

[Pit98] V. V. Piterbarg. Expansions and contractions of isotropic stochastic flows of homeomorphisms. *Ann. Probab.*, 26(2):479–499, 1998.

[RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[RZ] M. Röckner and X. Zhang. Well-posedness of distribution dependent SDEs with singular drifts. arXiv preprint arXiv:1809.02216, 2018.

[Sim87] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[Stu14] K.-T. Sturm. A monotone approximation to the Wasserstein diffusion. In *Singular phenomena and scaling in mathematical models*, pages 25–48. Springer, Cham, 2014.

[SV69] D. W. Stroock and S. R. S. Varadhan. Diffusion processes with continuous coefficients i&ii. *Comm. Pure Appl. Math.*, 22:345–400 & 479–530, 1969.

[SV79] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1979.

[Szn91] A.-S. Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[TH81] H. Tanaka and M. Hitsuda. Central limit theorem for a simple diffusion model of interacting particles. *Hiroshima Math. J.*, 11(2):415–423, 1981.

[Tha97] A. Thalmaier. On the differentiation of heat semigroups and Poisson integrals. *Stochastics Stochastics Rep.*, 61(3-4):297–321, 1997.

[TW98] A. Thalmaier and F.-Y. Wang. Gradient estimates for harmonic functions on regular domains in Riemannian manifolds. *J. Funct. Anal.*, 155(1):109–124, 1998.

[Vai88] J. Vaillancourt. On the existence of random McKean-Vlasov limits for triangular arrays of exchangeable diffusions. *Stochastic Anal. Appl.*, 6(4):431–446, 1988.

[Ver80] A. J. Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb. (N.S.)*, 111(153)(3):434–452, 480, 1980.

[Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Vil09] C. Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

[vRS09] M.-K. von Renesse and K.-T. Sturm. Entropic measure and Wasserstein diffusion. *Ann. Probab.*, 37(3):1114–1191, 2009.

[Wal86] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.

[Xio13] J. Xiong. Super-Brownian motion as the unique strong solution to an SPDE. *Ann. Probab.*, 41(2):1030–1054, 2013.

[Zha05] J. Zhang. Representation of solutions to BSDEs associated with a degenerate FSDE. *Ann. Appl. Probab.*, 15(3):1798–1831, 2005.

[Zvo74] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.