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THÈSE

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dans **l'École Doctorale Mathématiques, Sciences et
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Feuilletage isopériodique de l'espace de modules des surfaces de translation

Isoperiodic foliation on moduli space of translation surfaces

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Résumé

Les strates de l'espace de modules des différentielles abéliennes sont naturellement munies d'un feuilletage holomorphe, appelé feuilletage isopériodique (ou feuilletages des périodes absolues, ou encore feuilletage du noyau). Celui-ci a été introduit il y a 25 ans, d'abord par A. Eskin et M. Kontsevitch, puis par K. Calta et C. McMullen avant de devenir un objet important en dynamique de Teichmüller. La question générale abordée dans ce texte est la suivante :

Comment les feuilles du feuilletage isopériodique se répartissent-elles dans l'espace de module ?

McMullen a démontré l'ergodicité du feuilletage dans les strates principales (où toutes les singularités sont simples) en genre 2 et 3 en utilisant des techniques issues de la dynamique homogène. Calsamiglia, Deroin & Francaviglia ont ensuite étendu ce résultat et obtenu une classification à la Ratner des ensembles fermés saturés par le feuilletage. Simultanément, Hamenstädt a fourni une preuve alternative de l'ergodicité, toujours dans la strate principale. De façon étonnante, le seul résultat connu pour les autres strates est dû à P. Hooper et B. Weiss : les feuilles des surfaces de Arnoux-Yoccoz sont denses dans les strates qui les contiennent.

La question de la dynamique du feuilletage isopériodique peut être formulée dans le contexte plus général des sous-variétés affines. Avila, Eskin et Möller ont prouvé que la codimension des feuilles est alors paire. Le cas de la codimension 2, ou rang 1, est déjà riche. Nous établissons un critère de densité des feuilles et l'appliquons à différentes familles de variétés affines de rang 1. Parmi celles-ci, les lieux Prym occupent une place importante. Nous démontrons dans ce cadre que les feuilles sont soit fermées, soit denses, en fonction de l'arithméticité du lieu. Dans le cas non arithmétique, nous prouvons que le feuilletage est ergodique pour la mesure affine associée. Cela aboutit à la découverte de nouvelles feuilles denses dans des strates à singularités multiples. Ces résultats suggèrent une connection entre la géométrie des variétés affines et la dynamique isopériodique. L'exploitation de cette connection en genre 3 aboutit à la classification des variétés affines non arithmétiques ne provenant pas d'orbites fermées dans les strates à deux singularités.

Abstract

The strata of the moduli space of abelian differentials are endowed with a natural holomorphic foliation, known as the isoperiodic foliation (or absolute period foliation or kernel foliation). It has been introduced 25 years ago by A. Eskin and M. Kontsevich and later by K. Calta and C. McMullen before it became a central object in Teichmüller dynamics. The general question addressed in this text is the following:

How do the leaves of the isoperiodic foliation wander around in the moduli space ?

McMullen proved the ergodicity of the foliation in the principal stratum (where the singularities of the abelian differentials are all simple) in genus 2 and 3 using results from group actions on homogeneous space. Calsamiglia, Deroin & Francaviglia generalized this result in higher genera and obtained a Ratner-like classification of the closed saturated subsets. Simultaneously, Hamenstädt gave an alternative proof of the ergodicity. Surprisingly enough, for the strata where at least one zero is not simple, the only result available was due to Hooper and Weiss: the leaf of the Arnoux-Yoccoz surface is dense in the stratum in which it belongs.

The question of the dynamics of the isoperiodic foliation can be rephrased in the more general context of affine manifolds. Avila, Eskin, Möller proved that the codimension of the leaves is even. The codimension 2 case, also known as rank 1, already displays a rich and contrasted picture. We give a criterion for density of the leaves, and apply it to different families of rank one affine manifolds. Among those, special attention is dedicated to the Prym eigenform loci. We prove that the leaves are either compact or dense, depending on the arithmeticity of the locus. In the non arithmetic case, we prove that the foliation is ergodic with respect to the affine measure. In turn, this gives new examples of dense leaves in strata where at least one of the singularity is not simple. The aforementioned results suggest a connection between the dynamics of the isoperiodic foliation and the geometry of affine manifolds. This connection is analyzed in genus 3 and results in a classification of the proper non arithmetic affine manifolds in strata with 2 singularities.

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Chapter 1

Introduction

1.1 Framework

For a beautiful introduction to the subjects mentioned here and more, see for instance [Zor06] and [FM13] and the references therein.

1.1.1 Moduli space and strata

A stratum $\mathcal{H}_g(\kappa)$ of the moduli space of translation surfaces is the set isomorphism classes of pairs (X, ω) where X is a genus g Riemann surface and ω is a non vanishing holomorphic 1-form on X whose zeroes have multiplicities given by κ . This set admits a complex orbifoldic structure defined as follows. The Teichmüller space of abelian differentials $\tilde{\mathcal{H}}(\kappa)$ is the set of isomorphism classes of triplets (X, ω, f) , where X is a Riemann surface, ω is holomorphic 1-form non identically zero with singularities prescribed by κ , and $f : S \rightarrow X$ is an orientation preserving homeomorphism from fixed genus g topological surface such that the preimage of the singularities of ω are a given subset $\Sigma \subset S$. Two triplets $(X_i, \omega_i, f_i)_{i \in \{1,2\}}$ are isomorphic if there is biholomorphism $\varphi : X_1 \rightarrow X_2$ such that $\varphi^* \omega_2 = \omega_1$ and $f_2^{-1} \circ \varphi \circ f_1$ is isotopic to the identity of S . Define:

$$\Phi : \begin{array}{ccc} \tilde{\mathcal{H}}(\kappa) & \rightarrow & H^1(S, \Sigma, \mathbb{C}) \\ (X, \omega, f) & \mapsto & (\gamma \mapsto \int_{f \circ \gamma} \omega) \end{array}$$

This map, known as the period map, is a local homeomorphism, and the Teichmüller space $\tilde{\mathcal{H}}(\kappa)$ is endowed with the initial complex structure associated. The mapping class group of S acts by biholomorphisms on the Teichmüller space by:

$$\forall \varphi \in \Gamma_g, \varphi \cdot (X, \omega, f) = (X, \omega, f \circ \varphi^{-1})$$

Proposition 1.1.1. *The action of Γ_g on $\tilde{\mathcal{H}}(\kappa)$ is properly discontinuous, and there is a finite index subgroup that acts freely.*

A proof of that result can be found in [FM11]. The quotient $\tilde{\mathcal{H}}(\kappa)/\Gamma_g$ is in bijection with $\mathcal{H}_g(\kappa)$. This remark, together with proposition 1.1.1, allows to define a complex orbifold structure on $\mathcal{H}_g(\kappa)$ that turns the canonical projection $\pi : \tilde{\mathcal{H}}(\kappa) \rightarrow \mathcal{H}_g(\kappa)$ into a local biholomorphism.

1.1.2 The Masur-Veech measure

There is a natural measure $\tilde{\lambda}$ on $\tilde{\mathcal{H}}(\kappa)$ obtained by pulling back the Lebesgue measure on $H^1(S, \Sigma, \mathbb{C})$ by the period map Φ . Since this map is Γ_g -equivariant and that Γ_g acts as elements of $Sp_{2g}(\mathbb{Z})$, the measure $\tilde{\lambda}$ descends to a measure λ on $\mathcal{H}_g(\kappa)$. It turns out that this measure is not finite. This is essentially due to the fact that it is invariant by homothetic transformation. This is problematic in the perspective of ergodic theory. To bypass this difficulty one can restrict to the subset of area one surfaces. More precisely, the area map $\alpha : \mathcal{H}_g(\kappa) \rightarrow \mathbb{R}_{>0}$ is defined as follows:

$$\alpha(X, \omega) = \frac{i}{2} \int_X \omega \wedge \bar{\omega}$$

Disintegration of the measure λ along the level sets of α yields a family of measures $(\lambda_x)_{x \in \mathbb{R}_{>0}}$, such that for any Borel set $\mathcal{B} \in \mathcal{H}_g(\kappa)$:

$$\lambda(A) = \int_{\mathbb{R}_{>0}} \lambda_x(A) dx.$$

Masur and Veech proved independently in a famed result that the measure λ_1 defined on $\mathcal{H}_g^1(\kappa) := \alpha^{-1}(1)$ is finite. See [Mas82] and [Vee82].

1.1.3 The action of $GL_2^+(\mathbb{R})$

Any stratum $\mathcal{H}_g(\kappa)$ is endowed with a natural action of $GL_2^+(\mathbb{R})$. This action is a generalization of the action $GL_2^+(\mathbb{R})$ on the space of translation tori $GL_2^+(\mathbb{R})/SL_2(\mathbb{Z})$, and is defined as follows. Let $g \in GL_2^+(\mathbb{R})$ and let $(X, \omega, f) \in \tilde{\mathcal{H}}(\kappa)$. The form $\omega' = g \cdot \omega$ is an harmonic 1-form, thus there is a unique complex structure on X for which ω' is a holomorphic 1-form. This yields a new Riemann surface X' , and

$g \cdot (X, \omega, f)$ is defined to be the surface $(X', g \cdot \omega, f)$. This action is linear in period coordinates:

$$\forall g \in GL_2^+(\mathbb{R}), \forall \gamma \in H_1(S, \Sigma), \Phi(g \cdot (X, \omega, f))(\gamma) = g \cdot \Phi(X, \omega, f)(\gamma)$$

The moduli space $\mathcal{H}_g(\kappa)$ is endowed with the quotient action and the canonical projection $\pi : \tilde{\mathcal{H}}(\kappa) \rightarrow \mathcal{H}_g(\kappa)$ is $GL_2^+(\mathbb{R})$ -equivariant. Two subgroups will be of particular importance to us:

$$H = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

and

$$G = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R} \right\}$$

The action of H is known as the horocycle flow, while the action of G is known as the Teichmüller geodesic flow. Masur and Veech proved that those groups act ergodically on $\mathcal{H}_g^1(\kappa)$ with respect to the measure λ_1 . The classification of the orbit closures and ergodic measures for this action is a challenging problem in Teichmüller dynamic. A great leap has been made by Eskin, Mirzakhani and Mohammadi by describing the geometry of the closed $GL_2^+(\mathbb{R})$ invariant sets and the invariant measures associated. We will need the following definitions:

Definition 1.1.1 (Affine manifolds). *An affine manifold is a properly immersed closed connected $GL_2^+(\mathbb{R})$ -invariant manifold $f : \mathcal{M} \rightarrow \mathcal{H}_g(\kappa)$ such that for any $X \in \mathcal{M}$, there is a neighborhood \mathcal{U} of X in \mathcal{M} , a neighborhood \mathcal{V} of $f(X)$ in $\mathcal{H}_g(\kappa)$, a section s of π defined on a neighborhood \mathcal{V} and a subspace $V \in H^1(S, \Sigma, \mathbb{R})$ such that $\Phi \circ s \circ f(\mathcal{U}) = \Phi \circ s(\mathcal{V}) \cap V \otimes \mathbb{C}$. Such a V is called a local model of \mathcal{M} around X .*

Most of the time, we shall forget that \mathcal{M} is only immersed and will consider it is embedded in $\mathcal{H}_g(\kappa)$. This statement can be made rigorous by passing to a finite cover of the moduli space.

Definition 1.1.2 (Affine measures). *A $GL_2^+(\mathbb{R})$ -invariant measure μ on $\mathcal{H}_g(\kappa)$ is said to be affine if its support is an affine manifold \mathcal{M} such that for any local model V of \mathcal{M} associated to a section s , the measure $(\Phi \circ s)_* \mu$ is the lebesgue measure on V .*

Eskin, Mirzakhani and Mohammadi proved that any ergodic $GL_2^+(\mathbb{R})$ -invariant measure is an affine measure. See [EMM15]. If μ is an affine measure, the disintegration of μ along the level sets of the area map α yields a $SL_2(\mathbb{R})$ -invariant ergodic measure μ_1 on $\mathcal{H}_g^1(\kappa)$. Any measure obtained by this construction will also be called an affine measure. There is an important definition associated to any affine manifold, that we shall use later in this text:

Definition 1.1.3. *Let \mathcal{M} be an affine manifold. The field of definition of \mathcal{M} is the smallest subfield $k(\mathcal{M})$ of \mathbb{R} such that any local model of \mathcal{M} can be written as $V = V_0 \otimes_{k(\mathcal{M})} \mathbb{R}$, where V_0 is a $k(\mathcal{M})$ -vector space.*

1.1.4 Prym eigenforms

In this section, we recall a construction giving an infinite family of rank one proper affine manifolds discovered by McMullen. Let (X, ω) be a translation surface endowed with a holomorphic involution τ . We denote by $\Omega(X)$ the set of holomorphic 1-forms, and by $\Omega^-(X)$ the set of τ -anti-invariant holomorphic 1-forms. We say that (X, ω) is a Prym form if $\omega \in \Omega^-(X)$, that is $\tau^*\omega = -\omega$, and $\dim \Omega^-(X) = 2$. The Prym variety $\text{Prym}(X, \omega, \tau)$ is defined as the 2-dimensional abelian variety $(\Omega^-(X))^*/H_1^-(X, \mathbb{Z})$ endowed with the polarization coming from the intersection form on $H_1(X, \mathbb{Z})$. It is a factor of the jacobian of X .

Definition 1.1.4. *A Prym eigenform is a Prym form (X, ω) endowed with an involution τ such that there is a $\lambda \in \mathbb{R}$, $D \in \mathbb{N}$ and a faithful representation:*

$$\rho : \mathbb{Q}(\sqrt{D}) \rightarrow \text{End}(\text{Prym}(X, \omega, \tau)) \otimes \mathbb{Q}$$

Such that for any $x \in \mathbb{Q}(\sqrt{D})$, $\rho(x)$ is self-adjoint for the induced symplectic form on $H_1(X, \mathbb{Z})^- \otimes \mathbb{Q}$ and $\rho(x) \cdot \omega = \lambda \cdot \omega$. We say that $\text{Prym}(X, \omega, \tau)$ has real multiplication by the field $\mathbb{Q}(\sqrt{D})$.

We will denote by ΩE_D the Prym eigenform locus with multiplication by $\mathbb{Q}(\sqrt{D})$ in \mathcal{H}_g , and by $\Omega E_D(\kappa)$ its intersection with the strata $\mathcal{H}_g(\kappa)$. Notice that a Riemann surface X can be a Prym eigenform in more than one way. For more details, see [McM07b]. It is a beautiful theorem of McMullen that ΩE_D is closed and $GL_2^+(\mathbb{R})$ -invariant. Furthermore, we have the following:

Proposition 1.1.2. *The $GL_2^+(\mathbb{R})$ -orbit closure of any Prym eigenform is a rank one affine manifold.*

The Prym eigenform play a crucial role in McMullen's classification of affine manifolds in genus 2. It is proved in [McM07a] that if the orbit of a surface is neither closed nor dense in the stratum in which it belongs, then it is a Prym eigenform, the Prym involution being given by the hyperelliptic involution. In [McM07b], infinite families of Prym eigenform are constructed in genus up to 5, and it is a consequence Of Riemann Hurewitz formula that Prym eigenforms cannot exist for genus bigger than 5.

1.2 Isoperiodic foliations

In this section we define the isoperiodic foliation. We treat first the case of the strata, then extends the construction to arbitrary non absolute affine manifolds.

1.2.1 Foliation of the strata

Let $\rho : H^1(S, \Sigma, \mathbb{C}) \rightarrow H^1(S, \mathbb{C})$ be the canonical restriction map. We shall first define a foliation $\tilde{\mathcal{F}}$ on $\tilde{\mathcal{H}}(\kappa)$, and define the isoperiodic foliation as the quotient of the foliation. Recall that if M is any manifold, a p -distribution of TM is a section s of $\mathcal{G}_p(TM)$, the grassmanian of rank p of TM . Such a distribution is said to be integrable if there is a foliation $\mathcal{F}(s)$, such that at any point $x \in M$, $T_x \mathcal{F}_x(s) = s(x)$.

Proposition 1.2.1. *The $|\kappa| - 1$ -distribution $(X, \omega, f) \mapsto \text{Ker } d_X(\rho \circ \Phi)$ of $T\tilde{\mathcal{H}}(\kappa)$ is integrable, and the action of Γ_g preserves the leaves of the associated foliation.*

We denote by $\tilde{\mathcal{F}}$ the foliation given by the previous proposition. Since the action of Γ_g preserves the leaves there is an associated foliation \mathcal{F} on $\mathcal{H}_g(\kappa)$. This is not sensu strictu a foliation as $\mathcal{H}_g(\kappa)$ is not a manifold, but every issue arising through orbifoldic points can be resolved up to passing to a finite covering. The foliation is characterized by the following property:

Proposition 1.2.2. *Two surfaces (X_1, ω_1) and (X_2, ω_2) in $\mathcal{H}_g(\kappa)$ lie in the same leaf if, and only if, there is path γ in $\tilde{\mathcal{H}}(\kappa)$ such that:*

1. $\pi \circ \gamma(0) = (X_1, \omega_1)$ and $\pi \circ \gamma(1) = (X_2, \omega_2)$
2. $\rho \circ \Phi \circ \gamma$ is constant.

It is this characterization that we shall use later in this text.

1.2.2 Foliation of affine manifolds

We begin this section with an important definition for the remainder of this text.

Definition 1.2.1. *An affine manifold \mathcal{M} is said to be non absolute if for every $X \in \mathcal{M}$, any local model around X intersects $\ker \rho$ non trivially. Otherwise, it is said to be absolute.*

For instance, an absolute rank 1 affine manifold is the same thing as a closed orbit. Example of absolute rank 2 affine manifolds are constructed in [MMW17]. For connectedness reason, the dimension of the intersection in 1.2.1 is constant and the same construction as in the case of strata can be performed to obtain a foliation $\mathcal{F}^{\mathcal{M}}$ on \mathcal{M} . This foliation is characterized by the following property:

Proposition 1.2.3. *Let \mathcal{M} be a non absolute affine manifold. Two surfaces (X_1, ω_1) and (X_2, ω_2) in \mathcal{M} lie in the same leaf of $\mathcal{F}^{\mathcal{M}}$ if, and only if, there is path γ in $\hat{\mathcal{H}}(\kappa)$ such that:*

1. $\pi \circ \gamma(0) = (X_1, \omega_1)$ and $\pi \circ \gamma(1) = (X_2, \omega_2)$
2. $\rho \circ \Phi \circ \gamma$ is constant.
3. $\forall t \pi \circ \gamma(t) \in \mathcal{M}$

For a given non absolute affine manifold \mathcal{M} , deformations along the foliation $\mathcal{F}^{\mathcal{M}}$ preserve the absolute periods of the surfaces in \mathcal{M} as well as the symmetries defining \mathcal{M} . For example if \mathcal{M} is a locus of covering construction, moving along the leaf of $\mathcal{F}^{\mathcal{M}}$ preserves the symmetries of the covering. The isoperiodic foliation behaves nicely with respect to the action of $GL_2^+(\mathbb{R})$:

Proposition 1.2.4. *Let \mathcal{M} be a non absolute affine invariant manifold, let X be a translation surface in \mathcal{M} and let $g \in GL_2^+(\mathbb{R})$. The following formula holds :*

$$g \cdot \mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_{g \cdot X}^{\mathcal{M}}$$

1.2.3 Rel flow

The fact that Γ_g acts by permutation on Σ means that it generally does not exist integrable flows on the leaves. However, if (X, ω) is a translation surface in $\mathcal{H}_g(\kappa)$ one can still defined an action of $T_X \mathcal{F}_X \times \mathbb{R}$ on the leaf \mathcal{F}_X . It is defined as follow: Consider $\hat{\mathcal{H}}_g(\kappa)$ to the intermediate covering of $\mathcal{H}_g(\kappa)$ associated to the kernel of

the action by permutation of Γ_g on Σ , and denote by $\hat{\pi}$ the canonical projection associated. One can define on this space a foliation $\hat{\mathcal{F}}$ in the same way we did on $\mathcal{H}_g(\kappa)$. But now, the transition functions of $\hat{\mathcal{H}}_g(\kappa)$ acts by translation on $\ker(\rho)$, and this implies that the leaves $\hat{\mathcal{F}}$ are canonically endowed with a flat structure and there is an associated geodesic flow. Then, let $(u, t) \in T_X \mathcal{F}_X \times \mathbb{R}$, and chose a lift of (X, ω) in $\hat{\mathcal{H}}_g(\kappa)$ associated to a section s of $\hat{\pi}$ defined around (X, ω) . We define $Rel_u^t(X, \omega)$ to be the image by $\hat{\pi}$ of the surface in $\hat{\mathcal{H}}(\kappa)$ obtained by flowing for a time t along the geodesic flow in direction $d_X s(u)$. This surface is canonically defined is does not depend on the choice of the section we made. Notice that by construction $Rel_u^t(X, \omega)$ belongs to \mathcal{F}_X . This action is usually referred to as the Rel flow, even if it is not sensu stricto a flow, as we have already mentioned. Note that this action might not be defined for all t : this is due to the fact that singularities might collide along the orbits of this "flow". This discussion is summarized in the following commutative diagram.

$$\begin{array}{ccc} \hat{\mathcal{H}}_g(\kappa) & \xrightarrow{f_u^t} & \hat{\mathcal{H}}_g(\kappa) \\ \hat{\pi} \downarrow & Rel_{d\hat{\pi}(u)}^t & \downarrow \hat{\pi} \\ \mathcal{H}_g(\kappa) & \xrightarrow{\quad} & \mathcal{H}_g(\kappa) \end{array}$$

1.3 State of the art

1.3.1 The work of McMullen

To our knowledge, the first investigation on the dynamics of the isoperiodic foliation dates back to 2014 and appears in the article of C. McMullen "Moduli spaces of isoperiodic forms on Riemann surfaces". In this paper, the author studies the following space:

$$\mathcal{A}(L) = \{(X, \omega) \in \overline{\mathcal{H}}_g \mid Per(X, \omega) = L \text{ as a polarized module}\}$$

Where L is a given polarized module contained in \mathcal{L} and $Per(X, \omega)$ is the polarized module generated by the $\int_\gamma \omega$ for $\gamma \in H_1(X)$. The canonical projection $\pi : \mathcal{A}(L) \rightarrow \overline{\mathcal{H}}_g$ is related to the isoperiodic foliation in the sense that each leaf is swept out by a connected component of a $\mathcal{A}(L)$.

To continue.

1.3.2 The work of P. Hooper & B. Weiss

In their article [HW15], P. Hooper and B. Weiss study in details the topological dynamics of the leaves of the Arnoux-Yoccoz surfaces. The result that has been of particular interest to us is the following :

Theorem 1.3.1. *For any $g \geq 3$, the leaf of the genus g Arnoux Yoccoz surface is dense in $\mathcal{H}^{odd}(g-1, g-1)$.*

In this article, the authors prove that two usually mutually excluding phenomena coexist on the leaves of those surfaces. Indeed, the Arnoux-Yoccoz surfaces are periodic under the Teichmüller flow, ie endowed with a pseudo-Anosov, and thus the associated horizontal foliation is uniquely ergodic. Nonetheless, the authors noticed that an arbitrary small deformation of those surfaces in \mathcal{F} produces a surface whose associated horizontal foliation is periodic. It is this coexistence that results in density of the leaves. More precisely, let (X, ω) be one the Arnoux-Yoccoz surfaces, and let:

$$g = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \text{ such that } g \cdot (X, \omega) = (X, \omega)$$

We outline the proof:

1. There is a surface $(X_1, \omega_1) \in \mathcal{F}_X$ that is horizontally periodic.
2. Comparison of the horocycle orbit of (X_1, ω_1) and its orbit under $Rel_{(1,0)}^t$.
3. The horocycle orbit of (X, ω) is contained in the closure of \mathcal{F}_X .
4. The leaf of (X, ω) contains its $SL_2(\mathbb{R})$ -orbit.
5. The $SL_2(\mathbb{R})$ -orbit of (X, ω) is dense.

Step 1. The first step is a consequence of a careful analysis of the horizontal foliation on a suitable polygonal representation of the Arnoux Yoccoz surfaces.

Step 2. For the horizontally periodic surface (X_1, ω_1) , the horocycle and Rel flows in the horizontal direction amount to twisting the cylinders, the only difference being the speed of the twist. If (X_1, ω_1) has m cylinders, a vector in \mathbb{R}^m correspond to a deformation of the surface, each coordinate specifying the amount of twisting performed on the corresponding cylinder. Since twisting a cylinder by an amount of its circumference is the same as applying a Dehn twist and surfaces are regarded up to isomorphism, this factors through a torus. Therefore, the horocycle orbit

and the Rel flow read as linear flows on a torus of dimension m . For the surface (X_1, ω_1) it is proved that the subtorus corresponding the closure of the Rel orbit is big and almost contains the horocycle orbit. We remained voluntarily allusive here but precise statements will be given in chapter 3.

Step 3. Let $p_n = g^n \cdot (X_1, \omega_1)$. The surface (X_1, ω_1) is constructed so that it lies on the same strong stable leaf of the Teichmüller flow as (X, ω) , and thus the p_n converge to (X, ω) . Since g preserves the horizontal and vertical directions, the p_n are also horizontally periodic and Step 2 also applies to those surfaces. It is then proved that along this sequence, the closure of the Rel orbits get closer and closer to horocycle orbits. At the limit, this implies that the horocycle orbit of (X, ω) is contained in the closure of its leaf.

Step 4. Using results of Eskin, Mirzakhani and Mohammadi on affine measures, the authors prove that periodicity under the Teichmüller flow implies that the horocycle orbit of (X, ω) coincides with its $SL_2(\mathbb{R})$ orbit.

Step 5. This last step uses results on the field of definition of affine manifold proved by Wright in [Wri14]. It uses the fact that the holonomy field of (X, ω) has degree $2g$ over \mathbb{Q} where g is the genus of X , and that the only totally real subfield it contains \mathbb{Q} . This last step can also be deduced in genus 3 from results of Hubert, Lanneau and Möller in [HLM12]. Their proof does not rely on the structural results of Eskin, Mirzakhani and Mohammadi.

1.3.3 The work of G. Calsamiglia, B. Deroin & S. Francaviglia

In their article [CDF15]; the authors extend McMullen's aforementioned result in [McM14] to genus $g \geq 3$.

Theorem 1.3.2. *Let $(X, \omega) \in \mathcal{H}(1^{2g-2})$ be a translation surface of volume V . Denote by $\Lambda(\omega) \in \mathbb{C}$ the closure of the group generated by the periods of ω . Then, up to the action of $SL_2(\mathbb{R})$:*

1. *if $g > 2$, the leaf \mathcal{F}_X is either closed ($\Lambda(\omega)$ is discrete), dense in the set of translation surfaces of volume V with primitive imaginary part ($\Lambda(\omega) = \mathbb{R} \oplus i\mathbb{Z}$) or dense in the set of translation surfaces with volume V ($\Lambda = \mathbb{C}$).*
2. *if $g = 2$, then either the leaf \mathcal{F}_X falls in one of the three cases above or it is dense in a prym eigenform locus $\Omega E_D(1, 1)$.*

Moreover, the restriction of the foliation to any of those loci is ergodic.

Their theorem is actually stated in the whole moduli space \mathcal{H}_g , without restricting to the principal stratum $\mathcal{H}(1^{2g-2})$. However, the version mentioned above is equivalent as $\mathcal{H}(1^{2g-2})$ is dense in \mathcal{H}_g and the natural measure on \mathcal{H}_g is supported on $\mathcal{H}(1^{2g-2})$. To prove this result, the authors establish a transfer principle that allows them to recover the dynamical behaviour of the isoperiodic foliation from the action of $Sp_{2g}(\mathbb{Z})$ on \mathbb{C}^{2g} . The significant advantage of this approach is that the study of the closed invariants sets for this action had already been carried out by Kapovitch using Ratner's theory. To state their transfer principle, let us introduce some notations : let Γ_g^0 be the Torelli group, that is the subgroup of Γ_g that acts trivially at the level of the homology and let $\Omega\mathcal{S}_g(1^{2g-2})$ be the quotient of $\tilde{\mathcal{H}}(1^{2g-2})$ by Γ_g^0 , and denote by π_0 the canonical projection from $\Omega\mathcal{S}_g(1^{2g-2})$ to $\mathcal{H}_g(\kappa)$. Finally, define:

$$\Phi_0 : \begin{array}{ccc} \Omega\mathcal{S}_g(1^{2g-2}) & \rightarrow & H^1(S, \mathbb{C}) \\ (X, \omega, f) & \mapsto & (\gamma \mapsto \int_{f \circ \gamma} \omega) \end{array}$$

Notice the difference with the period map defined in 1.1.1: here the range of the map is the absolute cohomology. In other words, this is the map induced by $\rho \circ \Phi$ after passing to the quotient by the action of Γ_g^0 . The group Γ_g also acts on the homology of S in this usual way, and for this action the map Φ_0 is Γ_g -invariant. For any translation surface (X, ω) , the set $\Phi_0(\pi_0^{-1}(\mathcal{F}_X))$ is invariant by Γ_g . Reciprocally, any invariant set $A \in H^1(S, \mathbb{C})$ produces a \mathcal{F} -saturated subset of $\mathcal{H}(1^{2g-2})$ if the level sets of Φ_0 are connected. This connectedness property is the core of the transfer principle. In this perspective, the study of the dynamical behavior of the isoperiodic foliation is reduced to proving the connectedness of the level sets of Φ_0 , and this is actually the hard part of the paper of Calsamiglia, Deroin & Francaviglia.

Unfortunately, this line of logic cannot be reproduced without adaptation for other strata of the moduli space. The reason is that there exists in higher genera proper non arithmetic affine manifolds that are saturated by the isoperiodic foliation: Any surface (X, ω) in one of these loci verifies $\Lambda(\omega) = \mathbb{C}$, and should fall in the last case of theorem 1.3.2. The loci $\Omega E_D(\kappa)$ for $|\kappa| \geq 2$ provide example of such loci. Other examples are given by covering constructions over non arithmetic Veech surfaces in the minimal stratum $\mathcal{H}(2g-2)$ and ramification at the zero and non periodic points. This indicates that not all the fibers of the period map defined on other strata are connected. See Chapter 2 for more details.

We now sketch the proof of the connectedness of the level sets of the period map, as it appears in [CDF15]. Let $p \in H^1(S, \mathbb{C})$, and denote by $\mathcal{S}(p)$ the preimage of p by Φ_0 , and denote by $\Omega\bar{\mathcal{S}}_g(1^{2g-2})$ the compactification of $\Omega\mathcal{S}_g(1^{2g-2})$ obtained by adding the stable forms over stable curves. A stable form over a stable curve is a form which is

holomorphic, except for possibly simple poles at the nodes, such that the two residues at a single node are opposite. The period map Φ_0 can be extended to $\Omega\overline{\mathcal{S}}_g(1^{2g-2})$ and the preimage of p yields a compactification $\overline{\mathcal{S}}(p)$ of $\mathcal{S}(p)$. The authors actually only allow the underlying surfaces to have at most one simple node to deal only with mild singularities. A key observation is that the boundary components of $\overline{\mathcal{S}}(p)$ yield isoperiodic loci in strata of smaller genera. Therefore, it is natural to proceed to an induction, the base cases being given by [McM14]. The proof can be decomposed into 4 steps:

1. The boundary of any component of $\overline{\mathcal{S}}(p)$ is not empty.
2. The set $\mathcal{S}(p)$ is connected if, and only if, $\overline{\mathcal{S}}(p)$ is connected.
3. The boundary components of $\overline{\mathcal{S}}(p)$ are connected.
4. There is a path in $\overline{\mathcal{S}}(p)$ connecting any pair of boundary components of $\overline{\mathcal{S}}(p)$.

Step 1. The first step is easily proved when $\mathcal{S}(p)$ contains a surface (X, ω) that has a twin: a pair of saddle connections (σ_1, σ_2) joining the same pair of simple singularities and such that:

$$\int_{\sigma_1} \omega = \int_{\sigma_2} \omega.$$

Pinching the curve $\sigma_1 \star \sigma_2^{-1}$ yield a path that goes from (X, ω) to the boundary of $\overline{\mathcal{S}}(p)$. To conclude it remains to prove that any component of $\mathcal{S}(p)$ contains such a surface.

Step 2. The second step follows from a careful analysis of the foliation near a surface (X, ω) in the boundary of $\overline{\mathcal{S}}(p)$. For instance, if this boundary component is comprised of pairs of surfaces of smaller genera attached at one point, then, a neighborhood of (X, ω) in $\overline{\mathcal{S}}(p)$ is made of surfaces glued along a slit, and moving in $\overline{\mathcal{S}}(p)$ is the same as changing the slit. The singular locus is attained when the slit is reduced to a point: its codimension is 2. Therefore $\overline{\mathcal{S}}(p)$ is not disconnected when the singular locus is removed.

Step 3. Follows by induction.

Step 4 This step is the most delicate. Notice that the construction of step 1 can be reversed. There is thus two reciprocal operations called smoothing a node and pinching a twin, and those operations allow to move inside $\overline{\mathcal{S}}(p)$. When a nodal surface with period p has several simple nodes, one can smooth them to get to a

surface (X, ω) in $\mathcal{S}(p)$. Pinching back exactly one of the twins created gives a path from (X, ω) to a boundary component of $\overline{\mathcal{S}}(p)$. Applying this to the other twins of (X, ω) connects different boundary components. This idea drives the proof of step 4.

1.3.4 The work of U. Hamenstädt

In her paper [Ham18], U. Hamenstädt proved the following:

Theorem 1.3.3. *The isoperiodic foliation in $\mathcal{H}(1^{2g-2})$ is ergodic with respect to the Masur-Veech measure.*

For notational simplicity, we shall denote by $\mathcal{H}_g^{(a)}$ the locus of area a translation surfaces in the principal stratum $\mathcal{H}(1^{2g-2})$. The approach used to prove the theorem is, by many aspects, very similar to the one of the previous section. The author takes advantage of the existence of twins in almost every leaf to reduce the study of the isoperiodic foliation on $\mathcal{H}_g^{(a)}$ to the study of the product of the isoperiodic foliations on $\mathcal{H}_{g_1}^{(a_1)} \times \mathcal{H}_{g_2}^{(a_2)}$ with $g_1 + g_2 = g$ and $a_1 + a_2 = a$. This opens ways for an induction, the base case for genus 2 and 3 being given by [McM14]. This last set is related to a component of the boundary of the principal stratum. More precisely, let $\mathcal{H}_{g,1}^{(a)}$ be the moduli space of area a translation surfaces with a regular marked point. This last assertion means that if $(X, \omega, p) \in \mathcal{H}_{g,1}^{(a)}$, then $\omega_p \neq 0$. The set $\mathcal{H}_{g_2,1}^{(a_1)} \times \mathcal{H}_{g_2,1}^{(a_2)}$ is a boundary component of the stratum $\mathcal{H}_g^{(a)}$, eventually after taking the quotient by the map that exchanges the two factors if $g_1 = g_2$, but we will omit this technical difficulty. The period map Φ can be extended to this boundary component, and thus the isoperiodic foliation on $\mathcal{H}_g^{(a)}$ extends to a foliation on this boundary component. Define:

$$\partial_{g_1, g_2} \mathcal{H}_g^{(a)} = \bigcup_{a_1 + a_2 = a} \mathcal{H}_{g_1, 1}^{(a_1)} \times \mathcal{H}_{g_2, 1}^{(a_2)}$$

Before sketching the proof, one needs to define measures on those spaces. The set $\mathcal{H}_{g_1, p_1} \times \mathcal{H}_{g_2, p_2}$ is naturally endowed with the product ν of the Lebesgue measures on each of factor. Denote by A the continuous function that assigns to any pair of marked translation surface their area. Disintegration of ν on the level sets of A yields a measure ν_{a_1, a_2} on $\mathcal{H}_{g_1, 1}^{(a_1)} \times \mathcal{H}_{g_2, 1}^{(a_2)}$ as well as a measure ν_a on $\partial_{g_1, g_2} \mathcal{H}_g^{(a)}$ defined for any measurable set A by:

$$\nu_a(A) = \int_0^1 \nu_{x, a-x}(A) dx$$

Notice that the pushforward of ν_{a_1, a_2} by the node forgetting map is just the product measure on $\mathcal{H}_{g_1, 1}^{(a_1)} \times \mathcal{H}_{g_2, 1}^{(a_2)}$. Now, if $A \in \mathcal{H}_g(a)$, denote by $\partial_{g_1, g_2} A$ the set of all limit points of A in $\partial_{g_1, g_2} \mathcal{H}_g(a)$.

To prove that the isoperiodic foliation on $\mathcal{H}_g(a)$ is ergodic, one needs to show that any \mathcal{F} -saturated Borel set is measurably trivial. Let A be such a set. The proof can be decomposed into 4 steps:

1. If A has positive measure, then $\partial_{g_1, g_2} A$ has positive measure and is saturated by the isoperiodic foliation on $\partial_{g_1, g_2} \mathcal{H}_g(a)$.
2. By induction, the isoperiodic foliation on $\mathcal{H}_{g_1, 1}^{(a_1)} \times \mathcal{H}_{g_2, 1}^{(a_2)}$ is ergodic.
3. If $\partial_{g_1, g_2} A$ has positive measure, then there exists a Borel set C in $[0, 1]$ of positive measure such that $\partial_{g_1, g_2} A = \bigcup_{b \in C} \mathcal{H}_{g_1, 1}^{(a_1)}(b) \times \mathcal{H}_{g_2, 1}^{(a_2)}(1 - b)$.
4. The set A has full measure.

The subset $\mathcal{S}(g_1, g_2)$ of \mathcal{H}_g comprised of surfaces whose leaf accumulates on $\partial_{g_1, g_2} \mathcal{H}_g$ is of full measure: smoothing the node of surfaces in $\partial_{g_1, g_2} \mathcal{H}$ produces an open $SL_2(\mathbb{R})$ -invariant set contained in $\mathcal{S}(g_1, g_2)$. The ergodicity of the action of $SL_2(\mathbb{R})$ on \mathcal{H}_g shows that this set has full measure. Therefore, up to replacing A by $A \cap \mathcal{S}(g_1, g_2)$, it can be assumed further that the leaf of any element in A accumulates on $\partial_{g_1, g_2} \mathcal{H}_g$.

Step 1. The first claim follows from the fact that any point in $\partial_{g_1, g_2} A$ is obtained by pinching twins on elements of A .

Step 2. Notice that if the isoperiodic foliation on \mathcal{H}_{g_i} is ergodic, then the product foliation on $\mathcal{H}_{g_1}^{(a_1)} \times \mathcal{H}_{g_2}^{(a_2)}$ is ergodic for the product measure. Denote by $p : \mathcal{H}_{g_1, 1}^{(a_1)} \times \mathcal{H}_{g_2, 1}^{(a_2)} \rightarrow \mathcal{H}_{g_1}^{(a_1)} \times \mathcal{H}_{g_2}^{(a_2)}$ the node forgetting map. As we mentioned earlier $p_* \nu_{a_1, a_2}$ is exactly the product measure on $\mathcal{H}_{g_1}^{(a_1)} \times \mathcal{H}_{g_2}^{(a_2)}$. The important fact here is that if $\mathcal{B} \in \mathcal{H}_{g_1, 1}^{(a_1)} \times \mathcal{H}_{g_2, 1}^{(a_2)}$ is a Borel set saturated by the foliation, then $p(\mathcal{B})$ is measurable and $p^{-1}p(\mathcal{B}) = \mathcal{B}$. Step 2 then follows: if \mathcal{B} has positive measure, then $p(\mathcal{B})$ has positive measure. By ergodicity of the product foliation the measure of $p(\mathcal{B})$ is 1. By definition of $p_* \nu_{a_1, a_2}$, it implies that $\mathcal{B} = p^{-1}p(\mathcal{B})$ also has measure one.

Step 3. By step 1, the set $\partial_{g_1, g_2} A$ has positive measure. By definition of ν_1 , there exists of Borel set C in $[0, 1]$ such that $\nu_{x, 1-x}(A) > 0$ for any $x \in C$. For such a x , since $\partial_{g_1, g_2} A$ is saturated by the isoperiodic foliation, step 2 implies that up to a set of measure 0, $\partial_{g_1, g_2} A \cap (\mathcal{H}_{g_1, 1}^{(x)} \times \mathcal{H}_{g_2, 1}^{(1-x)}) = \mathcal{H}_{g_1, 1}^{(x)} \times \mathcal{H}_{g_2, 1}^{(1-x)}$. Step 3 follows.

Step 4. This last step is obtained by applying the previous steps for various values of g_1 and g_2 and further measure theoretic considerations.

1.4 Results

In this section we present the main results of this thesis.

1.4.1 Preliminary work

The article of P. Hooper and B. Weiss was the starting point of our work. On the one hand, their proof of the density of the leaf of the Arnoux-Yoccoz surfaces needs two essential ingredients: cylinder decomposition and periodicity under the Teichmüller flow. This coexistence implies the vanishing of a certain invariant of the horizontal foliation of the Arnoux-Yoccoz surfaces known as the SAF. This invariant measures the homological drift of the horizontal foliation. Strener proved in [Str16] that the SAF of pseudo-Anosov homeomorphism coming from lift of pseudo-Anosov on non orientable surfaces vanishes. This contains the case of the Arnoux-Yoccoz surfaces. On the other hand, Do and Schmidt describe in [DS16] an infinite family of translation surfaces endowed with a vanishing SAF pseudo-Anosov. It was thus natural to try and adapt the proof of P. Hooper and B. Weiss to this family. The description of these surfaces is concrete enough so that cylinder decomposition can be easily found. We prove:

Theorem A. *The leaves of the Do-Schmidt surfaces are dense in the stratum in which they belong.*

This result is actually consequence of a more general criterion that we establish. This criterion can be used to fix a minor issue arising in proposition 6.1 of [HW15].

1.4.2 Isoperiodic dynamic in rank one affine manifolds

The second part of our work concerns the more general problem of understanding the behavior of the isoperiodic leaves in affine manifolds. The advantage of this approach is twofold. Firstly, the leaves of the isoperiodic foliation have a symmetry under the action of $SL_2(\mathbb{R})$. This is 1.2.4. Working in the setting of affine manifolds takes better account of those symmetries, especially for the leaf of surfaces whose $SL_2(\mathbb{R})$ -orbit does not fill the strata in which it belongs. Secondly, because the rank is a natural measure of the complexity of the problem, as the codimension of the leaves gets bigger with the rank. Indeed, we recall that by definition $2rk(M) + dim\mathcal{F}^M = dim\mathcal{M}$. In this work we address the case of rank one affine manifolds. Here again, cylinder decomposition and periodicity under Teichmüller flow are very important. The first is provided by the complete periodicity property that prevails in rank 1 by

results of Wright in [Wri15], while the other is a consequence of a closing lemma for the Teichmüller flow that is stated, for example, in [Wri14], together with small codimension of the leaves. Particularly interesting rank one affine manifolds where to study the dynamics of the foliation are the Prym eigenform loci $\Omega E_D(\kappa)$. We prove :

Theorem B. *Let \mathcal{M} be a connected component of a genus 3 prym eigenform loci $\Omega E_D(\kappa)$, with D not a square. Then, the leaves of $\mathcal{F}^{\mathcal{M}}$ are either closed or dense in \mathcal{M} , depending on whether D is a square or not. In the latter case, the foliation is ergodic with respect to the affine invariant measure on \mathcal{M} .*

The density part in Theorem B is a consequence of a more general criterion that we establish in this thesis that holds in non arithmetic affine manifolds. The ergodicity part is obtained by adapting the transfer principle of Calsamiglia, Deroin and Francaviglia. We actually prove that the foliation is ergodic whenever at least one leaf is dense, without restriction on the genus. This result corroborates an assertion of U. Hamenstädt in [Ham18] that density of a leaf of the isoperiodic foliation in strata should imply ergodicity of the foliation.

Returning to the original problem of understanding the behavior of the leaves in strata, we prove:

Theorem C. *Let (X, ω) be a Prym eigenform in $\Omega E_D(2, 1, 1)$ or $\Omega E_D(1^4)$ with D not a square. Then, the isoperiodic leaf \mathcal{F}_X is dense in the stratum in which it belongs.*

We emphasize that Theorem C is not a direct consequence of Theorem B, as $\mathcal{F}_X^{\mathcal{M}}$ has codimension 1 in \mathcal{F}_X . This expresses the fact that deformation along $\mathcal{F}^{\mathcal{M}}$ preserves both the periods and the Prym involution while deformations along \mathcal{F}_X can break those symmetries. Notice that the case $\Omega E_D^{odd}(2, 2)$ is already included in Theorem B as in this case $\mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_X$. The contrast between the strata with two singularities and the other is worth noticing, and should be linked with the extra cases occurring in genus 2 in the classification of Calsamiglia, Deroin, Francaviglia. See Theorem 1.3.2 above. The proof of Theorem C relies heavily on the classification of higher rank affine manifold in genus 3 provided by Aulicino and Ngyuen for rank 2 and by Mirzakhani and Wright for rank 3. This explains why we couldn't establish the same result in genera 4 and 5 where Prym eigenform loci still exist, and emphasizes the statement made at the beginning of this section that the behavior of the leaves is better understood in the light of the geometry of $SL_2(\mathbb{R})$ -orbits.

1.4.3 Connection between isoperiodic dynamics and geometry of affine manifolds

The last part of our work is a contribution to the classification of affine manifolds in genus 3. The results presented in the previous section show how the geometry of affine manifolds dictate the behavior of the leaves. We explore the converse direction of this connection and investigate how the the dynamics of the isoperiodic foliation shapes affines manifolds. We prove:

Theorem D. *Let \mathcal{M} be a proper non arithmetic affine manifold in $\mathcal{H}^{odd}(2, 2)$. Then $k(\mathcal{M})$ is a totally real quadratic number field of discriminant D and \mathcal{M} is a connected component of $\Omega E_D(2, 2)$.*

In particular, we prove in our setting by a completely different approach that the field of definition is a totally real number field. The situation in the stratum $\mathcal{H}(3, 1)$ is different and we prove:

Theorem E. *There are no proper and non arithmetic affine manifolds in $\mathcal{H}(3, 1)$.*

This connection was also used by P. Apisa in [Api17] to establish his classification of rank one manifolds in hyperelliptic components.

1.5 Organization of the thesis

Our work lead to the submission of two articles. Chapter 2 is a reproduction of our first article "Isoperiodic dynamics in genus 3 prym eigenforms" and contains the proof of Theorems B and C. Chapter 3 is a reproduction of our second article "Non arithmetic affine manifolds in $\mathcal{H}(2, 2)$ & $\mathcal{H}(3, 1)$ " and contains the proof of Theorems D and E.

Chapter 2

Isoperiodic dynamics in genus 3 Prym eigenform loci

Isoperiodic dynamics in genus 3 Prym eigenform loci

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April 29, 2019

Abstract

We study the isoperiodic foliation in any genus 3 prym eigenform locus ΩE_D and prove that the leaves are either closed or dense, depending on the discriminant of $\mathbb{Q}(\sqrt{D})$. In the latter case, we show that the foliation is ergodic with respect to any affine invariant measure on ΩE_D . As a corollary, we give new examples of isoperiodic leaves that are dense in strata with at least one non simple singularity.

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1 Introduction

1.1 Context and results

A stratum $\mathcal{H}_g(\kappa)$ of the moduli space of translation surfaces is the set isomorphism classes of pairs (X, ω) where X is a genus g Riemann surface and ω is a non vanishing holomorphic 1-form on X whose zeroes have multiplicities given by κ . This space admits a natural action by the group $GL_2^+(\mathbb{R})$, which is a generalization of the action of $GL_2^+(\mathbb{R})$ on the space of flat tori $GL_2^+(\mathbb{R})/SL(2, \mathbb{Z})$. The classification of the closed invariant sets is a central problem in Teichmüller dynamics. Recently, a deep theorem by Eskin, Mirzakhani and Mohammadi has shed light on the structure of such sets: they are immersed manifolds cut out by linear equations in period coordinates. See [6].

Transverse to the $GL_2^+(\mathbb{R})$ -action, there is a local action by $\mathbb{C}^{|\kappa|-1}$ that fits into a natural holomorphic foliation of the (strata of the) moduli space. It is usually referred to as the isoperiodic foliation, or the kernel foliation. The leaf \mathcal{F}_X of a translation surface (X, ω) is locally described by modifying the periods of ω . This foliation has been introduced in the 90's by Kontsevich and Eskin, and later by McMullen and Calta before it became a central object in Teichmüller dynamics. See for example how it is involved in the classification of horocycle orbits of Prym eigenforms in $\mathcal{H}(1, 1)$ obtained by M. Bainbridge, J. Smillie and B. Weiss in [14].

Several papers have been devoted to understanding the dynamics of its leaves. McMullen showed that the foliation is ergodic in the principal stratum in genus 2 and 3 using Ratner's theory. Hooper and Weiss gave the first examples outside the principal stratum of dense leaves. Shortly after Calsamiglia, Deroin and Francaviglia have obtained a Ratner-like classification of the minimal sets in the principal stratum. Simultaneously, Hamenstädt proved the ergodicity for the Masur-Veech measure in the same setting, using a different

approach. Both this last two results used McMullen's result as a base case for an induction. Very surprisingly, apart from the serie of Hooper and Weiss, nothing is known for the dynamics of the isoperiodic foliation in strata where at least one zero is not simple. Understanding the behaviour of new examples was our original motivation.

To any affine manifold \mathcal{M} , there is natural subfoliation $\mathcal{F}^{\mathcal{M}}$ of the isoperiodic foliation associated, obtained by intersecting the isoperiodic leaves with the affine manifold: $\mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_X \cap \mathcal{M}$. For instance if \mathcal{M} is a locus of branch covering constructions, the foliation $\mathcal{F}^{\mathcal{M}}$ consists of all the isoperiodic deformations that preserve the symmetries of the covering. A more elaborated example is given by the foliation in the prym eigenform loci $\Omega E_D(\kappa)$. See section 2 for more details. Studying the dynamics of those subfoliations is a refined version of the aforementioned problem. In this paper, we prove:

Theorem A. *Let $(X, \omega) \in \Omega E_D(\kappa)$ be a prym eigenform of genus 3, and let \mathcal{M} be a connected component of $\Omega E_D(\kappa)$. Then, the leaf of $\mathcal{F}_X^{\mathcal{M}}$ is either closed or projectively dense in \mathcal{M} . The latter case occurs if, and only if, D is not a square.*

Here, projectively dense means that the leaf of any surface (X, ω) is dense in the locus of surfaces in \mathcal{M} that have the same area as (X, ω) . Equivalently, the projection of $\mathcal{F}_X^{\mathcal{M}}$ in $\mathbb{P}\mathcal{M}$ is dense. See definition 2.4 for more details. We also prove a similar result for the rank one affine manifolds in the hyperelliptic strata, and prove the same statement in the the genus 2 case. This was already known, for example as part of the transfer principle of Calsamiglia, Derion, Francaviglia but we provide an alternative proof. The techniques we use rely strongly on the flat geometry of translation surfaces, cylinder deformations and the geometry of affine manifolds. In particular, we do not make use of degenerations of translation surfaces to a partial compactification of the strata. In an effort to describe as precisely as possible the dynamical behavior of the isoperiodic leaves, we prove:

Theorem B. *Let \mathcal{M} be a connected component of a prym eigenform locus $\Omega E_D(\kappa)$ in genus 3 with D not a square. Then, the isoperiodic foliation $\mathcal{F}^{\mathcal{M}}$ is ergodic with respect to any affine invariant measure on \mathcal{M} .*

Here, the foliation $\mathcal{F}^{\mathcal{M}}$ is said to be ergodic if any $\mathcal{F}^{\mathcal{M}}$ -saturated measurable subset is measurably trivial. The theorem B is, in fact, the consequence of a more general result that holds in all the prym eigenform loci, without restriction on the genus. We prove:

Theorem C. *Let \mathcal{M} be a connected component of $\Omega E_D(\kappa)$. If $\mathcal{F}^{\mathcal{M}}$ is projectively minimal, then it is ergodic with respect to any affine invariant measure on \mathcal{M} .*

Here, projectively minimal means that any leaf is projectively dense. Theorem C is proved by adapting the transfer principle of Calsamiglia, Deroin and Francaviglia [10]. We defined the equivalent of their period map and a result on the connectedness of its level sets is also required. This connectedness result is a consequence of the minimality. Surprisingly

enough, the ergodicity it thus a consequence of the minimality, and Theorem B is a consequence of Theorems A and C. In rank one affine manifolds, having a projectively dense leaf is equivalent to projective minimality. See proposition 2.5. Theorem C corroborates in our setting an assertion of U. Hamenstädt in [20] that projective density of a leaf should imply ergodicity of the foliation. Returning to our original motivation, we prove the following two:

Theorem D. *Let (X, ω) be a prym eigenform in $\mathcal{H}^{odd}(2, 2)$, then the leaf \mathcal{F}_X is either closed or projectively dense in the connected component of $\Omega E_D^{odd}(2, 2)$ in which it belongs. The last case occurs if, and only if, D is not a square.*

Theorem E. *Let (X, ω) be a prym eigenform in $\mathcal{H}_3(\kappa)$ with $|\kappa| > 2$, then the leaf \mathcal{F}_X is either closed or projectively dense. The last case occurs if, and only if, D is not a square.*

It is interesting to note the dichotomy arising with the number of singularities. In particular, we emphasize that if the transfer principle of [10] were to exist in strata, its conclusion would be in contradiction with theorem D, as $\Omega E_D(2, 2)$ is saturated by the isoperiodic foliation and made of surfaces whose periods span a dense subgroup of \mathbb{C} .

1.2 Outline of the paper

In section 2, we collect relevant definitions and properties. We review the structures of the strata and of the closed $GL_2^+(\mathbb{R})$ -invariant sets, recall the definitions of the prym eigenform loci, and define the isoperiodic foliation and the subleaves associated. In section 3, we give a common framework for cylinder and isoperiodic deformations. In section 4, we establish a criterion for density of the leaves of subfoliations that preserves a rank one invariant manifold, and isolate a property, denoted by \mathcal{P} , required for this criterion. This result is key to prove the announced results. We also deduce structural results on the closure of those subleaves. In section 5, we provide a list of affine manifolds that have property \mathcal{P} , and deduce Theorem A. In section 6 we establish a variant of the transfer principle of [10] for ΩE_D and proceed to prove Theorem B and C. In section 7, we use the structural result established in section 4 to prove theorem D and E.

2 Framework

2.1 Period coordinates & the action of $GL_2^+(\mathbb{R})$

The Teichmüller space of translation surfaces $\tilde{\mathcal{H}}(\kappa)$ is the set of isomorphism classes of marked translation surfaces (X, ω, f) , where $f : S \rightarrow X$ is a homeomorphism from a fixed genus g surface S such that the preimage of the singularities of ω by f is a subset Σ . The following map is known as the period map :

$$\Phi : \begin{array}{ccc} \tilde{\mathcal{H}}(\kappa) & \rightarrow & H^1(S, \Sigma, \mathbb{C}) \\ (X, \omega, f) & \mapsto & (\gamma \mapsto \int_{f \circ \gamma} \omega) \end{array}$$

There is a complex structure on $\tilde{\mathcal{H}}(\kappa)$ that turns Φ into a local biholomorphism, and if Γ_g denotes the mapping class group of S , then Γ_g acts on $\tilde{\mathcal{H}}(\kappa)$ by precomposition: $\varphi \cdot (X, \omega, f) = (X, \omega, f \circ \varphi^{-1})$. The quotient set is isomorphic to $\mathcal{H}(\kappa)$ and the latter is endowed with the complex orbifold structure that turns the canonical projection $\pi : \tilde{\mathcal{H}}(\kappa) \rightarrow \mathcal{H}(\kappa)$ into a local biholomorphism. Most of the issues arising through the orbifoldic points can be resolved by passing to a finite cover of the moduli space and we will implicitly do so in the remainder of this text. $\tilde{\mathcal{H}}(\kappa)$ is endowed with a group action by $GL_2^+(\mathbb{R})$ defined by :

$$\forall g \in GL_2^+(\mathbb{R}) \quad \Phi(g \cdot (X, \omega, f)) = g \cdot \Phi(X, \omega, f)$$

That action descends to an action on $\mathcal{H}(\kappa)$ in a way that the canonical projection π is $GL_2^+(\mathbb{R})$ -equivariant. The action of the subgroup of diagonal matrices with determinant 1 is known as the Teichmüller geodesic flow, while the action of subgroup of the unipotent matrices of determinant 1 is known as the horocycle flow. More details on the structures of these spaces can be found in [13] or [15].

Definition 2.1 (affine manifold). *A closed connected subset \mathcal{M} of $\mathcal{H}(\kappa)$ is said to be an affine manifold if for any $X \in \mathcal{M}$, there is a section s of π defined on a neighborhood \mathcal{U} of X , and a subspace $V \in H^1(S, \Sigma, \mathbb{R})$ such that $\Phi \circ s(\mathcal{U} \cap \mathcal{M}) = \Phi \circ s(\mathcal{U}) \cap V \otimes \mathbb{C}$. Such a V is called a local model of \mathcal{M} around X .*

Affine manifolds are invariant under the action of $GL_2^+(\mathbb{R})$ and Eskin, Mirzakhani and Mohammadi proved in a celebrated result that the converse is true. More details can be found in [6]. An important numerical invariant associated to these loci is the rank, defined as follows: define $\rho : H^1(S, \Sigma, \mathbb{C}) \rightarrow H^1(S, \mathbb{C})$ to be the canonical restriction map, and for any X in \mathcal{M} , take a local model V of \mathcal{M} . Avila, Eskin and Möller proved in [16] that $\rho(V)$ is a symplectic subspace of $H^1(S, \mathbb{C})$. The rank of \mathcal{M} , denoted $rk(\mathcal{M})$, is then defined as half the dimension of this space. More details can be found in [1]. The following definition will be important for the remainder of this text :

Definition 2.2. *Let \mathcal{M} be a affine manifold. The field of definition of \mathcal{M} is the smallest subfield $k(\mathcal{M})$ of \mathbb{R} such that any local model of \mathcal{M} can be written as $V = V_0 \otimes_{k(\mathcal{M})} \mathbb{R}$, where V_0 is a $k(\mathcal{M})$ -vector space.*

Wright proved the following useful result in [1]:

Proposition 2.1. *Let \mathcal{M} be an affine manifold, and let (X, ω) be a periodic surface in \mathcal{M} with m cylinders. Then, $k(\mathcal{M})$ is contained in $\mathbb{Q}[c_2c_1^{-1}, \dots, c_m c_1^{-1}]$ where the c_i are the circumferences of the cylinders.*

Wright actually proved a stronger result and showed that reciprocal inclusion, provided one restrains to a subclass of cylinders. See [1] for more details.

2.2 The isoperiodic foliation

Definition 2.3. *An affine manifold \mathcal{M} is said to be non absolute if for every $X \in \mathcal{M}$, any local model around X intersects $\ker \rho$ non trivially.*

Notice that for a rank 1 affine manifolds \mathcal{M} , non absolute is equivalent to $\dim_{\mathbb{C}}(\mathcal{M}) \geq 3$. Since invariant manifolds are connected, it can easily be checked that the dimension of this intersection is constant, and since two sections differ by the action of Γ_g which preserves the Kernel of ρ , we can define a foliation on \mathcal{M} , called the isoperiodic foliation. If $X \in \mathcal{M}$, the leaf of X will be denoted by $\mathcal{F}_X^{\mathcal{M}}$. Intuitively, an invariant manifold is non absolute if deformations of surfaces that locally preserve the periods (the integral of ω along absolute cycles) can be performed inside \mathcal{M} . We say locally because the notion of periods is defined only up the choice of a local model of \mathcal{M} , and an other choice would produce different sets of periods. The two are related by the action of Γ_g on the homology of S . The following proposition characterizes the isoperiodic foliation $\mathcal{F}^{\mathcal{M}}$.

Proposition 2.2. *Let \mathcal{M} be a non absolute affine manifold. Two surfaces (X_1, ω_1) and (X_2, ω_2) in \mathcal{M} lie in the same leaf of $\mathcal{F}^{\mathcal{M}}$ if, and only if, there is path γ in $\mathcal{H}(\kappa)$ such that:*

1. $\pi \circ \gamma(0) = (X_1, \omega_1)$ and $\pi \circ \gamma(1) = (X_2, \omega_2)$
2. $\rho \circ \Phi \circ \gamma$ is constant.
3. $\forall t \pi \circ \gamma(t) \in \mathcal{M}$

The isoperiodic foliation behaves nicely with respect to the action of $GL_2^+(\mathbb{R})$:

Proposition 2.3. *Let \mathcal{M} be a non absolute affine invariant manifold, let X be a translation surface in \mathcal{M} and let $g \in GL_2^+(\mathbb{R})$. The following formula holds :*

$$g \cdot \mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_{g \cdot X}^{\mathcal{M}}$$

Proof. Let $g \in GL_2^+(\mathbb{R})$, $(X_0, \omega_0) \in \mathcal{M}$, and $(X_1, \omega_1) \in \mathcal{F}^{\mathcal{M}}$. By 2.2, there is a path γ in $\mathcal{H}(\kappa)$ such that $\pi \circ \gamma(i) = (X_i, \omega_i)$ and $\rho \circ \Phi \circ \gamma$ is constant. But since $\rho \circ \Phi$ is $GL_2^+(\mathbb{R})$ -equivariant, the path $\gamma_g = g \cdot \gamma$ is such that $\pi \circ \gamma_g(i) = g \cdot (X_i, \omega_i)$, and $\rho \circ \Phi \circ \gamma_g$ is constant. Consequently, the surfaces $g \cdot (X_i, \omega_i)$ belong to the same leaf. This proves the first inclusion. To prove the other, notice that $\mathcal{F}_{g \cdot X}^{\mathcal{M}} = g \cdot g^{-1} \mathcal{F}_{g \cdot X}^{\mathcal{M}}$ and the latter is contained in $g \cdot \mathcal{F}_X^{\mathcal{M}}$ by the first inclusion. \square

Recall that the area of a surface $(X, \omega) \in \mathcal{H}(\kappa)$ is defined as the integral of $\frac{i}{2} \cdot \omega \wedge \bar{\omega}$ over X .

Definition 2.4. *Let \mathcal{M} be a non absolute affine manifold, and let (X, ω) be a surface in \mathcal{M} . The leaf $\mathcal{F}_X^{\mathcal{M}}$ is said to be projectively minimal if it is dense in the set of surfaces in \mathcal{M} that have the same area as (X, ω) . The foliation is said to be projectively minimal if any leaf is projectively dense.*

We conclude this section with a useful lemma that describes the topology of rank one affine manifolds. Let \mathcal{M} be a non absolute rank one affine manifold, and let (X, ω) be a surface in \mathcal{M} . Let V be a local model of \mathcal{M} associated to a section s defined on a neighborhood \mathcal{U} of (X, ω) . If $g \in GL_2^+(\mathbb{R})$ and $v \in Ker(\rho) \cap V \otimes \mathbb{C}$ are small enough, we define $g(X, \omega) + v$ to be the surface $(\Phi \circ s)^{-1}(g \cdot p + v)$, where $p = \Phi(s(X, \omega))$.

Proposition 2.4. *The map $(g, v) \mapsto g \cdot (X, \omega) + v$ is a homeomorphism onto its range.*

Notice that by proposition 2.3, we have $g \cdot ((X, \omega) + v) = g \cdot (X, \omega) + g \cdot v$ whenever these expressions make sense.

Proposition 2.5. *Let \mathcal{M} be a non absolute rank one affine manifold. The foliation $\mathcal{F}^{\mathcal{M}}$ is projectively minimal if, and only if, there is a projectively dense leaf.*

Proof. Suppose (X, ω) is a surface in \mathcal{M} whose leaf $\mathcal{F}_X^{\mathcal{M}}$ is projectively dense, and let (X_1, ω_1) be a surface in \mathcal{M} . By ergodicity of the action of $GL_2^+(\mathbb{R})$ on \mathcal{M} and proposition 2.4, there is a matrix $g \in GL_2^+(\mathbb{R})$ such that $g \cdot \mathcal{F}_X^{\mathcal{M}} = \mathcal{F}_{X_1}^{\mathcal{M}}$. Since the action of $GL_2^+(\mathbb{R})$ is continuous, and g sends the locus of surfaces whose area is the same as the one of (X, ω) to the one whose area is the same as the one of (X_1, ω_1) , this proves that the leaf $\mathcal{F}_{X_1}^{\mathcal{M}}$ is projectively dense. Since the surface (X_1, ω_1) was chosen arbitrarily, this proves the claim. \square

2.3 Prym eigenform loci

We recall here the definition of the prym eigenforms. More information on those objects can be found in [7]. Let (X, ω) be a translation surface endowed with a holomorphic involution τ . We denote by $\Omega(X)$ the set of holomorphic 1-forms, and by $\Omega^-(X)$ the set of τ -anti invariant holomorphic 1-forms. We say that (X, ω) is a Prym form if $\omega \in \Omega^-(X)$, that is $\tau^*\omega = -\omega$, and $dim\Omega^-(X) = 2$. The Prym variety $Prym(X, \omega, \tau)$ is defined as the 2-dimensional abelian variety $\Omega^-(X)/H_1^-(X, \mathbb{Z})$ endowed with the polarization coming from the intersection form. We recall that a quadratic order of discriminant D is a ring \mathcal{O}_D isomorphic to $\mathbb{Z}[X]/(X^2 + bX + c)$, with $D^2 = b^2 - 4c$.

Definition 2.5. *A Prym eigenform is a Prym form such that $Prym(X, \omega, \tau)$ has real multiplication by \mathcal{O}_D for which ω is an eigenform, ie $End(Prym(X, \omega, \tau))$ contains a copy of \mathcal{O}_D acting by self adjoint endomorphisms, such that $\mathcal{O}_D \cdot \omega = \lambda \cdot \omega$, for a $\lambda \in \mathcal{O}_D$.*

3 Modifying the twist parameters

In this section, we establish a number of common results and tools to deform translation surfaces.

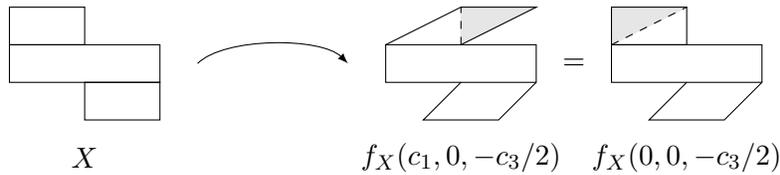


Figure 1: Twisting cylinders

3.1 The twist map

There is a particular class of translation surfaces for which deformations are easily seen : those are the periodic surfaces (also known as Jenkin-Strebel differentials). If (X, ω) is a translation surface, the quadratic form $\omega \otimes \bar{\omega}$ induces a flat metric on X , away from the singularities of ω . We say that a surface is periodic in direction θ if the geodesic rays in direction θ are either periodic, or saddle connections (they start and end at singularities of ω). It is well known that if a translation surface is periodic in a given direction, it is decomposed as a union a cylinders (those are subsets isometric to $[0, h] \times \mathbb{R}/c\mathbb{Z}$) whose boundary components are union of saddle connections in direction θ . We will denote by $h_{\mathcal{C}}$ the height of the cylinder \mathcal{C} , $c_{\mathcal{C}}$ its circumferences and by $\gamma_{\mathcal{C}}$ the inverse of $c_{\mathcal{C}}$. A classic and useful construction to navigate inside the moduli space is as follows : given a horizontally periodic surface X with m cylinders, one can cut open the surface along the core curve of any cylinder and glue back after having performed a twist. This construction yields a smooth embedding from the m -dimensional torus to $\mathcal{H}(\kappa)$. This embedding, called the twist map, is of particular interest as it allows to see specifics deformations of surfaces as linear flows, of which we have a detailed understanding.

More precisely, let (X_0, ω_0, f_0) be a representative of X in $\tilde{\mathcal{H}}(\kappa)$. Denote by $\gamma_{\mathcal{C}}$ the homology class of the core curve of the cylinder \mathcal{C} of X_0 , denote by $\eta_{\mathcal{C}}$ the Poincaré dual of $f^{-1}(\gamma_{\mathcal{C}})$, by E the linear subspace of $H^1(S, \Sigma, \mathbb{C})$ spanned by the $\eta_{\mathcal{C}}$, and finally set $\mathcal{C}(X_0)$ to be the arc-wise component of X_0 of $\{Y \in \tilde{\mathcal{H}}(\kappa) \mid \Phi(Y) \in \Phi(X_0) + E\}$.

Proposition 3.1. *The period map induces a diffeomorphism from $\mathcal{C}(X_0)$ to $\Phi(X_0) + E$.*

Proof. Firstly, we shall prove that the the imaginary part of any two abelian differentials in $\mathcal{C}(X_0)$ are isotopic. More precisely, if (X_1, ω_1, f_1) is a surface in $\mathcal{C}(X_0)$, we are going to construct a diffeomorphism (for the smooth structures induced by the complex atlases) from X_0 to X_1 that pulls back the imaginary part of ω_2 to the one of ω_1 , and such that $f_0 \circ \varphi \circ f_1^{-1}$ is isotopic to the identity of S . A particularly interesting consequence is that all the surfaces in $\mathcal{C}(X_0)$ will be horizontally periodic, as X_0 is. As $\mathcal{C}(X_0)$ is path-wise connected, there is a path γ going from X_0 to X_1 , and we chose a representative (X_t, ω_t, f_t) of $\gamma(t)$. Once we have chosen a smooth structure on S , this path can be chosen so that the f_t are smooth diffeomorphisms. Denote by α_t the pull back of the imaginary part of ω_t by f_t . We will use a variant of the Moser's homotopy trick to find a isotopy φ_t such that :

$$\varphi_t^* \alpha_t = \alpha_0$$

A sufficient condition for this equation to hold is that the time derivative of the left hand side vanishes. If we denote by X_t the flow associated to the isotopy, this derivative equals $d \circ \iota_{X_t}(\alpha_t) + \iota_{X_t} \circ d(\alpha_t) + \dot{\alpha}$. Since the α_t all have the same periods, the $\dot{\alpha}_t$ vanish as an element of the relative cohomology of X_0 . Therefore, there is a family of functions U_t whose derivative is $\dot{\alpha}_t$. Since the α_t are closed as they come from imaginary parts of holomorphic forms, it is enough to solve :

$$\iota_{X_t}(\alpha_t) = -U_t$$

Since U_t is defined up to a constant, we can require that it vanishes on Σ , using the fact that the relative periods are equal. Denote by V_t the pull back by f_t of the vertical vector field on X_t coming from the flat structure. This define a non autonomous vector field on $S - \Sigma$, and set $X_t = -U_t \cdot V_t$. This vector field can be extended to the whole surface S by setting $X_t(p) = 0$ for any $p \in \Sigma$. By definition, $\alpha_t(V_t) = 1$, and thus X_t is the desired vector field. The fact that this vector field is well defined at Σ can be checked in local coordinates using the fact that where the α_t vanish up to order k , the functions U_t vanish up to order $k + 1$.

Secondly, we will prove that the restriction of Φ to $\mathcal{C}(X_0)$ is injective. Suppose (X_1, ω_1, f_1) and (X_2, ω_2, f_2) have the same image. By what has been previously proved, there is a diffeomorphism φ from X_1 to X_2 that pulls back the imaginary part of ω_2 to the one of ω_1 , and such that $f_2 \circ \varphi \circ f_1^{-1}$ is isotopic to the identity of S . Such a diffeomorphism maps cylinders of X_1 to cylinders of X_2 . The assumptions on the periods of X_1 and X_2 implies the cylinders paired have same height, circumference and twist parameters. Since the mapping class group of the cylinders is generated by Dehn twists along the core curve of the cylinder, we get that the restriction to a cylinder of the map $f_2 \circ \varphi \circ f_1^{-1}$ is isotopic to the identity. Consequently, there is a vector field defined on each cylinder and the associated isotopy deforms φ into the identity of the cylinders. Those vector fields adds up to form a vector field on X_1 , and the resulting map from X_1 to X_2 is a diffeomorphism that maps the real and imaginary parts of ω_1 to the ones of ω_2 , while being isotopic to φ . It thus induces a biholomorphism ψ from X_1 to X_2 taking abelian differential to abelian differential, and such that $f_2^{-1} \circ \psi \circ f_1$ is isotopic to the identity of S . Therefore, the surfaces X_1 and X_2 represent the same point in $\tilde{\mathcal{H}}(\kappa)$, and this proves the injectivity.

It remains to prove the surjectivity. Let $u = \sum u_i \eta_i \in E$. Let $\gamma(t)$ be the surface obtained by cutting open the cylinder i of X_0 along its core curve and gluing back by performing a twist of amount $t \cdot u_i$. This defines a path from X_0 to a surface in $\mathcal{C}(X_0)$ whose periods are exactly $\Phi(X_0) + u$.

□

$$\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{\tilde{f}_X} & \tilde{\mathcal{H}}(\kappa) \\
p \downarrow & & \downarrow \pi \\
\mathbb{R}^m/\mathbb{Z}^m & \xrightarrow{f_X} & \mathcal{H}(\kappa)
\end{array}$$

Figure 2: The twist map

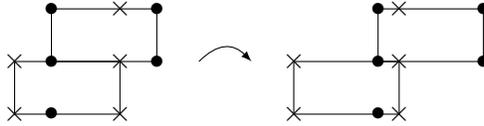
Now, define a map \tilde{f} from \mathbb{R}^m to $\tilde{\mathcal{H}}(\kappa)$ that sends (x_1, \dots, x_m) to the surface of $\mathcal{C}(X_0)$ whose image by the period map is $\Phi(X) + \sum c_i x_i \eta_{C_i}$. The action of \mathbb{Z}^m on \mathbb{R}^m by translation is intertwined by \tilde{f} with the action of the subgroup of Γ_g spanned the Dehn twists about the core curves of the cylinders. Consequently, there is a map $f : \mathbb{R}^m/\mathbb{Z}^m$ to $\mathcal{H}(\kappa)$ that fits into the commutative diagram depicted in figure 2. We shall refer to the map f_X as the twists map associated to X . Notice that by construction $\Phi \circ \tilde{f}$ is affine, and its constant part is $\Phi(X_0)$. If $x \in \mathbb{R}^m$, any section s of π can be chosen to map $f \circ p(x)$ to $\tilde{f}(x)$. In the sequel we will say that such a section is adapted to $\tilde{f}_X(x)$.

3.2 Navigating inside affine manifolds with twists

From now on, \mathcal{M} is a non absolute affine manifold that contains X . We denote by k the rank of \mathcal{M} and by r the dimension of the foliation $\mathcal{F}^{\mathcal{M}}$. This last quantity is also known as the Rel of \mathcal{M} .

Definition 3.1. *The cylinder decomposition of X is said to be $\mathcal{F}^{\mathcal{M}}$ -stable if every horizontal saddle connections of X vanishes as an element of $T_X^* \mathcal{F}_X^{\mathcal{M}}$.*

The condition of $\mathcal{F}^{\mathcal{M}}$ -stability means that there are no saddle connections joining different singularities along a boundary of a cylinder, or if there is it has to remain so along any isoperiodic deformation inside \mathcal{M} that preserves the cylinder decomposition. The stable decomposition is relevant in our context as they allow to recognize isoperiodic deformation as deformation in the twist space. The following Figure shows an example of isoperiodic deformation that is not expressed as a twist deformation :



Lemma 3.1. *There are two linear subspaces of \mathbb{R}^m denoted $V_{\mathcal{M}}$ and $K_{\mathcal{M}}$ such that $f^{-1}(\mathcal{M}) = p(V_{\mathcal{M}})$ and $f^{-1}(\mathcal{F}_X^{\mathcal{M}}) = p(K_{\mathcal{M}})$. The space $V_{\mathcal{M}}$ is rational and its dimension is at most $k + r$, the dimension of $K_{\mathcal{M}}$ is exactly r .*

Proof. Let V_0 be the local model of \mathcal{M} around X associated to a section adapted to X_0 . By definition, it means that $s(X) = \tilde{f}(0) = X_0$.

1. We define $V_{\mathcal{M}}$ to be the preimage of $V_0 + \Phi(X_0)$ by the affine map $\Phi \circ \tilde{f}$. This is a linear subspace since it contains 0 as the preimage of $\Phi(X_0)$. Set $Z = \{x \in \mathbb{R}^m, f \circ p(x) \in \mathcal{M}\}$. The image of this set by p is exactly $f^{-1}(\mathcal{M})$. Thus, it is enough to prove that Z coincides with $V_{\mathcal{M}}$ to get the equality between $f^{-1}(\mathcal{M})$ and $p(V_{\mathcal{M}})$. In that extent, we will prove first that Z contains all the line it intersects. Let x be in Z and define $A_x = \{a \geq 0 \mid \forall t \in B(1, a) \ t \cdot x \in Z\}$. Let us prove that this set is open. Let a be in A_x and take a local model V_a of \mathcal{M} around $f \circ p(a \cdot x)$ associated to a section s adapted to a . For any ε small enough, $a + \varepsilon$ belongs to Z if and only if $\Phi \circ \tilde{f}(a + \varepsilon \cdot x)$ belongs to V_a , and the latter equals $\Phi \circ \tilde{f}(a \cdot x) + \varepsilon \sum x_i c_i \eta_i$. By assumption, if ε is negative $a + \varepsilon$ belongs to A_x and that implies that the vector $\sum x_i c_i \eta_i$ is contained in V_a . By linearity, we deduce that $\Phi \circ \tilde{f}(a + \varepsilon \cdot x)$ is in V_a . Using the same argument for $-a$ we get that $a + \varepsilon$ is contained in A_x , hence this set is open. It is also closed and it is not empty since it contains 1. As a consequence, we deduce that Z contains the line spanned by x . Now, notice that there is a neighborhood \mathcal{V} of the 0 in \mathbb{R}^m such that $Z \cap \mathcal{V} = V_{\mathcal{M}} \cap \mathcal{V}$. This follows from the fact that V_0 is a local model of \mathcal{M} around X , and the commutativity of the diagram of figure 2. On the one hand, this remark, together with the previous claim, implies that the whole $V_{\mathcal{M}}$ is contained in V . Secondly, if $x \in Z$, there is a $t > 0$ so that $t \cdot x \in \mathcal{V} \cap Z$ and thus $t \cdot x$ is contained in $V_{\mathcal{M}}$. Since $V_{\mathcal{M}}$ is a linear space, we get that x is contained in $V_{\mathcal{M}}$ as well. We have prove the equality $f^{-1}(\mathcal{M}) = p(V_{\mathcal{M}})$. Notice that this implies that $p(V_{\mathcal{M}})$ is closed and thus that $V_{\mathcal{M}}$ is rational, see for example prop 1.4.1 of [9]. The bound on the dimension of $V_{\mathcal{M}}$ comes from the fact $\rho(E)$ is an isotropic subspace of $H^1(S, \mathbb{C})$. Indeed, the linear part of $\Phi \circ \tilde{f}$ maps injectively $V_{\mathcal{M}}$ to $E \cap V_0$ and the image of the latter by ρ is isotropic. Since V_0 is symplectic, this gives that the dimension of $E \cap V_0$ is at most k . By definition, the dimension of the kernel of ρ restricted to V_0 is r , hence the bound for the dimension of $V_{\mathcal{M}}$.

2. Similarly, we define $K_{\mathcal{M}}$ to be the preimage of $V_0 \cap \text{Ker} \rho + \Phi(X_0)$ by the map $\Phi \circ \tilde{f}$. As before, this is a linear subspace of \mathbb{R}^m , and we prove in the same fashion that the sets $f^{-1}(\mathcal{F}_X^{\mathcal{M}})$ and $p(K_{\mathcal{M}})$ are equal. Let us compute the dimension of $K_{\mathcal{M}}$. We shall prove that the linear part of $\Phi \circ \tilde{f}$ is onto $V_0 \cap \text{Ker} \rho$. Let u be in $V_0 \cap \text{Ker} \rho$. Since the cylinder decomposition of X is stable, it implies that u vanishes on the cycle forming the boundaries of the cylinders. It also vanishes on the core curves of the cylinders as they are absolute cycles. Consequently, the only cycles on which it may not vanish are the cross curves of the cylinders. Consequently, we get that $u = \sum u(\sigma_i) \eta_{c_i}$, and thus $u = \Phi \circ \tilde{f}(u(\sigma_1) c_{i_1}^{-1}, \dots, u(\sigma_m) c_m^{-1}) - \Phi(X_0)$. As the dimension of $V_0 \cap \text{Ker} \rho$ is r by definition and $\Phi \circ \tilde{f}$ is injective, we get that the dimension of $K_{\mathcal{M}}$ is r . \square

In the sequel, we shall denote the two linear spaces of lemma 3.1 by $V_{\mathcal{M}}(X)$ and $K_{\mathcal{M}}(X)$ to emphasize they are associated to the surface X . The case of equality for the dimension of $V_{\mathcal{M}}(X)$ in the previous proposition is obtained precisely when the two subspaces

$Pres(X, \mathcal{M})$ and $Twist(X, \mathcal{M})$ defined in [1] are equal. Reciprocally, if we do not require the decomposition of X to be $\mathcal{F}^{\mathcal{M}}$ -stable, the dimension of $K_{\mathcal{M}}(X)$ is only bounded above by r . This expresses the fact that some isoperiodic deformation will not be recognized as twist deformations.

Definition 3.2. *The support of $u \in K_{\mathcal{M}}(X)$ is the collection of indices i so that $u_i \neq 0$.*

The support records the information of which cylinders have been twisted on X to get to the surface $f(u)$.

Definition 3.3. *A minimal isoperiodic deformation is a vector $u \in K_{\mathcal{M}}$ whose support is minimal among vectors in K . Its degree $deg(u)$ is defined as $\dim \text{Span}_{\mathbb{Q}}\langle u_i \rangle - 1$. Finally, two minimal isoperiodic deformations are said to be transverse if their supports are disjoint.*

4 Property \mathcal{P} and density of the leaves

4.1 A criterion in rank 1

Proposition 4.1. *Let \mathcal{M} be a rank one non absolute affine manifold and $X \in \mathcal{M}$, a surface having a horizontal $\mathcal{F}^{\mathcal{M}}$ -stable cylinder decomposition. If $K_{\mathcal{M}}(X)$ contains a minimal deformation of positive degree, then the closure of $\mathcal{F}^{\mathcal{M}}$ contains the horocycle orbit of X .*

Proof. Let $u \in K_{\mathcal{M}}$ be a minimal deformation of positive degree. There is a subspace $V_u \subset \mathbb{R}^m$ of dimension $1 + deg(u)$ so that $p(V_u)$ is the closure of $\{p(t \cdot u), t \in \mathbb{R}\}$. Since \mathcal{M} is closed, V_u is actually contained in $V_{\mathcal{M}}$. However, it is not contained in $K_{\mathcal{M}}$. Indeed, the support of any vector in V_u is contained in the one of u , but since the degree of u is at least 2, there is a vector for which this inclusion is strict. By definition, such a vector can not be in $K_{\mathcal{M}}$. Now, the subspace $K_{\mathcal{M}} + V_u$ contains $K_{\mathcal{M}}$ and this inclusion is strict. Consequently, its dimension is at least $k + 1$. But, since \mathcal{M} is rank 1, the dimension of $V_{\mathcal{M}}$ is at most $k + 1$. Hence, $V_{\mathcal{M}} = K + V_u$ and the dimension of $V_{\mathcal{M}}$ is $k + 1$. As a consequence of this equality, we deduce that $f \circ p(V_{\mathcal{M}})$ is contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$. Indeed, let $w = k + v \in V_{\mathcal{M}}$, with $v \in V_u$. Then there is a growing sequence $t_n > 0$ so that $p(t_n \cdot u)$ converges to v . But then, by continuity, $f \circ p(k + t_n \cdot v)$ is a sequence of points in $\mathcal{F}_X^{\mathcal{M}}$ converging to $f \circ p(w)$. To conclude, notice that the horocycle orbit of X is contained in $f(V_{\mathcal{M}})$. \square

Proposition 4.2. *Let \mathcal{M} be a non absolute rank 1 affine invariant manifold, and $X \in \mathcal{M}$. If the horocycle orbit of X is contained in its isoperiodic leaf, then so is it $SL_2(\mathbb{R})$ -orbit*

Proof. We will need the following lemma :

Lemma 4.1. *The leaf of X contains a surface periodic under Teichmüller flow (in some direction). This surface can be chosen arbitrarily close to X .*

proof of the lemma. Pick a neighborhood \mathcal{U} of X . Up to choosing a smaller open set contained in this neighborhood, we can suppose by proposition 2.4 $\mathcal{U} = \mathbb{D}(0, \eta) \times B(Id, \epsilon)$. Set $\mathcal{U}_0 = \mathbb{D}(0, (1 - \epsilon)\eta) \times B(Id, \epsilon)$. Using a version of the closing lemma for the Teichmüller flow of [2], pick a surface $Z = g \cdot X + v \in \mathcal{U}_0$ which is periodic for the Teichmüller flow. Set $Y = g^{-1} \cdot Z = X + u$ where $g(u) = v$. Since $v \in \mathbb{D}(0, \frac{\eta}{1-\epsilon})$ and $g \in B(Id, \epsilon)$, we have that $u \in \mathbb{D}(0, \eta)$. (recall that the inverse of a matrix M close to the identity is invertible with inverse $\sum (Id - M)^n$. Thus if M is ϵ -close to the identity, the norm of M^{-1} is at most $\frac{1}{1-\epsilon}$.) \square

Now, pick a surface Y as in the previous lemma, with $g \cdot Y = Y$ If Y is close to X enough, the commutation relation between isoperiodic deformations and the $GL_2^+(\mathbb{R})$ -action (see proposition 2.3) shows that the horocycle orbit of Y is also contained in the closure of the leaf of X . Let θ_n be the angle between $g^n[1 : 0] \in \mathbb{R}P^1$ and the horizontal line. Denote by $H(\theta_n)$ the subgroup $\{r_{\theta_n} h_t r_{\theta_n}^{-1}, t \in \mathbb{R}\}$. If $H(\theta_n) \cdot X$ is contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$, using proposition 2.3, one gets that $g \cdot H(\theta_n) \cdot X$ is also contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$. But since $g \cdot H(\theta_n) = H(\theta_{n+1}) \cdot g$ the latter is exactly $H(\theta_{n+1}) \cdot g \cdot X$. As $g \cdot X = X$, this gives that for all $n > 0$ $H(\theta_n) \cdot X \subset \mathcal{F}_X^{\mathcal{M}}$. By continuity, this implies that $h_t^\theta \cdot X$ is also in the closure of the leaf. To conclude, use proposition 5.2 in [5] to deduce that the closure of $\mathcal{F}_X^{\mathcal{M}}$ contains the $SL_2(\mathbb{R})$ -orbit of Y . Once again, we use proposition 2.3 to deduce that the $SL_2(\mathbb{R})$ orbit of X is also contained in the closure of the leaf of X . \square

We introduce a useful notion :

Definition 4.1. *A non absolute rank one affine manifold \mathcal{M} is said to have property \mathcal{P} if for any $X \in \mathcal{M}$, there is a surface $Y \in \mathcal{F}_X^{\mathcal{M}}$ that has a $\mathcal{F}^{\mathcal{M}}$ -stable cylinder decomposition such that $K_{\mathcal{M}}(Y)$ contains a minimal isoperiodic deformation of positive degree.*

This property \mathcal{P} can actually be checked only on a single surface, as shows the following proposition.

Proposition 4.3. *A non absolute rank one affine manifold \mathcal{M} has property \mathcal{P} if, and only if, it contains a surface X which has a $\mathcal{F}^{\mathcal{M}}$ -stable cylinder decomposition such that $K_{\mathcal{M}}(X)$ contains a minimal isoperiodic deformation of positive degree.*

Proof. If (X, ω) has a decomposition as in property \mathcal{P} , note that this is also the case of any surface in a small enough neighborhood \mathcal{U} . This follows from proposition 2.4 together with the fact that this property is both invariant by small deformations along $\mathcal{F}^{\mathcal{M}}$ and by the action of $GL_2^+(\mathbb{R})$. Now, let X' be a surface in \mathcal{M} , and chose a neighborhood \mathcal{U}' around X' . By ergodicity of the action of $GL_2^+(\mathbb{R})$ on \mathcal{M} , there is a matrix $g \in GL_2^+(\mathbb{R})$ such that $g \cdot \mathcal{U} \cap \mathcal{U}' \neq \emptyset$. It thus contains a surface of the form $g' \cdot X' + u$ that is as in property \mathcal{P} . Up to shrinking again \mathcal{U}' , this implies that $g' \cdot (X' + g'^{-1}u)$ also is as in property \mathcal{P} , and thus so is $X' + g'^{-1}u \in \mathcal{F}_{X'}^{\mathcal{M}}$. \square

Theorem 4.1 (Criterion for density). *Let \mathcal{M} be a non absolute rank one affine manifold having property \mathcal{P} . Then, for any surface X in \mathcal{M} , the leaf $\mathcal{F}_X^{\mathcal{M}}$ is projectively dense in \mathcal{M} .*

Proof. Let (X, ω) be a surface in \mathcal{M} , and chose a surface (X_0, ω_0) in $\mathcal{F}_X^{\mathcal{M}}$ that is as in property \mathcal{P} . Proposition 4.1 implies that the closure of $\mathcal{F}_{X_0}^{\mathcal{M}}$ contains its horocycle orbit. By the commutation property of proposition 2.3, this implies that $\mathcal{F}_{X_0}^{\mathcal{M}}$ contains its horocycle orbit. By proposition 4.2, it implies that the closure of $\mathcal{F}_X^{\mathcal{M}}$ now contains the whole $SL_2(\mathbb{R})$ -orbit of (X, ω) . We conclude using the ergodicity of the action of $SL_2(\mathbb{R})$ on the subset of \mathcal{M} of surfaces whose area is the same as the one of (X, ω) and proposition 2.4. \square

4.2 Structure of the closure of the leaves

The criterion 4.1 allows us to prove structural result on the closure of the leaves.

Theorem 4.2. *Let (X, ω) be a translation surface whose $GL_2^+(\mathbb{R})$ -orbit closure is a non absolute rank one affine manifold with property \mathcal{P} , and let \mathcal{N} be an affine manifold that contains (X, ω) . Then the leaf $\mathcal{F}_X^{\mathcal{N}}$ is projectively dense in an affine manifold \mathcal{N}_0 . Furthermore, if $K_{\mathcal{N}}(X)$ contains m transverse isoperiodic deformations of degree $(d_i)_{i \leq m}$, then :*

$$rk(\mathcal{N}_0) \geq \sum d_i$$

Proof. Denote by \mathcal{M} the $GL_2^+(\mathbb{R})$ -orbit closure of (X, ω) . We shall prove first that the closure of $\mathcal{F}_X^{\mathcal{N}}$ is saturated by $\mathcal{F}^{\mathcal{N}}$: Let $Y \in \overline{\mathcal{F}_X^{\mathcal{N}}}$, and set $A = \{Z \in \mathcal{F}_Y^{\mathcal{N}}, Z \in \overline{\mathcal{F}_X^{\mathcal{N}}}\}$. This set is both closed and open as a subset of $\mathcal{F}_Y^{\mathcal{N}}$, its complementary being $\mathcal{F}_Y^{\mathcal{N}} - \overline{\mathcal{F}_X^{\mathcal{N}}} \cap \mathcal{F}_Y^{\mathcal{N}}$ and since $\mathcal{F}_Y^{\mathcal{N}}$ is connected, the claim is proved. Now, let $Y \in \mathcal{F}_X^{\mathcal{N}}$, and let $g \in SL_2(\mathbb{R})$. By proposition 2.3, $g \cdot Y$ is contained in $\mathcal{F}_{g \cdot X}^{\mathcal{N}}$, and by Theorem 4.1 $g \cdot X$ is contained in the closure of $\mathcal{F}_X^{\mathcal{M}}$, which in turn is contained in the closure of $\mathcal{F}_X^{\mathcal{N}}$. Since the latter is saturated by $\mathcal{F}^{\mathcal{N}}$, we deduce $\mathcal{F}_{g \cdot X}^{\mathcal{N}}$ is contained in the closure of $\mathcal{F}_X^{\mathcal{N}}$, and therefore gY belongs to the closure of $\mathcal{F}_X^{\mathcal{N}}$. This proves that $\mathcal{F}_X^{\mathcal{N}}$ is projectively dense in a affine manifold \mathcal{N}_0 . To prove the last claim of the statement, notice that $K_{\mathcal{N}_0}(X) = K_{\mathcal{N}}(X)$, and denote by u_i be the family of transverse minimal deformations. Denote also by V_i the associated subspace of $V_{\mathcal{N}_0}$ that gives the closure of $\{p(t \cdot u_i), t > 0\}$, and by p_0 the canonical projection from $V_{\mathcal{N}_0}$ to $V_{\mathcal{N}_0}/K_{\mathcal{N}_0}$. We claim that the $p_0(V_i)$ form a direct sum. Indeed, let $v_i \in V_i$ so that $\sum p_0(v_i) = 0$. Then, there is a $k \in K_{\mathcal{N}_0}$ so that $\sum v_i = k$. Let $k_i \in K_{\mathcal{N}_0} \cap V_i$ so that $k = \sum k_i$. We then have $\sum (v_i - k_i) = 0$. Notice that since the u_i are transverse, if two vectors belong to different V_i , their support are disjoint. this implies that $v_i = k_i$, and thus $p_0(v_i) = 0$. We are thus left with proving that the dimension of $p_0(V_i)$ is m_i . First, proposition 1.4.1 of [9] implies that there is a family $v_{i,j}$ of $m_i + 1$ deformations such that $v_{i,0} = v_i$ and the support of $v_{i,j+1}$ is strictly contained in the one of $v_{i,j}$. By definition of

being minimal, this implies that the $v_{i,j}$ are not contained in $K_{\mathcal{N}_0}$, for every $j \geq 1$. This gives that the dimension of $K_{\mathcal{N}_0}/V_{\mathcal{N}_0}$ is bounded below by $\sum d_i$, and since by lemma 3.1 the dimension of $K_{\mathcal{N}_0}/V_{\mathcal{N}_0}$ is bounded above by $rk(\mathcal{N}_0)$. \square

5 Examples of rank 1 affine manifolds with property \mathcal{P}

5.1 The genus 2 case

Proposition 5.1. *Let \mathcal{M} be a non arithmetic rank one affine manifold contained in $\mathcal{H}(1,1)$. Then \mathcal{M} has property \mathcal{P} .*

Proof. Let (X, ω) be a horizontally periodic surface in \mathcal{M} , and up to flowing along the isoperiodic foliation, we can assume that this decomposition is $\mathcal{F}^{\mathcal{M}}$ -stable. That means that (X, ω) is cylinder equivalent (meaning up to adjusting height and twist parameter of the cylinders) to the following surface:

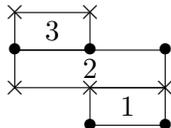


fig. The only stable cylinder decomposition in genus 2

This affirmation can be checked very easily as there are only two possibilities for the diagram of separatrices in this case, and they are isomorphic. See Appendix A for more details. Then the vector $(\gamma_1, -\gamma_2, \gamma_3)$ belongs to $K_{\mathcal{M}}(X)$, and it is necessarily of positive degree or else proposition would imply that \mathcal{M} is arithmetic. \square

We recall that the non absolute rank one affine manifolds have been classified by McMullen in [8], and they are all given by prym eigenform loci. This remark, together with the criterion 4.1, then implies:

Theorem 5.1. *Let (X, ω) be a prym eigenform in $\Omega E_D(1,1)$, then the leaf \mathcal{F}_X is either closed or projectively dense in $\Omega E_D(1,1)$. The latter case occurs if, and only if, D is not a square.*

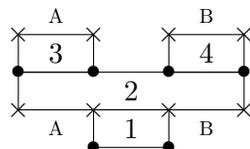
This result was already known prior to that work. It is for instance part of the classification established in [10]. However it was interesting to obtain an elementary proof of this result that does not make use of degeneration of translation surfaces as it is the case in the aforementioned paper.

5.2 Prym eigenform loci in genus 3

Proposition 5.2. *Let \mathcal{M} be a rank 1 non arithmetic affine manifold contained in a Prym eigenform locus of genus three surfaces ΩE_D . If $\dim \mathcal{M} \geq 3$, then \mathcal{M} has property \mathcal{P} .*

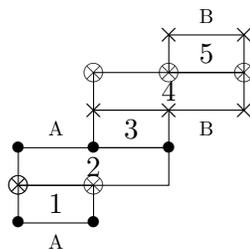
Proof. Notice that the non arithmeticity assumption is equivalent to the fact that D is not a square, as $k(\mathcal{M}) = \mathbb{Q}(\sqrt{D})$. We treat separately the different strata:

- The affine manifold \mathcal{M} cannot be contained in $\Omega E_D^{hyp}(2, 2)$ as this locus is a union a closed $GL_2^+(\mathbb{R})$ -orbits. This is proposition 2.3 of [11].
- If \mathcal{M} is contained in $\Omega E_D^{odd}(2, 2)$, theorem B of [11] states that \mathcal{M} contains a translation surface that cylinder equivalent to the following surface:



Clearly, the vector $v = (\gamma_1, -\gamma_2, \gamma_3, \gamma_4) \in K_{\mathcal{M}}(X)$, and it is minimal. If its degree is 0, that means that all the cylinders have a commensurable circumferences and thus that \mathcal{M} is arithmetic by 2.1, which is not. Then the degree of v is at least one, and \mathcal{M} has property \mathcal{P} .

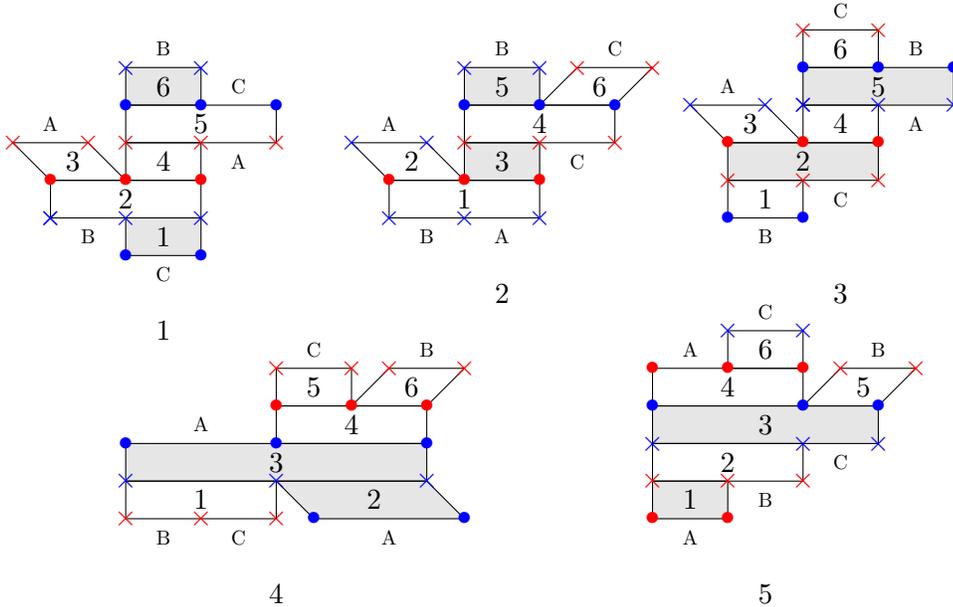
- If \mathcal{M} is contained in $\Omega E_D(2, 1, 1)$, then once again theorem B of [11] states that \mathcal{M} contains a translation surface that cylinder equivalent to the following surface:



In this case the vector $v = (\gamma_1, -\gamma_2, 2\gamma_3, -\gamma_4, \gamma_5)$ lies in $K_{\mathcal{M}}(X)$, and it is minimal. Note that the Prym involution exchanges 1 with 5 and 2 with 4. For the same reason as in the previous case, the degree of this deformation is at least one.

- If \mathcal{M} is contained in $\Omega E_D(1^4)$, let (X, ω) be a horizontally periodic surface in \mathcal{M} . It is well known that those surfaces always exists in affine manifolds, without any assumption on the rank. See for instance [SW]. Up to applying the Rel flow to (X, ω) , we

can assume that the corresponding cylinder decomposition is $\mathcal{F}^{\mathcal{M}}$ -stable. Appendix A, provides a list of such decompositions and thus (X, ω) is cylinder equivalent to one of the following surfaces:



We provide a minimal deformation of positive degree for each of those decompositions.

1. $(\gamma_1, 0, -\gamma_3, -\gamma_4, 0, \gamma_6)$
2. $(\gamma_1, -\gamma_2, 0, \gamma_4, -2\gamma_5, -\gamma_6)$
3. $(\gamma_1, 0, \gamma_3, \gamma_4, -2\gamma_5, \gamma_6)$
4. $(0, \gamma_2, -\gamma_3, 0, \gamma_5, \gamma_6)$
5. $(\gamma_1, 0, -\gamma_3, 0, \gamma_5, \gamma_6)$

□

Proposition 5.2 together with the criterion for density 4.1 imply the following:

Corollary 5.1 (Theorem A). *Let \mathcal{M} be an affine manifold contained in a Prym eigenform loci of genus three surfaces ΩE_D , and let $(X, \omega) \in \mathcal{M}$. Then, $\mathcal{F}_X^{\mathcal{M}}$ is either closed or projectively dense in \mathcal{M} . The latter case occurs if, and only if, D is not a square.*

5.3 The hyperelliptic case

Proposition 5.3. *Let \mathcal{M} be a non absolute rank one affine manifold in $\mathcal{H}(g-1^2)^{hyp}$. Then \mathcal{M} has property \mathcal{P} .*

Proof. Let (X, ω) be a translation surface in \mathcal{M} , and chose a periodic direction on X . Those directions always exist as rank 1 affine manifold have the complete periodicity property. Up to a small isoperiodic deformation, we can ensure that the direction is $\mathcal{F}^{\mathcal{M}}$ -stable (which is equivalent to \mathcal{F} -stable here as they are only two singularities). Note that the singularity present on a boundary component of a cylinder is different from the singularity present on the other boundary component. This is due to the fact that the hyperelliptic involution exchanges the singularities of ω while fixing all the cylinders. A consequence of that is that a minimal isoperiodic deformation is supported on all the cylinders, and is of the form $(\delta_1 c_1^{-1}, \dots, \delta_m c_m^{-1})$ with the δ_i belonging to $\{1, -1\}$. The fact that \mathcal{M} is non arithmetic is equivalent to the fact that this deformation has non vanishing degree. Therefore \mathcal{M} has property \mathcal{P} . \square

Theorem 5.2. *Let \mathcal{M} be a non absolute rank 1 affine manifold contained in $\mathcal{H}(g-1^2)^{hyp}$, and let (X, ω) be a surface in \mathcal{M} . Then the leaf of $\mathcal{F}_X^{\mathcal{M}}$ is either closed or projectively dense in \mathcal{M} . The latter case occurs if, and only if \mathcal{M} is non arithmetic.*

Paul Apisa classified the rank one affine manifolds in the hyperelliptic stratum and showed that if \mathcal{M} is a non arithmetic rank 1 affine manifold, then it is a translation cover of a surface in $\Omega E_D(1, 1)$. See [17].

5.4 A counterexample to property \mathcal{P}

Proposition 5.4. *Let $p : (X, \omega) \rightarrow (X', \omega')$ a translation covering where (X', ω') is a Veech surface in $\mathcal{H}(2g-2)$ and p ramifies over the singularity and a non periodic point. Then, the $GL_2^+(\mathbb{R})$ -orbit closure \mathcal{M} of (X, ω) has dimension 3 but does not have property \mathcal{P} .*

6 Ergodic theory of the isoperiodic foliation

6.1 Torelli covering of the Prym loci

let (X_0, ω_0) be a genus g Prym eigenform endowed with a marking f_0 , and denote by τ_0 the Prym involution. Define \mathfrak{P} be the set of isomorphism classes of tuples (X, ω, f, τ) , where X is a genus g Riemann surface, ω is a non vanishing holomorphic 1-form on X , f a homeomorphism from S to X , and τ is a holomorphic involution of X . $\tau^*(\omega) = -\omega$. We require that the map induced on homology by $f^{-1} \circ \tau \circ f$ is the same as the one induced by $f_0^{-1} \circ \tau_0 \circ f_0$. We say that $(X_1, \omega_1, f_1, \tau_1)$ and $(X_2, \omega_2, f_2, \tau_2)$ are isomorphic if there is a biholomorphism $\varphi : X_1 \rightarrow X_2$ such that:

1. $\varphi^*\omega_2 = \omega_1$
2. $f_2^{-1} \circ \varphi \circ f_1$ is isotopic to the identity of S .
3. $(\varphi \circ \tau_1)_* = (\tau_1 \circ \varphi)_*$

Denote by $\mathfrak{p} : \mathfrak{P} \rightarrow \mathcal{H}(\kappa)$ the canonical projection, by $\Gamma_{\mathfrak{P}}$ the subgroup of the mapping class group that commutes to $f_0^{-1} \circ \tau_0 \circ f_0$ at the level of the homology, and by $\Gamma_{\mathfrak{P}}^0$ the subgroup of $\Gamma_{\mathfrak{P}}$ the subgroup that acts as the identity on $H_1^-(S)$, where $H_1^-(S)$ is defined as the the subspace of $H_1(S)$ that is $f_0^{-1} \circ \tau_0 \circ f_0$ anti invariant. Notice that the range of \mathfrak{p} is contained in $\text{Prym}(\kappa)$. As the canonical projection from $\tilde{\mathcal{H}}(\kappa)$ to $\mathcal{H}(\kappa)$, the map \mathfrak{p} is a local homeomorphism, and it has the path lifting property. Finally, define the following map:

$$\Psi : \begin{array}{ccc} \mathfrak{P}/\Gamma_{\mathfrak{P}}^0 & \rightarrow & H^1(S, \mathbb{C})^- \\ (X, \omega, f, \tau) & \mapsto & (\gamma \mapsto \int_{f \circ \gamma} \omega) \end{array}$$

6.2 Transfer Principle for Prym eigenform loci

Let $\mathcal{M} \subset \Omega E_D(\kappa)$ be the affine manifolds generated by (X_0, ω_0) as in the previous subsection, where D is the discriminant of $k(\mathcal{M})$. Let $\sigma \in \text{Gal}(k(\mathcal{M})/\mathbb{Q})$, the non trivial element of the galois group of $k(\mathcal{M})/\mathbb{Q}$. Chose a symplectic basis \mathcal{B} of $H_1(S)$ and define $\mathcal{K}_{\mathcal{M}} = \{p \in H^1(S, \mathbb{C})^-, W_p^\sigma \subset W_p^\perp\}$ where W_p is the $k(\mathcal{M})$ -vector space generated by the entries of $\text{Re}(p)$ and $\text{Im}(p)$. Notice that $\Psi(\mathfrak{p}^{-1}\mathcal{M}) = \mathcal{K}_{\mathcal{M}}$. Denote by $\Psi_{\mathcal{M}}$ the restriction of Ψ to $\mathfrak{p}^{-1}\mathcal{M}$. This map is an analog of the Period map of Calsamiglia, Deroin and Francaviglia in [10].

Proposition 6.1. *Two surfaces (X_1, ω_1) and (X_2, ω_2) lie in the same leaf of $\mathcal{F}^{\mathcal{M}}$ if and only if, there is path γ in $\mathfrak{P}/\Gamma_{\mathfrak{P}}^0$ such that:*

1. $\forall t \mathfrak{p} \circ \gamma(t)$ belongs to \mathcal{M} .
2. $\mathfrak{p} \circ \gamma(0) = (X_1, \omega_1)$ and $\mathfrak{p} \circ \gamma(1) = (X_2, \omega_2)$
3. $\Psi_{\mathcal{M}} \circ \gamma$ is constant.

Lemma 6.1. *The following formula holds: $\mathfrak{p}(\Phi_{\mathcal{M}}^{-1}(\mathcal{K}_{\mathcal{M}})) = \mathcal{M}$.*

Proof. The proof is decomposed in three steps.

1. $\mathfrak{p}(\Psi_{\mathcal{M}}^{-1}(\mathcal{K}_{\mathcal{M}}))$ is saturated by the isoperiodic foliation.
2. $\mathfrak{p}(\Psi_{\mathcal{M}}^{-1}(\mathcal{K}_{\mathcal{M}}))$ is $GL_2^+(\mathbb{R})$ invariant
3. $\mathfrak{p}(\Psi_{\mathcal{M}}^{-1}(\mathcal{K}_{\mathcal{M}}))$ is closed.

The first claim is a direct consequence of proposition 6.1, while the second and the third claim are given by Theorem 3.2 of [7]. By definition, $\mathfrak{p}(\Phi_{\mathcal{M}}^{-1}\mathcal{K}_{\mathcal{M}})$ is contained in \mathcal{M} . Claims 1 and 2 together with 2.4 show that it is open in \mathcal{M} . Since it is non empty, this is \mathcal{M} , as a non empty closed and open subset. \square

Lemma 6.2. *If $\mathcal{F}^{\mathcal{M}}$ is projectively minimal, then the fibers of $\Phi_{\mathcal{M}}$ are connected.*

Proof. Let $p \in \mathcal{K}_{\mathcal{M}}$ be the periods of (X, ω, f, τ) , and chose $(X_1, \omega_1, f_1, \tau_1)$ in $\Psi_{\mathcal{M}}^{-1}(p)$. Since the foliation $\mathcal{F}^{\mathcal{M}}$ is projectively minimal and that the surface (X, ω) and (X_1, ω_1) have same area, there is a path in $\mathcal{F}_X^{\mathcal{M}}$ that starts at (X, ω) and ends arbitrarily close to (X_1, ω_1) . This path can be lifted to a path $\tilde{\gamma}$ on \mathfrak{P} starting from (X, ω, f, τ) , and it ends a surface close to $\varphi \cdot (X_1, \omega_1, f_1, \tau_1)$, for a $\varphi \in \Gamma_{\mathfrak{P}}$. By construction $\Phi_{\mathcal{M}} \circ \tilde{\gamma}(1) = p$ and $\Phi_{\mathcal{M}}(\varphi \cdot (X_1, \omega_1, f_1, \tau_1)) = \varphi \cdot p$. As these two are arbitrarily close, this is not possible unless $\varphi \cdot p = p$. This implies that φ acts as the identity on W_p , and since it commutes with the action of the Galois group, it also acts as the identity of W_p^{σ} . But, by construction $W_p \oplus W_p^{\sigma}$ is exactly the τ anti-invariant subspace of \mathbb{R}^{2g} , and thus φ acts as the identity of $H^1(S, \mathbb{C})^{-}$, that is $\varphi \in \Gamma_{\mathfrak{P}}^0$. In particular $(X_1, \omega_1, f_1, \tau_1)$ and $\varphi \cdot (X_1, \omega_1, f_1, \tau_1)$ are the same point in $\mathfrak{P}/\Gamma_{\mathfrak{P}}^0$, and $\tilde{\gamma}$ is a path in $\Psi_{\mathcal{M}}^{-1}(p)$ that connects (X, ω, f, τ) to $(X_1, \omega_1, f_1, \tau_1)$. \square

Corollary 6.1 (Theorem C). *If $\mathcal{F}^{\mathcal{M}}$ is projectively minimal, then it is $\lambda_{\mathcal{M}}$ -ergodic.*

Proof. Lemma 6.1 together with lemma 6.2 imply that there is a bijection between $\mathcal{F}^{\mathcal{M}}$ -saturated borel subsets of \mathcal{M} and $\Gamma_{\mathfrak{P}}$ -invariant borel subsets of $\mathcal{K}_{\mathcal{M}}$ given by $\mathcal{B} \mapsto \Psi_{\mathcal{M}}(\mathfrak{p}^{-1}\mathcal{B})$. Up to the choice of a base, the invariant sets of the action of $\Gamma_{\mathfrak{P}}$ on the area 1 locus of $H^1(X, \mathbb{C})^{-}$ is in correspondence with the action of $Sp_4(\mathbb{Z})$ on $Sp_4(\mathbb{R})/Sp_2(\mathbb{R})$, via $p \mapsto M_p$ where M_p is the matrix that maps $(1, 0, 0, 0)$ to $\mathcal{R}e(p)$ and $(0, 1, 0, 0)$ to $\mathcal{I}m(p)$. This matrix is well defined only up to the stabilizer of $((1, 0, 0, 0), (0, 1, 0, 0))$ which is isomorphic to $Sp_2(\mathbb{R})$. Note that $\mathcal{R}e(p)$ and $\mathcal{I}m(p)$ are seen as elements of the dimension 4 subspace $H^1(X, \mathbb{C})^{-}$. The study of these invariant sets has been carried out by Calsamiglia, Deroin and Francaviglia after Kapovitch in [10]. We deduce our result from their classification. \square

We conclude this section by noticing that Theorem B is implied by theorem A together with C.

7 Behavior of the full isoperiodic leaves

In this section we compute the closure of the leaves or Prym eigenforms of genus 3

7.1 The case $\mathcal{H}^{odd}(2, 2)$

Theorem 7.1. *Let (X, ω) be a prym eigenform in $\mathcal{H}^{odd}(2, 2)$, then the leaf \mathcal{F}_X either closed or projectively dense in the connected component of $\Omega E_D^{odd}(2, 2)$ in which it belongs. The last case occurs if, and only if, D is not a square.*

Proof. Notice that in this case $\mathcal{F}_X = \mathcal{F}_X^{\mathcal{M}}$ where \mathcal{M} is the connected component of $\Omega E_D^{odd}(2, 2)$ that contains (X, ω) . Therefore, the result is just a consequence of criterion 4.1. \square

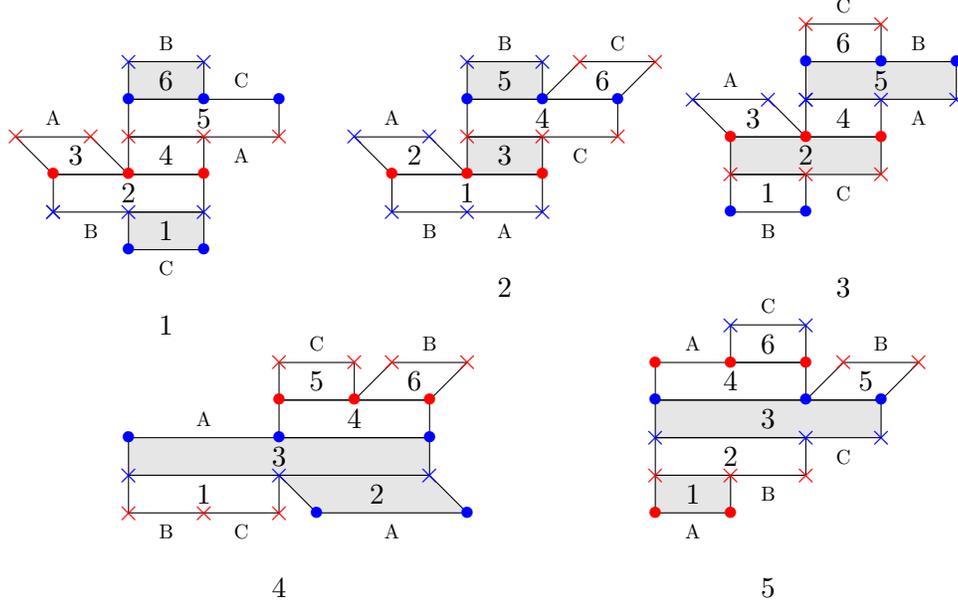
7.2 The case $\mathcal{H}_3(\kappa)$, $|\kappa| > 2$

Theorem 7.2. *Let (X, ω) be a prym eigenform in $\mathcal{H}_3(\kappa)$ with $|\kappa| > 2$, then the leaf \mathcal{F}_X is either closed or projectively dense. The last case occurs if, and only if, D is not a square.*

Proof. To prove this result, we are going to use theorem 4.2 and additional computations to prove that if the leaf of (X, ω) is not closed, it has to be projectively dense in an affine manifold \mathcal{M} of rank at least 2. However, the classification of Aulicino and Nguyen of rank 2 affine manifolds ([3]) shows that if an affine manifold is \mathcal{F} -saturated, as is \mathcal{M} , it cannot be rank 2. Thus the rank is 3, and by [19], it has to be the whole stratum, since the hyperelliptic locus is not saturated by \mathcal{F} neither.

If \mathcal{M} is contained in $\mathcal{H}(2, 1, 1)$, then we saw in the course of the proof of proposition 5.2, that we can assume that (X, ω) is horizontally periodic with 5 cylinders, and that $u = (\gamma_1, -\gamma_2, \gamma_3, 0, 0)$ and $v = (0, 0, \gamma_3, -\gamma_4, \gamma_5)$ belong to $K_{\mathcal{M}}(X)$ and have a positive degree, with $\gamma_1 = \gamma_5$, and $\gamma_2 = \gamma_4$. The projection of $K_{\mathcal{M}}(X)$ on the two first coordinates is a 2-dimensional space, or c_1 and c_2 would be \mathbb{Q} -dependant, and the relation $c_2 = c_1 + c_3$ would imply that the degree of u vanishes. Similarly, the projection of $K_{\mathcal{M}}(X)$ on the two last factors is also a 2-dimensional space. In particular, the dimension of $K_{\mathcal{M}}(X)$ is at least 4. By lemma 3.1, we have $r + 2 \geq 4$, and thus the rank of \mathcal{M} is at least 2.

If \mathcal{M} is contained in $\mathcal{H}(1^4)$, we saw that (X, ω) is cylinder equivalent to one of the following surfaces:



We provide a pair of transverse deformations for each of the first four decomposition:

1. $(\gamma_1, 0, 0, 0, -\gamma_5, \gamma_6)$ and $(0, \gamma_2, -\gamma_3, -\gamma_4, 0, 0)$
2. $(\gamma_1, -\gamma_2, 0, 0, -\gamma_5, 0)$ and $(0, 0, \gamma_3, -\gamma_4, 0, \gamma_6)$
3. $(\gamma_1, 0, 0, 0, -\gamma_5, \gamma_6)$ and $(0, \gamma_2, -\gamma_3, -\gamma_4, 0, 0)$
4. $(\gamma_1, 0, 0, 0, -\gamma_5, -\gamma_6)$ and $(0, \gamma_2, -\gamma_3, \gamma_4, 0, 0)$

Now, if (X, ω) is cylinder equivalent two the fifth surface, then $K_{\mathcal{M}}(X)$ contains the following three vectors $(\gamma_1, 0, 0, -\gamma_4, 0, \gamma_6)$, $(0, \gamma_2, 0, -\gamma_4, -\gamma_5, \gamma_6)$ and $(0, 0, \gamma_3, -\gamma_4, -\gamma_5, 0)$, and each of them is of positive degree. Using the same argument as in the case where $\mathcal{M} \subset \mathcal{H}(2, 1, 1)$, we conclude that $k + 3 \geq 5$, and thus the rank of \mathcal{M} has to be at least 2. \square

A Stable cylinder decompositions in $\text{Prym}(1, 1, 1, 1)$

A.1 diagram of separatrices

We recall from [12] the common framework for enumeration of cylinder decomposition.

Definition A.1. *A prediagram of separatrices is a quadruplet $\Gamma = (E, \sigma, \tau, \theta)$, where τ is a fixed point free involution of E , σ is a permutation of E and θ is a map from $E/\tau \rightarrow E$ such that $p \circ \theta = id$, where $p : E \rightarrow E/\tau$ is the canonical projection.*

The elements of E are called edges. An edge γ in E is said to be positively oriented if $\theta \circ p(\gamma) = \gamma$ and negatively oriented if $\theta \circ p(\gamma) = \tau(\gamma)$. The set of positively oriented edges will be denoted by E_+ and the set of negatively oriented edges will be denoted by E_- . A cylinder component is defined as an orbit of $\sigma_\infty := \sigma \circ \tau$. Such a cylinder component is said to be positively oriented if it corresponds to a positively oriented edge, and negatively oriented otherwise. We denote by \mathcal{C}_+ and \mathcal{C}_- the set of positively and negatively oriented cylinder components. A pairing of cylinder components is a bijection from \mathcal{C}_+ to \mathcal{C}_- . Finally, we define a metric on Γ as a strictly positive τ -invariant function l on E , and we consider its natural extension \hat{l} to the set of cylinder components defined by $\hat{l}(c) : \sum_n l(\sigma_\infty^n(\gamma))$, where c is the cylinder component associated to γ . Such an object can be encoded by a directed graph with additional information. Its set of vertices is the set of orbits of σ , and its set of edges is E/τ . Set $p_0 : E \rightarrow E/\sigma$ to be the canonical projection. The beginning of an edge is $p_0 \circ \theta(e)$, and its end is $p_0 \circ \tau \circ \theta(e)$. There is an cyclic ordering on the star of each vertex define by σ .

Definition A.2. *A diagram of separatrices (Γ, m, l) is the data of an alternating prediagram of separatrices Γ , together with a matching of its cylinder components and a metric l on Γ that is invariant by m , that is $\hat{l} \circ m = \hat{l}$.*

To any translation surface $(X, \omega) \in \mathcal{H}(\kappa)$, there is a canonical diagram of separatrices $\Gamma(X, \omega)$ associated. It is defined as follows : The set E is the collection of all geodesics rays $\gamma : [0, 1] \rightarrow X$, such that $\gamma^{-1}(\Sigma) = \{0, 1\}$, and $\gamma^*(Im(\omega)) = 0$, taken up to reparametrization. We define τ to be the orientation reversing map : $\tau(\gamma)(t) : \tau(\gamma)(1 - t)$. For any $\gamma \in E$, there is a chart around $\gamma(0)$ that takes ω to $z^k dz$; where k is the order of the singularity. The germs of geodesic ray that begins at $\gamma(0)$ is thus invariant by multiplication by $e^{\frac{2i\pi}{k+1}}$, and the germ of a geodesic ray at its beginning completely determines it. We thus define $\sigma(\gamma)$ to be the geodesic ray whose germ at its beginning is the one of $e^{\frac{2i\pi}{k+1}}\gamma$. Finally, to define θ we need to define a map that is τ -invariant. If γ is in E , we define $\theta(\gamma) = \gamma$ if $\gamma^*(Re(\omega)) > 0$, and $\theta(\gamma) = \tau(\gamma)$ otherwise. The metric on $\Gamma(X, \omega)$ is given by $l(\gamma) = |\int_\gamma \omega|$. Note that the orbits of σ_∞ are in correspondence with oriented core curves of cylinders. The matching m is then defined to map the orbit that corresponds to the positively oriented core curve to the one that corresponds to the negatively oriented core curves of the same cylinder. In the reverse direction, to any diagram of separatrices

Γ there is horizontally periodic translation surface $(X_\Gamma, \omega_\Gamma)$. It is defined in the following way: Replace any $\gamma \in \theta(E/\tau)$ by a strip of length $l(\gamma)$, and replace any element in E/σ by a disk. The permutation σ defines a cyclic ordering on any orbit. Using this ordering, one can glue the strips to the disks. We get a topological surface with boundary, and the boundary components correspond to orbits of σ_∞ . Since by requirement, the paired components have same length (defined by \hat{l}) we can glue those components using the pairing, and we get a horizontally periodic flat surface in $\mathcal{H}(\kappa)$, where $|\kappa|$ is the cardinal of E/σ and k_i is half the cardinal of the orbit i minus 1. The fact that the cardinal of an orbit is even comes from the alternating condition.

Two prediagrams of separatrices $\Gamma_1 = (E^1, \tau_1, \sigma_1, \theta_1)$ and $\Gamma_2 = (E^2, \tau_2, \sigma_2, \theta_2)$ are isomorphic if there is a map $\varphi : E^1 \rightarrow E^2$ such that $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$, $\varphi \circ \tau_1 = \tau_2 \circ \varphi$, and $\varphi(E_+^1) = E_+^2$, or equivalently $\varphi \circ \theta_1 = \theta_2$. Finally, two diagrams of separatrices (Γ_1, l_1, m_1) and (Γ_2, l_2, m_2) are isomorphic if there is an isomorphism of prediagram of separatrices φ between the two such that $l_2 \circ \varphi = l_1$, and if two cylinders components on Γ_1 associated to the orbits of γ_1 and γ_2 are paired by m_1 , then the cylinder components of Γ_2 associated to $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ are paired by m_2 .

Proposition A.1. *Two horizontally periodic translation surfaces (X_1, ω_1) and (X_2, ω_2) in $\mathcal{H}(\kappa)$ are isomorphic to cylinder equivalents surfaces if, and only if, the associated diagrams of separatrices are isomorphic.*

We refer to [18] for a proof of that result.

Definition A.3. *A connected component of a prediagram of separatrices of $\Gamma = (E, \sigma, \tau, \theta)$ is a subset E' of E that is preserved by both σ and τ .*

If E' is a connected component of Γ , there is an induced prediagram of separatrices $(E', \sigma|_{E'}, \tau|_{E'}, \theta|_{p(E')})$.

Definition A.4. *A prediagram of separatrices $(E, \tau, \sigma, \theta)$ is said to be stable if τ preserves the orbits of σ .*

In this setting a minimal connected component is said to be minimal if σ acts transitively on E . We can define the type a minimal connected component as follows : Let x be a positively oriented edge of the minimal component. Thus any positively oriented edges can be written as $\sigma^{2k}(x)$, and any negatively oriented edge can be written as $\sigma^{c_n(2l)}(x)$. Since τ reverses the orientation, there is a map $f \in \mathfrak{S}_n$ such that for any $k : \tau \circ \sigma^{2k}(x) = \sigma^{c_n(2f(k))}(x)$. The group $H = \langle c_n \rangle$ acts by conjugation on \mathfrak{S}_n and the type of the component is defined as the orbit of f under this action. The fact that we defined f up to conjugation by H comes from the fact we could have chosen any other even iteration of x as a generator of the orbit. More generally, we say that Γ is of type (f_i) if the type of the minimal connected components are given by the f_i .

Proposition A.2. *Two stable alternating prediagrams are isomorphic if, and only if, they have the same type.*

Proof. It is enough to prove the result for minimal connected components : Let $\Gamma_i = (E_i, \sigma_i, \tau_i, \theta_i)$ for $i \in \{1, 2\}$ be two minimal prediagrams of same type. Pick x_i in E_i that is positively oriented and define f_i such that $\tau_i(\sigma^{2k}(x_i)) = \sigma^{2(f_i(k)+1)}$. Saying that the diagrams have the same type means there is l such that $f_2 = c_n^l \circ f_1 \circ c_n^{-l}$. Define φ such that for all j $\varphi(\sigma_1^j(x_1)) = \sigma_2^{2l+j}(x_2)$. We claim that φ is an isomorphism of prediagram. Indeed, let $\zeta \in E_1$, and pick j such that $\sigma^j(x_1) = \zeta$. Thus $\varphi \circ \sigma_1(\zeta) = \varphi \circ \sigma^{j+1}(x_1) = \sigma_2^{2l+j+1}(x_2) = \sigma_2(\sigma_2^{2l+j}(x_2)) = \sigma_2 \circ \varphi(\sigma_1^j(x_1)) = \sigma_2 \circ \varphi(\zeta)$. Then, $\tau_2 \circ \varphi(\sigma_1^{2k}(x_1)) = \tau_2 \circ \sigma_2^{2(k+l)}(x_2) = \sigma_2^{2f_2(k+l)+1}(x_2) = \sigma_2^{2c_n^l \circ f_1 \circ c_n^{-l}(k+l)+1}(x_2) = \sigma_2^{2(f_1(k)+l)+1}(x_2) = \varphi \circ \sigma_1^{2f_1(k)+1} = \varphi \circ \tau_1(\sigma^{2k}(x_1))$. By construction, $\varphi(E_+^1) = E_+^2$. Reciprocally, if there is an isomorphism of prediagrams between Γ_1 and Γ_2 , then $\varphi(x_1) = \sigma_2^{2l}(x_2)$, and then $f_2 = c_n^l \circ f_1 \circ c_n^{-l}$. \square

We denote by $\bar{\Gamma}$ the prediagram $(E, \sigma, \tau, \tau \circ \theta)$. if Γ corresponds to a surface (X, ω) , then $\bar{\Gamma}$ corresponds to the surface $(X, -\omega)$. Note that if (X, ω) is represented by a polygon P, the surface $(X, -\omega)$ is represented by the image of the polygon by the rotation of angle π .

Proposition A.3. *Let Γ be a minimal stable alternating prediagram of type f . The type of $\bar{\Gamma}$ is given by $(f \circ c_n)^{-1}$.*

Proof. Let $x \in E$ be positively oriented, and such that $\tau \circ \sigma^{2k}(x) = \sigma^{2f(k)+1}$. Then $\sigma(x)$ is positively oriented on $\bar{\Gamma}$. And $\tau \circ \sigma^{2j}(\sigma(x)) = \sigma^{2f^{-1}(j)}(x) = \sigma^{2c_n^{-1}(f^{-1}(j))}(\sigma(x))$. \square

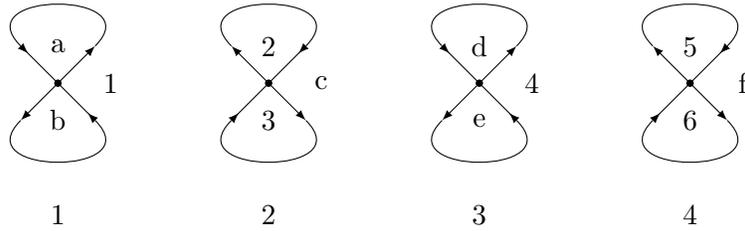
A.2 Combinatorial Prym involution

In this appendix we enumerate the possible cylinder decompositions for a surface in the Prym locus $Prym(1, 1, 1, 1)$. The following definition is the combinatorial version of the Prym involution.

Definition A.5. *A combinatorial Prym involution on a stable diagram of separatrices Γ is an isomorphism τ of diagram between Γ and $\bar{\Gamma}$ whose square the identity and such that $\#Fix(\tau) + 2\#Fix(\tau \circ m) = 10 - 2g$, where g is the genus of the surface associated to Γ .*

Proposition A.4. *If a horizontally periodic translation surface with a \mathcal{F} -stable decomposition has a Prym involution, then the morphism of diagram of separatrices it induces is a combinatorial Prym involution. Reciprocally, If a stable diagram of separatrices has a Prym involution, then the associated surface has a Prym involution.*

This proposition enables us to enumerate the cylinder decomposition in Prym loci. In $Prym(1^4)$, the Prym involution does not fix any singularity, so it fixes 2 cylinders. We will say that the diagram associated is of type 1 if both the cylinders are bordered by the same pair of exchanged singularities, and of type 2 if the cylinders are not bordered by singularity that are not exchanged by the involution. Here, the type of the diagram is necessarily $(id_{\mathfrak{S}_2}, id_{\mathfrak{S}_2}, id_{\mathfrak{S}_2}, id_{\mathfrak{S}_2})$, and the Prym involution forces the following orientation:

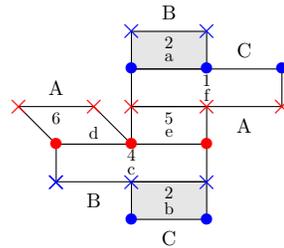


The prediagram of separatrices in $\text{Prym}(1^4)$

We can assume that the involution τ exchanges the two first singularity and the two last, that is $\tau(1) = 2$ and $\tau(3) = 4$, and that it maps 2 to a and 4 to f . Other choices would produce isomorphic diagrams.

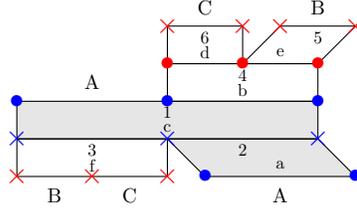
A.3 Stable decompositions of the first type in $\text{Prym}(1^4)$

- The first possible outcome is $m(a) = 2$ and $m(b) = 3$. Then $m(c)$ can be 4, 5 or 6. It cannot be 1 as this would give an extra fixed cylinder in contradiction with the Prym condition. The case 5 and 6 are isomorphic. So we only need to consider the case where $m(c)$ is 4 or 5. If it is 4, then, after applying τ_0 we get $m(f) = 1$. $m(d)$ is either 5 or 6, but the case $m(d) = 5$ produces one extra fixed cylinder, thus and then we get $m(d) = 6$, and then $m(e) = 5$. This gives that $m = (fabced)$. The other case is $m(c) = 5$. Then $m(d) = 1$. Now $m(e) = 4$ or 6 but only $m(e) = 6$ gives a metric solution, and this gives an extra fixed cylinder so this case is not possible.



the surface $(fabced)$

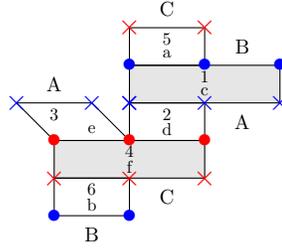
- The second possible outcome is $m(c) = 1$ and $m(a) = 2$ (or $m(b) = 3$ but this case is isomorphic). Then $m(f)$ cannot be 5 or 6 for metric reasons, and it cannot be 4 either as this would produce an extra fixed cylinder. This means that $m(f) = 3$. Thus by the Prym involution, $m(b) = 4$. Now, $m(d)$ is either 5 or 6 but 5 gives an extra cylinder. Thus $m(d) = 6$, and $m(e) = 5$. The only possibility here is $(cafbed)$.



the surface $(cafbcd)$

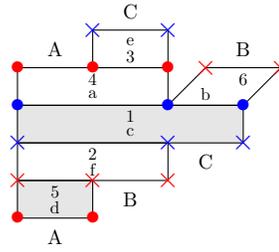
A.4 Stable decompositions of the second type in $\text{Prym}(1^4)$

- The first case is $m(c) = 1$ and $m(f) = 4$. Then $m(a)$ is either 3, 5 or 6. The last two cases are isomorphic. Let's consider first the case where $m(a) = 3$, then $m(2) = b$, and $m(d) = 6$ and $m(e) = 5$. So we get $m = (cbafed)$, but this doesn't give a connected surface, so this case does not exist. In the case where $m(a) = 5$. Then $m(d) = 2$, and then since there are not more than 2 fixed cylinders, $m(e) = 3$ and $m(b) = 6$. Thus we get $m = (cdefab)$.



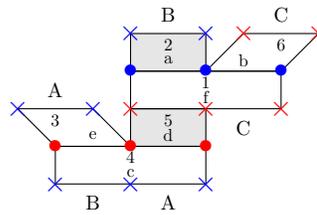
the surface $(cdefab)$

- Then $m(c) = 1$ and $m(d) = 5$ (the case where it $m(e) = 6$ instead of $m(d) = 5$ is isomorphic). If $m(b) = 3$ there is one extra fixed cylinder. If $m(b) = 2$ then $m(a) = 3$ and the surface is disconnected. Thus $m(b)$ is 4 or 6. In the first case, by applying the involution $m(f) = 3$. Likewise $m(a)$ cannot be 2 or there would be too many fixed cylinders thus $m(a) = 6$ and thus $m(e) = 2$. We get $m = (cefbda)$. If $m(b) = 6$, then $m(e) = 3$. $m(a)$ cannot be 2 as this would produce another fixed cylinder. thus $m(a) = 4$, and $m(f) = 2$. We get $m = (cfeadb)$. But this one is isomorphic to $(cefbda)$ by exchanging 2 with 3 and a with b.



the surface $(cfeadb)$

- Then $m(a) = 2$ and $m(d) = 5$. The other cases, like $m(f) = 4$ and $m(a) = 2$, are isomorphic, so we don't consider them. $m(c)$ cannot be neither 1, 2 nor 3. it is thus 4 or 6. If it is 4, then after applying the involution $m(f) = 1$. $m(b)$ cannot be 3, thus $m(b) = 6$ and $m(e) = 3$, and $m = (faecdb)$. If $m(c) = 6$, then $m(e) = 1$. $m(b)$ cannot be 3 thus $m(b) = 4$ and $m(f) = 3$. We get $m = (eafbdc)$. But this one do not have a metric solution. Indeed, we should have $l(e) = l(1)$ and $l(b) = l(4)$, but $l(1) > l(b) = l(4) > l(e)$.



the surface $(faecdb)$

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Chapter 3

Non arithmetic affine manifolds in $\mathcal{H}(2, 2)$ & $\mathcal{H}(3, 1)$

Non arithmetic affine manifolds in $\mathcal{H}(2, 2)$ and $\mathcal{H}(3, 1)$

Florent Ygouf

April 29, 2019

Abstract

We classify the non arithmetic rank one affine manifolds that do not arise from Veech surfaces in $\mathcal{H}(3, 1)$ and $\mathcal{H}^{odd}(2, 2)$. We also give rigidity results on the isoperiodic leaf of non arithmetic Veech surfaces.

1 Introduction

1.1 Context and results

The moduli space of pairs (X, ω) where X is a genus $g \leq 1$ Riemann surface and ω is a non vanishing holomorphic 1-form on X carries a natural action by $GL_2^+(\mathbb{R})$ which is a generalization of the action of $GL_2^+(\mathbb{R})$ on the space of flat tori $GL_2^+(\mathbb{R})/SL(2, \mathbb{Z})$. This action preserves the stratification of the moduli space induced by the combinatorics of the singularities, and the classification of the closed invariant sets of the strata is a central problem in Teichmüller dynamics. Such a classification in genus 2 has been initiated by K. Calta in [13] and by McMullen in [8]. In particular, the latter proved that the orbit of a surface (X, ω) is either closed, dense, or contained in a locus of surfaces whose jacobian have a special property called "Real multiplication by a quadratic order", with ω being an eigenform. See section 2 for more details. McMullen also discovered a generalization of those objects in higher genera and provided an infinite sequence of non trivial closed invariant sets ΩE_D , parametrized by their discriminant D . Those loci will play a central role in the remainder of this text. The question of the classification of closed orbit in genus 2 has as been addressed by P. Hubert and S. Lelièvre in [21], then later by E. Duryev in [20], but is still incomplete. E. Lanneau P. Hubert and M. Möller have contributed to the classification in genus 3 in [23]. Since, much effort has been made toward a classification in higher genera, but the question is still wide open. However, Eskin, Mirzakhani and Mohammadi proved a deep structural result on the closed invariant sets: they are immersed manifolds cut out by linear equations with real coefficients. Such objects are usually referred to affine manifolds. This result opened the way to new powerful tools, and has been the starting point of every classification result obtained so far. Wright strengthened the conclusion of this result and

proved that the coefficients of equations defining the manifold belongs to a number field whose degree is bounded above by the genus. This number field will be referred to as the field of definition. Wright also introduced an important numerical invariant called the rank $rk(\mathcal{M})$. This is a modified version of the dimension and it measures the size of affine manifolds up to isoperiodic deformations. See [1] for more details. Mirzhakani conjectured that arithmetic affine manifolds whose rank is bigger than 2 should arise from covering construction over quadratic differentials. Arithmetic means here that the field of definition is \mathbb{Q} . This conjecture is now known to be false due to work of A. Eskin, C. T. McMullen, R. E. Mukamel and A. Wright in [22] but counterexamples are expected to be rare. Mirzakhani and Wright proved in [18] that the only affine manifolds of maximal rank are the strata themselves and the hyperelliptic locus of those strata. Then, Apisa proved in a serie of papers [16] and [17] that the orbits of translation surfaces in the hyperelliptic strata are either closed, dense, or are contained in loci of branched covers. Finally, a classification of affine manifolds with $rk(\mathcal{M}) \geq 2$ in genus 3 has been obtained by Nguyen and Aulicino in [4] and [3]. In this paper, we pursue the classification in genus 3, and prove :

Theorem A. *Let \mathcal{M} be a proper non arithmetic affine manifold in $\mathcal{H}^{odd}(2, 2)$. Then $k(\mathcal{M})$ is a totally real quadratic number field of discriminant D and \mathcal{M} is a connected component of $\Omega E_D(2, 2)$.*

Here, an affine manifold is said to be proper if it is either a closed orbit or the whole statum. The connected components of $\Omega E_D^{odd}(2, 2)$ have been classified by Lanneau and Nguyen in [9]. In particular, this gives an alternative proof in this setting of the fact that $k(\mathcal{M})$ is totally real. In the stratum $\mathcal{H}(3, 1)$, the situation is different, and we prove :

Theorem B. *There are no proper non arithmetic affine manifold in $\mathcal{H}(3, 1)$.*

Notice that to establish theorem A and B, we only need to consider rank one affine manifolds as lemma 6.5 of [2] implies that if \mathcal{M} is non arithmetic, then $rk(\mathcal{M}) = 1$.

1.2 Outline of the proof

The techniques we use rely on deformation, known as cylinder deformations, of the flat geometry of translation surfaces initiated by Wright in [1]. For a horizontally periodic translation surface (X, ω) , the horocycle flow (the action of the one parameter subgroup of unipotent matrices in $SL_2(\mathbb{R})$ and the Rel flow (see section 3 for more details) read as linear flows on a torus. Their orbit closures are thus given by rational subtori and their equations are linked to the equations defining the affine manifold \mathcal{M} . The non arithmeticity of the latter is a necessary condition for non trivial intersection of the subtori. This non triviality implies that equations defining the two are not independent. In particular, since the equations given by the Rel flow only involve the circumferences of the cylinders while the equations induced by the horocycle flow are expressed with the circumferences and the height of the cylinders, one would expect equations on the height. This is the

object of proposition 3.2. To implement this strategy, we crucially need the existence of cylinder decompositions. This is obtained through the complete periodicity property that prevails in rank 1. More details can be found in section 3. However, this strategy fails if there are what we call non mixed cylinders: those are cylinders not affected by the isoperiodic deformations. The circumferences of such cylinders therefore do not intervene in the equations obtained. In appendix A, we give a list of the cylinder decompositions in the strata at stakes. This allows us to show that there is always a decomposition with only mixed cylinders in $\mathcal{H}(2, 2)$. We then look at how the equations on the heights given by proposition 3.2 transpose for the surfaces in this list: they rule out all the surfaces except one. We repeatedly apply the same argument in different direction keeping in mind that equations on the heights in one direction give equations on the circumferences in other directions. After having collected all the equations obtained, we then show that the surface has a prym involution. We conclude with a criterion given by McMullen to recognizes Prym eigenforms. For the stratum $\mathcal{H}(3, 1)$, the situation is different and we cannot rule out the existence of non mixed cylinders. We show that this translates in a trivial intersection of the subtori, which in turn implies the commensurability of the mixed cylinders in a way that is not compatible with non arithmeticity.

1.3 Organisation of the paper

We start by recalling basic definitions for the moduli space of translation surfaces in Section 2. In section 3 we collect the important results we will use in the course of our proofs. Section 3.4 is dedicated to the proof of Theorem A. We also draw a corollary on the leaf of Veech surfaces in $\mathcal{H}^{odd}(2, 2)$. Section 5 is dedicated to the proof Theorem B, and we draw the same corollary as in the previous section in the stratum $\mathcal{H}(3, 1)$. Finally, Appendix A provides a list of the possible stable cylinder decompositions in $\mathcal{H}(2, 2)$ and $\mathcal{H}(3, 1)$.

2 Framework

Let $g \geq 1$ and let κ be an integer partition of $2g - 2$. We define $\mathcal{H}(\kappa)$ to be the set of isomorphism classes of pairs (X, ω) where X is a genus g Riemann surface and ω is a non vanishing holomorphic 1-form on X whose zeroes have multiplicities given by α . We also define $\tilde{\mathcal{H}}(\kappa)$ to be the set of isomorphism classes of marked translation surfaces (X, ω, f) by adding a marking $f : S \rightarrow X$, where S is a fixed genus g surface, and such that the preimage of the singularity of ω by f is a subset Σ . Define the following map, called the period map :

$$\Phi : \begin{array}{l} \tilde{\mathcal{H}}(\kappa) \quad \rightarrow \quad H^1(S, \Sigma, \mathbb{C}) \\ (X, \omega, f) \quad \mapsto \quad (\gamma \mapsto \int_{f \circ \gamma} \omega) \end{array}$$

There is a complex structure on $\tilde{\mathcal{H}}(\kappa)$ that turns Φ into a local biholomorphism. The

set $\mathcal{H}(\kappa)$ is then endowed with the complex orbifold structure that turns the canonical projection $\pi : \tilde{\mathcal{H}}(\kappa) \rightarrow \mathcal{H}(\kappa)$ into local biholomorphism. Most of the issues arising through the orbifoldic points can be resolved by passing to a finite cover of the moduli space and we will implicitly do so in the remainder of this text. $\tilde{\mathcal{H}}(\kappa)$ is endowed with an group action by $GL_2^+(\mathbb{R})$ defined by :

$$\forall g \in GL_2^+(\mathbb{R}) \quad \Phi(g \cdot (X, \omega, f)) = g \cdot \Phi(X, \omega, f)$$

That action descends to an action on $\mathcal{H}(\kappa)$ in a way that the canonical projection π is $GL_2^+(\mathbb{R})$ -equivariant. More details on the structures of these spaces can be found in [12] or [14].

Definition 2.1 (affine manifold). *A closed connected subset \mathcal{M} of $\mathcal{H}(\kappa)$ is said to be an affine manifold if for any $X \in \mathcal{M}$, there is a section s of π defined on a neighborhood \mathcal{U} of X , and a subspace $V \in H^1(S, \Sigma, \mathbb{R})$ such that $\Phi \circ s(\mathcal{U} \cap \mathcal{M}) = \Phi \circ s(\mathcal{U}) \cap V \otimes \mathbb{C}$. Such a V is called a local model of \mathcal{M} around X .*

Affine manifolds are invariant under the action of $GL_2^+(\mathbb{R})$ and Eskin, Mirzakhani and Mohammadi proved in a celebrated result that the converse is true. More details can be found in [6]. An important numerical invariant associated to these loci is the rank, defined as follows: define $\rho : H^1(S, \Sigma, \mathbb{C}) \rightarrow H^1(S, \mathbb{C})$ to be the canonical restriction map, and for any X in \mathcal{M} , take a local model V of \mathcal{M} . Avila, Eskin and Moëller proved in [24] that $\rho(V)$ is a symplectic subspace of $H_1(S)$. The rank of \mathcal{M} , denoted $rk(\mathcal{M})$, is then defined as half the dimension of this space. More details can be found in [1]. The following definition will be important for the remainder of this text :

Definition 2.2. *Let \mathcal{M} be a affine manifold. The field of definition of \mathcal{M} is the smallest subfield $k(\mathcal{M})$ of \mathbb{R} such that any local model of \mathcal{M} can be written as $V = V_0 \otimes_{k(\mathcal{M})} \mathbb{R}$, where V_0 is a $k(\mathcal{M})$ -vector space.*

A particularly interesting family of rank 1 affine manifolds has been discovered by McMullen in [7]. We recall here the definition. Let (X, ω) be a translation surface endowed with a holomorphic involution τ . We denote by $\Omega(X)$ the set of holomorphic 1-forms, and by $\Omega^-(X)$ the set of τ -anti invariant holomorphic 1-forms. We say that (X, ω) is a Prym form is $\omega \in \Omega^-(X)$, that is $\tau^*\omega = -\omega$, and $\dim \Omega^-(X) = 2$. The Prym variety $Prym(X, \omega, \tau)$ is defined as the 2-dimensional abelian variety $\Omega^-(X)/H_1^-(X, \mathbb{Z})$ endowed with the polarization coming from the intersection form. We recall that a quadratic order of discriminant D is a ring \mathcal{O}_D isomorphic to $\mathbb{Z}[X]/(X^2 + bX + c)$, with $D^2 = b^2 - 4c$.

Definition 2.3. *A Prym eigenform is a Prym form such that $Prym(X, \omega, \tau)$ has real multiplication by \mathcal{O}_D for which ω is an eigenform, ie $End(Prym(X, \omega, \tau))$ contains a copy of \mathcal{O}_D action by self adjoint endomorphisms, such that $\mathcal{O}_D \cdot \omega = \lambda \cdot \omega$, for a $\lambda \in \mathcal{O}_D$.*

3 Preparation of a toolkit

In this section, we collect the tools we will use.

3.1 Isoperiodic foliation and Rel flow

The stratum $\mathcal{H}(\kappa)$ is endowed with a foliation which we describe in this section. It is usually referred to as the isoperiodic foliation or the kernel foliation. It is defined as follows: let ρ be the canonical restriction map from the relative cocycles $H^1(S, \Sigma, \mathbb{C})$ to the absolute cocycles $H^1(S, \mathbb{C})$. The action of Γ_g on $H^1(S, \Sigma, \mathbb{C})$ preserves the kernel of this map and since the transition functions on the moduli space are given by the action of Γ_g , the pull backs of the kernel of ρ by the charts can be pieced together to form a foliation \mathcal{F} of dimension $\dim(\ker \rho) = |\kappa| - 1$. The fact that Γ_g acts by permutation on Σ means that it generally does not exist integrable flows on the leaves. However, if (X, ω) is a translation surface in $\mathcal{H}(\kappa)$ one can still define an action of $T_X \mathcal{F}_X \times \mathbb{R}$ on the leaf \mathcal{F}_X . It is defined as follows: Consider $\hat{\mathcal{H}}(\kappa)$ to be the intermediate covering of $\mathcal{H}(\kappa)$ associated to the kernel of the action by permutation of Γ_g on Σ , and denote by $\hat{\pi}$ the canonical projection associated. One can define on this space a foliation $\hat{\mathcal{F}}$ in the same way we did on $\mathcal{H}(\kappa)$. But now, the transition functions of $\hat{\mathcal{H}}(\kappa)$ acts by translation on $\ker(\rho)$, and this implies that the leaves $\hat{\mathcal{F}}$ are canonically endowed with a flat structure and there is an associated geodesic flow. Then, let $(u, t) \in T_X \mathcal{F}_X \times \mathbb{R}$, and choose a lift of (X, ω) in $\hat{\mathcal{H}}(\kappa)$ associated to a section s of $\hat{\pi}$ defined around (X, ω) . We define $Rel_u^t(X, \omega)$ to be the image by $\hat{\pi}$ of the surface in $\hat{\mathcal{H}}(\kappa)$ obtained by flowing for a time t along the geodesic flow in direction $d_X s(u)$. This surface is canonically defined and does not depend on the choice of the section we made. Notice that by construction $Rel_u^t(X, \omega)$ belongs to \mathcal{F}_X . This action is usually referred to as the Rel flow, even if it is not sensu stricto a flow, as we have already mentioned. Note that this action might not be defined for all t : this is due to the fact that singularities might collide along the orbits of this "flow".

3.2 Modifying the twist parameters

Let \mathcal{M} be an affine manifold, and let (X, ω) be a translation surface in \mathcal{M} . Suppose (X, ω) has a cylinder decomposition into m cylinders. Denote by $\mathcal{C}_1, \dots, \mathcal{C}_m$ the collection of its cylinders. For a given cylinder \mathcal{C}_i , we denote by h_i its height and c_i its circumference. We also denote $\gamma_i = c_i^{-1}$, and $\mu_i = h_i \gamma_i$. This last quantity is known as the modulus of the cylinder. If \mathcal{C} is a horizontal cylinder on X , we denote by $\eta_{\mathcal{C}}$ the element of $H^1(S, \Sigma, \mathbb{C})$ that is dual to the homology class of the core curve of the cylinder, and denote by E the subspace spanned by the $\eta_{\mathcal{C}}$. Finally, denote by $\mathcal{C}(X)$ the connected component of $\{Y \in \hat{\mathcal{H}}(\kappa) \mid \Phi(Y) \in \Phi(X) + E\}$. It is proved in [19] the following result:

Proposition 3.1. *The period map induces a diffeomorphism from $\mathcal{C}(X)$ to $\Phi(X) + E$.*

We thus define a map \tilde{f}_X from \mathbb{R}^m to $\tilde{\mathcal{H}}(\kappa)$ that takes a tuple (x_1, \dots, x_m) to the surface of $\mathcal{C}(X_0)$ whose periods are given by $\Phi(X) + \sum c_i x_i \eta_i$. The action of \mathbb{Z}^m on \mathbb{R}^m by translation is intertwined by \tilde{f} with the action of the subgroup of Γ_g spanned by the Dehn twists about the core curves of the cylinders. Consequently, there is a map $f : \mathbb{R}^m / \mathbb{Z}^m$ to $\mathcal{H}(\kappa)$ that fits into the following commutative diagram :

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\tilde{f}_X} & \tilde{\mathcal{H}}(\kappa) \\ p \downarrow & & \downarrow \pi \\ \mathbb{R}^m / \mathbb{Z}^m & \xrightarrow{f_X} & \mathcal{H}(\kappa) \end{array}$$

We shall refer to the map f_X as the twists map associated to X . We will need the following result. We refer to [19] for a proof.

Lemma 3.1. *There is a linear subspace $V_{\mathcal{M}}(X)$ of \mathbb{R}^m such that $f^{-1}(\mathcal{M}) = p(V_{\mathcal{M}})$. The space $V_{\mathcal{M}}$ is rational and its dimension is at most $\dim(\mathcal{M}) - rk(\mathcal{M})$.*

From now on, we assume that $\mathcal{H}(\kappa)$ is a stratum of surfaces with exactly two singularities, and that the decomposition of X is stable: we mean by that that any component of the boundaries of cylinders contain only one singularity. A cylinder on X is said to be mixed if any cross-section is not an absolute cycle. By extension, we shall say that a decomposition on X is mixed if all the cylinders are mixed. Denote by \mathcal{C}_m the collection of the mixed cylinders of X , and by γ_i the inverse of the circumference. denote by d the algebraic degree of the $\delta_i \gamma_i$ (we mean the dimension of the \mathbb{Q} -vector space spanned by these numbers). There is an equivalence relation on \mathcal{C}_m given by the sign of boundary of cross sections. More precisely, let \mathcal{C}_1 and \mathcal{C}_2 be two mixed cylinder and chose a cross section σ_1 and σ_2 for each. Denote by ∂ the boundary map from $H_1(X, \Sigma)$ to $H_1(\Sigma)$. The image of this map is one dimensional. Thus \mathcal{C}_1 and \mathcal{C}_2 are equivalent if $\partial(\sigma_1)$ is positively colinear to $\partial(\sigma_2)$. There are two equivalence classes which we denote by \mathcal{C}^+ and \mathcal{C}^- . This relation does not depend on the choice of representative we chose for (X, ω) . Let δ_i be 1 if $i \in \mathcal{C}^+$, -1 if $i \in \mathcal{C}^-$ and 0 if i does not correspond to a mixed cylinder.

Proposition 3.2. *The vectors (μ_1, \dots, μ_m) and $(\delta_1 \gamma_1, \dots, \delta_m \gamma_m)$ belong to $V_{\mathcal{M}}(X)$.*

Proof. Set $\mu = (\mu_1, \dots, \mu_m)$ and $u = (\delta_1 \gamma_1, \dots, \delta_m \gamma_m)$. The fact that the vector μ belongs to $V_{\mathcal{M}}(X)$ simply traduces the fact that $f(t \cdot \mu) = h_t \cdot (X, \omega)$. Then, notice that $\Phi \circ \tilde{f}(u) - \Phi(X_0)$ evaluates to 0 on $H_1(S)$. It means that $f(u)$ belongs to \mathcal{F}_X which is included in \mathcal{M} by assumption. \square

Remark. Taking the notation of the proof, if we set $u_0 = d_0 f(u)$, then $f(t \cdot v) = Rel_{u_0}^t(X)$.

Proposition 3.3. *If $d \geq 2$, any rational relation satisfied by the $\delta_i \gamma_i$ are also satisfied by the coordinates of the elements of $V_{\mathcal{M}}(X)$.*

Proof. Set $u = (\delta_1\gamma_1, \dots, \delta_m\gamma_m)$. It is a classic result in dynamical systems that there is a rational subspace V of \mathbb{R}^m such that $p(u)$ is the closure of the set $\{p(t \cdot u) \mid t \in \mathbb{R}\}$, and V is the smallest rational subset that contains u , and its dimension equals the algebraic degree of u . See [10] for more details. The assumptions imply that V has dimension at least two, and is contained in $V_{\mathcal{M}}(X)$ since the latter is rational by 3.1. But the dimension of $V_{\mathcal{M}}(X)$ is also bounded above by 2, hence $V = V_{\mathcal{M}}(X)$. This implies that any rational relation satisfied by the $\delta_i\gamma_i$ must define $V_{\mathcal{M}}(X)$. \square

Corollary 3.1. *If the horizontal decomposition of X is not mixed, then all the mixed cylinders have commensurable circumference.*

Proof. Let \mathcal{C}_i be a non mixed cylinder and suppose the circumference were not commensurable. Then $\delta_i\gamma_i = 0$ is a rational relation and it must be satisfied by the coordinates of the vectors of $V_{\mathcal{M}}(X)$. This cannot hold as μ is an element of $V_{\mathcal{M}}(X)$ by proposition 3.2, and its i -th coordinate does not vanish. \square

Proposition 3.4. *If $d \geq 2$, two non equivalent cylinders can not have commensurable circumference.*

Proof. Suppose to a contradiction that two non equivalent cylinders \mathcal{C}_1 and \mathcal{C}_2 have a commensurable circumference, and say \mathcal{C}_1 belongs to \mathcal{C}^+ . There is thus a positive rational number p such that $\gamma_1 = q\gamma_2 = -q\delta_2\gamma_2$. By proposition 3.3, it implies that $\mu_1 = -q\mu_2$. This is a contradiction as moduli are positive numbers. \square

Corollary 3.2. *If $d \geq 2$, any pair of mixed cylinders with commensurable circumference have same height.*

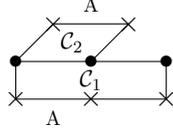
Proof. Let \mathcal{C}_1 and \mathcal{C}_2 be two mixed cylinders with commensurable circumference. Since $d \geq 2$, proposition 3.4 implies these cylinders are equivalent. Take a rational number q such that $\gamma_1 = q\gamma_2$. By 3.3, it implies that $\mu_1 = q\mu_2 = \frac{\gamma_1}{\gamma_2}\mu_2$. This equation is exactly $h_1 = h_2$. \square

Now, we say that two cylinders are adjacent if they share a saddle connection on their boundaries, and that they are 2-adjacent if they share two saddle connections, one on each boundary. Finally, denote by \mathcal{C}_0^\pm the collection of cylinders whose height is $\min\{h_{\mathcal{C}} \mid \mathcal{C} \in \mathcal{C}^\pm\}$. Denote by \mathcal{C}_1^\pm the complementary of \mathcal{C}_0^\pm in \mathcal{C}^\pm .

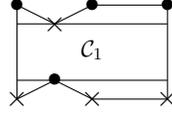
Proposition 3.5. *Suppose $d \geq 2$. If there is a cylinder in \mathcal{C}_0^+ that is 2-adjacent to a cylinder in \mathcal{C}_0^- , but not adjacent to any cylinder in \mathcal{C}_1^+ , then the circumferences of the cylinders in \mathcal{C}_1^+ are commensurable.*

Proof. Suppose \mathcal{C}_1 and \mathcal{C}_2 are as in the statement with \mathcal{C}_1 in \mathcal{C}^+ and denote by h the height of the cylinders in \mathcal{C}_0^+ . Up to replacing the δ_i by $-\delta_i$, we can suppose $\delta_1 = 1$, and define $X' = Rel_{iu}^{h+\varepsilon}(X)$. Notice that on X' , the cylinders \mathcal{C}_1 and $(\mathcal{C}_i)_{i \in \mathcal{C}_1^+}$ have persisted, but

\mathcal{C}_1 is now in \mathcal{C}^- , while the cylinders in \mathcal{C}_1^+ in X stay in \mathcal{C}^+ on X' . This comes from the adjacency conditions. The following picture depicts the deformation of the cylinders 1 and 2.



On the surface X



On the surface X'

Suppose that the circumference of the cylinders in \mathcal{C}_1^+ are not commensurable. Then, in particular $d \geq 2$, and since the dimension of $V_{\mathcal{M}}$ is 2, there are rational numbers p_i so that $\gamma_1 = \sum_{i \in \mathcal{C}_1^+} p_i \gamma_i$. Now, proposition 3.3 implies the following two equations :

$$\begin{aligned} \mu_1 &= \sum_{i \in \mathcal{C}_1^+} p_i \delta_i \mu_i \\ -\mu'_1 &= \sum_{i \in \mathcal{C}_1^+} p_i \delta_i \mu'_i \end{aligned}$$

Adding this two equations yields, using the fact $\mu'_i = \mu_i - \frac{h+\varepsilon}{c_i}$ if $i \geq 2$, and $\mu'_1 = \mu_1 + \frac{h-\varepsilon}{c_1}$:

$$(\varepsilon - h)\gamma_1 = \mu_1 - \mu'_1 = \sum_{i \in \mathcal{C}_1^+} p_i \delta_i (\mu_i + \mu'_i) = \sum_{i \in \mathcal{C}_1^+} p_i \delta_i (2\mu_i - (h + \varepsilon)\gamma_i) = 2\mu_1 - (h + \varepsilon)\gamma_1$$

This equation simplifies to $\varepsilon = h_1$, which is a contradiction. \square

3.3 The field of definition

In this section we recall a very useful formula for the field of definition of an affine manifold \mathcal{M} . It has been proved by Wright in [2].

Proposition 3.6. *Let (X, ω) be a translation surface in \mathcal{M} that is decomposed into m cylinders whose circumferences are denoted by c_i . Then, the following formula holds :*

$$k(\mathcal{M}) \subseteq \mathbb{Q}[c_2 c_1^{-1}, \dots, c_m c_1^{-1}]$$

Wright actually proved a stronger version and established the reciprocal inclusion if one restrain to a subclass of cylinders, but we wont need such generality.

3.4 Complete periodicity

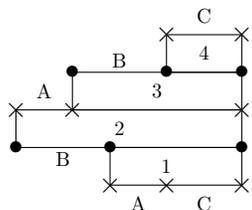
The last tool we will need is the complete periodicity property. See ?? for more details.

Proposition 3.7. *Let \mathcal{M} be a rank one affine manifold, and let (X, ω) be a translation surface in \mathcal{M} . If there is a saddle connection on X in direction θ that joins a singularity to itself, then X is periodic in direction θ .*

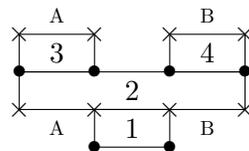
4 The stratum $\mathcal{H}^{odd}(2, 2)$

Theorem 4.1. *Let \mathcal{M} non absolute rank one affine manifold in $\mathcal{H}(2, 2)^{odd}$. If \mathcal{M} is non arithmetic, then $k(\mathcal{M})$ is a totally real quadratic field, and \mathcal{M} is a component of $\Omega E_D^{odd}(2, 2)$, where D is the discriminant of $k(\mathcal{M})$.*

Proof. Appendix A provides a list of the cylinder decomposition in $\mathcal{H}^{odd}(2, 2)$. Notice that only two of them have only mixed cylinders: the decompositions 3 and 4 in proposition 6.1. It can easily be checked that up to flowing along the leaf of X , we can suppose that X is cylinder equivalent to one of the following two surfaces :

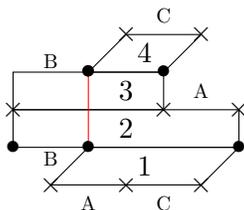


Decomposition A

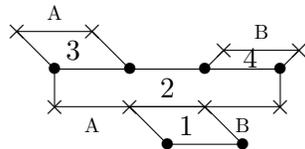


Decomposition B

We shall denote by τ_i , h_i and c_i respectively the twist, height and circumference of the cylinder \mathcal{C}_i (The twist is the period of a cross section, it is defined only mod c_i). Suppose (X, ω) has a cylinder decomposition as in A, and set $\gamma_i = c_i^{-1}$. By proposition 3.2, the vector $(\gamma_1, -\gamma_2, \gamma_3, -\gamma_4)$ belongs to $V_{\mathcal{M}}(X)$, and proposition 3.4 shows that the circumferences of the cylinders 1 and 2 are not commensurable. Consequently, the twists of the cylinders 2 and 3 can be chosen independently. More precisely, there is a vector $v \in V_{\mathcal{M}}$ so that $f_X(v)$ is the following surface :

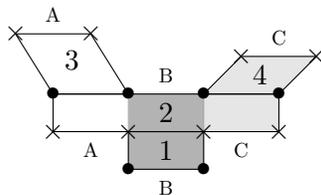


On that surface, the saddle connection drawn in red starts and ends at the same singularity. By proposition 3.7 that implies that the vertical direction is completely periodic. There is a small ε such that on $Rel_u^\varepsilon(X)$ the vertical decomposition is stable, and the decomposition must appear in the list provided in appendix A. On this surface the red saddle connection borders a cylinders, and notice that the total angle along this cylinder is at least 3π . The only cylinder decomposition that presents this feature is the decomposition A. That means we can only consider this decomposition. From now on, (X, ω) is the following surface:



The surface (X, ω)

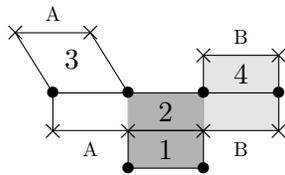
We shall start by proving that that two of the cylinders among $\{1, 3, 4\}$ have same height. By 3.2, $u = (\gamma_1, -\gamma_2, \gamma_3, \gamma_4)$ belongs to $V_{\mathcal{M}}(X)$. Its algebraic degree is 2 or else proposition 3.6 would imply that \mathcal{M} is arithmetic. If \mathcal{C}_0^+ has at least two elements, the claim is proved. If not, the two remaining cylinders are commensurable by proposition 3.5 and corollary 3.2 implies that $h_1 = h_4$. Now, for the same reason as before, γ_1 and γ_2 are not commensurable, and thus there is a $v \in V_{\mathcal{M}}(X)$ such that on $f_X(v)$ the twists of the cylinder 1 and 2 equal to zero.



The surface $f_X(v)$

If the twist of \mathcal{C}_1 on the surface $f_X(v)$ equals 0, that means that $v_1 = -\frac{\tau_1}{c_1}$. Since γ_1 and γ_4 are rationally dependent, proposition 3.3 implies that $u_4 = \frac{\tau_4}{\gamma_1}u_1 = \frac{c_1}{c_4}u_1 = -\frac{\tau_1}{c_4}$, and the twist of \mathcal{C}_4 is now $\tau_4 - \tau_1$. Notice that the dark grey part of the surface is crossed by a vertical geodesic. By proposition 3.7, that implies that the vertical direction is periodic. This is possible only if $\tau_4 - \tau_1$ is a rational multiple of c_4 : let p, q two coprime integers such that $\tau_4 - \tau_1 = \frac{p}{q}c_4$. The vertical cylinder decomposition is thus made out of three cylinders: the white one, and the two grey ones. The circumference of the dark grey cylinder is $h_1 + h_2$ and its height is c_1 , while the circumference of the light grey cylinder is $q(h_2 + h_4)$ and its height is $\frac{c_4}{q}$. Applying proposition 3.2 to $Rel_u^\varepsilon(X)$ gives $c_1 = \frac{c_4}{q}$, that is to say $\tau_4 - \tau_1 = pc_1$.

Up to applying powers of the Dehn twist about the core curve of the cylinder \mathcal{C}_1 , the twist of the cylinders 1 and 4 are equal. We can replace (X, ω) with the following surface:



The surface (X, ω) now has a vertical cylinder decomposition into 3 cylinders that we have colored once more in white, dark grey and light grey. The two greys cylinders have the same circumference given by $h_1 + h_2 = h_2 + h_4$, and their heights are c_1 and c_4 . Applying proposition 3.2 to $Rel_u^\varepsilon(X)$ gives that $c_1 = c_4$. Consequently, there is an involution that takes the light grey cylinder to the dark grey one, while fixing the white one. Notice that $\tau^*\omega = -\omega$ and that the genus of X/τ is 1. That means that τ is a Prym involution. To conclude, we invoke theorem 3.5 in [7]. This theorem states that if the veech group of a Prym form contains a hyperbolic element, then it is a Prym eigenform. The fact that any invariant manifold contains a surface that has a hyperbolic element in its veech group is a consequence of a closing lemma for the Teichmüller flow that we can find, for instance, in [2]. This concludes the proof. □

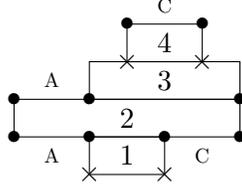
Proposition 4.1. *If X is a non arithmetic Veech surface in $\mathcal{H}^{odd}(2, 2)$ that is not contained in the Prym locus, then the subset $G \cdot \mathcal{F}_X$ is dense in $\mathcal{H}^{odd}(2, 2)$.*

Proof. Let \mathcal{M} be the closure of $G \cdot \mathcal{F}_X$. This is an affine manifold as it is connected, closed, and $GL_2^+(\mathbb{R})$ invariant, and it is non absolute. Its rank is at least two, as otherwise theorem A would imply that X is contained in a Prym locus. The work of Aulicino and Nguyen (Theorem 1.1 in [4]) implies it can not be rank two either as the Prym locus is the only non absolute rank two affine manifold. Its rank is thus 3, and as it is saturated by the isoperiodic foliation, this is actually the whole stratum $\mathcal{H}^{odd}(2, 2)$. □

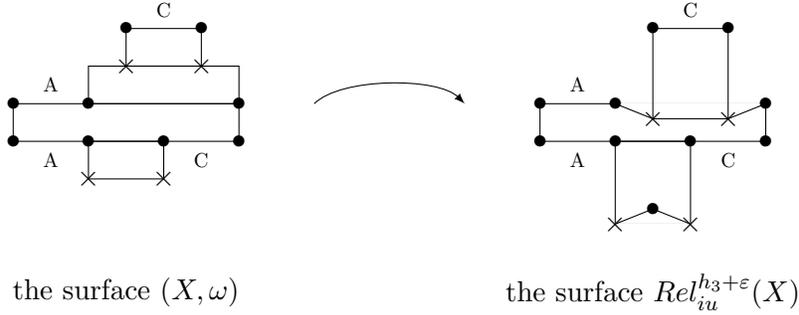
5 The stratum $\mathcal{H}(3, 1)$

Theorem 5.1. *Any non absolute rank one affine manifold in $\mathcal{H}(3, 1)$ is arithmetic.*

Proof. Let \mathcal{M} be a non absolute rank 1 affine manifold, and let (X, ω) be a horizontally periodic surface in \mathcal{M} . Notice that the dimension of $K_{\mathcal{M}}(X)$ is one, and the dimension of $V_{\mathcal{M}}(X)$ is two. Appendix A provides a list of all the stable cylinder decompositions in $\mathcal{H}(3, 1)$. We consider the first decomposition of the list. Up to modifying the twists and height of the cylinder, the surface (X, ω) is as follows :



Denote by c_1 the circumference of the cylinder \mathcal{C}_i , and by $\gamma_i = c_i^{-1}$. By proposition 3.2, $u = (\gamma_1, 0, -\gamma_2, \gamma_3)$ is an element of $K_{\mathcal{M}}(X)$. The algebraic degree of this deformation is at most 2. However, if it were to be 2, then by proposition ??, every vector in $V_{\mathcal{M}}(X)$ would have a vanishing second coordinate, and that is not the case of (μ_1, \dots, μ_4) . Therefore the circumference of the cylinders 1, 2 and 4 are pairwise rationally dependent. Note that proposition 3.6 is of no help at this stage to prove that \mathcal{M} is arithmetic as we do not know anything yet on the circumference of the second cylinder. The remaining of the argument is to prove that c_2 is indeed commensurable to the circumference of the other cylinders. To do so, consider the surface $Y = Rel_{iu}^{h_3+\varepsilon}(X)$, where h_3 is the height of the cylinder 3 and ε is small enough. The deformation is depicted in the following picture:



For the same reason as previously, on the surface Y the cylinder 1, 2 and 4 are pairwise commensurable. But notice that the circumferences of the cylinder 2 or 3 are the same on X and Y . This concludes that all the cylinders of (X, ω) are commensurable. Proposition 3.6 shows that \mathcal{M} is thus arithmetic. The scheme of proof can be used for the decomposition 1, 2, 3, 4, 5 and 6. To deal with the other two cases, note that the surface can be deformed through the horocycle flow and the Rel flow so that the vertical direction is periodic with one non mixed cylinder. It means that the vertical decomposition falls in the previous list. This concludes the proof. \square

Corollary 5.1. *If (X, ω) is a non arithmetic Veech surface in $\mathcal{H}(3, 1)$, then the subset $GL_2^+(\mathbb{R}) \cdot \mathcal{F}_X$ is dense in $\mathcal{H}(3, 1)$.*

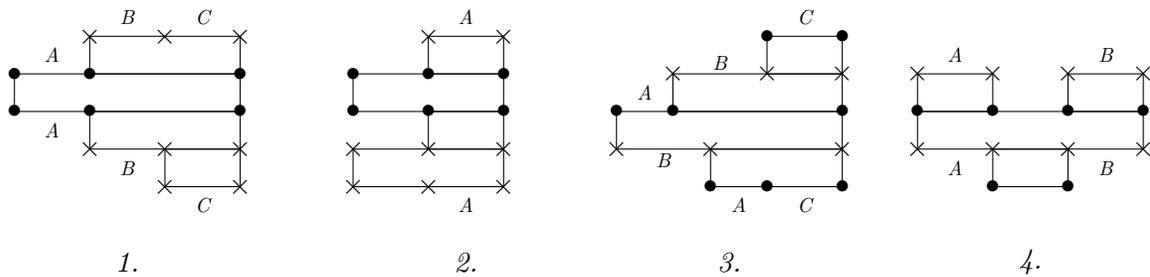
Proof. Let \mathcal{M} be the closure of $GL_2^+(\mathbb{R}) \cdot \mathcal{F}_X$. It is an affine manifold. If its rank were one, then \mathcal{M} would be non arithmetic and this is a contradiction with Theorem B. The

rank of \mathcal{M} is then at least 2, but Nguyen and Aulicino proved that there are no rank 2 affine manifold in $\mathcal{H}(3,1)$. Therefore the rank of \mathcal{M} is 3, and \mathcal{M} is the whole stratum $\mathcal{H}(3,1)$. \square

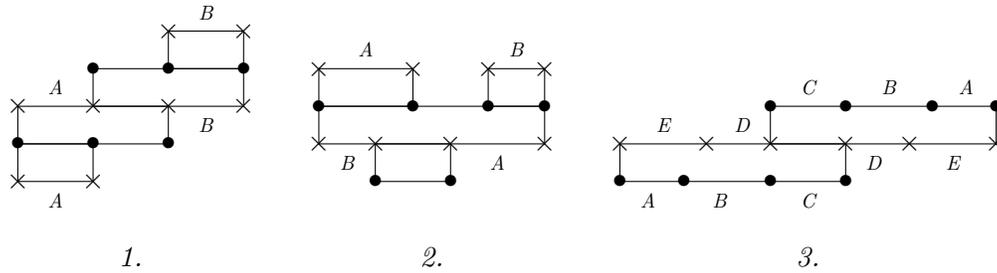
6 Appendix : Stable Cylinder decompositions in $\mathcal{H}(2, 2)$ and $\mathcal{H}(3, 1)$

In this appendix, we give the possible stable cylinder decompositions in $\mathcal{H}(2, 2)$, and $\mathcal{H}(3, 1)$. More formally, two periodic translation surfaces are said to be cylinder equivalent if one is obtained from the other by modifying the height and twist parameter of the cylinders. We prove the following three propositions :

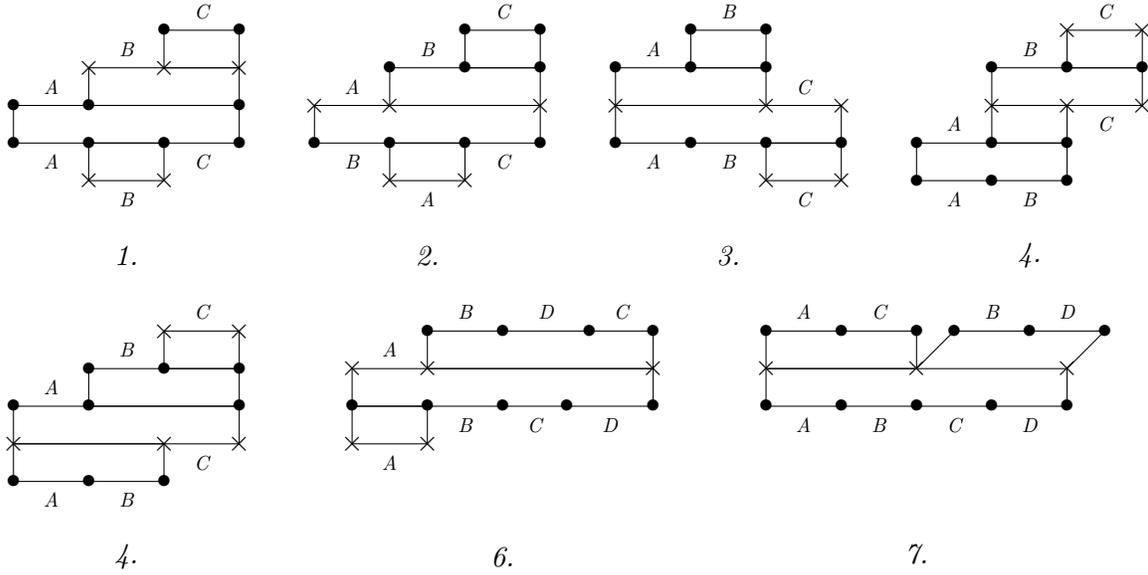
Proposition 6.1. *If (X, ω) is a horizontally periodic translation surface in $\mathcal{H}^{odd}(2, 2)$, then it is cylinder equivalent to one of the following translation surfaces:*



Proposition 6.2. *If (X, ω) is a horizontally periodic translation surface in $\mathcal{H}^{hyp}(2, 2)$, then it is cylinder equivalent to one of the following translation surfaces:*



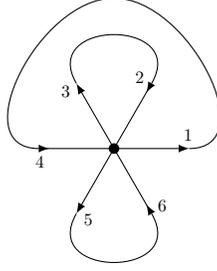
Proposition 6.3. *If (X, ω) is a horizontally periodic translation surface in $\mathcal{H}(3, 1)$, then it is cylinder equivalent to one of the following translation surfaces:*



We start by giving a common framework to study cylinder decompositions of translation surfaces.

Definition 6.1. A *prediagram of separatrices* is a quadruplet $\Gamma = (E, \sigma, \tau, \theta)$, where τ is a fixed point free involution of E , σ is a permutation of E and θ is a map from $E/\tau \rightarrow E$ such that $p \circ \theta = id$, where $p : E \rightarrow E/\tau$ is the canonical projection.

The elements of E are called edges. An edge γ in E is said to be positively oriented if $\theta \circ p(\gamma) = \gamma$ and negatively oriented if $\theta \circ p(\gamma) = \tau(\gamma)$. The set of positively oriented edges will be denoted by E_+ and the set of negatively oriented edges will be denoted by E_- . A cylinder component is defined as an orbit of $\sigma_\infty := \sigma \circ \tau$. Such a cylinder component is said to be positively oriented if it corresponds to a positively oriented edge, and negatively oriented otherwise. We denote by \mathcal{C}_+ and \mathcal{C}_- the set of positively and negatively oriented cylinder components. A pairing of cylinder components is a bijection from \mathcal{C}_+ to \mathcal{C}_- . Finally, we define a metric on Γ as a strictly positive τ -invariant function l on E , and we consider its natural extension \hat{l} to the set of cylinder components defined by $\hat{l}(c) := \sum_n l(\sigma_\infty^n(\gamma))$, where c is the cylinder component associated to γ . Such an object can be encoded by a directed graph with additional information. Its set of vertices is the set of orbits of σ , and its set of edges is E/τ . Set $p_0 : E \rightarrow E/\sigma$ to be the canonical projection. The beginning of an edge is $p_0 \circ \theta(e)$, and its end is $p_0 \circ \tau \circ \theta(e)$. There is a cyclic ordering on the star of each vertex define by σ . In figure ? is depicted the graph associated to the alternating prediagram of separatrices $(\{1, \dots, 6\}, id, (14)(23)(56), \theta_0)$, where $\theta_0(i) = i$ for any $i \in \{1, 2, 5\}$.



The graph associated to $(\{1, \dots, 6\}, id, (14)(23)(56), \theta_0)$

Definition 6.2. A diagram of separatrices (Γ, m, l) is the data of an alternating prediagram of separatrices Γ , together with a matching of its cylinder components and a metric l on Γ that is invariant by m , that is $\hat{l} \circ m = \hat{l}$.

To any translation surface $(X, \omega) \in \mathcal{H}(\kappa)$, there is a canonical diagram of separatrices $\Gamma(X, \omega)$ associated. It is defined as follows : The set E is the collection of all geodesics rays $\gamma : [0, 1] \rightarrow X$, such that $\gamma^{-1}(\Sigma) = \{0, 1\}$, and $\gamma^*(Im(\omega)) = 0$, taken up to reparametrization. We define τ to be the orientation reversing map : $\tau(\gamma)(t) : \tau(\gamma)(1 - t)$. For any $\gamma \in E$, there is a chart around $\gamma(0)$ that takes ω to $z^k dz$; where k is the order of the singularity. The germs of geodesic ray that begins at $\gamma(0)$ is thus invariant by multiplication by $e^{\frac{2i\pi}{k+1}}$, and the germ of a geodesic ray at its beginning completely determines it. We thus define $\sigma(\gamma)$ to be the geodesic ray whose germ at its beginning is the one of $e^{\frac{2i\pi}{k+1}}\gamma$. Finally, to define θ we need to define a map that is τ -invariant. If γ is in E , we define $\theta(\gamma) = \gamma$ if $\gamma^*(Re(\omega)) > 0$, and $\theta(\gamma) = \tau(\gamma)$ otherwise. The metric on $\Gamma(X, \omega)$ is given by $l(\gamma) = |\int_{\gamma} \omega|$. Note that the orbits of σ_{∞} are in correspondence with oriented core curves of cylinders. The matching m is then defined to map the orbit that corresponds to the positively oriented core curve to the one that corresponds to the negatively oriented core curves of the same cylinder. In the reverse direction, to any diagram of separatrices Γ there is horizontally periodic translation surface $(X_{\Gamma}, \omega_{\Gamma})$. It is defined in the following way: Replace any $\gamma \in \theta(E/\tau)$ by a strip of length $l(\gamma)$, and replace any element in E/σ by a disk. The permutation σ defines a cyclic ordering on any orbit. Using this ordering, one can glue the strips to the disks. We get a topological surface with boundary, and the boundary components correspond to orbits of σ_{∞} . Since by requirement, the paired components have same length (defined by \hat{l}) we can glue those components using the pairing, and we get a horizontally periodic flat surface in $\mathcal{H}(\kappa)$, where $|\kappa|$ is the cardinal of E/σ and k_i is half the cardinal of the orbit i minus 1. The fact that the cardinal of an orbit is even comes from the alternating condition.

Two prediagrams of separatrices $\Gamma_1 = (E^1, \tau_1, \sigma_1, \theta_1)$ and $\Gamma_2 = (E^2, \tau_2, \sigma_2, \theta_2)$ are isomorphic if there is a map $\varphi : E^1 \rightarrow E_2$ such that $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$, $\varphi \circ \tau_1 = \tau_2 \circ \varphi$, and $\varphi(E_+^1) = E_+^2$, or equivalently $\varphi \circ \theta_1 = \theta_2$. Finally, two diagrams of separatrices (Γ_1, l_1, m_1) and (Γ_2, l_2, m_2) are isomorphic if there is an isomorphism of prediagram of separatrices φ

between the two such that $l_2 \circ \varphi = l_1$, and if two cylinders components on Γ_1 associated to the orbits of γ_1 and γ_2 are paired by m_1 , then the cylinder components of Γ_2 associated to $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ are paired by m_2 .

Proposition 6.4. *Two horizontally periodic translation surfaces (X_1, ω_1) and (X_2, ω_2) in $\mathcal{H}(\kappa)$ are isomorphic to cylinder equivalents surfaces if, and only if, the associated diagrams of separatrices are isomorphic.*

We refer to [15] for a proof of that result.

Definition 6.3. *A connected component of a prediagram of separatrices of $\Gamma = (E, \sigma, \tau, \theta)$ is a subset E' of E that is preserved by both σ and τ .*

If E' is a connected component of Γ , there is an induced prediagram of separatrices $(E', \sigma|_{E'}, \tau|_{E'}, \theta|_{p(E')})$.

Definition 6.4. *A prediagram of separatrices $(E, \tau, \sigma, \theta)$ is said to be stable if τ preserves the orbits of σ .*

In this setting a minimal connected component is said to be minimal if σ acts transitively on E . We can define the type a minimal connected component as follows : Let x be a positively oriented edge of the minimal component. Thus any positively oriented edges can be written as $\sigma^{2k}(x)$, and any negatively oriented edge can be written as $\sigma^{c_n(2l)}(x)$. Since τ reverses the orientation, there is a map $f \in \mathfrak{S}_n$ such that for any k : $\tau \circ \sigma^{2k}(x) = \sigma^{c_n(2f(k))}(x)$. The group $H = \langle c_n \rangle$ acts by conjugation on \mathfrak{S}_n and the type of the component is defined as the orbit of f under this action. The fact that we defined f up to conjugation by H comes from the fact we could have chosen any other even iteration of x as a generator of the orbit. More generally, we say that Γ is of type (f_i) if the type of the minimal connected components are given by the f_i .

Proposition 6.5. *Two stable alternating prediagrams are isomorphic if, and only if, they have the same type.*

Proof. It is enough to prove the result for minimal connected components : Let $\Gamma_i = (E_i, \sigma_i, \tau_i, \theta_i)$ for $i \in \{1, 2\}$ be two minimal prediagrams of same type. Pick x_i in E_i that is positively oriented and define f_i such that $\tau_i(\sigma^{2k}(x_i)) = \sigma^{2(f_i(k)+1)}$. Saying that the diagrams have the same type means there is l such that $f_2 = c_n^l \circ f_1 \circ c_n^{-l}$. Define φ such that for all j $\varphi(\sigma_1^j(x_1)) = \sigma_2^{2l+j}(x_2)$. We claim that φ is an isomorphism of prediagram. Indeed, let $\zeta \in E_1$, and pick j such that $\sigma^j(x_1) = \zeta$. Thus $\varphi \circ \sigma_1(\zeta) = \varphi \circ \sigma^{j+1}(x_1) = \sigma_2^{2l+j+1}(x_2) = \sigma_2(\sigma_2^{2l+j}(x_2)) = \sigma_2 \circ \varphi(\sigma_1^j(x_1)) = \sigma_2 \circ \varphi(\zeta)$. Then, $\tau_2 \circ \varphi(\sigma_1^{2k}(x_1)) = \tau_2 \circ \varphi(\sigma_2^{2(k+l)}(x_2)) = \sigma_2^{2f_2(k+l)+1}(x_2) = \sigma_2^{2c_n^l \circ f_1 \circ c_n^{-l}(k+l)+1}(x_2) = \sigma_2^{2(f_1(k)+l)+1}(x_2) = \varphi \circ \sigma_1^{2f_1(k)+1} = \varphi \circ \tau_1(\sigma^{2k}(x_1))$. By construction, $\varphi(E_1^+) = E_2^+$. Reciprocally, if there is an isomorphism of prediagrams between Γ_1 and Γ_2 , then $\varphi(x_1) = \sigma_2^{2l}(x_2)$, and then $f_2 = c_n^l \circ f_1 \circ c_n^{-l}$. \square

We denote by $\bar{\Gamma}$ the prediagram $(E, \sigma, \tau, \tau \circ \theta)$. If Γ corresponds to a surface (X, ω) , then $\bar{\Gamma}$ corresponds to the surface $(X, -\omega)$. Note that if (X, ω) is represented by a polygon P , the surface $(X, -\omega)$ is represented by the image of the polygon by the rotation of angle π .

Proposition 6.6. *Let Γ be a minimal stable alternating prediagram of type f . The type of $\bar{\Gamma}$ is given by $(f \circ c_n)^{-1}$.*

Proof. Let $x \in E$ be positively oriented, and such that $\tau \circ \sigma^{2k}(x) = \sigma^{2f(k)+1}$. Then $\sigma(x)$ is positively oriented on $\bar{\Gamma}$. And $\tau \circ \sigma^{2j}(\sigma(x)) = \sigma^{2f^{-1}(j)}(x) = \sigma^{2c_n^{-1}(f^{-1}(j))}(\sigma(x))$. \square

These propositions enable us to enumerate all the possible stable alternating prediagrams up to isomorphism, we shall represent only the surface associated to Γ or $\bar{\Gamma}$ as one is obtained from the other by a rotation of angle π .

7 Stable cylinder decompositions in $\mathcal{H}^{odd}(2, 2)$

Let (X, ω) be a stable horizontally periodic translation surface in $\mathcal{H}_g(2, 2)$, and denote by $\Gamma(X, \omega)$ the prediagram associated. The types of the minimal connected components of $\Gamma(X, \omega)$ are, up to reversing the orientation, either $id_{\mathfrak{S}_3}$, (123) or (12)(3).

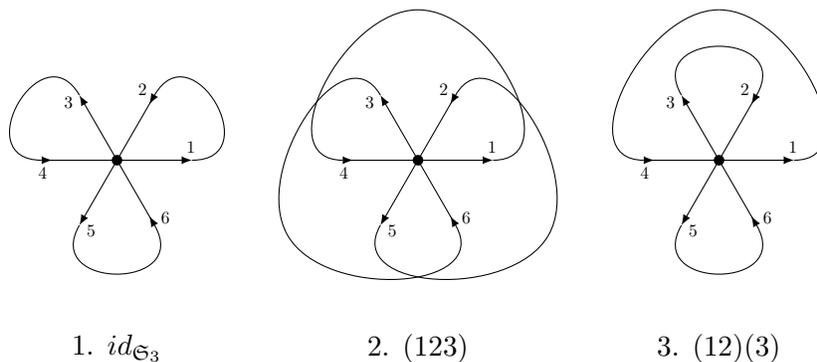


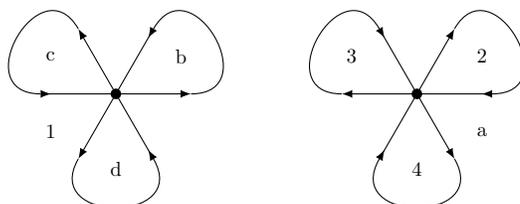
Fig. The possible types for the components of $\Gamma(X, \omega)$

Indeed, the permutation given in the previous picture generates the group \mathfrak{S}_3/H_3 . The isomorphism class of the prediagram $\Gamma(X, \omega)$ is thus completely determined by the choice of one of the three types for its two minimal connected components, together with an orientation on those minimal components. This orientation cannot be determined arbitrarily as there is a pairing on the orbits of σ_∞ . The components of type 1 and 3 both have 4 cylinder components, while the type 2 has 2 components. A pairing is possible only if there an even number of cylinder components, half of them being positively oriented, and the other half

being negatively oriented. Hence the type of $\Gamma(X, \omega)$ is, up to reversing the orientation on minimal components, either $(1, 1)$, $(2, 2)$ or $(2, 3)$. The reason $\Gamma(X, \omega)$ can not be of type $(1, 2)$ is the fact the type 1 minimal components has the same number of positively and negatively cylinder components, while the type 2 one has 3 cylinders components with the same orientation and only one with the opposite orientation.

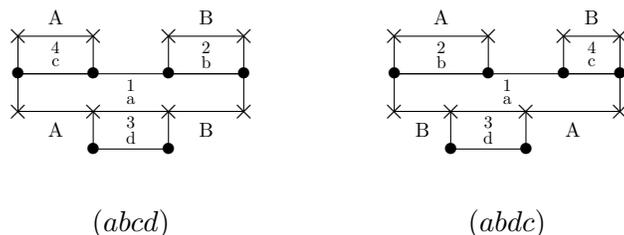
7.1 Stable cylinder decompositions of type $(1, 1)$

The two minimal components need to be oppositely oriented so that a pairing is possible. The associated graph is planar and we can identify cylinder components to connected components of the plane once the graphs are removed. We have named those orbits by letters when they are positively oriented, and with a digit when they are negatively oriented.



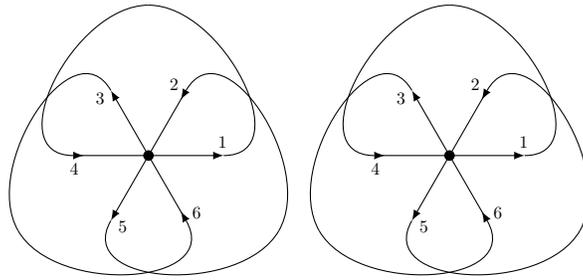
The prediagram of $\Gamma(X, \omega)$ in the type $(2, 2)$ case

We will denote a pairing by an ordered quadruple of letters. The cylinder components whose label appears first is matched with the cylinder components labeled by 1 etc. For instance, in the pairing labeled by $(abcd)$, the first components a is paired by the components 1, the component c with the components 2, etc. Here, for metric reason, the cylinder component denoted by 1 is necessarily paired with a . We can also assume that 2 is paired with b . We depict the two corresponding surfaces associated:



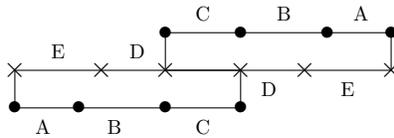
7.2 Stable cylinder decomposition of type $(2, 2)$

Here, any choice of orientation produces isomorphic prediagrams, so we can chose an arbitrary orientation on each minimal component.



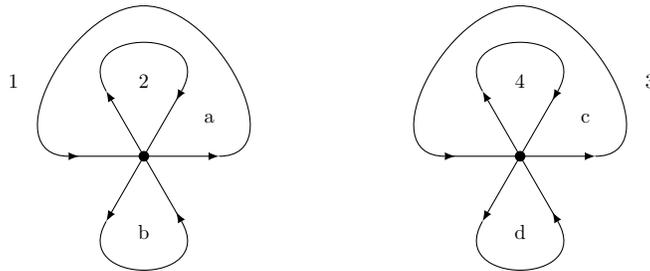
The prediagram of $\Gamma(X, \omega)$ in the type (2, 2) case

In this case there is only one possible pairing that produces a connected translation surface. The associated surface is represented in the following picture.



7.3 Stable cylinder decomposition of type (3, 3)

Here, Any choice of orientation produces isomorphic prediagrams, so we can chose an arbitrary orientation on each minimal connected component. Here again, the graph associated to the minimal connected components is planar. We use the same notation as in the previous section.

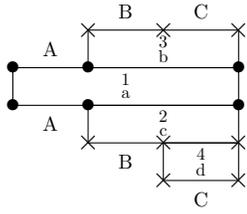


The prediagram of $\Gamma(X, \omega)$ in the type (1, 1) case

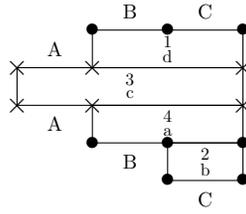
There are a priori 24 pairing possible. However, the following 4 represent non connected surfaces : $(abcd)$, $(abdc)$, $(bacd)$, $(badc)$, and on the following 13 there are compatible metric :

1. $(acdb)$. If $l_a = l_1$, then $l_2 = l_b$. But l_b must be equal to l_4 , while l_2 must be equal to l_c , and thus $l_c = l_4$. However, l_c should be strictly greater than l_4 .
2. $(adbc)$. The pairing would give $l_c = l_4$, while we should get that l_2 is strictly greater than l_4 .
3. $(bdca)$ The pairing would give $l_1 = l_b$, while we should get that l_1 is strictly greater than l_b .
4. $(bcad)$. The pairing would give $l_1 = l_b$, while we should get that l_1 is strictly greater than l_b .
5. $(cbda)$. The pairing would give $l_d = l_3$, while we should get that l_3 is strictly greater than l_d .
6. $(cabd)$. The pairing would give $l_2 = l_a$, while we should get that l_a is strictly greater than l_2 .
7. $(bcda)$. The pairing would give $l_1 = l_b$, while we should get that l_1 is strictly greater than l_b .
8. $(dabc)$. The pairing would give $l_c = l_4$, while we should get that l_2 is strictly greater than l_4 .
9. $(dbac)$. The pairing would give $l_c = l_4$, while we should get that l_2 is strictly greater than l_4 .
10. $(bdac)$. The pairing would give $l_1 = l_b$, while we should get that l_1 is strictly greater than l_b .
11. $(cadb)$. The pairing would give $l_d = l_3$, while we should get that l_3 is strictly greater than l_d .
12. $(dacb)$. The pairing would give $l_2 = l_a$, while we should get that l_a is strictly greater than l_2 .
13. $(dcba)$ For metrical reason, we have l_a is greater than l_2 , and likewise l_c greater than l_4 . How ever the pairing requires $l_2 = l_c$, thus we get that l_4 is greater than l_a . But the pairing would imply that l_4 equals l_a .

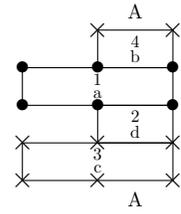
We depict the surfaces associated to the other diagrams:



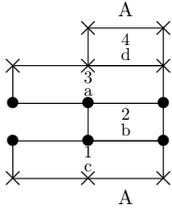
(acbd)



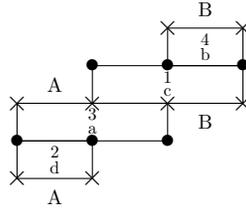
(dbca)



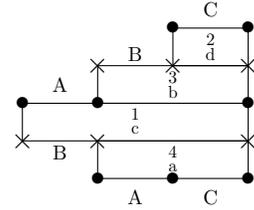
(adcb)



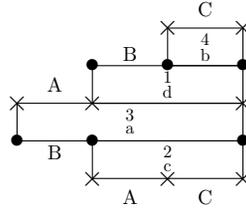
(cbad)



(cdab)



(cdba)



(dcab)

Finally, notice that the first two diagrams are isomorphic, as well as the last two. The isomorphism is, in both case the one that exchanges the two components while commuting with both σ and τ . Intuitively, the corresponding isomorphism at the level of the surfaces swap the singularities.

8 Stable cylinder decompositions in $\mathcal{H}(3, 1)$

The types of the minimal connected components of order 3 of $\Gamma(X, \omega)$ are, up to reversing the orientation, either $(1)(243)$, $(1)(3)(24)$, $(1)(234)$, $(13)(24)$ or $id_{\mathfrak{S}_4}$, and the type of the minimal component of order 1 is, up to reversing the orientation, $id_{\mathfrak{S}_2}$.

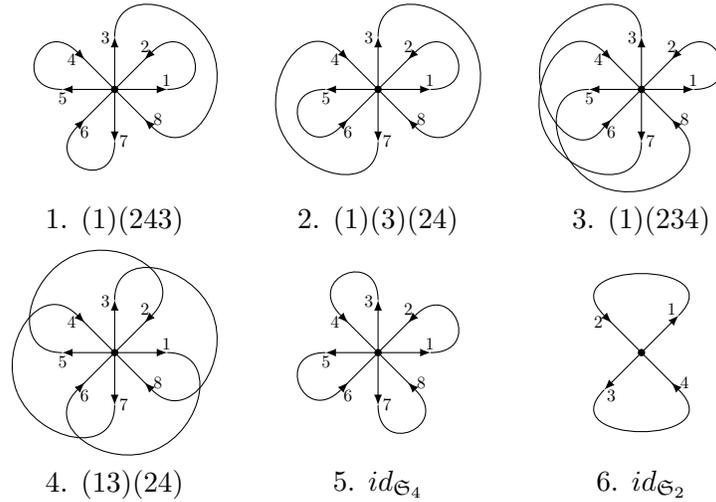
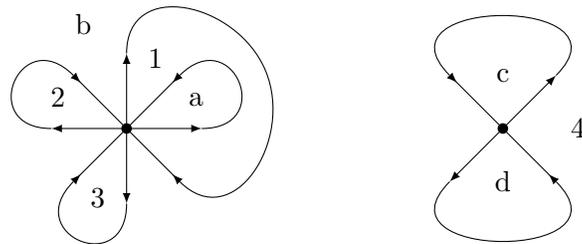


Fig. The possible types for the components of $\Gamma(X, \omega)$

The types 1 and 3 both have 3 cylinder components with the same orientation and 2 with the opposite orientation, while types 3 and 4 have both 2 cylinder components with the same orientation and only one with the reverse orientation. The type 5 has 4 cylinder components with one orientation and 1 with the opposite orientation. The type 6 has two cylinder component with the same orientation, and one with the reverse orientation. With the pairing condition, we deduce that $\Gamma(X, \omega)$ can be type (1, 6), (2, 6), (3, 6) or (4, 6).

8.1 Stable cylinder decomposition of type (1, 6)

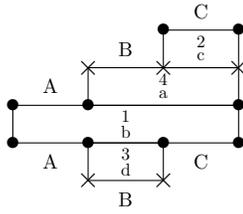
The following picture depicts the only possible orientation. Here again, the associated graphs are planar, and we label the components with letters and digits.



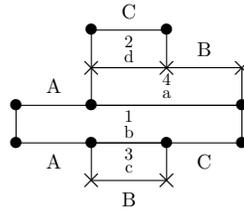
Here as well, we will denote a pairing by an ordered quadruple of letters. There are a priori 24 pairing possible. However, the following 18 cannot be endowed with a positive metric :

1. $(abcd)$. does not have a metric solution since $\lambda_a > \lambda_1$
2. $(abdc)$. does not have a metric solution since $\lambda_a > \lambda_1$
3. $(acbd)$. does not have a metric solution since $\lambda_a > \lambda_1$
4. $(acbd)$. does not have a metric solution since $\lambda_a > \lambda_1$
5. $(adbc)$. does not have a metric solution since $\lambda_a > \lambda_1$
6. $(adcb)$. does not have a metric solution since $\lambda_a > \lambda_1$
7. $(bacd)$. does not have a metric solution since $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$ and $\lambda_2 = \lambda_a$ together with $\lambda_1 = \lambda_b$ imply $\lambda_3 = 0$.
8. $(badc)$. does not have a metric solution since $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$ and $\lambda_2 = \lambda_a$ together with $\lambda_1 = \lambda_b$ imply $\lambda_3 = 0$.
9. $(bcad)$. does not have a metric solution since $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$ and $\lambda_3 = \lambda_a$ together with $\lambda_1 = \lambda_b$ imply $\lambda_2 = 0$.
10. $(bdac)$. does not have a metric solution since $\lambda_b - \lambda_2 - \lambda_3 = \lambda_1 - \lambda_a$ and $\lambda_2 = \lambda_a$ together with $\lambda_1 = \lambda_b$ imply $\lambda_3 = 0$.
11. $(cabd)$. does not have a metric solution since $\lambda_3 < \lambda_b$
12. $(cbad)$. does not have a metric solution since $\lambda_2 < \lambda_b$.
13. $(cbda)$. does not have a metric solution since $\lambda_2 < \lambda_b$.
14. $(cdba)$. does not have a metric solution since $\lambda_3 < \lambda_b$.
15. $(dabc)$. does not have a metric solution since $\lambda_2 < \lambda_b$.
16. $(dbac)$. does not have a metric solution since $\lambda_2 < \lambda_b$.
17. $(dbca)$. does not have a metric solution since $\lambda_2 < \lambda_b$.
18. $(dcba)$. does not have a metric solution since $\lambda_3 < \lambda_b$.

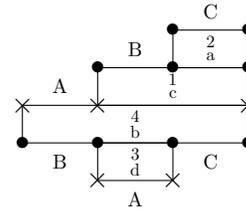
We depict the surfaces associated to the other diagrams:



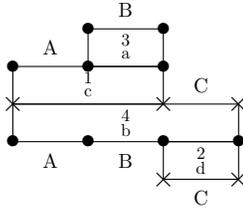
1. $(bcda)$



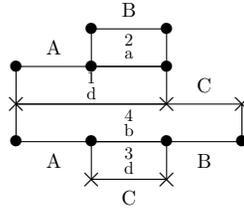
2. $(bdca)$



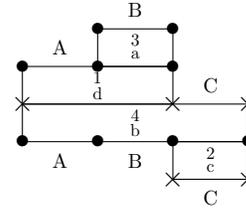
3. $(cadb)$



4. $(cdab)$



5. $(dacb)$

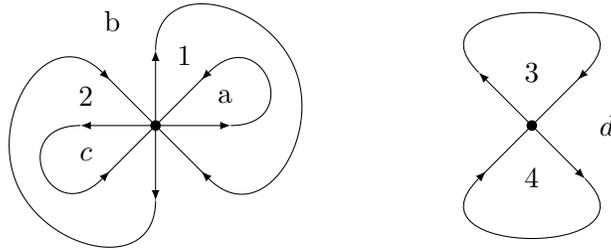


6. $(dcab)$

To conclude, notice that the diagrams 1 and 2 are isomorphic, as well as 3 and 5, and 4 and 6.

8.2 Stable cylinder decompositions of type $(2, 6)$

The following picture depicts the only possible orientation. Here again, the associated graphs are planar, and we label the components with letters and digits.



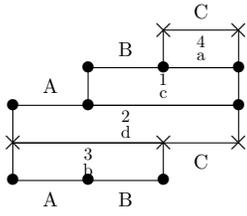
The prediagram of $\Gamma(X, \omega)$ in the type $(2, 6)$ case

We continue to denote a pairing by an ordered quadruple of letters. There are a priori 24 pairings possible. However, the following 18 cannot be endowed with a positive metric :

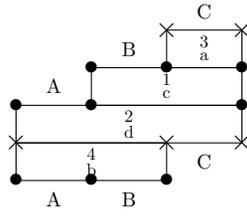
1. $(abcd)$. Does not have a metric solution as $\lambda_1 > \lambda_a$.
2. $(abdc)$. Does not have a metric solution as $\lambda_1 > \lambda_a$.

3. $(acbd)$. Does not have a metric solution as $\lambda_1 > \lambda_a$.
4. $(acbd)$. Does not have a metric solution as $\lambda_1 > \lambda_a$.
5. $(adbcb)$. Does not have a metric solution as $\lambda_1 > \lambda_a$.
6. $(adcb)$. Does not have a metric solution as $\lambda_1 > \lambda_a$.
7. $(bacd)$. Does not have a metric solution as $\lambda_d > \lambda_4$.
8. $(badc)$. Does not have a metric solution as $\lambda_d > \lambda_4$.
9. $(bcad)$. Does not have a metric solution as $\lambda_d > \lambda_4$.
10. $(bcda)$. Does not have a metric solution as $\lambda_2 > \lambda_c$.
11. $(bdac)$. Does not have a metric solution as $\lambda_1 - \lambda_a = \lambda_b - \lambda_2$ together with $\lambda_b = \lambda_1$ imply $\lambda_a = \lambda_2$. But $\lambda_a = \lambda_3 + \lambda_4$; and thus if $\lambda_a = \lambda_3$ then $\lambda_4 = 0$.
12. $(bdca)$. Does not have a metric solution as $\lambda_1 - \lambda_a = \lambda_b - \lambda_2$ together with $\lambda_b = \lambda_1$ imply $\lambda_a = \lambda_2$. But if $\lambda_2 = \lambda_d$ then $\lambda_a = \lambda_3 + \lambda_4$; and thus if $\lambda_a = \lambda_4$ then $\lambda_3 = 0$.
13. $(cabd)$. Does not have a metric solution as $\lambda_d > \lambda_4$.
14. $(cadb)$. Does not have a metric solution as $\lambda_d > \lambda_4$.
15. $(cbad)$. Does not have a metric solution as $\lambda_d > \lambda_4$.
16. $(cbda)$. Does not have a metric solution as $\lambda_d > \lambda_3$.
17. $(dcab)$. Does not have a metric solution as $\lambda_d > \lambda_4$.
18. $(dcba)$. Does not have a metric solution as $\lambda_d > \lambda_4$.

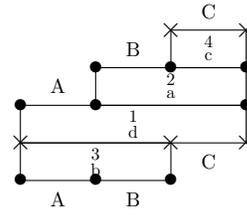
We depict the surfaces associated to the other diagrams:



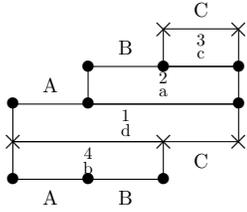
1. $cdab$



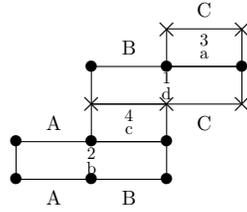
2. $cdba$



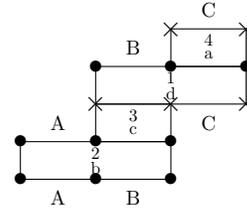
3. $dabc$



4. $dacb$



5. $dbac$

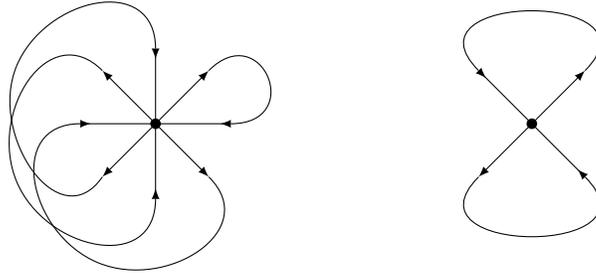


6. $dbca$

To conclude, notice that the four first diagrams are isomorphic, as well as the last two.

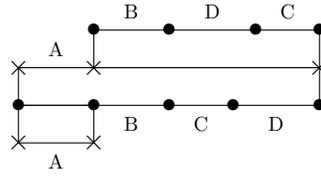
8.3 Stable cylinder decomposition of type $(3, 6)$

In this case, only one choice of orientation is valid. There are two possible diagrams but one is obtain from the other by rotating the order one singularity and thus the diagrams are isomorphic.



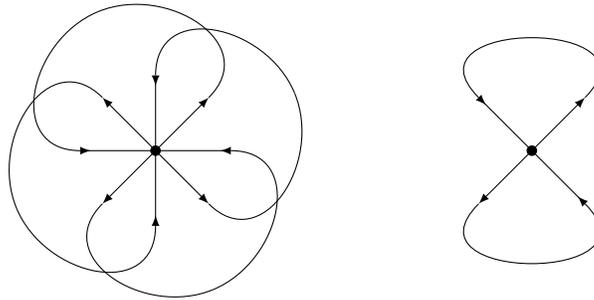
The prediagram of $\Gamma(X, \omega)$ in the type $(2, 6)$ case

We depict the only surface associated :



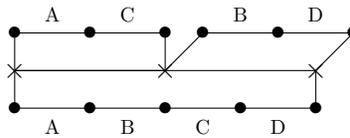
8.4 Stable cylinder decomposition of type (4, 6)

In this case, only one choice of orientation is valid. There are two diagrams but one is obtained from the other by rotating the order one singularity.



The preimage of $\Gamma(X, \omega)$ in the type (2, 6) case

We depict the only surface associated:



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