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# Compressible-incompressible transitions in fluid mechanics : waves-structures interaction and rotating fluids

Edoardo Bocchi

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École doctorale  
**Mathématiques  
et informatique** / université  
de **BORDEAUX**

## THÈSE DE DOCTORAT

présentée par

**EDOARDO BOCCHI**

sous la direction de DAVID LANNES et CHRISTOPHE PRANGE

pour obtenir le grade de

DOCTEUR DE L' UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET INFORMATIQUE

SPÉCIALITÉ : MATHÉMATIQUES APPLIQUÉES

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# Compressible-incompressible transitions in fluid mechanics: waves-structures interaction and rotating fluids

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Soutenue le 23 Septembre 2019 à l'Institut de Mathématiques de Bordeaux  
devant le jury composé de :

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**Title :** Compressible-incompressible transitions in fluid mechanics:  
waves-structures interaction and rotating fluids

**Abstract :** This manuscript deals with compressible-incompressible transitions arising in partial differential equations of fluid mechanics. We investigate two problems: floating structures and rotating fluids. In the first problem, the introduction of a floating object into water waves enforces a constraint on the fluid and the governing equations turn out to have a compressible-incompressible structure. In the second problem, the motion of geophysical compressible fluids is affected by the Earth's rotation and the study of the high rotation limit shows that the velocity vector field tends to be horizontal and with an incompressibility constraint. Floating structures are a particular example of fluid-structure interaction, in which a partially immersed solid is floating at the fluid surface. This mathematical problem models the motion of wave energy converters in sea water. In particular, we focus on heaving buoys, usually implemented in the near-shore zone, where the shallow water asymptotic models describe accurately the motion of waves. We study the two-dimensional nonlinear shallow water equations in the axisymmetric configuration in the presence of a floating object with vertical side-walls moving only vertically. The assumptions on the solid permit to avoid the free boundary problem associated with the moving contact line between the air, the water and the solid. Hence, in the domain exterior to the solid the fluid equations can be written as an hyperbolic quasilinear initial boundary value problem. This couples with a nonlinear second order ODE derived from Newton's law for the free solid motion. Local in time well-posedness of the coupled system is shown provided some compatibility conditions are satisfied by the initial data in order to generate smooth solutions.

Afterwards, we address a particular configuration of this fluid-structure interaction: the return to equilibrium. It consists in releasing a partially immersed solid body into a fluid initially at rest and letting it evolve towards its equilibrium position. A different hydrodynamical model is used. In the exterior domain the equations are linearized but the nonlinear effects are taken into account under the solid. The equation for the solid motion becomes a nonlinear second order integro-differential equation which rigorously justifies the Cummins equation, assumed by engineers to govern the motion of floating objects. Moreover, the equation derived improves the linear approach of Cummins by taking into account the nonlinear effects. The global existence and uniqueness of the solution is shown for small data using the conservation of the energy of the fluid-structure system.

In the second part of the manuscript, highly rotating fluids are studied. This mathematical problem models the motion of geophysical flows at large scales affected by the Earth's rotation, such as massive oceanic and atmospheric currents.

The motion is also influenced by the gravity, which causes a stratification of the density in compressible fluids. The rotation generates anisotropy in viscous flows and the vertical turbulent viscosity tends to zero in the high rotation limit. Our interest lies in this singular limit problem taking into account gravitational and compressible effects. We study the compressible anisotropic Navier-Stokes-Coriolis equations with gravitational force in the horizontal infinite slab with no-slip boundary condition. Both this condition and the Coriolis force cause the apparition of Ekman layers near the boundary. They are taken into account in the analysis by adding corrector terms which decay in the interior of the domain. In this work well-prepared initial data are considered. A stability result of global weak solutions is shown for power-type pressure laws. The limit dynamics is described by a two-dimensional viscous quasi-geostrophic equation with a damping term that accounts for the boundary layers.

**Keywords :** Fluid Mechanics - Floating structures - Nonlinear shallow water equations - Hyperbolic PDEs - Rotating fluids - Boundary layers

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**Titre :** Transitions compressible-incompressible en mécanique des fluides: interaction vagues-structures et fluides en rotation

**Résumé :** Ce manuscrit porte sur les transitions compressible-incompressible dans les équations aux dérivées partielles de la mécanique des fluides. On s'intéresse à deux problèmes : les structures flottantes et les fluides en rotation. Dans le premier problème, l'introduction d'un objet flottant dans les vagues induit une contrainte sur le fluide et les équations gouvernant le mouvement acquièrent une structure compressible-incompressible. Dans le deuxième problème, le mouvement de fluides géophysiques compressibles est influencé par la rotation de la Terre. L'étude de la limite à rotation rapide montre que le champ vectoriel de vitesse tend vers une configuration horizontale et incompressible.

Les structures flottantes constituent un exemple particulier d'interaction fluide-structure, où un solide partiellement immergé flotte à la surface du fluide. Ce problème mathématique modélise le mouvement de convertisseurs d'énergie marine. En particulier, on s'intéresse aux bouées pilonnantes, installées proche de la côte où les modèles asymptotiques en eaux peu profondes sont valables. On étudie les équations de Saint-Venant axisymétriques en dimension deux avec un objet flottant à murs verticaux se déplaçant seulement verticalement. Les hypothèses sur le solide permettent de supprimer le problème à bord libre associé avec la ligne de contact entre l'air, le fluide et le solide. Les équations pour le fluide dans le domaine extérieur au solide sont donc écrites comme un problème au bord quasi-linéaire hyperbolique. Celui-ci est couplé avec une EDO non-linéaire du second ordre qui est dérivée de l'équation de Newton pour le mouvement libre du solide. On montre le caractère bien posé localement en temps du système couplé lorsque que les données initiales satisfont des conditions de compatibilité afin de générer des solutions régulières.

Ensuite on considère une configuration particulière: le retour à l'équilibre. Il s'agit de considérer un solide partiellement immergé dans un fluide initialement au repos et de le laisser retourner à sa position d'équilibre. Pour cela, on utilise un modèle hydrodynamique différent, où les équations sont linearisées dans le domaine extérieur, tandis que les effets non-linéaires sont considérés en dessous du solide. Le mouvement du solide est décrit par une équation intégro-différentielle non-linéaire du second ordre qui justifie rigoureusement l'équation de Cummins, utilisée par les ingénieurs pour les mouvements des objets flottants. L'équation que l'on dérive améliore l'approche linéaire de Cummins en tenant compte des effets non-linéaires. On montre l'existence et l'unicité globale de la solution pour des données petites en utilisant la conservation de l'énergie du système fluide-structure.

Dans la deuxième partie du manuscrit, on étudie les fluides en rotation rapide. Ce problème mathématique modélise le mouvement des flots géophysiques à grandes

échelles influencés par la rotation de la Terre. Le mouvement est aussi affecté par la gravité, ce qui donne lieu à une stratification de la densité dans les fluides compressibles. La rotation génère de l'anisotropie dans les flots visqueux et la viscosité turbulente verticale tend vers zéro dans la limite à rotation rapide. Notre intérêt porte sur ce problème de limite singulière en tenant compte des effets gravitationnels et compressibles. On étudie les équations de Navier-Stokes-Coriolis anisotropes compressibles avec force gravitationnelle dans la bande infinie horizontale avec une condition au bord de non glissement. Celle-ci et la force de Coriolis donnent lieu à l'apparition des couches d'Ekman proche du bord. Dans ce travail on considère des données initiales bien préparées. On montre un résultat de stabilité des solutions faibles globales pour des lois de pression particulières. La dynamique limite est décrite par une équation quasi-géostrophique visqueuse en dimension deux avec un terme d'amortissement qui tient compte des couches limites.

**Mots-clés :** Mécanique des fluides - Structures flottantes - Équations de Saint-Venant - EDPs hyperboliques - Fluides en rotation - Couches limites

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*Alla mia famiglia*





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## Résumé

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L'objectif de ce manuscrit est l'analyse des modèles mathématiques avec transitions compressible-incompressible qui apparaissent en mécanique des fluides. En océanographie et en météorologie, le mouvement des fluides est décrit par différents modèles, dépendant des caractéristiques et des propriétés du flot. D'une part, il est connu qu'en mécanique des fluides, les flots compressibles satisfont l'équation de continuité, dérivée du principe physique de conservation de la masse. D'autre part, le mouvement des fluides incompressibles est décrit par un champ vectoriel de vitesse avec une contrainte de divergence nulle.

Malgré cette différence qualitative, il se trouve que parfois l'introduction d'un acteur extérieur, comme des structures flottantes ou la rotation de la Terre, donne lieu à une "transition" du caractère compressible au caractère incompressible. Cette transition peut être interprétée de différentes façons selon la situation considérée: structure compressible-incompressible des équations, limite incompressible de flots compressibles pour des paramètres qui tendent vers zéro, etc.

L'intérêt de ce manuscrit porte sur deux problèmes. Dans la première partie du manuscrit, on étudie l'interaction des vagues avec des structures flottantes, qui est un problème particulier d'interaction fluide-structure où le solide est partiellement immergé dans le fluide. Ce problème mathématique décrit le mouvement des bateaux ou des convertisseurs d'énergie marine. Dans un régime particulier, la présence de l'objet donne lieu à une contrainte sur le fluide et les équations qui gouvernent son mouvement acquièrent une structure compressible-incompressible. Dans la deuxième partie du manuscrit, on étudie les fluides en rotation. Cette branche de la mécanique des fluides étudie des modèles qui décrivent les flots géophysiques, dans les océans ou dans l'atmosphère, à larges échelles en considérant

l'influence de la rotation de la Terre sur leur mouvement. Quand la limite en rotation rapide est considérée, le mouvement de ces fluides compressibles devient horizontale, avec une contrainte d'incompressibilité.

## Structures flottantes

Le problème mathématique des structures flottantes a été formulé en premier par John en [75, 76]. Il s'agit d'un exemple particulier d'interaction fluide-structure, où un corps partiellement immergé flotte à la surface du fluide. Deux problèmes à bord libre sont à traiter. Le premier vient de la surface libre du fluide et le deuxième est dû au fait que la portion de l'objet en contact avec le fluide varie en temps et la ligne de contact est donc un bord libre. Le but de la première partie du manuscrit est de décrire les caractéristiques de cette interaction avec un modèle mathématique rigoureux et qui puisse tenir compte des effets non-linéaires et des variations de la ligne de contact entre l'eau et le solide.

On s'inspire du travail de Lannes, qui a modélisé dans [83] le problème de structures flottantes en considérant ces deux derniers aspects et en utilisant une formulation à moyenne verticale, déjà utilisée pour des modèles asymptotiques des équations des vagues en océanographie côtière. Motivés par son approche, on veut étendre l'analyse non-linéaire de ce problème à une configuration axisymétrique en deux dimensions horizontales (un modèle 2+1-D). On considère un régime d'eaux peu profondes (en anglais *shallow water*) où le rapport entre la profondeur typique et la longueur d'onde typique de la vague est très petit.

### Chapitre 1

Dans le chapitre 1, on s'intéresse au caractère bien posé des équations de Saint-Venant avec une bouée pilonnante. Ce type de convertisseur d'énergie marine est souvent mis en place proche de la côte. En cette zone, les modèles asymptotiques en eaux peu profondes décrivent d'une manière précise les mouvements des vagues. On considère une structure avec symétrie cylindrique et des murs verticaux et qui a un mouvement seulement vertical dans un fluide homogène, non-visqueux, incompressible et irrotationnel. Les hypothèses sur la forme et le mouvement du solide permettent de supprimer le problème à bord libre associé à la ligne de contact et le problème se simplifie. On suppose que le flot soit axisymétrique et sans swirl, c'est à dire qu'on considère un champ vectoriel de vitesse invariant par rotation et avec une composante azimutale nulle. Pour simplifier, le fond du domaine du fluide est supposé plat. Les équations de Saint-Venant avec une structure flottante sont écrites en forme axisymétrique. Les inconnues du problème sont l'élévation de surface  $\zeta$ , la composante radiale du débit horizontal  $q$ , définie comme l'intégrale

sur la profondeur du champ de vitesse horizontale, et la pression de surface en dessous de l'objet. Cette dernière est un multiplicateur de Lagrange associé à la contrainte de contact entre le fluide et solide qui est imposé dans le modèle et qui tient en compte la présence du solide dans les équations. Cela donne une structure compressible-incompressible aux équations. Les hypothèses sur le flot font du débit horizontal une inconnue scalaire d'une seule dimension. On note  $u$  le couple  $(\zeta_e, q_e)$  des inconnues dans le domaine extérieur au solide et par  $\delta_G$  le déplacement de la position verticale du centre de masse de sa position d'équilibre. Les équations pour le fluide dans le domaine extérieur  $(R, +\infty)$ , où  $R$  est le rayon du solide, peut être écrit comme le problème quasi-linéaire hyperbolique à valeur initiale et au bord

$$\begin{cases} \partial_t u + A(u)\partial_r u + B(u, r)u = 0 & \text{dans } (R, +\infty) \\ \mathbf{e}_2 \cdot u|_{r=R} = -\frac{R}{2}\dot{\delta}_G(t) \\ u(t=0) = u_0, \end{cases} \quad (1)$$

avec

$$A(u) = \begin{pmatrix} 0 & 1 \\ gh_e - \frac{q_e^2}{h_e^2} & \frac{2q_e}{h_e} \end{pmatrix}, \quad B(u, r) = \begin{pmatrix} 0 & \frac{1}{r} \\ 0 & \frac{q_e}{rh_e} \end{pmatrix}$$

où  $h_e = h_0 + \zeta_e$  est la hauteur du fluide, égale à  $h_0$  au repos. De plus, à cause de la présence des murs verticaux, un terme correcteur doit être ajouté dans la condition au bord pour la pression intérieure afin que le système couplé fluide-structure soit conservatif. D'autre part, la loi de Newton pour la conservation de la quantité de mouvement linéaire peut être écrite comme

$$\begin{cases} (m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathbf{c}\delta_G(t) + \mathbf{c}\mathbf{e}_1 \cdot u|_{r=R} + (\mathbf{b}(u) + \beta(\delta_G))\dot{\delta}_G^2(t), \\ \delta_G(0) = \delta_0 \\ \dot{\delta}_G(0) = \delta_1 \end{cases} \quad (2)$$

avec  $\mathbf{c} = \rho g \pi R^2$ ,  $b = \frac{\pi \rho R^4}{8}$  et

$$m_a(\delta_G) = \frac{b}{h_w(\delta_G)}, \quad \beta(\delta_G) = \frac{b}{2h_w^2(\delta_G)}, \quad \mathbf{b}(u) = \frac{b}{(\mathbf{e}_1 \cdot u|_{r=R} + h_0)^2},$$

où  $h_w(\delta_G) = h_{w,eq} + \delta_G$  est la hauteur du fluide en dessous du solide. Le résultat principal du Chapitre 1 est le caractère bien posé localement en temps du système couplé (1) - (2) dans des espaces de Sobolev pour données initiales régulières et compatibles. On introduit un symétriseur à la Kreiss pour transformer le système



en un système symétrique avec une condition au bord maximal dissipative, qui permet de contrôler la trace de la solution au bord par l'estimation d'énergie standard pour des systèmes hyperboliques quasi-linéaires. Ce symétriseur est construit grâce à la dérivation d'une formulation équivalente de la condition de Kreiss-Lopatinskiĭ uniforme.

## Chapitre 2

Dans le chapitre 2, on étudie un cas particulier du problème de structures flottantes, le *retour à l'équilibre*. Il s'agit de considérer un solide partiellement immergé dans un fluide initialement au repos et le laisser "retourner" à sa position d'équilibre. L'intérêt du problème vient du fait qu'il peut être facilement reproduit expérimentalement et qu'il est utilisé en ingénierie pour déterminer plusieurs caractéristiques importantes des objets flottants. D'abord, on montre que l'équation différentielle pour le mouvement du solide peut être écrite sous une forme fermée en introduisant un opérateur d'extension-trace, qui prend  $q_e|_{r=R}$  et retourne  $\zeta_e|_{r=R}$ . Ensuite on considère la configuration du retour à l'équilibre dans un cadre axisymétrique avec les mêmes hypothèses sur l'objet que dans le chapitre 1 et les conditions initiales appropriées sont données. Les conditions de compatibilité nécessaires pour appliquer le résultat du Chapitre 1 ne sont pas satisfaites avec ces conditions initiales particulières. On propose donc un modèle hydrodynamique linéaire-nonlinéaire pour le problème de structures flottantes. Les équations pour le fluide dans le domaine extérieur deviennent les équations de Saint Venant linéaires

$$\begin{cases} \partial_t \zeta_e + \partial_r q_e + \frac{q_e}{r} = 0 \\ \partial_t q_e + gh_0 \partial_r \zeta_e = 0 \end{cases} \quad (3)$$

alors que dans le domaine intérieur les mêmes équations nonlinéaires et la même condition de transition sont considérés.

Le résultat principal du chapitre 2 est la dérivation de l'équation intégral-différentielle non-linéaire de deuxième ordre

$$(m + m_a(\delta_G)) \ddot{\delta}_G = -\mathbf{c} \delta_G - \nu \dot{\delta}_G + \mathbf{c} \int_0^t F(s) \dot{\delta}_G(t-s) ds + (\mathbf{b}(\dot{\delta}_G) + \beta(\delta_G)) \dot{\delta}_G^2, \quad (4)$$

avec  $\mathbf{c}$ ,  $b$ ,  $m_a(\delta_G)$ ,  $\beta(\delta_G)$  comme dans (2),  $\nu = \frac{\mathbf{c}R}{2v_0}$  pour  $v_0 = \sqrt{gh_0}$ ,

$$\mathbf{b}(\dot{\delta}_G) = \frac{b}{\left( \int_0^t F(s) \dot{\delta}_G(t-s) ds - \frac{R}{2v_0} \dot{\delta}_G(t) + h_0 \right)^2}$$

et le noyau de convolution  $F(t)$  défini par

$$F(t) = \lim_{v \rightarrow +\infty} \frac{1}{2\pi} \int_{-v}^v \left( \frac{iRH_0^{(1)}\left(\frac{i(c+i\omega)R}{v_0}\right)}{2v_0H_1^{(1)}\left(\frac{i(c+i\omega)R}{v_0}\right)} + \frac{R}{2v_0} \right) e^{(c+i\omega)t} d\omega$$

pour tout  $c > 0$  avec  $H_0^{(1)}, H_1^{(1)}$  des fonctions d'Hankel. Un résultat d'existence pour le problème de Cauchy associé aux données initiales du retour à l'équilibre est aussi obtenu. La linéarisation de (4) est l'équation de Cummins, une équation linéaire intégral-différentielle largement utilisée par les ingénieurs pour décrire le mouvement verticale du solide. Elle a été dérivée empiriquement par Cummins dans [34] à partir de la conservation d'énergie et de l'équilibre des forces. L'équation que l'on dérive est la première justification rigoureuse de l'équation de Cummins avec terme de retard et, de plus, elle améliore son approche linéaire en tenant compte des effets non-linéaires sur le mouvement du solide. Ceux-là ne sont plus négligeables dans des mouvements à grandes amplitudes.

## Fluides en rotation

Les océans et l'atmosphère sont liés d'un point de vue de dynamique de fluide. Ils présentent des analogies et propriétés communes qui ont été constatées dans des expériences réelles. En particulier, c'est dans les flots à grande échelle que ces similarités entre flots océanographiques et atmosphériques sont les plus évidentes et faciles à comprendre. Dans ce manuscrit, les flots à grande échelle sont ceux influencés par la rotation de la Terre. La présence de cette rotation affecte la vitesse du fluide à travers la *force de Coriolis*. Il s'agit d'une force fictive qui apparaît dans les équations qui décrivent le mouvement du fluide quand elles sont considérées dans un système de référence en rotation. Le paramètre dont la taille décrit l'influence de la rotation planétaire sur le mouvement de fluides géophysiques est le nombre de Rossby.

Le but de la deuxième partie du manuscrit est l'étude du comportement asymptotique de solutions des équations qui gouvernent le mouvement de fluides visqueux compressibles en rotation lorsque le nombre de Rossby et autres nombres sans dimension (nombre de Mach, nombre de Froude et nombre d'Ekman) tendent vers zéro. Ce fait donne lieu à des problèmes à limite singulière.

## Chapitre 3

Le travail présenté dans le chapitre 3 s'agit d'une collaboration avec Francesco Fanelli et Christophe Prange. Le problème mathématique traité ici permet de

décrire le plus précisément les propriétés physiques des mouvements de fluides à grande échelle sous l'influence de la rotation terrestre et de la force gravitationnelle. On prend en considération le fait que l'anisotropie a lieu pour des fluides en rotation rapide, en particulier la viscosité verticale est très petite et du même ordre que le nombre de Rossby. On considère la condition de non-glissement au bord du domaine  $\mathbb{R}^2 \times (0, 1)$ . Cela cause l'apparition de couches limites horizontales proches du fond et du sommet du domaine. De plus, on tient compte aussi des effets de stratification sur le profil de densité dus à la force gravitationnelle. Le cas bien préparé est étudié, en négligeant l'analyse de la propagation des ondes soniques. On considère le système suivant

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t (\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \Delta_{\mu, \varepsilon} u^\varepsilon - \lambda \nabla (\nabla \cdot u^\varepsilon) \\ \quad + \frac{1}{\varepsilon} e_3 \times (\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon^2} \nabla p(\rho^\varepsilon) = \frac{1}{\varepsilon^2} \rho^\varepsilon \nabla G \end{cases} \quad (5)$$

où  $\rho^\varepsilon$  est la densité du fluide,  $u^\varepsilon$  est la vitesse du fluide,  $p$  est la pression du fluide et  $G$  est le potentiel gravitationnel. Ici  $\Delta_{\mu, \varepsilon}$  désigne l'opérateur de Laplace anisotrope. Les nombres de Rossby, de Mach et de Froude sont proportionnels à un petit paramètre  $\varepsilon$ . On s'intéresse au comportement asymptotique des solutions à la limite  $\varepsilon \rightarrow 0$ . Le résultat principal du chapitre 3 est constitué d'un résultat de stabilité pour des solutions faibles de (5) qui rend la convergence quantitative et montre la structure des solutions, pour des lois de pression du type  $p \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^3((0, \infty))$  satisfaisant

$$p(0) = 0, \quad p'(\rho) > 0 \quad \forall \rho > 0, \quad \lim_{\rho \rightarrow \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = a > 0$$

avec  $\gamma \geq 3/2$ . En particulier, on utilise la méthode de l'entropie relative pour dériver une estimation quantitative sur la différence entre la solution de (5) et une solution approchée, qui contient de termes de couches limites correctifs. De plus, on montre que la vitesse limite formelle est du type  $u = (\nabla_h^\perp q, 0)$  où  $q$  est l'unique solution régulière de l'équation quasi-géostrophique visqueuse en dimension deux

$$\partial_t \left( \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle q - \langle \bar{\rho} \rangle \Delta_h q \right) - \langle \bar{\rho} \rangle \nabla_h^\perp q \cdot \nabla_h \Delta_h q + \mu \Delta_h^2 q - \frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} \Delta_h q = 0 \quad (6)$$

avec donnée initiale  $q_0 \in H^5(\mathbb{R}^2)$ . Cette régularité est nécessaire pour contrôler le terme de source dans l'inégalité d'entropie relative.

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# Preface

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The objective of this manuscript is the analysis of mathematical models with compressible-incompressible transitions arising in fluid mechanics. In oceanography and in meteorology the motion of fluids is described via different models depending on the characteristics and the properties of the flow. On the one hand, it is known in fluid mechanics that compressible flows satisfy the continuity equation, which is derived from the physical principle of the conservation of mass. On the other hand, the motion of incompressible fluids is described via a velocity vector field which is constrained to be divergence free.

In spite of this qualitative distinction, sometimes the introduction of an external actor, such as floating structures or the Earth's rotation, in the system governing the fluid motion causes some "transition" from the compressible to the incompressible property. This transition can be understood in different ways with respect to the situation considered: compressible-incompressible structure of the equations, incompressible limit of compressible flows as some parameter goes to zero, etc. Moreover, this phenomenon can happen regardless of the scale of the problem investigated. Indeed, this effect may appear either in water waves motions at the coastal (beach) scale or in large-scale oceanic and atmospheric currents.

The focus of this manuscript is twofold. In the first part of the manuscript, we study the floating structures problem, a particular fluid-structure interaction problem in which the solid is partially immersed in the fluid. This mathematical problem describes the motion of boats or wave energy converters floating on the sea water. In a particular regime, the presence of the object causes a constraint on the fluid and the governing equations turn out to have a compressible-incompressible structure.

In the second part of the manuscript, we address rotating fluids. This mathematical subject provides models which describe geophysical flows, in oceans or in the atmosphere, at large scales taking into account the influence of the Earth's rotation on their motion. In the high rotation limit, the motion of these compressible fluids tends to be horizontal and with an incompressibility constraint.

# **FLOATING STRUCTURES**



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# Introduction

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## I Wave energy converters

In recent years, the need of free and renewable energies has brought hydrodynamical engineers to focus on the implementation of wave energy converters (WEC) which are devices that allow to convert the kinetic and the potential energies associated with a moving wave into useful mechanical or electrical energy. A large variety of examples of WECs can be found in hydrodynamical engineering. However, they are substantially classified in three different ways: by the distance from the shore, by the size and orientation with respect to the wavelength and by how they act (see Figure 1 for the different families of WECs). In the case when the implementation is in the coastal region we call them *onshore*. Close to the shore,



they are called *nearshore* and typically exploit the ocean bottom. When they are far from the coast, for instance when the water depth is 100 m or more, they are called *offshore*.

The second classification deals with the size and the orientation of the structures. When their dimensions are equal or bigger than the order of magnitude of the wavelength, we call them *terminators* and *attenuator* depending on the fact that the principal direction of the structures is transversal to the waves motion or parallel. If the converter is small compared to the wave we call it *point-absorber*. We remark that the terminators can also be seen as a coastal defense technique: the waves coming towards the coast are absorbed and the environmental impact of the coastal erosions can be reduced.

Finally, the systems are divided into three macro-families according to how they act. We distinguish wave energy converters in *oscillating water columns*, which behaves like an oscillating liquid piston compressing the air in a closed chamber, *overtopping devices*, which transform the kinetic energy of overwhelming waves into potential energy, and *wave-activated bodies*, whose motion is created by the wave. This last group covers a wide range of WECs and can be separated into two subgroups. The *oscillating wave surge converters* move essentially horizontally and the *heaving buoys* move essentially vertically. The focus of this manuscript lies in heaving buoys. For the other wave energy converters, we refer the interested reader to the recent monograph [9], from which this presentation takes inspiration. Let us focus on heaving buoys and in particular on the case when they are implemented in regions where the water depth is relatively small, which normally happens in the nearshore region. The exploitation of the vertical motion of the floating structures, which is called *heave*, is the simplest and most direct way to absorb waves energy. Moreover, the costs of their implementation are sustainable and a direct intervention or maintenance is possible. These buoys are cylindrically symmetric with a vertical axis of symmetry and their diameter is usually between 5 and 10 m. While they are floating at the water surface, cables connect them to anchorage points at the bottom of the ocean. Here, the energy conversion system transforms the kinetic energy of the structure into electric energy. In contrast with their simple implementation, heaving buoys do not provide a satisfactory amount of energy. Indeed, since they are usually small compared to the wavelengths, their periods of resonance are so much shorter than the periods of the wave that they have a weak energetic performance (10 to 20% for structures of 5 to 10m). The introduction of an ad hoc acting system, though its effective application is not clear at this day, permits to control the responses and significantly increase the performance of heaving buoys (see [21]).

A better comprehension of the interaction between WECs and the water waves may guarantee an effective implementation (prediction of the surface) and maxi-

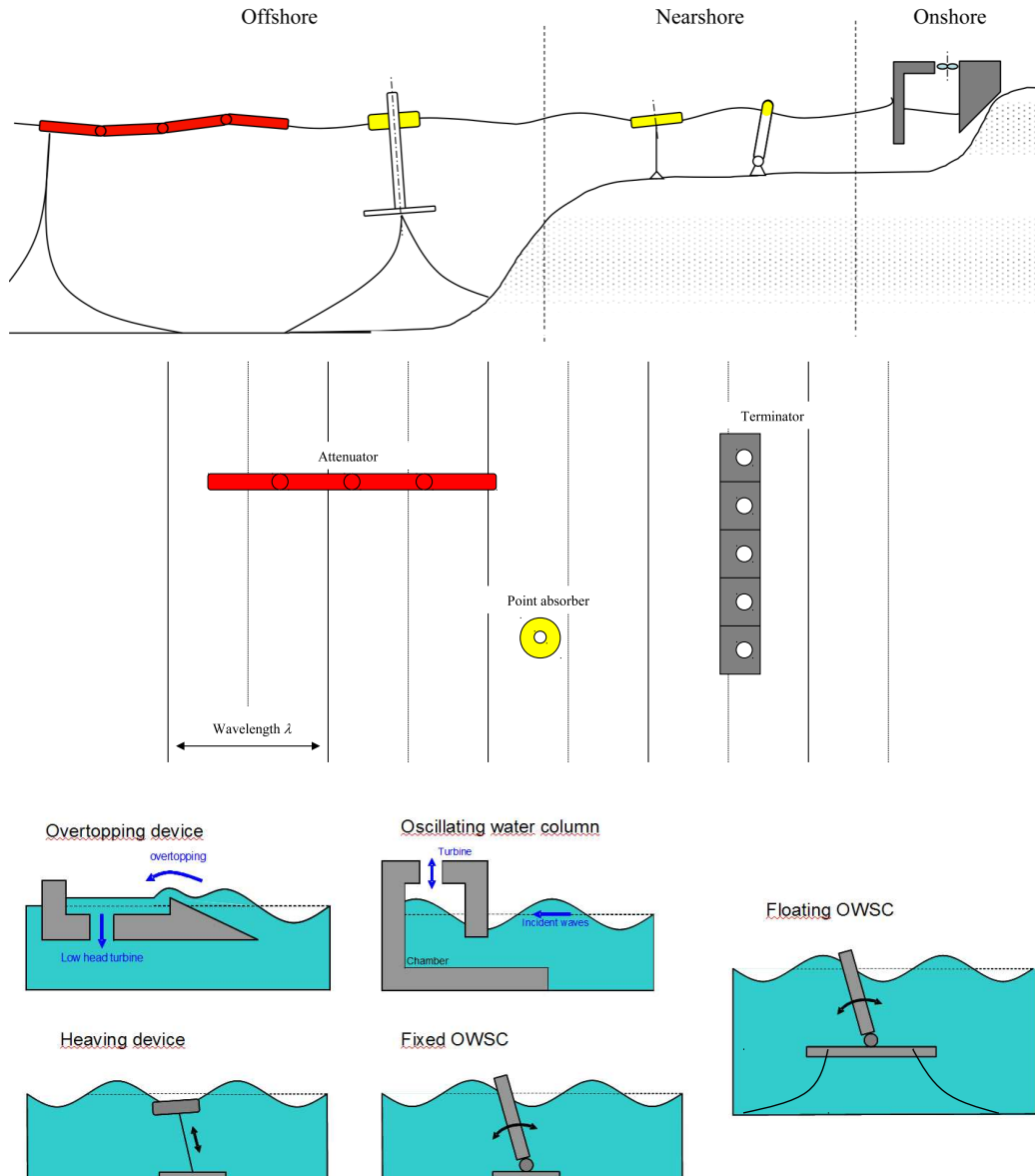


Figure 1 – (from Babarit [9]) Classification of WECs by the distance from the shore (top), by the size and the orientation with respect to the wavelength (middle) and by how they act (bottom).

mize the ratio of the produced energy to the energy spent for the implementation. Hence, engineers and oceanographers reproduced tests in wave tanks and did numerical simulations. In order to understand the behavior of the wave-structure interaction and to have a general description of the phenomenon, a rigorous mathematical model is needed. The goal is to describe the features of this interaction, taking into account nonlinear effects and the variations of the contact line between the water and the solid.

Although the analysis of WECs is quite recent, floating structures have been already studied in the last century in order to describe the motion of ships. Froude in 1861 discussed in [55] the rolling of ships and Krylov in 1898 presented a method to compute the hydrodynamic loads for ship motions in waves, neglecting the influence of the ship on the waves.

The first formulation of the *floating body problem* appeared in two famous papers [75, 76] of John. It is a particular example of fluid-structure interaction, when a partially immersed body is floating at the fluid free surface. In this problem two free boundary problems need to be treated. The first free boundary problem is the one coming from the free surface elevation. The second free boundary problem is given by the fact that the portion of the body in contact with the fluid depends on time, so that the contact line is a free boundary.

The difficulty of the problem brought John to study a more simplified problem. He considered a linear one-dimensional model in order to describe the evolution of the free surface waves and he used a potential velocity formulation. Then, he assumed that the motion of the solid is of small amplitude and he neglected the variations of the contact line in time. These assumptions permitted him to avoid the free boundary problem associated with the contact line. The linear approach was used also by Ursell in [120] and Maskell and Ursell in [90] to describe the heave of a floating body.

Though the model of John is oversimplified, it was extensively adopted in hydrodynamic engineering. In particular, we refer to Cummins who, dealing with ship motion, proposed in [34] his celebrated delay differential equation on the six modes of response: surge, sway, heave, roll, pitch and yaw. We will comment on this equation in Section III.2. The linear potential model is still used in present days and represents the principal method in the literature related to floating structures, such as wave energy converters [94]. For instance, the well-known solvers WAMIT and NEMOH, used for the analysis of wave interactions with offshore structures, are based on this model.

Recently, Lannes addressed the modeling of floating structures in [83], where he took into consideration the nonlinear effects and the evolution of the contact line. The author showed that the resulting equations have a compressible form in the domain where the waves are not in contact with the solid and an incompressible

form in the contact region. The same structure was shown for reduced asymptotic models. Moreover, a reformulation of the Newton's laws for the free solid motion was given. His work motivated the first part of this manuscript. Section III follows the analysis in [83].

To have a better comprehension of this fluid-structure interaction, let us first deal only with the motion of the waves, in the absence of a floating body. In this case the problem reduces to the well-known water waves problem, whose theory and results are briefly presented in the next section.

## II Water waves problem

The water waves problem consists in describing the motion, under the influence of the gravity, of a fluid in a domain with a fixed bottom and a free surface at the top that separates it from vacuum or another fluid whose density can be neglected. One particular example of this mathematical problem is the motion of ocean waves, where the free surface is the interface between the water and the air. We consider a homogeneous inviscid incompressible fluid with an irrotational flow. The particles do not cross neither the surface nor the bottom of the domain, which can be both parametrized as graphs of some functions. Since the domain considered is infinite, the fluid velocity is assumed to vanish at infinity. The water depth is always bounded from below by a nonnegative constant. Moreover, the surface tension is neglected and the external pressure assumed constant.

Let us comment on the previous assumptions. Here we are interested at the scale of a beach. The density of seawater does not show significant variations at this scale and the homogeneity assumption is justified. On the other hand, the density of the air is neglected. In general it is small but not zero and we should deal with a two-fluids interface problem, in which two fluids of different densities are separated by an interface and Kelvin- Helmholtz instabilities may appear. In [81] Lannes showed that the density of the air can be neglected in the asymptotic models used in coastal oceanography to describe the propagation of waves and our assumption is justified.

The inviscidity assumption is introduced to neglect energy dissipations at the bottom, in the interior of the domain and at the surface. The presence of viscosity causes the apparition of two different type of boundary layers, one close to the bottom and one close to the surface. While handling the bottom boundary layer is still an open problem, the uniform (with respect to the viscosity) boundedness of the Lipschitz norm of the velocity in the surface boundary layer allowed Masmoudi and Rousset to justify in [93] the inviscid limit of the free-surface Navier-Stokes equations to the water waves equations in infinite depth. This result was extended

to the case with surface tension by Elgindi and Lee in [42]. Moreover, this assumption is motivated by the typical scales of the configuration we are interested in. Indeed, the typical size of the boundary layer due to the molecular viscosity between water and a solid boundary is approximately  $10^{-3}\text{m}$ , which is much smaller than the scales we are interested here. For instance, the damping effects experienced by a freely floating object are mainly due to the energy it loses by creating waves, and not to viscous effects.

The incompressibility assumption comes from the fact that the ratio of the typical wave celerity to the speed of sound is very small unless the characteristic depth is of the order of 200 km, which is not the case.

The irrotationality assumption is introduced here since rotational effects are often negligible up to the breaking point of the waves. The vorticity was taken into account by P. Zhang and Z. Zhang in [131] and by Constantin in [30] using Lagrangian coordinates. Recently, Castro and Lannes derived a new formulation of the rotational case in [23], showing its local well-posedness, the Hamiltonian structure and uniform bounds allowing the shallow-water limit.

The assumptions on the fluid behavior at the top and the bottom provide boundary conditions. The first is a kinematic condition at the surface and the second one is that the normal component of the velocity must vanish at the bottom. We will show later the explicit expressions.

Since we are interested to the description of the wave behavior as it approaches the shore, the wave does not enter the breaking zone and hence can be assumed to be parametrized a graph. However, the wave can be considered as a parametrized curve or surface as Wu did in [124] and [125]. This approach is required for the study of splash singularities by Castro, Cordoba, Fefferman, Gancedo and Gómez-Serrano in [22]. It is not necessary that the bottom is a graph either. Indeed, the bottom affects the dynamics of the surface only at lower order. For instance, rough bottoms (with any regularity) were considered by Alazard, Burq and Zuily in [3].

The bound on the water depth excludes vanishing depths configurations such as beaches. The analysis of the vanishing depth problem was tackled by de Poyferré in [37] and by Ming in [99] for the Euler equations. Lannes and Métivier recently addressed the shoreline problem in [84] for reduced asymptotic models.

The surface tension was considered in many works, for instance by Coutand and Shkoller in [31], by Ming and Zhang in [100] and by Alazard, Burq and Zuily in [3], but for the same reason as the viscosity (see above) we neglect it here.

## II.1 Free surface Euler equations

The previous assumptions can be reformulated using a mathematical language. Let us first introduce the following notation:

- The domain occupied by the fluid at time  $t$  is denoted  $\Omega(t) \subseteq \mathbb{R}^{d+1}$ , where  $d$  is the horizontal dimension (for oceanography applications  $d = 1, 2$ ).
- The velocity of the fluid at  $(X, z) \in \Omega(t)$  at time  $t$  is  $\mathbf{U}(t, X, z) \in \mathbb{R}^{d+1}$ . The horizontal component is  $V(t, X, z) \in \mathbb{R}^d$  and the vertical one is  $w(t, X, z) \in \mathbb{R}$ .
- $P(t, X, z) \in \mathbb{R}$  is the pressure at time  $t$  at the point  $(X, z) \in \Omega(t)$ .
- The acceleration of gravity (constant) is denoted by  $-g\mathbf{e}_z$ , where  $g > 0$  and  $\mathbf{e}_z$  is the upward unit vector in the vertical direction.
- The constant fluid density is  $\rho$ .

The motion of a homogeneous, inviscid, incompressible, and irrotational fluid is governed by the Euler equation, with constraints on the divergence and the curl of the velocity field:

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z}) \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g\mathbf{e}_z \quad \text{in } \Omega(t), \quad (\text{II.1})$$

$$\nabla \cdot \mathbf{U} = 0 \quad \text{in } \Omega(t), \quad (\text{II.2})$$

$$\nabla \times \mathbf{U} = 0 \quad \text{in } \Omega(t). \quad (\text{II.3})$$

We assume that there exist two functions  $b : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\zeta : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$  for  $T > 0$  such that

$$\forall t \in [0, T), \quad \Omega(t) = \{(X, z) \in \mathbb{R}^{d+1}, -h_0 + b(X) < z < \zeta(t, X)\}, \quad (\text{II.4})$$

where  $h_0 > 0$  is a constant reference depth; the hyperplane  $z = 0$  corresponds to the still water level. Denoting by  $\mathbf{N}_b$  and by  $\mathbf{N}$  the normal unit vector pointing upwards to the bottom and to the surface, the boundary conditions at the bottom and at the free surface read

$$\mathbf{U} \cdot \mathbf{N}_b = 0, \quad \mathbf{N}_b = (-\nabla b, 1)^T \quad \text{at } z = -h_0 + b(X), \quad (\text{II.5})$$

$$\partial_t \zeta - \mathbf{U} \cdot \mathbf{N} = 0, \quad \mathbf{N} = (-\nabla \zeta, 1)^T \quad \text{at } z = \zeta(t, X), \quad (\text{II.6})$$

where (II.5) and (1.1.4) are called the *non-permeability* and the *kinematic* equations respectively. Denoting by  $P_{\text{atm}}$  the constant atmospheric pressure, the pressure at the surface is given by

$$P = P_{\text{atm}} \quad \text{at } z = \zeta(t, X), \quad (\text{II.7})$$

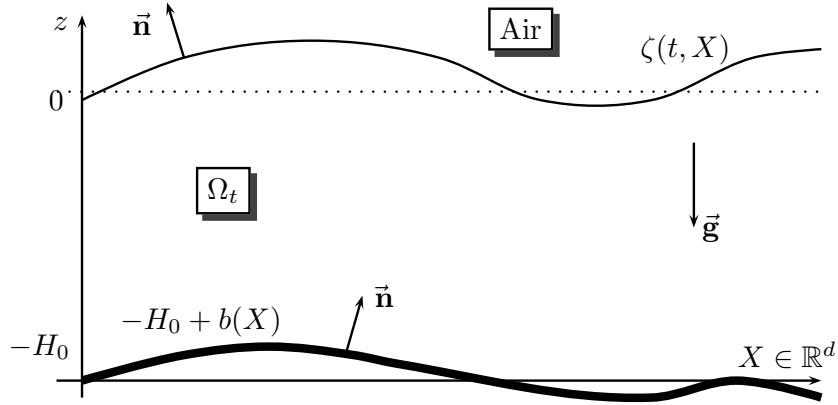


Figure 2 – (from Lannes [82]) Main notations:  $X$  is the horizontal variable,  $z$  is the vertical variable,  $\Omega(t)$  is the domain,  $-h_0 + b(X)$  is the bottom,  $\zeta$  is the free surface,  $\mathbf{n}$  is the normal unit vector and  $\mathbf{g}$  is the gravity.

while the non-vanishing shoreline and the rest-at-infinity assumption are

$$\forall t \in [0, T], \quad \lim_{(X,z) \in \Omega(t), |(X,z)| \rightarrow \infty} |\zeta(t, X)| + |\mathbf{U}(t, X, z)| = 0, \quad (\text{II.8})$$

$$\exists h_{\min} > 0, \quad \forall (t, X) \in [0, T] \times \mathbb{R}^d, \quad h_0 + \zeta(t, X) - b(X) \geq h_{\min}. \quad (\text{II.9})$$

The equations (II.1) - (II.9) are called *free surface Euler equations*. They conserve the fluid energy  $E_{\text{fluid}}$  defined as

$$E_{\text{fluid}} = \frac{1}{2} \int_{\Omega(t)} |\mathbf{U}|^2 + \frac{1}{2} \int_{\mathbb{R}^d} g \zeta^2, \quad (\text{II.10})$$

where the first term of the sum is the kinetic energy and the second is the potential energy.

### Zakharov-Craig-Sulem formulation

We present in this section the Zakharov-Craig-Sulem formulation of the water waves equations. Zakharov [130] remarked that the free surface elevation  $\zeta$  and the trace of the velocity potential at the surface  $\Phi|_{z=\zeta}$  fully define the flow. Craig, Sulem and Sulem [32] and Sulem [33] gave a formulation of the equations introducing the Dirichlet-Neumann operator. More precisely, from assumption (II.3), there exists a mapping  $\Phi : [0, T] \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  such that  $\mathbf{U} = \nabla_{X,z} \Phi$ . Replacing the potential expression in the incompressibility condition (II.2), in the non-permeability condition (II.5) and denoting by  $\psi$  the trace of the velocity potential at the surface, we get the following Laplace equation

$$\begin{cases} \Delta_{X,z} \Phi = 0 & \text{in } \Omega(t) \\ \Phi|_{z=\zeta} = \psi, \quad \partial_{\mathbf{n}} \Phi|_{z=-h_0+b} = 0. \end{cases} \quad (\text{II.11})$$

with Dirichlet and Neumann boundary conditions respectively at the free surface and at the bottom. For  $\zeta, b$  regular enough satisfying the non-vanishing shoreline assumption (II.9) and  $\mathbf{U} = \nabla_{x,z} \Phi$  satisfying the rest-at-infinity assumption (II.8), the system (II.11) admits a unique variational solution  $\Phi$  for any Dirichlet condition  $\psi$  using the Lax-Milgram theorem (see Chapter 2 of [83] for details). Then the velocity field  $\mathbf{U}$  is recovered and the pressure  $P$  follows from (II.1). Therefore, the problem can be formulated via two evolution equations on  $\zeta$  and  $\psi$ , since these are all the relevant quantities. For this purpose, Craig and Sulem introduced the Dirichlet-Neumann operator, defined as

$$G[\zeta, b] : \psi \quad \mapsto \quad \sqrt{1 + |\nabla\zeta|^2} \partial_{\mathbf{n}} \Phi|_{z=\zeta}, \quad (\text{II.12})$$

where  $\Phi$  solves (II.11). This operator is linear with respect to  $\psi$  but highly non-linear with respect to the free surface  $\zeta$  and the bottom  $b$ . We refer to Chapter 3 of [83] for all the details and the properties about this operator. Therefore, the water waves problem can be written as a system of two scalar evolution equations:

$$\begin{cases} \partial_t \zeta - G[\zeta, b] \psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2} |\nabla\psi|^2 - \frac{(G[\zeta, b] \psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + |\nabla\zeta|^2)} = 0. \end{cases} \quad (\text{II.13})$$

Note that, written under this form, the problem ceases to be a free boundary problem in terms of  $(\zeta, \psi)$  and the domain of the problem is  $d$ -dimensional, while the primitive problem is a free boundary  $(d+1)$ -dimensional problem in terms of  $(\zeta, \mathbf{U})$ . Moreover, the system (II.13) has a Hamiltonian structure: indeed it can be written under the form

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_\zeta E_{\text{fluid}} \\ \partial_\psi E_{\text{fluid}} \end{pmatrix},$$

with the fluid energy  $E_{\text{fluid}}$ , which is conserved by the equations, given by

$$E_{\text{fluid}} = \frac{1}{2} \int_{\mathbb{R}^d} \psi G[\zeta, b] \psi + \frac{1}{2} \int_{\mathbb{R}^d} g\zeta^2. \quad (\text{II.14})$$

### Well-posedness of the problem

The well-posedness theory of the water waves problem is well-known for different formulations. All the early works were set in a Lagrangian framework which allows to consider the surface waves that are not graphs. Nalimov in [101] showed first the local well-posedness result for Sobolev small initial data in the one-dimensional case using a formulation in which the velocity is a vector field complex-valued. The author considered the case of infinite depth and the result was extended to the finite



depth case by Yosihara in [129]. The smallness condition on the initial data was removed by Wu in [124]. She modified the Nalimov formulation in the infinite depth case and she was able to treat also the two-dimensional case in [125]. Moreover, using the same formulation the author proved an almost global existence result in [126] for the one-dimensional case and a global well-posedness result in [127] for the two-dimensional case. Differently from the previous works, Lannes considered the Eulerian framework proving in [80] the well-posedness of the Zakharov-Craig-Sulem formulation (II.13) with finite depth (flat or space-dependent bottoms) in any dimension. The author established the well-posedness result without assuming any smallness condition on the initial data. Alazard, Burq and Zuily in [4] required less regularity for the initial data in any dimension using a paradifferential approach. More precisely, they showed the well-posedness of (II.13) for initial surface with unbounded curvature and for initial velocities with only Lipschitz regularity. Recently in [5] the same authors improved the result to cover also non-Lipschitz initial velocities using Strichartz estimates.

The interest of this manuscript does not lie in the analysis of the full water waves equations, but in the study of asymptotic models deriving from it. To do this, we write the dimensionless water waves equations and look at the different regimes depending the size of some dimensionless parameters.

## II.2 Dimensionless equations

We introduce here the dimensionless parameters, derived from the characteristic scales of the problem, in order to study the qualitative properties of the solutions to (II.13). For the sake of simplicity, the bottom of the domain is assumed to be flat, *i.e.*  $b(X) = 0$ . Thus, the main length scales of the water waves problem are:

1. The typical water depth  $h_0$
2. The typical wavelength  $L$
3. The order of the free surface amplitude  $a_{\text{surf}}$

Then, we can define the following independent dimensionless parameters

$$\varepsilon = \frac{a_{\text{surf}}}{h_0}, \quad \mu = \frac{h_0^2}{L^2} \quad (\text{II.15})$$

called respectively the *nonlinearity*, *shallowness* parameters. According to the regime considered, assumptions on these parameters are made. For example, in

the case of small amplitude waves the nonlinearity parameter  $\varepsilon$  is assumed to be small. The dimensionless Dirichlet-Neumann operator becomes

$$\mathcal{G}_\mu[\varepsilon\zeta]\psi = \sqrt{1 + |\nabla(\varepsilon\zeta)|^2} \partial_{\mathbf{n}} \Phi|_{z=\varepsilon\zeta} \quad (\text{II.16})$$

with  $\Phi$  satisfying the nondimensionalized version of (II.11)

$$\begin{cases} \mu \partial_x^2 \Phi + \partial_z \Phi = 0 & \text{for } -1 \leq z \leq \varepsilon\zeta, \\ \Phi|_{z=\varepsilon\zeta} = \psi, \quad \partial_{\mathbf{n}} \Phi|_{z=-1} = 0. \end{cases} \quad (\text{II.17})$$

Rewriting (II.13) with the new dimensionless quantities, the water waves equations in the dimensionless form become

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu\nu} \mathcal{G}_\mu[\varepsilon\zeta]\psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2\nu} |\nabla\psi|^2 - \frac{\varepsilon\mu}{\nu} \frac{\left(\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \nabla(\varepsilon\zeta) \cdot \nabla\psi\right)^2}{2(1 + \varepsilon^2\mu|\nabla\zeta|^2)}, \end{cases} \quad (\text{II.18})$$

with  $\nu \sim 1$  if  $\mu$  is small. In [6] Alvarez-Samaniego and Lannes proved the well-posedness of (II.18) uniformly with respect to  $\varepsilon$ ,  $\mu$  and other dimensionless parameters related to the topography and the transversal character of the waves. The authors established a large time existence result (of order  $O(1/\varepsilon)$ ), in the sense that it does not shrink as  $\mu$  goes to zero, which provides a uniform energy control. They exploited this fact to rigorously justify the main asymptotic models used in coastal oceanography. We write here the regimes and the approximation orders corresponding to some of the different reduced models they justified:

- Shallow water models  $\mu \ll 1$ :
  - Large amplitude  $\varepsilon \sim 1$ :
    - Nonlinear shallow water equations  $O(\mu)$
    - Green-Naghdi equations  $O(\mu^2)$
  - Medium amplitude  $\varepsilon \sim \sqrt{\mu}$ :
    - Serre equations  $O(\mu^2/\varepsilon)$
  - Small amplitude  $\varepsilon \sim \mu$ :
    - Boussinesq systems  $O(\varepsilon^2)$
- Deep water models  $\mu \geq 1$ 
  - Small steepness  $\varepsilon\sqrt{\mu} \ll 1$ :
    - Full dispersion model  $O(\varepsilon\sqrt{\mu})$

Let us focus on the nonlinear shallow water equations, also called Saint-Venant<sup>1</sup> equations. They read

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t \bar{V} + \nabla \zeta + \varepsilon(\bar{V} \cdot \nabla)\bar{V} = 0, \end{cases}$$

where  $\bar{V} = \int_{-1}^{\varepsilon\zeta} \nabla \Phi$ , with  $\Phi$  satisfying (II.17), is the vertically averaged horizontal component of the velocity and  $h = -1 + \varepsilon\zeta$  is the fluid depth. Although the latter is a reduced model, the first equation in (II.2) is exact, in the sense that no approximations with respect to  $\mu$  or the other parameters are made. Hence, it appears also in the other models listed before. To derive it, we only use the incompressibility condition (II.2) and the boundary condition (II.5). We notice that the irrotationality assumption (II.3) does not play any role here. Moreover, it is evident that the divergence operator obtained is easier to work with than the complex Dirichlet-Neumann operator. The aim at closing the equations in terms of the free surface and this new vertical integrated quantity motivates the derivation of a different formulation, which can deal also with the rotational case. It is called the *depth averaged formulation*.

### II.3 Depth averaged formulation

In the previous section we have presented the Zakharov-Craig-Sulem formulation for the water waves equations, where the dimension reduction has been done using the irrotational condition  $\nabla \times \mathbf{U} = 0$  which permits to write  $\mathbf{U} = \nabla_{X,z} \Phi$  and introduce the trace of the potential velocity  $\psi = \Phi|_{z=\zeta}$ . A set of evolution equations (II.13) on  $(\zeta, \psi)$  has been successively derived. Though this formulation has been widely used in the analysis of the water waves equations, the asymptotic models employed for applications in oceanography, as the ones cited at the end of the previous section, are generally not expressed via this couple of unknowns. They are derived using a formulation, present also in numerical simulations, which relies on the integration along the fluid depth and permits, as in the Zakharov-Craig-Sulem formulation, to get rid of the vertical dimension and reduce the problem to a  $d$ -dimensional one.

Another reason for which it is useful to introduce the depth averaged formulation is that, as we shall see later, in the presence of a floating structure it is easy to reformulate the contact between the fluid and the object as a constraint on the new unknown under the solid.

Let us introduce the new unknown, which couples with the free surface elevation  $\zeta$  in this formulation, as follows:

<sup>1</sup>Barré de Saint-Venant derived these equations in 1871 in [38, 39].

**Definition II.1.** Given  $\mathbf{U}(t, X, z)$  the fluid velocity field, the horizontal discharge  $Q$  is defined as

$$Q(t, X) := \int_{-h_0}^{\zeta(t, X)} V(t, X, z) dz, \quad (\text{II.19})$$

where  $V$  is the ( $d$ -dimensional) horizontal component of  $\mathbf{U} = (V, w)^T$ .

It is immediate that  $Q = h\bar{V}$  and writing the quantities with dimensions, we get the following set of evolution equations on  $(\zeta, Q)$ :

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0 \\ \partial_t Q + \nabla \cdot \left( \int_{-h_0}^{\zeta} V \otimes V \right) + gh \nabla \zeta + \frac{1}{\rho} \int_{-h_0}^{\zeta} \nabla P_{\text{NH}} = 0, \end{cases} \quad (\text{II.20})$$

where  $h$  is the fluid depth defined as  $h(t, X) = \zeta(t, X) + h_0$ ,  $\rho$  is the fluid density and the non-hydrostatic pressure  $P_{\text{NH}}$  reads

$$P_{\text{NH}}(t, X, z) = \rho \int_z^{\zeta(t, X)} (\partial_t w + \mathbf{U} \cdot \nabla_{X,z} w).$$

At this point, the reader may doubt if the system (II.20) is in some sense a real "reformulation" of the original water waves problem, meaning that the velocity field  $\mathbf{U}$  in  $\Omega(t)$  can be fully recovered if  $\zeta$  and  $Q$  are given. A rigorous justification of this reconstruction was given by Lannes in [83]. He defined the average operator

$$\text{Av}[\zeta] : \quad \mathbf{U} \quad \mapsto \quad \bar{V}$$

and the reconstruction operator

$$\text{Rec}[\zeta] : \quad \bar{V} \quad \mapsto \quad \nabla_{X,z} \Phi$$

where  $\Phi$  satisfies the Laplace system (II.11) with  $-G[\zeta]^{-1}(\nabla \cdot Q)$  as Dirichlet boundary condition. He showed that they are well-defined and that  $\text{Rec}[\zeta]$  is a right-inverse of  $\text{Av}[\zeta]$ . In other words, given the horizontal discharge  $Q$  and the free surface elevation  $\zeta$ , the primitive velocity field  $\mathbf{U}$  can be recovered by applying  $\text{Rec}[\zeta]$  to  $\bar{V}$ . This fact permits to write the system (II.20) in a closed form, with all quantities depending on  $\zeta$  or  $Q$ . Indeed, using (1.1.4) and integrating along the depth the horizontal component of (II.1), we can write the water waves equations in the dimensionless form

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \varepsilon \nabla \cdot \left( \frac{1}{h} Q \otimes Q \right) + h \nabla \zeta + \varepsilon \nabla \cdot \mathbf{R}(\zeta, Q) + h \mathbf{a}_{\text{NH}}(\zeta, Q) = 0. \end{cases} \quad (\text{II.21})$$

where

$$\mathbf{R}(\zeta, Q) = \int_{-1}^{\varepsilon\zeta} (V - \bar{V}) \otimes (V - \bar{V}), \quad \bar{V} = \frac{1}{h} \int_{-1}^{\varepsilon\zeta} V$$

$$h\mathbf{a}_{\text{NH}}(\zeta, Q) = \int_{-1}^{\varepsilon\zeta} \nabla \left( \int_z^{\varepsilon\zeta} \partial_t w + \varepsilon V \cdot \nabla w + \frac{\varepsilon}{\mu} w \partial_z w \right).$$

with  $(V, w)^T = \text{Rec}[\zeta] \bar{V}$  and  $\bar{V} = Q/h$ . It can be shown that the horizontal and the vertical component of the velocity vector field take the form

$$V = \bar{V} + O(\mu), \quad w = O(\mu),$$

hence we have

$$\mathbf{R}(\zeta, Q) = O(\mu^2), \quad h\mathbf{a}_{\text{NH}}(\zeta, Q) = O(\mu).$$

At order  $O(\mu)$  these two terms can be neglected and we obtain the *nonlinear shallow water equations in the depth averaged formulation*

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot \left( \frac{1}{h} Q \otimes Q \right) + h \nabla \zeta = 0. \end{cases} \quad (\text{II.22})$$

For the sake of simplicity we have assumed  $\varepsilon = 1$ .

### III Water waves equations with floating objects

Let us introduce here a floating object in the configuration considered in the previous section. A relevant difference with respect to the water waves problem relies on the structure of the problem. In the absence of the structure, the fluid pressure at the surface is constrained to be equal to the atmospheric pressure and the surface is free. Conversely, in the presence of the structure, the fluid pressure under the solid is free, in the sense that is not constrained but it is an unknown of the problem, while the surface is constrained by the solid. Several issues come from the presence of the solid. One is the choice of appropriate boundary conditions: indeed, according to the geometric form of the solid (vertical walls “boat-type” shape), different conditions need to be assumed to write a valid model and ensure the energy conservation of the fluid-structure coupled system. A very intricate point is the evolution of the contact line between the fluid, the object and the air: this increases the difficulty of the problem, which contains two free boundary problems (the first is the one associated with the free surface not in contact with the fluid).

Let us introduce some notations which will be used later. We call  $C(t)$  the region occupied by the solid at time  $t$ ,  $\partial C(t)$  the boundary and  $\partial_w C(t)$  the portion

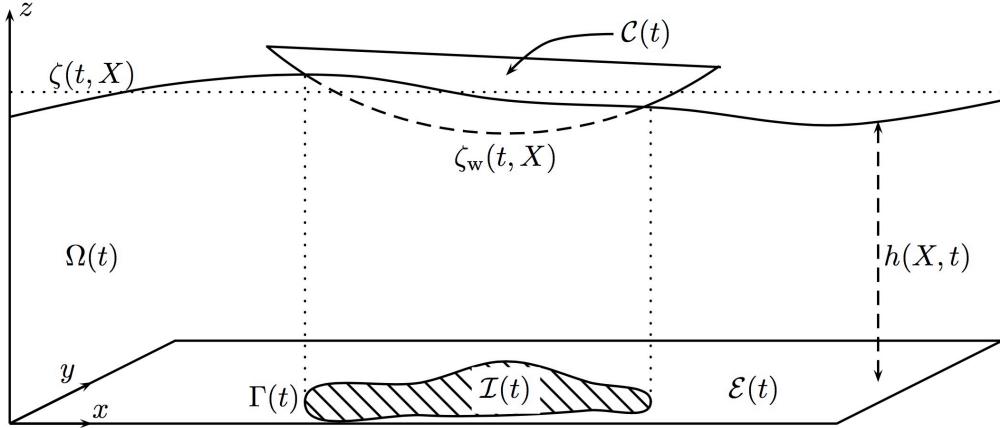


Figure 3 – (from Lannes [83]) A floating object on water waves

of the boundary in contact with the fluid, called the *wetted surface*. As for the fluid free surface, we assume that the wetted surface can be parametrized as the graph of some function  $\zeta_w(t, X)$  for  $X \in \mathcal{I}(t)$ . The velocity of the solid on the wetted surface is denoted by  $\underline{U}_w$ . The presence of the solid naturally allows to divide the horizontal hyperplane  $\mathbb{R}^d$  into two regions, the projection  $\mathcal{I}(t)$  of the wetted surface on it, and  $\mathcal{E}(t) := \mathbb{R}^d \setminus \mathcal{I}(t)$  (see Figure 3). We call them *interior* and *exterior domain* respectively. The boundary  $\Gamma(t) := \partial\mathcal{I}(t) = \partial\mathcal{E}(t)$  is called the projection of the *contact line*, where the solid, the fluid and the exterior air interact. For simplicity we call  $\Gamma(t)$  itself the contact line.

As written before, John treated a simplified problem: he considered the following linear equations for the evolution of the free surface in the exterior domain:

$$\begin{cases} \partial_t \zeta - \partial_z \Phi|_{z=0} = 0, \\ \partial \Phi|_{z=0} + g\zeta = 0. \end{cases}$$

In addition, he assumed the solid motion to be of small amplitude and he neglected the time variations of the wetted surface. In the interior domain, the continuity of the normal velocity at the wetted surface was assumed,

$$\partial_{\mathbf{n}} \Phi = \underline{U}_w \cdot \mathbf{n}.$$

In the case of a free motion, in order to compute  $\underline{U}_w$  via Newton's laws, the pressure exerted by the fluid on the wetted surface has to be known. Denoting it by  $\underline{P}_i$ , John was capable to recover this quantity via the linearized Bernoulli equation

$$-\frac{\underline{P}_i - P_{\text{atm}}}{\rho} = \partial_t \Phi|_{z=\zeta_w} + g\zeta_w$$

where  $P_{\text{atm}}$  is the constant atmospheric pressure.

The main issue of modeling floating structures is to take into account the non-linear effects and the evolution of the contact line. As already said, John's model does not answer these questions. In recent years, these aspects have been considered in several numerical studies, such as [67] and [77]. The concern expressed is the large computational cost, coming from the resolution of a  $(d+1)$ -dimensional elliptic problem in the fluid domain to recover the interior pressure  $\underline{P}_i$  through the Bernoulli equation.

In the next section we present a more accurate model based on the depth averaged formulation in Section II.3. It was introduced by Lannes in [83] and it aims at taking into account nonlinear effects, the evolution of the contact line. The pressure is recovered by solving a  $d$ -dimensional problem, reducing the computational cost for numerical applications. In particular we are interested in the shallow water regime with a precision of order  $O(\mu)$ . It will be shown that in this regime the transition conditions between the interior domain and the exterior domain are easy to write down in terms of the discharge  $Q$ . This fact can be seen as a confirmation that this unknown is more suitable than the velocity potential  $\Phi$  to shallow water models when a floating structure is considered.

### III.1 Depth averaged nonlinear shallow waters equations with a floating structure

Let us consider the same assumptions on the fluid as in Section II and (II.1) - (1.1.4) as the equations governing the fluid motion. Differently from the water waves problem, the fluid surface in the interior domain is constrained to coincide with the parametrization of the solid wetted surface  $\zeta_w$ . From a physical point of view, this constraint represents the situation of a fluid completely attached to the solid, that is a quite intuitive configuration. It reads

$$\zeta = \zeta_w \quad \text{in} \quad \mathcal{I}(t). \quad (\text{III.1})$$

On the other hand, in the interior domain the pressure at the free surface is an unknown of the problem, depending on the dynamics of the solid. Contrarily, in the exterior domain the same structure as in the water waves problem holds: the surface of the fluid is free, an unknown of the problem and the pressure at the surface is assumed to coincide with the atmospheric pressure, namely

$$\underline{P} = P_{\text{atm}} \quad \text{in} \quad \mathcal{E}(t). \quad (\text{III.2})$$

We distinguish now two different configurations, depending on the characteristics of the solid side-walls: when they are non-vertical and when they are vertical.

### Non-vertical side-walls

In a configuration as the one showed in Figure 3, two boundary conditions at the contact line  $\Gamma(t)$  are assumed: the continuity of the free surface elevation  $\zeta$ , namely

$$\zeta_e = \zeta_i \quad \text{at} \quad \Gamma(t), \quad (\text{III.3})$$

and the continuity of the pressure at the free surface  $\underline{P}$ , namely

$$\underline{P}_i = P_{\text{atm}}, \quad \text{at} \quad \Gamma(t). \quad (\text{III.4})$$

The equations (II.1) - (1.1.4) together with (1.1.7), the constraint (1.1.6) and the boundary conditions (III.3) - (III.4) are called *constrained free surface Euler equations*.

As done for the case without a floating object, we can derived the dimensionless equations from (II.2) and (II.1) taking into account the new constraints and boundary conditions. After having neglected the terms of order  $O(\mu)$ , we dimensionalize the equations obtaining the *averaged nonlinear shallow water equation with a floating structure*

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot \left( \frac{1}{h} Q \otimes Q \right) + gh \nabla \zeta = -\frac{h}{\rho} \nabla \underline{P} \end{cases} \quad (\text{III.5})$$

The surface pressure  $\underline{P}$  is given by

$$\underline{P}_e = P_{\text{atm}} \quad \text{and} \quad \begin{cases} -\nabla \cdot \left( \frac{h_w}{\rho} \nabla \underline{P}_i \right) = -\partial_t^2 \zeta_w + \mathbf{a}_{FS}(\zeta_w, Q_i) & \text{in} \quad \mathcal{I}(t) \\ \underline{P}_i|_{\Gamma(t)} = P_{\text{atm}} \end{cases} \quad (\text{III.6})$$

where

$$\mathbf{a}_{FS}(\zeta_w, Q_i) = \nabla \cdot \left( \nabla \cdot \left( \frac{1}{h_w} Q_i \otimes Q_i \right) + gh_w \nabla \zeta_w \right),$$

and the transition conditions at the contact line read

$$\zeta_e = \zeta_i, \quad Q_e = Q_i \quad \text{at} \quad \Gamma(t). \quad (\text{III.7})$$

The interior pressure  $\underline{P}_i$  is a Lagrange multiplier associated with the contact constraint (1.1.6). Injecting the solution to the elliptic problem (III.6) into the momentum equation of (III.5) the constraint in the interior domain can be removed. The same technique is used in the incompressible Euler equations which can be reduced to an unconstrained equation by using the pressure to remove the incompressible constraint. Thus, the system (III.5) has an incompressible behavior in the



interior domain, while in the exterior domain the absence of the pressure gradient ( $P_e$  is constant) provides a compressible form to the equations. This compressible-incompressible structure of the equations typically characterizes congested flows arising in different contexts. In the case when  $g = 0$  this model appears in traffic jams [13], crowds motions [40], granular flows [105, 106], and compressible-low Mach coupling in gaz dynamics [104]. The transition conditions at the congested zone, in our case the contact line, play a key role and the computation of the evolution of the free boundary is intricate.

We remark that in the case with a floating structure, at this day the rigorous justification of the nonlinear shallow water equations (III.5) as an asymptotic model of the water waves equations, as Alvarez-Samanego and Lannes in [6] and Iguchi in [72] did without objects, is not known in the literature.

### Vertical side-walls

The boundary conditions (III.3) - (III.4) obviously do not permit to model the interaction between water waves and structures with vertical walls at the contact line. These particular structures are often considered in numerical simulations and, as described in Section I, heaving buoys have this geometrical property. The main difference with the situation in Figure 3 is the jump of the surface elevation at the contact line. In order to justify the boundary condition chosen later, let us show how to derive the transition condition on the discharge in (III.7). Using the continuity of the free surface at the contact line  $\zeta_e = \zeta_i = \zeta$ , it yields

$$Q_e - Q_i = \int_{-h_0}^{\zeta} (V_e - V_i) dz \quad \text{at} \quad \Gamma(t). \quad (\text{III.8})$$

Due to the incompressibility and irrotationality conditions (II.2) and (II.3), elliptic estimates provide the smoothness of the vector field  $\mathbf{U}$  in the interior of the domain  $\Omega(t)$  and (III.7) follows. On the other hand, in the presence of vertical walls, the smoothness of the fluid velocity can be used only under the solid and we have

$$Q_e - Q_i = \int_{\zeta_i}^{\zeta_e} V dz \quad \text{at} \quad \Gamma(t). \quad (\text{III.9})$$

On the wetted part of the vertical walls, we assume the continuity of normal component of the velocities, namely

$$V \cdot \nu = V_C \cdot \nu \quad (\text{III.10})$$

where  $\nu$  is the unit normal vector to  $\Gamma(t)$  pointing outside the solid and  $V, V_C$  are the horizontal component of the fluid and solid velocity respectively. Therefore, taking the scalar product of (III.9) with  $\nu$  and using (III.10), we get

$$(Q_e - Q_i) \cdot \nu = (\zeta_e - \zeta_i) V_C \cdot \nu \quad \text{at} \quad \Gamma(t) \quad (\text{III.11})$$

since  $V_C \cdot \nu$  does not depend on the vertical variable. In the two-dimensional axisymmetric case, the transition condition takes a simpler form. Indeed, the normal component of the discharge  $Q$  reduces to the radial component of the discharge  $q$  and, assuming that the structure is moving only vertically ( $V_C = 0$ ), the transition condition becomes the continuity condition

$$q_e = q_i \quad \text{at} \quad \Gamma.$$

where the contact line  $\Gamma$  is fixed.

The boundary condition for the interior pressure at the contact line is not known since (III.4) ceases to hold in the vertical side-walls case. Hence, one more condition is needed to recover the appropriate boundary condition, such as the conservation of the energy of the system. This property is required in the axisymmetric case considered in this manuscript, see Section IV.1 for more details.

### State of the art

The well-posedness theory of the complete water waves equations with a floating object is not known in the literature. It does not seem evident to provide an existence result and a fortiori one that provides bounds allowing to handle the shallow water limit. The difficulty relies on the presence of the tensor  $\mathbf{R}$  and the acceleration  $\mathbf{a}_{\text{NH}}$ , which account respectively for the “turbulent” (understood as the nonlinear interaction of the fluctuation of the velocity around its average  $\bar{V}$ ) and the non-hydrostatic effects. However, the same approach (consider the pressure as a Lagrange multiplier associated with the constraint on the surface elevation under the object) can be carried out for asymptotic models. Here we present the recent works who dealt with floating structures using the depth averaged formulation for different configurations and models. The one-dimensional nonlinear shallow water equations with a floating object were showed by Lannes in [83]. Although the author did not investigate the well-posedness of the equations, he modeled the problem in general dimensions and considered both the cases of a solid with vertical and non-vertical walls. Considering vertical walls he wrote the one-dimensional equations under a conservative form with explicit source terms. In the case of non-vertical walls, the evolution of the contact line was written explicitly and its singular nature was discussed.

As a continuation of this study, Iguchi and Lannes in [73] solved this free boundary problem by developing a general theory of one-dimensional hyperbolic initial boundary value problems with fully nonlinear boundary conditions. Their approach was used to solve other problems such as the case of vertical walls and other transmission problems and to improve some results on the stability of one-dimensional shocks. We will detail their work in the next section.

We refer to [88] for the one-dimensional viscous case. In this paper Maity, San

Martín, Takahashi and Tucsnak studied the motion of a floating structure with vertical walls constraint to move only vertically. With these assumptions the contact line is a fixed boundary. Using a Hamiltonian formalism, they modeled the fluid motion with the viscous nonlinear shallow water equations and the well-posedness of the coupled fluid-solid system was established, local for any initial data and global for small initial data.

The one-dimensional Boussinesq regime was addressed by Bresch, Lannes and Métivier in [20], where the configuration of a fixed partially immersed structure with vertical walls was considered. It corresponds to an initial boundary value problem for a dispersive perturbation of hyperbolic systems and the apparition of a new phenomenon, dispersive boundary layers, was exhibited. The authors reformulated the problem as a transmission problem which is written as one ODE which is locally well-posed with a blow-up criterion dependent on some parameters. In order to get a uniform bound on the quantity appearing in the criterion and prove the local well-posedness of the problem, a control on the boundary layer and the oscillations generated was required.

To the best of our knowledge, the two-dimensional nonlinear shallow water case has not been studied yet. The difficulty of working with two dimensions relies on the loss of different properties (Riemann invariants, construction of a Kreiss symmetrizer, ...). As a first step, it can be interesting to address the axisymmetric configuration in order to reduce the two-dimensional problem to a one-dimensional problem and adapt the existent theory to provide a well-posedness result. Differently from the one-dimensional case, in the two-dimensional axisymmetric case dispersion occurs and this permits to recover the delay terms present in the Cummins equation. More details on this delay effect are given in Section IV.2. Moreover, the axisymmetry appears in several experimental data from hydrodynamical engineering and the study of this configuration can improve the comprehension of the wave-structure interaction reproduced in tests.

### One-dimensional problem

From a physical point of view, the one-dimensional model describes the case when water waves are moving mostly along one direction, for instance the longitudinal one. This problem has also a mathematical interest because in this case the computations are simpler and the quantities can be made explicit. In particular, the evolution of the contact line and the well-posedness of the system can be investigated. In one dimension the interior domain  $\mathcal{I}(t)$  and the contact line  $\Gamma(t)$  take respectively the form of an interval and two points, namely

$$\mathcal{I}(t) = (x_-(t), x_+(t)), \quad \Gamma(t) = \{x_-(t), x_+(t)\}$$

with  $x_-(t) < x_+(t)$ . The equation that governs the time evolution of the contact line in the one-dimensional case, which reduces to only two contact points  $x_\pm(t)$ , is derived from the implicit condition

$$\zeta(t, x_\pm(t)) = \zeta_w(t, x_\pm(t)).$$

Using the subscript  $\pm$  for the evaluation at  $x_\pm(t)$ , the time evolution is given by the ODE

$$\begin{aligned} \dot{x}_\pm(t) = & - \frac{(\partial_x \zeta_w)_\pm(t) (u_G(t) + \omega(t)(\zeta_{w\pm}(t) - z_G(t)))}{(\partial_x \zeta_e)_\pm(t) - (\partial_x \zeta_w)_\pm(t)} \\ & + \frac{w_G(t) - \omega(x_\pm(t) - x_G(t)) + (\partial_x q_e)_\pm(t)}{(\partial_x \zeta_e)_\pm(t) - (\partial_x \zeta_w)_\pm(t)}. \end{aligned} \quad (\text{III.12})$$

It depends on the position  $(x_G, z_G)$  and the velocity  $(u_G, w_G)$  of the center of mass, on the angular velocity  $\omega$  of the solid and on the traces of the spatial derivative of  $\zeta_e$ ,  $q_e$  and  $\zeta_w$ . The presence of these derivatives makes (III.12) more singular than the kinematic equation describing the evolution of the shoreline. The evolution equation for the contact line in the two-dimensional case, which can be parametrized by a closed curve in the plane, was written explicitly by Lannes in [83]. However, no existence results are known in the literature for the initial boundary value problem with a free boundary of this type.

As already said before, Iguchi and Lannes established in [73] a well-posedness result for the one-dimensional nonlinear shallow water equations with a floating structure solving the free boundary problem (III.12). More precisely, the authors proposed a general approach to one-dimensional hyperbolic initial boundary value problems with two types of evolution equations at the boundary.

In the first one, called of “kinematic type”, the velocity of the interface has the same regularity as the trace of the solution at the interface. The interaction of waves with a lateral piston was shown to fall into this category. Moreover, two examples of transmission problems were considered. The first one, with a fixed interface, is given by a conservation law with a flux which is discontinuous across the interface. This is the case of the nonlinear shallow water equations over a discontinuous topography. The second application is a free boundary problem: the stability of shocks. Using the specificities of the one-dimensional case, the regularity threshold from the general theory was improved.

The second type of evolution equations considered is the “fully nonlinear” case, when the velocity of the interface is one derivative more singular than the trace of the solution. In this case loss of two derivatives occurs and the standard procedure in free boundary hyperbolic systems does not work. A second-order Alinhac good unknown is introduced to remove this loss and new sharp estimates are established

for one-dimensional hyperbolic initial boundary values problems. Then, the result was applied to the one-dimensional nonlinear shallow water equations with a floating structures since, as shown before, the evolution equation of the contact line have is fully nonlinear.

### III.2 The Cummins equation and the return to equilibrium problem

The return to equilibrium problem is a particular configuration of the floating structure problem. It consists in releasing a partially submerged solid body into a fluid initially at rest and letting it evolve towards its equilibrium position. The interest of this problem is that it can easily be done experimentally and it is used in engineering to determine several important characteristics of floating objects. More precisely, engineers assume that the solid satisfies a linear integro-differential equation, the Cummins equation. It was empirically derived in [34], in which the author dealt with ship motions using a linear model, supposing the conservation of the energy and the balance of forces. The experimental data coming from the return to equilibrium problem, called decay test in the hydrodynamical engineering context, are then used to identify the coefficients of this linear equation.

John in [75] studied the problem in shallow water in one horizontal dimension for an object with flat bottom: he considered the linearized fluid equations for small amplitude waves and he wrote an explicit expression for the solid motion under linear approximation. Ursell in [120] and Maskell and Ursell in [90], using like John the linear approach, obtained an explicit solution in integral form for the vertical displacement of the object. From Wehausen and Laitone [123] we know that also Sretenskii, several years before Cummins, obtained an integro-differential equation for the vertical displacement which he solved numerically. The Cummins equation for the vertical displacement reads

$$(m + a_\infty) \ddot{\delta}_G(t) = -c\delta_G(t) - \int_0^t K(\tau) \dot{\delta}_G(t - \tau) d\tau, \quad (\text{III.13})$$

where  $\delta_G(t) = z_G(t) - z_{G,eq}$  is the displacement from the equilibrium position of the vertical position of the center of mass,  $m$  is the mass of the structure,  $a_\infty$  is the added mass at infinity frequency,  $c$  is the hydrostatic coefficient and  $K$  is the impulse response function (also known as retardation function and fluid memory). It appears in naval architecture and hydrodynamical engineering literature and it is used to study the motion of ships or wave energy converters. In [83] Lannes modeled the return to equilibrium problem using the depth averaged formulation presented before taking into account nonlinear effects. He wrote the explicit equations of the one-dimensional nonlinear shallow water model and he showed that

the position of the solid is fully determined by the nonlinear second order damped ODE

$$(m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathbf{c}\delta_G(t) - \nu(\dot{\delta}_G) + \beta(\delta_G)\dot{\delta}_G^2(t), \quad (\text{III.14})$$

where  $\nu(\dot{\delta}_G)$  is the nonlinear damping term. Numerical simulations for the one dimensional model proposed by Lannes are made in [121]. An important point is the presence of the added mass term  $m_a(\delta_G)$  which can be explained by the fact that, in order to move, the solid has to accelerate itself and also a portion of fluid around it. The added mass effect can also be found in other fluid-structure interaction problems: for instance, in the case of a totally submerged solid it was shown by Glass, Sueur and Takahashi in [63] and by Glass, Munnier and Sueur in [62]. Moreover, this effect has an important role for the stability of numerical simulations as show in [24] by Causin, Gerbeau and Nobile.

In (III.14) there is no delay effect on the motion of the solid due to the absence of dispersion in the one-dimensional case. Contrarily, in the two-dimensional axisymmetric configuration considered in Chapter 2 it will be shown that the dispersion of waves provides a retarding effect on the motion of the solid. Indeed, in this case a convolution appear in the damping term (as in (III.13)) and the motion of the solid is governed by a delay differential equation.

## IV Contributions of the thesis

Motivated by the works [83] and [73], we aim at extending the nonlinear analysis of the floating structures problem to the two dimensional axisymmetric configuration. In particular, the goal of the first part of this manuscript is to investigate the well-posedness of the nonlinear shallow water equations in the presence of a heaving buoy. This is a type of wave energy converters frequently implemented in the nearshore zone (see Section I) in which the shallow water asymptotic model accurately describe the motion of the waves.

In the second part, we propose an equation for the vertical motion of a floating structure which justifies the Cummins equation, which was derived empirically from the conservation of the energy and the balance of forces, via a rigorous mathematical analysis. The equation derived here is the first rigorous justification of the Cummins equation with delay term and, moreover, it improves the linear approach of Cummins by taking into account the nonlinear effects on the motion of the solid, which are shown to be non-negligible in large amplitude motions.

### IV.1 Chapter 1

In Chapter 1 we consider a cylindrically symmetric structure with vertical side-walls moving only on the vertical direction on an homogeneous, invicid, incom-

pressible, irrotational fluid. The assumptions on the shape and the motion of the solid permit to avoid the free boundary problem associated with the contact line and to simplify the problem. Indeed, in this case the projection of the solid wetted surface does not depend on time. We suppose the flow to be *axisymmetric* and *without swirl*, i.e. we consider a rotation-invariant velocity field with no azimuthal component. For simplicity, the bottom of the fluid domain is assumed to be flat. The nonlinear shallow water equations with a floating structure are written in an axisymmetric form. From the assumptions on the flow, the horizontal discharge reduces to a scalar one-dimensional (radial) unknown. Let us denote by  $u$  the couple  $(\zeta_e, q_e)$  and by  $\delta_G$  the displacement of the vertical position of the solid center of mass from its equilibrium position. The fluid equations in the exterior domain  $(R, +\infty)$ , where  $R$  is the radius of the solid, can be written as the quasilinear hyperbolic initial boundary value problem

$$\begin{cases} \partial_t u + A(u) \partial_r u + B(u, r) u = 0 & \text{in } (R, +\infty) \\ \mathbf{e}_2 \cdot u|_{r=R} = -\frac{R}{2} \dot{\delta}_G(t) \\ u(t=0) = u_0, \end{cases} \quad (\text{IV.1})$$

with

$$A(u) = \begin{pmatrix} 0 & 1 \\ gh_e - \frac{q_e^2}{h_e^2} & \frac{2q_e}{h_e} \end{pmatrix}, \quad B(u, r) = \begin{pmatrix} 0 & \frac{1}{r} \\ 0 & \frac{q_e}{rh_e} \end{pmatrix}$$

where  $h_e = h_0 + \zeta_e$  is the fluid height. Moreover, due to the presence of vertical walls, it is shown that a corrector term must be added in the interior pressure boundary condition to make the coupled fluid-structure system conservative. This addition does not influence the existence result in Theorem IV.1, but it is necessary for the global existence of the solution to the return to equilibrium problem, shown in Chapter 2. On the other hand, using the elliptic system for the interior pressure and adapting the elementary potentials argument used in [83] to the radial case, Newton's law for the conservation of the linear momentum can be written as

$$\begin{cases} (m + m_a(\delta_G)) \ddot{\delta}_G(t) = -\mathbf{c} \delta_G(t) + \mathbf{c} \mathbf{e}_1 \cdot u|_{r=R} + (\mathbf{b}(u) + \beta(\delta_G)) \dot{\delta}_G^2(t), \\ \delta_G(0) = \delta_0 \\ \dot{\delta}_G(0) = \delta_1 \end{cases} \quad (\text{IV.2})$$

with  $\mathbf{c} = \rho g \pi R^2$ ,  $b = \frac{\pi \rho R^4}{8}$  and

$$m_a(\delta_G) = \frac{b}{h_w(\delta_G)}, \quad \beta(\delta_G) = \frac{b}{2h_w^2(\delta_G)}, \quad \mathbf{b}(u) = \frac{b}{(\mathbf{e}_1 \cdot u|_{r=R} + h_0)^2},$$

where  $h_w(\delta_G) = h_{w,eq} + \delta_G$  is the fluid height under the solid.

**Main result** We state here the main result of Chapter 1. It is a local in time well-posedness result for the coupled system (IV.1) - (IV.2) in Sobolev spaces for regular and compatible initial data (see Theorem 2.3.4 for the rigorous statements).

**Theorem IV.1.** *For  $k \geq 2$ , let  $u_0 = (\zeta_{e,0}, q_{e,0}) \in H_r^k((R, +\infty))$ ,  $\delta_0$  and  $\delta_1$  satisfy some compatibility conditions. Assume that there exists some  $c_{\text{sub}}, h_{\text{min}} > 0$  such that*

$$\forall r \in (R, +\infty) \quad h_{e,0}(r) \geq h_{\text{min}}, \quad \left( gh_{e,0} - \frac{q_{e,0}^2}{h_{e,0}^2} \right)(r) \geq c_{\text{sub}}.$$

with  $h_{e,0} = h_0 + \zeta_{e,0}$ . Then, the coupled problem (IV.1) - (IV.2) admits a unique solution

$$(u, \delta_G) \in \bigcap_{j=0}^k C^j([0, T], H_r^{k-j}((R, +\infty))) \times H^{k+1}((0, T)),$$

where  $H_r^k((R, +\infty))$  denotes the weighted Sobolev space  $H^k((R, +\infty), r dr)$ .

The proof of the previous result is based on the study of the linearized system and an iterative scheme. Differently from the well-posedness theory for quasilinear hyperbolic systems on the whole line, here the Friedrichs symmetrizability does not guarantee that the initial boundary value problem (IV.1) is well-posed. To ensure that, a Kreiss symmetrizer is introduced. It transforms the system into a symmetric system with a maximal dissipative boundary condition, which, roughly speaking, means that the boundary terms that appear in the energy estimate have good sign properties so that the trace of the solution at the boundary is controlled by the standard energy estimate for quasilinear hyperbolic systems. This symmetrizer is constructed under the derivation of an equivalent formulation of the so-called uniform Kreiss-Lopatinskiĭ. We remark that this procedure is made possible by the axisymmetric configuration, while it does not happen in the general two-dimensional case. In addition, as usual for hyperbolic problems, the initial data are required to satisfy compatibility conditions to generate smooth solutions and this must be preserved by the iterative scheme.

**Perspectives** One first perspective is to take into account the vorticity in the water waves equations without a floating structure. Adding an evolution equation for the vorticity  $\omega$ , the depth averaged formulation  $(\zeta, Q)$  considered in this manuscript can be generalized to the rotational case showing that the equations are closed in terms of  $(\zeta, Q, \omega)$ . This work is motivated by the study made by Castro and Lannes in [23], where the authors generalized the Zakharov-Craig-Sulem formulation to the case with vorticity. Afterwards, the influence of the vorticity on the solid motion can be investigated.

The study of shallow water asymptotic models for the floating structures problem



in the presence of vorticity is more intricate. In the nonlinear shallow water equations vorticity does not appear for the size of the vorticity considered in [23] for which the asymptotics are fully justified. Hence, we may consider the next order of approximation and study the Green-Naghdi equations, where the vorticity effects become apparent, but using these equations the wave-structure interaction is not understood neither in the irrotational case. Another possibility is to consider vorticity that go beyond the scope of [23], in which case the rotational effects appear at the level of the shallow water equations that must be extended to a system of three equations, the third equation being an equation on a "turbulent" tensor. This model has not been rigorously justified but is of interest because it has been proposed to model wave breaking by Gavriluk and Richard in [109, 110], and validated experimentally in [109].

## Outline

Chapter 1 is based on [14]. In Section 1.1 we write the free surface Euler equations with the constraint that the solid must be in contact with the fluid during all the motion, avoiding air holes between them. In Section 1.2 we introduce the nonlinear shallow water approximation for this floating structure problem using the depth averaged formulation. In Section 1.3 we introduce the axisymmetric configuration and the problem is reformulated to get a one-dimensional set of equations. In Section 1.4 we write the fluid equations in the exterior domain as a quasilinear hyperbolic initial boundary value problem, namely. The local existence and uniqueness of the solution to the associated linear problem is established via a standard iterative scheme. We address the solid motion in Section 1.5. We write the vertical component of Newton's law for the conservation of linear momentum as a nonlinear second order ODE on the displacement of the solid vertical position from its equilibrium position. We show that the ODE can be written under the form (IV.2). In Section 1.6 we write the coupled system modeling the problem and we show the local in time existence and uniqueness of the solution introducing an iterative scheme and using a fixed point argument. In Appendix A we show the details in the case of a non-flat solid bottom, considering that the contact between the solid and the fluid is still on the vertical side-walls, and we derive the corresponding solid motion ODE. In Appendix B we show the proof of a product estimate.

## IV.2 Chapter 2

In Chapter 2 we study the return to equilibrium problem. First we show that the differential equation for the solid motion can be written in a closed form by introducing an extension-trace operator, which takes  $q_e|_{r=R}$  and gives  $\zeta_e|_{r=R}$ . After-

wards, we consider the return to equilibrium configuration, giving the particular initial conditions on the unknowns of the coupled problem. It turns out that the compatibility conditions, which are necessary in order to apply Theorem IV.1 are not satisfied for these particular initial conditions. Therefore, we propose a linear-nonlinear hydrodynamical model for the floating structures problem. The fluid equations in the exterior domain become the linear shallow water equations

$$\begin{cases} \partial_t \zeta_e + \partial_r q_e + \frac{q_e}{r} = 0 \\ \partial_t q_e + gh_0 \partial_r \zeta_e = 0 \end{cases} \quad (\text{IV.3})$$

while in the interior domain the same nonlinear equations are considered as well as the transition condition.

**Main result** We state here the main result of Chapter 2. It consists in the derivation of a nonlinear second order integro-differential equation for the solid motion and a well-posedness result for the Cauchy problem associated with the return to equilibrium initial data (see Proposition 2.3.1 and Theorem 2.3.4 for the rigorous statements).

**Theorem IV.2.** *Considering the linear shallow water equations (IV.3) for the fluid motion in the exterior domain, the solid motion equation (IV.2) can be written as the following second order nonlinear integro-differential equation:*

$$(m + m_a(\delta_G))\ddot{\delta}_G = -\mathbf{c}\delta_G - \nu\dot{\delta}_G + \mathbf{c} \int_0^t F(s)\dot{\delta}_G(t-s)ds + (\mathbf{b}(\dot{\delta}_G) + \beta(\delta_G))\dot{\delta}_G^2, \quad (\text{IV.4})$$

with  $\mathbf{c}$ ,  $b$ ,  $m_a(\delta_G)$ ,  $\beta(\delta_G)$  as in (IV.2),  $\nu = \frac{\mathbf{c}R}{2v_0}$  for  $v_0 = \sqrt{gh_0}$ ,

$$\mathbf{b}(\dot{\delta}_G) = \frac{b}{\left(\int_0^t F(s)\dot{\delta}_G(t-s)ds - \frac{R}{2v_0}\dot{\delta}_G(t) + h_0\right)^2}$$

and the convolution kernel  $F(t)$  defined by

$$F(t) = \lim_{v \rightarrow +\infty} \frac{1}{2\pi} \int_{-v}^v \left( \frac{iRH_0^{(1)}\left(\frac{i(c+i\omega)R}{v_0}\right)}{2v_0H_1^{(1)}\left(\frac{i(c+i\omega)R}{v_0}\right)} + \frac{R}{2v_0} \right) e^{(c+i\omega)t} d\omega$$

for any  $c > 0$  with  $H_0^{(1)}, H_1^{(1)}$  some Hankel functions. Moreover, let us denote by  $\rho_m$  the density of the solid and by  $H$  its height. Then, the Cauchy problem associated with the initial data

$$\delta_0 \neq 0, \quad \delta_1 = 0,$$

admits a unique global solution  $\delta_G \in C^2([0, +\infty), \mathbb{R})$  provided

$$|\delta_0| < \min \left( h_0 - \frac{\rho_m H}{\rho}, -\frac{2\rho_m H}{\rho} + \sqrt{\frac{4\rho_m^2 H^2}{\rho^2} + \frac{h_0^3 \rho_m H}{\rho R^2}} \right) \quad (\text{IV.5})$$

To derive the integro-differential equation in Theorem IV.2, we use the fact that, considering (IV.3) in the exterior domain, the free surface elevation satisfies a radial wave equation. Using a Laplace transform argument, the extension-trace operator can be explicitly described. It becomes a convolution operator on the time derivative of the displacement with the kernel given by the inverse Laplace transform of some Hankel function. The linearization of (IV.4) around the equilibrium position gives a reformulation of the Cummins equation for the vertical displacement (III.13) shown in the previous section. A similar linear integro-differential equation was derived by Maity, San Martín, Takahashi and Tucsnak in [88], where they dealt with the vertical motion of a floating structure in the linear viscous case.

By introducing an assumption on the convolution kernel justified by numerical results and extending it to zero in the past, the solid motion equation (IV.4) becomes an infinite delay differential equation. This type of equation appears usually in the context of population dynamics, for instance in [112] and in [8] and it was studied using a semi-group formulation by Liu and Magal in [87]. We use their theory to get the global well-posedness for small data using the conservation of the coupled fluid-structure energy.

**Perspectives** One first perspective is to relax the assumption on the convolution kernel and to investigate the global well-posedness of the nonlinear integro-differential equation with a non-integrable kernel. Another interesting future work is the study of other configurations of the wave-structure interaction using the hydrodynamical linear model derived in our work. For instance, the case where incoming waves are sent towards the solid with a wave creator or considering a bounded cylindrical domain in which outgoing waves can be reflected and resonances may appear. Our analysis can be applied in the case when a prescribed motion is enforced to the solid in order to kill waves. This problem may have applications in coastal oceanography when the size of the structure is significant, as for terminators and attenuators presented in Section I. From a mathematical point of view, it represents

a particular example of control problems for floating structures, which started to be investigated recently but it is not yet completely understood.

### Outline

Chapter 2 is based on [15]. In Section 2.1.5 we show that the differential equation for the solid motion can be written in a closed form by introducing an extension-trace operator and we show the local well-posedness using a fixed point argument. Then, we consider the return to equilibrium configuration, giving the initial conditions on the fluid and solid unknowns. It turns out that the compatibility conditions, which are necessary in order to apply the existence theorem in Chapter 1, are not satisfied for these particular initial conditions. In Section 2.2 we neglect the nonlinear effects in the exterior region, but we keep them under the object provided it does not touch the bottom of the fluid domain. We write a linear-nonlinear model for the floating structure problem linearizing the equations in the exterior domain but keeping the nonlinearities in the interior domain. In Section 2.3 we write the equation of the solid motion as a nonlinear second order integro-differential equation. Its linearization around the equilibrium gives a reformulation of the Cummins equation for the vertical displacement (III.13). We write the integro-differential equation as a functional differential equation with infinite delay and the results of Liu and Magal [87] are applied to get the global existence and uniqueness of the solution provided a smallness condition on the initial datum. Moreover, we show that the equilibrium position is locally asymptotically stable. In Section 2.4 we explain the numerical method used to plot the time evolution of the the solid and we compare the numerical solution with the solution to the Cummins equation, showing that nonlinearities are not negligible for large amplitude motions. In Appendix C the Hankel functions are defined and some properties and results are shown.



## CHAPTER 1

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# Floating structures in axisymmetric shallow water

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*This work is based on [14].*

### 1.1 Constrained free surface Euler equations

Let us consider a floating body, typically a wave energy converter, with vertical side-walls and a cylindrical symmetry, forced to move only in the vertical direction. We call  $C(t)$  the region occupied by the solid at time  $t$ ,  $\partial C(t)$  the boundary and  $\partial_w C(t)$  the portion of the boundary in contact with the fluid, called the *wetted surface*. The presence of the solid naturally allows to divide the horizontal plane  $\mathbb{R}^2$  into two regions, the projection  $\mathcal{I}$ , of the wetted surface on it, and  $\mathcal{E} := \mathbb{R}^2 \setminus \overline{\mathcal{I}}$ . We call them *interior* and *exterior domain* respectively. The boundary  $\Gamma := \partial \mathcal{I} = \partial \mathcal{E}$

is called the projection of the *contact line*, where the solid, the fluid and the exterior air interact. For simplicity we call  $\Gamma$  itself the contact line. These domains do not depend on time since the solid is moving only vertically and is assumed to have vertical side-walls. We consider a wetted surface that can be parametrized as graph of some function  $\zeta_w(t, X)$  for  $X \in \mathcal{I}$  and, like in the water waves theory, we assume that the surface of the fluid is the graph of a function  $\zeta(t, X)$  for  $X \in \mathbb{R}^2$ , as shown in Figure 1.1.

We assume that the fluid is incompressible, irrotational, with constant density  $\rho$  and inviscid. For simplicity we consider a flat bottom which can be parametrized by  $-h_0$  with  $h_0 > 0$  and the fluid domain is

$$\Omega(t) = \{(X, z) \in \mathbb{R}^{2+1} \mid -h_0 < z < \zeta(t, X)\}.$$

Then, the motion of the fluid is given by the incompressible Euler equation

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z \quad \text{in } \Omega(t) \quad (1.1.1)$$

$$\nabla \cdot \mathbf{U} = 0 \quad (1.1.2)$$

$$\nabla \times \mathbf{U} = 0. \quad (1.1.3)$$

The boundary conditions for the Euler equation on the velocity field  $\mathbf{U}$  in the fluid domain are the traditional kinematic equation at the surface and the impermeability condition at the bottom, respectively

$$z = \zeta, \quad \partial_t \zeta - \mathbf{U} \cdot N = 0 \quad \text{with } N = \begin{pmatrix} -\nabla \zeta \\ 1 \end{pmatrix} \quad (1.1.4)$$

$$z = -h_0, \quad \mathbf{U} \cdot \mathbf{e}_z = 0 \quad (1.1.5)$$

We consider a configuration when the fluid is completely attached to the solid. Hence we have the following contact constraint:

$$\zeta(t, X) = \zeta_w(t, X) \quad \text{in } \mathcal{I}.$$

Let us denote the restrictions to the interior domain and the exterior domain of a function  $f$  defined on  $\mathbb{R}^2$  as

$$f_i := f|_{\mathcal{I}} \quad f_e := f|_{\mathcal{E}}.$$

According to this notation the contact constraint becomes the following

$$\zeta_i = \zeta_w \quad \text{in } \mathcal{I} \quad (1.1.6)$$

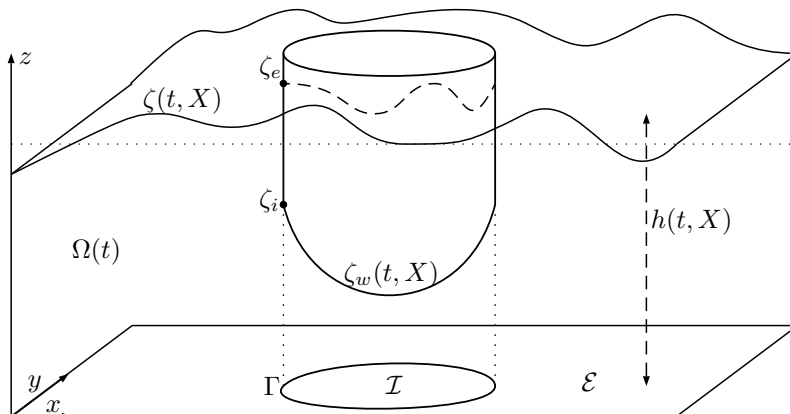


Figure 1.1 – A cylindrically symmetric floating structure with vertical side-walls

**Remark 1.1.1.** *Since the solid has vertical side-walls, the free surface is not continuous across  $\Gamma$ , i.e.*

$$\zeta_e \neq \zeta_i \quad \text{on } \Gamma.$$

In the presence of a floating structure we have to change the standard condition on the value of the pressure on the free surface. In the exterior domain it is given by the constant atmospheric pressure  $P_{\text{atm}}$ , i.e.

$$\underline{P}_e = P_{\text{atm}}. \quad (1.1.7)$$

with  $\underline{P} = P|_{z=\zeta}$ . In the interior domain the pressure on the free surface is an unknown of the problem, depending on the dynamics of the solid but we know its value on  $\Gamma$ . Indeed, by integrating the vertical component of Euler's equation (1.1.1) between  $z = \zeta_i$  and  $z = \zeta_e$ , we have

$$\underline{P}_i(t, \cdot) = P_{\text{atm}} + \rho g(\zeta_e - \zeta_i) + \rho \int_{\zeta_i}^{\zeta_e} (\partial_t w + \mathbf{U} \cdot \nabla_{X,z} w) \quad \text{on } \Gamma, \quad (1.1.8)$$

where  $w$  is the vertical component of the velocity field  $\mathbf{U}$ . The second and the third term do not vanish due to the discontinuity of the free surface on  $\Gamma$  (see Remark 1.1.1).

Moreover one has the continuity of the normal velocity at the vertical side-walls, i.e.

$$V \cdot \nu = V_C \cdot \nu \quad (1.1.9)$$

where  $\nu$  is the unit normal vector to  $\Gamma$  pointing towards  $\mathcal{E}$ ,  $V$  and  $V_C$  are the horizontal velocities of the fluid and the solid respectively.



As in the standard water waves theory we suppose that the height of the fluid  $h_e(t)$  does not vanish during all the motion. Hence we have the following assumption:

$$\exists h_m > 0 : h_e(t, X) \geq h_m \quad \forall t \in [0, T], \forall X \in \mathcal{E}. \quad (1.1.10)$$

From the physics of the floating structure problem we suppose also that the solid does not touch the bottom of the domain during its motion. This is equivalent to assuming that the height of the fluid  $h_i(t, X)$  under the solid does not vanish, i.e.

$$\exists h_{min} > 0 : h_w(t, X) \geq h_{min} \quad \forall t \in [0, T], \forall X \in \mathcal{I}. \quad (1.1.11)$$

with  $h_w(t, X) = h_i(t, X)$  in  $\mathcal{I}$  due to (1.1.6). This assumption is completely relevant for the situation investigated here; we refer to [37, 99] (Euler equation) and [84] (nonlinear shallow water and Green-Naghdi equations) for the analysis of the vanishing depth problem.

## 1.2 The shallow water regime

We are interested here in shallow water regime, when the horizontal scale of the problem is much larger than the depth. In the water waves problem the horizontal scale is given by the typical wavelength of the waves.

**Remark 1.2.1.** *In the floating structure problem a third relevant length in the floating structures problem is the solid width. In particular the ratio of the solid width and the wavelength naturally appears in the adimensionalized equations. Here we consider this quantity as a parameter independent of the other parameters, such as the shallowness and the nonlinear parameters, and it takes no role in the derivation of the asymptotic model.*

Following [83], we consider the following shallow water asymptotic model for the floating structures problem:

**Proposition 1.2.2.** *The nonlinear shallow water equations with a floating structure in the  $(\zeta, Q)$ -formulation are*

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot \left( \frac{1}{h} Q \otimes Q \right) + gh \nabla \zeta = -\frac{h}{\rho} \nabla \underline{P}. \end{cases} \quad (1.2.1)$$

with the surface pressure  $\underline{P}$  given by

$$\underline{P}_e = P_{\text{atm}} \quad \text{and} \quad \begin{cases} -\nabla \cdot \left( \frac{h}{\rho} \nabla \underline{P}_i \right) = -\partial_t^2 \zeta_w + \mathbf{a}_{FS}(h, Q) & \text{in } \mathcal{I} \\ \underline{P}_i|_{\Gamma} = P_{\text{atm}} + \rho g (\zeta_e - \zeta_i)|_{\Gamma} + P_{\text{cor}}, \end{cases} \quad (1.2.2)$$

where

$$\mathbf{a}_{FS}(h, Q) = \nabla \cdot \left( \nabla \cdot \left( \frac{1}{h} Q \otimes Q \right) + gh \nabla \zeta \right),$$

coupled with the transition condition at the contact line

$$Q_e \cdot \nu = Q_i \cdot \nu \quad \text{on } \Gamma. \quad (1.2.3)$$

Differently from the case considered by Lannes in [83], where the jump of pressure  $\underline{P}_i|_{\Gamma(t)} - P_{\text{atm}}$  at the boundary of the object is assumed to be only due to the hydrostatic pressure, *i.e.*

$$\underline{P}_i|_{\Gamma(t)} - P_{\text{atm}} = \rho g (\zeta_e - \zeta_i)|_{\Gamma(t)},$$

we add here a non-hydrostatic correction term  $P_{\text{cor}}$ . This corrector is determined later in Proposition 2.2.1 below to ensure exact energy conservation (the mathematical analysis if we remove this term can be performed in the same way).

As for the kinematic condition (1.1.4), we have that

$$\partial_t \zeta_w - \underline{U}_w \cdot N_w \quad \text{in } \mathcal{I} \quad \text{with} \quad N_w = \begin{pmatrix} -\nabla \zeta_w \\ 1 \end{pmatrix} \quad (1.2.4)$$

where  $\underline{U}_w$  is the velocity of the solid on the wetted surface. Let us denote the center of mass of the solid  $G(t) = (X_G(t), z_G(t))$  and  $\mathbf{U}_G(t) = (V_G(t), w_G(t))$  its velocity and  $\omega$  the angular velocity of the solid. From the solid mechanics we have

$$\underline{U}_w = \mathbf{U}_G + \omega \times \mathbf{r}_G \quad \text{with} \quad \mathbf{r}_G(t, X) = \begin{pmatrix} X - X_G(t) \\ \zeta_w(t, X) - z_G(t) \end{pmatrix}.$$

Then, (1.2.4) gives

$$\partial_t \zeta_w = (\mathbf{U}_G + \omega \times \mathbf{r}_G) \cdot N_w \quad \text{in } \mathcal{I}. \quad (1.2.5)$$

Because of the linearity of the elliptic problem we can decompose the interior pressure as  $\underline{P}_i = \underline{P}_i^I + \underline{P}_i^{II} + \underline{P}_i^{III}$  where:

- $\underline{P}_i^I$  is the pressure we would have in the case of a fixed solid, solution to

$$\begin{cases} -\nabla \cdot \left( \frac{h}{\rho} \nabla \underline{P}_i^I \right) = \mathbf{a}_{FS}(h, Q) & \text{in } \mathcal{I}, \\ \underline{P}_i^I|_{\Gamma} = P_{\text{atm}}, \end{cases} \quad (1.2.6)$$

where  $\mathbf{a}_{FS}(h, Q)$  is the free surface acceleration in the absence of a floating structure;

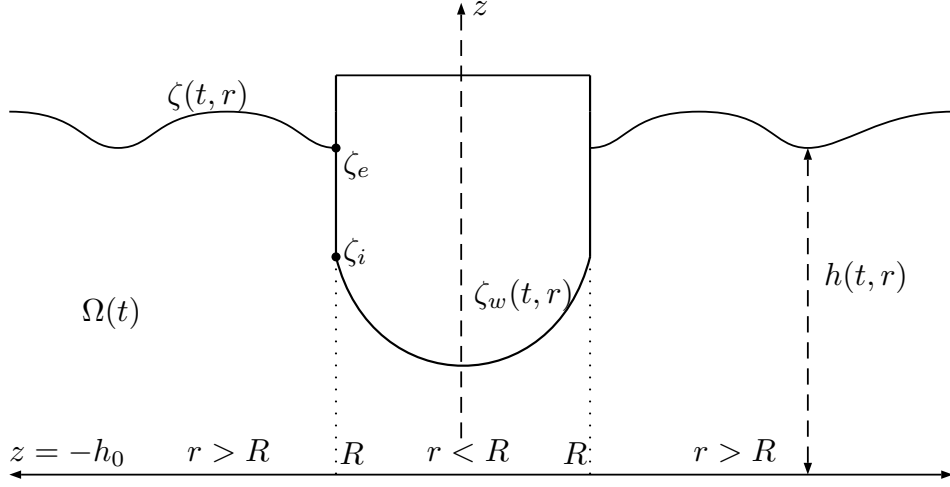


Figure 1.2 – Vertical cross section of the axisymmetric configuration

- $\underline{P}_i^{\text{II}}$  is the part of the pressure due to the acceleration of the solid

$$\begin{cases} -\nabla \cdot \left( \frac{h}{\rho} \nabla \underline{P}_i^{\text{II}} \right) = -\partial_t^2 \zeta_w, & \text{in } \mathcal{I}, \\ \underline{P}_i^{\text{II}}|_{\Gamma} = 0, \end{cases} \quad (1.2.7)$$

where  $w_G$  is the vertical component of the velocity of the center of mass  $G(t)$  of the solid;

- $\underline{P}_i^{\text{III}}$  is the part of the pressure due to the pressure discontinuity at the contact line

$$\begin{cases} -\nabla \cdot \left( \frac{h}{\rho} \nabla \underline{P}_i^{\text{III}} \right) = 0 & \text{in } \mathcal{I}, \\ \underline{P}_i^{\text{III}}|_{\Gamma} = \rho g (\zeta_e - \zeta_i)|_{\Gamma} + P_{\text{cor}}. \end{cases} \quad (1.2.8)$$

### 1.3 Axisymmetric without swirl setting

Without loss of generality we suppose the center of mass to have coordinates  $G(t) = (0, 0, z_G(t))$  and let  $R$  be the radius of the interior domain  $\mathcal{I}$ . Introducing a cylindrical coordinates system with the  $z$ -axis coincident with the axis of symmetry of the solid (see Figure 1.2) we write the velocity field  $\mathbf{U}$  as

$$\mathbf{U}(t, r, \theta, z) = (u_r(t, r, \theta, z), u_\theta(t, r, \theta, z), u_z(t, r, \theta, z)).$$

From now on and throughout the paper we consider an axisymmetric flow without swirl, which means that the flow has no dependence on the angular variable  $\theta$ , i.e.  $\mathbf{U} = \mathbf{U}(t, r, z)$ , and  $u_\theta = 0$  respectively. Hence the horizontal discharge can be written as

$$Q(t, r) = (q_r(t, r), 0)$$

with

$$q_r(t, r) = \int_{-h_0}^{\zeta} u_r(t, r, z) dz$$

and the tangential component vanishes since

$$q_\theta(t, r) = \int_{-h_0}^{\zeta} u_\theta(t, r, z) dz = 0.$$

For simplicity we write  $q$  instead of  $q_r$  for the radial component of the horizontal discharge. Moreover, since the solid moves only vertically and the swirl is neglected in the flow,  $V_G = 0$  and  $\omega = 0$ . Hence from (1.2.5) we have

$$\partial_t^2 \zeta_w = \dot{w}_G.$$

In the new system of reference the shallow water model (1.2.1) - (1.2.3) becomes

$$\begin{cases} \partial_t h + \partial_r q + \frac{q}{r} = 0 \\ \partial_t q + \partial_r \left( \frac{q^2}{h} \right) + \frac{q^2}{rh} + gh \partial_r h = -\frac{h}{\rho} \partial_r \underline{P} \end{cases} \quad \text{in } (0, +\infty) \quad (1.3.1)$$

coupled with the transition condition

$$q_{e|_{r=R}} = q_{i|_{r=R}}. \quad (1.3.2)$$

We have  $\underline{P}_e = P_{\text{atm}}$  and (1.2.6) - (1.2.8) become

$$\begin{cases} -\left( \partial_r + \frac{1}{r} \right) \left( \frac{h_w}{\rho} \partial_r \underline{P}_i^I \right) = \\ \left( \partial_r + \frac{1}{r} \right) \left( \partial_r \left( \frac{q_i^2}{h_w} \right) + \frac{q_i^2}{rh_w} + gh_w \partial_r h_w \right) \quad \text{in } (0, R) \\ \underline{P}_i^I|_{r=R} = P_{\text{atm}}, \end{cases} \quad (1.3.3)$$

$$\begin{cases} -\left( \partial_r + \frac{1}{r} \right) \left( \frac{h_w}{\rho} \partial_r \underline{P}_i^{\text{II}} \right) = -\dot{w}_G \quad \text{in } (0, R) \\ \underline{P}_i^{\text{II}}|_{r=R} = 0, \end{cases} \quad (1.3.4)$$

$$\begin{cases} -\left(\partial_r + \frac{1}{r}\right)\left(\frac{h_w}{\rho}\partial_r P_i^{\text{III}}\right) = 0 & \text{in } (0, R) \\ P_i^{\text{III}}|_{r=R} = \rho g(\zeta_e - \zeta_i)|_{r=R} + P_{\text{cor}}, \end{cases} \quad (1.3.5)$$

where we replace  $h_i = \zeta_i + h_0$  with  $h_w = \zeta_w + h_0$  due to the contact constraint (1.1.6). Using axisymmetry and absence of swirl together with this change of coordinates we pass from a two-dimensional to a one-dimensional problem, where explicit calculations can be done (see Section 1.3). With these assumptions, the horizontal discharge is no more a vectorial quantity but a scalar quantity, making the problem easier to handle.

**Remark 1.3.1.** *Under the shallow water approximation and in the axisymmetric without swirl setting the fluid energy is defined by*

$$E_{SW} = 2\pi \frac{\rho}{2} g \int_0^{+\infty} \zeta^2 r dr + 2\pi \frac{\rho}{2} \int_0^{+\infty} \frac{q^2}{h} r dr. \quad (1.3.6)$$

In the presence of a floating structure the fluid energy  $E_{SW}$  is no more conserved by the equations in (1.2.1). Let us define the energy for a solid moving only vertically as

$$E_{sol} = \frac{1}{2} m w_G^2 + m g z_G. \quad (1.3.7)$$

and the total fluid-structure energy

$$E_{tot} := E_{SW} + E_{sol}. \quad (1.3.8)$$

**Notation 1.3.2.**

$$\llbracket f \rrbracket := f_{e|_{r=R}} - f_{i|_{r=R}}$$

*is the jump between the exterior and the interior domain at the contact line  $r = R$ .*

We can now state the following proposition:

**Proposition 1.3.3.** *Choosing  $P_{\text{cor}} = \frac{\rho}{2} q_{i|_{r=R}}^2 \left[ \left[ \frac{1}{h^2} \right] \right]$ , the total fluid-structure energy is conserved, i.e.*

$$\frac{d}{dt} E_{tot} = 0. \quad (1.3.9)$$

*Proof.* Multiplying the first equation in (1.3.1) by  $\rho g \zeta r$  and the second by  $\frac{qr}{h}$  and summing up, we use the fact that  $\partial_t h = \partial_t \zeta$  to write the system under the conservative form

$$\partial_t \mathbf{e} + \partial_r F = -r q \partial_r P, \quad (1.3.10)$$

where  $\mathbf{e}$  is the local fluid energy

$$\mathbf{e} = \frac{\rho}{2}g\zeta^2r + \frac{\rho}{2}\frac{q^2}{h}r$$

and  $F$  is the flux

$$F = \rho \left( \frac{q^3}{2h^2}r + g\zeta qr \right).$$

The conservative form (1.3.10) reads in the interior domain  $(0, R)$

$$\partial_t \mathbf{e}_i + \partial_r F_i = -rq_i \partial_r \underline{P}_i \quad (1.3.11)$$

and in the exterior domain  $(R, +\infty)$

$$\partial_t \mathbf{e}_e + \partial_r F_e = 0 \quad (1.3.12)$$

We integrate (1.3.11) on  $(0, R)$  and (1.3.12) on  $(R, +\infty)$  and multiplying by  $2\pi$  we obtain

$$\frac{d}{dt} E_{SW} - 2\pi\rho R \left[ \frac{q^3}{2h^2} + g\zeta q \right] = -2\pi \int_0^R rq_i \partial_r (\underline{P}_i - P_{\text{atm}}) dr, \quad (1.3.13)$$

By integration by parts we get

$$\begin{aligned} \frac{d}{dt} E_{SW} &= 2\pi\rho R \left[ \frac{q^3}{2h^2} + g\zeta q \right] - 2\pi R (\underline{P}_i - P_{\text{atm}})|_{r=R} q_{i|r=R} \\ &\quad + 2\pi \int_0^R (\underline{P}_i - P_{\text{atm}}) \partial_r (rq_i) dr. \end{aligned}$$

On the other hand, from the definition of  $E_{sol}$ , we have

$$\begin{aligned} \frac{d}{dt} E_{sol} &= mw_G \dot{w}_G + mgw_G = w_G (m\dot{w}_G + mg) \\ &= w_G 2\pi \int_0^R (\underline{P}_i - P_{\text{atm}}) r dr \\ &= 2\pi \int_0^R (\underline{P}_i - P_{\text{atm}}) \partial_t \zeta_w r dr \end{aligned}$$

where we have used Newton's law for the conservation of the linear momentum in the axisymmetric configuration

$$m\dot{w}_G = -mg + 2\pi \int_0^R (\underline{P}_i - P_{\text{atm}}) r dr$$

and (1.2.5). From the contact constraint (1.1.6) and the mass conservation equation in (1.3.1) the following yields:

$$\frac{d}{dt} E_{sol} = -2\pi \int_0^R (\underline{P}_i - P_{\text{atm}}) \partial_r(rq_i) dr. \quad (1.3.14)$$

Therefore

$$\frac{d}{dt} E_{sol} = -\frac{d}{dt} E_{SW} + 2\pi\rho R \left[ \frac{q^3}{2h^2} + g\zeta q \right] - 2\pi R (\underline{P}_i - P_{\text{atm}})|_{r=R} q_{i|_{r=R}}.$$

Using the expression of the interior pressure  $\underline{P}_i$  on the boundary  $r = R$  in (1.3.3) - (1.3.5) and the transition condition (1.3.2) we obtain

$$\frac{d}{dt} (E_{SW} + E_{sol}) = 2\pi R \left( \frac{\rho}{2} q_{i|_{r=R}}^3 \left[ \frac{1}{h^2} \right] - q_{i|_{r=R}} P_{\text{cor}} \right).$$

If  $q_{i|_{r=R}} = 0$  the result follows directly. Otherwise we choose the pressure corrector term in (1.3.5) as

$$P_{\text{cor}} = \frac{\rho}{2} q_{i|_{r=R}}^2 \left[ \frac{1}{h^2} \right], \quad (1.3.15)$$

and we get (1.3.9).  $\square$

## 1.4 The fluid equations

In this section we focus on the “fluid part” of the coupled problem. We show that the exterior part of (1.3.1) can be written as a one-dimensional quasilinear hyperbolic initial boundary value problem in an exterior domain and we shall prove the local in time well-posedness. Like frequently in the literature, throughout this paper we also use the term *mixed* problem: this comes from the fact that we have, as data of the problem, both the initial (in time) and the boundary (in space) values.

Hyperbolic problems in exterior domains have been treated in many works. Métivier [97], Benzoni and Serre [12] have studied hyperbolic initial boundary value problems in exterior domains with constant coefficients and maximally dissipative boundary condition. Isozaki [74] and Alazard [2] have studied the singular incompressible limit for the compressible Euler equation in an exterior domain. Concerning the quasilinear hyperbolic mixed problems, Schochet [113] has proved the local in time existence in the case of bounded domains and Shibata and Kikuchi [115] have showed the local in time existence for some second order problem in bounded and unbounded domains. Differentiability of solutions to hyperbolic mixed problems has also been studied by Rauch and Massey [108].

The case we are considering here has not been treated in the literature yet. We consider a two-dimensional problem, but the axisymmetry keeps the boundary condition maximally dissipative which in general for a two-dimensional problem is not true. This property is essential for the coupling with the solid motion as it provides us better trace estimates than the ones for the general two-dimensional shallow water equations, but in other cases it is not necessary (see [89] for elementary examples). Hence we reduce the problem to a one-dimensional radial problem, then we must adapt the classical theory.

Let us recall that in the exterior domain we have

$$\begin{cases} \partial_t h_e + \partial_r q_e + \frac{q_e}{r} = 0 \\ \partial_t q_e + \partial_r \left( \frac{q_e^2}{h_e} \right) + \frac{q_e^2}{r h_e} + g h_e \partial_r h_e = 0. \end{cases} \quad \text{in } (R, +\infty) \quad (1.4.1)$$

coupled with the boundary condition

$$q_e|_{r=R} = q_i|_{r=R} \quad (1.4.2)$$

Defining  $u = (\zeta_e, q_e)^T$  and adding the Cauchy data we can write (1.4.1) - (1.4.2) as the following quasilinear hyperbolic mixed problem

$$\begin{cases} \partial_t u + A(u) \partial_r u + B(u, r) u = 0 & \text{in } (R, +\infty) \\ \mathbf{e}_2 \cdot u|_{r=R} = q_i|_{r=R} \\ u(0) = u_0 \end{cases} \quad (1.4.3)$$

with

$$A(u) = \begin{pmatrix} 0 & 1 \\ g h_e - \frac{q_e^2}{h_e^2} & \frac{2q_e}{h_e} \end{pmatrix}, \quad B(u, r) = \begin{pmatrix} 0 & \frac{1}{r} \\ 0 & \frac{q_e}{r h_e} \end{pmatrix}$$

and

$$u_0 = (\zeta_{e,0}, q_{e,0})^T.$$

### 1.4.1 The linear hyperbolic mixed problem

In order to construct the solution to the floating structure problem, which is a quasilinear mixed problem of the form (1.4.3) coupled with Newton's equation for the solid motion, we shall use an iterative scheme based on the following linearization of (1.4.3),

$$\begin{cases} L(\bar{u})u = \partial_t u + A(\bar{u}) \partial_r u + B(\bar{u}, r) u = f, \\ \mathbf{e}_2 \cdot u|_{r=R} = g, \\ u(0) = u_0, \end{cases} \quad (1.4.4)$$



with some  $\bar{u}$  and  $g = q_{i|_{r=R}}$ . Since the coefficients of  $A$  and  $B$  are rational fractions, we have  $A(\cdot), B(\cdot, r) \in C^\infty(\mathcal{U})$  for some open set  $\mathcal{U} \in \mathbb{R}^2$  where  $\bar{h}_e$  does not vanish, which represents a phase space of  $\bar{u}$ .

Let us assume that we are in the subsonic regime, which means that for  $\bar{u} = (\bar{\zeta}_e, \bar{q}_e)$  the following holds:

$$\frac{\bar{q}_e^2}{\bar{h}_e^2} < g\bar{h}_e \quad (1.4.5)$$

with  $\bar{h}_e = h_0 + \bar{\zeta}_e$ . In the case of water waves in oceans, where the water depth is much bigger than the water velocity, this assumption is satisfied. Then, for the linear initial boundary problem (1.4.4) we have the following result:

**Proposition 1.4.1.** *Assume that  $\bar{u}$  satisfies (1.4.5). Then, the linear exterior hyperbolic mixed problem (1.4.4), which is the linearization of the floating structure equations (1.4.3), satisfies the following properties :*

(P3)  $A(\bar{u})$  has one strictly positive eigenvalue  $\lambda_+(\bar{u})$  and one strictly negative eigenvalue  $\lambda_-(\bar{u})$ ,

(P4)  $P_-(\bar{u})\mathbf{e}_2^\perp \neq (0, 0)$  except in  $(0, 0)$ , where  $P_-(\bar{u})$  is the projector on the eigenspace associated with the negative eigenvalue of  $A(\bar{u})$  and  $\mathbf{e}_2^\perp$  is the orthogonal complement of  $\mathbf{e}_2$ .

*Proof.* The eigenvalues of  $A(\bar{u})$  are  $\lambda_\pm(\bar{u}) = \pm\sqrt{g\bar{h}_e} + \frac{\bar{q}_e}{\bar{h}_e}$  and the associated unit eigenvectors are

$$\mathbf{e}_\pm(\bar{u}) = \frac{1}{\sqrt{1 + \lambda_\pm^2(\bar{u})}}(1, \lambda_\pm(\bar{u})).$$

The assumption (1.4.5) gives property (P3). We prove now property (P4). Let us denote  $P_+(\bar{u})$  and  $P_-(\bar{u})$  the projectors on the eigenspaces associated with  $\lambda_+(\bar{u})$  and  $\lambda_-(\bar{u})$  respectively. They are given explicitly by

$$P_+(\bar{u}) = \frac{A(\bar{u}) - \lambda_-(\bar{u})\text{Id}}{\lambda_+(\bar{u}) - \lambda_-(\bar{u})} \quad P_-(\bar{u}) = -\frac{A(\bar{u}) - \lambda_+(\bar{u})\text{Id}}{\lambda_+(\bar{u}) - \lambda_-(\bar{u})} \quad (1.4.6)$$

Since  $\mathbf{e}_2^\perp$  is of the form  $a\mathbf{e}_1$  with  $a \in \mathbb{R}$ , from the definition of  $P_-(\bar{u})$  in (1.4.6) we have

$$P_-(\bar{u})\mathbf{e}_2^\perp = -\frac{a}{\lambda_+(\bar{u}) - \lambda_-(\bar{u})}(\lambda_+(\bar{u}), -\lambda_+(\bar{u})\lambda_-(\bar{u}))^T, \quad a \in \mathbb{R} \quad (1.4.7)$$

which is different from zero, except for  $a = 0$ , since  $\lambda_+(\bar{u}) \neq 0$ .  $\square$

The following lemma is a direct consequence of Proposition 1.4.1:

**Lemma 1.4.2.** *Assume that  $\bar{u}$  satisfies (1.4.5). The linear hyperbolic exterior mixed problem (1.4.4) satisfies the following properties:*

- (P1) *The system is Friedrichs symmetrizable, i.e. there exists a symmetric matrix  $S(\bar{u})$ , called the symmetrizer, such that there exist  $\alpha > 0$  such that  $S(\bar{u}) \geq \alpha \text{Id}$  and  $S(\bar{u})A(\bar{u})$  is symmetric.*
- (P2) *The boundary condition is maximally dissipative:  $S(\bar{u})A(\bar{u})$  is negative definite on the kernel of the boundary condition  $\mathbf{e}_2^\perp$ .*

*Proof.* From (P3) we have that  $\lambda_+(\bar{u}) > 0$  and  $\lambda_-(\bar{u}) < 0$ . We define the symmetrizer  $S(\bar{u}) := MP_-^T(\bar{u})P_-(\bar{u}) + P_+^T(\bar{u})P_+(\bar{u})$  for some constant  $M > 0$ . We compute that

$$\begin{aligned} (S(\bar{u})v, v) &= ((MP_-^T(\bar{u})P_-(\bar{u}) + P_+^T(\bar{u})P_+(\bar{u}))v, v) \\ &= M(P_-(\bar{u})v, P_-(\bar{u})v) + (P_+(\bar{u})v, P_+(\bar{u})v). \end{aligned}$$

Hence, from the decomposition  $v = P_+(\bar{u})v + P_-(\bar{u})v$  we get

$$(S(\bar{u})v, v) \geq \alpha(v, v) \quad (1.4.8)$$

with  $\alpha = \min(M, 1)/2$ . The symmetry of  $S(\bar{u})$  is trivial. We have the following spectral decomposition

$$A(\bar{u}) = \lambda_+(\bar{u})P_+(\bar{u}) + \lambda_-(\bar{u})P_-(\bar{u}). \quad (1.4.9)$$

By the definition of the projectors (1.4.6),  $S(\bar{u})A(\bar{u})$  reads

$$S(\bar{u})A(\bar{u}) = \lambda_-MP_-^T(\bar{u})P_-(\bar{u}) + \lambda_+P_+^T(\bar{u})P_+(\bar{u}) \quad (1.4.10)$$

which is clearly symmetric and we get property (P1). We refer to Taylor (see Prop. 2.2 of [119]) for a different proof with a general notion of symmetrizer involving pseudo-differential operators.

Let us consider  $\mathbf{e}_2^\perp$ , the one dimensional orthogonal complement of  $\mathbf{e}_2 \in \mathbb{R}^2$ , which is the kernel of the boundary condition. Then, we compute that

$$\begin{aligned} (S(\bar{u})A(\bar{u})\mathbf{e}_2^\perp, \mathbf{e}_2^\perp) &= \lambda_-(\bar{u})M(P_-(\bar{u})\mathbf{e}_2^\perp, P_-(\bar{u})\mathbf{e}_2^\perp) \\ &\quad + \lambda_+(\bar{u})(P_+(\bar{u})\mathbf{e}_2^\perp, P_+(\bar{u})\mathbf{e}_2^\perp). \end{aligned}$$

Due to property (P4), we obtain property (P2) choosing

$$M > -\frac{\lambda_+(\bar{u})(P_+(\bar{u})\mathbf{e}_2^\perp, P_+(\bar{u})\mathbf{e}_2^\perp)}{\lambda_-(\bar{u})(P_-(\bar{u})\mathbf{e}_2^\perp, P_-(\bar{u})\mathbf{e}_2^\perp)}.$$

□

**Remark 1.4.3.** *Property (P<sub>4</sub>) is a reformulation of the uniform Kreiss-Lopatinskiĭ condition. Then, we have just proved that the system (1.4.4) admits a Kreiss symmetrizer, which transforms the system into a symmetric one with the additional property that the boundary condition for this symmetric system is maximally dissipative. This property will permit to control the trace of the solution at the boundary by the standard energy estimate.*

Let us now introduce the following space:

$$X^k(T) := \bigcap_{j=0}^k C^j([0, T], H_r^{k-j}((R, +\infty))) \quad (1.4.11)$$

endowed with the norm

$$\|u\|_{X^k(T)} := \sup_{t \in [0, T]} \|u(t)\|_{X^k} \quad , \quad \|u(t)\|_{X^k} = \sum_{j=0}^k \|\partial_t^j u(t)\|_{H_r^{k-j}((R, +\infty))}$$

with  $\|\cdot\|_{H_r^k((R, +\infty))}$  the norm of the weighted Sobolev space  $H^k((R, +\infty), r dr)$ . We first show the following a priori estimate useful to find strong solutions of the problem (1.4.4).

**Proposition 1.4.4.** *Let  $T > 0$  and  $\bar{u} \in X^2(T)$  be such that (1.4.5) is satisfied. With  $\alpha$  as in Lemma 1.4.2 there are constants  $c_{\alpha, R}$  and a non-decreasing function  $C_R(\cdot)$  on  $[0, +\infty)$  such that all the solutions  $u \in H_r^1((0, T) \times (R, +\infty))$  solving (1.4.4) satisfy*

$$\begin{aligned} \|u(t)\|_{L_r^2((R, +\infty))}^2 + \|u|_{r=R}\|_{L^2((0, t))}^2 &\leq c_{\alpha, R} e^{tC_{\alpha, R}(\bar{u})} \times \\ &\times \left( \|u_0\|_{L_r^2((R, +\infty))}^2 + \|g\|_{L^2((0, t))}^2 + \int_0^t \|f(\tau)\|_{L_r^2((R, +\infty))}^2 d\tau \right) \end{aligned} \quad (1.4.12)$$

for all  $t \in [0, T]$ , with  $C_{\alpha, R}(\bar{u}) = 1 + \alpha^{-1} C_R(\|\bar{u}\|_{X^2(T)})$ .

*Proof.* Following Proposition 2.2 of [12], we have from property (P1) and by integrations by parts

$$\begin{aligned} \frac{d}{dt} (S(\bar{u})u, u)_{L_r^2((R, +\infty))} &= -2(S(\bar{u})A(\bar{u})\partial_r u, u)_{L_r^2((R, +\infty))} \\ &\quad - 2(S(\bar{u})B(\bar{u}, r)u, u)_{L_r^2((R, +\infty))} + ((\partial_t S(\bar{u}))u, u)_{L_r^2((R, +\infty))} + 2(S(\bar{u})f, u)_{L_r^2((R, +\infty))} \\ &= S(\bar{u})A(\bar{u})u|_{r=R} \cdot u|_{r=R} R + (W(\bar{u})u, u)_{L_r^2((R, +\infty))} + (S(\bar{u})u, f)_{L_r^2((R, +\infty))} . \end{aligned}$$

with

$$W(\bar{u}) = \partial_t S(\bar{u}) + \partial_r (S(\bar{u})A(\bar{u})) + \frac{1}{r} S(\bar{u})A(\bar{u}) - 2S(\bar{u})B(\bar{u}, r) \quad (1.4.13)$$

Property (P2) permits us to control the first term on the right-hand side of the inequality, using the following Lemma from Métivier [97]:

**Lemma 1.4.5.** *The symmetric matrix  $S(\bar{u})A(\bar{u})$  is negative definite on  $\mathbf{e}_2^\perp$ , the set of all vectors orthogonal to  $\mathbf{e}_2$ , if and only if there are constants  $c_1, c_2 > 0$  such that for each vector  $h \in \mathbb{C}^2$ :*

$$-(S(\bar{u})A(\bar{u})h, h) \geq c_1|h|^2 - c_2|\mathbf{e}_2 \cdot h|^2.$$

Choosing  $h = u|_{r=R}$ , integrating in time and using property (H1) we have

$$\begin{aligned} & (S(\bar{u})u(t), u(t))_{L^2((R, +\infty))} \leq (S(\bar{u}(0))u_0, u_0)_{L^2((R, +\infty))} \\ & + \int_0^t (S(\bar{u})f(\tau), f(\tau))_{L^2((R, +\infty))} + c_2R|g(\tau)|^2 - c_1R|u|_{r=R(\tau)}|^2 d\tau \\ & + (1 + \alpha^{-1}\|W(\bar{u})\|_{L^\infty((0, T) \times (R, +\infty))}) \int_0^t (S(\bar{u})u(\tau), u(\tau))_{L^2((R, +\infty))} d\tau. \end{aligned} \quad (1.4.14)$$

where we have used the fact that  $S(\bar{u}) \geq \alpha \text{Id}$ . Moreover, we get the following estimate:

$$\begin{aligned} & \|W(\bar{u})\|_{L^\infty((0, T) \times (R, +\infty))} \\ & = \|S'(\bar{u})\partial_t \bar{u} + S'(\bar{u})\partial_r \bar{u} A(\bar{u}) + S(\bar{u})A'(\bar{u})\partial_r \bar{u} \\ & \quad + \frac{1}{r}S(\bar{u})A(\bar{u}) - 2S(\bar{u})B(\bar{u}, r)\|_{L^\infty((0, T) \times (R, +\infty))} \\ & \leq C_R(\|\bar{u}\|_{W^{1, \infty}((0, T) \times (R, +\infty))}) \leq C_R(\|\bar{u}\|_{X^2(T)}) \end{aligned} \quad (1.4.15)$$

for some non-decreasing function  $C_R(\cdot)$  on  $[0, +\infty)$ , where we have used the fact that  $r \in (R, +\infty)$ , the embedding  $H_r^2((R, +\infty)) \hookrightarrow W^{1, \infty}((R, +\infty))$  and the definition of  $X^2(T)$ . By Gronwall's Lemma we have

$$\begin{aligned} \alpha\|u(t)\|_{X^0}^2 + c_1R\|u|_{r=R}\|_{L^2((0, t))}^2 & \leq e^{t(1 + \alpha^{-1}C_R(\|\bar{u}\|_{X^2(T)})} \times \\ & \times \left( c(\|\bar{u}(0)\|_{X^2})\|u_0\|_{X^0}^2 + c_2R\|g\|_{L^2((0, t))}^2 + \alpha^{-1} \int_0^t \|f(\tau)\|_{X^0}^2 d\tau \right). \end{aligned}$$

We get (1.4.12) for

$$C_{\alpha, R}(\bar{u}) = 1 + \alpha^{-1}C_R(\|\bar{u}\|_{X^2(T)}), \quad c_{\alpha, R} = \frac{\max(c(\|\bar{u}(0)\|_{X^2}), c_2R)}{\min(\alpha, c_1R)}.$$

□

Then, following Theorem 2.4.5 of [97], one can show that there is a unique solution  $u \in C^0([0, T], L_r^2((R, +\infty)))$  for the initial datum  $u_0$  in  $L_r^2((R, +\infty))$  and boundary value  $g$  in  $L^2((0, T))$ . This solution satisfies the energy estimate (1.4.12).

**Regular solutions.** To solve the mixed problem in Sobolev spaces we need some compatibility conditions. For instance, the initial and the boundary conditions imply that necessarily

$$\mathbf{e}_2 \cdot u_0|_{r=R} = g|_{t=0} = \mathbf{e}_2 \cdot u|_{t=0, r=R}, \quad (1.4.16)$$

if the traces are defined. Let us consider the generic equation

$$\partial_t u = f - A(\bar{u})\partial_r u - B(\bar{u}, r)u.$$

Then, we can formally (this is the meaning of the brackets “”) define

$$“\partial_t u|_{t=0}” = -A(\bar{u}_0)|_{t=0}\partial_r u_0 - B(\bar{u}_0, r)|_{t=0}u_0 + f|_{t=0}.$$

Hence, provided traces are defined,

$$\mathbf{e}_2 \cdot “\partial_t u|_{t=0}”|_{r=R} = \mathbf{e}_2 \cdot (f_0 - A(\bar{u}_0)\partial_r u_0 - B(\bar{u}_0, r)u_0)|_{r=R} = g_1$$

where  $g_1 := \partial_t g|_{t=0}$ . These conditions are necessary for the existence of a smooth solution. We can continue the expansion to higher orders looking for more compatibility conditions. Let us introduce the following notation (as in [96] and [113]):

**Notation 1.4.6.** *Let us write the linear equation in (1.4.4)*

$$\partial_t u = F(\bar{u}, \partial)u + f$$

with

$$F(\bar{u}, \partial) = -A(\bar{u})\partial_r - B(\bar{u}, r).$$

We formally define the traces  $u_j := “\partial_t^j u|_{t=0}”$  as functions of  $u_0$  determined inductively by

$$u_0 = u|_{t=0} \quad u_{j+1} = \bar{\mathcal{F}}_j(u_0, \dots, u_j) + f_j \quad (1.4.17)$$

with

$$\bar{\mathcal{F}}_j(u_0, \dots, u_j) = \sum_{p+|k|\leq j} A_{j,p,k}(\bar{u}_0)\bar{u}_{(k)}\partial_r u_p + \sum_{p+|k|\leq j} B_{j,p,k}(\bar{u}_0, r)\bar{u}_{(k)}u_p. \quad (1.4.18)$$

where we use the notation

$$\text{for } k = (k_1, \dots, k_r), \quad u_{(k)} = u_{k_1} \dots u_{k_r}.$$

Note that  $u_j$  is not the derivative of a known function but rather the value that the derivative of  $u$  will have if  $u$  exists. Therefore necessarily smooth enough solutions to (1.4.4) must satisfy

$$\mathbf{e}_2 \cdot u_j|_{r=R} = g_j \quad (1.4.19)$$

with  $g_j := \partial_t^j g|_{t=0}$ .

**Definition 1.4.7.** *The data  $u_0 \in H_r^k(\mathbb{R}_+)$ ,  $g \in H^k((0, T))$  and  $f \in H_r^k((0, T) \times \mathbb{R}_+)$  satisfy the compatibility conditions up to order  $s \leq k - 1$  if (1.4.19) holds for each  $j = 0, \dots, s$ .*

For the linear floating structure mixed problem (1.4.4) the compatibility conditions (1.4.19) on the initial data  $u_0 = (\zeta_{e,0}, q_{e,0})$  and the boundary value  $g = q_{i|_{r=R}}$  can be written as:

$$q_{e|_{r=R,t=0}} = q_{i|_{r=R,t=0}}, \quad \overline{\mathcal{F}}_{j_2}(\zeta_{e,0}, \dots, \zeta_{e,j}, q_{e,0}, \dots, q_{e,j})|_{r=R} = \partial_t^j q_{i|_{r=R,t=0}}, \quad j \geq 1. \quad (1.4.20)$$

where  $\overline{\mathcal{F}}_{j_2}$  is the second component of  $\overline{\mathcal{F}}_j$ . As in the  $L^2$  case, our goal is to find an a priori estimate for the linear problem (1.4.4) in order to get existence and uniqueness of the solution in some more regular space.

**Proposition 1.4.8.** *Let  $T > 0$  and  $k \geq 1$  be an integer. Assume that  $\bar{u} \in X^s(T)$  with  $s = \max(k, 2)$  satisfies (1.4.5). With  $\alpha$  as in Lemma 1.4.2 there are a constant  $c_{\alpha,R}$  and non-decreasing functions  $C_R(\cdot)$ ,  $C_{1,k,R}(\cdot)$  and  $C_{2,k,R}(\cdot)$  on  $[0, +\infty)$  such that all the solutions  $u \in H_r^{k+1}((0, T) \times (R, +\infty))$  solving (1.4.4) satisfy:*

$$\begin{aligned} \|u(t)\|_{X^k}^2 + \|u|_{r=R}\|_{H^k((0,t))}^2 &\leq c_{\alpha,R} e^{tC_{\alpha,R,k}(\|\bar{u}\|_{X^s(T)})} \times \\ &\times \left( \|u(0)\|_{X^k}^2 + \|g\|_{H^k((0,t))}^2 + K_{k,R}(\|\bar{u}\|_{X^s(T)}) \int_0^t \|f(\tau)\|_{X^k}^2 d\tau \right) \end{aligned} \quad (1.4.21)$$

for all  $t \in [0, T]$  with

$$C_{\alpha,R,k}(\|\bar{u}\|_{X^s(t)}) = 1 + \alpha^{-1} C_R(\|\bar{u}\|_{X^2(t)}) + \alpha^{-1}(k+1) C_{1,k,R}(\|\bar{u}\|_{X^s(t)})$$

and

$$K_{k,R}(\|\bar{u}\|_{X^s(T)}) = \frac{C_{2,k,R}(\|\bar{u}\|_{X^s(T)})}{\max(c(\|\bar{u}(0)\|_{X^2}), c_2 R)}.$$

*Proof.* We adapt here the argument presented in [97]. We denote by  $u^i$  the tangential derivative  $\partial_t^i u$  for  $i \leq k$ , which in the one dimensional case is simply the time derivative, and we introduce the tangential norm

$$\|u(t)\|'_{X^k} := \sum_{i=0}^k \|\partial_t^i u(t)\|_{L_r^2((R, +\infty))}$$

We apply  $\partial_t^i$  to the equation of (1.4.4) and we get

$$\partial_t u^i + A(\bar{u}) \partial_r u^i + B(\bar{u}, r) u^i = [A(\bar{u}), \partial_t^i] \partial_r u + [B(\bar{u}, r), \partial_t^i] u + f^i, \quad (1.4.22)$$

$$\mathbf{e}_2 \cdot u|_{r=R}^i = g^i.$$

As we have done in the previous  $L^2$  case we consider  $\frac{d}{dt}(S(\bar{u})u^i, u^i)_{L^2_r((R, +\infty))}$ . The only difference from the previous case is the presence of the two commutator terms in (1.4.22). We need to control their  $L^2_r$  norms in a different way.

The first term can be written under the form  $\sum_{\alpha=1, \dots, i} \|\partial_t^\alpha(A(\bar{u}))\partial_t^{i-\alpha}\partial_r u\|_{L^2_r}$ . For  $\alpha \leq k-1$  every term of the sum is controlled by

$$\begin{aligned} \|\partial_t^\alpha(A(\bar{u}))\|_{L^\infty} \|\partial_t^{i-\alpha}\partial_r u\|_{L^2_r} &\leq c_R \|\partial_t^\alpha(A(\bar{u}))\|_{H_r^{k-\alpha}} \|u\|_{X^{k-\alpha+1}} \\ &\leq c_R \|A(\bar{u})\|_{X^k} \|u\|_{X^k} \end{aligned} \quad (1.4.23)$$

using the fact that  $1 \leq \alpha \leq k-1$  for the Sobolev embedding and that  $\|\cdot\|_{X^\lambda} \leq \|\cdot\|_{X^\delta}$  if  $\lambda < \delta$ . Recall that Sobolev embeddings  $H_r^k \hookrightarrow W^{s, \infty}$  still hold for the weighted spaces  $H_r^k$  since we are considering the exterior domain  $(R, +\infty)$ . For  $\alpha = k$  we directly have

$$\|\partial_t^k(A(\bar{u}))\|_{L^2_r} \|\partial_r u\|_{L^\infty} \leq c_R \|A(\bar{u})\|_{X^k} \|u\|_{X^k}$$

since  $u \in H_r^{k+1}$  with  $k \geq 1$ . We can find the same estimate for the commutator term with  $B(\bar{u}, r)$ .

We recall the following Moser-type estimate for the  $\|\cdot\|_{X^k}$  norm:

**Lemma 1.4.9** (Schochet). *For  $A(\cdot)$  smooth enough, the following holds*

$$\|A(\bar{u})\|_{X^k} \leq C_k (1 + \|\bar{u}\|_{X^k}^k) \quad (1.4.24)$$

with  $k \geq 1$ .

We refer to the Appendix B of [113] for the details of the proof based on Gagliardo-Nirenberg inequalities. Better estimates can be used, in particular if one wants to derive blow up conditions as Métivier in [96], but here we are not interested in this problem. Hence we get the following inequality:

$$\begin{aligned} &\left( S(\bar{u})u^i(t), u^i(t) \right)_{L^2((R, +\infty))} \leq \left( S(\bar{u}(0))u_0^i, u_0^i \right)_{L^2((R, +\infty))} \\ &+ \int_0^t \left[ \left( S(\bar{u})f^i(\tau), f^i(\tau) \right)_{L^2((R, +\infty))} + c_2 R \left| g^i(\tau) \right|^2 - c_1 R \left| u_{|r=R}^i(\tau) \right|^2 \right] d\tau \\ &+ \left( 1 + \alpha^{-1} C_R (\|\bar{u}\|_{X^2(T)}) \right) \int_0^t \left( S(\bar{u})u^i(\tau), u^i(\tau) \right)_{L^2((R, +\infty))} d\tau \\ &+ C_{k,R} (\|\bar{u}\|_{X^s(T)}) \int_0^t \|u(\tau)\|_{X^k}^2 d\tau \end{aligned} \quad (1.4.25)$$

with  $s = \max(k, 2)$  and some non-decreasing function  $C_{k,R}(\cdot)$  on  $[0, +\infty)$ . Here  $u_0^i = u_i$  where the  $u_i$  are defined in (1.4.17). We note that for solutions to (1.4.4), we have

$$\|u(0)\|'_{X^k} = \sum_{j=0}^k \|u_j\|_{L^2_r}, \quad \|u(0)\|_{X^k} = \sum_{j=0}^k \|u_j\|_{H_r^{k-j}}.$$

For  $u$  satisfying the equation (1.4.4), we have

$$\partial_r u = A^{-1}(\bar{u})(f - \partial_t u - B(\bar{u}, r)u)$$

and we can show that the  $X^k$  norm is controlled by the tangential one. The following holds for  $0 \leq t \leq T$ :

$$\begin{aligned} \|u(t)\|_{X^k}^2 &\leq C_1(\|\bar{u}\|_{X^s(T)})\|u(t)\|_{X^k}'^2 + C_2(\|\bar{u}\|_{X^s(T)})\|f(t)\|_{X^k}^2 \\ &\leq C_1(\|\bar{u}\|_{X^s(T)})\alpha^{-1} \sum_{i=0}^k \left( S(\bar{u})u^i, u^i \right)_{L^2((R, +\infty))} + C_2(\|\bar{u}\|_{X^s(T)})\|f(t)\|_{X^k}^2 \end{aligned} \quad (1.4.26)$$

with some non-decreasing functions  $C_1(\cdot), C_2(\cdot)$  on  $[0, +\infty)$ . In the second inequality we have used the fact that  $S(\bar{u}) \geq \alpha \text{Id}$ . By taking the sum for  $i$  from 0 to  $k$  and by applying Gronwall's Lemma we obtain

$$\begin{aligned} \alpha\|u(t)\|_{X^k}'^2 + c_1 R \|u|_{r=R}\|_{H^k((0,t))}^2 &\leq e^{t(1+\alpha^{-1}C_R(\|\bar{u}\|_{X^2(T)})+\alpha^{-1}(k+1)C_{1,k,R}(\|\bar{u}\|_{X^s(T)}))} \times \\ &\times \left( c(\|\bar{u}(0)\|_{X^2})\|u(0)\|_{X^k}'^2 + c_2 R \|g\|_{H^k((0,t))}^2 + C_{2,k,R}(\|\bar{u}\|_{X^s(T)}) \int_0^t \|f(\tau)\|_{X^k}^2 d\tau \right) \end{aligned} \quad (1.4.27)$$

with some non-decreasing functions  $C_{1,k,R}(\cdot), C_{2,k,R}(\cdot)$  on  $[0, +\infty)$ . By definition of the tangential norm we have  $\|u_0\|_{X^k}'^2 \leq \|u_0\|_{X^k}^2$ . We use (1.4.26) and the estimate (1.4.21) follows with

$$C_{\alpha,R} = \frac{\max(c(\|\bar{u}(0)\|_{X^2}), c_2 R)}{\min(\alpha, c_1 R)},$$

$$C_{\alpha,R,k}(\|\bar{u}\|_{X^s(T)}) = 1 + \alpha^{-1}C_R(\|\bar{u}\|_{X^2(T)}) + \alpha^{-1}(k+1)C_{1,k,R}(\|\bar{u}\|_{X^s(T)})$$

and

$$K_{k,R}(\|\bar{u}\|_{X^s(T)}) = \frac{C_{2,k,R}(\|\bar{u}\|_{X^s(T)})}{\max(c(\|\bar{u}(0)\|_{X^2}), c_2 R)}.$$

□

Equivalently to the  $L^2$ -case, we can state the following theorem:

**Theorem 1.4.10.** *Let  $k \geq 1$  be an integer and  $T > 0$ . Suppose  $u_0 \in H_r^k((R, +\infty))$ ,  $g \in H^k((0, T))$  and  $f \in H_r^k((0, T) \times (R, +\infty))$  satisfy the compatibility conditions (1.4.19) up to the order  $k-1$ . Assume that  $\bar{u} \in X^s(T)$  with  $s = \max(k, 2)$  satisfies (1.4.5). Then, there is a unique solution  $u \in X^k(T)$  to (1.4.4). Its trace on  $r = R$  belongs to  $H^k((0, T))$  and  $u$  satisfies the energy estimate (1.4.21).*

*Proof.* We show only the idea of the proof of the existence. For more details and the proof of uniqueness see [96]. First we solve the equation with a loss



of smoothness. We consider the data  $u_0$ ,  $f$  and  $g$  in  $H_r^{k+2}$  satisfying the compatibility conditions up to order  $k$ . One can prove that there is a solution in  $H_r^{k+1}((0, T) \times (R, +\infty)) \subseteq X^k(T)$ , by extending the data by 0 for  $t < 0$  and then by applying the existence result for the mixed problem in  $(-\infty, T] \times (R, +\infty)$  of [97].

The second step is to consider  $H^k$ -data: we use the compatibility conditions up to order  $k - 1$  to approximate  $u_0$ ,  $f$  and  $g$  in  $H_r^k$  and  $H^k$  with sequences  $u_0^n \in H_r^{k+2}((R, +\infty))$ ,  $f^n \in H_r^{k+2}((0, T) \times (R, +\infty))$  and  $g^n \in H^{k+2}((0, T))$  satisfying the compatibility conditions up to order  $k + 1$ . From the previous argument and the energy estimate (1.4.21) we have that  $u^n$  is a Cauchy sequence in  $X^k(T)$  and therefore converges to the limit  $u \in X^k(T)$ , which is a solution to (1.4.4) since  $k \geq 1$ .  $\square$

## 1.4.2 The quasilinear problem and application to the case of a solid with prescribed motion

In the particular case of the floating structure problem, the boundary condition in (1.4.3) is  $g = q_{i|_{r=R}}$ , the value of the horizontal discharge in the interior domain at the boundary  $r = R$ . We will see in the next section that this quantity is strictly linked to the solid motion, in particular to the vertical component of the velocity of the center of mass  $w_G(t)$ .

In the case of a solid with prescribed motion, the boundary condition  $g$  is a datum of the problem. Hence, after having studied the linear problem (1.4.4), one can use a standard iterative scheme argument in order to get existence and uniqueness of the solution to (1.4.3).

**Theorem 1.4.11.** *Consider a solid with a prescribed motion. For  $k \geq 2$ , let  $u_0 \in H_r^k((R, +\infty))$  and  $w_G \in H^k((0, T))$  satisfy the compatibility conditions in Definition 1.6.2 up to order  $k - 1$ . Assume that  $u_0$  satisfies (1.4.5). Then, the fluid problem (1.4.3) with boundary condition  $-\frac{R}{2}w_G(t)$  admits a unique solution  $u \in X^k(T)$  with  $X^k(T)$  as in (1.4.11).*

*Sketch of the proof.* We introduce the iterative scheme by defining the sequence  $(u^n)_n$  with  $u^n$  solution to the linear problem  $L(u^{n-1})u^n = 0$ . The existence of such a sequence is given by Theorem 1.4.10. Once we have showed the control of the sequence in some “big norm” and the convergence in some “small norm”, the limit  $u$  of  $(u^n)_n$  is the solution to (1.4.3). For more details we refer to [96]. We will show a detailed proof in the case of a free motion in Theorem 1.6.3 below.  $\square$

From this point on we consider a solid with free motion. Therefore the boundary condition is still an unknown of the problem and we must adapt the classical argument used in Theorem 1.4.11 to our problem introducing an iterative scheme

for the fluid-structure coupled system. The details of this coupled iterative scheme argument are given in Section 1.6. Before, we deal with the solid problem and we deduce an ordinary differential equation describing the motion of its center of mass.

## 1.5 The solid equation

In this section we address the motion of the solid. We recall that we are considering a floating structure moving only vertically.

Denoting  $m$  the mass of the body,  $g$  the gravity acceleration and  $z_G$  the vertical position of the center of mass, we consider only the vertical component of Newton's law for the conservation of linear momentum:

$$m\ddot{z}_G = -mg + F_{\text{fluid}} \quad (1.5.1)$$

where  $F_{\text{fluid}} = 2\pi \int_0^R (\underline{P}_i - P_{\text{atm}})rdr$  is the resulting vertical force exerted by the fluid on the solid.

Let us introduce the displacement  $\delta_G(t) := z_G(t) - z_{G,eq}$  between the vertical position of the center of mass at time  $t$  and at its equilibrium. In the case of vertical motion  $h_w(t, r) = h_{w,eq}(r) + \delta_G(t)$ , where  $h_{w,eq}$  is the fluid height at the equilibrium. For simplicity we consider a cylindrically symmetric solid with flat bottom, which means that the wetted surface  $\zeta_w$  (hence  $h_w$ ) does not depend on the spatial coordinate in the interior domain  $(0, R)$ . See Appendix A for the general case with a cylindrically symmetric solid with a non-flat bottom.

**Proposition 1.5.1.** *Newton's law (1.5.1) can be written under the following form:*

$$(m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathbf{c}\delta_G(t) + \mathbf{c}\zeta_e(t, R) + \left( \frac{b}{h_e^2(t, R)} + \beta(\delta_G) \right) \delta_G^2(t) \quad (1.5.2)$$

with

$$\begin{aligned} \mathbf{c} &= \rho g \pi R^2, & b &= \frac{\pi \rho R^4}{8}, \\ m_a(\delta_G) &= \frac{b}{h_w(\delta_G)} = \frac{b}{h_{w,eq} + \delta_G(t)}, \\ \beta(\delta_G) &= \frac{b}{2h_w^2(\delta_G)} = \frac{b}{2(h_{w,eq} + \delta_G(t))^2}. \end{aligned}$$

**Remark 1.5.2.** *In (1.5.2)  $m_a(\delta_G)$  is called the added mass term and it represents the fact that, in order to move in the fluid, the solid has to accelerate itself but also the portion of fluid next to it. This effect appears in other hydrodynamical*

configurations, in particular for totally submerged solids studied by Glass, Sueur and Takahashi [63] and Glass, Munnier and Sueur [62]. It has an important role for the stability of numerical simulations of fluid-structure interactions [24].

The coupling with the fluid motion is given by the term  $\zeta_e(t, R)$  and  $\frac{b}{h_e^2(t, R)}$ , which means that the solid motion depends on the value of the elevation of the exterior free surface at the boundary  $r = R$ .

*Proof.* We decompose  $F_{\text{fluid}}$  according to the decomposition (1.3.3) - (1.3.5) of the pressure

$$F_{\text{fluid}} = F_{\text{fluid}}^{\text{I}} + F_{\text{fluid}}^{\text{II}} + F_{\text{fluid}}^{\text{III}}$$

with

$$F_{\text{fluid}}^{\text{I}} = 2\pi \int_0^R (\underline{P}_i^{\text{I}} - P_{\text{atm}}) r dr, \quad F_{\text{fluid}}^j = 2\pi \int_0^R \underline{P}_i^j r dr, \quad j = \text{II}, \text{III}.$$

Using the elementary potential  $\Phi_{\mathcal{I}}^r$  defined in [83] we can write

$$F_{\text{fluid}}^{\text{II}} = -2\pi \int_0^R \underline{P}_i^{\text{II}} \left( \partial_r + \frac{1}{r} \right) (h_w \partial_r \Phi_{\mathcal{I}}^r) r dr,$$

and, after integration by parts,

$$\begin{aligned} F_{\text{fluid}}^{\text{II}} &= -2\pi \int_0^R \left( \partial_r + \frac{1}{r} \right) (h_w \partial_r \underline{P}_i^{\text{II}}) \Phi_{\mathcal{I}}^r r dr \\ &= -2\pi \rho \int_0^R \dot{w}_G \Phi_{\mathcal{I}}^r r dr, \end{aligned}$$

where the second equality comes from the definition (1.3.4) of  $\underline{P}_i^{\text{II}}$ . Using again the definition of elementary potential we obtain

$$\begin{aligned} F_{\text{fluid}}^{\text{II}} &= 2\pi \rho \left[ \int_0^R \Phi_{\mathcal{I}}^r \left( \partial_r + \frac{1}{r} \right) (h_w \Phi_{\mathcal{I}}^r) \right] \dot{w}_G r dr \\ &= -2\pi \rho \left[ \int_0^R \frac{1}{h_w} (h_w \partial_r \Phi_{\mathcal{I}}^r)^2 \right] \dot{w}_G r dr. \end{aligned}$$

From the definition of the elementary potential we explicitly have that

$$h_w \partial_r \Phi_{\mathcal{I}}^r = -\frac{r}{2}.$$

It follows that

$$F_{\text{fluid}}^{\text{II}} = -m_a (h_w) \dot{w}_G,$$

with  $m_a(h_w)$  as in (1.5.2). Proceeding similarly we can write also

$$F_{\text{fluid}}^{\text{I}} = -2\pi\rho \int_0^R \frac{r}{2h_w} \frac{h_w}{\rho} \partial_r P_i^{\text{I}} r dr.$$

Then, (1.5.1) becomes

$$(m + m_a(h_w)) \dot{w}_G = -mg + F_{\text{fluid}}^{\text{I}} + 2\pi \int_0^R P_i^{\text{III}} r dr. \quad (1.5.3)$$

Moreover, (1.3.3) can be written as

$$\partial_r y(r) + \frac{y(r)}{r} = -\frac{b(r)}{r} \quad \text{in } (0, R)$$

with

$$y(r) = \frac{h_w}{\rho} \partial_r P_i^{\text{I}} + \partial_r \left( \frac{q_i^2}{h_w} \right) \quad \text{and} \quad b(r) = \partial_r \left( \frac{q_i^2}{h_w} \right).$$

Hence we have

$$y(r) = -\frac{1}{r} \left( \frac{q_i(r)^2}{h_w} - \frac{q_i(0)^2}{h_w(0)} \right). \quad (1.5.4)$$

Because of the constraint (1.1.6), the mass conservation equation of (1.3.1) in the interior domain becomes

$$\partial_r q_i + \frac{1}{r} q_i = -\dot{\delta}_G(t) \quad \text{in } (0, R)$$

then we have

$$q_i(t, r) = -\frac{r}{2} \dot{\delta}_G(t). \quad (1.5.5)$$

Hence  $q_i(t, 0) = 0$  and (1.5.4) becomes

$$\frac{h_w}{\rho} \partial_r P_i^{\text{I}} = -\partial_r \left( \frac{q_i^2}{h_w} \right) - \frac{1}{r} \frac{q_i^2}{h_w} = -\frac{3}{4h_w} r \dot{\delta}_G^2.$$

Replacing the expression of  $\frac{h_w}{\rho} \partial_r P_i^{\text{I}}$  in  $F_{\text{fluid}}^{\text{I}}$  we get

$$F_{\text{fluid}}^{\text{I}} = \frac{3\pi\rho R^4}{16h_w^2} \dot{\delta}_G^2$$

and, by definition of the equilibrium state, we have

$$-mg - 2\pi\rho g \int_0^R \zeta_{w,eq} r dr = 0. \quad (1.5.6)$$

Since the solid has vertical side-walls the following equality holds

$$2\pi\rho g \int_0^R \zeta_w(t)rdr - 2\pi\rho g \int_0^R \zeta_{w,eq}rdr = \mathbf{c}\delta_G(t). \quad (1.5.7)$$

These two equalities, together with the constraint  $\zeta_i = \zeta_w$ , give

$$-mg = \mathbf{c}\zeta_i(t, R) - \mathbf{c}\delta_G(t).$$

Solving the elliptic problem (1.3.5) whose solution is the constant (in space) boundary value, we obtain the nonlinear second order ordinary differential equation (1.5.2).  $\square$

**Remark 1.5.3.** *All the computations in the proof of Proposition 1.5.1 reduce to particular cases of the Green's identity:*

$$\int_0^R p\nabla_r \cdot (h_w\partial_r q)rdr = \int_{r=R} h_w p\partial_r q r - \int_{r=R} h_w q\partial_r p r + \int_0^R q\nabla_r \cdot (h_w\partial_r p)rdr \quad (1.5.8)$$

with particular  $p$  and  $q = \Phi_{\mathcal{I}}^r$ , where  $\nabla_r \cdot = \partial_r + \frac{1}{r}$  is the divergence operator in the axisymmetric configuration.

Recall that in (1.5.5) we have

$$q_i(t, R) = -\frac{R}{2}\dot{\delta}_G(t).$$

This term is the boundary value in the fluid mixed problem (1.4.1). It follows that this is the coupling term between the fluid and the solid motion in the fluid system, as  $\zeta_e(t, R)$  has the same property in the solid equation (see Remark 1.5.2).

## 1.6 Fluid-structure coupling

From the previous two sections, it follows that the fluid-structure interaction problem considered in this paper is described by the following mathematical model:

**Proposition 1.6.1.** *The nonlinear shallow water equations with a floating structure for an axisymmetric flow without swirl take the form*

$$\begin{cases} \partial_t u + A(u)\partial_r u + B(u, r)u = 0 & \text{in } (R, +\infty), \\ \mathbf{e}_2 \cdot u|_{r=R} = -\frac{R}{2}\dot{\delta}_G(t), \\ u(0) = u_0, \end{cases} \quad (1.6.1)$$

with  $A(u), B(u, r)$  as in (1.4.3). Moreover the solid motion is given by the Cauchy problem

$$\begin{cases} (m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathbf{c}\delta_G(t) + \mathbf{c}\mathbf{e}_1 \cdot u|_{r=R} + (\mathbf{b}(u) + \beta(\delta_G))\dot{\delta}_G^2(t), \\ \delta_G(0) = \delta_0, \\ \dot{\delta}_G(0) = \delta_1, \end{cases} \quad (1.6.2)$$

with  $\mathbf{c}, b, m_a(\delta_G), \beta(\delta_G)$  as in (1.5.2),

$$\mathbf{b}(u) = \frac{b}{(\mathbf{e}_1 \cdot u|_{r=R} + h_0)^2} = \frac{b}{h_e(t, R)^2},$$

using the fact that

$$\zeta_e(t, R) = \mathbf{e}_1 \cdot u|_{r=R} \quad \text{and} \quad h_e(t, R) = h_0 + \zeta_e(t, R).$$

Let us give the notion of compatibility conditions in the case of this particular fluid-structure coupled problem. We recall the equation in (1.6.1)

$$\partial_t u = F(u, \partial)u$$

with

$$F(u, \partial) = A(u)\partial_r + B(u, r) \quad (1.6.3)$$

We define formally the traces  $u_j := \partial_t^j u|_{t=0}$  as functions of  $u_0$  determined inductively by

$$u_0 = u|_{t=0} \quad u_{j+1} = \mathcal{F}_j(u_0, \dots, u_j) \quad (1.6.4)$$

with

$$\mathcal{F}_j(u_0, \dots, u_j) = \sum_{p+|k|\leq j} A_{j,p,k}(u_0)u_{(k)}\partial_r u_p + \sum_{p+|k|\leq j} B_{j,p,k}(u_0, r)u_{(k)}u_p.$$

where we use the notation

$$\text{for } k = (k_1, \dots, k_r), \quad u_{(k)} = u_{k_1} \dots u_{k_r}.$$

**Definition 1.6.2.** *The data  $u_0 \in H_r^k((R, +\infty))$ ,  $\delta_0 \in \mathbb{R}$  and  $\delta_1 \in \mathbb{R}$  of the floating structure coupled system (1.6.1) - (1.6.2) satisfy the compatibility conditions up to order  $k - 1$  if, for  $0 \leq j \leq k - 1$ , the following holds:*

$$\mathbf{e}_2 \cdot u_j|_{r=R} = -\frac{R}{2}\delta_{j+1},$$

where  $\delta_{j+1}$  are the formal traces “ $\frac{d^{j+1}}{dt^{j+1}}\delta_G(0)$ ” defined from the ODE in (1.6.2) as

$$\frac{d^{j-1}}{dt^{j-1}} \left[ \frac{1}{(m + m_a(\delta_G))} \left( -\mathbf{c}\delta_G + \mathbf{c}\mathbf{e}_1 \cdot u|_{r=R} + (\mathbf{b}(u) + \beta(\delta_G))\dot{\delta}_G^2 \right) \right] (0).$$

In the following theorem we prove that the coupled model (1.6.1) - (1.6.2) is locally in time well-posed:

**Theorem 1.6.3.** *For  $k \geq 2$ , let  $u_0 = (\zeta_{e,0}, q_{e,0}) \in H_r^k((R, +\infty))$ ,  $\delta_0$  and  $\delta_1$  satisfy the compatibility conditions in Definition 2.1.1 up to order  $k - 1$ . Assume that there exist some constants  $h_{\min}, c_{\text{sub}} > 0$  such that*

$$\forall r \in (R, +\infty) : \quad h_{e,0}(r) \geq h_{\min}, \quad \left( gh_{e,0} - \frac{q_{e,0}^2}{h_{e,0}^2} \right)(r) \geq c_{\text{sub}},$$

with  $h_{e,0} = h_0 + \zeta_{e,0}$ , and that

$$\delta_0 > -h_{w,eq}$$

with the constant  $h_{w,eq}$  as in Section 1.5. Then, the coupled problem (1.6.1) - (1.6.2) admits a unique solution  $(u, \delta_G) \in X^k(T) \times H^{k+1}((0, T))$  with  $X^k(T)$  as in (1.4.11).

**Remark 1.6.4.** *Considering an initial datum  $u_0 \in H_r^2((R, +\infty))$ , we need the following compatibility conditions satisfied:*

$$q_e(0, R) = -\frac{R}{2}\delta_1$$

and

$$\begin{aligned} & -\partial_r \left( \frac{q_e^2}{h_e} \right)(0, R) - \frac{1}{R} \frac{q_e^2}{h_e}(0, R) - gh_e(0, R) \partial_r \zeta_e(0, R) \\ &= -\frac{R}{2(m + m_a(\delta_0))} \left( -\mathbf{c}\delta_0 + \mathbf{c}\zeta_e(0, R) + \left( \frac{\mathbf{b}}{h_e^2(0, R)} + \beta(\delta_0) \right) \delta_1^2 \right). \end{aligned}$$

For instance let us take the initial configuration of the fluid-structure interaction as the following: the solid displaced from its equilibrium position with no initial velocity, which means

$$\delta_0 \neq 0 \quad \text{and} \quad \delta_1 = 0,$$

and the fluid such that

$$h_e(0, R) = h_0, \quad q_e(0, R) = 0, \quad \partial_r \zeta_e(0, R) = -\frac{\mathbf{c}\delta_0 R}{2(m + m_a(\delta_0)) gh_0}.$$

Then, the initial conditions are compatible and we can apply Theorem 1.6.3.

*Proof.* We adapt here the argument that Métivier used in [96] for the existence and uniqueness of the solution to the fluid mixed problem and we couple it with the solid equation. Similar techniques are used in [73] by Iguchi and Lannes.

We introduce the following iterative scheme for the coupled system (1.6.1) - (1.6.2). For  $u_0 \in H_r^k((R, +\infty))$ , let us consider the linear mixed problem

$$\begin{cases} \partial_t u^n + A(u^{n-1})\partial_r u^n + B(u^{n-1}, r)u^n = 0, & \text{in } (R, +\infty) \\ \mathbf{e}_2 \cdot u^n|_{r=R} = -\frac{R}{2}\delta_G^{n-1}(t) \\ u^n(0) = u_0. \end{cases} \quad (1.6.5)$$

and the linear ODE

$$\begin{cases} (m + m_a(\delta_G^{n-1}))\ddot{\delta}_G^n = -\mathbf{c}\delta_G^n + \mathbf{c}\mathbf{e}_1 \cdot u^n|_{r=R} + (\mathbf{b}(u^{n-1}) + \beta(\delta_G^{n-1}))\dot{\delta}_G^{n-1}\dot{\delta}_G^n, \\ \delta_G^n(0) = \delta_0, \\ \dot{\delta}_G^n(0) = \delta_1, \end{cases} \quad (1.6.6)$$

Our goal is to find the solution of the coupled system as the limit of the previous iterative scheme. Hence we need to show the existence and the convergence of the sequence  $V^n = (u^n, \delta_G^n)$ . We consider the product space  $X^k(T) \times H^{k+1}((0, T))$  endowed with the norm

$$\|V^n\|_{\text{coup},k} := \|u^n\|_{X^k(T)} + \|\delta_G^n\|_{H^{k+1}((0,T))}$$

with the  $X^k(T)$  norm defined as in (1.4.11). We denote by  $E$  the subspace

$$E = \{V = (u, \delta_G) \in X^k(T) \times H^{k+1}((0, T)) \mid \|V\|_{\text{coup},k} \leq \tilde{R}\},$$

for some  $\tilde{R} > 0$  to determine later, such that

$$\forall t \in [0, T], \forall r \in (R, +\infty) : \quad h_e(t, r) \geq C_0, \quad \left(gh_e - \frac{q_e^2}{h_e^2}\right)(t, r) \geq c_0$$

and

$$\|\delta_G - \delta_0\|_{L^\infty((0,T))} \leq M_0$$

for some constants  $0 < C_0 \leq h_{\min}$ ,  $0 < c_0 \leq c_{\text{sub}}$  and  $M_0 = \frac{\delta_0 + h_{w,eq}}{2} > 0$ . We choose the first element of the sequence  $V^0 = (u^0, \delta_G^0)$  with  $u^0 \in H_r^{k+\frac{1}{2}}(\mathbb{R} \times (R, +\infty))$  such that

$$\partial_t^j u^0|_{t=0} = u_j, \quad 0 \leq j \leq k$$

with  $u_j$  as in (1.6.4). We can assume that  $u^0$  vanishes for  $|t| \geq 1$ , hence  $u^0 \in X^k(T)$  for all  $T$ . There exists a constant  $K_0 = K_0(u_0, \delta_0, \delta_1)$  depending only on the data such that

$$\|u^0\|_{X^k(T)} + \|\delta_G^0\|_{H^k((0,T))} \leq K_0. \quad (1.6.7)$$



We have that  $V^0 \in E$  choosing  $\tilde{R} \geq K_0$ . We suppose that  $V^{n-1} = (u^{n-1}, \delta_G^{n-1})$  is constructed in  $E \subseteq X^k(T) \times H^{k+1}((0, T))$  for some  $T > 0$  with

$$\partial_t^j u_{|t=0}^{n-1} = u_j, \quad j \leq k. \quad (1.6.8)$$

For  $n = 1$  this is true. By the definition (1.4.18) of  $\overline{\mathcal{F}}_j$  and by (1.6.8),

$$\partial_t^j (F(u^{n-1}, \partial)u^n)_{|t=0} = \overline{\mathcal{F}}_j(u_0, \dots, u_j)$$

with  $\bar{u} = u^{n-1}$ . Now we consider the linear problem (1.6.5). We compute  $u_j^n$  using (1.4.17). We can see that  $u_j^n = u_j$  with the  $u_j$  defined before. Then, the compatibility conditions  $\mathbf{e}_2 \cdot u_j|_{r=R} = -\frac{R}{2}\delta_{j+1}$  imply that the data  $\delta_G^{n-1}$  and  $u_0$  are compatible for the linear problem. From Theorem 1.4.10 the system (1.6.5) has a unique solution  $u^n \in X^k(T)$  and  $\partial_t^j u_{|t=0}^n = u_j^n = u_j$ . Moreover,

$$\|u^n(0)\|_{X^k} = \sum_{j \leq k} \|u_j\|_{H_r^k} \leq K_0.$$

Therefore we can continue the construction and this permits to define a sequence  $u^n \in X^k(T)$  solving the linear problem (1.6.5) and, from (2.1.4), satisfying

$$\|u^n\|_{X^k(T)} + \|u_{|r=R}^n\|_{H^k((0, T))} \leq C(K_0)e^{TC(\|u^{n-1}\|_{X^k(T)})} \left( K_0 + \|\delta_G^{n-1}\|_{H^k((0, T))} \right). \quad (1.6.9)$$

The existence and uniqueness of  $\delta_G^n \in W^{2, \infty}((0, T))$  is given by the Cauchy-Lipschitz-Picard theorem since the coefficients in (1.6.6) are bounded when  $V^{n-1} = (u^{n-1}, \delta_G^{n-1}) \in E$ . We want to show that  $V^n = (u^n, \delta_G^n) \in E$ . To do that, we now provide a control of product estimates in Sobolev spaces in time; of course one has the standard estimate

$$\|fg\|_{H^k((0, T))} \leq C(T)\|f\|_{H^k((0, T))}\|g\|_{H^k((0, T))}$$

but the constant  $C(T)$  blows up as  $T \rightarrow 0$  which raises some issues since we are led to choose  $T$  small enough in the proof. We therefore use the following more precise lemma where the time dependence of the constants is made explicit (see Proposition ?? in Appendix B for the proof):

**Lemma 1.6.5.** *Let  $k \geq 1$  be an integer. For  $f, g \in H^k((0, T))$  the following holds:*

$$\begin{aligned} \|fg\|_{H^k((0, T))} &\lesssim \sqrt{T}\|f\|_{H^k((0, T))}\|g\|_{H^k((0, T))} \\ &+ (|f(0)| + \left|\frac{d}{dt}f(0)\right| + \dots + \left|\frac{d^{k-1}}{dt^{k-1}}f(0)\right|)\|g\|_{H^k((0, T))} \\ &+ (|g(0)| + \left|\frac{d}{dt}g(0)\right| + \dots + \left|\frac{d^{k-1}}{dt^{k-1}}g(0)\right|)\|f\|_{H^k((0, T))} \end{aligned} \quad (1.6.10)$$

The previous lemma yields the following estimate for the solution  $\delta_G^n$  to (1.6.6):

**Proposition 1.6.6.** *The solution  $\delta_G^n$  to (1.6.6) satisfies*

$$\|\delta_G^n\|_{H^{k+1}((0,T))} \leq \alpha(T, \tilde{R}) + \beta(T, \tilde{R}) \|u_{|_{r=R}}^n\|_{H^k((0,T))} \quad (1.6.11)$$

with  $\alpha(T, \tilde{R}), \beta(T, \tilde{R}) \rightarrow 0$  as  $T \rightarrow 0$ .

*Proof.* It is immediate to derive

$$\|\delta_G^n\|_{H^{k+1}((0,T))} \leq C_1(T) + C_2(T) \|\ddot{\delta}_G^n\|_{H^{k-1}((0,T))} \quad (1.6.12)$$

with  $C_1(T), C_2(T) \rightarrow 0$  as  $T \rightarrow 0$ . Using the equation on  $\ddot{\delta}_G^n$  we can estimate  $\ddot{\delta}_G^n$  in the following way:

$$\begin{aligned} \|\ddot{\delta}_G^n\|_{H^{k-1}} &\lesssim \sqrt{T} \left\| \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})} \right\|_{H^{k-1}} \left( \|\delta_G^n\|_{H^{k-1}} + \|\mathbf{e}_1 \cdot u_{|_{r=R}}^n\|_{H^{k-1}} \right) \\ &\quad + \sqrt{T} \left\| \frac{\mathbf{b}(u^{n-1}) + \beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})} \dot{\delta}_G^{n-1} \right\|_{H^{k-1}} \|\dot{\delta}_G^n\|_{H^{k-1}} \\ &\quad + C_0 \left\| \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})} \right\|_{H^{k-1}} + D_0 \left( \|\delta_G^n\|_{H^{k-1}} + \|\mathbf{e}_1 \cdot u_{|_{r=R}}^n\|_{H^{k-1}} \right) \\ &\quad + C_1 \left\| \frac{\mathbf{b}(u^{n-1}) + \beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})} \dot{\delta}_G^{n-1} \right\|_{H^{k-1}} + D_1 \|\dot{\delta}_G^n\|_{H^{k-1}} \end{aligned}$$

with

$$C_0 = C_0 \left( |\delta_0^n|, \dots, |\delta_{k-2}^n|, |\mathbf{e}_1 \cdot u_{|_{r=R}}^n(0)|, \dots, \left| \frac{d^{k-2}}{dt^{k-2}} \mathbf{e}_1 \cdot u_{|_{r=R}}^n(0) \right| \right),$$

$$D_0 = D_0 \left( \left| \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})}(0) \right|, \dots, \left| \frac{d^{k-2}}{dt^{k-2}} \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})}(0) \right| \right),$$

$$C_1 = C_1 \left( |\delta_1^n|, \dots, |\delta_{k-1}^n| \right),$$

$$D_1 = D_1 \left( \left| \frac{\mathbf{b}(u^{n-1}) + \beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})} \dot{\delta}_G^{n-1}(0) \right|, \dots, \left| \frac{d^{k-2}}{dt^{k-2}} \frac{\mathbf{b}(u^{n-1}) + \beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})} \dot{\delta}_G^{n-1}(0) \right| \right).$$

By applying Lemma 1.6.5 several times to the products in the previous estimate and using the fact that for  $V^{n-1} = (u^{n-1}, \delta_G^{n-1}) \in E$  the denominators are bounded from below, we get

$$\|\ddot{\delta}_G^n\|_{H^{k-1}} \lesssim C(T, \|\delta_G^{n-1}\|_{H^{k+1}}, \|u^{n-1}\|_{H^k}) \|\delta_G^n\|_{H^{k+1}} + C(T, \|\delta_G^{n-1}\|_{H^{k+1}}) \|u_{|_{r=R}}^n\|_{H^k} \quad (1.6.13)$$

Here the constants may not tend to zero as  $T$  goes to zero but they are bounded. Then, using the control for  $\|V^{n-1}\|_{\text{coup},k}$ , for  $T$  small enough we can move the first term in the right hand side of (1.6.13) to the left of the inequality (1.6.12) and (1.6.11) follows.  $\square$

From (1.6.9) and (1.6.11), we get the following estimate for the coupled norm:

$$\begin{aligned} \|V^n\|_{\text{coup},k} &\leq \|u^n\|_{X^k(T)} + \|\delta_G^n\|_{H^{k+1}((0,T))} \\ &\leq \|u^n\|_{X^k(T)} + \alpha(T, \tilde{R}) + \beta(T, \tilde{R}) \|u^n\|_{H^k((0,T))} \\ &\leq C(T, \tilde{R}, K_0, \|u^{n-1}\|_{X^k(T)}) (K_0 + \|\delta_G^{n-1}\|_{H^{k+1}((0,T))}) + \alpha(T, \tilde{R}). \end{aligned}$$

Using again the control for  $\|V^{n-1}\|_{\text{coup},k}$ , we can find some  $\tilde{R} \geq K_0$  such that for  $T$  small enough

$$\|V^n\|_{\text{coup},k} \leq \tilde{R}.$$

For  $u_0 = (\zeta_{e,0}, q_{e,0})$  we have

$$h_e^n(t) = h_{e,0} + \int_0^t \partial_t h_e^n$$

with

$$\left| \int_0^t \partial_t h_e^n \right| \leq T \|\partial_t h_e^n\|_{L^\infty((0,T))L_r^\infty((R,+\infty))} \leq T \|u^n\|_{X^k(T)} \leq T \tilde{R}.$$

Moreover,

$$gh_e^n - \left(\frac{q_e^n}{h_e^n}\right)^2 = gh_{e,0} - \left(\frac{q_{e,0}}{h_{e,0}}\right)^2 + \int_0^t \partial_t \left(gh_e^n - \left(\frac{q_e^n}{h_e^n}\right)^2\right)$$

with

$$\begin{aligned} \left| \int_0^t \partial_t \left(gh_e^n - \left(\frac{q_e^n}{h_e^n}\right)^2\right) \right| &\leq T \left\| \partial_t \left(gh_e^n - \left(\frac{q_e^n}{h_e^n}\right)^2\right) \right\|_{L^\infty((0,T))L_r^\infty((R,+\infty))} \\ &\leq T \|A(u^n)\|_{X^k(T)} \leq TC(1 + \tilde{R}^k) \end{aligned}$$

where in the last inequality we have used (1.4.24). Finally,

$$\|\delta_G^n - \delta_0\|_{L^\infty((0,T))} \leq \sqrt{T} \|\delta_G^n\|_{H^{k+1}((0,T))} \leq \sqrt{T} \tilde{R}.$$

Hence, using the assumption on the initial data, the time existence  $T$  can be shorten to get

$$\forall t \in [0, T], \forall r \in (R, +\infty) \quad h_e^n(t, r) \geq, \quad \left(gh_e^n - \left(\frac{q_e^n}{h_e^n}\right)^2\right)(t, r) \geq c_0$$

and

$$\|\delta_G^n - \delta_0\|_{L^\infty((0,T))} \leq M_0.$$

Now we look for the convergence of the sequence  $V^n$  in a “smaller” norm. We consider the space

$$X^0(T) \times H^1((0, T)) = C^0([0, T], L_r^2((R, +\infty))) \times H^1((0, T)).$$

We have that  $u^n - u^{n-1}$  satisfies

$$\begin{cases} \partial_t (u^n - u^{n-1}) + A(u^{n-1}) \partial_r (u^n - u^{n-1}) + B(u^{n-1}, r) (u^n - u^{n-1}) \\ \quad = - (A(u^{n-1}) - A(u^{n-2})) \partial_r u^{n-1} - (B(u^{n-1}, r) - B(u^{n-2}, r)) u^{n-1}, \\ \mathbf{e}_2 \cdot (u^n - u^{n-1})|_{r=R} = -\frac{R}{2} (\dot{\delta}_G^{n-1}(t) - \dot{\delta}_G^{n-2}(t)), \\ (u^n - u^{n-1})(0) = 0. \end{cases}$$

Using the embedding  $H_r^k \hookrightarrow W^{1,\infty}$  for  $k \geq 2$  it yields

$$\begin{aligned} & \left\| \left( A(u^{n-1}) - A(u^{n-2}) \right) \partial_r u^{n-1} + \left( B(u^{n-1}, r) - B(u^{n-2}, r) \right) u^{n-1} \right\|_{X^0}^2 \\ & \leq K (\|u^{n-1}\|_{X^k(T)}) \|u^{n-1} - u^{n-2}\|_{X^0}^2. \end{aligned} \quad (1.6.14)$$

Then, by (??) it follows

$$\begin{aligned} & \|u^n - u^{n-1}\|_{X^0(T)}^2 + \|(u^n - u^{n-1})|_{r=R}\|_{L^2((0,T))}^2 \leq C(K_0) e^{TC(\|u^{n-1}\|_{X^2(T)})} \times \\ & \times \left( \|\dot{\delta}_G^{n-1} - \dot{\delta}_G^{n-2}\|_{L^2((0,t))}^2 + K(\|u^{n-1}\|_{X^k(T)}) \int_0^T \|(u^{n-1} - u^{n-2})(\tau)\|_{X^0}^2 d\tau \right). \end{aligned} \quad (1.6.15)$$

On the other hand, we have

$$\|\delta_G^n - \delta_G^{n-1}\|_{H^1((0,T))}^2 \leq C_2(T) \|\ddot{\delta}_G^n - \ddot{\delta}_G^{n-1}\|_{L^2((0,T))}^2, \quad (1.6.16)$$

with  $C_2(T) \rightarrow 0$  as  $T \rightarrow 0$ . Since in the ODE (1.6.2) the terms

$$\frac{\mathbf{c}}{m + m_a(\delta_G)}, \quad \frac{\mathbf{b}(u)}{m + m_a(\delta_G)}, \quad \frac{\beta(\delta_G)}{m + m_a(\delta_G)}$$

are Lipschitz continuous on  $(u, \delta_G) \in E$  from  $L^2$  to  $L^2$  and considering the equation for  $\delta_G^n$  and  $\delta_G^{n-1}$ , we obtain the following estimate for  $T$  small enough:

$$\begin{aligned} \|\delta_G^n - \delta_G^{n-1}\|_{H^1((0,T))} & \leq \tilde{\alpha}(T, \tilde{R}) \|(u^n - u^{n-1})|_{r=R}\|_{L^2((0,T))} \\ & \quad + \tilde{\beta}(T, \tilde{R}) \|\delta_G^{n-1} - \delta_G^{n-2}\|_{L^2((0,T))} \end{aligned} \quad (1.6.17)$$

for some constants  $\tilde{\alpha}(T, \tilde{R}), \tilde{\beta}(T, \tilde{R})$ . Therefore, using (1.6.15) and (1.6.17), we get

$$\begin{aligned} \|V^n - V^{n-1}\|_{coup,0} & = \|u^n - u^{n-1}\|_{X^0(T)} + \|\delta_G^n - \delta_G^{n-1}\|_{H^1((0,T))} \\ & \leq \|u^n - u^{n-1}\|_{X^0(T)} + \tilde{\alpha}(T, \tilde{R}) \|(u^n - u^{n-1})|_{r=R}\|_{L^2((0,T))} \\ & \quad + \tilde{\beta}(T, \tilde{R}) \|\delta_G^{n-1} - \delta_G^{n-2}\|_{L^2((0,T))} \\ & \leq C(T, K_0, \tilde{R}) \left( \|\delta_G^{n-1} - \delta_G^{n-2}\|_{H^1((0,T))} + \int_0^T \|(u^{n-1} - u^{n-2})(t)\|_{X^0} dt \right) \\ & \leq K(T, K_0, \tilde{R}) \|V^{n-1} - V^{n-2}\|_{coup,0}. \end{aligned}$$

where we have used  $\|u^{n-1}\|_{X^2(T)} \leq \|u^{n-1}\|_{X^k(T)}$  and the inductive hypothesis (??). Then, we can choose  $T$  small enough such that  $K(T, K_0, \tilde{R}) < 1$  and we obtain that  $V^n$  is a convergent sequence in  $X^0(T) \times H^1((0, T))$  with limit  $V = (u, \delta_G)$ . By standard arguments (see [113]) we have that  $V \in E \subseteq X^k(T) \times H^{k+1}((0, T))$  is the unique solution of the coupled problem (1.6.1) - (1.6.2). □

## CHAPTER 2

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# Return to equilibrium problem in axisymmetric shallow water

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*This work is based on [15].*

## 2.1 Extension-trace operator for the coupling with the exterior domain

In the previous chapter, we have shown that floating structures problem is described by the coupled PDE-ODE system

$$\begin{cases} \partial_t h_e + \partial_r q_e + \frac{q_e}{r} = 0 \\ \partial_t q_e + \partial_r \left( \frac{q_e^2}{h_e} \right) + \frac{q_e^2}{r h_e} + g h_e \partial_r h_e = 0 \\ q_e|_{r=R} = -\frac{R}{2} \dot{\delta}_G, \end{cases} \quad (2.1.1)$$

$$(m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathbf{c}\delta_G(t) + \mathbf{c}\zeta_e(t, R) + (\mathbf{b}(h_e) + \beta(\delta_G))\dot{\delta}_G^2(t) \quad (2.1.2)$$

Moreover, denoting by  $u$  the couple  $(\zeta_e, q_e)^T$ , the exterior quasilinear hyperbolic initial boundary value problem (2.1.1) takes the form

$$\begin{cases} \partial_t u + A(u)\partial_r u + B(u, r)u = 0 \\ q_e|_{r=R} = -\frac{R}{2}\dot{\delta}_G \\ u(0) = u_0 \end{cases} \quad (2.1.3)$$

with

$$A(u) = \begin{pmatrix} 0 & 1 \\ gh_e - \frac{q_e^2}{h_e^2} & \frac{2q_e}{h_e} \end{pmatrix}, \quad B(u, r) = \begin{pmatrix} 0 & \frac{1}{r} \\ 0 & \frac{q_e}{rh_e} \end{pmatrix}$$

and

$$u_0 = (\zeta_{e,0}, q_{e,0})^T.$$

Let us recall the functional space considered in the well-posedness theory presented in Chapter 1

$$X^k(T) := \bigcap_{j=0}^k C^j([0, T], H_r^{k-j}((R, +\infty)))$$

endowed with the norm

$$\|u\|_{X^k(T)} = \sup_{[0, T]} \|u(t)\|_{X^k}, \quad \|u(t)\|_{X^k} = \sum_{j=0}^k \|\partial_t^j u(t)\|_{H_r^{k-j}((R, +\infty))},$$

where  $H_r^k = H^k(rdr)$  is the weighted Sobolev space.

In this section we want to show that, in the ODE for the solid part of the coupled system (2.1.1) - (2.1.2), we can write the coupling term  $\zeta_e(t, R)$  (also  $h_e^2(t, R)$ ), the trace of the free surface elevation in the exterior domain at the boundary  $r = R$ , as an extension-trace operator applied to the trace of the horizontal discharge in the interior domain at the boundary  $r = R$ , that is  $-\frac{R}{2}\dot{\delta}_G$ .

In Theorem 1.6.3 of Chapter 1 we have shown that, for  $k \geq 2$ , there exists  $T > 0$  and a unique solution  $u = (\zeta_e, q_e)^T \in X^k(T)$  to (2.1.3), provided the initial data  $u_0 \in H_r^k((R, +\infty))$ , the boundary condition  $q_e|_{r=R} \in H^k((0, T))$  and compatibility conditions are satisfied up to order  $k - 1$ . Moreover,  $u$  satisfies the following energy estimate:

$$\|u\|_{X^k(T)}^2 + \|u|_{r=R}\|_{H^k((0, T))}^2 \leq C \left( T, \|u_0\|_{H_r^k((R, \infty))}^2, \|q_e|_{r=R}\|_{H^k((0, T))}^2 \right) \quad (2.1.4)$$

for all  $t \in (0, T)$ . Then, we can define an operator  $\mathcal{B}$  such that

$$\begin{aligned} \mathcal{B} : \quad H^k((0, T)) \times H_r^k((R, \infty)) &\rightarrow H^k((0, T)) \\ (\dot{\delta}_G \quad , \quad u_0) &\mapsto \mathcal{B} [\dot{\delta}_G, u_0] = \zeta_e|_{r=R}. \end{aligned} \quad (2.1.5)$$

We call it an extension-trace operator since it takes the trace of  $q_e$ , that is  $-\frac{R}{2}\dot{\delta}_G$ , the initial data  $u_0$  and it extends to the couple  $(\zeta_e, q_e)$  by solving the initial boundary value problem (2.1.3) and then it takes the trace of  $h_e$ . One can easily note that  $\mathcal{B}$  is nonlinear. Then, using the fact that  $h_e = \zeta_e + h_0$  and assuming  $u_0$  to be given, we can write the equation (2.1.2) for the solid motion as a second order delay differential equation only in terms of  $\delta_G$ , namely

$$\begin{aligned} &(m + m_a(\delta_G))\ddot{\delta}_G(t) \\ &= -\mathbf{c}\delta_G(t) + \mathbf{c}\mathcal{B} [\dot{\delta}_G, u_0] (t) + \left( \frac{b}{(\mathcal{B} [\dot{\delta}_G, u_0] (t) + h_0)^2} + \beta(\delta_G) \right) \delta_G^2(t). \end{aligned} \quad (2.1.6)$$

It is a delay differential equation since we need to know  $\dot{\delta}_G$  for all  $t' \in [0, t]$  in order to know the value of  $\mathcal{B} [\dot{\delta}_G, u_0]$  at time  $t$ . This equation can be solved by a standard fixed point argument. Let us first recall the compatibility conditions on the initial data:

**Definition 2.1.1.** *The data  $u_0 \in H_r^k((R, +\infty))$ ,  $\delta_0 \in \mathbb{R}$  and  $\delta_1 \in \mathbb{R}$  of the floating structure coupled system (2.1.3) - (2.1.6) satisfy the compatibility conditions up to order  $k - 1$  if, for  $1 \leq j \leq k - 1$ , the following holds:*

- $\mathbf{e}_2 \cdot u_0|_{r=R} = -\frac{R}{2}\delta_1$ ,
- $\mathbf{e}_2 \cdot \left( \left( \partial_t^{j-1} (-A(u)\partial_r u - B(u, r)u) \right)|_{t=0} \right)|_{r=R} = \left( \partial_t^{j-1} \left[ -\frac{R}{2(m + m_a(\delta_G))} \times \left( -\mathbf{c}\delta_G + \mathbf{c}\mathcal{B} [\dot{\delta}_G, u_0] + \left( \frac{b}{(\mathcal{B} [\dot{\delta}_G, u_0] + h_0)^2} + \beta(\delta_G) \right) \delta_G^2 \right) \right] \right)|_{t=0}$ .

where the brackets “ $\cdot$ ” mean that the derivatives are formally written.

Then, we can state the following existence result:

**Theorem 2.1.2.** *For  $k \geq 2$ , let  $u_0 = (\zeta_{e,0}, q_{e,0})$ ,  $\delta_0$  and  $\delta_1$  satisfy the compatibility conditions in Definition 2.1.1 up to order  $k - 1$ . Assume that there exist some constants  $h_{\min}, c_{\text{sub}} > 0$  such that*

$$\forall r \in (R, +\infty) : \quad h_{e,0}(r) \geq h_{\min}, \quad \left( gh_{e,0} - \frac{q_{e,0}^2}{h_{e,0}^2} \right) (r) \geq c_{\text{sub}},$$



with  $h_{e,0} = h_0 + \zeta_{e,0}$ , and that

$$\delta_0 > - \inf_{(0,R)} h_{w,eq}.$$

Then, there exists  $T > 0$  such that the Cauchy problem for (2.1.6) with initial data

$$\delta_G(0) = \delta_0, \quad \dot{\delta}_G(0) = \delta_1,$$

admits a unique solution  $\delta_G \in H^{k+1}((0, T))$ .

*Proof.* By defining  $U(t) = (\delta_G(t), \dot{\delta}_G(t))^T$  we can reduce (2.1.6) to the first order delay differential equation

$$\begin{cases} \frac{d}{dt}U(t) = \Pi(U)U(t) + M(U)(t) + G(U)(t) \\ U(0) = U_0 \end{cases} \quad (2.1.7)$$

with

$$\begin{aligned} \Pi(U) &= \begin{pmatrix} 0 & 1 \\ -\frac{\mathbf{c}}{m + m_a(\delta_G)} & 0 \end{pmatrix}, \\ M(U) &= \begin{pmatrix} 0 \\ \frac{\mathbf{c}\mathcal{B}[\dot{\delta}_G, u_0]}{m + m_a(\delta_G)} \end{pmatrix}, \\ G(U) &= \begin{pmatrix} 0 \\ \frac{b}{(\mathcal{B}[\dot{\delta}_G, u_0] + h_0)^2 + \beta(\delta_G)} \delta_G^2 \end{pmatrix} \end{aligned}$$

and

$$U_0 = (\delta_0, \delta_1)^T.$$

We write the equation in (2.1.7) under the integral form

$$U(t) = U_0 + \int_0^t (\Pi(U)U(\tau) + M(U)(\tau) + G(U)(\tau)) d\tau := L(U)(t). \quad (2.1.8)$$

We look for the solution as the limit of the sequence  $U^n$  defined by

$$U^{n+1}(t) = L(U^n)(t).$$

Let us denote by  $H_T^k$  the space  $H^k((0, T))$ . In the same way as in Theorem 1.6.3 of Chapter 1, we consider  $E$  the subspace of  $H_T^k$  such that for  $U \in E$  the heights  $h_w + \delta_G$  and  $\mathcal{B}[\dot{\delta}_G, u_0] + h_0$  are bounded from below and the coefficients of the equations are bound (we refer to the proof of Theorem 1.6.3 for the details). We suppose that  $U^n \in E$  with

$$\|U^n - U^0\|_{H_T^k} \leq K \quad (2.1.9)$$

for some  $K > 0$  with the first iterative step  $U^0 = U_0 \in E$ . For  $n = 0$  the previous inequality is trivial. We want to show that the inductive assumption is true also for  $n + 1$ . Then, we consider  $\|U^{n+1} - U^0\|_{H_T^k} = \|L(U^n) - U^0\|_{H_T^k}$ . Using the inequality

$$\left\| \int_0^t F \right\|_{H_T^k} \leq C_1(T) + C_2(T) \|F\|_{H_T^{k-1}}$$

with  $C_1(T), C_2(T) \rightarrow 0$  as  $T \rightarrow 0$  and Lemma 1.6.5 of Chapter 1, we have

$$\begin{aligned} \left\| \int_0^t \Pi(U^n)U^n(\tau) d\tau \right\|_{H_T^k} &\leq C_1(T, \delta_0, \dots, \delta_{k-2}, \|\delta_G^n\|_{H_T^k}, \|\dot{\delta}_G^n\|_{H_T^k}) \\ &\leq C_1(T, K, \delta_0, \dots, \delta_{k-2}) \end{aligned}$$

with  $\delta_{k-2} := \frac{d^{k-2}}{dt} \delta_G^n(0)$  where in the last inequality we have used (2.1.9). The constant  $C_1(T, K, \delta_0, \dots, \delta_{k-2}) \rightarrow 0$  as  $T \rightarrow 0$ . In the same way we have

$$\begin{aligned} \left\| \int_0^t M(U^n)(\tau) d\tau \right\|_{H_T^k} &\leq C_2 \left( T, \delta_0, \dots, \delta_{k-2}, \mathcal{B}_0, \dots, \mathcal{B}_{k-2}, \|\delta_G\|_{H_T^k}, \|\mathcal{B}[\dot{\delta}_G^n, u_0]\|_{H_T^k} \right) \\ &\leq C_2 \left( T, \delta_0, \dots, \delta_{k-2}, \mathcal{B}_0, \dots, \mathcal{B}_{k-2}, \|\delta_G\|_{H_T^k}, \|u_0\|_{H_T^k}, \|\dot{\delta}_G^n\|_{H_T^k} \right) \\ &\leq C_2 \left( T, K, \delta_0, \dots, \delta_{k-2}, \mathcal{B}_0, \dots, \mathcal{B}_{k-2}, \|u_0\|_{H_T^k} \right) \end{aligned}$$

with  $\mathcal{B}_{k-2} := \frac{d^{k-2}}{dt} \mathcal{B}[\dot{\delta}_G^n, u_0](0)$ . We have used the estimate (2.1.4) in the last inequality for  $\|u_0\|_{H_T^k} := \|u_0\|_{H_T^k((R, +\infty))}$ . The constant

$$C_2(T, K, \delta_0, \dots, \mathcal{B}_{k-2}, \|u_0\|_{H_T^k}) \rightarrow 0$$

as  $T \rightarrow 0$ . Moreover, we have

$$\begin{aligned} \left\| \int_0^t G(U^n)(\tau) d\tau \right\|_{H_T^k} &\leq C_3 \left( T, \delta_0, \dots, \mathcal{B}_{k-2}, \|\delta_G\|_{H_T^k}, \|\mathcal{B}[\dot{\delta}_G^n, u_0]\|_{H_T^k}, \|(\dot{\delta}_G^n)^2\|_{H_T^{k-1}} \right) \\ &\leq C_3 \left( T, \delta_0, \dots, \mathcal{B}_{k-2}, \|\delta_G\|_{H_T^k}, \|u_0\|_{H_T^k}, \|\dot{\delta}_G^n\|_{H_T^k} \right) \\ &\leq C_3 \left( T, K, \delta_0, \dots, \mathcal{B}_{k-2}, \|u_0\|_{H_T^k} \right) \end{aligned}$$

where we have used  $\|(\dot{\delta}_G^n)^2\|_{H_T^{k-1}} \leq C \left( T, \|\delta_G^n\|_{H_T^k}^2 \right)$  (by Lemma 1.6.5). The constant

$$C_3 \left( T, K, \delta_0, \dots, \mathcal{B}_{k-2}, \|u_0\|_{H_T^k} \right) \rightarrow 0$$

as  $T \rightarrow 0$ . In all three estimates we have used the fact that, since  $U^n \in E$ , the coefficients of the ODE are bounded. Then, choosing  $T > 0$  small enough such that

$$\begin{aligned} C_1 \left( T, K, \delta_0, \dots, \delta_{k-2} \right) + C_2 \left( T, K, \delta_0, \dots, \mathcal{B}_{k-2}, \|u_0\|_{H_T^k} \right) \\ + C_3 \left( T, K, \delta_0, \dots, \mathcal{B}_{k-2}, \|u_0\|_{H_T^k} \right) \leq K \end{aligned} \quad (2.1.10)$$

we have  $\|U^{n+1} - U^0\|_{H_T^k} \leq K$  for some  $K$ . In an analogous way as in Theorem 1.6.3, it can be shown that  $U^{n+1} \in E$ . We omit the details. We have also the convergence of  $U^n$  in  $L_T^2 := L^2((0, T))$ :

**Lemma 2.1.3.** *There exists a constant  $\Theta = \Theta(T, K) < 1$  such that*

$$\|U^{n+1} - U^n\|_{L_T^2} \leq \Theta \|U^n - U^{n-1}\|_{L_T^2}.$$

*Proof.* We show only the control on the term with the extension-trace operator  $\mathcal{B}$ . For the other terms in (2.1.8) the control is classical since the coefficients are locally Lipschitz. We have

$$\begin{aligned} \left\| \int_0^t \mathcal{B} [\dot{\delta}_G^n, u_0] - \mathcal{B} [\dot{\delta}_G^{n-1}, u_0] \right\|_{L_T^2}^2 &\leq \frac{T^2}{2} \left\| \mathcal{B} [\dot{\delta}_G^n, u_0] - \mathcal{B} [\dot{\delta}_G^{n-1}, u_0] \right\|_{L_T^2}^2 \\ &\leq \frac{T^2}{2} C \left( T, K, \|\dot{\delta}_G^n - \dot{\delta}_G^{n-1}\|_{L_T^2}^2 \right) \end{aligned}$$

where  $C$  depends exponentially on  $T$ . The second inequality comes from the  $L^2$  a priori estimate of Proposition (1.4.4) in Chapter 1 for the hyperbolic system

$$\begin{cases} \partial_t(u^{n+1} - u^n) + A(u^n)\partial_r(u^{n+1} - u^n) + B(u^n, r)(u^{n+1} - u^n) \\ \quad = -(A(u^n) - A(u^{n-1}))\partial_r u^n - (B(u^n, r) - B(u^{n-1}, r))u^n, \\ q_e^{n+1} - q_e^n|_{r=R} = -\frac{R}{2}(\dot{\delta}_G^n - \dot{\delta}_G^{n-1}), \\ (u^{n+1} - u^n)(0) = 0. \end{cases} \quad (2.1.11)$$

We control the source term in (2.1.11) using the fact that  $k \geq 2$  and

$$\|\partial_r u^n\|_{L^\infty((R, +\infty))} \leq \|u^n\|_{X^k(T)} \leq C \left( T, \|u_0\|_{H_T^k((R, +\infty))}, K \right),$$

where the second inequality comes from (2.1.4) and (2.1.9).  $\square$

By an interpolation argument we have the convergence also in  $H^k((0, T))$ . So we get the existence and uniqueness of the solution  $U$  to the Cauchy problem (2.1.7) in  $H^k((0, T))$ . Hence the Cauchy problem for (2.1.6) admits a unique solution  $\delta_G \in H^{k+1}((0, T))$ .  $\square$

### 2.1.1 The return to equilibrium configuration

We want to focus now on a particular configuration of the floating structure problem, the return to equilibrium problem. It consists in dropping the solid, with no initial velocity, into a fluid initially at rest from a non-equilibrium position. By the definition of this particular configuration, we have specific initial conditions for the coupled problem (2.1.1) - (2.1.2).

The initial conditions for the solid equation are

$$\delta_G(0) = \delta_0 \neq 0, \quad \dot{\delta}_G(0) = \delta_1 = 0,$$

and for the fluid equations are

$$h_e(0, r) = h_0, \quad q_e(0, r) = 0 \quad \text{in } (R, +\infty).$$

In order to apply the theory of the initial boundary value problem we need these specific initial data to satisfy the compatibility conditions defined in Chapter 1. The compatibility conditions of order 0 and 1 are respectively:

- $q_e(0, R) = -\frac{R}{2}\delta_1,$
- $-\partial_r \left( \frac{q_e^2}{h_e} \right) (0, R) - \frac{1}{R} \frac{q_e^2}{h_e} (0, R) - gh_e(0, R) \partial_r \zeta_e(0, R)$   
 $= -\frac{R}{2(m + m_a(\delta_0))} \left( -\mathbf{c}\delta_0 + \mathbf{c}\zeta_e(0, R) + \left( \frac{b}{h_e^2(0, R)} + \beta(\delta_0) \right) \delta_1^2 \right).$

Due to the nature of the return to equilibrium configuration, we have

$$\partial_r \zeta_e(0, R) = 0, \quad \zeta_e(0, R) = 0, \quad q_e(0, R) = 0. \quad (2.1.12)$$

Therefore the compatibility condition of order 0 is satisfied but not the one of order 1. Then, Theorem 1.6.3 in Chapter 1 can not be applied since one hypothesis required is that the initial and boundary data must satisfy the compatibility conditions at least up to order 1. When the compatibility conditions at order 1 are not satisfied, sonic waves propagate (we refer to Métivier [95] for the existence of such waves).

**Remark 2.1.4.** *One can choose a different value for  $\delta_1$  in order to satisfy the compatibility conditions and be able to apply the results of Theorem 1.6.3 in Chapter 1.*

## 2.2 Linear-nonlinear model for floating structures

The impossibility to apply the mixed problem theory to the particular configuration of the return to equilibrium brings us to consider a linearization of the equations (1.3.1) in the exterior domain, which describes the case of small amplitude waves. We generalize however the works by Cummins and other authors in the literature by keeping the nonlinear effects in the interior domain. We only assume that the solid does not touch the bottom of the fluid domain. In this section we introduce the linear-nonlinear model for the floating structure problem, we prove the conservation of the total energy for this model and then we show that with this linear approximation we can write the extension-trace operator  $\mathcal{B}[\dot{\delta}_G, u_0]$  (simply written  $\mathcal{B}[\dot{\delta}_G]$  from now on) as a linear convolution operator. Then, the delay differential equation (2.1.6) for the solid motion becomes a nonlinear second order integro-differential equation.

### 2.2.1 An energy conserving linear-nonlinear model

We consider the following linear-nonlinear model for the floating structure problem:

- in the exterior domain  $(R, +\infty)$

$$\begin{cases} \partial_t \zeta_e + \partial_r q_e + \frac{q_e}{r} = 0, \\ \partial_t q_e + gh_0 \partial_r \zeta_e = 0, \end{cases} \quad (2.2.1)$$

- in the interior domain  $(0, R)$

$$\begin{cases} \partial_t h_i + \partial_r q_i + \frac{q_i}{r} = 0, \\ \partial_t q_i + \partial_r \left( \frac{q_i^2}{h_i} \right) + \frac{q_i^2}{rh_i} + gh_i \partial_r h_i = -\frac{h_i}{\rho} \partial_r \underline{P}_i, \end{cases} \quad (2.2.2)$$

and the boundary conditions

$$q_e|_{r=R} = q_i|_{r=R}, \quad (2.2.3)$$

$$\underline{P}_i|_{r=R} = P_{atm} + \rho g (\zeta_e - \zeta_i)|_{r=R} + P_{cor}, \quad (2.2.4)$$

with  $P_{cor} = -\frac{\rho}{2} \frac{q_i^2}{h_i^2}|_{r=R}$ . As in the full nonlinear case,  $q_i$  can be written in terms of the solid vertical displacement  $\delta_G$  (solving the first equation in (2.2.2)) and the transition condition (2.2.3) becomes

$$q_e|_{r=R} = -\frac{R}{2} \dot{\delta}_G. \quad (2.2.5)$$

Furthermore we have the conservation of the energy for the new linear-nonlinear model (see Chapter 1 for the conservation of the energy in the full nonlinear model):

**Proposition 2.2.1.** *Let us define the shallow water fluid energy for the linear-nonlinear shallow water equations (2.2.1) - (2.2.2)*

$$E_{SW} = 2\pi \frac{\rho}{2} g \int_0^{+\infty} \zeta^2 r dr + 2\pi \frac{\rho}{2} \int_0^R \frac{q_i^2}{h_i} r dr + 2\pi \frac{\rho}{2} \int_R^{+\infty} \frac{q_e^2}{h_0} r dr \quad (2.2.6)$$

and the solid energy (only with vertical motion)

$$E_{sol} = \frac{1}{2} m w_G^2 + m g z_G.$$

Then, the total fluid-structure energy  $E_{tot} = E_{SW} + E_{sol}$  is conserved, i.e.

$$\frac{d}{dt} E_{tot} = 0.$$

*Proof.* By multiplying the first equation of (2.2.1) by  $\rho g \zeta_e r$  and the second equation by  $\frac{q_e r}{h_0}$  we have local conservation of the energy

$$\partial_t e_{ext} + \partial_r F_{ext} = 0, \quad (2.2.7)$$

where  $e_{ext}$  is the local fluid energy in the exterior domain

$$e_{ext} = \frac{\rho}{2} g \zeta_e^2 r + \frac{\rho}{2} \frac{q_e^2}{h_0} r$$

and  $F_{ext}$  is the flux in the exterior domain

$$F_{ext} = \rho g \zeta_e q_e r.$$

We consider the equations (2.2.2) in the interior domain:

$$\begin{cases} \partial_t \zeta_i + \partial_r q_i + \frac{q_i}{r} = 0, \\ \partial_t q_i + \partial_r \left( \frac{q_i^2}{h_i} \right) + \frac{q_i^2}{r h_i} + g h_i \partial_r \zeta_i = -\frac{h_i}{\rho} \partial_r \underline{P}_i. \end{cases} \quad (2.2.8)$$

By multiplying the first equation of (2.2.8) by  $\rho g \zeta_i r$  and the second equation by  $\frac{q_i r}{h_i}$  we obtain

$$\partial_t e_{int} + \partial_r F_{int} = -r q_i \partial_r \underline{P}_i, \quad (2.2.9)$$

where  $e_{int}$  is the local fluid energy in the interior domain

$$e_{int} = \frac{\rho}{2} g \zeta_i^2 r + \frac{\rho}{2} \frac{q_i^2}{h_i} r$$

and  $F_{int}$  is the flux in the interior domain

$$F_{int} = \frac{\rho q_i^3}{2 h_i^2} r + \rho g \zeta_i q_i r.$$

We integrate (2.2.7) on  $[R, +\infty)$  and (2.2.9) on  $[0, R]$  and by multiplying by  $2\pi$  we obtain

$$\frac{d}{dt} E_{SW} - 2\pi \rho R g \llbracket \zeta q \rrbracket + 2\pi \rho R \frac{q_i^3}{2 h_i^2} \Big|_{r=R} = -2\pi \int_0^R r q_i \partial_r (\underline{P}_i - P_{atm}) dr, \quad (2.2.10)$$

where  $\llbracket f \rrbracket$  is the jump of a function  $f$  at the boundary  $r = R$  defined by

$$\llbracket f \rrbracket := f_{e|_{r=R}} - f_{i|_{r=R}}.$$

By integration by parts we get

$$\begin{aligned} \frac{d}{dt} E_{SW} &= 2\pi \rho R g \llbracket \zeta q \rrbracket - 2\pi \rho R \frac{q_i^3}{2 h_i^2} \Big|_{r=R} - 2\pi R (\underline{P}_i - P_{atm}) \Big|_{r=R} q_i \Big|_{r=R} \\ &\quad + 2\pi \int_0^R (\underline{P}_i - P_{atm}) \partial_r (r q_i) dr. \end{aligned} \quad (2.2.11)$$

On the other hand, from the definition of  $E_{sol}$ , we have

$$\begin{aligned} \frac{d}{dt} E_{sol} &= m w_G \dot{w}_G + m g w_G = w_G (m \dot{w}_G + m g) \\ &= w_G 2\pi \int_0^R (\underline{P}_i - P_{atm}) r dr \\ &= 2\pi \int_0^R (\underline{P}_i - P_{atm}) \partial_t \zeta_w r dr \end{aligned}$$

where we used Newton's law for the conservation of the linear momentum and, since the structure moves only vertically,

$$\partial_t \zeta_w = w_G$$

coming from standard solid mechanics. From the contact constraint (1.1.6) and the first equation in (2.2.2) we get

$$\frac{d}{dt} E_{sol} = -2\pi \int_0^R (\underline{P}_i - P_{atm}) \partial_r (r q_i) dr. \quad (2.2.12)$$

Therefore

$$\frac{d}{dt}E_{SW} = -\frac{d}{dt}E_{sol} + 2\pi\rho Rg \llbracket \zeta q \rrbracket - 2\pi\rho R \frac{q_i^3}{2h_i^2} \Big|_{r=R} - 2\pi R (P_i - P_{atm}) \Big|_{r=R} q_i \Big|_{r=R}.$$

Using the expression of the interior pressure  $P_i$  on the boundary  $r = R$  in (2.2.4) and the transition condition (2.2.3) we get the conservation of the total energy.  $\square$

### 2.2.2 Linear equations in the exterior domain

In this subsection we focus on the linear shallow water equations in the exterior domain

$$\begin{cases} \partial_t \zeta_e + \partial_r q_e + \frac{q_e}{r} = 0, \\ \partial_t q_e + v_0^2 \partial_r \zeta_e = 0, \end{cases} \quad (2.2.13)$$

with  $v_0 = \sqrt{gh_0}$ , coupled with the transition condition

$$q_e \Big|_{r=R} = -\frac{R}{2} \dot{\delta}_G(t). \quad (2.2.14)$$

Taking the derivative of the first equation in (2.2.13) with respect to time and replacing the value of  $\partial_t q_e$  with the expression in the second equation we find the linear wave equation

$$\partial_{tt} \zeta_e - v_0 \Delta_r \zeta_e = 0$$

with  $\Delta_r := \partial_{rr} + \frac{1}{r} \partial_r$ .

We consider only positive time  $t$  (we can treat  $\zeta_e$  as a causal function, i.e.  $\zeta_e = 0$  for  $t < 0$ ). In the same way as John did in [75], we apply the Laplace transform

$$\mathcal{L}(\zeta_e)(r, s) = \int_0^{+\infty} \zeta_e(t, r) e^{-st} dt \quad \text{Re}(s) > 0$$

to the wave equation and we get the following Helmholtz equation with complex coefficients:

$$s^2 \mathcal{L}(\zeta_e) - v_0 \Delta_r \mathcal{L}(\zeta_e) = 0. \quad (2.2.15)$$

We have  $\mathcal{L}(\partial_{tt} \zeta_e) = s^2 \mathcal{L}(\zeta_e) + \partial_t \zeta_e(0) + s \zeta_e(0)$  but in this configuration we have in addition  $\partial_t \zeta_e(0) = 0$  and  $\zeta_e(0) = 0$  from (2.1.12). The general solution of (2.2.15) is

$$\mathcal{L}(\zeta_e)(r, s) = a_1(s) H_0^{(1)}\left(\frac{isr}{v_0}\right) + a_2(s) H_0^{(2)}\left(\frac{isr}{v_0}\right),$$

where  $H_0^{(1)}$  and  $H_0^{(2)}$  are the Hankel functions of first order and second order respectively with index 0.



**Remark 2.2.2.** Let us consider the Bessel functions of the first kind and of the second kind, respectively  $J_n$  and  $Y_n$ , solutions to

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - n^2)w = 0, \quad z \in \mathbb{C}.$$

The Hankel functions of first order with index  $n$  are defined by

$$H_n^{(1)} = J_n + iY_n,$$

and the Hankel functions of second order with index  $n$  as

$$H_n^{(2)} = J_n - iY_n.$$

From the asymptotic behavior of the Hankel functions (see Appendix A) we know that  $H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{iz}$  and  $H_0^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-iz}$  for large  $|z|$  and  $0 < \arg z < \pi$ . Therefore for large  $|s|r$  and  $-\frac{\pi}{2} < \arg s < \frac{\pi}{2}$

$$H_0^{(1)}\left(\frac{isr}{v_0}\right) \sim \sqrt{\frac{2v_0}{\pi isr}} e^{\frac{-sr}{v_0}},$$

$$H_0^{(2)}\left(\frac{isr}{v_0}\right) \sim \sqrt{\frac{2v_0}{\pi isr}} e^{\frac{sr}{v_0}}.$$

Thus for  $\operatorname{Re}(s) > 0$  and large  $r$

$$a_1(s) H_0^{(1)}\left(\frac{isr}{v_0}\right) e^{st} \sim a_1(s) \sqrt{\frac{2v_0}{\pi isr}} e^{s\left(t - \frac{r}{v_0}\right)},$$

$$a_2(s) H_0^{(2)}\left(\frac{isr}{v_0}\right) e^{st} \sim a_2(s) \sqrt{\frac{2v_0}{\pi isr}} e^{s\left(t + \frac{r}{v_0}\right)}.$$

These terms represent respectively an outgoing progressive wave and an incoming progressive wave. Since in this problem we consider only outgoing waves, we impose  $a_2(s) = 0$ .

Applying the Laplace transform to the second equation of (2.2.13), we get the following boundary condition for the exterior Helmholtz problem:

$$\partial_r \mathcal{L}(\zeta_e)|_{r=R} = -\frac{s \mathcal{L}(q_e)}{v_0^2}|_{r=R} = \frac{sR}{2v_0^2} \mathcal{L}(\dot{\delta}_G),$$

using the transition condition (2.2.14). Therefore we finally have

$$\mathcal{L}(\zeta_e)(s, R) = \frac{iRH_0^{(1)}\left(\frac{isR}{v_0}\right)}{2v_0H_1^{(1)}\left(\frac{isR}{v_0}\right)} \mathcal{L}(\dot{\delta}_G)(s), \quad (2.2.16)$$

using the relation  $(H_0^{(1)})' = -H_1^{(1)}$  between the derivative of  $H_0^{(1)}$  and the Hankel function of first order with index 1. From Appendix A we have  $\frac{H_0^{(1)}(s)}{H_1^{(1)}(s)} \rightarrow i$  for large  $|s|$ . Adding and subtracting this limit we have

$$\mathcal{L}(\zeta_e)(s, R) = f(s)\mathcal{L}(\dot{\delta}_G)(s) - \frac{R}{2v_0}\mathcal{L}(\dot{\delta}_G)(s) \quad (2.2.17)$$

with

$$f(s) = \frac{iRH_0^{(1)}\left(\frac{isR}{v_0}\right)}{2v_0H_1^{(1)}\left(\frac{isR}{v_0}\right)} + \frac{R}{2v_0}$$

with  $f(s) \rightarrow 0$  as  $|s| \rightarrow +\infty$ . It turns out that we can write  $f$  as a Laplace transform of some function:

**Lemma 2.2.3.** *There exists a unique function  $F \in L^2(\mathbb{R}_+) \cap C([0, +\infty))$  such that  $f(s) = \mathcal{L}(F)(s)$ , with either*

$$F(t) = \lim_{v \rightarrow +\infty} \frac{1}{2\pi} \int_{-v}^v f(c + i\omega) e^{(c+i\omega)t} d\omega,$$

independent of  $c > 0$ , in the sense of  $L^2$  Fourier transforms and

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ f(c + i\omega) - \frac{\lambda}{c + i\omega} \right] e^{(c+i\omega)t} d\omega + \lambda,$$

with  $\lambda = \frac{1}{4}$ , in the sense of Lebesgue integral.

*Proof.* We know that both  $H_0^{(1)}(is)$ ,  $H_1^{(1)}(is)$  are holomorphic functions on  $\mathbb{C}_+$ , and  $H_1^{(1)}(is) \neq 0$  in  $\mathbb{C}_+$  (see [1],[41]), then  $f(s)$  is holomorphic on  $\mathbb{C}_+$ . Moreover,  $f$  is bounded in  $\mathbb{C}_+$  since  $f \rightarrow 0$  at infinity and  $f$  is bounded around the boundary  $i\mathbb{R}$  (from Appendix A we have  $\frac{H_0^{(1)}(is)}{H_1^{(1)}(is)} \sim -is \log(is)$  for  $s \rightarrow 0$ ). Hence  $f \in H^\infty(\mathbb{C}_+)$ .

Now we want to show that  $f \in L^2(i\mathbb{R})$ :  $f$  is defined also in  $\overline{\mathbb{C}_+}$  if we consider the one-valued functions  $H_0^{(1)}$  and  $H_1^{(1)}$  (considering the one-valued logarithm in the definition of the Hankel functions in Appendix A). Moreover, we have that

$$f(s) = \frac{1}{4s} + O\left(\frac{1}{s^2}\right) \quad (2.2.18)$$

as  $|s| \rightarrow +\infty$ , hence

$$\int_{-\infty}^{+\infty} |f(i\omega)|^2 d\omega < +\infty.$$

Therefore by the Smirnov theorem (see [102])  $f \in H^2(\mathbb{C}_+)$ , where  $H^2(\mathbb{C}_+)$  is the so-called Hardy space, and by the Paley-Wiener theorem (see [69, 128]) there exists a unique function  $F \in L^2(\mathbb{R}_+)$  such that  $\mathcal{L}(F)(s) = f(s)$  with

$$F(t) = \lim_{v \rightarrow +\infty} \frac{1}{2\pi} \int_{-v}^v f(c + i\omega) e^{(c+i\omega)t} d\omega$$

is to be understood in the sense of  $L^2$  Fourier transforms for any  $c > 0$ . On the other hand, from (2.2.18) we have  $g(s) = f(s) - \frac{1}{4s}$  is Lebesgue integrable on the line  $\operatorname{Re} s = c$  for any  $c > 0$ . From Lemma 3.9. of [111] there exists a function  $\tilde{F} \in C([0, +\infty))$  such that  $\mathcal{L}(\tilde{F})(s) = g(s)$ , with

$$\tilde{F}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ f(c + i\omega) - \frac{\lambda}{c + i\omega} \right] e^{(c+i\omega)t} d\omega$$

independent of  $c > 0$ . Hence, writing  $f(s) = g(s) + \frac{1}{4s}$  and using the fact that  $\mathcal{L}(\lambda) = \frac{\lambda}{s}$  for all complex constant  $\lambda$ , we have that  $\mathcal{L}(F)(s) = f(s)$  with

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ f(c + i\omega) - \frac{\lambda}{c + i\omega} \right] e^{(c+i\omega)t} d\omega + \lambda$$

and  $\lambda = \frac{1}{4}$ . □

Then, we can write the coupling term with the fluid motion  $\zeta_e(t, R)$  as an explicit function of the solid velocity  $\dot{\delta}_G$  under convolution form:

**Proposition 2.2.4.** *Considering the linearized shallow water equations in the exterior domain, the following holds:*

$$\zeta_e(t, R) = \int_0^t F(s) \dot{\delta}_G(t-s) ds - \frac{R}{2v_0} \dot{\delta}_G(t) \quad (2.2.19)$$

with  $F(t)$  as in Lemma 2.2.3.

*Proof.* From (2.2.17) and Lemma 2.2.3 we have that

$$\mathcal{L}(\zeta)(s, R) = \mathcal{L}(F)(s) \mathcal{L}(\dot{\delta}_G)(s) - \frac{R}{2v_0} \mathcal{L}(\dot{\delta}_G)(s) \quad (2.2.20)$$

Using the convolution theorem for the Laplace transform,

$$\mathcal{L}(F)(s) \mathcal{L}(\dot{\delta}_G)(s) = \mathcal{L} \left( \int_0^t F(s) \dot{\delta}_G(t-s) ds \right)$$

and we apply the inverse Laplace transform to (2.2.20) to get (2.2.19). □

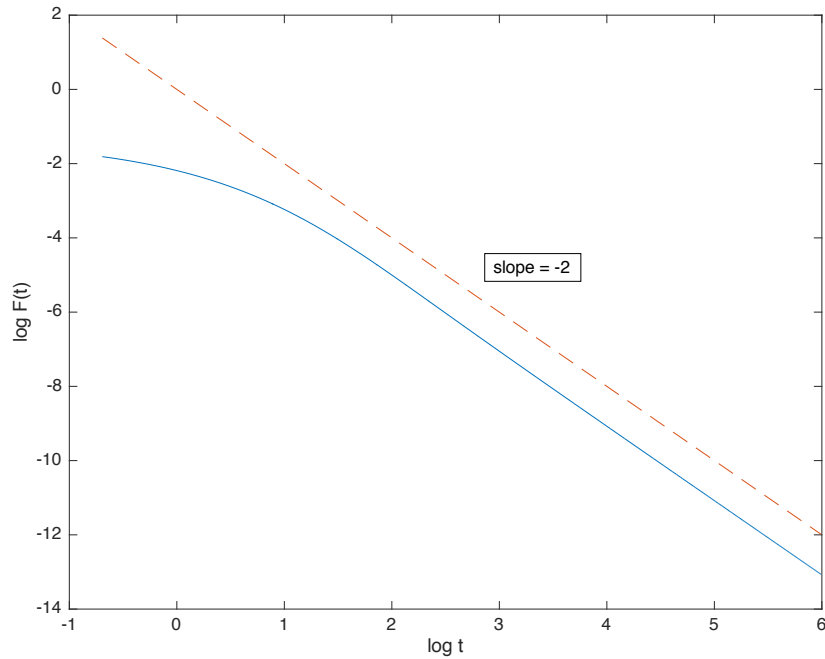


Figure 2.1 – Assumption 2.2.5 is numerically justified: here  $F(t)$  (full) is compared to  $t^{-2}$  (dash) using two logarithmic scales, for  $R = 10$  m and  $h_0 = 5$  m.

From the numerical behavior of  $F$  shown in Figure 2.1, the following assumption on  $F$  is justified:

**Assumption 2.2.5.**  $F$  is a positive function and there exist  $M > 0$  and  $t_0 > 0$  such that

$$F(t) \leq Mt^{-2}$$

for all  $t \geq t_0$ .

This hypothesis on the behavior of  $F$  is independent of the parameters of the problem  $R$  and  $v_0$ . In fact, there exists a function  $F_0$

$$F(t) = F_0\left(\frac{v_0}{R}t\right)$$

with

$$\mathcal{L}(F_0)(s) = \frac{iH_0^{(1)}(is)}{2H_1^{(1)}(is)} + \frac{1}{2}.$$

Moreover, the kernel  $F$  satisfies the following equality, which will be used in the proof of the Theorem 2.3.4:

**Lemma 2.2.6.** *The convolution kernel  $F$  is such that*

$$\int_0^{+\infty} F(\theta) d\theta = \frac{R}{2v_0}. \quad (2.2.21)$$

*Proof.* By the definition of the Laplace transform and by Lemma 2.2.3,

$$\int_0^{+\infty} F(t) e^{-st} dt = \frac{iRH_0^{(1)}\left(\frac{isR}{v_0}\right)}{2v_0H_1^{(1)}\left(\frac{isR}{v_0}\right)} + \frac{R}{2v_0} \quad \text{for } \operatorname{Re} s > 0. \quad (2.2.22)$$

From Appendix A, we have that, as  $s \rightarrow 0$ ,

$$\frac{H_0^{(1)}(is)}{H_1^{(1)}(is)} \sim -is \log(is) \rightarrow 0.$$

Hence, taking the limit  $s \rightarrow 0^+$  in (2.2.22) we get

$$\int_0^{+\infty} F(t) dt = \frac{R}{2v_0},$$

where we have used Lebesgue's dominated convergence theorem due to Assumption 2.2.5.  $\square$

## 2.3 An integro-differential equation governing the solid motion

From now on we suppose for simplicity that the bottom of the structure is flat, then  $\zeta_w$  (as well as  $h_w$ ) does not depend on the space variable  $r$ , but Proposition 2.3.1 holds for a structure with non-flat bottom as well. We know from Proposition 2.2.4 that, considering the linear shallow water equations (2.2.1) in the exterior domain, we can write the trace of the surface elevation  $\zeta_e$  at the boundary  $r = R$  as a function of the time derivative of the displacement  $\delta_G$ . Then, the nonlinear differential equation (2.1.2) describing the solid motion can be written as a nonlinear delay differential equation.

**Proposition 2.3.1.** *Considering the linear shallow water equations (2.2.13) for the fluid motion in the exterior domain, the solid motion is described by the following second order nonlinear integro-differential equation:*

$$(m + m_a(\delta_G))\ddot{\delta}_G = -\mathbf{c}\delta_G - \nu\dot{\delta}_G + \mathbf{c} \int_0^t F(s)\dot{\delta}_G(t-s)ds + (\mathbf{b}(\dot{\delta}_G) + \beta(\delta_G))\dot{\delta}_G^2, \quad (2.3.1)$$

with  $\mathbf{c} = \rho g \pi R^2$ ,  $\nu = \frac{\mathbf{c}R}{2v_0}$ ,  $m_a(\delta_G) = \frac{b}{h_w(\delta_G)}$ ,  $\beta(\delta_G) = \frac{b}{2h_w^2(\delta_G)}$ ,

$$F(t) = \lim_{v \rightarrow +\infty} \frac{1}{2\pi} \int_{-v}^v \left( \frac{iRH_0^{(1)}\left(\frac{i(c+i\omega)R}{v_0}\right)}{2v_0H_1^{(1)}\left(\frac{i(c+i\omega)R}{v_0}\right)} + \frac{R}{2v_0} \right) e^{(c+i\omega)t} d\omega$$

for any  $c > 0$  and

$$\mathfrak{b}(\dot{\delta}_G) = \frac{b}{\left(\int_0^t F(s)\dot{\delta}_G(t-s)ds - \frac{R}{2v_0}\dot{\delta}_G(t) + h_0\right)^2} \quad \text{with} \quad b = \frac{\pi\rho R^4}{8}.$$

**Remark 2.3.2.** In the integro-differential equation (2.3.1)  $m_a(\delta_G)$  is the time dependent added mass,  $\mathbf{c}$  is the hydrostatic coefficient,  $\nu$  is the damping coefficient and the convolution integral. The retardation term whose kernel  $F$  is the so-called impulse response function, accounts for fluid-memory effects that incorporate the energy dissipation due to the radiated waves coming from the motion of the structure. Moreover, linearizing (2.3.1) around the equilibrium state, we get

$$(m + m_a(0))\ddot{\delta}_G(t) = -\mathbf{c}\delta_G(t) - \nu\dot{\delta}_G(t) + \mathbf{c} \int_0^t F(s)\dot{\delta}_G(t-s)ds \quad (2.3.2)$$

This linear equation is nothing but the well-known Cummins equation for the vertical displacement (III.13). Proposition 2.3.1 therefore provides a rigorous justification of the Cummins equation and generalizes it to take into account the nonlinear effects in the interior domain.

**Remark 2.3.3.** Recall that in Proposition 2.2.4 we show that

$$\zeta_e(t, R) = \int_0^t F(s)\dot{\delta}_G(t-s)ds - \frac{R}{2v_0}\dot{\delta}_G(t). \quad (2.3.3)$$

Therefore, considering the linear equations (2.2.1), the extension-trace operator (2.1.5) becomes a linear convolution operator, that is

$$\mathcal{B}[\dot{\delta}_G](t) = \int_0^t K(s)\dot{\delta}_G(t-s)ds, \quad (2.3.4)$$

with the convolution kernel  $K(s) = F(s) - \frac{R}{2v_0}\mathfrak{d}_{s=0}$ , where  $\mathfrak{d}_{s=0}$  is the Dirac delta distribution.

We state now the following global existence and uniqueness result of the solution to the solid motion equation in the case of linear shallow water equations for the fluid motion:

**Theorem 2.3.4.** *The Cauchy problem for the nonlinear second order integro-differential equation (2.3.1) with initial data*

$$\delta_G(0) = \delta_0 \neq 0, \quad \dot{\delta}_G(0) = 0,$$

*admits a unique solution  $\delta_G \in C^2([0, +\infty), \mathbb{R})$  provided*

$$|\delta_0| < \min \left( h_0 - \frac{\rho_m H}{\rho}, -\frac{2\rho_m H}{\rho} + \sqrt{\frac{4\rho_m^2 H^2}{\rho^2} + \frac{h_0^3 \rho_m H}{\rho R^2}} \right) \quad (2.3.5)$$

**Remark 2.3.5.** *One needs to consider the parameters of the problem such that*

$$h_{w,eq} = h_0 - \frac{\rho_m H}{\rho} > 0,$$

*which means that the fluid height under the solid at the equilibrium position is positive, or equivalently that the volume of the solid is less than the volume of the portion of fluid fulfilling the column projected from the solid along the symmetry axis at the equilibrium position.*

*Proof.* We adapt here the analysis made by Liu and Magal in [87]. First let us consider the weighted space

$$BUC_\eta = \{ \varphi \in C((-\infty, 0], \mathbb{R}^2) : \theta \mapsto |\theta - 1|^{-\eta} \varphi(\theta) \text{ is bounded and uniformly continuous in } (-\infty, 0] \}$$

for  $\eta > 0$ , which is a Banach space endowed with the norm

$$\|\varphi\|_\eta := \sup_{\theta \leq 0} |\theta - 1|^{-\eta} |\varphi(\theta)|.$$

**Remark 2.3.6.** *The weighted space  $BUC_\eta$  satisfies the axioms of the phase space given by Hale and Kato in [68] (see also the monograph [70]). When the kernel  $F$  has an exponential decay, we choose the weight  $e^{\eta\theta}$ . This case is considered in Appendix D.*

From the nature of the return to equilibrium problem,  $\dot{\delta}_G(t) = 0$  for  $t < 0$  and we can write the convolution term as the infinite delay term

$$\int_{-\infty}^0 F(-\theta) \dot{\delta}_G(t + \theta) d\theta.$$

Recall that for any map  $x \in C((-\infty, \tau], \mathbb{R}^2)$  (for some  $\tau \geq 0$ ) and each  $t \leq \tau$  the map  $x_t \in C((-\infty, 0], \mathbb{R}^2)$  is defined by

$$x_t(\theta) = x(t + \theta), \quad \forall \theta \leq 0.$$

Moreover, define the trace functional

$$Tr : C((-\infty, 0], \mathbb{R}^2) \rightarrow \mathbb{R}^2$$

$$x_t \mapsto Tr(x_t) = x_t(0) = x(t)$$

with components  $Tr_1(x_t)$  and  $Tr_2(x_t)$ . Then, we consider  $x(t) = (\delta_G(t), \dot{\delta}_G(t))^T$ . We can write (2.3.1) as the following functional differential equation

$$\begin{cases} \frac{dx(t)}{dt} = \mathcal{F}(x_t) \quad \forall t \geq 0 \\ x_0 = \varphi_0 \in BUC_\eta. \end{cases} \quad (2.3.6)$$

with  $\varphi_0 = (\delta_0, 0)^T$  and  $\mathcal{F}(x_t) = (Tr_2(x_t), S(x_t))^T$  where

$$S(x_t) = \frac{-\mathbf{c}Tr_1(x_t) - \nu Tr_2(x_t) + \mathbf{c}Conv(x_t) + (\mathbf{b}(x_t, Tr_2(x_t)) + \beta(Tr_1(x_t)))Tr_2^2(x_t)}{m + m_a(Tr_1(x_t))}$$

with

$$Conv(x_t) = \int_{-\infty}^0 F(-s)x_{2t}(s)ds,$$

$$(m + m_a(Tr_1(x_t)))^{-1} = \left(m + \frac{b}{h_w(Tr_1(x_t))}\right)^{-1} = \left(m + \frac{b}{h_{w,eq} + Tr_1(x_t)}\right)^{-1},$$

$$\beta(Tr_1(x_t)) = \frac{b}{2h_w^2(Tr_1(x_t))} = \frac{b}{2(h_{w,eq} + Tr_1(x_t))^2}$$

and

$$\mathbf{b}(x_t, Tr_2(x_t)) = \frac{b}{\left(Conv(x_t) - \frac{R}{2v_0}Tr_2(x_t) + h_0\right)^2}.$$

Let us give the following definition:

**Definition 2.3.7.**  $\mathcal{F}$  is Lipschitz on bounded sets if for each  $\xi > 0$  there exists a constant  $\kappa(\xi)$  such that

$$\|\mathcal{F}(u) - \mathcal{F}(v)\|_\eta \leq \kappa(\xi)\|u - v\|_\eta$$

with  $u, v \in BUC_\eta$  and  $\|u\|_\eta, \|v\|_\eta \leq \xi$ .



It is clear that the functional  $\mathcal{F}$  is not Lipschitz on bounded sets due to the singularities that occur when the denominator of the ratios vanish. Recall that  $h_{w,eq} = h_0 - \frac{\rho_m H}{\rho}$ . We define three functions  $\chi_0, \chi_1, \chi_2 : \mathbb{R} \rightarrow \mathbb{R}$  with

$$\chi_0(\psi) = \psi \quad \text{for} \quad \psi \leq \frac{h_{w,eq} + \delta_0}{m(h_{w,eq} - \delta_0) + b} \quad (2.3.7)$$

$$\chi_1(\psi) = \psi \quad \text{for} \quad \psi \leq \frac{b}{2(h_{w,eq} - \delta_0)^2} \quad (2.3.8)$$

$$\chi_2(\psi) = \psi \quad \text{for} \quad \psi \leq \frac{b}{\left(-\frac{R}{v_0}C(|\delta_0|) + h_0\right)^2} \quad (2.3.9)$$

where

$$C(|\delta_0|) = \sqrt{\frac{g\rho}{\rho_m H} \left(|\delta_0|^2 + \frac{4\rho_m H}{\rho}|\delta_0|\right)}, \quad (2.3.10)$$

such that they are Lipschitz continuous on any compact set in  $\mathbb{R}$ . The condition (2.3.5) guarantees that the denominators of the fractions in (2.3.7) - (2.3.9) are strictly positive. We consider now the functional  $\tilde{\mathcal{F}} : BUC_\eta \rightarrow \mathbb{R}^2$  defined by

$$\tilde{\mathcal{F}}(x_t) = (Tr_2(x_t), \tilde{S}(x_t))^T \quad (2.3.11)$$

where

$$\begin{aligned} \tilde{S}(x_t) = & \chi_0 \left( \frac{1}{m + m_a(Tr_1(x_t))} \right) \times \\ & \left[ -\mathbf{c}Tr_1(x_t) - \nu Tr_2(x_t) + \mathbf{c}Conv(x_t) + (\chi_1(\beta(Tr_1(x_t))) + \chi_2(\mathbf{b}(x_t, Tr_2(x_t)))) Tr_2^2(x_t) \right] \end{aligned}$$

Then, we have the following property:

**Lemma 2.3.8.**  $\tilde{\mathcal{F}} : BUC_\eta \rightarrow \mathbb{R}^2$  is Lipschitz on bounded sets for  $\eta$  small enough.

*Proof.* From the definition of  $Conv$ , using Assumption 2.2.5 we have

$$\begin{aligned} |Conv(u) - Conv(v)| & \leq \int_{-\infty}^0 |F(-s)| |s - 1|^\eta ds \|u - v\|_\eta \\ & \leq \left( C_0 + M \int_{t_0}^{+\infty} |s|^{-2} |s + 1|^\eta ds \right) \|u - v\|_\eta \end{aligned}$$

where  $C_0$  comes from the fact that  $F$  is continuous on the interval  $[0, t_0]$ . Choosing  $\eta$  such that  $-2 + \eta < -1$ , by definition of the function  $\chi_0, \chi_1$  and  $\chi_2$  it is clear that  $\tilde{\mathcal{F}}$  is Lipschitz on bounded sets in  $BUC_\eta$ .  $\square$

Then, an equivalent version of the Theorem 7.4 of [87] can be applied to

$$\begin{cases} \frac{dx(t)}{dt} = \tilde{\mathcal{F}}(x_t) \quad \forall t \geq 0 \\ x_0 = \varphi_0 \in BUC_\eta. \end{cases} \quad (2.3.12)$$

and we have that (2.3.12) admits a unique solution  $x_{\varphi_0} \in C((-\infty, \tau), \mathbb{R}^2)$  with initial data  $\varphi_0$ . From the continuity of  $\tilde{\mathcal{F}}$  we get  $x_{\varphi_0} \in C^1((-\infty, \tau_{\varphi_0}), \mathbb{R}^2)$ . Furthermore the theorem gives an explosion condition on the solution, i.e. if  $\tau_{\varphi_0} < +\infty$  then

$$\lim_{t \nearrow \tau_{\varphi_0}^-} \|x_{\varphi_0}(t)\| = +\infty. \quad (2.3.13)$$

We show in the following lemma that the solution is bounded:

**Lemma 2.3.9.** *The displacement  $\delta_G$  and its derivative  $\dot{\delta}_G$  are both bounded.*

*Proof.* From Proposition 2.2.1 we know that the energy of the coupled floating structure system considering the linear shallow water equations for the fluid motion

$$E_{tot}(t) = \frac{1}{2} m \dot{\delta}_G^2(t) + mg \delta_G(t) + E_{SW}(t) \quad (2.3.14)$$

is conserved. Moreover,  $E_{SW}(t)$  can be written as the sum of the fluid energy in the interior domain,

$$E_{int}(t) = \frac{1}{2} \rho g 2\pi \int_0^R \zeta_w^2(t) r dr - \frac{1}{2} \rho g 2\pi \int_0^R \zeta_{w,eq}^2 r dr + \frac{1}{2} \rho 2\pi \int_0^R \frac{q_i^2(t, r)}{h_w(t)} r dr,$$

and the fluid energy in the exterior domain,

$$E_{ext}(t) = \frac{1}{2} \rho g 2\pi \int_R^{+\infty} \zeta_e^2(t, r) r dr + \frac{1}{2} \frac{\rho}{h_0} 2\pi \int_R^{+\infty} q_e^2(t, r) r dr.$$

To get the expression of the fluid energy in the interior domain we use the constraint (1.1.6) and we add the constant term  $\frac{1}{2} \rho g 2\pi \int_0^R \zeta_{w,eq}^2 r dr$  in order to have zero energy at the equilibrium. From Archimedes' principle we have

$$-\rho_m H - \rho \zeta_{w,eq} = 0 \quad (2.3.15)$$

and, since the bottom of the solid is flat, we have

$$z_{G,eq} = \zeta_{w,eq} + \frac{H}{2}.$$

Then,

$$z_{G,eq} = \left( \frac{1}{2} - \frac{\rho_m}{\rho} \right) H$$

and

$$\zeta_w(t) = z_G(t) - \frac{H}{2} = \delta_G(t) + z_{G,eq} - \frac{H}{2} = \delta_G(t) - \frac{\rho_m}{\rho} H. \quad (2.3.16)$$

Using (2.3.16) and the fact that  $q_i(t, r) = -\frac{r}{2}\dot{\delta}_G(t)$  (see Section 1.5), the fluid energy in the interior domain  $E_{int}(t)$  becomes

$$E_{int}(t) = \frac{1}{2}g\rho\pi R^2 \left( \delta_G(t) - \frac{\rho_m}{\rho} H \right)^2 - \frac{1}{2}g\rho\pi R^2 \frac{\rho_m^2}{\rho^2} H^2 + \frac{\pi\rho R^4}{16h_w(t)} \dot{\delta}_G^2(t).$$

In particular the total energy at instant  $t = 0$  is

$$E_{tot}(0) = mg\delta_0 + \frac{1}{2}g\rho\pi R^2 \left( \delta_0 - \frac{\rho_m}{\rho} H \right)^2 - \frac{1}{2}g\rho\pi R^2 \frac{\rho_m^2}{\rho^2} H^2$$

using  $\delta_G(0) = \delta_0$  and  $\dot{\delta}_G(0) = 0$ . By the conservation of the energy we have

$$\begin{aligned} \left( \frac{m}{2} + \frac{\pi\rho R^4}{16h_w(t)} \right) \dot{\delta}_G^2(t) &= mg\delta_0 + \frac{1}{2}g\rho\pi R^2 \left( \delta_0 - \frac{\rho_m}{\rho} H \right)^2 - mg\delta_G(t) \\ &\quad - \frac{1}{2}g\rho\pi R^2 \left( \delta_G(t) - \frac{\rho_m}{\rho} H \right)^2 - E_{ext}(t). \end{aligned} \quad (2.3.17)$$

Consider  $t^* = \sup\{t \in (-\infty, \tau_{\varphi_0}) \mid h_w(s) > 0 \text{ for } s \in (-\infty, t)\}$ . From condition (2.3.5) we have  $h_w(t) = h_{w,eq} + \delta_0 > 0$  for  $t \in (-\infty, 0]$ , hence  $t^* > 0$ . Suppose  $t^* < \tau_{\varphi_0}$ . Then, for  $t \in (-\infty, t^*)$  the right hand side of (2.3.17) has to be non-negative. By solving the inequality with respect to  $\delta_G(t)$  and writing  $m = \rho_m\pi R^2 H$ , we have

$$-\sqrt{\delta_0^2 - \frac{2E_{ext}(t)}{g\rho\pi R^2}} \leq \delta_G(t) \leq \sqrt{\delta_0^2 - \frac{2E_{ext}(t)}{g\rho\pi R^2}}.$$

By the non-negativity of  $E_{ext}(t)$  we get the bound

$$-|\delta_0| \leq \delta_G(t) \leq |\delta_0|. \quad (2.3.18)$$

Combining (2.3.18) with (2.3.17), we have

$$\begin{aligned} \frac{m}{2}\dot{\delta}_G^2 &\leq mg(\delta_0 - \delta_G) + \frac{1}{2}g\rho\pi R^2(|\delta_0|^2 - \delta_G^2) + g\rho\pi R^2 \frac{\rho_m H}{\rho} (\delta_G - \delta_0) \\ &\leq mg(\delta_0 + |\delta_0|) + \frac{1}{2}g\rho\pi R^2 \delta_0^2 + g\rho\pi R^2 \frac{\rho_m H}{\rho} (|\delta_0| - \delta_0) \\ &= g\rho\pi R^2 \left( \frac{|\delta_0|^2}{2} + \frac{2\rho_m H}{\rho} |\delta_0| \right). \end{aligned}$$

It yields the bound

$$|\dot{\delta}_G(t)| \leq C(|\delta_0|), \quad (2.3.19)$$

with  $C(|\delta_0|)$  as in (2.3.10). Using condition (2.3.5) on  $\delta_0$ , by continuity we have  $h_w(t^*) \geq h_{w,eq} - \delta_0 > 0$  and there exists  $\epsilon > 0$  small enough such that  $h_w(t^* + \epsilon) > 0$ , where  $t^*$  is the maximal time such that  $h_w(t) > 0$  for  $t \in (-\infty, \tau_{\varphi_0})$ . Then, necessarily  $t^* = \tau_{\varphi_0}$ , which implies that the bound (2.3.18) holds in the existence interval  $(-\infty, \tau_{\varphi_0})$ .  $\square$

Hence the solution  $x_{\varphi_0}$  to (2.3.12) is bounded in  $(-\infty, \tau_{\varphi_0})$ , then from the explosion condition (2.3.13) we have  $\tau_{\varphi_0} = +\infty$ . The bounds (2.3.18) - (2.3.19) give

$$\begin{aligned} h_w(t) &= h_{w,eq} + \delta_G(t) \geq h_{w,eq} - |\delta_0| \\ h_e(t, R) &= \int_{-\infty}^0 F(-\theta) \dot{\delta}_G(t + \theta) d\theta - \frac{R}{2v_0} \dot{\delta}_G(t) + h_0 \\ &\geq \left( - \int_{-\infty}^0 F(-\theta) d\theta - \frac{R}{2v_0} \right) C(|\delta_0|) + h_0 \end{aligned}$$

using the positivity of  $F$  by Assumption 2.2.5. The admissibility condition (2.3.5) on  $\delta_0$  and the equality (2.2.21) guarantee that for all  $t \geq 0$

$$h_w(t) \geq h_{w,eq} - |\delta_0| > 0, \quad (2.3.20)$$

$$h_e(t, R) \geq -\frac{R}{v_0} C(|\delta_0|) + h_0 > 0. \quad (2.3.21)$$

Therefore  $\tilde{\mathcal{F}}(x_t)$  coincides with  $\mathcal{F}(x_t)$  in (2.3.6) since, for all the values of  $x_t$ , the arguments of  $\chi_0, \chi_1$  and  $\chi_2$  stay in the region where the three functions are identities. Therefore we get the global existence of the solution to (2.3.6) which implies the global existence of the solution  $\delta_G$  to (2.3.1).  $\square$

**Remark 2.3.10.** *The conditions (2.3.20) - (2.3.21) express the physical fact that, during all the motion, both the solid and the fluid trace at the solid walls do not touch the bottom of domain.*

## 2.4 Numerical method

In order to solve numerically the delay differential equation (2.3.1) we write it under the form

$$\frac{dy}{dt}(t) = f(t, y(t), y(d_1(t)), \dots, y(d_k(t))),$$

with  $d_1(t), \dots, d_k(t)$  the components of the non-constant delays vector  $d(t)$ . In our case we have chosen  $d_k(t) = t - kdt$  with  $dt = 0.1$  and  $k = 1, \dots, N$  for  $N = 100$ .

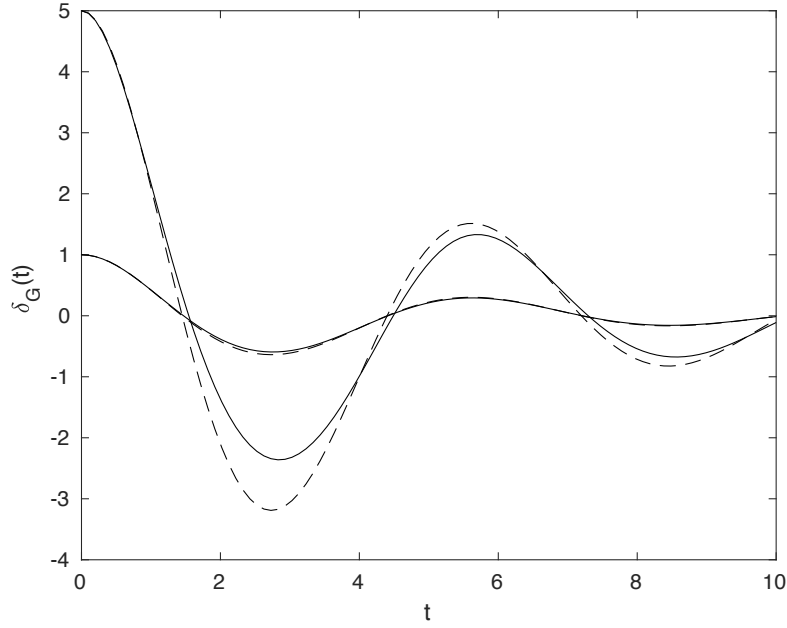


Figure 2.2 – Time evolution of the displacement  $\delta_G$  given by the nonlinear integro-differential (2.3.1) (full) and by the linear Cummins equation (2.3.2) (dash) for two different initial data.

Then, we implement in our code the MATLAB solver `ddesd`, which integrates with the explicit Runge-Kutta (2,3) pair and interpolant of `ode23`. For more details on the solver we refer to Shampine [114]. Moreover, we compute the convolution integral applying the trapezoidal integration method following Armesto et al. [7]. In an analogous way, we compute the convolution kernel  $F$  for a given set of time steps  $n\Delta t$  with  $n = 1, \dots, N$  since the influence of the Kernel is negligible after some time  $t^* = N\Delta t$ . Then, we compare the numerical result given by the nonlinear integro-differential equation (2.3.1) with the one obtained from its linear approximation. In Figure 2.2 we consider  $h_0 = 15$  m,  $R = 10$  m,  $H = 10$  m,  $\rho = 1000$  kg/m<sup>3</sup> and the volume density of the solid  $\rho_m = 0.5 \rho$ . We choose two different initial data:  $\delta_0 = 1$  m and  $\delta_0 = 5$  m. One can see that for large amplitudes the nonlinear effects should not be neglected in order to better describe the solid motion. This difference justifies the approach to keep nonlinearities in the equation of the floating body problem in the interior domain. Moreover, one can note that the displacement goes to zero but the structure definitely does not reach its equilibrium position: this is due to the motion of the fluid which makes the solid constantly move.

# ROTATING FLUIDS



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# Introduction

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## V Geophysical fluids

Oceans and the atmosphere are related from a fluid dynamical point of view. They have analogies and common properties which have been shown in real experiences. Geophysical fluid dynamics is the subject which deals with the dynamical features of both atmospheric and oceanic motions. These motions vary in different spatial and temporal scales, from micrometers to hundred of kilometers, from microseconds to months, and include different scale-specific phenomena, from internal gravity waves to massive atmospheric and oceanic currents. Moreover, when interaction across scales are studied, multiple scales models are introduced to describe the phenomena. We refer to Klein [78] for a discussion of the different scales and the associated scale-dependent models.



Though the range of geophysical fluid dynamics is wide, it is in large-scale flows that the common points for oceanographic and atmospheric flows are more evident and easiest to understand. In this manuscript large-scale flows are the ones influenced by the Earth's rotation. Let us give an example. The typical horizontal scale of oceans is 5000 km and it is known that velocities are of the order of meters per second; hence a particle of water takes approximately 50 days to move across the ocean, while the Earth has already rotated around its axis 50 times. This fact shows that, in order to study oceanic motions at large scale, the Earth's rotation cannot be neglected.

Other physical effects are relevant for a more accurate description of the reality, for instance temperature, salinity and stratification. However, in a first step the analysis of rotating fluids is sufficient to explain some physical phenomena and processes, such as the so-called "western intensification" for currents with western boundaries. Two examples are the Gulf Stream in the North Atlantic Ocean and the Kuroshio in the North Pacific Ocean.

Let us now discuss the equations governing the motion of large-scale rotating fluids. We consider a motion with characteristic length  $L$  and characteristic horizontal velocity  $U$ ; the time a particle takes to cover the distance  $L$  moving at velocity  $U$  is  $T = L/U$ . If this time is much bigger than the period of rotation of the Earth, or if the dimensionless parameter

$$\text{Ro} = \frac{U}{2\Omega L} \ll 1, \quad (\text{V.1})$$

then the effects of rotation are not negligible. Here  $\Omega = 7.3 \times 10^{-5} \text{s}^{-1}$  is the module of the Earth's angular velocity  $\boldsymbol{\Omega}$ , which is assumed to be locally parallel to the vertical axis. This fact means that here only mid-latitudes and high-latitudes motions are considered. This parameter is called the *Rossby number*<sup>1</sup>. The smaller Ro is, the more relevant the Earth's rotation is in the study of geophysical flows. This regime can be satisfied also considering different length and velocity scales. For example in the case of the Gulf Stream, though the length  $L = 100 \text{ km}$  and  $U = 1 \text{ m s}^{-1}$  are smaller than the typical oceanic scales, the Rossby number is  $\text{Ro} = 0.07$ ; in the core of the Earth,  $L = 3000 \text{ km}$  and  $U = 0.1 \text{ cm s}^{-1}$  give a very small Rossby number  $\text{Ro} = 2 \times 10^{-7}$ . As can be seen, the influence of the Earth's rotation is taken into account by the size of the Rossby number.

On the other side, the presence of rotation affects the velocity of the fluid via the *Coriolis force*. It is a fictitious force that appears in the equations of the

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<sup>1</sup>In the exact Rossby number the module of the angular velocity  $\Omega$  is replaced by the module of the component of the angular velocity perpendicular to the Earth's local tangent plane, namely  $\Omega \sin \phi$  where  $\phi$  is the angle between the rotation axis and the planet surface vector. Obviously, this approximation is not consistent when dealing with low-latitudes and equatorial motions, where  $\phi$  tends to zero.

motion when they are considered in a rotating coordinate frame. More precisely, Newton's law for the conservation of the linear momentum for an incompressible homogeneous fluid in an inertial coordinate frame reads

$$\rho \frac{Du}{Dt} = -\nabla p + \rho \nabla \phi + F(u) \quad (\text{V.2})$$

coupled with the incompressibility condition

$$\nabla \cdot u = 0, \quad (\text{V.3})$$

where  $u$  is the velocity vector field,  $\rho$  is the density,  $p$  is the pressure,  $\phi$  is the potential of conservative body forces,  $F(u)$  is the frictional force in the fluid and  $\frac{D}{Dt}$  denotes the convective derivative  $\partial_t + u \cdot \nabla$ . We focus now on the momentum equations (V.2) and we consider incompressible fluids. The case of compressible fluids, together with comments on the incompressibility condition (V.3), will be discussed in Section V.2. Introducing the rotating frame with the angular velocity  $\Omega$ , which is constant in the time scales considered (months), the equations (V.2) take the form

$$\rho \left( \frac{Du}{Dt} + 2\Omega \times u \right) = -\nabla p + \rho \nabla \Phi + F(u) \quad (\text{V.4})$$

where  $u$  is the relative velocity observed in the rotating frame and

$$\Phi = \phi + \frac{|\Omega \times x|^2}{2}$$

with the second term in the right-hand side representing the centrifugal force potential. The term  $2\Omega \times u$  is called the *Coriolis acceleration*. In the case when the frictional forces are not considered or are negligible, the source term  $F(u)$  is assumed to vanish and the equations (V.4) become the *Euler-Coriolis* equations

$$\partial_t u + u \cdot \nabla u + 2\Omega \times u + \frac{\nabla p}{\rho} = \nabla \Phi. \quad (\text{V.5})$$

We will discuss in the next section the case when friction occurs in the motion. It will be shown that different choices of  $F(u)$  can be more or less accurate according to the physical situation investigated.

We want to emphasize now a well-known phenomenon arising in geophysical fluid dynamics, the *Taylor-Proudman theorem* [116, 107]: in a high rotation regime large-scale fluid motions become horizontal. Let us neglect here the effect of an external force and of the friction. We consider the dimensionless version of (V.5), namely

$$\partial_t u + u \cdot \nabla u + \frac{1}{\text{Ro}} \times u + \frac{1}{\text{Ro}} \nabla p = 0. \quad (\text{V.6})$$

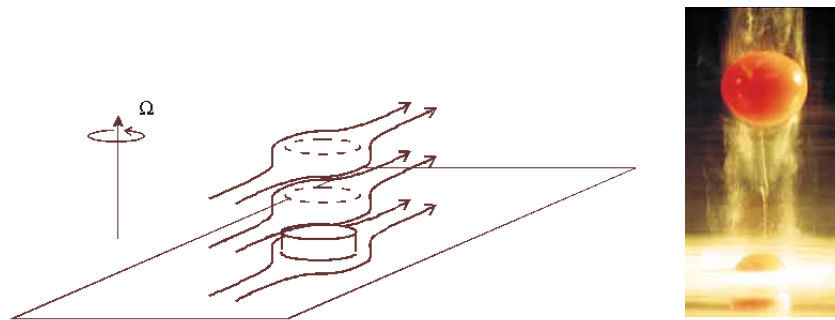


Figure 2.3 – When an object moves in a rotating flow, it drags along with it a column of fluid parallel to the rotation axis.

For a small Rossby number, the equations (V.4) at the leading order read

$$\begin{aligned} -2\Omega u_2 &= -\frac{\partial_1 p}{\rho} \\ 2\Omega u_1 &= -\frac{\partial_2 p}{\rho} \\ 0 &= -\partial_3 p. \end{aligned} \tag{V.7}$$

and it is called the *geostrophic balance*: the pressure gradient must everywhere compensate the Coriolis force. Using the incompressibility condition, the velocity does not depend on the variable related to the direction of the rotation axis and the horizontal vector field is divergence-free. The flow associated with this horizontal velocity is called *geostrophic flow*. Hence the motion of the fluid particles occur in vertical columns, called the *Taylor-Proudman columns*. This effect can be seen in experiments for fluids with uniform density by dragging a solid on a path perpendicular to the direction of the rotation axis. As Taylor showed in several experiments described in [117, 118], the portion of fluid above and below the solid follows the motion of the fluid divided by the solid. It creates a phantom body, formed by the fluid fulfilling the Taylor-Proudman column projected from the solid along the rotation axis, which moves through the bulk fluid (see Figure 2.3).

## V.1 Viscous flows and the Reynolds number

For a better description of the reality, friction and the associated dissipation of mechanical energy must be considered in the model of the fluid motion. In both the atmosphere and oceans fluid motions are excited by different type of external forces, for example the solar heating acting on the atmosphere and the wind stress on the ocean's surface. Considering newtonian fluids such as the air or the water,

the frictional force takes the form

$$F(u) = \mu\Delta u + \frac{\mu}{3}\nabla\nabla\cdot u,$$

where the molecular viscosity  $\mu$  is a constant. Considering incompressible fluids ( $\nabla\cdot u = 0$ ), the equations (V.4) become the *incompressible Navier-Stokes-Coriolis equations*

$$\partial_t u + u\cdot\nabla u - \nu\Delta u + 2\Omega\times u + \frac{\nabla p}{\rho} = \nabla\Phi. \quad (\text{V.8})$$

The constant  $\nu = \mu/\rho$  is called the kinematic viscosity. The force  $F$  is responsible for the dissipation of the kinetic energy that creates chaotic molecular motions, but its influence on the large-scale motion is negligible. On the other hand, many turbulent motions at small scales appear in the atmosphere and oceans, draining energy from the larger scale flows. One of the most intricate<sup>2</sup> issue in geophysical fluid dynamics is to understand this energy exchange, which is a cascade of energy from the largest to the smallest scales of motion. These dissipative effects are taken into account by the *Reynolds stresses*, which appear when the small-scale and the large-scale velocities are separated. One simple way (not rigorously justified) to close the governing equations in terms of the large-scale velocity is to assume that the Reynolds stresses are linearly dependent on the derivatives of the large-scale velocity via a turbulent viscosity. We denote its horizontal and vertical components  $\nu_h$  and  $\nu_3$  respectively. Therefore, considering the case of an incompressible fluid, the equation (V.4) takes the form

$$\partial_t u + u\cdot\nabla u - \nu_h\Delta_h u - \nu_3\partial_3^2 u - \nu\Delta u + 2\Omega\times u + \frac{\nabla p}{\rho} = \nabla\Phi. \quad (\text{V.9})$$

In the case  $\nu_h = \nu_3$  the previous equations become the standard isotropic incompressible Navier-Stokes equations with a potential source term. More precise models describe better the physics of the motion by taking in consideration non-constant stresses, depending on the temperature of the fluid. In general the two turbulent viscosities are not equal, in particular when dealing with large-scale motions. For instance in the ocean  $\nu_h$  ranges from  $10^3$  to  $10^8$   $\text{cm}^2\text{s}^{-1}$  and  $\nu_3$  from 1 to  $10^3$   $\text{cm}^2\text{s}^{-1}$ . A justification of this fact can be seen in the anisotropy between the horizontal and the vertical scales of the flows. The turbulent viscosities are larger than the molecular viscosity  $\mu$ , which can be neglected for the large-scale

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<sup>2</sup>Pedlosky in his celebrated book [103] wrote: "...the situation with regard to the representation of the turbulent interactions of small-scale and large-scale motions is considerably less satisfactory. Indeed this problem is one of the less satisfactory...there seems to be no tractable theory of turbulence that provides a practical and accurate description of the effective frictional force due to the cascade of energy by turbulent fluctuations."

flow. A parameter showing the importance of the inertial forces with respect to the viscous forces is the *Reynolds number* defined as

$$\text{Re} = \frac{LU}{\nu}. \quad (\text{V.10})$$

The smaller Re is, the more relevant are the viscous effects on the fluid motion. In geophysical fluid dynamics and for large-scale flows, another dimensionless parameter is relevant in the description of the motion. The vertical *Ekman number*, defined by

$$\text{Ek} = \frac{\nu_3}{2\Omega H^2}, \quad (\text{V.11})$$

with  $H$  the vertical length, describes the ratio of viscous forces and of the Coriolis force coming from the planetary rotation. It characterizes also the thickness of the so-called *Ekman layers*, which are layers where the viscous diffusion is balanced by the effects of the Coriolis force. We remark that  $\text{Ek} = \frac{\text{Ro}}{\text{Re}}$  when in the expression of the Rossby number  $\text{L}$  is the vertical length  $H$  and in the expression of the Reynolds number  $\nu$  is the vertical viscosity  $\nu_3$ .

## V.2 Compressible flows and the Mach number

In the previous sections we wrote Newton's law for the motion of a fluid considering the rotation and the friction. For the sake of simplicity, we have neglected the dependence on time and space of the fluid density. In fluid mechanics, the evolution of a compressible fluid is given by two evolution equations on the momentum and on the density. In the compressible case the momentum equation (V.8), showed in the constant density case, becomes

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) - \mu \Delta u - \frac{\mu}{3} \nabla \nabla \cdot u + 2\Omega \times (\rho u) + \nabla p(\rho) = \rho \nabla \Phi. \quad (\text{V.12})$$

where the pressure term now depends on the density. This dependence is given by an *equation of state*, which is not unique and depends on the possible applications. The most known are for instance the Dalton's law, the ideal gas law, Van der Waals equation of state and the virial equation of state. In general the pressure law may depend on both density and temperature of the fluid. It is called barotropic when only the density dependence is assumed. The simplest example is the one for isothermal flows, with a power-type pressure law [45].

The second evolution equation is called the continuity equation and it is obtained under the physical consideration that mass must be conserved. It reads

$$\partial_t \rho + \nabla \cdot (\rho u) = 0. \quad (\text{V.13})$$

In oceans and in the lower layers of the atmosphere the density of the fluid does not vary substantially and it oscillates around a positive constant reference value. Let us investigate small oscillations. Thus, we can write the density

$$\rho(t, x) = \rho_0 + \delta\rho(t, x), \quad (\text{V.14})$$

where  $\delta\rho$  is the displacement from a characteristic density  $\rho_0$ . The oscillating part of the density  $\delta\rho$  is of order  $O(\text{Ma}^2)$  (see [79]), where  $\text{Ma}$  is a dimensionless parameter called the *Mach number* and defined by

$$\text{Ma} = \frac{U}{c}. \quad (\text{V.15})$$

The constant  $c$ , called *speed of sound*, is the velocity of propagation in the medium of oscillatory motions with small amplitude, called *sound* or *acoustic waves*. These oscillating waves cause alternate compression and rarefaction. The ratio  $\text{Ma}$  represents the influence of these waves on the global flow in which they propagate. Considering a planetary scale, for both air and water this parameter is generally small. The continuity equation can be written in terms of  $\text{Ma}$ , namely

$$\nabla \cdot u = \frac{1}{\rho_0} O(\text{Ma}^2),$$

and for very small values ( $\text{Ma} < 0.3$  in applications) the incompressibility condition

$$\nabla \cdot u = 0 \quad (\text{V.16})$$

is derived. It represents the fact that in incompressible fluids the speed of sound is very large with respect to the fluid velocity. This justifies the term incompressibility for a fluid. In section VI singular limit problems for both compressible and incompressible fluids will be presented but in Chapter 3 only compressible fluids will be treated.

### V.3 Gravitational and centrifugal forces

The Earth's rotation has a significant impact on the fluid motions at large scales and creates non-intuitive phenomena, such as the Taylor-Proudman columns. In addition, the gravity plays equally an important role in the characteristics of the motion. In geophysical flows its principal effect is the stratification of fluids with space-dependent density, called non-homogeneous fluids. The gravitational force tends to lower regions of fluid with higher density and raise regions of fluid with lower density. It affects more fluids with high density. As shown in Figure 2.4, it tends to the equilibrium configuration in which the density profile decreases with

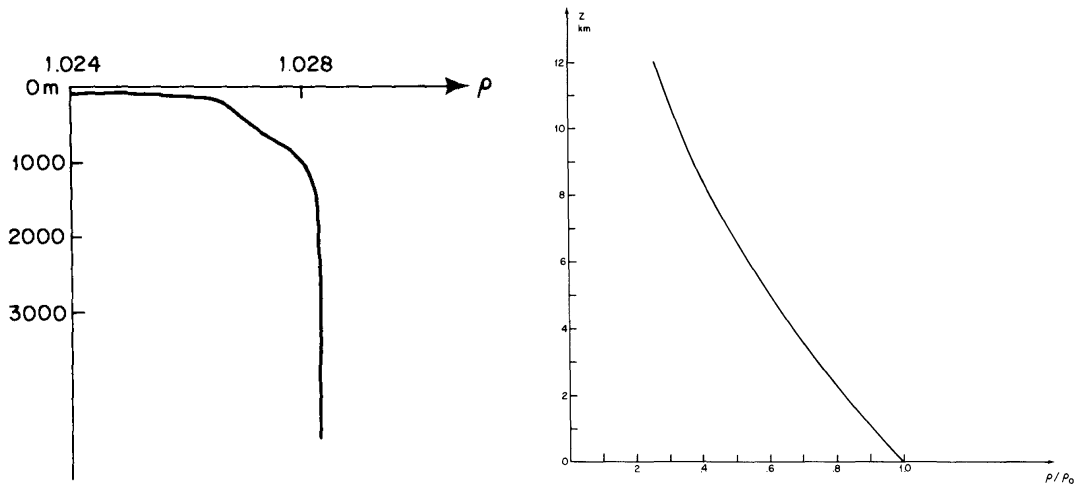


Figure 2.4 – Distribution of density (from Pedlosky [103]): in the ocean (left) with depth and in the atmosphere (right) with height. The reference density  $\rho_0 = 1.225 \times 10^{-3} \text{g cm}^{-3}$ .

respect to the vertical direction. In the case when there is a gravitational force, the *Froude number*, defined as

$$\text{Fr} = \frac{U}{\sqrt{gH}}, \quad (\text{V.17})$$

is a relevant dimensionless parameter since it measures the ratio of inertial forces of a fluid element to its weight.

The primary consequence of the centrifugal forces on the Earth is to flatten the spherical shape. The direction of the resultant gravitational and centrifugal force is always orthogonal to the Earth's local tangent plane. The vertical component of the centrifugal force at distance  $R$  from the rotation axis <sup>3</sup> is

$$R\Omega^2. \quad (\text{V.18})$$

It represents the amount by which local gravity is reduced by the centrifugal force, which acts as anti-gravity. The ratio of the centrifugal acceleration to gravity is  $\Omega^2 R/g \sim 3.5 \times 10^{-3}$ , hence the effects of this force are often neglected.

## V.4 Ekman layers

As we have seen before the Taylor-Proudman theorem, which constrains the fluid motion to be horizontal, is one the main characteristics of geophysical flows caused

<sup>3</sup>As for the Rossby number, the exact vertical component of the centrifugal force is  $R\Omega^2 \sin^2 \phi$ , where  $\phi$  is the latitude, *i.e.* the angle between the rotation axis and the Earth's radial vector.

by the high rotation. It seems natural to couple this phenomenon with the viscous effects due to friction, in particular when the fluid is considered to move in domains with boundaries. From the anisotropic nature of large-scale flows, horizontal boundaries naturally appear to be a good representation of the reality: they can be either rigid or free boundaries.

We discuss here the first case. Let us consider a motion of a homogeneous, incompressible fluid in a domain with a wall  $x_3 = 0$ , perpendicular to the angular velocity  $\Omega$ . The friction can be assumed to be so prominent that it forces the fluid velocity to vanish at this boundary, *i.e.*  $u = 0$  at  $x_3 = 0$ . This condition is called the *no-slip boundary condition*. Other conditions can be assumed such as the complete slip condition or the Navier-slip condition. The no-slip condition is able to capture a very interesting consequence of the interaction between rotating fluids and boundaries: the apparition of boundary layers, which are called *Ekman layers*. More precisely, the Taylor-Proudman theorem is not compatible with the fact that the fluid velocity must vanish at the boundary (the velocity vector field is the trivial one). In the interior of the domain, the flow is horizontal and does not depend on the vertical variable, but close to the boundary the same property ceases to hold. In this region the velocity departs from the horizontal geostrophic flow  $u^{int}$  due to friction. The departure  $u^{bl} = u - u^{int}$  satisfies

$$\begin{aligned} -2\Omega u_2^{bl} &= \nu_3 \partial_3^2 u_1^{bl} \\ 2\Omega u_1^{bl} &= \nu_3 \partial_3^2 u_2^{bl}. \end{aligned} \tag{V.19}$$

Assuming that the boundary layer term  $u^{bl}$  is bounded when approaching the interior of the domain, the solution of (V.19) is called the *Ekman spiral* because of the shape of the velocity vector field. As shown in Figure 2.5, as the wall is approached the vector field slowly turns due to the Earth's rotation. On the other hand, both components  $u_1^{bl}$  and  $u_2^{bl}$  decrease exponentially to zero far from the boundary. Hence, the total velocity tends exponentially to the horizontal geostrophic flow when leaving the boundary. The thickness of these boundary layers, where friction is dominant, is

$$\delta_E = \left( \frac{\nu_3}{\Omega} \right)^{1/2}, \tag{V.20}$$

which does not depend on the velocity. It is immediate to remark that the faster the rotation the smaller is the layer affected by viscosity, which becomes a thin region of size  $\delta_E$  close to the wall  $x_3 = 0$  for highly rotating fluids. Hence, in the study of rotating fluids moving in domains with horizontal boundaries, the mutual existence of two different vertical scales needs to be taken into consideration: the large scale  $x_3$  and the small scale  $x_3/\delta_E$ .

In the case of incompressible fluids, enforcing the incompressibility property given



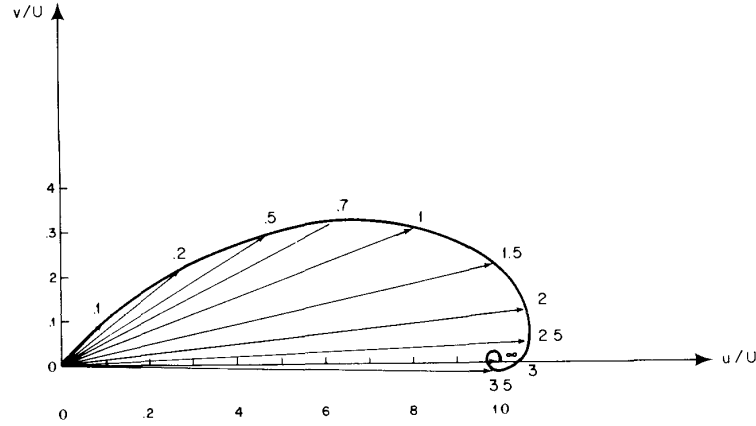


Figure 2.5 – (Pedlosky [103]) The velocity vector field in the Ekman layer forms a spiral. Here the value of  $u = u_1^{bl}$  and  $v = u_2^{bl}$  are scaled by the value of the internal velocity  $U = u^{int}$ .

by  $\nabla \cdot u = 0$  yields the fact that the vertical velocity does not go to zero as approaching the interior. If the velocity in the interior is not constant a small quantity of fluid enters the bulk of the fluid or vice-versa. This phenomenon, called the Ekman suction, causes a three-dimensional global circulation in the whole domain, not only in the boundary layers, and has an effect on the energy balance. Indeed, the Ekman layer dissipates an important amount of energy and damps the interior velocity. This phenomenon, called the Ekman pumping, represents a linear dissipation. The interior velocity satisfies the following two-dimensional Navier-Stokes equations with a damping term (see [66] for more details):

$$\begin{cases} \partial_t u + u \cdot \nabla_h u - \nu_h \Delta u + \sqrt{2}u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases} \quad (\text{V.21})$$

The presence of the frictional Ekman layer requires an energy source to keep the geostrophic horizontal flow. The presence of the frictional Ekman layer requires an energy source to keep the geostrophic horizontal flow. For a layer of height  $H$ , the ratio between the characteristic decay time and the rotation period is given by

$$\sqrt{\frac{H^2 \Omega}{\nu_3}} = \frac{1}{\sqrt{2\text{Ek}}}. \quad (\text{V.22})$$

Hence, considering flows with low Ekman number, the frictional forces are weak enough such that the time of the decay caused by the viscous dissipation is large in comparison with the rotation period.

The physical introduction presented in this section has been written following

the two celebrated monographs [103] and [64] on geophysical fluid dynamics, respectively by Pedlosky and by Greenspan. The reader interested in more physical motivations and details is referred to these books and references therein.

## VI Singular limit problems

In mathematical geophysical fluid dynamics a central question is to study the behavior of solutions to partial differential equations when some parameter tends to zero or infinity. This creates singularities, hence the name *singular limits*. Indeed, it turns out often that the solutions converge to solutions of a limit system which is qualitatively different from the original equations. Moreover, singular limits are strongly connected with the analysis of asymptotic models. Due to the complexity of many mathematical problems, simplified and reduced models based on asymptotic limit equations are studied. These may provide a deeper comprehension of the dynamics of the original problem. Nevertheless these reduced models come from singular asymptotic limits of the full governing equations. This has a significant influence on the behavior of the solutions. For instance degeneracies may appear.

Previously in this chapter, several dimensionless numbers relevant in the geophysical fluid dynamics have been introduced. Depending on their size, the governing equations describe different physical situations and the solutions may have different qualitative behaviors. For example we have seen that, if the Rossby number  $Ro$  is very small, the influence of the planetary rotation is so important that it constrains the fluid motion to be horizontal, as shown by the Taylor-Proudman theorem. The incompressibility itself is a concept derived by a singular limit, namely when the Mach number  $Ma$  goes to zero (see Section V.2). Viscosity is strictly related to the Reynolds number  $Re$ . For high values of this number the importance of viscous effects diminishes and the fluid tends to a completely inviscid configuration. In the case of anisotropic viscosity, if the vertical Ekman number  $Ek$  tends to zero Coriolis forces exceed the vertical viscous forces.

Let us now give an overview of the main mathematical results known for singular limit problems for geophysical fluids in rotation. This presentation is not exhaustive, since the literature is rich and covers many different cases. However, the results and the techniques presented here represent a good background for the analysis presented in Chapter 3. Since viscosity is considered, the governing equations are the Navier-Stokes-Coriolis equations. We can consider the incompressible or the compressible case.

For general mathematical singular limits problems in thermodynamics of viscous fluids we refer to the celebrated monograph [49] by Feireisl and Novotný, where the

authors give an introduction to the analysis of singular limits. Taking into consideration heat conductivity and the governing equations are the Navier-Stokes-Fourier equations, describing the time evolution of the density, the velocity and the absolute temperature of the fluid.

## VI.1 High rotation limit of the incompressible Navier-Stokes-Coriolis equations

The motion of a homogeneous, incompressible fluid subjected to the planetary rotation is governed by the incompressible dimensionless Navier-Stokes-Coriolis equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \frac{1}{\text{Re}} \Delta u + \frac{1}{\text{Ro}} \mathbf{e}_3 \times u + \frac{1}{\text{Ro}} \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases} \quad (\text{VI.1})$$

in a domain  $\Omega$  which will be specified later. The angular velocity of the Earth is assumed to be the unit vector in the vertical direction  $\mathbf{e}_3$ . Here we are interested in one particular type of solutions to the incompressible Navier-Stokes equations, the so-called *Leray weak solutions*. Let us give a definition of the Leray weak solutions to (VI.1) (for  $\text{Re} = \text{Ro} = 1$ ):

**Definition VI.1.** *Let  $\Omega$  be  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . We say that  $u$  is a finite energy weak solution or a Leray weak solution to (VI.1) on  $[0, T] \times \Omega$  with initial data  $u_0 \in L^2_\sigma(\Omega)$ , where  $L^2_\sigma(\Omega)$  denotes the space of  $L^2(\Omega)$  divergence-free functions, if for some  $T > 0$*

$$u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_\sigma(\Omega))$$

with

$$\|u(t, \cdot) - u_0(\cdot)\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow 0$$

and, for any function  $\phi \in \mathcal{C}_0^\infty([0, T] \times \Omega)$  with  $\text{div} \phi = 0$ , the following holds for almost all  $t \in (0, T)$ :

$$\begin{aligned} & \int_\Omega (u \cdot \phi)(t, x) dx + \int_0^t \int_\Omega (\nu \nabla u : \nabla \phi - u \otimes u : \nabla \phi - u \cdot \partial_t \phi)(\tau, x) dx d\tau \\ &= \int_0^t \int_\Omega (\mathbf{e}_3 \times u \cdot \phi)(\tau, x) dx d\tau + \int_\Omega u_0(x) \cdot \phi(0, x) dx. \end{aligned} \quad (\text{VI.2})$$

Moreover,  $u$  satisfies the energy inequality

$$\frac{1}{2} \|u(t, \cdot)\|_{L^2}^2 + \nu \int_0^t \|\nabla u(\tau, \cdot)\|_{L^2}^2 d\tau \leq \frac{1}{2} \|u_0\|_{L^2}^2 \quad (\text{VI.3})$$

for almost all  $t \in (0, T)$ . The weak solution  $u$  is said to be global if (VI.2) holds for every  $T > 0$ .

The weak formulation (VI.2) reduces to the Navier-Stokes weak formulation in the absence of the Coriolis term  $\mathbf{e}_3 \times u$  and  $u$  is the Leray-type weak solutions to the incompressible Navier-Stokes equations. Leray in his celebrated paper [85] proved the existence of global weak solutions to the Navier-Stokes equations introducing the  $L^2$  projector onto divergence-free vector fields

$$\mathbb{P} = \text{Id} - \Delta^{-1} \nabla \nabla \cdot .$$

The fact that the operator  $\mathbb{P}(\mathbf{e}_3 \times \cdot)$  is skew symmetric implies that  $\int \mathbb{P}(\mathbf{e}_3 \times u) \cdot u = 0$ . Hence, if the initial data belongs to  $L^2_\sigma(\mathbb{R}^3)$ , then there exists a Leray-type solution  $u$  to (VI.1) in  $L^\infty(\mathbb{R}_+; L^2_\sigma(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^1_\sigma(\mathbb{R}^3))$ .

Since we are interested in the behavior of the solutions in the high rotation limit, we assume now that the Rossby number is very small, *i.e.*  $\text{Ro} = \varepsilon$  and we deal with the limit  $\varepsilon \rightarrow 0$ . The mathematical results presented in this section have been obtained considering as governing equations either (VI.1) with  $\text{Ro} = \varepsilon$  and  $1/\text{Re}$  equal to a numerical constant  $\nu$  or

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon + \frac{1}{\varepsilon} \mathbf{e}_3 \times u^\varepsilon + \nabla p = 0 \\ \nabla \cdot u = 0. \end{cases} \quad (\text{VI.4})$$

The difference lies in the scaling introduced for the pressure term, which can be chosen differently if the compressible effects compensate the Coriolis force or not. We treat separately the cases when the initial data are *well-prepared* and *ill-prepared*. In the first case, the initial data  $u_0^\varepsilon$  converge to a limit  $u_0$  which belongs to the kernel of  $\mathbb{P}(\mathbf{e}_3 \times \cdot)$ . An equivalent of formulation is that the limit  $u_0$  depends only on the horizontal variables. Roughly speaking, the independence on the vertical variable of  $u_0$  permits to get rid of the oscillation part of the solutions.

In the ill-prepared case this is no more possible and fast waves, called *Rossby waves* for the atmosphere and *Poincaré waves* for the oceans, propagate. The difficulty of this case lies in the fact that two time scales need to be considered. For more details we refer to the monograph [28] by Chemin, Desjardins, Gallagher and Grenier.

### The well-prepared case

Colin and Fabrie considered in [29] the problem (VI.4) assuming periodic boundary conditions and anisotropic viscosity with vanishing vertical viscosity of the order of the Rossby number. They established long time existence to (VI.4) and the convergence to the two-dimensional Navier-Stokes equations. Moreover, the authors treated also the horizontal periodic case  $\mathbb{T}^2 \times (0, 1)$  with a stress on the boundary

due to the wind. They showed the convergence of the solutions to the ones of the two-dimensional Navier-Stokes equations with a stress-dependent source term for well-prepared initial data.

In the pioneering paper [66] Grenier and Masmoudi addressed the convergence of the solution as the Rossby and the vertical Ekman numbers go to zero. They showed the apparition of the Ekman boundary layers  $u_{bl}^\varepsilon(x_h, x_3/\varepsilon)$  due to the no-slip boundary condition and the convergence of the solutions to the ones of the two-dimensional damped Navier-Stokes equations (V.21).

The case of rough horizontal boundaries was treated by Gérard-Varet in [60]. The roughness and the viscosity was assumed of the same order of the Rossby number. He showed that the limit system is a two-dimensional Euler equation with a nonlinear damping term due to boundary layers.

### The ill-prepared case

The *ill-prepared* case, in which the initial data do not belong to the kernel of  $\mathbb{P}$ , was studied in the periodic space  $\mathbb{T}^3$  by Grenier in [65]. He showed that solutions to (VI.4) converge weakly to solutions of the 2D-3C (two dimensions and three components) Navier-Stokes equations. He used the *filtering method*, which consists in getting rid of the penalization term by filtering out the oscillations. This was done by composing the solution with the Coriolis semigroup  $\exp(tL/\varepsilon)$  with  $L = \mathbb{P}(\mathbf{e}_3 \times \cdot)$ .

Babin, Mahalov and Nikolaenko in [10] also addressed the periodic case with a non-resonance assumption on the domain. They showed that the solutions can be approximated by the sum of a solution of the 2D-3C Navier-Stokes equations and a vector field associated with the solution of some three-dimensional extended Navier-Stokes equations. The global existence of regular solutions was showed and this result was extended to resonant domain in [11].

The study of the singular limit when the domain is the whole space  $\mathbb{R}^3$  was principally investigated by Chemin, Desjardins, Gallagher and Grenier in [26] and [28]. In the first work they considered anisotropic viscosity (with  $\nu_3 \geq 0$ ) and in the second work isotropic viscosity. In both cases the ill-prepared case is considered. The global well-posedness is shown for small  $\varepsilon$  in anisotropic Sobolev spaces introduced by Iftimie in [71]. The convergence to the solution to the two-dimensional Navier-Stokes equations is proved by introducing the associated linear problem. The result is obtained using Strichartz-type anisotropic dispersive estimates which are derived by writing the solution of the linearized problem in Fourier space.

Gallagher and Saint-Raymond in [59] address a generalization of the problem by taking into account the variations of the rotation angular velocity. This has geophysical applications since the vertical component of the planetary rotation depends on the latitude. The authors considered the domain  $\Omega_h \times \Omega_3$  with  $\Omega_h =$

$\mathbb{R}^2, \mathbb{T}^2$  and  $\Omega_3 = \mathbb{R}, \mathbb{T}$ . In the case where some non-degeneracy condition on the inhomogeneous angular velocity holds, the solutions weakly converge to zero if the domain is  $\mathbb{R}^3$  and to a vector field whose horizontal component satisfies a two-dimensional heat equation if the domain is  $\mathbb{R}^2 \times \mathbb{T}$ . Since the singular perturbation operator has variable coefficients, the authors had to resort to compensated compactness arguments to prove the convergence of the nonlinear terms. We refer also to [35] for the study of rotating incompressible fluids with an inhomogeneous rotation axis in the thin layer  $\mathbb{T} \times \mathbb{R} \times (0, 1)$ . In [35] Dalibard and Saint-Raymond showed that the stability of stationary solutions amounts to describing the behavior of waves under a particular horizontal variation of the axis, the so-called  $\beta$ -plane model, with a thin layer effect. Differently from the whole-space model (see [58]), dispersion occurs and takes place on a time scale much larger (group velocity of order  $O(1)$ ) than usual (group velocity of order  $O(1/\varepsilon)$ ).

The convergence result in [66], detailed in the previous paragraph, was extended to the ill-prepared case by Masmoudi in [91] and by Chemin, Desjardins, Gallagher and Grenier in [27]. The first work considered the horizontal periodic domain  $\mathbb{T}^2 \times (0, 1)$ . In addition, the author showed the inviscid limit provided a smallness condition and this result was generalized by Masmoudi and Rousset in [92]. The second work treated the infinite slab  $\mathbb{R}^2 \times (0, 1)$ .

The cylindrical configuration  $S \times (0, 1)$ , with  $S$  a bounded domain in  $\mathbb{R}^2$ , has been studied by Bresch, Desjardins and Gérard-Varet in [18]. Assuming no-slip at the boundaries and vanishing vertical viscosity, the authors showed the convergence to the solution of the two-dimensional damped Navier-Stokes equations (V.21) in  $S$  with no-slip boundary condition. A spectral condition on the operator  $\mathbb{P}(\mathbf{e}_e \times \cdot)$  and a geometrical assumption on the cylinder section are considered. This second assumption ensures the damping of the oscillations in the viscous boundary layer near the lateral sides of the cylinder.

Let us remark that the horizontal infinite slab  $\mathbb{R}^2 \times (0, 1)$  is particularly adapted to the study of large-scale flows. Indeed, as we have presented before, the motion of fluids in oceans and the atmosphere at large scales takes place in configurations where the horizontal length scale is much bigger than the vertical one. This anisotropic structure motivates the choice of this domain.

## VI.2 Low Mach and high rotation limit of the compressible Navier-Stokes-Coriolis equations

The study of the double low Mach and high rotation limit for the compressible Navier-Stokes-Coriolis equations is more difficult. Only in recent years some progress has been made. The motion of a compressible barotropic viscous rotating

fluid, after a suitable scaling, is governed by the dimensionless equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \frac{1}{\text{Re}} \nabla \cdot \mathbb{S}(u) + \frac{1}{\text{Ro}} \mathbf{e}_3 \times (\rho u) + \frac{1}{\text{Ma}^2} \nabla p(\rho) = \frac{1}{\text{Fr}^2} \rho \nabla \Phi. \end{cases} \quad (\text{VI.5})$$

where  $\mathbb{S}(u)$  is the viscous *stress tensor* and  $\Phi$  is the potential of some external force, such as the gravitational or the centrifugal force. We distinguish the cases with isotropic and anisotropic viscosity.

### Isotropic viscosity

In the case when the viscosity is isotropic, the stress tensor is assumed to be the *symmetric isotropic stress tensor*

$$\mathbb{S}(u) = \mu (\nabla u + \nabla^T u) + \lambda \nabla \cdot u \text{Id}, \quad (\text{VI.6})$$

where the constant  $\lambda = -\frac{2}{3}\mu + \eta$  contains the so-called bulk viscosity  $\eta \geq 0$ , which is related to the vibrational energy of the molecules. Let us deal with the theory of existence of weak solutions to (VI.5) without the Coriolis term and using the stress tensor (VI.6). The isotropic compressible barotropic Navier Stokes equations (for  $\text{Re} = \text{Ma} = \text{Fr} = 1$ ) in  $\Omega = \mathbb{R}^2, \mathbb{R}^3$  read

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \nabla \cdot u + \nabla p(\rho) = \rho \nabla \Phi. \end{cases} \quad (\text{VI.7})$$

The theory developed by Lions states that, assuming a power-law for the pressure, *i.e.*  $p(\rho) = a\rho^\gamma$  with  $a > 0$ , there exist global weak solutions to (VI.7) satisfying the energy inequality

$$\begin{aligned} & \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + E(\rho, \bar{\rho}) \right) (t, x) dx + \int_0^t \int_{\Omega} \left( \mu |\nabla u|^2 + (\lambda + \mu) |\nabla \cdot u|^2 \right) (\tau, x) dx d\tau \\ & \leq \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + E(\rho_0, \bar{\rho}) \right) (x) dx + \int_0^t \int_{\Omega} \rho \nabla \Phi \cdot u(\tau, x) dx d\tau. \end{aligned} \quad (\text{VI.8})$$

for almost all  $t \in (0, T)$  and for all  $T > 0$  with a limitation on the adiabatic constant  $\gamma$ . The relative energy  $E(\rho, \bar{\rho})$ , where  $\bar{\rho}$  stands for the (constant) density limit at infinity, is defined by

$$E(\rho, \bar{\rho}) = H(\rho) - H'(\bar{\rho})(\rho - \bar{\rho}) - H(\bar{\rho}) \quad (\text{VI.9})$$

with the entropy  $H(\rho)$  given by

$$H(\rho) = \rho \int_{\rho_{\text{ref}}}^{\rho} \frac{p(s)}{s^2} ds.$$

The rigorous definition of Leray-type weak solutions for the compressible case will be stated in Chapter 3. Lions in [86] established the global existence asking  $\gamma > 9/5$ . In [52] Feireisl, Novotný and Petzeltova introduced the necessary modifications to extend the approach of Lions in order to cover the range  $\gamma > 3/2$  and to include the larger class of physically relevant pressure laws

$$p \in \mathcal{C}^1([0, +\infty)), \quad p(0) = 0, \quad p'(\rho) > 0 \text{ for } \rho > 0, \quad \lim_{\rho \rightarrow \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = p_\infty > 0. \quad (\text{VI.10})$$

Let us now focus on highly rotating compressible viscous fluids. We assume now the Rossby and the Mach number to be comparably small, *i.e.*  $\text{Ro} = \text{Ma} = \varepsilon$ , and we deal with the limit  $\varepsilon \rightarrow 0$ . The equations (VI.5) take the form

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0 \\ \partial_t (\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) \\ - \frac{1}{\text{Re}} (\mu \Delta u^\varepsilon + (\lambda + \mu) \nabla \nabla \cdot u^\varepsilon) + \frac{1}{\varepsilon} \mathbf{e}_3 \times (\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon^2} \nabla p(\rho^\varepsilon) = \frac{1}{\text{Fr}^2} \rho^\varepsilon \nabla \Phi. \end{cases} \quad (\text{VI.11})$$

We consider the equations in the domain  $\mathbb{R}^2 \times (0, 1)$ . In all the works presented here the authors considered complete slip boundary conditions eliminating the effect of the Ekman layers. As for the incompressible case, we treat separately the well-prepared and the ill-prepared case.

**The ill-prepared case** Feireisl, Gallagher and Novotný studied in [47] the singular limit problem (VI.11) for a pressure of the type (VI.10). They neglected the action of both gravitational and centrifugal forces. They showed the convergence of solutions to (VI.11) towards solutions to a two-dimensional viscous quasi-geostrophic equation for the limit density. They used the decomposition of the density in an *essential* and a *residual part*, already introduced in [49]. In the limit  $\varepsilon \rightarrow 0$  the residual part disappears and only the essential part is relevant. In [51, 50] Feireisl and Novotny treated also the inviscid limit, assuming  $1/\text{Re} = \varepsilon$  and using the same viscous stress tensor and complete slip boundary conditions. In [51], the authors showed that the limit dynamics is given by a horizontal motion for an incompressible inviscid fluid governed by a damped variant of the planar incompressible Euler system. In [50] the stratification was introduced, with the Froude number of the same order of the Rossby number, and multiple scalings were investigated. In both works the approach used is based on the concept of finite energy weak solutions satisfying the relative entropy inequality (see [48, 61]). The effects of the centrifugal force have been considered in [46]. The authors considered the periodic domain  $\mathbb{R}^2 \times \mathbb{T}$  and the configuration when the Rossby and



Froude number are comparably small but the Mach number can be at a different order of magnitude. One of the most difficult issue is that acoustic waves do not decay at space infinity due to the strong centrifugal force and therefore Strichartz-type dispersive estimates in [28] can not be used. In the case when the Rossby number dominates the Mach number, the limit problem is given by the two-dimensional incompressible Navier-Stokes equations describing the horizontal motion of the vertically averaged velocity. If they are at the same order, the singular perturbation operator has variable coefficients and compensated compactness arguments (as in [59]) are used to pass to the limit. The convergence to radially symmetric solutions to a linear two-dimensional equation was shown.

Recently, Fanelli in [43] proved a similar result to the one established in [47] for the scaled Navier-Stokes-Korteweg equations with Coriolis force, which differs from (VI.11) by a capillarity term. The author considered different rates wherewith the capillarity coefficient goes to zero.

**The well-prepared case** As a continuation of the works [51, 50] in the context of strong stratification, Feireisl, Lu and Novotny in [54] studied the limit  $\varepsilon \rightarrow 0$  and  $\text{Re} \rightarrow \infty$  when the Mach and the Froude number are of the same order. Although the static density profile depends on the vertical direction, in the well-prepared case the limit system is the same as in the problem without stratification in [51]. Moreover, they obtained an exact convergence rate in terms of the singular parameters  $\varepsilon$  and  $1/\text{Re}$ .

### Anisotropic viscosity

As we have explained in Section V.1 small-scale turbulent flows drain energy from large-scale flows and these dissipative effects are taken into account by the Reynolds stresses. They are assumed to be linearly dependent of the derivatives of the large-scale velocity and some turbulent (or eddy) viscosity is added in the equations. Hence the viscosity is anisotropic. This is more relevant than the molecular viscosity  $\mu$  which can be neglected. Hence, in order to capture most properties of large-scale motions, the previous stress tensor needs to be replaced by the anisotropic stress tensor

$$\mathbb{S}(u) = A_{\nu_h, \nu_3} \nabla u + \lambda \nabla \cdot u \text{Id} \quad (\text{VI.12})$$

with the anisotropic matrix

$$A_{\nu_h, \nu_3} = \begin{pmatrix} \nu_h & 0 & 0 \\ 0 & \nu_h & 0 \\ 0 & 0 & \nu_3 \end{pmatrix}. \quad (\text{VI.13})$$

Using this expression for  $\mathbb{S}(u)$  and neglecting the Coriolis force and the influence of any external force, equations (VI.5) take the form (for  $\text{Re} = \text{Ro} = \text{Ma} = \text{Fr} = 1$ )

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) - \nu_h \Delta_h u - \nu_3 \partial_3^2 u - \lambda \nabla \nabla \cdot u + \nabla p(\rho) = 0 \end{cases} \quad (\text{VI.14})$$

One may choose the fully symmetric<sup>4</sup> anisotropy replacing  $\nabla u$  by  $\nabla u + \nabla u^T$  in (VI.12), but in the momentum equation the diffusion terms take the form  $\nu_h \Delta_h u + \nu_3 \partial_3^2 u + \nu_3 \nabla \partial_3 u_3$ . It is immediate that the choice of the non-symmetric anisotropic stress tensor permits to recover the same viscous terms as in (V.9).

Although one may consider power pressure laws, the theory of Lions and Feireisl fails in the anisotropic case (VI.14) due to a loss of structure. More precisely, the failure happens in the arguments to control the oscillations of the density. The positivity of some term can not be guaranteed in the anisotropic case and the strong compactness of the density can not be obtained. For more details we refer to Section 3.1 of [18] and Section 2.2 of [19]. The unique result of well-posedness of (VI.14) was recently established by Bresch and Jabin in [19]. Considering the domain  $\mathbb{T}^3$  (this choice is only to avoid discussing boundary conditions and the result can be extended to other cases), they proved the existence of global weak solutions to (VI.14) in the three-dimensional case for

$$\gamma > 2 + \frac{\sqrt{10}}{2}$$

assuming some restrictions on the viscosities. In their paper, the authors proved the result in any dimension<sup>5</sup>. Moreover, they also extended the theory of Lions for an isotropic viscosity to a larger class of barotropic pressure laws than the power laws.

Let us deal with singular limit problems in the anisotropic case. Writing the dimensionless equations (VI.5) using (VI.12), we get

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) \\ -\frac{\nu_h}{\text{Re}} \Delta_h u - \frac{\nu_3}{\text{Re}} \partial_3^2 u - \frac{\lambda}{\text{Re}} \nabla \nabla \cdot u + \frac{1}{\text{Ro}} \mathbf{e}_3 \times (\rho u) + \frac{1}{\text{Ma}^2} \nabla p(\rho) = \frac{1}{\text{Fr}^2} \rho \nabla \Phi. \end{cases} \quad (\text{VI.15})$$

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<sup>4</sup>This is the anisotropic case in linear elasticity and the theory of Bresch and Jabin in [19] can be extended to this symmetric case.

<sup>5</sup>Bresch and Jabin proved in [19] the existence of global weak solutions for  $\gamma > \frac{d}{2} \left( 1 + \frac{1}{d} + \sqrt{1 + \frac{1}{d^2}} \right)$  where  $d$  is the dimension of the problem

Recalling that  $\text{Ek} = \frac{\text{Ro}}{\text{Re}}$ , the coefficient of  $\partial_3^2 u$  takes the form  $\nu_3 \frac{\text{Ek}}{\text{Ro}}$ . In the high rotation case, expressed by taking  $\text{Ro} = \varepsilon$  for a small parameter  $\varepsilon$ , the Ekman number is very small and of order  $O(\varepsilon^2)$  (see [66]). The ratios containing the horizontal and bulk viscosities are both of order  $O(1)$  and can be assumed equal to some positive constants. We denote them by  $\tilde{\mu}$  and  $\tilde{\lambda}$  respectively. Therefore, assuming that the Mach and the Froude numbers are at the same order as the Rossby number, the equations (VI.15) become

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0 \\ \partial_t (\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) \\ -\tilde{\mu} \Delta_h u^\varepsilon - \varepsilon \partial_3^2 u^\varepsilon - \tilde{\lambda} \nabla \nabla \cdot u^\varepsilon + \frac{1}{\varepsilon} \mathbf{e}_3 \times (\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon^2} \nabla p(\rho^\varepsilon) = \frac{1}{\varepsilon^2} \rho^\varepsilon \nabla \Phi. \end{cases} \quad (\text{VI.16})$$

To the best of our knowledge, the singular limit problem for the anisotropic case (VI.14) when  $\varepsilon$  goes to zero was only investigated by Bresch, Desjardins and Gérard-Varet in [18] in the case when the domain is a cylinder  $S \times (0, 1)$  and the effects of external forces are neglected. Considering a power pressure law, they proved a stability result for  $\gamma = 2$  in the well-prepared case. More precisely, they showed that global weak solutions to (VI.16) (whose existence for small enough  $\varepsilon$  is assumed) are a priori close to the unique regular solution of a two-dimensional quasi-geostrophic equation, with a term taking into account the compressibility of the fluid. Since the difference between the solution to the original equations and the solution to the limit system is small, the convergence can be obtained following [57]. In the ill-prepared case, they showed that strong convergence can be expected under the same assumptions as in the incompressible case.

## VII Contributions of the thesis

Motivated by the works [18] and [54], we aim at going deeper into the analysis of highly rotating compressible fluids. The mathematical problem treated in this manuscript permits to describe the most physical features of large-scale fluid motions under the influence of the Earth's rotation and the gravitational force. We take into account the fact that for highly rotating fluids anisotropy occurs, in particular the vertical viscosity is very small and of the same order as the Rossby number. We consider the no-slip boundary condition. This causes the apparition of horizontal boundary layers close to the bottom and the top of the domain  $\mathbb{R}^2 \times (0, 1)$ . We are interested in more general pressure laws than the ones considered in [18], namely  $p \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^3((0, \infty))$  satisfying

$$p(0) = 0, \quad p'(\rho) > 0 \quad \forall \rho > 0, \quad \lim_{\rho \rightarrow \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = a > 0.$$

Moreover, we take into account the stratification effects of the gravitational force on the density profile as in [54], taking the Froude number of the same order as the Rossby number. As a first step, we treat the well-prepared case avoiding the analysis of the propagation of acoustic waves. A perspective of this work is to handle also ill-prepared initial data.

Let us introduce the equations we are considering. The motion of compressible highly rotating viscous fluids is governed by the compressible barotropic Navier-Stokes-Coriolis equations

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t (\rho^\varepsilon u^\varepsilon) + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \Delta_{\mu,\varepsilon} u^\varepsilon - \lambda \nabla (\nabla \cdot u^\varepsilon) \\ \quad + \frac{1}{\varepsilon} e_3 \times (\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon^2} \nabla p(\rho^\varepsilon) = \frac{1}{\varepsilon^2} \rho^\varepsilon \nabla G. \end{cases} \quad (\text{VII.1})$$

The small parameter  $\varepsilon$  denotes the Rossby number. The Mach and the Froude number are assumed to be of the same order as the Rossby number, *i.e.*  $\text{Ma} = \text{Fr} = \varepsilon$ , meaning that the effects of the gravity are balanced by the effects of the compressibility. We denote by  $\Delta_{\mu,\varepsilon}$  the anisotropic Laplace operator  $\mu \Delta_h + \varepsilon \partial_3^2$ . The fluid moves in the horizontal infinite slab  $\Omega = \mathbb{R}^2 \times (0, 1)$  under the influence of the gravitational force  $\nabla G = (0, 0, -1)^T$ . We emphasize that no-slip boundary conditions are supposed at the bottom and the top. We are interested in the asymptotic behavior of the solutions in the limit  $\varepsilon \rightarrow 0$ .

**Main result of Chapter 3** We state here the main result of Chapter 3. This is a work in collaboration with Francesco Fanelli and Christophe Prange. It consists in a stability result for weak solutions to (VII.1) that makes the convergence quantitative and shows the structure of the solutions.

**Theorem VII.1.** *Let  $\rho_0^\varepsilon = \bar{\rho}(x_3) + \varepsilon r_0^\varepsilon$  with  $(r_0^\varepsilon)_\varepsilon$  bounded in  $L^2 \cap L^\infty$  and  $(u_0^\varepsilon)_\varepsilon$  bounded in  $L^2$ . Assume that there exists a  $q_0 \in H^5(\mathbb{R}^2)$  such that  $(r_0^\varepsilon, u_0^\varepsilon) \rightarrow (r_0, u_0)$  in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$  with*

$$r_0 = \frac{\bar{\rho}}{p'(\bar{\rho})} q_0 \quad \text{and} \quad u_0 = \left( \nabla_h^\perp q_0, 0 \right).$$

*Let  $(\rho^\varepsilon, u^\varepsilon)$  be a finite energy weak solution (see Chapter 3) with initial data  $(\rho_0^\varepsilon, u_0^\varepsilon)$  and  $(\rho_{app}^\varepsilon, u_{app}^\varepsilon)$  an approximated solution to (VII.1). We denote the difference by  $\delta u^\varepsilon$ . For  $\gamma \geq 3/2$  there exist functions  $C_1(t), C_2(t) \in L^1([0, T])$  for all  $T > 0$  and constants  $C > 0$  and  $\varepsilon_0 \in (0, 1)$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ , the following estimate*

holds:

$$\begin{aligned} & \int_{\Omega} \rho^\varepsilon(t) |\delta u^\varepsilon(t)|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon(t), \rho_{app}^\varepsilon(t)) dx \\ & \quad + \int_0^t \int_{\Omega} \mu |\nabla_h \delta u^\varepsilon|^2 + \varepsilon |\partial_3 \delta u^\varepsilon|^2 + \lambda |\nabla \cdot \delta u^\varepsilon|^2 dx \\ & \leq C e^{\int_0^t C_1(s) ds} \left( \int_{\Omega} \rho_0^\varepsilon |\delta u_0^\varepsilon|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho_0^\varepsilon, \rho_{0,app}^\varepsilon) dx + \varepsilon \int_0^t e^{\int_\tau^t C_1(s) ds} C_2(\tau) d\tau \right). \end{aligned} \tag{VII.2}$$

for almost every  $t > 0$ , where  $E(\rho^\varepsilon, \rho_{app}^\varepsilon)$  is the relative energy (VI.9). Moreover, the formal limit velocity is  $u = (\nabla_h^\perp q, 0)$  where  $q$  is the unique regular solution to the quasi-geostrophic equation

$$\partial_t \left( \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle q - \langle \bar{\rho} \rangle \Delta_h q \right) - \langle \bar{\rho} \rangle \nabla_h^\perp q \cdot \nabla_h \Delta_h q + \mu \Delta_h^2 q - \frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} \Delta_h q = 0 \tag{VII.3}$$

with initial datum  $q_0$ .

We remark that the condition assumed on the initial data means that initial data are well-prepared. This result extends the analysis presented in [18] to the range  $\gamma \geq 3/2$ . We remark that surprisingly the critical exponent  $\gamma = 3/2$  is the same as the critical exponent of the weak solutions theory of Lions and Feireisl.

The approximate solution  $(\rho_{app}^\varepsilon, u_{app}^\varepsilon)$  is constructed by making an ansatz on the solutions. The term  $u_{app}^\varepsilon$  contains corrective boundary layers terms. In order to derive the quantitative estimate (VII.2), we resort to the relative entropy method. The decomposition of the density into essential and residual parts in [47, 49] is adapted.

The main issue is the impossibility to provide a good control for  $\partial_3 u_h$  due to the anisotropy of the viscosity. The key point to overcome this difficulty is the introduction of an anisotropic version of the standard Sobolev embedding  $\dot{H}^1 \hookrightarrow L^6$  when  $\gamma$  is smaller than the critical value  $\gamma = 2$ . This enables to handle the anisotropy of the viscosity. We refer to Lemma 3.3.5 for the statement of the anisotropic embedding.

The limit system (VII.3) is the viscous quasi-geostrophic equation shown in [18] with a damping term describing the phenomenon of the Ekman pumping due to the presence of Ekman boundary layers. The  $H^5$ -regularity of the initial datum  $q_0$  is assumed in order to have the necessary regularity of  $q$  required for the control of the source term in the relative entropy inequality.

**Perspectives** The perspective of this work is to extend the same analysis to the case when the effects of both Ekman layers and centrifugal force are considered and to compare the limit dynamics with the one shown in [46]. In this case we

expect the limit density profile to depend only on the horizontal variables, the limit velocity to be radially symmetric and the limit equation to be linear with a damping term which takes into account the boundary layers.

### Outline

Chapter 3 is based on [16]. In Section 3.1 the problem is presented. The definition of suitable weak solutions and the definition of the entropy inequality are given. In Section 3.2 the formal ansatz is constructed by expanding the solution in powers of the parameter  $\varepsilon$  and taking into consideration boundary layers terms for the density and for the velocity vector field. The equations for the different profiles are formally derived by injecting the ansatz in the equations (VII.1) and the interior terms are treated separately from the boundary layers terms. The profiles are identified and the quasi-geostrophic equation (VII.3) for the limit velocity is derived. The ansatz for the structure of the solutions being obtained, the equations satisfied by the approximate solutions are written. Section contains the core of our work. After recalling the uniform bounds for weak solutions, the stability result based on the relative entropy estimate is proved. The estimates on the difference between the solutions and the approximate solutions are proved. The rest of the section is devoted to the control of the source terms appearing in the inequality, which permits to get the quantitative result (VII.2). In Appendix E the well-posedness of the quasi-geostrophic equation (VII.3) is shown and the global regularity is proved for data in Sobolev spaces with integer order.



## CHAPTER 3

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# Ekman layers for highly rotating compressible fluids

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*This work is in collaboration with Francesco Fanelli and Christophe Prange.  
It is based on [16].*

### 3.1 Presentation of the problem

This chapter is devoted to the study of the compressible barotropic Navier-Stokes system with fast rotation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \frac{1}{\varepsilon} e_3 \times \rho u + \frac{1}{\varepsilon^2} \nabla p(\rho) = \Delta_{\mu, \varepsilon} u + \lambda \nabla (\nabla \cdot u) + \frac{1}{\varepsilon^2} \rho \nabla G, \end{cases} \quad (3.1.1)$$



with the gravitational potential  $\nabla G = (0, 0, -1)^T$ . Here we consider the scaling  $\text{Re} = 1$ ,  $\text{Ek} = \varepsilon^2$  and  $\text{Ro} = \text{Ma} = \text{Fr} = \varepsilon$  with  $\varepsilon > 0$  a small parameter. The previous dimensionless numbers, the physical aspects and motivations related to the equations (3.1.1) are discussed in the introduction of the second part of this manuscript.

The previous system of equations is considered on  $(0, \infty) \times \Omega$ , where  $\Omega := \mathbb{R}^2 \times (0, 1)$ , and we supplement it with no-slip boundary conditions for  $u$  on  $\partial\Omega$ :

$$u = 0 \quad \text{at} \quad x_3 = 0, 1. \quad (3.1.2)$$

We will specify later on the hypotheses on the initial conditions at time  $t = 0$ :

$$\rho|_{t=0} = \rho_0 \quad \text{and} \quad u|_{t=0} = u_0.$$

For the time being, let us just say that we impose (in a sense to be clarified later) the following *far field conditions*, for  $|x| \rightarrow \infty$ :

$$\rho_0(x) \longrightarrow \bar{\rho}(x_3) \quad \text{and} \quad u_0(x) \longrightarrow 0,$$

where  $\bar{\rho}$  is a strictly positive function, and we consider initial densities which are far away from vacuum.

In System (3.1.1), the function  $\rho = \rho(x, t) \in \mathbb{R}$  represents the density of the fluid,  $u = u(x, t) \in \mathbb{R}^3$  denotes its velocity field and  $p$  the pressure of the fluid. The first two terms on the right hand side of the second equation in (3.1.1) represent the effect of the viscosity. We have denoted by  $\Delta_{\mu, \varepsilon}$  the anisotropic Laplace operator

$$\mu\Delta_h + \varepsilon\partial_3^2 = \mu(\partial_1^2 + \partial_2^2) + \varepsilon\partial_3^2.$$

The parameters  $\mu$  and  $\lambda$  are dimensionless numbers which satisfy  $\mu, \lambda > 0$  (see Introduction). The vector  $e_3$  is the third unit vector of the canonical basis of  $\mathbb{R}^3$ , and the notation  $e_3 \times v = (-v_2, v_1, 0)$  represents the usual cross product in  $\mathbb{R}^3$ . Let us introduce the following notation:

- For a vector  $x \in \mathbb{R}^3$ , the notation  $x = (x_h, x_3) \in \mathbb{R}^3$  denotes the horizontal component  $x_h \in \mathbb{R}^2$  and the vertical component  $x_3 \in \mathbb{R}$
- The differential operators  $\nabla_h$ ,  $\Delta_h$  and  $\nabla_h \cdot$  as the usual operators, but acting just with respect to the  $x_h$  variables.
- We set  $\nabla_h^\perp := (-\partial_2, \partial_1)$  and, more in general, given a two-dimensional vector  $v = (v_1, v_2)$ , we define  $v^\perp := (-v_2, v_1)$ .
- Given a Banach space  $X$  and a sequence  $(a^\varepsilon)_\varepsilon$  of elements of  $X$ ,  $(a^\varepsilon)_\varepsilon \subseteq X$  means that  $(a^\varepsilon)_\varepsilon$  is uniformly bounded in  $X$ .

- For any  $p \in [1, \infty]$  and any Banach space  $X$ ,  $L_T^p(X) := L^p((0, T); X(\Omega))$ .

We consider here barotropic flows, for which the pressure function  $p$  is supposed to be a smooth function of the density only. More precisely, we assume (see e.g. [53], [47], [48], [50]) that  $p \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^3((0, \infty))$  satisfies

$$p(0) = 0, \quad p'(\rho) > 0 \quad \forall \rho > 0, \quad \lim_{\rho \rightarrow \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = a > 0, \quad (3.1.3)$$

for some  $\gamma > 3/2$ . Given the pressure function, we define the *internal energy function*  $H$  by the formula

$$H(\rho) := \rho \int_1^\rho \frac{p(z)}{z^2} dz \quad \text{for all } \rho \in (0, \infty).$$

Notice that, with the previous definition, one has, for all  $\rho > 0$ , the relation

$$\rho H''(\rho) = p'(\rho).$$

Although all the quantities  $\rho = \rho^\varepsilon$  and  $u = u^\varepsilon$  depend on  $\varepsilon$  we often omit to write this dependence explicitly. Let  $\bar{\rho} = \bar{\rho}(x_3) > 0$  satisfy the logistic equation

$$\nabla p(\bar{\rho}) = \bar{\rho} \nabla G. \quad (3.1.4)$$

The positivity of  $\bar{\rho}$  is required in order to avoid vacuum.

The goal is to study the asymptotics of the solutions of (3.1.1) when  $\varepsilon \rightarrow 0$  in the *well-prepared case*. Namely, we consider an initial density  $(\rho_0^\varepsilon)_\varepsilon$  and initial velocity fields  $(u_0^\varepsilon)_\varepsilon$  such that the following requirements are satisfied:

- for all  $\varepsilon \in (0, 1]$ , one has

$$\rho_0^\varepsilon = \bar{\rho} + \varepsilon r_0^\varepsilon, \quad \text{with } (r_0^\varepsilon)_\varepsilon \subseteq (L^2 \cap L^\infty)(\Omega);$$

- we have  $(u_0^\varepsilon)_\varepsilon \subseteq L^2(\Omega)$ ;
- there exists  $q_0 \in H^5(\mathbb{R}^2)$  such that for

$$r_0 := \frac{\bar{\rho}}{p'(\bar{\rho})} q_0, \quad u_0 := (-\partial_2 q_0, \partial_1 q_0, 0) \quad (3.1.5)$$

we have

$$r_0^\varepsilon \longrightarrow r_0 \quad \text{in } L^2(\Omega), \quad (3.1.6)$$

$$u_0^\varepsilon \longrightarrow u_0 \quad \text{in } L^2(\Omega). \quad (3.1.7)$$

**Remark 3.1.1.** Notice that the condition (3.1.5) implies that

$$\bar{\rho} \begin{pmatrix} u_{0,h}^\perp \\ 0 \end{pmatrix} + \begin{pmatrix} p'(\bar{\rho}) \nabla_h r_0 \\ \partial_3(p'(\bar{\rho}) r_0) \end{pmatrix} = r_0 \nabla G. \quad (3.1.8)$$

The fact that  $\rho_{0,\varepsilon}$  has to be a perturbation of  $\bar{\rho}$ , defined by (3.1.4), will be justified by the computations of Section 3.2. The fact that this perturbation has to be of order  $\varepsilon$  is related to the size of the Mach number, and is required in order to have the initial energy uniformly bounded. See Definition 3.1.2 and Section 3.3.1 below for more details.

### 3.1.1 Suitable weak solutions and entropy inequality

Following [47] and [46], we start by giving the definition of weak solutions we are interested in. These are weak solutions *à la Leray*: namely we include in the definition a finite energy condition, which will be important to derive (in Section 3.3.1) useful bounds, uniformly in  $\varepsilon \in (0, 1]$ , for the given family of solutions  $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$ .

**Definition 3.1.2.** We say that  $(\rho, u)$  is a finite energy weak solution to system (3.1.1) on  $[0, T] \times \Omega$ , related to the initial datum  $(\rho_0, u_0)$ , with

$$u_0 \in L^2(\Omega, \mathbb{R}^3), \quad \rho_0 - \bar{\rho} \in L^2(\Omega) \cap L^\infty(\Omega), \quad \rho_0 > 0,$$

if the following conditions are satisfied:

- $\rho > 0$ , with  $\rho - \bar{\rho} \in L^\infty((0, T); (L^2 + L^\gamma)(\Omega))$  for  $\gamma$  given by (3.1.3), and  $u \in L^2((0, T); H^1(\Omega; \mathbb{R}^3))$ ;
- the mass equation, i.e. the first equation in (3.1.1), is satisfied in the weak sense: namely, for any test-function  $\varphi \in C_0^\infty([0, T] \times \Omega)$ , one has

$$-\int_0^T \int_\Omega (\rho \partial_t \varphi + \rho u \cdot \nabla \varphi) dx dt = \int_\Omega \rho_0 \varphi(0) dx;$$

- $p(\rho) \in L_{loc}^1((0, T) \times \Omega)$ , and the momentum equation, i.e. the second relation in (3.1.1), is verified in the weak sense: for any test-function  $\psi \in C_0^\infty([0, T] \times \Omega; \mathbb{R}^3)$ , one has

$$\begin{aligned} & \int_0^T \int_\Omega \left( -\rho u \cdot \partial_t \psi - \rho u \otimes u : \nabla \psi + \frac{1}{\varepsilon} e^3 \times (\rho u) \cdot \psi - \frac{1}{\varepsilon^2} p(\rho) \nabla \cdot \psi \right. \\ & \left. + \nabla_{\mu,\varepsilon} u : \nabla_{\mu,\varepsilon} \psi + \lambda \nabla \cdot u \nabla \cdot \psi - \frac{1}{\varepsilon^2} \rho \nabla G \cdot \psi \right) dx dt = \int_\Omega \rho_0 u_0 \cdot \psi(0), \end{aligned}$$

where  $\nabla_{\mu,\varepsilon}$  denotes the anisotropic gradient operator  $(\sqrt{\mu} \partial_1, \sqrt{\mu} \partial_2, \sqrt{\varepsilon} \partial_3)$ ;

- *the energy inequality*

$$\begin{aligned} & \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\varepsilon^2} E(\rho(t), \bar{\rho}) \right) dx + \int_0^T \int_{\Omega} \left( \mu |\nabla_h u|^2 + \varepsilon |\partial_3 u|^2 + \lambda |\nabla \cdot u|^2 \right) dx dt \\ & \leq C \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{1}{\varepsilon^2} E(\rho_0, \bar{\rho}) \right) dx \end{aligned}$$

holds for almost every  $t \in (0, T)$ , where we have defined the relative energy functional

$$E(\rho, \bar{\rho}) := H(\rho) - H(\bar{\rho}) - H'(\bar{\rho})(\rho - \bar{\rho}). \quad (3.1.9)$$

The solution is said to be global if the previous conditions hold for all  $T > 0$ .

We also remark here that (see Section 2.3 of [50], [53] and [48] for details) any finite energy weak solution can be proven to satisfy a relative entropy inequality. Only recently the global existence of weak solutions to (3.1.1) with an anisotropic viscous stress tensor was obtained by Bresch and Jabin in [19]. The result holds for  $\gamma \geq 2 + \sqrt{10}/2$  and under the assumption that  $\lambda + \frac{2}{3}\mu > \mu - \varepsilon$ , which means that  $\lambda$  is large enough with respect to  $\mu$ . In this chapter we are interested in the study of the asymptotic behavior of the solutions and their structure. Hence, we suppose the existence of weak solutions to (3.1.1) without any particular assumption on the parameters  $\gamma$  and  $\lambda$ . However, the result of Bresch and Jabin does not apply here in full generality but their assumption on  $\lambda$  is reasonable and compatible with the anisotropic viscosity. To the best of our knowledge, the existence of weak solutions without these two hypotheses is still an open problem. To this end, in the last months Bresch and Burtea proved in [17] the global existence of weak solutions for the quasi-stationary compressible Stokes equations with an anisotropic viscous tensor, using an approach based on the control of some defect measure associated with the pressure.

## 3.2 Formal asymptotic expansions in the limit $\varepsilon \rightarrow 0$

In this section we perform formal computations in order to have a grasp on the structure of the solutions to system (3.1.1). Because of the no-slip boundary conditions (3.1.2), boundary layers appear in the limit  $\varepsilon \rightarrow 0$  both in the vicinity of the top boundary  $\mathbb{R}^2 \times \{1\}$  and of the bottom boundary  $\mathbb{R}^2 \times \{0\}$ .

### 3.2.1 Construction of the ansatz

We start with expanding the solution  $(u^\varepsilon, \rho^\varepsilon)$  to (3.1.1) as

$$\begin{aligned}
 u^\varepsilon &= u_0(x_h, x_3, t) + u_{0,b}^{bl}(x_h, \frac{x_3}{\varepsilon}, t) + u_{0,t}^{bl}(x_h, \frac{1-x_3}{\varepsilon}, t) \\
 &\quad + \varepsilon \left( u_1(x_h, x_3, t) + u_{1,b}^{bl}(x_h, \frac{x_3}{\varepsilon}, t) + u_{1,t}^{bl}(x_h, \frac{1-x_3}{\varepsilon}, t) \right) + O(\varepsilon^2) \\
 \rho^\varepsilon &= \rho_0(x_h, x_3, t) + \rho_{0,b}^{bl}(x_h, \frac{x_3}{\varepsilon}, t) + \rho_{0,t}^{bl}(x_h, \frac{1-x_3}{\varepsilon}, t) \\
 &\quad + \varepsilon \left( \rho_1(x_h, x_3, t) + \rho_{1,b}^{bl}(x_h, \frac{x_3}{\varepsilon}, t) + \rho_{1,t}^{bl}(x_h, \frac{1-x_3}{\varepsilon}, t) \right) \\
 &\quad + \varepsilon^2 \left( \rho_2(x_h, x_3, t) + \rho_{2,b}^{bl}(x_h, \frac{x_3}{\varepsilon}, t) + \rho_{2,t}^{bl}(x_h, \frac{1-x_3}{\varepsilon}, t) \right) + O(\varepsilon^3).
 \end{aligned} \tag{3.2.1}$$

The superscript *bl* stands for “boundary layer”, while the subscripts *b* and *t* stand for “bottom” and “top” respectively. For simplicity of the presentation, in the next computations we are going to consider only the boundary layer near the bottom, since the terms related to the top boundary layer are dealt with in the exact same way. Therefore, from now on we omit the subscript *b* for the boundary layer terms. However, when needed, we will explicitly write *t* or *b* subscripts to avoid confusion. Below we denote by  $\zeta = \frac{x_3}{\varepsilon}$  the fast vertical variable in the boundary layer. The boundary layer profiles are supposed to decay to 0 at exponential rate when  $\zeta \rightarrow \infty$ , since their effect is almost negligible in the interior of the domain: we will use this fact repeatedly in the following computations.

We remark that, at this level, (3.2.1) is just a formal ansatz. As is usual, we will first formally derive the equations for the profiles: this is the purpose of the present section. After that, we will prove quantitative estimates for the difference between the solution and the profiles we have constructed, using the relative entropy method: this will be done in Section 3.3.

### 3.2.2 Identification of the profiles

In order to identify the profiles, we plug the ansatz (3.2.1) into (3.1.1) and identify the terms of the same order of magnitude in  $\varepsilon$ . We immediately notice that the highest order term is a term of order  $\varepsilon^{-3}$ , which appears in the third component of the momentum equation:

$$p'(\rho_0 + \rho_0^{bl}) \partial_\zeta \rho_0^{bl} = 0.$$

We assume that  $\rho_0 + \rho_0^{bl}$  stays bounded away from zero. This hypothesis is fully justified here below. In view of the hypothesis (3.1.3) on the pressure and the fact that  $\rho_0^{bl}$  has to vanish for  $\zeta \rightarrow \infty$ , we immediately deduce that  $\rho_0^{bl} \equiv 0$ . Thanks to that property, and ignoring the terms of order  $O(\varepsilon^2)$ , which have been neglected in (3.2.1) in the expansion of the velocity fields, we find the following cascade of

equations: from the conservation of mass equation, we get

$$\begin{aligned} \rho_0 \partial_\zeta u_{0,3}^{bl} &= 0 && (\text{mass-}\varepsilon^{-1}) \\ \partial_t \rho_0 + \nabla_h \cdot (\rho_0 (u_{0,h} + u_{0,h}^{bl})) + \partial_3(\rho_0 u_{0,3}) + \partial_3 \rho_0 u_{0,3}^{bl} &&& (\text{mass-}\varepsilon^0) \\ &+ \rho_1 \partial_\zeta u_{0,3}^{bl} + \partial_\zeta(\rho_1^{bl} u_{0,3}^{bl}) + \partial_\zeta \rho_1^{bl} u_{0,3} + \rho_0 \partial_\zeta u_{1,3}^{bl} = 0, \end{aligned}$$

and from the momentum equation we get

$$\nabla p(\rho_0) + \begin{pmatrix} 0 \\ p'(\rho_0) \partial_\zeta \rho_1^{bl} \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \partial_\zeta^2 u_{0,3}^{bl} \end{pmatrix} + \rho_0 \nabla G \quad (\text{mom-}\varepsilon^{-2})$$

$$\begin{aligned} \rho_0 (u_{0,3} + u_{0,3}^{bl}) \cdot \partial_\zeta u_0^{bl} + e_3 \times \rho_0 (u_0 + u_0^{bl}) + &&& (\text{mom-}\varepsilon^{-1}) \\ \begin{pmatrix} \nabla_h(p'(\rho_0)(\rho_1 + \rho_1^{bl})) \\ \partial_3(p'(\rho_0)\rho_1) + \partial_3(p'(\rho_0))\rho_1^{bl} + p''(\rho_0)\rho_1^{bl} \partial_\zeta \rho_1^{bl} + p'(\rho_0) \partial_\zeta \rho_2^{bl} \end{pmatrix} \\ = \partial_\zeta^2 \begin{pmatrix} u_{0,h}^{bl} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \nabla_h \partial_\zeta u_{0,3}^{bl} \\ \partial_\zeta \nabla_h \cdot u_{0,h}^{bl} + \partial_\zeta^2 u_{1,3}^{bl} \end{pmatrix} + (\rho_1 + \rho_1^{bl}) \nabla G. \end{aligned}$$

We will examine the equation at order  $O(\varepsilon^0)$  coming from the momentum equation later. Let us first infer some properties for the profiles.

### The terms in the interior

Recall that the boundary layer profiles are expected to go to zero when  $\zeta \rightarrow \infty$ . Therefore, it follows from (mom- $\varepsilon^{-2}$ ) that

$$\nabla p(\rho_0) = \rho_0 \nabla G, \quad (3.2.2)$$

which yields

$$H'(\rho_0) = G + c(t) \quad (3.2.3)$$

and, by using (3.1.3), that  $\nabla_h \rho_0 = 0$ . Hence  $\rho_0$  is independent of  $x_h$ , namely  $\rho_0 = \rho_0(x_3, t)$ , and satisfies the ODE

$$p'(\rho_0) \partial_3 \rho_0 = -\rho_0. \quad (3.2.4)$$

Since  $p' \in \mathcal{C}^1((0, \infty))$  and non-zero, we can use Cauchy-Lipschitz theorem to get that  $\rho_0(t) \in \mathcal{C}^1((0, 1))$ , and hence bounded. Moreover, from (mom- $\varepsilon^{-1}$ ) we gather that

$$\rho_0 \begin{pmatrix} u_{0,h}^{bl} \\ 0 \end{pmatrix} + \begin{pmatrix} p'(\rho_0) \nabla_h \rho_1 \\ \partial_3(p'(\rho_0)\rho_1) \end{pmatrix} = \rho_1 \nabla G.$$

This equation is called *geostrophic balance*; it implies the Taylor-Proudman theorem (see Section V). In particular, its third component reads

$$\partial_3(p'(\rho_0)\rho_1) = -\rho_1.$$

Using the previous relation together with (3.2.2), we get

$$\partial_3\left(\frac{p'(\rho_0)}{\rho_0}\rho_1\right) = 0, \quad (3.2.5)$$

and hence the quantity

$$q := \frac{p'(\rho_0)}{\rho_0}\rho_1 \quad (3.2.6)$$

is independent of the vertical variable, *i.e.*  $q = q(t, x_h)$ . From the horizontal component, instead, we get (recall that  $\nabla_h \rho_0 = 0$ )

$$u_{0,h} = \nabla_h^\perp \left( \frac{p'(\rho_0)}{\rho_0} \rho_1 \right). \quad (3.2.7)$$

In particular, we deduce that  $u_{0,h} = u_{0,h}(x_h, t)$ , which justifies the introduction of boundary layer terms in order to enforce the no-slip boundary conditions on  $x_3 = 0, 1$ . In addition, applying the horizontal divergence we obtain

$$\nabla_h \cdot u_{0,h} = \nabla_h \cdot \nabla_h^\perp \left( \frac{p'(\rho_0)}{\rho_0} \rho_1 \right) = 0,$$

so that  $u_{0,h}$  is a 2-D horizontal divergence-free vector field.

We now exploit (mass- $\varepsilon^0$ ): considering it in the interior of the domain (*i.e.*, neglecting the boundary terms) and using the inequalities just proved, after an integration in the vertical variable we infer that

$$\int_0^1 \partial_t \rho_0 dx_3 = - \int_0^1 \partial_3 (\rho_0 u_{0,3}) dx_3 = 0. \quad (3.2.8)$$

By taking the time derivative of (3.2.3) and using (3.1.3) we have

$$\partial_t \rho_0 = \frac{\partial_t c}{H''(\rho_0)}$$

We integrate in the vertical variable and, from (3.2.8), we get  $\partial_t c = 0$ , hence  $\partial_t \rho_0 = 0$ . This implies that  $\rho_0$  has to be independent also of time, and hence it is equal to a positive function  $\bar{\rho}(x_3)$ , solution of (3.2.2). Thanks to this fact, we have now that  $\partial_3(\rho_0 u_{0,3}) = 0$ . Using the no-slip boundary condition and the positivity of  $\bar{\rho}$ , it yields  $u_{0,3} \equiv 0$ .

From now on  $\bar{\rho}$  denotes  $\rho_0$ . Let us now consider the equations outside the boundary layers, at order  $O(\varepsilon)$  in the mass equation,

$$\partial_t \rho_1 + \nabla \cdot (\bar{\rho} u_1) + \nabla_h \cdot (\rho_1 u_{0,h}) = 0, \quad (\text{mass-}\varepsilon^1)$$

and at order  $O(\varepsilon^0)$  in the momentum equation,

$$\begin{aligned} \bar{\rho} \partial_t u_0 + \nabla \cdot (\bar{\rho} u_0 \otimes u_0) & \quad (\text{mom-}\varepsilon^0) \\ + e_3 \times (\rho_1 u_0 + \bar{\rho} u_1) + \nabla \left( \frac{p''(\bar{\rho})}{2} \rho_1^2 + p'(\bar{\rho}) \rho_2 \right) & = \mu \Delta_h u_0 + \lambda \nabla (\nabla \cdot u_0) + \rho_2 \nabla G. \end{aligned}$$

Recall that  $u_0 = (u_{0,h}(t, x_h), 0)$ . Taking the curl of the horizontal component in (mom- $\varepsilon^0$ ), we obtain an equation for the horizontal vorticity  $\omega_0 = \nabla_h^\perp \cdot u_{0,h}$ :

$$\bar{\rho} \partial_t \omega_0 + \bar{\rho} u_{0,h} \cdot \nabla_h \omega_0 + \nabla_h \cdot (\rho_1 u_{0,h}) + \nabla_h \cdot (\bar{\rho} u_{1,h}) - \mu \Delta_h \omega_0 = 0.$$

Notice that, by (3.2.7), we get

$$\omega_0 = \omega_0(t, x_h) = \Delta_h q,$$

where  $q$  has been defined in (3.2.6); from the previous relation it follows that

$$\bar{\rho} \partial_t \Delta_h q + \bar{\rho} \nabla_h^\perp q \cdot \nabla_h \Delta_h q + \nabla_h \cdot (\bar{\rho} u_{1,h}) - \mu \Delta_h^2 q = 0, \quad (3.2.9)$$

where we have used the cancellation

$$\nabla_h \cdot (\rho_1 \nabla_h^\perp \rho_1) = \frac{1}{2} \nabla_h \cdot \nabla_h^\perp (\rho_1^2) = 0 \quad (3.2.10)$$

in order to get rid of the term  $\nabla_h \cdot (\rho_1 u_{0,h})$ . In order to compute the term  $\nabla_h \cdot (\bar{\rho} u_{1,h})$  in (3.2.9), we use equation (mass- $\varepsilon^1$ ) and the cancellation (3.2.10) again: we find

$$\nabla_h \cdot (\bar{\rho} u_{1,h}) = -\partial_t \rho_1 - \nabla_h \cdot (\rho_1 u_{0,h}) - \partial_3 (\bar{\rho} u_{1,3}) = -\partial_t \rho_1 - \partial_3 (\bar{\rho} u_{1,3}). \quad (3.2.11)$$

After integrating in  $x_3$  both (3.2.9) and (3.2.11) and summing up the resulting expressions, we eventually obtain

$$\begin{aligned} \partial_t \left( \langle \bar{\rho} \rangle \Delta_h q - \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle q \right) + \langle \bar{\rho} \rangle \nabla_h^\perp q \cdot \nabla_h \Delta_h q - \mu \Delta_h^2 q \\ = \bar{\rho}(1) u_{1,3}(x_h, 1, t) - \bar{\rho}(0) u_{1,3}(x_h, 0, t), \end{aligned} \quad (3.2.12)$$

where  $\langle f \rangle = \int_0^1 f(x_3) dx_3$  denotes the vertical mean of  $f$ .



### Boundary layer terms

We now consider the boundary layer terms. These terms are crucial to compute the right hand side of (3.2.12): indeed

$$u_{j,3,b}^{bl}(x_h, 0, t) = -u_{j,3}(x_h, 0, t) \quad \text{and} \quad u_{j,3,t}^{bl}(x_h, 0, t) = -u_{j,3}(x_h, 1, t) \quad (3.2.13)$$

for  $j = 0, 1$ , in order to enforce the no-slip boundary condition on the bottom and top boundaries.

First of all, (mass- $\varepsilon^{-1}$ ) yields  $u_{0,3}^{bl} = u_{0,3}^{bl}(x_h, t)$ , and hence

$$u_{0,3}^{bl} \equiv 0. \quad (3.2.14)$$

Using (3.2.14), we obtain from (mom- $\varepsilon^{-2}$ )

$$p'(\bar{\rho}) \partial_\zeta \rho_1^{bl} = \lambda \partial_\zeta^2 u_{0,3}^{bl} = 0.$$

Hence, thanks to (3.1.3),  $\rho_1^{bl} = \rho_1^{bl}(x_h, t)$  is constant in the boundary layer and goes to zero when  $\zeta \rightarrow \infty$ , therefore  $\rho_1^{bl} \equiv 0$ . Taking into account this last equality and reading the horizontal component of (mom- $\varepsilon^{-1}$ ), one has

$$\bar{\rho} (u_{0,h}^{bl})^\perp = \partial_\zeta^2 u_{0,h}^{bl}. \quad (3.2.15)$$

Notice that, in (3.2.15),  $x_h$  is a parameter. We use Taylor formula at first order

$$\bar{\rho}(x_3) = \bar{\rho}(0) + x_3 \int_0^1 \partial_3 \bar{\rho}(s x_3) ds$$

to write (3.2.15) as

$$\bar{\rho}(0) (u_{0,h}^{bl})^\perp + \left( x_3 \int_0^1 \partial_3 \bar{\rho}(s x_3) ds \right) (u_{0,h}^{bl})^\perp = \partial_\zeta^2 u_{0,h}^{bl}, \quad (3.2.16)$$

Let us now consider the equation

$$\bar{\rho}(0) (u_{0,h}^{bl})^\perp = \partial_\zeta^2 u_{0,h}^{bl}, \quad (3.2.17)$$

supplemented with the boundary condition

$$u_{0,h}^{bl}(x_h, 0, t) = -u_{0,h}(x_h, t) \quad (3.2.18)$$

at  $\zeta = 0$ , in view of (3.1.2) and (3.2.13). We remark that the system of ODEs (3.2.17)-(3.2.18) is the same (here in general  $\bar{\rho}(0) \neq 1$ ) as in the incompressible

case, see e.g. Chapter 7 of [28] and references therein. Its solutions are exponentially decaying and have a spiral structure. Indeed, we have the following formula:

$$u_{0,h,b}^{bl}(x_h, \zeta, t) = - \begin{pmatrix} e^{-\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}} \left[ u_{0,1}(x_h, t) \cos\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) + u_{0,2}(x_h, t) \sin\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) \right] \\ e^{-\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}} \left[ -u_{0,1}(x_h, t) \sin\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) + u_{0,2}(x_h, t) \cos\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) \right] \end{pmatrix}.$$

Let us move further. The vertical component in (mom- $\varepsilon^{-1}$ ) is

$$0 = \lambda \left( \partial_\zeta \nabla_h \cdot u_{0,h}^{bl} + \partial_\zeta^2 u_{1,3}^{bl} \right) + p'(\bar{\rho}) \partial_\zeta \rho_2^{bl}. \quad (3.2.19)$$

Equation (mass- $\varepsilon^0$ ), together with the fact the  $\bar{\rho}$  is positive, yields

$$\nabla_h \cdot u_{0,h}^{bl} + \partial_\zeta u_{1,3}^{bl} = 0. \quad (3.2.20)$$

Hence  $p'(\bar{\rho}) \partial_\zeta \rho_2^{bl} = 0$  and, similarly to the argument used for  $\rho_1^{bl}$ , we get  $\rho_2^{bl} \equiv 0$ . The previous equality determines  $u_{1,3}^{bl}$  up to a constant in  $\zeta$ , which we take so that  $u_{1,3}^{bl}$  converges to zero when  $\zeta \rightarrow \infty$ :

$$u_{1,3,b}^{bl}(x_h, \zeta, t) = - \frac{e^{-\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}}}{\sqrt{2\bar{\rho}(0)}} \left( \cos\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) + \sin\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) \right) \nabla_h^\perp \cdot u_{0,h}(x_h, t).$$

Similar computations can be done for the top boundary layers. Indeed, denoting by  $\eta = \frac{1-x_3}{\varepsilon}$  the fast vertical variable in the upper boundary layer, we use Taylor formula at first order

$$\bar{\rho}(x_3) = \bar{\rho}(1) - (1-x_3) \int_0^1 \partial_s \bar{\rho}(1-s(1-x_3)) ds \quad (3.2.21)$$

to define  $u_{0,h,t}^{bl}$  as the solution to the equation

$$\bar{\rho}(1) (u_{0,h,t}^{bl})^\perp = \partial_\eta^2 u_{0,h,t}^{bl}, \quad (3.2.22)$$

supplemented with the boundary condition

$$u_{0,h,t}^{bl}(x_h, 0, t) = -u_{0,h}(x_h, t) \quad (3.2.23)$$

at  $\eta = 0$ , recall (3.2.13). We have

$$u_{0,h,t}^{bl}(x_h, \eta, t) = - \begin{pmatrix} e^{-\eta\sqrt{\frac{\bar{\rho}(1)}{2}}} \left[ u_{0,1}(x_h, t) \cos\left(\eta\sqrt{\frac{\bar{\rho}(1)}{2}}\right) + u_{0,2}(x_h, t) \sin\left(\eta\sqrt{\frac{\bar{\rho}(1)}{2}}\right) \right] \\ e^{-\eta\sqrt{\frac{\bar{\rho}(1)}{2}}} \left[ -u_{0,1}(x_h, t) \sin\left(\eta\sqrt{\frac{\bar{\rho}(1)}{2}}\right) + u_{0,2}(x_h, t) \cos\left(\eta\sqrt{\frac{\bar{\rho}(1)}{2}}\right) \right] \end{pmatrix}.$$

and, from (3.2.20) with  $\partial_\zeta$  replaced by  $-\partial_\eta$ ,

$$u_{1,3,t}^{bl}(x_h, \eta, t) = \frac{e^{-\eta\sqrt{\frac{\bar{\rho}(1)}{2}}}}{\sqrt{2\bar{\rho}(1)}} \left( \cos\left(\eta\sqrt{\frac{\bar{\rho}(1)}{2}}\right) + \sin\left(\eta\sqrt{\frac{\bar{\rho}(1)}{2}}\right) \right) \nabla_h^\perp \cdot u_{0,h}(x_h, t).$$

Hence, using the boundary conditions (3.2.13), one can compute the right hand side of (3.2.12):

$$\begin{aligned} \bar{\rho}(1) u_{1,3}(x_h, 1, t) - \bar{\rho}(0) u_{1,3}(x_h, 0, t) &= -\bar{\rho}(1) u_{1,3,t}^{bl}(x_h, 0, t) + \bar{\rho}(0) u_{1,3,t}^{bl}(x_h, 0, t) \\ &= -\frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} \omega_0 = -\frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} \Delta_h q. \end{aligned} \quad (3.2.24)$$

This is the so-called *Ekman pumping term*, which represents the secondary (global) circulation created by the boundary layer. It appears as a damping term for the quasigeostrophic dynamics, described by equation (3.2.12).

### Final choices

It remains to choose the functions  $\rho_2$ ,  $u_1$  and  $u_{1,h}^{bl}$ . These terms are auxiliary terms which do not appear in the final result.

We choose the interior terms in order to make the terms of order  $O(\varepsilon)$  in the mass equation and the terms of order  $O(\varepsilon^0)$  in the momentum equation vanish identically. Notice that (mom- $\varepsilon^0$ ) determines  $u_{1,h}$  in terms of  $u_0$ ,  $\rho_1$  and  $\rho_2$ , and hence, through relation (3.2.7), in terms of  $\rho_1$  and  $\rho_2$  only. Specifically,

$$\begin{aligned} u_{1,h} := \frac{1}{\bar{\rho}} \left( -\mu \Delta_h u_{0,h}^\perp + \bar{\rho} \partial_t u_{0,h}^\perp \right. \\ \left. + \bar{\rho} u_{0,h} \cdot \nabla_h u_{0,h}^\perp - u_{0,h} \rho_1 + \nabla_h^\perp \left( p'(\bar{\rho}) \rho_2 + \frac{p''(\bar{\rho})}{2} \rho_1^2 \right) \right). \end{aligned} \quad (3.2.25)$$

Next, the vertical component of (mom- $\varepsilon^0$ ) reads

$$\partial_3 \left( p'(\bar{\rho}) \rho_2 + \frac{p''(\bar{\rho})}{2} \rho_1^2 \right) = -\rho_2 \quad (3.2.26)$$

where we have used that  $u_{0,3} \equiv 0$ . Since, by (3.1.3),  $p'(\bar{\rho}) > 0$ ,  $\rho_2$  can be defined as the solution of the ODE

$$\partial_3 \rho_2 + \frac{\partial_3 (p'(\bar{\rho})) + 1}{p'(\bar{\rho})} \rho_2 = -\frac{\partial_3 (p''(\bar{\rho}) \rho_1^2)}{2 p'(\bar{\rho})}, \quad (3.2.27)$$

up to an arbitrary constant  $c(x_h, t)$  that we take equal to zero for simplicity. We remark that this choice does not affect the choice of the other quantities since  $\rho_2$

appears only in (mom- $\varepsilon^0$ ) or higher order equations. Moreover, since  $\rho_1$  and  $\nabla\rho_1$  are bounded in time and space ( $q$ , defined in (3.2.6), satisfies the quasi-geostrophic equation (3.2.12), which admits regular solutions, see Proposition 3.2.3 later),  $\rho_2$  and  $\nabla\rho_2$  are bounded in time and space.

Moreover, equation (mass- $\varepsilon^1$ ) determines  $u_{1,3}$  up to a constant in  $x_3$ , which we take equal to  $-u_{1,3}^{bl}(x_h, 0, t)$  in order to enforce the no-slip boundary condition for the vertical component at order  $O(\varepsilon)$ . Therefore, thanks to (3.2.11) we get

$$\bar{\rho}(x_3) u_{1,3}(x_h, x_3, t) = -\bar{\rho}(0) u_{1,3}^{bl}(x_h, 0, t) - \int_0^{x_3} (\partial_t \rho_1 + \bar{\rho} \nabla_h \cdot u_{1,h}) dz. \quad (3.2.28)$$

Differently from the case without the gravitational potential, this term does not have an affine structure as in the incompressible case (see again Chapter 7 of [28]), since  $u_{1,h}$  does not depend only on  $x_h$ .

At this point, in order to enforce the no-slip boundary condition at order  $O(\varepsilon)$  for the horizontal component, we impose

$$u_{1,h}^{bl}(x_h, 0, t) = -u_{1,h}(x_h, 0, t) \quad (3.2.29)$$

at  $\zeta = 0$ . It remains to choose the boundary layer term  $u_{1,h}^{bl}$ . The specifications for the boundary layer term  $u_{1,h}^{bl}$  are that it is exponentially decaying to 0 for  $\zeta \rightarrow \infty$  and satisfies (3.2.29) at the boundary  $\zeta = 0$ . Hence, we define  $u_{1,h,b}^{bl}$  in the following way: for all  $\zeta \in [0, \infty)$  and  $x_h \in \mathbb{R}^2$ ,

$$u_{1,h,b}^{bl}(x_h, \zeta, t) := -u_{1,h}(x_h, 0, t) e^{-\zeta \sqrt{\frac{\bar{\rho}(0)}{2}}}.$$

Analogously,  $u_{1,h,t}^{bl}$  is defined for all  $\eta \in [0, \infty)$  and  $x_h \in \mathbb{R}^2$  by

$$u_{1,h,t}^{bl}(x_h, \eta, t) := -u_{1,h}(x_h, 1, t) e^{-\eta \sqrt{\frac{\bar{\rho}(1)}{2}}}.$$

**Remark 3.2.1.** *Contrary to the interior terms, it is not possible to make the terms of order  $O(\varepsilon)$  in the mass equation and the terms of order  $O(\varepsilon^0)$  in the momentum equation vanish identically. Indeed that would come down to imposing*

$$\begin{aligned} \bar{\rho} \nabla_h \cdot u_{1,h}^{bl} &= -\nabla_h \rho_1 \cdot u_{0,h}^{bl} - \partial_3 \bar{\rho} u_{1,3}^{bl} \\ \lambda \partial_\zeta (\nabla_h \cdot u_{1,h}^{bl}) &= -\partial_\zeta^2 u_{1,3}^{bl} = \partial_\zeta (\nabla_h \cdot u_{0,h}^{bl}), \end{aligned} \quad (3.2.30)$$

*which is overdetermined. This fact is due to the lack of higher-order correctors, since we truncate the expansion at order one in  $\varepsilon$ .*

Notice that, due to exponential decay to zero in the interior of the domain, the boundary layer terms will be small. Moreover, we can exploit their decay by

relying on Hardy's inequality (see the computations in Section 3.3). The final estimate, though, will be worse (see for instance term  $I_2$  below) than in the absence of boundary layer phenomena (absence of boundaries or complete slip boundary conditions). Improving this estimate would require to consider higher-order correctors in the ansatz (3.2.1).

We still have to add the corrector terms correcting the exponentially small terms on the boundaries due to the boundary layers on the opposite boundaries. This is a technical point, but needed to apply Hardy's inequality later. Using (3.2.13), we have at the bottom  $x_3 = 0$

$$\begin{aligned} u_{0,h}(x_h, t) + u_{0,h,b}^{bl}(x_h, 0, t) + u_{0,h,t}^{bl}(x_h, \frac{1}{\varepsilon}, t) &= u_{0,h,t}^{bl}(x_h, \frac{1}{\varepsilon}, t) \\ u_1(x_h, 0, t) + u_{1,b}^{bl}(x_h, 0, t) + u_{1,t}^{bl}(x_h, \frac{1}{\varepsilon}, t) &= u_{1,t}^{bl}(x_h, \frac{1}{\varepsilon}, t), \end{aligned} \quad (3.2.31)$$

and at the top  $x_3 = 1$

$$\begin{aligned} u_{0,h}(x_h, t) + u_{0,h,b}^{bl}(x_h, \frac{1}{\varepsilon}, t) + u_{0,h,t}^{bl}(x_h, 0, t) &= u_{0,h,b}^{bl}(x_h, \frac{1}{\varepsilon}, t) \\ u_1(x_h, 1, t) + u_{1,b}^{bl}(x_h, \frac{1}{\varepsilon}, t) + u_{1,t}^{bl}(x_h, 0, t) &= u_{1,b}^{bl}(x_h, \frac{1}{\varepsilon}, t). \end{aligned} \quad (3.2.32)$$

It means that we have a trace of the top boundary layer on the bottom boundary and vice-versa. Hence we will add corrector terms in the ansatz in Paragraph 3.2.2 below, to keep homogeneous boundary conditions.

### The ansatz

To put it in a nutshell, we have obtained the following ansatz for the structure of the solutions to (3.1.1)-(3.1.2):

$$\begin{aligned} \rho_{app}^\varepsilon(x_h, x_3, t) &= \bar{\rho}(x_3) + \varepsilon \rho_1(x_h, x_3, t) + \varepsilon^2 \rho_2(x_h, x_3, t) \\ u_{app}^\varepsilon(x_h, x_3, \zeta, \eta, t) &= \\ &\left( \begin{array}{c} \nabla_h^\perp \left( \frac{p'(\bar{\rho})}{\bar{\rho}} \rho_1 \right) (x_h, t) + u_{0,h,b}^{bl}(x_h, \zeta, t) + u_{0,h,t}^{bl}(x_h, \eta, t) - u_{0,h,1/\varepsilon}^{bl}(x_h, x_3, t) \\ 0 \end{array} \right) \\ &+ \varepsilon \left( \begin{array}{c} u_{1,h}(x_h, x_3, t) + u_{1,h,b}^{bl}(x_h, \zeta, t) + u_{1,h,t}^{bl}(x_h, \eta, t) - u_{1,h,1/\varepsilon}^{bl}(x_h, x_3, t) \\ u_{1,3}(x_h, x_3, t) + u_{1,3,b}^{bl}(x_h, \zeta, t) + u_{1,3,t}^{bl}(x_h, \eta, t) - u_{1,3,1/\varepsilon}^{bl}(x_h, x_3, t) \end{array} \right) \end{aligned} \quad (3.2.33)$$

with

$$\begin{aligned} u_{0,h,1/\varepsilon}^{bl}(x_h, x_3, t) &= x_3 u_{0,h,b}^{bl}(x_h, \frac{1}{\varepsilon}, t) + (1 - x_3) u_{0,h,t}^{bl}(x_h, \frac{1}{\varepsilon}, t) \\ u_{1,1/\varepsilon}^{bl}(x_h, x_3, t) &= x_3 u_{1,b}^{bl}(x_h, \frac{1}{\varepsilon}, t) + (1 - x_3) u_{1,t}^{bl}(x_h, \frac{1}{\varepsilon}, t) \end{aligned} \quad (3.2.34)$$

Recalling the definition of  $q$  in (3.2.6), it follows from (3.2.12) that

$$\partial_t \left( -\langle \bar{\rho} \rangle \Delta_h q + \left\langle \frac{\bar{\rho}}{\bar{\rho}'(\bar{\rho})} \right\rangle q \right) - \langle \bar{\rho} \rangle \nabla_h^\perp q \cdot \nabla_h \Delta_h q + \mu \Delta_h^2 q - \frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} \Delta_h q = 0. \quad (3.2.35)$$

This is the *quasi-geostrophic equation*. Similar limit equations without damping term have been shown in e.g. [47] and [54], where the boundary layers do not appear due to the complete slip condition. Notice that in [54] the parabolic term disappears, since the authors consider also the inviscid limit. We state here the well-posedness and the regularity results for the quasi-geostrophic equation (3.2.35), whose detailed proofs are given in Appendix E.

**Theorem 3.2.2.** *Let  $q_0 \in H^1(\mathbb{R}^2)$ . Then, there exists a unique global weak solution  $q$  to the quasi-geostrophic equation (3.2.35) such that*

$$q \in \mathcal{C}(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+; H^2(\mathbb{R}^2))$$

with initial datum  $q_0$ .

**Proposition 3.2.3.** *Let  $n \geq 1$  be an integer and  $q_0 \in H^n(\mathbb{R}^2)$ . Then, there exists a constant  $C_{n-1} > 0$  such that any weak solution to (3.2.35) with initial datum  $q_0$  satisfies the following inequality for all  $t \geq 0$ :*

$$\begin{aligned} \sum_{j=0}^{n-1} \left( \|\nabla_h^j q(t)\|_{L^2}^2 + \|\nabla_h^{j+1} q(t)\|_{L^2}^2 \right) + \sum_{j=0}^{n-1} \left( \int_0^t \|\nabla_h^{j+1} q\|_{L^2}^2 + \|\nabla_h^{j+2} q\|_{L^2}^2 \right) \\ \leq C_{n-1} \sum_{j=0}^{n-1} \left( \|\nabla_h^j q_0\|_{L^2} + \|\nabla_h^{j+1} q_0\|_{L^2}^2 \right). \end{aligned} \quad (3.2.36)$$

with  $C_0 = C_1 = 1$  and  $C_{n-1} = C_{n-1}(\|q_0\|_{H^{n-1}})$  for  $n - 1 \geq 2$ .

The boundary layer profiles  $u_{0,h,b}^{bl}$  and  $u_{0,h,t}^{bl}$  are solutions of the systems (3.2.17) - (3.2.18) and (3.2.22) - (3.2.23) respectively. We refer to the previous computations for the precise definitions of the higher-order terms.

We conclude this part by remarking that, according to the previous computations, we have that  $(\rho_{app}^\varepsilon, u_{app}^\varepsilon)$  solves the following system:

$$\left\{ \begin{aligned} \partial_t \rho_{app}^\varepsilon + \nabla \cdot (\rho_{app}^\varepsilon u_{app}^\varepsilon) &= \varepsilon R^{bl} + \varepsilon^2 R^\varepsilon \\ \rho_{app}^\varepsilon \partial_t u_{app}^\varepsilon + \rho_{app}^\varepsilon u_{app}^\varepsilon \cdot \nabla u_{app}^\varepsilon + \frac{1}{\varepsilon} e_3 \times \rho_{app}^\varepsilon u_{app}^\varepsilon + \frac{1}{\varepsilon^2} \nabla p(\rho_{app}^\varepsilon) &= \frac{1}{\varepsilon^2} \rho_{app}^\varepsilon \nabla G \\ + \frac{x_3}{\varepsilon} \int_0^1 \partial_3 \bar{\rho}(s x_3) ds e_3 \times u_{0,h,b}^{bl} - \frac{1-x_3}{\varepsilon} \int_0^1 \partial_3 \bar{\rho}(1-s(1-x_3)) ds e_3 \times u_{0,h,t}^{bl} \\ + \Delta_{\mu,\varepsilon} u_{app}^\varepsilon + \lambda \nabla(\nabla \cdot u_{app}^\varepsilon) + S^{bl} + \varepsilon S^\varepsilon \end{aligned} \right. \quad (3.2.37)$$

in the slab  $\Omega$  with no-slip boundary conditions (3.1.2). The remainder terms  $R^\varepsilon$  and  $S^\varepsilon$  are of the form

$$R^\varepsilon = R^\varepsilon(x_h, x_3, \frac{x_3}{\varepsilon}, \frac{1-x_3}{\varepsilon}, t) \quad \text{and} \quad S^\varepsilon = S^\varepsilon(x_h, x_3, \frac{x_3}{\varepsilon}, \frac{1-x_3}{\varepsilon}, t),$$

while the boundary layer terms are

$$\begin{aligned} R^{bl}(x_h, x_3, \frac{x_3}{\varepsilon}, \frac{1-x_3}{\varepsilon}, t) &= \bar{\rho} \nabla_h \cdot (u_{1,h,b}^{bl} + u_{1,h,t}^{bl}) \\ &\quad + \nabla_h \rho_1 \cdot (u_{0,h,b}^{bl} + u_{0,h,t}^{bl}) + \partial_3 \bar{\rho} (u_{1,3,b}^{bl} + u_{1,3,t}^{bl}) \end{aligned}$$

and

$$\begin{aligned} S^{bl}(x_h, x_3, \frac{x_3}{\varepsilon}, \frac{1-x_3}{\varepsilon}, t) &= \bar{\rho} \partial_t (u_{0,b}^{bl} + u_{0,t}^{bl}) + \bar{\rho} u_{0,h} \cdot \nabla_h (u_{0,b}^{bl} + u_{0,t}^{bl}) \\ &\quad + \bar{\rho} (u_{0,b}^{bl} + u_{0,t}^{bl}) \cdot \nabla_h u_0 + \bar{\rho} (u_{0,b}^{bl} + u_{0,t}^{bl}) \cdot \nabla_h (u_{0,b}^{bl} + u_{0,t}^{bl}) \\ &\quad + \bar{\rho} (u_{1,3} + u_{1,3,b}^{bl} + u_{1,3,t}^{bl}) \partial_\eta (u_{0,b}^{bl} + u_{0,t}^{bl}) - \mu \Delta_h (u_{0,t}^{bl} + u_{0,b}^{bl}) \\ &\quad - \partial_\eta^2 (u_{1,b}^{bl} + u_{1,t}^{bl}) - \lambda \left( \begin{array}{c} 0 \\ \partial_\eta \nabla_h \cdot (u_{1,h,b}^{bl} + u_{1,h,t}^{bl}) \end{array} \right) \\ &\quad + e_3 \times \left( \rho_1 (u_{0,b}^{bl} + u_{0,t}^{bl}) + \bar{\rho} (u_{1,b}^{bl} + u_{1,t}^{bl}) \right). \end{aligned}$$

The remainders  $\varepsilon^2 R^\varepsilon$  and  $\varepsilon S^\varepsilon$  contain also the terms of order  $O(e^{-1/\varepsilon})$  coming from the addition of the boundary layers correctors  $u_{0,h,1/\varepsilon}^{bl}$  and  $u_{1,1/\varepsilon}^{bl}$ , defined in (3.2.34). Notice that  $S^{bl}$  appears at order  $O(1)$ , but has fast, exponential, decay in the inside  $\Omega$ : more precisely, we have  $\|S^{bl}\|_{L^p} \leq C \varepsilon^{\frac{1}{p}}$  for all  $p \in [1, \infty]$ .

The choice of the regularity of the initial datum  $q_0$  guarantees enough regularity for the approximated solution  $(\rho_{app}^\varepsilon, u_{app}^\varepsilon)$  in order to derive the stability estimates later in Section 3.3, as stated in the following lemma:

**Lemma 3.2.4.** *The approximated density  $\rho_{app}^\varepsilon$  can be written as*

$$\rho_{app}^\varepsilon(x_h, x_3, t) = \bar{\rho}(x_3) + q(t, x_h) \frac{\bar{\rho}}{p'(\bar{\rho})}(x_3) + q(t, x_h) l(x_3)$$

and the approximated velocity  $u_{app}^\varepsilon$  can be written as a finite sum

$$u_{app}^\varepsilon(x_h, x_3, \zeta, \eta, t) = \sum_{i=1}^N f_i(t, x_h) g_i(x_3) h_i(\zeta) w_i(\eta),$$

with  $\bar{\rho}, l, g_i \in C^1([0, 1])$  and  $h_i, w_i \in C^\infty(\mathbb{R}_+)$ , for some  $N \geq 1$ . From Proposition 3.2.3, we have that for  $q_0 \in H^5(\mathbb{R}^2)$

$$q \in L^\infty(\mathbb{R}_+; H^5(\mathbb{R}^2)), \quad f_i \in L^\infty(\mathbb{R}_+; H^{k_i}(\mathbb{R}^2)) \quad \text{with } k_i \geq 1.$$

### 3.2.3 Large-scale quasi-geostrophic equation

We recover here the equation for  $u_0$  from (3.2.35). For this we need the following standard lemma, which gives the Helmholtz decomposition for two-dimensional vector fields.

**Lemma 3.2.5.** *Let  $p \in (1, \infty)$ . Let  $a$  and  $b$  be two scalar fields in  $L^p(\mathbb{R}^2)$ .*

*Then there exists a unique vector field  $F$ , belonging to the homogeneous Sobolev space  $\dot{W}^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$ , which solves the system*

$$\begin{cases} \nabla_h^\perp \cdot F = a \\ \nabla_h \cdot F = b. \end{cases} \quad (3.2.38)$$

Moreover, the following formula holds:

$$F = -\nabla_h^\perp (-\Delta_h)^{-1} a - \nabla_h (-\Delta_h)^{-1} b.$$

The previous result being classical, we do not give the proof here: we rather refer to [25] (see Sections 1.2 and 1.3) and [49] (see Section 10.6) for details. We just give some explanations about the uniqueness, which will be needed here below. By linearity of system (3.2.38), let us suppose that  $F$  solves (3.2.38) with  $a = b = 0$ . In particular  $\nabla \times F = 0$ , and hence (see Corollary 1.2.1 of [25])  $F = \nabla q$ , for some  $q \in L^p$ . But since we also have  $\nabla \cdot F = 0$ , we deduce the equation  $-\Delta q = 0$ , which admits the only solution  $q = 0$  in  $L^p$ , due to the decay condition at infinity. Now, let  $\pi \in \dot{H}^1(\mathbb{R}^2)$  solve

$$-\Delta_h \pi = \langle \bar{\rho} \rangle \nabla_h \cdot (u_{0,h} \cdot \nabla_h u_{0,h}) = \langle \bar{\rho} \rangle \nabla_h u_{0,h} : \nabla_h u_{0,h}. \quad (3.2.39)$$

Notice that, by the previous argument, such a  $\pi$  is unique. We then define  $F(\cdot, t) \in L^2(\mathbb{R}^2; \mathbb{R}^2)$  for almost every  $t$  by the formula

$$F := \langle \bar{\rho} \rangle \partial_t u_{0,h} + \langle \bar{\rho} \rangle u_{0,h} \cdot \nabla_h u_{0,h} - \mu \Delta_h u_{0,h} + \frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} u_{0,h} + \nabla_h \pi.$$

Notice that, thanks to equations (3.2.39) and (3.2.35) and the divergence-free condition  $\nabla_h \cdot u_{0,h} = 0$ , we have

$$\nabla_h^\perp \cdot F = \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle \partial_t q \quad \text{and} \quad \nabla_h \cdot F = 0.$$

Therefore, the uniqueness part of Lemma 3.2.5 implies that

$$F = \nabla_h^\perp (\Delta_h)^{-1} \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle \partial_t q = \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle \partial_t (\Delta_h)^{-1} u_{0,h},$$



where we have also used (3.2.7). Eventually, we find that  $u_{0,h}$  solves the system

$$\begin{cases} \partial_t \left( \langle \bar{\rho} \rangle - \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle (\Delta_h)^{-1} \right) u_{0,h} \\ \quad + \langle \bar{\rho} \rangle u_{0,h} \cdot \nabla_h u_{0,h} - \mu \Delta_h u_{0,h} + \frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} u_{0,h} + \nabla \pi = 0 \\ \nabla_h \cdot u_{0,h} = 0 \end{cases} \quad (3.2.40)$$

in  $\mathbb{R}^2$ . The second term appearing in the time derivative is a consequence of the combination of the effects due to density stratification and fast rotation. Notice that both (3.2.12) and (3.2.40) are averaged (in  $x_3$ ) versions of (mom- $\varepsilon^0$ ).

### 3.3 Stability estimates

The goal of this section is twofold. First, we recall uniform bounds which are available for the family of weak solution  $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$ . Second, we prove the stability result shown in Theorem VII.1 via relative entropy estimates, performed in Section 3.3.2.

#### 3.3.1 Uniform bounds for the family of weak solutions

We collect here some uniform bounds verified by the family of finite energy weak solutions  $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$ . We refer e.g. to [47] and [46] for details on their derivation. These bounds will be important in the next subsection, when proving stability estimates.

Observe that, by assumption, for any  $\varepsilon \in (0, 1]$  the energy inequality

$$\begin{aligned} & \int_{\Omega} \left( \frac{1}{2} \rho_\varepsilon |u^\varepsilon|^2 + \frac{1}{\varepsilon^2} E(\rho^\varepsilon(t), \bar{\rho}) \right) + \int_0^T \int_{\Omega} \left( \mu |\nabla_h u^\varepsilon|^2 + \varepsilon |\partial_3 u^\varepsilon|^2 + \lambda |\nabla \cdot u^\varepsilon|^2 \right) \\ & \leq C \int_{\Omega} \left( \rho_{0,\varepsilon} |u_0^\varepsilon|^2 + \frac{1}{\varepsilon^2} E(\rho_0^\varepsilon, \bar{\rho}) \right) \end{aligned} \quad (3.3.1)$$

holds for almost every  $t > 0$ . According to inequality (4.15) of [53], we have the following control, which holds for any positive scalar functions  $\rho(x, t)$  and  $r(x, t)$ , with  $0 < r_- \leq r(x, t) \leq r_+$ , for some real numbers  $r_-, r_+$ : there exist constants  $c_1, c_2 > 0$  such that, for almost all  $(x, t) \in \Omega \times \mathbb{R}_+$ , one has

$$\begin{aligned} & c_1 \left( |\rho(x, t) - r(x, t)|^2 \mathbf{1}_{\{|\rho(x,t)-r(x,t)| < 1\}} + |\rho(x, t) - r(x, t)|^\gamma \mathbf{1}_{\{|\rho(x,t)-r(x,t)| \geq 1\}} \right) \\ & \leq E(\rho(x, t), r(x, t)) \leq \\ & c_2 \left( |\rho(x, t) - r(x, t)|^2 \mathbf{1}_{\{|\rho(x,t)-r(x,t)| < 1\}} + |\rho(x, t) - r(x, t)|^\gamma \mathbf{1}_{\{|\rho(x,t)-r(x,t)| \geq 1\}} \right), \end{aligned} \quad (3.3.2)$$

where the notation  $\{|\rho(x, t) - r(x, t)| < 1\}$  stands for the set of  $x \in \Omega$  such that  $|\rho(x, t) - r(x, t)| < 1$  (and analogously for the  $\geq$  symbol) and  $\mathbf{1}_A$  denotes the characteristic function of a set  $A \subseteq \Omega$ . Notice that the same inequalities hold if we replace 1 by any constant  $M > 0$ , up to change the value of the constants  $c_1$  and  $c_2$ .

Now, following Chapters 4 and 5 of [49], let us introduce the *essential set* and the *residual set* as follows: for almost every  $t > 0$ , we set

$$\Omega_{\text{ess}}(t) := \left\{ x \in \Omega \mid |\rho^\varepsilon(x, t) - \bar{\rho}(x_3)| < \sigma \right\} \quad \text{and} \quad \Omega_{\text{res}}(t) := \Omega \setminus \Omega_{\text{ess}}(t), \quad (3.3.3)$$

for some  $\sigma$  such that

$$0 < \sigma < \inf_{(0,1)} \bar{\rho}.$$

Then, given any function  $h$ , we define

$$[h]_{\text{ess}} := h \mathbf{1}_{\Omega_{\text{ess}}} \quad \text{and} \quad [h]_{\text{res}} := h \mathbf{1}_{\Omega_{\text{res}}} = h - [h]_{\text{ess}}.$$

The notation *ess* stands for *essential part* of the function, while *res* for its *residual part*. We refer to Section 4.7 of [49] for more details. One should keep in mind that such a decomposition depends on  $\rho^\varepsilon$ .

After this preparation, let us establish uniform bounds for  $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$ . First of all, notice that, in view of our assumptions on the initial data  $(\rho_0^\varepsilon, u_0^\varepsilon)_\varepsilon$ , the right hand side of (3.3.1) is uniformly bounded for  $\varepsilon \in (0, 1]$ . Therefore, using (3.3.2), we deduce the existence of a constant  $C > 0$  such that, for all  $T > 0$  fixed and all  $0 < \varepsilon \leq 1$ , one has

$$\left\| \sqrt{\rho^\varepsilon} u^\varepsilon \right\|_{L_T^\infty(L^2)} \leq C \quad (3.3.4)$$

$$\frac{1}{\varepsilon} \left\| [\rho^\varepsilon - \bar{\rho}]_{\text{ess}} \right\|_{L_T^\infty(L^2)} \leq C \quad (3.3.5)$$

$$\sup_{t \in [0, T]} \mathcal{L}(\Omega_{\text{res}}(t)) + \left\| [\rho^\varepsilon]_{\text{res}} \right\|_{L_T^\infty(L^\gamma)}^\gamma \leq C \varepsilon^2, \quad (3.3.6)$$

where  $\mathcal{L}(A)$  denotes the Lebesgue measure of a set  $A \subseteq \Omega$ . We refer to Section 2 of [47] and Section 4 of [50] for details.

Next, let us consider the viscosity terms: recalling that  $\mu > 0$  and  $\lambda > 0$ , from (3.3.1) we immediately get

$$\left\| \nabla_h u^\varepsilon \right\|_{L_T^2(L^2)} + \left\| \nabla \cdot u^\varepsilon \right\|_{L_T^2(L^2)} \leq C, \quad (3.3.7)$$

for some “universal” constant  $C > 0$  independent of  $\varepsilon$  and of the fixed time  $T > 0$ . In addition, owing to the identity

$$\partial_3 u_3^\varepsilon = \nabla \cdot u^\varepsilon - \nabla_h \cdot u_h^\varepsilon,$$

we also deduce that

$$\|\partial_3 u_3^\varepsilon\|_{L_T^2(L^2)} \leq C. \quad (3.3.8)$$

Moreover, using (3.3.1) we also gather

$$\sqrt{\varepsilon} \|\partial_3 u_h^\varepsilon\|_{L_T^2(L^2)} \leq C. \quad (3.3.9)$$

Let us come back to estimates for the density. Arguing as in Section 2 of [47], from (3.3.5) and (3.3.6) we infer the following property:

$$\int_{\Omega} \left[ \left| \frac{\rho^\varepsilon(t) - \bar{\rho}}{\varepsilon} \right|^\gamma \right]_{\text{res}} dx \leq C \varepsilon^{2-\gamma}. \quad (3.3.10)$$

It still remains to establish some uniform bounds for the velocity fields  $u_\varepsilon$ . This is the goal of the next computations, which are borrowed from [47]. First of all, using the positivity of  $\bar{\rho}$ , we can estimate

$$\int_{\Omega} |u^\varepsilon|^2 dx \leq C \left( \int_{\Omega} \rho^\varepsilon |u^\varepsilon|^2 dx + \int_{\Omega} |\rho^\varepsilon - \bar{\rho}| |u^\varepsilon|^2 dx \right), \quad (3.3.11)$$

where the first term in the right hand side is uniformly bounded in  $L_T^\infty(L^2)$  in view of (3.3.4). For the second term, we can use the decomposition  $\rho^\varepsilon - \bar{\rho} = [\rho^\varepsilon - \bar{\rho}]_{\text{ess}} + [\rho^\varepsilon - \bar{\rho}]_{\text{res}}$ . Let us focus on the term localised in the essential part: by Hölder's inequality, we have

$$\begin{aligned} \int_{\Omega} [[\rho^\varepsilon - \bar{\rho}]_{\text{ess}} |u^\varepsilon|^2] &\leq \|[\rho^\varepsilon - \bar{\rho}]_{\text{ess}}\|_{L^2} \|u^\varepsilon\|_{L^4}^2 \leq C \varepsilon \|u^\varepsilon\|_{L^2}^{1/2} \|\nabla u^\varepsilon\|_{L^2}^{3/2} \\ &\leq C \varepsilon^{1/4} \left( \|u^\varepsilon\|_{L^2}^2 \right)^{1/4} \left( \varepsilon \|\nabla u^\varepsilon\|_{L^2}^2 \right)^{3/4}, \end{aligned} \quad (3.3.12)$$

where we have used estimate (3.3.5), the interpolation inequality

$$\|f\|_{L^4} \leq \|f\|_{L^2}^{1/4} \|f\|_{L^6}^{3/4}$$

and the Sobolev embedding  $\dot{H}^1 \hookrightarrow L^6$ . For the residual part, we argue in a very similar way: after defining  $\gamma'$  such that  $1/\gamma + 1/\gamma' = 1$ , thanks to (3.3.10) we infer

$$\int_{\Omega} [[\rho^\varepsilon - \bar{\rho}]_{\text{res}} |u^\varepsilon|^2] \leq \|[\rho^\varepsilon - \bar{\rho}]_{\text{res}}\|_{L^\gamma} \|u^\varepsilon\|_{L^{2\gamma'}}^2 \leq C \varepsilon^{2/\gamma} \|u^\varepsilon\|_{L^{2\gamma'}}^2.$$

Notice that, since  $\gamma > 3/2$ , we have  $2\gamma' \in (2, 6)$ , so that we can apply interpolation again and estimate  $\|f\|_{L^{2\gamma'}} \leq \|f\|_{L^2}^\theta \|f\|_{L^6}^{1-\theta}$ , where  $\theta = (3-\gamma')/(2\gamma')$ . Combining this bound with Sobolev embeddings and arguing as above, we finally find

$$\int_{\Omega} [[\rho^\varepsilon - \bar{\rho}]_{\text{res}} |u_\varepsilon|^2] \leq C \varepsilon^{2/\gamma - (1-\theta)} \left( \|u^\varepsilon\|_{L^2}^2 \right)^\theta \left( \varepsilon \|\nabla u^\varepsilon\|_{L^2}^2 \right)^{1-\theta}. \quad (3.3.13)$$

We remark that  $2/\gamma - (1 - \theta) > 0$ . Therefore, inserting (3.3.12) and (3.3.13) into (3.3.11) and applying Young's inequality, we finally deduce that there exists a constant  $C > 0$  such that, for all  $\varepsilon > 0$  and all  $T > 0$ , one has

$$\|u^\varepsilon\|_{L_T^2(L^2)} \leq C. \quad (3.3.14)$$

### 3.3.2 Relative entropy estimates

From the previous uniform bounds, it is classical to derive that, up to extraction of a suitable subsequence,  $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$  converges to a limit state  $(\rho, u)$  which belongs to the kernel of the singular perturbation operator. We refer e.g. [47], [46], [51], [50] for details.

The goal of the present subsection is to make this convergence quantitative, showing also the general structure of our solutions and taking into account the correctors due to Ekman's boundary layers.

We aim at proving the following result. The following proposition implies Theorem VII.1. Recall that  $E$  has been defined in (3.1.9).

**Proposition 3.3.1.** *For  $\gamma \geq 3/2$ , suppose that there exists a finite energy weak solution  $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$  to (3.1.1) with initial data  $(\rho_0^\varepsilon, u_0^\varepsilon)_\varepsilon$  verifying the hypotheses in Section 3.1. Let  $(\rho_{app}^\varepsilon, u_{app}^\varepsilon)_\varepsilon$  be defined as in (3.2.33), and define  $\delta u^\varepsilon = u^\varepsilon - u_{app}^\varepsilon$ . Then, there exist functions  $C_1(t), C_2(t) \in L^1([0, T])$  for all  $T > 0$  and constants  $C > 0$  and  $\varepsilon_0 \in (0, 1)$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ , the following estimate holds:*

$$\begin{aligned} & \int_{\Omega} \rho^\varepsilon(t) |\delta u^\varepsilon(t)|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon(t), \rho_{app}^\varepsilon(t)) dx \\ & \quad + \int_0^t \int_{\Omega} \mu |\nabla_h \delta u^\varepsilon|^2 + \varepsilon |\partial_3 \delta u^\varepsilon|^2 + \lambda |\nabla \cdot \delta u^\varepsilon|^2 dx \\ & \leq C e^{\int_0^t C_1(s) ds} \left( \int_{\Omega} \rho_0^\varepsilon |\delta u_0^\varepsilon|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho_0^\varepsilon, \rho_{0,app}^\varepsilon) dx + \varepsilon \int_0^t e^{\int_\tau^t C_1(s) ds} C_2(\tau) d\tau \right) \end{aligned} \quad (3.3.15)$$

for almost every  $t > 0$ .

**Remark 3.3.2.** *The lower bound for the exponent  $\gamma$  comes from the control of the source term in the relative entropy inequality (3.3.22): in particular, in (3.3.41) we need  $\gamma \geq 3/2$  to apply Hölder's inequality and get the estimate (3.3.43).*

In order to prove the previous result, we resort to the technique of the *relative entropy/relative energy* inequality, see e.g. [61], [53] (see equation (2.5) therein), [48] and [50]. The relative entropy estimate of those works is directly applicable in our framework, but it is not immediately clear how to take advantage of the small remainders in (3.2.37). Instead, we directly derive the entropy inequality on

the system for  $(\delta\rho^\varepsilon, \delta u^\varepsilon)$ . Hence, we take into account from the beginning that  $(\rho_{app}^\varepsilon, u_{app}^\varepsilon)$  is almost a solution to (3.1.1). On the contrary, the relative entropy inequality of e.g. [50] holds for a much wider class of smooth functions.

### Formulating the relative entropy inequality

Let us set  $\delta\rho^\varepsilon := \rho^\varepsilon - \rho_{app}^\varepsilon$  and  $\delta u^\varepsilon := u^\varepsilon - u_{app}^\varepsilon$ . Taking the difference between the first equations in (3.1.1) and (3.2.37), we find that  $\delta\rho^\varepsilon$  solves

$$\partial_t \delta\rho^\varepsilon + \nabla \cdot (u_{app}^\varepsilon \delta\rho^\varepsilon) = -\nabla \cdot (\rho^\varepsilon \delta u^\varepsilon) - \varepsilon R^{bl} - \varepsilon^2 R^\varepsilon, \quad (3.3.16)$$

while, taking the difference of the second equations in the same systems, after setting  $\delta p^\varepsilon := p(\rho^\varepsilon) - p(\rho_{app}^\varepsilon)$ , we get that  $\delta u^\varepsilon$  solves

$$\begin{aligned} & \rho^\varepsilon \partial_t \delta u^\varepsilon + \rho^\varepsilon u^\varepsilon \cdot \nabla \delta u^\varepsilon + \frac{1}{\varepsilon} e_3 \times \rho^\varepsilon \delta u^\varepsilon + \frac{1}{\varepsilon^2} \nabla \delta p^\varepsilon - \Delta_{\mu, \varepsilon} \delta u^\varepsilon - \lambda \nabla \nabla \cdot \delta u^\varepsilon = \\ & \frac{1}{\varepsilon^2} \delta\rho^\varepsilon \nabla G - \frac{1}{\varepsilon} e_3 \times \delta\rho^\varepsilon u_{app}^\varepsilon - \delta\rho^\varepsilon \partial_t u_{app}^\varepsilon + (\rho_{app}^\varepsilon u_{app}^\varepsilon - \rho^\varepsilon u^\varepsilon) \cdot \nabla u_{app}^\varepsilon - S^{bl} - \varepsilon S^\varepsilon \\ & - \frac{x_3}{\varepsilon} \int_0^1 \partial_3 \bar{\rho}(s x_3) ds e_3 \times u_{0,h,b}^{bl} + \frac{1-x_3}{\varepsilon} \int_0^1 \partial_3 \bar{\rho}(1-s(1-x_3)) ds e_3 \times u_{0,h,t}^{bl}. \end{aligned} \quad (3.3.17)$$

From the point of view of energy estimates, the main term to work on is the difference of the pressure terms. Testing it against  $\delta u^\varepsilon$  yields

$$\begin{aligned} \int_{\Omega} \nabla \delta p^\varepsilon \cdot \delta u^\varepsilon dx &= \int_{\Omega} \nabla p(\rho^\varepsilon) \cdot u^\varepsilon dx - \int_{\Omega} \nabla p(\rho_{app}^\varepsilon) \cdot u_{app}^\varepsilon dx \\ &+ \int_{\Omega} \nabla \cdot u_{app}^\varepsilon \delta p^\varepsilon dx - \int_{\Omega} \nabla p(\rho_{app}^\varepsilon) \cdot \delta u^\varepsilon dx. \end{aligned} \quad (3.3.18)$$

By standard computations, using the mass equation in (3.1.1), we get

$$\int_{\Omega} \nabla p(\rho^\varepsilon) \cdot u^\varepsilon dx = \int_{\Omega} \nabla (H'(\rho^\varepsilon)) \cdot \rho^\varepsilon u^\varepsilon dx = \frac{d}{dt} \int_{\Omega} H(\rho^\varepsilon) dx.$$

Similarly, from the first equation in (3.2.37) we gather

$$\int_{\Omega} \nabla p(\rho_{app}^\varepsilon) \cdot u_{app}^\varepsilon dx = \frac{d}{dt} \int_{\Omega} H(\rho_{app}^\varepsilon) dx - \varepsilon \int_{\Omega} H'(\rho_{app}^\varepsilon) (R^{bl} + \varepsilon R^\varepsilon) dx.$$

In the identity (3.3.18) we now add and subtract the term  $\frac{d}{dt} \int_{\Omega} H'(\rho_{app}^\varepsilon) \delta\rho^\varepsilon dx$ , in order to make the relative entropy  $E(\rho^\varepsilon(t), \rho_{app}^\varepsilon(t))$  appear. Then, from (3.3.18)

and the previous computations we infer

$$\begin{aligned} \int_{\Omega} \nabla \delta p^\varepsilon \cdot \delta u^\varepsilon dx &= \frac{d}{dt} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) dx + \int_{\Omega} \nabla \cdot u_{app}^\varepsilon \delta p^\varepsilon dx - \int_{\Omega} \nabla p(\rho_{app}^\varepsilon) \cdot \delta u^\varepsilon dx \\ &\quad + \frac{d}{dt} \int_{\Omega} H'(\rho_{app}^\varepsilon) \delta \rho^\varepsilon dx + \varepsilon \int_{\Omega} H'(\rho_{app}^\varepsilon) (R^{bl} + \varepsilon R^\varepsilon) dx. \end{aligned}$$

Using again the mass equations in (3.1.1) and (3.2.37), we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} H'(\rho_{app}^\varepsilon) \delta \rho^\varepsilon dx &= \int_{\Omega} \partial_t H'(\rho_{app}^\varepsilon) \delta \rho^\varepsilon dx + \int_{\Omega} H'(\rho_{app}^\varepsilon) \partial_t \delta \rho^\varepsilon dx \\ &= \int_{\Omega} \partial_t H'(\rho_{app}^\varepsilon) \delta \rho^\varepsilon dx + \int_{\Omega} \nabla H'(\rho_{app}^\varepsilon) \cdot (\rho^\varepsilon u^\varepsilon - \rho_{app}^\varepsilon u_{app}^\varepsilon) dx \\ &\quad - \varepsilon \int_{\Omega} H'(\rho_{app}^\varepsilon) (R^{bl} + \varepsilon R^\varepsilon) dx. \end{aligned}$$

This relation yields

$$\begin{aligned} \int_{\Omega} \nabla \delta p^\varepsilon \cdot \delta u^\varepsilon &= \frac{d}{dt} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) dx + \int_{\Omega} \nabla \cdot u_{app}^\varepsilon \delta p^\varepsilon - \int_{\Omega} \nabla p(\rho_{app}^\varepsilon) \cdot \delta u^\varepsilon \\ &\quad + \int_{\Omega} \partial_t H'(\rho_{app}^\varepsilon) \delta \rho^\varepsilon + \int_{\Omega} \nabla H'(\rho_{app}^\varepsilon) \cdot (\rho^\varepsilon u^\varepsilon - \rho_{app}^\varepsilon u_{app}^\varepsilon). \end{aligned} \tag{3.3.19}$$

Let us now define

$$I := \int_{\Omega} \nabla \cdot u_{app}^\varepsilon \delta p^\varepsilon dx + \int_{\Omega} \partial_t H'(\rho_{app}^\varepsilon) \delta \rho^\varepsilon dx + \int_{\Omega} \nabla H'(\rho_{app}^\varepsilon) \cdot (\rho^\varepsilon u^\varepsilon - \rho_{app}^\varepsilon u_{app}^\varepsilon) dx$$

and work on it for a while. We use the following Taylor expansion,

$$\begin{aligned} P(\rho^\varepsilon, \rho_{app}^\varepsilon) &:= p(\rho^\varepsilon) - p(\rho_{app}^\varepsilon) - p'(\rho_{app}^\varepsilon) \delta \rho^\varepsilon \\ &= \frac{1}{2} (\delta \rho^\varepsilon)^2 \int_0^1 (1-s) p''(\rho_{app}^\varepsilon + s \delta \rho^\varepsilon) ds, \end{aligned} \tag{3.3.20}$$

to obtain the next series of equalities (recall that  $H''(z) = p'(z)/z$ ):

$$\begin{aligned}
I &= \int_{\Omega} \nabla \cdot u_{app}^{\varepsilon} p'(\rho_{app}^{\varepsilon}) \delta \rho^{\varepsilon} dx + \int_{\Omega} \nabla \cdot u_{app}^{\varepsilon} P(\rho^{\varepsilon}, \rho_{app}^{\varepsilon}) dx \\
&\quad + \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \partial_t \rho_{app}^{\varepsilon} \delta \rho^{\varepsilon} dx + \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \nabla \rho_{app}^{\varepsilon} \cdot (\rho^{\varepsilon} u^{\varepsilon} - \rho_{app}^{\varepsilon} u_{app}^{\varepsilon}) dx \\
&= \int_{\Omega} \nabla \cdot u_{app}^{\varepsilon} p'(\rho_{app}^{\varepsilon}) \delta \rho^{\varepsilon} dx - \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \rho_{app}^{\varepsilon} \nabla \cdot u_{app}^{\varepsilon} \delta \rho^{\varepsilon} dx \\
&\quad + \int_{\Omega} H''(\rho_{app}^{\varepsilon}) (\partial_t \rho_{app}^{\varepsilon} + \nabla \cdot (\rho_{app}^{\varepsilon} u_{app}^{\varepsilon})) \delta \rho^{\varepsilon} dx \\
&\quad + \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \nabla \rho_{app}^{\varepsilon} \cdot \delta u^{\varepsilon} \rho^{\varepsilon} dx + \int_{\Omega} \nabla \cdot u_{app}^{\varepsilon} P(\rho^{\varepsilon}, \rho_{app}^{\varepsilon}) dx \\
&= \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \nabla \rho_{app}^{\varepsilon} \cdot \delta u^{\varepsilon} \rho^{\varepsilon} dx + \varepsilon \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \delta \rho^{\varepsilon} (R^{bl} + \varepsilon R^{\varepsilon}) dx \\
&\quad + \int_{\Omega} \nabla \cdot u_{app}^{\varepsilon} P(\rho^{\varepsilon}, \rho_{app}^{\varepsilon}) dx.
\end{aligned}$$

The last two terms in the above identity are small (in a sense to be made precise later). So, let us focus on the first term in the right hand side: we have

$$\int_{\Omega} H''(\rho_{app}^{\varepsilon}) \nabla \rho_{app}^{\varepsilon} \cdot \delta u^{\varepsilon} \rho^{\varepsilon} dx - \int_{\Omega} \nabla p(\rho_{app}^{\varepsilon}) \cdot \delta u^{\varepsilon} dx = \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \nabla \rho_{app}^{\varepsilon} \cdot \delta u^{\varepsilon} \delta \rho^{\varepsilon} dx.$$

Inserting this expression into the last equality for  $I$ , from (3.3.19) we finally find

$$\begin{aligned}
\int_{\Omega} \nabla \delta p^{\varepsilon} \cdot \delta u^{\varepsilon} dx &= \frac{d}{dt} \int_{\Omega} E(\rho^{\varepsilon}, \rho_{app}^{\varepsilon}) dx + \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \nabla \rho_{app}^{\varepsilon} \cdot \delta u^{\varepsilon} \delta \rho^{\varepsilon} dx \\
&\quad + \varepsilon \int_{\Omega} H''(\rho_{app}^{\varepsilon}) \delta \rho^{\varepsilon} (R^{bl} + \varepsilon R^{\varepsilon}) dx + \int_{\Omega} \nabla \cdot u_{app}^{\varepsilon} P(\rho^{\varepsilon}, \rho_{app}^{\varepsilon}) dx.
\end{aligned} \tag{3.3.21}$$

At this point, we can perform energy estimates directly on equations (3.3.16)-

(3.3.17). Using (3.3.21) above, we finally obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho^\varepsilon |\delta u^\varepsilon|^2 + \frac{1}{\varepsilon^2} E(\rho^\varepsilon, \rho_{app}^\varepsilon) \right) dx \\
& + \mu \int_{\Omega} |\nabla_h \delta u^\varepsilon|^2 dx + \varepsilon \int_{\Omega} |\partial_3 \delta u^\varepsilon|^2 dx + \lambda \int_{\Omega} |\nabla \cdot \delta u^\varepsilon|^2 dx \\
& \leq \frac{1}{\varepsilon^2} \int_{\Omega} \delta \rho^\varepsilon \nabla G \cdot \delta u^\varepsilon dx - \frac{1}{\varepsilon^2} \int_{\Omega} H''(\rho_{app}^\varepsilon) \nabla \rho_{app}^\varepsilon \cdot \delta u^\varepsilon \delta \rho^\varepsilon dx \\
& - \frac{1}{\varepsilon} \int_{\Omega} H''(\rho_{app}^\varepsilon) \delta \rho^\varepsilon (R^{bl} + \varepsilon R^\varepsilon) dx - \frac{1}{\varepsilon^2} \int_{\Omega} \nabla \cdot u_{app}^\varepsilon P(\rho^\varepsilon, \rho_{app}^\varepsilon) dx \\
& - \frac{1}{\varepsilon} \int_{\Omega} e_3 \times \delta \rho^\varepsilon u_{app}^\varepsilon \cdot \delta u^\varepsilon dx - \int_{\Omega} \delta \rho^\varepsilon \partial_t u_{app}^\varepsilon \cdot \delta u^\varepsilon dx \\
& + \int_{\Omega} (\rho_{app}^\varepsilon u_{app}^\varepsilon - \rho^\varepsilon u^\varepsilon) \cdot \nabla u_{app}^\varepsilon \cdot \delta u^\varepsilon dx - \int_{\Omega} S^{bl} \cdot \delta u^\varepsilon dx - \varepsilon \int_{\Omega} S^\varepsilon \cdot \delta u^\varepsilon dx \\
& - \frac{1}{\varepsilon} \int_{\Omega} x_3 \int_0^1 \partial_3 \bar{\rho}(s x_3) ds (u_{0,h,b}^{bl})^\perp \cdot \delta u_h^\varepsilon dx \\
& + \frac{1}{\varepsilon} \int_{\Omega} (1-x_3) \int_0^1 \partial_3 \bar{\rho}(1-s(1-x_3)) ds (u_{0,h,t}^{bl})^\perp \cdot \delta u_h^\varepsilon dx = \sum_{j=1}^{11} I_j.
\end{aligned} \tag{3.3.22}$$

**Remark 3.3.3.** A standard regularization argument (see for instance [53] and [61]) must be used in order to derive the relative entropy inequality (3.3.22), where the equality holds if the solutions are regular enough.

Our next goal is to bound each term appearing in the sum  $\sum_{j=1}^{11} I_j$  in the right hand side of (3.3.22). Before doing that, let us remark that, since  $\rho_1 \in L^\infty(\Omega \times \mathbb{R}_+)$ , and consequently  $\rho_2 \in L^\infty(\Omega \times \mathbb{R}_+)$ , up to restrict our attention to all  $\varepsilon \leq \varepsilon_0$ , with  $\varepsilon_0$  depending on  $\|\rho_1\|_{L_{t,x}^\infty}$  and  $\|\rho_2\|_{L_{t,x}^\infty}$ , we can assume that  $-\frac{\sigma}{2} \leq \varepsilon \rho_1 + \varepsilon^2 \rho_2 \leq \frac{\sigma}{2}$  with  $\sigma > 0$  as in (3.3.3). Consequently, we can assume that  $0 < \rho_{app}^- \leq \rho_{app}^\varepsilon(x, t) \leq \rho_{app}^+$  for all  $\varepsilon > 0$  with

$$\rho_{app}^- = \inf_{(0,1)} \bar{\rho} - \sigma, \quad \rho_{app}^+ = \sup_{(0,1)} \bar{\rho} + \sigma.$$

Then, in view of (3.3.2), we have the following control:

$$E(\rho^\varepsilon(x, t), \rho_{app}^\varepsilon(x, t)) \geq c \left( |\delta \rho^\varepsilon(x, t)|^2 \mathbf{1}_{\{|\delta \rho^\varepsilon(x, t)| < 1\}} + |\delta \rho^\varepsilon(x, t)|^\gamma \mathbf{1}_{\{|\delta \rho^\varepsilon(x, t)| \geq 1\}} \right). \tag{3.3.23}$$

Resorting to the definitions (3.3.3), from (3.3.23) we derive the following lower bound.



**Lemma 3.3.4.** *For all  $(x, t) \in \Omega \times \mathbb{R}_+$  the following holds*

$$E\left(\rho^\varepsilon(x, t), \rho_{app}^\varepsilon(x, t)\right) \geq c\left([\delta\rho^\varepsilon]_{\text{ess}}^2(x, t) + \mathbf{1}_{\Omega_{\text{res}}(t)}(x)\right) \quad (3.3.24)$$

where  $c$  is a positive constant independent of  $\varepsilon$ ,  $t$  and  $x$ .

*Proof.* We divide the proof of the inequality into two steps: first we show that

$$E\left(\rho^\varepsilon(x, t), \rho_{app}^\varepsilon(x, t)\right) \geq c|\delta\rho^\varepsilon(x, t)|^2 \mathbf{1}_{\{|\delta\rho^\varepsilon(x, t)| < 1\}}$$

implies

$$E\left(\rho^\varepsilon(x, t), \rho_{app}^\varepsilon(x, t)\right) \geq c[\delta\rho^\varepsilon]_{\text{ess}}^2(x, t). \quad (3.3.25)$$

To prove (3.3.25), we just need to show that  $\Omega_{\text{ess}}(t) \subseteq \{|\delta\rho^\varepsilon(x, t)| < 1\}$ . Let  $x \in \Omega_{\text{ess}}(t)$ , then

$$-\frac{3}{2}\sigma \leq -\sigma - \varepsilon\rho_1(x, t) - \varepsilon^2\rho_2(x, t) < \delta\rho^\varepsilon(x, t) < -\varepsilon\rho_1(x, t) - \varepsilon^2\rho_2(x, t) + \sigma \leq \frac{3}{2}\sigma,$$

where we have used that  $-\frac{\sigma}{2} \leq \varepsilon\rho_1(x, t) + \varepsilon^2\rho_2(x, t) \leq \frac{\sigma}{2}$ . Choosing

$$\sigma < \min\left(\frac{2}{3}, \inf_{(0,1)} \bar{\rho}\right),$$

we have that  $|\delta\rho^\varepsilon(x, t)| < 1$ . Afterwards, we prove that for  $x \in \Omega_{\text{res}}(t)$

$$E\left(\rho^\varepsilon(x, t), \rho_{app}^\varepsilon(x, t)\right) \geq c, \quad (3.3.26)$$

where  $c$  is a positive constant independent of  $\varepsilon$ ,  $t$  and  $x$ . By the definition of  $\Omega_{\text{res}}(t)$ , either  $\rho^\varepsilon(x, t) \leq \bar{\rho}(x_3) - \sigma$  or  $\rho^\varepsilon(x, t) \geq \bar{\rho}(x_3) + \sigma$ . Hence, since  $E(\cdot, \rho_{app}^\varepsilon(x, t))$  is strictly decreasing before  $\rho_{app}^\varepsilon(x, t)$  and strictly increasing after  $\rho_{app}^\varepsilon(x, t)$ , we get

$$E\left(\rho^\varepsilon(x, t), \rho_{app}^\varepsilon(x, t)\right) \geq E\left(\bar{\rho}(x_3) - \sigma, \rho_{app}^\varepsilon(x, t)\right)$$

and

$$E\left(\rho^\varepsilon(x, t), \rho_{app}^\varepsilon(x, t)\right) \geq E\left(\bar{\rho}(x_3) + \sigma, \rho_{app}^\varepsilon(x, t)\right).$$

respectively in each part of  $\Omega_{\text{res}}(t)$ . By Taylor's formula, up to taking a smaller  $\sigma$  (which amounts to choosing a smaller  $\varepsilon_0$ ),

$$\begin{aligned} E\left(\bar{\rho}(x_3) - \sigma, \rho_{app}^\varepsilon(x, t)\right) &\geq \frac{H''(\rho_{app}^\varepsilon(x, t))}{4}(-\sigma - \varepsilon\rho_1(x, t) - \varepsilon^2\rho_2(x, t))^2 \\ &\geq \frac{H''(\rho_{app}^\varepsilon(x, t))\sigma^2}{16} \end{aligned}$$

and

$$\begin{aligned} E\left(\bar{\rho}(x_3) + \sigma, \rho_{app}^\varepsilon(x, t)\right) &\geq \frac{H''(\rho_{app}^\varepsilon(x, t))}{4} (\sigma - \varepsilon \rho_1(x, t) - \varepsilon^2 \rho_2(x, t))^2 \\ &\geq \frac{H''(\rho_{app}^\varepsilon(x, t)) \sigma^2}{16}. \end{aligned}$$

Then, using the uniform boundedness in time and space of  $\rho_{app}^\varepsilon$  and the hypothesis (3.1.3), we get (3.3.26) and, together with (3.3.25), the lemma is proved.  $\square$

Notice that  $[[\delta\rho^\varepsilon]]_{\text{ess}}$  is uniformly bounded. Next, we claim that there exists a constant  $C > 0$  such that, for all  $T > 0$  fixed, one has

$$\|[\delta\rho^\varepsilon]_{\text{res}}\|_{L_T^\infty(L^p)}^p \leq C \varepsilon^2 \quad \forall p \in [1, \gamma]. \quad (3.3.27)$$

Indeed, concerning the  $L^1$  norm, by Hölder's inequality and the  $L^\infty$  control on  $\rho_{app}^\varepsilon$  we deduce

$$\begin{aligned} \int_{\Omega} |[\delta\rho^\varepsilon]_{\text{res}}| &\leq \int_{\Omega} [\rho^\varepsilon]_{\text{res}} + \int_{\Omega} [\rho_{app}^\varepsilon]_{\text{res}} \\ &\leq \left( \int_{\Omega} (\rho^\varepsilon)^\gamma \mathbf{1}_{\Omega_{\text{res}}} \right)^{1/\gamma} (\mathcal{L}(\Omega_{\text{res}}))^{1/\gamma'} + C \mathcal{L}(\Omega_{\text{res}}) \leq C \varepsilon^2, \end{aligned}$$

where, in the last step, we have used estimates (3.3.6). As for the  $L^p$  norm, for any  $1 < p \leq \gamma$  we start by decomposing

$$\Omega_{\text{res}}(t) = \{0 < \rho^\varepsilon(x, t) \leq \bar{\rho}(x_3) - \sigma\} \cup \{\rho^\varepsilon(x, t) \geq \bar{\rho}(x_3) + \sigma\}. \quad (3.3.28)$$

For the first set, it is just a matter of applying (3.3.6) again, since  $\rho^\varepsilon$  is bounded on that set. For the second set, we use the fact that, for  $a \geq 1$  and  $b \geq 0$ , with  $b \leq b^* \leq a$ , one has  $|a - b|^p \leq (a + b)^p \leq C_p (a^p + 1) \leq C_p (a^\gamma + 1)$ .

### 3.3.3 Estimates of source terms

Below, we estimate every term  $I_j$  appearing in (3.3.22), for  $0 < \varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0$  is given by Lemma 3.3.4. On the one hand, for the terms  $I_1, I_2, I_3, I_5$  and  $I_6$  we need to treat separately the cases  $\gamma \geq 2$  and  $3/2 \leq \gamma < 2$  since we use different estimates. On the other hand, the terms  $I_4, I_8, I_9, I_{10}$  and  $I_{11}$  can be controlled in the same way for any  $\gamma$ . The term  $I_7$  is more intricate. It is written as a sum of five terms: for some terms we distinguish the case  $\gamma \geq 2$  and  $3/2 \leq \gamma < 2$ , for the others we use one estimate for both cases.

The easiest terms to handle are  $I_3, I_4, I_6$  and  $I_9$ . The terms  $I_1$  and  $I_2$  need to be combined with the Coriolis term  $I_5$ . For the remaining part of  $I_5$ , we rely on

Hardy's inequality, which is also useful to deal with  $I_7$ ,  $I_8$ ,  $I_{10}$  and  $I_{11}$ . This idea is borrowed from [18]: let us briefly explain it. Whenever there is a boundary layer term  $G^{bl}(\frac{x_3}{\varepsilon})$  such as  $u_{0,h}^{bl}(\frac{x_3}{\varepsilon})$ , we gain one additional  $\varepsilon$  by using the decay of  $G^{bl}$  in  $\zeta$ . The price to pay is a  $\partial_3$  derivative on  $\delta u^\varepsilon$ , which however can be swallowed by the third term in the left hand side of (3.3.22).

For every term, we decompose  $u_{app}^\varepsilon$  according to (3.2.33). The terms which require more care are those of order  $O(1)$ , which involve in general  $u_{0,h}$  and  $u_{0,h}^{bl}$ , except for  $I_7$  where the product  $u_{app}^\varepsilon \cdot \nabla u_{app}^\varepsilon$  also involves  $u_{1,3}$  and  $u_{1,3}^{bl}$  at order  $O(1)$ . On the other hand, for the terms which are not of order  $O(1)$  the analysis can be always reduced to the case  $I_3$ ,  $I_4$ ,  $I_6$  and  $I_9$  and the same estimates are used.

We introduce now an anisotropic version of the standard Sobolev embedding  $\dot{H}^1 \hookrightarrow L^6$ . This estimate enables to handle the anisotropy of the viscosity and it will be used later in the control for  $I_7$ .

**Lemma 3.3.5** (anisotropic Sobolev embedding). *Let  $\Omega = \mathbb{R}^2 \times (0, 1)$ . There exists a universal constant  $C > 0$  such that for all  $\kappa > 0$ , for all  $u \in H_0^1(\Omega)$ ,*

$$\|u\|_{L^6(\Omega)} \leq C \left( \kappa^{-\frac{1}{2}} \|\nabla_h u\|_{L^2(\Omega)} + \kappa \|\partial_3 u\|_{L^2(\Omega)} \right). \quad (3.3.29)$$

*Proof of Lemma 3.3.5.* Let  $\kappa > 0$  and  $u \in H_0^1(\Omega)$ . We first extend  $u$  by zero on  $\mathbb{R}^3 \setminus \Omega$  and still denote the extended function by  $u$ . Now  $u \in H^1(\mathbb{R}^3)$ . We then consider the rescaled function

$$u_\kappa(y_h, y_3) = u \left( \frac{y_h}{\kappa^{\frac{1}{2}}}, \kappa y_3 \right), \quad (y_h, y_3) \in \mathbb{R}^3.$$

By Sobolev's inequality [56, estimate (II.3.7)] for the whole space, there exists a universal constant  $C \in (0, \infty)$  such that

$$\|u_\kappa\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla u_\kappa\|_{L^2(\mathbb{R}^3)}.$$

Estimate (3.3.29) then follows by a change of variables and the fact that  $u$  is zero outside the strip  $\mathbb{R}^2 \times (0, 1)$ ,

$$\begin{aligned} \|u_\kappa\|_{L^6(\mathbb{R}^3)} &= \|u\|_{L^6(\Omega)}, \\ \|\nabla u_\kappa\|_{L^2(\mathbb{R}^3)} &= \kappa^{-\frac{1}{2}} \|\nabla_h u\|_{L^2(\Omega)} + \kappa \|\partial_3 u\|_{L^2(\Omega)}. \end{aligned}$$

This concludes the proof.  $\square$

Of course, for every term involving a boundary layer, one has to equally consider the top and bottom boundary layers; again, for simplicity, we focus on the boundary layer at the bottom only. In the computations below,  $U$  and  $U^{bl}$  generically denote remainder terms in the expansion for  $u_{app}^\varepsilon$  or its derivatives. The

definition of these remainder terms may change from the estimate of one  $I_i$  to another  $I_j$ .

First, we deal with the terms for which estimates hold for any  $\gamma$ .

*Term  $I_4$ .* We start by considering  $I_4$ , and more precisely when restricted to the essential set. Using (3.3.20), the assumptions on the pressure function and the fact that  $[\delta\rho^\varepsilon]_{\text{ess}}$  is uniformly bounded, we can estimate

$$\begin{aligned} \left| \frac{1}{\varepsilon^2} \int_{\Omega} \nabla \cdot u_{app}^\varepsilon [P(\rho^\varepsilon, \rho_{app}^\varepsilon)]_{\text{ess}} \right| &\leq \frac{1}{\varepsilon^2} \|\nabla \cdot u_{app}^\varepsilon\|_{L^\infty} \|[\delta\rho^\varepsilon]_{\text{ess}}\|_{L^2}^2 \\ &\leq C \varepsilon \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon), \end{aligned}$$

where we have used also that  $\nabla \cdot u_{app}^\varepsilon = \varepsilon (\nabla \cdot u_1 + \nabla \cdot_h u_{1,h}^{bl})$ .

Let us consider the integral over the residual set. By (3.3.20) again, we have  $[P]_{\text{res}} = [p(\rho^\varepsilon) - p(\rho_{app}^\varepsilon)]_{\text{res}} - p'(\rho_{app}^\varepsilon)[\delta\rho^\varepsilon]_{\text{res}}$ . The second term can be easily controlled, in view of the uniform boundedness of  $\rho_{app}^\varepsilon$  and of the  $L^1$  estimate in (3.3.27). For the first term, we use decomposition (3.3.28): when  $\rho^\varepsilon$  is bounded, the same argument as above applies. On the set  $\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}$ , instead, we use hypothesis (3.1.3), the uniform boundedness for  $\rho_{app}^\varepsilon$  and the controls in (3.3.6) to get

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\Omega} |\nabla \cdot u_{app}^\varepsilon| |p(\rho^\varepsilon) - p(\rho_{app}^\varepsilon)| \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} &\leq \frac{C}{\varepsilon} \int_{\Omega} |p(\rho^\varepsilon) - p(\rho_{app}^\varepsilon)| \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} \\ &\leq \frac{C}{\varepsilon} \left( \|[\rho^\varepsilon]_{\text{res}}\|_{L^\gamma}^\gamma + \mathcal{L}(\Omega_{\text{res}}) \right) \leq C \varepsilon. \end{aligned}$$

Putting everything together, we finally infer that

$$|I_4| \leq C \varepsilon + C \varepsilon \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon), \quad (3.3.30)$$

where the last term will be handled by Grönwall's lemma.

*Term  $I_9$ .* Let us switch our attention to  $I_9$ : its control is direct, as no  $\rho^\varepsilon$  or  $\delta\rho^\varepsilon$  enter into play in its definition. We get

$$|I_9| \leq \varepsilon \|S^\varepsilon\|_{L^2} \|\delta u^\varepsilon\|_{L^2} \leq C \varepsilon K_2(t), \quad (3.3.31)$$

where the function  $K_2(t) = \|u^\varepsilon(t)\|_{L^2} + \|u_{app}^\varepsilon(t)\|_{L^2}$  belongs to  $L^2([0, T])$  for all  $T > 0$ .

Terms  $I_8$ ,  $I_{10}$ , and  $I_{11}$ . We deal with  $I_8$  using Hardy's inequality. This gives

$$\begin{aligned} |I_8| &= \varepsilon \left| \int_{\Omega} \frac{x_3}{\varepsilon} S^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot \frac{\delta u^\varepsilon}{x_3} dx \right| \\ &\leq C_\delta \varepsilon \left\| \zeta S^{bl} \right\|_{L^2}^2 + \delta \varepsilon \left\| \partial_3 \delta u^\varepsilon \right\|_{L^2}^2 \leq C_\delta \varepsilon^2 + \delta \varepsilon \left\| \partial_3 \delta u^\varepsilon \right\|_{L^2}^2, \end{aligned} \quad (3.32)$$

for some small  $\delta > 0$ , to be chosen later. The same holds for  $I_{10}$  and  $I_{11}$  and we get the bounds

$$\begin{aligned} |I_{10}| &= \varepsilon \left| \int_{\Omega} \frac{x_3^2}{\varepsilon^2} \left( \int_0^1 \partial_3 \bar{\rho}(s x_3) ds \right) (u_{0,h,b}^{bl})^\perp \cdot \frac{\delta u_h^\varepsilon}{x_3} dx \right| \\ &\leq C_\delta \varepsilon \left\| \zeta^2 u_{0,h,b}^{bl} \right\|_{L^2}^2 + \delta \varepsilon \left\| \partial_3 \delta u_h^\varepsilon \right\|_{L^2}^2 \leq C_\delta \varepsilon^2 + \delta \varepsilon \left\| \partial_3 \delta u_h^\varepsilon \right\|_{L^2}^2 \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} |I_{11}| &= \varepsilon \left| \int_{\Omega} \frac{(1-x_3)^2}{\varepsilon^2} \left( \int_0^1 \partial_3 \bar{\rho}(1-s(1-x_3)) ds \right) (u_{0,h,t}^{bl})^\perp \cdot \frac{\delta u_h^\varepsilon}{1-x_3} dx \right| \\ &\leq C_\delta \varepsilon \left\| \eta^2 u_{0,h,b}^{bl} \right\|_{L^2}^2 + \delta \varepsilon \left\| \partial_3 \delta u_h^\varepsilon \right\|_{L^2}^2 \leq C_\delta \varepsilon^2 + \delta \varepsilon \left\| \partial_3 \delta u_h^\varepsilon \right\|_{L^2}^2, \end{aligned} \quad (3.34)$$

where we have used the uniform boundedness of  $\partial_3 \bar{\rho}$ . In the second bound, Hardy's inequality has been used via a change of variable  $\tilde{x}_3 = 1 - x_3$ . In the three estimates we have used the fact that  $\left\| \zeta S^{bl} \right\|_{L^2}^2$ ,  $\left\| \zeta^2 u_{0,h,b}^{bl} \right\|_{L^2}^2$  and  $\left\| \eta^2 u_{0,h,t}^{bl} \right\|_{L^2}^2$  are  $O(\varepsilon^2)$ . Then, for  $\delta > 0$  small enough, we can swallow the last terms on the right hand side of (3.32), (3.33) and (3.34) into the left hand side of (3.32).

We consider now the terms whose bounds must be treated differently if  $\gamma \geq 2$  or  $3/2 \leq \gamma < 2$ .

*Term  $I_3$ .* Let us deal with  $I_3$  now. First of all, observe that  $\left\| R^{bl}(x_3/\varepsilon) \right\|_{L_x^2}^2 = O(\varepsilon)$ : thanks to this, we can estimate

$$\begin{aligned} \frac{1}{\varepsilon} \left| \int_{\Omega} H''(\rho_{app}^\varepsilon) [\delta \rho^\varepsilon]_{\text{ess}} \left( R^{bl} + \varepsilon R^\varepsilon \right) \right| &\leq C \frac{1}{\varepsilon} \left\| R^{bl} + \varepsilon R^\varepsilon \right\|_{L^2} \left\| [\delta \rho^\varepsilon]_{\text{ess}} \right\|_{L^2} \\ &\leq C \varepsilon + \frac{C}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon), \end{aligned}$$

where we have used also (3.34). As for the residual part, in view of (3.327), we can argue in exactly the same way if  $\gamma \geq 2$ . If  $3/2 \leq \gamma < 2$ , instead, we put the  $L^\infty$  norm on the remainder terms and use the  $L^1$  bound of (3.327) to get

$$\frac{1}{\varepsilon} \left| \int_{\Omega} H''(\rho_{app}^\varepsilon) [\delta \rho^\varepsilon]_{\text{res}} \left( R^{bl} + \varepsilon R^\varepsilon \right) \right| \leq C \varepsilon.$$

Finally, we arrive at

$$|I_3| \leq C \varepsilon + \frac{C}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon). \quad (3.3.35)$$

*Term  $I_6$ .* For  $I_6$ , we use once again the decomposition of  $\delta\rho^\varepsilon$  into essential and residual parts. For the term involving the essential part, thanks to Young's inequality and to the controls (3.3.14) and (3.3.24), one has

$$\begin{aligned} \left| \int_{\Omega} [\delta\rho^\varepsilon]_{\text{ess}} \partial_t u_{app}^\varepsilon \cdot \delta u^\varepsilon \right| &\leq \|[\delta\rho^\varepsilon]_{\text{ess}}\|_{L^2} \|\partial_t u_{app}^\varepsilon\|_{L^\infty} \|\delta u^\varepsilon\|_{L^2} \\ &\leq C \varepsilon^2 K_1(t) + \frac{1}{\varepsilon^2} E(\rho^\varepsilon, \rho_{app}^\varepsilon), \end{aligned}$$

where the function  $K_1$  (here  $K_1 = \|u^\varepsilon\|_{L^2}^2 + \|u_{app}^\varepsilon\|_{L^2}^2$ ) belongs to  $L^1([0, T])$  for all  $T > 0$ .

Next, let us consider the term involving the residual part: in the case when  $\gamma \geq 2$ , it can be dealt with in exactly the same way as for the term involving the essential part, in view of (3.3.27). If instead  $3/2 \leq \gamma < 2$ , we have to argue in a different way. First of all, we write

$$\int_{\Omega} [\delta\rho^\varepsilon]_{\text{res}} \partial_t u_{app}^\varepsilon \cdot \delta u^\varepsilon = \int_{\Omega} [\rho^\varepsilon]_{\text{res}} \partial_t u_{app}^\varepsilon \cdot \delta u^\varepsilon - \int_{\Omega} [\rho_{app}^\varepsilon]_{\text{res}} \partial_t u_{app}^\varepsilon \cdot \delta u^\varepsilon.$$

For the second term, we use the uniform boundedness of  $\rho_{app}^\varepsilon$  and estimate (3.3.6) to gather, for some function  $K_2 \in L^2([0, T])$  for all  $T > 0$ , the inequality

$$\left| \int_{\Omega} [\rho_{app}^\varepsilon]_{\text{res}} \partial_t u_{app}^\varepsilon \cdot \delta u^\varepsilon \right| \leq C \|\delta u^\varepsilon\|_{L^2} (\mathcal{L}(\Omega_{\text{res}}))^{1/2} \leq C \varepsilon K_2(t).$$

For the term involving the residual part of  $\rho^\varepsilon$ , we use decomposition (3.3.28) for the residual set, where the integral over the first set can be treated exactly as just done for  $\rho_{app}^\varepsilon$  (because therein  $\rho^\varepsilon$  is uniformly bounded). Concerning the integral over the second set, we get instead

$$\begin{aligned} \left| \int_{\Omega} \rho^\varepsilon \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} \partial_t u_{app}^\varepsilon \cdot \delta u^\varepsilon \right| &\leq C \|\sqrt{\rho^\varepsilon} \delta u^\varepsilon\|_{L^2} \left( \int_{\Omega} \rho^\varepsilon \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} \right)^{1/2} \\ &\leq C \|\sqrt{\rho^\varepsilon} \delta u^\varepsilon\|_{L^2}^2 + C \varepsilon^2, \end{aligned}$$

since the last integral in the first line can be bounded by the integral over the residual set, for which we can use (3.3.6).

Let us introduce the following notation: we set  $\delta_{2-}(\gamma) = 1$  if  $3/2 \leq \gamma < 2$ ,  $\delta_{2-}(\gamma) = 0$  otherwise. In the end, from the previous computations we get

$$|I_6| \leq C \varepsilon (\varepsilon K_1(t) + \delta_{2-}(\gamma) K_2(t) + \delta_{2-}(\gamma) \varepsilon) \quad (3.3.36)$$

$$+ C \left( \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) + \delta_{2-}(\gamma) \left\| \sqrt{\rho^\varepsilon} \delta u^\varepsilon \right\|_{L^2}^2 \right).$$

*Terms  $I_1$ ,  $I_2$  and  $I_5$ .* Terms  $I_1$ ,  $I_2$  and  $I_5$  have to be combined together, enabling to see a cancellation at the highest order in  $\varepsilon$ . Such a cancellation is already a key point in [50]. Namely, after setting  $U := u_{app}^\varepsilon - (u_{0,h} + u_{0,h}^{bl}, 0)$ , we can write

$$\begin{aligned} I_1 + I_2 + I_5 &= \frac{1}{\varepsilon^2} \int_{\Omega} \delta \rho^\varepsilon \nabla G \cdot \delta u^\varepsilon dx - \frac{1}{\varepsilon^2} \int_{\Omega} H''(\rho_{app}^\varepsilon) \partial_3 \bar{\rho} \delta u_3^\varepsilon \delta \rho^\varepsilon dx \\ &\quad - \frac{1}{\varepsilon} \int_{\Omega} H''(\rho_{app}^\varepsilon) \nabla \rho_1 \cdot \delta u^\varepsilon \delta \rho^\varepsilon dx \\ &\quad - \frac{1}{\varepsilon} \int_{\Omega} \delta \rho^\varepsilon \frac{p'(\bar{\rho})}{\bar{\rho}} (\nabla_h^\perp \rho_1)^\perp \cdot \delta u_h^\varepsilon dx - \frac{1}{\varepsilon} \int_{\Omega} \delta \rho^\varepsilon (u_{0,h}^{bl})^\perp \cdot \delta u_h^\varepsilon dx \\ &\quad - \int_{\Omega} \delta \rho^\varepsilon e_3 \times U(x_h, x_3, \frac{x_3}{\varepsilon}, \frac{1-x_3}{\varepsilon}, t) \cdot \delta u^\varepsilon dx. \end{aligned}$$

Notice that  $H''(\bar{\rho}) = \frac{p'(\bar{\rho})}{\bar{\rho}}$  and that  $(\nabla_h^\perp \rho_1)^\perp = -\nabla_h \rho_1$ . Moreover, from (3.2.2) we have

$$\bar{\rho} \nabla G = p'(\bar{\rho}) \nabla \bar{\rho},$$

so that

$$\begin{aligned} I_1 + I_2 + I_5 &= -\frac{1}{\varepsilon^2} \int_{\Omega} (H''(\rho_{app}^\varepsilon) - H''(\bar{\rho})) \partial_3 \bar{\rho} \delta u_3^\varepsilon \delta \rho^\varepsilon dx - \frac{1}{\varepsilon} \int_{\Omega} H''(\bar{\rho}) \partial_3 \rho_1 \delta u_3^\varepsilon \delta \rho^\varepsilon dx \\ &\quad - \frac{1}{\varepsilon} \int_{\Omega} (H''(\rho_{app}^\varepsilon) - H''(\bar{\rho})) \nabla \rho_1 \cdot \delta u^\varepsilon \delta \rho^\varepsilon dx - \frac{1}{\varepsilon} \int_{\Omega} \delta \rho^\varepsilon (u_{0,h}^{bl})^\perp \cdot \delta u_h^\varepsilon dx \\ &\quad - \int_{\Omega} \delta \rho^\varepsilon U_h^\perp(x_h, x_3, \frac{x_3}{\varepsilon}, \frac{1-x_3}{\varepsilon}, t) \cdot \delta u_h^\varepsilon dx = J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Using a Taylor expansion for  $h(z) = H''(z)$  with integral remainder, we can write

$$\begin{aligned} J_1 + J_2 &= -\frac{1}{\varepsilon} \int_{\Omega} (h'(\bar{\rho}) \rho_1 \partial_3 \bar{\rho} + h(\bar{\rho}) \partial_3 \rho_1) \delta u_3^\varepsilon \delta \rho^\varepsilon dx \\ &\quad - \int_{\Omega} \rho_1^2 \left( \int_0^1 (1-s) h''(\bar{\rho} + s\varepsilon \rho_1) ds \right) \partial_3 \bar{\rho} \delta u_3^\varepsilon \delta \rho^\varepsilon dx \\ &= - \int_{\Omega} \rho_1^2 \left( \int_0^1 (1-s) h''(\bar{\rho} + s\varepsilon \rho_1) ds \right) \partial_3 \bar{\rho} \delta u_3^\varepsilon \delta \rho^\varepsilon dx \end{aligned}$$

where we have used (3.2.5) in the last equality. Since  $\rho_1$ ,  $\bar{\rho}$  and  $\partial_3 \bar{\rho}$  are uniformly bounded in time and space and from (3.1.3), the control of  $J_1 + J_2$  becomes similar to the one exhibited for  $I_6$ . In the same way, after noticing that both  $\nabla_h \rho_1$  and  $\varepsilon^{-1} (H''(\rho_{app}^\varepsilon) - H''(\bar{\rho}))$  are uniformly bounded both in time and space, the control of  $J_3$  is obtained. Then,  $J_1 + J_2$  and  $J_3$  verify estimate (3.3.36). The same can be said about  $J_5$ , also because  $U_h$  belongs to  $L_{t,x}^\infty$ .

Therefore, it remains to deal with  $J_4$ , for which we rely on Hardy's inequality. More precisely, let us start, as usual, by dealing with the essential part: we have

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_{\Omega} [\delta \rho^\varepsilon]_{\text{ess}} (u_{0,h}^{bl})^\perp \cdot \delta u_h^\varepsilon \right| &= \left| \int_{\Omega} [\delta \rho^\varepsilon]_{\text{ess}} \frac{x_3}{\varepsilon} (u_{0,h}^{bl})^\perp \cdot \frac{\delta u_h^\varepsilon}{x_3} \right| \\ &\leq \| [\delta \rho^\varepsilon]_{\text{ess}} \|_{L^2} \left\| \zeta u_{0,h}^{bl}(t, x_h, \zeta) \right\|_{L_{t,x,\zeta}^\infty} \| \partial_3 \delta u_h^\varepsilon \|_{L^2} \\ &\leq \frac{C}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) + C \varepsilon^2 \left\| \zeta u_{0,h}^{bl} \right\|_{L_{t,x,\zeta}^\infty}^2 \| \partial_3 \delta u_h^\varepsilon \|_{L^2}^2. \end{aligned}$$

Notice that, for  $\varepsilon$  small enough, the second term can be swallowed by the third term in the left hand side of (3.3.22). As for the control of the residual part, suppose that  $\gamma \geq 2$  for a while: in this case, we can argue in the exact same way and obtain, in view of (3.3.27), that

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_{\Omega} [\delta \rho^\varepsilon]_{\text{res}} (u_{0,h}^{bl})^\perp \cdot \delta u_h^\varepsilon \right| &\leq \| [\delta \rho^\varepsilon]_{\text{res}} \|_{L^2} \left\| \zeta u_{0,h}^{bl} \right\|_{L_{t,x,\zeta}^\infty} \| \partial_3 \delta u_h^\varepsilon \|_{L^2} \\ &\leq \frac{C}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) + C \varepsilon^2 \left\| \zeta u_{0,h}^{bl} \right\|_{L_{t,x,\zeta}^\infty}^2 \| \partial_3 \delta u_h^\varepsilon \|_{L^2}^2. \end{aligned}$$

The case  $3/2 \leq \gamma < 2$  is slightly more involved. The control over  $\{0 < \rho^\varepsilon \leq \bar{\rho} - \sigma\}$  does not present any special difficulty, since we have on that set uniform bounds for  $\rho_\varepsilon$  (and obviously for  $\rho_{app}^\varepsilon$ ): then, we can argue as for controlling the essential part. Hence, let us focus on  $\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}$ . First of all, using that  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ , we notice that

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_{\Omega} [\delta \rho^\varepsilon]_{\text{res}} (u_{0,h}^{bl})^\perp \cdot \delta u_h^\varepsilon \right| &\leq \\ \frac{1}{\varepsilon} \int_{\Omega} \sqrt{\delta \rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} |u_{0,h}^{bl}| \sqrt{\rho_\varepsilon} |\delta u_h^\varepsilon| + \frac{1}{\varepsilon} \int_{\Omega} \sqrt{\delta \rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} |u_{0,h}^{bl}| \sqrt{\rho_{app}^\varepsilon} |\delta u_h^\varepsilon|. \end{aligned} \quad (3.3.37)$$

For the first term in the right hand side of (3.3.37), we proceed in the following way:

$$\begin{aligned} \int_{\Omega} \sqrt{\delta \rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} |u_{0,h}^{bl}| \sqrt{\rho_\varepsilon} |\delta u_h^\varepsilon| \\ \leq \| [\delta \rho^\varepsilon]_{\text{res}} \|_{L^\gamma}^{1/2} \left\| u_{0,h}^{bl} \right\|_{L^\infty} \| \sqrt{\rho_\varepsilon} \delta u_h^\varepsilon \|_{L^2} (\mathcal{L}(\Omega_{\text{res}}))^{1/q}, \end{aligned}$$



where  $1/(2\gamma) + 1/2 + 1/q = 1$ . Using (3.3.23) and (3.3.24), we deduce that

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{\Omega} \sqrt{\delta\rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} |u_{0,h}^{bl}| \sqrt{\rho^\varepsilon} |\delta u_h^\varepsilon| \\ & \leq \varepsilon^{1/\gamma + 2/q - 1} \left( \frac{1}{\varepsilon^2} E(\rho^\varepsilon, \rho_{app}^\varepsilon) \right)^{1/(2\gamma) + 1/q} \|u_{0,h}^{bl}\|_{L^\infty} \|\sqrt{\rho^\varepsilon} \delta u_h^\varepsilon\|_{L^2} \\ & = \left( \frac{1}{\varepsilon^2} E(\rho^\varepsilon, \rho_{app}^\varepsilon) \right)^{1/2} \|u_{0,h}^{bl}\|_{L^\infty} \|\sqrt{\rho^\varepsilon} \delta u_h^\varepsilon\|_{L^2}. \end{aligned}$$

After an application of Young's inequality, this term can be controlled by Grönwall's lemma in the final estimate. For the last term in (3.3.37), we can argue in the following way:

$$\begin{aligned} & \int \sqrt{\delta\rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} |u_{0,h}^{bl}| \sqrt{\rho_{app}^\varepsilon} |\delta u_h^\varepsilon| = \varepsilon \int \sqrt{\delta\rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} \left| \frac{x_3}{\varepsilon} u_{0,h}^{bl} \right| \sqrt{\rho_{app}^\varepsilon} \left| \frac{\delta u_h^\varepsilon}{x_3} \right| \\ & \leq \varepsilon \int \left( \sqrt{\rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} \left| \frac{x_3}{\varepsilon} u_{0,h}^{bl} \right| \sqrt{\rho_{app}^\varepsilon} \left| \frac{\delta u_h^\varepsilon}{x_3} \right| + \rho_{app}^\varepsilon \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} \left| \frac{x_3}{\varepsilon} u_{0,h}^{bl} \right| \left| \frac{1}{x_3} \delta u_h^\varepsilon \right| \right) \\ & \leq \varepsilon \|\partial_3 \delta u_h^\varepsilon\|_{L^2} \|\zeta u_{0,h}^{bl}\|_{L^\infty} \times \\ & \quad \left( \|\rho_{app}^\varepsilon\|_{L^\infty}^{1/2} \|[\rho^\varepsilon]_{res}\|_{L^\gamma}^{1/2} (\mathcal{L}(\Omega_{res}))^{1/q} + \|\rho_{app}^\varepsilon\|_{L^\infty} (\mathcal{L}(\Omega_{res}))^{1/2} \right), \end{aligned}$$

where  $q$  is defined as above. At this point, we notice that, in view of (3.3.6), we have that  $\|[\rho^\varepsilon]_{res}\|_{L^\gamma} = O(\varepsilon^{2/\gamma})$  and that  $\mathcal{L}(\Omega_{res}) = O(\varepsilon^2)$ . Therefore, we finally find

$$\frac{1}{\varepsilon} \int \sqrt{\delta\rho^\varepsilon} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}} |u_{0,h}^{bl}| \sqrt{\rho_{app}^\varepsilon} |\delta u_h^\varepsilon| \leq C_\delta \varepsilon + \delta \varepsilon \|\partial_3 \delta u_h^\varepsilon\|_{L^2}^2,$$

where, for  $\delta > 0$  small enough, the last term can be absorbed in the left hand side of the energy inequality. In the end, we deduce the following control for  $\varepsilon$  small enough

$$\begin{aligned} |I_1 + I_2 + I_5| & \leq \frac{C}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) + C\delta_{2-(\gamma)} \|\sqrt{\rho^\varepsilon} \delta u_h^\varepsilon\|_{L^2}^2 \\ & \quad + C\varepsilon(\varepsilon K_1(t) + \delta_{2-(\gamma)} K_2(t) + \delta_{2-(\gamma)}) + (C\varepsilon^2 + \delta_{2-(\gamma)}\delta\varepsilon) \|\partial_3 \delta u_h^\varepsilon\|_{L^2}^2 \end{aligned} \quad (3.3.38)$$

where we recall that the last term in the right hand side can be absorbed into the left hand side of (3.3.22). Finally, let us deal with  $I_7$ .

Term  $I_7$ . We start by considering the following decomposition:

$$\begin{aligned} I_7 &= - \int_{\Omega} \delta \rho^\varepsilon u_{app}^\varepsilon \cdot \nabla u_{app}^\varepsilon \cdot \delta u^\varepsilon - \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon \delta u_3^\varepsilon \partial_\zeta u_{0,h}^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot \delta u_h^\varepsilon \\ &\quad - \int_{\Omega} \rho^\varepsilon \delta u_3^\varepsilon \partial_\zeta u_1^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot \delta u^\varepsilon - \int_{\Omega} \rho^\varepsilon \delta u_h^\varepsilon \cdot \nabla_h (u_{0,h} + u_{0,h}^{bl}) \cdot \delta u_h^\varepsilon - \varepsilon \int_{\Omega} \rho^\varepsilon \delta u^\varepsilon \cdot U \cdot \delta u^\varepsilon \\ &= J_6 + J_7 + J_8 + J_9 + J_{10}, \end{aligned}$$

where  $\varepsilon U = \varepsilon \left( \nabla u_1 + \begin{pmatrix} \nabla_h \\ 0 \end{pmatrix} u_1^{bl} \right)$  is the remainder term in the expansion for  $\nabla u_{app}^\varepsilon$ . The first term  $J_6$  can be handled as done with  $I_6$ , after noticing that

$$u_{app}^\varepsilon \cdot \nabla u_{app}^\varepsilon = (u_{0,h} + u_{0,h}^{bl}) \cdot \nabla_h (u_{0,h} + u_{0,h}^{bl}) + (u_{1,3} + u_{1,3}^{bl}) \partial_\zeta u_{0,h}^{bl} \left( \frac{x_3}{\varepsilon} \right) + h.o.t.,$$

where *h.o.t.* represents higher order terms in  $\varepsilon$ . Then  $u_{app}^\varepsilon \cdot \nabla u_{app}^\varepsilon$  is uniformly bounded in  $L_{t,x}^\infty$ . Therefore,  $J_6$  also verifies an inequality similar to (3.3.36) above. The terms  $J_8$ ,  $J_9$  and  $J_{10}$  can be simply bounded as follows:

$$\begin{aligned} |J_8| &\leq C \left\| \partial_\zeta u_1^{bl} \right\|_{L_{t,x}^\infty} \left\| \sqrt{\rho^\varepsilon} \delta u^\varepsilon \right\|_{L^2}^2, \\ |J_9| &\leq C \left\| \nabla_h (u_{0,h} + u_{0,h}^{bl}) \right\|_{L_{t,x}^\infty} \left\| \sqrt{\rho^\varepsilon} \delta u^\varepsilon \right\|_{L^2}^2, \\ |J_{10}| &\leq C \varepsilon \|U\|_{L_{t,x}^\infty} \left\| \sqrt{\rho^\varepsilon} \delta u^\varepsilon \right\|_{L^2}^2. \end{aligned}$$

We remark that these estimates holds for  $\gamma \geq 3/2$ .

We now focus on the remaining term  $J_7$ , which is the most difficult one to deal with. The difficulties come from the need to gain smallness in  $\varepsilon$  by using Hardy's inequality as above, from the low integrability of the residual part and from the fact that this term is quadratic in  $\delta u^\varepsilon$ . We first decompose

$$\rho^\varepsilon = [\delta \rho^\varepsilon]_{\text{ess}} + [\delta \rho^\varepsilon]_{\text{res}} + \rho_{app}^\varepsilon. \quad (3.3.39)$$

The essential part is easy to bound, owing to the boundedness of  $\rho^\varepsilon$  on that set and to an application of Hardy's inequality: more precisely, one has

$$\begin{aligned} \frac{1}{\varepsilon} \left| \int_{\Omega} [\delta \rho^\varepsilon]_{\text{ess}} \delta u_3^\varepsilon \partial_\zeta u_{0,h}^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot \delta u_h^\varepsilon \right| &\leq C \varepsilon \left\| \zeta^2 \partial_\zeta u_{0,h}^{bl} \right\|_{L_{t,x}^\infty} \left\| \frac{\delta u_3^\varepsilon}{x_3} \right\|_{L^2} \left\| \frac{\delta u_h^\varepsilon}{x_3} \right\|_{L^2} \\ &\leq C \varepsilon \left\| \partial_3 \delta u_3^\varepsilon \right\|_{L^2} \left\| \partial_3 \delta u_h^\varepsilon \right\|_{L^2} \\ &\leq C \varepsilon^{3/2} \left\| \partial_3 \delta u_h^\varepsilon \right\|_{L^2}^2 + C \varepsilon^{1/2} \left\| \partial_3 \delta u_3^\varepsilon \right\|_{L^2}^2, \end{aligned}$$

where both terms in the right hand side can be absorbed into the left hand side of (3.3.22) for  $\varepsilon > 0$  small enough. The control of the part involving  $\rho_{app}^\varepsilon$  is similar, so let us turn to the residual part. Two different estimates are computed if  $\gamma$  is

larger or smaller than the critical exponent 2. For  $\gamma \geq 2$ , we write, for  $\alpha \in (0, 1)$  to be chosen later on,

$$\delta u_3^\varepsilon = (\delta u_3^\varepsilon)^{1-\alpha} \frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha} x_3^\alpha$$

and then apply Sobolev's and Hardy's inequalities which yields

$$\|(\delta u_3^\varepsilon)^{1-\alpha}\|_{L^{\frac{6}{1-\alpha}}} \leq C \|\nabla \delta u_3^\varepsilon\|_{L^2}^{1-\alpha} \quad \text{and} \quad \left\| \frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha} \right\|_{L^{\frac{2}{\alpha}}} \leq C \|\partial_3 \delta u_3^\varepsilon\|_{L^2}^\alpha. \quad (3.3.40)$$

We use the same technique for  $\delta u_h^\varepsilon$  with  $\beta \in (0, 1)$ . Then, choosing  $\alpha, \beta$  such that

$$\alpha + \beta = \frac{1}{2},$$

we have, for all  $\delta > 0$  to be chosen later,

$$\begin{aligned} & \frac{1}{\varepsilon} \left| \int_{\Omega} [\delta \rho^\varepsilon]_{\text{res}} \delta u_3^\varepsilon \partial_\zeta u_{0,h}^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot \delta u_h^\varepsilon \right| \\ &= \varepsilon^{\alpha+\beta-1} \left| \int_{\Omega} [\delta \rho^\varepsilon]_{\text{res}} (\delta u_3^\varepsilon)^{1-\alpha} \frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha} \frac{x_3^{\alpha+\beta}}{\varepsilon^{\alpha+\beta}} \partial_\zeta u_{0,h}^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot (\delta u_h^\varepsilon)^{1-\beta} \frac{(\delta u_h^\varepsilon)^\beta}{x_3^\beta} \right| \\ &\leq \varepsilon^{\alpha+\beta-1} \| [\delta \rho^\varepsilon]_{\text{res}} \|_{L^2} \|\nabla \delta u_3^\varepsilon\|_{L^2}^{1-\alpha} \|\partial_3 \delta u_3^\varepsilon\|_{L^2}^\alpha \|\nabla \delta u_h^\varepsilon\|_{L^2}^{1-\beta} \|\partial_3 \delta u_h^\varepsilon\|_{L^2}^\beta \|\zeta^{\alpha+\beta} \partial_\zeta u_{0,h}^{bl}\|_{L^\infty} \\ &\leq C \varepsilon^{1/2} \left( \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) \right)^{1/2} \|\nabla \delta u_3^\varepsilon\|_{L^2} \|\nabla \delta u_h^\varepsilon\|_{L^2} \\ &\leq \frac{C_\delta}{\varepsilon^2} K_1(t) \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) + \delta \varepsilon \|\nabla \delta u_h^\varepsilon\|_{L^2}^2 \\ &= \frac{C_\delta}{\varepsilon^2} K_1(t) \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) + \delta \varepsilon \|\nabla_h \delta u_h^\varepsilon\|_{L^2}^2 + \delta \varepsilon \|\partial_3 \delta u_h^\varepsilon\|_{L^2}^2, \end{aligned}$$

where  $K_1 \in L^1([0, T])$  for all  $T > 0$  (here  $K_1(t) = \|\nabla u_3^\varepsilon(t)\|_{L^2}^2 + \|\nabla u_{app,3}^\varepsilon(t)\|_{L^2}^2$ ). In the second inequality we have used the lower bound

$$E(\rho^\varepsilon(t), \rho_{app}^\varepsilon(t)) \geq c |\delta \rho^\varepsilon(t)|^2$$

which comes from (3.3.23) when  $\gamma \geq 2$ . It is clear that, for  $\delta > 0$  small enough, we can absorb the last two terms into the left hand side of (3.3.22).

For  $3/2 \leq \gamma < 2$ , we use the same argument as in the case  $\gamma \geq 2$  for  $\delta u_3^\varepsilon$ . The control of  $\delta u_h^\varepsilon$ , instead, is via the anisotropic Sobolev embedding given in Lemma 3.3.5. Hence, for  $\alpha \in [0, 1]$  such that  $2 - \frac{3}{\gamma} = \alpha$ , Hölder's inequality gives, using

(3.3.40) for  $\delta u_3^\varepsilon$  and (3.3.29) for  $\delta u_h^\varepsilon$  with  $\kappa = \varepsilon^{(-\frac{1}{2} + \frac{1}{\gamma})_+}$ ,

$$\begin{aligned}
& \frac{1}{\varepsilon} \left| \int_{\Omega} [\delta \rho^\varepsilon]_{\text{res}} \delta u_3^\varepsilon \partial_\zeta u_{0,h}^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot \delta u_h^\varepsilon \right| \\
&= \varepsilon^{\alpha-1} \left| \int_{\Omega} [\delta \rho^\varepsilon]_{\text{res}} (\delta u_3^\varepsilon)^{1-\alpha} \frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha} \frac{x_3^\alpha}{\varepsilon^\alpha} \partial_\zeta u_{0,h}^{bl} \left( \frac{x_3}{\varepsilon} \right) \cdot \delta u_h^\varepsilon \right| \\
&\leq C \varepsilon^{\alpha-1} \| [\delta \rho^\varepsilon]_{\text{res}} \|_{L^\gamma} \| \nabla \delta u_3^\varepsilon \|_{L^2}^{1-\alpha} \| \partial_3 \delta u_3^\varepsilon \|_{L^2}^\alpha \\
&\quad \times \left( \kappa^{-\frac{1}{2}} \| \nabla_h \delta u_h^\varepsilon \|_{L^2} + \kappa \| \partial_3 \delta u_h^\varepsilon \|_{L^2} \right) \| \zeta^\alpha \partial_\zeta u_{0,h}^{bl} \|_{L^\infty} \\
&\leq C \varepsilon^{1-\frac{1}{\gamma}} \| \nabla \delta u_3^\varepsilon \|_{L^2} \left( \kappa^{-\frac{1}{2}} \| \nabla_h \delta u_h^\varepsilon \|_{L^2} + \kappa \| \partial_3 \delta u_h^\varepsilon \|_{L^2} \right) \\
&\leq \frac{\min(\mu, \lambda)}{10} \| \nabla \delta u_3^\varepsilon \|_{L^2}^2 + C(\mu, \lambda) \varepsilon^{2-\frac{2}{\gamma}} \left( \kappa^{-1} \| \nabla_h \delta u_h^\varepsilon \|_{L^2}^2 + \kappa^2 \| \partial_3 \delta u_h^\varepsilon \|_{L^2}^2 \right) \\
&\leq \frac{\min(\mu, \lambda)}{10} \| \nabla \delta u_3^\varepsilon \|_{L^2}^2 + C(\mu, \lambda) \varepsilon^{(\frac{5}{2} - \frac{3}{\gamma})_-} \| \nabla_h \delta u_h^\varepsilon \|_{L^2}^2 + C(\mu, \lambda) \varepsilon^{1+} \| \partial_3 \delta u_h^\varepsilon \|_{L^2}^2.
\end{aligned} \tag{3.3.41}$$

Hence we see that we can swallow the whole right hand side on condition that  $\gamma > 6/5$  (which is the case since  $\gamma \geq 3/2$  so that  $\alpha \in [0, 1]$ , see above) and  $\varepsilon$  is sufficiently small. To put it in a nutshell, we obtain the following bound on  $J_7$ :

$$\begin{aligned}
|J_7| &\leq \frac{C_\delta}{\varepsilon^2} K_1(t) \int_{\Omega} E(\rho^\varepsilon, \rho_{\text{app}}^\varepsilon) + (\delta \varepsilon + C \varepsilon^{\frac{3}{2}} + \delta_{2-}(\gamma) C(\mu, \lambda) \varepsilon^{1+}) \| \partial_3 \delta u_h^\varepsilon \|_{L^2}^2 \\
&\quad + \left( \delta \varepsilon + \delta_{2-}(\gamma) C(\mu, \lambda) \varepsilon^{(\frac{5}{2} - \frac{3}{\gamma})_-} \right) \| \nabla_h \delta u_h^\varepsilon \|_{L^2}^2 \\
&\quad + \left( \delta_{2-}(\gamma) \frac{\min(\mu, \lambda)}{10} + C \varepsilon^{\frac{1}{2}} \right) \| \nabla \delta u_3^\varepsilon \|_{L^2}^2.
\end{aligned} \tag{3.3.42}$$

Therefore, we finally get the following estimate for  $I_7$ :

$$\begin{aligned}
|I_7| &\leq C \varepsilon \left( \varepsilon K_1(t) + \delta_{2-}(\gamma) K_2(t) + \delta_{2-}(\gamma) \varepsilon \right) + \frac{C + C_\delta K_1(t)}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{\text{app}}^\varepsilon) \\
&\quad + (\delta_{2-}(\gamma) C + C_1 + C_2 \varepsilon) \| \sqrt{\rho^\varepsilon} \delta u^\varepsilon \|_{L^2}^2 \\
&\quad + (\delta \varepsilon + C \varepsilon^{\frac{3}{2}} + \delta_{2-}(\gamma) C(\mu, \lambda) \varepsilon^{1+}) \| \partial_3 \delta u_h^\varepsilon \|_{L^2}^2 \\
&\quad + \left( \delta \varepsilon + \delta_{2-}(\gamma) C(\mu, \lambda) \varepsilon^{(\frac{5}{2} - \frac{3}{\gamma})_-} \right) \| \nabla_h \delta u_h^\varepsilon \|_{L^2}^2 \\
&\quad + \left( \delta_{2-}(\gamma) \frac{\min(\mu, \lambda)}{10} + C \varepsilon^{\frac{1}{2}} \right) \| \nabla \delta u_3^\varepsilon \|_{L^2}^2.
\end{aligned} \tag{3.3.43}$$

**Remark 3.3.6.** *The anisotropic Sobolev embedding can be used to provide better estimates only for  $\gamma$  small. Indeed, in the control of  $J_2$  with the residual part of  $\delta \rho^\varepsilon$  (3.3.37), using Lemma 3.3.5 we get a remainder term of order  $\varepsilon^\alpha$  with  $0 < \alpha < 1$  for  $3/2 \leq \gamma < 2$  and  $\varepsilon^\alpha$  with  $\alpha > 1$  only for  $\gamma < 12/11$ , while by using the smallness of the Lebesgue measure of  $\Omega_{\text{res}}$  we get a remainder term of order  $\varepsilon$  for  $3/2 \leq \gamma < 2$ .*

In the end, summing up our estimates, we get from (3.3.22) the following differential inequality: there exist functions  $C_1(t)$ ,  $C_2(t) \in L^1([0, T])$ , and constants  $C_3 > 0$ ,  $\varepsilon_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ , for all  $t \in (0, T)$ , for all  $\delta > 0$ ,

$$\begin{aligned}
& \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \rho^\varepsilon |\delta u^\varepsilon|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) dx \right) \\
& \quad + \mu \int_{\Omega} |\nabla_h \delta u^\varepsilon|^2 dx + \varepsilon \int_{\Omega} |\partial_3 \delta u^\varepsilon|^2 dx + \lambda \int_{\Omega} |\nabla \cdot \delta u^\varepsilon|^2 dx \\
& \leq C_1(t) \left( \int_{\Omega} \rho^\varepsilon |\delta u^\varepsilon|^2 dx + \frac{1}{\varepsilon^2} \int_{\Omega} E(\rho^\varepsilon, \rho_{app}^\varepsilon) dx \right) + \varepsilon C_2(t) \\
& \quad + \left( \frac{\min(\mu, \lambda)}{10} + C \varepsilon^{(\frac{5}{2} - \frac{3}{\gamma})^-} \right) \|\nabla_h \delta u^\varepsilon\|_{L^2}^2 + C_3 (\delta \varepsilon + \varepsilon^{1+}) \|\partial_3 \delta u^\varepsilon\|_{L^2}^2 \\
& \quad + \left( \frac{\min(\mu, \lambda)}{10} + C \varepsilon^{\frac{1}{2}} \right) \|\partial_3 \delta u_3^\varepsilon\|_{L^2}^2.
\end{aligned} \tag{3.3.44}$$

Let us stress that  $C_1(t)$ ,  $C_2(t)$ ,  $C_3$  and  $\varepsilon_0$  do not depend on  $\varepsilon$ . The quantities these constants depend on have been written explicitly in the computations above. In particular,  $C_1(t)$  and  $C_2(t)$  contains the functions  $K_1(t)$  and  $K_2(t)$ . Choosing  $\delta$  small enough and using the identity  $\partial_3 \delta u_3^\varepsilon = \nabla \cdot \delta u^\varepsilon - \nabla_h \cdot \delta u_h^\varepsilon$ , the last three terms in (3.3.44) can be swallowed in the left hand side. The estimate in Proposition 3.3.1 follows from the Grönwall's lemma.

## APPENDIX A

---

# Solid with non-flat bottom

---

We derive here the equation for the solid motion, as in Section 4, in the more general case of a solid with non-flat bottom. Due to the fact that the interior and exterior domains do not change during the motion, we suppose that the contact between the free surface and the floating structure takes place on the vertical side-walls during all the motion. Then we can state the following proposition:

**Proposition A.1.** *In the case of a solid with non-flat bottom, Newton's law (1.5.1) can be written under the following form:*

$$(m + m_a^{NF}(\delta_G))\ddot{\delta}_G(t) = -\mathbf{c}\delta_G(t) + \mathbf{c}\zeta_\epsilon(t, R) + \left( \frac{\mathbf{b}}{h_\epsilon^2(t, R)} + \beta^{NF}(\delta_G) \right) \dot{\delta}_G^2(t) \quad (\text{A.1})$$

with

$$\mathbf{c} = \rho g \pi R^2, \quad \mathbf{b} = \frac{\pi \rho R^4}{8}, \quad m_a^{NF}(\delta_G) = \frac{\rho \pi}{2} \int_0^R \frac{r^3}{h_w(\delta_G, r)} dr,$$

$$\beta^{NF}(\delta_G) = \frac{\mathbf{b}}{2h_w^2(\delta_G, R)} + \frac{\pi \rho}{8} \int_0^R \frac{r^4}{h_w^3(\delta_G, r)} \partial_r h_w(\delta_G, r) dr,$$

and the dependence on  $\delta_G$  given by

$$h_w(\delta_G, r) = \delta_G(t) + h_{w,eq}(r).$$

**Remark A.2.** *One can note that in the case of a solid with flat bottom ( $\partial_r h_w(\delta_G, r) = 0$ )*

$$m_a^{NF}(\delta_G) = m_a(\delta_G), \quad \beta^{NF}(\delta_G) = \beta(\delta_G),$$

and (A.1) coincides with (1.5.2).

*Proof.* We derive only the expression of  $F_{fluid}^I$  and  $F_{fluid}^{III}$  in the case of a solid with a non-flat bottom. The added mass term comes from the fact that  $F_{fluid}^{II}$  can be written as

$$F_{fluid}^{II} = -m_a^{NF}(h_w)\dot{w}_G$$

with

$$m_a^{NF}(h_w) = \frac{\rho\pi}{2} \int_0^R \frac{r^3}{h_w} dr.$$

By definition,

$$F_{fluid}^I = 2\pi\rho \int_0^R \frac{r}{2h_w} \left( -\frac{h_w}{\rho} \partial_r \underline{P}_i \right) r dr$$

with  $\underline{P}_i$  defined as the solution to (1.3.3). Since we want

$$\underline{P}_i - P_{atm} \in H_{0,r}^1((0, R))$$

we get

$$-\frac{h_w}{\rho} \partial_r \underline{P}_i = \partial_r \left( \frac{q_i^2}{h_w} \right) + \frac{q_i^2}{rh_w} + gh_w \partial_r h_w.$$

Using the formula for the horizontal discharge in the interior domain  $q_i(t, r) = -\frac{r}{2} \dot{\delta}_G(t)$ , we obtain that

$$F_{fluid}^I = \pi\rho \int_0^R \left( \frac{3r^3}{4h_w^2} \dot{\delta}_G^2 + (gr^2 - \frac{r^4}{4h_w^3} \dot{\delta}_G^2) \partial_r h_w \right) dr.$$

Also in the case of a solid with non-flat bottom, (1.3.5) admits the unique constant solution

$$\underline{P}_i^{III}(t, r) = \rho g(\zeta_e(t, R) - \zeta_i(t, R)) + \frac{\rho}{2} q_i^2(t, R) \left( \frac{1}{h_e^2(t, R)} - \frac{1}{h_w^2(t, R)} \right),$$

with  $q_i(t, R) = -\frac{R}{2} \dot{\delta}_G(t)$ . By definition of  $F_{fluid}^{III}$  we have

$$F_{fluid}^{III} = \mathfrak{c}(\zeta_e(t, R) - \zeta_i(t, R)) + \mathfrak{b} \left( \frac{1}{h_e^2(t, R)} - \frac{1}{h_w^2(t, R)} \right) \dot{\delta}_G^2(t).$$

The relations (1.5.6) and (1.5.7) still hold but in this case we obtain

$$-mg = 2\pi\rho g \int_0^R r \zeta_i(t, r) dr - \mathfrak{c} \dot{\delta}_G(t).$$

Now we observe that the term  $\pi\rho \int_0^R gr^2 \partial_r h_w dr$  can be written by integration by parts as

$$\pi\rho \int_0^R gr^2 \partial_r h_w dr = \mathfrak{c} \zeta_i(t, R) - 2\pi\rho g \int_0^R r \zeta_i(t, r) dr$$

where  $\partial_r h_w = \partial_r \zeta_w = \partial_r \zeta_i$  using the constraint (1.1.6). Putting all these expressions in Newton's law (1.5.1) and integrating by parts, we get (A.1).  $\square$





## APPENDIX B

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### Proof of Lemma 1.6.5

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We prove here the product estimate in Lemma 1.6.5:

**Proposition B.1.** *Let  $k \geq 1$  be an integer. For  $f, g \in H^k((0, T))$  the following estimate holds:*

$$\begin{aligned} \|fg\|_{H^k((0, T))} &\lesssim \sqrt{T} \|f\|_{H^k((0, T))} \|g\|_{H^k((0, T))} \\ &\quad + (|f(0)| + |\frac{d}{dt}f(0)| + \dots + |\frac{d^{k-1}}{dt^{k-1}}f(0)|) \|g\|_{H^k((0, T))} \\ &\quad + (|g(0)| + |\frac{d}{dt}g(0)| + \dots + |\frac{d^{k-1}}{dt^{k-1}}g(0)|) \|f\|_{H^k((0, T))} \end{aligned} \quad (\text{B.1})$$

*Proof.* We write  $f(t)$  as

$$f(t) = f(0) + \int_0^t \frac{d}{dt}f(s)ds,$$

hence

$$\|f\|_{L^\infty((0, T))} \leq |f(0)| + \sqrt{T} \|f\|_{H^1((0, T))} \quad (\text{B.2})$$

We prove (B.1) by induction. For  $k = 1$  we have

$$\begin{aligned} \|fg\|_{H^1((0, T))} &\lesssim \|fg\|_{L^2((0, T))} + \|\frac{df}{dt}g\|_{L^2((0, T))} + \|f\frac{dg}{dt}\|_{L^2((0, T))} \\ &\leq (\|f\|_{L^2((0, T))} + \|\frac{df}{dt}\|_{L^2((0, T))}) \|g\|_{L^\infty((0, T))} + \|f\|_{L^\infty((0, T))} \|g\|_{H^1((0, T))} \\ &\leq \sqrt{2} \|f\|_{H^1((0, T))} \|g\|_{L^\infty((0, T))} + \|f\|_{L^\infty((0, T))} \|g\|_{H^1((0, T))} \end{aligned}$$

and using (B.2) we get

$$\begin{aligned} \|fg\|_{H^1((0,T))} &\lesssim \sqrt{T}\|f\|_{H^1((0,T))}\|g\|_{H^1((0,T))} \\ &\quad + |f(0)|\|g\|_{H^1((0,T))} + |g(0)|\|f\|_{H^1((0,T))} \end{aligned} \quad (\text{B.3})$$

Let us suppose that (B.1) is true for  $k-1$ . Then we have

$$\begin{aligned} \|fg\|_{H^k((0,T))} &\leq \|fg\|_{H^{k-1}((0,T))} + \left\| \frac{d^k}{dt^k}(fg) \right\|_{L^2((0,T))} \\ &\lesssim \|fg\|_{H^{k-1}((0,T))} + \left\| \frac{d^k}{dt^k}f \right\|_{L^2((0,T))}\|g\|_{L^\infty((0,T))} + \|f\|_{L^\infty((0,T))}\left\| \frac{d^k}{dt^k}g \right\|_{L^2((0,T))} \\ &\quad + \left\| \frac{d}{dt}f \right\|_{L^2((0,T))}\left\| \frac{d^{k-1}}{dt^{k-1}}g \right\|_{L^\infty((0,T))} + \sum_{i=2}^{k-1} C_{k,i} \left\| \frac{d^i}{dt^i}f \right\|_{L^\infty((0,T))}\left\| \frac{d^{k-i}}{dt^{k-i}}f \right\|_{L^2((0,T))}. \end{aligned}$$

From the estimate (B.2) for  $f, g, \frac{d^{k-1}}{dt^{k-1}}g$  and  $\frac{d^i}{dt^i}f$  we get

$$\begin{aligned} \|fg\|_{H^k((0,T))} &\lesssim \|fg\|_{H^{k-1}((0,T))} + 3\sqrt{T}\|f\|_{H^k((0,T))}\|g\|_{H^k((0,T))} \\ &\quad + (|f(0)| + \left| \frac{d^2}{dt^2}f(0) \right| + \dots + \left| \frac{d^{k-1}}{dt^{k-1}}f(0) \right|)\|g\|_{H^k((0,T))} \\ &\quad + (|g(0)| + \left| \frac{d^{k-1}}{dt^{k-1}}g(0) \right|)\|f\|_{H^k((0,T))} \end{aligned}$$

and (B.1) follows using the inductive hypothesis.  $\square$

## APPENDIX C

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# Hankel functions

---

In this appendix we show some results and properties for the Hankel functions. Let us consider the following differential equation:

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0, \quad z \in \mathbb{C}.$$

This differential equation is called Bessel equation of index  $\nu$ . Solutions to this equation are called Bessel functions. Let us consider the case when  $\nu = n$ , with  $n \in \mathbb{Z}$ . Bessel functions of the first kind, denoted by  $J_n(z)$ ,

$$J_n(z) = \left(\frac{1}{2}z\right)^n \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(n+k+1)}.$$

are entire in  $z$ .

Bessel functions of the second kind, denoted by  $Y_n(z)$

$$\begin{aligned} Y_n(z) = & -\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi} \log\left(\frac{1}{2}z\right) J_n(z) \\ & - \frac{\left(\frac{1}{2}z\right)^n}{\pi} \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!}, \end{aligned}$$

where  $\psi = \frac{\Gamma'}{\Gamma}$ , with  $\Gamma$  the Gamma function, have a branch point in  $z = 0$ . Both  $J_n$  and  $Y_n$  are real valued if  $z$  is real. Let us define

$$H_n^{(1)}(z) := J_n(z) + iY_n(z),$$

$$H_n^{(2)}(z) := J_n(z) - iY_n(z).$$

We call them respectively *Hankel functions* of first order and second order with index  $n$ , and they are solutions to the Bessel equation. Each solution has a branch point at  $z = 0$  for all  $n$ . The principal branches of  $H_n^{(1)}(z)$  and  $H_n^{(2)}(z)$  are two-valued and discontinuous on the cut along the negative real axis. They are holomorphic functions of  $z$  throughout the complex plane cut (see Chapter 9 of [1]).

Now let us show some representations of these functions useful for our problem. From [122] we have an integral representation for  $z = x > 0$ :

$$H_n^{(1)}(x) = \frac{2e^{-n\pi i/2}}{\pi i} \int_0^{+\infty} e^{ix \cosh(s)} \cosh(ns) ds$$

and

$$H_n^{(2)}(x) = -\frac{2e^{n\pi i/2}}{\pi i} \int_0^{+\infty} e^{-ix \cosh(s)} \cosh(ns) ds,$$

and a series representation for large  $|z|$  and  $0 < \arg z < \pi$ :

$$H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\pi}{4} - n\frac{\pi}{2})} \left[ \sum_{k=0}^{p-1} \frac{(-)^k a_k(n)}{z^k} + O(z^{-p}) \right]$$

$$H_n^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\pi}{4} - n\frac{\pi}{2})} \left[ \sum_{k=0}^{p-1} \frac{a_k(n)}{z^k} + O(z^{-p}) \right]$$

with

$$a_0(n) = 1,$$

$$a_k(n) = \frac{\{4n^2 - 1^2\}\{4n^2 - 3^2\} \cdots \{4n^2 - (2k-1)^2\}}{8^k k! (i)^k}, \quad k > 0.$$

Last we recall analytic continuation formulas for  $m \in \mathbb{Z}$  (see [41]):

$$H_n^{(1)}(ze^{m\pi i}) = (-1)^{mn-1} ((m-1) H_n^{(1)}(z) + m H_n^{(2)}(z)),$$

$$H_n^{(2)}(ze^{m\pi i}) = (-1)^{mn} (m H_n^{(1)}(z) + (m+1) H_n^{(2)}(z)).$$

$$H_n^{(1)}(\bar{z}) = \overline{H_n^{(2)}(z)}, \quad H_n^{(2)}(\bar{z}) = \overline{H_n^{(1)}(z)}.$$

## APPENDIX D

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# The case with an exponentially decreasing kernel

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We address here the case where the convolution kernel  $F$  of the nonlinear integro-differential equation (2.3.1) has an exponential decay instead of the polynomial decay  $t^{-2}$  assumed in Section 2.2. We recall that in that case the assumption was justified by the numerical computation of the function  $F$ . However, interested in this type of integro-differential equation (widely used in population dynamics), we address the case of the exponential decay for the kernel, which may occur in other contexts.

Let us make the following assumption on the long time behavior of the convolution kernel:

**Assumption D.1.**  *$F$  is a positive exponentially decreasing function, that is there exist  $M > 0$ ,  $\gamma > 0$  and  $t_0$  such that*

$$F(t) \leq Me^{-\gamma t}$$

for all  $t \geq t_0$ .

Differently from the case studied in Chapter 2, here Lemma 2.2.6 does not hold since  $F$  does not have the particular structure that allowed to prove the equality (2.2.21). Instead, we suppose the following property on  $F$ :

**Assumption D.2.** *The convolution kernel  $F$  satisfies*

$$\int_0^{+\infty} F(\theta) d\theta \leq \frac{R}{2v_0}. \quad (\text{D.1})$$

The inequality (D.1) is of the same type as the one given in [98]. These assumptions will be used not only to prove the existence theorem but also to get a stability result. We state now the global existence and uniqueness of the solution to the solid motion equation considering the linear shallow water equations in the exterior domain:

**Theorem D.3.** *The Cauchy problem for the nonlinear second order integro-differential equation (2.3.1) with initial data*

$$\delta_G(0) = \delta_0 \neq 0, \quad \dot{\delta}_G(0) = 0,$$

admits a unique solution  $\delta_G \in C^2([0, +\infty), \mathbb{R})$  provided

$$\delta_0 < \min \left( h_0, h_0 \sqrt{\frac{h_0 \rho_m H}{\rho R^2}} \right) - \frac{\rho_m H}{\rho}. \quad (\text{D.2})$$

*Proof.* The proof is analogous to the one of Theorem 2.3.4. However, in this case the weighted space of uniformly continuous functions  $BUC_\eta$  is replaced by .

$$BUC_{e^\eta} = \left\{ \varphi \in C((-\infty, 0], \mathbb{R}^2) : \theta \rightarrow e^{\eta\theta} \varphi(\theta) \text{ is bounded and uniformly continuous} \right\}$$

for  $\eta > 0$ . This is a Banach space endowed with the norm

$$\|\varphi\|_{e^\eta} := \sup_{\theta \leq 0} e^{\eta\theta} |\varphi(\theta)|.$$

We write (2.3.1) as

$$\begin{cases} \frac{dx(t)}{dt} = \mathcal{F}(x_t) \quad \forall t \geq 0 \\ x_0 = \varphi_0 \in BUC_{e^\eta}. \end{cases} \quad (\text{D.3})$$

with  $\varphi_0 = (\delta_0, 0)^T$  and  $\mathcal{F}$  as in (2.3.6). Then, the same analysis can be carried out and we can show that  $\tilde{\mathcal{F}}$ , defined in(2.3.11) is Lipschitz on the new space  $BUC_{e^\eta}$ .

**Lemma D.4.**  $\tilde{\mathcal{F}} : BUC_{e^\eta} \rightarrow \mathbb{R}^2$  is Lipschitz on bounded sets for  $\eta$  small enough.

*Proof.* We just prove the Lipschitz continuity of the convolution term. From the expression of  $Conv$  and using Assumption D.1 we have

$$\begin{aligned} |Conv(u) - Conv(v)| &\leq \int_{-\infty}^0 |F(-s)| e^{-\eta s} ds \|u - v\|_{e^\eta} \\ &\leq \left( C_0 + M \int_{t_0}^{+\infty} e^{(-\gamma+\eta)s} ds \right) \|u - v\|_{e^\eta} \end{aligned}$$

where  $C_0$  comes from the fact that  $F$  is continuous on  $[0, t_0]$ . Choosing  $\eta$  such that  $-\gamma + \eta < 0$ , from the same properties of the cut-off functions  $\chi_0, \chi_1$  and  $\chi_2$  the result follows.  $\square$

Then, we can apply Theorem 7.4 of [87] to

$$\begin{cases} \frac{dx(t)}{dt} = \tilde{\mathcal{F}}(x_t) \quad \forall t \geq 0 \\ x_0 = \varphi_0 \in BUC_{e^\eta}. \end{cases} \quad (\text{D.4})$$

and we have that (D.4) admits a unique solution  $x_{\varphi_0} \in C((-\infty, \tau), \mathbb{R}^2)$  with initial data  $\varphi_0$ . From the continuity of  $\tilde{\mathcal{F}}$  we get  $x_{\varphi_0} \in C^1((-\infty, \tau_{\varphi_0}), \mathbb{R}^2)$ . Furthermore the theorem gives an explosion condition on the solution, i.e. if  $\tau_{\varphi_0} < +\infty$  then

$$\lim_{t \nearrow \tau_{\varphi_0}^-} \|x_{\varphi_0}(t)\| = +\infty. \quad (\text{D.5})$$

The global existence is derived from the conservation of the fluid-structure problem energy.  $\square$

The difference with respect to the  $BUC_\eta$ -case is that, considering the space  $BUC_{e^\eta}$ , we can state the following local stability result:

**Proposition D.5.** *The equilibrium  $\delta_G \equiv 0, \dot{\delta}_G \equiv 0$  of (2.3.1) is exponentially asymptotically stable, i.e. there exist  $M \geq 1, \omega > 0$  and  $\epsilon > 0$  such that*

$$|\delta_G(t)|^2 + |\dot{\delta}_G(t)|^2 \leq M e^{-\omega t} |\delta_0|^2 \quad \forall t \geq 0 \quad (\text{D.6})$$

for  $|\delta_0| \leq \epsilon$ .

*Proof.* Since  $\mathcal{F}(0_{BUC_{e^\eta}}) = 0$ ,  $x_t \equiv 0$  is an equilibrium solution of (D.3). Moreover, let us consider the linearized equation of (D.3)

$$\begin{cases} \frac{dx(t)}{dt} = \mathcal{L}(x_t) \quad \forall t \geq 0 \\ x_0 = \varphi_0 \in BUC_{e^\eta} \end{cases} \quad (\text{D.7})$$

where  $\mathcal{L}(x_t) = (Tr_2(x_t), \hat{S}(x_t))^T$  with

$$\hat{S}(x_t) = \frac{-\mathbf{c}Tr_1(x_t) - \nu Tr_2(x_t) + Conv(x_t)}{m + m_a(0)}. \quad (\text{D.8})$$

Let  $\lambda \in \Omega := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > -\eta\}$  and consider

$$\Delta(\lambda) = \lambda I - \mathcal{L}(e^{\lambda \cdot} I) \in M_2(\mathbb{C}).$$

Suppose  $\det(\Delta(\lambda)) = 0$ . Then  $\lambda$  satisfies the following equation

$$\lambda^2 - \frac{\lambda}{m + m_a(0)} \left( -\nu + \mathbf{c} \int_{-\infty}^0 F(-\theta) e^{\lambda \theta} d\theta \right) = -\frac{\mathbf{c}}{m + m_a(0)}. \quad (\text{D.9})$$



The left-hand side of (D.9) must have real part negative and imaginary part equal to zero. Combining this with Assumption D.2, necessarily  $\operatorname{Re}(\lambda) < 0$ . Therefore we can apply Theorem 8.1 of [87] to get the following local stability result for the semiflow  $U(t)\varphi_0 = x_{\varphi_0}(t + \theta)$  in the  $BUC_{e^\eta}$ -norm: there exists  $M \geq 1$ ,  $\delta > 0$  and  $\epsilon > 0$  such that

$$\|x_{\varphi_0}(t + \cdot)\|_\eta \leq M e^{-\delta t} \|\varphi_0\|_{e^\eta} \quad \forall t \geq 0 \quad (\text{D.10})$$

for  $\|\varphi_0\|_{e^\eta} \leq \epsilon$ . By definition of the  $BUC_{e^\eta}$ -norm we get (D.6).

□

## APPENDIX E

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# Quasi-geostrophic equation: global well-posedness

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In this appendix we discuss the well-posedness result for the quasi-geostrophic equation (3.2.35):

$$\partial_t \left( \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle q - \langle \bar{\rho} \rangle \Delta_h q \right) - \langle \bar{\rho} \rangle \nabla_h^\perp q \cdot \nabla_h \Delta_h q + \mu \Delta_h^2 q - \frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} \Delta_h q = 0.$$

Different versions of this equation appear in the literature as the limit dynamics of highly rotating fluids systems, see for instance [47], [54], [43] and [44]. The well-posedness of a higher order quasi-geostrophic equation was also investigated in [36]. For the sake of simplicity, in this section we assume

$$\langle \bar{\rho} \rangle = \left\langle \frac{\bar{\rho}}{p'(\bar{\rho})} \right\rangle = 1, \quad \frac{\sqrt{\bar{\rho}(0)} + \sqrt{\bar{\rho}(1)}}{\sqrt{2}} = \alpha,$$

for some  $\alpha > 0$ . Let us give the weak formulation of the equation:

**Definition E.1.** *Let  $q_0 \in H^1(\mathbb{R}^2)$ . A function  $q$  is a weak solution to (3.2.35) with initial datum  $q_0$  if*

$$q \in L^\infty([0, T]; H^1(\mathbb{R}^2)) \cap L^2([0, T]; H^2(\mathbb{R}^2))$$

and it solves, for  $\phi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}^2)$ ,

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{R}^2} (q \partial_t \phi - \Delta_h q \cdot \partial_t \phi) dx_h dt + \int_0^T \int_{\mathbb{R}^2} \Delta_h q \nabla_h^\perp q \cdot \nabla_h \phi dx_h dt \\
& + \mu \int_0^T \int_{\mathbb{R}^2} \Delta_h q \Delta_h \phi dx_h dt + \alpha \int_0^T \int_{\mathbb{R}^2} \Delta_h q \cdot \phi dx_h dt \\
& = \int_{\mathbb{R}^2} (q_0 \phi(0) + \nabla_h q_0 \cdot \nabla_h \phi(0)) dx_h
\end{aligned} \tag{E.1}$$

In order to show the existence and uniqueness of the solution to (3.2.35), we need the following a priori estimate.

**Proposition E.2.** *Let  $q_0 \in H^1(\mathbb{R}^2)$ . Then, there exists a constant  $C > 0$  such that any smooth solution to (3.2.35) with initial datum  $q_0$  satisfies the following equality for all  $t \geq 0$ :*

$$\|q(t)\|_{H^1}^2 + 2 \int_0^t (\alpha \|\nabla_h q(\tau)\|_{L^2}^2 + \mu \|\Delta_h q(\tau)\|_{L^2}^2) d\tau = \|q_0\|_{H^1}^2. \tag{E.2}$$

*Proof.* Using an approximation argument, we can test the weak formulation (E.1) with  $q$  and we get

$$\begin{aligned}
& \frac{1}{2} (\|q(t)\|_{L^2}^2 + \|\nabla_h q(t)\|_{L^2}^2) + \int_0^t (\alpha \|\nabla_h q(\tau)\|_{L^2}^2 + \mu \|\Delta_h q(\tau)\|_{L^2}^2) d\tau \\
& = \frac{1}{2} (\|q_0\|_{L^2}^2 + \|\nabla_h q_0\|_{L^2}^2),
\end{aligned}$$

where the convective term

$$\int_{\mathbb{R}^2} \Delta_h q \nabla_h^\perp q \cdot \nabla_h q = 0,$$

and the result follows.  $\square$

We can now state the following well-posedness result:

**Theorem E.3.** *Let  $q_0 \in H^1(\mathbb{R}^2)$ . Then, there exists a unique global weak solution  $q$  to the quasi-geostrophic equation (3.2.35) such that*

$$q \in \mathcal{C}(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+; H^2(\mathbb{R}^2))$$

with initial datum  $q_0$ . Moreover the solution  $q$  satisfies the energy equality (E.2) for all  $t \geq 0$ .

*Proof.* Following [36], the rigorous existence result is established using the Friedrichs smoothing method. The main idea is to construct smooth approximate solution  $(q^n)_n$  introducing the Fourier cutoff operator defined on  $L^2$

$$\mathcal{J}_n f = \mathcal{F}^{-1} \left( \mathbf{1}_{\{|\xi|^2 \leq n\}}(\xi) \widehat{f}(\xi) \right) \quad (\text{E.3})$$

which is the orthogonal projection of  $L^2$  on the closed subspace  $L_n^2$  of the  $L^2$  functions whose Fourier transforms are supported in the ball  $B_\xi(0, \sqrt{n})$ . It is a generalization to the whole space of the spectral projector for the Laplace operator in bounded domains (see Section 2.1 of [28]). The functions in  $L_n^2$  are smooth and, after applying the cutoff operator to the initial datum, we get the global existence and uniqueness of the smooth solution  $q^n$  to an approximate system. The solution  $q^n$  satisfies the energy estimate (E.2) uniformly with respect to  $n$ ; therefore, up to extraction of a subsequence we get the weak-\* convergence to a function  $q \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+; H^2(\mathbb{R}^2))$ . After deriving some compactness properties for the approximated solutions  $(q^n)_n$  in order to overcome the convergence problem for the nonlinear term, we can pass to the limit in the weak formulation of the equation solved by  $q^n$  and show that  $q$  is a weak solution of (3.2.35), which gives the existence result.

Let us show the uniqueness of the solution. Denoting  $w = q_1 - q_2$  where  $q_1$  and  $q_2$  are two solutions to (3.2.35), we subtract the weak formulations (E.1) for  $q_1$  and  $q_2$  and testing with  $w$  it yields

$$\begin{aligned} & \frac{1}{2} \left( \|w(t)\|_{L^2}^2 + \|\nabla_h w(t)\|_{L^2}^2 \right) + \int_0^t \left( \alpha \|\nabla_h w(\tau)\|_{L^2}^2 + \mu \|\Delta_h w(\tau)\|_{L^2}^2 \right) d\tau \\ &= \frac{1}{2} \left( \|w_0\|_{L^2}^2 + \|\nabla_h w_0\|_{L^2}^2 \right) - \int_0^t \int_{\mathbb{R}^2} \nabla_h^\perp q_1 \cdot \nabla_h w \Delta_h w - \int_0^t \int_{\mathbb{R}^2} \Delta_h q_2 \nabla_h^\perp w \cdot \nabla_h w, \end{aligned}$$

where the last term in the right-hand side obviously vanishes. By using the Gagliardo-Nirenberg estimate for  $u \in H^1(\mathbb{R}^2)$

$$\|u\|_{L^4} \leq C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}$$

we have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \nabla_h^\perp q_1 \cdot \nabla_h w \Delta_h w \right| &\leq \|\nabla_h w\|_{L^4} \|\nabla_h^\perp q_1\|_{L^4} \|\Delta_h w\|_{L^2} \\ &\leq C \|\nabla_h w\|_{L^2}^{\frac{1}{2}} \|\nabla q_1\|_{L^2}^{\frac{1}{2}} \|\Delta_h w\|_{L^2}^{\frac{3}{2}} \\ &\leq C \|\nabla_h w\|_{L^2}^2 \|\nabla q_1\|_{L^2}^2 \|\Delta_h w\|_{L^2}^2 + \frac{\mu}{2} \|\Delta_h w\|_{L^2}^2. \end{aligned}$$

Hence we get

$$\begin{aligned} & \frac{1}{2} \left( \|w(t)\|_{L^2}^2 + \|\nabla_h w(t)\|_{L^2}^2 \right) + \int_0^t \left( \alpha \|\nabla_h w(\tau)\|_{L^2}^2 + \frac{\mu}{2} \|\Delta_h w(\tau)\|_{L^2}^2 \right) d\tau \\ & \leq \frac{1}{2} \left( \|w_0\|_{L^2}^2 + \|\nabla_h w_0\|_{L^2}^2 \right) + C \int_0^t \|\nabla q_1(\tau)\|_{L^2}^2 \|\Delta q_1(\tau)\|_{L^2}^2 \|\nabla_h w(\tau)\|_{L^2}^2 d\tau \end{aligned}$$

By (E.2) the function  $\|\nabla q_1(t)\|_{L^2}^2 \|\Delta q_1(t)\|_{L^2}^2$  is integrable in  $(0, T)$  for all  $T \geq 0$  and we can apply Grönwall's lemma to get

$$\begin{aligned} \|w(t)\|_{L^2}^2 + \|\nabla_h w(t)\|_{L^2}^2 & + \int_0^t \left( 2\alpha \|\nabla_h w(t)\|_{L^2}^2 + \mu \|\Delta_h w(t)\|_{L^2}^2 \right) \\ & \leq \left( \|w_0\|_{L^2}^2 + \|\nabla_h w_0\|_{L^2}^2 \right) e^{\int_0^t 2C \|\nabla q_1\|_{L^2}^2 \|\Delta q_1\|_{L^2}^2} \end{aligned}$$

and the uniqueness follows directly. For the proof of the time continuity we refer to Section 4 of [36].  $\square$

We actually need more regularity for the solution  $q$  to compute estimates in subsection (3.3.2). For example in the control of the term  $I_6$  we require  $\partial_t u_{app}^\varepsilon$  to be bounded in time and space; if we consider the term  $\partial_t u_{0,h} = \partial_t \nabla_h^\perp q$ , using the linear part of the equation (the non-linear part is easily checked to be somehow lower order in this computation), we need  $q \in L^\infty(\mathbb{R}_+; H^5(\mathbb{R}^2))$  to control  $\partial_t \nabla_h^\perp q \sim \nabla_h (\text{Id} - \Delta_h)^{-1} \Delta_h^2 q$  in  $L_{t,x}^\infty$ . Higher regularity estimates can be established in a general way using the same parilinearization argument as in [36]. In the following proposition we establish higher regular estimates in the case of Sobolev spaces with integer order via an inductive argument. Here, the notation  $\|\nabla^m u\|_{L^2}$  stands for the sum of the  $L^2$  norms of all  $D^\beta u = \partial_1^{\beta_1} \partial_2^{\beta_2} u$ , for  $\beta \in \mathbb{N}^2$  such that  $|\beta| = m$ .

**Proposition E.4.** *Let  $n \geq 1$  be an integer and  $q_0 \in H^n(\mathbb{R}^2)$ . Then, there exists a constant  $C_{n-1} > 0$  such that any weak solution to (3.2.35) with initial datum  $q_0$  satisfies the following inequality for all  $t \geq 0$ :*

$$\begin{aligned} \sum_{j=0}^{n-1} \left( \|\nabla_h^j q(t)\|_{L^2}^2 + \|\nabla_h^{j+1} q(t)\|_{L^2}^2 \right) & + \sum_{j=0}^{n-1} \left( \int_0^t \|\nabla_h^{j+1} q\|_{L^2}^2 + \|\nabla_h^{j+2} q\|_{L^2}^2 \right) \\ & \leq C_{n-1} \sum_{j=0}^{n-1} \left( \|\nabla_h^j q_0\|_{L^2} + \|\nabla_h^{j+1} q_0\|_{L^2} \right)^2. \end{aligned} \tag{E.4}$$

with  $C_0 = C_1 = 1$  and  $C_{n-1} = C_{n-1}(\|q_0\|_{H^{n-1}})$  for  $n - 1 \geq 2$ .

*Proof.* The proof of the proposition is by induction. For  $n = 1$ , the inequality (E.4) is trivially derived from (E.2). Let us prove the estimate for  $n = 2$ . Omitting a

standard approximation argument as before, we can test the weak formulation (E.1) with  $-\Delta_h q$  and we find

$$\begin{aligned} & \frac{1}{2} \left( \|\nabla_h q(t)\|_{L^2}^2 + \|\Delta_h q(t)\|_{L^2}^2 \right) + \int_0^t \left( \alpha \|\Delta_h q(\tau)\|_{L^2}^2 + \mu \|\nabla_h \Delta_h q(\tau)\|_{L^2}^2 \right) d\tau \\ & \leq \frac{1}{2} \left( \|\nabla_h q_0\|_{L^2}^2 + \|\Delta_h q_0\|_{L^2}^2 \right). \end{aligned} \quad (\text{E.5})$$

Summing up (E.5) and (E.2), the inequality (E.4) for  $n = 2$  follows.

Let  $n \geq 2$ , and let us suppose the inductive hypothesis at any rank  $j \leq n$ . We want to prove the estimate for rank  $n + 1$ . We start by testing (E.1) with  $(-1)^n \Delta_h^n q$ . We have to distinguish between two cases, depending on whether  $n = 2m$  or  $n = 2m + 1$ , for some  $m \in \mathbb{N}$ . Let us detail the computations only for the former case, the latter being analogous. So, assume  $n = 2m$ : concerning the linear part of the equation, straightforward manipulations lead to

$$\begin{aligned} & \int_{\mathbb{R}^2} \left( \partial_t (\text{Id} - \Delta_h) q - \alpha \Delta_h q + \mu \Delta_h^2 q \right) \Delta_h^{2m} q \\ & = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \left( |\Delta_h^m q|^2 + |\nabla_h \Delta_h^m q|^2 \right) + \alpha \int_{\mathbb{R}^2} |\nabla_h \Delta_h^m q|^2 + \mu \int_{\mathbb{R}^2} |\Delta_h^{m+1} q|^2. \end{aligned} \quad (\text{E.6})$$

The final estimate follows then from this equality and Calderón-Zygmund's theory, provided we are able to show a global  $L^2$  bound in time on the integral involving the convective term

$$- \int_{\mathbb{R}^2} \nabla_h^\perp q \cdot \nabla_h \Delta_h q \Delta_h^{2m} q = - \int_{\mathbb{R}^2} \Delta_h^{m-1} \left( \nabla_h^\perp q \cdot \nabla_h \Delta_h q \right) \Delta_h^{m+1} q.$$

Observe that

$$\Delta_h^{m-1} \left( \nabla_h^\perp q \cdot \nabla_h \Delta_h q \right) = \nabla_h^\perp q \cdot \nabla_h \Delta_h^m q + \sum_{\substack{|\beta|+|\eta|=2(m-1) \\ |\beta|>0}} D_h^\beta \nabla_h^\perp q \cdot \nabla_h D_h^\eta \Delta_h q,$$

where the sum is performed for  $|\beta| > 0$  (at least one derivative falls on  $\nabla_h^\perp q$ ). First of all, Hölder's, Gagliardo-Nirenberg's and Young's inequalities (together with Calderón-Zygmund's theory) yield

$$\begin{aligned} \left| \int \nabla_h^\perp q \cdot \nabla_h \Delta_h^m q \Delta_h^{m+1} q \right| & \leq \|\nabla_h q\|_{L^4} \|\nabla_h \Delta_h^m q\|_{L^4} \|\Delta_h^{m+1} q\|_{L^2} \\ & \leq \|\nabla_h q\|_{L^2}^{1/2} \|\Delta_h q\|_{L^2}^{1/2} \|\nabla_h \Delta_h^m q\|_{L^2}^{1/2} \|\Delta_h^{m+1} q\|_{L^2}^{3/2} \\ & \leq C_\mu \|\nabla_h q\|_{L^2}^2 \|\Delta_h q\|_{L^2}^2 \|\nabla_h \Delta_h^m q\|_{L^2}^2 + \frac{\mu}{4} \|\Delta_h^{m+1} q\|_{L^2}^2. \end{aligned}$$

Of course, the last term can obviously be swallowed in the left-hand side of the inequality, thanks to the last term in (E.6). On the other hand, the term  $\nabla_h \Delta_h^m q$  contains  $2m + 1 = n + 1$  derivatives, so it is in  $L^2([0, T])$  for all  $T > 0$ , by the inductive hypothesis. Moreover, since  $n \geq 2$ , both  $\|\nabla_h q\|_{L^2}^2$  and  $\|\Delta_h q\|_{L^2}^2$  belong to  $L^\infty([0, T])$  for all  $T > 0$ . Hence, we deduce that

$$\int_0^T \|\nabla_h q\|_{L^2}^2 \|\Delta_h q\|_{L^2}^2 \|\nabla_h \Delta_h^m q\|_{L^2}^2 dt \leq C_{n-1} \|q_0\|_{H^n}^2 .$$

For the sake of conciseness, we omit here to give the details of the control for the term

$$\sum_{\substack{|\beta|+|\eta|=2(m-1) \\ |\beta|>0}} D_h^\beta \nabla_h^\perp q \cdot \nabla_h D_h^\eta \Delta_h q ,$$

which can be performed resorting once again to Hölder's and Young's inequalities and is actually easier, since each term appearing in the product is of order strictly smaller than  $n$ . Notice that, after the application of Young's inequality, the remaining term will be always  $L^1([0, T])$  for all  $T > 0$  because  $D_h^\beta \nabla_h^\perp q$  contains at least two derivatives, and then we can apply the bound (E.2) or the inductive hypothesis.  $\square$

We conclude by noticing that, from the previous estimate, a well-posedness result in  $H^n(\mathbb{R}^2)$  similar to Theorem E.3 easily follows.

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