Weak type inequalities in noncommutative Lp-spaces

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0.1 Introduction (Français)

Cette thèse traite deux problèmes issues de la théorie de l'intégration non-commutative : les inégalités de Khintchine et les opérateurs de Calderón-Zygmund. Elle se concentre sur l'étude d'inégalités de type faible et plus généralement fait souvent intervenir des espaces quasi-Banach. La théorie de l'interpolation y joue un rôle prépondérant et multiple. Le premier chapître est d'ailleurs entièrement dédié à des questions d'interpolation des espaces L_p classiques. La majeur partie du travail présentée est extraite de deux articles ([5], [4]) et d'une prépublication [3]. Le chapitre 2 est principalement dédié aux définitions de base de l'intégration non-commutative et à des résultats techniques tandis que les chapitres 3 et 4 contiennent les résultats principaux de cette thèse, sur les inégalités de Khintchine non-commutatives et les opérateurs de Calderón-Zygmund respectivement.

Analyse harmonique non-commutative

Les espaces L_p non-commutatifs ont été définies pour la première fois au début du développement de la théorie des algèbres de von Neumann par Segal [56] et Dixmier. Cependant, seuls les espaces L_1, L_2 et L_∞ sont réellement pertinents au sein de cette théorie et il fallut attendre une vingtaine d'années pour que le sujet reçoive une véritable attention, venue tout d'abord de la physique mathématique ([18],[40]). Depuis lors, ils ont été l'objet d'un intérêt soutenu, à la fois pour étudier le parallèle que l'on peut établir avec l'analyse classique et pour comprendre leurs propriétés spécifiques. Bien que ceci ne soit pas immédiat, ils sont similaires en de nombreux points aux espaces L_p classiques. Ils vérifient les même propriétés de dualité, de convexité, d'interpolation et certaines propriétés géométriques, justifiant le fait de les considérer, sur un plan intuitif, comme des espaces L_p où les fonctions auraient été remplacées par des opérateurs et par conséquent ne commutent pas toujours. Notons donc que les espaces d'intégration non-commutatifs ne proviennent pas en général d'espaces mesurés au sens classique, la notion de point et d'ensemble mesurable y disparait et tout est exprimé au niveau purement fonctionnel. Cette idée se retrouve par exemple dans la définition des groupes quantiques (généralisation non-commutative de l'algèbre des fonctions sur un groupe), permet aussi de considérer les $\mathbb{C}^*$-algèbres comme des espaces topologiques non-commutatifs et apparaît de façon générale comme un des concepts unificateurs de la géométrie non-commutative de Connes. Se placer dans un cadre plus général a évidemment un prix et des inégalités évidentes dans le cas classique deviennent fausses dans le cadre non-commutatif. Par exemple, le fait que $|x+y| \leq |x| + |y|$ n'est plus vrai, en général, pour des opérateurs x et y. Par conséquent, généraliser des résultats classiques, en commençant par l'inégalité de Hölder, demande souvent d'inventer de nouvelles techniques.

Rappelons maintenant quelques résultats et concepts fondamentaux qui permettent de contourner les difficultés posées par la non-commutativité. Premièrement, la décomposition polaire énonce que tout opérateur x peut s'écrire de façon essentiellement unique sous forme d'un produit $u\,|x|$ où u est unitaire et $|x|$ est positif. Cela est une façon par exemple de réduire de nombreux problèmes au cas des opérateurs positifs. Deuxièmement, la théorie spectrale et le calcul fonctionnel permettent de manipuler les opérateurs normaux, considérés individuellement, comme des fonctions et par exemple de définir la distribution d'un opérateur ou sa norme dans L_p. Le troisième point est un concept général de la théorie des algèbres de von Neumann : les projections orthogonales y jouent un rôle analogue à celui des fonctions indicatrices d'ensembles mesurables dans la théorie de l'intégration classique. Une partie de l'intuition spatiale de la théorie de la mesure classique est ainsi transférée à ce cadre purement fonctionnel. De façon beaucoup plus concrète, cela est aussi un moyen de construire des fonctions étagées non-commutatives (combinaisons linéaires de projections orthogonales) qui forment un ensemble dense dans les opérateurs normaux (pour n'importe quelle norme L_p) et sont bien plus faciles à manipuler que des opérateurs quelconques. Le dernier point est la méthode d'interpolation. Contrairement aux trois premiers, elle ne provient pas de la théorie des opérateurs mais de celle des espaces L_p classiques. Elle sera présentée dans plus de détails par la suite. Grossièrement, elle permet d'obtenir des résultats sur tous les espaces L_p à partir d'inégalités dans des espaces spécifiques (souvent L_1, L_2 ou L_∞).

Parmi différents points de vue que l'on peut choisir de prendre pour l'étude des espaces L_p non-commuattifs, cette thèse se place dans le cadre de l'analyse harmonique non-commutative. Le but de ce domaine est de comprendre le comportement et en particulier les propriétés de bornitude de certains opérateurs naturels sur les espaces L_p non-commutatifs : la transformée de Fourier, les multiplicateurs de Fourier et de Schur, le calcul fonctionnel ... Une première étape est de généraliser au monde non-commutatif les outils utilisés en analyse harmonique classique. Parmi ceux-ci, nous sommes particulièrement intéressés par les opérateurs de Calderón-Zygmund et les inégalités de Khintchine. Ces deux notions ont été vastement étudiées en analyse réelle et en probabilité, et admettent même des généralisations à valeurs vectorielles. Cependant, elles possèdent aussi des comportements spécifiques dans le contexte non-commutatif : les inégalités de Khintchine prennent une nouvelle forme introduite par Lust-Piquard dans [36] et les opérateurs de Calderón - Zygmund admettent une inégalité de type faible particulière [41].

Interpolation des espaces L_p classiques

A partir de cette sous-section et tout au long du texte, on manipulera de nombreuses inégalités mais on ne sera intéressé que par le caractère qualitatif des résulats obtenus et on ne tentera pas de calculer les constantes qui apparaissent dans ces inégalités. Ainsi, pour deux quantités $A(x)$ et $B(x)$, on notera $A(x) \lesssim B(x)$ ou $A(x) \lesssim_C B(x)$ si il existe une constante C telle que pour tout x, $A(x) \leq CB(x)$ et de même, $A(x) \approx B(x)$ ou $A(x) \approx_C B(x)$ si il existe une constante C telle que pour tout x, $\frac{1}{C}B(x) \leq A(x) \leq CB(x)$.

L'étude d'inégalités de type faible, alors même que le comportement des objets considérés est bien connu dans les espaces L_p, est principalement motivée par la théorie de l'interpolation. Pour expliquer cette idée, commençons avec un théorème d'interpolation fondamental dans la théorie des espaces L_p. Soit Ω un espace mesuré σ-fini et T un opérateur sur les fonctions mesurables sur Ω.

Théorème 0.1.1 (Riesz-Thorin)**.** *Soit $p < q \in (O, \infty]$. Si T est borné de $L_p(\Omega)$ vers $L_p(\Omega)$ et de $L_q(\Omega)$ vers $L_q(\Omega)$ alors T est borné de $L_r(\Omega)$ vers $L_r(\Omega)$ pour tout r dans (p, q).*

Cela signifie que l'ensemble des valeurs p pour lesquelles T est borné de $L_p(\Omega)$ vers $L_p(\Omega)$ est un interval. Une motivation pour l'introduction de nouvelles normes et espaces en analyse harmonique est l'étude d'opérateurs T aux bornes de cet interval. Par exemple, si T est un opérateur de Calderón-Zygmund ou plus particulièrement la transformée de Hilbert, T est borné sur $L_p(\mathbb{R})$ pour $p \in (1, \infty)$ mais pas pour $p = 1$ ou $p = \infty$. Cependant, T est borné de $L_1(\mathbb{R})$ vers $L_{1,\infty}(\mathbb{R})$ (on dit que T est de type $(1, 1)$-faible) et de L_∞ vers BMO. Connaître ce type d'inégalités aux bornes est ensuite traduit par la théorie de l'interpolation en des inégalités sur les espaces L_p. L'exemple premier d'un tel résultat est le théorème suivant :

Théorème 0.1.2 (Marcinkiewicz)**.** *Soit $p < q \in (0, \infty)$. Si T est de type (p, p)-faible et de type (q, q)-faible alors T est borné de $L_r(\Omega)$ vers $L_r(\Omega)$ pour tout $r \in (p, q)$.*

Ainsi, les inégalités de type faible peuvent être suffisantes pour montrer la bornitude de certains opérateurs. C'est par exemple le schéma de preuve qui a été utilisé par Calderón et Zygmund pour démontrer la bornitude des intégrales singulières. Une autre application possible qui motive la recherche d'inégalités à une borne p est qu'elles peuvent permettre d'obtenir un contrôle optimal sur la norme de T sur L_q lorsque q tend vers p.

Définie abstraitement, la théorie de l'interpolation traite de couples généraux (A, B) d'espaces quasi-Banach et, sans donner de définition rigoureuse, un espace interpolé E entre A et B est un espace quasi-Banach tel que pour tout opérateur T borné sur A et sur B, T est automatiquement borné sur E. Pour appliquer les résultats de cette théorie, il est donc utile de savoir décrire les espaces interpolés des couples d'espaces qui nous intéressent. Ceci n'est pas possible en général mais peut être accompli de façon relativement satisfaisante pour le couple $(L_p(0, \infty), L_q(0, \infty))$ pour $p, q \in (0, \infty]$ en utilisant la notion de K-fonctionelle et en particulier de K-monotonicité.

Définition 0.1.1 (K-fonctionnelle)**.** *Soit (A, B) un couple d'espaces quasi-Banach telle que l'on puisse définir leur somme $A + B$. Soit $x \in A + B$ alors la K-fonctionelle de x pour le couple (A, B) est une fonction $(0, \infty) \to \mathbb{R}$ définie pour tout $t > 0$ par :*

$$K_t(x, A, B) = \inf_{y+z=x} \left\{ \|y\|_A + t\|z\|_B \right\}.$$

Cette notion a plusieurs applications en théorie de l'interpolation. Elle permet de définir la méthode d'interpolation réelle et ainsi de voir les espaces de

Lorentz comme des interpolés d'espaces L_p. Nous sommes plus intéressés pour le moment par ses applications à la théorie de l'interpolation abstraite.

Définition 0.1.2 (K-monotone)**.** *Soient A, B, E des espaces quasi-Banach tels que $E \subset A+B$. On dit que E est K-monotone pour le couple (A, B) si pour tout x dans E et tout y dans $A+B$,*

$$K(y, A, B) \lesssim K(x, A, B) \Rightarrow y \in E, \left\|y\right\|_E \lesssim \left\|x\right\|_E.$$

Il est facile de voir que tout espace K-monotone pour un couple (A, B) est une espace interpolé pour ce couple. Dans certains cas, la réciproque est aussi vraie. Le théorème suivant rassemble certains résultats que l'on peut trouver dans la littérature allant dans cette direction.

Théorème 0.1.3. *Tous les espaces interpolés des couples suivants sont K-monotones :*

- *(Lorentz-Shimogaki) $p \geq 1$, $(L_p(0, \alpha), L_\infty(0, \alpha))$ et $(L_1(0, \alpha), L_p(0, \alpha))$,*
- *(Sparr) $p, q \geq 1$, $(L_p(\Omega), L_q(\Omega))$ pour tout espace mesuré σ-fini Ω,*
- *(Sparr, Cwikel) $p, q \in (0, \infty)$, $(L_p(0, \alpha), L_q(0, \alpha))$,*
- *(Cwikel) $p \in (0, \infty]$, (ℓ^p, ℓ^∞).*

Nous nous sommes intéressés à une généralisation au cas quasi-Banach du théorème de Lorentz et Shimogaki et avons obtenu le résultat suivant :

Théorème 0.1.4. *Soit $\alpha \in (0, \infty]$ et $p \in (0, \infty)$. Alors tous les espaces interpolés du couple $(L_p(0, \alpha), L_\infty(0, \alpha))$ sont K-monotones.*

La démonstration ne pose pas de difficultés majeures. Elle consiste, pour tout couple de fonctions f, g tel que $K(f, L_p(0, \alpha), L_\infty(0, \alpha)) \leq K(g, L_p(0, \alpha), L_\infty(0, \alpha))$, à construire explicitement un opérateur T borné sur L_p et L_∞ tel que $Tf = g$. Notons que dans ce cas, la K-fonctionnelle peut se calculer explicitement [21]. En effet, on a, pour tout $t > 0$:

$$K_t(f, L_p(0, \alpha), L_\infty(0, \alpha)) \approx \left(\int_0^{t^p} f^p \right)^{1/p}.$$

Nous donnons aussi une caractérisation des espaces interpolés pour le couple (L_p, L_q), $p, q \in (0, \infty)$ qui fait intervenir deux notions de monotonicité.

Définition 0.1.3. *Soit E un espace de fonctions sur $(0, \infty)$. On dit que E est p-monotone à gauche si pour toutes fonctions $f \in E$ et $g \in L_0(0, \infty)$ telles que pour tout $t > 0$, $\int_0^t (f^*)^p \geq \int_0^t (g^*)^p$ alors $g \in E$ et $\left\|g\right\|_E \lesssim \left\|f\right\|_E$.*

De même, on dit que E est p-monotone à droite si pour toutes fonctions $f \in E$ et $g \in L_0(0, \infty)$ telles que pour tout $t > 0$, $\int_t^\infty (f^*)^p \geq \int_t^\infty (g^*)^p$ alors $g \in E$ et $\left\|g\right\|_E \lesssim \left\|f\right\|_E$.

Nous prouvons le théorème suivant :

Théorème 0.1.5. *Soit E un espace de fonctions sur $(0, \alpha)$, $\alpha > 0$ et soit $0 < p < q < \infty$. Alors E est p-monotone à gauche et q-monotone à droite si et seulement si E est un espace interpolé entre $L_p(0, \alpha)$ et $L_q(0, \alpha)$.*

Notre preuve se base sur le résultat de Sparr. Nous prouvons en réalité qu'un espace est p-monotone à gauche et p-monotone à droite si et seulement si il est K-monotone pour le couple (L_p, L_q). Nous montrons aussi que si l'on suppose de plus que l'espace E a la propriété de Fatou alors si E est p-convexe, E est p-monotone à gauche et si E est q-concave, E est q-monotone à droite. Rappelons qu'un espace E est dit naturel si il existe $p > 0$ tel que E est p-convexe. Nous obtenons le corollaire suivant :

Corollaire 0.1.6. *Soit E un espace naturel de fonctions sur $(0, \alpha)$, ayant la propriété de Fatou. Alors E est q-monotone à droite si et seulement si il existe $0 < p < q$ tel que $E \in (L_p(0, \alpha), L_q(0, \alpha))$.*

Ceci résoud, sous des hypothèses qui son mineures en pratique, une conjecture de Levitina, Sukochev et Zanin [34] dans le cadre des espaces de fonctions. Notons que le problème reste pleinement ouvert pour les espaces de suites. En effet, on ne sait même pas si tous les espaces interpolés du couple (ℓ^p, ℓ^q) sont K-monotones.

Inégalités de Khintchine non-commutatives

La motivation originale pour étudier les problèmes d'interpolation abstraite présentés dans la section précédente vient des inégalités de Khintchine non-commutatives. Celles-ci sont essentielles pour comprendre les suites inconditionnelles dans les espaces L_p non-commutatifs. Elles ont joué un rôle important dans le développement de l'analyse harmonique non-commutative ces trente dernières années, à la fois en tant qu'outil et en apportant la bonne définition des fonctions carré dans ce contexte qui est à l'origine de développements majeurs comme les inégalités de martingales non-commutatives [49]. Commençons par rappeler les inégalités de Khintchine classiques. Soit (ξ_i) une suite de variable de Rademacher inépendentes définies dans $L_\infty(0,1)$ et $(a_i)_{i\in\mathbb{N}}$ une suite finie de nombres complexes. Alors pour tout $p \in (0, \infty)$:

$$\big\| \sum\nolimits_{i\geq 0} a_i\xi_i \big\|_p \approx_{c_p} \big\| \sum\nolimits_{i\geq 0} a_i\xi_i \big\|_2.$$

Cette équivalence s'étend immédiatement à des sommes dont les coefficients prennent leurs valeurs dans des espaces L_p classiques. Soit Ω un espace mesuré, $p \in (0, \infty)$ et $(f_i)_{i\in\mathbb{N}}$ une suite finie dans $L_p(\Omega)$. Alors :

$$\big\| \sum\nolimits_{i\geq 0} f_i\xi_i \big\|_{L_p(\Omega\otimes(0,1))} \approx_{c_p} \Big\| \Big(\sum\nolimits_{i\geq 0} |f_i|^2\Big)^{1/2} \Big\|_{L_p(\Omega)}. \tag{1}$$

Sous cette seconde forme, les inégalités de Khintchine apparaissent fréquemment en analyse harmonique, pour évaluer des fonctions carrés ou étudier la

$\mathcal{R}$-bornitude de familles de fonctions, un problème qui est essentiel lorsque l'on essaie de comprendre les propriétés du calcul fonctionnel sur les espaces L_p. Notons que le terme de gauche de l'équivalence peut être défini dans tout espace de Banach X plutôt que $L_p(\Omega)$ en voyant $\sum_{i\geq 0} f_i\xi_i$ comme un élément de $L_p((0,1),X)$, et une version des inégalités de Khintchine (inégalités de Khintchine-Kahane) s'appliquent à ce niveau de généralité. Plus précisément, pour toute suite finie $(x_i)_{i\geq 0}$ dans X, et tout $p \in (1,\infty)$,

$$\|\sum_{i\geq 0} x_i\xi_i\|_{L_p((0,1),X)} \approx_{c_p} \|\sum_{i\geq 0} f_i\xi_i\|_{L_2((0,1),X)}$$

Cependant, nous sommes d'avantage intéressés, dans le contexte de l'intégration non-commutative, à trouver un analogue du terme de droite de (1), qui est plus facile à calculer ou majorer que le terme de gauche. Notons tout de suite que la même formule ne peut pas fonctionner dans le cadre non-commutatif car il y a deux notions possibles de fonctions carrés (ligne et colonne) qui ne sont pas équivalentes dans L_p dès que $p \neq 2$. Soit $\mathcal{M}$ un espace mesuré non-commutatif, $\mathcal{M}_c \subset \mathcal{M}$ les opérateurs à support fini et $S(\mathcal{M})$ l'ensemble des suites finies de $\mathcal{M}_c$. On rappelle qu'un espace symétrique est un espace quasi-Banach de fonctions tel que la (quasi-)norme ne dépende que de la distribution du module de la fonction. Puisque l'on peut définir la distribution du module d'un opérateur, ces espaces admettent des généralisations non-commutatives [31] similaires à celles des espaces L_p. En effet, soit E un espace symétrique de fonctions sur $(0,\infty)$, notons par $\mu(x)$ la réarrangée décroissante d'un opérateur x (i.e. une fonction sur $(0,\infty)$ ayant la même distribution que $|x|$). On peut alors définir :

$$\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_E,$$

et ainsi l'espace $E(\mathcal{M})$. Par abus de notation, on notera $\|x\|_E := \|x\|_{E(\mathcal{M})}$.

Définition 0.1.4 (Normes ligne et colonne)**.** *Soit $x \in S(\mathcal{M})$, E un espace symétrique. On définit les normes colonne et ligne comme suit :*

$$\|x\|_{C_E} := \|\left(\sum_{i\in\mathbb{N}} x_i^* x_i\right)^{1/2}\|_E \textit{ and } \|x\|_{R_E} := \|\left(\sum_{i\in\mathbb{N}} x_i x_i^*\right)^{1/2}\|_E.$$

Un équivalent peut tout de même être trouvé en combinant les deux fonctions carrés.

Théorème 0.1.7 (Inégalités de Khintchine non-commutatives)**.** *Soit $p \in [2,\infty)$. Alors :*

$$\|\sum_{i\in\mathbb{N}} x_i \otimes \xi_i\|_p \approx \max\left(\|x\|_{R_p}, \|x\|_{C_p}\right).$$

Soit $p \in (0,2]$. Alors :

$$\|\sum_{i\in\mathbb{N}} x_i \otimes \xi_i\|_p \approx \inf_{y+z=x}\left\{\|z\|_{C_p} + \|y\|_{R_p}\right\}.$$

Ce résultat est du à Lust-Piquard pour $p \in (1,\infty)$ [36], Lust-Piquard et Pisier pour $p=1$ [37] et plus de vingt ans plus tards, Pisier et Ricard [47] pour

$p < 1$. Par la méthode d'interpolation réelle, le résultat s'étend à la plupart des espaces de Lorentz qui sont des espaces de Banach mais pas à $L_{2,\infty}$ ou $L_{1,\infty}$. Pour $L_{2,\infty}$, cela est du au fait que pour écrire $L_{2,\infty}$ comme un espace d'interpolation de L_p et L_q, il est nécessaire que $p < 2$ et $q > 3$ et donc d'utiliser les deux formules qui ne s'interpolent pas bien. De même, pour écrire $L_{1,\infty}$ comme un espace d'interpolation de L_p et L_q, il est nécessaire que $p < 1$ ce qui est problématique puisque la fermeture de l'espace vectoriel généré par les ξ_i n'est pas complémenté dans $L_p(0,1)$ pour $p < 1$. Ces deux exemples, qui sont des espaces relativement communs, furent la motivation originale de notre travail. Supposons maintenant que les ξ_i sont une suite de variables aléatoires orthonormales arbitraires dans un espace de probabilité non-commutatif. Nous obtenons le théorème suivant :

Théorème 0.1.8. *Soit $p < 1$. Supposons que pour tout $x \in S(\mathcal{M})$,*

$$\|\sum\nolimits_{i\geq 0} x_i \otimes \xi_i\|_p \gtrsim \inf_{y+z=x} \left\{\|z\|_{C_p} + \|y\|_{R_p}\right\}.$$

Alors pour tout espace interpolé E entre L_p et L_∞ et $x \in S(\mathcal{M})$,

$$\|\sum\nolimits_{i\geq} x_i \otimes \xi_i\|_E \gtrsim \inf_{y+z=x} \left\{\|z\|_{C_E} + \|y\|_{R_E}\right\}.$$

De plus, la décomposition dans le terme de gauche peut être choisie indépendante de E.

Ce théorème implique que pour des variables de Rademacher indépendantes ou des unitaires de Haar libres, les inégalités de Khintchine sont vérifiées dans $L_{1,\infty}$, puisque, comme nous l'avont précédemment mentionné, les inégalités de Khintchine non-commutatives sont vraies dans L_p pour $p < 1$ pour ces variables (par [48]).

Corollaire 0.1.9. *Soit $(\xi_i)_{i\in\mathbb{N}}$ une suite de variables de Rademacher inépendantes et $x \in S(\mathcal{M})$. Alors :*

$$\|\sum\nolimits_{i\geq 0} x_i \otimes \xi_i\|_{1,\infty} \approx \inf_{y+z=x} \left\{\|z\|_{C_{1,\infty}} + \|y\|_{R_{1,\infty}}\right\}.$$

La preuve du théorème est basée sur l'idée de considérer des décompositions $x = y+z$ qui sont presques optimales dans le sens où elles minimisent la quantité $\|y\|_{R_p}^p + \|z\|_{C_p}^p$. Pour ces décompositions, il n'est pas difficile de montrer, en utilisant les inégalités de Khintchine dans L_p pour des suites de la forme $(ex_i)_{i\in\mathbb{N}}$ où e est une projection dans $\mathcal{M}$, que la K-fonctionnelle en ligne de y pour le couple (L_p, L_∞) est dominée par la K-fonctionnelle de $\sum_{i\in\mathbb{N}} x_i \otimes \xi_i$ pour le couple (L_p, L_∞). De même, par dualité, on majore la K-fonctionnelle en colonne de z par la K-fonctionnelle de $\sum_{i\in\mathbb{N}} x_i \otimes \xi_i$. Une fois une majoration de la K-fonctionnelle obtenue, le théorème de Lorentz-Shimogaki généralisé permet de l'étendre à tous les espaces interpolés entre L_p et L_∞.

Dans le cas particulier de L_1, on peut montrer qu'il existe toujours des décompositions optimales. Cela simplifie la preuve du théorème précédent et permet aussi de pousser plus loin l'étude de ces décompositions et d'obtenir une certaine indentité algébrique qui les caractérise. A l'aide de cette dernière, qui semble jouer le rôle d'une décomposition polaire pour les suites, nous retrouvons la plupart des résultats connus sur les inégalités de Khintchine dans le cas libre.

Théorème 0.1.10. *Soit $x = (x_i)_{i\geq 0}$ une suite finie d'éléments de $\mathcal{M}_c$. Il existe $\alpha, \beta \in \mathcal{M}_c^+$, et $u = (u_i)_{i\geq 0} \in \mathcal{M}_c^{\mathbb{N}}$ tels que :*

- $s(\alpha) \leq \sum_{i\geq 0} u_i u_i^* \leq 1,$
- $s(\beta) \leq \sum_{i\geq 0} u_i^* u_i \leq 1,$
- *pour tout $i \geq 0$, $x_i = u_i\beta + \alpha u_i$.*

De plus, supposons que :

- *les ξ_i forment une suite de variables orthonormées dans $L_2(\mathcal{A})$*
- *les ξ_i vérifient l'inégalité de Khintchine dans L_∞ i.e pour toute suite finie $x = (x_i)_{i\geq 0} \in \mathcal{M}^{\mathbb{N}}$:*

$$\max\left(\left\|x\right\|_{R_\infty}, \left\|x\right\|_{C_\infty}\right) \approx_{c_\infty} \left\|\sum_{i\geq 0} x_i \otimes \xi_i\right\|_\infty.$$

Alors pour tout $p \in (0,\infty)$ et E un espace interpolé entre L_p et L_∞:

$$\left\|\alpha\right\|_E + \left\|\beta\right\|_E \approx_c \left\|\sum_{i\geq 0} x_i \otimes \xi_i\right\|_E,$$

où c dépend seulement de p et c_∞.

La preuve du théorème s'effectue en deux étapes. Premièrement, l'identité algébrique est obtenue par un argument de dualité. On considére une décomposition optimale $x = y + z$ et l'on écrit les décompositions polaires de y et z. L'optimalité se traduit par le fait que ces décompositions saturent certaines inégalités, et donc vérifient une relation qui donne l'identité attendue. Dans un second temps, on utilise cette identité pour obtenir une nouvelle fois des majorations en terme de K-fonctionnelle, qui impliquent le théorème ci-dessus. Notons que pour $p \geq 1$, les calculs sont élémentaires mais pour $p < 1$, nous devons utiliser une inégalité de Hölder pour les commutateurs, prouvée récemment par Ricard [54]. Il s'agit d'un résultat technique très similaire à celui utilisé par Pisier et Ricard dans [48] pour obtenir les inégalités de Khintchine pour $p < 1$. Il est important de remarquer que le théorème donne un équivalent déterministe de la norme de $\sum_{i\geq 0} x_i \otimes \xi_i$ dans $L_{2,\infty}$ et répond ainsi d'une certaine façon à une question posée par Lust-Piquard et Xu dans [38]. Cet équivalent semble difficile à manipuler en vue d'applications mais il a l'avantage de prendre la même forme dans tous les interpolés d'espaces L_p. En particulier, les cas $p < 2$ et $p > 2$ sont traités de la même façon, ce qui est une nouveauté dans l'étude des inégalités de Khintchine non-commutatives.

Finalement, nous donnons, sous des conditions techniques sur les espaces considérés et dans le cadre des unitaires de Haar libres, une charactérisation des espaces dans lesquels les formules usuelles pour les inégalités de Khintchine sont valides.

Théorème 0.1.11. *Soit E un espace symmétrique ayant la propriété de Fatou et $\mathcal{M} = \mathcal{B}(\ell^2)\overline{\otimes}L_\infty(0,1)$. Alors E est 2-monotone à gauche si et seulement si pour toute suite finie $x \in \mathcal{M}_c^{\mathbb{N}}$, $\|x\|_{R_E \cap C_E} \approx \|\sum_{i\geq 0} x_i \otimes \xi_i\|_E$.*

Supposons de plus qu'il existe $p > 0$ tel que E est un espace interpolé entre L_p et L_∞. Alors E est 2-monotone à droite si et seulement si pour toute suite finie $x \in \mathcal{M}_c^{\mathbb{N}}$, $\|x\|_{R_E + C_E} \approx \|\sum_{i\geq 0} x_i \otimes \xi_i\|_E$.

Ceci vient en complément à la formule déterministe donnée dans le théorème précédent, montrant par exemple que dans le cas de $E = L_{2,\infty}$, aucune des deux formules usuelles ne fonctionnent. La preuve du théorème se base essentiellement sur l'utilisation du théorème de Schur-Horn.

Inégalités de type faible pour les intégrales singulières

Dans le dernier chapître, nous présentons le travail qui apparaît dans l'article [5]. Il est presque indépendant du reste de la thèse, si ce n'est pour les définitions de base et une petite application des inégalités de Khintchine dans $L_{1,\infty}$ donnée à la fin du chapître. Il traite des opérateurs de Calderón-Zygmund ou intégrales singulières i.e. un certain type d'opérateurs T définis de façon informelle par des expressions de la forme :

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy,$$

où $n \geq 1$, $x \in \mathbb{R}^n$ et $k : \mathbb{R}^{2n} \to \mathbb{C}$ est appelé le noyau de T et vérifie des conditions (que nous appellerons "conditions de Calderón-Zygmund" bien que cette terminologie ne soit pas standard) qui permettent de contrôler les variations de k mais laissent la possibilité d'avoir une singularité sur la diagonale. Une source d'exemples de tels noyaux est donnée par les noyaux de convolution, c'est-à-dire ceux pour lesquels il existe une fonction $K : \mathbb{R}^n \to \mathbb{C}$ telle que $k(x,y) = K(x-y)$. Dans ce cas, une condition de Calderón-Zygmund typique est :

$$|\nabla K(x)| \lesssim \frac{1}{|x|^{n+1}}.$$

Cette condition est rarement optimale mais facile à vérifier en pratique. Elle est vérifiée par les noyaux de la transformée de Riesz et de la transformée de Hilbert, qui sont des exemples fondamentaux de la théorie des intégrales singulières. Nous sommes intéressés par les potentielles généralisations non-commutatives du théorème suivant, que nous citons de façon informelle.

Theorem 0.1.12 (Calderón, Zygmund)**.** *Si T est borné sur $L_2(\mathbb{R}^n)$ et est associé à un noyau k vérifiant des conditions de Calderón-Zygmund alors T s'étend en un opérateur borné de $L_1(\mathbb{R}^n)$ vers $L_{1,\infty}(\mathbb{R}^n)$.*

Ce théorème implique par interpolation - plus précisément par le théorème de Marcinkiewicz cité plus haut - et par dualité que les opérateurs de Calderón-Zygmund sont bornés de $L_p(\mathbb{R}^n)$ vers $L_p(\mathbb{R}^n)$ pour tout $p \in (1,\infty)$. Par la même expression, T pourrait définir un opérateur de $L_p(\mathbb{R}^n, X)$ vers $L_p(\mathbb{R}^n, X)$

pour chaque espace de Banach X. Les conditions qu'il faut imposer sur X pour que la bornitude de T soit alors conservée ont été étudiée, la condition clé étant que X soit UMD. Les espaces L_p non-commutatifs sont justement UMD pour $p \in (0, \infty)$ et donc les intégrales singulières peuvent y être utilisées. Cependant, la théorie des opérateurs de Calderón-Zygmund à valeurs vectorielles ne produit pas un analogue non-commutatif du théorème de Calderón-Zygmund qui permettrait d'obtenir des bornes optimales pour la norme de T dans L_p vers L_p lorsque p tend vers 1. Dans [41], Parcet a généralisé le théorème 0.1.12 au cadre non-commutatif. Notons que dans le cadre non-commutatif, les méthodes directes ou d'interpolation faisant intervenir les espaces BMO peuvent être préférées aux inégalités de type faible qui sont parfois techniques. Toutefois, le résultat de Parcet a trouvé de nombreuses applications, parmi lesquels une en théorie de la perturbation.

Les interactions entre la théorie de la perturbation et l'anayse harmonique non-commutative ont débuté lorsque Potapov et Sukochev ont résolu un problème posé par Krein et resté longtemps intouché sur le calcul fonctionnel Lipschitzien dans les classes de Schatten [51]. Ils ont prouvé que pour tout p dans $(1, \infty)$, il existe une constante c_p telle que pour toute fonction Lipschitzienne $f : \mathbb{R} \to \mathbb{R}$ et tous opérateurs auto-adjoints x et y,

$$\big\|f(x) - f(y)\big\|_p \leq c_p \big\|f\big\|_{Lip} \big\|x - y\big\|_p, \tag{2}$$

où l'on suppose que $x - y \in L_p(\mathcal{M})$ et on définit :

$$\big\|f\big\|_{Lip} := \sup_{t \neq s} \frac{|f(t) - f(s)|}{|t - s|}.$$

Récemment, les deux auteurs, dans une collaboration avec Caspers et Zanin, ont donné une nouvelle preuve de (2) qui passe par une inégalité de type faible et décrit le comportement optimal de la constante c_p quand p tend vers 1. Elle est basée sur la théorie de Calderón-Zygmund et en particulier sur le théorème de Parcet. Le schéma de preuve employé - de réduire un problème à la bornitude d'un opérateur de Calderón-Zygmund - pourrait être appliquée à d'autres questions. C'est la raison principale pour laquelle le théorème de Parcet reçoit ces dernières années une attention importante. Cependant, la démonstration donnée par Parcet dans [41] est difficile et par conséquent difficile à adapter si l'on rencontre une situation dans laquelle le théorème ne s'applique pas immédiatement. Ceci a motivé notre travail. Nous présentons une preuve simplifiée de l'inégalité de type faible pour les intégrales singulières noncommutatives, en s'appuyant sur les mêmes idées que Parcet.

Commençons par donner un énoncé précis du théorème. Les conditions de Calderón-Zygmund nécessaires pour faire fonctionner la démonstration dans le cadre non-commutatif sont les suivantes :

Définition 0.1.5 (Lissité et Taille). *On dit que T a pour paramètre de Lipschitz γ ($0 < \gamma \leq 1$) si pour tout x,y et z dans $\mathbb{R}^n$ tels que $|x - z| \leq \frac{1}{2} |y - z|$, les inégalités de lissité suivantes sont vérifiées :*

$$|k(x,y)-k(z,y)| \lesssim \frac{|x-z|^\gamma}{|y-z|^{n+\gamma}},$$
$$|k(y,x)-k(y,z)| \lesssim \frac{|x-z|^\gamma}{|y-z|^{n+\gamma}}.$$

On dit que T vérifie l'hypothèse de taille si pour tout $x, y \in \mathbb{R}^n$,

$$\big\|k(x,y)\big\|_{\widetilde{\mathcal{M}}} \lesssim \frac{1}{|x-y|^n}.$$

Théorème 0.1.13. *On suppose que T a pour paramètre de Lipschitz γ ($0 < \gamma \leq 1$), vérifie l'hypothèse de taille et est $L_2(\mathbb{R}^n)$ borné sur $L_2(\mathbb{R}^n)$. Alors T s'étend en un opérateur borné de $L_1(\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}^n))$ vers $L_{1,\infty}(\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}^n))$.*

La preuve de ce théorème, dans le cas classique, repose sur une décomposition de fonction $f = g + b$, qui peut être interprétée dans le language des martingales. Cette interprétation permet de construire une version non-commutative de la décomposition, en utilisant la théorie bien développée des martingales non-commutatives et en particulier les projections de Cuculescu. Cependant, la décomposition obtenue dans le cadre non-commutatif ne satisfait pas toutes les bonnes propriétés de la décomposition classique. De nouveaux termes apparaissent et pour les majorer, le théorème de pseudolocalisation L_2 suivant est nécessaire. On définit $\mathcal{N} = \mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}^n)$ et $\mathcal{N}_k$ la sous-algèbre de $\mathcal{N}$ composée des fonctions constantes sur les cubes dyadiques de côté de longueur 2^{-k}.

Théorème 0.1.14 (Pseudolocalisation dans L_2)**.** *Soit $f \in L_2(\mathcal{N})$ and $s \in \mathbb{N}$. Pour tout $k \in \mathbb{Z}$, soient A_k et B_k des projections dans $\mathcal{N}_k$ telles que $A_k^\perp df_{k+s} = df_{k+s}B_k^\perp = 0$. Pour tout entier positif impair d, on définit :*

$$A_k = \textstyle\sum_{Q\in\mathcal{Q}_k} A_Q \mathcal{I}_Q,\ A_Q \in \mathcal{M} \text{ and define } dA_k := \bigvee_{Q\in\mathcal{Q}_k} A_Q \mathcal{I}_{dQ}.$$

De même, on définit dB_k. Soit :

$$A_{f,s} := \bigvee_{k\in\mathbb{Z}} 5A_k \text{ and } B_{f,s} := \bigvee_{k\in\mathbb{Z}} 5B_k.$$

Soit T un opérateur associé à un noyau k de paramètre de Lipschitz γ, vérifiant la condition de taille, et borné de $L_2(\mathcal{N})$ vers $L_2(\mathcal{N})$. Alors pour tout $s \in \mathbb{N}$ et $f \in L_2(\mathcal{N})$ on a :

$$\big\|A_{f,s}^\perp(Tf)\big\|_2 \lesssim 2^{-\frac{\gamma s}{2}}\big\|f\big\|_2 \text{ and } \big\|(Tf)B_{f,s}^\perp\big\|_2 \lesssim 2^{-\frac{\gamma s}{2}}\big\|f\big\|_2.$$

Ce théorème est cité pour des fonctions prenant leurs valeurs dans un espace mesuré non-commutatif mais il est en réalité de nature commutative. Il quantifie la vitesse de décroissance des intégrales singulières loin de la diagonale. Ceci n'avait jamais été utilisé auparavant en analyse harmonique classique mais est crucial lorsque l'on traite le cas non-commutatif. La preuve de la pseudolocalisation L_2 revient à démontrer un résultat de presque-orthogonalité. Dans cette thèse, seules des techniques élémentaires sont employées : le lemme de Schur pour contrôler la norme sur L_2 d'un opérateur associé à un noyau et une décomposition judicieuse de T. La principale nouveauté de notre approche, par rapport à celle de Parcet, est que nous interprétons les calculs fait en terme de "modification de noyau". Par exemple, si le calcul montre que :

$$A_{f,s}(x)Tf(x) = A_{f,s}(x)\int_{\mathbb{R}^n} k(x,y)f(y)dy = A_{f,s}(x)\int_{\mathbb{R}^n} k'(x,y)f(y)dy,$$

nous introduisons un nouvel opérateur T' associé au noyau k'. Ceci permet de travailler à la fois au niveau des opérateurs et au niveau des noyaux et ainsi d'améliorer la lisibilité de la démonstration et de la raccourcir. Comme conséquence de cette simplification, nous pouvons rapidement retrouver une version de la pseudolocalisation L_p, un résultat similaire à celui obtenu par Hytönen dans [22].

Théorème 0.1.15. *Avec les même hypothèses que le théorème précédent, pour tout $1 < p < \infty$, $s \in \mathbb{N}$ et $f \in L_p(\mathcal{N})$, il existe $\theta_p > 0$ tel que :*

$$\big\|A_{f,s}^{\perp}(Tf)\big\|_p \lesssim 2^{-\gamma s\theta_p}\big\|f\big\|_p \text{ et } \big\|(Tf)B_{f,s}^{\perp}\big\|_p \lesssim 2^{-\gamma s\theta_p}\big\|f\big\|_p,$$

Les simplifications de la preuve permettent aussi de repérer précisément l'utilisation de chaque hypothèse pendant la démonstration. On en déduit ainsi une version du théorème pour des noyaux à valeur opérateur, à condition que les valeurs prises par le noyau commutent avec celles prisent par la fonction. Ceci n'est qu'une petite amélioration du résultat original de Parcet mais, en la combinant avec les inégalités de Khintchine dans $L_{1,\infty}$, on retrouve directement l'inégalité de type faible pour les intégrales singulières avec un noyau à valeur dans un espace de Hilbert, obtenues pour la première fois par Mei et Parcet dans [39].

Théorème 0.1.16. *Soit T un opérateur associé à un noyau k à valeurs dans ℓ^2. On suppose de plus que T est borné de $L_2(\mathcal{N})$ vers $R_2(\mathcal{N})$ et que k satisfait les hypothèses de lissité et de taille. Alors T est borné de $L_1(\mathcal{N})$ vers $R_{1,\infty}(\mathcal{N}) + C_{1,\infty}(\mathcal{N})$ et de $L_p(\mathcal{N})$ vers $R_p(\mathcal{N}) + C_p(\mathcal{N})$ pour $1 < p < 2$.*

0.2 Introduction (English)

This thesis mainly deals with two different topics in the theory of noncommutative integration: Khintchine inequalities and singular integrals. It focuses on weak type inequalities and the quasi-Banach theory. Interpolation theory plays an important role in this text, as a mean of proof or as an incentive and is even the main topic of the first chapter. The major part of the material presented is extracted from two articles ([5],[4]) and a preprint [3]. Chapter 2 is mainly dedicated to basic definitions in noncommutative integration and technical results whereas chapter 3 and 4 contain the main results of the thesis on noncommutative Khintchine inequalities and singular integrals, respectively.

Noncommutative harmonic analysis

Noncommutative L_p-spaces were introduced as a byproduct of the theory of von Neumann algebras by Segal [56] and Dixmier in the 50s. They serve so far no purpose as an invariant of von Neuman algebras but found applications in mathematical physics in the 70's ([18],[40]). From there on, they have been the object of sustained interest, both to investigate the parallel they hold with classical integration and harmonic analysis and to understand their specific properties. Though not at first glance, noncommutative L_p-spaces bear remarkable similarities with classical L_p-spaces. Their duality, convexity, interpolation and some geometrical properties remain the same and so they can be intuitively thought as classical L_p-spaces, where the functions are replaced by operators and therefore do not commute. However, some basic identities that are true in the classical setting become false in the noncommutative world. For example, the fact that $|x+y| \leq |x| + |y|$ is not longer true in general for operators x and y. Hence, generalising classical results - starting with Hölder's inequality - to the noncommutative setting, often requires to invent new techniques.

Let us mention a few tools which are fundamental in doing so. The first one is the polar decomposition, which states that any operator x can be written as a product $u\,|x|$ of a unitary u and a positive operator $|x|$. It allows for example to reduce many problems to the case of positive operators. The second one is spectral theory and functional calculus, which enables to manipulate a single normal operator as a function and for example to define its distribution. The third one is more of a general idea in the theory of von Neumann algebras that orthogonal projections are the noncommutative equivalent of measurable sets (more precisely, characteristic functions of measurable sets since noncommutative integration only exists on the functional level). This allows to translate some of the intuition gained in classical integration in the noncommutative world or, on a purely technical level, provides a useful dense subset of normal operators given by linear combination of orthogonal projections (noncommutative step functions) which is used as a starting point for numerous proofs. Finally, the method of interpolation: contrary to the three other points, it does not come from the operator world but from the classical theory of L_p-spaces and it enables to obtain results on all L_p-spaces from the study of a few. More will be said about it later in this introduction.

Among various points of view that can be taken when studying noncommutative L_p-spaces, this thesis belongs to noncommutative harmonic analysis. The goal of this branch is to understand the behaviour and in particular the boundedness properties of certain natural operators on noncommutative L_p-spaces: Fourier transform, Schur and Fourier multipliers, functional calculus ... A first step in doing so is to generalize tools from the classical setting. As mentioned before, we are particularly interested in singular integrals and Khintchine inequalities. Note that vector-valued versions which apply to noncommutative L_p-spaces have been inverstigated. But in this noncommutative context, more can be said: Khintchine inequalities take a new form first introduced by Lust-Piquard in [36] and singular integrals admit a particular weak type inequality proved by Parcet in [41].

Interpolation theory of L_p-spaces

A motivation to push further the investigation of these tools, whose behaviour is already well-understood on L_p-spaces, comes from interpolation theory. To explain this idea, let us start with a fundamental interpolation theorem in the theory of L_p-spaces. Let Ω be a σ-finite measure space and T an operator acting on measurable functions on Ω.

Theorem 0.2.1 (Riesz-Thorin)**.** *Let $p < q \in (0, \infty]$. If T is bounded from $L_p(\Omega)$ to $L_p(\Omega)$ and from $L_q(\Omega)$ to $L_q(\Omega)$ then T is bounded from $L_r(\Omega)$ to $L_r(\Omega)$ for all $r \in (p, q)$.*

This means the set of values p such that T is bounded from $L_p(\Omega)$ to $L_p(\Omega)$ is an interval. One motivation for the introduction of various spaces in harmonic analysis is to study the behaviour of T at the endpoints of this interval. If T is a Calderón-Zygmund operator for example or more particularly the Hilbert transform then T is bounded on $L_p(\mathbb{R})$ for $p \in (1, \infty)$ but not for $p = 1$ and $p = \infty$. However, T is bounded from L_1 to $L_{1,\infty}$ (we say that T is of weak type $(1, 1)$) and from L_∞ to BMO. Knowing such endpoint inequalities is then translated by interpolation theory into inequalities on L_p. The prime example of such interpolation theorem is the following.

Theorem 0.2.2 (Marcinkiewicz)**.** *Let $p < q \in (0, \infty)$. If T is of weak type (p, p) and of weak type (q, q) then T is bounded on $L_r(\Omega)$, $r \in (p, q)$.*

This means that weak type inequalities can be enough to prove the boundedness of certain operators. This is for example the way the boundedness of singular integrals was originally proved by Calderón and Zygmund. Another motivation for obtaining endpoint estimates at a point p is that they often give an optimal control on the norm of T on L_q when q goes to p.

Abstractly, the theory of interpolation deals with couples (A, B) of quasi-Banach spaces and, informally, an interpolation space between A and B is a quasi-Banach space E such that if an operator T is bounded on A and B, it is automatically bounded on E. To apply the machinery of interpolation, it is therefore useful to know how to describe the interpolation spaces of a given couple. There is no effective way to accomplish this in full generality but it can be done for the couple $L_p(0,\infty)$, $L_q(0,\infty)$ with $p, q \in (0,\infty]$ by using the notion of K-functional and in particular K-monotonicity (1.2.4). For the remainder of this text, for two quantities $A(x)$ and $B(x)$, we will write $A(x) \lesssim B(x)$ or $A(x) \lesssim_C B(x)$, if there is a universal constant C such that for all x, $A(x) \leq CB(x)$, and similarly $A(x) \approx B(x)$ or $A(x) \approx_C B(x)$ if there exists C such that for all x, $\frac{1}{C}B(x) \leq A(x) \leq CB(x)$. Roughly, we say that a space $E \subset A+B$ is K-monotone for a couple (A,B) if for all $x, y \in E$, $x \succ_{A,B} y$ implies that $\|x\|_E \gtrsim \|y\|_E$, where $\succ_{A,B}$ is an order depending on (A,B). It is a basic fact that for any couple, every K-monotone space is an interpolation space. In certain cases, the converse is also true.

Theorem 0.2.3. *Every interpolation space of the following couples of quasi-Banach spaces is K-monotone:*

- *(Lorentz-Shimogaki)* $p \geq 1$, $(L_p(0,\alpha), L_\infty(0,\alpha))$ *and* $(L_1(0,\alpha), L_p(0,\alpha))$,
- *(Sparr)* $p, q \geq 1$, $(L_p(\Omega), L_q(\Omega))$ *for any σ-finite measure space* Ω.
- *(Sparr, Cwikel)* $p, q \in (0,\infty)$, $(L_p(0,\alpha), L_q(0,\alpha))$,
- *(Cwikel)* $p \in (0,\infty]$, (ℓ^p, ℓ^∞).

We extend Lorentz and Shimogaki's result to the quasi-Banach setting with the following theorem.

Theorem 0.2.4. *Let $\alpha \in (0,\infty]$ and $p \in (0,\infty)$. Then every interpolation space of the couple $(L_p(0,\alpha), L_\infty(0,\alpha))$ is K-monotone.*

The proof does not present any major difficulty, it consists, for any two functions f, g such that $f \succ_{L_p,L_\infty} g$, in explicitely constructing an operator T bounded on L_p and L_∞ such that $Tf = g$. Note that it is also known by Sparr [57], that every interpolation space of the couple $(L_p(0,\alpha), L_q(0,\alpha))$ is K-monotone for $p, q \in (0,\infty)$. We give an alternative characterization of those spaces in terms of two notions of monotonicity.

Definition 0.2.5 (p-monotonicity)**.** *Let E be a function space on $(0,\infty)$, we say that E is left-p-monotone if for all functions $f \in E$ and $g \in L_0(0,\infty)$ such that for all $t > 0$, $\int_0^t (f^*)^p \geq \int_0^t (g^*)^p$ then $g \in E$ and $\|g\|_E \lesssim \|f\|_E$.*

Similarly, we say that E is right-p-monotone if for all functions $f \in E$ and $g \in L_0(0,\infty)$ such that for all $t > 0$, $\int_t^\infty (f^*)^p \geq \int_t^\infty (g^*)^p$ then $g \in E$ and $\|g\|_E \lesssim \|f\|_E$.

We prove the following.

Theorem 0.2.6. *Let E be a function space on $(0,\alpha)$, $\alpha > 0$, and let $0 < p < q < \infty$. Then E is left-p-monotone and right-q-monotone if and only if E is an interpolation space between $L_p(0,\alpha)$ and $L_q(0,\alpha)$.*

Our proof relies on Sparr's result. We concentrate on proving that a space E is left-p-monotone and right-q-monotone if and only if it is K-monotone for the couple (L_p, L_q). We also show that for a function space with the Fatou property, p-convexity implies left-p-monotoncity and similarly q-concavity implies right-q-monotonicity. Recall that a space E is said to be natural if there exists a $p > 0$ such that E is p-convex. Then we obtain the following corollary.

Corollary 0.2.7. *Let E be a natural function space on $(0,\alpha)$ with the Fatou property. Then E is right-q-monotone if and only if there exists $0 < p < q$ such that $E \in (L_p(0,\alpha), L_q(0,\alpha))$.*

This solves, under assumptions which are minor in practice, a conjecture of Levitina, Sukochev and Zanin [34] in the setting of function spaces. Note the K-monotonicity of the couple (ℓ^p, ℓ^q) is not known and the conjecture remains fully open for sequence spaces.

Noncommutative Khintchine inequalities in symmetric spaces

The original motivation for this work on abstract interpolation theory comes from the study of noncommutative Khintchine inequalities. Noncommutative Khintchine inequalities are essential to study unconditional sequences in noncommutative L_p-spaces. They played an important role in the development of noncommutative harmonic analysis over the last three decades, both as a tool and by providing the "right" definition for square functions in this context which is at the origin of major developments such as noncommutative martingale inequalities [49]. Let us start by recalling the statement of classical Khintchine inequalities. Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of Rademacher variables in $L_\infty(0,1)$ and $(a_i)_{i\in\mathbb{N}}$ a finite sequence of complex numbers. Then for all $p \in (0,\infty)$:

$$\big\|\sum_{i\geq 0} a_i\xi_i\big\|_p \approx_{c_p} \big\|\sum_{i\geq 0} a_i\xi_i\big\|_2.$$

This equivalence extends immediately to sums whose coefficients take values in classical L_p-spaces. Let Ω be a measure space, $p \in (0,\infty)$ and $(f_i)_{i\in\mathbb{N}}$ a finite sequence in $L_p(\Omega)$. Then:

$$\big\|\sum_{i\geq 0} f_i\xi_i\big\|_{L_p(\Omega\otimes(0,1))} \approx_{c_p} \Big\|\Big(\sum_{i\geq 0} |f_i|^2\Big)^{1/2}\Big\|_{L_p(\Omega)}. \tag{3}$$

Under this second form, Khintchine inequalities appear frequently in harmonic analysis to estimate square functions or study the $\mathcal{R}$-boundedness of families of functions, a problem which is essential when trying to understand the properties of functional calculus on L_p-spaces. Note that the expression on the left hand side of the equivalence can in fact be defined for any Banach space X instead of $L_p(\Omega)$, by seeing $\sum_{i\geq 0} f_i\xi_i$ as an element of $L_p((0,1),X)$, and a version of Khintchine inequalities called Khintchine-Kahane inequality holds at that level of generality. More precisely, for any finite sequence $(x_i)_{i\geq 0}$ in X, and any $p \in (1,\infty)$,

$$\big\|\sum_{i\geq 0} x_i\xi_i\big\|_{L_p((0,1),X)} \approx_{c_p} \big\|\sum_{i\geq 0} f_i\xi_i\big\|_{L_2((0,1),X)}$$

However, we are more interested, in the context of noncommutative integration, in finding an analog of the right hand side of (3), which is more easily computed or estimated. Note right away that the same expression cannot work in the noncommutative context since there are two possible "square functions" (row and column) which are not equivalent in L_p for $p \neq 2$. Let $\mathcal{M}$ be a noncommutative measure space, $\mathcal{M}_c$ the finitely supported operators in $\mathcal{M}$ and $S(\mathcal{M})$ the set of finite sequences of $\mathcal{M}_c$. Recall that a symmetric function space E is a (quasi-)Banach space of functions such that the norm only depends on the distribution of the functions. These spaces can be transfered to the noncommutative setting with the same definition [31]. This construction generalizes the one of noncommutative L_p-spaces.

Definition 0.2.8 (Row and column norms)**.** *Let $x \in S(\mathcal{M})$, E a symmetric space. Define the column and row norms as follows:*

$$\|x\|_{C_E} := \| \left(\sum_{i\in\mathbb{N}} x_i^* x_i\right)^{1/2} \|_E \text{ and } \|x\|_{R_E} := \| \left(\sum_{i\in\mathbb{N}} x_i x_i^*\right)^{1/2} \|_E.$$

Theorem 0.2.9 (Noncommutative Khintchine inequalities)**.** *Let $p \in [2,\infty)$ then:*

$$\| \sum_{i\in\mathbb{N}} x_i \otimes \xi_i \|_p \approx \max\left(\|x\|_{R_p}, \|x\|_{C_p}\right).$$

Let $p \in (0,2]$ then:

$$\| \sum_{i\in\mathbb{N}} x_i \otimes \xi_i \|_p \approx \inf_{y+z=x} \left\{\|z\|_{C_p} + \|y\|_{R_p}\right\}.$$

This result is due to Lust-Piquard for $p \in (1,\infty)$ [36], Lust-Piquard and Pisier for $p = 1$ [37] and more than twenty years later Pisier and Ricard [47] for $p < 1$. By the real interpolation method, the result extends to most Banach Lorentz spaces but not to $L_{2,\infty}$ or $L_{1,\infty}$. For $L_{2,\infty}$, it is due to the fact that to write $L_{2,\infty}$ as an interpolation space of L_p and L_q, one must have $p < 2$ and $q > 2$ and therefore use the two formulas which do not interpolate nicely. Similarly, to write $L_{1,\infty}$ as an interpolation space of L_p and L_q, one must have $p < 1$ which is problematic since the closed vector space generated by the ξ_i is not complemented in $L_p(0,1)$ for $p < 1$. These two examples which are relatively common spaces prompted our investigation. Suppose now that the ξ_i are an arbitrary orthonormal sequence of elements in a noncommutative probability space. We obtain the following theorem:

Theorem 0.2.10. *Let $p < 1$. Suppose that for all $x \in S(\mathcal{M})$,*

$$\| \sum_{i\geq 0} x_i \otimes \xi_i \|_p \gtrsim \inf_{y+z=x} \left\{\|z\|_{C_p} + \|y\|_{R_p}\right\}.$$

Then for any interpolation space E between L_p and L_∞ and $x \in S(\mathcal{M})$,

$$\| \sum_{i\geq} x_i \otimes \xi_i \|_E \gtrsim \inf_{y+z=x} \left\{\|z\|_{C_E} + \|y\|_{R_E}\right\}.$$

Furthermore, the decomposition on the left hand side can be chosen to be independent of E.

This theorem implies the noncommutative Khintchine inequalities in $L_{1,\infty}$ for Rademacher variables as a corollary, since for those variables, as mentioned before, the noncommutative Khintchine inequalities are valid for $p < 1$.

Corollary 0.2.11. *Suppose that the ξ_i are Rademacher variables (or free Haar unitaries). Then for all $x \in S(\mathcal{M})$:*

$$\| \sum_{i\geq 0} x_i \otimes \xi_i \|_{1,\infty} \approx \inf_{y+z=x} \left\{ \|z\|_{C_{1,\infty}} + \|y\|_{R_{1,\infty}} \right\}.$$

The idea of the proof of the theorem is to take a decomposition $x = y + z$ which is close to being optimal in the sense that it almost minimizes the quantity $\|y\|_{R_p}^p + \|z\|_{C_p}^p$ and to show by applying the noncommutative Khintchine inequalities in L_p to sequences of the form $(ex_i)_{i\in\mathbb{N}}$, where e is a projection, that the row K-functional of y for the couple (L_p, L_∞) and the column K-functional of z for the couple (L_p, L_∞) are dominated by the K-functional of $\sum_{i\in\mathbb{N}} x_i \otimes \xi_i$ for the couple (L_p, L_∞). Then, by applying the extension of the Lorentz-Shimogaki theorem, we obtain a control in every interpolation space.

By pushing further the idea of studying optimal decomposition in the particular case of L_1, we obtain an algebraic decomposition for sequences which seems to play the role of a polar decomposition and enables us to recover most known results on Khintchine inequalities in the case of free Haar unitaries.

Theorem 0.2.12. *Let $x = (x_i)_{i\geq 0}$ a finite sequence of elements in $\mathcal{M}_c$. There exist $\alpha, \beta \in \mathcal{M}_c^+$, and $u = (u_i)_{i\geq 0} \in \mathcal{M}_c^{\mathbb{N}}$ such that:*

- $s(\alpha) \leq \sum_{i\geq 0} u_i u_i^* \leq 1$,
- $s(\beta) \leq \sum_{i\geq 0} u_i^* u_i \leq 1$,
- *for all $i \geq 0$, $x_i = u_i\beta + \alpha u_i$.*

Furthermore, suppose that:

- *the ξ_i are orthonormal in $L_2(\mathcal{A})$*
- *the ξ_i verify the Khintchine inequality in L_∞ i.e for any finite sequence $x = (x_i)_{i\geq 0} \in \mathcal{M}^{\mathbb{N}}$:*

$$\max\left(\|x\|_{R_\infty}, \|x\|_{C_\infty}\right) \approx_{c_\infty} \| \sum_{i\geq 0} x_i \otimes \xi_i \|_\infty.$$

Then for any $p \in (0, \infty)$ and E an interpolation space between L_p and L_∞:

$$\|\alpha\|_E + \|\beta\|_E \approx_c \| \sum_{i\geq 0} x_i \otimes \xi_i \|_E,$$

where c only depends on p and c_∞.

This theorem is proved in two independent steps. First, the algebraic decomposition is found by considering an optimal decomposition of $x = y + z$ in L_1. By writing the polar decomposition of y and z and their optimality in terms of duality, we obtain the algebraic identities we need. Then, using these algebraic properties of the decomposition, we prove once again a control in terms of K-functionals which delivers the result claimed in the theorem. For $p \geq 1$, the computations do not require any complex tool but to treat the case of $p < 1$, we crucially need a form of Hölder's inequality for commutators recently proved by Ricard [54]. Note that the theorem gives a deterministic equivalent of the norm of $\sum_{i\geq 0} x_i \otimes \xi_i$ in $L_{2,\infty}$ though its very particular form may not make it suitable for applications. Another notable fact is that the formula for this deterministic equivalent, contrary to previously known ones, unifies the cases $p < 2$ and $p > 2$.

Finally, we give, under some technical assumptions on the spaces and in the setting of free Haar unitaries, a characterisation of spaces in which the usual formulas for noncommutative Khintchine inequalities hold.

Theorem 0.2.13. *Let E be a symmetric space with the Fatou property and $\mathcal{M} = \mathcal{B}(\ell^2)\overline{\otimes}L_\infty(0,1)$. Then E is left-2-monotone if and only if for all finite sequences $x \in \mathcal{M}_c^{\mathbb{N}}$, $\left\|x\right\|_{R_E \cap C_E} \approx \left\| \sum_{i\geq 0} x_i \otimes \xi_i \right\|_E$.*

Suppose furthermore there exists $p > 0$ such that E is an interpolation space between L_p and L_∞. Then E is right-2-monotone if and only if for all finite sequences $x \in \mathcal{M}_c^{\mathbb{N}}$, $\left\|x\right\|_{R_E + C_E} \approx \left\| \sum_{i\geq 0} x_i \otimes \xi_i \right\|_E$.

This provides a complement to the deterministic formula given by the previous theorem. Showing for example that in the case of $E = L_{2,\infty}$, none of the usual formulas can work.

Weak type inequality for singular integrals

In the last chapter, we present the work which appears in [5]. It is almost independent from the rest of the text, apart from a short application of the noncommutative Khintchine inequalities appearing at the end. It deals with singular integrals i.e. a certain type of operator T informally defined by an expression of the form:

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy,$$

where $n \geq 1$, $x \in \mathbb{R}^n$, $k : \mathbb{R}^{2n} \to \mathbb{C}$ is said to be the kernel of T and verifies conditions that allow a singularity on the diagonal (near $x = y$) but control its speed of variation (which we will call Calderón-Zygmund type conditions). If there exists a function $K : \mathbb{R}^n \to \mathbb{C}$ such that $k(x,y) = K(x-y)$, k is said to be of convolution type and in this case, if $K \in C^1(\mathbb{R}^n \setminus \{0\})$, a typical Calderón-Zygmund condition would be:

$$|\nabla K(x)| \lesssim \frac{1}{|x|^{n+1}}.$$

This condition is verified by the Riesz and the Hilbert transform which are some of the motivating examples for introducing the notion of singular integral. We are interested in noncommutative generalisations of the following theorem, that we state here informally.

Theorem 0.2.14 (Calderón, Zygmund)**.** *If T is bounded on $L_2(\mathbb{R}^n)$ and is associated to a kernel k verifying Calderón-Zygmund type conditions then T extends to a bounded operator from $L_1(\mathbb{R}^n)$ to $L_{1,\infty}(\mathbb{R}^n)$.*

This theorem implies by interpolation and duality the boundedness of singular integrals from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ for all $p \in (1,\infty)$. By the same expression T could define an operator from $L_p(\mathbb{R}^n, X)$ to $L_p(\mathbb{R}^n, X)$ for any Banach space X. Conditions on X for the boundedness of T to remain true when considering X-valued functions have been studied, the key condition being that X is UMD. Noncommutative L_p-spaces are UMD for $p \in (1,\infty)$ and hence singular integrals can be used in this context, however a noncommutative analog of Calderón and Zygmund theorem was missing to obtain optimal bounds for the norm of T from L_p to L_p when p tends to 1. In [41], Parcet generalized theorem 0.2.14 to the noncommutative context. Note that in the noncommutative setting, when studying singular integrals, BMO methods or direct proofs can be prefered to with weak type inequalities which can be technical. Nonetheless, the theorem has now found various applications, among which an important one in perturbation theory.

The interaction between perturbation theory and noncommutative harmonic analysis started in [51] where Potapov and Sukochev solved a longstanding problem on operator Lipshitz functions in the Schatten classes. They proved that for any $p \in (1,\infty)$, there exists a constant c_p such that for any Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and self-adjoint operators x and y,

$$\big\|f(x) - f(y)\big\|_p \leq c_p \big\|f\big\|_{Lip} \big\|x - y\big\|_p, \tag{4}$$

where we assume that $x - y \in L_p(\mathcal{M})$ and:

$$\big\|f\big\|_{Lip} := \sup_{t \neq s} \frac{|f(t) - f(s)|}{|t - s|}.$$

Recently, the two authors, joined by Caspers and Zanin, gave a new proof of (4) which describes the optimal behaviour of the constant c_p when p tends to 1. It relies on Calderón-Zygmund theory and in particular on Parcet's theorem. The scheme of proof they employ - to reduce a question to the boundedness of a Calderón-Zygmund operator - could be applicable to other problems. This is the main reason why a particular interest has sprung towards Parcet's theorem in the last few years. However, the proof given in [41] is difficult and therefore not easy adapt if one should encounter a situation where the theorem does not immediately apply. This is what motivated our work. We present a new simplified proof of the weak boundedness of singular integrals, relying on the same idea as Parcet's.

Let us first give a precise statement of the theorem. We start by introducing the conditions we impose on the kernel.

Definition 0.2.15 (Smoothness and Size)**.** *Say that T has Lipschitz parameter γ ($0 < \gamma \leq 1$) if for all x,y and z in $\mathbb{R}^n$ verifying $|x-z| \leq \frac{1}{2}|y-z|$, the following* smoothness estimates *hold:*

$$|k(x,y) - k(z,y)| \lesssim \frac{|x-z|^\gamma}{|y-z|^{n+\gamma}},$$
$$|k(y,x) - k(y,z)| \lesssim \frac{|x-z|^\gamma}{|y-z|^{n+\gamma}}.$$

We say that T verifies the size estimate if for all $x, y \in \mathbb{R}^n$,

$$\|k(x,y)\|_{\widetilde{\mathcal{M}}} \lesssim \frac{1}{|x-y|^n}.$$

Theorem 0.2.16. *Assuming that T has a Lipschitz parameter γ ($0 < \gamma \leq 1$), verifies the size estimate and is bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$. Then T extends as a bounded operator from $L_1(\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}^n))$ to $L_{1,\infty}(\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}^n))$.*

The proof of this theorem, in the classical setting, relies on a decomposition of functions $f = g + b$, which can be interpreted in the language of martingales. This interpretation allows to construct a noncommutative version of this decomposition, by using the well-developed theory of noncommutative martingales and in particular Cuculescu's projections. The decomposition obtained in the noncommutative setting however, does not enjoy all the good properties of the classical one. New terms appear and to estimate them, the following L_2-pseudolocalisation theorem is needed. Denote by $\mathcal{N}$ the noncommutative measure space $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}^n)$ and by $\mathcal{N}_k$ the subalgebra of $\mathcal{N}$ composed of functions constant on dyadic cubes of edge length 2^{-k}, $k \in \mathbb{Z}$.

Theorem 0.2.17 (L_2-pseudolocalisation)**.** *Let $f \in L_2(\mathcal{N})$ and $s \in \mathbb{N}$. For all $k \in \mathbb{Z}$, let A_k and B_k be projections in $\mathcal{N}_k$ such that $A_k^\perp df_{k+s} = df_{k+s}B_k^\perp = 0$. For any odd positive integer d, write:*

$$A_k = \textstyle\sum_{Q\in\mathcal{Q}_k} A_Q\mathcal{I}_Q,\ A_Q \in \mathcal{M} \text{ and define } dA_k := \bigvee_{Q\in\mathcal{Q}_k} A_Q\mathcal{I}_{dQ}.$$

Define dB_k the same way. Let:

$$A_{f,s} := \bigvee_{k\in\mathbb{Z}} 5A_k \text{ and } B_{f,s} := \bigvee_{k\in\mathbb{Z}} 5B_k.$$

Let T be a Calderón-Zygmund operator associated with a kernel k with Lipschitz parameter γ, verifying the size condition, taking values in $\mathbb{C}$ and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$. Then for all $s \in \mathbb{N}$ and $f \in L_2(\mathcal{N})$ we have:

$$\|A_{f,s}^\perp(Tf)\|_2 \lesssim 2^{-\frac{\gamma s}{2}}\|f\|_2 \text{ and } \|(Tf)B_{f,s}^\perp\|_2 \lesssim 2^{-\frac{\gamma s}{2}}\|f\|_2.$$

This theorem is stated here for functions taking values in a noncommutative measure space but it is commutative in nature. It gives a quantified estimate of the rate of decrease of singular integrals outside of the diagonal. This had never been needed before in the classical theory but such estimates are crucial when dealing with the noncommutative case. The proof of L_2-pseudolocalisation boils down to showing an almost orthogonality result. In this text, only elementary techniques are employed: Schur's lemma to estimate the $L_2 \to L_2$ norm of a kernel operator and a careful decomposition of T. The main novelty of our approach compared to Parcet's is that we interpret the computations made in terms of "kernel modifications" i.e. if we manage to prove that

$$A_{f,s}(x)Tf(x) = A_{f,s}(x) \int_{\mathbb{R}^n} k(x,y)f(y)dy = A_{f,s}(x) \int_{\mathbb{R}^n} k'(x,y)f(y)dy,$$

we introduce a new operator T' associated to the kernel k'. This allows to work both on the level of operators and kernels thus to improve the readability of the proof and to shorten it. As a byproduct of this simplification, we can quickly recover a version of L_p-pseudolocalisation, a result similar to the one obtained by Hytönen in [22].

Theorem 0.2.18. *Using the notations introduced above, let T be a Calderón-Zygmund operator associated with a kernel k with Lipschitz parameter γ, verifying the size condition, taking values in $\mathbb{C}$ and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$. Then for all $1 < p < \infty$, $s \in \mathbb{N}$ and $f \in L_p(\mathcal{N})$, there exists $\theta_p > 0$ such that :*

$$\left\| A_{f,s}^{\perp}(Tf) \right\|_p \lesssim 2^{-\gamma s \theta_p} \left\| f \right\|_p \text{ and } \left\| (Tf) B_{f,s}^{\perp} \right\|_p \lesssim 2^{-\gamma s \theta_p} \left\| f \right\|_p,$$

where the implied constant depends on p.

The simplifications made also allow us to precisely track the use of each hypothesis during the proof. We obtain this way a version of the theorem for operator-valued kernels, as long as the values taken by the kernel commute with the values taken by the function. It is but a small improvement of the original paper but combining it with noncommutative Khintchine inequalities in $L_{1,\infty}$, we recover without effort the boundedness of singular integrals with a Hilbert-valued kernel, originally obtained by Mei and Parcet in [39].

Theorem 0.2.19. *Let T be a Calderón-Zygmund operator associated to a kernel k taking values in ℓ^2. Suppose furthermore that T is bounded from $L_2(\mathcal{N})$ to $R_2(\mathcal{N})$ and that k satisfies the smoothness and size condition. Then T is bounded from $L_1(\mathcal{N})$ to $R_{1,\infty}(\mathcal{N}) + C_{1,\infty}(\mathcal{N})$ and from $L_p(\mathcal{N})$ to $R_p(\mathcal{N}) + C_p(\mathcal{N})$ for $1 < p < 2$.*

Chapter 1

Interpolation spaces between L_p-spaces

1.1 Introduction

In this chapter, we study characterizations of interpolation spaces between L_p-spaces. One of the first results of this kind is due to Calderón [6] and deals with the couple (L_1, L_∞). Lorentz and Shimogaki extended the result to the couples (L_1, L_p) and (L_p, L_∞) [35]. Since then, tools have been developed to give, when possible, systematic descriptions of the interpolation spaces of a given couple of Banach spaces, see for example [2]. In the simplest cases, the knowledge of the K-functional associated to the couple is enough to provide satisfying results.

One of the motivations behind many of the statements compiled in this chapter is that we encountered some difficulties in finding suitable references that deal with the case of quasi-Banach spaces. Some results can be generalised without modifications to the proofs, others require more work and some become false. In particular, we generalise the Lorentz-Shimogaki theorem to the couple $(L_p(0,\alpha), L_\infty(0,\alpha))$, $p < 1$, $\alpha \in (0,\infty]$. A result that does not require new techniques but that we could not find in the literature.

The second motivation for this chapter is to provide a new characterisation of interpolation spaces for the couple $(L_p(0,\alpha), L_\infty(0,\alpha))$, which will be used in chapter 3. It is expressed in terms of *p-monotonicity*, a notion that will appear naturally when studying noncommutative Khintchine inequalities. Further motivation is given by a recent conjecture of Levitina, Sukochev and Zanin (LSZ). When investigating Cwikel estimates, they stumbled upon a condition for a sequence space that we will call *right-2-monotonicity* and which will be defined later. In those terms, their conjecture can be formulated as follows:

Conjecture 1.1.1 (Levitina, Sukochev, Zanin)**.** *A symmetric sequence space E is right-2-monotone if and only if there exists $p \in (0,2)$ such that E is an interpolation space for the couple (ℓ^p, ℓ^2).*

Neither of the two implications are known. In the Banach case however, the problem was almost completely solved by Cwikel and Nielson. Indeed, they showed that if E a Banach sequence space with the Fatou property then E is left-2-monotone if and only if it is an interpolation space for the couple (ℓ^1, ℓ^2). Substantial difficulties appear when considering quasi-Banach sequence spaces. We are more interested in function spaces. In particular, if they are supposed to be *natural*, a notion introduced by Kalton and later defined in the text, awe obtain:

Proposition 1.1.2. *Let $\alpha \in (0, \infty]$ and E a natural symmetric function space on $(0, \alpha)$ with the Fatou property. E is right-2-monotone if and only if there exists $p \in (0, 2)$ such that E is an interpolation space for the couple $(L^p(0, \alpha), L^2(0, \alpha))$.*

1.2 Some results from abstract interpolation

In this section we give a brief introduction to the theory of interpolation. A detailed exposition of this topic can be found in various books [1]. Nonetheless, we will recall some definitions and basic properties both to obtain a more self-contained text and to fix some potentially non-standard terminology.

1.2.1 First definitions

Interpolation theory allows on one hand to construct new spaces which lie "in between" two given quasi-Banach spaces and on the other to deduce properties of these "in between" spaces from those of the end-spaces. It is one of the main techniques employed in L_p-spaces theory. Indeed, since for any $p < q < r \in (0, \infty]$, L_q is an interpolation space between L_p and L_r, many inequalities for L_p-norms can be deduced by studying only the cases of L_1, L_2 and L_∞ which are often more tractable.

We will say that a couple of quasi-Banach spaces (A, B) is *compatible* if it is associated to a topological vector space C and embeddings of A and B into C. This enables us to consider the sum $A + B$ and the intersection $A \cap B$ of A and B.

Definition 1.2.1. *Let (A, B) be a compatible couple of quasi-Banach spaces. We say that a quasi-Banach space E is an interpolation space (with constant C) for this couple if $A \cap B \subset E \subset A + B$ and there exists a constant $C > 0$ such that for every bounded operator $T : A + B \to A + B$ whose restriction to A (resp.B) is a contraction from A to A (resp. B to B), T is bounded from E to E with norm less than C. E is said to be an exact interpolation space if $C = 1$.*

Both for explicit constructions of interpolation spaces with the real method and for the general theory of interpolation, the K-functional is a fundamental tool. It is defined as follows:

Definition 1.2.2. *Let (A, B) be a compatible couple of quasi-Banach spaces and $x \in A + B$. For all $t > 0$ define the K-functional of x by:*

$$K_t(x, A, B) = \inf\{\|y\|_A + t\|z\|_B : y \in A, z \in B, y + z = x\}.$$

Remark 1.2.3. Let (A, B) be a compatible couple of Banach spaces and $t > 0$. Then $A + B$ equipped with the quasi-norm $\|.\|_{A+tB} = K_t(., A, B)$ is an exact interpolation space for the couple (A, B) that we will denote by $A + tB$.

It enables to state a simple sufficient condition for a space to be an interpolation space.

Definition 1.2.4 (K-monotone)**.** *Let (A, B) be a compatible couple of quasi-Banach spaces. We say that a quasi-Banach space E is K-monotone for (A, B) if $A \cap B \subset E \subset A + B$ and there exists a constant C such that for all $x \in E$ and $y \in A + B$, verifying*

$$\forall t > 0, K_t(x, A, B) \geq K_t(y, A, B),$$

then $y \in E$ and

$$\|y\|_E \leq C\|x\|_E.$$

The following fact is well-known, we prove it here in the context of quasi-Banach spaces for completion.

Proposition 1.2.5. *If E is K-monotone with constant C for the couple (A, B) then E is an interpolation space between A and B with the same constant.*

Proof. Let $T : A + B \to A + B$ be a bounded operator such that its restriction to A (resp. B) is a bounded operator with norm 1 on A (resp. B). Let $x \in E$ and let $y = Tx$, $y \in A + B$. Let $t > 0$, as noted in remark 1.2.3, $A + tB$ is an exact interpolation space for the couple (A, B). This means that for all $t > 0$, $K_t(x, A, B) \geq K_t(y, A, B)$. Hence, by K-monotonicity of E, $y \in E$ and $\|y\|_E \leq C\|x\|_E$. So T defines a bounded operator of norm less than C on E. $\square$

If, reciprocally, every interpolation space of a couple (A, B) is K-monotone, (A, B) is said to be a *Calderón* couple. If the unit ball of A and the unit ball of B are closed in $A + B$ (for its natural norm i.e $K_1(., A, B)$) then the couple (A, B) is said to be a *Gagliardo couple* or *Gagliardo complete.*

1.3 K-monotonicity for the couple (L_p, L_q)

1.3.1 Definitions and overview of the literature

We start by introducing some notations. Fix $\alpha \in (0, \infty]$. Denote by $L_0(0, \alpha)$ the set of measurable functions f on $(0, \alpha)$, such that $\mathbf{1}_{|f|>t}$ is finite for some $t \in \mathbb{R}$. For $f \in L_0(O, \alpha)$, denote by f^* the non-increasing rearrangement of f.

Definition 1.3.1 (Symmetric space). *We call* quasi-Banach symmetric function space *a nonzero subspace of $L_0(0,\alpha)$ which is rearrangement invariant (the quasi-norm of a function only depends on its distribution) and equipped with an increasing quasi-norm. Similarly, a quasi-Banach symmetric sequence space is a subspace of ℓ^∞ enjoying the same properties.*

A symmetric space E is said to have the *Fatou property* if for every increasing net $(f_n)_{n\in I}$ of elements of E such that $(\|f_n\|_E)_{n\in I}$ is bounded and $f_n \uparrow f$ a.e., then $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$. An introduction to symmetric spaces can be found in [32].

Recall that E is said to be p-convex with constant C if for all $n \in \mathbb{N}$ and $(x_i)_{i\leq n} \in E^n$,

$$\| \left(\sum\nolimits_{i=1}^n |x_i|^p\right)^{1/p} \|_E \leq C \left(\sum\nolimits_{i=1}^n \|x_i\|_E^p\right)^{1/p}.$$

Similarly, E is said to be q-concave with constant C if for all $n \in \mathbb{N}$ and $(x_i)_{i\leq n} \in E^n$,

$$\left(\sum\nolimits_{i=1}^n \|x_i\|_E^q\right)^{1/q} \leq C\| \left(\sum\nolimits_{i=1}^n |x_i|^q\right)^{1/q} \|_E.$$

A symmetric space is said to be *natural* if it is p-convex for some p. This includes Lorentz spaces and Orlicz spaces. There exists symmetric function spaces which are not natural but constructing one is not easy, see for example [29] for an idea of how to do it. Remark that all Banach spaces are natural since they are 1-convex.

Characterizations of symmetric function spaces which are interpolation spaces of the couple (L_p, L_q) relie on its K-functional. It was computed by Krée and can be found in [21].

Proposition 1.3.2. *Let $p < q \in (0,\infty]$, then for all $f \in L_p(0,\alpha) + L_q(0,\alpha)$:*

$$K_t(f, L_p, L_q) \approx \left(\int_0^{t^\theta} (f^*)^p\right)^{1/p} + t\left(\int_{t^\theta}^\infty (f^*)^q\right)^{1/q}. \tag{1.1}$$

with $\theta = (p^{-1} - q^{-1})^{-1}$. If $q = \infty$, the formula is to be understood as follows:

$$K_t(f, L_p, L_\infty) \approx_{A_p} \left(\int_0^{t^p} (f^*)^p\right)^{1/p}. \tag{1.2}$$

In view of this second formula, we introduce the following definitions.

Definition 1.3.3 (Majorization). *For two elements $x, y \in L_0(0,\alpha)$, we write $x \succ y$ and we say that x left-majorizes y if and only if for all $t > 0$, $\int_0^t x^* \geq \int_0^t y^*$.*

Similarly, we write $x \rhd y$ and we say that x right-majorizes y if and only if for all $t > 0$, $\int_t^\infty x^* \geq \int_t^\infty y^*$.

These notions already appear in the literature [9]. The order given by left-majorization is particularly useful since for two functions f, g, there exists an integral preserving positive subunital operator T on $L_1(0,\infty) + L_\infty(0,\infty)$ such that $Tf \geq g$ if and only if $f \succ g$ [6].

In many cases, the couple (L_p, L_q) is known to be a Calderón couple. This is summarized in the following proposition ([35],[57],[9],[10]).

Theorem 1.3.4. *The following couples of quasi-Banach spaces are Calderón couples:*

- *(Lorentz-Shimogaki)* $p \geq 1$, $(L_p(0,\alpha), L_\infty(0,\alpha))$ *and* $(L_1(0,\alpha), L_p(0,\alpha))$,
- *(Sparr)* $p, q \geq 1$, $(L_p(\Omega), L_q(\Omega))$ *for any* σ*-finite measure space* Ω.
- *(Sparr, Cwikel)* $p, q \in (0,\infty)$, $(L_p(0,\alpha), L_q(0,\alpha))$,
- *(Cwikel)* $p \in (0,\infty]$, (ℓ^p, ℓ^∞).

1.3.2 The couple $(L_p(0,\alpha), L_\infty(0,\alpha))$, $p < 1$

For the remainder of this section, fix $p \in (0,\infty)$.

Theorem 1.3.5. *Every interpolation space of the couple* $(L_p(0,\alpha), L_\infty(0,\alpha))$ *is* K*-monotone.*

We will deduce the theorem above from the following proposition:

Proposition 1.3.6. *Let* $f, g \in L_p(0,\alpha) + L_\infty(0,\alpha)$ *and* $d > 0$. *Suppose that* f *and* g *are non-negative, non-increasing, left-continuous, take values in* $\mathbb{D} = \{d^n\}_{n\in\mathbb{Z}} \cup \{0\}$ *and* $f^p \succ g^p$. *Then there exists an operator* $T : L_p(0,\alpha) + L_\infty(0,\alpha) \to L_p(0,\alpha) + L_\infty(0,\alpha)$ *and a positive function* h *such that:*

- $h \leq 2^{1/p} f$
- $Th^* = g$,
- *the restrictions of* T *on* $L_p(0,\alpha)$ *and* $L_\infty(0,\alpha)$ *are contractions.*

Proof. Define $h := 2^{1/p} f \mathbf{1}_{2f^p \geq g^p}$. Let $t > 0$, let us show that $\int_0^t h^p \geq \int_0^t g^p$. Note that $g^p > 2f^p \Leftrightarrow g^p < 2(g^p - f^p)$.

$$\begin{aligned}
\int_0^t h^p - \int_0^t g^p &= \int_0^t \mathbf{1}_{2f^p \geq g^p}(2f^p - g^p) - \mathbf{1}_{2f^p < g^p} g^p \\
&\geq \int_0^t \mathbf{1}_{2f^p \geq g^p} 2(f^p - g^p) - \mathbf{1}_{2f^p < g^p} 2(g^p - f^p) \\
&= 2\int_0^t f^p - g^p \\
&\geq 0.
\end{aligned}$$

Denote $X := \|g\|_p \in (0,\infty]$. Define:

$$H : t \mapsto \int_0^t h^p \text{ and } G : t \mapsto \int_0^t g^p.$$

Let $x \in [0, X)$, note that $G^{-1}(x)$ is well-defined since G is continuous and increasing on $[0, \beta]$, the support of g. Define also $H^{-1}(x) = \min H^{-1}(\{x\})$ (H is continuous). Note that $h(H^{-1}(x)) \neq 0$. Indeed, by definition of $t_x := H^{-1}(x)$, for all $\varepsilon > 0$, $\int_{t_x-\varepsilon}^{t_x} h \neq 0$ so there exists an increasing sequence (t_n) converging to t_x and such that $h(t_n) \neq 0$. This means that $2f(t_n) \geq g(t_n)$. Using the fact that f is left-continuous and g is non-increasing, we obtain $2f(t_x) \geq g(t_x)$. And since we showed that $H \geq G$, $H^{-1}(x) \leq G^{-1}(x)$, hence $g^p(H^{-1}(x)) \geq g^p(G^{-1}(x))$. In conclusion:

$$h^p(H^{-1}(x)) \geq g^p(G^{-1}(x)). \tag{1.3}$$

Consider now h^* and $H_* : t \mapsto \int_0^t (h^*)^p$. By construction, h restricted to $\{h > 0\}$ is non-increasing and h^* is non-increasing too. Since h and h^* also have a countable range, they are of the form $\sum_{i\in\mathbb{N}} a_i \mathbf{1}_{I_i}$ where the I_i are intervals. Let t in the support of h so that t is not an end point of one of those intervals then since h is non-increasing:

$$\int_{h>h(t)} h < \int_0^t h < \int_{h\geq h(t)} h.$$

Similarly, if t' is in the support of h^* and not at an end point,

$$\int_{h^*>h^*(t')} h^* < \int_0^{t'} h^* < \int_{h^*\geq h^*(t')} h^*.$$

Since h and h^* have the same distribution, this means that for all $s > 0$,

$$\int_{h>s} h = \int_{h^*>s} h^* \text{ and } \int_{h\geq s} h = \int_{h^*\geq s} h^*.$$

By the previous inequalities, we deduce that except for a countable number of t and t', if $H(t) = H_*(t')$ then $h(t) = h^*(t')$. Since H_* is strictly increasing on $H_*^{-1}((0, X))$, this implies by (1.3) that for all $x \in (0, X)$, except maybe a countable number,

$$h^*(H_*^{-1}(x)) = h(H^{-1}(x)) \geq g(G^{-1}(x)). \tag{1.4}$$

Let $\phi = H_*^{-1} \circ G$ on $(0, \beta)$. Remark that since h^* and g have range in $2^{1/p}\mathbb{D}$ and $\mathbb{D}$ respectively, H_* and G are piecewise affine on every compact of $(0, \beta)$. Since $H_* \circ \phi = G$, by derivating, for all $t \in (0, \alpha)$, except maybe a countable number,

$$\phi'(t)(h^* \circ \phi)^p(t) = g^p(t). \tag{1.5}$$

Hence define $T : L_p(0, \alpha) + L_\infty(0, \alpha) \to L_p(0, \alpha) + L_\infty(0, \alpha)$ by $T(u) = (\phi'^{1/p} \cdot u \circ \phi) \circ \mathbf{1}_{(0,\beta)}$. By the change of variables formula, T is a contraction on L_p. Indeed, one can apply the formula on any compact of $(0, \beta)$ and obtain the p-norm as a limit. Let $t \in (0, \alpha)$ and write $t = G^{-1}(x)$, then by (1.4):

$$\phi'(t) = (H_*^{-1} \circ G)'(t) = \frac{g^p(t)}{(h^*)^p(H_*^{-1}(G(t))} = \left(\frac{g(G^{-1}(x))}{h^*(H_*^{-1}(x))}\right)^p \leq 1,$$

except for a countable number of values of t. So T is also a contraction on $L_\infty(0,\alpha)$. Finally, by (1.5), $Th^* = g$. □

Proof of theorem 1.3.5. Let E be a interpolation space for the couple $(L_p(0,\alpha), L_\infty(0,\alpha))$ with constant C. Let $f \in E$ and $g \in L_p + L_\infty$ such that for all $t > 0$, $K_t(f, L_p(0,\alpha), L_\infty(0,\alpha) \geq K_t(g, L_p(0,\alpha), L_\infty(0,\alpha)$. By (1.2), we can replace this condition by $f^p \succ g^p$. Let $d \in (1,\infty)$ and define $f_2 = d^{\lfloor \log_d(f^*) \rfloor + 1}$ and $g_2 = d^{\lfloor \log_d(g^*) \rfloor}$. We have $f^* \leq f_2 \leq df^*$, $g^* \leq dg_2 \leq dg^*$, hence $f_2^p \succ g_2^p$. Furthermore, f_2 and g_2 take values in $\mathbb{D} = \{d^n\}_{n\in\mathbb{Z}} \cup \{0\}$. Take h and T given by proposition 1.3.6. Since $h \leq 2^{1/p} f_2$, $h \in E$ so by the definition of an interpolation space, $Th = g_2 \in E$ and finally $g \in E$. Furthermore:

$$\begin{aligned}\|g\|_E = \|g^*\|_E \leq d\|g_2\|_E \leq dC\|h\|_E \\ \leq dC2^{1/p}\|f_2\|_E \leq d^2C2^{1/p}\|f^*\|_E = d^2C2^{1/p}\|f\|_E.\end{aligned}$$

This is true for any $d \in (1,\infty)$ so, $\|g\|_E \leq C2^{1/p}\|f\|_E$. □

1.4 p-monotonicity

In this section, we introduce the notions of left-p-monotonicity and right-p-monotonicity. This allows us to find an equivalent formulation for the K-monotonicity of the couple $(L_p(0,\alpha), L_q(0,\alpha))$.

Definition 1.4.1 (p-monotonicity)**.** *A symmetric quasi-Banach space E is said to be* left-p-monotone *if there exists $C > 0$ such that for all $f \in E$ and $g \in L_0(0,\alpha)$,*

$$f^p \succ g^p \Rightarrow g \in E, \|g\|_E \leq C\|f\|_E.$$

It is said to be right-p-monotone *if there exists $C > 0$ such that for all $f \in E$ and $g \in L_0(0,\alpha)$,*

$$f^p \triangleright g^p \Rightarrow g \in E, \|g\|_E \leq C\|f\|_E.$$

Remark that by (1.1), left-p-monotonicity is (almost) a particular case of K-monotonicity i.e. K-monotonicity for the couple (L_p, L_∞) (the only difference lies in the fact that we do not suppose E to be an intermidiate space).

We will denote by F the set of non-negative dyadic step functions:

$$F := \left\{ f \in L_\infty(0,\infty) : \exists n \in \mathbb{N}, \exists N \in \mathbb{N}, \exists a \in (\mathbb{R}^+)^N, f = \sum_{i=1}^N a_i \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})} \right\}.$$

Note that this space will also be useful later in this text.

Lemma 1.4.2. Let $f, g \in F$ and $p \in (0,\infty)$.

- If $f^p \succ g^p$ then there exists $h \in F$ such that $h \geq g$, $f^p \succ h^p$ and $\|f\|_p = \|h\|_p$.

- If $f^p \triangleright g^p$ then there exists $h \in F$ such that $h \geq g$, $h^p \succ f^p$ and $\|f\|_p = \|h\|_p$.

Proof. We believe that it is not difficult to convince oneself that this lemma is true. However, for completeness, we construct explicit expressions for h. Write:

$$f = \sum_{i=1}^{N} a_i \mathbf{1}_{(i-1)\varepsilon, i\varepsilon)} \text{ and } g = \sum_{i=1}^{M} b_i \mathbf{1}_{[(i-1)\varepsilon, i\varepsilon)}.$$

By changing the order of the a_i and b_i, we can suppose that f and g are non-increasing and replacing f by f^p and g by g^p we can suppose that $p = 1$. Suppose also that none of the a_i nor b_i are 0

Suppose that $f \succ g$. Let $\delta = \min(a_N, b_M)$ and $D = \|f\|_1 - \|g\|_1$. Choose $n \in \mathbb{N}$ such that $\varepsilon n \delta > D$ and let $b = D/\varepsilon n$. Define $h = \sum_{i=1}^{M+n} c_i \mathbf{1}_{[(i-1)\varepsilon, i\varepsilon)}$ with $c_i = b_i$ for $i \leq M$ and $c_i = b$ for $i \geq M+1$. Since $b < \delta$, h is non-increasing.

$$\|h\|_1 = \varepsilon \sum_{i=0}^{M+n} c_i = \|b\|_1 + n\varepsilon b = \|b\|_1 + D = \|f\|_1.$$

To prove that $f \succ g$ define:

$$F : t \mapsto \int_0^t f,\ G : t \mapsto \int_0^t g \text{ and } H : t \mapsto \int_0^t h.$$

For $t \leq M\varepsilon$, $F(t) \geq G(t) = H(t)$. Since F and H are affine on intervals of the form $[i\varepsilon, (i+1)\varepsilon]$,$i \in \mathbb{N}$, it suffices to prove that $F(i\varepsilon) \geq H(i\varepsilon)$ for all $i \in \mathbb{N}$. Let us argue by induction on i. We can initialise at $i = M$. Let $i \geq M$ and suppose that $F(i\varepsilon) \geq G(i\varepsilon)$ then they are two possibilities: either $a_i = 0$ then $F((i+1)\varepsilon) = \|f\|_1 = \|h\|_1 \geq H((i+1)\varepsilon)$ or $a_i \neq 0$ then by hypothesis on b, $a_i \geq b$ and $F((i+1)\varepsilon) = F(i\varepsilon) + a_i\varepsilon \geq G(i\varepsilon) + b = G((i+1)\varepsilon)$.

Suppose now that $f \triangleright g$. Define $h = \sum_{i=1}^{N} c_i \mathbf{1}_{[(i-1)\varepsilon, i\varepsilon)}$. With $c_i = b_i$ for all $i \geq 2$ and $c_1 := \sum_{i=1}^{N} a_i - \sum_{i=2}^{N} b_i$ is chosen so that $\sum_{i=0}^{N} a_i = \sum_{i=0}^{N} b_i$. To verify the majorization condition, define:

$$F : t \mapsto \int_t^\infty f,\ G : t \mapsto \int_t^\infty g \text{ and } H : t \mapsto \int_0^t h.$$

For $t \geq \varepsilon$, $F(t) \geq G(t) = H(t)$. Furthermore, $F(0) = H(0)$ by construction, $F(\varepsilon) \geq H(\varepsilon)$ and F and G are affine on $[0, \varepsilon]$ so $F(t) \geq H(t)$ for all $t \in [0, \varepsilon]$. This completes the construction of h. Note that for all $t > 0$:

$$\int_0^t h = \|h\|_2 - H(t) \geq \|f\|_1 - F(t) = \int_0^t f.$$

Hence, $h \succ f$. □

1.4.1 Sufficient condition for p-monotonicity

In this subsection we prove some lemmas linking p-convexity (resp. q-concavity) and left-p-monotonicity (resp. right-q-monotonicity). First, we show that p-convexity implies some weak form of the left-p-monotonicity. Then, we show that under the assumption that E has the Fatou property, this weak form of left-p-monotonicity is, in fact, equivalent to left-p-monotonicity by a simple approximation argument.

Convexity implies left-monotonicity

Our first lemma is a direct consequence of the geometric form of the Schur-Horn theorem.

Lemma 1.4.3. Let E be a quasi-Banach symmetric function space, p-convex with constant C. Let $f, g \in F$,

$$f^p \succ g^p \Rightarrow \|g\|_E \leq C\|f\|_E.$$

Proof. Let $f, g \in F$ suppose that $f^p \succ g^p$. Since $\|.\|_E$ is increasing, we can also assume that $\|f\|_p = \|g\|_p$ by lemma 1.4.2. Since f and g belong to F, there exist $N, n \in \mathbb{N}$ and vectors $a = (a_i)_{i \leq N}$ and $b = (b_i)_{i \leq N}$ in $(\mathbb{R}^+)^N$ such that:

$$f = \sum_{i=1}^N a_i \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})} \text{ and } g = \sum_{i=1}^N b_i \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})}.$$

The hypothesis on f and g means that $a^p \succ b^p$ and $\|a\|_p = \|b\|_p$. This means by the Schur-Horn lemma that b^p is in the convex hull of the permutations a^p_σ of a^p where for $\sigma \in \mathfrak{S}_N$, we write $a^p_\sigma := (a^p_{\sigma(i)})_{1 \leq i \leq N}$. Let

$$f_\sigma = \sum_{i=1}^N a_{\sigma(i)} \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})}.$$

This means that there exist nonnegative coefficients $(\lambda_\sigma)_{\sigma \in \mathfrak{S}_N}$ adding up to 1 such that

$$g^p = \sum_{\sigma \in \mathfrak{S}_N} \lambda_\sigma f^p_\sigma = \sum_{\sigma \in \mathfrak{S}_N} \left|\lambda^{1/p} f_\sigma\right|^p.$$

Hence, by p-convexity of E and noting that since E is rearrangment invariant, $\|f_\sigma\|_E = \|f\|_E$:

$$\|g\|_E \leq C\left(\sum_{\sigma \in \mathfrak{S}_N} \|\lambda_\sigma^{1/p} f_\sigma\|_E^p\right)^{1/p} = C\|f\|_E.$$

□

Let us now state the key approximation lemma.

Lemma 1.4.4. Let $p > 0$ and $\varepsilon > 0$. Let $f, g \in L_0(0, \alpha)$ be positive non-increasing functions such that $f^p \succ g^p$. There exist sequences (f_n) and (g_n) of functions in F such that $f_n \uparrow f$ a.e. and $g_n \uparrow g$ a.e. and $(1+\varepsilon)f^p_n \succ g^p_n$ for all $n \in \mathbb{N}$.

Proof. For any $t \geq 0$ and $n \in \mathbb{N}$, consider the interval of the form $[i2^{-n}, (i+1)2^{-n})$ containing t and denote by $a_n(t)$ its right extremity. Define

$$g_n(t) = \begin{cases} g(a_n(t)) & \text{if } t < 2^n \\ 0 & \text{if } t \geq 2^n \end{cases}$$

By construction, $g_n \in F$ and g_n is non-increasing. Moreover, $g_n(t) \uparrow g(t)$ for all points of continuity of g. Since g is non-increasing, it has only countably many points of discontinuity and $g_n \uparrow g$ a.e. Consider also the sequence $(f_n)_{n\in\mathbb{N}}$ similarly defined. Now fix n and let us find m_n such that $(1+\varepsilon)f_{m_n} \succ g_n$. To do so, note that since for all m:

$$F_m : t \mapsto \int_0^t f_m^p$$

is concave and increasing and

$$G_n : t \mapsto \int_0^t g_n^p$$

is affine on intervals of the form $[i2^{-n}, (i+1)2^{-n})$ and constant for t large enough, it suffices to check that $(1+\varepsilon)F_m(i2^n) \geq G_n(i2^n)$ for finitely many $i \in \mathbb{N}$. Since

$$F_m(t) \to \int_0^t f^p$$

and

$$G_n(t) \leq \int_0^t g^p \leq \int_0^t f^p,$$

for m_n large enough, we have $(1+\varepsilon)f_{m_n} \succ g_n$. Let $i_n = \max(m_n, n)$, the sequences (f_{i_n}) and (g_n) verify the requirements of the lemma. □

Lemma 1.4.5. Let $p > 0$ and $C > 0$. Let E be a quasi-Banach symmetric function space with the Fatou property such that:

$$\forall f, g \in F, f^p \succ g^p \Rightarrow \|g\|_E \leq C\|f\|_E.$$

Then E is left-p-monotone.

Proof. Since E is rearrangement invariant, we can suppose that f and g are positive non-increasing functions by considering f^* and g^*. Let $\varepsilon > 0$. Take (f_n) and (g_n) two sequences given by lemma 1.4.4. For all $n \in \mathbb{N}$: $\|g_n\|_E \leq (1+\varepsilon)C\|f_n\|_E \leq (1+\varepsilon)C\|f\|_E$. So by the Fatou property, $g \in E$ and $\|g\|_E \leq (1+\varepsilon)C\|f\|_E$. Since this is true for all $\varepsilon > 0$, $\|g\|_E \leq C\|f\|_E$. □

By combining the results above, we obtain the following:

Corollary 1.4.6. *Let E be a quasi-Banach symmetric sequence space with the Fatou property. If E is p-convex then E is left-p-monotone.* □

Concavity implies right-monotonicity

The same kind of proofs can be reproduced to obtain results on concavity and right-monotonicity. The approximation lemma, similar to lemma 1.4.4, is given by:

Lemma 1.4.7. Let $q > 0$ and $\varepsilon > 0$. Let $f, g \in L_0(0,\infty)$ be positive non-increasing functions such that $f^q \triangleright g^q$. There exist sequences (f_n) and (g_n) of functions in F such that $f_n \uparrow f$ a.e., $g_n \uparrow g$ a.e. and $(1+\varepsilon)f_n^q \triangleright g_n^q$ for all $n \in \mathbb{N}$.

And we have the following:

Lemma 1.4.8. Let E be a quasi-Banach symmetric function space.

- if E is q-concave with constant C then for all $f, g \in F$,

$$f^q \triangleright g^q \Rightarrow \|g\|_E \leq C\|f\|_E.$$

- if E has the Fatou property and there exists $C > 0$ such that for all $f, g \in F$,

$$f^q \triangleright g^q \Rightarrow \|g\|_E \leq C\|f\|_E,$$

  then E is right-q-monotone.

- if E is q-concave and has the Fatou property, E is right-q-monotone.

Remark 1.4.9. The reverse of the lemma above and corollary 1.4.6 are not true. Indeed, consider for example the space $L_{1,\infty}$. It will be proved later that it is right-q-monotone for all $q > 1$ but it is not q-concave for any q. Constructing symmetric spaces which are not p-convex for any p is not an easy task. We are indebted to F. Sukochev for indicating the following reference to us ([30]). It is possible that similar techniques can be used to construct a left-1-monotone space which is not p-convex for any p.

1.4.2 Equivalence between K-monotonicity and p-monotonicity

Function spaces

In this section, we prove the main theorem of this chapter, a characterization of interpolation spaces between $L_p(0,\infty)$ and $L_q(0,\infty)$. To lighten the notations, we set $L_r := L_r(0,\infty)$ for all $r \in (0,\infty]$. The proof uses a strategy similar as [8]. Let us first state the result precisely.

Theorem 1.4.10. *Let $0 < p \leq q < \infty$ and E be a quasi-Banach symmetric function space on $(0,\infty)$. Then E is left-p-monotone and right-q-monotone if and only if E is an interpolation space between L_p and L_q.*

Proof of the reverse implication. Suppose that E is an interpolation for the pair (L_p, L_q). Then by reiteration, E is an interpolation space for the pair (L_p, L_∞) and by proposition 1.3.5, E is left-p-monotone. Let $f \in E$ and $g \in L_p + L_q$ such that $f^q \triangleright g^q$. Assume that f and g are positive by taking if necessary their modules. We will show that $g \in E$ and $\|g\|_E \lesssim \|f\|_E$. Since E is K-monotone by 1.3.4, it is enough to show that for all $t > 0$,

$$K_t(g, L_p, L_q) \lesssim K_t(f, L_p, L_q).$$

Recall also that by (1.1), for all $h \in L_p + L_q$ and $t > 0$:

$$K_t(h,p,q) \approx \left(\int_0^{t^\alpha} (h^*)^p\right)^{1/p} + t\left(\int_{t^\alpha}^\infty (h^*)^q\right)^{1/q},$$

where $\alpha = (p^{-1} - q^{-1})^{-1}$. Let $t > 0$. First, since $f^q \triangleright g^q$,

$$t\left(\int_{t^\alpha}^\infty (g^*)^q\right)^{1/q} \leq t\left(\int_{t^\alpha}^\infty (f^*)^q\right)^{1/q} \leq K_t(f,p,q).$$

We now have to estimate the second term i.e to prove that:

$$\left(\int_0^{t^\alpha} (g^*)^p\right)^{1/p} \lesssim K_t(f, L_p, L_q).$$

Suppose now that f and g are bounded so that f^q and g^q belong to L_1. Since $f^q \triangleright g^q$, by lemma 1.4.2, there exists $g' \geq g$ such that $\|g'\|_q = \|f\|_q$ and $g'^q \succ f^q$. Then by [6], there exists a subunital, positive, integral preserving operator T such that $T((g'^*)^q) = (f^*)^q$. Hence $T((g^*)^q) \leq (f^*)^q$. Define $e = \mathbf{1}_{(0,t^\alpha)}$ and write,

$$\left(\int_0^{t^\alpha} (g^*)^p\right)^{1/p} = \left(\int_0^\infty e(g^*)^p\right)^{1/p} = \left(\int_0^\infty T(e(g^*)^p)\right)^{1/p}.$$

Note that there is an Hölder type inequality for positive unital operators i.e, for r, r' such that $1 = r^{-1} + r'^{-1}$, $T(ab) \leq T(a^r)^{1/r} T(b^{r'})^{1/r'}$. We apply it to our situation with $r = q/p$ and $r' = \alpha/p$ and obtain:

$$\begin{aligned}\left(\int_0^{t^\alpha} (g^*)^p\right)^{1/p} &\leq \left(\int_0^\infty T(e)^{p/\alpha} T((g^*)^q)^{p/q}\right)^{1/p} \\ &\lesssim \left(\int_0^{t^\alpha} (f^*)^p T(e)^{p/\alpha}\right)^{1/p} + \left(\int_{t^\alpha}^\infty (f^*)^p T(e)^{p/\alpha}\right)^{1/p}\end{aligned}$$

To estimate the left summand, we use the fact that $\|Te\|_\infty \leq 1$ and for the right summand, we apply the usual Hölder's inequality.

$$\begin{aligned}&\leq \left(\int_0^{t^\alpha} (f^*)^p\right)^{1/p} + \left(\int_{t^\alpha}^\infty (f^*)^q\right)^{1/q} \left(\int_{t^\alpha}^\infty T(\mathbf{1}_{(0,t^\alpha)})\right)^{1/\alpha} \\ &\lesssim K_t(f, L_p, L_q)\end{aligned}$$

where we used that $\left(\int_{t^\alpha}^\infty T(\mathbf{1}_{(0,t^\alpha)})\right)^{1/\alpha} \leq \left(\int_0^\infty T(\mathbf{1}_{(0,t^\alpha)})\right)^{1/\alpha} = t$. To conclude for unbounded functions, it suffices to approximate g^* and f^*. A way to do it is to apply the previous inequality to $g^*(. - \varepsilon)$ and $f^*(. - \varepsilon)$ and let ε go to 0.

□

Proving the direct implication requires more work. The first lemma is an easy check. We need to verify that E is an intermediate space for the couple (L_p, L_q).

Lemma 1.4.11. Let $q \geq p > 0$ and $C > 0$. Let E be a quasi-Banach symmetric function space. Suppose that E is left-p-monotone and right-q-monotone. Then $L_p \cap L_q \subset E \subset L_p + L_q$.

Proof. Let us prove the first inclusion. First note that since E is a symmetric function space, it contains characteristic functions of sets of finite measure ([11]). Let $f \in L_p \cap L_q$. Decompose f into $f_1 = f\mathbf{1}_{|f|>1}$ and $f_2 = f\mathbf{1}_{|f|\leq 1}$. Let $h_1 = \mathbf{1}_{\left(0,\|f_1\|_q^q\right)}$. Since $f_1 \in L_q$, $h_1 \in E$ and it is straightforward to check that $h_1^q \triangleright f_1^q$. Indeed, for all $t \in [0, \|f_1\|_q^q)$,

$$\int_t^\infty (f_1^*)^q = \|f_1\|_q^q - \int_0^t f_1^* \geq \|f_1\|_q^q - \int_0^t 1 = \int_t^\infty (h_1^*)^q.$$

And for $t \geq \|f_1\|_q^q$, $\int_t^\infty (h_1^*)^q = 0 \leq \int_t^\infty (f_1^*)^q$. So $h_1^q \triangleright f_1^q$, hence $f_1 \in E$. Similarly let $h_2 = \mathbf{1}_{\left(0,\|f_2\|_p^p\right)}$, h_2 is in E and $h_2^p \succ f_2^p$ so $f_2 \in E$. Hence $f \in E$.

Let us now prove the second inclusion. Let $f \notin L_p + L_q$. Decompose f into f_1 and f_2 as above. By assumption, $f_1 \notin L_p + L_q$ or $f_2 \notin L_p + L_q$.

case 1: $f_1 \notin L_p + L_q$. Then f_1 is not in L_p, this means that $f_1^p \succ a\mathbf{1}_{[0,1]}$ for all $a > 0$, hence $\|f_1\|_E = \infty$, $f_1 \notin E$.

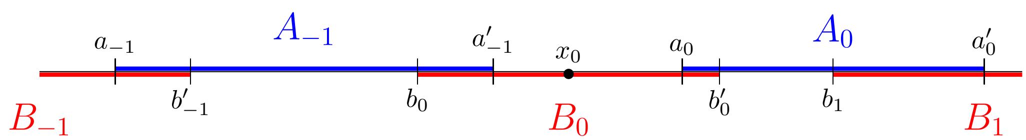
case 2: $f_2 \notin L_p + L_q$. Similarly, $f_2 \notin L_q$ so $f_2^q \triangleright a\mathbf{1}_{[0,1]}$ for all $a > 0$ so $f_2 \notin E$. □

The following two lemmas constitute the heart of the proof. They will be used to prove that left-p-monotonicity and right-q-monotonicity imply K-monotonicity for the pair (L_p, L_q). For A a topological space and $x \in A$, we will denote by $A(x)$ the connected component of A containing x.

Lemma 1.4.12. Let A and B be two open subsets of $(0, \infty)$ such that $A \cup B = (0, \infty)$. Then there exist intervals $I, J \subset \mathbb{Z}$ and sequences of open intervals $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ in $(0, \infty)$ such that:

- for all $i \in I$, $A_i = (a_i, a_i')$ is a connected component of A and for all $j \in J$, $B_j = (b_j, b_j')$ is a connected component of B,
- for all $n \in \mathbb{Z}$, if $n \in I \cap J$ then $a_n \in B_n$ and if $n \in I$ and $n+1 \in J$ then $b_{n+1} \in A_n$,
- if $n \in I$ and $n-1 \in I$, then $n \in J$ and $a_{n-1} < b_n < a_n$. Similarly, if $n \in J$ and $n+1 \in J$ then $n \in A$ and $b_n < a_n < b_{n+1}$.
- $\bigcup_{i\in I} A_i \cup \bigcup_{j\in J} B_j = (0, \infty)$.

These conditions, expressed in terms of indices, can be difficult to picture at first. What we want to prove is essentially that the situation is similar to the figure below.



Proof. Let us construct (A_n), (B_n), I and J by induction. If $A = (0, \infty)$, there is nothing to do so suppose that there exists $x_0 \in B \backslash A$ and define $B_0 = B(x_0)$. There is now essentially one possible construction sastisfying the conditions above.

Let us detail the first step: if $b_0 \neq 0$, then $b_0 \in A$ and the second condition implies that it belongs to A_{-1}. Then, by the first condition, we have to define, $A_{-1} = A(b_0)$. If $a_{-1} \neq 0$, the extension to the left continues with $B_{-1} = B(a_{-1})$...

This leads to the following procedure, if $a_n \neq 0$ or $b_n \neq 0$ is the point the more to the left that has already been defined, extend to the left by defining $A_{n-1} = A(b_n)$ and $B_n = B(a_n)$. The procedure stops if at some point $a_n = 0$ or $b_n = 0$. Else, we always have $a_n \notin A$ so $a_n \in B$ (similarly, $b_n \in A$) and the procedure is well-defined. The extension to the right is similar. If $b'_n \neq \infty$ or $a'_{n-1} \neq \infty$ is the point the more to the right, define $A_n = A(b'_n)$ and $B_n = B(a'_{n-1})$. This time, the procedure stops if $a'_n = \infty$ or $b'_n = \infty$ and it is still well defined. Let us check that the properties are verified.

Condition 1. It is true by definition.

Condition 2. We only give details for the first property. If $n \in I \cap J$ this means that $a_n \neq 0$. If $n < 0$, we have $B_n = B(a_n)$ so $a_n \in B_n$. If $n = 0$, $A_0 = A(b'_0)$. So $a_0 < b'_0$ and by assumption on x_0, $a_0 > x_0 > b_0$ so $a_0 \in B_0$. Finally, if $n > 0$ then $B_n = B(a'_{n-1})$ and $A_n = A(b'_n)$ so $b_n < a'_{n-1} < a_n < b'_n$, $a_n \in B_n$.

Condition 3. Suppose that $n \in I$ and $n-1 \in I$. This implies that $a_n \neq 0$ and $a'_{n-1} \neq \infty$. So $a_n \in B$ so, using the two previous properties, $B_n = B(a_n)$ so $b_n < a_n$. And $A_{n-1} = A(b_n)$ so $a_{n-1} < b_n$. The second property is proved by a similar argument.

Condition 4. We only consider the case where $I = J = \mathbb{Z}$ since the other cases are easier. It suffices to prove that the $a'_i \to \infty$ when $i \to \infty$ and $a_i \to 0$ when $i \to -\infty$ since $\bigcup_{n \in \mathbb{N}} A_n \cup B_n$ is connected. Assume by contradiction that the $(a'_n)_{n \geq 0}$ accumulates at some point x. Then the (b'_n) also accumulates at x. But $x \in A \cup B$ which are open set so for n large enough we either have $a'_n \in A(x)$ or $b'_n \in B(x)$ which is a contradiction. Similarly, we get a contradiction if we suppose that the sequence $(a_n)_{n \leq 0}$ accumulates at a point $x > 0$. $\square$

We use the previous construction to obtain the following key lemma.

Lemma 1.4.13. Let $0 < p \leq q < \infty$, $\alpha > 0$ and $f, g \in (L_p + L_q)^+$ be non-increasing right-continuous functions such that for all $t > 0$:

$$\left(\int_0^{t^\alpha} f^p\right)^{1/p} + t\left(\int_{t^\alpha}^\infty f^q\right)^{1/q} > \left(\int_0^{t^\alpha} g^p\right)^{1/p} + t\left(\int_{t^\alpha}^\infty g^q\right)^{1/q},$$

then there exist nonegative functions h and l such that $h + l = g$, h is non-increasing, $f^p \succ h^p$ and $f^q \rhd l^q$. More precisely, for all $t > 0$,

$$\int_t^\infty f^q \geq \int_t^\infty l^q.$$

Proof. Let $A := \{t \in \mathbb{R}^+ : \int_0^t f^p > \int_0^t g^p\}$ and $B := \{t \in \mathbb{R}^+ : \int_t^\infty f^q > \int_t^\infty g^q\}$.

A and B are open sets and $A \cup B = (0, \infty)$. Let $(A_n)_{n\in\mathbb{Z}}$ and $(B_n)_{n\in\mathbb{Z}}$ be obtained by the lemma above. We only consider the case of infinite families of intervals in both directions, the other cases are similar. Define h and l as follows. For all $n \in \mathbb{Z}$,

$$h(t) = \begin{cases} g(a_n) & \text{if } t \in [b_n, a_n) \\ g(t) & \text{if } t \in [a_n, b_{n+1}) \end{cases}$$

Let $l = g - h$. g and l are clearly nonegative.

Note that for all $n \in \mathbb{Z}$, $f(a_n)^p \geq g(a_n)^p$. Indeed, since a_n is an extremity of a connected component of A, $\int_0^{a_n} f^p = \int_0^{a_n} g^p$ and for ε small enough so that $a_n + \varepsilon < a'_n$, $\int_0^{a_n+\varepsilon} f^p > \int_0^{a_n+\varepsilon} g^p$. Hence, $\int_{a_n}^{a_n+\varepsilon} f^p > \int_{a_n}^{a_n+\varepsilon} g^p$. Since f and g are right-continuous this implies that $f(a_n)^p \geq g(a_n)^p$.

Let us now check that the decomposition verifies what we claimed. For all $n \in \mathbb{N}$, for all $t \in [b_n, a_n)$:

$$\begin{aligned}\int_0^t h^p &= \int_0^{b_n} h^p + \int_{b_n}^t h^p \\ &\leq \int_0^{b_n} g^p + \int_{b_n}^t g(a_n) \\ &\leq \int_0^{b_n} f^p + \int_{b_n}^t f(a_n)^p \\ &\leq \int_0^t f^p,\end{aligned}$$

and since $a_n \in B_n$, $t \in \overline{B}$ and:

$$\int_t^\infty l^q \leq \int_t^\infty g^q \leq \int_t^\infty f^q.$$

For all $t \in [a_n, b_{n+1})$, since $b_{n+1} \in A_n$, $t \in \overline{A}$ which implies that:

$$\int_0^t h^p \leq \int_0^t g^p \leq \int_0^t f^p.$$

and

$$\int_t^\infty l^q = \int_{b_{n+1}}^\infty l^q \leq \int_{b_{n+1}}^\infty g^q \leq \int_{b_{n+1}}^\infty f^q.$$

Moreover, note that h is non-increasing, so for all $t \geq 0$:

$$\int_0^t (h^*)^p = \int_0^t h^p \leq \int_0^t f^p = \int_0^t (f^*)^p.$$

And since l is positive, for all $t \geq 0$:

$$\int_t^\infty (l^*)^q \leq \int_t^\infty l^q \leq \int_t^\infty f^q = \int_t^\infty (f^*)^q.$$

□

We are now ready to finish the proof of the main theorem.

End of the proof of theorem 1.4.10. By proposition 1.2.5, it suffices to prove that E is K-monotone. We already know from lemma 1.4.11 that $L_p \cap L_q \subset E \subset L_p + L_q$. Assume that $f \neq 0$. Suppose that for all $t > 0$, $K_t(f, L_p, L_q) \geq K_t(g, L_p, L_q)$. By (1.1), this means that there exists a constant $c > 0$ independent of f and g such that for all $t > 0$:

$$\left(\int_0^{t^\alpha} (cf^*)^p\right)^{1/p} + t\left(\int_{t^\alpha}^\infty (cf^*)^q\right)^{1/q} > \left(\int_0^{t^\alpha} (g^*)^p\right)^{1/p} + t\left(\int_{t^\alpha}^\infty (g^*)^q\right)^{1/q},$$

So by lemma 1.4.13 applied to f^* and g^*, there exists two functions h and l such that $h + l = g^*$, $(cf)^p \succ h^p$ and $(cf)^q \rhd l^q$. Hence, by the assumptions on E, $h, l \in E$ so $g \in E$ and

$$\|g\|_E = \|g^*\|_E \lesssim \|h\|_E + \|l\|_E \leq 2cK\|f\|_E.$$

□

Remark 1.4.14. The result can be extended, with the same proof, to symmetric function spaces on the interval $(0, \alpha)$, $\alpha \in (0, \infty)$.

Remark 1.4.15. It is known that a q-concave space is automatically p-convex for some $p > 0$, it is for example a consequence of theorem 4.1 in [28]. Conjecture 1.1.1 for function spaces, is now reduced to proving a similar fact for right-q-monotonicity and left-p-monotonicity.

Remark 1.4.16. As a corollary of theorem 1.4.10 and corollary 1.4.6, we obtain theorem 1.1.2 mentionned in the introduction.

Sequence spaces

We will now prove the main theorem in the context of symmetric sequence spaces rather than symmetric function spaces. In this setting, it is easy to see that the approximation arguments made above still hold. We will also use the fact that ℓ^∞ can be embedded into L_∞ by:

$$u \mapsto \sum_{i=1}^{\infty} u_i \mathbf{1}_{[i-1,i)}.$$

Hence, we will identify ℓ^∞ with the subalgebra of L_∞ of functions a.e. constant on intervals of the form $[i, i+1)$, $i \in \mathbb{N}$. Denote by $\mathbb{E}$ the conditional expectation from L_∞ to ℓ^∞. The notions of left and right majorization can be extended without modifications to sequences and coincide with the usual notion. Let u and v in ℓ^∞:

$$u \succ v \Leftrightarrow \forall n \in \mathbb{N}, \sum_{i=0}^{n} u_i^* \geq \sum_{i=0}^{n} v_i^* \text{ and } u \rhd v \Leftrightarrow \forall n \in \mathbb{N}, \sum_{i=n}^{\infty} u_i^* \geq \sum_{i=n}^{\infty} v_i^*.$$

Let us now state the result of this subsection:

Theorem 1.4.17. *Let $0 < p \leq q < \infty$ and E be a quasi-Banach symmetric sequence space. If E is left-p-monotone and right-q-monotone then E is an interpolation space between ℓ_p and ℓ_q.*

Proof. By proposition 1.2.5, it suffices to check that E is K-monotone. Similarly to the case of function spaces, the hypothesis of the theorem imply that $\ell^p = \ell^p \cap \ell^q \subset E \subset \ell^p + \ell^q = \ell^q$ and that for any $u \in E$ and $v \in \ell^q$:

$$u^q \rhd v^q \Rightarrow v \in E, \|v\|_E \leq C\|u\|_E \text{ and } u^p \succ v^p \Rightarrow v \in E, \|v\|_E \leq C\|u\|_E.$$

Let $u \in E$ and $v \in \ell^q$, $u \neq 0$. Assume that for all $t > 0, K_t(u, \ell^p, \ell^q) \leq K_t(v, \ell^p, \ell^q)$. The formula for the K-functional used previously still holds in this context. Indeed, for any sequence $w \in \ell^q$ and $t > 0$:

$$K_t(w, p, q) \approx \left(\int_0^{t^\alpha} (w^*)^p\right)^{1/p} + t\left(\int_{t^\alpha}^{\infty} (w^*)^q\right)^{1/q}.$$

So there exists a constant c independent of u and v so that for all $t > 0$:

$$\left(\int_0^{t^\alpha} (cu^*)^p\right)^{1/p} + t\left(\int_{t^\alpha}^{\infty} (cu^*)^q\right)^{1/q} > \left(\int_0^{t^\alpha} (v^*)^p\right)^{1/p} + t\left(\int_{t^\alpha}^{\infty} (v^*)^q\right)^{1/q},$$

By lemma 1.4.13, there exists h and l non-negative functions such that, $h + l = v^*$, h is non-increasing, $(cu)^p \succ h^p$ and $(cu)^q \rhd l^q$. Let $\overline{h} = (\mathbb{E}(h^p))^{1/p}$ and $\bar{l} = \mathbb{E}(l^q)^{1/q}$.

We will start by showing that $(cu)^p \succ \overline{h}^p$ and $(cu)^q \rhd \overline{g}^q$. The functions:

$$t \mapsto \int_0^t (cu^*)^p \text{ and } t \mapsto \int_0^t \overline{h}^p$$

are affine on intervals of the form $[i, i+1]$, $i \in \mathbb{N}$. So it suffices to check the inequality at integers. Let $n \in \mathbb{N}$,

$$\int_0^n (cu^*)^p \geq \int_0^n h^p = \int_0^n \overline{h}^p.$$

The other inequality is proven in the same manner.

Recall that $\overline{h}$ and $\overline{l}$ can be seen as sequences. Note that $a_p := \max(1, 2^{p-1})$ is the optimal constant such that for all $a, b \in \mathbb{R}^+$, $(a+b)^p \leq a_p(a^p + b^p)$. We have:

$$v^* = (\mathbb{E}((v^*)^p))^{1/p} \leq a_p \mathbb{E}(h^p + l^p)^{1/p} \leq a_p a_{1/p} \left(\mathbb{E}(h^p)^{1/p} + \mathbb{E}(l^p)^{1/p}\right) \leq a_p a_{1/p}(\overline{h} + \overline{l}).$$

And by hypothesis on E, $\overline{h} \in E$ and $\overline{l} \in E$ with

$$\left\|\overline{h}\right\|_E \leq cK\left\|u\right\|_E \text{ and } \left\|\overline{l}\right\|_E \leq cK\left\|u\right\|_E.$$

So $v \in E$ and $\left\|v\right\|_E \lesssim \left\|u\right\|_E$ with an implicit constant depending on E, p and q. □

Remark 1.4.18. Note that K-monotonicity is not known in general for couples of ℓ^p-spaces. This is why we cannot show the reverse implication in theorem 1.4.17. However, note that for $1 \leq p \leq q \leq \infty$, the couple (ℓ^p, ℓ^q) is a Calderón couple (1.3.4) and we can recover the following:

Corollary 1.4.19. *Let $1 \leq p \leq q < \infty$ and E be a quasi-Banach symmetric sequence space. Then E is left-p-monotone and right-q-monotone if and only if E is an interpolation space between ℓ_p and ℓ_q.*

This is close to the main result proved by Cwikel and Nilsson in [8].

Chapter 2

Preliminaries in Noncommutative Integration Theory

This chapter is meant as a toolbox in view of the two following ones. In the first section, we introduce the basic definitions of noncommutative integration. In the second, we recall and prove properties of the K-functional for couples of noncommutative L_p-spaces. In the third, we introduce ultraproducts and prove some of their properties. Finally, we detail the proof of an inequality which is a consequence of [54] and will be crucial in the next chapter.

2.1 Definition and basic properties of noncommutative L_p-spaces

2.1.1 Definition

A *noncommutative measure space* is the data of a von Neumann algebra $\mathcal{M}$ equipped with a semifinite normal faithful trace τ. Any element x in $\mathcal{M}$ admits a polar decomposition $x = u\,|x|$ where u is a unitary and $|x|$ is positive. More precisely, $|x| = (x^*x)^{1/2}$. For any $p \in (0, \infty)$, we can define the L_p-norm (or quasi-norm) on this space by Borel functional calculus and the following formula:

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

The completion of $\{x \in \mathcal{M} : \|x\|_p < \infty\}$ with respect to $\|.\|_p$ is denoted by $L_p(\mathcal{M})$ and verifies properties similar to those of classical L_p-spaces. We refer to [50] for a more complete introduction of noncommutative L_p-spaces.

The noncommutative analog of measurable functions is denoted by $L_0(\mathcal{M})$ and is the space of unbounded operators x affilated with $\mathcal{M}$ which are τ-measurable (i.e there exists $\lambda \in \mathbb{R}^+$ such that $\tau(\mathbf{1}_{(\lambda,\infty)}(|x|)) < \infty$, where

$\mathbf{1}_{(\lambda,\infty)}(|x|)$ is defined by the Borel functional calculus). This space $L_0(\mathcal{M})$ equipped with the measure topology continuously contains $L_p(\mathcal{M})$ for all p. This makes any couple $(L_p(\mathcal{M}),\ L_q(\mathcal{M}))$ compatible in the sense of interpolation [60].

The *support* of any self-adjoint element $x \in L_0(\mathcal{M})$ is defined as the least projection $s(x)$ such that $s(x)x = x$. Let

$$\mathcal{M}_c = \{x \in \mathcal{M} : \tau(s(|x|)) < \infty\}$$

be the space of bounded, finitely supported operators in $\mathcal{M}$. For any $p \in (0, \infty)$, $\mathcal{M}_c$ is dense in $L_p(\mathcal{M})$.

Denote by $\mathcal{P}(\mathcal{M})$ the set of orthogonal projections in $\mathcal{M}$ and by $\mathcal{P}_c(\mathcal{M})$ the set of finite projections in $\mathcal{M}$.

2.1.2 Generalised singular values

A crucial tool to understand and noncommutative L_p-spaces is the *generalised singular numbers* $\mu(x)$ associated to any $x \in L_0(\mathcal{M})$. They can be defined by the following formula [16]:

$$\begin{aligned} \mu(x) :& \mathbb{R}^+ \to \mathbb{R}^+ \\ & t \mapsto \mu_t(x) = \inf\{\|ex\|_\infty : e \in \mathcal{P}(\mathcal{M}), \tau(1-e) \le t\}, \\ & \qquad\qquad = \sup\{a \in \mathbb{R}^+ : \tau(\mathbf{1}_{(a,\infty)}(|x|)) \ge t\}. \end{aligned}$$

The meaning of these formulas may not be immediately clear but $\mu(x)$ is to be thought as a non-increasing non-negative function which has the same distribution as x, in particular $\|\mu(x)\|_p = \|x\|_p$ for all p. The generalised singular values satisfy the following properties:

Proposition 2.1.1. *Let $x, y \in L_0(\mathcal{M})$, $t, s > 0$, then:*

1. *$\forall p \in (0, \infty)$, if x is positive, $\mu(x^p) = \mu(x)^p$,*
2. *$\mu_{s+t}(x + y) \le \mu_s(x) + \mu_t(y)$,*
3. *$\mu(x)$ is right-continuous.*

Thanks to the generalised singular values, various classical notions can be naturally translated to the noncommutative context. The interpolation of noncommutative L_p-spaces for example, can be reduced to the classical case (see remark 2.2.7). The construction of noncommutative L_p-spaces can also be generalised to any symmetric space [31] . Let E be a symmetric function space on $(0, \infty)$. Define:

$$E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu(x) \in E\}.$$

$E(\mathcal{M})$ equipped with the (quasi-)norm $\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_E$ is a (quasi-)Banach space. We say that E is a *noncommutative symmetric space*.

2.1.3 Tensor product

Let $(\mathcal{M}_1, \tau_1)$ and $(\mathcal{M}_2, \tau_2)$ be two noncommutative measure spaces, we can define their tensor product $(\mathcal{M}, \tau)$ in a natural way. First, consider the algebraic tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$. It is naturally equipped with a trace $\tau := \tau_1 \otimes \tau_2$ which induces a scalar product on $\mathcal{M}_1 \otimes \mathcal{M}_2$. This allows, by taking the completion with respect to the norm associated to this tensor product, to define a Hilbert space $L_2(\mathcal{M})$ on which $\mathcal{M}_1 \otimes \mathcal{M}_2$ acts by multiplication on the right. Define $\mathcal{M}$ to be the weak closure of $\mathcal{M}_1 \otimes \mathcal{M}_2$ in $\mathcal{B}(L_2(\mathcal{M}))$, $\tau_1 \otimes \tau_2$ extends uniquely to a positive semifinite normal trace τ on $\mathcal{M}$. In general, the tracial tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$ is denoted by $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.

This construction provides a general setting for various objects that will appear in the following chapters. For example, some spaces of functions taking values in noncommutative L_p-spaces can be seen as noncommutative L_p-spaces themselves. More precisely, for $p \in (0, \infty)$, and $n \in \mathbb{N}$, the spaces $L_p(\mathbb{R}^n,\ L_p(\mathcal{M}))$ is canonically isometric to $L_p(\mathcal{M} \overline{\otimes} L_\infty(\mathbb{R}^n))$.

2.1.4 Noncommutative martingales

Martingales admit a noncommutative counterpart. We start by defining *conditional expectations*. Let $(\mathcal{M}, \tau)$ be a noncommutative measure space and $\mathcal{A}$ a subalgebra of $\mathcal{M}$. If τ is semifinite when restricted to $\mathcal{A}$ then there exists a conditional expectation $\mathbb{E} : \mathcal{M} \to \mathcal{A}$. $\mathbb{E}$ is unital, positive, trace preserving and for any $a, b \in \mathcal{A}$ and $x \in \mathcal{M}$, $\mathbb{E}(axb) = a\mathbb{E}(x)b$. It can be defined as the adjoint of the inclusion map $i : L_1(\mathcal{A}) \to L_1(\mathcal{M})$. Note that for all $p \geq 1$, $\mathbb{E}$ can be extended to a contraction from $L_p(\mathcal{M})$ to $L_p(\mathcal{A})$. This fact is no longer true when $p < 1$.

A filtration $\mathcal{F}$ on $\mathcal{M}$ is an increasing sequence (potentially two-sided) of subalgebras $(\mathcal{M}_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, τ restricted to $\mathcal{M}_n$'s is semifinite and $\bigcup_n \mathcal{M}_n$ is weakly dense in $\mathcal{M}$. Denote by $\mathbb{E}_n$ the conditional expectation onto $\mathcal{M}_n$. Let $p \in [1, \infty]$, a martingale in $L_p(\mathcal{M})$, with respect to the filtration $\mathcal{F}$ is a sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $x_n \in L_p(\mathcal{M}_n)$ and $\mathbb{E}_n(x_n) = x_{n-1}$. We say that a martingale converges in $L_p(\mathcal{M})$ if the sequence x_n converges to an element x_∞ in $L_p(\mathcal{M})$. Note that in this case, $\mathbb{E}_n(x_\infty) = x_n$. We say that $(x_n)_{n\in\mathbb{N}}$ is bounded in L_p if $\sup_n \left\|x_n\right\|_p < \infty$. Similarly to the classical case, we have the following proposition:

Proposition 2.1.2. *Let $p \in (1, \infty)$. Every bounded martingale in $L_p(\mathcal{M})$ converges.*

Finally, if x is a martingale, we denote by $(dx_n) = (x_n - x_{n-1})$ the sequence of *martingale differences* of x. We will also use the notation $\Delta_n = \mathbb{E}_n - \mathbb{E}_{n-1}$.

2.1.5 Rows and columns

In classical harmonic analysis, a considerable amount of attention has been given to square functions. They appear for example in Littlewood-Paley theory and in the theory of functional calculus for semigroups. These are expressions of the form $\left(\sum_{i\in\mathbb{N}}|f_i|^2\right)^{1/2}$. In the noncommutative setting however, it is not clear at first how to associate a square function to a sequence $(x_i)_{i\in\mathbb{N}}\in\mathcal{M}^{\mathbb{N}}$. Indeed, the naive expression $\left(\sum_{i\in\mathbb{N}}|x_i|^2\right)^{1/2}$ has no reason to be chosen over $\left(\sum_{i\in\mathbb{N}}|x_i^*|^2\right)^{1/2}$ but the two do not have the same distribution in general. They can, in fact, if $\mathcal{M}$ is a factor, be any two positive elements with the same norm in L_2 (as a consequence of [59], theorem 3.5). The answer to constructing noncommutative square functions is given by the noncommutative Khintchine inequalities which will be presented in the next chapter. For now, we will simply introduce some useful terminology.

Denote by $S(\mathcal{M})$ the set of sequences of elements of $\mathcal{M}_c$. For $a,b\in\mathcal{M}$ and $x\in S(\mathcal{M})$ we can naturally define the product axb by:

$$axb=(ax_nb)_{n\geq 0}.$$

Let $x\in S(\mathcal{M})$. Denote by $N\in\mathbb{N}$ its length and consider the projection,

$$e=s\left(\sum_{i=0}^{N}x_i^*x_i\right)\vee s\left(\sum_{i=0}^{N}x_ix_i^*\right).$$

The fact that $x\in(e\mathcal{M}e)^N$ will be useful in various proofs in the next chapter since it enables us to work with only finite sequences of finitely supported elements.

Recall that the elements of $\mathcal{B}(\ell^2)$ can be identified with infinite matrices and that $\mathcal{B}(\ell^2)$ is endowed with its canonical trace. We will denote by $e_{n,m}$ the element $(\delta_{i,n}\delta_{j,m})_{i,j\in\mathbb{N}}\in\mathcal{B}(\ell^2)$. For x in $S(\mathcal{M})$, we define $Rx=\sum_{n\geq 0}x_n\otimes e_{1,n}$, $Cx=\sum_{n\geq 0}x_n\otimes e_{n,1}$ which are understood as elements of the von Neumann algebra $\mathcal{M}_{\mathcal{B}}:=\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$. Both algebras are equipped with the tensor product trace. When it is convenient to us, $\mathcal{M}$ will be identified with $\mathcal{M}\otimes e_{1,1}$ in $\mathcal{M}_{\mathcal{B}}$. With this in mind, note that the two "square functions" can be obtained as the moduli of Rx and Cx:

$$|(Rx)^*|=\left(\sum_{n\geq 0}x_nx_n^*\right)^{1/2}\text{ and }|Cx|=\left(\sum_{n\geq 0}x_n^*x_n\right)^{1/2}.$$

Fix E a symmetric space. Define $\|x\|_{R_E}:=\|Rx\|_{E(\mathcal{M}_{\mathcal{B}})}$ (resp. $\|x\|_{C_E}:=\|Cx\|_{E(\mathcal{M}_{\mathcal{B}})}$) and R_E (resp. C_E) the completion of $S(\mathcal{M})$ with respect to $\|.\|_{R_E}$ (resp. $\|.\|_{C_E}$). To lighten the notations, for $p\in(0,\infty]$, we write $C_p=C_{L_p}$ and $R_p=R_{L_p}$.

Note that R_p and C_p are continuously included in $L_0(\mathcal{M})^{\mathbb{N}}$. Therefore, the spaces R_p+C_p and $R_p\cap C_p$ are well-defined. They both contain $S(\mathcal{M})$ as a dense subspace (weak-$*$ dense for $p=\infty$).

Remark 2.1.3. Note that R_r and C_r can be embedded into $L_r(\mathcal{M}_{\mathcal{B}})$ by extending R and C. They can by this mean be identified with closed, 1-complemented subspaces of $L_r(\mathcal{M}_{\mathcal{B}})$. This also makes the couples (R_p,R_q) and (C_p,C_q) compatible in the sense of interpolation.

Let us now prove a lemma that will be useful when manipulating rows and columns.

Lemma 2.1.4. Let $p > 0$, $y, z \in S(\mathcal{M})$ and $e, f \in \mathcal{P}(\mathcal{M})$ such that e commutes with $|(Ry)^*|$. Then,

i. $e\,|(Ry)^*|^p = |(R(ey))^*|^p$,

ii. $\|Ry\|_p^p = \|R(ey)\|_p^p + \|R(e^\perp y)\|_p^p$,

iii. $|C(fz)|^2 \le |Cz|^2$ and consequently $\|C(fz)\|_p \le \|Cz\|_p$.

Proof. i. This is a direct computation:

$$\begin{aligned} e\,|(Ry)^*|^p &= e^{p/2}(|(Ry)^*|^2)^{p/2} = (e\,|(Ry)^*|^2)^{p/2} = (e(\textstyle\sum_i y_i y_i^*)e)^{p/2} \\ &= (\textstyle\sum_i (ey_i)(ey_i)^*)^{p/2} = |R(ey)^*|^p \,. \end{aligned}$$

ii. Write $f := e^\perp$. Using i., this is again a direct computation:

$$\begin{aligned} \|Ry\|_p^p &= \tau(|(Ry)^*|^p) = \tau((e+f)\,|(Ry)^*|^p) = \tau(e\,|(Ry)^*|^p) + \tau(f\,|(Ry)^*|^p) \\ &= \tau(|(R(ey))^*|^p) + \tau(|(R(fy))^*|^p) = \|R(ey)\|_p^p + \|R(fy)\|_p^p. \end{aligned}$$

iii. Indeed, $|C(fz)|^2 = \sum_i z_i^* f z_i \le \sum_i z_i^* z_i = |Cz|^2$. □

2.2 Some properties of the K-functional

In the context of noncommutative integration, the K-functional does not depend, up to universal constants, on the von Neumann algebra in which it is calculated. This result was proved by Xu in his unpublished lecture notes ([62]). We will give an alternative proof here using the following proposition which we will also need in the remainder of the text. It is a version of the power theorem (see [1]) in the particular case of noncommutative L_p-spaces.

Proposition 2.2.1. *Let $\alpha > 0$, $x \in (L_p(\mathcal{M}) + L_q(\mathcal{M}))^+$ and $p, q \in (0, \infty]$ then:*

$$K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})) \le c_{p,\alpha} \left[K_{t^{1/\alpha}}(x^{1/\alpha}, L_{p\alpha}(\mathcal{M}), L_{q\alpha}(\mathcal{M}))\right]^\alpha ,$$

where $c_{p,\alpha} = \max(1, 2^{1/p\alpha - 1}) \max(2^{\alpha-1}, 2^{1-\alpha})$.

Remark 2.2.2. We will use this proposition under the following form. Let $\alpha > 0$, $x \in (L_p(\mathcal{M}) + L_q(\mathcal{M}))^+$ and $p, q \in (0, \infty]$ then:

$$K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})) \approx \left[K_{t^{1/\alpha}}(x^{1/\alpha}, L_{p\alpha}(\mathcal{M}), L_{q\alpha}(\mathcal{M}))\right]^\alpha ,$$

We use the following routine operator inequalities, proved at the end of this section.

Lemma 2.2.3. Let $a, b \in L_0(\mathcal{M})^+$, $\alpha \geq 1$ and $\theta \leq 1$.

i. if $0 \leq a \leq b$ then there exists a contraction c such that $a = cbc^*$,

ii. if $0 \leq a \leq b$ then there exists a partial isometry u such that $a^2 \leq ub^2u^*$,

iii. there exists a partial isometry $u \in \mathcal{M}$ such that:
$$(a+b)^\alpha \leq 2^{\alpha-1} u(a^\alpha + b^\alpha)u^*,$$

iv. there exist two partial isometries u and v such that:
$$(a+b)^\theta \leq u a^\theta u^* + v b^\theta v^*.$$

Proof of proposition 2.2.1. For all $r > 0$, define $a_r := \max(1, 2^{1/r-1})$. Take $a \in L_{p\alpha}(\mathcal{M})$ and $b \in L_{q\alpha}(\mathcal{M})$ such that $a + b = x^{1/\alpha}$. We start by considering the case $\alpha < 1$. We need to find $a' \in L_p$ and $b' \in L_q$ such that $a' + b' = x$ and:
$$\big\|a'\big\|_p + t\big\|b'\big\|_q \leq a_{p\alpha} 2^{1-\alpha} \left(\big\|a\big\|_{p\alpha} + t^{1/\alpha}\big\|b\big\|_{q\alpha}\right)^\alpha.$$
Since x is positive, we can suppose that a and b are positive. Indeed, first note that:
$$x = \frac{a+a^*}{2} + \frac{b+b^*}{2},$$
and:
$$\Big\|\frac{a+a^*}{2}\Big\|_{p\alpha} + t\Big\|\frac{b+b^*}{2}\Big\|_{q\alpha} \leq a_{p\alpha}(\big\|a\big\|_{p\alpha} + t\big\|b\big\|_{q\alpha})$$
using the triangular inequality for $p\alpha \geq 1$ and the $p\alpha$-triangular inequality for $p\alpha \leq 1$. So we can suppose that a and b are selfadjoint and write $a = a_+ - a_-$ and $b = b_+ - b_-$ their decompositions into positive and negative parts. It follows that $x \leq a_+ + b_+$ so there exists a contraction c such that $x = ca_+c^* + cb_+c^*$ which yields a better decomposition than a, b.

Using lemma 2.2.3, we can find two contractions u and v such that:
$$x = ua^\alpha u^* + vb^\alpha v^* =: a' + b'.$$
And we have:
$$\begin{aligned}
\big\|a'\big\|_p + t\big\|b'\big\|_q &\leq \big\|a^\alpha\big\|_p + \big\|b^\alpha\big\|_q \\
&= \big\|a\big\|_{p\alpha}^\alpha + t^\alpha\big\|b\big\|_{q\alpha}^\alpha \\
&\leq 2^{1-\alpha}\left(\big\|a\big\|_{p\alpha} + t\big\|b\big\|_{q\alpha}\right)^\alpha.
\end{aligned}$$
Taking the infimum over all decompositions $x^{1/\alpha} = a + b$, and recalling that we had to add a factor $a_{p\alpha}$ to consider only a and b positive, we obtain:
$$K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})) \leq a_{p\alpha} 2^{1-\alpha} K_{t^{1/\alpha}}(x^{1/\alpha}, L_{p\alpha}(\mathcal{M}), L_{q\alpha}(\mathcal{M}))^\alpha.$$

Let us now consider $\alpha \geq 1$. Continue to assume a and b to be nonnegative and recall that we lose a constant $a_{p\alpha}$ by doing so. Using lemma 2.2.3 there exists a contraction u such that:

$$x = 2^{\alpha-1}u(a^\alpha + b^\alpha)u^* = 2^{\alpha-1}ua^\alpha u^* + 2^{\alpha-1}ub^\alpha u^* =: a' + b'.$$

Now, we compute:

$$\begin{aligned}\|a'\|_p + t\|b'\|_q &\leq 2^{\alpha-1}(\|a\|_{p\alpha}^\alpha + t\|b\|_{q\alpha}^\alpha) \\ &\leq 2^{\alpha-1}(\|a\|_{p\alpha} + t^{1/\alpha}\|b\|_{q\alpha})^\alpha.\end{aligned}$$

Taking the infimum over all decompositions, the proof is complete. □

We will now begin to prove the result mentioned at the beginning of this section i.e. that the K-functional of an element only depends on its distribution. The finite cas can be proved without much trouble by using conditional expectations. For the semifinite case however, we need the following two lemmas.

Lemma 2.2.4. Let $p, q \in (1, \infty]$ and $t > 0$. Let $x \in L_p(\mathcal{M}) + L_q(\mathcal{M})$, then

$$K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})) = \sup_{e \in \mathcal{P}_c(\mathcal{M})} K_t(exe, L_p(\mathcal{M}), L_q(\mathcal{M})).$$

Proof. Define $E = L_{p'}(\mathcal{M}) \cap t^{-1}L_{q'}(\mathcal{M})$ with $p' = (1-p^{-1})^{-1}$ and $q' = (1-q^{-1})^{-1}$ and note that $L_p(\mathcal{M}) + tL_q(\mathcal{M}) = E^*$. Since p' and q' belong to $[1, \infty)$, $\mathcal{M}_c$ is dense in E. Hence,

$$\begin{aligned}K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})) &= \sup\{|\tau(xy)| : y \in \mathcal{M}_c, \|y\|_E = 1\} \\ &= \sup\{|\tau(xeye)| : e \in \mathcal{P}_c(\mathcal{M}), y \in \mathcal{M}_c, \|y\|_E = 1\} \\ &= \sup_{e \in \mathcal{P}_c(\mathcal{M})} \|exe\|_{L_p(\mathcal{M})+tL_q(\mathcal{M})} \\ &= \sup_{e \in \mathcal{P}_c(\mathcal{M})} K_t(exe, L_p(\mathcal{M}), L_q(\mathcal{M})).\end{aligned}$$

□

Lemma 2.2.5. Let $x \in L_0^+(\mathcal{M})$, $\epsilon > 0$ and $f \in \mathcal{P}_c(L_\infty(0, \infty))$. There exists a finite projection $e \in \mathcal{M}$ such that:

$$\mu(\mu(x)f) \leq \mu(exe) + \varepsilon.$$

Proof. Let $f \in \mathcal{P}_c(L_\infty(0, \infty))$ and define $T := \int_{\mathbb{R}^+} f$, then

$$\mu(\mu(x)f) \lesssim \mu(x)\mathbf{1}_{(0,T)} = \mu(\mu(x)\mathbf{1}_{(0,T)}).$$

Hence, it suffices to consider projections of the form $\mathbf{1}_{(0,T)}$, $T > 0$. Let $a = \mu_T(x)$ and let $e_1 = \mathbf{1}_{(a,\infty)}(x)$. By definition of μ, e_1 is finite, write $t_1 = \tau(e_1)$. Note that $\mu_t(x) = a$ for $t_1 \leq t < T$. Since $a = \mu(T)$, $\tau(\mathbf{1}_{(a-\varepsilon,\infty)}) \geq T$, hence $\tau(\mathbf{1}_{(a-\varepsilon,a]}) \geq T - t_1$. Let e_2 be a finite projection such that $e_2 \leq \mathbf{1}_{(a-\varepsilon,a]}$ and

$\tau(e_2) \geq T - t_1$. It is possible to find such a projection since τ is semifinite. Define $e = e_1 + e_2$. By construction, for $t \in (0, t_1)$, $\mu_t(exe) = \mu_t(x) = f(t)\mu_t(x)$ and for $t \in [t_1, T)$, $\mu_t(exe) \geq a - \varepsilon = f(t)\mu_t(x) - \varepsilon$. So e satisfies the conditions of the lemma. □

Proposition 2.2.6. *Let $x \in L_p(\mathcal{M}) + L_q(\mathcal{M})$, $p, q \in \mathbb{R}$ such that $0 < p < q$ and $t > 0$:*

$$K_t(\mu(x), L_p(0,\infty), L_q(0,\infty)) \approx_{c_p} K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})),$$

where c_p only depends on p.

Proof of proposition 2.2.6. Suppose that x is positive. We do not lose generality here since multiplying by a partial isometry does not change the K-functional.

First, assume that $p > 1$. Let $e \in \mathcal{P}_c(\mathcal{M})$ be a finite projection. Let us prove the proposition for $y = exe$. To do so, we can work in the finite algebra $e\mathcal{M}e$ which clearly does not modify the K-functional of y. Denote by $\mathcal{M}_y$ the von Neumann algebra generated by $y \in \mathcal{M}$ which is abelian. There are two conditional expectations $E_1 : \mathcal{M} \to \mathcal{M}_y$ and $E_2 : L_\infty(0, \tau(e)) \to \mathcal{M}_{\mu(y)}$ and $\mathcal{M}_y$ is canonically isomorphic to $\mathcal{M}_{\mu(y)}$ by $y \mapsto \mu(y)$. Since conditional expectations extend to contractions on L_p, we have,

$$K_t(y, L_p(\mathcal{M}), L_q(\mathcal{M})) = K_t(\mu(y), L_p(0,\infty), L_q(0,\infty)).$$

Let us consider x once again. Combining the equality above with lemma 2.2.4, we obtain:

$$K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})) = \sup_{e \in \mathcal{P}_c(\mathcal{M})} K_t(\mu(exe), L_p(0,\infty), L_q(0,\infty)) \tag{2.1}$$

and

$$K_t(\mu(x), L_p(0,\infty), L_q(0,\infty)) = \sup_{f \in \mathcal{P}_c(L_\infty(0,\infty))} K_t(\mu(f\mu(x)f), L_p(0,\infty), L_q(0,\infty)). \tag{2.2}$$

Since for all $e \in \mathcal{P}(\mathcal{M})$, $\mu(exe) \leq \mu(x)$ (see [16]):

$$K_t(x, L_p(\mathcal{M}), L_q(\mathcal{M})) \leq K_t(\mu(x), L_p(0,\infty, L_q(0,\infty)).$$

Conversely, by lemma 2.2.5, for every projection $f \in \mathcal{P}_c(L_\infty(0,\infty))$ and $\varepsilon > 0$, there exists a finite projection e in $\mathcal{M}$, such that $\mu(exe) + \varepsilon \geq \mu(f\mu(x))$. This implies that

$$\sup_{e \in \mathcal{P}_c(\mathcal{M})} K_t(\mu(exe), L_p(0,\infty), L_q(0,\infty)) \geq \sup_{f \in \mathcal{P}_c(L_\infty(0,\infty))} K_t(f\mu(x)f, L_p(0,\infty), L_q(0,\infty)),$$

and enables us to conclude using (2.1) and (2.2).

Now take $0 < p \leq 1$ and $q > p$. The result follows from proposition 2.2.1:

$$\begin{aligned} K_t(y, L_p(\mathcal{M}), L_q(\mathcal{M})) &\approx (K_{t^{p/2}}(y^{p/2}, L_2(\mathcal{M}), L_{2q/p}(\mathcal{M}))^{2/p} \\ &= K_{t^{p/2}}(\mu(y^{p/2}), L_2(\mathbb{R}^+), L_{2q/p}(\mathbb{R}^+))^{2/p} \\ &\approx K_t(\mu(y), L_p(\mathbb{R}^+), L_q(\mathbb{R}^+)). \end{aligned}$$

□

Remark 2.2.7. In [62], the result above is obtained by first proving that the couple $(L_1(\mathcal{M}), L_\infty(\mathcal{M}))$ is, in interpolation language, a partial retract of the couple $(L_1(0,\infty), L_\infty(0,\infty))$.

This enables us to define: $K_t(x,p,q) := K_t(\mu(x), L_p(\mathbb{R}^+), L_q(\mathbb{R}^+))$.

Remark 2.2.8. The equivalence above also applies to the row and column spaces. More precisely, for $x \in S(\mathcal{M})$ and $p, q > 0$,

$$K_t(x, R_p, R_q) \approx K_t(Rx, p, q) \text{ and } K_t(x, C_p, C_q) \approx K_t(Cx, p, q).$$

Proof. Using the fact that for all $r > 0$, R_r and C_r are complemented in $L_r(\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2))$ and proposition 2.2.6,

$$K_t(x, R_p, R_q) \approx K_t(Rx, L_p(\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)), L_q(\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2))) \approx K_t(Rx, p, q)$$

and similarly for columns.

□

As shown in the next proposition, the K-functional, $K_t(x,p,\infty)$, can be expressed as a supremum over projections. This is crucial for the next chapter.

Proposition 2.2.9. *Suppose that $\mathcal{M}$ is diffuse. Let $p > 0$. Then for all $x \in \mathcal{M}_c$:*

$$\sup\{\|ex\|_p : \tau(e) \le t^p, e \in \mathcal{P}(\mathcal{M})\} \approx_{A_p} K_t(x, p, \infty).$$

Proof. If $t^p > \tau(1)$, we have:

$$\sup\{\|ex\|_p : \tau(e) \le t^p, e \in \mathcal{P}(\mathcal{M})\} = \|x\|_p$$

and the proposition is verified by proposition 1.3.2. So from now on, assume that $t^p \le \tau(1)$.

Since $\mathcal{M}$ is diffuse, there is a projection e in $\mathcal{M}$ with trace t^p, commuting with $|x^*|$, such that:

$$\tau(e\,|x^*|^p) = \int_0^{t^p} \mu_s(x)^p ds \ge \frac{1}{A_p} K_t(x, p, \infty)^p.$$

Furthermore, since e and $|x^*|$ live in a commutative von Neumann algebra, they can be represented as functions in a space $L_\infty(\Omega)$, thus:

$$\tau(e\,|x^*|^p) = \tau(e^{p/2}(xx^*)^{p/2}e^{p/2}) = \tau((exx^*e)^{p/2}) = \|ex\|_p^p.$$

Hence:

$$K_t(x, p, \infty) \le A_p\|ex\|_p \le A_p \sup\{\|ex\|_p : \tau(e) = t^p, e \in \mathcal{P}(\mathcal{M})\}.$$

To prove the converse inequality, take $e \in \mathcal{P}(\mathcal{M})$ such that $\tau(e) = t^p$. Note that:

$$\|ex\|_p^p = \int_0^\infty \mu(|ex|^p) = \int_0^{t^p} \mu(|ex|)^p \le A_p K_t(ex, p, \infty)^p.$$

Furthermore for all $s \in \mathbb{R}^+$, $\mu_s(ex) \leq \mu_s(x)$, see for example [16]. Hence, $K_t(ex,p,\infty) \leq K_t(x,p,\infty)$. Combining the two previous inequalities, we obtain:

$$\|ex\|_p \leq A_p K_t(ex,p,\infty) \leq A_p K_t(x,p,\infty).$$

□

Remark 2.2.10. The proof yields a bit more than the proposition. Indeed, it suffices to consider the supremum over projections e commuting with $|x^*|$ to obtain the left hand side inequality. This will be of importance later on.

Let us mention one final lemma before moving on to the next section.

Lemma 2.2.11. For any $x \in \mathcal{M}$, $t \in \mathbb{R}_{\geq 0}$, $y, z \in \mathcal{M}_c$ such that $y + z = x$, there exists $y', z' \in \mathcal{M}_c$ such that $y' + z' = x$, $\|y'\|_\infty \leq 2\|x\|_\infty$, $\|z'\|_\infty \leq 2\|x\|_\infty$ and:

$$\|y'\|_p + t\|z'\|_q \leq \|y\|_p + t\|z\|_q.$$

Proof. Suppose that x is positive, which is possible by multiplying by a partial isometry. First, note that if $p, q \geq 1$, we can start by considering the decomposition $x = \mathbb{E}_x(y) + \mathbb{E}_x(z)$ where $\mathcal{E}_x$ is the conditional expection onto $\mathcal{M}_x$, a semifinite commutative subalgebra of $\mathcal{M}$ containing x. Since $\mathcal{M}_x$ is commutative, we are reduced to the commutative case which is direct so the argument is complete.

Then, suppose for example that $p < 1$ ($q < 1$ is similar). Let $A = \|x\|_\infty$ and define $e = \mathbf{1}_{[0,A]}(|y|)$. Consider $y' = ye + xe^\perp$ and $z' = ze$. We have $y' + z' = x$, $\|y'\|_\infty \leq \|ye\|_\infty + \|xe^\perp\|_\infty \leq 2A$ and $\|z'\|_\infty = \|x - y'\|_\infty = \|xe - ye\|_\infty \leq 2A$. Since e is a contraction, $\|z'\|_q \leq \|z\|_q$. Finally, recall that $e = \mathbf{1}_{[0,A]}(|y|)$ so $\|ye^\perp\|_p^p \geq \tau(e^\perp)A^p \geq \|xe^\perp\|_p^p$ and e commutes with $|y|$ so:

$$\begin{aligned}
\|y\|_p^p &= \tau(|y|^p) = \tau(|y|^p\, e) + \tau(|y|^p\, e^\perp) \\
&= \tau((|y|)^p e) + \tau((|y|)^p e^\perp) \\
&= \tau(|ye|^p) + \tau(|ye^\perp|^p) \\
&= \|ye\|_p^p + \|ye^\perp\|_p^p.
\end{aligned}$$

And we can conclude, by the p-triangular inequality:

$$\|y'\|_p^p \leq \|ye\|_p^p + \|xe^\perp\|_p^p \leq \|ye\|_p^p + \|ye^\perp\|_p^p = \|y\|_p^p.$$

□

We conclude this section by proving 2.2.3.

Proof. *i.* Define $b_\varepsilon = b + \varepsilon$. Set $c_\varepsilon = a^{1/2} b_\varepsilon^{-1/2}$. We only have to check that c is a contraction.

$$c^* c = b_\varepsilon^{-1/2} a^{1/2} a^{1/2} b_\varepsilon^{-1/2} = b_\varepsilon^{-1/2} a b_\varepsilon^{-1/2} \leq b_\varepsilon^{-1/2} b_\varepsilon b_\varepsilon^{-1/2} = 1$$

Hence, c_ε is a contraction. Let c be a weak*-limit of the c_ε. c is a contraction and $a = cbc^*$.

ii. If $a \leq b$ then $a^2 \leq a^{1/2} b a^{1/2}$. Now define u to be a partial isometry appearing in the polar decomposition of $a^{1/2} b^{1/2}$ i.e $a^{1/2} b^{1/2} = u \left| a^{1/2} b^{1/2} \right|$. Then $u^* b^{1/2} a^{1/2} u = a^{1/2} b^{1/2}$ and $a^{1/2} b a^{1/2} = u b^{1/2} a b^{1/2} u^* \leq u b^2 u^*$.

iii. For $\alpha \in [1, 2]$, by operator convexity of the function $x \mapsto x^\alpha$, the result holds with $u = 1$. Now we proceed by induction. Suppose that iii. is true for a $\alpha \geq 1$, we will show that it holds for $\alpha' = 2\alpha$. By hypothesis, there exists a partial isometry u such that:

$$(a+b)^\alpha \leq 2^{\alpha-1} u (a^\alpha + b^\alpha) u^*.$$

And by the ii. there exists v such that:

$$\begin{aligned} (a+b)^{2\alpha} &\leq 2^{2\alpha-2} v (u(a^\alpha + b^\alpha) u^*)^2 v^* \\ (a+b)^{\alpha'} &\leq 2^{\alpha'-2} vu (a^\alpha + b^\alpha)^2 (vu)^* \\ &\leq 2^{\alpha'-1} vu (a^{\alpha'} + b^{\alpha'}) (vu)^* \end{aligned}$$

where the last inequality is a consequence of the operator convexity of $x \mapsto x^2$ or more precisely the inequality $(c+d)^2 \leq 2c^2 + 2d^2$ applied to $c = a^\alpha$ and $d = b^\alpha$.

iv. Define $x = a+b$ and take two contractions α and β such that $a^{1/2} = \alpha x^{1/2}$ and $b^{1/2} = \beta x^{1/2}$. We can suppose that $\alpha^* \alpha + \beta^* \beta = s(x)$. The operator $T : \mathcal{M} \to \mathcal{M}$ defined by $T(c) = \alpha c \alpha^*$ is positive and subunital. Hence, we can apply the Jensen inequality for the operator-concave function $t \mapsto t^\theta$ and we obtain $T(x)^\theta \geq T(x^\theta)$. Similarly, $\beta x^\theta \beta^* \leq (\beta x \beta^*)^\theta$. Note also that, as we have used in the previous proof, there exist partial isometries u and v such that $x^{\theta/2} \alpha^* \alpha x^{\theta/2} = u \alpha x^\theta \alpha^* u^*$ and $x^{\theta/2} \beta^* \beta x^{\theta/2} = v \beta x^\theta \beta^* v^*$ since adjoints are always conjugated by a partial isometry. Now, we can conclude by the following computation:

$$\begin{aligned} (a+b)^\theta &= x^\theta = x^{\theta/2} (\alpha^* \alpha + \beta^* \beta) x^{\theta/2} \\ &= x^{\theta/2} \alpha^* \alpha x^{\theta/2} + x^{\theta/2} \beta^* \beta x^{\theta/2} \\ &= u \alpha x^\theta \alpha^* u^* + v \beta x^\theta \beta^* v^* \\ &\leq u (\alpha x \alpha^*)^\theta u^* + v (\beta x \beta^*)^\theta v^* \end{aligned}$$

$$= \leq u a^\theta u^* + v b^\theta v^*.$$

$\square$

2.3 Ultraproducts

Suppose in this section that $\mathcal{M}$ is finite. We will denote by $\mathcal{M}_\omega$ the ultraproduct of $\mathbb{N}$ copies of $\mathcal{M}$. Here, ω is an ultrafilter on $\mathbb{N}$. See [44] for the detailed construction of tracial ultraproducts, we only recall the definition here. Let $B = \ell_\infty(\mathcal{M})$ and $I = \{x \in B : \lim_\omega \tau(x_n) = 0\}$. I is a closed ideal of B and we define $\mathcal{M}_\omega = B/I$ equipped with the trace $\tau_\omega(x+I) = \lim_\omega \tau(x_n)$. $(\mathcal{M}_\omega, \tau_\omega)$ is a noncommutative measure space and $(\mathcal{M}, \tau)$ can be isometrically identified with the subalgebra of classes of constant sequences. In the next chapter, we will need the following lemma to manipulate the K-functional in an ultraproduct.

Lemma 2.3.1. Let $x = \overline{(x_n)_{n\in\mathbb{N}}} \in \mathcal{M}_\omega$, $p, q \in (0,\infty]$ with $p \neq q$, then $K(x_n, L_p(\mathcal{M}), L_q(\mathcal{M}))$ converges pointwise to $K(x, L_p(\mathcal{M}_\omega), L_q(\mathcal{M}_\omega))$ along ω. Furthermore if $p \neq \infty$, the convergence is uniform.

Proof. We treat the case $p < q$, the case $q > p$ is similar. Choose a decomposition $x = y + z$ such that $\|y\|_p + t\|z\|_q < K_t(x,p,q) + \varepsilon$. Take a representative $(z_n)_{n\geq 0}$ of z such that $\lim_{n\to\infty} \|z_n\|_\infty = \|z\|_\infty$. We take this precaution in case $q = \infty$. We now have a natural representative for y which is $(y_n)_{n\in\mathbb{N}}$ with $y_n = x_n - z_n$. Furthermore, by definition:

$$\|y\|_p = \tau(|y|^p)^{1/p} = \lim_\omega \tau(|y_n|^p)^{1/p} = \lim_\omega \|y_n\|_p.$$

Since the same is true for z and the q-norm (with the exeption of $q = \infty$ which has been dealt with already), this means that there exists $A_1 \in \omega$ such that for all $n \in A_1$,

$$\Big| \|y\|_p + t\|z\|_q - \|y_n\|_p - t\|z_n\|_q \Big| \leq \varepsilon.$$

This implies that:

$$K_t(x_n, p, q) \leq K_t(x,p,q) + 2\varepsilon.$$

Reciprocally, for all $n \in \mathbb{N}$ take a decomposition $x_n = y_n + z_n \in \mathcal{M}$ such that $\|y_n\|_p + t\|z_n\|_q < K_t(x_n,p,q) + \varepsilon$. By construction, there exists a certain constant M such that for all $n \in \mathbb{N}$, $\|x_n\|_\infty \leq M$. By lemma 2.2.11, we can suppose that for all $n \in \mathbb{N}$, $\|y_n\|_\infty \leq 2M$ and $\|z_n\|_\infty \leq 2M$. So $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ define elements y and $z \in \mathcal{M}_\omega$. We have $x = y + z$, and we can find $A_2 \in \omega$ such that for any $n \in A_2$:

$$\Big| \|y\|_p + t\|z\|_q - \|y_n\|_p - t\|z_n\|_q \Big| \leq \varepsilon$$

since even for $q = \infty$, $\|z\|_\infty \leq \lim_\omega \|z_n\|_\infty$. Hence,

$$K_t(x,p,q) \leq K_t(x_n,p,q) + 2\varepsilon.$$

Which concludes the proof of pointwise convergence.

If $p \neq \infty$, extend $K(x_n,p,q)$ at ∞ by letting $K_\infty(x_n,p,q) = \|x_n\|_p$. The same can be done for $K(x,p,q)$. Since $p \neq \infty$, $K(x_n,p,q)$ converges to $K(x,p,q)$

at ∞. Note that the K-functional are increasing continuous functions and that $\mathbb{R}^+ \cup \{\infty\}$ is compact. Consequently, by Dini's theorem, the convergence is uniform.

$\square$

2.4 K-functional of commutators

This section is devoted to proving the following lemma.

Lemma 2.4.1. Let $p \in (0, \infty)$, $q \in (0, \infty]$, $\theta \in (0, 1)$. For all $t > 0$, $\alpha, \beta \in \mathcal{M}_c^+$ and $b \in \mathcal{M}$:

$$K_{t^\theta}(\alpha^\theta b + b\beta^\theta, p/\theta, q/\theta) \lesssim K_t(\alpha b + b\beta, p, q)^\theta \|b\|_\infty^{1-\theta},$$

where the implicit constant only depends on p, q and θ.

We recall the following proposition that is the main result of [54] (proposition 4.3).

Proposition 2.4.2. *Let $0 < p < q \leq \infty$, $\theta \in (0, 1)$ and $x, y \in (L_p(\mathcal{M}) + L_q(\mathcal{M}))^{sa}$. Then for all $t > 0$ and $f : x \mapsto |x|^\theta$ or $f : x \mapsto sgn(x)\,|x|^\theta$, we have:*

$$K_{t^\theta}(f(x) - f(y), p/\theta, q/\theta) \lesssim K_t(x - y, p, q)^\theta.$$

The relationship between Mazur maps and anti-commutators has already been explicited in [53]. The arguments rely on 2×2 matrix tricks and the Cayley transform. We reproduce them in our context for completion but the following proofs do not contain any new idea. We break it down into two steps, the first one is given by the following lemma and uses the Cayley transform.

Lemma 2.4.3. Let $p \in (0, \infty)$, $q \in (0, \infty]$, $\theta \in (0, 1)$. For all $t > 0$, $f : x \mapsto |x|^\theta$ or $f : x \mapsto \mathrm{sgn}(x)\,|x|^\theta$, $\alpha \in \mathcal{M}^{sa}$ and $b \in \mathcal{M}^{sa}$:

$$K_{t^\theta}(bf(\alpha) - f(\alpha)b, p/\theta, q/\theta) \lesssim K_t(b\alpha - \alpha b, p, q)^\theta \|b\|_\infty^{1-\theta},$$

where the implicit constant only depends on p, q and θ.

Proof. We can assume that $\|b\|_\infty = 1$ by homogeneity. Since $b = b^*$, the Cayley transform is defined by

$$u = (b + i)(b - i)^{-1},\ b = 2i(1 - u)^{-1} - i.$$

Note that u is unitary and by the second formula $\|(1 - u)^{-1}\|_\infty \leq \frac{1}{\sqrt{2}}$. These facts allow us to conclude by a computation using 2.4.2.

$$\begin{aligned} K_{t^\theta}(bf(\alpha) - f(\alpha)b, p/\theta, q/\theta) &\leq 2K_{t^\theta}((1 - u)^{-1}f(\alpha) - f(\alpha)(1 - u)^{-1}, p/\theta, q/\theta) \\ &\leq 2\|(1 - u)^{-1}\|_\infty^2 K_{t^\theta}((1 - u)f(\alpha) - f(\alpha)(1 - u), p/\theta, q/\theta) \\ &\leq K_{t^\theta}(f(u^*\alpha u) - f(\alpha), p/\theta, q/\theta) \end{aligned}$$

using proposition 2.4.2,

$$
\begin{aligned}
&\lesssim K_t(\alpha u - u\alpha, p, q)^\theta \\
&\lesssim \left\|(b-i)^{-1}\right\|_\infty^{2\theta} K_t((b-i)\alpha(b+i) - (b+i)\alpha(b-i), p, q)^\theta \\
&\lesssim K_t(\alpha b - b\alpha, p, q)^\theta.
\end{aligned}
$$

□

We can now conclude the main proof using a matrix trick.

Proof of lemma 2.4.1. Let $b \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{M}^+$. Consider:

$$
b' = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}, \ \alpha' = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}.
$$

Note that in general, the matrix b' has the same distribution as $b \oplus b$. Hence, for all $t > 0$,

$$
K_t(b', p, q) \approx K_t(b, p, q).
$$

Indeed, $x \mapsto K_t(x, p, q)$ is a quasi-norm so $K_t(b, p, q) \lesssim K_t(b', p, q)$ and $\mu(b') = \mu(b \oplus b) \geq \mu(b)$ so $K_t(b', p, q) \geq K_t(b, p, q)$. It is now straightforward to check that for all $t > 0$,

$$
K_t(\alpha b + b\beta, p, q) \approx K_t(\alpha' b' - b'\alpha', p, q)
$$

and that for $f : x \mapsto \operatorname{sgn}(x)\,|x|^\theta$,

$$
K_t(f(\alpha)b + bf(\beta), p, q) \approx K_t(f(\alpha')b' - b'f(\alpha'), p, q).
$$

Hence, it suffices to apply lemma 2.4.3 to b', α', θ and f to conclude the proof.

□

Chapter 3

Noncommutative Khintchine inequalities

3.1 Introduction

The work presented in this chapter is intended as a step towards completing the study of noncommutative Khintchine inequalities in interpolation spaces between L_p-spaces. No satisfying results were known in $L_{1,\infty}$ and $L_{2,\infty}$ in spite of the extensive literature on the subject which includes some variants in general symmetric spaces. The remarkable growth of this topic in the last decades is to be attributed to the central role Khintchine inequalities play in noncommutative analysis. Similarly to their classical counterparts, they appear constantly when the norm of an unconditional sequence has to be estimated, and they allow us to describe the Banach space structure of the span of independent or free random variables. They were a stepping stone to develop noncommutative martingale inequalities which are essential and powerful tools to translate classical notions to the noncommutative setting, as illustrated by the last chapter of this thesis.

The seminal result of Lust-Piquard [36] and then Lust-Piquard and Pisier [37] who first fomulated and proved Khintchine inequalities in the setting of noncommutative integration (for Rademacher variables and in L_p-spaces) led to generalisations spreading into different directions. In the context of free probability or analysis on the free group, they were introduced by Pisier and Haagerup in [20], and studied further in different works (see for example [42] and [55]). As mentioned before, Khintchine inequalities are also the precursors of noncommutative martingale inequalities ([49], [24], [26]), another pillar of the theory, see for example [27], [12] and [52]. Closely related to our subject, for more than a decade, attention has been given towards general symmetric spaces. One can mention, for example, the work of Lust-Piquard and Xu ([38]), Le Merdy and Sukochev ([33]) and Dirksen, de Pagter, Potapov, Sukochev ([13]). But only recently the case of quasi-Banach spaces was tackled by Pisier and Ricard in [48] who proved Khintchine inequalities in noncommutative L_p-spaces for $p < 1$. The latter paper gives the final key inequality to apply the method found in [45] by Pisier. Our method takes inspiration from [14] where Dirksen and Ricard proved the upper Khintchine inequalities in a very efficient way. We give a similar proof of the lower Khintchine inequality which is usually obtained by duality. This partially explains the difficulty of proving Khintchine inequalities in L_p-spaces for $p < 1$ or in any quasi-Banach space.

We present two different results (Sections 3.2 and 3.3) with independent proofs though they partially rely on the same idea. In the first one, we show that the lower Khintchine inequality in L_p for $p < 2$ implies the lower Khintchine inequalities for all interpolation spaces between L_p and L_∞ with a decomposition that does not depend on the space. This, combined with known results, directly implies Khintchine inequalities in $L_{1,\infty}$, which could not be reached before due to the inapplicability of interpolation or duality techniques in this case. A motivation to prove this last result was that it allows us to prove the weak-1-boundedness of Calderón-Zygmund operators in the noncommutative setting for Hilbert-valued kernels ([39]) directly from the scalar-valued kernel case ([41]), see chapter 4., corollary 4.4.2.

The second theorem gives a deterministic equivalent for "free averages" in every interpolation spaces between L_p-spaces. The remarkable feature here is that its formulation does not depend on whether $p < 2$ or $p > 2$. In particular, it holds in $L_{2,\infty}$ which is a tricky case since neither of the two usual formulas for Khintchine inequalities work (see Section 3.5). Note that a deterministic formula was already found using interpolation methods by Pisier in [46]. Our equivalent is less tractable than the usual formulas. It is obtained by first proving that any sequence of operators (x_i) admits a factorisation of the form $\alpha(u_i) + (u_i)\beta$ where α and β are positive operators and (u_i) has good properties. Then, if (ξ_i) is a sequence of free Haar unitaries, the norm of $\sum x_i \otimes \xi_i$ happens to coincide with the norm of $\alpha \oplus \beta$ in all interpolation spaces between L_p-spaces. This yields a new proof of free Khintchine inequalities for $p < 1$. It does not apply to Rademacher variables since it relies on Haagerup's inequality ([19]) together with a Hölder inequality for anti-commutators (2.4.1) found in [54].

Let us now state the main results of this chapter. We refer to the previous chapter, for the definitions of most notations appearing thereafter.

Consider $(\mathcal{A}, \tau_{\mathcal{A}})$ a noncommutative probability space ($\tau_{\mathcal{A}}(1) = 1$) and $\xi = (\xi_i)_{i\in\mathbb{N}}$ a sequence of elements in $\mathcal{A}$. For x in $S(\mathcal{M})$, we define $Gx = \sum_{n\geq 0} x_n \otimes \xi_n$ in $\mathcal{M}_{\mathcal{A}} := \mathcal{M}\overline{\otimes}\mathcal{A}$. Recall that $\mathcal{M}$ will be identified with $\mathcal{M} \otimes 1$ in $\mathcal{M}_{\mathcal{A}}$.

Fix E a symmetric space. The quantity we want to estimate is $\|x\|_{\mathscr{H}_E} := \|Gx\|_{E(\mathcal{M}_{\mathcal{A}})}$ which is a quasi-norm on $S(\mathcal{M})$. Denote by $\mathscr{H}_E(\mathcal{M})$ ($\mathscr{H}_E$ if there is no ambiguity) the completion of $S(\mathcal{M})$ for $\|.\|_{\mathscr{H}_E}$ (for $p \in (0, \infty]$, we write $\mathscr{H}_p := \mathscr{H}_{L_p}$).

Remark 3.1.1. Let $p, q \in (0, \infty]$, the couple $(\mathscr{H}_p, \mathscr{H}_q)$ is a compatible couple of Banach spaces in the sense of interpolation. Indeed, for all $r > 0$, $\mathscr{H}_r$ can be identified with a closed subspace of $L_r(\mathcal{M}_{\mathcal{A}})$ by extending the map G.

With this notations, the first noncommutative Khintchine inequalities ([36],[37]) state that, assuming that ξ is a sequence of independent Rademacher variables,

$$\mathscr{H}_p = \begin{cases} R_p + C_p & \text{if } p \in [1,2] \\ R_p \cap C_p & \text{if } p \in [2,\infty) \end{cases}$$

with equivalent norms. This motivates the following definition.

Definition 3.1.2. *Let E be a symmetric function space. Define the following two properties of E, depending on the noncommutative measure space $\mathcal{M}$:*

$$Kh_+(E,\mathcal{M}) \Leftrightarrow \forall x \in S(\mathcal{M}), \|x\|_{\mathscr{H}_E} \approx \|x\|_{R_E \cap C_E},$$

and:

$$Kh_-(E,\mathcal{M}) \Leftrightarrow \forall x \in S(\mathcal{M}), \|x\|_{\mathscr{H}_E} \approx \|x\|_{R_E + C_E}.$$

The main results of sections 3.2 and 3.3 come from the study of optimal decompositions. Let us give their definition right away.

Definition 3.1.3. *For any $x \in S(\mathcal{M})$ and $p \in (0,1]$. Define*

$$m_p(x) = \inf\{\|Ry\|_p^p + \|Cz\|_p^p : x = z + y, y, z \in S(\mathcal{M})\}.$$

We say that $y, z \in S(\mathcal{M})$ is an optimal decomposition of x in L_p if $y + z = x$ and $\|Ry\|_p^p + \|Cz\|_p^p = m_p(x)$.

The intuition behind this definition is that distributions of optimal decompositions of x should somehow approach the distribution of Gx. Though very vague, this idea is partially confirmed by the theorem 3.1.8.

Our first result is a negative one, we exhibit counterexamples to prove the following proposition:

Proposition 3.1.4. *Suppose that the ξ_i are free Haar unitaries or Rademacher variables and that $\mathcal{M} = B(\ell^2)$. There is no constant c such that for every finite sequence $x = (x_n)_{n\geq 0} \in \mathcal{M}^{\mathbb{N}}$:*

$$\Big\|\sum_{i\geq 0} x_i \otimes \xi_i\Big\|_{2,\infty} \leq c\|x\|_{R_{2,\infty}+C_{2,\infty}}.$$

Similarly, there is no constant c such that for every finite sequence $x = (x_n)_{n\geq 0} \in \mathcal{M}^{\mathbb{N}}$:

$$\|x\|_{R_{2,\infty}\cap C_{2,\infty}} \leq c\Big\|\sum_{i\geq 0} x_i \otimes \xi_i\Big\|_{2,\infty}.$$

We present two proofs of this proposition. In the first, we construct explicit counter-examples. Proving the required estimate in this case is not immediate and relies in paticular on [14]. In the second, we crucially use the Schur-Horn lemma for the construction. The method works in a much more general setting and combined with theorem 3.1.8, enables us to prove the following:

Theorem 3.1.5. *Let E be a symmetric function space with the Fatou property. Suppose that the ξ_i are free Haar unitaries and $\mathcal{M} = B(\ell^2)\overline{\otimes}L_\infty(0,1)$. Then,*

$$Kh_+(E,\mathcal{M}) \Leftrightarrow E \textit{ is right-2-monotone}$$

and if furthermore there exists $p > 0$ such that E is an interpolation space between L_p and L_∞,

$$Kh_-(E,\mathcal{M}) \Leftrightarrow E \textit{ is left-2-monotone.}$$

The next theorem has the advantage of requiring very little conditions on the variables considered. We prove that if the lower Khintchine inequality holds for some p, it also holds for $q > p$ with the same decomposition.

Theorem 3.1.6. *Let $p \in (0,1]$. Suppose that:*

- *the ξ_i are orthonormal in $L_2(\mathcal{A})$,*
- *the ξ_i verify the lower bound of the Khintchine inequality in L_p i.e $Kh_-(L_p,\mathcal{M})$ holds.*

Then there is a decomposition y, z such that $x = y + z$ and for all interpolation space E between L_p and L_∞:

$$\| \Big(\sum\nolimits_{i\geq 0} y_i y_i^*\Big)^{1/2} \|_{E(\mathcal{M})} + \| \Big(\sum\nolimits_{i\geq 0} z_i^* z_i\Big)^{1/2} \|_{E(\mathcal{M})} \lesssim_c \| \sum\nolimits_{i\geq 0} x_i \otimes \xi_i \|_{E(\mathcal{M}_\mathcal{A})}.$$

The constant c only depends on A and p.

The proof is a dual version of the argument used in [14]. We use the Khintchine inequality in L_p not only for an element $x \in S(\mathcal{M})$ but also for elements of the form ex where e is a well chosen projection in $\mathcal{M}$. This enables us to obtain a control in terms of K-functionals which, by proposition 1.3.5, immediately implies the theorem above as a corollary. For this idea to work the decomposition y, z has to be close enough to an optimal decomposition.

Remark 3.1.7. The second condition on ξ is actually independent of p for $p < 2$. This is a consequence of [48] (to go from p to $q < p$) and theorem 3.2.1 (to go from p to $q > p$). This means that applying the theorem above we can prove Khintchine inequalities in $L_{1,\infty}$ for any sequence of variables ξ_i that verifies Khintchine inequalities in L_p for some $p < 2$. More details are given in Corollary 3.2.2.

Note also that by remark 3.2.9, theorem 3.1.6 also holds for $p \in (1,2)$.

The second theorem summarizes the results found in section 3.3. By pushing further the properties of optimal decompositions in L_1 and together with the key inequality due to Ricard (2.4.1) we obtain a new proof of Khintchine inequalities in all interpolation spaces between L_p-spaces if the variables are for example free unitaries. We also have a "deterministic" equivalent of $\big\| \sum_{i\geq 0} x_i \otimes \xi_i \big\|_{2,\infty}$ which is however less explicit than the usual Khintchine inequalities.

Theorem 3.1.8. *Let $x = (x_i)_{i\geq 0} \in S(\mathcal{M})$. There exist $\alpha, \beta \in \mathcal{M}_c^+$ and $u = (u_i)_{i\geq 0} \in S(\mathcal{M})$ such that:*

- $s(\alpha) \leq \sum_{i\geq 0} u_i u_i^* \leq 1$,
- $s(\beta) \leq \sum_{i\geq 0} u_i^* u_i \leq 1$,
- *for all* $i \geq 0$, $x_i = u_i\beta + \alpha u_i$.

Furthermore, suppose that:

- *the ξ_i are orthonormal in $L_2(\mathcal{A})$*
- *the ξ_i verify the Khintchine inequality in L_∞ i.e $Kh_+(L_\infty, E)$ holds.*

Then for all $p \in (0, \infty)$ and E an interpolation space between L_p and L_∞:

$$\|\alpha\|_{E(\mathcal{M})} + \|\beta\|_{E(\mathcal{M})} \approx_c \|\sum_{i\geq 0} x_i \otimes \xi_i\|_{E(\mathcal{M}_\mathcal{A})},$$

where c only depends on p and the constant c_∞ appearing in (**??**).

Apart from the inequality found in [54], the proof is based on a short duality argument and the fact that optimal decompositions always exist in L_1.

3.2 Some properties of optimal decompositions in L_p for $p < 2$

3.2.1 Main result and consequences

In this section, the variables ξ_i will always satisfy the following conditions:

1. the ξ_i are orthonormal in $L_2(\mathcal{A})$,
2. the ξ_i verify the lower Khintchine inequality for some $p \leq 1$. More precisely, there exists a constant B_p such that for all $x \in S(\mathcal{M})$,

$$m_p(x)^{1/p} \leq B_p \|Gx\|_p. \tag{3.1}$$

Typical examples of such variables include free Haar unitaries or Rademacher variables. We focus on lower Khintchine inequalities i.e of the type:

$$\|x\|_{R_E+C_E} \lesssim \|Gx\|_E,$$

for E a symmetric space and $x \in S(\mathcal{M})$. The converse inequality presents no difficulty in the motivating example of $L_{1,\infty}$ as we will see later. In [14], it is shown that by applying multiple times Khintchine inequality in L_∞ for an element x, one can obtain a majoration of $\mu(Gx)$. Though it is less direct, our method is similar and by using the Khintchine inequality in L_p for $p < 2$ we obtain a minoration of the K-functional of Gx. The main theorem of this section is the following.

Theorem 3.2.1. *Let $p \in (0,1]$ such that (3.1) holds. There exists a constant C_p such that for all $x \in S(\mathcal{M})$ there exist $y, z \in S(\mathcal{M})$ such that $y + z = x$ and for all $t \geq 0$,*

$$K_t(Ry, p, \infty) \leq C_p K_t(Gx, p, \infty) \text{ and } K_t(Cz, p, \infty) \leq C_p K_t(Gx, p, \infty).$$

The constant C_p only depends on A_p and B_p which appear in (1.2) and (3.1) respectively.

Let us highlight some consequences of the theorem. Since we have a control on the K-functional, it extends to all interpolation spaces between L_p-spaces by theorem 1.3.5 and thus we obtain theorem 3.1.6 mentioned in the preliminaries as a corollary. It also allows us to prove the Khintchine inequalities in $L_{1,\infty}$.

Corollary 3.2.2 (The Khintchine inequality in $L_{1,\infty}$)**.** *For any $x \in S(\mathcal{M})$,*

$$\|Gx\|_{1,\infty} \approx \|x\|_{R_{1,\infty}+C_{1,\infty}}.$$

Proof. By remark 3.1.7, we can suppose that $p < 1$. The inequality:

$$\inf\{\|Ry\|_{1,\infty} + \|Cz\|_{1,\infty} : x = y + z, y, z \in S(\mathcal{M})\} \lesssim \|Gx\|_{1,\infty}$$

is given by theorem 3.1.6 applied to $E = L_{1,\infty}$. The converse inequality is classical. We know that the map $Ry \mapsto Gy$ is a contraction on L_p and L_2, hence, by interpolation, it is bounded on $L_{1,\infty}$. By adjunction, the same is true for $Cz \mapsto Gz$. Hence:

$$\|Gx\|_{1,\infty} \lesssim \|Gy\|_{1,\infty} + \|Gz\|_{1,\infty} \lesssim \|Ry\|_{1,\infty} + \|Cz\|_{1,\infty}.$$

□

Remark 3.2.3. We emphasized the previous corollary because it concerns a special case that motivated our work but the exact same proof works for a general interpolation space between L_p, $p < 2$ and L_2 using interpolation for the upper bound and theorem 3.1.6 for the lower bound. More precisely, let $p < 2$ and E an interpolation space between p and 2 then:

$$\mathscr{H}_E = R_E + C_E$$

with equivalent quasi-norms.

Remark 3.2.4. As a consequence of the preceding remark and theorem 3.2.1, we obtain the following. Let $p, q \in (0,2]$, $p \leq q$. Then for all $x \in S(\mathcal{M})$,

$$K_t(x, \mathscr{H}_p, \mathscr{H}_q) \lesssim K_t(Gx, p, q).$$

Using the terminology introduced in [43], the couple $(\mathscr{H}_p, \mathscr{H}_q)$ is K-closed in $(L_p(\mathcal{M}_\mathcal{A}), L_q(\mathcal{M}_\mathcal{A}))$ (recall that $\mathscr{H}_p$ and $\mathscr{H}_q$ are identified with subspaces of $L_p(\mathcal{M}_\mathcal{A})$ and $L_q(\mathcal{M}_\mathcal{A})$ by remark 3.1.1). This means that the $\mathscr{H}_p$-spaces behave well with respect to interpolation for $p \leq 2$.

Proof. Let $t > 0$ and note that $L_p + tL_q$ is an interpolation space between L_p and L_q. Using the fact that L_q is an interpolation space between L_p and L_∞ and by applying the definition of an interpolation space, this means that $L_p + tL_q$ is an interpolation space between L_p and L_∞. Let $x \in S(\mathcal{M})$. This means by theorem 3.2.1 that there exists $y, z \in S(\mathcal{M})$ such that $x = y + z$ and:

$$K_t(Ry, p, q) + K_t(Cz, p, q) \lesssim K_t(Gx, p, q).$$

Hence, by remark 2.2.8, we have:

$$K_t(y, R_p, R_q) + K_t(z, C_p, C_q) \lesssim K_t(Gx, p, q).$$

Let $y_1, y_2, z_1, z_2 \in S(\mathcal{M})$ such that $y = y_1 + y_2$, $z = z_1 + z_2$,

$$\|y_1\|_{R_p} + t\|y_2\|_{R_q} \lesssim K_t(y, R_p, R_q) \text{ and } \|z_1\|_{R_p} + t\|z_2\|_{R_q} \lesssim K_t(z, C_p, C_q)$$

Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$, combining the previous inequalities:

$$\|x_1\|_{R_p+C_p} + t\|x_2\|_{C_p+C_q} \lesssim K_t(Gx, p, q).$$

Since, by the previous remark $R_p + C_p \approx \mathscr{H}_p$ and $R_q + C_q \approx \mathscr{H}_q$, we obtain:

$$K_t(x, \mathscr{H}_p, \mathscr{H}_q) \lesssim K_t(Gx, p, q).$$

□

3.2.2 First steps towards the proof

In this part, we present the main ideas that will allow us to prove theorem 3.2.1. The central one is contained in proposition 3.2.5. Starting with an element x, we use the Khintchine inequality on $e(Gx)$ for well chosen projections $e \in \mathcal{P}(\mathcal{M})$ and thanks to proposition 2.2.9 we deduce the expected control on K-functionals. There are, however, two technical difficulties. The first one is that we could not prove that an optimal decomposition always exists when $p < 1$. To skirt this problem, in the next part, we will prove that the argument also works for decompositions that are close enough to being optimal but in this case we need an additional control on the operator norm of the decomposition which is given by lemma 3.2.6. The second difficulty is that to use propositon 2.2.9, we need to work in a diffuse algebra. To that effect, we simply tensor our base algebra $\mathcal{M}$ by $L_\infty(0,1)$ which fixes the proof immediately thanks to lemma 3.2.7.

Proposition 3.2.5. *Suppose that $\mathcal{M}$ is diffuse. Let $p \in (0, 1]$, if $x \in S(\mathcal{M})$ and $x = y + z$ is an optimal decomposition in L_p then for all $t \geq 0$, $K_t(Ry, p, \infty) \leq A_p^2 B_p K_t(Gx, p, \infty)$ and $K_t(Cz, p, \infty) \leq A_p^2 B_p K_t(Gx, p, \infty)$.*

Proof. Let e be a projection commuting with $|(Ry)^*|$ and $f = 1 - e$. Take $\varepsilon > 0$ and $y_1, y_2, z_1, z_2 \in S(\mathcal{M})$ such that $ex = y_1 + z_1$, $fx = y_2 + z_2$, $\|Ry_1\|_p^p +$

$\|Cz_1\|_p^p \leq m_p(ex) + \varepsilon$ and $\|Ry_2\|_p^p + \|Cz_2\|_p^p \leq m_p(fx) + \varepsilon$. We can write $x = ex + fx = y_1 + z_1 + y_2 + z_2$. Then by minimality of y and z:

$$\|Ry\|_p^p + \|Cz\|_p^p \leq \|R(y_1 + y_2)\|_p^p + \|C(z_1 + z_2)\|_p^p.$$

We use the p-triangular inequality and obtain that:

$$\|R(y_1 + y_2)\|_p^p + \|C(z_1 + z_2)\|_p^p \leq \|Ry_1\|_p^p + \|Cz_1\|_p^p + \|Ry_2\|_p^p + \|Cz_2\|_p^p.$$

Combined with lemma 2.1.4, we get:

$$\|R(ey)\|_p^p + \|R(fy)\|_p^p + \|Cz\|_p^p \leq \|Ry_1\|_p^p + \|Cz_1\|_p^p + \|Ry_2\|_p^p + \|Cz_2\|_p^p.$$

Now using the almost minimality of the couple (y_2, z_2) and lemma 2.1.4, we obtain:

$$\|Ry_2\|_p^p + \|Cz_2\|_p^p \leq \|R(fy)\|_p^p + \|C(fz)\|_p^p + \varepsilon \leq \|R(fy)\|_p^p + \|Cz\|_p^p + \varepsilon.$$

Hence, combining the last two inequalities and then using the Khintchine inequality for p:

$$\|R(ey)\|_p^p \leq \|Ry_1\|_p^p + \|Cz_1\|_p^p + \varepsilon \leq B_p^p \|G(ex)\|_p^p + 2\varepsilon.$$

This is true for all $\varepsilon > 0$ so:

$$\|R(ey)\|_p^p \leq B_p^p \|G(ex)\|_p^p.$$

The previous inequality holds for all projections e commuting with $|(Ry)^*|$. Taking the supremum over all such e with $\tau(e) \leq t^p$ and using proposition 2.2.9 and remark 2.2.10, we obtain:

$$K_t(Ry, p, \infty) \leq A_p^2 B_p K_t(Gx, p, \infty).$$

The case of z is exactly symmetrical by taking adjoints and so the proof is complete. $\square$

To prove the theorem without making any assumptions, the following lemma is crucial.

Lemma 3.2.6. Let $\varepsilon > 0$ and $x \in S(\mathcal{M})$. There exist y and z in $S(\mathcal{M})$ such that $x = y + z$, $\|Ry\|_p^p + \|Cz\|_p^p \leq m_p(x) + \varepsilon$, $\|Ry\|_\infty \leq 2\|Gx\|_\infty$ and $\|Cz\|_\infty \leq 2\|Gx\|_\infty$.

Proof. With the notations of the lemma, choose $y, z \in S(\mathcal{M})$ such that $y + z = x$ and $\|Ry\|_p^p + \|Cz\|_p^p \leq m_p(x) + \varepsilon$ and denote $\|Gx\|_\infty = A$. Let $e = \mathbf{1}_{[A,\infty)}(|(Ry)^*|)$, $f = \mathbf{1}_{[A,\infty)}(|Cz|)$. We write $x = e^\perp x f^\perp + ex + e^\perp x f$ and deduce the new decomposition: $y' = e^\perp y f^\perp + ex$ and $z' = e^\perp z f^\perp + e^\perp x f$. Let us check that it satisfies the conditions of the lemma.

Note that $\|R(ex)\|_\infty \leq \|R(x)\|_\infty \leq A$. The last inequality follows from the fact that the ξ_i are orthonormal and that consequently the map $Id \otimes \tau_{\mathcal{A}} : \mathcal{M}\overline{\otimes}\mathcal{A} \to \mathcal{M}$ sends $|(Gx)^*|^2$ to $|(Rx)^*|^2$. Moreover, the left support of $R(ex)$ is less than e, indeed: $|(R(ex))^*|^2 = e(\sum_i x_i x_i^*)e$. Hence $|R(ex)^*| \leq Ae$. Note also that $|(R(ey))^*| \geq Ae$. Indeed, since e and $|(Ry)^*|$ commute by lemma 2.1.4, $|(R(ey))^*| = e\,|(Ry)^*| \geq Ae$ by definition of e. By symmetry, we have the same kind of estimates for the columns i.e $|C(e^\perp xf)| \leq Af$ and $|C(zf)| \geq Af$. For rows, we get:

$$\begin{aligned}\|Ry'\|_p^p &\leq \|R(e^\perp y f^\perp)\|_p^p + \|R(ex)\|_p^p \\ &\leq \|R(e^\perp y)\|_p^p + \tau(e)A^p \\ &\leq \|R(e^\perp y)\|_p^p + \|R(ey)\|_p^p \\ &= \|Ry\|_p^p\end{aligned}$$

where the last inequality is given by lemma 2.1.4. Similarly, for columns, we get:

$$\|Cz'\|_p^p \leq \|Cz\|_p^p.$$

Consequently:

$$\|Ry'\|_p^p + \|Cz'\|_p^p \leq \|Ry\|_p^p + \|Cz\|_p^p \leq m_p(x) + \varepsilon.$$

The control in L_∞ also follows quickly:

$$\begin{aligned}\|Ry'\|_\infty &\leq \|R(e^\perp y f^\perp)\|_\infty + \|R(ex)\|_\infty \\ &\leq \|R(e^\perp y)\|_\infty + \|Rx\|_\infty \\ &\leq 2A\end{aligned}$$

where the last inequality is a consequence of the definition of e. The case of columns is as always symmetrical which concludes the proof. □

The following lemma is the key to remove the hypothesis that $\mathcal{M}$ is diffuse.

Lemma 3.2.7. Consider the noncommutative measure space $\mathcal{N} = \mathcal{M}\overline{\otimes}L_\infty([0,1])$ equipped with the tensor product trace and identify $\mathcal{M}$ with $\mathcal{M}\otimes 1 \subset \mathcal{N}$. Then for any $p > 0$ and $x \in S(\mathcal{M})$:

$$m_p(x) = \inf\{\|Rf\|_p^p + \|Cg\|_p^p : f + g = x, f, g \in S(\mathcal{N})\}.$$

Proof. Since $\mathcal{M} \subset \mathcal{N}$ the inequality:

$$m_p(x) \geq \inf\{\|Rf\|_p^p + \|Cg\|_p^p : f + g = x, f, g \in S(\mathcal{N})\}$$

is clear.

Let $f, g \in S(\mathcal{N})$ such that $f + g = x$. Seeing f and g as functions from $[0,1]$ to $S(\mathcal{M})$ we write:

$$\|Rf\|_p^p + \|Cg\|_p^p = \int_0^1 \|R(f(t))\|_p^p + \|C(g(t))\|_p^p dt \geq \int_0^1 m_p(x)dt = m_p(x).$$

□

Extend the notation $m_p(h)$ to elements $h \in S(\mathcal{N})$ by:

$$m_p(h) := \inf\{\|Rf\|_p^p + \|Cg\|_p^p : f + g = h, f, g \in S(\mathcal{N})\}.$$

When $x \in S(\mathcal{M})$ is considered as an element of $S(\mathcal{N})$, there are now two definitions of $m_p(x)$. Either the decompositions can be taken is $S(\mathcal{M})$ or in $S(\mathcal{N})$. Note that by the lemma above, these two definitions coincide.

3.2.3 Proof of the theorem in full generality

In this section, we present a proof of the main result using decompositions that are close to be optimal rather than optimal. We essentially follow the proof of proposition 3.2.5 but we need some additional care and lemma 3.2.6 to get the final estimate. An alternative approach is to work in an ultraproduct where optimal decompositions always exist, it will developed in the next subsection. Note that the two strategies yield the same constants.

Lemma 3.2.8. Let $0 < p \leq 1$. For all $x \in S(\mathcal{M})$, $\eta > 0$ and decompositions $x = y + z$ such that $\|Ry\|_p^p + \|Cz\|_p^p \leq m_p(x) + \eta^p$, we have:

$$K_t(Ry, p, \infty) \leq A_p \left(A_p^p B_p^p K_t(Gx, p, \infty)^p + \eta^p\right)^{1/p}$$

for all $t > 0$.

Proof. Let $p \in (0, \infty)$ and $x \in S(\mathcal{M})$. Take $\eta > 0$ and $y, z \in S(\mathcal{M})$ such that $y + z = x$ and $\|Ry\|_p^p + \|Cz\|_p^p \leq m_p(x) + \eta^p$. To be able to use lemma 2.2.9, we need to work in a diffuse algebra so we will now consider x, y and z as elements of $S(\mathcal{N})$.

We can now repeat the argument of the proof of proposition 3.2.5. Take e a projection commuting with $|(Ry)^*|$ in $S(\mathcal{N})$ and $f = 1 - e$. Take $\varepsilon > 0$ and $ex = y_1 + z_1$, $fx = y_2 + z_2$ such that $\|Ry_1\|_p^p + \|Cz_1\|_p^p \leq m_p(ex) + \varepsilon$ and $\|Ry_2\|_p^p + \|Cz_2\|_p^p \leq m_p(fx) + \varepsilon$. By definition of y and z, $\|Ry\|_p^p + \|Cz\|_p^p \leq m_p(x) + \eta^p$, hence :

$$\|Ry\|_p^p + \|Cz\|_p^p \leq \|R(y_1 + y_2)\|_p^p + \|C(z_1 + z_2)\|_p^p + \eta^p.$$

By the p-triangular inequality and lemma 2.1.4:

$$\|R(ey)\|_p^p + \|R(fy)\|_p^p + \|Cz\|_p^p \leq \|Ry_1\|_p^p + \|Cz_1\|_p^p + \|Ry_2\|_p^p + \|Cz_2\|_p^p + \eta^p.$$

Now using the almost minimality of the couple (y_2, z_2) and lemma 2.1.4, we obtain:

$$\|Ry_2\|_p^p + \|Cz_2\|_p^p \leq \|R(fy)\|_p^p + \|C(fz)\|_p^p + \varepsilon \leq \|R(fy)\|_p^p + \|Cz\|_p^p + \varepsilon.$$

And we conclude using the Khintchine inequality:

$$\|R(ey)\|_p^p \leq \|Ry_1\|_p^p + \|Cz_1\|_p^p + \varepsilon + \eta^p \leq B_p^p \|G(ex)\|_p^p + 2\varepsilon + \eta^p.$$

This is true for all $\varepsilon > 0$ so:

$$\|R(ey)\|_p^p \leq B_p^p \|G(ex)\|_p^p + \eta^p.$$

Taking the supremum over all projections $e \in \mathcal{N}$ commuting with $|(Ry)^*|$, by proposition 2.2.9, we obtain that:

$$K_t(Ry, p, \infty) \leq A_p (A_p^p B_p^p K_t(Gx, p, \infty)^p + \eta^p)^{1/p}.$$

□

Proof of theorem 3.2.1. Let $x \in S(\mathcal{M})$. We need to find y and z in $S(\mathcal{M})$ such that $y + z = x$,

$$K_t(Ry, p, \infty) \lesssim K_t(Gx, p, \infty)$$

and similarly for Cz. Write again $A = \|Gx\|_\infty$. Define $\delta := \tau(\mathbf{1}_{|Gx|>A/2})^{1/p}$. For $t < \delta$, (1.1) gives:

$$A_p K_t(Gx, p, \infty) \geq (\int_0^{t^p} \mu_u(Gx)^p du)^{1/p} \geq \frac{tA}{2}.$$

Let $\eta = K_\delta(Gx, p, \infty)$.

Take y, z such that $\|Ry\|_p^p + \|Cz\|_p^p \leq m_p(x) + \eta^p$, $\|Ry\|_\infty \leq 2A$ and $\|Cz\|_\infty \leq 2A$. This is possible by lemma 3.2.6. For $t \geq \delta$, using lemma 3.2.8 and since $t \mapsto K_t(Gx, p, \infty)$ is increasing, we have:

$$\begin{aligned} K_t(Ry, p, \infty) &\leq A_p (A_p^p B_p^p K_t(Gx, p, \infty)^p + \eta^p)^{1/p} \\ &\leq A_p (A_p^p B_p^p K_t(Gx, p, \infty)^p + K_\delta(Gx, p, \infty)^p)^{1/p} \\ &\leq A_p (A_p^p B_p^p + 1)^{1/p} K_t(Gx, p, \infty). \end{aligned}$$

For $t < \delta$, we know that $A_p K_t(Gx, p, \infty) \geq \dfrac{tA}{2}$ and since $\|Ry\|_\infty \leq 2A$, we also have $K_t(Ry, p, \infty) \leq 2At$ so $K_t(Ry, p, \infty) \leq 4A_p K_t(Gx, p, \infty)$.

Let $C_p = \max(A_p(A_p^p B_p^p + 1)^{1/p}, 4A_p)$. We have just proven that for all $t > 0$, $K_t(Ry, p, \infty) \leq C_p K_t(Gx, p, \infty)$. Since the argument is perfectly symmetrical, we also have $K_t(Cz, p, \infty) \leq C_p K_t(Gx, p, \infty)$. □

Remark 3.2.9. The theorem still holds when $p \in (1, 2]$. Indeed, the p-triangular inequality is false in general but here, we only use it to prove inequalities of the type:

$$\|ex + fy\|_p^p \leq \|ex\|_p^p + \|fy\|_p^p \tag{3.2}$$

where e is a projection, $f = 1 - e$ and $x, y \in \mathcal{M}$. This still holds for $p \in [1, 2]$ using the inequality, for all $a, b \in \mathcal{M}$:

$$\frac{\|a - b\|_p^p + \|a + b\|_p^p}{2} \leq \|a\|_p^p + \|b\|_p^p. \tag{3.3}$$

Note that $ex + fy = (e - f)(ex - fy)$ and that $e - f$ is a unitary. Hence $\|ex + fy\|_p = \|ex - fy\|_p$. Now take $a = ex$ and $b = fy$ in (3.3) to obtain (3.2). To prove (3.3) one can for example apply the Riesz-Thorin interpolation theorem to the application $T : (x, y) \mapsto (x + y, x - y)$.

3.2.4 Proof using an ultraproduct argument

In this section, we prove the main result by reducing it to proposition 3.2.5. We have already seen that we can get rid of the hypothesis that $\mathcal{M}$ is diffuse by considering $\mathcal{N} = \mathcal{M}\overline{\otimes}L_\infty([0,1])$. Here, we reduce the problem to the case where x has an optimal decomposition by placing ourselves in an ultraproduct. We will denote by $\mathcal{M}_\omega$ the ultraproduct of $\mathbb{N}$ copies of $\mathcal{M}$. Since we only consider sequences $x \in S(\mathcal{M})$, as mentioned before, once x is fixed, we can work in $(e\mathcal{M}e)^N$ where $e = s(|(Rx)^*|) \vee s(|Cx|)$. Hence, to simplify the exposition of this subsection, we will suppose right away that $\mathcal{M}$ is finite. We also fix the following convention:

- if $x \in S(\mathcal{M})$, since $\mathcal{M}$ is canonically embedded in $\mathcal{M}_\omega$, x can also be considered as an element of $S(\mathcal{M}_\omega)$,
- consider y a finite sequence of elements in B, $y = (y^{(1)}, ..., y^{(N)})$ with $N \in \mathbb{N}$. For $i \leq N$, write $y^{(i)} = (y_n^{(i)})_{n\in\mathbb{N}}$ and define, for all $n \in \mathbb{N}$, $y_n = (y_n^{(1)}, ..., y_n^{(N)}) \in S(\mathcal{M})$.

Lemma 3.2.10. Let $p > 0$ and $x \in S(\mathcal{M})$. Then:

$$m_p(x) = \inf\{\|Ry\|_p^p + \|Cz\|_p^p : x = \overline{y} + \overline{z}, \overline{y}, \overline{z} \in S(\mathcal{M}_\omega)\}.$$

Furthermore, there exists $\overline{y}, \overline{z}$ in $S(\mathcal{M}_\omega)$ such that $m_p(x) = \|R\overline{y}\|_p^p + \|C\overline{z}\|_p^p$ and representatives $y, z \in S(B)$ such that for all $n \in \mathbb{N}$, $y_n + z_n = x$, $\|Ry_n\|_\infty \leq 2\|Gx\|_\infty$ and $\|Rz_n\|_\infty \leq 2\|Gx\|_\infty$

Proof. Let $\overline{y}, \overline{z} \in S(\mathcal{M}_\omega)$ and choose representatives y and z in B such that $y_n + z_n = x$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
\|R\overline{y}\|_p^p + \|C\overline{z}\|_p^p &= \lim_\omega \tau(|Ry_n|^p) + \lim_\omega \tau(|Cz_n|^p) \\
&= \lim_\omega \tau(|Ry_n|^p) + \tau(|Cz_n|^p) \\
&\geq m_p(x).
\end{aligned}$$

The other inequality is immediate since $L_p(\mathcal{M})$ is isometrically identified to a subspace of $L_p(\mathcal{M}_\omega)$.

To prove the second part of the lemma take $(y_n, z_n)_{n\geq 0}$ a bounded sequence of elements in $S(\mathcal{M})$ such that for all $n \in \mathcal{N}$, $y_n + z_n = x$, $\|Ry_n\|_\infty \leq 2\|Gx\|_\infty$, $\|Rz_n\|_\infty \leq 2\|Gx\|_\infty$ and $\lim_{n\to\infty} \|Ry_n\|_p^p + \|Cz_n\|_p^p = m_p(x)$. This is possible thanks to lemma 3.2.6. Denote by $\overline{y}$ and $\overline{z}$ the corresponding elements in $S(\mathcal{M}_\omega)$. We have $\overline{y} + \overline{z} = x$ and:

$$\begin{aligned}\|R\overline{y}\|_p^p + \|C\overline{z}\|_p^p &= \lim_\omega \tau(|Ry_n|^p) + \lim_\omega \tau(|Cz_n|^p) \\ &= \lim_\omega \tau(|Ry_n|^p) + \tau(|Cz_n|^p) \\ &= m_p(x).\end{aligned}$$

□

Remark 3.2.11. Note that for all $p > 0$, $x \in \mathcal{M}$, $\lim_{t\to\infty} K_t(x, \infty, p) = \|x\|_\infty$. As usual, we can suppose that $x > 0$. Since $K(x, \infty, p)$ is increasing and bounded by $\|x\|_\infty$, it has a finite limit l less or equal than $\|x\|_\infty$. Suppose by contradiction that $l < \|x\|_\infty$. Then for all t, there exists $y \in \mathcal{M}$ such that $\|y\|_\infty \le l$ and $\|x - y\|_p \le l/t$. Let $e = \mathbf{1}_{(l,\infty)}(x)$. $\|x - y\|_p \ge \|e(x - y)e\|_p$. Now $exe > el$ by definition of e and $eye < el$ so $\|e(x - y)e\|_p > \|e(x - l)e\|_p$. Hence for all t, $l/t \ge \|e(x - l)e\|_p > 0$ which yields a contractiction taking limits when t goes to infinity.

Proof of theorem 3.2.1. By the previous lemma and lemma 3.2.7, we can apply proposition 3.2.5 to x in the algebra $\mathcal{N} = \mathcal{M}_\omega \overline{\otimes} L_\infty([0, 1])$. Hence, an optimal decomposition $\overline{y}, \overline{z}$ given by the previous lemma verifies, for all $t \ge 0$, $K_t(R\overline{y}, p, \infty) \le A_p^2 B_p K_t(Gx, p, \infty)$ and $K_t(C\overline{z}, p, \infty) \le A_p^2 B_p K_t(Gx, p, \infty)$. Now, by lemma 2.3.1, applied to $|R\overline{y}^*|$, we know that $K(Ry_n, L_p(\mathcal{M}), \mathcal{M})$ converges uniformly to $K(Ry, L_p(\mathcal{M}_\omega), \mathcal{M}_\omega)$. Note that $\lim_{t\to\infty} K_t(Gx, \infty, p) = \|Gx\|_\infty$ so there exists $t_0 \in \mathbb{R}^+$ such that for all $t \ge t_0$, $K_t(Gx, \infty, p) \ge \frac{1}{2}\|Gx\|_\infty$. Now let $\varepsilon = K_{1/t_0}(Ry, L_p(\mathcal{M}_\omega), \mathcal{M}_\omega)$ and chose N such that for all $t \ge 0$,

$$|K_t(Ry_N, L_p(\mathcal{M}), \mathcal{M}) - K_t(Ry, L_p(\mathcal{M}_\omega), \mathcal{M}_\omega)| \le \varepsilon$$

Since the K-functionnal is increasing this means that for all $t \ge 1/t_0$, $K_t(Ry_N, L_p(\mathcal{M}), \infty) \le 2K_t(Ry, L_p(\mathcal{M}_\omega), \mathcal{M}_\omega)$. Now, note that $K_t(y_N, L_p(\mathcal{M}), \mathcal{M}) = tK_{1/t}(Ry_N, \mathcal{M}, L_p(\mathcal{M}))$ and recall that $\|Ry_N\|_\infty \le 2\|Gx\|_\infty$. For $t < 1/t_0$:

$$\begin{aligned}K_t(y_N, L_p(\mathcal{M}), \mathcal{M}) &= tK_{1/t}(Ry_N, \mathcal{M}, L_p(\mathcal{M})) \\ &\le t\|Ry_N\|_\infty \\ &\le 2t\|Gx\|_\infty \\ &\le 4tK_{1/t}(Gx, \infty, p) \\ &\le 4K_t(Gx, p, \infty).\end{aligned}$$

Now, by applying theorem 2.2.6, we know that $K_t(Ry_N, L_p(\mathcal{M}), \mathcal{M}) \approx K_t(Ry_N, p, \infty)$ and that $K_t(Ry_N, L_p(\mathcal{M}_\omega), \mathcal{M}_\omega) \approx K_t(Ry, p, \infty)$. So, for $t \le 1/t_0$:

$$K_t(Ry_N, p, \infty) \le c_p K_t(Ry_N, L_p(\mathcal{M}), \mathcal{M}) \le 4c_p K_t(Gx, p, \infty).$$

And for $t \geq 1/t_0$:

$$\begin{aligned} K_t(Ry_N, p, \infty) &\leq c_p K_t(Ry_N, L_p(\mathcal{M}), \mathcal{M}) \leq 2c_p K_t(Ry, L_p(\mathcal{M}_\omega), \mathcal{M}_\omega) \\ &\leq 2c_p^2 K_t(Ry, p, \infty) \leq 2c_p^2 A_p^2 B_p K_t(Gx, p, \infty). \end{aligned}$$

□

3.3 Further results on optimal decompositions in L_1

In this section, we investigate further the properties of optimal decompositions in L_1. The first notable fact is that in this case, we can prove that an optimal decomposition always exists (see lemma 3.3.1). Knowing this, a simple duality argument yields a factorisation for elements in $S(\mathcal{M})$ (theorem 3.3.2). Remarkably, this result is of purely algebraic nature and combined with [54] which provides the necessary estimate on anticommutators, produces a new proof of Khintchine inequalities. The main novelty is the emergence of elements $\alpha, \beta \in \mathcal{M}^+$ associated to $x \in S(\mathcal{M})$ which play the role of a "modulus" in $\mathscr{H}_p$-spaces (theorem 3.1.8). In particular, there is no need in the proofs to distinguish between $p \leq 2$ or $p \geq 2$. The drawback of the method is that it relies on Khintchine inequalities in L_∞ and therefore does not apply, at least directly, to Rademacher variables. Let us already assume that for all $x \in S(\mathcal{M})$,

$$\|Gx\|_\infty \lesssim_{c_\infty} \max(\|Rx\|_\infty, \|Cx\|_\infty). \tag{3.4}$$

We start by proving the existence of an optimal decomposition in L_1. The argument is straightforward by taking a limit of a minimising sequence of decompositions.

Lemma 3.3.1. Let $x \in S(\mathcal{M})$. There exist $y, z \in S(\mathcal{M})$ such that $y + z = x$ and $\|Ry\|_1 + \|Cz\|_1 = m_1(x)$.

Proof. Consider a sequence $(y^{(i)}, z^{(i)})_{i \geq 0}$ such that:

$$\lim_{i \to \infty} \|Ry^{(i)}\|_1 + \|Cz^{(i)}\|_1 = m_1(x),$$

and for all $i \geq 0$, $y^{(i)} + z^{(i)} = x$. By lemma 3.2.6, we can suppose that the $y_n^{(i)}$ and $z_n^{(i)}$ are uniformly bounded in $L_\infty(\mathcal{M})$. Recall that the sequence x is finite, say of length N and let:

$$e = s(|(Rx)^*|) \vee s(|Cx|).$$

By considering the sequences $(ey_n^{(i)}e)_{n \leq N}$ and $(ez^{(i)}e)_{n \leq N}$ we can also suppose that $y^{(i)}$ and $z^{(i)}$ are in $(e\mathcal{M}e)^N$. Since the $y_n^{(i)}$ and $z_n^{(i)}$ are uniformly bounded in $L_\infty(e\mathcal{M}e)$, by the criterion of weak compacity in L_1 ([58]), up to extraction, we can suppose that for all $n \geq 0$, the sequence $(y_n^{(i)})_{i\geq 0}$ converges weakly in $L_1(e\mathcal{M}e)$ to an element $y_n \in e\mathcal{M}e$ and using Mazur's lemma, taking convex combinations we can even assume the norm-convergence in $L_1(\mathcal{M})$.

Let $y = (y_n)_{0\leq n\leq N}$ and $z = (z_n)_{0\leq n\leq N}$. Since the sequences y and z belong to $(e\mathcal{M}e)^N$, they belong to $S(\mathcal{M})$. Note that $\{a, b : a + b = x\}$ is closed and convex, so for all $n \geq 0$, $y_n + z_n = x_n$. Moreover, for $0 \leq n \leq N$, we have $\lim_{i\to\infty} y_n^{(i)} \otimes e_{1,n} = y_n \otimes e_{1,n}$ in $L_1(e\mathcal{M}e\overline{\otimes}\mathcal{B}(\ell^2))$ and similarly $\lim_{i\to\infty} z_n^{(i)} \otimes e_{n,1} = z_n \otimes e_{n,1}$. Hence, by summing over n, we obtain: $m_1(x) = \lim_{i\to\infty} \|Ry^{(i)}\|_1 + \|Cz^{(i)}\|_1 = \|Ry\|_1 + \|Cz\|_1$.

□

Theorem 3.3.2. *Let $x \in S(\mathcal{M})$. There exist $\alpha, \beta \in \mathcal{M}_c^+$ and $u \in S(\mathcal{M})$ such that $x = \alpha u + u\beta$, $s(\alpha) \leq |(Ru)^*| \leq 1$ and $s(\beta) \leq |Cu| \leq 1$.*

Proof. Using the same notations as in the previous proof, we can work in $(e\mathcal{M}e)^N$ which garantees that all sequences considered are in $S(\mathcal{M})$ and operators are finitely supported. Let $y, z \in S(\mathcal{M})$ be the elements given by lemma 3.3.1 i.e $y+z = x$ and $\|Ry\|_1 + \|Cz\|_1 = m_1(x)$. Denote $\alpha = |(Ry)^*|$, $e = s(\alpha)$, $\beta = |Cz|$ and $f = s(\beta)$. Write $y = \alpha v$, $v \in S(\mathcal{M})$ such that αRv is a polar decomposition of Ry, in particular v can be chosen such that $ev = v$. Similarly, $z = w\beta$ with $w \in S(\mathcal{M})$ and $wf = w$.

By duality, there exists an element $u \in (e\mathcal{M}e)^N$ such that $\|u\|_{R_\infty\cap C_\infty} = 1$ and

$$\tau(\sum_{i\geq 0} u_i^* x_i) = \|x\|_{R_1+C_1} = m_1(x).$$

Let us rewrite the previous equality with the notations introduced previously:

$$m_1(x) = \tau\big(\sum_{i\geq 0} u_i^*(\alpha v_i + w_i\beta)\big)$$

$$\tau(\alpha) + \tau(\beta) = \tau\big(\sum_{i\geq 0} \alpha v_i u_i^* e\big) + \tau\big(\sum_{i\geq 0} f u_i^* w_i \beta\big)$$

taking the real part on both sides of the equality, we get:

$$\tau(\alpha) + \tau(\beta) = \tau\big(\Re(\sum_{i\geq 0} \alpha v_i u_i^* e)\big) + \tau\big(\Re(\sum_{i\geq 0} f u_i^* w_i \beta)\big)$$

and by the tracial property of τ:

$$\tau(\alpha) + \tau(\beta) = \tau\big(\alpha\Re(\sum_{i\geq 0} v_i u_i^* e)\big) + \tau\big(\Re(\sum_{i\geq 0} f u_i^* w_i)\beta)\big).$$

Moreover,

$$\|\Re(\sum_{i\geq 0} v_i u_i^* e)\|_\infty = \|\Re(R(v)C(u^*e))\|_\infty \leq 1$$

and since $ev = v$:

$$s(\Re(\sum_{i\geq 0} v_i u_i^* e)) = s(\Re(e(\sum_{i\geq 0} v_i u_i^*)e)) = s(e\Re(\sum_{i\geq 0} v_i u_i^*)e) \leq e$$

so $\Re(\sum_{i\geq 0} v_i u_i^* e) \leq e$. Similarly, since $wf = w$, $\Re(\sum_{i\geq 0} f u_i^* w_i) \leq f$. Hence, we must have

$$\Re(\sum_{i\geq 0} v_i u_i^* e) = e.$$

This means that

$$|\tau(R(v)C(u^*e))| = \|R(v)\|_2 \|C(u^*e)\|_2.$$

There is equality in the Cauchy-Schwarz inequality in $L_2(\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2))$, so there exists $\lambda \in \mathbb{C}$, $eu = \lambda v$. The only possibility is that $eu = v$. Hence, $\alpha u = \alpha eu = \alpha v = y$. Similarly, $u\beta = z$. Therefore, $x = \alpha u + u\beta$. Let us now verify the other required properties. First, since $\|u\|_{R_\infty \cap C_\infty} = 1$, $|(Ru)^*| \leq 1$. Secondly, since $eu = v$, $e = |(R(eu))^*|$ and note that $\|Ru\|_\infty \leq 1$ implies that $\|e(Ru)(Ru)^*\|_2^2 \leq \tau(e)$. Moreover, by orthogonality,$\|e(Ru)(Ru)^*\|_2^2 = \|e(Ru)(Ru)^*e\|_2^2 + \|e(Ru)(Ru)^*(1-e)\|_2^2$. Hence, $\tau(e) \geq \tau(e) + \|e(Ru)(Ru)^*(1-e)\|_2^2$, which means that $e(Ru)^*(Ru)(1-e) = 0$. Symmetrically, $(1-e)(Ru)(Ru)^*e = 0$ so

$$(Ru)(Ru)^* - e = (Ru)(Ru)^* - e(Ru)(Ru)^*e = (1-e)(Ru)^*(Ru)(1-e) \geq 0.$$

By adjunction, we obtain similar inequalities for columns. $\square$

We are now going to show that we can obtain Khintchine-type inequalities in a very general sense from the factorization found above.

Theorem 3.3.3. *Let $x \in S(\mathcal{M})$, $\alpha, \beta \in \mathcal{M}_c^+$ and $u \in S(\mathcal{M})$ such that $x = \alpha u + u\beta$, $s(\alpha) \leq |(Ru)^*| \leq 1$ and $s(\beta) \leq |Cu| \leq 1$. Then, for all $t > 0$ and $p > 0$:*

$$K_t(\alpha, p, \infty) + K_t(\beta, p, \infty) \approx_c K_t(Gx, p, \infty),$$

where c only depends on p and (ξ_i) (more precisely on the constant c_∞ appearing in (3.4)*).*

Proof. Upper estimate. Let $p > 0$ and $t > 0$. Note that $Gx = \alpha(Gu) + (Gu)\beta$ where α is identified with $\alpha \otimes 1$ and β with $\beta \otimes 1$ in $\mathcal{M}\overline{\otimes}\mathcal{A}$. By the Khintchine inequalities in L_∞ (i.e. estimate (3.4)), since $\|u\|_{R_\infty \cap C_\infty} \leq 1$ we have $\|Gu\|_\infty \leq c_\infty$. Hence,

$$\begin{aligned} K_t(Gx, p, \infty) &\lesssim K_t(\alpha(Gu), p, \infty) + K_t((Gu)\beta, p, \infty) \\ &\leq c_\infty (K_t(\alpha, p, \infty) + K_t(\beta, p, \infty)). \end{aligned}$$

Lower estimate. Let $t > 0$. We only prove the theorem for $p < 1$, to obtain the result for $p \geq 1$ it suffices to take $\theta = 1$ in the argument. Let us rewrite lemma 2.4.1, with $\theta = p$, $q = \infty$:

$$K_{t^p}(\alpha^p(Gu) + (Gu)\beta^p, 1, \infty) \lesssim K_t(Gx, p, \infty)^p \|Gu\|_\infty^{1-p}.$$

With this inequality, we can conclude without too much effort. Indeed:

$$K_{t^p}(\alpha^p(Gu)+(Gu)\beta^p,1,\infty)\geq \|Gu\|_\infty^{-1}K_{t^p}(((Gu)\beta^p+\alpha^p(Gu))(Gu)^*,1,\infty).$$

By (3.4), $\|Gu\|_\infty \lesssim \|u\|_{R_\infty\cap C_\infty}=1$. Let us now consider the conditional expectation $Id\otimes\tau_{\mathcal{A}}$.

$$\begin{aligned}(Id\otimes\tau_{\mathcal{A}})(((Gu)\beta^p+\alpha^p(Gu))(Gu)^*) &= (Id\otimes\tau_{\mathcal{A}})(\sum_{i,j\geq 0}u_i\beta^p u_j^*\otimes\xi_i\xi_j^*+\sum_{i,j\geq 0}\alpha^p u_i u_j^*\otimes\xi_i\xi_j^*)\\ &= \sum_{i=0}^\infty u_i\beta^p u_i^*+\sum_{i=0}^\infty \alpha^p u_i u_i^*.\end{aligned}$$

As a conditional expectation, $Id\otimes\tau_{\mathcal{A}}$ is bounded on L_1 and L_∞ so

$$K_{t^p}(((Gu)\beta^p+\alpha^p(Gu))(Gu)^*,1,\infty)\gtrsim K_{t^p}(\sum_{i=0}^\infty u_i\beta^p u_i^*+\alpha^p(\sum_{i=0}^\infty u_iu_i^*),1,\infty),$$

note that $\sum_{i=0}^\infty u_i\beta^p u_i^*\geq 0$, and that since $s(\alpha)\leq\sum_{i=0}^\infty u_iu_i^*\leq 1$, we get $\alpha^p(\sum_{i=0}^\infty u_iu_i^*)=\alpha^p\geq 0$. Hence,

$$K_{t^p}(\sum_{i=0}^\infty u_i\beta^p u_i^*+\alpha^p(\sum_{i=0}^\infty u_iu_i^*),1,\infty)\gtrsim K_{t^p}(\alpha^p,1,\infty)\gtrsim K_t(\alpha,p,\infty)^p,$$

where we used the power theorem (proposition 2.2.1) for the last inequality. The same tricks work for β by multiplying Gx by $(Gu)^*$ on the left. $\square$

Remark 3.3.4. Theorem 3.1.8 claimed in the introduction is obtained by a combination of the two previous theorems and the characterisation of interpolation spaces between L_p-spaces given by proposition 1.3.5.

Remark 3.3.5. If we start with an $x\in S(\mathcal{M})$ such that for all $i\geq 0$, $x_i=x_i^*$, the factorisation given by theorem 3.3.2 takes the following form. There exists $\alpha\in\mathcal{M}^+$ and $u\in S(\mathcal{M})$ such that

- $x=u\alpha+\alpha u$,
- for all $i\geq 0$, $u_i=u_i^*$,
- $s(\alpha)\leq\sum_{i\geq 0}u_i^2\leq 1$.

Remark 3.3.6. From the results of this section, it is easy to recover the upper and lower Khintchine inequalities. More precisely, if E is an interpolation space between L_p-spaces, $p\in(0,\infty]$ and $x\in S(\mathcal{M})$:

$$\|x\|_{R_E+C_E}\lesssim\|Gx\|_{E(\mathcal{M}_{\mathcal{A}})}\lesssim\|x\|_{R_E\cap C_E}.$$

Proof. Let $x\in S(\mathcal{M})$. The left hand side inequality is obtained directly by considering the decomposition $y=\alpha u$ and $z=u\beta$ and applying the lower estimate in theorem 3.1.8.

To show the right hand side inequality, we make a computation similar to what appeared in the proof of theorem 3.3.3. First, using again theorem 3.1.8, we know that

$$\|Gx\|_{E(\mathcal{M}_{\mathcal{A}})}\lesssim\max(\|\alpha\|_{E(\mathcal{M})},\|\beta\|_{E(\mathcal{M})}).$$

Moreover, note that

$$\|Rx\|_{E(\mathcal{M}_\mathcal{B})} \gtrsim \|Rx(Ru)^*\|_{E(\mathcal{M}_\mathcal{B})} = \|\alpha + (Ru)\beta(Ru)^*\|_{E(\mathcal{M}_\mathcal{B})} \geq \|\alpha\|_{E(\mathcal{M})}.$$

By adjunction, we also obtain

$$\|Cx\|_{E(\mathcal{M}_\mathcal{B})} \gtrsim \|\beta\|_{E(\mathcal{M})}.$$

Hence,

$$\|Gx\|_{E(\mathcal{M}_\mathcal{A})} \lesssim \max\left(\|Rx\|_{E(\mathcal{M}_\mathcal{B})}, \|Cx\|_{E(\mathcal{M}_\mathcal{B})}\right).$$

□

3.4 A remark about martingale inequalities

In this section, we recover a variant of a result first proved in [52] on martingale inequalities. The novelty compared to the original noncommutative martingale inequalities ([49]) is that we show that the decomposition appearing for $1 < p < 2$ can be chosen to be independent of p. Note that this result has also been obtained, using a constructive approach, in a recent preprint ([23]). The setting is the following, let $\mathcal{F} = (\mathcal{M}_n)_{n\geq 0}$ be a filtration on $\mathcal{M}$, and assume that for all $n \in \mathbb{N}$, the conditional expectation $\mathbb{E}_n : \mathcal{M} \to \mathcal{M}_n$ exists. Denote by $\mathcal{M}_\infty^- := \cup_{n\geq 0}\mathcal{M}_n$ the set of finite bounded martingales for the filtration $\mathcal{F}$. For any x in $\mathcal{M}_\infty^-$, denote by dx the associated martingale differences. For the remainder of the section, suppose that the ξ_i are independent Rademacher variables.

Theorem 3.4.1. *There exist constants $(k_p)_{p>1}$ such that for any $x \in \mathcal{M}_\infty^-$, there exists y and $z \in \mathcal{M}_\infty^-$ such that $x = y + z$ and for all $p > 1$:*

$$\|R(dy)\|_p + \|C(dz)\|_p \leq k_p\|x\|_p.$$

Proof. By theorem 3.2.1 applied for $p = 1$, there exists y' and $z' \in S(\mathcal{M})$ such that $dx = y' + z'$ and for all $t \geq 0$, $K_t(Ry', 1, \infty) \leq C_1 K_t(Gx, 1, \infty)$ and $K_t(Cz', 1, \infty) \leq C_1 K_t(Gx, 1, \infty)$. By real interpolation, this means that for all $p > 1$, $\|Ry'\|_p + \|Cz'\|_p \leq 2C_1\|Gx\|_p$. Define $\Delta_0 = \mathbb{E}_0$ and for $n \geq 1$, $\Delta_n = \mathbb{E}_n - \mathbb{E}_{n-1}$. Let $dy = (\Delta_n(y'_n))_{n\geq 0}$ and $dz = (\Delta_n(z'_n))_{n\geq 0}$. This way y and z belong to $\mathcal{M}_\infty^-$ and they keep the same properties. Indeed, for all $n \in \mathbb{N}$, $dx_n = \Delta_n(dx_n) = \Delta_n(y'_n + z'_n) = dy_n + dz_n$ and by Stein's inequality ([49], [62]):

$$\|R(dy)\|_p + \|C(dz)\|_p \lesssim \|Ry'\|_p + \|Cz'\|_p \lesssim \|G(dx)\|_p.$$

Moreover, by the the unconditionality of martingal differences [49], $\|G(dx)\|_p \approx \|x\|_p$. Hence, there exists a constant k_p (independant of x) such that:

$$\|R(dy)\|_p + \|C(dz)\|_p \leq k_p\|x\|_p.$$

□

Remark 3.4.2. Since by interpolation, Burkholer-Gundy inequality stays true in all interpolation space E between L_p-spaces, $\infty > p > 1$, the argument above can be reproduced and the decomposition y, z constructed in the proof verifies

$$\|Ry\|_E + \|Cz\|_E \leq k_E \|x\|_E.$$

3.5 On the properties Kh_+ and Kh_-

3.5.1 Explicit counter-examples in $L_{2,\infty}$

In this section, we construct two counter-examples to prove proposition 3.1.4.

3.5.2 Counter-example for $H_{2,\infty} = R_{2,\infty} + C_{2,\infty}$

The first counter-example is very simple to define. Fix $N \in \mathbb{N}$, we are going to construct a sequence of matrices $x = (x_i)_{1 \leq i \leq N}$ such that $\|Gx\|_{2,\infty} \approx \sqrt{\ln N}$ and $\|Rx\|_{2,\infty} = 1$. In consequence, we get:

$$\|x\|_{R_{2,\infty}+C_{2,\infty}} \leq \|Rx\|_{2,\infty} \leq \frac{1}{\sqrt{\ln N}} \|Gx\|_{2,\infty}.$$

Lemma 3.5.1. Let $x = (x_i)_{1 \leq i \leq N} \in \mathcal{B}(H)$ with $x_i = \dfrac{1}{\sqrt{i}} e_{1,i}$ for $1 \leq i \leq N$. Then, $\|Gx\|_{2,\infty} \geq \sqrt{\ln(N-1)}$ and $\|Cx\|_{2,\infty} = 1$.

Proof. We have:

$$|Gx^*|^2 = \sum_{i=1}^{N} \frac{1}{i} e_{1,1} \otimes 1_{\mathcal{A}} \geq \ln(N-1) e_{1,1} \otimes 1_{\mathcal{A}}$$

and:

$$|Cx|^2 = \sum_{i=1}^{N} \frac{1}{i} e_{i,i}.$$

Hence $\| \, |Gx^*|^2 \, \|_{1,\infty} \geq ln(N-1)$ and $\| \, |Cx|^2 \, \|_{1,\infty} = 1$.

□

3.5.3 Counter-example for $H_{2,\infty} = R_{2,\infty} \cap C_{2,\infty}$

In this subsection, we will always assume that the $(\xi_i)_{i \geq 0}$ are free Haar unitaries. We start by making the following observation that is easily deduced from [14]:

Lemma 3.5.2. Let $x \in S(\mathcal{M})$. If all eigenvalues of $|Rx|$ are less are equal to the positive eigenvalues of $|Cx|$, then

$$\mu(Gx) \leq 2\mu(Cx).$$

Proof. Consider the element $Rx \oplus Cx \in (\mathcal{M} \otimes B(\ell^2)) \oplus (\mathcal{M} \otimes B(\ell^2))$ which is no longer a noncommutative probability space but is still a noncommutative integration space equipped with the sum trace. In [14] (p.3), it is proved that $\lambda_{2t}(Gx) \leq \lambda_t(Rx \oplus Cx)$ for all $t \geq 0$. Symmetrically, for all $t \geq 0$, $\mu_t(Gx) \leq 2\mu_t(Rx \oplus Cx)$ which is the key inequality we will need.

Furthermore, recall that the conditionnal expectation $Id \otimes \tau$ sends $|Gx|^2$ to $|Cx|^2$ in $\mathcal{M} \otimes \mathcal{A}$. This implies that for all $t \geq 0$:

$$\int_0^t \mu(Cx)^2 \leq \int_0^t \mu(Gx)^2.$$

In particular, for $t_0 = \tau(\operatorname{supp}|Cx|)$, we obtain:

$$\tau(|Cx|^2) = \int_0^{t_0} \mu(Cx)^2 \leq \int_0^{t_0} \mu(Gx)^2 \leq \int_0^{\infty} \mu(Gx)^2 = \tau(|Gx|^2) = \tau(|Cx|^2).$$

Therefore, all terms appearing above are equal and $\mu_t(Gx) = 0$ for all $t > t_0$. Hence, $\mu_t(Gx) \leq \mathbf{1}_{t \leq t_0} 2\mu_t(Rx \oplus Cx)$.

Finally, by hypothesis, all eigenvalues of $|Rx|$ are less are equal to the positive eigenvalues of $|Cx|$. In this special case, $\mu(Rx \oplus Cx)$ is very easy to compute. Indeed, consider the function f defined by:

$$f = \begin{cases} \mu_t(Cx) & \text{if } t \leq t_0 \\ \mu_{t-t_0}(Rx) & \text{if } t > t_0. \end{cases}$$

One can easily see that f has the same distribution as $Rx \oplus Cx$, f is nonincreasing and left-continuous so $f = \mu(Rx \oplus Cx)$. In consequence, for all $t \geq 0$:

$$\mu_t(Gx) \leq \mathbf{1}_{t \leq t_0} 2f(t) = 2\mu_t(Cx).$$

$\square$

Using the previous lemma, we will be able to control the norm of Gx. What is left to do now is to find a sequence such that the eigenvalues of Cx dominate those of Rx and $\|Cx\|_{2,\infty} \ll \|Rx\|_{2,\infty}$. In the remainder of the section, we construct an explicit example with once again $\mathcal{M} = B(\ell^2)$.

Fix $N \in \mathbb{N}$. Consider the sequence:

$$u_n = \sum\nolimits_{k=1}^{n} k\left(\left\lfloor \frac{N}{k} \right\rfloor - \left\lfloor \frac{N}{k+1} \right\rfloor\right)$$

and define $n_j = \inf\{n : u_n \geq j\}$ with the convention $n_j = \infty$ if $j > \sup_{n \geq 1} u_n = u_N$.

Lemma 3.5.3. For all i,

$$\sum\nolimits_{j \geq 1} \mathbf{1}_{i \leq n_j} \frac{1}{n_j} = \left\lfloor \frac{N}{i} \right\rfloor.$$

And for all $j \geq 1$,

$$\sum\nolimits_{i \geq 1} \mathbf{1}_{i \leq n_j} \frac{1}{n_j} = \mathbf{1}_{j \leq u_N}.$$

Proof. By definition of n_j, $(i \leq n_j) \Leftrightarrow (j > u_{i-1})$. Hence,

$$\begin{aligned}\sum_{j>0} \mathbf{1}_{i\leq n_j} \frac{1}{n_j} &= \sum_{j>u_{i-1}} \frac{1}{n_j} \\ &= \sum_{k\geq i} \sum_{j=u_{k-1}+1}^{u_k} \frac{1}{n_j} \\ &= \sum_{k\geq i} k(\left\lfloor \frac{N}{k} \right\rfloor - \left\lfloor \frac{N}{k+1} \right\rfloor)\frac{1}{k} \\ &= \left\lfloor \frac{N}{i} \right\rfloor.\end{aligned}$$

The other equality is direct with the convention $\frac{1}{\infty} = 0$. □

Let $x_{i,j} = \mathbf{1}_{i\leq n_j} \dfrac{1}{\sqrt{n_j}} e_{i,j} \in B(\ell^2)$ for $i, j \geq 1$.

Proposition 3.5.4. *Let $(\xi_{i,j})_{i,j}$ be free Haar unitaries. Consider the $x_{i,j}$ defined above. We have:*

$$\| \sum_{i,j} x_{i,j} \otimes \xi_{i,j} \|_{2,\infty} \leq 2\sqrt{N},$$

and:

$$\| \sum_{i,j} |x_{i,j}|^2 \| \approx \sqrt{N \ln(N)}.$$

Proof. Using the previous lemma, we have:

$$\sum_{i,j} \left|x_{i,j}^*\right|^2 = \sum_{i,j\geq 1} \mathbf{1}_{i\leq n_j} \frac{1}{n_j} e_{i,i} = \sum_{i\geq 1} \left\lfloor \frac{N}{i} \right\rfloor e_{i,i}.$$

Consequently, for all $t \geq 0$, $\mu_t(\sum_{i,j} \left|x_{i,j}^*\right|^2) = \left\lfloor \dfrac{N}{\lfloor t+1 \rfloor} \right\rfloor \leq \dfrac{1}{t}$. Similarly,

$$\sum_{i,j} |x_{i,j}|^2 = \sum_{i,j\geq 1} \mathbf{1}_{i\leq n_j} \frac{1}{n_j} e_{j,j} = \sum_{j\geq 1} \mathbf{1}_{j\leq u_N} e_{j,j}.$$

So we have $\mu(\sum_{i,j} |x_{i,j}|^2) = \mathbf{1}_{0,u_N}$.

The conditions of lemma 3.5.2 are satisfied so:

$$\mu(\left|\sum_{i,j} x_{i,j} \otimes \xi_{i,j}\right|) \leq 2\mu((\sum_{i,j} \left|x_{i,j}^*\right|^2)^{1/2}).$$

Hence, $\| \sum_{i,j} x_{i,j} \otimes \xi_{i,j} \|_{2,\infty} \leq 2\sqrt{\| \sum_{i,j} \left|x_{i,j}^*\right|^2 \|_{1,\infty}} \leq 2\sqrt{N}$.

Furthermore, note that:

$$u_N = \sum_{k=1}^{\infty} k(\left\lfloor \frac{N}{k} \right\rfloor - \left\lfloor \frac{N}{k+1} \right\rfloor) = \sum_{k=1}^{\infty} \left\lfloor \frac{N}{k} \right\rfloor \approx N \ln(N).$$

So, as expected, we have $\| \sum_{i,j} |x_{i,j}|^2 \|_{1,\infty} \approx N \ln(N)$. □

3.5.4 The systematic construction

In this section, we give a second proof of proposition 3.1.4 and more importantly, we prove theorem 3.1.5. The proofs are based on the Schur-Horn theorem which enables us to show that the distributions of Gx and Rx can be prescribed. For now, the ξ_i are only supposed to be orthonormal.

A finite sequence $a = (a_1, ..., a_n)$ will be identified with the infinite sequence $a = (a_1, ..., a_n, 0, 0, 0, ...)$. To any sequence a we associate the function

$$f_a := \sum_{i=1}^{\infty} a_i \mathbf{1}_{(i-1,i]}.$$

Proposition 3.5.5. *Let $\mathcal{M} = \mathcal{B}(\ell^2)$. Let a and b be two finite nonincreasing sequences of positive reals such that for all $n \in \mathbb{N}$,*

$$\sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} b_i^2 \text{ and } \sum_{i=1}^{\infty} a_i^2 = \sum_{i=1}^{\infty} b_i^2.$$

Then, there exists $x \in S(\mathcal{M})$ such that

$$\mu(Gx) = \mu(Cx) = f_a \text{ and } \mu(Rx) = f_b.$$

Proof. Let $N \in \mathbb{N}$ be the length of a and b. The Schur-Horn theorem applied to (a_i^2) and (b_i^2) produces a positive matrix $M \in \mathbb{M}_N(\mathbb{C})$ such that the eigenvalues of M are given by (a_i^2) and the diagonal of M is given by (b_i^2). Consider $x = (e_{i,i} M^{1/2})_{0 \leq i \leq N}$. Then

$$|Gx|^2 = \sum_{i,j} M^{1/2} e_{i,i} e_{j,j} M^{1/2} \otimes \xi_i^* \xi_j = \sum_i M^{1/2} e_{i,i} M^{1/2} \otimes 1 = M \otimes 1.$$

Similar computations give: $|Cx|^2 = M \otimes e_{1,1}$ and $|(Rx)^*|^2 = Diag(M) \otimes e_{1,1}$. $\square$

Second proof of proposition 3.1.4. Fix $N \in \mathbb{N}$. Denote by u_N the quantity $u_N := \sum_{i=1}^{N} 1/i$. Let $a = (a_i)_{i \leq 1}$ and $b = (b_i)_{i \leq N}$ be defined by $a_1 = \sqrt{u_N}$, $a_i = 0$ for $i \geq 2$, $b_i = \sqrt{1/i}$ for $i \leq N$ and $b_i = 0$ for $i > N$. Applying the previous proposition, we obtain an element $x \in S(\mathcal{M})$ such that $\|Gx\|_{2,\infty} = \sqrt{u_N} \approx \sqrt{\ln(N)}$ and $\|Rx\|_{2,\infty} = 1$. Hence, we cannot have $\|.\|_{\mathscr{H}_{2,\infty}} \lesssim \|.\|_{R_{2,\infty}+C_{2,\infty}}$.

Now define $v_N := \sum_{i=1}^{N} \lfloor N/i \rfloor$. Let $a = (\sqrt{\lfloor N/i \rfloor})_{i \leq N}$ and $b = (1)_{i \leq v_N}$. By applying, the previous proposition again, we obtain an element $x \in S(\mathcal{M})$ such that $\|Gx\|_{2,\infty} = \sqrt{N}$ and $\|Rx\|_{2,\infty} = \sqrt{v_N} \approx \sqrt{N \ln(N)}$ which denies the possibility of having $\|.\|_{\mathscr{H}_{2,\infty}} \gtrsim \|.\|_{R_{2,\infty} \cap C_{2,\infty}}$. $\square$

Let us now prove theorem 3.1.5. We divide the proof into three lemmas.

Recall the definition of the set F of non-negative dyadic step functions:

$$F := \{f \in L_\infty(0, \infty) : \exists n \in \mathbb{N}, \exists N \in \mathbb{N}, \exists a \in (\mathbb{R}^+)^N, f = \sum_{i=1}^{N} a_i \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})}\}.$$

Lemma 3.5.6. Let $\mathcal{M} = \mathcal{B}(\ell^2) \overline{\otimes} L_\infty(0, 1)$. Let $f, g \in F$ such that $f^2 \succ g^2$ and $\|f\|_2 = \|g\|_2$ then there exists $x \in S(\mathcal{M})$ such that $\mu(Gx) = f$ and $\mu(Rx) = g$.

Proof. This is a direct corollary of proposition 3.5.5. Write $f = \sum_{i=1}^N a_i \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})}$ and $g = \sum_{i=1}^N b_i \mathbf{1}_{[(i-1)2^{-n}, i2^{-n})}$. Since $f^2 \succ g^2$, $a^2 \succ b^2$. Let $x \in S(\mathcal{B}(\ell^2))$ be given by proposition 3.5.5 and define $x' = (x_i \otimes \mathbf{1}_{(0,2^{-n})})_{i \leq N}$. Then $\mu(Gx') = f$ and $\mu(Rx') = g$. □

Lemma 3.5.7. Let E be a symmetric function space and $\mathcal{M} = \mathcal{B}(\ell^2)\overline{\otimes}L_\infty(0,1)$. Then,

$$Kh_+(E,\mathcal{M}) \Rightarrow \left(\forall f,g \in F, f^2 \succ g^2 \Rightarrow \|g\|_E \lesssim \|f\|_E\right),$$

and

$$Kh_-(E,\mathcal{M}) \Rightarrow \left(\forall f,g \in F, f^2 \triangleright g^2 \Rightarrow \|g\|_E \lesssim \|f\|_E\right).$$

Proof. Suppose $Kh_+(E,\mathcal{M})$. Let $f, g \in F$ such that $f^2 \succ g^2$. We can find $h \geq g \in F$ such that $f^2 \succ h^2$ and $\|f^2\| = \|h^2\|$ by lemma 1.4.2. Then by lemma 3.5.6 there exists $x \in S(\mathcal{M})$ such that $\mu(Gx) = f$ and $\mu(Rx) = h$. Hence, by $Kh_+(E,\mathcal{M})$,

$$\|f\|_E = \|Gx\|_E \gtrsim \|Rx\|_E = \|h\|_E.$$

Suppose $Kh_-(E,\mathcal{M})$. Let $f, g \in F$ such that $f^2 \triangleright g^2$. We can find $h \geq g \in F$ such that $f^2 \triangleright h^2$ and $\|h\|_2 = \|f\|_2$ again by lemma 1.4.2. So by lemma 3.5.6, there exists $x \in S(\mathcal{M})$ such that $\mu(Gx) = h$ and $\mu(Rx) = f$. By $Kh_-(E,\mathcal{M})$, $\|Gx\|_E \lesssim \|x\|_{R_E + C_E}$. In particular, $\|Gx\|_E \lesssim \|Rx\|_E$. Hence, $\|h\|_E \lesssim \|f\|_E$. Recall that $g \leq h$ so $\|g\|_E \leq \|h\|_E$ and the proof is complete. □

Lemma 3.5.8. Let E be a symmetric function space. If E is left-2-monotone then $Kh_+(E,\mathcal{M})$ holds. If furthermore, there exists $p > 0$ such that $E \in Int(L_p, L_\infty)$, E is right-2-monotone then $Kh_-(E,\mathcal{M})$ holds.

Proof. Consider the conditional expectation $\mathbb{E} : \mathcal{M}_\mathcal{A} \to \mathcal{M}$ defined by $\mathbb{E} := Id \otimes \tau_\mathcal{A}$. Note that for any $x \in S(\mathcal{M})$,

$$\mathbb{E}(|Gx|^2) = |Cx|^2 \text{ and } \mathbb{E}(|(Gx)^*|^2) = |(Rx)^*|^2. \tag{3.5}$$

Suppose that now that E is left-2-monotone. By (3.5), $|Cx|^2 \prec |Gx|^2$ and $|Rx|^2 \prec |Gx|^2$ so $\|Gx\|_E \geq \|Cx\|_E$ and $\|Gx\|_E \geq \|Rx\|_E$. Reciprocally, by remark 3.3.6 or [14], $\|Gx\|_E \lesssim \|x\|_{R_E \cap C_E}$.

Suppose that there exists $p > 0$ such that $E \in Int(L_p, L_\infty)$ and that E is right-2-monotone. Fix $x \in S(\mathcal{M})$, then by theorem 3.1.8, $\|x\|_{R_E + C_E} \lesssim \|Gx\|_E$. Reciprocally, by (3.5), $|Cy|^2 \triangleright |Gy|^2$ and $|Ry|^2 \triangleright |Gy|^2$ for all $y \in S(\mathcal{M})$. Now fix $y, z \in S(\mathcal{M})$ such that $y + z = x$. Using right-2-monotonicity,

$$\|Gx\|_E \lesssim \|Gy\|_E + \|Gz\|_E \leq \|Ry\|_E + \|Cz\|_E.$$

So E has $Kh_-(E,\mathcal{M})$. □

Proof of theorem 3.1.5. The implications from right to left are given by lemma 3.5.8. The implications from left to right are obtained by lemma 3.5.7 combined with the Fatou property and in particular its consequences given by lemmas 1.4.5 and 1.4.8. □

Chapter 4

Weak type inequality for singular integrals

4.1 Introduction

The main result of this chapter belongs to the now well developped theory of singular integrals. The latter was initiated in the 1950's by Calderón and Zygmund who found a sufficient condition for a kernel operator to be bounded on L_p for $1 < p < \infty$. It can be expressed (without details about definition) by the following theorem:

Theorem 4.1.1. *Let $n \in \mathbb{N}$, $p \in (1, \infty)$ and k a measurable function from $\mathbb{R}^{2n}$ to $\mathbb{C}$ verifying the size and smoothness conditions. Then if the formal expression :*

$$Tf(x) = \int_{y \in \mathbb{R}^n} k(x, y) f(y) dy$$

defines a bounded operator T on $L_2(\mathbb{R}^n)$, it also defines a bounded operator (still noted T) on $L_p(\mathbb{R}^n)$. T is called a Calderón-Zygmund operator and k its kernel.

Size and *smoothness* will be defined later in a more general context. For the $p = 1$ case, only weak boundedness is true in general:

Theorem 4.1.2. *With the same conditions and notations, T defines an operator on $L_1(\mathbb{R}^n)$ and there exists a constant C such that for all $f \in L_1(\mathbb{R}^n)$:*

$$\sup_{t>0} t\mu\{Tf > t\} \leq C\|f\|_1$$

where μ is the Lebesgue measure.

A motivation to show this second theorem is that it directly implies the first one by real interpolation and duality.

The ideas behind these two theorems remain valid for kernels and functions taking values in different vector spaces, which makes them a great tool to show boundedness of certain operators such as generalised Hilbert transform or Littlewood-Paley inequalities. What we prove in this chapter is a generalisation of the second theorem to noncommutative integration. Furthermore, the result has already proven to be useful since it is the main ingredient used in [7] in which it is shown that there exists a constant c such that for any Lipshitz function f and for any self-adjoints operators x and y,

$$\|f(x) - f(y)\|_{1,\infty} \leq c\|f'\|_\infty \|x-y\|_1.$$

This inequality does not directly express the weak $(1,1)$ boundedness of a Calderón-Zygmund operator but is reduced to it in the mentionned papers. We also hope that our main result is a way to tackle generalisations of classical inequalities on L_p whose proof relies on Calderón-Zygmund theory. Another approach for this kind of problem is to show BMO boundedness rather than weak boundedness and conclude thanks to interpolation theory. This strategy is often easier. It first appears in [25] to show the boundedness of some Fourier multipliers. And has also been applied in [61], where L_p-boundedness of Calderón-Zygmund operators with operator valued kernels and column valued functions is used to study Hardy spaces on quantum tori.

4.1.1 Main theorem

We will now introduce the notations that will allow us to state the main theorem of this chapter. Let $(\widetilde{\mathcal{M}}, \tau_{\widetilde{\mathcal{M}}})$ be a noncommutative measure space and $\mathcal{M}$ a von Neumann subalgebra of $\widetilde{\mathcal{M}}$ such that $\tau_{\widetilde{\mathcal{M}}}$ restricted to $\mathcal{M}$ (denoted $\tau_{\mathcal{M}}$) is semifinite. $L_p(\mathcal{M})$ is naturally included in $L_p(\widetilde{\mathcal{M}})$ for all $1 \leq p \leq \infty$. Let $\mathcal{M}'$ be the commutant of $\mathcal{M}$. We will consider $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ the von Neumann algebras of $*$-weakly measurable $\mathcal{M}$- and $\widetilde{\mathcal{M}}$-valued functions on $\mathbb{R}^n$ i.e the von Neumann tensor products $\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}^n)$ and $\widetilde{\mathcal{M}}\overline{\otimes}L_\infty(\mathbb{R}^n)$, τ will denote their natural trace (with no ambiguity since $\mathcal{N}$ is naturally included in $\widetilde{\mathcal{N}}$). Let T be a Calderón-Zygmund operator associated with a kernel $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{M}' \cap \widetilde{\mathcal{M}}$, formally given by the expression:

$$Tf(x) = \int_{y\in\mathbb{R}^n} k(x,y)f(y)dy$$

Definition 4.1.3. *Say that T has Lipschitz parameter γ ($0 < \gamma \leq 1$) if for all x,y and z in $\mathbb{R}^n$ verifying $|x-z| \leq \frac{1}{2}|y-z|$, the following* smoothness estimates *hold:*

$$\|k(x,y) - k(z,y)\|_{\widetilde{\mathcal{M}}} \leq \frac{|x-z|^\gamma}{|y-z|^{n+\gamma}},$$
$$\|k(y,x) - k(y,z)\|_{\widetilde{\mathcal{M}}} \leq \frac{|x-z|^\gamma}{|y-z|^{n+\gamma}}.$$

We will also suppose that T verifies the *size condition*, for all $x, y \in \mathbb{R}^n$:

$$\|k(x,y)\|_{\widetilde{\mathcal{M}}} \leq \frac{1}{|x-y|^n}.$$

Remark 4.1.4. This size condition will never appear throughout the proof. It is however used in remark 4.1.6. Moreover, it is implied by the smoothness condition provided that k goes to 0 at infinity.

On the contrary, the smoothness condition will be crucial to many computations. A Hörmander type condition:

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|>2|x-x'|} |k(x,y)-k(x',y)|\, dy < C$$

would not suffice to use the proof we present. We would at least need a decay of the following type:

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|>\alpha|x-x'|} |k(x,y)-k(x',y)|\, dy < \frac{C}{\alpha^n},\ \alpha > 2.$$

But we do not know if it is a strong enough condition since part of the proof of pseudo-localisation relies on pointwise estimates of the kernel.

Define, for all $t>0$ and $f \in \widetilde{\mathcal{N}}$:

$$\lambda_t(f) = \tau(\{f>t\}).$$

The main theorem we are aiming to prove in this chapter is the following.

Theorem 4.1.5. *There exists a constant $c_{n,\gamma}$ depending only on n and γ such that for all Calderón-Zygmund operator T with $(\mathcal{M}'\cap\widetilde{\mathcal{M}})$-valued kernel and Lipschitz parameter $0<\gamma\leq 1$, verifying the size estimate and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$, for all $f\in L_1(\mathcal{N})$:*

$$\sup\nolimits_{t>0} t\lambda_t(Tf) \leq c_{n,\gamma}\|f\|_1$$

To prove this theorem, we will need a pseudo-localisation result which constitutes the first part of the chapter. This is where the most important simplifications are made compared to J. Parcet's work. In particular, our proof is elementary and does not explicitely use the size condition. We decide to directly show the result with noncommutative variables but the pseudo-localisation theorem is essentially a commutative one. The second part, which is the proof of the main theorem has been shortened and clarified thanks to more efficient organisation and computations but the underlying ideas all appear in [41]. In particular, we use the same decomposition which relies on Cuculescu's projections. It is a natural noncommutative counterpart to the decomposition used for the classical proof but unfortunately, new kind of terms appear which will necessitate more refined estimates and in particular, the pseudo-localisation theorem. In a third section, we show a boundedness result for singular integrals with Hilbert valued kernels and operator valued functions. It was already shown in [39] by J. Parcet and T. Mei but follows directly from theorem 4.1.5 and Khintchine inequalities. We conclude the chapter by providing an L_p-pseudo-localisation theorem similar to the one of P. Hytönen in [22]. The result follows mainly from our proof of L_2-pseudo-localisation, martingale inequalities and interpolation.

4.1.2 A technical remark and notations

The following remark allows us to manipulate the integral expression of T, in particular to use the smoothness condition.

Remark 4.1.6. For any Calderón-Zygmund operator T there exists a Calderón-Zygmund operator T' such that $Tf = T'f + gf$ for all f in $L_2(\mathcal{N})$ and T' is the strong limit in $\mathcal{B}(L_2(\mathcal{N}))$ of operators T_i given by:

$$T_i f(x) = \int_{y \in \mathbb{R}^n} k_i(x,y) f(y) dy,$$

for any $f \in L_2(\mathcal{N})$. Here, k_i can be taken as a truncation of k so that the integral makes sense and g is a bounded function in $\widetilde{\mathcal{N}}$. This fact is explained in more details in [17], proposition 8.1.11. As a consequence, it suffices to prove the theorem for operators given by a converging integral formula on $L_2(\mathcal{N})$.

The proof of the main theorem will rely on the use of dyadic martingales which will require a few notations.

- Since we deal with cubes, the ∞-norm will be easier to manipulate than the euclidean one. Hence, for all $x \in \mathbb{R}^n$, we set $|x| = \left\|x\right\|_\infty$ in the remainder of the text.
- $\mathcal{Q}$ will denote the set of all dyadic cubes and $\mathcal{Q}_k$ the set of dyadic cubes of edge length 2^{-k}. Let $V_k = 2^{-nk}$ be the volume of such a cube.
- For all x in $\mathbb{R}^n$, $Q_{x,k}$ will denote the cube in $\mathcal{Q}_k$ containing x and $c_{x,k}$ its center.
- Let $(\mathbb{E}_k)_{k\in\mathbb{Z}}$ be the martingale associated with dyadic filtration, i.e for all f :

  $$\mathbb{E}_k(f)(x) = \tfrac{1}{V_k} \textstyle\int_{Q_{x,k}} f(t)dt$$

  For convenience, we will write $f_k := \mathbb{E}_k(f)$ and $\Delta_k(f) := f_k - f_{k-1} =: df_k$. The filtration associated with these expectations will be denoted by $(\mathcal{N}_k)_{k\in\mathbb{Z}}$ where $\mathcal{N}_k$ is the von Neumann subalgebra of $\mathcal{N}$ constituted of functions that are constant on cubes of edge length 2^{-k}.
- For any odd positive integer i and Q in $\mathcal{Q}_k$, iQ will designate the image of Q by the homothety of center c_Q and parameter i such that iQ is the union of i^n cubes in $\mathcal{Q}_k$.
- Notice that for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, $x \in iQ_{y,k} \Leftrightarrow y \in iQ_{x,k}$.

The notation $A \lesssim B$ will stand for "there exists a constant c depending only on n and γ such that $A \leq cB$".

In the next section, we will frequently use "polar" changes of coordinates with respect to the norm $|.|$ since it is more adapted to our problem. The spherical element of volume is replaced by the border of a cube which leads to a similar formula:

$$\textstyle\int_{\mathbb{R}^n} f(|x|)dx = \int_{\mathbb{R}^+} 2n(2r)^{n-1} f(r)dr.$$

4.2 Pseudo-localisation

4.2.1 Theorem

A localisation result would be of the form supp $Tf \approx$ supp f which is ideal to show weak boundedness. The theorem that follows expresses in a way the fact that singular integrals rapidly vanish outside of the support of the function to which they are applied. A simpler result of this type appears in the commutative proof in the L_1-context and is enough to conclude in this case. But, as mentionned before, the noncommutative case requires new tools such as the L_2-version of pseudo-localisation that follows. Note that the proof is written with operator-valued functions because we will need this result later but is almost a copy of the proof we had with scalar functions. So, suprisingly, the most technical part of the proof of our main theorem is purely commmutative.

Theorem 4.2.1. *Let $f \in L_2(\mathcal{N})$ and $s \in \mathbb{N}$. F or all $k \in \mathbb{Z}$, let A_k and B_k be projections in $\mathcal{N}_k$ such that $A_k^\perp df_{k+s} = df_{k+s}B_k^\perp = 0$. For any odd positive integer d, write:*

$$A_k = \textstyle\sum_{Q\in\mathcal{Q}_k} A_Q\mathcal{I}_Q,\ A_Q \in \mathcal{M} \text{ and define } dA_k := \bigvee_{Q\in\mathcal{Q}_k} A_Q\mathcal{I}_{dQ}.$$

Define dB_k the same way. Let:

$$A_{f,s} := \bigvee_{k\in\mathbb{Z}} 5A_k \text{ and } B_{f,s} := \bigvee_{k\in\mathbb{Z}} 5B_k. \tag{4.1}$$

Let T be a Calderón-Zygmund operator associated with a kernel k with Lipschitz parameter γ, verifying the size condition, taking values in $\mathcal{M}' \cap \widetilde{\mathcal{M}}$ and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$. Then for all $s \in \mathbb{N}$ and $f \in L_2(\mathcal{N})$ we have:

$$\big\|A_{f,s}^\perp(Tf)\big\|_2 \lesssim 2^{-\frac{\gamma s}{2}}\big\|f\big\|_2 \text{ and } \big\|(Tf)B_{f,s}^\perp\big\|_2 \lesssim 2^{-\frac{\gamma s}{2}}\big\|f\big\|_2.$$

Throughout the course of the proof we will often use the following lemma:

Lemma 4.2.2 (Schur)**.** Let T be an operator on $L_2(\mathcal{N})$ given by a $\widetilde{\mathcal{M}}$-valued kernel:

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy.$$

Let $S_1(x) = \int_{\mathbb{R}^n}\big\|k(x,y)\big\|_{\widetilde{\mathcal{M}}}dy$ and $S_2(y) = \int_{\mathbb{R}^n}\big\|k(x,y)\big\|_{\widetilde{\mathcal{M}}}dx$ then :

$$\big\|T\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \le \sqrt{\big\|S_1\big\|_\infty\big\|S_2\big\|_\infty}.$$

Proof. This is not different from the commutative case (see [41]). Let $f \in L_2(\mathcal{N})$:

$$\begin{aligned}
\|Tf\|_2^2 &= \int_{\mathbb{R}^n} \Big\| \int_{\mathbb{R}^n} k(x,y)f(y)dy\Big\|_2^2 dx \\
&\leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \|k(x,y)f(y)\|_2 dy\Big)^2 dx \\
&\leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \|k(x,y)\|_{\widetilde{\mathcal{M}}} \|f(y)\|_2 dy\Big)^2 dx \\
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|k(x,y)\|_{\widetilde{\mathcal{M}}} dy \int_{\mathbb{R}^n} \|k(x,y)\|_{\widetilde{\mathcal{M}}} \|f(y)\|_2^2 dy dx \\
&\leq \|S_1\|_\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|k(x,y)\|_{\widetilde{\mathcal{M}}} \|f(y)\|_2^2 dy dx \\
&\leq \|S_1\|_\infty \|S_2\|_\infty \int_{\mathbb{R}^n} \|f(y)\|_2^2 dy = \|S_1\|_\infty \|S_2\|_\infty \|f\|_2^2
\end{aligned}$$

□

It will also be important to note that by construction, for all $x, y \in \mathbb{R}^n$ such that $x \in 5Q_{k,y}$, we have $A_k(y) \leq (5A_k)(x) \leq A_{f,s}(x)$ and $A_k^\perp(y) df_{k+s}(y) = 0$. Consequently:

$$A_{f,s}^\perp(x) df_{k+s}(y) = (5A_k)^\perp(x) df_{k+s}(y) = 0. \tag{4.2}$$

We will only show the pseudo-localisation theorem in the case of left multiplications since the exact same proof can be writen for right multiplication. Another way of seeing it is that left and right multiplication are equivalent by taking adjoints. The general strategy will be to find an operator T' verifying $A_{f,s}^\perp Tf = A_{f,s}^\perp T'f$ and such that we can control $\|T'\|_{\mathcal{B}(L_2(\mathcal{N}))}$ thanks to the lemma above. By remark 4.1.6, we can also suppose that the integral defining T converges. The result for any T follows then directly by approximations.

4.2.2 The s shift

Let $f \in L_2(\mathcal{N})$, fixed throughout the proof. Let $s \in \mathbb{N}$. To make use of the construction of $A_{f,s}$ we immediately write $T = \sum_{k\in\mathbb{Z}} T\Delta_{k+s}$ where the sum converges for the strong operator topology on $\mathcal{B}(L_2(\mathcal{N}))$. We will show in this section that the constant $2^{-\gamma s}$ appears quite easily thanks to the smoothness condition when estimating the norm of $T\Delta_{k+s}f$. The tricky part will be to glue these pieces back together in the following sections. Let $k \in \mathbb{Z}$, note that :

$$\int_{\mathbb{R}^n} k(x, c_{y,k+s-1}) df_{k+s}(y) dy = \int_{\mathbb{R}^n} \mathbb{E}_{k+s-1}\big(k(x, c_{.,k+s-1}) df_{k+s}(.)\big)(y) dy = 0.$$

Therefore we can write:

$$\begin{aligned}
A_{f,s}^\perp(x) T df_{k+s}(x) &= A_{f,s}^\perp(x) \int_{\mathbb{R}^n} k(x,y) df_{k+s}(y) dy \\
&= A_{f,s}^\perp(x) \int_{\mathbb{R}^n} (k(x,y) - k(x, c_{y,k+s-1})) df_{k+s}(y) dy
\end{aligned}$$

$A_k^\perp(y)$ commutes with $k(x,y)$ and $k(x,c_{y,k+s-1})$ since k is $\mathcal{M}'$-valued so:

$$A_{f,s}^\perp(x)Tdf_{k+s}(x) = A_{f,s}^\perp(x)\int_{\mathbb{R}^n}(k(x,y)-k(x,c_{y,k+s-1}))(5A_k)^\perp(x)df_{k+s}(y)dy$$

and by (4.2):

$$= A_{f,s}^\perp(x)\int_{\mathbb{R}^n}\mathcal{I}_{x\notin 5Q_{k,y}}(k(x,y)-k(x,c_{y,k+s-1}))df_{k+s}(y)dy.$$

Denote by T_k the operator associated with the kernel

$$k_k : (x,y) \mapsto \mathcal{I}_{x\notin 5Q_{k,y}}k(x,y) \tag{4.3}$$

and $T_{k,s}$ the operator associated with the kernel

$$k_{k,s} : (x,y) \mapsto \mathcal{I}_{x\notin 5Q_{k,y}}(k(x,y)-k(x,c_{y,k+s-1})). \tag{4.4}$$

It will be useful to express the result of the previous computation in terms of these operators:

Lemma 4.2.3. For all $k \in \mathbb{Z}$ and $s \in \mathbb{N}$:

$$A_{f,s}^\perp T\Delta_{k+s}f = A_{f,s}^\perp T_k\Delta_{k+s}f = A_{f,s}^\perp T_{k,s}\Delta_{k+s}f, \tag{4.5}$$

and more precisely:

$$T_k\Delta_{k+s}f = T_{k,s}\Delta_{k+s}f. \tag{4.6}$$

The introduction of $T_{k,s}$ is motivated by the following lemma.

Lemma 4.2.4. For all $k \in \mathbb{Z}$ and $s \in \mathbb{N}$:

$$\left\|T_{k,s}\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-s\gamma}. \tag{4.7}$$

Proof. The smoothness hypothesis on T (see definition 4.1.3) gives:

$$\begin{aligned}\left\|k_{k,s}(x,y)\right\|_{\widetilde{\mathcal{M}}} &= \mathcal{I}_{x\notin 5Q_{k,y}}\left\|k(x,y)-k(x,c_{y,k+s-1})\right\|_{\widetilde{\mathcal{M}}}\\ &\leq \mathcal{I}_{x\notin 5Q_{k,y}}\frac{\left|y-c_{y,k+s-1}\right|^\gamma}{\left|y-x\right|^{n+\gamma}}\\ &\lesssim \mathcal{I}_{x\notin 5Q_{k,y}}\frac{2^{-(k+s)\gamma}}{\left|y-x\right|^{n+\gamma}}.\end{aligned} \tag{4.8}$$

The condition $|y-c_{y,k+s-1}| \leq \frac{1}{2}|x-y|$ is verified even for $s=0$ as long as $x \notin 5Q_{k,y}$. This estimate is enough to apply Schur's lemma and we obtain the result by a direct computation using a "polar" change of coordinates.

□

4.2.3 One more cancellation property

We introduce another kernel modification whose purpose is not immediately clear but will be crucial in obtaining the estimates to apply Schur's lemma later.

Proposition 4.2.5. *For all $k \in \mathbb{Z}$ and $s \in \mathbb{N}$ there exists an operator $S_{k,s}$ associated with a kernel $s_{k,s}$ such that the three following conditions hold:*

1. $A^{\perp}_{f,s} T \Delta_{k+s} f = A^{\perp}_{f,s} S_{k,s} \Delta_{k+s} f$,

2. $\int_{y \in \mathbb{R}^n} s_{k,s}(x,y)dy = 0$ *for all* $x \in \mathbb{R}^n$,

3. $\left\| s_{k,s}(x,y) \right\|_{\widetilde{\mathcal{M}}} \lesssim \mathcal{I}_{|x-y|>2^{-k-1}} \dfrac{2^{-(k+s)\gamma}}{|y-x|^{n+\gamma}}$.

Proof. Consider $R_{k,s}$ an operator associated with the kernel $r_{k,s}$ defined by:

$$r_{k,s}(x,y) := \mathcal{I}_{\mathcal{A}_k} \frac{K(x) 2^{-(k+s)\gamma}}{I_{n,\gamma} |y-x|^{n+\gamma}}$$

where $\mathcal{A}_k := \{(x,y) : x \in 5Q_{y,k}, x \notin 3Q_{y,k}\}$, $K(x) = -\int_{\mathbb{R}^n} k_{k,s}(x,t)dt$ and $I_{n,\gamma} = \int_{\mathbb{R}^n} \mathcal{I}_{\mathcal{A}_k}(0,t) \dfrac{2^{-(k+s)\gamma}}{|t|^{n+\gamma}} dt$.

We claim that $S_{k,s} := T_{k,s} + R_{k,s}$ satisfies the conditions above.

Condition 1. $A^{\perp}_{f,s} R_{k,s} \Delta_{k+s} f = 0$ is verified since:

$$\text{supp } r_{k,s} \subset \{(x,y) : x \in 5Q_{y,k}\} \text{ i.e } r_{k,s} = \mathcal{I}_{x \in 5Q_{y,k}} r_{k,s}.$$

Indeed, the shift section shows that to compute $A^{\perp}_{f,s} R_{k,s} \Delta_{k+s} f$, and in particular $(5A_k)^{\perp} R_{k,s} \Delta_{k+s} f$, we might as well replace the kernel $r_{k,s}$ by $\mathcal{I}_{x \notin 5Q_{k,y}} r_{k,s} = \mathcal{I}_{x \notin 5Q_{k,y}} \mathcal{I}_{x \in 5Q_{y,k}} r_{k,s} = 0$.

Condition 2. It is direct by definition of K and $I_{n,\gamma}$. Let $x \in \mathbb{R}^n$:

$$\int_{y \in \mathbb{R}^n} s_{k,s}(x,y)dy = \int_{\mathbb{R}^n} k_{k,s}(x,y)dy + K(x) = 0.$$

Condition 3. It suffices to show that $\left\| K(x) \right\|_{\widetilde{\mathcal{M}}}$ is bounded by a constant depending only on n and γ. By a "polar" change of coordinates, there exists a constant c_n such that:

$$\left\| K(x) \right\|_{\widetilde{\mathcal{M}}} \leq \int_{|t|>4.2^{-k}} \frac{2^{-(k+s)\gamma}}{|t|^{n+\gamma}} dt = c_n 2^{-\gamma s}.$$

The bound does not depend on x, as we needed.

□

4.2.4 The decomposition

We are ready to decompose T into operators whose norms in $\mathcal{B}(L_2(\mathcal{N}))$ are controlled. Fix $s \in \mathbb{N}$, write:

$$\begin{aligned} A_{f,s}^{\perp} T f &= A_{f,s}^{\perp} \sum_{k\in\mathbb{Z}} T\Delta_{k+s} f = A_{f,s}^{\perp} \sum_{k\in\mathbb{Z}} (T_k + R_{k,s})\Delta_{k+s} f \\ &= A_{f,s}^{\perp} \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \Delta_{k+i}(T_k + R_{k,s})\Delta_{k+s} f \end{aligned}$$

Note that for $i \geq 1$, $(5A_k)^{\perp}$ commutes with Δ_{k+i} and recall that the first point of $R_{k,s}$'s construction implies that $(5A_k)^{\perp} R_{k,s}\Delta_{k+s} f = 0$, then:

$$\begin{aligned} A_{f,s}^{\perp}\Delta_{k+i}(T_k + R_{k,s})\Delta_{k+s} f &= A_{f,s}^{\perp}\Delta_{k+i}(5A_k)^{\perp}(T_k + R_{k,s})\Delta_{k+s} f \\ &= A_{f,s}^{\perp}\Delta_{k+i} T_k \Delta_{k+s} f \end{aligned}$$

For any $i \in \mathbb{Z}$, we have:

$$\begin{aligned} A_{f,s}^{\perp}\Delta_{k+i}(T_k + R_{k,s})\Delta_{k+s} f &= A_{f,s}^{\perp}\Delta_{k+i}(T_{k,s} + R_{k,s})\Delta_{k+s} f \\ &= A_{f,s}^{\perp}\Delta_{k+i} S_{k,s} \Delta_{k+s} f \end{aligned}$$

Now we can write our final decomposition:

$$A_{f,s}^{\perp} T f = A_{f,s}^{\perp}\big(\sum\nolimits_{i=0}^{\infty} \Phi_i f + \sum\nolimits_{i=1}^{\infty} \Psi_i f\big) \tag{4.9}$$

where:

$$\Phi_i = \sum\nolimits_{k\in\mathbb{Z}} \Delta_{k+i} T_k \Delta_{k+s} \text{ and } \Psi_i = \sum\nolimits_{k\in\mathbb{Z}} \Delta_{k-i} S_{k,s} \Delta_{k+s}$$

We have then reduced our pseudo-localisation theorem to the following proposition.

Proposition 4.2.6. *The following estimates hold for $i \geq 0$:*

$$\big\|\Phi_i\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma\frac{i+s}{2}} \textit{ and } \big\|\Psi_i\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i}\,2^{-\gamma\frac{i+s}{2}}$$

The proof of this result will occupy the next two parts.

4.2.5 Estimate for Φ_i

Let $i \geq 0$. Note that T^* is also a Calderón-Zygmund operator, associated with the kernel $k^\star$ where: $k^\star(x,y) = k(y,x)^*$.

So $k^\star$ satisfies the smoothness condition of parameter γ, which means by (4.7) that:

$$\big\|(T^*)_{k,s}\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-s\gamma}.$$

Consequently :

$$\left\|\Delta_{k+i}T_k\right\|_{\mathcal{B}(L_2(\mathcal{N}))} = \left\|T_k^*\Delta_{k+i}\right\|_{\mathcal{B}(L_2(\mathcal{N}))} = \left\|(T^*)_{k,i}\Delta_{k+i}\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma i},$$

where we used that $(T_k)^* = (T^*)_k$ and (4.6).

By orthogonality of the Δ_k, we have on one hand:

$$\left\|\Phi_i\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \left\|\Delta_{k+i}T_k\Delta_{k+s}\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \left\|\Delta_{k+i}T_k\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma i},$$

and on the other hand :

$$\left\|\Phi_i\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \left\|\Delta_{k+i}T_k\Delta_{k+s}\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \left\|T_k\Delta_{k+s}\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma s}.$$

By combining the two, $\left\|\Phi_i\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma\frac{s+i}{2}}$.

4.2.6 Estimate for Ψ_i

Lemma 4.2.7. For all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, the following estimate holds:

$$\int_{t\in Q_{x,k-i}} \int_{y\in Q^c_{x,k-i}} \mathcal{I}_{|t-y|>2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dydt \lesssim (1+i)2^{(\gamma-n)(k-i)}.$$

Proof. Fix $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $i \in \mathbb{N}$. Denote $Q = Q_{x,k-i}$, c the center of Q and:

$$X = \int_{t\in Q} \int_{y\in Q^c} \mathcal{I}_{|t-y|>2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dydt.$$

For every t in Q let δ_t be the distance from t to Q^c and notice that for any t:

$$\int_{Q^c} \mathcal{I}_{|t-y|>2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dy \leq \int_{|y-t|>\delta_t} \mathcal{I}_{|t-y|>2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dy =: f(\delta_t)$$

Indeed, the term on the right only depends on δ_t. For $r \leq 2^{-(k-i+1)}$:

$$\{t \in Q : \delta_t = r\} = \{t \in Q : |t-c| = 2^{-(k-i+1)} - r =: r'\},$$

So by a "polar" change of coordinates:

$$X \leq \int_0^{2^{-(k-i+1)}} 2n(2r')^{n-1} f(r)dr \lesssim 2^{-(k-i)(n-1)} \int_0^{2^{-(k-i+1)}} f(r)dr.$$

By a direct computation:

$$f(r) \lesssim \begin{cases} r^{-\gamma} & \text{for all } r \\ 2^{\gamma(k-1)} & \text{if moreover } r \leq 2^{-(k-1)} \end{cases}$$

For $0 < \gamma < 1$ we only need the first estimate:

$$
\begin{aligned}
X &\lesssim 2^{-(k-i)(n-1)} \int_0^{2^{-(k-i+1)}} f(r)dr & &\lesssim 2^{-(k-i)(n-1)} \int_0^{2^{-(k-i+1)}} r^{-\gamma} dr \\
&\lesssim 2^{-(k-i)(n-1)} 2^{-(k-i+1)(1-\gamma)} & &\lesssim 2^{-n(k-i)} 2^{\gamma(k-i)}
\end{aligned}
$$

For $\gamma = 1$ we have to decompose the integral according to the distinction made above when estimating f:

$$
\begin{aligned}
\int_0^{2^{-(k-i+1)}} f(r)dr &= \int_0^{2^{-(k-1)}} f(r)dr + \int_{2^{-(k-1)}}^{2^{-(k-i+1)}} f(r)dr \\
&\lesssim 2^{-(k-1)} 2^{\gamma(k-1)} + \int_{2^{-(k-1)}}^{2^{-(k-i+1)}} \frac{1}{r} dr \\
&\lesssim 1 + i
\end{aligned}
$$

Therefore:

$$
X \lesssim 2^{-(k-i)(n-1)}(1+i) = (1+i)2^{-n(k-i)}2^{\gamma(k-i)}.
$$

$\square$

Proposition 4.2.8. *For all k in $\mathbb{Z}$ and $i > 0$, we have the following estimate:*

$$
\left\|\mathcal{E}_{k-i} S_{k,s}\right\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i}\, 2^{-\gamma(s+i/2)}
$$

Proof. For any function g : $\mathcal{E}_{k-i} S_{k,s} g(x) = \int_{Q_{x,k-i}} \frac{1}{V_{k-i}} \int_{\mathbb{R}^n} s_{k,s}(t,y) g(y) dy dt$. So $\mathcal{E}_{k-i} S_{k,s}$ corresponds to the kernel:

$$
E : (x,y) \mapsto \frac{1}{V_{k-i}} \int_{Q_{x,k-i}} s_{k,s}(t,y)dt = \frac{1}{V_{k-i}} \int_{Q^c_{x,k-i}} s_{k,s}(t,y)dt,
$$

where we used the cancellation property on $S_{k,s}$. We are looking to estimate both integrals in order to apply Schur's lemmma. Fix x:

$$
\begin{aligned}
\int_{\mathbb{R}^n} \left\|E(x,y)\right\|_{\widetilde{\mathcal{M}}} dy &= \int_{Q_{x,k-i}} \left\|E(x,y)\right\|_{\widetilde{\mathcal{M}}} dy + \int_{Q^c_{x,k-i}} \left\|E(x,y)\right\|_{\widetilde{\mathcal{M}}} dy \\
&= \int_{Q_{x,k-i}} \Big\| \int_{Q^c_{x,k-i}} \frac{1}{V_{k-i}} s_{k,s}(t,y)dt \Big\|_{\widetilde{\mathcal{M}}} dy \\
&+ \int_{Q^c_{x,k-i}} \Big\| \int_{Q_{x,k-i}} \frac{1}{V_{k-i}} s_{k,s}(t,y)dt \Big\|_{\widetilde{\mathcal{M}}} dy \\
&\lesssim \frac{1}{V_{k-i}} \int_{Q_{x,k-i}} \int_{Q^c_{x,k-i}} \mathcal{I}_{|t-y|>2^{1-k}} \frac{2^{-\gamma(k+s)}}{|t-y|^{n+\gamma}} dt dy
\end{aligned}
$$

Using lemma 4.2.7 :

$$
\begin{aligned}
\int_{\mathbb{R}^n} \left\|E(x,y)\right\|_{\widetilde{\mathcal{M}}} dy &\lesssim \frac{1}{V_{k-i}} (1+i) 2^{-n(k-i)} 2^{\gamma(k-i)} 2^{-\gamma(k+s)} \\
&\lesssim (1+i) 2^{-\gamma(i+s)}
\end{aligned}
$$

The other estimate, with y fixed, is straightforward:

$$\begin{aligned}\int_{\mathbb{R}^n} \big\|E(x,y)\big\|_{\widetilde{\mathcal{M}}} dx &= \int_{\mathbb{R}^n} \Big\|\frac{1}{V_{k-i}} \int_{Q_{x,k-i}} s_{k,s}(t,y)dt\Big\|_{\widetilde{\mathcal{M}}} dx \\ &\leq \int_{\mathbb{R}^n} \frac{1}{V_{k-i}} \int_{Q_{x,k-i}} \big\|s_{k,s}(t,y)\big\|_{\widetilde{\mathcal{M}}} dt dx \\ &= \int_{\mathbb{R}^n} \big\|s_{k,s}(t,y)\big\|_{\widetilde{\mathcal{M}}} dt \\ &\lesssim 2^{-\gamma s}\end{aligned}$$

The proposition follows directly from the two previous computations and Schur's lemma.

□

We now have:

$$\big\|\Delta_{k-i}S_{k,s}\Delta_{k+s}\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \big\|\mathcal{E}_{k-i}S_{k,s}\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i}\ 2^{-\gamma\frac{s+i}{2}}.$$

We conclude once more thanks to the orthogonality between the $\Delta_{k-i}S_{k,s}\Delta_{k+s}$:

$$\big\|\Psi_i\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sup_k \big\|\Delta_{k-i}S_{k,s}\Delta_{k+s}\big\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i}\ 2^{-\gamma\frac{s+i}{2}}$$

This concludes the proof of the pseudo-localisation theorem.

4.3 Proof of the main theorem

4.3.1 The good and the bad functions

The idea of the following decomposition comes from the classical proof of the weak type inequality for singular integrals. It has been noticed that the commutative decomposition can be expressed in terms of martingales which is well suited for a translation in the noncommutative setting. However, even with this idea, the construction is not immediate and the estimates are more difficult to obtain due to the appearance of new "off-diagonal" terms.

Fix $t > 0$ and $f \in L_1(\mathcal{N})$. We will suppose that f is positive to avoid unnecessary computations. This is possible because f can always be written as:

$$f = f_1 - f_2 + if_3 - if_4$$

with $\big\|f_k\big\|_1 \leq \big\|f\big\|_1$ for $k = 1,2,3,4$. Suppose also that $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ is bounded (in other words, f, considered as a function, has a compact support), and that f is in $L_2(\mathcal{N}) \cap \mathcal{N}$. We can make these assumptions since the functions satisfying them form a dense subspace $\mathcal{S}$ of $L_1(\mathcal{N})$. Once the theorem is proven, T can be defined on $L_1(\mathcal{N})$ as the only bounded extension of its restriction to $\mathcal{S}$.

Denote by $(f_n)_{n\in\mathbb{Z}}$ the martingale associated with f and the filtration $(\mathcal{N}_n)_{n\in\mathbb{Z}}$. The main tools to decompose f into a good and a bad part are Cuculescu's projections:

Theorem 4.3.1 (Cuculescu)**.** *Let $x = (x_n)$ be a bounded positive L_1-martingale and $t \geq 0$. Then there exists a decreasing sequence (q_n) of projections in $\mathcal{N}$ such that for every $n \geq 1$:*

1. *$q_n \in \mathcal{N}_n$.*
2. *q_n commutes with $q_{n-1}x_nq_{n-1}$.*
3. *$q_nx_nq_n \leq t$.*
4. *moreover, if $q = \bigwedge q_n$ then :*

$$qx_nq \leq t \text{ for } n \geq 1 \text{ and } \tau(q^\perp) \leq \frac{\|x\|_1}{t}.$$

The boundedness hypothesis on f and its support imply that there exists n_0 such that for all $n \leq n_0$, $f_n \leq t$. Without loss of generality, we will suppose that $n_0 = 0$. From now on, let q_n denote the projections given by Cuculescu's theorem, associated with t and $(f_n)_{n\geq 0}$. To complete this definition let $q_n = 1$ for all $n \leq 0$. Notice that for all $n \in \mathbb{Z}$, $q_nf_nq_n \leq t$.

Define :

$$\forall n \in \mathbb{Z}, p_n = q_{n-1} - q_n \text{ and } p_\infty = q.$$

Let $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$. By definition, $p_n \in \mathcal{N}_n$ and:

$$\textstyle\sum_{n\in\overline{\mathbb{Z}}} p_n = 1.$$

Which allows us to define the good and bad parts as follows :

$$g = \textstyle\sum_{i\in\overline{\mathbb{Z}}}\sum_{j\in\overline{\mathbb{Z}}} p_if_{i\vee j}p_j \text{ and } b = \sum_{i\in\overline{\mathbb{Z}}}\sum_{j\in\overline{\mathbb{Z}}} p_i(f - f_{i\vee j})p_j.$$

By properties of the distribution function (see [16]):

$$\lambda_t(Tf) \lesssim \lambda_{t/2}(Tg) + \lambda_{t/2}(Tb)$$

So it suffices to prove estimates of the form:

$$\lambda_t(Tx) \lesssim \frac{\|f\|_1}{t}$$

for both $x = g$ and $x = b$. This is the purpose of the next sections.

Remark 4.3.2. The same formula for b and g works in the commutative case except that only the diagonal terms are non zero. This explains why the commutative proof can only be repeated for the diagonal terms and "shifted" diagonals. What makes this decomposition work is that, due to the pseudo-localisation lemma, the estimates for "shifted" diagonals get exponentially better as the shift increases.

We will use the following notation: for all k and $Q \in \mathcal{Q}_k$, $p_Q := p_k(x)$ for any $x \in Q$. We will also need two lemmas which are directly deduced from the construction.

Lemma 4.3.3. For all $k \in \mathbb{Z}$, we have : $f_{k+1} \leq 2^n f_k$.

Proof. This is straightforward from the definition of f_k and positivity of f. Let $x \in \mathbb{R}^n$:

$$2^n f_k(x) = \frac{2^n}{V_k} \int_{Q_{x,k}} f(t)dt \geq \frac{2^n}{V_k} \int_{Q_{x,k+1}} f(t)dt = f_{k+1}(x).$$

□

Lemma 4.3.4. Let d be an odd positive integer. Define:

$$\zeta_d = \big(\bigvee_{Q \in \mathcal{Q}} p_Q \mathcal{I}_{dQ}\big)^{\perp}.$$

Then :

1. $\tau(\zeta_d^{\perp}) \leq d^n \dfrac{\|f\|_1}{t}$
2. For all cubes $Q \in \mathcal{Q}$, we have the following cancellation property:

$$x \in dQ \to \zeta_d(x) p_Q = p_Q \zeta_d(x) = 0.$$

Proof. The first estimate is a consequence of Cuculescu's inequality:

$$\tau(\zeta_d^{\perp}) \leq \sum_{Q \in \mathcal{Q}} d^n \tau(p_Q \mathcal{I}_Q) = d^n \sum_{k=1}^{\infty} \tau(p_k) = d^n \tau(q^{\perp}) \leq d^n \frac{\|f\|_1}{t}.$$

Let $Q \in \mathcal{Q}$ and $x \in dQ$. By construction, $\zeta_d^{\perp}(x) \geq p_Q$ so $\zeta_d(x) \leq p_Q^{\perp}$. This concludes the proof.

□

From now on, we fix $d = 5$ and denote ζ_d by ζ.

Remark 4.3.5. This projection ζ is to be thought as a dilatation of the support of the bad function which already plays a crucial role in the commutative setting.

4.3.2 Estimate for the bad function

The strategy of proof is to write:

$$Tb = \zeta Tb\zeta + (1 - \zeta)Tb\zeta + \zeta Tb(1 - \zeta) + (1 - \zeta)Tb(1 - \zeta).$$

Therefore, lemma 4.3.4 and Tchebychev's inequality give:

$$\lambda_t(Tb) \lesssim \tau(1 - \zeta) + \lambda_t(\zeta Tb\zeta) \lesssim \frac{\|f\|_1}{t} + \frac{\|\zeta Tb\zeta\|_1}{t}.$$

The estimate for the bad function is now reduced to the following proposition:

Proposition 4.3.6. *We have the estimate:* $\|\zeta Tb\zeta\|_1 \lesssim \|f\|_1$.

The proof will require three intermediate lemmas.

Define, for all $i,j\in\mathbb{Z}: b_{i,j}=p_i(f-f_{i\vee j})p_j$.

Lemma 4.3.7. For all $s\in\mathbb{Z}:\sum_{i-j=s}\|b_{i,j}\|_1\lesssim\|f\|_1$

Proof. Let $i,j\in\mathbb{Z}$:

$$\|b_{i,j}\|_1\le\|p_ifp_j\|_1+\|p_if_{i\vee j}p_j\|_1$$

By Holder's inequality :

$$\begin{aligned}\|b_{i,j}\|_1&\le\|f^{1/2}p_i\|_2\|f^{1/2}p_j\|_2+\|f_{i\vee j}^{1/2}p_i\|_2\|f_{i\vee j}^{1/2}p_j\|_2\\&\le\frac{1}{2}(\|p_ifp_i\|_1+\|p_jfp_j\|_1+\|p_if_{i\vee j}p_i\|_1+\|p_jf_{i\vee j}p_j\|_1)\\&\le\tau(p_if)+\tau(p_jf)\end{aligned}$$

Consequently, for all $s\in\mathbb{Z}$:

$$\sum_{i-j=s}\|b_{i,j}\|_1\le 2\sum_{i\in\mathbb{Z}}\tau(p_if)\lesssim\|f\|_1$$

□

Lemma 4.3.8. The following cancellation properties hold:

- for all $i,j\in\mathbb{Z}$ and $Q\in\mathcal{Q}_{i\vee j}$: $\int_Q b_{i,j}=0$;
- for all $x,y\in\mathbb{R}^n$ such that $y\in 5Q_{x,i\wedge j}$: $\zeta(x)b_{i,j}(y)\zeta(x)=0$.

Proof. The first point is straightforward. Since $Q\in\mathcal{Q}_{i\vee j}$: $\int_Q b_{i,j}=\int_Q\mathbb{E}_{i\vee j}(b_{i,j})=0$.

The second one is a consequence of the construction of ζ expressed in property 4.3.4. Indeed, for all $x,y\in\mathbb{R}^n$ and $k\in\mathbb{Z}$ such that $y\in 5Q_{x,k}$ we know that $\zeta(x)p_k(y)=p_k(y)\zeta(x)=0$. Recall that $b_{i,j}=p_i(f-f_{i\vee j})p_j$, it is now clear that $\zeta(x)b_{i,j}(y)\zeta(x)=0$ for $y\in 5Q_{x,k}$ and $k=i,j$ which concludes the proof of the lemma. Note that $Q_{x,k}\subset Q_{x,k'}$ for $k\ge k'$, so we do not lose anything by taking $k=i\wedge j$.

□

The following lemma is the core of the bad function estimate, it relies on a computation which allows us to make use of the smoothness condition.

Lemma 4.3.9. For all $i,j\in\mathbb{Z}$: $\|\zeta Tb_{i,j}\zeta\|_1\lesssim 2^{-|i-j|\gamma}\|b_{i,j}\|_1$.

Proof. Note that $b_{i,j}$ is in $L_2(\mathcal{N})$. Fix $i,j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, recall that $k(x,y)$ commutes with $\zeta(x)$ since ζ takes values in $\mathcal{M}$ and k in $\mathcal{M}'$:

$$\begin{aligned}\zeta(x)Tb_{i,j}(x)\zeta(x)&=\int_{y\in\mathbb{R}^n}k(x,y)\zeta(x)b_{i,j}(y)\zeta(x)dy\\&=\int_{y\in 5Q^c_{x,i\wedge j}}\big(k(x,y)-k(x,c_{y,i\vee j})\big)\zeta(x)b_{i,j}(y)\zeta(x)dy\end{aligned}$$

where we used both cancellation properties of $b_{i,j}$ (lemma 4.3.8) the first one to make the term $k(x, c_{y,i\vee j})$ appear and the second one to reduce the domain of integration. Therefore, the smoothness condition (definition 4.1.3) applies and gives :

$$\begin{aligned}\left\|\zeta(x)Tb_{i,j}(x)\zeta(x)\right\|_1 &\leq \int_{y\in 5Q^c_{x,i\wedge j}} \left\|k(x,y)-k(x,c_{y,i\vee j})\right\|_{\widetilde{\mathcal{M}}}\left\|b_{i,j}(y)\right\|_1 dy\\ &\leq \int_{y\in\mathbb{R}^n} \mathcal{I}_{y\notin 5Q_{x,i\wedge j}} \frac{2^{-\gamma(i\vee j)}}{|x-y|^{n+\gamma}}\left\|b_{i,j}(y)\right\|_1 dy\end{aligned}$$

It follows that :

$$\begin{aligned}\left\|\zeta Tb_{i,j}\zeta\right\|_1 &= \int_{x\in\mathbb{R}^n} \left\|\zeta(x)Tb_{i,j}(x)\zeta(x)\right\|_1 dx\\ &\leq \int_{y\in\mathbb{R}^n}\int_{x\in\mathbb{R}^n} \mathcal{I}_{x\notin 5Q_{y,i\wedge j}} \frac{2^{-\gamma(i\vee j)}}{|x-y|^{n+\gamma}}\left\|b_{i,j}(y)\right\|_1 dxdy\\ &\lesssim \int_{y\in\mathbb{R}^n} 2^{-\gamma(i\vee j-i\wedge j)}\left\|b_{i,j}(y)\right\|_1 dy\\ &\lesssim 2^{-\gamma|i-j|}\left\|b_{i,j}\right\|_1\end{aligned}$$

□

Proof of property 4.3.6. The only thing left to do is to glue the pieces together thanks to lemmas 4.3.7 and 4.3.9:

$$\begin{aligned}\left\|\zeta Tb\zeta\right\|_1 &\leq \sum_{i,j\in\mathbb{Z}}\left\|\zeta Tb_{i,j}\zeta\right\|_1 \leq \sum_{s\in\mathbb{Z}}\sum_{i-j=s}\left\|\zeta Tb_{i,j}\zeta\right\|_1\\ &\leq \sum_{s\in\mathbb{Z}} 2^{-\gamma|s|}\sum_{i-j=s}\left\|b_{i,j}\right\|_1 \lesssim \sum_{s\in\mathbb{Z}} 2^{-\gamma|s|}\left\|f\right\|_1 \lesssim \left\|f\right\|_1.\end{aligned}$$

4.3.3 Estimate for the good function

This one is more involved and requires the L_2-pseudo-localisation theorem. The same trick as for the bad function allows us to write:

$$\lambda_t(Tg) \lesssim \tau(1-\zeta) + \lambda_t(\zeta Tg\zeta) \lesssim \frac{\left\|f\right\|_1}{t} + \frac{\left\|\zeta Tg\zeta\right\|_2^2}{t^2}.$$

Define the diagonal, left and right parts of g as follows:

$$g^{(d)} = \sum_{i\in\mathbb{Z}} p_i f_i p_i\ ,\ g^{(l)} = \left(\sum_{i<j\in\mathbb{Z}} p_i f_j p_j\right) + q^\perp f q \text{ and}$$
$$g^{(r)} = \left(\sum_{i<j\in\mathbb{Z}} p_j f_j p_i\right) + q f q^\perp.$$

Note that the estimate for the good function can easily be deduced from the following property:

Proposition 4.3.10. *The following estimates hold :*

$$\|Tg^{(d)}\|_2^2 \lesssim t\|f\|_1 \ , \ \|\zeta Tg^{(l)}\|_2^2 \lesssim t\|f\|_1 \ \textit{and} \ \|Tg^{(r)}\zeta\|_2^2 \lesssim t\|f\|_1$$

Proof of the $g^{(d)}$ estimate. Since T is bounded on $L_2(\mathcal{N})$ it sufices to prove that $\|g^{(d)}\|_2^2 \lesssim t\|f\|_1$.

Notice that $g^{(d)}$ is positive.

$$\|g^{(d)}\|_1 = \sum_{i\in\overline{\mathbb{Z}}} \tau(p_i f_i p_i) = \sum_{i\in\overline{\mathbb{Z}}} \tau(p_i f) = \tau(\sum_{i\in\overline{\mathbb{Z}}} p_i f) = \|f\|_1.$$

By orthogonality, we have $\|g^{(d)}\|_\infty = \sup_{k\in\overline{\mathbb{Z}}} \|p_k f_k p_k\| \leq 2^n t$. Indeed, for $k < \infty$, we have, by Lemma 4.3.3:

$$p_k f_k p_k \leq 2^n p_k f_{k-1} p_k \leq 2^n q_{k-1} f_{k-1} q_{k-1} \leq 2^n t$$

For $k = \infty$, reasoning in $L_2(\mathcal{N})$: $t - qfq = \lim_{k\to\infty} t - qf_k q \geq 0$ since $L_2(\mathcal{N})^+$ is closed. So $\|g_d\|_\infty \leq 2^n t$.

We conclude by Hölder's inequality:

$$\|g_d\|_2^2 \leq \|g_d\|_1 \|g_d\|_\infty \leq 2^n t\|f\|_1.$$

□

Proof of the $g^{(l)}$ estimate. This will conclude the proof of proposition 4.3.10 since the argument for $g^{(r)}$ is similar.

Lemma 4.3.11. We have the following expression for $g^{(l)}$:

$$g^{(l)} = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} p_k df_{k+s} q_{k+s-1} =: \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} g_{s,k}^{(l)}.$$

Proof. This is obtained by an Abel's transform. First, fix j_0:

$$\begin{aligned}\sum_{i<j\leq j_0} p_i f_j p_j &= \sum_{i<j\leq j_0} p_i f_j (q_{j-1} - q_j) = \sum_{i\leq j<j_0} p_i df_{j+1} q_j - \sum_{i<j_0} p_i f_{j_0} q_{j_0} \\ &= \sum_{i\leq j<j_0} p_i df_{j+1} q_j - q_{j_0-1}^{\perp} f_{j_0} q_{j_0}\end{aligned}$$

Letting j_0 go to infinity, we obtain:

$$\sum_{i<j\in\mathbb{Z}} p_i f_j p_j = \big(\sum_{i\leq j\in\mathbb{Z}} p_i df_{j+1} q_j\big) - q^{\perp} fq,$$

which is exactly what we needed.

□

Proposition 4.3.12. *Define, for all $s \geq 1$:*

$$g_s^{(l)} = \sum_{k=1}^{\infty} p_k df_{k+s} q_{k+s-1}$$

Then: $\|\zeta T g_s^{(l)}\|_2^2 \lesssim 2^{-\gamma s} t\|f\|_1$.

This is, as for the bad function, the core of the proof and will require some work.

Lemma 4.3.13. We have the following properties:

1. $\Delta_{k+s}(g_s^{(l)}) = g_{s,k}^{(l)}$.

2. Fix s, the $g_{s,k}^{(l)}$ are orthogonal in $L_2(\mathcal{N})$.

3. $\|g_s^{(l)}\|_2^2 \lesssim t\|f\|_1$.

Proof. 1 is straightforward and 2 follows directly from 1 since martingale differences are always orthogonal. Let us prove 3.

$$\begin{aligned}\|g_{s,k}^{(l)}\|_2^2 &\leq 2(\|p_k f_{k+s} q_{k+s-1}\|_2^2 + \|p_k f_{k+s-1} q_{k+s-1}\|_2^2)\\ &\leq 2\tau(p_k f_{k+s} q_{k+s-1} f_{k+s} p_k) + 2\tau(p_k f_{k+s-1} q_{k+s-1} f_{k+s-1} p_k)\end{aligned}$$

By Cuculescu's theorem and recalling that $f_i \leq 2^n f_{i-1}$ from lemma 4.3.3:

$$\begin{aligned}\|f_{k+s}^{1/2} q_{k+s-1} f_{k+s}^{1/2}\|_\infty &= \|q_{k+s-1} f_{k+s} q_{k+s-1}\|_\infty \lesssim t\\ \|f_{k+s}^{1/2} q_{k+s-1} f_{k+s-1}^{1/2}\|_\infty &= \|q_{k+s-1} f_{k+s-1} q_{k+s-1}\|_\infty \leq t\end{aligned}$$

Therefore, for all $s > 0$:

$$\|g_s^{(l)}\|_2^2 = \textstyle\sum_{k>0} \|g_{k,s}^{(l)}\|_2^2 \lesssim \sum_{k>0} t\tau(p_k f) = t\|f\|_1.$$

□

Lemma 4.3.14. We have the following estimate:

$$\|\zeta T g_s^{(l)}\|_2 \lesssim 2^{-\gamma s/2}\|g_s^{(l)}\|_2$$

Proof. This is where we apply pseudo-localisation i.e theorem 4.2.1 to $g_s^{(l)}$. Using the notations introduced for this theorem, we can take $A_k = p_k$ since $p_k^\perp g_{s,k}^{(l)} = 0$. Then:

$$A_{f,s} = \textstyle\bigvee_{k>0} 5p_k = \zeta^\perp.$$

The theorem gives:

$$\|A_{f,s}^\perp T g_s^{(l)}\|_2 \lesssim 2^{-\gamma s/2}\|g_s^{(l)}\|_2$$

which is exactly the expected estimate.

□

The proposition 4.3.12 is clear from the two previous lemmas.
It follows that :

$$\begin{aligned}\|\zeta T g^{(l)}\|_2^2 &\leq \Big(\sum_{s=1}^\infty \|\zeta T g_s^{(l)}\|_2\Big)^2 \leq \Big(\sum_{s=1}^\infty 2^{-\frac{\gamma s}{2}}\|g_s^{(l)}\|_2\Big)^2\\ &\lesssim \Big(\sum_{s=1}^\infty 2^{-\frac{\gamma s}{2}}\sqrt{t\|f\|_1}\Big)^2 \lesssim t\|f\|_1\end{aligned}$$

which is the expected estimate for $g^{(l)}$ and concludes the proof of the main theorem.

4.4 Application to Hilbert-valued kernels

Let $(\mathcal{M},\tau)$ be a noncommutative measure space, $\mathcal{N} = L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$. Let T be a Calderón-Zygmund operator associated to a kernel k taking values in ℓ^2. For all $1 \le p \le \infty$, we have a multiplication from $\ell^2 \times L_p(\mathcal{M})$ to $L_p(\mathcal{M})^{\mathbb{N}}$: for all $h = (h_i)_{i\in\mathbb{N}} \in \ell^2$ and $x \in \mathcal{M}$, define $hx = (h_i x)_{i\in\mathbb{N}}$.

Thanks to the multiplication we defined, T can be seen formally as an operator from $L_p(\mathcal{N})$ to $L_p(\mathcal{N})^{\mathbb{N}}$ by the usual expression :

$$Tf(x) = \int_{\mathbb{R}} k(x,y)f(y)dy.$$

We will show in this section that T is bounded from $L_1(\mathcal{N})$ to $R_{1,\infty}(\mathcal{N}) + C_{1,\infty}(\mathcal{N})$.

To apply the theorem, we have to include $\mathcal{M}$ and ℓ^2 in a von Neumann algebra such that their images commute. Let us use the setting introduced in the previous chapter. Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of free Haar unitaries defined on a noncommutative probability space $\mathcal{A}$. Define $\mathcal{M}_{\mathcal{A}} = \mathcal{M}\overline{\otimes}\mathcal{A}$, $\mathcal{N}_{\mathcal{A}} = \mathcal{N}\overline{\otimes}\mathcal{A}$ and the inclusion maps $i_1 : \mathcal{N} \to \mathcal{N}_{\mathcal{A}}$ and $i_2 : \ell^2 \to \mathcal{M}_{\mathcal{A}}$ by:

$$i_1(x) = 1 \otimes x \text{ and } i_2(h) = \textstyle\sum_{k=0}^{\infty} h_k \otimes \lambda(g_k).$$

From now on $\mathcal{M}$ will be identified with $\mathcal{M}\otimes 1 \subset \mathcal{M}_{\mathcal{A}}$ and can thus be considered as a subalgebra of $\widetilde{\mathcal{M}}$ since $\tau_{\widetilde{\mathcal{M}}}$ coincides with $\tau_{\mathcal{M}}$ on $\mathcal{M}$. In this context, we can again consider the map $G : S(\mathcal{N}) \to \mathcal{N}_{\mathcal{A}}$ defined for any finite sequence $x = (x_k)_{k\in\mathbb{N}}$ by $G(x) = \sum_{k=0}^{\infty} x_k \otimes \lambda(g_k)$. Note that for any symmetric space E , G extends to $\mathscr{H}_E$ and is an isomorphism onto its image that we will denote by V_E and i_1 extends to $E(\mathcal{N})$. Note that the image of i_2 is in $\mathcal{M}' \cap \mathcal{M}_{\mathcal{A}}$. Let $\widetilde{k} : (x,y) \mapsto i_2(k(x,y))$ be the kernel of a Calderón-Zygmund operator $\widetilde{T} : L_p(\mathcal{N}) \to L_p(\mathcal{N}_{\mathcal{A}})$. Note that, since the range of $\widetilde{T}$ is included included in the range of G,

$$\widetilde{T} = G \circ T \text{ and } T = G^{-1} \circ \widetilde{T} \tag{4.10}$$

Proposition 4.4.1. *Let T be a Calderón-Zygmund operator associated to a kernel k taking values in ℓ^2. Suppose furthermore that T is bounded from $L_2(\mathcal{N})$ to $R_2\mathcal{N}$ and that k satisfies the smoothness and size conditions then the operator $\widetilde{T}$ defined above is bounded from $L_1(\mathcal{N})$ to $L_{1,\infty}(\mathcal{N}_{\mathcal{A}})$.*

Proof. We only have to check that Theorem 4.1.5 can be applied to $\widetilde{T}$. By Haagerup's inequality [19], for all $h \in \ell^2$, $\|h\| \approx \|i_2(h)\|_{\widetilde{\mathcal{M}}}$ so $\widetilde{k}$ verifies the size and smoothness conditions. Furthermore, by (4.10), $\widetilde{T}$ is bounded on $L_2(\mathcal{N})$. So all the hypothesis are verified and $\widetilde{T}$ is bounded from $L_1(\mathcal{N})$ to $L_{1,\infty}(\widetilde{\mathcal{N}})$. □

Corollary 4.4.2. *Let T be a Calderón-Zygmund operator associated to a kernel k taking values in ℓ^2. Suppose furthermore that T is bounded from $L_2(\mathcal{N})$ to $R_2(\mathcal{N})$ and that k satisfies the smoothness and size condition. Then T is bounded from $L_1(\mathcal{N})$ to $R_{1,\infty}(\mathcal{N}) + C_{1,\infty}(\mathcal{N})$ and from $L_p(\mathcal{N})$ to $R_p(\mathcal{N}) + C_p(\mathcal{N})$ for $1 < p < 2$.*

Proof. Let $p \in (1,2)$. By real interpolation, since $\widetilde{T}$ is bounded from $L_1(\mathcal{N})$ to $L_{1,\infty}(\mathcal{N}_{\mathcal{A}})$ and from $L_2(\mathcal{N})$ to $L_2(\mathcal{N}_{\mathcal{A}})$, $\widetilde{T}$ is bounded from $L_p(\mathcal{N})$ to $L_p(\mathcal{N}_{\mathcal{A}})$. This means by (4.10) that T is bounded from $L_1(\mathcal{N})$ to $\mathscr{H}_{1,\infty}(\mathcal{N})$ and from $L_p(\mathcal{N})$ to $\mathscr{H}_p(\mathcal{N})$. Recall that by the free Khintchine inequalities ([20] and corollary 3.2.2), $\mathscr{H}_{1,\infty}(\mathcal{N}) \approx R_{1,\infty}(\mathcal{N}) + C_{1,\infty}(\mathcal{N})$ and $\mathscr{H}_p(\mathcal{N}) \approx R_p(\mathcal{N}) + C_p(\mathcal{N})$, concluding the proof of the corollary. □

Remark 4.4.3. T is not bounded from $L_p(\mathcal{N})$ to $C_p(\mathcal{N})$, for $1 < p < 2$. It is necessary to consider $R_p(\mathcal{N}) + C_p(\mathcal{N})$.

Proof. We will construct a counter-example thanks to the Littlewood-Paley kernel (the same general idea can again be found in [41]).

Let ψ be a C^∞ function supported in $(1,2)$, bounded by 1 and constant equal to one on $(5/4, 7/4)$. For any $i \in \mathbb{N}^*$, let $\phi_i : t \mapsto \psi(it)$. Consider the kernel $k : \mathbb{R}^2 \to \ell^2$ defined by $k_i(x,y) = \widehat{\phi}_i(x-y)$ where $\widehat{\phi}_i$ denotes the Fourrier transform of ϕ_i. It is standard that k verifies the smoothness estimate for $\gamma = 1$ and the size condition. The Calderón-Zygmund operator T associated to k is bounded on L_2 by Plancherel's theorem.

Let $\mathcal{M} = \mathcal{B}(\ell^2)$. As always, let $\mathcal{N} = L_\infty(\mathbb{R}) \otimes \mathcal{M}$. For all $p \geq 1$, T can be seen as an operator from $L_p(\mathcal{N})$ to $C_p(\mathcal{N})$. For all $k > 0$, let g_k be the Fourier transform of $\mathcal{I}_{(\frac{5}{4}2^{k-1}, \frac{5}{4}2^{k-1}+1/2)}$. Notice that $\widehat{\phi}_i * g_k = \delta$. Fix $m > 0$ an integer. Let $f_m = \sum_{k=1}^m g_k \otimes e_{1,k}$ then $Tf_m = (g_k \otimes e_{1,k})_{e_k \in \mathbb{N}^*}$. It results that $\|f_m\|_{L_p(\mathcal{N})} = m^{1/2}\|g_1\|_p$ and $\|Tf_m\|_{C_p(\mathcal{N})} = m^{1/p}\|g_1\|_p$. So T is not bounded for $p < 2$. □

4.5 L_p-pseudo-localisation

In this section, we will recover in this context the main result from [22] which is an L_p-version of pseudo-localisation. The proof heavily relies on estimates and definitions introduced in section 2. Precisely, we will show the following theorem, using notations from theorem 4.2.1.

Theorem 4.5.1. *Let T be a Calderón-Zygmund operator associated with a kernel k with Lipschitz parameter γ, verifying the size condition, taking values in $\mathcal{M}' \cap \widetilde{\mathcal{M}}$ and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$. Then for all $1 < p < \infty$, $s \in \mathbb{N}$ and $f \in L_p(\mathcal{N})$, there exists $\theta_p > 0$ such that :*

$$\|A_{f,s}^\perp(Tf)\|_p \lesssim 2^{-\gamma s\theta_p}\|f\|_p \text{ and } \|(Tf)B_{f,s}^\perp\|_p \lesssim 2^{-\gamma s\theta_p}\|f\|_p,$$

where the implied constant depends on p.

We will use the decomposition we obtained in the L_2-case in (4.9):

$$A_{f,s}^\perp Tf = A_{f,s}^\perp\big(\sum_{i=0}^\infty \Phi_i f + \sum_{i=1}^\infty \Psi_i f\big).$$

The crucial result, proved in the next sections, is the following :

Proposition 4.5.2. *For all $p \in (1,\infty)$ the sequences of operators $(\Phi_i)_{i\in\mathbb{N}}$ and $(\Psi_i)_{i\in\mathbb{N}}$ are uniformly bounded on $L_p(\mathcal{N})$.*

Proof of the theorem. Fix $p \in (1,\infty)$. By complex interpolation, there exist $q \in (1,\infty)$ and $\theta_p > 0$ such that $\|\Phi_i\|_{\mathcal{B}(L_p(\mathcal{N}))} \leq \|\Phi_i\|^{2\theta_p}_{\mathcal{B}(L_2(\mathcal{N}))}\|\Phi_i\|^{1-2\theta_p}_{\mathcal{B}(L_q(\mathcal{N}))}$.

Now, it suffices to plug in the previous proposition and proposition 4.2.6 to get the expected estimate. We obtain: $\|\Phi_i\|_{\mathcal{B}(L_p(\mathcal{N}))} \lesssim 2^{-\theta_p(i+s)}$. A similar estimate is true for Ψ_i. This is enough to conclude thanks to the decomposition (4.9) mentionned above. □

4.5.1 L_p-boundedness of Φ_i

Lemma 4.5.3. Consider a Calderón-Zygmund operator T with kernel k, and an integer m. Then, for any $p \in (1,\infty)$, the kernel $k' : (x,y) \mapsto \mathcal{I}_{x\in 5Q_{y,m}} k(x,y)$ defines a bounded operator T' on $L_p(\mathcal{N})$.

Proof. We introduce a sequence of Rademacher variables indexed by the set of dyadic cubes of size 2^{-m} to express the kernel k' in a suitable way. Let Ω be the probability space $\{-1,1\}^{\mathcal{Q}_m}$ where $\mathcal{Q}_m$ is the set of dyadic cubes of size 2^{-m} and $\widetilde{\mathcal{N}} = \mathcal{N}\overline{\otimes}L_\infty(\Omega)$. Consider $\mathcal{N}$ as a subalgebra of $\widetilde{\mathcal{N}}$. We will take a functional approach and work in $\widetilde{\mathcal{N}}$ from now on. Denote by ε_Q the Q-coordinate in Ω. Define $g_m(x,\omega) = \sum_{Q\in\mathcal{Q}_m} \mathcal{I}_Q(x)\varepsilon_Q(\omega)$, $l_m(x,\omega) = \sum_{Q\in\mathcal{Q}_m} \mathcal{I}_{5Q}(x)\varepsilon_Q(\omega)$. Let G_m (resp. L_m) be the multiplication by g_m (resp. l_m) and $\mathcal{E}$ be the conditionnal expectation onto $L_p(\mathcal{N})$. Note that $\|G_m\|_{\mathcal{B}(L_p(\mathcal{N}))} = \|g_m\|_\infty = 1$ and $\|L_m\|_{\mathcal{B}(L_p(\mathcal{N}))} = \|l_m\|_\infty = 5^n$.

It is easily checked that $T' = \mathcal{E}L_mTG_m$ since

$$k'(x,y) = \int_\Omega l_m(x,\omega)k(x,y)g_m(y,\omega)d\omega.$$

So T' can be extended to a bounded operator on $L_p(\mathcal{N})$. □

Proposition 4.5.4. *Let $p \in (1,\infty)$. The operators Φ_i are uniformly bounded on $L_p(\mathcal{N})$.*

Proof. We identify $L_p(\mathcal{N})$ quasi-isometrically with a subspace of $RC_p(\widetilde{\mathcal{N}})$, thanks to Burkholder-Gundy inequality (theorem 2.1 in [49]): $\|f\|_p \approx \|df\|_{RC_p(\widetilde{\mathcal{N}})}$. Recall that $\Phi_i = \sum_{k\in\mathbb{Z}} \Delta_{k-s+i}T_{k-s}\Delta_k = \sum_{k\in\mathbb{Z}} \Delta_{k-s+i}\big(T - \mathcal{E}L_{k-s}TG_{k-s}\big)\Delta_k$, where the L_k and G_k are given by the lemma above. We can write $\Phi_i = \Delta T - \Delta\mathcal{E}LTG$ where $Gx = (G_{k-s}x_k)_{k\in\mathbb{Z}}$, $Lx = (L_{k-s}x_k)_{k\in\mathbb{Z}}$, $\Delta x = (\Delta_{k-s+i}x_k)_{k\in\mathbb{Z}}$ and $Tx = (Tx_k)_{k\in\mathbb{Z}}$. L and G are still bounded multiplications by unconditionnality of $RC_p(\widetilde{\mathcal{N}})$. T is completely bounded on $RC_p(\widetilde{\mathcal{N}})$ as a diagonal Calderón-Zygmund operator, $\mathcal{E}$ as a conditional expectation from $RC_p(\widetilde{\mathcal{N}})$ to $RC_p(\mathcal{N})$ and Δ by Stein's inequality (Theorem 2.3 in [49]):

$\|(\mathbb{E}_n(x_n))\|_{L_p(\widetilde{\mathcal{N}},\ell_2^c)} \leq \|(x_n)\|_{L_p(\widetilde{\mathcal{N}},\ell_2^c)}$ and $\|(\mathbb{E}_n(x_n))\|_{L_p(\widetilde{\mathcal{N}},\ell_2^r)} \leq \|(x_n)\|_{L_p(\widetilde{\mathcal{N}},\ell_2^r)}$.

Hence, the Φ_i are uniformly bounded.

□

4.5.2 L_p-boundedness of Ψ_i

Recall that,

$$\Psi_i = \sum\nolimits_{k\in\mathbb{Z}} \Delta_{k-i}S_{k,s}\Delta_{k+s} = \sum\nolimits_{k\in\mathbb{Z}} \Delta_{k-i}(T_{k,s}+R_{k,s})\Delta_{k+s} = A_i + B_i$$

where $A_i = \sum_{k\in\mathbb{Z}} \Delta_{k-i}T_{k,s}\Delta_{k+s} = \sum_{k\in\mathbb{Z}} \Delta_{k-i}T_k\Delta_{k+s}$ (by 4.6) and $B_i = \sum_{k\in\mathbb{Z}} \Delta_{k-i}R_{k,s}\Delta_{k+s}$. The previous section tells us that A_i is bounded from $L_p(\mathcal{N})$ to $L_p(\mathcal{N})$, with a control on the norm independant from i. So what is left to prove is the following proposition.

Proposition 4.5.5. *Let $p \in (1,\infty)$, then B_i is bounded from $L_p(\mathcal{N})$ to $L_p(\mathcal{N})$, uniformly in i.*

Proof. Using notations from the previous section, we will again consider B_i as a partial operator from $RC_p(\widetilde{\mathcal{N}})$ to $RC_p(\widetilde{\mathcal{N}})$. Let ϕ be a Schwarz function from $\mathbb{R}^n$ to $\mathbb{R}$ such that $\mathcal{I}_{\mathcal{A}_0}(x,y) \le \phi(x-y) \le \mathcal{I}_{|x-y|>1/2}$ for all $x,y\in\mathbb{R}^n$. Let U be a smooth function from $\mathbb{S}^{n-1}$ to the unit circle of $\mathbb{C}$ with average 0. Such U exists, take for example $U_\lambda(x) = e^{2i\arctan(\lambda x_1)}$. Since arctan is an odd function, the average of U_λ is real. Now, note that U_λ goes to -1 when λ goes to ∞ and to 1 when $\lambda = 0$ so there must exist a suitable U_λ. Extend U to $\mathbb{R}^n$ by $U(0) = 0$ and $U(x) = U(x/|x|)$ otherwise. Write,

$$\begin{aligned} r_{k,s}(x,y) &= \mathcal{I}_{\mathcal{A}_k}(x,y)\frac{K(x)2^{-(k+s)\gamma}}{I_{n,\gamma}|x-y|^{n+\gamma}} \\ &= \frac{K(x)2^{-s\gamma}}{I_{n,\gamma}}.\mathcal{I}_{\mathcal{A}_k}(x,y).\frac{\overline{U}(x-y)\phi(2^k(x-y))}{|2^k(x-y)|^\gamma}.\frac{U(x-y)}{|x-y|^n} \end{aligned}$$

Denote by M_K the multiplication by $\dfrac{K(.)2^{-s\gamma}}{I_{n,\gamma}}$, M_K is bounded (see section 2.3). Define:

$$F(x) = \frac{\overline{U}(x)\phi(x)}{|x|^\gamma}.$$

Since ϕ is zero in a neighbourhood of the origin, F has no singularity and is a Schwarz function. So F is the Fourier transform of an L_1 function $\widehat{F}$. Note also that $\mathcal{I}_{\mathcal{A}_k} = \mathcal{I}_{x\in 5Q_{y,k}} - \mathcal{I}_{x\in 3Q_{y,k}}$, we will only prove the boundedness replacing $\mathcal{I}_{\mathcal{A}_k}$ by $\mathcal{I}_{x\in 5Q_{y,k}}$ since $\mathcal{I}_{x\in 3Q_{y,k}}$ is similar, denote the associated operator B_i'. Define also:

$$h(x) = \frac{U(x)}{|x|^n}$$

and H the convolution operator associated to h. Since U has 0 average, H is bounded from $RC_p(\widetilde{\mathcal{N}})$ to $RC_p(\widetilde{\mathcal{N}})$ for any $p\in(1,\infty)$. Indeed, H can then be written as an average of directionnal Hilbert operators (see [15]). Using notations from the previous proposition and lemma:

$$\begin{aligned}\mathcal{I}_{x\in 5Q_{y,k}}F(2^k(x-y))h(x-y) &= \int_\Omega l_k(x,\omega)F(2^k(x-y))h(x-y)g_k(y,\omega)d\omega \\ &= \int_\Omega\int_{\mathbb{R}^n}\widehat{F}(t)l_k(x,\omega)e^{2^k itx}h(x-y)e^{-2^k ity}g_k(y,\omega)dtd\omega.\end{aligned}$$

We will now translate this equality in terms of operators. Take again L and G from the previous part, the diagonal operators associated to the multiplication by $(l_k)_{k\in\mathbb{Z}}$ and $(g_k)_{k\in\mathbb{Z}}$, M_t defined by $M_t x = (e^{2^k it\cdot}x_k)_{k\in\mathbb{Z}}$, $\mathcal{E}$ the expectation from $RC_p(\widetilde{\mathcal{N}})$ to $RC_p(\mathcal{N})$, and Δ such that $\Delta x = (\Delta_{k-s-i}x_k)_{k\in\mathbb{Z}}$. Then,

$$B_i' = M_K\int_{\mathbb{R}^n}\widehat{F}(t)\Delta\mathcal{E}LM_tHM_{-t}Gdt.$$

As we have seen in the previous proof, M_t,M_{-t},L and G are bounded by unconditionnality of $RC_p(\widetilde{\mathcal{N}})$. More precisely, $\|M_t\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = \|M_{-t}\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = \|G\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = 1$ and $\|L\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = 5^n$. Δ is bounded thanks to Stein's inequality and we have have constructed F and H such that $\widehat{F}$ is L_1 and H is bounded on $RC_p(\widetilde{\mathcal{N}})$. Consequently,

$$\|B_i\|_{\mathcal{B}(L_p(\mathcal{N}))} \lesssim 2\|\Delta\|_{\mathcal{B}(RC_p(\mathcal{N}))}5^n\|\widehat{F}\|_1\|H\|_{\mathcal{B}(RC_p(\mathcal{N}))}.$$

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Inégalités de type faible dans les espaces L_p non-commutatifs

Résumé : Cette thèse vise à développer des outils d'analyse harmonique non-commutative. Elle porte plus précisément sur les inégalités de Khintchine non-commutatives et les intégrales singulières à valeurs opérateur. La première partie est dédiée à des questions d'interpolation des espaces L_p classiques. On généralise et on énonce de nouvelles caractérisations des espaces interpolés entre espaces L_p. Dans une seconde partie, on démontre une forme des inégalités de Khintchine non-commutatives valides dans tous les espaces interpolés entre espace Lp. Celle-ci permet d'unifier les cas $p < 2$ et $p > 2$ ainsi que de traiter les espaces Lp faibles, même pour $p = 1$ ou 2. En s'appuyant sur la première partie, on caractérise les espaces dans lesquels les formules usuelles pour les inégalités de Khintchine sont valides. Dans une dernière partie, on donne une preuve simplifiée de l'inégalité de type $(1, 1)$ faible pour les intégrales singulières non-commutatives, un résultat précédemment obtenu par Parcet. Cette simplification nous permet de retrouver rapidement deux autres résultats connus : la pseudolocalisation L_p et l'inégalité de type faible pour les intégrales singulières non-commutatives dont le noyau est à valeurs dans un espace de Hilbert.

⋆⋆⋆

Weak type inequalities in noncommutative L_p-spaces

Abstract: The purpose of this thesis is to develop tools of noncommutative harmonic analysis. More precisely, it deals with noncommutative Khintchine inequalities and operator-valued singular integrals. The first part is dedicated to questions of interpolation between classical L_p-spaces. We generalize and state new characterisations of interpolation spaces between L_p-spaces. In a second part, we introduce a form of the noncommutative Khintchine inequalities which holds in every interpolation space between two L_p-spaces. It enables us to unify the cases $p < 2$ and $p > 2$ and to deal with weak L_p-spaces even when $p = 1$ or 2. By relying on the first part, we characterize spaces in which the usual formulas for Khintchine inequalities hold. In a last part, we give a simplified proof of the weak boundedness of noncommutative singular integrals, a result previously obtained by Parcet. This simplification allows us to recover quickly two results: the L_p pseudolocalisation and the weak type inequality for noncommutative singular integrals associated to Hilbert-valued kernels.