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Benedikt Hurle

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Benedikt Hurle. Deformation and Quantization of color Lie bialgebras and alpha-type cohomologies for Hom-algebras. General Mathematics [math.GM]. Université de Haute Alsace - Mulhouse, 2018. English. NNT : 2018MULH1819 . tel-02375683

HAL Id: tel-02375683

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UNIVERSITÉ DE HAUTE ALSACE
École Doctorale de Mathématiques, Sciences de l'Information et de l'Ingénieur
(ED269)
IRIMAS, Département de Mathématiques

THÈSE
pour obtenir le grade de
Docteur en Mathématiques

présentée par

Benedikt HURLE

DÉFORMATION ET QUANTIFICATION
DE BIALGÈBRES DE LIE COLORÉES ET
COHOMOLOGIES DE HOM-ALGÈBRES
DE TYPE α

Thèse dirigée par Abdenacer Makhlouf
soutenue le 04/10/2018 devant le jury composé de :

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Beauty is the first test: there is no permanent place in the world for ugly mathematics.

G. H. Hardy

REMERCIEMENTS

I would like to thank my advisor Abdenacer Makhlouf for accepting me as his student, for our discussions and his support. I also want to thank Martin Bordemann, who suggest going to Mulhouse to me, for our many discussions and his constant support. I want to thank Stefan Waldman and Peter Schauenburg for agreeing to review this work and also Camille Laurent Gengoux for taking the time and being part of the “jury”.

Next, I want to thank the other PhD students, especially Diana Gillicq-Hirtz, Mourad Mehidi, Ahmed Zahari, Othman Yakhlef and Hamilton Araujo, with whom I shared the office, for our conversations and their friendliness. I also want to thank the other members of the department.

Last but not least I would like to thank my family and my friends for supporting me at any time.

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INTRODUCTION

THIS thesis consists of two parts, which are mostly independent, but both focus on homological algebra and deformations of different algebraic structures. In the first part we study deformations and quantization of color Lie bialgebras and in the second part we deal with cohomology and deformations of Hom-type algebras and bialgebras.

A color vector space is a graded vector space with a commutation factor, which is an antisymmetric bicharacter. A color Lie bialgebra is a Lie bialgebra on a color vector space. Examples of color spaces are superalgebras, which have been studied since a long time and have many important applications in physics. Lie bialgebras appeared first as classical limits when studying quantum integrable systems. The theory of Lie bialgebras was developed largely by Drinfeld [Dri89, Dri90, Dd86]. The quantization of Lie bialgebras is well studied and there are several approaches. The first one was provided by Etingof and Kazhdan [EK96]. One may also mention the papers by Ševera [Šev16] and Merkulov-Willwacher [MW16]. Moreover quantization of quasi-Lie bialgebras has been considered by Enriquez in [EH10] and Ševera in [SŠ15]. In this thesis, we aim to study quantization of color Lie bialgebras following the Etingof and Kazhdan approach. This is a generalization of the Lie superbialgebra quantization studied by Geer [Gee06]. To better understand the situation in the colored setting, we first give a description of the category of color vector spaces as an enriched functor category. For this we study the symmetric monoidal structures on the discrete category on a group Γ and the induced monoidal structure on the category of functors from Γ to the category of vector spaces. There is a much general setting in which one gets a symmetric monoidal structure on a functor category. For this one only needs that the source category is promonoidal [Day70]. We also consider deformations of color Lie bialgebras, for this we generalize the big bracket or grand crochet introduced by Lecomte and Roger [LR90] to the color setting. With this bracket we define a cohomology for color Lie bialgebras. Indeed the Maurer-Cartan elements are color Lie bialgebras. Similar to the original work by Gerstenhaber [Ger64], we relate this cohomology to formal deformations.

This part is structured as follows: in the first chapter, we describe the category of graded vector spaces and consider color vector spaces as enriched functors. In Chapter 2, we provide various definitions of color algebraic structures, properties and basic constructions like semidirect products for color Lie algebras. Then we

discuss color Lie bialgebras in more detail. For example we generalize the notion of Manin triples to the color setting. In Chapter 3, we extend the big bracket, introduced by Roger and Lecomte, to the the color setting. This leads to a definition of a cohomology for color Lie bialgebras. Moreover, we study deformations of color Lie bialgebras in terms of cohomology. Finally Chapter 4 is dedicated to quantization of color Lie bialgebras, we give a generalization of the proof of Etingof and Kazhdan in the color setting.

In the second part of this thesis, we aim to study cohomologies and deformations of Hom-algebras. Hom-Lie algebras have been introduced in the study of quantum deformations of Witt and Virasoro algebras by σ -derivations. Such examples appear in various contexts in physics. Formal definition of Hom-Lie and a systematic study was given in [HLS06]. Since then many different types of Hom-algebras have been introduced and studied [MS08]. In this thesis we will focus on Hom-associative and Hom-Lie algebras, which are generalization of associative and Lie algebras respectively. A Hom-algebra consists of a multiplication and a linear map, called structure map and usually denoted by α . In the case of Hom-associative algebras the associative condition is twisted by α and in the Lie case the Jacobi identity. Many different aspects of these algebras have been studied. Hochschild and Chevalley-Eilenberg cohomologies have been extended to Hom-setting, see [MS08, AEM11], with the disadvantage that they can only be used to study deformations following Gerstenhaber [Ger64], where the structure map is fixed. In this thesis, we provide a new type of cohomologies, which we call α -type cohomologies, for Hom-associative algebras and Hom-Lie algebras. However with the α -type cohomologies, we give a formal deformation theory for Hom-associative algebras and Hom-Lie algebras, where both the multiplication and the map α are deformed. The α -type Hochschild cohomology is extended to an α -type cohomology for Hom-bialgebras, generalizing the known Gerstenhaber-Schack cohomology for bialgebras [GS90] and Hom-bialgebras [DM17]. This cohomology is related via the Yau twist to the cohomology of an algebra endomorphism, which is a special case of the cohomology of an algebra morphism e.g. considered in [FMY09, FZ15, NR67, GS83].

We start this part by Chapter 5 which includes the definitions of Hom-associative and Hom-Lie algebras and corresponding bialgebras. We also recall several known constructions for which we give some operadic interpretation. In Chapter 6 we recall some cohomology constructions and α -type Hochschild cohomology and Chevalley-Eilenberg cohomology. Various properties and constructions are generalized and discussed. In Section 6.3, we give the definition of the α -type Hochschild cohomology. Moreover, we show several properties of this new cohomology and calculate it for associative algebras. It is also related to the known Hochschild cohomology. The Yau twist gives a way of constructing Hom-algebras starting from usual algebra of the same type and an algebra morphism. The procedure may give a new Hom-algebra from a given Hom-algebra and an algebra morphism. In the associa-

tive case, one needs an associative or Hom-associative algebra A and a morphism γ . We denote the Yau twist of A with respect to γ by A_γ . There exists a cohomology for a morphism between associative algebras, which we specialize to the case of an endomorphism. This can be related to the α -type cohomology of A_γ . We also give some concrete examples for the cohomology of low dimensional algebras. In Section 6.7, we extend the α -type Hochschild cohomology to Hom-bialgebras and get a generalization of the well known Gerstenhaber-Schack cohomologies for bialgebras. In the case of Hom-Lie bialgebras, we define a generalization of the big bracket in the Hom-setting, which is used to define a cohomology for Hom-Lie bialgebras, where the structure maps for the product and coproduct are independent. In Section 6.9, we give an α -type Chevalley-Eilenberg cohomology for Hom-Lie algebras similar to the α -type Hochschild cohomology. We also give an α -type cohomology for Hom-Lie bialgebras but only in the case, where the two structure maps are the same. In Chapter 7, we construct an L_∞ structure, which is used to define the low degrees of the α -type Hochschild differential. Also the Maurer-Cartan elements of this L_∞ structure are Hom-associative algebras. Similar we obtained L_∞ structure for Hom-Lie algebras, Hom-bialgebras and Hom-Lie bialgebras in the following sections. In Chapter 8, we consider formal deformations of different types of Hom-algebras and show how they are related to the α -type cohomologies defined before. We begin with Hom-associative algebras. We prove that the deformation equations lie in the third cohomology group. Also the equivalence of deformation gives rise to equations in the second cohomology group. So for example the first order of the deformation is a 2-cocycle which is invariant under equivalence. We furthermore study the deformation of commutative Hom-associative algebras. Similar to the associative case deformations give rise to Hom-Poisson algebras. Similarly, we study the deformation of Hom-Lie algebras, and Hom-bialgebras in the following sections. Finally, in Section 8.6 we consider the quantization of Hom-Lie bialgebras. We mostly deal with the case that the structure map is invertible.

The thesis is ended with 4 appendices, the first one recalls the basic category theory, the second one is dedicated to enriched categories and the third one provides some basics about operads and properads. The final one gives the differentials of the free dg operads considered in Chapter 7.

VERSION FRANÇAISE

1 INTRODUCTION

Cette thèse est constituée de deux parties relativement distinctes. Le but de la première partie est d'étudier les déformations et quantification des bialgèbres de Lie colorée et la deuxième partie est consacrée à la cohomologie des Hom-algèbres.

Dans le cas des bialgèbres de Lie, l'existence d'une quantification pour chaque bialgèbre de Lie a été démontrée par Etingof et Kazhdan [EK96]. Cette preuve a été généralisée par Geer au cas \mathbb{Z}_2 -gradué [Gee06]. Dans ce travail, on s'intéresse au cas encore plus général des bialgèbres de Lie colorées, c'est à dire la graduation est donnée par un groupe quelconque et un bicaractère. A cet effet, on adapte la preuve de Etingof et Kazhdan en conséquence. Pour ça nous regardons la catégorie des modules colorés comme une catégorie de foncteurs enrichies par **Vec** entre la groupe et la catégorie **Vec**. On définit les triples de Manin dans le cas coloré et on introduit une généralisation au cas coloré du grand crochet introduit par Lecomte et Roger [LR90]. Par ailleurs nous définissons une cohomologie pour les algèbres et bialgèbres de Lie colorées.

Les premiers exemples des algèbres de type "Hom" sont apparus en physique dans l'étude des déformations quantiques de certaines algèbres de champs de vecteurs, comme par exemple les algèbres de Witt et de Virasoro. Une déformation quantique consiste à remplacer la dérivation usuelle par une σ -dérivation. Il s'avère que les algèbres obtenues ne satisfont pas l'identité de Jacobi, mais une version modifiée qui implique un homomorphisme. Ces algèbres sont appelées algèbres Hom-Lie et étudiées par Hartwig, Larson et Silvestrov [HLS06]. Une propriété fondamentale des algèbres de Lie est qu'elles sont liées à des algèbres associatives par la construction du crochet commutateur. Les algèbres de type associative correspondantes aux algèbres Hom-Lie, appelées algèbres Hom-associatives, ont été introduites par Makhoul et Silvestrov [MS08]. Une algèbre Hom-associative A est définie par une multiplication et une application linéaire α modifiant l'associativité $\alpha(x)(yz) = (xy)\alpha(z)$. Ils ont montré que le crochet commutateur défini par la multiplication d'une algèbre Hom-associative conduit naturellement à définir une algèbre Hom-Lie. Durant la dernière décennie de nombreux résultats et structures classiques ont été généralisés aux cas Hom. Un des objectifs de cette thèse est de généraliser le concept de cohomologie et de déformation formelle. Des généralisations de type classique où seul la multiplication est déformée ont été proposées par Ammar, Ejbhi, Makhoul et

Silvestrov [AEM11]. Dans ce travail, on profite de la richesse de la structure des Hom-algèbres, pour proposer un nouveau type de cohomologie qui tient compte fortement de l'application linéaire. Ces cohomologies qu'on appelle de type α sont adaptées aux déformations formelles des Hom-algèbres où la multiplication et l'application linéaire sont toutes les deux déformées. On commence la première partie par rappeler les définitions et propriétés des algèbres Hom-associatives et Hom-Lie, ainsi que les bialgèbres de type Hom. Ensuite, on définit la cohomologie de Hochschild de type α , en donnant ses propriétés et en la calculant pour une algèbre associative. Par ailleurs, on établit le lien avec la cohomologie de Hochschild usuelle ainsi que son comportement par le Twist de Yau. Une étude similaire est faite dans le cas des algèbres Hom-Lie et la cohomologie de Chevalley-Eilenberg, ainsi que pour les Hom-bialgèbres et bialgèbres Hom-Lie. La théorie de déformations formelles introduite par Gerstenhaber met en lien les déformations et la cohomologie. Dans cette thèse on établit une théorie de déformations des algèbres Hom-associatives basée sur la cohomologie de Hochschild de type α . Il s'agit de déformer simultanément la multiplication et l'application linéaire. On montre par exemple que le premier terme est un 2-cocycle et que les obstructions sont des 3-cocycles. Une étude similaire est faite pour les algèbres Hom-Lie. Par ailleurs, on explore la structure L_∞ correspondante, tel que les éléments de Maurer-Cartan sont des Hom-algèbres.

2 DEFORMATION ET QUANTIFICATION DES ALGÈBRÉS LIE COLORÉE

2.1 Les modules gradués comme catégorie des foncteurs

Dans cette sous-section nous regardons la catégorie monoïdale des modules colorés comme une catégorie enrichie, pour la théorie des catégories enrichies voir par exemple [Kel82] pour les détails et a construction complète.

Pour un groupe commutatif Γ , on définit une **Vec**-catégorie, tel que les objets sont les éléments de Γ et les morphisms sont

$$\text{Hom}(i, j) = \mathbb{K} \text{ si } i = j, 0 \text{ si } i \neq j. \quad (1)$$

On considère le produit tensoriel associatif

$$i \otimes j = i + j \text{ for } i, j \in \Gamma. \quad (2)$$

Un associateur pour Γ est donné par un 3-cocycle dans la cohomologie de Γ . Il y a un foncteur monoïdal entre deux associateurs s'ils sont cohomologue.

Si α est trivial, une symétrie pour Γ est donnée par un bicaractère symétrique ou facteur de commutation, i.e. une application $\epsilon : \Gamma \times \Gamma \rightarrow \mathbb{K}^\times$ telle que

$$\epsilon(a, c)\epsilon(b, c) = \epsilon(a + b, c), \quad \epsilon(a, b)\epsilon(a, c) = \epsilon(a, b + c), \epsilon(a, b) = \epsilon^{-1}(b, a). \quad (3)$$

On peut définir un module Γ -gradué par un foncteur $\Gamma \rightarrow \mathbf{Vec}$ et la catégorie des modules Γ -gradués **Modgr** est donnée par la catégorie des **Vec**-foncteur $\Gamma \rightarrow \mathbf{Vec}$.

La catégorie **Modgr** est une catégorie monoïdale symétrique avec produit tensoriel

$$F \hat{\otimes} G = \int^{a,b} F(a) \otimes G(b) \otimes \text{Hom}(a+b, \cdot). \quad (4)$$

L'associateur et le symétrique est induit par α et ϵ .

2.2 Algèbres colorées

Dans ce section nous définissons les différents types d'algèbres colorées.

Pour une transposition $\tau \in S_n$ qui échange i contre $i+1$, on définit une action $V^{\otimes n} \rightarrow V^{\otimes n}$ par $\tau \cdot (v_1 \otimes \dots \otimes v_n) = \epsilon(v_i, v_{i+1}) v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n$. Ceci induit une action de S_n .

Définition 2.1 Une multiplication n -aire est une application $\mu : V^{\otimes n} \rightarrow V$ de degré zéro. Elle est dite symétrique colorée si

$$\mu(v_1 \otimes \dots \otimes v_n) = \mu(\sigma \cdot v_1 \otimes \dots \otimes v_n) \quad (5)$$

quel que soit $\sigma \in S_n$ et antisymétrique colorée si

$$\mu(v_1 \otimes \dots \otimes v_n) = -\mu(\tau \cdot v_1 \otimes \dots \otimes v_n) \quad (6)$$

pour toutes transpositions $\tau \in S_n$.

Une algèbre colorée associative est un espace vectoriel avec une multiplication associative.

Un morphisme d'algèbres colorées binaires est automatiquement de degré zéro.

Une dérivation d'une algèbre colorée est une application linéaire homogène $d : A \rightarrow A$ telle que

$$d(ab) = d(a)b + \epsilon(d, a)ad(b) \quad (7)$$

ou

$$d \circ \mu = \mu \circ (\text{id} \otimes d + d \otimes \text{id}), \quad (8)$$

où μ est la multiplication de A .

Dualement, on peut définir des coalgèbres et des codérivations.

Un élément $x \in C$ s'appelle primitif si $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Etant données une algèbre colorée A et une coalgèbre colorée C , l'espace $\text{Homgr}(A, C)$ est une algèbre colorée associative par

$$(\varphi * \psi)(x) = \mu(\varphi \otimes \psi)\Delta(x) \quad (9)$$

quelque soient $\varphi, \psi \in \text{Hom}(C, A)$ et $x \in C$.

Définition 2.2 Une algèbre de Hopf colorée $(H, \mu, \Delta, \mathbb{1}, \varepsilon, S)$ est une algèbre colorée unitaire $(H, \mu, \mathbb{1})$, une coalgèbre counitaire (H, Δ, ε) et un antipode S , tels que μ et $\mathbb{1}$ sont des morphismes de coalgèbres. Un antipode est un inverse de l'unité pour la convolution.

Définition 2.3 Une algèbre quasi-Hopf colorée est une algèbre colorée associative H avec une multiplication μ , une comultiplication Δ , une unité 1 , une counité ε , un antipode et un associateur inversible $\Phi \in H^{\otimes 3}$ tel que

$$\begin{aligned} \forall x, y \in H : \Delta(xy) &= \Delta(x)\Delta(y) && \text{(compatibilité),} \\ (\varepsilon \otimes \text{id})\Delta &= \text{id} = (\text{id} \otimes \varepsilon)\Delta && \text{(counit),} \\ \Phi(\Delta \otimes \text{id})\Delta &= (\text{id} \otimes \Delta)\Delta\Phi && \text{(quasi- coassociativité),} \\ \Phi_{1,2,34}\Phi_{12,3,4} &= \Phi_{2,3,4}\Phi_{1,23,4}\Phi_{1,2,3} && \text{(condition Pentagon),} \\ (\text{id} \otimes \varepsilon \otimes \text{id})\Phi &= 1 \otimes 1. \end{aligned}$$

Une algèbre quasi-Hopf colorée s'appelle quasi-triangulaire s'il existe une R -matrice $R \in H^{\otimes 2}$, de degré 0, telle que

$$(\text{id} \otimes \Delta)R = \Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}\Phi_{123}^{-1} \quad (10)$$

$$(\Delta \otimes \text{id})R = \Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi_{123}R\Delta^{\text{opp}} = \Delta R. \quad (11)$$

La matrice R a degré 0.

Deux algèbres quasi-Hopf quasi-triangulaire colorée sont twist équivalentes s'il existe un élément inversible $J \in H^{\otimes 2}$ de degré 0 tel que

$$\begin{aligned} (\varepsilon \otimes \text{id})J &= 1 = (\text{id} \otimes \varepsilon)J, \\ \Delta' &= J^{-1}(\Delta(x))J, \\ \Phi' &= J_{2,3}^{-1}J_{1,23}^{-1}(\Phi)J_{12,3}J_{1,2}, \\ R' &= J_{21}^{-1}RJ. \end{aligned}$$

Si J satisfait la première identité, on peut alors définir une algèbre quasi-Hopf quasi-triangulaire.

Théorème 2.4 *La catégorie des modules sur une algèbre quasi-Hopf colorée est une catégorie monoïdale tressée. Si deux algèbres sont équivalentes par twist les deux catégories sont tenseur équivalentes.*

Bialgèbres et Algèbres de Lie colorées

Définition 2.5 Soient Γ un groupe et ε un bicaractère, une algèbre de Lie (Γ, ε) -colorée est un espace vectoriel Γ -gradué \mathfrak{g} avec une application bilinéaire $[\cdot, \cdot] : \mathfrak{g} \times$

$\mathfrak{g} \rightarrow \mathfrak{g}, (x, y) \mapsto [x, y]$ tel que

$$[a, b] = -\epsilon(a, b)[b, a], \quad (12)$$

$$j(a, b, c) := \epsilon(c, a)[a, [b, c]] + \epsilon(a, b)[b, [c, a]] + \epsilon(b, c)[c, [a, b]] = 0. \quad (13)$$

Soit A une algèbre associative Γ -graduée, alors

$$[a, b] = ab - \epsilon(a, b)ba \quad (14)$$

est un crochet de Lie colorée.

Proposition 2.6 *Les dérivations colorées sont une sous-algèbre de Lie colorée d'algèbre de Lie $\text{Homgr}(A, A)$.*

Une algèbre de Lie colorée \mathfrak{g} admet une algèbre enveloppante $U(\mathfrak{g})$ qui est analogue au cas non gradué. Elle admet une structure d'algèbre de Hopf colorée.

Définition 2.7 Une algèbre de Poisson colorée $(P, \{\cdot, \cdot\}, \cdot)$ est une algèbre de Lie colorée $(P, \{\cdot, \cdot\})$ avec une multiplication associative commutative \cdot , tel que

$$\{a, bc\} = \{a, b\}c + \epsilon(a, b)b\{a, c\}. \quad (15)$$

Le produit tensoriel de deux algèbres de Poisson colorée est encore une algèbre de Poisson colorée.

Nous considérons l'algèbre libre commutative $S(V)$ pour un espace colorée V . Soit V un espace coloré, on définit $\text{Pol}(V) := T(V^*)/I$, où I est l'idéal engendré par $x \otimes y - \epsilon(x, y)y \otimes x$. Soit $\{x_i\}_i$ une base graduée de V^* , alors on peut définir une dérivation ∂_i sur les générateurs par

$$\partial_i x_j := \delta_{ij}. \quad (16)$$

Si V est de dimension finie, un crochet de Poisson sur $A = \text{Pol}(V)$ est de la forme

$$\{f, g\} = \epsilon(f, j)r^{ij}\partial_i f \partial_j g = r^{ij}(\partial_i \otimes \partial_j)(f \otimes g), \quad (17)$$

où $r^{ij} \in A$ est de degré $-\text{deg}(i) - \text{deg}(j)$, si le crochet est de degré zéro.

Proposition 2.8 *Le crochet (17) satisfait l'identité de Leibniz et*

i) *il est antisymétrique coloré si $r^{ij} = -\epsilon(i, j)^{-1}r^{ji}$ (sic!),*

ii) *il satisfait l'identité de Jacobi si $J(r) = 0$.*

Ici $J(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$. Alors si les deux conditions sont satisfaites, on obtient un crochet de Poisson coloré.

Chaque bialgèbre de Lie colorée peut être transformée en une superbialgèbre de Lie. En effet, on a :

Proposition 2.9 *Il y a un foncteur monoïdale symétrique de la catégorie des espaces (Γ, ϵ) -gradués dans la catégorie des superspaces.*

On peut considérer différents types de produits semi-directs de (bi)algèbres de Lie colorées.

Triplets de Manin et r -matrices

Comme pour le cas non-gradué, on peut définir les bialgèbres de Lie cobord, triangulaires et quasi-triangulaires.

Proposition 2.10 *Soit \mathfrak{g} une algèbre de Lie, $r \in \Lambda^2(\mathfrak{g})$ de degré 0 et $\delta(x) := \text{ad}_x r = [x, r] = [\Delta x, r]$. Alors on a une bialgèbre de Lie colorée si et seulement si*

$$J(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \quad (18)$$

est \mathfrak{g} -invariant, i.e. $[\Delta^{(3)}x, J(r)] = 0$ quel que soit $x \in \mathfrak{g}$.

Définition 2.11 Un triplet de Manin est un triplet (ρ, ρ_+, ρ_-) , où ρ est une algèbre de Lie colorée, ρ_{\pm} sont des sous-algèbres de ρ et $\rho = \rho_+ \oplus \rho_-$ comme espace gradué, avec une forme bilinéaire symétrique non-dégénérée $(\cdot, \cdot) : \rho \times \rho \rightarrow \mathbb{K}$ telle que ρ_{\pm} sont isotropes.

Théorème 2.12 *Soit \mathfrak{g} une bialgèbre de Lie colorée. Alors (ρ, ρ_+, ρ_-) avec $\rho_+ = \mathfrak{g}, \rho_- = \mathfrak{g}^*$ et $\rho = \rho_+ \oplus \rho_-$ est une triplet de Manin coloré. Inversement chaque triplet de Manin coloré de dimension finie donne une bialgèbre de Lie colorée.*

Soit \mathfrak{g} une bialgèbre de Lie colorée alors le triplet de Manin associé est une bialgèbre de Lie triangulaire avec $r = x_i \otimes \alpha^i$, où $\{x_i\}_i$ est une base de \mathfrak{g} et $\{\alpha_i\}_i$ la base duale.

2.3 Cohomologie et Déformations

Grande Crochet

Nous notons par ΛV l'algèbre extérieure pour un espace vectoriel V donné. Nous posons

$$B := \text{Homgr}(\Lambda V, \Lambda V), B^k := \text{Homgr}^k(\Lambda V, \Lambda V), B^{i,j} := \text{Homgr}(\Lambda^i V, \Lambda^j V). \quad (19)$$

On définit une application bilinéaire $\odot : B \otimes B \rightarrow B$ par

$$f \odot g = \mu \circ (f \otimes \text{id}) \circ Q \circ (\text{id} \otimes g) \circ \Delta, \quad (20)$$

où

$$Q = (\mu \otimes \text{id}) \circ (\text{id} \otimes \text{pr}_V \otimes \text{id}) \circ (\text{id} \otimes \Delta). \quad (21)$$

Ainsi, on définit le grand crochet par :

Définition 2.13 (grand crochet) Soient $f, g \in B$, on définit

$$\{f, g\}_{\text{BB}} = f \odot g - \epsilon(f, g)g \odot f. \quad (22)$$

Proposition 2.14 *Le grand crochet est un crochet de Poisson coloré pour la convolution. Pour $\varphi \in B^{i,j}, \psi \in B^{k,l}$, on a $\{\varphi, \psi\}_{\text{BB}} \in B^{i+k-1, j+l-1}$.*

Proposition 2.15 *Soient $\beta \in B^{2,1}, \delta \in B^{1,2}$, alors*

- $\{\beta, \beta\}_{\text{BB}} = 0$ si et seulement si β est un crochet de Lie coloré.
- $\{\delta, \delta\}_{\text{BB}} = 0$ si et seulement si δ est un cocrochet de Lie coloré.
- $\{\beta + \delta, \beta + \delta\}_{\text{BB}} = 0$ si et seulement si $(\mathfrak{g}, \beta, \delta)$ est une bialgèbre de Lie colorée.

Nous définissons maintenant une cohomologie pour une bialgèbre de Lie colorée (clba). Le complexe est donné par

$$\mathcal{B}^k(\mathfrak{g}) = \bigoplus_{i+j=k} B^{i,j}(\mathfrak{g}) = \bigoplus_{i+j=k} \text{Homgr}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}). \quad (23)$$

est la différentielle par

$$\partial(\varphi) := \{\delta + \beta, \varphi\}_{\text{BB}}. \quad (24)$$

Déformations

Soit A une algèbre associative commutative colorée. Une déformation de A est une algèbre associative (non-commutative) sur $A[[t]]$, tel que $a \star b = ab + \sum_{r=1}^{\infty} t^r C_r(a, b)$. Ici C_r sont des applications bilinéaires de degré 0. On appelle le produit déformé un star produit.

Deux déformations \star, \star' sont dites équivalentes s'il existe une application linéaire $S = \text{id} + \sum_{r=1}^{\infty} S_r t^r$ de degré 0, telle que

$$S(a \star' b) = S(a) \star S(b). \quad (25)$$

Proposition 2.16 *Soit \star une déformation d'une algèbre commutative colorée, alors*

$$\{a, b\} = C_1(a, b) - \epsilon(a, b)C_1(b, a) \quad (26)$$

est un crochet de Poisson coloré.

Le crochet de Poisson de deux déformations équivalentes est le même.

Définition 2.17 Une déformation d'une clba $(\mathfrak{g}, \beta, \delta)$ est une $\mathbb{K}[[t]]$ -bialgèbre de Lie $(\mathfrak{g}[[t]], \beta_\star, \delta_\star)$, telle que $\beta_\star = \beta + \sum_{i=1}^{\infty} \beta_i t^i$ et $\delta_\star = \delta + \sum_{i=1}^{\infty} \delta_i t^i$. Ici t est de degré zéro.

Proposition 2.18 La construction ordre par ordre de la déformation donne des équations en lien avec la cohomologie de la clba. Alors les obstructions pour l'existence d'une déformation sont dans $H_r^3(\mathfrak{g})$. Si $H_r^3(\mathfrak{g}) = \{0\}$, on peut étendre toute déformation d'ordre n en une déformation d'ordre $n + 1$.

Définition 2.19 Deux déformations $([\ , \], \delta)$ et $([\ , \]', \delta')$ sont équivalentes s'il existe une série formelle $S = \text{id} + \sum_{i=1}^{\infty} t^i S_i$ d'applications linéaire $S_i : \mathfrak{g} \rightarrow \mathfrak{g}$ de Γ -degré 0, telle que

$$\begin{aligned} S^{-1}([S(a), S(b)]) &= [a, b]', \\ (S^{-1} \otimes S^{-1})\delta(S(a)) &= \delta'(a). \end{aligned}$$

Proposition 2.20 Si deux déformations sont équivalentes jusqu'à l'ordre k , alors $\partial(\beta'_{k+1} + \delta'_{k+1} - \beta_{k+1} - \delta_{k+1}) = 0$ et il y a une équivalence d'ordre $k + 1$ si la différence est exacte. Donc l'obstruction pour l'équivalence est dans $H_r^2(\mathfrak{g})$ et si $H_r^2(\mathfrak{g}) = \{0\}$ toute déformation est équivalente à la déformation triviale.

2.4 Quantification de bialgèbres de Lie colorées

Une algèbre enveloppante quantique colorée (cQUE) est une algèbre de Hopf colorée A sur $\mathbb{K}[[t]]$, telle que A/tA est l'algèbre enveloppante $U(\mathfrak{g})$ pour une algèbre de Lie colorée \mathfrak{g} .

Si H est une cQUE, alors $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ définie par

$$\delta(x) = \frac{\Delta(x) - \Delta^{\text{opp}}(x)}{t} \pmod{t} \quad (27)$$

détermine une bialgèbre de Lie colorée sur \mathfrak{g} . Dans ce cas, on appelle H une quantification de \mathfrak{g} .

Suivant la preuve de Etingof-Kazhdan, il est possible de démontrer que chaque bialgèbre de Lie colorée admet une quantification.

Pour cela, nous considérons les transformations naturelles $\text{End}(F, F)$ enrichies sur **Modgr**, qui sont isomorphes à $U(\mathfrak{g})$ comme algèbre. Ici F est le foncteur $V \mapsto \text{Hom}_{\mathfrak{g}}(U(\mathfrak{g}), V)$ pour V un \mathfrak{g} -module.

3 HOM-ALGÈBRES

3.1 Définitions

Dans cette section, nous nous intéressons aux (bi)algèbres Hom-associative et Hom-Lie.

Définition 3.1 (Hom-module) Un Hom-module (V, α) est un espace vectoriel V avec une application linéaire $\alpha : V \rightarrow V$, appelé twist.

Soient (V, α) et (W, β) deux Hom-modules. Un morphisme de Hom-modules est une application linéaire $\varphi : V \rightarrow W$ telle que $\varphi \circ \alpha = \beta \circ \varphi$. On note l'ensemble de ces morphismes par $\text{Hom}((V, \alpha), (W, \beta))$.

Un Hom-module est dit régulier si α est inversible.

Définition 3.2 Une algèbre Hom-associative (A, μ, α) est un Hom-module (A, α) avec une application bilinéaire $\mu : A \times A \rightarrow A : (x, y) \mapsto \mu(x, y) = xy$, telle que

$$\mu \circ (\mu \otimes \alpha) = \mu \circ (\alpha \otimes \mu). \quad (28)$$

Nous supposons aussi

$$\mu(\alpha \otimes \alpha) = \alpha \mu. \quad (29)$$

Une unité est un élément $\mathbb{1} \in A$ tel que $\mathbb{1}x = \alpha(x) = x\mathbb{1}$.

Un morphisme de Hom-algèbres $(A, \mu, \alpha), (B, \nu, \beta)$ est une application linéaire $\varphi : A \rightarrow B$ tel que $\varphi\alpha = \beta\varphi$ et $\mu(\varphi \otimes \varphi) = \nu\mu$. Pour les algèbres unitaires, on a aussi $\varphi(\mathbb{1}_A) = \mathbb{1}_B$.

Proposition 3.3 (Yau twist) Soit (A, μ, α) une algèbre Hom-associative et $\gamma : A \rightarrow A$ un morphisme, alors $A_\gamma = (A, \gamma\mu, \gamma\alpha)$ est une algèbre Hom-associative.

Définition 3.4 Une algèbre Hom-associative A , qui a la forme \tilde{A}_γ pour une algèbre associative \tilde{A} , est dite de type associative.

Par exemple une algèbre Hom-associative (A, μ, α) régulière est de type associative, car l'algèbre $A_{\alpha^{-1}}$ est associative.

On peut aussi définir les coalgèbres Hom-coassociatives.

Définition 3.5 Soient (A, μ, α) une algèbre Hom-associative et (M, β) un Hom-module. Une application linéaire $\rho : A \times M \rightarrow M, (a, m) \mapsto a \cdot m$, définit un A -module (à gauche) si

$$(ab) \cdot \beta(m) = \alpha(a)(b \cdot m) \quad (30)$$

$$\beta(a \cdot m) = \alpha(a) \cdot \beta(m). \quad (31)$$

De même on peut définir un A -module à droite.

Un A -bimodule est un Hom-module (M, β) , avec deux applications linéaires $\rho : A \otimes M \rightarrow M, a \otimes m \mapsto a \cdot m$ et $\lambda : M \otimes A \rightarrow M, m \otimes a \mapsto m \cdot a$, telles que ρ définit un module à gauche, λ un module à droite et

$$\alpha(a) \cdot (m \cdot b) = (a \cdot m) \cdot \alpha(b). \quad (32)$$

Naturellement, A est un A -bimodule, l'action étant donnée par la multiplication.

Pour une coalgèbre Hom-coassociative, on peut définir de manière analogue des comodules.

Définition 3.6 Une Hom-bialgèbre est un septuplet $(A, \mu, \Delta, \mathbb{1}, \epsilon, \alpha, \beta)$ tel que $(A, \mu, \mathbb{1}, \alpha)$ soit une algèbre Hom-associative unitaire, $(A, \Delta, \epsilon, \beta)$ est une coalgèbre Hom-coassociative counitaire et elles sont compatibles. Cela veut dire

$$\begin{aligned} \alpha\beta &= \beta\alpha, & \Delta(xy) &= \Delta(x)\Delta(y), \\ \Delta\alpha &= (\alpha \otimes \alpha)\Delta, & \beta\mu &= \mu(\beta \otimes \beta), \\ \Delta(\mathbb{1}) &= \mathbb{1} \otimes \mathbb{1}, & \epsilon(xy) &= \epsilon(x)\epsilon(y) \\ \epsilon\alpha &= \epsilon & \beta(\mathbb{1}) &= \mathbb{1}, \epsilon(\mathbb{1}) = 1. \end{aligned}$$

Soit A une Hom-bialgèbre et M_1, \dots, M_n des A -modules, nous construisons une action de A sur $M_1 \otimes \dots \otimes M_n$. Pour ça on définit $\Delta_\beta^2 = \Delta$ et

$$\Delta_\beta^{n+1} = (\Delta \otimes \beta^{\otimes(n-1)})\Delta_\beta^n. \quad (33)$$

qui satisfait

$$\Delta_\beta^{n+m} = (\Delta_\beta^n \beta^{m-1} \otimes \Delta_\beta^m \beta^{n-1})\Delta. \quad (34)$$

L'action de A sur $M_1 \otimes \dots \otimes M_n$ est définie par

$$(\rho_1 \otimes \dots \otimes \rho_n)\tau_{(2,n)}(\Delta_\beta^n \otimes \text{id}^n). \quad (35)$$

Ici $\tau_{(2,n)}$ est la permutation $(1, \dots, 2n) \mapsto (1, n+1, 2, n+2, \dots, n, 2n)$.

On a aussi

$$x \cdot (v_1 \otimes \dots \otimes v_{m+n}) = \beta^{m-1}(x^{(1)}) \cdot (v_1 \otimes \dots \otimes v_n) \otimes \beta^{n-1}(x^{(2)}) \cdot (v_{n+1} \otimes \dots \otimes v_{n+m}). \quad (36)$$

Définition 3.7 Une algèbre Hom-Lie $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ est un Hom-module avec une application bilinéaire antisymétrique $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ telle que l'identité de Hom-Jacobi soit satisfaite, i.e.

$$[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)] = 0 \quad (37)$$

et $[\alpha(x), \alpha(y)] = \alpha([x, y])$ quel que soient $x, y, z \in \mathfrak{g}$.

Proposition 3.8 Soit $(\mathfrak{g}, \nu, \alpha)$ une algèbre Hom-Lie et $\gamma : \mathfrak{g} \rightarrow \mathfrak{g}$ un morphisme, alors $(\mathfrak{g}, \gamma\nu, \gamma\alpha)$ est une algèbre Hom-Lie.

Définition 3.9 Soit \mathfrak{g} une algèbre Hom-Lie. Un \mathfrak{g} -module ou une représentation de \mathfrak{g} est un Hom-module (V, β) , avec une action $\rho : \mathfrak{g} \otimes V \rightarrow V, \mathfrak{g} \otimes v \mapsto x \cdot v$, telle que

$$[x, y] \cdot \beta(v) = \alpha(x) \cdot (y \cdot v) - \alpha(y) \cdot (x \cdot v) \beta(x \cdot v) = \alpha(x) \cdot \beta(v). \quad (38)$$

Etant donné une algèbre Hom-Lie, on peut définir la représentation adjointe de \mathfrak{g} sur lui-même par $x \cdot y = [x, y]$.

Etant donné n \mathfrak{g} -modules $(M_1, \alpha_1), \dots, (M_n, \alpha_n)$, on peut définir une \mathfrak{g} -action sur $M_1 \otimes \dots \otimes M_n$ par

$$x \cdot (y_1 \otimes \dots \otimes y_n) = \sum_{i=1}^k \alpha_1(y_1) \otimes \dots \otimes [x, y_i] \otimes \dots \otimes \alpha_n(y_n) \quad (39)$$

quelque soit $y_i \in M_i$.

L'action coadjointe de \mathfrak{g} sur \mathfrak{g}^* est définie par

$$(\text{ad}_x \varphi)(y) = \varphi([x, y]). \quad (40)$$

Pour une algèbre Hom-Lie, on peut construire une algèbre enveloppante $U(\mathfrak{g})$, voir [Yau08]. C'est une algèbre Hom-associative, telle que pour toutes les algèbres Hom-associatives

$$\text{Hom}_{HLie}(\mathfrak{g}, A_L) \cong \text{Hom}_{HAss}(U(\mathfrak{g}), A). \quad (41)$$

Définition 3.10 (Bialgèbre Hom-Lie) Une bialgèbre Hom-Lie est un quintuplet $(\mathfrak{g}, \nu, \delta, \alpha, \beta)$, tel que $(\mathfrak{g}, \nu, \alpha)$ est une algèbre Hom-Lie, $(\mathfrak{g}, \delta, \beta)$ est une coalgèbre Hom-Lie et elles sont compatibles, c'est à dire

$$\begin{aligned} \delta([x, y]) &= \alpha(x^{(1)}) \otimes [x^{(2)}, \beta(y)] + [x^{(1)}, \beta(y)] \otimes \alpha(x^{(1)}) \\ &+ [\beta(x), y^{(1)}] \otimes \alpha(y^{(1)}) + \alpha(y^{(1)}) \otimes [\beta(x), y^{(2)}]. \end{aligned} \quad (42)$$

Proposition 3.11 (Yau twist) Soit \mathfrak{g} une bialgèbre Hom-Lie et $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ un morphisme, alors $(\mathfrak{g}, \varphi\nu, \delta, \varphi\alpha, \beta)$, $(\mathfrak{g}, \nu, \varphi\delta, \alpha, \varphi\beta)$ et $(\mathfrak{g}, \varphi\nu, \varphi\delta, \varphi\alpha, \varphi\beta)$ sont encore des bialgèbres Hom-Lie.

Définition 3.12 Une algèbre Hom-Poisson est un quadruplet $(A, \mu, \{\cdot, \cdot\}, \alpha)$ tel que (A, μ, α) est une algèbre Hom-associative commutative, $(A, \{\cdot, \cdot\}, \alpha)$ est un algèbre Hom-Lie et elles satisfont l'identité de Hom-Leibniz

$$\{\alpha(a), bc\} = \alpha(b)\{a, c\} + \{a, b\}\alpha(c). \quad (43)$$

3.2 Cohomologie de type α

Endomorphismes

Dans cette section nous définissons une cohomologie pour algèbre associative munie d'un endomorphisme.

Soit A une algèbre associative et $\gamma : A \rightarrow A$ un morphisme. On définit un complexe par

$$C^n(\gamma) = C_\mu^n(\gamma) \oplus C_\gamma^n(A) = C^n(A) \oplus C^{n-1}(A, \tilde{A}) \quad (44)$$

où $C(A)$ est le complexe de Hochschild pour A et \tilde{A} est A comme espace vectoriel mais avec l'action à gauche $a \cdot x = \gamma(a)x$ quel que soit $a \in A, x \in \tilde{A}$ et de manière similaire pour l'action à droite. Nous posons $C^0(A, \cdot) = 0$.

La différentielle est donnée par

$$\partial(\varphi, \psi) = (\partial_A \varphi, \partial_\gamma \varphi - \partial_A \psi), \quad (45)$$

où ∂_A est la différentielle de Hochschild. On écrit $H(A, \gamma)$ pour la cohomologie associée.

On peut faire une construction similaire pour une algèbre de Lie et un endomorphisme.

Pour une bialgèbre de Lie \mathfrak{g} et un morphisme $\gamma : \mathfrak{g} \rightarrow \mathfrak{g}$ on peut définir un complexe par

$$C^n(\mathfrak{g}, \gamma) = C_\mu^n(\mathfrak{g}) \oplus C_\gamma^n = C^n(\mathfrak{g}) \oplus C^{n-1}(\mathfrak{g}). \quad (46)$$

La différentielle est

$$\partial\varphi = (\partial_{CE}\varphi, \partial_\gamma\varphi, \partial_{CE}^c\varphi) \in C_\mu^{i+1,j} \oplus C_\alpha^{i,j} \oplus C_\mu^{i,j+1} \quad (47)$$

$$\partial\psi = (-\partial_{CE}\psi, -\partial_{CE}^c\psi) \in C_\alpha^{i+1,j} \oplus C_\alpha^{i,j+1} \quad (48)$$

pour tous $\varphi \in C_\mu^{i,j}(\gamma)$ et $\psi \in C_\gamma^{i,j}(\gamma)$. Ici ∂_{CE} est la différentielle de Chevalley-Eilenberg. L'action à gauche de \mathfrak{g} sur ψ est donnée par $x \cdot \psi = \gamma(x)\psi$ et de même pour la coaction. On écrit $H(\mathfrak{g}, \gamma)$ pour la cohomologie associée.

Pour une bialgèbre de Lie et deux morphismes $\gamma_1, \gamma_2 : \mathfrak{g} \rightarrow \mathfrak{g}$, on définit un complexe par

$$\begin{aligned} C^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) &= C_\mu^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \oplus C_{\gamma_1}^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \oplus C_{\gamma_2}^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \oplus C_{com}^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \\ &= C^{i,j}(\mathfrak{g}) \oplus C^{i-1,j}(\mathfrak{g}) \oplus C^{i,j-1}(\mathfrak{g}) \oplus C^{i-1,j-1}(\mathfrak{g}). \end{aligned}$$

La différentielle horizontale pour $(\varphi, \psi, \chi, \xi) \in C^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2)$ est donnée par

$$\partial(\varphi, \psi, \chi, \xi) = (\partial_{CE}\varphi, \partial_{\gamma_1}\varphi - \partial_{CE}\psi, \partial_{\gamma_2}\varphi - \partial_{CE}\chi, -\partial_{\gamma_2}\psi - \partial_{\gamma_1}\chi + \partial_{CE}\xi).$$

De façon duale, on peut définir la différentielle verticale.

Cohomologie de Hochschild de type α

Soit (A, μ, α) une algèbre Hom-associative et (M, ρ, α_M) un A -module. On définit un complexe par

$$\tilde{\text{HC}}^n(A, M) = \tilde{\text{HC}}_\mu^n(A, M) \oplus \tilde{\text{HC}}_\alpha^n(A, M) = \text{Hom}(A^{\otimes n}, M) \oplus \text{Hom}(A^{\otimes n-1}, M) \quad (49)$$

pour $n \geq 2$, $\tilde{\text{HC}}^1(A, M) = \text{Hom}(A, M)$, $\tilde{\text{HC}}^n(A, M) = 0$ pour $n \leq 0$ et des applications linéaires

$$\begin{array}{ccc} \tilde{\text{HC}}_\mu^n & \xrightarrow{\partial_{\mu\mu}} & \tilde{\text{HC}}_\mu^{n+1} \\ & \searrow \partial_{\mu\alpha} & \nearrow \partial_{\alpha\mu} \\ \oplus & & \oplus \\ \tilde{\text{HC}}_\alpha^n & \xrightarrow{\partial_{\alpha\alpha}} & \tilde{\text{HC}}_\alpha^{n+1} \end{array}$$

$$\begin{aligned} \partial_{\mu\mu}\varphi(x_1, \dots, x_{n+1}) &= \alpha^{n-1}(x_1) \cdot \varphi(x_2, \dots, x_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i \varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1})), \\ &+ (-1)^{n+1} \varphi(x_1, \dots, x_{n-1}) \cdot \alpha^{n-1}(x_{n+1}) \end{aligned} \quad (50)$$

$$\begin{aligned} \partial_{\alpha\alpha}\psi(x_1, \dots, x_n) &= \alpha^{n-1}(x_1) \cdot \psi(x_2, \dots, x_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i \psi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n)) \\ &+ (-1)^n \psi(x_1, \dots, x_{n-1}) \cdot \alpha^{n-1}(x_n), \end{aligned} \quad (51)$$

$$\partial_{\mu\alpha}\varphi(x_1, \dots, x_n) = \alpha_M(\varphi(x_1, \dots, x_n)) - \varphi(\alpha(x_1), \dots, \alpha(x_n)), \quad (52)$$

$$\partial_{\alpha\alpha}\psi(x_1, \dots, x_{n+1}) = \alpha^{n-2}(x_1 x_2) \cdot \psi(x_3, \dots, x_{n+1}) - \psi(x_1, \dots, x_{n-1}) \cdot \alpha^{n-2}(x_n x_{n+1}). \quad (53)$$

Finalement, la différentielle est définie par

$$\partial(\varphi + \psi) = (\partial_{\mu\mu} + \partial_{\mu\alpha})\varphi - (\partial_{\alpha\mu} + \partial_{\alpha\alpha})\psi \quad (54)$$

$$= (\partial_{\mu\mu}\varphi - \partial_{\alpha\mu}\psi, \partial_{\mu\alpha}\varphi - \partial_{\alpha\alpha}\psi). \quad (55)$$

Définition 3.13 On appelle la cohomologie $\tilde{\text{HC}}(A, M)$, la cohomologie de Hochschild de type α de A à valeurs dans M , et on la note $\tilde{\text{HH}}(A, M)$.

L'ensemble $\text{HC}^n(A) = \{(\varphi, 0) \in \tilde{\text{HC}}^n(A) \mid \alpha\varphi = \varphi^{\otimes n}\}$ est un sous-complexe de $\tilde{\text{HC}}^n(A)$ avec la différentielle $\partial_{\mu\mu}$. C'est le complexe considéré dans [AEM11].

Théorème 3.14 Soit A une algèbre associative, alors $\tilde{\mathrm{HH}}^k(A) \cong \mathrm{HH}^k(A) \oplus \mathrm{HH}^{k-1}(A)$ pour $k \geq 1$, où $\mathrm{HH}(A)$ est la cohomologie de Hochschild de A avec $\mathrm{HH}^1(A)$ remplacé par $\mathrm{Der}(A)$ et $\mathrm{HH}^0(A)$ par $\{0\}$.

Soit A une algèbre Hom-associative et $\gamma : A \rightarrow A$ un morphisme. Nous considérons l'algèbre A_γ obtenue par le twist de Yau. On définit une application $\Phi : C(A, \gamma) \rightarrow \tilde{\mathrm{HC}}(A_\gamma)$ pour $(\varphi, \psi) \in C^n(A, \gamma)$ par

$$(\varphi, \psi) \mapsto (\gamma^{n-1}\varphi + \gamma^{n-2}\psi \circ \mu, \gamma^{n-2}\psi), \quad (56)$$

où $\psi \circ \mu = \sum_{i=0}^{n-2} (-1)^i \psi \circ_i \mu$ avec $\varphi \circ_i \mu := \varphi(\mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes n-i-2})$.

Théorème 3.15 L'application Φ est un morphisme de complexes. Si γ est inversible, Φ est un isomorphisme et notamment les cohomologies correspondantes sont également isomorphes.

Cohomologie de Gerstenhaber-Schack de type α

Soit A une Hom-bialgèbre. Nous utilisons l'abréviation $H^{i,j} = \mathrm{Hom}(A^{\otimes i}, A^{\otimes j})$ pour $i, j \in \mathbb{N}$ et $H^{i,j} = 0$ pour i ou $j \leq 0$. On définit un bicomplexe $C_{GS}^{\bullet, \bullet}(A)$ par

$$\begin{aligned} C_{GS}^{n,m}(A) &= C_{GS}^{n,m}_{\mu\Delta}(A) \oplus C_{GS}^{n,m}_{\alpha\Delta}(A) \oplus C_{GS}^{n,m}_{\mu\beta}(A) \oplus C_{GS}^{n,m}_{\alpha\beta}(A) \\ &= H^{n,m} \oplus H^{n-1,m} \oplus H^{n,m-1} \oplus H^{n-1,m-1}. \end{aligned} \quad (57)$$

Nous notons un élément de $C_{GS}^{n,m}(A)$ par $(\varphi, \psi, \chi, \xi)$. La différentielle horizontale $\partial_\mu : C_{GS}^{n,m}(A) \rightarrow C_{GS}^{n+1,m}(A)$ est définie par

$$\partial_\mu(\varphi, \psi, \chi, \xi) = (\partial_{\mu\mu}\varphi - \partial_{\mu\alpha}\psi, \partial_{\mu\alpha}\varphi - \partial_{\alpha\alpha}\psi, \partial_{\mu\mu}\chi - \partial_{\alpha\mu}\xi, \partial_{\mu\alpha}\chi - \partial_{\alpha\alpha}\xi), \quad (58)$$

où l'action à gauche sur $A^{\otimes m}$ dans $C_{GS}^{n,m}_{\alpha\Delta}(A)$ et $C_{GS}^{n,m}_{\mu\Delta}(A)$ est l'action usuelle et l'action à gauche sur $A^{\otimes m-1}$ dans $C_{GS}^{n,m}_{\alpha\beta}(A)$ et $C_{GS}^{n,m}_{\mu\beta}(A)$ est $\beta(x) \cdot y$, où \cdot est l'action usuelle sur $A^{\otimes m-1}$, et de manière similaire pour l'action à droite.

Proposition 3.16 L'espace $C_{GS}(A)$ avec les différentielles définies avant est un bicomplexe différentiel. On appelle la cohomologie associée $H_{GS}(A)$ la cohomologie de Gerstenhaber-Schack de type α .

On peut aussi définir une cohomologie pour les Hom-bialgèbres où $\alpha = \beta$, utilisant un sous-complexe de $C_{GS}(A)$.

Grand Crochet

Nous définissons un grand crochet pour le cas Hom, qui généralise les crochets dans [LR90, CS16].

Soient V un espace vectoriel et $\alpha, \beta : V \rightarrow V$ deux applications linéaires, qui commutent. Soit $\varphi \in \text{Hom}(\Lambda V, \Lambda V)$. On définit la α - et β -hauteur de φ par $\text{hgt}_\alpha \varphi \in \mathbb{N}$ et $\text{hgt}_\beta \varphi \in \mathbb{N}$ resp. On écrit aussi $\text{hgt} \varphi = (\text{hgt}_\alpha \varphi, \text{hgt}_\beta \varphi)$. On pose $\text{hgt} \alpha = (1, 0)$ et $\text{hgt} \beta = (0, 1)$ et pour une bialgèbre Hom-Lie $(V, \nu, \delta, \alpha, \beta)$ aussi $\text{hgt} \nu = (1, 0)$ et $\text{hgt} \delta = (0, 1)$. La hauteur de $\varphi\psi$ est $\text{hgt} \varphi + \text{hgt} \psi$.

Le produit et coproduit dans ΛV ont la hauteur zéro. On note aussi la projection $\Lambda V \rightarrow V$ par pr .

Définition 3.17 Soient $\varphi, \psi \in \text{Hom}(\Lambda V, \Lambda V)$ des applications avec des hauteurs associées, alors on définit un produit par

$$\psi \circ \varphi = \mu(\psi \otimes \alpha^\psi)(\mu \otimes \text{id})(\text{id} \otimes \text{pr} \otimes \text{id})(\text{id} \otimes \Delta)(\alpha^\varphi \otimes \varphi)\Delta \quad (59)$$

et

$$\{\varphi, \psi\}_{\text{BB}} = \varphi \circ \psi - (-1)^{\text{deg}(\varphi)\text{deg}(\psi)} \psi \circ \varphi. \quad (60)$$

Ici nous utilisons $\alpha^\varphi = \alpha^{\text{hgt}_\alpha \varphi} \beta^{\text{hgt}_\beta \varphi}$, qui a la même hauteur que φ . Le crochet $\{\cdot, \cdot\}_{\text{BB}}$ s'appelle grand crochet.

Nous considérons le complexe

$$B^{i,j} = \text{Hom}_{\alpha, \beta}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) = \{\varphi \in \text{Hom}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) \mid \alpha^{\otimes j} \varphi = \varphi \alpha^{\otimes i}, \beta^{\otimes j} \varphi = \varphi \beta^{\otimes i}\}$$

pour $i, j \geq 1$ et $\{0\}$ autrement.

La hauteur de $\varphi \in B^{i,j}$ est $\text{hgt}(\varphi) = (i-1, j-1)$.

Théorème 3.18 *Le grand crochet définit une algèbre de Lie graduée sur $B^{\bullet\bullet}$.*

Proposition 3.19 *Soit \mathfrak{g} un espace vectoriel avec deux applications linéaires commutantes $\alpha, \beta : \mathfrak{g} \rightarrow \mathfrak{g}$, alors :*

- Si $\nu \in B^{2,1}$ satisfait $\{\nu, \nu\}_{\text{BB}} = 0$ alors $(\mathfrak{g}, \nu, \alpha)$ est une algèbre Hom-Lie.
- Si $\delta \in B^{1,2}$ satisfait $\{\delta, \delta\}_{\text{BB}} = 0$ alors $(\mathfrak{g}, \delta, \beta)$ est une coalgèbre Hom-Lie.
- Si un couple (ν, δ) satisfait $\{\nu + \delta, \nu + \delta\}_{\text{BB}} = 0$ alors $(\mathfrak{g}, \nu, \delta, \alpha, \beta)$ est une bialgèbre Hom-Lie.

On peut restreindre le grand crochet sur $\text{Hom}_\alpha(\Lambda \mathfrak{g}, \mathfrak{g})$ et obtenir le crochet de Nijenhuis-Richardson pour la cas Hom [AAM15].

Bialgèbres Hom-Lie

Etant donnée une bialgèbre Hom-Lie, on définit le complexe $B^i(\mathfrak{g}) = \bigoplus_{j=1}^i B^{j, i-j}(\mathfrak{g})$.

Proposition 3.20 Soit \mathfrak{g} une bialgèbre Hom-Lie. L'application $\partial : B^\bullet \rightarrow B^\bullet, \partial\varphi = \{\nu + \delta, \varphi\}_{\text{BB}}$ est une différentielle.

Avec ça on peut définir une cohomologie pour les bialgèbres Hom-Lie, qu'on appelle cohomologie de Chevalley-Eilenberg $C_{\text{CE}}(\mathfrak{g})$ de \mathfrak{g} .

Cohomologie de Chevalley-Eilenberg pour les algèbres Hom-Lie de type α

Nous définissons une cohomologie de type α pour les algèbres Hom-Lie. Soient $(\mathfrak{g}, \nu, \alpha)$ une algèbre Hom-Lie et (M, β) un \mathfrak{g} -module. On note par $\Lambda^k \mathfrak{g}$ la k -ème puissance extérieure de \mathfrak{g} . Alors le complexe pour la cohomologie de \mathfrak{g} à valeurs dans M est donné par

$$\tilde{C}_{\text{CE}}^n(\mathfrak{g}, M) = \tilde{C}_{\text{CE } \mu}^n(\mathfrak{g}, M) \oplus \tilde{C}_{\text{CE } \alpha}^n(\mathfrak{g}, M) = \text{Hom}(\Lambda^n \mathfrak{g}, M) \oplus \text{Hom}(\Lambda^{n-1} \mathfrak{g}, M). \quad (61)$$

Ici $\text{Hom}(\Lambda^0 \mathfrak{g}, M) = \{0\}$ comme dans le cas précédant. La hauteur de $\varphi \in \tilde{C}_{\text{CE } \mu}^n(\mathfrak{g}, M)$ et $\psi \in \tilde{C}_{\text{CE } \alpha}^n(\mathfrak{g}, M)$ est donnée par $\text{hgt}(\varphi) = \text{hgt}(\psi) = (i - 1)$. On définit quatre applications linéaires

$$\begin{array}{ccc} \tilde{C}_{\text{CE } \mu}^n & \xrightarrow{\partial_{\mu\mu}} & \tilde{C}_{\text{CE } \mu}^{n+1} \\ & \searrow \partial_{\mu\alpha} & \nearrow \partial_{\alpha\mu} \\ \oplus & & \oplus \\ \tilde{C}_{\text{CE } \alpha}^n & \xrightarrow{\partial_{\alpha\alpha}} & \tilde{C}_{\text{CE } \alpha}^{n+1} \end{array}$$

$$\begin{aligned} (\partial_{\mu\mu}\varphi)(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \alpha^{n-1}(x_i) \cdot \varphi(x_1, \dots, x_{n+1}) \\ &\quad - \sum_{i < j} (-1)^{i+j-1} \varphi([x_i, x_j], \alpha(x_1), \dots, \hat{x}_i, \hat{x}_j, \dots, \alpha(x_n)) \end{aligned} \quad (62)$$

$$\begin{aligned} (\partial_{\alpha\alpha}\psi)(x_1, \dots, x_n) &= \sum_{i=1}^n (-1)^{i+1} \alpha^{n-1}(x_i) \cdot \psi(x_1, \dots, x_n) \\ &\quad - \sum_{i < j} (-1)^{i+j-1} \psi([x_i, x_j], \alpha(x_1), \dots, \hat{x}_i, \hat{x}_j, \dots, \alpha(x_n)) \end{aligned} \quad (63)$$

$$(\partial_{\mu\alpha}\varphi)(x_1, \dots, x_n) = \beta(\varphi(x_1, \dots, x_n)) - \varphi(\alpha(x_1), \dots, \alpha(x_n)) \quad (64)$$

$$(\partial_{\alpha\mu}\psi)(x_1, \dots, x_{n+1}) = \sum_{i \leq j} (-1)^{i+j-1} [\alpha^{n-2}(x_i), \alpha^{n-2}(x_j)] \cdot \psi(x_1, \dots, \hat{x}_i, \hat{x}_j, x_{n+1}). \quad (65)$$

Théorème 3.21 *L'application $\partial : \tilde{C}_{CE}^n(\mathfrak{g}, M) \rightarrow \tilde{C}_{CE}^{n+1}(\mathfrak{g}, M)$ définie par $\partial(\varphi, \psi) = (\partial_{\mu\mu}\varphi - \partial_{\alpha\mu}\psi, \partial_{\mu\alpha}\varphi - \partial_{\alpha\alpha}\psi)$ est une différentielle, c.-à-d. $\partial \circ \partial = 0$.*

On appelle le complexe $\tilde{C}_{CE}^\bullet(\mathfrak{g}, M)$ avec différentielle ∂ le complexe de Chevalley-Eilenberg de type α . La cohomologie correspondante $\tilde{H}_{CE}^\bullet(\mathfrak{g}, M)$ est la cohomologie de Chevalley-Eilenberg de type α de \mathfrak{g} à valeurs dans M .

Proposition 3.22 *Pour $(\varphi, \psi) \in \tilde{C}_{CE}(V, V) = \text{Hom}(\Lambda V, V)$ et $\chi_1, \chi_2 \in \text{Hom}(\Lambda V, \Lambda V)$ avec des hauteurs arbitraires on a :*

- $\partial_{\mu\mu}\varphi = (-1)^{k-1}\{\nu, \varphi\}_{BB}$.
- $\partial_{\alpha\alpha}\psi = (-1)^{k-1}\{\nu, \psi\}_{BB}$.
- $\partial_{\mu\alpha}(\chi_1 \circ \chi_2) = (\partial_{\mu\alpha}\chi_1 \circ \chi_2) + (\chi_1 \circ \partial_{\mu\alpha}\chi_2)$

Nous étudions l'algèbre Hom-Lie \mathfrak{g}_γ obtenue par le twist de Yau de \mathfrak{g} par γ . On définit une application linéaire $\Phi : C(\mathfrak{g}, \gamma) \rightarrow \tilde{C}_{CE}(\mathfrak{g}_\gamma)$ pour $\varphi \in C_\mu^n(\gamma)$ et $\psi \in C_\gamma^n(\gamma)$ par

$$(\varphi, \psi) \mapsto (\gamma^{n-1}\varphi + \gamma^{n-2}\psi \circ \nu, \gamma^{n-2}\psi), \quad (66)$$

où

$$\begin{aligned} (\varphi_1 \circ \varphi_2)(x_1, \dots, x_{k+l-1}) &= \varphi_1(\varphi_2 \wedge \text{id}^{\wedge k-1})(x_1, \dots, x_{k+l-1}) \\ &= \sum_{\sigma \in \text{Sh}(l, k-1)} \text{sign}(\sigma) \varphi_1(\varphi_2(x_{\sigma(1)}, \dots, x_{\sigma(l)}, x_{\sigma(l+1)}, \dots, x_{\sigma(k+l-1)})) \end{aligned}$$

pour $\varphi_1 \in \text{Hom}(\Lambda^k \mathfrak{g}, \mathfrak{g})$ et $\varphi_2 \in \text{Hom}(\Lambda^l \mathfrak{g}, \mathfrak{g})$.

Théorème 3.23 *Soit \mathfrak{g} une algèbre de Lie et $\gamma \mathfrak{g} \rightarrow \mathfrak{g}$ un endomorphisme, alors Φ est un morphisme des complexes et inversible si γ est inversible. En particulier, il induit un morphisme en cohomologie.*

Théorème 3.24 *Soit $(\mathfrak{g}, \nu, \text{id})$ une algèbre de Lie. Alors*

$$\tilde{H}_{CE}^n(\mathfrak{g}) = H_{CE}^n(\mathfrak{g}) \oplus H_{CE}^{n-1}(\mathfrak{g}) \quad (67)$$

avec $H_{CE}^1(\mathfrak{g}) = \text{Der}(\mathfrak{g})$ et $H_{CE}^0(\mathfrak{g}) = \{0\}$.

Proposition 3.25 (Théorème de Whitehead) *Soit $(\mathfrak{g}, \nu, \alpha)$ une algèbre Hom-Lie simple de dimension finie, alors $\tilde{H}_{CE}^1(\mathfrak{g}) = \text{Der}_\alpha(\mathfrak{g})$, $\tilde{H}_{CE}^2(\mathfrak{g}) = \alpha \text{Der}(\mathfrak{g}) / \partial_\gamma \text{Der}(\mathfrak{g})$ et $\tilde{H}_{CE}^k(\mathfrak{g}) = 0$ for $k \geq 2$.*

Cohomologie de bialgèbres Hom-Lie de type α

Nous considérons ici seulement le cas $\alpha = \beta$.

Pour une bialgèbre Hom-Lie $(\mathfrak{g}, \nu, \delta, \alpha)$, on définit le complexe suivant

$$\tilde{C}_{CE}^k(\mathfrak{g}) = \tilde{C}_{CE\mu}^k(\mathfrak{g}) \oplus \tilde{C}_{CE\alpha}^k(\mathfrak{g}) = \bigoplus_{l=1}^k \tilde{C}_{CE\mu}^{l, k-l+1}(\mathfrak{g}) \oplus \bigoplus_{l=1}^{k-1} \tilde{C}_{CE\alpha}^{l, k-l}(\mathfrak{g}) \quad (68)$$

$$= \bigoplus_{l=1}^k \text{Hom}(\Lambda^l \mathfrak{g}, \Lambda^{k-l+1} \mathfrak{g}) \oplus \bigoplus_{l=1}^{k-1} \text{Hom}(\Lambda^l \mathfrak{g}, \Lambda^{k-l} \mathfrak{g}). \quad (69)$$

Pour $\varphi_{ij} \in \text{Hom}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) \subset \tilde{C}_{CE}^{i+j-1}{}_{\mu}$ et $\psi_{ij} \in \text{Hom}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) \subset \tilde{C}_{CE\alpha}^{i+j}$, on pose

$$\partial\varphi = (\partial_{\mu\mu}\varphi, \partial_{\mu\alpha}\varphi, \partial_{\mu\mu}^c\varphi) \in \tilde{C}_{CE\mu}^{i+1, j} \oplus \tilde{C}_{CE\alpha}^{i, j} \oplus \tilde{C}_{CE\mu}^{i, j+1}, \quad (70)$$

$$\begin{aligned} \partial\psi &= (\partial_{\alpha\mu}\psi, -\partial_{\alpha\alpha}\psi, \partial_b\psi, -\partial_{\alpha\alpha}^c\psi, \partial_{\alpha\mu}^c\psi) \\ &\in \tilde{C}_{CE\mu}^{i+2, j} \oplus \tilde{C}_{CE\alpha}^{i+1, j} \oplus \tilde{C}_{CE\mu}^{i+1, j+1} \oplus \tilde{C}_{CE\alpha}^{i, j+1} \oplus \tilde{C}_{CE\mu}^{i, j+2}. \end{aligned} \quad (71)$$

Ici $\partial_{\alpha\alpha} = \{\nu, \cdot\}_{BB}$, $\partial_{\alpha\alpha}^c = \{\delta, \cdot\}_{BB}$, $\partial_{\mu\alpha}\varphi = \alpha\varphi - \varphi\alpha$ et $\partial_{\mu\alpha}\varphi = \mu(\alpha^{\text{hgt}\varphi-1}\nu \wedge \varphi)\Delta$. La hauteur de $\varphi \in \tilde{C}_{CE}^n(\mathfrak{g})$ est donnée par $\text{hgt}(\varphi) = n - 1$. Ici on a $\text{hgt}\varphi \in \mathbb{N}$, parce que $\alpha = \beta$.

L'application ∂_b est définie par

$$\begin{aligned} \partial_b\psi &= \partial_{b1} + \partial_{b2} \\ &= \mu^3(\text{id} \otimes \nu \otimes \text{id})(\text{pr} \otimes \mu \otimes \text{id})(\Delta \otimes \Delta)(\delta\alpha^{\text{hgt}\psi-1} \otimes \psi)\Delta \\ &\quad + \mu(\alpha^{\text{hgt}\psi-1}\nu \otimes \psi)(\mu \otimes \mu)(\text{pr} \otimes \Delta \otimes \text{id})(\text{id} \otimes \delta \otimes \text{id})\Delta^3. \end{aligned} \quad (72)$$

Cela veut dire que pour $\psi \in \tilde{C}_{CE\alpha}^{k, l}$

$$\partial_{b1}\psi(x_1, \dots, x_{k+1}) = \sum_{i=1}^{k+1} (-1)^i \delta(\alpha^{k+l} x_i) \cdot \psi(x_1, \dots) \quad (73)$$

$$= \sum_{i=1}^{k+1} (-1)^i \alpha^{k+l}(x^{[1]}) \wedge (\alpha^{k+l}(x^{[1]}) \cdot \psi(x_1, \dots)). \quad (74)$$

On a le diagramme suivant :

$$\begin{array}{ccccc}
 & & C_{CE\mu}^{i+2,j} & & \\
 & & \uparrow & & \\
 & & C_{CE\alpha}^{i,j+1} & & \\
 & & \uparrow & & \\
 & & C_{CE\mu}^{i,j+1} & \longrightarrow & C_{CE\mu}^{i+1,j+1} \\
 & & \uparrow & & \uparrow \\
 & & C_{CE\alpha}^{i,j} & \longrightarrow & C_{CE\alpha}^{i+1,j} \\
 & & \uparrow & & \uparrow \\
 & & C_{CE\mu}^{i+1,j} & \longrightarrow & C_{CE\mu}^{i,j+2}
 \end{array}$$

Théorème 3.26 *L'application ∂ est une différentielle pour le complexe $\tilde{C}_{CE}(\mathfrak{g})$.*

Proposition 3.27 *Soient \mathfrak{g} une bialgèbre de Lie et $\gamma : \mathfrak{g} \rightarrow \mathfrak{g}$ un endomorphisme. Alors $\Phi : C(\gamma) \rightarrow \tilde{C}_{CE}(\mathfrak{g}_\gamma)$ définie par*

$$\Phi(\varphi) = (\gamma^{i-1} \varphi \gamma^{j-1}) \quad (75)$$

$$\Phi(\psi) = (\gamma^{i-1}(\psi \circ \nu) \gamma^{j-1}, \gamma^{i-1} \psi \gamma^{j-1}, \gamma^{i-1}(\delta \circ \psi) \gamma^{j-1}) \quad (76)$$

est un morphisme de complexes. Si γ est inversible alors Φ est inversible.

Homomorphisme de Hom-algèbres

On définit une cohomologie de type α pour un morphisme de Hom-algèbres, qui généralise la cohomologie dans [AFM17]. Nous considérons seulement le cas Hom-associative ici. Le cas Hom-Lie est similaire.

Soient (A, μ, α) et (B, μ, β) deux algèbres Hom-associatives et $\gamma : A \rightarrow B$ un morphisme. On définit

$$\tilde{H}C(\gamma) = \tilde{H}C(A) \oplus \tilde{H}C(B) \oplus \tilde{H}C(A, B), \quad (77)$$

où B est considéré comme un A -bimodule par γ . Cela veut dire que l'action est donnée par $a \cdot b = \gamma(a)b$ et $b \cdot a = b\gamma(a)$ pour $a \in A$ et $b \in B$.

Nous écrivons $\chi_A = (\varphi_A, \psi_A)$ pour un élément dans $\tilde{H}C(A) \subset \tilde{H}C(\gamma)$ et de manière analogue pour $\chi_B \in \tilde{H}C(B)$ et $\chi_\gamma \in \tilde{H}C(A, B)$.

Proposition 3.28 *Soit $(\chi_A, \chi_\gamma, \chi_B) \in \tilde{H}C^\bullet(\gamma)$, alors l'application $\partial : \tilde{H}C^\bullet(\gamma) \rightarrow \tilde{H}C^{\bullet+1}(\gamma)$ définie par*

$$\partial(\chi_A, \chi_\gamma, \chi_B) = (\partial_H \chi_A, \gamma \chi_A - \chi_B \gamma - \partial_H \chi_\gamma, \partial_H \chi_B), \quad (78)$$

où ∂_H est la différentielle de Hochschild de type α , est une différentielle.

Le complexe $\text{Hom}_\alpha(A, A) \oplus \text{Hom}_\alpha(A, B) \oplus \text{Hom}_\alpha(A, B)$ est un sous-complexe de $\check{H}C(\gamma)$ et c'est celui ci qui est considéré dans [AFM17].

On peut construire un complexe $\check{C}(\gamma)$ et une cohomologie pour un endomorphisme $\gamma : A \rightarrow A$ et obtenir un morphisme de complexes $\check{C}(\gamma) \rightarrow \check{H}C(A_\gamma)$.

3.3 Algèbres L_∞

En utilisant un complexe de graphes, qui correspond à une opérade libre, on peut construire une algèbre L_∞ (seulement dans les petits degrés), telles que les algèbres Hom-associatives sont les éléments de Maurer-Cartan.

3.4 Déformations de Hom-algèbres

Dans cette section, nous considérons les déformations formelles introduites par M. Gerstenhaber [Ger64]. Soit V un espace vectoriel V , on note par $V[[t]]$ l'espace des séries formelles, i.e. l'ensemble des éléments de la forme $v = \sum_{i=0}^{\infty} v_i t^i$ avec $v_i \in V$. Tout morphisme $\varphi : V[[t]] \rightarrow W[[t]]$, où V et W sont deux espaces vectoriels, peut être écrit comme $\varphi = \sum_{i=0}^{\infty} \varphi_i t^i$, où $\varphi_i : V \rightarrow W$ sont des applications linéaires.

Dans la suite, nous considérons seulement le cas des algèbres Hom-associatives. Le cas Hom-Lie est similaire. On peut aussi considérer de la même manière le cas des déformations de Hom-bialgèbres et trouver une relation entre les cohomologies de type α et les déformations.

Définition 3.29 Soit (A, μ, α) une algèbre Hom-associative sur \mathbb{K} . Une déformation de A est une algèbre Hom-associative $(A[[t]], \star = \mu_\star, \alpha_\star)$ sur $\mathbb{K}[[t]]$ telle que $\mu_0 = \mu$ et $\alpha_0 = \alpha$.

Définition 3.30 Deux déformations (\star, α_\star) et (\star', α'_\star) d'une algèbre Hom-associative A sont dites équivalentes s'il existe une application linéaire $T : A[[t]] \rightarrow A[[t]]$, qui a la forme $T = \text{id} + \sum_{i=0}^{\infty} T_i t^i$, telle que $T(a \star' b) = T(a) \star T(b)$ et $T(\alpha'_\star(a)) = \alpha(T(a))$, quelque soient $a, b \in A$.

La condition de Hom-associativité $\mu_\star(\mu_\star(a, b), \alpha_\star(c)) - \mu_\star(\alpha_\star(a), \mu_\star(b, c))$ peut être écrite sous la forme

$$\sum_{i,j,k=0}^{\infty} t^{i+j+k} (\mu_i(\alpha_j(a), \mu_k(b, c)) - \mu_i(\mu_k(a, b), \alpha_j(c))) = 0. \quad (79)$$

Cette équation s'appelle équation de déformation.

L'équation d'ordre n s'écrit

$$(\partial_{\mu\mu}\mu_n + \partial_{\alpha\mu}\alpha_n)(a, b, c) = \sum_{\substack{i,j,k=0,\dots,n-1 \\ i+j+k=n}} \mu_i(\alpha_j(a), \mu_k(b, c)) - \mu_i(\mu_k(a, b), \alpha_j(c)) =: R_n^1, \quad (80)$$

où $\partial_{\mu\mu}\mu_n(a, b, c) = \alpha_0(a)\mu_n(b, c) - \mu_n(ab, \alpha_0(c)) + \mu_n(\alpha_0(a), bc) - \mu_n(a, b)\alpha_0(c)$ et $\partial_{\alpha\mu}\alpha_n(a, b, c) = \alpha_n(a)(bc) - (ab)\alpha_n(c)$.

Par ailleurs, la condition de multiplicativité peut s'écrire

$$\sum_{i,j=0}^{\infty} t^{i+j} \alpha_i(\mu_j(a, b)) - \sum_{i,j,k=0}^{\infty} t^{i+j+k} \mu_i(\alpha_j(a), \alpha_k(b)) = 0. \quad (81)$$

L'équation d'ordre n est

$$(\partial_{\alpha\alpha}\alpha_n - \partial_{\alpha\mu}\mu_n)(a, b) = \sum_{\substack{i,j=0,\dots,n-1 \\ i+j=n}} \alpha_i(\mu_j(a, b)) - \sum_{\substack{i,j,k=0,\dots,n-1 \\ i+j+k=n}} \mu_i(\alpha_j(a), \alpha_k(b)) =: R_n^2, \quad (82)$$

où $\partial_{\alpha\alpha}\alpha_n(a, b) = \alpha_0(a)\alpha(b) - \alpha_n(ab) - \alpha_n(a)\alpha_0(b)$ et $\partial_{\alpha\mu}\mu_n(a, b) = \alpha_0(\mu_n(a, b)) - \mu_n(\alpha_0(a), \alpha_0(b))$.

Le couple (R_n^1, R_n^2) est appelé n -ième obstruction.

Etant donné qu'il y a une structure L_∞ , on a

Théorème 3.31 Soit (A, μ, α) une algèbre Hom-associative et $(A[[t]], \star, \alpha_\star)$ une déformation de A . Alors

1. $\partial(\mu_1, \alpha_1) = 0$ est un 2-cocycle et sa classe de cohomologie est invariante par équivalence.
2. La n -ième équation relative à la Hom-associativité et multiplicativité sont équivalentes à $\partial(\mu_n, \alpha_n) = (R_n^1, R_n^2)$. De plus, (R_n^1, R_n^2) est un 3-cocycle, i.e. $\partial(R_n^1, R_n^2) = 0$.

Proposition 3.32 Si deux déformations $(\mu_\star, \alpha_\star)$ et $(\mu'_\star, \alpha'_\star)$ ont les mêmes termes jusqu'à l'ordre $n-1$, alors $\partial(\mu_n - \mu'_n, \alpha_n - \alpha'_n) = 0$ et il y a une équivalence jusqu'à l'ordre n s'il y a une application linéaire $S_n : A \rightarrow A$ tel que $\partial(S_n, 0) = (\mu_n - \mu'_n, \alpha_n - \alpha'_n)$.

Nous avons le résultat suivant concernant les déformations des Hom-algèbres associatives commutatives.

Proposition 3.33 Soit A une algèbre Hom-associative commutative et A_\star une déformation de A . Alors $\{a, b\} = \frac{1}{2t}(a \star b - b \star a) \pmod t$ est un crochet de Hom-Poisson sur A .

Proposition 3.34 Deux déformations équivalentes d'une algèbre Hom-associative commutative A ont le même crochet Hom-Poisson.

Part I.

Deformation and Quantization of
color Lie bialgebras

1. GRADED MODULES AS FUNCTOR CATEGORY

CONTENTS

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IN this chapter we give a description of the category of graded modules over a commutative ring \mathbb{K} for a given grading group as a functor category from this grading group to the category of modules over this ring. For this we need to consider enriched categories. For more on enriched categories, see [Kel82]. Some brief overview is also given in Appendix B. In this chapter \mathbb{K} can be an arbitrary commutative ring with unit, but we are especially interested in the case where \mathbb{K} is a field of characteristic 0, e.g. the complex numbers. By \mathbb{K}^\times we denote the multiplicative group of \mathbb{K} , i.e. the set of invertible elements. If \mathbb{K} is a field we have $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$.

1.1 THE CATEGORY Γ

For an (abelian) group Γ we consider the \mathbb{K} -**Mod** category $\Gamma_{\mathbb{K}}$ with the set of objects Γ and

$$\text{Hom}(i, j) = \begin{cases} \mathbb{K} & \text{if } i = j \\ \{0\} & \text{otherwise,} \end{cases} \quad (1)$$

where \mathbb{K} is a commutative ring, and the composition on morphisms is given by the multiplication in \mathbb{K} . This is the discrete \mathbb{K} -**Mod**-category on Γ as defined in Appendix B. We will sometimes simply write Γ instead of $\Gamma_{\mathbb{K}}$.

We consider the tensor product

$$i \otimes j = i + j \text{ for } i, j \in \Gamma. \quad (2)$$

This is clearly associative, since Γ is a group. The unit object is the unit element of the group. We want to find (all) monoidal structures on Γ for this tensor product.

Note that a tensor product on a discrete category gives always an associative composition on the objects, so for this it would be enough if Γ were a monoid.

On morphisms we define

$$f \otimes g = fg, \quad (3)$$

where we identify the different Hom-sets, for the equation to make sense. Since we consider Γ to be enriched over \mathbb{K} -**Mod** all structures have to be linear. By this condition and the fact that the identity must be mapped to the identity, \otimes is uniquely defined on morphisms.

Note that if we considered ordinary categories, the tensor product on morphisms could be of a different form, but we do not know which exactly. We also do not have an example showing that this is possible. It only has to satisfy certain conditions coming from naturality of α .

The internal homs are given by

$$\text{HOM}(a, b) = b - a, \quad (4)$$

since

$$\text{Hom}(a + b, c) \cong \text{Hom}(a, c - b). \quad (5)$$

To be precise one must choose an invertible morphism for all $a, b \in \Gamma$, but since all choices are equivalent we simply choose the identity.

Note that this shows that a closed monoidal structure on a discrete small category gives the structure of a group on the objects, since the existence of the internal homs gives the existence of inverse elements.

Next we consider possible associators for this category, this is we are searching a natural transformation

$$\alpha(a, b, c) : (a \otimes b) \otimes c \rightarrow a \otimes (b \otimes c), \quad (6)$$

which satisfy the pentagon identity. It turns out that the condition for naturality is empty. The pentagon identity gives

$$\alpha(a, b, c + d)\alpha(a + b, c, d) = \alpha(b, c, d)\alpha(a, b + c, d)\alpha(a, b, c), \quad (7)$$

which is equivalent to

$$d\alpha = 0 \quad (8)$$

in group cohomology. We briefly recall its definition in Section 1.2.1.

We also need a left and right unitor, this is, maps

$$\rho(a) : a \otimes 0 = a \rightarrow a, \quad (9)$$

$$\lambda(a) : 0 \otimes a = a \rightarrow a. \quad (10)$$

We define

$$\rho(0) = \lambda(0) =: \nu. \quad (11)$$

Given α the unitors λ and ρ are related by

$$\rho(a)\alpha(a, 0, c) = \lambda(c). \quad (12)$$

Setting $c = 0, a = 0$ resp. one gets

$$\rho(a)\alpha(a, 0, 0) = \nu, \quad (13)$$

$$\lambda(c) = \nu\alpha(0, 0, c). \quad (14)$$

So given α and ν the unitors λ and ρ are completely determined. So we get

Proposition 1.1.1 *A monoidal structure on the discrete \mathbb{K} -Mod category Γ for a group Γ is given by a 3-cocycle $\alpha : \text{Hom}(\Gamma \times \Gamma \times \Gamma, \mathbb{K}^\times)$ in group cohomology and an element $\nu \in \Gamma$.*

1.2 MONOIDAL FUNCTORS ON Γ

We consider a strict monoidal functor $(F, F_0, F_2) : \Gamma \rightarrow \mathbb{H}$, for two categories coming from two abelian groups Γ, \mathbb{H} as in the previous section.

We have that F_2 consists of maps $F_2(a, b) \in \text{Hom}(F(a)+F(b), F(a+b))$. Since $\text{Hom}(a, b)$ is only non zero for $a = b$, one must have $F(a) + F(b) = F(a + b)$ for a strict monoidal functor, which means that F is a group morphism.

Since F is a functor, it must satisfy

$$F(fg) = F(f)F(g) \quad (15)$$

for all morphisms f, g . But since the category is discrete, F is determined on morphisms, by requiring that $F(\text{id}_a) = \text{id}_{F(a)}$ for all $a \in \Gamma$ and linearity, and this condition is automatic.

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Naturality for F_2 gives

$$F_a(f)F_b(g)F_2(a, b) = F_2(a, b)F_{a+b}(fg), \quad (16)$$

here $a, b \in \Gamma$, $f : a \rightarrow a$, $g : b \rightarrow b$ and F_a denotes the map $\text{Hom}(a, a) \rightarrow \text{Hom}(Fa, Fa)$. But again this is always satisfied, since F_a for all $a \in \Gamma$ is the trivial identification of $\text{Hom}(a, a) \cong \mathbb{K}$ with $\text{Hom}(Fa, Fa) \cong \mathbb{K}$.

We need F_2 to be compatible with the tensor product, which means the following diagram must commute:

$$\begin{array}{ccccc} F(a) \otimes (F(b) \otimes F(c)) & \xrightarrow{\text{id} \otimes F_2} & F(a) \otimes F(b \otimes c) & \xrightarrow{F_2} & F(a \otimes (b \otimes c)) \\ \downarrow \alpha_H^{-1} & & & & \downarrow F(\alpha_\Gamma)^{-1} \\ (F(a) \otimes F(b)) \otimes F(c) & \xrightarrow{F_2 \otimes \text{id}} & F(a \otimes b) \otimes F(c) & \xrightarrow{F_2} & F((a \otimes b) \otimes c) \end{array}$$

This can be written as

$$F_2(b, c)F_2(a, b + c)F(\alpha(a, b, c)) = \alpha(F(a), F(b), F(c))F_2(a, b)F_2(a + b, c), \quad (17)$$

$$(\text{d}F_2)(a, b, c)F(\alpha(a, b, c)) = \alpha(F(a), F(b), F(c)). \quad (18)$$

For the unit and F_0 we have

$$\rho'(a) = F(\rho(a))F_2(a, 0)(\text{id} \otimes F_0), \quad (19)$$

which given ρ, ρ' and F_2 determines F_0 .

If F is bijective it is possible to define a new associator and unitors on Γ by the above formulas.

This can be summarized as:

Proposition 1.2.1 *The associators on Γ for a fixed v up to isomorphisms are given by classes in the third group cohomology $H^3(\Gamma, \mathbb{K}^\times)$.*

1.2.1 Group cohomology

We want to briefly recall the definition of group cohomology, see e.g. [Bro82], and give it for some simple examples, which are useful here.

Let Γ be an (abelian) group and M a Γ -module, this is, an abelian group with a linear Γ action. We set $C^n(\Gamma, M) = \text{Hom}(\Gamma^n, M)$ and define $\text{d} : C^n(\Gamma, M) \rightarrow C^{n+1}(\Gamma, M)$

by

$$\begin{aligned}
 (d\varphi)(g_1, \dots, g_{n+1}) &= g_1 \varphi(g_2, \dots, g_n) + \sum_{i=1}^n (-1)^i \varphi(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\
 &\quad + (-1)^{n+1} \varphi(g_1, \dots, g_n).
 \end{aligned}$$

One can check that $d \circ d = 0$ and define the cohomology $H^\bullet(\Gamma, M)$.

We need the cohomology $H(\Gamma, \mathbb{K}^\times)$, where we will consider Γ to be a finitely generated abelian group and \mathbb{K} the complex or real numbers for simplicity.

It is well known that in general for a trivial group action

$$H^n(\mathbb{Z}, M) = \begin{cases} M & \text{if } n = 0, 1 \\ 0 & \text{else.} \end{cases} \quad (20)$$

So in our case we get

$$H^n(\mathbb{Z}, \mathbb{K}^\times) = \begin{cases} \mathbb{K}^\times & \text{if } n = 0, 1 \\ 0 & \text{else.} \end{cases} \quad (21)$$

So in this case there are no non-trivial associators.

For the cyclic group C_n with n elements we have in general

$$H^p(C_n, M) = \begin{cases} M & \text{if } p = 0 \\ T_n & \text{if } p = 1, 3, 5, \dots \\ M/nM & \text{if } p = 2, 4, 6, 8, \dots \end{cases} \quad (22)$$

Here T_n denotes the n -torsion module of M , i.e. the elements $m \in M$ such that $n \cdot m = 0$.

In our case $C(C_n, \mathbb{K}^\times)$, we get T_n is the set of n -th roots of unity in \mathbb{K} . For $\mathbb{K} = \mathbb{C}$ this is isomorphic to C_n . For $\mathbb{K} = \mathbb{R}$ this is C_2 for n even and $\{1\}$ for n odd.

Further we have $\mathbb{K}^\times / (\mathbb{K}^\times)^n = \{1\}$ for $\mathbb{K} = \mathbb{C}$, and for $\mathbb{K} = \mathbb{R}$ we get C_2 for n even and $\{1\}$ for n odd.

So we get $H^p(C_n, \mathbb{C}) = C_n$ for p odd and $\{1\}$ for p even. Further we get $H^p(C_n, \mathbb{R}^\times) = \{1\}$ for n odd and $H^p(C_n, \mathbb{R}^\times) = C_2$ for n even.

1.3 SYMMETRY

We want to study the possible symmetries on the category Γ for a given associator α .

A braiding γ must satisfy

$$\gamma(a, b)\alpha(b, a, c)\gamma(a, c) = \alpha(a, b, c)\gamma(a, b + c)\alpha(b, c, a) \quad (23)$$

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and

$$\gamma(b, c)\alpha^{-1}(a, c, b)\gamma(a, c) = \alpha^{-1}(a, b, c)\gamma(a + b, c)\alpha^{-1}(c, a, b). \quad (24)$$

For the identity one needs

$$\gamma(a, 0)\lambda(a) = \rho(a), \quad \gamma(0, a)\rho(a) = \lambda(a).$$

To be a symmetry γ must also satisfy

$$\gamma(a, b)\gamma(b, a) = 1, \quad (25)$$

which is equivalent to

$$\gamma(a, b) = \gamma^{-1}(b, a). \quad (26)$$

If α and ρ, λ are trivial Eq. (23) simplifies to

$$\gamma(a, b)\gamma(a, c) = \gamma(a, b + c), \quad \gamma(a, c)\gamma(b, c) = \gamma(a + b, c). \quad (27)$$

So γ is a group homomorphism in each argument. This is also called a bicharacter.

For a finitely generated abelian group it is possible to classify all bicharacters, since it is enough to define it on pairs of generators. For details see [Sch79, Sect. 5]

A strict monoidal functor (F, F_2, F_0) between two such categories is a braided monoidal functor if in addition

$$\gamma'(F(a), F(b))F_2(b, a) = F_2(a, b)F(\gamma(a, b)). \quad (28)$$

If F is the identity this gives

$$\gamma'(a, b) = F_2(a, b)F_2^{-1}(b, a)\gamma(a, b) \quad (29)$$

So given a 2-cocycle in group cohomology it is possible to change the symmetry of Γ by the above formula without changing the associator.

Proposition 1.3.1 *For $\mathbb{K} = \mathbb{C}$, every symmetry with trivial associator is equivalent to one with values in $\{-1, +1\}$.*

Proof. Given a symmetry γ , one can define the subgroup $\Gamma_+ = \{i \in \Gamma \mid \gamma(i, i) = +1\}$. This is a subgroup since $\gamma(i + j, i + j) = \gamma(i, i)\gamma(i, j)\gamma(j, i)\gamma(j, j) = \gamma(i, i)\gamma(j, j)$. Similar one can define $\Gamma_- := \{i \in \Gamma \mid \gamma(i, i) = -1\}$. Then $\Gamma = \Gamma_+ \cup \Gamma_-$. Then σ is equivalent, to σ' given by $\sigma(i, j) = -1$ for $i, j \in \Gamma_-$ and 1 else. This can be seen using [Sch79, Pop97]. \square

1.4 GRADED MODULES AS FUNCTOR CATEGORY

We want \mathbf{Modgr} to be a functor category. One can view a Γ -graded module as a functor from the discrete category on Γ to the category $\mathbb{K}\text{-Mod}$ of \mathbb{K} -modules. But since we want to encode the monoidal data for \mathbf{Modgr} in this functor category, this does not work. Instead we need the category $\Gamma_{\mathbb{K}}$ defined as above, which has non trivial monoidal structures. Note that we need to consider enriched functors, since otherwise $\mathbf{Mod}^{\mathbb{K}}$ has too many objects and is not what we want. (The functor also operates on the morphisms and there are different possibilities). This is why we view the category Γ enriched over \mathbf{Mod} . We denote this category from now on simply by Γ . Here we consider \mathbf{Mod} equipped with the normal tensor product and symmetry, and the associator to be the identity, this is possible due to [Sch01].

So we now study the functor category \mathbf{Mod}^{Γ} . Note that since this is now a functor category between enriched categories, it is defined differently than ordinary functor categories to get again the structure of an enriched category, see Appendix B. However, in our case it consists simply of ordinary functors which are additionally \mathbb{K} -linear.

To be precise a functor $G \in \mathbf{Mod}^{\Gamma}$ consists of a module $F(i)$ for all $i \in \Gamma$, and on morphisms is determined by the fact that F applied to the identity is again the identity and \mathbb{K} -linearity.

The morphisms $\text{Hom}(V, W)$ of two objects $V, W \in \mathbf{Mod}^{\Gamma}$ are the \mathbf{Mod} -natural transformations between them. Here again \mathbf{Mod} -natural simply means natural and \mathbb{K} -linear.

But since the category Γ is discrete the naturality condition is empty and a morphisms $\varphi \in \text{Hom}(V, W)$ simply consists of a family of \mathbb{K} linear maps $\varphi(i) : V(i) \rightarrow W(i)$.

So it should be clear that this category describes precisely the Γ -graded modules with the degree 0 maps as morphisms.

Next we want to define a tensor product on \mathbf{Mod}^{Γ} . Generally, in any \mathcal{V} -functor category $\mathcal{V}^{\mathcal{C}}$, where \mathcal{C} is a (small) symmetric monoidal \mathcal{V} -category, one can define a tensor product by

$$F \hat{\otimes} G = \int^{a,b} F(a) \otimes G(b) \otimes \text{Hom}(a+b, \cdot). \quad (30)$$

This is a special form of the so called Day convolution [Day70].

Again in our case the formula simplifies due to the fact that Γ is discrete, and we get

$$(V \hat{\otimes} W)(c) = \int^{a,b} U(a) \otimes V(b) \otimes \text{Hom}(a, c-b)$$

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$$\begin{aligned}
&\cong \bigoplus_{a,b \in \Gamma} U(a) \otimes V(b) \otimes \text{Hom}(a+b, c) \cong \bigoplus_{a+b=c} U(a) \otimes V(b) \\
&\cong \int^b U(c-b) \otimes V(b) \cong \bigoplus_b U(c-b) \otimes V(b).
\end{aligned}$$

Which gives back the usual formula for the tensor product of two graded modules. Note however that for the following the original formula is better, since it allows us to see the effects of the associator of Γ on $\hat{\otimes}$.

We show the associativity of the tensor product. Using Proposition B.4.4, we get

$$\begin{aligned}
((U \hat{\otimes} V) \hat{\otimes} W)(e) &= \int^{c,d} \int^{a,b} U(a) \otimes V(b) \otimes \text{Hom}(a+b, c) \otimes W(d) \otimes \text{Hom}(c+d, e) \\
&= \int^{a,b,d} U(a) \otimes V(b) \otimes W(d) \otimes \int^c \text{Hom}(a+b, c) \otimes \text{Hom}(c+d, e) \\
&= \int^{a,b,d} U(a) \otimes V(b) \otimes W(d) \otimes \text{Hom}((a+b)+d, e). \tag{31}
\end{aligned}$$

And similarly

$$(U \hat{\otimes} (V \hat{\otimes} W))(e) = \int^{a,b,d} U(a) \otimes V(b) \otimes W(d) \otimes \text{Hom}(a+(b+d), e). \tag{32}$$

Since we consider Γ to be monoidal, we have the associator α , which gives an isomorphisms between this and (31) using Proposition B.4.3.

As unit we take the representable functor $I := \text{Hom}(0, \cdot)$. This is simply the ring \mathbb{K} considered as graded module concentrated in degree 0. We get

$$\begin{aligned}
U \hat{\otimes} I &= \int^{a,b} U(a) \otimes \text{Hom}(0, b) \otimes \text{Hom}(a+b, \cdot) \\
&\cong \int^a U(a) \otimes \text{Hom}(0+a, \cdot) \\
&\cong \int^a U(a) \otimes \text{Hom}(a, \cdot) = U(\cdot),
\end{aligned}$$

where in the third step we need λ for Γ , which may be non trivial. A similar calculation shows that $I \hat{\otimes} U \cong U$, so I is in fact a unit for the tensor product $\hat{\otimes}$.

One can show that the new associator satisfies the pentagon identity, since the associators on Γ and \mathbf{Mod} do so.

For the symmetry we get

$$U \hat{\otimes} V = \int^{a,b} U(a) \otimes V(b) \otimes \text{Hom}(a+b, \cdot) \quad (33)$$

$$\cong \int^{a,b} V(b) \otimes U(a) \otimes \text{Hom}(b+a, \cdot) \quad (34)$$

$$= \int^{a,b} V(a) \otimes U(b) \otimes \text{Hom}(a+b, \cdot) = V \hat{\otimes} U, \quad (35)$$

where in the first step we used the symmetry γ on Γ .

For the internal homs we have

$$\begin{aligned} \text{Hom}(U \hat{\otimes} V, W) &= \text{Hom}\left(\int^{a,b} U(a) \otimes V(b) \otimes \text{Hom}(a+b, \cdot), W(c)\right) \\ &\cong \int_c \int_{a,b} \text{Hom}(U(a) \otimes \text{Hom}(a+b, c) \otimes V(b), W(c)) \\ &\cong \int_c \int_{a,b} \text{Hom}(U(a), \text{Hom}(\text{Hom}(a+b, c), \text{Hom}(V(b), W(c)))) \\ &\cong \int_a \text{Hom}(U(a), \int_{b,c} \text{Hom}(\text{Hom}(a+b, c), \text{Hom}(V(b), W(c)))) \\ &\cong \text{Hom}(U, \text{Homgr}(V, W)). \end{aligned}$$

If we define $\text{Homgr}(V, W)(a) = \int_{b,c} \text{Hom}(\text{Hom}(a+b, c), \text{Hom}(V(b), W(c)))$. One can simplify this using

$$\begin{aligned} \text{Homgr}(V, W)(a) &= \int_{b,c} \text{Hom}(\text{Hom}(a+b, c), \text{Hom}(V(b), W(c))) \\ &\cong \int_{b,c} \text{Hom}(\text{Hom}(a+b, c) \otimes V(b), W(c)) \\ &\cong \int_c \text{Hom}\left(\int^b \text{Hom}(a+b, c) \otimes V(b), W(c)\right) \\ &\cong \int_c \text{Hom}\left(\int^b \text{Hom}(b, c-a) \otimes V(b), W(c)\right) \\ &\cong \int_c \text{Hom}(V(c-a), W(c)). \end{aligned}$$

Note that in this calculation we use that in Γ the tensor functor $a + \cdot$ has a right adjoint and we also use the symmetry in Γ .

Again in our case this can be written as

$$\text{Homgr}(V, W)(a) = \prod_c \text{Hom}(V(c-a), W(c)), \quad (36)$$

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which shows that $\text{Homgr}(V, W)(a)$ are precisely the maps of degree a .

The representable functors $\mathbb{K}[a] := \text{Hom}(a, \cdot)$, generate \mathbf{Mod}^Γ . One has $\mathbb{K}[a] \otimes \mathbb{K}[b] \cong \mathbb{K}[a + b]$. With this one can also define functors $\mathbf{Modgr} \rightarrow \mathbf{Modgr}$ by $V \mapsto V[i] := \mathbb{K}[i] \otimes V$.

One can define the sum and product in \mathbf{Modgr} componentwise. Also it is possible to compute other limits and colimits, so in fact \mathbf{Modgr} is a complete closed monoidal category.

One can also define the dual of a graded module V by $V^* = \text{Homgr}(V, \mathbb{K})$ using (36) we have

$$V^*(a) = \text{Homgr}(V, \mathbb{K})(a) = \prod_{c \in \Gamma} \text{Hom}(V(c - a), \mathbb{K}(c)) = \text{Hom}(V(-a), \mathbb{K}) = V(-a)^*.$$

If one considers the category of finite dimensional vector spaces over a field instead of the category $\mathbb{K}\text{-Mod}$ for a commutative ring, the category \mathbf{Modgr} becomes a rigid monoidal category similar to the category of finite dimensional vector spaces.

1.4.1 Functors between \mathbf{Modgr}

The idea is to extend a functor or tensor functor between the (monoidal) categories Γ, H of two groups to a functor on the functor categories $\mathbf{Modgr}_\Gamma, \mathbf{Modgr}_H$, which describe the graded modules. So consider a functor $F : \Gamma \rightarrow H$. Then one can define the pullback for $U : H \rightarrow \mathbf{Mod}$ along F by

$$(F^*U)(a) = U(F(a)). \quad (37)$$

This clearly gives a functor $\mathbf{Modgr}_H \rightarrow \mathbf{Modgr}_\Gamma$. But unless F is bijective this is not what one wants. (Consider e.g. $F : a \mapsto 0$)

So we also define a pushforward $\mathbf{Modgr}_\Gamma \rightarrow \mathbf{Modgr}_H$ for $U \in \mathbf{Mod}^\Gamma$ by

$$(F_*U)(a) = \int^{b \in \Gamma} U(b) \otimes \text{Hom}(F(b), a). \quad (38)$$

This is equivalent to

$$(F_*U)(a) = \bigoplus_{F(b)=a} U(b). \quad (39)$$

We want to consider, whether F_* is again a monoidal functor if F is a monoidal

functor. For this we compute

$$\begin{aligned}
 F_*(U \otimes V)(e) &= \int^{a,b \in \Gamma} \int^{c \in \Gamma} U(a) \otimes V(b) \otimes \text{Hom}(a+b, c) \otimes \text{Hom}(F(c), e) \\
 &= \int^{a,b \in \Gamma} U(a) \otimes V(b) \otimes \text{Hom}(F(a+b), e) \\
 &\cong \int^{a,b \in \Gamma} U(a) \otimes V(b) \otimes \text{Hom}(F(a) + F(b), e).
 \end{aligned}$$

In the third line we used the morphisms $F_2 : F(a+b) \rightarrow F(a) + F(b)$, of the monoidal functor. On the other side we have

$$\begin{aligned}
 (F_*U) \otimes (F_*V)(e) &= \int^{a,b \in H} \int^{c \in \Gamma} U(a) \otimes \text{Hom}(F(c), a) \\
 &\quad \otimes \int^{d \in \Gamma} U(d) \otimes \text{Hom}(F(d), b) \otimes \text{Hom}(a+b, e) \\
 &= \int^{c,d \in \Gamma} U(c) \otimes U(d) \otimes \text{Hom}(F(c) + F(d), e)
 \end{aligned}$$

So we get a morphism from $F_*(U \otimes V)$ to $(F_*U) \otimes (F_*V)$.

For the identity F_0 gives a functor $\text{Hom}(F_0, \cdot) : I_\Gamma \rightarrow I_H$.

In general the pushforward is not a (symmetric) monoidal functor. It is monoidal if $\alpha_H(F(a), F(b), F(c)) = \alpha_\Gamma(a, b, c)$ and symmetric if further $\gamma_H(F(a), F(b)) = \gamma_\Gamma(a, b)$.

If F is bijective the pushforward agrees with the pullback and F^* is fully faithful and an equivalence of categories. The pushforward is always faithful but only full if F is a group isomorphism.

1.4.2 Direct sum functor

One can define a functor \oplus from the category **Mod_{gr}** to the category **Mod** by taking the direct sum. So for $V \in \mathbf{Mod}_{\mathbf{gr}}$ we define $\oplus V := \bigoplus_{i \in \Gamma} V(i)$.

This is right adjoint to the diagonal functor, so for $U \in \mathbf{Mod}^\Gamma, V \in \mathbf{Mod}$ one has

$$\text{Hom}\left(\bigoplus U, V\right) \cong \text{Hom}(U, \Delta V). \tag{40}$$

The functor \oplus is a monoidal functor if **Mod_{gr}** has the trivial associator, or an equivalent one. It is a braided monoidal functor if the symmetry on **Mod_{gr}** is trivial, or equivalent to the trivial one. It is always faithful but in general not full. It can also be seen as the pushforward of the canonical morphism $\Gamma \rightarrow \{e\}$.

1.5 CONNECTION TO TRADITIONAL DEFINITION

We recall some facts about graded modules in the normal formulation and relate them to the ones given before.

Definition 1.5.1 (Commutation factor) Let Γ be an abelian group. A map $\epsilon : \Gamma \times \Gamma \rightarrow \mathbb{K}^\times$ is called an anti-symmetric bicharacter or commutation factor, if

$$\epsilon(f + g, h) = \epsilon(f, h)\epsilon(g, h), \quad (41)$$

$$\epsilon(f, g + h) = \epsilon(f, g)\epsilon(f, h), \quad (42)$$

$$\epsilon(f, g)\epsilon(g, f) = 1. \quad (43)$$

The product of two commutation factors is again a commutation factor.

So a commutation factor is precisely a symmetry for the monoidal category Γ with trivial associator.

Remark 1.5.2 We only consider the case of a commutative group because a commutation factor on a noncommutative group always factors through the abelianization of that group.

If \mathbb{K} is a field or more general a connected ring, we have $\epsilon(a, a) = \pm 1$ for all $a \in \Gamma$, since the symmetry of ϵ gives $\epsilon(a, a)\epsilon(a, a) = 1$. We will assume this in the following. So we can define

$$\Gamma_\pm := \{a \in \Gamma \mid \epsilon(a, a) = \pm 1\}, \quad (44)$$

and we get that Γ is the disjoint union $\Gamma = \Gamma_+ \sqcup \Gamma_-$. In fact, $a \mapsto \epsilon(a, a)$ is a group homomorphism from Γ to $\{-1, 1\} \cong \mathbb{Z}_2$. This is true since

$$\epsilon(a + b, a + b) = \epsilon(a, a)\epsilon(a, b)\epsilon(b, a)\epsilon(b, b) = \epsilon(a, a)\epsilon(b, b), \quad (45)$$

and because of the antisymmetry of ϵ .

In the sequel, ϵ will always denote a commutation factor.

Definition 1.5.3 (Color vector space) A Γ -graded vector space $V = \bigoplus_{i \in \Gamma} V^i$ together with a commutation factor ϵ will be called a color vector space.

We write $\deg(x) = a$ for the degree a of an element $x \in V^a$, and write $\epsilon(x, y)$ instead of $\epsilon(\deg x, \deg y)$ for $x, y \in V$. In this context, $\epsilon(xy, z)$ for $x, y, z \in V$ means $\epsilon(\deg x + \deg y, \deg z)$.

A graded subspace of a color vector space V is a vector subspace U , such that $U = \bigoplus U^i$, with $U^i \subset V^i$.

The tensor product of two graded vector spaces V, W is given by

$$(V \otimes W)^i = \bigoplus_{j+k=i} V^j \otimes W^k. \quad (46)$$

So the degree of $v \otimes w$ for $v \in V, w \in W$ is $\deg v + \deg w$. This agrees with the definition as Day-convolution if the associator on Γ is trivial.

Let a commutation factor be fixed, we consider the symmetric monoidal category of Γ -graded vector spaces $\mathbf{Modgr} = \mathbb{K}\text{-Modgr}_\Gamma$, with the standard tensor product, and symmetry $\tau : V \otimes W \rightarrow W \otimes V$ given for $v \in V, w \in W$ by

$$\tau(v \otimes w) = \epsilon(v, w)(w \otimes v). \quad (47)$$

Using this, it is possible to define the action of the symmetric group on $V^{\otimes n}$ for a color vector space V , since every permutation can be written as a product of transpositions of neighboring elements. For $\sigma \in S_n$ we denote this action by $\sigma \cdot (v_1 \otimes \cdots \otimes v_k)$.

It can be explicitly given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_k) = \epsilon(\sigma; v_1, \dots, v_k) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \quad (48)$$

where we use

$$\epsilon(\sigma; v_1, \dots, v_n) = \prod_{\substack{i < j \\ \sigma(i) > \sigma(j)}} \epsilon(v_i, v_j). \quad (49)$$

If the color vector space V is concentrated in degree 1, we have $\epsilon(\sigma; v_1, \dots, v_n) = \text{sign}(\sigma)$.

We define the graded homomorphisms between two color vector spaces by

$$\text{Homgr}(V, W) = \bigoplus_{i \in \Gamma} \text{Homgr}^i(V, W), \quad (50)$$

where $\text{Homgr}^i(V, W) = \{\varphi : V \rightarrow W \text{ linear} \mid \varphi(V^j) \subset W^{j+i}\}$. This agrees with the internal homs obtained before.

So $\text{Homgr}(V, W)$ is also a color vector space, and $\text{Homgr}(V, V)$ a color algebra with respect to composition of linear maps, see Definition 2.2.1. Note that in the category \mathbf{Modgr} the morphisms are only the linear maps of degree 0, which is Homgr^0 .

We also use the Koszul sign rule, so for $f \in \text{Homgr}(V, W), g \in \text{Homgr}(V', W'), a \in V, b \in V'$, we have

$$(f \otimes g)(a \otimes b) = \epsilon(g, a) f(a) \otimes g(b), \quad (51)$$

or more generally, if two graded objects change their order one gets a sign determined by epsilon and the degrees of the objects involved.

For a color vector space V , one can also define the linear dual $V^* = \text{Homgr}(V, \mathbb{K})$, which is again a color vector space. If \mathbb{K} is a field and $\{b_i\}$ a basis of V there exists a dual basis $\{b^i\}$, such that $b^i(b_j) = \delta_j^i$. Here δ is the Kronecker symbol. Note that since \mathbb{K} is assumed to be of degree zero, we have $\deg b^i = -\deg b_i$.

2. COLOR ALGEBRAS

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IN this chapter we study color algebras, that is, algebras on color vector spaces. We are mostly interested in color Lie bialgebras. We study the deformation of color Lie bialgebras for this we define a generalization of the big bracket or grande crochet invented by Lecomte and Roger [LR90]. With this we can define a cohomology for color Lie bialgebras which can be used to study formal deformations as in the non-graded case. In a second part we study the quantization of color Lie bialgebras. This has been introduced in the non graded case by Drinfeld. Since then different methods to prove the existence of quantizations have been found [EK96, SŠ15, MW16]. We will follow here the first one due to Etingof and Kazhdan [EK96].

2.1 GENERAL DEFINITIONS

In this chapter we define different type of color algebras. This means algebras on color vector spaces. However we will assume that the operations are of degree zero.

Definition 2.1.1 An n -ary multiplication is a map $\mu : V^{\otimes n} \rightarrow V$, which is considered to be of degree zero. It is called color symmetric if

$$\mu(v_1 \otimes \cdots \otimes v_n) = \mu(\sigma \cdot v_1 \otimes \cdots \otimes v_n) \quad (1)$$

for all $\sigma \in S_n$ and color skew-symmetric if

$$\mu(v_1 \otimes \cdots \otimes v_n) = -\mu(\tau \cdot v_1 \otimes \cdots \otimes v_n) \quad (2)$$

for all transpositions $\tau \in S_n$. Note that the action of τ (resp. σ) contains the commutation factor ϵ .

A morphism φ between two n -ary multiplications $\mu : V^{\otimes n} \rightarrow V$ and $\mu' : V'^{\otimes n} \rightarrow V'$ is a linear map $\varphi : V \rightarrow V'$, often considered of degree zero, such that

$$\varphi \circ \mu = \mu' \circ \varphi^{\otimes n}. \quad (3)$$

Similarly, one can consider comultiplications $\Delta : V \rightarrow V^{\otimes n}$.

A comultiplication Δ on a color vector space V always gives a corresponding multiplication Δ^c on its dual, by $\Delta^c(f_1 \otimes \cdots \otimes f_n)(x) = (f_1 \otimes \cdots \otimes f_n)\mu(x)$ for $f_i \in V^*, x \in V$. The other way round, a multiplication μ only gives a comultiplication μ^c on V^* if V is finite dimensional, since then $(V^{\otimes n})^*$ is isomorphic to $(V^*)^{\otimes n}$. In this case it is given by $(\mu^c(f))(x_1, \dots, x_n) = f(\mu(x_1, \dots, x_n))$ for $f \in V^*, x_i \in V$.

2.2 COLOR ASSOCIATIVE AND HOPF ALGEBRAS

We give the definition of color associative algebras and bialgebras. Note that in the algebra case the commutation factor would not be needed, in the bialgebra case however it is.

Definition 2.2.1 (Color (associative) algebra) A color (associative) algebra is a color vector space A equipped with a structure of an associative algebra, such that $A^i \cdot A^j \subset A^{i+j}$. This means that there is a multiplication $\mu : A \otimes A \rightarrow A, a \otimes b \mapsto \mu(a \otimes b) = ab$ of degree zero, which satisfies

$$a(bc) = (ab)c. \quad (4)$$

It is called color commutative if μ is color symmetric, which here means

$$ab = \epsilon(a, b)ba, \text{ for all } a, b \in A.$$

A color algebra morphism between two color algebras (A, \cdot) and (B, \cdot') is a graded morphism φ , such that $\varphi(x) \cdot' \varphi(y) = \varphi(x \cdot y)$. We will normally assume that φ is of degree 0. If A and B are unital algebras, the unit has to be of degree 0, and we assume that a morphism respects the unit. This also implies that morphisms are of degree 0.

By a color derivation on a color algebra A , we mean a homogeneous map $d : A \rightarrow A$ such that

$$d(ab) = d(a)b + \epsilon(d, a)ad(b). \quad (5)$$

In terms of operations and tensor products, it may be written as

$$d \circ \mu = \mu \circ (\text{id} \otimes d + d \otimes \text{id}), \quad (6)$$

where μ denotes the multiplication on A . Note that here the ϵ is implicitly contained in the Koszul rule.

Similarly, we define color coalgebras.

Definition 2.2.2 (Color (coassociative) coalgebra) A color (coassociative) coalgebra is a color vector space C with a comultiplication $\Delta : C \rightarrow C \otimes C$, which is of degree 0 and coassociative that is

$$(\Delta \otimes \text{id})\Delta(x) = (\text{id} \otimes \Delta)\Delta(x) \quad (7)$$

for all $x \in C$. It is called cocommutative if $\tau\Delta(x) = \Delta(x)$. It is called counital if it has a counit ϵ , i.e. a linear map $C \rightarrow \mathbb{K}$ such that $(\text{id} \otimes \epsilon)\Delta = \text{id} = (\epsilon \otimes \text{id})\Delta$.

It is called coaugmented if there exists a morphism of coalgebras $\eta : \mathbb{K} \rightarrow C$, so in particular $\epsilon\eta = \text{id}_{\mathbb{K}}$. Then C is isomorphic to $\bar{C} \oplus 1\mathbb{K}$, where $\bar{C} := \ker \epsilon$ and $1 := \eta(1)$. In this case we define the reduced coproduct $\bar{\Delta} : \bar{C} \otimes \bar{C} \rightarrow \bar{C}$ by

$$\bar{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1. \quad (8)$$

When writing down formulas containing a coproduct, we will often make use of the following Sweedler notations, $\Delta(x) = x^{(1)} \otimes x^{(2)}$, or $(\Delta \otimes \text{id})\Delta(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$. To be precise one should write $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$, since in general it is a sum, but for simplicity we will omit to write this sum.

A morphism of color coalgebras C and D is a graded homomorphism $\varphi : C \rightarrow D$, such that $\Delta_D(\varphi(x)) = (\varphi \otimes \varphi)\Delta(x)$ for all $x \in C$.

A color coderivation is a graded homomorphism $d : C \rightarrow C$ such that

$$\Delta \circ d = (\text{id} \otimes d + d \otimes \text{id}) \circ \Delta.$$

A coideal of a color coalgebra C is a graded subspace I , such that $\Delta(C) \subset I \otimes C + C \otimes I$, and a subcoalgebra A is a graded subspace such that $\Delta(A) \subset A \otimes A$. Note that different from the algebra case a subcoalgebra is a coideal, and not the other way round.

2. Color algebras

An element $x \in C$ of a coaugmented coalgebra is called primitive, if $\Delta(x) = x \otimes 1 + 1 \otimes x$. We denote the set of all primitive elements by $\text{Prim}(C)$.

Given a coaugmented color coalgebra C one can define an ascending filtration by $F_0 C = \mathbb{K}1$, and $F_i C = \mathbb{K}1 \oplus \ker \bar{\Delta}^{n+1}$, where $\bar{\Delta}^{(2)} = \bar{\Delta}$ and $\bar{\Delta}^{(n+1)} = (\bar{\Delta} \otimes \text{id}^{\otimes n-1})\bar{\Delta}^{(n)}$. It is clear from the definition that $F_1 C = \mathbb{K}1 \oplus \text{Prim}(C)$. We call a coalgebra connected (or conilpotent) if this filtration is exhaustive, i.e. $\bigcup_{r \in \mathbb{N}} F_r C = C$.

Given a color algebra A and a color coalgebra C , one can define the convolution product on the set of graded morphism $\text{Homgr}(C, A)$ by

$$(\varphi * \psi)(x) = \mu(\varphi \otimes \psi)\Delta(x). \quad (9)$$

for $\varphi, \psi \in \text{Hom}(C, A)$ and $x \in C$.

Proposition 2.2.3 (Convolution product) *Let A be a color algebra and C a color coalgebra, with the same grading group and commutation factor, then $\text{Homgr}(C, A)$ with the convolution product defined in Eq. (9) is a color algebra. If A is commutative and C is cocommutative, then it is commutative. If A and C are unital (resp. counital) then 1ε is a unit for the convolution product.*

Proof. We have

$$(f * g) * h = \mu(\mu \otimes \text{id})((f \otimes g) \otimes h)(\Delta \otimes \text{id})\Delta,$$

so the associativity of $*$ follows from the (co)associativity of the (co)algebra. For the color commutativity, we consider

$$f * g = \mu\tau^2(f \otimes g)\Delta = \mu\tau\varepsilon(f, g)(g \otimes f)\tau\Delta = \mu(g \otimes f)\Delta = \varepsilon(f, g)g * f. \quad (10)$$

For the unit we have $\varphi * 1\varepsilon = \mu(\varphi \otimes 1\varepsilon)\Delta = \varphi$. \square

Definition 2.2.4 (Color Hopf algebra) A color Hopf algebra $(H, \mu, \Delta, \eta, \varepsilon, S)$ is a unital color algebra (H, μ, η) , a counital coassociative coalgebra (H, Δ, ε) and an antipode S such that the comultiplication Δ and the counit ε are algebra morphisms. If there is no antipode, we say that it is a bialgebra.

The homogeneous map Δ of degree 0 is an algebra morphism means

$$\Delta(ab) = \Delta(a)\Delta(b) \text{ and } \Delta\eta = \eta \otimes \eta, \quad (11)$$

where the product on $H \otimes H$ is given by $(a \otimes b)(c \otimes d) = \varepsilon(b, c)(ac \otimes bd)$.

The linear map ε is an algebra morphism means

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \text{ and } \varepsilon\eta = \text{id}_{\mathbb{K}}.$$

The linear map $S : H \rightarrow H$ is an antipode means that it is a convolution inverse of the identity, i.e. $S * \text{id} = \mu(S \otimes \text{id})\Delta = \varepsilon\eta = \mu(\text{id} \otimes S)\Delta = \text{id} * S$.

The compatibility condition between the algebra and coalgebra structures may also be stated as the multiplication and the unit being coalgebra morphisms.

A Hopf algebra is a coaugmented coalgebra, where the coaugmentation is given by η , dually it is an augmented algebra, where the augmentation is given by ε .

In the next section we will consider color Lie algebras, but we already note:

Proposition 2.2.5 *The set of primitive elements of a Hopf algebra H is a color Lie sub-algebra.*

Proof. We only have to show that $\text{Prim}(H)$ is closed under the graded commutator. For this we compute

$$\begin{aligned}\Delta([x, y]) &= \Delta(xy - \varepsilon(x, y)yx) = \\ &= xy \otimes 1 + x \otimes y + \varepsilon(x, y)y \otimes x + 1 \otimes xy \\ &\quad - \varepsilon(x, y)(yx \otimes 1 + y \otimes x + \varepsilon(y, x)x \otimes y + 1 \otimes yx) \\ &= [x, y] \otimes 1 + 1 \otimes [x, y].\end{aligned}$$

□

A Hopf ideal of a Hopf algebra H is an ideal I of the underlying algebra, which is also a coideal for the coalgebra structure and respects the unit, counit and antipode. Given an ideal I of the Hopf algebra H , one can define the quotient Hopf algebra H/I .

2.2.1 Quasitriangular color quasi-Hopf algebras

Definition 2.2.6 (Color quasi-Hopf algebra) A color quasi-Hopf algebra is an associative color algebra H with a multiplication μ , a coproduct Δ , a unit 1 , a counit ε and an invertible associator $\Phi \in H^{\otimes 3}$ all of degree 0, which satisfy

$$\begin{aligned}\forall x, y \in H : \Delta(xy) &= \Delta(x)\Delta(y) && \text{(compatibility),} \\ (\varepsilon \otimes \text{id})\Delta &= \text{id} = (\text{id} \otimes \varepsilon)\Delta && \text{(counit),} \\ \Phi(\Delta \otimes \text{id})\Delta &= (\text{id} \otimes \Delta)\Delta\Phi && \text{(quasi-coassociativity),} \\ \Phi_{1,2,34}\Phi_{12,3,4} &= \Phi_{2,3,4}\Phi_{1,23,4}\Phi_{1,2,3} && \text{(Pentagon identity),} \\ (\text{id} \otimes \varepsilon \otimes \text{id})\Phi &= 1 \otimes 1.\end{aligned}$$

It also has an antipode S , which satisfies $\mu(\text{id} \otimes S)\Delta = \text{id} = \mu(\text{id} \otimes S)\Delta$.

One could also allow for left and right unitors, but we will not do so.

Note that since these equations use the product in $H \otimes H$, defined by $(a \otimes b)(c \otimes d) = \varepsilon(b, c)ac \otimes bd$ for $a, b, c, d \in H$, they depend on the commutation factor ε .

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We assume also the operations to be of degree zero for two reasons, it is easier to handle this way categorically, and actually all operations have to be of degree zero, because they respect the unit.

A color quasi-Hopf algebra, where $\Phi = 1 \otimes 1 \otimes 1$ is simply a color Hopf algebra. A color quasi-Hopf algebra H is called quasitriangular, if there exists also an R-matrix $R \in H^{\otimes 2}$, of degree 0, such that

$$(\text{id} \otimes \Delta)R = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1} \quad (12)$$

$$(\Delta \otimes \text{id})R = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123} \quad (13)$$

$$R \Delta^{\text{opp}} = \Delta R. \quad (14)$$

Here $\Delta^{\text{opp}}(x) = \tau \Delta(x)$, and $\Phi_{312} = \tau_{H, H \otimes H} \Phi$ and similar for other permutations. It is called triangular if $R_{21} R = \text{id}$.

From Eq. (13) and the fact the R is invertible it follows that $(\varepsilon \otimes \text{id})R = 1 = (\text{id} \otimes \varepsilon)R$, so R is automatically of degree 0.

Two quasitriangular color quasi-Hopf algebras H and H' are called twist equivalent if there exists an invertible element $J \in H^{\otimes 2}$ of degree 0 and an algebra isomorphism $\theta : H \rightarrow H'$, such that

$$\begin{aligned} (\varepsilon \otimes \text{id})J &= 1 = (\text{id} \otimes \varepsilon)J, \\ \Delta' &= J^{-1}((\theta \otimes \theta)\Delta\theta^{-1}(x))J, \\ \Phi' &= J_{2,3}^{-1} J_{1,23}^{-1} \theta(\Phi) J_{12,3} J_{1,2}, \\ R' &= J_{21}^{-1} R J. \end{aligned}$$

In the following we will be mostly interested in the case, where θ is the identity.

If J satisfies the first identity above one can define a new twist equivalent quasitriangular quasi-Hopf algebra by using the identities as definitions for Δ' , μ' and R' .

Theorem 2.2.7 *The category of modules over a quasitriangular color quasi-Hopf algebra is a braided monoidal category, enriched over \mathbf{grVec} .*

If two color Hopf algebras are twist equivalent the category of modules are tensor equivalent, i.e. it exists an invertible monoidal functor between them.

Proof. The proof is similar to the non-graded case. It is clear that the modules of a color Hopf-algebra H form a category. Further for two H -modules V, W the tensor product $V \otimes W$ is again a H -module with action $x \cdot (v \otimes w) = (\rho_V \otimes \rho_W) \tau_{34} \Delta(x) \otimes (V \otimes w) = \varepsilon(x^{(2)}, v) x^{(1)} \cdot v \otimes x^{(2)} \cdot w$. Here ρ_V denotes the action on V and similar for W . The associator is given by the action of Φ . The braiding is given by $v \otimes w \mapsto \tau R \cdot (v \otimes w)$. \square

There is a category of ((quasi)-triangular) color quasi-Hopf algebras, where the morphisms preserve the product, coproduct and the associator. Here the morphism are automatically of degree zero.

2.2.2 The Hopf algebra $T(V)$

In the following we discuss an important example of a color Hopf algebra. We endow the tensor algebra of a color vector space with a color Hopf algebra structure.

Let V be a color vector space and ϵ its commutation factor, then we define the tensor algebra $T(V)$ of V by

$$T(V) = \bigoplus_{k \in \mathbb{N}} V^{\otimes k}, \text{ with } V^0 := \mathbb{K} \text{ and } V^{\otimes k+1} := V^{\otimes k} \otimes V.$$

In the following we write xy for $x \otimes y$ for simplicity and because \otimes is used for other tensor products. The tensor algebra $T(V)$ is also Γ -graded and additionally \mathbb{Z} -graded, so in total it is $(\Gamma \times \mathbb{Z})$ -graded. We define $\epsilon'(a, b) = \epsilon(a, b)(-1)^{|a||b|}$, where $|\cdot|$ denotes the tensor degree. Of course ϵ' is again a commutation factor.

Let V be a (graded) vector space then we define a comultiplication on $T(V)$ by

$$\Delta(x) = 1 \otimes x + x \otimes 1 \text{ for } x \in V \quad (15)$$

and the comultiplication is extended to the whole space $T(V)$ using the compatibility condition with the multiplication. This gives the unshuffle coproduct.

Note that the coproduct depends on the chosen symmetry, this means on ϵ .

For example for $x, y, z \in V$, we get

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y) = (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) \\ &= 1 \otimes xy + x \otimes y - \epsilon(x, y)y \otimes x + 1 \otimes xy. \end{aligned} \quad (16)$$

$$\Delta(xyz) = 1 \otimes xyz + x \otimes yz - \epsilon(x, y)y \otimes xz + \epsilon(xy, z)z \otimes xy + xyz \otimes 1. \quad (17)$$

The counit is defined by $\epsilon(x) = 0$ for $x \in V^{\otimes k}$ for $k \geq 1$ and $\epsilon(1) = 1$ for $1 \in \mathbb{K} = V^0$. We define an antipode by $S(x) = -x$ and extend it to $T(V)$ by using the antihomomorphism property with respect to the multiplication.

We want to give a closed formula for computing the coproduct. For this we need:

Definition 2.2.8 (Shuffle permutations) We define the set of (p, a) -shuffle permutations by

$$\text{Sh}_{p,q} = \{\sigma \in S_{p+q} \mid \sigma(i) < \sigma(j) \text{ if } i, j \leq p \vee i, j > p\}. \quad (18)$$

With this one can write

$$\Delta(x_1 x_n) = \sum_{\substack{p,q \\ p+q=n}} \sum_{\sigma^{-1} \in \text{Sh}(p,q)} \epsilon(\sigma^{-1}; x_1, \dots, x_n) x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_{\sigma(n)}. \quad (19)$$

We construct the Grassmann algebra for a graded vector space V . For this we consider $T(V)$ and define the ideal I generated by elements of the form $ab - \epsilon'(a, b)ba$ for all $a, b \in V$.

2. Color algebras

Lemma 2.2.9 *The ideal I defined above is also a coideal with respect to Δ_0 .*

Proof. We have

$$\Delta(xy - \epsilon'(x, y)yx) = 1 \otimes xy + xy \otimes 1 - \epsilon'(x, y)(1 \otimes yx + yx \otimes 1) \in T(V) \otimes I + I \otimes T(V).$$

□

Definition 2.2.10 (Grassmann algebra) We get a bialgebra structure on the quotient $\Lambda(V) = T(V)/I$. For the product on $\Lambda(V)$ we write \wedge . We even get a color Hopf algebra, which we call the Grassmann algebra of V .

The Grassmann algebra is color commutative with respect to ϵ' .

2.3 COLOR LIE ALGEBRAS AND COLOR LIE BIALGEBRAS

In this section we give the basic definitions of color Lie algebras and color Lie bialgebras.

Definition 2.3.1 (Color Lie algebra) For a group Γ and a commutation factor ϵ a $(\Gamma; \epsilon)$ -color Lie algebra is a Γ -graded vector space \mathfrak{g} with a graded bilinear map of degree zero $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, such that for any homogeneous elements $a, b, c \in \mathfrak{g}$

$$[a, b] = -\epsilon(a, b)[b, a], \quad (20)$$

$$j(a, b, c) := \epsilon(c, a)[a, [b, c]] + \epsilon(a, b)[b, [c, a]] + \epsilon(b, c)[c, [a, b]] = 0. \quad (21)$$

For simplicity, we write cla for color Lie algebra.

The second equation is called Jacobi identity and can also be written as

$$[a, [b, c]] = [[a, b], c] + \epsilon(a, b)[b, [a, c]], \quad (22)$$

which shows that the adjoint representation $\text{ad}_a(b) := [a, b]$ for $a \in \mathfrak{g}$ is a color Lie algebra derivation.

Using $\sigma_c(a \otimes b \otimes c) = \epsilon(a, bc)b \otimes c \otimes a$ and $\beta(a, b) = [a, b]$, it can also be written as

$$\beta(\beta \otimes \text{id})(\text{id} + \sigma_c + \sigma_c^2) = 0. \quad (23)$$

A morphism φ of color Lie algebras $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{h}, [\cdot, \cdot]')$ is a morphism of color vector spaces such that

$$[\varphi(x), \varphi(y)]' = \varphi([x, y]). \quad (24)$$

An ideal of a color Lie algebra \mathfrak{g} is a graded subspace \mathfrak{i} such that $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$. We say that a color Lie algebra is simple if it has no proper color Lie ideal. Note that it can

have a non graded ideal. If $\mathfrak{i} \subset \mathfrak{g}$ is a color Lie ideal then the quotient $\mathfrak{g}/\mathfrak{i}$ is again a color Lie algebra.

A color Lie subalgebra is a graded subspace \mathfrak{h} such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

Let A be a Γ -graded associative algebra and ϵ a commutation factor then the bracket

$$[a, b] = ab - \epsilon(a, b)ba \quad (25)$$

defines a color Lie algebra structure. We denote the corresponding cla by A_L . So we get especially a Lie bracket on the graded homomorphisms of a color vector space, for which we have:

Proposition 2.3.2 *The color derivations $\text{Der}(A)$ of a color algebra A form a color Lie subalgebra of $\text{Homgr}(A)$.*

Proof. Let d, f be derivations and $a, b \in A$, then

$$\begin{aligned} [d, f](ab) &= d(f(ab)) - \epsilon(d, f)f(d(ab)) \\ &= d(f(a)b + \epsilon(f, a)af(b)) - \epsilon(d, f)f(d(a)b + \epsilon(d, a)ad(b)) \\ &= d(f(a)b + \epsilon(d, f a)f(a)d(b) + \epsilon(f, a)d(a)f(b) + \epsilon(df, a)ad(f(b)) \\ &\quad - \epsilon(d, f)f(d(a))b - \epsilon(f, a)d(a)f(b) - \epsilon(d, f a)f(a)d(b) - \epsilon(d, f a)\epsilon(f, a)af(d(b)) \\ &= [d, f](a)b + \epsilon(df, a)a[d, f](b). \end{aligned}$$

This shows that $[d, f]$ is a derivation of degree $\deg(d) + \deg(f)$. □

Proposition 2.3.3 *Let \mathfrak{g} be a cla and A be a color commutative algebra then the tensor product $\mathfrak{g} \otimes A$ is a cla, with Lie bracket*

$$[x \otimes a, y \otimes b] := \epsilon(a, y)[x, y] \otimes ab. \quad (26)$$

Proof. The color skew-symmetry is clear and the Jacobi identity is a simple calculation using

$$[[x \otimes a, y \otimes b], z \otimes c] = \epsilon(a, yz)\epsilon(b, z)[[x, y], z] \otimes abc.$$

□

Remark 2.3.4 By this construction and also similar ones, one can consider \mathfrak{g} to be Γ -graded and A to be H -graded, then the tensor product is $(\Gamma \times H)$ -graded, and as commutation factor one can use $\epsilon(g + h, g' + h') = \epsilon_\Gamma(g, g')\epsilon_H(h, h')$, so $\epsilon(g, h) = 1$ for $g, g' \in \Gamma, h, h' \in H$. But one can also construct different commutation factors. Indeed, \mathfrak{g} and A can be considered to be $(\Gamma \times H)$ -graded, by defining $\mathfrak{g}^{(g, 0)} = \mathfrak{g}^g$ and $\mathfrak{g}^{(g, h)} = 0$ else, and similarly for A .

2. Color algebras

Definition 2.3.5 (Universal enveloping algebra) For a color Lie algebra \mathfrak{g} one defines the universal enveloping algebra $U(\mathfrak{g})$ by the tensor algebra $T(\mathfrak{g})$ modulo the ideal generated by elements of the form

$$xy - \epsilon(x, y)yx - [x, y] \text{ for } x, y \in \mathfrak{g}. \quad (27)$$

Theorem 2.3.6 ([Sch79]) *The Universal enveloping algebra $U(\mathfrak{g})$ is a (filtered) associative color algebra. With the graded commutator it is a color Lie algebra, with \mathfrak{g} as a Lie subalgebra. It has the universal property, that is for any color algebra A and cla homomorphism $f : \mathfrak{g} \rightarrow A_L$ there exists a unique algebra homomorphism such that $f = g|_{\mathfrak{g}}$.*

In fact one has the structure of a color Hopf algebra on $U(\mathfrak{g})$. One can consider the Hopf algebra structure defined in Section 2.2.2 and just has to verify that it passes to the quotient $U(\mathfrak{g})$. For the product and unit this is clear by definition. For the coproduct this follows from

$$\Delta([x, y]) = [\Delta(x), \Delta(y)]. \quad (28)$$

The Lie algebra \mathfrak{g} is precisely formed by the primitive elements in $U(\mathfrak{g})$.

For the following definition we need the action of a cla \mathfrak{g} on $\mathfrak{g} \otimes \mathfrak{g}$. So for $a \in \mathfrak{g}$ and $b_1 \otimes b_2 \in \mathfrak{g} \otimes \mathfrak{g}$, we define

$$a \cdot (b_1 \otimes b_2) = [a, b_1] \otimes b_2 + \epsilon(a, b_1) b_1 \otimes [a, b_2]. \quad (29)$$

This can be generalized to an action on any tensor power of \mathfrak{g} .

Definition 2.3.7 (Color Lie bialgebra) A color Lie bialgebra, clba for short, is a color Lie algebra \mathfrak{g} with a cobracket $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ of degree 0, such that the compatibility condition

$$\delta([a, b]) = a \cdot \delta(b) - \epsilon(a, b) b \cdot \delta(a) \quad (30)$$

holds and δ satisfies the co-Jacobi identity given by

$$(\text{id} + \sigma_c + \sigma_c^2)(\delta \otimes \text{id})\delta = 0. \quad (31)$$

The second condition (31) is equivalent to the fact that $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ defines a color Lie algebra on \mathfrak{g}^* .

A subcoalgebra \mathfrak{h} of a Lie coalgebra \mathfrak{g} is a graded subspace such that $\delta(\mathfrak{h}) \subset \mathfrak{h} \otimes \mathfrak{h}$ and a coideal is a graded subspace \mathfrak{h} such that $\delta(\mathfrak{h}) \subset \mathfrak{g} \otimes \mathfrak{h} + \mathfrak{h} \otimes \mathfrak{g}$. Note that in contrast to the algebra or Lie algebra cases the notion of a subcoalgebra is stronger than the one of a coideal, so every subcoalgebra is also a coideal. Given a graded subspace \mathfrak{h} of a color Lie coalgebra \mathfrak{g} , the quotient $\mathfrak{g}/\mathfrak{h}$ is again a color Lie coalgebra

if \mathfrak{h} is a coideal. One defines a Lie coalgebra to be cosimple if it has no nontrivial subcoalgebra.

Observe that the dual of an ideal is a subcoalgebra.

An ideal for a clba is an ideal for the Lie bracket, which is also a coideal for the cobracket. We call a clba simple if it has no nontrivial ideal. So a clba is for example simple if the underlying color Lie algebra is simple.

On the other hand, we call a clba cosimple if it has no nontrivial subalgebra. Again this is the case, when the color Lie coalgebra is cosimple.

Observe that the dual of a simple color Lie (bi)algebra is cosimple and the other way round.

Remark 2.3.8 Given a finite dimensional color Hopf algebra H (or a bialgebra is enough), one can define the structure of a cla on H by the commutator and dually the structure of a color Lie coalgebra, but this gives no clba. Instead, we get the following relation

$$2\delta([a, b]) = [\delta a, \delta b] - \tau[\delta a, \delta b], \quad (32)$$

between the Lie bracket and the Lie cobracket. Here $H \otimes H$ has to be equipped with the Lie algebra structure coming from the associative product. So this relation cannot be seen of another type of a Lie bialgebra, since $H \otimes H$ in general is not a cla, if H is only a color Lie algebra.

Now, we aim to give some calculations in coordinates. Here and also later on, we will always use the Einstein summation convention. This means, one has to sum over all indices, which appear twice, once as lower and once as upper index.

Let \mathfrak{g} be a clba and $\{x_i\}_i$ an ordered basis of \mathfrak{g} , then we define the structure constants b, c of the Lie bracket and Lie cobracket respectively by

$$[x_i, x_j] = b_{ij}^k x_k, \quad \delta(x_i) = c_i^{jk} x_j \otimes x_k.$$

The skew-symmetry is equivalent to

$$b_{ij}^k = -\epsilon(i, j)b_{ji}^k, \quad c_i^{jk} = -\epsilon(j, k)c_i^{kj},$$

where $\epsilon(i, j)$ is short for $\epsilon(x_i, x_j)$. The Jacobi identity is given by

$$b_{ij}^l b_{lk}^m + \epsilon(i, jk)b_{jk}^l b_{li}^m + \epsilon(ij, k)b_{ki}^l b_{lj}^m = 0, \quad (33)$$

and similarly the co-Jacobi identity by

$$c_i^{jk} c_j^{lm} + \epsilon(k, lm)c_i^{jl} c_j^{mk} + \epsilon(kl, m)c_i^{jm} c_j^{kl} = 0. \quad (34)$$

The compatibility condition is equivalent to

$$b_{ij}^k c_k^{lm} = c_j^{kl} b_{ik}^m + \epsilon(i, m)c_j^{mk} b_{ik}^l - \epsilon(i, j)c_i^{kl} b_{jk}^m - \epsilon(i, j)\epsilon(j, m)c_j^{mk} b_{jk}^l. \quad (35)$$

2. Color algebras

Example 2.3.9 We consider the three dimensional color \mathfrak{sl}_2 Lie algebra \mathfrak{g} and variants of it, see also [PS07]. It is \mathbb{Z}_2^3 -graded, but the only nonvanishing summands are $\mathfrak{g}_{(1,1,0)} = e_1, \mathfrak{g}_{(1,0,1)} = e_2, \mathfrak{g}_{(0,1,1)} = e_3$. The commutation factor can be then given by a 3×3 matrix $(\epsilon(e_i, e_j))_{i,j}$

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \quad (36)$$

The Lie bracket with respect to the basis $\{e_1, e_2, e_3\}$ is given by

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = e_2, \quad [e_2, e_3] = \lambda e_1.$$

The other brackets are given by skew-symmetry. We only consider the cases $\lambda = 1$ and $\lambda = 0$ since all other cases are isomorphic to one of these by $\varphi(e_i) = \sqrt{\lambda}e_i$ for $i = 1, 2$ and $\varphi(e_3) = e_3$. For $\lambda = 1$, it is called the color \mathfrak{sl}_2 . We aim now to find a cobracket on \mathfrak{g} such that we get a color Lie bialgebra. Because of the grading, it must hold that

$$\delta(e_1) = \gamma_1 e_2 \wedge e_3, \quad \delta(e_2) = \gamma_2 e_3 \wedge e_1, \quad \delta(e_3) = \gamma_3 e_1 \wedge e_2.$$

For every choice of the γ_i the co-Jacobi identity is satisfied, but as a calculation shows for $\lambda = 1$ this only gives a clba if $\gamma_i = 0$ for $i = 1, 2, 3$. In the case $\lambda = 0$, we get a clba for $\gamma_2 = \gamma_3 = 0$ and γ_1 arbitrary.

2.3.1 Classification in dimension 3

Notes on color algebras morphisms

The simplest definition of color algebra morphisms (similar for Lie, ...) needs to assume that morphisms are of degree 0.

But for example, considering two Lie algebras \mathfrak{g} and \mathfrak{h} , spanned by e_1, e_2, e_3 with grading and ϵ given as in Example 2.3.9. The nontrivial brackets are

$$\text{for } \mathfrak{g}: [e_1, e_2] = e_3, \quad \text{and for } \mathfrak{h}: [e_2, e_3] = e_1.$$

There is an obvious isomorphism between these two color lie algebras, given by $e_i \mapsto e_{i+1}$ for $i = 1, 2$ and $e_3 \mapsto e_1$. It turns out that this isomorphism is not of degree 0, so we define:

Definition 2.3.10 A linear map f between color algebras A, B is said to preserve ϵ , if there exists a group homomorphism φ , such that $f(A_i) \subset A_{\varphi(i)}$ and

$$\epsilon(\varphi(i), \varphi(j)) = \epsilon(i, j).$$

Observe that such isomorphisms are in general not homogeneous, but they have to be graded homomorphisms.

So, we consider the following definition:

Definition 2.3.11 Two clbas $(\mathfrak{g}, [\cdot, \cdot], \delta)$ and $(\mathfrak{g}', [\cdot, \cdot]', \delta')$ are said to be isomorphic if there exists a linear map $f : \mathfrak{g} \rightarrow \mathfrak{g}'$, which preserves ϵ and satisfies, for all $x, y \in \mathfrak{g}$,

$$f([x, y]) = [f(x), f(y)]' \text{ and } \delta'(f(x)) = (f \otimes f)\delta(x). \quad (37)$$

Classification

We consider the 3-dimensional classification of clas from [PS07] and construct cobrackets in order to get clbas. There the ϵ is given as we did in Eq. (36), and we set $g_i = \deg(e_i) \in \Gamma$. It only considers commutation factors containing only ± 1 , which are injective. This means no two columns or rows in ϵ are the same.

For simplicity, we also consider only the clba which are neither commutative nor cocommutative, since these are easy to obtain from the classification of clas.

We begin with the case that g_1 is the identity. So we have the following commutation factors:

$$\epsilon_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \epsilon_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \epsilon_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

We begin with ϵ_1 and only give the nonzero brackets. Also all given clbas are non-isomorphic for all given parameters. The \pm indicates that if one only considers real clbas there are two non-isomorphic ones, which are isomorphic in the complex case.

| Nr. | bracket | cobacket |
|---------|--|---|
| A_1^1 | $[e_1, e_2] = ae_2 \quad [e_1, e_3] = e_3$ | $\delta(e_2) = be_1 \wedge e_2 \quad \delta(e_3) = \frac{b}{a}e_1 \wedge e_3$ |
| A_2^1 | $[e_1, e_2] = e_2$ | $\delta(e_2) = be_1 \wedge e_2$ |

For ϵ_2 we have

| Nr. | bracket | cobacket |
|---------|--|--|
| A_1^2 | $[e_1, e_2] = ae_2 \quad [e_1, e_3] = e_3$ | $\delta(e_1) = \pm e_3 \wedge e_3$ |
| A_2^2 | $[e_1, e_2] = e_2$ | $\delta(e_1) = \pm e_3 \wedge e_3$ |
| A_3^2 | $[e_1, e_2] = ae_2$ | $\delta(e_2) = e_1 \wedge e_2$ |
| A_4^2 | $[e_1, e_3] = e_3$ | $\delta(e_1) = e_3 \wedge e_3$ |
| A_5^2 | $[e_3, e_3] = \pm e_1$ | $\delta(e_2) = ae_1 \wedge e_2 \quad \delta(e_3) = e_1 \wedge e_3$ |

2. Color algebras

For ϵ_3 we have

| | | |
|---------|---|--|
| A_1^3 | $[e_1, e_2] = ae_2 \quad [e_1, e_3] = e_3$ | $\delta(e_1) = \pm e_2 \wedge e_2 \pm e_3 \wedge e_3$ |
| A_2^3 | $[e_1, e_2] = e_2$ | $\delta(e_1) = \pm e_2 \wedge e_2 \pm e_3 \wedge e_3$ |
| A_3^3 | $[e_1, e_2] = ae_2 \quad [e_1, e_3] = e_3$ | $\delta(e_1) = \pm e_2 \wedge e_2$ |
| A_3^3 | $[e_2, e_2] = \pm e_1 \quad [e_3, e_3] = \pm e_1$ | $\delta(e_2) = e_1 \wedge e_2 \quad \delta(e_3) = ae_1 \wedge e_3$ |
| A_4^3 | $[e_3, e_3] = \pm e_1$ | $\delta(e_2) = e_1 \wedge e_2 \quad \delta(e_3) = ae_1 \wedge e_3$ |

Next we consider that case where $g_1 + g_2 = g_3$, here we have:

$$\epsilon_4 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad \epsilon_5 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

This corresponds to ϵ_{18} (resp. ϵ_{21}) in the classification established in [PS07].

For ϵ_4

| | | |
|---------|--|---|
| | $[e_1, e_2] = e_3 \quad [e_2, e_3] = e_1 \quad [e_1, e_3] = e_2$ | only $\delta = 0$ |
| A_1^4 | $[e_1, e_2] = e_3 \quad [e_1, e_3] = e_2$ | $\delta(e_1) = e_2 \wedge e_3$ |
| A_2^4 | $[e_1, e_2] = e_3$ | $\delta(e_1) = e_2 \wedge e_3$ |
| A_3^4 | $[e_1, e_2] = e_3$ | $\delta(e_2) = e_1 \wedge e_3 \quad \delta(e_1) = e_3 \wedge e_2$ |

For ϵ_5

| | | |
|---------|---|---|
| A_1^5 | $[e_1, e_2] = e_3 \quad [e_1, e_3] = e_2$ | $\delta(e_1) = e_2 \wedge e_3$ |
| A_2^5 | $[e_2, e_3] = e_1$ | $\delta(e_2) = e_1 \wedge e_3 \quad \delta(e_3) = e_1 \wedge e_2$ |
| A_3^5 | $[e_2, e_3] = e_1$ | $\delta(e_2) = e_1 \wedge e_3$ |
| A_4^5 | $[e_1, e_2] = e_3$ | $\delta(e_1) = e_2 \wedge e_3$ |

2.4 FROBENIUS ALGEBRAS

In this section we define color Frobenius algebras and a tensor product construction dealing with a color Frobenius algebra and clba.

Definition 2.4.1 (Frobenius algebra) A color Frobenius algebra A is an associative

color algebra and coalgebra such that the product and coproduct satisfy the following compatibility condition

$$a^{(1)} \otimes a^{(2)}b = \delta(ab) = ab^{(1)} \otimes b^{(2)}, \quad (38)$$

where the product corresponds to the concatenation and δ denotes the coproduct.

We call a color Frobenius algebra commutative if the multiplication is commutative and the comultiplication is cocommutative.

An example of a Frobenius algebra is given by the group algebra $\mathbb{K}[G]$ for a finite group G with coproduct

$$\Delta(a) = \sum_{cd=a} c \otimes d, \quad (39)$$

where $a \in G$ and the sum runs over the elements of G , which form a basis of $\mathbb{K}[G]$. In fact, the coproduct is just the dual of the product in $\mathbb{K}[G]$.

Proposition 2.4.2 *Let \mathfrak{g} be a clba and A be a commutative Frobenius algebra. Then one can define a clba on $\mathfrak{g} \otimes A$ by*

$$\begin{aligned} [x \otimes a, y \otimes b] &:= \epsilon(a, y)[x, y] \otimes ab \\ \delta(x \otimes a) &:= \epsilon(x^{(2)}, a^{(1)})(x^{(1)} \otimes a^{(1)}) \otimes (x^{(2)} \otimes a^{(2)}). \end{aligned}$$

Proof. Since a Frobenius algebra is a commutative algebra and cocommutative coalgebra, we have that $\mathfrak{g} \otimes A$ is a color Lie algebra and coalgebra. So we only need to check the compatibility. First we compute

$$\begin{aligned} \delta([x \otimes a]) &= \delta([x, y] \otimes ab) \\ &= \tau_{23}(\delta([x, y]) \otimes \delta(ab)) \\ &= \tau_{23}((x \cdot \delta(y) - y \cdot \delta(x)) \otimes a^{(1)} \otimes a^{(2)}b). \end{aligned}$$

On the other side, we have

$$\begin{aligned} (x \otimes a)\delta(y \otimes b) &= (x \otimes a) \cdot (\delta(y) \otimes \delta(b)) \\ &= [x, y^{(1)}] \otimes y^{(2)} \otimes ab^{(1)} \otimes b^{(2)} + y^{(1)} \otimes [x, y^{(2)}] \otimes b^{(1)} \otimes ab^{(2)}. \end{aligned}$$

Using the Frobenius property and commutativity of A , we get

$$a^{(1)} \otimes a^{(2)}b = \delta(ab) = \delta(ba) = ab^{(1)} \otimes b^{(2)} = b^{(1)} \otimes b^{(2)}a.$$

□

2.5 COLOR POISSON ALGEBRAS

In this section, we discuss color Poisson algebras.

Definition 2.5.1 (Color Poisson algebra) A color Poisson algebra $(P, \{\cdot, \cdot\}, \cdot)$ is a color Lie algebra, with an additional color commutative associative algebra structure, such that the graded Leibniz rule holds, i.e.

$$\{a, bc\} = \{a, b\}c + \epsilon(a, b)b\{a, c\}. \quad (40)$$

Because of the symmetry of the bracket, one also has

$$\{ab, c\} = a\{b, c\} + \epsilon(b, c)\{a, c\}b. \quad (41)$$

Further the condition (40) can be written as

$$\{\cdot, \cdot\}(\text{id} \otimes \mu) = \mu(\{\cdot, \cdot\} \otimes \text{id} + \text{id} \otimes \{\cdot, \cdot\})\tau_{12}, \quad (42)$$

such that ϵ only appears in the permutation τ_{12} .

Proposition 2.5.2 Let A, B be two color Poisson algebras. Then, one can define the tensor product by

$$\begin{aligned} (a \otimes b)(a' \otimes b') &= \epsilon(b, a')aa' \otimes bb', \\ \{a \otimes b, a' \otimes b'\} &= \epsilon(b, a')(\{a, a'\} \otimes bb' + aa' \otimes \{b, b'\}), \end{aligned}$$

and this is again a color Poisson algebra.

Proof. One easily sees that the product is color commutative and the bracket is color anticommutative. Also the product is again associative. So we check the Leibniz identity:

$$\begin{aligned} \{(a \otimes b)(a' \otimes b'), a'' \otimes b''\} &= \epsilon(b, a'a'')\epsilon(b', a'')(\{aa', a'\} \otimes bb'b'' + aa'a'' \otimes \{bb', b''\}) \\ &= \epsilon(b, a'a'')\epsilon(b', a'')(a\{a', a''\} \otimes bb'b'' + \epsilon(a', a'')\{a, a''\}a' \otimes bb'b'' \\ &\quad + aa'a'' \otimes b\{b', b''\} + \epsilon(b', b'')aa'a'' \otimes \{b, b''\}b') \\ &= (a \otimes b)\{(a \otimes b)(a' \otimes b'), a'' \otimes b''\} + \epsilon(a'b', a''b'')\{(a \otimes b), a'' \otimes b''\}(a' \otimes b'). \end{aligned}$$

For the Jacobi identity, we compute

$$\begin{aligned} \{a \otimes b, \{a' \otimes b', a'' \otimes b''\}\} &= \\ \epsilon(b, a')\epsilon(bb', a'')(\{a, \{a', a''\}\} \otimes bb'b'' + aa'a'' \otimes \{b, \{b', b''\}\}). \end{aligned}$$

Taking the cyclic sum with the appropriate commutation factors gives clearly 0, since the bracket on A and B satisfies already the color Jacobi identity and the product is color commutative. This shows the color Jacobi identity. \square

We want to consider something like color manifolds, which should be a generalization of supermanifolds, but we do not know precisely how to do this so we start with polynomials. We consider a (finite dimensional) color vector space V and the graded commutative polynomial algebra $A = \text{Pol}(V)$ over it. We define here $\text{Pol}(V) := T(V^*)/I$, where I is the ideal generated by $x \otimes y - \epsilon(x, y)y \otimes x$. We write $\epsilon(i, j)$ as short for $\epsilon(x_i, x_j)$. So, in fact, it is just the free color commutative algebra over V^* . Let $\{x_i\}_i$ be a graded basis of V^* , then we can define derivations ∂_i on $\text{Pol}(V)$ on generators by

$$\partial_i x_j := \delta_{ij} \text{ for } x \in V. \quad (43)$$

This can be extended to a color derivation on the whole space $\text{Pol}(V)$. One can show that these derivations span the set $\text{Der}(A)$ as an A -module.

The derivation ∂_i has degree $\deg(\partial_i) = -\deg(x_i)$.

Proposition 2.5.3 *The derivations satisfy*

$$[\partial_i, \partial_j] = 0, \quad (44)$$

where we use the color commutator of linear maps.

Proof. Since it is a color derivation it is uniquely defined by its action on V^* , where it obviously vanishes. \square

A Poisson bracket on A can be written as

$$\{f, g\} = \epsilon(f, j)r^{ij}\partial_i f \partial_j g = r^{ij}(\partial_i \otimes \partial_j)(f \otimes g), \quad (45)$$

where $r^{ij} \in A$ is of degree $-\deg(i) - \deg(j)$, when the bracket is of degree zero.

Proposition 2.5.4 *A bracket as defined in Eq. (45) satisfies the Leibniz rule and*

- i) is color skew-symmetric if $r^{ij} = -\epsilon(i, j)^{-1}r^{ji}$ (sic!),*
- ii) satisfies the Jacobi identity if $J(r) = 0$.*

Here $J(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ denotes the left hand side of the classical Yang Baxter equation. So if both conditions are satisfied, it is a color Poisson bracket.

Proof. We first prove the Leibniz identity in the first argument:

$$\begin{aligned} \{fg, h\} &= r^{ij}\partial_i(fg)\partial_j h \\ &= \epsilon(fg, j)r^{ij}((\partial_i f)g + \epsilon^{-1}(i, f)f\partial_i g)\partial_j h \\ &= \epsilon(fg, j)\epsilon(g, j^{-1}h)r^{ij}\partial_i f \partial_j hg + \epsilon(fg, j)\epsilon(j, f)f r^{ij}\partial_i g \partial_j h \\ &= f\{g, h\} + \{f, h\}g. \end{aligned}$$

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For the skew-symmetry, we have

$$\begin{aligned}
\{f, g\} &= \epsilon(f, j)r^{ij}\partial_i f \partial_j g \\
&= \epsilon(f, j)r^{ij}\epsilon(i^{-1}f, j^{-1}g)\partial_j g \partial_i f && \text{renaming } i \text{ and } j \\
&= \epsilon(f, i)r^{ji}\epsilon(j^{-1}f, i^{-1}g)\partial_i g \partial_j f \\
&= -\epsilon(f, i)\epsilon^{-1}(i, j)r^{ij}\epsilon(f, g)\epsilon^{-1}(j, g)\epsilon^{-1}(f, i)\epsilon(i, j)\partial_i g \partial_j f \\
&= -\epsilon(f, g)\{g, f\}.
\end{aligned}$$

For the last statement, we compute with $\beta = \mu \circ r^{ij}\partial_i \otimes \partial_j$ the Poisson bracket and μ the multiplication

$$\beta(\beta \otimes \text{id}) = r^{ij}\mu(\partial_i \otimes \partial_j)[\mu(r^{st}\partial_s \otimes \partial_t) \otimes \text{id}] \quad (46)$$

$$= \mu(\mu \otimes \text{id})[r^{ij}(\partial_i r^{st})\partial_s \otimes \partial_t \otimes \partial_j] \quad (47)$$

$$+ r^{ij}r^{st}\epsilon^{-1}(i, st)\partial_i \partial_s \otimes \partial_t \otimes \partial_j \quad (48)$$

$$+ r^{ij}r^{st}\epsilon^{-1}(i, st)\epsilon(i, s)\partial_s \otimes \partial_i \partial_t \otimes \partial_j]. \quad (49)$$

On the other side

$$[r^{12}, r^{13}] = [r^{ij}\partial_i \otimes \partial_j \otimes \text{id}, r^{st}\partial_s \otimes \text{id} \otimes \partial_t] \quad (50)$$

$$\begin{aligned}
&= r^{ij}(\partial_i r^{st})\partial_s \otimes \epsilon(-j, t)\partial_j \otimes \partial_t + r^{ij}r^{st}[\partial_i, \partial_s] \otimes \epsilon(-j, t)\partial_j \otimes \partial_t \epsilon(-i, st) \\
&\quad + r^{st}(\partial_s r^{ij})\partial_i \otimes \partial_j \otimes \partial_t
\end{aligned} \quad (51)$$

$$[r^{12}, r^{23}] = [r^{ij}\partial_i \otimes \partial_j \otimes \text{id}, \text{id} \otimes r^{ij}\partial_i \otimes \partial_j] \quad (52)$$

$$= r^{ij}\partial_i \otimes (\partial_j r^{st})\partial_s \otimes \partial_t + r^{ij}\partial_i \otimes r^{st}\epsilon(-j, st)[\partial_j, \partial_s] \otimes \partial_t \quad (53)$$

$$[r^{13}, r^{23}] = [r^{ij}\partial_i \otimes \text{id} \otimes \partial_j, \text{id} \otimes r^{ij}\partial_i \otimes \partial_j] \quad (54)$$

$$= r^{ij}\partial_i \otimes \epsilon(-j, t)r^{st}\partial_s \otimes [\partial_j, \partial_t]. \quad (55)$$

We want now to show that $\beta(\beta \otimes \text{id})(\text{id} + \sigma_c + \sigma_c^2) = J(r)$. Since both sides are skew-symmetric it is enough to see that the first term in Eq. (49) equals the third term in Eq. (51), and the second term in Eq. (49) equals the second term in Eq. (51). And finally the third term in Eq. (49) equals part of the second term in Eq. (53). \square

2.6 CHANGING THE COMMUTATION FACTOR

In this section we show that, at least over the complex numbers, every color (Lie) algebra can be transformed into a (Lie) superalgebra. This construction relies on [Sch79]. We also relate this to Section 1.3.

Lemma 2.6.1 *Let $\sigma : \Gamma \times \Gamma \rightarrow \mathbb{K}^\times$ be a 2-cocycle in group cohomology, this means it satisfies $\sigma(b, c)\sigma(a, bc) = \sigma(ab, c)\sigma(a, b)$, then $\sigma(a, b)\sigma(b, a)^{-1}$ is a commutation factor.*

Proof. This is a straightforward calculation:

$$\begin{aligned}
 \epsilon(ab, c) &= \sigma(ab, c)\sigma^{-1}(c, ab) \\
 &= \sigma(b, c)\sigma(a, bc)\sigma^{-1}(a, b)\sigma^{-1}(c, a)\sigma^{-1}(ca, b)\sigma(a, b) \\
 &= \sigma(b, c)\sigma(a, cb)\sigma^{-1}(c, a)\sigma^{-1}(ac, b) \\
 &= \sigma(b, c)\sigma^{-1}(c, b)\sigma(ac, b)\sigma(a, c)\sigma^{-1}(c, a)\sigma^{-1}(ac, b) \\
 &= \sigma(b, c)\sigma^{-1}(c, b)\sigma(a, c)\sigma^{-1}(c, a) \\
 &= \epsilon(a, c)\epsilon(b, c).
 \end{aligned}$$

□

This corresponds to the fact given a 2-cocycle for a group Γ there is a symmetric monoidal functor intertwining the different symmetries.

One can also consider the image of a color Lie bialgebra under this functor.

Proposition 2.6.2 *Let σ be a 2-cocycle then if \mathfrak{g} is an ϵ -color Lie algebra so*

$$[a, b]_\sigma = \sigma(a, b)[a, b] \quad (56)$$

is an ϵ_σ -color Lie algebra, with $\epsilon_\sigma(a, b) := \epsilon(a, b)\sigma(a, b)\sigma(b, a)^{-1}$.

If it is a color Lie bialgebra then $(\mathfrak{g}, \delta_\sigma)$ is an ϵ_σ -color Lie bialgebra, where

$$\delta_\sigma(b) = \sigma(b_1, b_2)^{-1}\delta(b), \quad (57)$$

with $\delta(b) = b_1 \otimes b_2$.

Proof. We have to check

$$\delta_\sigma([a, b]_\sigma) = a \cdot_\sigma \delta_\sigma(b) - \epsilon(a, b)b \cdot_\sigma \delta_\sigma(a),$$

where \cdot_σ denotes the action of the new Lie algebra. We get, setting $c = [a, b]$ and $\delta(c) = c_1 \otimes c_2$,

$$\begin{aligned}
 &\sigma(a, b)\sigma^{-1}(c_1, c_2)\delta(c) \\
 &= \sigma^{-1}(b_1, b_2)\sigma(a, b_1)[a, b_1] \otimes b_2 + \sigma^{-1}(b_1, b_2)\sigma(a, b_2)\epsilon_\sigma(a, b_1)b_1 \otimes [a, b_2] \\
 &\quad - \epsilon_\sigma(a, b)\sigma(b, a_1)\sigma(a_1, a_2)[b, a_1] \otimes a_2 \\
 &\quad - \epsilon_\sigma(a, b)\sigma(b, a_2)\sigma(a_1, a_2)\epsilon_\sigma(b, a_1)a_1 \otimes [b, a_2] \\
 &= \sigma(a, b)\sigma^{-1}(ab_1, b_2)[a, b_1] \otimes b_2 + \sigma(a, b)\sigma^{-1}(b_1, ab_2)\epsilon(a, b_1)b_1 \otimes [a, b_2] \\
 &\quad - \epsilon(a, b)\sigma(a, b)\sigma^{-1}(ba_1, a_2)[b, a_1] \otimes a_2 \\
 &\quad - \epsilon(a, b)\sigma(a, b)\sigma^{-1}(a_1, ba_2)\epsilon(b, a_1)a_1 \otimes [b, a_2].
 \end{aligned}$$

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Using in the last equality for each term respectively

$$\begin{aligned}\sigma(a, b) &= \sigma(ab_1, b_2)\sigma(a, b_1)\sigma^{-1}(b_1, b_2), \\ \sigma(b_1, ab_2) &= \sigma(a, b)\sigma(b_1, b_2)\sigma^{-1}(a, b_1)\sigma(b_1, a)\sigma^{-1}(a, b_2), \\ \sigma(b, a) &= \sigma(ba_1, a_2)\sigma(b, a_1)\sigma^{-1}(a_1, a_2), \\ \sigma(b, a) &= \sigma(a_1, ba_2)\sigma^{-1}(a_1, b)\sigma(b, a_1)\sigma(a_1, a_2),\end{aligned}$$

and comparing terms with the same degree, one gets that this is equivalent to the compatibility condition of the old Lie bialgebra. \square

Theorem 2.6.3 [Sch79, Pop97] *For every commutation factor ϵ , with values in the complex numbers \mathbb{C} , such that $\Gamma = \Gamma_+$, there is a 2-cocycle σ such that $\epsilon(a, b) = \sigma(a, b)\sigma^{-1}(b, a)$.*

Given a commutator factor ϵ , we can define a commutator factor ϵ_0 by

$$\epsilon_0(a, b) = \begin{cases} -1 & \text{if } a, b \in \Gamma_- \\ +1 & \text{otherwise} \end{cases} \quad (58)$$

Then $\epsilon_0\epsilon$ is again a commutation factor with $\Gamma = \Gamma_+$.

So we get that every clba can be transformed into a Lie superbialgebra.

In fact, one even has in terms of categories.

Proposition 2.6.4 *There is a monoidal functor from the category of (Γ, ϵ) color vector spaces over \mathbb{C} to the category of vector superspaces.*

Proof. First we define a map $\varphi : \Gamma \rightarrow \mathbb{Z}_2$ by $\varphi(g) = 1$ if $g \in \Gamma_-$ and 0 else. So for a Γ -graded vector space V , one can define a \mathbb{Z}_2 -graded vector space by $V^0 = \bigoplus_{g \in \Gamma_+} V^g$ and $V^1 = \bigoplus_{g \in \Gamma_-} V^g$.

Using the 2-cocycle σ from Theorem 2.6.3, one can define

$$F(v \otimes w) = \sigma(v, w)v \otimes w. \quad (59)$$

\square

This corresponds to first changing the symmetry with a 2-cocycle, where the underlying symmetric monoidal categories are isomorphic. Then one takes the push-forward with respect to the map $\Gamma \rightarrow \mathbb{Z}_2$ defined above, which is again a symmetric monoidal functor. But with this functor one loses the finer structure of the grading on V .

On the other hand, if one has for example a Lie superalgebra or bialgebra with a Γ -grading it is possible with a Γ 2-cocycle to construct a color Lie algebra or bialgebra with non-trivial commutation factor ϵ .

There is a classification for the different gradings one can put and different types of Lie algebras, see e.g. Bahturin and Kochetov's work [BK10].

2.7 REPRESENTATION AND SEMIDIRECT PRODUCTS

Definition 2.7.1 (Representation of cla) By a representation α of a cla \mathfrak{g} on a color vector space V , we mean a cla morphism α from \mathfrak{g} into $\text{Hom}_L(V)$ of degree 0, that is we have

$$[x, y] = \alpha(x)\alpha(y) - \epsilon(x, y)\alpha(y)\alpha(x) \text{ for all } x, y \in \mathfrak{g}. \quad (60)$$

A representation clearly gives a map $\rho : \mathfrak{g} \otimes V \rightarrow V$ given by $x \otimes v \mapsto \alpha(x)(v)$. For simplicity we often write $x \cdot v$ instead of $\rho(x \otimes v)$.

Definition 2.7.2 (Direct sum of color Lie algebras) Given two color $(\Gamma; \epsilon)$ -Lie algebras \mathfrak{g} and \mathfrak{h} , one can define a cla structure on $\mathfrak{g} \oplus \mathfrak{h}$ by

$$[a + x, b + y] = [a, b] + [x, y] \quad (61)$$

for $a, b \in \mathfrak{g}, x, y \in \mathfrak{h}$ and $\deg(x) = \deg(a), \deg(b) = \deg(y)$. This is called the direct sum of the clas \mathfrak{g} and \mathfrak{h} .

Proposition 2.7.3 (semidirect sum of cla) Let \mathfrak{g} and \mathfrak{h} be two clas with the same ϵ , and let $\cdot : \mathfrak{g} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ be an action, i.e. for $a, b \in \mathfrak{g}, x, y \in \mathfrak{h}$ we have

$$[a, b] \cdot x = a \cdot (b \cdot x) - \epsilon(a, b)b \cdot (a \cdot x), \quad (62)$$

$$a \cdot [x, y] = [a \cdot x, y] + \epsilon(a, x)[x, a \cdot y]. \quad (63)$$

Then one can define a cla structure on $\mathfrak{g} \oplus \mathfrak{h}$ by

$$[a + x, b + y] = [a, b] + [x, y] + a \cdot y - \epsilon(x, b)b \cdot x. \quad (64)$$

Since we assume that elements are homogeneous, we have $\deg(x) = \deg(a)$. We call this cla the semidirect product of \mathfrak{g} and \mathfrak{h} and denote it by $\mathfrak{g} \rtimes \mathfrak{h}$.

Proof. The skew-symmetry is clear, so we only need to check the Jacobi identity. We have

$$\begin{aligned} [[a + x, b + y], c + z] &= [[a, b], c] + [[x, y], z] + [a, b] \cdot z - \epsilon(ab, c)c \cdot ([x, y] \\ &\quad + a \cdot y - \epsilon(x, b)b \cdot x) + [a \cdot y, z] - \epsilon(x, b)[b \cdot x, z]. \end{aligned}$$

If one calculates the cyclic sum the first two terms drop out due to the Jacobi identity for \mathfrak{g} and \mathfrak{h} . For the rest we get

$$\begin{aligned} &\epsilon(ab, c)(-c \cdot (a \cdot y) + \epsilon(x, b)c \cdot (b \cdot x) + [a, b] \cdot z) \\ &+ \epsilon(bc, a)(-a \cdot (b \cdot z) + \epsilon(y, c)a \cdot (c \cdot y) + [b, c] \cdot x) \\ &+ \epsilon(ca, b)(-b \cdot (c \cdot x) + \epsilon(z, a)b \cdot (a \cdot z) + [c, a] \cdot y) = 0, \end{aligned}$$

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using Eq. (62) (e.g. the first two terms drop with the last one) and using Eq. (63) one gets

$$\begin{aligned} &+[a \cdot y, z] - \epsilon(x, b)[b \cdot x, z] - \epsilon(xy, c)c \cdot [x, y] \\ &+[b \cdot z, x] - \epsilon(y, c)[c \cdot y, x] - \epsilon(yz, a)a \cdot [y, z] \\ &+[c \cdot x, y] - \epsilon(z, a)[a \cdot z, y] - \epsilon(yx, b)b \cdot [z, x] = 0. \end{aligned}$$

□

Proposition 2.7.4 (Semidirect product) *Let \mathfrak{g} and \mathfrak{h} be two clba, and $\cdot : \mathfrak{g} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ be a Lie algebra action such that additionally*

$$\delta(a \cdot y) = a \cdot \delta(y) - \delta(a) \cdot y. \quad (65)$$

Then the Lie algebra structure from Proposition 2.7.3 gives rise to a clba on $\mathfrak{g} \rtimes \mathfrak{h}$, with cobracket $\delta(a+x) = \delta(a) + \delta(x)$. Here with $\delta(a) \cdot y$ we mean $\epsilon(a^{(2)}, y)(a^{(1)} \cdot y) \otimes a^{(2)} + a^{(1)} \otimes (a^{(2)} \cdot y)$.

Proof. The product $\mathfrak{g} \rtimes \mathfrak{h}$ is clearly a color Lie algebra and a color Lie coalgebra, so we only need to check the compatibility condition. For the left hand side we get

$$\delta([a+x, b+y]) = \delta([a, b]) + \delta([x, y]) + \delta(a \cdot y) - \epsilon(x, b)\delta(b \cdot x). \quad (66)$$

On the other side we get

$$(a+x) \cdot \delta(b+y) = a \cdot \delta(b) + x \cdot \delta(y) + a \cdot \delta(y) - \epsilon(x, b)\delta(b) \cdot x. \quad (67)$$

Notice that since we assume $a+x$ to be homogeneous, a and x must have the same degree.

Comparing Eq. (66) with Eq. (67) minus $\epsilon(a, b)$ Eq. (67) with $a+x$ and $b+y$ exchanged, gives the compatibility condition. After dropping the terms, which cancel since \mathfrak{g} and \mathfrak{h} are clbas, it remains

$$\begin{aligned} &\delta(a \cdot y) - \epsilon(x, b)\delta(b \cdot x) \\ &= a \cdot \delta(y) - \epsilon(x, b)\delta(b) \cdot x - \epsilon(a, b)y \cdot \delta(a) - \epsilon(a, b)\epsilon(b, x)\delta(x) \cdot b, \end{aligned}$$

which is true by definition of δ . □

Observe that \mathfrak{h} is an ideal in $\mathfrak{g} \rtimes \mathfrak{h}$ and \mathfrak{g} is a subalgebra.

Now, we deal with the dual construction.

Proposition 2.7.5 *Let \mathfrak{g} and \mathfrak{h} be two clba, and $\rho : \mathfrak{h} \rightarrow \mathfrak{g} \otimes \mathfrak{h}$ be a Lie algebra coaction, i.e.*

$$(\text{id} \wedge \delta_{\mathfrak{h}})\rho(x) + (\rho \wedge \text{id})\delta(x) + (\text{id} \wedge \rho)\delta(x), \quad (68)$$

such that additionally

$$\rho([x, y]) = x \cdot \rho(y) - y \cdot \rho(x) \text{ and } a \cdot \rho(x), \quad (69)$$

where $x, y \in \mathfrak{h}, a \in \mathfrak{g}$ and \cdot denotes the action on itself by the Lie bracket. Then the Lie coalgebra structure

$$\delta(a + x) = \delta(a) + \rho(x) + \delta(x) \quad (70)$$

gives rise to a clba on $\mathfrak{g} \rtimes \mathfrak{h}$, with a bracket as in Definition 2.7.2.

Definition 2.7.6 (Lie bialgebra action) *Let \mathfrak{g} be a color Lie bialgebra and V a color vector space, on which \mathfrak{g} acts as a Lie algebra by $\cdot : \mathfrak{g} \otimes V \rightarrow V$ and coacts as a Lie coalgebra by $\rho : V \rightarrow \mathfrak{g} \otimes V$. We call V a bialgebra module if*

$$\begin{aligned} \bar{\rho}(a \cdot w) &= \delta(a) \cdot w + a \cdot \bar{\rho}(w), \\ \bar{\rho}(w) \cdot v &= \epsilon(v, w)\bar{\rho}(v) \cdot w. \end{aligned}$$

Here by $\bar{\rho}$ we mean $\rho - \tau\rho$, where τ is the flip. More explicitly this can be written as

$$\begin{aligned} \rho(a \cdot w) &= a^{(1)} \otimes (a^{(2)} \cdot w) + [a, w^{(1)}] \otimes w^{(0)} + w^{(1)} \otimes \epsilon(a, w^{(1)})(a \cdot w^{(0)}), \\ \epsilon(w^{(0)}, v)(w^{(1)} \cdot v) \otimes w^{(0)} - w^{(0)} \otimes (w^{(1)} \cdot v) &= \epsilon(v^{(0)}, w)(v^{(1)} \cdot w) \otimes v^{(0)} - v^{(0)} \otimes (v^{(1)} \cdot w), \end{aligned}$$

where we used $\rho(v) = v^{(1)} \otimes v^{(0)}$.

Proposition 2.7.7 *Let V be a \mathfrak{g} color Lie bialgebra module, then the semidirect sum $\mathfrak{g} \rtimes V$ defined by $\mathfrak{g} \oplus V$ as vector space with operations*

$$\begin{aligned} \delta(a + v) &= \delta(a) + \bar{\rho}(v), \\ [a + v, b + w] &= [a, b] + a \cdot w - b \cdot v, \end{aligned}$$

is again a color Lie bialgebra.

Proof. Firstly $\mathfrak{g} \rtimes V$ is clearly a Lie algebra and coalgebra, since this can be considered as a special case of Proposition 2.7.3. So we only need to check the compatibility condition. On the left side, we get

$$\begin{aligned} \delta([a + v, b + w]) &= \delta([a, b]) + \bar{\rho}(a \cdot w) - \epsilon(a, b)\bar{\rho}(b \cdot v) \\ &= \delta(a) \cdot b - \delta(b) \cdot a + \delta(a) \cdot w + a \cdot \bar{\rho}(w) - \epsilon(a, b)b \cdot \bar{\rho}(v) - \delta(b) \cdot v. \end{aligned}$$

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For the right hand side, we compute

$$\begin{aligned}
& (a+v) \cdot \delta(b+w) = (a+v) \cdot (\delta(b) + \bar{\rho}(w)) \\
& = [a+v, b^{(1)}] \otimes b^{(2)} + \epsilon(a, b^{(1)}) b^{(1)} \otimes [a+v, b^{(2)}] + [a+v, w^{(1)}] \otimes w^{(0)} \\
& \quad - \epsilon(aw^{(1)}, w^{(0)}) w^{(0)} \otimes [a+v, w^{(1)}] + \epsilon(a, w^{(1)}) w^{(1)} \otimes (a \cdot w^{(0)}) \\
& \quad - \epsilon(w^{(1)}, w^{(0)}) (a \cdot w^{(0)}) \otimes w^{(1)} \\
& = [a, b^{(1)}] \otimes b^{(2)} + \epsilon(v, b^{(1)}) (b^{(1)} \cdot v \otimes b^{(2)} + b^{(1)} \otimes [a, b^{(2)}]) + \epsilon(v, b) b^{(1)} \otimes b^{(2)} \cdot v \\
& \quad + [a, w^{(1)}] \otimes w^{(0)} - \epsilon(a, w^{(0)}) w^{(0)} \otimes [a, w^{(1)}] - \epsilon(w^{(1)}, v) w^{(1)} \cdot v \otimes w^{(0)} \\
& \quad + \epsilon(v, w) w^{(0)} \otimes w^{(1)} \cdot v + \epsilon(a, w^{(1)}) w^{(1)} \otimes a \cdot w^{(0)} - (a \cdot w^{(0)}) \otimes w^{(1)} \\
& = a \cdot \delta(b) + \epsilon(v, b) \delta(b) \cdot v + a \cdot \bar{\rho}(w) \\
& \quad - \epsilon(v, w^{(1)}) w^{(1)} \cdot v \otimes w^{(0)} + \epsilon(v, w) \epsilon(x^{(1)}, w^{(0)}) w^{(0)} \otimes w^{(1)} \cdot v \\
& = a \cdot \delta(b) + \epsilon(v, b) \delta(b) \cdot v + a \cdot \bar{\rho}(w) - \epsilon(v, w) \bar{\rho}(w) \cdot v.
\end{aligned}$$

Exchanging $a+v$ and $b+w$ in this computation and adding up everything gives the desired compatibility condition, if one uses the condition that V is a Lie bialgebra module. \square

Proposition 2.7.4 and Proposition 2.7.7 give the same notion if one considers \mathfrak{h} to have the trivial Lie bracket, i.e. $[\cdot, \cdot] = 0$ in Proposition 2.7.4 and $\rho = 0$ in Proposition 2.7.7.

Example 2.7.8 Consider the 3-dimensional clba \mathfrak{g} , with notations and ϵ_3 from Section 2.3.1, given by

$$[e_1, e_2] = e_2, \quad \delta(e_2) = e_1 \wedge e_2 \quad (71)$$

and all other operations determined by symmetry or zero, and \mathfrak{h} with

$$[f_1, f_2] = f_2. \quad (72)$$

An action of \mathfrak{g} on \mathfrak{h} as clba is given by $e_3 \cdot f_1 = af_3$ for $a \in \mathbb{K}$. This is easy to see since all terms appearing in the definition are identically zero.

Let us also consider the color vector space V , with the same grading and ϵ like \mathfrak{g} and a basis $\{v_1, v_2, v_3\}$ such that $\deg(v_i) = \deg(e_i)$. Then a clba action of \mathfrak{g} on V is given by

$$\begin{aligned}
\rho(v_1) &= -e_1 \otimes v_1 - c_1 e_3 \otimes v_3, & \rho(v_3) &= -e_1 \otimes v_3 - c_2 e_3 \otimes v_1, \\
e_1 \cdot v_1 &= v_1, & e_1 \cdot v_3 &= v_3, \\
e_3 \cdot v_1 &= c_2 v_3, & e_3 \cdot v_3 &= c_1 v_1,
\end{aligned}$$

for $c_1, c_2 \in \mathbb{C}$. Proving this is a straightforward calculation, and was done using Mathematica Software.

2.8 MANIN TRIPLES AND R-MATRICES

Next, we study color Lie bialgebras (CLBAs) in more detail. We give the description by color Manin triples in an analogue way to the classical or super case, and we also study the notion of r-matrix in the color case.

2.8.1 Coboundary, Quasi-triangular and Triangular color Lie bialgebras

Proposition 2.8.1 *Let \mathfrak{g} be a color Lie algebra, $r \in \Lambda^2(\mathfrak{g})$ be of degree 0 and define $\delta(x) := \text{ad}_x r = [x, r] = [\Delta x, r]$. Then this gives a color Lie bialgebra if and only if*

$$J(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \quad (73)$$

is \mathfrak{g} invariant, i.e. $[\Delta^{(3)}x, J(r)] = 0$ for all $x \in \mathfrak{g}$.

Remark 2.8.2 Notice that in the nongraded case

$$J(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad (74)$$

is called the (classical) Yang-Baxter equation, and a solution r provides a Lie bialgebra.

Proof. Following [And93]. We write $r = r_i \otimes r^i$. Because r is of degree 0, we have $\deg r^i = -\deg r_i$. We claim

$$\text{Alt}(\delta \otimes \text{id})\delta(x) = -\text{ad}_x J(r). \quad (75)$$

This gives immediately the result since $\text{ad}_x J(r) = 0$ is equivalent to the Jacobi identity.

We compute

$$\text{Alt}(\delta \otimes \text{id})\delta(x) = \text{Alt}(\delta \otimes \text{id}) \left([x, r_i] \otimes r^i + \epsilon(x, r_i) r_i \otimes [x, r^i] \right) \quad (76)$$

$$= \text{Alt} \left(\underbrace{[[x, r_i], r_j] \otimes r^j \otimes r^i}_{A} + \underbrace{\epsilon(x r_i, r_j) r_j \otimes [[x, r_i], r^j]}_{B} \otimes r^i \right) \quad (77)$$

$$+ \epsilon(x, r_i) \underbrace{[r_i, r_j] \otimes r^j \otimes [x, r^i]}_{C} + \epsilon(x, r_i) \underbrace{\epsilon(r_i, r_j) r_j \otimes [r_i, r^j]}_{D} \otimes [x, r^i]. \quad (78)$$

Here Alt denotes the total antisymmetrization, i.e. for $x_1, \dots, x_n \in \mathfrak{g}$ we have

$$\text{Alt}(x_1 \otimes \dots \otimes x_n) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \frac{1}{n!} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}.$$

Notice that we have $\deg r^i = -\deg r_i$, because of $\deg(r) = 0$, so $\epsilon(x, r^i) = \epsilon(x, r_i)^{-1}$.

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We have $\tau_{12}\delta(x) \otimes y = -\delta(x) \otimes y$ and with this $\text{Alt}(\delta(x) \otimes y) = 2(\delta(x) \otimes y - \tau_{13}\delta(x) \otimes y - \tau_{23}\delta(x) \otimes y)$, where τ_{ij} denotes the flip of the i -th and j -th factor in the tensor product, with the appropriate ϵ . So we can arrange the terms of $\text{Alt}(\delta \otimes \text{id})\delta(x)$ in the following way:

$$\begin{aligned}
A - \tau_{23}A &= ([x, r_i, r_j] - \epsilon(r^i, r^j)[x, r_j, r_i]) \otimes r^j \otimes r^i \\
&= ([x, r_i, r_j] - \epsilon(r^i, r^j)\epsilon(xr_j, r_i)[r_i, [x, r_j]]) \otimes r^j \otimes r^i \\
&= [x, [r_i, r_j]] \otimes r^j \otimes r^i \\
&= -(\text{ad}_x \otimes \text{id} \otimes \text{id})[r^{12}, r^{13}]. \\
-\tau_{13}A - \tau_{23}B &= -\epsilon(xr_i r_j, r^j r^i)\epsilon(r^i, r^j)r^i \otimes r^j \otimes [x, r_i, r_j] - \epsilon(xr_i, r_j)r_j \otimes r^i \otimes [x, r_i, r_j]\epsilon(xr_i r^j, r^i) \\
&= r_i \otimes r_j \otimes (-\epsilon(x, r^j r^i)[x, r^i, r^j] + \epsilon(x, r_i)\epsilon(r_j, r_i)\epsilon(r^j, r_j)\epsilon(x, r^j)\epsilon(r_j r^i, r^j)[x, r^j, r^i]) \\
&= -\epsilon(x, r^j r^i)\epsilon(r_i r_j, r^j r^i)\epsilon(r^j, r^i)\epsilon(xr_i, r_j)\epsilon(r^j, xr_i) \\
&= -(\text{id} \otimes \text{id} \otimes \text{ad}_x)[r^{13}, r^{23}]. \\
B - \tau_{13}B &= \epsilon(xr_i, r_j)r_j \otimes [x, r_i, r^j] \otimes r^i - \epsilon(xr_i, r_j)\epsilon(r_j, xr^j)\epsilon(xr_i r^j, r^i)r^i \otimes [x, r_i, r^j] \otimes r_j \\
&= \epsilon(xr_j, r_i)r_i \otimes [x, r_j, r^i] \otimes r^j - \epsilon(x, r_j)\epsilon(r_j r^i, x)r_i \otimes [x, r^i, r_j] \otimes r^j \\
&= -(\text{id} \otimes \text{ad}_x \otimes \text{id})[r^{12}, r^{23}]. \\
C &= \epsilon(x, r_i)[r_i, r_j] \otimes r^j \otimes [x, r^i] \\
&= -\epsilon(x, r_i r_j r^i)\epsilon(r^i, r_j)[r_i, r_j] \otimes r^i \otimes [x, r^j] \\
&= -(\text{id} \otimes \text{id} \otimes \text{ad}_x)[r^{12}, r^{13}]. \\
-\tau_{23}C &= -(\text{id} \otimes \text{ad}_x \otimes \text{id})[r^{12}, r^{13}] \\
-\tau_{13}C &= -(\text{ad}_x \otimes \text{id} \otimes \text{id})[r^{13}, r^{23}]. \\
D &= -(\text{id} \otimes \text{id} \otimes \text{ad}_x)[r^{12}, r^{23}]. \\
-\tau_{23}D &= -(\text{id} \otimes \text{ad}_x \otimes \text{id})[r^{13}, r^{23}]. \\
-\tau_{13}D &= -(\text{ad}_x \otimes \text{id} \otimes \text{id})[r^{12}, r^{23}].
\end{aligned}$$

This proves the statement. □

Proposition 2.8.3 *If $r + \tau(r)$ is invariant for $r \in \mathfrak{g} \otimes \mathfrak{g}$, and r satisfies the Yang-Baxter equation (73), then $\delta(x) := \text{ad}_x r$ defines a color Lie bialgebra.*

Proof. It is easy to see that δ is skew-symmetric. Consider $r_0 := \frac{1}{2}(r - \tau(r))$, then $\text{ad}_x r = \text{ad}_x r_0$ and by Proposition 2.8.1 it is enough to check that $J(r_0)$ is \mathfrak{g} invariant.

$$\begin{aligned}
J(r_0) &= [r^{12} + r^{21}, r^{23} + r^{32}] - (\text{id} \otimes \tau)J(r) - (\tau \otimes \text{id})J(r) \\
&\quad + (\tau \otimes \text{id})(\text{id} \otimes \tau)J(r) + (\text{id} \otimes \tau)(\tau \otimes \text{id})J(r).
\end{aligned}$$

Note that here we set $r^{21} = (\tau \otimes \text{id})r$, so there is an ϵ included. Now by assumptions, we have that $r^{12} + r^{21}, r^{23} + r^{32}$ and $J(r)$ are ad-invariant. So using $\text{ad}_x(\tau \otimes \text{id}) = (\tau \otimes \text{id})\text{ad}_x$ and the fact that ad is a derivation, we get that $J(r_0)$ is invariant. □

Definition 2.8.4 If the cobracket of a clba is given by $\delta(x) = \text{ad}_x r$, the clba is called coboundary. If r satisfies the Yang-Baxter equation, that is $J(r) = 0$ with J defined in (73), and is not necessarily skew-symmetric, the clba is called quasi-triangular and if it is also skew-symmetric it is called triangular.

Example 2.8.5 We consider the color \mathfrak{sl}_2 Lie algebra as above in Example 2.3.9.

It turns out that there are no quasi-triangular r-matrices in both cases. Because of the grading, an r-matrix has to be of the form

$$r = \rho_1 e_1 \otimes e_1 + \rho_2 e_2 \otimes e_2 + \rho_3 e_3 \otimes e_3, \quad (79)$$

for some $\rho_i \in \mathbb{C}$ and an easy calculation shows that r satisfies the Yang-Baxter equation if and only if it is 0.

Example 2.8.6 (Simple Lie algebras) Take a simple Lie algebra graded by the root system with triangular cobracket, see e.g. [BD84], which is of degree zero and modify the ϵ , as described in Section 2.6, which is constant 1, with a 2-cocycle. This way, one gets a simple color Lie bialgebra. Since the grading is a \mathbb{Z}^k -grading, where k is the rank of the Lie algebra, there are always non-trivial cocycles, if $k \geq 1$. For this ϵ , one of course has $\Gamma_+ = \Gamma$.

Similarly one can modify a simple Lie superalgebra, again graded by the root system, with a triangular cobracket, which is of degree zero. There are especially these coming from the standard r-matrices, see [Kar04].

We denote by Δ^+ the set of positive roots with respect to a chosen Cartan subalgebra \mathfrak{h} , and for each root α we consider an $e_\alpha \in \mathfrak{g}$ in the root space of α such that $(e_\alpha, e_{-\alpha}) = 1$, where (\cdot, \cdot) denotes the Killing form. We also consider an element $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$, such that $r_0 + \tau r_0 = \Omega_0$, where Ω is the Casimir element of \mathfrak{g} and Ω_0 its projection to \mathfrak{h} . With this, the standard r-matrices are given by

$$r = r_0 + \sum_{\alpha \in \Delta^+} e_\alpha \otimes e_{-\alpha}. \quad (80)$$

Note that only the standard r-matrices are compatible with the root grading, for the other r-matrices discussed in [Kar04], one has to consider a coarser grading.

The first non trivial example, this means of rank greater than one, is the Lie superalgebra $\mathfrak{sl}(2, 1)$ of supertraceless matrices on $\mathbb{C}^{(2,1)}$, with the normal supercommutator of matrices.

2.8.2 Color Manin triples

Definition 2.8.7 (color Manin triple) A color Manin triple is a triple (ρ, ρ_+, ρ_-) , where ρ is a color Lie algebra, ρ_\pm are color Lie subalgebras and $\rho = \rho_+ \oplus \rho_-$ as color vector

2. Color algebras

spaces, with a non-degenerate invariant symmetric inner product $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$, such that ρ_{\pm} are isotropic. Invariant here means that

$$([a, b], c) + \epsilon(a, b)(b, [a, c]) = 0, \quad (81)$$

and symmetric means that

$$(a, b) = \epsilon(a, b)(b, a). \quad (82)$$

Note that the invariance can also be written as

$$([b, a], c) = (b, [a, c]). \quad (83)$$

Theorem 2.8.8 *Let \mathfrak{g} be a color Lie bialgebra and set $\rho_+ = \mathfrak{g}$, $\rho_- = \mathfrak{g}^*$ and $\rho = \rho_+ \oplus \rho_-$. Then (ρ, ρ_+, ρ_-) is a color Manin triple. Conversely any finite-dimensional color Manin triple ρ gives rise to a Lie bialgebra structure on ρ_+ .*

Proof. Let $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$ then the bilinear form is given by

$$(x + \alpha, y + \beta) = \langle \alpha, y \rangle + \langle x, \beta \rangle = \langle \alpha, y \rangle + \epsilon(x, \beta)\langle \beta, x \rangle, \quad (84)$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between \mathfrak{g} and \mathfrak{g}^* . Moreover we must have

$$([x, \alpha], \beta) = (x, [\alpha, \beta]) \text{ and } ([x, \alpha], y) = -\epsilon(x, \alpha)(\alpha, [x, y]), \quad (85)$$

which completely determines $[x, \alpha]$. We write $[\cdot, \cdot] = [\cdot, \cdot]_+ + [\cdot, \cdot]_-$, with $[\cdot, \cdot]_{\pm} \in \rho_{\pm}$. So $[x, \alpha]_- = \text{ad}_x^* \alpha$ and $[x, \alpha]_+ = \text{ad}_\alpha^* x$. Here ad^* is the coadjoint action defined by $\langle \text{ad}_x(\alpha), y \rangle = -\epsilon(x, \alpha)(\alpha, [x, y])$.

Now we need to verify the color Jacobi identity. It is enough to check the cases when one or two arguments are in \mathfrak{g} the others follow from the Jacobi identity on \mathfrak{g} resp. \mathfrak{g}^* . In fact it is enough to check it when two elements are in \mathfrak{g} , since the other case follows by duality. We compute for $u \in \mathfrak{g}$:

$$\begin{aligned} & \langle j(x, y, \alpha), u \rangle \\ &= \langle \epsilon(\alpha, x)[[x, y], \alpha] + \epsilon(y, \alpha)[[\alpha, x], y] + \epsilon(x, y)[[y, \alpha], x], u \rangle \\ &= -\epsilon(\alpha, x)\epsilon(xy, \alpha)\langle \alpha, [[x, y], u] \rangle + \epsilon(y, \alpha)\langle \alpha, [x, [y, u]] \rangle - \epsilon(x, y)\epsilon(y, \alpha)\langle \alpha, [y, [x, u]] \rangle \\ &= \epsilon(y, \alpha)\langle \alpha, -[[x, y], u] + [x, [y, u]] - \epsilon(x, y)[y, [x, u]] \rangle = 0, \end{aligned}$$

and for β in \mathfrak{g}^* :

$$\begin{aligned}
 \langle j(x, y, \alpha), \beta \rangle &= \langle \epsilon(\alpha, x)[[x, y], \alpha] + \epsilon(y, \alpha)[[\alpha, x], y] + \epsilon(x, y)[[y, \alpha], x], \beta \rangle \\
 &= \epsilon(\alpha, x)\langle [x, y], [\alpha, \beta] \rangle - \epsilon(y, \alpha)\epsilon(\alpha, x)\langle x, [\alpha, \text{ad}_y \beta] \rangle \\
 &\quad + \epsilon(y, \alpha)\epsilon(\alpha, x)\epsilon(\alpha x, y)\langle y, [\text{ad}_x \alpha, \beta] \rangle \\
 &\quad + \epsilon(x, y)\langle y, [\alpha, \text{ad}_x \beta] \rangle + \epsilon(x, y)\epsilon(y\alpha, x)\langle x, [\text{ad}_y \alpha, \beta] \rangle \\
 &= \epsilon(\alpha, x)\langle \delta([x, y]), \alpha \otimes \beta \rangle - \epsilon(y, \alpha)\epsilon(\alpha, x)\langle \delta(x), \alpha \otimes \text{ad}_y \beta \rangle \\
 &\quad + \epsilon(\alpha, x)\epsilon(x, y)\langle \delta(y), \text{ad}_x \alpha \otimes \beta \rangle + \epsilon(x, y)\langle \delta(y), \alpha \otimes \text{ad}_x \beta \rangle \\
 &\quad + \epsilon(\alpha, x)\langle \delta(x), \text{ad}_y \alpha \otimes \beta \rangle \\
 &= \epsilon(\alpha, x)\langle \delta([x, y]) - \delta(x)(\text{id} \otimes \text{ad}_y) + \epsilon(x, y)\delta(y)(\text{ad}_x \otimes \text{id}) \\
 &\quad + (\text{id} \otimes \text{ad}_x)\delta(y) + \epsilon(x, y)(\text{ad}_y \otimes \text{id})\delta(x), \alpha \otimes \beta \rangle = 0.
 \end{aligned}$$

□

2.8.3 Double of a color Lie bialgebra

In this section, we construct the double of a given finite dimensional clba. Let \mathfrak{g} be a clba and $(\rho = \mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ be the corresponding Manin triple. Let $\{x_i\}_i$ be a basis of \mathfrak{g} and $\{\alpha^i\}_i$ the dual basis on \mathfrak{g}^* , that is $\langle \alpha^i, x_j \rangle = \delta_j^i$, where δ denotes the Kronecker delta. Note that $\deg(\alpha^i) = -\deg(x_i)$.

First we write the bracket and the cobracket, in terms of coordinates with respect to this basis, as

$$[x_i, x_j] = c_{ij}^k x_k, \quad [\alpha^i, \alpha^j] = \gamma_k^{ij} \alpha^k,$$

where c and γ are the corresponding structure constants. For simplicity we write for example $\epsilon(i, j)$ for $\epsilon(x_i, x_j)$. Because of the skew-symmetry we have that $c_{jk}^i = -\epsilon(j, k)c_{kj}^i$, and the invariance of the inner product leads to

$$([\alpha^i, x_j], x_k) = (\alpha^i, [x_j, x_k]) = (\alpha^i, c_{jk}^l x_l) = c_{jk}^i,$$

which gives $[\alpha^i, x_j]_+ = c_{jk}^i \alpha^k$. Similarly we have

$$([\alpha_i, x_j], \alpha^k) = -(x_j, [\alpha^i, \alpha^k])\epsilon(\alpha^i, x_j) = -\gamma_j^{ik}\epsilon(\alpha^i, x_j)\epsilon(j, j),$$

which gives $[\alpha^i, x_j]_- = -\gamma_j^{ik} x_k \epsilon^{-1}(i, k)$, using

$$\epsilon^{-1}(i, j)\epsilon(j, j)\epsilon(k, k) = \epsilon^{-1}(i, i+k)\epsilon(i+k, i+k)\epsilon(k, k) = \epsilon^{-1}(i, k).$$

So

$$[\alpha^i, x_j] = c_{jk}^i \alpha^k - \gamma_j^{ik} x_k \epsilon^{-1}(i, k).$$

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Proposition 2.8.9 *Let \mathfrak{g} be a clba and $(\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ be the corresponding Manin triple and assume that \mathfrak{p} is finite dimensional. Then there is a structure of color Lie bialgebra on \mathfrak{p} given by the r -matrix $r = x_i \otimes \alpha^i$. We have $\delta(x) = x \cdot r = [x, x_i] \otimes \alpha^i + \epsilon(i, x) x_i \otimes [x, \alpha^i]$. This clba is called the double of \mathfrak{g} .*

Proof. Using Proposition 2.8.1 it is enough to show that $r + \tau(r)$ is skew-symmetric and r satisfies the Yang-Baxter equation. Using the formulas above one gets

$$\begin{aligned} J(r) &= [x_i, x_j] \otimes \alpha^i \otimes \alpha^j \epsilon(\alpha^i, x_j) + x_i \otimes [\alpha^i, x_j] \otimes \alpha^j + x_i \otimes x_j \otimes [\alpha^i, \alpha^j] \epsilon(\alpha^i, x_j) \\ &= c_{ij}^k x_k \otimes \alpha^i \otimes \alpha^j \epsilon(\alpha^i, x_j) + c_{jk}^i x_i \otimes \alpha^k \otimes \alpha^j \\ &\quad - \gamma_j^{ik} x_i \otimes x_k \otimes \alpha^j \epsilon^{-1}(i, k) + \gamma_k^{ij} x_i \otimes x_j \otimes \alpha^k \epsilon(\alpha^i, x_j) = 0. \end{aligned}$$

and a similar calculation shows that

$$[x, r + \tau(r)] = 0.$$

Observe that $\epsilon(\alpha^i, x_j) = \epsilon^{-1}(i, j)$. □

3. COHOMOLOGY AND DEFORMATIONS

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IN this section we study a cohomology for color Lie-bialgebras, induced by the big bracket, and how it can be used to study formal deformations in the sense of Gerstenhaber. To do this we first recall and extend to the color setting the definition and properties of the so called big bracket or grand crochet [LR90]. We follow [Hof06] in the construction of the big bracket.

3.1 BIG BRACKET

In this section, we aim to define a structure of a color Poisson algebra of linear operators on the Grassmann algebra of a color vector space. With this bracket it is easy to define a differential graded complex and a cohomology for color Lie (bi-)algebras and study their deformations.

3.1.1 Construction of the big bracket

For $\lambda \in \mathbb{K}$, we define a function $F_\lambda : \Lambda(V) \rightarrow \Lambda(V)$ by

$$f_\lambda(x) = \begin{cases} \lambda^{|x|}x & \text{if } \lambda \neq 0 \\ 1\varepsilon(x) & \text{if } \lambda = 0 \end{cases}, \quad (1)$$

where $|x|$ is the tensor degree of x .

Lemma 3.1.1 *The map F_λ is a Hopf algebra morphism.*

Proof. For the product we have if $\lambda \neq 0$

$$F_\lambda(x \cdot y) = \lambda^{|x|+|y|}x \cdot y = \lambda^{|x|}x \cdot \lambda^{|y|}y = F_\lambda(x) \cdot F_\lambda(y).$$

We do similar calculations for the coproduct and antipode. □

We define

$$B := \text{Homgr}(\Lambda V, \Lambda V), \quad B^k := \text{Homgr}^k(\Lambda V, \Lambda V), \quad B^{i,j} := \text{Homgr}(\Lambda^i V, \Lambda^j V), \quad (2)$$

where here the degree is given only by the tensor degree. Of course every B^i and $B^{i,k}$ are also Γ -graded.

Next we define two functions $\rho_\lambda, \sigma_\lambda : B \rightarrow B$ for $\lambda \in \mathbb{K}$ (resp. $\lambda \in \mathbb{K}^\times$) by

$$\rho_\lambda(\varphi) := (\varphi F_\lambda) * \text{id}, \quad \sigma_\lambda(\varphi) := (\varphi * S)F_{1/\lambda}. \quad (3)$$

Lemma 3.1.2 *For $\lambda \in \mathbb{K}^\times$, the functions ρ_λ and σ_λ are inverse to each other.*

Proof. We compute

$$(\rho_\lambda \sigma_\lambda)(\varphi) = \rho_\lambda((\varphi * S)F_{1/\lambda}) = (\varphi * S)F_{1/\lambda}F_\lambda * \text{id} = (\varphi * S) * \text{id} = \varphi$$

using $F_{1/\lambda}F_\lambda = \text{id}$ and similarly

$$(\sigma_\lambda \rho_\lambda)(\varphi) = ((\varphi F_\lambda * \text{id}) * S)F_{1/\lambda} = \varphi.$$

This proves the statement. □

With the lemma above we can define a new associative product \star_λ by

$$\rho_\lambda(\varphi)\rho_\lambda(\psi) = \rho_\lambda(\varphi \star_\lambda \psi). \quad (4)$$

Lemma 3.1.3 *If $\lambda \neq 0$, we have*

$$\varphi \star_\lambda \psi = \mu(\varphi \otimes \text{id})(\mu \otimes \text{id})(\text{id} \otimes F_\lambda \otimes \text{id})(\text{id} \otimes \Delta)(\text{id} \otimes \psi)\Delta.$$

Proof. This is the same as in [Hof06, Lemme 3.2.4]. \square

To study this product further we define two maps

$$H : \Lambda(V) \otimes \Lambda(V) \rightarrow \Lambda(V) \otimes \Lambda(V) : e^{\lambda Q}, \quad (5)$$

where

$$Q = (\mu \otimes \text{id}) \circ (\text{id} \otimes \text{pr}_V \otimes \text{id}) \circ (\text{id} \otimes \Delta), \quad (6)$$

and

$$G : \Lambda(V) \otimes \Lambda(V) \rightarrow \Lambda(V) \otimes \Lambda(V) : (\mu \otimes \text{id})(\text{id} \otimes F_\lambda \otimes \text{id})(\text{id} \otimes \Delta). \quad (7)$$

Lemma 3.1.4 *The map Q is a derivation and a coderivation. Moreover, it is color since $\deg Q = 0$.*

Proof. For simplicity we write pr for the projection pr_V of $T(V)$ on V . Then Q is a derivation since for $a, b \in \Lambda(V)$ we have:

$$\begin{aligned} Q(a \otimes b)(a' \otimes b') &= Q(aa' \otimes bb')\epsilon(b, a') \\ &= aa' \wedge \text{pr}(b^{(1)}b'^{(1)}) \otimes b^{(2)}b'^{(2)}\epsilon(b, a')\epsilon(b^{(2)}, b'^{(1)}) \\ Q(a \otimes b) \wedge (a' \otimes b') + (a \otimes b) \wedge Q(a' \otimes b') & \\ &= a \wedge \text{pr} b^{(1)} \otimes b^{(2)}(a' \otimes b') + (a \otimes b)(a' \text{pr} b'^{(1)} \otimes b'^{(2)}) \\ &= aa' \text{pr} b^{(1)} \otimes b^{(2)}b' \epsilon(b, a') + qq' b^{(1)} \otimes bb'^{(2)} \epsilon(b, a'b'^{(1)}). \end{aligned}$$

We have equality since $\text{pr} ab = (\text{pr} a)\epsilon(b) + \epsilon(a)\text{pr}(b)$. Also Q is a coderivation since:

$$\begin{aligned} \Delta Q(a \otimes b) &= \Delta(a \wedge \text{pr} b^{(1)} \otimes b^{(2)}) \\ &= \tau_{23}[a^{(1)} \text{pr} b^{(1)} \otimes a^{(2)} \otimes b^{(2)} \otimes b^{(3)}\epsilon(a^{(2)}, b^{(1)}) + a^{(1)} \otimes a^{(2)} \text{pr} b^{(1)} \otimes b^{(2)} \otimes b^{(3)}] \\ &= a^{(1)} \text{pr} b^{(1)} \otimes b^{(2)} \otimes a^{(2)} \otimes b^{(3)}\epsilon(a^{(2)}, b^{(1)})\epsilon(a^{(2)}, b^{(2)}) \\ &\quad + a^{(1)} \otimes b^{(2)} \otimes a^{(2)} \text{pr} b^{(1)} \otimes b^{(3)}\epsilon(a^{(2)}b^{(1)}, b^{(2)}), \end{aligned}$$

and

$$(\text{id} \otimes Q)\Delta(a \otimes b) = a^{(1)} \otimes b^{(1)} \otimes a^{(2)} \text{pr} b^{(2)} \otimes b^{(3)}\epsilon(a^{(2)}, b^{(1)}).$$

This equals the first term in the previous equation, because of the symmetry of Δ , which gives $b^{(1)} \otimes b^{(2)} \otimes b^{(3)} = \epsilon(b^{(1)}, b^{(2)})b^{(2)} \otimes b^{(1)} \otimes b^{(3)}$. \square

3. Cohomology and Deformations

Lemma 3.1.5 *The maps H and G are algebra homomorphisms and $H = G$.*

Proof. The map H is an algebra morphism since Q is a derivation, and G since all involved maps are algebra homomorphisms. So it is enough to check the equality on generators, where we have

$$H(x \otimes 1) = x \otimes \text{id}, \quad H(1 \otimes x) = 1 \otimes x + \lambda x \otimes 1,$$

using $Q(x \otimes 1) = 0$ and $Q(1 \otimes x) = x \otimes 1$, and the same for G . \square

Proposition 3.1.6 *We have*

$$\varphi \star_\lambda \psi = \mu(\varphi \otimes \text{id})e^{\lambda Q}(\text{id} \otimes \psi)\Delta. \quad (8)$$

Proof. This follows immediately from Lemma 3.1.3 and Lemma 3.1.5. \square

Next we define

$$f \odot g = \mu \circ (f \otimes \text{id}) \circ Q \circ (\text{id} \otimes g) \circ \Delta, \quad (9)$$

so that we can finally define the big bracket, see [KS92, LR90].

Definition 3.1.7 (Big bracket) For $f, g \in B$, we define

$$\{f, g\}_{\text{BB}} = f \odot g - \epsilon(f, g)g \odot f. \quad (10)$$

Proposition 3.1.8 *The big bracket gives a color Poisson algebra, with respect to the convolution.*

Proof. In Eq. (8) one can replace λ by a formal parameter. This gives a formal deformation of the algebra B with the convolution product. Then the claim follows from Definition and Proposition 3.3.1. \square

3.1.2 Properties of the Big Bracket

Now, we provide some elementary properties of the big bracket and compute it explicitly for some important cases.

Proposition 3.1.9 *For $\varphi \in B^{i,j}, \psi \in B^{k,l}$, we have $\{\varphi, \psi\}_{\text{BB}} \in B^{i+k-1, j+l-1}$.*

Proof. In fact, we show $\odot : B^{i,j} \otimes B^{k,l} \rightarrow B^{i+k-1, j+l-1}$ from which the statement immediately follows. For $t \in \mathbb{Z}$, we have

$$\begin{aligned} (\varphi \odot \psi)(\Lambda^t V) &\subset \sum_{r+s=t} \mu(\varphi \otimes \text{id})Q(\Lambda^r V \otimes \psi(\Lambda^s V)) \\ &\subset \mu(\varphi \otimes \text{id})Q(\Lambda^{t-k} V \otimes \Lambda^l V) \\ &\subset \mu(\varphi \otimes \text{id})(\Lambda^{t-k+1} V \otimes \Lambda^{l-1} V). \end{aligned}$$

Here in the second step, we consider that φ is in $B^{i,j}$ and using $\psi \in B^{k,l}$ this vanishes unless $t - k + 1 = i$ or $t = i + k - 1$. So we can continue

$$\begin{aligned} (\varphi \odot \psi)(\Lambda^t V) &\subset \mu(\varphi(\Lambda^i V) \otimes \Lambda^{t-1}) \\ &\subset \mu(\Lambda^j V \otimes \Lambda^{t-1}) \\ &\subset \Lambda^{j+t-1}. \end{aligned}$$

This proves the statement. \square

Lemma 3.1.10 For $\psi \in B^{k,1}, \varphi \in B$, we have

$$\varphi \odot \psi = \varphi(\psi * \text{id}). \quad (11)$$

Proof. For $x \in \Lambda V$, we get

$$\begin{aligned} (\varphi \odot \psi)x &= \mu(\varphi \otimes \text{id})Q(\text{id} \otimes \psi)\Delta(x) \\ &= \mu(\varphi \otimes \text{id})Q\epsilon'(\psi, x^{(1)})x^{(1)} \otimes \psi(x^{(2)}) \\ &= \varphi(x^{(1)} \wedge \psi(x^{(2)})) \\ &= \varphi\mu(\text{id} \otimes \psi)\Delta x, \end{aligned}$$

using $(\text{pr} \otimes \text{id})\Delta y = y \otimes 1$ for $|y| = 1$. \square

Proposition 3.1.11 For $\beta \in B^{2,1}$ and $\psi \in B^{k,l}$, we have

$$\begin{aligned} \{\beta, \psi\}_{\text{BB}} &= \epsilon'(\beta, \psi) \left(\sum_{i=1}^{k+1} \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(\psi, x_i) x_i \cdot \psi(x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_{k+1}) \right. \\ &\quad \left. + \sum_{1 \leq i < j \leq k} \epsilon'(\beta, x_1 \cdots x_{i-1}) \epsilon'(x_{i+1} \cdots x_{j-1}, x_j) \psi(x_1 \wedge \cdots \beta(x_i, x_j) \cdots \wedge x_{k+1}) \right), \end{aligned} \quad (12)$$

which equals the differential in Section 3.2.2, if we consider \mathfrak{g} -action on $\Lambda^k \mathfrak{g}$ for some fixed k .

Proof. By the definition of the big bracket we have

$$\{\beta, \psi\}_{\text{BB}} = \beta \odot \psi - \epsilon'(\beta, \psi) \psi \odot \beta.$$

3. Cohomology and Deformations

So we first compute

$$\begin{aligned}
& \beta \odot \psi(x_1 \wedge \cdots \wedge x_{k+1}) = \mu(\beta \otimes \text{id})Q(\text{id} \otimes \psi)\Delta(x_1 \wedge \cdots \wedge x_{k+1}) \\
& = \mu(\beta \otimes \text{id})Q \sum_{i=1}^{k+1} \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(\psi, x_i) x_i \otimes \psi(x_1 \wedge \hat{x}_i \wedge \cdots \wedge x_{k+1}) \\
& = \sum_{i=1}^{k+1} \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(\psi, x_i) \mu(\beta \otimes \text{id})(\mu \otimes \text{id}) x_i \otimes \text{pr}_V(\psi_i^{(1)}) \otimes \psi_i^{(2)} \\
& = \sum_{i=1}^{k+1} \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(\psi, x_i) \beta(x_i \wedge \text{pr}_V(\psi_i^{(1)})) \wedge \psi_i^{(2)} \\
& = \sum_{i=1}^{k+1} \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(\psi, x_i) x_i \cdot \psi(x_1 \wedge \hat{x}_i \cdots \wedge x_{k+1}),
\end{aligned}$$

using

$$\psi_i^{(1)} \otimes \psi_i^{(2)} := \Delta\psi(x_1 \wedge \hat{x}_i \cdots \wedge x_{k+1}). \quad (13)$$

We recall that for $y = y_1 \wedge \cdots \wedge y_k$,

$$x \cdot y = [\Delta^{(|y|)}x, y] = \sum_{i=1}^k \epsilon(x, y_1 \cdots y_{i-1}) y_i \wedge \cdots \wedge [x, y_i] \cdots \wedge y_k. \quad (14)$$

Next, we use Lemma 3.1.10 to get the other term. So, we have

$$\begin{aligned}
& (\beta * \text{id})(x_1 \wedge \cdots \wedge x_{k+1}) \\
& = \mu(\beta \otimes \text{id})\Delta(x_1 \wedge \cdots \wedge x_{k+1}) \\
& = \mu(\beta \otimes \text{id})\epsilon'(x_1 \cdots x_{j-1}, x_j) \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(x_j, x_i) x_i \wedge x_j \otimes x_1 \wedge \cdots \wedge \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_{k+1} \\
& = \epsilon'(x_1 \cdots x_{j-1}, x_j) \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(x_j, x_i) \beta(x_i \wedge x_j) \wedge x_1 \wedge \cdots \wedge \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_{k+1},
\end{aligned}$$

$$\begin{aligned}
& (\psi \odot \beta)(x_1 \wedge \cdots \wedge x_{k+1}) \\
& = \epsilon'(x_1 \cdots x_{j-1}, x_j) \epsilon'(x_1 \cdots x_{i-1}, x_i) \epsilon'(x_j, x_i) \psi(\beta(x_i \wedge x_j) \wedge x_1 \wedge \cdots \wedge \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_{k+1}) \\
& = \epsilon'(\beta, x_1 \cdots x_{i-1}) \epsilon'(x_{i+1} \cdots x_{j-1}, x_j) \psi(x_1 \wedge \cdots \wedge \beta(x_i, x_j) \cdots \hat{x}_j \cdots \wedge x_{k+1}).
\end{aligned}$$

For the Lie bracket β , we have $\deg(\beta) = 0$ and $|\beta| = 1$, also $|\psi| = 0$, because we can compare the differentials only in this case, so the formula simplifies to

$$\begin{aligned}
 \{\beta, \psi\}_{\text{BB}} = & \\
 (-1)^{|\psi|} & \left(\sum_{i=1}^{k+1} (-1)^{(i-1)} (-1)^{|\psi|} \epsilon(x_1 \cdots x_{i-1}, x_i) \epsilon(\psi, x_i) x_i \cdot \psi(x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_{k+1}) \right. \\
 & \left. + \sum_{1 \leq i < j \leq k} (-1)^{i-1} (-1)^{j-i-1} \epsilon(x_{i+1} \cdots x_{j-1}, x_j) \psi(x_1 \wedge \cdots \beta(x_i, x_j) \cdots \wedge x_{k+1}) \right).
 \end{aligned}$$

□

The following proposition shows that the Maurer-Cartan elements for the big bracket and the zero differential are precisely color Lie algebras, color Lie coalgebras and color Lie bialgebras.

Proposition 3.1.12 *Let $\beta \in B^{2,1}, \delta \in B^{1,2}$ then*

- $\{\beta, \beta\}_{\text{BB}} = 0$ if and only if β defines a color Lie algebra structure,
- $\{\delta, \delta\}_{\text{BB}} = 0$ if and only if δ defines a color Lie coalgebra structure,
- $\{\beta + \delta, \beta + \delta\}_{\text{BB}} = 0$ if and only if β, δ define a color Lie bialgebra structure.

Proof. For $x \in V$ and $\delta(x) = x^{[1]} \wedge x^{[2]}$, we have

$$\begin{aligned}
 (\delta \odot \delta)(x) &= \mu(\delta \otimes \text{id})Q(\text{id} \otimes \delta)\Delta(x) \\
 &= \mu(\delta \otimes \text{id})Q(1 \otimes \delta(x)) \\
 &= \mu(\delta \otimes \text{id})Q(1 \otimes x^{[1]} \wedge x^{[2]}) \\
 &= \mu(\delta \otimes \text{id})x^{[1]} \otimes x^{[2]} - \epsilon(x^{[1]}, x^{[2]})x^{[2]} \otimes x^{[1]} \\
 &= 2\delta(x^{[1]}) \wedge x^{[2]}.
 \end{aligned}$$

Using Lemma 3.1.10, we have

$$\begin{aligned}
 (\beta \odot \beta)(x \wedge y \wedge z) &= \beta(\beta * \text{id})(x \wedge y \wedge z) \\
 &= \beta \mu(\text{id} \otimes \beta)(x \otimes y \wedge z + \epsilon'(x, y)y \otimes x \wedge z + \epsilon'(xy, z)z \otimes x \wedge y) \\
 &= \beta(x \wedge \beta(y \wedge z)) - \epsilon(x, y)\beta(y \wedge \beta(x \wedge z)) + \epsilon(xy, z)\beta(z \wedge \beta(x \wedge y)).
 \end{aligned}$$

For the third statement, we calculate

$$\{\beta + \delta, \beta + \delta\}_{\text{BB}} = \{\beta, \beta\}_{\text{BB}} + \{\delta, \delta\}_{\text{BB}} + 2\{\delta, \beta\}_{\text{BB}}.$$

Because the terms are of different degrees, they have to vanish separately. Using Proposition 3.1.11, we get that $\{\delta, \beta\}_{\text{BB}}$ gives the compatibility condition, and the other two terms say that β defines a color Lie algebra and δ defines a color Lie coalgebra. □

3. Cohomology and Deformations

Similarly to color Lie algebras and bialgebras, one can also describe the different semidirect products using the big bracket.

Proposition 3.1.13 *Let $(\mathfrak{g}, \beta_{\mathfrak{g}}, \delta_{\mathfrak{g}})$ and $(\mathfrak{h}, \beta_{\mathfrak{h}}, \delta_{\mathfrak{h}})$ be two color Lie bialgebras. Then $\rho : \mathfrak{g} \wedge \mathfrak{h} \rightarrow \mathfrak{g}$ is a Lie bialgebra action and so gives a semidirect product, if and only if $\{\beta_{\mathfrak{g}} + \delta_{\mathfrak{g}} + \beta_{\mathfrak{h}} + \delta_{\mathfrak{h}} + \frac{1}{2}\rho, \rho\}_{\text{BB}} = 0$.*

Proof. We consider the vector space $V = \mathfrak{g} \oplus \mathfrak{h}$, then e.g. $\delta_{\mathfrak{g}}$ can be extended to a map $V \rightarrow V \wedge V$ and similarly for the other operations. So using $\beta = \beta_{\mathfrak{g}} + \beta_{\mathfrak{h}} + \rho$ and $\delta = \delta_{\mathfrak{g}} + \delta_{\mathfrak{h}}$, we have that V is a Lie bialgebra if $\{\beta + \delta, \beta + \delta\}_{\text{BB}} = 0$, which gives the claim, since the other terms vanish because \mathfrak{g} and \mathfrak{h} are already color Lie bialgebra. \square

A similar result holds for a coaction or a combination of them.

The most general case would be to have maps

$$\begin{aligned} \rho_{\mathfrak{g}} : \mathfrak{h} \wedge \mathfrak{g} &\rightarrow \mathfrak{g}, & \rho_{\mathfrak{h}} : \mathfrak{g} \wedge \mathfrak{h} &\rightarrow \mathfrak{h}, \\ \sigma_{\mathfrak{g}} : \mathfrak{h} &\rightarrow \mathfrak{h} \wedge \mathfrak{g}, & \sigma_{\mathfrak{h}} : \mathfrak{g} &\rightarrow \mathfrak{h} \wedge \mathfrak{g}, \\ \tau_{\mathfrak{g}} : \mathfrak{h} &\rightarrow \mathfrak{g} \wedge \mathfrak{g}, & \tau_{\mathfrak{h}} : \mathfrak{g} &\rightarrow \mathfrak{h} \wedge \mathfrak{h}, \\ \chi_{\mathfrak{h}} : \mathfrak{h} \wedge \mathfrak{h} &\rightarrow \mathfrak{g}, & \chi_{\mathfrak{g}} : \mathfrak{g} \wedge \mathfrak{g} &\rightarrow \mathfrak{h}. \end{aligned}$$

Then one can define the bracket and cobracket by $\beta = \beta_{\mathfrak{g}} + \beta_{\mathfrak{h}} + \rho_{\mathfrak{g}} + \rho_{\mathfrak{h}} + \chi_{\mathfrak{g}} + \chi_{\mathfrak{h}}$ (resp. $\delta = \delta_{\mathfrak{g}} + \delta_{\mathfrak{h}} + \sigma_{\mathfrak{g}} + \sigma_{\mathfrak{h}} + \tau_{\mathfrak{h}} + \tau_{\mathfrak{g}}$). Note that in this generality neither \mathfrak{g} nor \mathfrak{h} is an ideal any more and not even a subalgebra. But if one wants to find conditions for this being again color Lie bialgebras, one can simply compute the big bracket and gets the corresponding conditions.

Remark 3.1.14 In fact one could also consider color quasi-and-coquasi-Lie bialgebra, that is one has, in addition to the bracket and cobracket, an associator $\varphi \in \Lambda^3 \mathfrak{g}$ and a coassociator $\psi : \Lambda^3 \mathfrak{g} \rightarrow \mathbb{K}$, such that $\{\varphi + \delta + \beta + \psi, \varphi + \delta + \beta + \psi\}_{\text{BB}} = 0$. One may consider color quasi-Lie bialgebra i.e. $\psi = 0$ in the previous definition, see [Dri89] for the non-graded case. Also, one may consider coquasi-Lie bialgebras which are defined by $\varphi = 0$.

At last, we provide relationships between r-matrices and the big bracket. Note that since everything here is considered to be color skew-symmetric, one can only treat the triangular case, i.e. skew-symmetric r-matrices.

For $r \in B^{2,0}$, one can define $\delta := \{r, \beta\}_{\text{BB}}$.

Proposition 3.1.15 *In this case δ, β is a color Lie bialgebra if*

$$\{\{\{r, \beta\}_{\text{BB}}, r\}_{\text{BB}}, \beta\}_{\text{BB}} = 0. \quad (15)$$

Proof. We have

$$\{\beta + \delta, \beta + \delta\}_{\text{BB}} = \{\beta, \beta\}_{\text{BB}} + 2\{\{r, \beta\}_{\text{BB}}, \beta\}_{\text{BB}} + \{\{r, \beta\}_{\text{BB}}, \{r, \beta\}_{\text{BB}}\}_{\text{BB}}.$$

The first term is zero, because we assume β to be a color Lie algebra. The second term is zero because of the color Jacobi identity. For the third term, we get

$$\begin{aligned} \{\{r, \beta\}_{\text{BB}}, \{r, \beta\}_{\text{BB}}\}_{\text{BB}} &= \{\{\{r, \beta\}_{\text{BB}}, r\}_{\text{BB}}, \beta\}_{\text{BB}} + \epsilon'(r\beta, r)\{r, \{\{r, \beta\}_{\text{BB}}, \beta\}_{\text{BB}}\}_{\text{BB}} \\ &= \{\{\{r, \beta\}_{\text{BB}}, r\}_{\text{BB}}, \beta\}_{\text{BB}}. \end{aligned}$$

□

Therefore $\{\{r, \beta\}_{\text{BB}}, r\}_{\text{BB}} = 0$ corresponds to the classical Yang-Baxter equation.

3.2 COHOMOLOGY FOR COLOR LIE BIALGEBRAS

In this section, we derive from the big bracket a cohomology complex and a reduced cohomology complex for clbas.

3.2.1 Clba cohomology and reduced clba cohomology

To define the cohomology of a Lie bialgebra \mathfrak{g} , we first need to define the cochains complex. So we set

$$\mathcal{B}^k(\mathfrak{g}) = \bigoplus_{i+j=k} B^{i,j}(\mathfrak{g}) = \bigoplus_{i+j=k} \text{Homgr}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}). \quad (16)$$

This is in fact a bicomplex. Note however that the degree is given here by the sum of the tensor degrees and not the difference, so the grading is not given by the degree of the morphisms, as for B^i . The commutation factor is independent of this, since the gradings agree modulo 2.

Definition 3.2.1 (clba cohomology) For a given clba $(\mathfrak{g}, \beta, \delta)$ we define a differential $\partial : \mathcal{B}^k \rightarrow \mathcal{B}^{k+1}$ on $\mathcal{B}^\bullet(\mathfrak{g})$ by

$$\partial(\varphi) := \{\delta + \beta, \varphi\}_{\text{BB}}. \quad (17)$$

The cohomology for this differential is called clba cohomology.

Hence $\partial^2 = 0$ follows from the color Jacobi identity.

We will write ∂_β or ∂_δ for the two differentials given by $\{\beta, \varphi\}_{\text{BB}}$ (resp. $\{\delta, \varphi\}_{\text{BB}}$). So $\partial = \partial_\beta + \partial_\delta$ and in total we have a bicomplex.

Proposition 3.2.2 The differential ∂ is a derivation of the big bracket, i.e.

$$\partial\{\varphi, \psi\}_{\text{BB}} = \{\partial\varphi, \psi\}_{\text{BB}} + (-1)^{|\varphi|}\{\varphi, \partial\psi\}_{\text{BB}}. \quad (18)$$

3. Cohomology and Deformations

Proof. This follows directly from the color Jacobi identity for the big bracket, and the fact that β has Γ -degree 0. \square

We will also need a “reduced” cohomology complex \mathcal{B}_r^\bullet which starts by 1 in both direction, so we make the following

Definition 3.2.3 (Reduced clba cohomology) The reduced clba cohomology has the same bicomplex and differential as the clba cohomology but with $B_r^{i,0} = B_r^{0,i} = \{0\}$. We denote it by $H_r^\bullet(\mathfrak{g})$.

This is clearly a sub-bicomplex and we again get a complex \mathcal{B}_r^i . Notice that the Lie bracket as well as the Lie cobracket lie in this reduced complex. We also note that the big bracket can be reduced to this reduced complex.

Now, we provide more explicit formula for the bicomplex differentials, the one for ∂_μ , up to a sign, can be found in Proposition 3.1.11. For ∂_δ we get:

Proposition 3.2.4 For $\varphi \in B^{k,l}$ one has

$$\begin{aligned} \partial_\delta(\varphi)(x_1 \wedge \dots \wedge x_k) &= \sum_{i=0}^l (-1)^i \delta(\varphi_i) \wedge \varphi_1 \wedge \dots \wedge \hat{\varphi}_i \dots \wedge \varphi_l \\ &\quad - \sum_{i=0}^k (-1)^{|\varphi|} (-1)^{i+k} \epsilon(x_i, x_{i+1} \dots x_k) \varphi(x_1 \wedge \dots \wedge \hat{x}_i \dots \wedge x_k \wedge x_i^{(1)}) \wedge x_i^{(2)}, \end{aligned}$$

where we use $\varphi(x_1 \wedge \dots \wedge x_k) = \varphi_1 \wedge \dots \wedge \varphi_k$.

Proof. We have

$$\delta \odot \varphi = \sum_{i=0}^l (-1)^i \delta(\varphi_i) \wedge \varphi_1 \wedge \dots \wedge \hat{\varphi}_i \dots \wedge \varphi_l$$

and

$$\begin{aligned} &\varphi \odot \delta(x_1 \wedge \dots \wedge x_k) \\ &= (-1)^{i-1} (-1)^{k-1} \epsilon(x_i, x_{i+1} \dots x_k) \\ &\quad \mu(\varphi \otimes \text{id})(\mu \otimes \text{id})(\text{id} \otimes \text{pr}_{\mathfrak{g}} \otimes \text{id})(\text{id} \otimes \Delta)(x_1 \wedge \dots \wedge \hat{x}_i \dots \wedge x_k \otimes \delta(x_i)) \\ &= (-1)^{i+k} \epsilon(x_i, x_{i+1} \dots x_k) \varphi(x_1 \wedge \dots \wedge \hat{x}_i \dots \wedge x_k \wedge x_i^{(1)}) \wedge x_i^{(2)}. \end{aligned}$$

The graded commutator of these two expressions gives the result. \square

It is more difficult to define the cohomology of a clba with values in a clba module. So let \mathfrak{g} be a clba and V a clba module. The idea is to consider the direct sum

$W = \mathfrak{g} \oplus V$, which according to Proposition 2.7.4 is again a clba, so one can consider its cohomology. The differential is given by

$$\partial\varphi = \{\delta + \sigma + \rho + \beta, \varphi\}_{\text{BB}}, \quad (19)$$

where ρ and σ denote the action (resp. coaction) of \mathfrak{g} on V . Note that since $\Lambda^i W \cong \Lambda^j \mathfrak{g} \wedge \Lambda^{i-j} V$, we have in fact a \mathbb{Z}^4 -grading on $B(W)$. We denote it by

$$B^{ij,kl} := \text{Homgr}(\Lambda^i \mathfrak{g} \wedge \Lambda^j V, \Lambda^k \mathfrak{g} \wedge \Lambda^l V).$$

The idea is to find a subcomplex of morphisms of this which contain e.g. only one V as output, as it can be done for e.g. Lie algebras. The problem here is that this does not form a subcomplex, so we take the smallest subcomplex, which contains maps of the form $\mathfrak{g} \wedge V$ and $V \rightarrow \mathfrak{g}$. The resulting subcomplex consists of all $B^{ij,kl}$, such that $|j - l| = 1$.

3.2.2 Comparison with Scheunert's color Lie algebra cohomology

We aim to compare the cohomology described above with the one in [SZ98] for color Lie algebras.

Let \mathfrak{g} be a color Lie algebra and V a color \mathfrak{g} -module. We denote the graded vector space of graded n -linear maps by $\text{Homgr}_n(\mathfrak{g}, V) = \text{Homgr}(\underbrace{\mathfrak{g}, \dots, \mathfrak{g}}_n; V)$. One can define

an action of \mathfrak{g} on it by

$$(x \cdot \varphi)(y_1, \dots, y_n) = x \cdot (\varphi(y_1, \dots, y_n)) - \sum_i \epsilon(x, y_1 \cdots y_{i-1}) \varphi(y_1, \dots, [x, y_i], \dots, y_n),$$

for $x, y_i \in \mathfrak{g}$ and $\varphi \in \text{Homgr}_n(\mathfrak{g}, V)$. One has $\text{Homgr}_n(\mathfrak{g}, V) \cong \text{Homgr}(\mathfrak{g}^n, V)$ and the action of S_n on the tensor product gives by pullback an S_n -action on $\text{Homgr}_n(\mathfrak{g}, V)$, which we denote by $\pi \cdot \varphi$ for $\pi \in S_n$ and $\varphi \in \text{Homgr}_n(\mathfrak{g}, V)$.

We define

$$\begin{aligned} C^{-n}(\mathfrak{g}, V) &= 0, \quad C^0(\mathfrak{g}, V) = V, \\ C^n(\mathfrak{g}, V) &= \{\varphi \in \text{Homgr}_n(\mathfrak{g}, V) \mid \forall \pi \in S_n : \pi\varphi = \text{sign}(\pi)\varphi\}. \end{aligned}$$

One can identify $C^n(\mathfrak{g}, V)$ with $\text{Homgr}(\Lambda^n \mathfrak{g}, V)$.

Now one can define a differential on the complex C_n . The explicit formula for the differential is given by

$$\begin{aligned} (d^n \varphi)(x_0, \dots, x_n) &= \sum_i (-1)^i \epsilon(\varphi x_0 \cdots x_{i-1}, x_i) x_i \cdot \varphi(x_0, \dots, \hat{x}_i, \dots, x_n) \\ &\quad + \sum_{i < j} (-1)^j \epsilon(x_i \cdots x_{j-1}, x_j) \varphi(x_0, \dots, x_{i-1}, [x_i, x_j], \dots, \hat{x}_j, \dots, x_n). \end{aligned}$$

As stated in Proposition 3.1.11, this corresponds with our definition of the clba cohomology.

3.2.3 Some cohomology computations

In this section, we compute explicitly the first reduced cohomologies of some of the clbas from the previous classification. We first give explicit formulas for the first and second degree of the differential. In degree one for $\varphi \in B^{1,1}$, we have

$$\begin{aligned} (\partial\varphi)(a \wedge b) &= \epsilon(\varphi, a)\beta(a \wedge \varphi(b)) - \epsilon(a, b)\epsilon(\varphi, b)\beta(b \wedge \varphi(a)) - \varphi(\beta(a \wedge b)) \\ (\partial\varphi)(a) &= \delta(\varphi(a)) - \varphi(a^{(1)}) \wedge a^{(2)} + \epsilon(\varphi, a^{(1)})a^{(1)} \wedge \varphi(a^{(2)}). \end{aligned}$$

So the exact one-cochains are precisely the derivations and coderivations.

In second degree, we get for $\varphi \in B^{1,2}$,

$$\begin{aligned} (\partial\varphi)(a \wedge b) &= \epsilon(\varphi, a)\beta(a \wedge \varphi_1(b)) \wedge \varphi_2(b) + \epsilon(\varphi, b)\epsilon(a, b)\beta(b \wedge \varphi_1(a)) \wedge \varphi_2(a) + \varphi(\beta(a \wedge b)) \\ (\partial\varphi)(a) &= \delta(\varphi_1(a)) \wedge \varphi_2(a) - \epsilon(\varphi_1(a), \varphi_2(a))\delta(\varphi_2(a)) \wedge \varphi_1(a) \\ &\quad + \varphi(\delta_1(a)) \wedge \delta_2(a) - \epsilon(\delta_1(a), \delta_2(a))\varphi(\delta_2(a)) \wedge \delta_1(a). \end{aligned}$$

Here we use $\varphi(x) = \varphi_1(x) \wedge \varphi_2(x)$ and similar for δ , where a sum on the right side is to be understood.

For $\varphi \in B^{2,1}$ one gets

$$\begin{aligned} (\partial\varphi)(a \wedge b \wedge c) &= -\beta(a \wedge \varphi(b \wedge c))\epsilon(\varphi, a) \mp \text{cyclic} - \varphi(a \wedge \beta(b \wedge c))\epsilon(\beta, a) \mp \text{cyclic} \\ (\partial\varphi)(a \wedge b) &= \delta(\varphi(a \wedge b)) + \varphi(a \wedge \delta_1(b)) \wedge \delta_2(b) - \epsilon(\delta_1(b), \delta_2(b))\varphi(a \wedge \delta_2(b)) \wedge \delta_1(b) \\ &\quad + \varphi(a \wedge \delta_2(b)) \wedge \delta_1(b) - \epsilon(\delta_1(a), \delta_2(a))\epsilon(a, b)\varphi(b \wedge \delta_1(a)) \wedge \delta_2(a). \end{aligned}$$

Cohomology of A_1^2

We consider here the following degrees:

$$\deg(e_1) = (0, 0), \quad \deg(e_2) = (1, 0), \quad \deg(e_3) = (0, 1).$$

3.2. Cohomology for color Lie bialgebras

| Degree | Derivations | Closed 2-cochains | Exact 2-cochains | Second cohomology |
|--------|---|--|---|--|
| 0 | $\varphi(e_2) = c_2 e_2$ | $\varphi(e_1 \wedge e_2) = c_1 e_2$ $\varphi(e_1 \wedge e_3) = c_2 e_3$ $\varphi(e_1) = d_1 e_3 \wedge e_3$ | $\varphi(e_1 \wedge e_2) = a c_1 e_2$ $\varphi(e_1 \wedge e_3) = c_1 e_3$ $\varphi(e_1) = d_1 e_3 \wedge e_3$ | $\varphi(e_1 \wedge e_2) = c_1 e_2$ |
| (1,0) | $\varphi(e_1) = c_1 e_2$ | if $a \neq 2$: $\varphi(e_1 \wedge e_2) = -a c_1 e_1$ $\varphi(e_2 \wedge e_3) = c_1 e_3$ $\varphi(e_2) = c_1 e_3 \wedge e_3$ if $a = 2$: $\varphi(e_1 \wedge e_2) = -2 c_1 e_1$ $\varphi(e_2 \wedge e_3) = c_1 e_3$ $\varphi(e_3 \wedge e_3) = -c_2 e_2$ $\varphi(e_1) = c_2 e_1 \wedge e_2$ $\varphi(e_2) = d_1 e_3 \wedge e_3$ | $\varphi(e_1 \wedge e_2) = -a c_1 e_1$ $\varphi(e_2 \wedge e_3) = c_1 e_3$ $\varphi(e_2) = c_1 e_3 \wedge e_3$ | if $a \neq 2$: {0} if $a = 2$: $\varphi(e_3 \wedge e_3) = -c_2 e_2$ $\varphi(e_1) = c_2 e_1 \wedge e_2$ $\varphi(e_2) = d_1 e_3 \wedge e_3$ |
| (0,1) | $\varphi(e_1) = c_1 e_3$ | $\varphi(e_1 \wedge e_3) = -c_1 e_1$ $\varphi(e_2 \wedge e_3) = a c_1 e_2$ $\varphi(e_3 \wedge e_3) = 2 c_1 e_3$ $\varphi(e_1) = -c_1 e_3 \wedge e_3$ $\varphi(e_3) = c_1 e_1 \wedge e_3$ | $\varphi(e_1 \wedge e_3) = -c_1 e_1$ $\varphi(e_2 \wedge e_3) = a c_1 e_2$ $\varphi(e_3 \wedge e_3) = 2 c_1 e_3$ $\varphi(e_1) = -c_1 e_3 \wedge e_3$ $\varphi(e_3) = c_1 e_3 \wedge e_3$ | {0} |
| (1,1) | $\varphi(e_2) = c_2 e_3$ only if $a = 1$ | $\varphi(e_1 \wedge e_2) = c_1 e_3$ $\varphi(e_1 \wedge e_3) = c_2 e_2$ $\varphi(e_1) = d_1 e_2 \wedge e_3$ | $\varphi(e_1 \wedge e_2) = (a-1) c_1 e_3$ $\varphi(e_1 \wedge e_3) = (a-1) c_2 e_2$ $\varphi(e_1) = -c_1 e_2 \wedge e_3$ | $\varphi(e_1 \wedge e_2) = c_1 e_3$ if $a = 1$ also: $\varphi(e_1 \wedge e_3) = c_2 e_2$ |

Cohomology of A_2^4

We consider here the following degrees:

$$\deg(e_1) = (1, 1), \deg(e_2) = (1, 0), \deg(e_3) = (0, 1).$$

3. Cohomology and Deformations

| Degree | Derivations | closed 2-cochains | exact 2-cochains | 2-cohomology |
|--------|--|--|--|--|
| (1,1) | $\varphi(e_2) = c_1 e_3$ $\varphi(e_3) = c_1 e_2$ | $\varphi(e_1 \wedge e_2) = c_1 e_2$ $\varphi(e_1 \wedge e_3) = c_2 e_3$ | $\varphi(e_1 \wedge e_2) = c_1 e_2$ $\varphi(e_1 \wedge e_3) = -c_1 e_3$ | $\varphi(e_1 \wedge e_2) = c_1 e_2$ |
| (1,0) | $\varphi(e_1) = c_1 e_2$ | $\varphi(e_1) = c_1 e_1 \wedge e_2$ $\varphi(e_3) = c_2 e_2 \wedge e_3$ $\varphi(e_1 \wedge e_2) = \frac{1}{2}(c_1 - c_2)e_1$ $\varphi(e_2 \wedge e_3) = \frac{1}{2}(c_2 - c_1)e_3$ | $\varphi(e_1) = -c_1 e_1 \wedge e_2$ $\varphi(e_3) = c_1 e_2 \wedge e_3$ $\varphi(e_1 \wedge e_2) = -c_1 e_1$ $\varphi(e_2 \wedge e_3) = c_1 e_2$ | $\varphi(e_1) = c_1 e_1 \wedge e_2$ $\varphi(e_3) = c_1 e_2 \wedge e_3$ |
| (0,1) | $\varphi(e_1) = c_1 e_3$ | $\varphi(e_1) = c_1 e_1 \wedge e_3$ $\varphi(e_2) = c_2 e_2 \wedge e_2$ $\varphi(e_1 \wedge e_3) = \frac{1}{2}(c_1 - c_2)e_1$ $\varphi(e_2 \wedge e_3) = \frac{1}{2}(c_2 - c_1)e_2$ | $\varphi(e_1) = -c_1 e_1 \wedge e_3$ $\varphi(e_2) = c_1 e_2 \wedge e_3$ $\varphi(e_1 \wedge e_3) = -c_1 e_1$ $\varphi(e_2 \wedge e_3) = c_1 e_3$ | $\varphi(e_1) = c_1 e_1 \wedge e_3$ $\varphi(e_2) = c_1 e_2 \wedge e_3$ |
| (0,0) | 0 | $\varphi(e_1) = c_3 e_2 \wedge e_3$ $\varphi(e_1 \wedge e_2) = c_1 e_3$ $\varphi(e_1 \wedge e_3) = c_2 e_2$ | $\varphi(e_1) = c_1 e_2 \wedge e_3$ $\varphi(e_1 \wedge e_2) = c_2 e_3$ $\varphi(e_1 \wedge e_3) = c_3 e_2$ | 0 |

3.3 DEFORMATION THEORY

In this section, we study deformations of clbas using Gerstenhaber's approach based on formal power series [Ger64]. We also describe a quantum universal enveloping algebra of a color Lie algebra. The main ingredient for one-parameter formal deformations is to extend the scalar field \mathbb{K} to formal power series $\mathbb{K}[[t]]$, where t is an indeterminate. Given a vector space V , we denote by $V[[t]] = \mathbb{K}[[t]] \hat{\otimes} V$ the space of formal power series. Here $\hat{\otimes}$ denotes the completed tensor product. The main results, due first to Gerstenhaber, connect these deformations to suitable cohomology groups.

3.3.1 Deformations of associative algebras and star products

Let \mathcal{A} be a color algebra, then the commutator defines a Lie structure on \mathcal{A} , which also satisfies the Leibniz identity

$$[a, bc] = [a, b]c + \epsilon(a, b)b[a, c], \quad (20)$$

$$[ab, c] = \epsilon(b, c)[a, c]b + a[b, c]. \quad (21)$$

So it gives the structure of a non-commutative color Poisson algebra on \mathcal{A} .

Let \mathcal{A} be a color commutative associative algebra. By a deformation of \mathcal{A} , we mean a (noncommutative) associative algebra structure \star on $\mathcal{A}[[t]]$, such that $a \star b = ab + \sum_{r=1}^{\infty} t^r C_r(a, b)$. Here the C_r are bilinear maps of degree 0. We call the deformed product a star product.

Definition and Proposition 3.3.1 *For a deformation of a color commutative algebra, we define a color Poisson bracket by*

$$\{a, b\} = C_1(a, b) - \epsilon(a, b)C_1(b, a). \quad (22)$$

Proof. The Leibniz identity for the Poisson bracket follows directly from the Leibniz identity of the commutator of the product \star in first order with respect to t .

The Jacobi identity of the Poisson bracket follows from the Jacobi identity of the commutator in second order. Precisely in order t^2 we have

$$\begin{aligned} [[a, b], c] &= \epsilon(a \star b) \star c - \epsilon(a, b)(b \star a) \star c - \epsilon(ab, c)c \star (a \star b) + \epsilon(ab, c)\epsilon(a, b)c \star (b \star a) \\ &= C_1(C_1(a, b), c) + C_2(a, b)c + C_2(ab, c) - \epsilon(a, b)(C_1(C_1(b, a), c) + C_2(b, a)c + C_2(ba, c)) \\ &\quad - \epsilon(ab, c)(C_1(c, C_1(a, b)) + C_2(c, ab) + cC_2(a, b)) \\ &\quad + \epsilon(ab, c)\epsilon(a, b)(C_1(c, C_1(b, a)) + C_2(c, ba) + cC_2(b, a)) \\ &= \{\{a, b\}, c\}. \end{aligned}$$

□

Two deformations \star, \star' are called equivalent if there exists a linear transformation $S = \text{id} + \sum_{r=1}^{\infty} t^r S_r$ of degree 0, such that

$$S(a \star' b) = S(a) \star S(b). \quad (23)$$

Proposition 3.3.2 *The color Poisson bracket of two equivalent deformations is the same.*

Proof. Let \star and \star' be the two star products, since they are equivalent we have that $a \star' b = S^{-1}(S(a) \star S(b))$ for some series $S = \sum S_i t^i$. Then a simple computation gives

$$a \star' b = ab + t[S_1(a)b + aS_1(b) + C_1(a, b) - S_1(ab)] + \mathcal{O}(t^2). \quad (24)$$

So the corresponding Poisson bracket is the same, since the terms involving S are symmetric. □

3. Cohomology and Deformations

Similarly to the non-color case, given a set of color commuting derivations $\{X_i\}$ and a matrix π , one can define a star product by

$$f \star g = \mu \circ e^{\pi^{ij} X_i \otimes X_j} (f \otimes g). \quad (25)$$

Here μ denotes the multiplication in \mathcal{A} . In order that \star is of order 0, π^{ij} can only be different from 0 if $\deg(X_i) + \deg(X_j) = 0$. The product is associative, because

$$\begin{aligned} f \star (g \star h) &= \mu \circ e^{t\pi_{ij} X_i \otimes X_j} (f \otimes \mu \circ e^{t\pi^{kl} X_k \otimes X_l} (g \otimes h)) \\ &= \mu \circ e^{t\pi^{ij} X_i \otimes X_j} \circ (\text{id} \otimes \mu) \circ (\text{id} \otimes e^{t\pi^{kl} X_k \otimes X_l}) (f \otimes g \otimes h) \\ &= \mu \circ (\mu \otimes \text{id}) \circ (e^{t\pi^{ij} X_i \otimes X_j \otimes \text{id} + X_i \otimes \text{id} \otimes X_j + \text{id} \otimes X_i \otimes X_j}) (f \otimes g \otimes h). \end{aligned}$$

Note that in the color case $e^{a+b} = e^a e^b$ only if $\deg a = \deg b = 0$, which here is the case since the product should be of order 0.

3.3.2 Deformations of color Lie bialgebras

We consider formal deformations of a color Lie bialgebra $(\mathfrak{g}, \beta, \delta)$.

Definition 3.3.3 A (formal) deformation of a color Lie bialgebra $(\mathfrak{g}, \beta, \delta)$ is a color $\mathbb{K}[[t]]$ -Lie bialgebra structure $(\beta_\star, \delta_\star)$ on the formal power series $\mathfrak{g}[[t]]$, such that $\beta_\star = \beta + \sum_{i=1}^{\infty} \beta_i t^i$ and $\delta_\star = \delta + \sum_{i=1}^{\infty} \delta_i t^i$. Here t has degree zero.

One can construct deformations order by order, meaning that if one knows a deformation $\delta_\star = \sum_{i=0}^n \delta_i$, $\beta_\star = \sum_{i=0}^n \beta_i$ up to a certain order n , one can search for β_{n+1} and δ_{n+1} such that $\delta_\star = \sum_{i=0}^{n+1} \delta_i$, $\beta_\star = \sum_{i=0}^{n+1} \beta_i$ is a deformation up to order $n+1$. Here a deformation up to order n means that the defining relations are satisfied mod t^{n+1} .

Proposition 3.3.4 *The order by order construction gives equations in the clba cohomology defined in Definition 3.2.1 and the obstructions for the existence of a clba deformation are in $H_r^3(\mathfrak{g})$. This means that given a deformation $\delta_\star = \sum_{i=0}^n \delta_i$, $\beta_\star = \sum_{i=0}^n \beta_i$ up to order n , $\delta_\star = \sum_{i=0}^{n+1} \delta_i$, $\beta_\star = \sum_{i=0}^{n+1} \beta_i$ is a deformation up to order $n+1$ if $\partial(\beta_{n+1} + \delta_{n+1}) = R_n := \sum_{i=1}^{n-1} [\delta_i + \beta_i, \delta_{n+1-i} + \beta_{n+1-i}]$, and R_n is cocycle. Furthermore if $H_r^3(\mathfrak{g}) = \{0\}$ then any deformation up to a certain order can be extended to a full deformation.*

Proof. We have

$$\{\beta_\star + \delta_\star, \beta_\star + \delta_\star\}_{\text{BB}} := \sum_{i=1}^{\infty} t^i \sum_{j+k=i} \{\beta_j + \delta_j, \beta_k + \delta_k\}_{\text{BB}} = \sum_{i=1}^{\infty} t^i \{\beta_\star + \delta_\star, \beta_\star + \delta_\star\}_{\text{BB}i}.$$

So we get

$$\partial(\beta_k + \delta_k) = \sum_{j=1}^{k-1} \{\beta_j + \delta_j, \beta_{k-j} + \delta_{k-j}\}_{\text{BB}}. \quad (26)$$

The graded Jacobi identity gives

$$0 = \{\beta_\star + \delta_\star, \{\beta_\star + \delta_\star, \beta_\star + \delta_\star\}_{\text{BB}}\}_{\text{BB}}. \quad (27)$$

So in order k we have

$$\begin{aligned} & \{\beta_\star + \delta_\star, \{\beta_\star + \delta_\star, \beta_\star + \delta_\star\}_{\text{BB}}\}_{\text{BB}} = \\ & \sum_j \{\beta_j + \delta_j, \{\beta_\star + \delta_\star, \beta_\star + \delta_\star\}_{\text{BB}k-j}\}_{\text{BB}} + \{\beta_0 + \delta_0, \{\beta_\star + \delta_\star, \beta_\star + \delta_\star\}_{\text{BB}k}\}_{\text{BB}} = 0. \end{aligned}$$

Since $\beta_\star, \delta_\star$ determine already a clba up to order $k-1$, and so the bracket of $\beta_\star + \delta_\star$ with itself vanishes up to order $k-1$. So finding β_k and δ_k is a problem in the cohomology $H_r^3(\mathfrak{g})$. \square

So especially $\partial(\beta_1 + \delta_1) = 0$. So $\beta_1 + \delta_1$ is a cocycle, which also means that β is a cocycle in clba cohomology and δ a cocycle in color Lie coalgebra cohomology.

Definition 3.3.5 (Equivalence of deformations) Two deformations $([\ , \], \delta)$ and $([\ , \]', \delta')$ are said to be equivalent if there exists a formal power series $S = \text{id} + \sum_{i=1}^{\infty} t^i S_i$ of linear maps $S_k : \mathfrak{g} \rightarrow \mathfrak{g}$ of Γ -degree 0, such that

$$\begin{aligned} S^{-1}([S(a), S(b)]) &= [a, b]', \\ (S^{-1} \otimes S^{-1})\delta(S(a)) &= \delta'(a). \end{aligned}$$

Proposition 3.3.6 *If two deformations are equivalent up to order k then $\partial(\beta'_{k+1} + \delta'_{k+1} - \beta_{k+1} - \delta_{k+1}) = 0$ and there exists an equivalence of order $k+1$ if the difference is exact. So the obstruction for equivalence is in $H_r^2(\mathfrak{g})$ and if $H_r^2(\mathfrak{g}) = \{0\}$ all deformations are equivalent to trivial deformations.*

Proof. We have that $\partial(\beta'_{k+1} + \delta'_{k+1})$ only depends on terms up to order k . Since these are the same in both clba structures, the difference vanishes.

Consider $S = \text{id} + t^k S_k$ then

$$\begin{aligned} S^{-1}([S(a), S(b)]) &= [a, b] + t^k ([S_k(a), b] + [a, S_k(b)] - S_k([a, b])) + \mathcal{O}(t^{k+1}) \\ &= [a, b] + t^k (\partial_\beta S_k)(a, b) + \mathcal{O}(t^{k+1}). \end{aligned}$$

Similarly for δ one gets

$$\begin{aligned} (S^{-1} \otimes S^{-1})\delta(S(a)) &= \delta(a) + t^k ((-S_k \otimes \text{id} - \text{id} \otimes S_k)\delta(a) + \delta(S_k(a))) + \mathcal{O}(t^{k+1}) \\ &= \delta(a) + t^k (\partial_\delta S_k)(a) + \mathcal{O}(t^{k+1}). \end{aligned}$$

3. Cohomology and Deformations

This shows that if the difference of $\delta + \beta$ and $\delta' + \beta'$ is exact there exists an equivalence S , as given above, such that $\delta + \beta$ and $\delta' + \beta'$ are equivalent up to order $k + 1$. \square

If all deformations are equivalent the clba is called rigid, this is for example the case when $H_r^2(\mathfrak{g}) = 0$.

The maps of Γ -degree 0 form a subcomplex, since the differential has Γ -degree 0. Note that only the cohomology of this subcomplex, which is the Γ -degree 0 part of the cohomology, is important for the deformation theory, since all operations are considered to be of degree 0.

Remark 3.3.7 In the case of quasi-Lie bialgebras, coquasi-Lie bialgebras or quasi-and-coquasi-Lie bialgebras, one has to consider different subcomplexes of the full double complex B^{ij} . The obstructions for deformations are, similarly to previous propositions, in these complexes. Together with the big bracket and the zero differential, these complexes are again differential graded Lie algebras and the Maurer-Cartan elements are precisely the corresponding algebra structures as defined in Remark 3.1.14.

To be precise for color quasi-Lie bialgebras, which is the most interesting case, one has to consider the complex, where the terms of the form $B^{0,i}$ are set to zero. The differential is again defined as the bracket with the Maurer-Cartan element (β, δ, φ) and the statements connecting deformations and cohomology stay true.

4. QUANTIZATION

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IN this section we give a generalization of the proof of Etingof and Kazhdan for the quantization of Lie bialgebras to the color case [EK96]. This has already been done for the super-case in [Gee06]. For this we use to some extent the description of graded modules as enriched category **Mod \mathfrak{g}** and the category of \mathfrak{g} -modules for a color Lie bialgebra \mathfrak{g} as enriched over **Mod \mathfrak{g}** . This is useful here, since this way one can consider the natural transformation to be graded.

4.1 QUANTUM UNIVERSAL ENVELOPING ALGEBRAS

4.1.1 Topologically free modules

In this section we consider \mathbb{K} to be a field of characteristic 0. We give some remarks on $\mathbb{K}[[t]]$ -modules, where $\mathbb{K}[[t]]$ denotes the ring of formal power series over \mathbb{K} . A $\mathbb{K}[[t]]$ -module is called topologically free if it is isomorphic to one of the form $V \hat{\otimes} \mathbb{K}[[t]]$ for a \mathbb{K} -module V . We will simply denote it by $V[[t]]$. Here $\hat{\otimes}$ denotes the completed tensor product with respect to the filtration by t or the t -adic topology.

A graded $\mathbb{K}[[t]]$ -module V is called free, if all V_i are free. In general this is not equivalent to the statement that $\bigoplus_{i \in \Gamma} V_i$ is free. But it is equivalent if only finitely many V_i are nonzero.

4.1.2 Quantum universal enveloping algebras

A color quantum universal enveloping algebra (cQUE) is a topological color Hopf algebra A over $\mathbb{K}[[t]]$, complete with respect to the t -adic topology, such that A/tA is the universal enveloping algebra $U(\mathfrak{g})$ of a color Lie algebra \mathfrak{g} , with the color Hopf algebra structure given after Definition 2.3.5. This means, it is a deformation of the universal enveloping algebra by a formal parameter t .

In the sequel, we write $\Delta = \sum_{i=0}^{\infty} t^i \Delta_i$ and $x \cdot y = \sum_{i=0}^{\infty} t^i x \cdot_i y$. We also need the opposite coproduct $\Delta^{\text{opp}} = \tau \circ \Delta$.

Proposition 4.1.1 *Let H be a cQUE then $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ given by*

$$\delta(x) = \frac{\Delta(x) - \Delta^{\text{opp}}(x)}{t} \pmod{t} \quad (1)$$

determines a color Lie bialgebra structure on \mathfrak{g} .

Proof. We have

$$\Delta([x, y]) = \Delta(x)\Delta(y) - \epsilon(x, y)\Delta(y)\Delta(x),$$

which in first order in t gives

$$\begin{aligned} \Delta_1([x, y]_0) + \Delta_0([x, y]_1) &= \Delta_1(x) \cdot_0 \Delta_0(y) + \Delta_0(x) \cdot_0 \Delta_1(y) + \Delta_0(x) \cdot_1 \Delta_0(y) \\ &\quad - \epsilon(x, y)(\Delta_1(y) \cdot_0 \Delta_0(x) + \Delta_0(y) \cdot_0 \Delta_1(x) + \Delta_0(x) \cdot_1 \Delta_0(y)). \end{aligned}$$

Doing the same for Δ^{opp} and subtracting it using $\Delta_0 = \Delta_0^{\text{opp}}$ gives

$$\delta_1([x, y]_0) = \delta_1(x) \cdot_0 \Delta_0(y) + \Delta_0(x) \cdot_0 \delta_1(y) - \epsilon(x, y)(\delta_1(y) \cdot_0 \Delta_0(x) + \Delta_0(y) \cdot_0 \delta_1(x)),$$

which is precisely the compatibility for δ and the Lie-bracket.

The cobracket δ also defines a Poisson coalgebra, the proof of this is dual to the proof of Definition and Proposition 3.3.1. This also shows that it satisfies the co-Jacobi identity.

We also have to show that δ , as defined above, has values in $\mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$. For this, we first note that $\mathfrak{g} \subset U(\mathfrak{g})$ consists precisely of the primitive elements, i.e. the elements x which satisfy $\Delta(x) = x \otimes 1 + 1 \otimes x$. This follows from the PBW-Theorem for color Lie algebras [Sch79]. We compute

$$\begin{aligned} (\text{id} \otimes \Delta_0)\delta(x) &= -(\text{id} \otimes \Delta_1)\Delta_0(x) + (\Delta_0 \otimes \text{id})\Delta_1(x) + (\Delta_1 \otimes \text{id})\Delta_0(x) \\ &\quad + (\text{id} \otimes \Delta_1^{\text{opp}})\Delta_0(x) - (\Delta_0 \otimes \text{id})\Delta_1^{\text{opp}}(x) - (\Delta_1^{\text{opp}} \otimes \text{id})\Delta_0(x) \\ &= (\Delta_0 \otimes \text{id})\delta(x) - 1 \otimes \Delta_1(x) + \Delta_1(x) \otimes 1 + 1 \otimes \Delta_1^{\text{opp}} - \Delta_1^{\text{opp}}(x) \otimes 1 \\ &= (\Delta_0 \otimes \text{id})\delta(x) - 1 \otimes \delta(x) + \delta(x) \otimes 1. \end{aligned}$$

Using that δ satisfies the co-Leibniz rule gives

$$\begin{aligned} (\text{id} \otimes \Delta_0)\delta(x) &= (\text{id} \otimes \delta)(x \otimes 1 + 1 \otimes x) - \tau_{23}(\delta \otimes \text{id})(x \otimes 1 + 1 \otimes x) + \delta(x) \otimes 1 - 1 \otimes \delta(x) \\ &= \tau_{23}(\delta(x) \otimes 1) + \delta(x) \otimes 1. \end{aligned}$$

So the second tensor factor of $\delta(x)$ is again primitive, which means it lies in \mathfrak{g} and by symmetry, we get $\delta(x) \in \mathfrak{g} \otimes \mathfrak{g}$. \square

So the semi-classical limit of the coproduct gives a color Lie cobracket, which is compatible with the bracket of the underlying color Lie algebra.

Definition 4.1.2 Let H be a cQUEA and \mathfrak{g} a color Lie bialgebra. Then H is a quantization of \mathfrak{g} if $H/tH \cong U(\mathfrak{g})$ and in addition

$$\delta = \frac{1}{t}(\Delta - \Delta^{\text{opp}}) \pmod{t}. \quad (2)$$

Further if (H, R) is quasitriangular and (\mathfrak{g}, r) a quasitriangular clba then (H, R) is called a quasitriangular quantization if $R \equiv 1 \otimes 1 + tr \pmod{t^2}$.

4.2 DRINFELD CATEGORY

4.2.1 Associators

We recall some facts about associators following [EK96]. Let T_n be the algebra over \mathbb{K} generated by symmetric elements t_{ij} for $1 \leq i, j \leq n, i \neq j$, satisfying the relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{lm}] = 0$ if i, j, l, m are distinct and $[t_{ij}, t_{ik} + t_{jk}] = 0$. For disjoint sets $P_1, \dots, P_n \subset \{1, \dots, m\}$, there exists a unique homomorphism $T_n \rightarrow T_m$ defined on

4. Quantization

generators by $t_{ij} \mapsto \sum_{p \in P_i, q \in P_j} t_{pq}$. We denote it by $X \mapsto X_{P_1, \dots, P_n}$. For $\Phi \in T_3$ the relation

$$\Phi_{1,2,34} \Phi_{12,3,4} = \Phi_{2,3,4} \Phi_{1,23,4} \Phi_{1,2,3} \quad (3)$$

is called the pentagon relation, and for $R = e^{\frac{t}{2} t_{12}} \in T_2[[t]]$ the relations

$$R_{12,3} = \Phi_{3,1,2} R_{1,3} \Phi_{1,2,2}^{-1} R_{2,3} \Phi_{1,2,3} \quad (4)$$

$$R_{1,23} = \Phi_{2,3,1}^{-1} R_{12,3} \Phi_{2,1,3} R_{12,3} \Phi_{1,2,3}^{-1} \quad (5)$$

are called the hexagon relations.

Definition 4.2.1 A (Drinfeld) associator is an element $\Phi \in T_3[[t]]$, which satisfies the pentagon identity Eq. (3) and the hexagon identities Eqs. (4) and (5).

There is the following well known theorem due to Drinfeld [Dri90]:

Theorem 4.2.2 *There exists an associator over \mathbb{Q} .*

This means that there is also an associator for every field which contains the rational numbers. In the following we will fix such an associator for \mathbb{K} .

For a color Lie algebra \mathfrak{g} , with a symmetric invariant element $\Omega = \Omega_1 \otimes \Omega_2$, we can define a map from T_n to $\text{End}(M_1 \otimes \dots \otimes M_n)$, by setting $t_{ij} \mapsto \Omega_{ij}$. Here Ω_{ij} is $1 \otimes \dots \otimes \Omega_1 \otimes \dots \otimes \Omega_2 \otimes \dots \otimes 1$ with the components of Ω in the i -th and j -th factor in the tensor product. If $i > j$, we have that $\Omega_{i,j} = \tau \Omega_{j,i}$. So the Ω_{ij} satisfy the same relations as the t_{ij} since Ω is invariant. As in the non-color case we get the following theorem.

One can show that the Drinfeld associator can be written involving only t_{12} and t_{23} [Dri90]. This allows us to consider the Drinfeld associator as an element of $F(x, y)$ the free group in two elements. To get the associator defined in the previous paragraph one sets $x = \Omega_{12}$ and $y = \Omega_{23}$.

Theorem 4.2.3 *Let \mathfrak{g} be a color Lie bialgebra and $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ a symmetric invariant element. We get a quasi-triangular color quasi-Hopf algebra $(U(\mathfrak{g})[[t]], \Delta, \varepsilon, \Phi, R)$, which we denote by $A_{\mathfrak{g}, \Omega}$.*

4.2.2 Drinfeld category

Let \mathfrak{g}_+ be a finite dimensional color Lie bialgebra, and $\mathfrak{g} = D(\mathfrak{g}_+) = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be the Drinfeld double of \mathfrak{g}_+ , with its Casimir Ω . Since Ω is invariant and symmetric, we get a quasi-triangular color quasi-Hopf algebra $A_{\mathfrak{g}, \Omega}$.

We define the category $\mathcal{M}_{\mathfrak{g}}$, whose objects are \mathfrak{g} -modules and whose morphisms are given by

$$\text{Hom}_{\mathcal{M}_{\mathfrak{g}}}(V, W) = \text{Hom}_{\mathfrak{g}}(V, W)[[t]]. \quad (6)$$

Note that we consider this category to be enriched over \mathbf{grVec} so the homomorphisms here are graded in general.

We equip $\mathcal{M}_{\mathfrak{g}}$ with the usual tensor product and symmetry given by $\beta_{V,W} : V \otimes W \rightarrow W \otimes V, v \otimes w \mapsto \tau \exp(\frac{\hbar t}{2})v \otimes w$ and associator

$$\Phi_{V,W,U} : (V \otimes W) \otimes U \rightarrow V \otimes (W \otimes U), v \otimes w \otimes u \mapsto \Phi \cdot v \otimes w \otimes u. \quad (7)$$

This is called Drinfeld category. In fact it is just the category of modules over the quasi-triangular color quasi-Hopf algebra $A_{\mathfrak{g},\Omega}$.

4.3 QUANTIZATION OF COLOR LIE BIALGEBRAS

Let \mathcal{A} be the category of topological free graded $\mathbb{K}[[\hbar]]$ -modules, considered as \mathbf{grVec} -category.

Let \mathfrak{g}_+ be a finite dimensional color Lie bialgebra and $\mathfrak{g} = D(\mathfrak{g}_+)$ be its double. We consider the Verma modules

$$M_+ = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_+)} c_+ \quad \text{and} \quad M_- = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_-)} c_-,$$

where c_{\pm} is the trivial one dimensional $U(\mathfrak{g}_{\pm})$ -module, concentrated in degree 0. The module structure on M_{\pm} comes from its definition as Verma module by acting on $U(\mathfrak{g})$.

Remark 4.3.1 It should be clear that given a graded algebra A , a right A -module M and a left A -module N , the tensor product $M \otimes_A N$ is well defined and again a graded vector space. If M , was in fact a B - A -bimodule, it is a left B -module, and similarly for N .

There is also a nice definition for the tensor product over A using a coend. A graded algebra A can be considered as a category \mathcal{A} enriched over \mathbf{grVec} with only one object, which we denote by $*$. A left (resp. right) A -module is precisely a functor from \mathcal{A} (resp. \mathcal{A}^{opp}) to \mathbf{grVec} . Then the tensor product over A can be defined as

$$M \otimes_A N := \int^{a \in \mathcal{A}} M(a) \otimes N(a), \quad (8)$$

where the integrals symbol denotes a coend.

Note that by the PBW theorem, we have an isomorphism $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-) \cong U(\mathfrak{g})$ of vector spaces, which is given by the multiplication in $U(\mathfrak{g})$ and in general $U(\mathfrak{g} \oplus \mathfrak{h}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{h})$ as vector space.

This implies that

$$M_{\pm} = U(\mathfrak{g}_{\mp})1_{\pm} \quad (9)$$

with 1_{\pm} in M_{\pm} . So M_{\pm} are free $U(\mathfrak{g}_{\mp})$ -modules.

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Lemma 4.3.2 *There exists an isomorphism $\varphi : U(\mathfrak{g}) \rightarrow M_+ \otimes M_-$ of $U(\mathfrak{g})$ -modules of degree 0 given on generators by $1 \mapsto 1_+ \otimes 1_-$.*

Proof. It is well defined by the universal property of $U(\mathfrak{g})$, as the extension of $x \mapsto x1_+ \otimes 1_- + 1_+ \otimes x1_-$. It is an isomorphism since $U(\mathfrak{g})$ and M_{\pm} can be regarded as free connected coalgebras and φ clearly is an isomorphism on the primitive elements. \square

Next we define a **grVec**-functor $F : \mathcal{M}_{\mathfrak{g}} \rightarrow \mathcal{A}$ by

$$F(V) = \text{Hom}_{\mathcal{M}_{\mathfrak{g}}}(M_+ \otimes M_-, V). \quad (10)$$

Since this is just a Hom-functor, its definition on morphisms is clear.

The isomorphism from Lemma 4.3.2 gives an isomorphism

$$\Psi_V : F(V) \rightarrow V[[t]], f \mapsto f(1_+ \otimes 1_-). \quad (11)$$

So the functor F is naturally isomorphic to the “forgetful” functor.

We want to show that F is a tensor functor, for this we need a natural transformation $J : \otimes_{\mathbb{K}[[t]]} \circ (F \otimes F) \rightarrow F \circ \otimes_{\mathcal{M}_{\mathfrak{g}}}$, which also satisfies $J_{U \otimes V, W} \circ (J_{U, V} \otimes \text{id}_W) = J_{U, V \otimes W} \circ (\text{id}_U \otimes J_{V, W})$.

Define $i_{\pm} : M_{\pm} \rightarrow M_{\pm} \otimes M_{\pm}$ by $1_{\pm} \mapsto 1_{\pm} \otimes 1_{\pm}$ and extended as \mathfrak{g} -module morphism. Clearly i_{\pm} is of degree 0.

Lemma 4.3.3 *The maps i_{\pm} are coassociative, i.e. $\Phi \circ (i_{\pm} \otimes \text{id}) \circ i_{\pm} = (\text{id} \otimes i_{\pm}) \circ i_{\pm}$.*

Proof. Following [EK96, Lemma 2.3]. We only prove the identity for i_+ , since the proof for i_- is analog.

Let $x \in M_+$. Then since the comultiplication in $U(\mathfrak{g}_-)$ is coassociative we have

$$(i_+ \otimes \text{id})i_+x = (\text{id} \otimes i_+)i_+x. \quad (12)$$

It is enough to show

$$\Phi \cdot (i_+ \otimes \text{id})i_+x = (i_+ \otimes \text{id})i_+x, \quad (13)$$

but since Φ is \mathfrak{g} invariant by definition, it is enough to show this for $x = 1_+$. This means $\Phi \cdot 1_+ \otimes 1_+ \otimes 1_+ = 1_+ \otimes 1_+ \otimes 1_+$. Which follows directly from the fact that Ω annihilates $1_+ \otimes 1_+$ and the definition of Φ . \square

We define J by

$$J_{V, W}(v, w) = (v \otimes w) \circ \Phi_{1,2,3,4}^{-1} \circ (\text{id} \otimes \Phi_{2,3,4}) \circ \beta_{2,3} \circ (\text{id} \otimes \Phi_{2,3,4}^{-1}) \circ \Phi_{1,2,3,4} \otimes (i_+ \otimes i_-) \quad (14)$$

for $v \in F(V), w \in F(W)$. Since all involved maps are of degree 0, J is also of degree 0.

The maps can be given by a diagram

$$\begin{aligned} M_+ \otimes M_- &\rightarrow (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \rightarrow M_+ \otimes ((M_+ \otimes M_-) \otimes M_-) \\ &\rightarrow M_+((\otimes M_- \otimes M_+) \otimes M_-) \rightarrow (M_+ \otimes M_-) \otimes (M_+ \otimes M_-) \rightarrow V \otimes W. \end{aligned} \quad (15)$$

Actually in the graded case there is a better definition:

$$\begin{aligned} J_{V,W} : \text{Hom}(M_+ \otimes M_-, V) \otimes \text{Hom}(M_+ \otimes M_-, W) &\xrightarrow{\otimes} \text{Hom}(M_+ \otimes M_- \otimes M_+ \otimes M_-, V \otimes W) \\ &\xrightarrow{\text{Hom}(\beta_{23}, \cdot)} \text{Hom}(M_+ \otimes M_+ \otimes M_- \otimes M_-, V \otimes W) \xrightarrow{\text{Hom}(i_+ \otimes i_-, \cdot)} \text{Hom}(M_+ \otimes M_-, V \otimes W). \end{aligned}$$

Theorem 4.3.4 *The functor F together with the natural transformation J forms a tensor functor.*

Proof. One only needs to check the equation

$$F(\Phi_{UVW}) \circ J_{U \otimes V, W}(J_{U, V} \otimes \text{id}) = J_{U, V \otimes W} \circ (\text{id} \otimes J_{V, W}). \quad (16)$$

The proof given for this in [ES02] is diagrammatically, so it also holds in the color case. \square

Let $\text{End}(F)$ be the color algebra of natural endomorphisms of F . In fact what we mean here is

$$\text{End}(F) = \int_{V \in \mathcal{M}_{\mathfrak{g}}} \text{Hom}(F(V), F(V)), \quad (17)$$

as stated in more detail in Appendix B. This is the end in the category \mathbf{grVec} enriched over itself, so it is a color vector space. We need to use this definition since in the ‘‘classical’’ one the natural transformations must consist of morphisms of degree 0. A natural transformation in this sense consist of a family η_V of graded maps $V \rightarrow V$, which satisfy the ‘‘normal’’ relation for a natural transformation but with additional signs, this means

$$\eta_W F(f) = \epsilon(\eta, f) F(f) \eta_V. \quad (18)$$

We say that $\eta \in \text{End}(F)$ is of degree i if all η_V are of degree i . So $\text{End}(F) = \bigoplus_{g \in \Gamma} \text{End}_g(F)$ is a graded vector space, where $\text{End}_g(F)$ consists of the natural transformations of degree g .

Proposition 4.3.5 *There is a canonical color algebra isomorphism*

$$\Theta : U(\mathfrak{g})[[t]] \rightarrow \text{End}(F), x \mapsto x \cdot, \quad (19)$$

where $x \cdot$ on the right denotes the action induced on every $U(\mathfrak{g})$ -module.

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Proof. $\text{End}(F)$ and $F(M_+ \otimes M_-)$ are isomorphic due to the Yoneda lemma for enriched categories. Since $M_+ \otimes M_-$ is isomorphic to $U(\mathfrak{g})$ as $U(\mathfrak{g})$ -module, $\text{End}(F)$ is in fact isomorphic to $U(\mathfrak{g})$.

There is a more explicit way of proving this following [Gee06]:

We can identify $F(V)$ with $V[[t]]$. The map Θ is injective since the action of $U(\mathfrak{g})$ on itself is injective. We want to show that is surjective. Let $\eta \in \text{End}(F)$, and we identify η_V with a map $V[[t]] \rightarrow V[[t]]$. We define $x := \eta_{U(\mathfrak{g})}(1)$. We claim that $\eta_V = x \cdot$. For $y \in U(\mathfrak{g})$ we define $r_y \in \text{End}(U(\mathfrak{g}))$ by $r_y(z) = \epsilon(y, z)zy$ for $z \in U(\mathfrak{g})$. We have

$$\eta_{U(\mathfrak{g})}(y) = \eta_{U(\mathfrak{g})}(r_y 1) = \epsilon(x, y)r_y \eta_{U(\mathfrak{g})}(1) = \epsilon(x, y)r_y x = xy. \quad (20)$$

So we get $\eta_{U(\mathfrak{g})} = x \cdot$. Similarly one shows that $\eta_V = x \cdot$ for any free \mathfrak{g} -modules V and since any \mathfrak{g} -module is a quotient of a free one the claim follows. \square

We define $J \in U(\mathfrak{g}) \otimes U(\mathfrak{g})[[t]]$ by

$$J = (\varphi^{-1} \otimes \varphi^{-1})(\Phi_{1,2,3,4}^{-1}(1 \otimes \Phi_{2,3,4})\beta_{2,3}(\text{id} \otimes \Phi_{2,3,4})\Phi_{1,2,3,4}(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-)). \quad (21)$$

This means $J_{U(\mathfrak{g}), U(\mathfrak{g})}(\varphi^{-1} \otimes \varphi^{-1})(1_+ \otimes 1_-)$. With this the natural transformation J can be identified with the action of the element J .

Proposition 4.3.6 *Using Ψ as defined in Eq. (11), we have*

$$J \cdot (v \otimes w) = \Psi_{V \otimes W}(J_{V,W}(\Psi_V^{-1}(v) \otimes \Psi_W^{-1}(w))) \quad (22)$$

for $v \in V[[t]]$, $w \in W[[t]]$.

Proof. For each $v \in V[[t]]$ we define $f_v : M_+ \otimes M_- \rightarrow V$ by $f_v(x) = \epsilon(v, x)x \cdot v$. Then $f_v(1_+ \otimes 1_-) = v$, since $1_+ \otimes 1_-$ is the unit in $M_+ \otimes M_-$. So we have $f_v = \Psi^{-1}(v)$ Let $\theta_1 \otimes \theta_2 := (\varphi \otimes \varphi)J \in (M_+ \otimes M_-)^{\otimes 2}$. Then the right hand side gives

$$(J_{V,W}(\Psi_V^{-1}(v) \otimes \Psi_W^{-1}(w)))(1_+ \otimes 1_-) = (f_v \otimes f_w)(\theta_1 \otimes \theta_2) = \epsilon(w, \theta_1)f_v(\theta_1) \otimes f_w(\theta_2) \quad (23)$$

$$= \epsilon(w, \theta_1)\epsilon(v, \theta_1)\epsilon(w, \theta_2)\varphi^{-1}(\theta_1) \cdot v \otimes \varphi^{-1}(\theta_2) \cdot w \quad (24)$$

and the left hand side gives

$$\varphi^{-1}(\theta_1) \otimes \varphi^{-1}(\theta_2)(v \otimes w) = \epsilon(\theta_2, v)\varphi^{-1}(\theta_1) \cdot v \otimes \varphi^{-1}(\theta_2) \cdot w.$$

These two are equal since J and with this $\theta_1 \otimes \theta_2$ are of degree 0. \square

Lemma 4.3.7 *We have*

$$J \equiv 1 + \frac{t}{2}r \pmod{t^2}. \quad (25)$$

Proof. Recall that $r = \sum m_i \otimes p_i$, where the p_i form a basis of \mathfrak{g}_+ and the m_i form the corresponding dual basis of \mathfrak{g}_- . We have $\tau(r) \cdot (1_- \otimes 1_+) = 0$, since p_i acts trivially on 1_+ . So we have using $\Phi \equiv 1 \pmod{t^2}$

$$\begin{aligned} J &\equiv (\varphi^{-1} \otimes \varphi^{-1}) e^{t\Omega_{23}/2} (1_+ \otimes 1_- \otimes 1_+ \otimes 1_-) \pmod{t^2} \\ &\equiv 1 + \frac{t}{2} (\varphi^{-1} \otimes \varphi^{-1}) (r_{23} + \tau(r_{23})) (1_+ \otimes 1_- \otimes 1_+ \otimes 1_-) \pmod{t^2} \\ &\equiv 1 + \frac{t}{2} (\varphi^{-1} \otimes \varphi^{-1}) (1_+ \otimes p_i 1_- \otimes m_i 1_+ \otimes 1_-) \pmod{t^2} \\ &\equiv 1 + \frac{t}{2} p_i \varphi^{-1} (1_+ \otimes 1_-) \otimes m_i \varphi^{-1} (1_+ \otimes 1_-) \pmod{t^2} \\ &\equiv 1 + \frac{t}{2} r \pmod{t^2}. \end{aligned}$$

□

Definition 4.3.8 We can now define a color Hopf algebra H on $U(\mathfrak{g})[[t]]$ by

$$\Delta = J^{-1} \Delta_0 J, \quad \varepsilon = \varepsilon_0 \text{ and } S = Q S_0 Q^{-1},$$

with $Q = \mu(S_0 \otimes \text{id})J$.

We want to give the corresponding color Hopf algebra structure on $\text{End}(F)$ under the isomorphisms Θ . For this we first need:

Lemma 4.3.9 *We have*

$$\text{End}(F) \otimes \text{End}(F) = \text{End}(F \otimes F) \quad (26)$$

as algebras. Here $F \otimes F : \mathcal{M}_{\mathfrak{g}} \otimes \mathcal{M}_{\mathfrak{g}} \rightarrow \mathcal{A}$ denotes the functor defined by $(F \otimes F)(V, W) = F(V) \otimes F(W)$.

Proof. This can be seen using the enriched Yoneda Lemma for $\mathcal{M}_{\mathfrak{g}}$ and $\mathcal{M}_{\mathfrak{g}} \otimes \mathcal{M}_{\mathfrak{g}}$ since we have $\text{End}(F) = \text{Hom}(U(\mathfrak{g}), U(\mathfrak{g})) \cong U(\mathfrak{g})$ and $\text{End}(F \otimes F) = \text{Hom}(U(\mathfrak{g}), U(\mathfrak{g})) \otimes \text{Hom}(U(\mathfrak{g}), U(\mathfrak{g})) \cong \text{Hom}(U(\mathfrak{g}), U(\mathfrak{g}))$. This shows that $\text{End}(F) \otimes \text{End}(F) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g}) \cong \text{End}(F \otimes F)$. □

There is a natural coproduct on $\text{End}(F)$ given by

$$\Delta(a)_{V,W} = J_{V,W}^{-1} a_{V \otimes W} J_{V,W} \quad (27)$$

and $\text{End}(F)$ becomes a bialgebra with it. In fact it is a Hopf algebra. This follows directly from the fact that (F, J) is a tensor functor, which gives that J is a twist. The twisted coproduct is precisely the one given here and the twisted Φ is trivial. We have that $H \cong \text{End}(F)$ as Hopf algebra.

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Proposition 4.3.10 *The Hopf algebra H is a quantization of the color Lie bialgebra \mathfrak{g} . Moreover we define an R -matrix on H by*

$$R = (J^{\text{opp}})^{-1} e^{\frac{t}{2}\Omega}. \quad (28)$$

Then (H, R) is a quasitriangular quantization of (\mathfrak{g}, r) .

Proof. By definition H/tH is isomorphic to $U(\mathfrak{g})$ as color Hopf algebra. From Lemma 4.3.7 and the definition of the coproduct it follows that $\Delta_1(x) = \frac{1}{2}r\Delta_0(x) + \frac{1}{2}\Delta_0(x)r = \frac{1}{2}[\Delta_0(x), r]$, so we have $\delta(x) = \Delta_1(x) - \Delta_1^{\text{opp}}(x) = \frac{1}{2}[\Delta_0(x), r - \tau(r)] = [\Delta_0(x), r]$, since $r + \tau(r) = \Omega$, which is \mathfrak{g} invariant.

The element R is an R -matrix, because it is obtained by twist. From Lemma 4.3.7 we immediately get $R \equiv 1 \otimes 1 + t(\frac{1}{2}r - \frac{1}{2}\tau(r) + \frac{1}{2}\Omega) \equiv 1 \otimes 1 + rt \pmod{t^2}$. \square

4.3.1 Quantization of \mathfrak{g}_+ and \mathfrak{g}_-

As shown before, we have $\text{End}(F) \cong \text{End}(M_+ \otimes M_-)$, since both are isomorphic to $U(\mathfrak{g})$ as color algebras. So we can define $U_t(\mathfrak{g}_+) = F(M_-)$ and embed it into H via the map $i : F(M_-) \rightarrow \text{End}(M_+ \otimes M_-)$ given by

$$i(x) = (\text{id} \otimes x) \circ \Phi \circ (i_+ \otimes \text{id}) \quad (29)$$

for $x \in F(M_-)$. Then i is injective, and satisfies

$$i(x) \circ i(y) = i(z) \quad (30)$$

for $x, y \in F(M_-)$ and $z = x \circ (\text{id} \otimes y) \circ \Phi \circ (i_+ \otimes \text{id}) \in F(M_-)$. So $U_t(\mathfrak{g}_+)$ is a color subalgebra of H .

We next want to show that it is indeed a color Hopf subalgebra, for this we need:

Proposition 4.3.11 *The R -matrix of H is polarized which means that $R \in U_t(\mathfrak{g}_+) \otimes U_t(\mathfrak{g}_-) \subset H \otimes H$.*

Proof. Following [ES02, Lemma 19.4]. The defining equation of R is equivalent to

$$R \circ \beta_{23}^{-1} \circ (i_+ \otimes i_-) = \beta_{23} \circ (i_+ \otimes i_-) \quad (31)$$

in $\text{Hom}(M_+ \otimes M_-, M_+ \otimes M_- \otimes M_+ \otimes M_-)$, where we regard R as an element in $\text{Hom}(M_+ \otimes M_- \otimes M_+ \otimes M_-, M_+ \otimes M_- \otimes M_+ \otimes M_-)$.

Using the counit we have

$$\beta_{23}(i_+ \otimes i_-) = (\text{id} \otimes \varepsilon \text{id} \otimes \text{id} \otimes \varepsilon \text{id}) \beta_{34}(\text{id} \otimes i_+ \otimes i_- \otimes \text{id})(i_+ \otimes i_-)$$

and using Eq. (31) in the middle and then the coassociativity of i_+ and i_- we further get

$$\begin{aligned}\beta_{23}(i_+ \otimes i_-) &= (\text{id} \otimes \varepsilon \text{id} \otimes \text{id} \otimes \varepsilon \text{id} \otimes \text{id})(\text{id} \otimes R \otimes \text{id})\beta_{34}(\text{id} \otimes i_+ \otimes i_- \otimes \text{id})(i_+ \otimes i_-) \\ &= (\text{id} \otimes \varepsilon \text{id} \otimes \text{id} \otimes \varepsilon \text{id} \otimes \text{id})(\text{id} \otimes R \otimes \text{id})\beta_{34}(i_+ \otimes \text{id} \otimes \text{id} \otimes i_-)(i_+ \otimes i_-).\end{aligned}$$

So we have proved

$$R = (\text{id} \otimes \varepsilon \otimes \text{id} \otimes \text{id} \otimes \varepsilon \otimes \text{id}) \circ (\text{id} \otimes R \otimes \text{id}) \circ (i_+ \otimes \text{id} \otimes \text{id} \otimes i_-).$$

□

We define maps $p_{\pm} : U_t(\mathfrak{g}_{\mp})^* \rightarrow U_t(\mathfrak{g}_{\pm})$ by

$$p_+(f) = (\text{id} \otimes f)(R) \text{ resp. } p_-(g) = (g \otimes \text{id})(R). \quad (32)$$

Let $\text{im } p_{\pm}$ be the image of p_{\pm} and U_{\pm} be the color algebra generated by it, then we have:

Lemma 4.3.12 *The algebra U_{\pm} is closed under the coproduct and the antipode, so it is a Hopf-algebra.*

Proof. Using the hexagon identity, it is easy to set that for $x = (\text{id} \otimes f)(R) \in \text{im } p_{\pm}$ the element $\Delta(x)$ is in $U_{\pm} \otimes U_{\pm}$. □

Lemma 4.3.13 *We have $U_t(\mathfrak{g}_{\pm}) \otimes_{\mathbb{K}[[t]]} \mathbb{K}((t)) = U_{\pm} \otimes_{\mathbb{K}[[t]]} \mathbb{K}((t))$.*

Proof. The proof in [ES02, Lemma19.5] works also in the color case. □

With this we can now prove the main theorem of this section, which shows that the construction given can be used to quantize every finite dimensional Lie bialgebra.

Theorem 4.3.14 *$U_t(\mathfrak{g}_{\pm})$ is a color Hopf algebra and a quantization of \mathfrak{g}_{\pm} .*

Proof. From the previous lemmas it follows that, it is a color Hopf subalgebra of U_t . □

4.4 QUANTIZATION OF TRIANGULAR COLOR LIE BIALGEBRAS

Let \mathfrak{a} be a not necessarily finite dimensional triangular color Lie bialgebra, then we define $\mathfrak{g}_+ := \{(1 \otimes f)(r), f \in \mathfrak{a}^*\}$, $\mathfrak{g}_- := \{(f \otimes 1)(r), f \in \mathfrak{a}^*\}$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. One can identify \mathfrak{g}_- with \mathfrak{g}_+ via the map $\chi(f) = (f \otimes \text{id})(r)$. Then one can define a Lie bracket

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on \mathfrak{g} , such that for $x, y \in \mathfrak{g}_\pm$ it is the Lie bracket in \mathfrak{g}_\pm and for $x \in \mathfrak{g}_+, y \in \mathfrak{g}_-$ it is defined by

$$[x, y] := (\text{ad}^* x)(y) - \epsilon(x, y)(\text{ad}^* y)(x). \quad (33)$$

One can define a map $\pi : \mathfrak{g} \rightarrow \mathfrak{a}$, such that the restriction to \mathfrak{g}_+ and \mathfrak{g}_- is the embedding. With this one has

$$\pi([x, y]) = [\pi(x), \pi(y)]. \quad (34)$$

Proposition 4.4.1 *The bracket on \mathfrak{g} defined above actually is a color Lie bracket, and the natural pairing gives an invariant inner product, so in fact a color Manin triple.*

Let $\mathcal{M}_{\mathfrak{a}}$ be the category of \mathfrak{a} -modules, with morphisms given by $\text{Hom}(V, W) = \text{Hom}_{\mathfrak{a}}(V, W)[[t]]$, this again can be viewed as a category enriched over \mathbf{grVec} . Using the morphisms π one can define the pullback functor $\pi^* : \mathcal{M}_{\mathfrak{a}} \rightarrow \mathcal{M}_{\mathfrak{g}}$ to the Drinfeld category of \mathfrak{g} . One can also pullback the monoidal structure along this functor.

$\Omega := r + \tau r$ is \mathfrak{g} invariant, this is needed to pullback the monoidal structure.

Using the pullback functor π^* and the Verma modules defined before one can define a functor $F : \mathcal{M}_{\mathfrak{a}} \rightarrow \mathscr{A}$ by

$$F(V) := \text{Hom}_{\mathcal{M}_{\mathfrak{b}}}(M_+ \otimes M_-, \pi^*(V)). \quad (35)$$

The functor F is again isomorphic to the forgetful functor, and so we have $H := \text{End}(F) = U(\mathfrak{a}[[t]])$. In the same way as before one can define a tensor structure on F , and with this a deformed bialgebra on H .

Note that if \mathfrak{a} was in fact triangular then, we have $\Omega = 0$ and the Hopf algebra is also triangular.

So essentially following the construction in the previous section, one gets the following

Theorem 4.4.2 *Any quasitriangular color Lie bialgebra admits a quasitriangular quantization $U_t(\mathfrak{g})$, and if \mathfrak{g} is triangular, the quantization is also triangular.*

4.5 SECOND QUANTIZATION OF COLOR LIE BIALGEBRAS

4.5.1 Topological spaces

We recall some basic facts about topological vector spaces in the non-graded setting, which we will need in the following.

Let F be a space of functions into a topological space, then the weak topology is the initial topology with respect to the evaluation maps. Let V and W be topological vector spaces. We use on $\text{Hom}(V, W)$ the weak topology, for which a basis is given by

$$\{f \in \text{Hom}(V, W) \mid f(v_i) \in U_i, i = 1, \dots, n\}_{U_1, \dots, U_n, v_1, \dots, v_n}, \quad (36)$$

where U_i are open sets in W and v_i are elements in V . Let \mathbb{K} be a field of characteristic zero, with the discrete topology, and V a topological vector space over \mathbb{K} . Then its topology is called linear if the open subspaces form a basis of neighborhoods of 0.

Let V be a topological vector space with a linear topology, then V is called separated if the map $V \rightarrow \text{projlim}_{U \text{ open subspace}} (V/U)$ is injective, this is e.g. the case when V is discrete, i.e. 0 is an open set. All topological vector spaces we consider will be linear and separated, so we will just call them topological vector spaces.

If V is finite dimensional than the weak topology on $\text{Hom}(V, \mathbb{K})$ is the discrete topology. In general a neighborhood basis of zero is given by cofinite dimensional subspaces.

A topological vector space is called complete if the canonical map $V \rightarrow \text{projlim}_{U \text{ open subspace}} (V/U)$ is surjective.

Let V and W be topological vector spaces, we define the topological tensor product by

$$V \hat{\otimes} W = \text{projlim } V/V' \otimes W/W', \quad (37)$$

where V', W' run over open subspaces of V resp. W . With this we have:

Proposition 4.5.1 *Complete vector spaces with continuous linear maps form a symmetric monoidal category.*

Topology and grading

A topological color vector space is a color vector space $V = \bigoplus_{i \in \Gamma} V_i$, where each space V_i is a topological vector space. A linear map between topological color vector spaces is continuous if each homogeneous part is continuous.

For a graded vector space we say that it is complete, separated or has a given property if every space V_i has this property. The tensor product can also be defined by replacing the usual tensor product over vector spaces by the completed one. We note that the tensor product involves the direct sum $\bigoplus_{j \in \Gamma} V_i \otimes W_{i-j}$, for which a priori it is not clear whether it is complete. But it turns that here since the considered topologies are linear, this is the case.

In fact using the construction, which defines graded vector spaces as functors, one just replaces the category of vector spaces by the category of complete vector spaces and gets a category of graded complete vector spaces, which is again monoidal due to Proposition 4.5.1.

4.5.2 Manin triples

Let \mathfrak{a} be a color Lie bialgebra with discrete topology, i.e. each \mathfrak{a}_i is equipped with the discrete topology, and \mathfrak{a}^* its dual with the weak topology. Since \mathfrak{a} is discrete \mathfrak{a}^*

is the full graded dual. The cocommutator defines a continuous Lie bracket on \mathfrak{a}^* . We have a natural topology on $\mathfrak{a} \oplus \mathfrak{a}^*$ and the above defines a continuous Lie bracket with respect to this topology.

Let \mathfrak{g} be a Lie algebra, with a nondegenerate inner product $\langle \cdot, \cdot \rangle$ and \mathfrak{g}_+ and \mathfrak{g}_- be two isotropic Lie subalgebras, i.e. $\langle \mathfrak{g}_+, \mathfrak{g}_+ \rangle = 0$, such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Then the inner product defines an embedding $\mathfrak{g}_- \rightarrow \mathfrak{g}_+^*$. To get a topology on \mathfrak{g} we equip \mathfrak{g}_+ with the discrete topology and \mathfrak{g}_- with the weak topology. If the Lie bracket on \mathfrak{g} is continuous in this topology we call $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ a Manin triple.

To every color Lie bialgebra \mathfrak{a} one can associate a color Manin triple by $(\mathfrak{a} \oplus \mathfrak{a}^*, \mathfrak{a}, \mathfrak{a}^*)$, and conversely every color Manin triple gives a Lie bialgebra on \mathfrak{g}_+ . Note that here different from in the finite dimensional case the Manin triple is no longer symmetric in \mathfrak{g}_+ and \mathfrak{g}_- .

4.5.3 Equicontinuous \mathfrak{g} -modules

Let M be a topological vector space and $\{A_x\}_{x \in X}$ a family in $\text{End } M$ indexed by an arbitrary set X . Then $\{A_x\}_{x \in X}$ is equicontinuous if for all open neighborhoods U of 0 in M there exists a open neighborhood V such that $A_x V = U$ for all $x \in X$.

Definition 4.5.2 Let M be a complete topological color vector space then we call M an equicontinuous \mathfrak{g} -module if there is a continuous color Lie algebra morphism $\pi : \mathfrak{g} \rightarrow \text{End}(M)$ such that $\{\pi(\mathfrak{g})\}$ is an equicontinuous family.

For two equicontinuous \mathfrak{g} -modules M, N , we have that $M \hat{\otimes} N$ is again an equicontinuous \mathfrak{g} -module. Further $(V \otimes W) \otimes U$ can be identified with $V \otimes (W \otimes U)$ and $V \otimes W$ with $W \otimes V$ by the flip. So we can define the symmetric monoidal category \mathcal{M}_0^e of equicontinuous \mathfrak{g} -modules.

We define again the Verma modules M_{\pm} by $M_{\pm} = \text{Ind}_{\mathfrak{g}_{\pm}}^{\mathfrak{g}} 1 = U(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g}_{\pm})} 1$.

Lemma 4.5.3 *The module M_- equipped with the discrete topology is an equicontinuous \mathfrak{g} -module.*

Proof. This is true in the non-graded case, see e.g. [EK96], so it also holds in the color case since it can be checked in each degree. \square

We want to define a topology on M_+ , for this we first define a topology on $U(\mathfrak{g}_-)$. We have $U_n(\mathfrak{g}_-) \cong \bigoplus_{k \leq n} S^k \mathfrak{g}_-$. We equip $S^k \mathfrak{g}_-$ with the weak topology coming from the embedding into $(\mathfrak{g}_+^{\otimes k})^*$. This gives a topology on $U_n(\mathfrak{g}_-)$. Finally we put on $U(\mathfrak{g}_-)$, and with this on M_+ , the topology coming from the inductive limit $\lim U_n(\mathfrak{g}_-) = U(\mathfrak{g}_-)$. Similar to [EK96] we get:

Lemma 4.5.4 *For all $g \in \mathfrak{g}$ the map $\pi_{M_+}(g) : M_+ \rightarrow M_+$ is continuous.*

Next we need a topology on M_+^* . For this we note that $U_n(\mathfrak{g}_-)^* \cong \bigoplus_{k \leq n} S^k \mathfrak{g}_+$, so we equip $U_n(\mathfrak{g}_-)^*$ with the discrete topology. Since $U(\mathfrak{g}_-)^*$ is the projective limit of $U_n(\mathfrak{g}_-)^*$, it carries the corresponding topology.

Lemma 4.5.5 M_+^* is an equicontinuous \mathfrak{g} -module.

However M_+ is not equicontinuous in general.

Let $r \in \mathfrak{a} \hat{\otimes} \mathfrak{a}^*$ be the element corresponding to the identity under the isomorphism $\mathfrak{a} \hat{\otimes} \mathfrak{a}^* \rightarrow \text{End}(\mathfrak{a})$. Further we have $r^{\text{opp}} \in \mathfrak{a}^* \hat{\otimes} \mathfrak{a}$ then one can define a Casimir element $\Omega = r + r^{\text{opp}} \in \mathfrak{a} \hat{\otimes} \mathfrak{a}^* \oplus \mathfrak{a}^* \hat{\otimes} \mathfrak{a} \subset \mathfrak{g} \otimes \mathfrak{g}$.

Let $\mathcal{M}_{\mathfrak{g}}^t$ be the category of equicontinuous \mathfrak{g} -modules and morphisms

$$\text{Hom}_{\mathcal{M}_{\mathfrak{g}}^t}(V, W) = \text{Hom}_{\mathfrak{g}}(V, W)[[t]], \quad (38)$$

where the Hom on the right denotes the continuous \mathfrak{g} -module morphisms.

We define a natural transformation γ by $\gamma_{V,W} = \beta_{W,V}^{-1} \in \text{Hom}(V \otimes W, W \otimes V)$ for $V, W \in \mathcal{M}^e$.

Using the completed tensor product, we can define the structure of a braided monoidal category on $\mathcal{M}_{\mathfrak{g}}^t$ using Φ and γ similarly to Section 4.3.

4.5.4 Tensor functor F

Let V be a complete color space over \mathbb{K} then the space $V \hat{\otimes} \mathbb{K}[[t]]$ is again a complete color space, and carries a natural structure of a topological color $\mathbb{K}[[t]]$ -module. A $\mathbb{K}[[t]]$ -module is called complete if it is isomorphic to $V[[t]]$ for a complete color space V .

Let \mathcal{A}^c be the category of complete color $\mathbb{K}[[t]]$ -modules. On complete $\mathbb{K}[[t]]$ -modules we define a tensor product $V \tilde{\otimes} W = V \hat{\otimes} W / \langle 1 \otimes t - t \otimes 1 \rangle$. And with this a color $\mathbb{K}[[t]]$ -modules by $(V \tilde{\otimes} W)_i = \bigoplus_{j \in \Gamma} (V_j \tilde{\otimes} W_{i-j})$.

We define a functor F from the category $\mathcal{M}_{\mathfrak{g}}^t$ of equicontinuous \mathfrak{g} -modules to the category of complete color vector spaces \mathcal{A}^c , by $V \mapsto \text{Hom}(M_-, M_+^* \otimes V)$.

We can define a comultiplication on M_+^* by

$$i_+^* : M_+^* \otimes M_+^* \rightarrow M_+^* : i_+^*(f \otimes g)(x) := (f \otimes g)(i_+(x)) \quad (39)$$

for $f, g \in M_+^*$. This is continuous.

Proposition 4.5.6 There are isomorphisms $\Psi_V : F(V) \rightarrow V$ for all $V \in \mathcal{M}_{\mathfrak{g}}^t$ natural in V , given by

$$f \mapsto (\text{ev}(1_+) \otimes \text{id})f(1_-) \quad (40)$$

Proof. This follows from Frobenius reciprocity. \square

4. Quantization

This shows that F is natural isomorphic to the forgetful functor.

We now define a tensor structure on the functor F . Similar to (21) we define, $J_{V \otimes W}(v \otimes w)$ by

$$\begin{aligned} M_- &\xrightarrow{i_-} M_- \otimes M_- \xrightarrow{v \otimes w} (M_+^* \otimes V) \otimes (M_+^* \otimes W) \xrightarrow{\Phi} M_+^* \otimes ((V \otimes M_+^*) \otimes W) \\ &\xrightarrow{\gamma_{23}} M_+^* \otimes ((M_+^* \otimes V) \otimes W) \xrightarrow{\Phi} (M_+^* \otimes M_+^*) \otimes (V \otimes W) \xrightarrow{i_+^* \otimes \text{id} \otimes \text{id}} M_+^* \otimes (V \otimes W). \end{aligned}$$

That is (without associators)

$$J_{VW}(v \otimes w) = (i_+^* \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \gamma \otimes \text{id}) \circ (v \otimes w) \circ i_-. \quad (41)$$

Again one can write down a version without using explicitly the maps v and w , which do not exist in the categorical sense for the graded case.

Lemma 4.5.7 *We have*

$$\Phi \circ (i_- \otimes \text{id}) \circ i_- = (\text{id} \otimes i_-) \circ i_- \quad (42)$$

and

$$\Phi \circ (i_+^* \otimes \text{id}) \circ i_+^* = (\text{id} \otimes i_+^*) \circ i_+^*, \quad (43)$$

i.e. i_- and i_+^* are coassociative in $\text{Hom}(M_-, (M_-)^{\otimes 3})$ and $\text{Hom}(M_+^*, (M_+^*)^{\otimes 3})$ resp.

Proposition 4.5.8 *The maps J_{VW} are isomorphisms and define a tensor structure on F .*

Proof. They are isomorphisms because they are isomorphisms modulo t . \square

4.5.5 Quantization of color Lie bialgebras

Let $H = \text{End}(F)$ be the algebra of endomorphisms of the fiber functor F , where $\text{End}(F)$ is again to be understood in the enriched sense. Let H_0 be the endomorphism algebra of the forgetful functor from \mathcal{M}_0^e to the category of complete color vector spaces. The algebra H is naturally isomorphic to $H_0[[t]]$.

Let $F^2 : \mathcal{M}^e \times \mathcal{M}^e \rightarrow \mathcal{A}^c$ be the bifunctor defined by $F^2(V, W) = F(V \otimes W)$ and $H^2 = \text{End}(F^2)$ then $H \otimes H \subset H^2$ but not necessarily $H^2 = H \otimes H$.

H has a ‘‘comultiplication’’ $\Delta : H \rightarrow H^2$, defined by

$$\Delta(a)_{VW}(v \otimes w) = J_{VW}^{-1} a_{V \otimes W} J_{VW}(v \otimes w). \quad (44)$$

For $x \in F(M_-)$ we define $m_+(x) \in \text{End}(F)$ by

$$m_+(x)(v) = \epsilon(x, v) i_+^* \otimes \text{id} \circ \Phi^{-1} \circ (\text{id} \otimes v) \circ x \quad (45)$$

for $v \in F(V)$.

We define $U_t(g_+) \subset H$ as the image of m_+ . The map m_+ is an embedding since it is so modulo t .

Proposition 4.5.9 $U_t(\mathfrak{g}_+)$ is a subalgebra of H .

Proof. Since i_+^* is coassociative we get

$$\begin{aligned}
 m_+(x)m_+(y)(v) &= \epsilon(x, yv)\epsilon(y, v)(i_+^* \otimes \text{id})\Phi^{-1}(\text{id} \otimes i_+^* \otimes \text{id})\Phi_{2,3,4}^{-1}(\text{id} \otimes \text{id} \otimes v)(\text{id} \otimes y)x \\
 &= \epsilon(x, y)\epsilon(x, v)\epsilon(y, v)(i_+^* \otimes \text{id})(i_+^* \otimes \text{id} \otimes \text{id})\Phi_{1,2,3}^{-1}\Phi_{1,23,4}^{-1}\Phi_{2,3,4}^{-1}(\text{id} \otimes \text{id} \otimes v)(\text{id} \otimes y)x \\
 &= \epsilon(x, y)\epsilon(xy, v)(i_+^* \otimes \text{id})(i_+^* \otimes \text{id} \otimes \text{id})\Phi_{12,3,4}^{-1}\Phi_{12,3,4}^{-1}(\text{id} \otimes \text{id} \otimes v)(\text{id} \otimes y)x \\
 &= \epsilon(x, y)\epsilon(xy, v)(i_+^* \otimes \text{id})\Phi^{-1}(\text{id} \otimes v)(\otimes i_+^* \otimes \text{id})\Phi^{-1}(\text{id} \otimes y)x \\
 &= \epsilon(xy, v)(i_+^* \otimes \text{id})\Phi^{-1}(\text{id} \otimes v)z \\
 &= i(z)v,
 \end{aligned}$$

where $z = (\otimes i_+^* \otimes \text{id})\Phi^{-1}(\text{id} \otimes y)x$. □

Proposition 4.5.10 The algebra $U_t(\mathfrak{g}_+)$ is closed under the coproduct.

Proof. The proof in [ES02, Section 21.2] is pictorial so it can also be used in the color case.

The element $\Delta(m_+(x))$ is uniquely defined by the equation

$$\begin{aligned}
 (i_+^* \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes i_+^* \otimes \text{id} \otimes \text{id}) \circ \gamma_{34} \circ (\text{id} \otimes v \otimes w) \circ (\text{id} \otimes i_-) = \\
 (i_+^* \otimes \text{id} \otimes \text{id}) \circ \gamma_{23} \circ \Delta(m_+(x))(v \otimes w) \circ i_-
 \end{aligned} \tag{46}$$

for $v \in F(V), w \in F(W)$.

We want to get

$$\begin{aligned}
 (i_+^* \otimes \text{id} \otimes \text{id}) \circ \gamma_{23} \circ (i_+^* \otimes \text{id} \otimes i_+^* \otimes \text{id}) \circ (\text{id} \otimes v \otimes \text{id} \otimes w) = \\
 (i_+^* \otimes \text{id} \otimes \text{id}) \circ (i_+^* \otimes \text{id} \otimes \text{id} \otimes \text{id}) \circ \gamma_{34} \circ (\text{id} \otimes v \otimes w) \circ (i_+^* \otimes \text{id} \otimes \text{id}) \circ \gamma_{23}.
 \end{aligned}$$

□

This finishes the proof of the main theorem of this chapter and we can state:

Theorem 4.5.11 $U_t(\mathfrak{g}_+)$ is a quantization of \mathfrak{g}_+ , so for every Lie bialgebra there exists a QUE.

4.6 SIMPLE COLOR LIE BIALGEBRAS OF CARTAN TYPE

In the case of super Lie algebras, there are the so called classical simple ones of type A-G, see e.g. [Kac77].

Let $A = (A_{ij})_{i,j \in I}$ be a Cartan matrix, $I = \{1, \dots, s\}$ and $\tau \subset I$ the set corresponding to odd roots.

4. Quantization

Let \mathfrak{g} be the Lie superalgebra generated by h_i, e_i and f_i for $i \in I$. We can put a \mathbb{Z}^s -grading on it as follows. We denote by z_i the i -th generator. The elements h_i are all of degree zero and $\deg(f_i) = -\deg(e_i) = z_i$. So we consider \mathfrak{g} to be graded by the root system. The commutation factor is given by $\epsilon_0(z_i, z_j) = -1$ if either $i \in \tau$ or $j \in \tau$ and $\epsilon(z_i, z_j) = 1$ else.

The generators satisfy the relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = A_{ij}e_j, \quad [h_i, f_j] = -A_{ij}f_j, \quad [e_i, e_j] = \delta_{ij}h_i \quad (47)$$

and the so called super classical Serre-type relations

$$\begin{aligned} [e_i, e_j] &= [f_i, f_j] = 0 \\ (\text{ad } e_i)^{1+|A_{ij}|}e_j &= (\text{ad } f_i)^{1+|A_{ij}|}f_j = 0 \text{ if } i \neq j, i \notin \tau \\ [e_m, [e_{m-1}, [e_m, e_{m+1}]]] &= [f_m, [f_{m-1}, [f_m, f_{m+1}]]] = 0 \text{ for } m-1, m, m+1 \in I, A_{mm} = 0 \\ [[[e_{m-1}, e_m], e_m], e_m] &= [[[f_{m-1}, f_m], f_m], f_m] = 0 \\ &\text{if the Cartan matrix is of type B and } \tau = m, s = m. \end{aligned}$$

The relations respect the \mathbb{Z}^s -grading so \mathfrak{g} can be considered as a \mathbb{Z}^s -graded algebra.

For a set of constants $\epsilon_{ij} \in \mathbb{K}^\times, i, j = 1, \dots, s$. We define $\sigma(a, b) = \prod_{i,j=1}^s \epsilon_{ij} a_i b_j$ for $a, b \in \mathbb{Z}^s$. Then $\sigma : \mathbb{Z}^s \times \mathbb{Z}^s \rightarrow \mathbb{K}^\times$ is a bicharacter and $\epsilon'(a, b) = \sigma(a, b)\sigma^1(b, a)$ a commutation factor. We set $\epsilon = \epsilon'\epsilon_0$. And for $x, y \in \mathfrak{g}$, we define $[x, y]' = \sigma(x, y)[x, y]$. With this bracket \mathfrak{g} becomes a ϵ -color Lie algebra. This is precisely the construction given in Section 2.6.

There are different r-matrices on these Lie algebras, see e.g. [Kar04]. Not all of them given there respect the \mathbb{Z}^s -grading, but the standard r-matrices given by $r = \sum h_i \otimes h_i + \sum_{\alpha \in \Delta^+} e_\alpha \otimes f_\alpha$ do. Here Δ^+ denotes the set of positive roots.

For these super-Lie bialgebras there is a well known quantization given by the so called Drinfeld-Jimbo-type superalgebras. To define them we first need:

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_t = \prod_{i=0}^{n-1} \frac{t^{m+n-i} - t^{-m-n+i}}{t^{i+1} - t^{-i-1}}. \quad (48)$$

Assume that the Cartan matrix A is symmetrizable that is there are non-zero rationals number d_1, \dots, d_s such that $d_i A_{ij} = d_j A_{ji}$. Set $q = e^{t/2}$ and $q_i = e^{d_i}$.

Let $U(\mathfrak{g})$ be the $\mathbb{C}[[t]]$ superalgebra generated by h_i, e_i and $f_i, i = 1, \dots, s$ and rela-

tions

$$[h_i, h_j] = 0, [h_i, e_j] = A_{ij}e_j, [h_i, f_j] = -A_{ij}f_j$$

$$[e_i, e_j] = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q_i - q_i^{-1}}$$

$$e_i^2 = 0 \text{ for } i \in I, A_{ii} = 0$$

$$[e_i, e_j] = 0, i, j \in I, i \neq j, A_{ij} = 0$$

$$\sum_{k=0}^{1+|A_{ij}|} (-1)^k \begin{bmatrix} 1+|A_{ij}| \\ k \end{bmatrix}_t e_i^{1+|A_{ij}|-k} e_j e_i^k = 0, 1 \leq i, j \leq s, i \neq j, i \notin \tau$$

$$\begin{aligned} e_m e_{m-1} e_m e_{m+1} + e_m e_{m+1} e_m e_{m-1} + e_{m-1} e_m e_{m+1} e_m \\ + e_{m+1} e_m e_{m-1} e_m (q + q^{-1}) e_m e_{m-1} e_{m+1} e_m = 0, m-1, m, m+1 \in I, A_{mmm} = 0 \\ e_{m-1} e_{m-1}^3 - (q + q^{-1} - 1) e_m e_{m-1} e_m^2 - (q + q^{-1} - 1) e_m^2 e_{m-1} e_m + e_m^3 e_{m-1} = 0 \\ \text{if the Cartan matrix is of type B and } \tau = m, s = m, \end{aligned}$$

and the same relations where e_i is replaced by f_i . For more details on these super QUE algebras see e.g. [Yam94].

Note that again all relations are compatible with the \mathbb{Z}^s -grading. Again we set $xy = \sigma(x, y)xy$, and get so a ϵ -color algebra.

We can define a comultiplication on $U(\mathfrak{g})$ by specifying it on generators as

$$\Delta(e_i) = e_i \otimes q^{d_i h_i} + 1 \otimes e_i$$

$$\Delta(f_i) = f_i \otimes q^{-d_i h_i} + 1 \otimes f_i$$

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i$$

$$\epsilon(h_i) = \epsilon(f_i) = \epsilon(e_i) = 0$$

Here the ϵ does not appear, but in does if one computes the comultiplication of other elements. Since it appears in the definition of the multiplication on $U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

We this we get a color Hopf algebra.

Part II.

α -type cohomologies and
deformations of Hom-algebras
and bialgebras

5. DEFINITIONS

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IN this section we recall the basic definitions of Hom-associative and Hom-Lie algebras, as well as their modules. We will also construct the corresponding free algebras. We also give some operadic consideration. Here we give some morphism between operads and their corresponding left adjoint functors on the category of algebras if possible. We also define Hom-bialgebra and Hom-Lie bialgebras. For the later we also define Hom-Manin triple.

This is mostly well known, except the Hom-Manin triples and maybe some small remarks concerning operads.

Let \mathbb{K} be a field of characteristic zero, but note that most constructions should also work in other characteristics, not 2, or if \mathbb{K} is a nice (containing the rational numbers) commutative ring. Especially all definitions of the different type of algebras in this chapter can be used in this more general setting and we will make use of this later.

By \mathbf{Vec} we denote the category of vector spaces over \mathbb{K} . We consider it to be symmetric monoidal with the trivial associator and symmetry $\tau : V \otimes W \rightarrow W \otimes V$ the usual flip.

5.1 HOM-MODULES AND HOM-MAGMATIC ALGEBRAS

Definition 5.1.1 (Hom-module) A Hom-module (V, α) is a vector space V together with a linear map $\alpha : V \rightarrow V$, called structure map. A morphism between Hom-modules (V, α) and (W, β) is a linear map $\varphi : V \rightarrow W$ such that $\varphi \circ \alpha = \beta \circ \varphi$. We denote the set of these by $\text{Hom}((V, \alpha), (W, \beta))$.

Sometimes we will also write $\text{Hom}_\alpha(V, W)$ for the set of Hom-module morphisms.

A Hom-submodule U of a Hom-module (V, α) is a subspace U of V , which is preserved by α , i.e. $\alpha(U) \subset U$.

There is an operad describing Hom-modules, it is the free operad generated by one operation α in arity 1. We denote this operad by \mathbf{HMod} .

The class of all Hom-modules obviously forms a category, which we denote by \mathbf{HomMod} . There also clearly is a direct sum and product of arbitrary sets of Hom-modules. One can also define the tensor product of two Hom-modules in the obvious way, i.e. it is given by $(V \otimes W, \alpha \otimes \beta)$.

There is a forgetful functor $\mathcal{O} : \mathbf{HomMod} \rightarrow \mathbf{Vec}$ which forgets the structure map α . It has a right adjoint $\mathbb{F}_{\mathbf{HMod}} : \mathbf{Vec} \rightarrow \mathbf{HomMod}$, and for a vector space V we call $\mathbb{F}_{\mathbf{HMod}}(V)$ the free Hom-module on V . It is given by $\mathbb{F}_{\mathbf{HMod}}(V) = \bigoplus_{\mathbb{N}_0} V$, and α is given by the right shift, i.e. $\alpha(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. It is clear that any element in $\mathbb{F}_{\mathbf{HMod}}(V)$ is of the form $\sum_{k=1}^n \alpha^{i_k}(x_k)$ for some $i_k \in \mathbb{N}_0$ and $x_i \in V$. Since this is just the polynomial algebra in one variable, we will often write $V[\alpha]$ instead of $\mathbb{F}_{\mathbf{HMod}}(V)$.

Definition 5.1.2 (regular Hom-module) We call a Hom-module (V, α) regular if α is invertible.

There is again an operad describing regular Hom-modules, it is given by the operad generated by two operations α, α^{-1} in arity one, which satisfy $\alpha \circ \alpha^{-1} = \text{id} = \alpha^{-1} \circ \alpha$. We denote it by \mathbf{rHMod} . The free regular Hom-module is given by $\mathbb{F}_{\mathbf{rHMod}}(V) = \bigoplus_{\mathbb{Z}} V$ with structure map given by the right shift. We also write $V[\alpha, \alpha^{-1}]$ for it. In this case the structure map is given by the multiplication with α .

Definition 5.1.3 (Hom-magmatic algebra) A Hom-magmatic (or non-associative) algebra is a Hom-module A together with a linear map $\mu : A \otimes A \rightarrow A$, called the multiplication. It is called multiplicative if it satisfies $\alpha \circ \mu = \mu \circ (\alpha \otimes \alpha)$.

We will in this thesis mostly work with multiplicative Hom-algebras and assume that all Hom-algebras are multiplicative if not stated otherwise.

A morphism of two Hom-magmatic algebras A and B is a morphism of the underlying Hom-modules, which respects the product. So there is a category of multiplicative Hom-magmatic algebras which we denote by \mathbf{HomMag} .

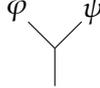
Definition 5.1.4 (Ideal, Subalgebra) A subalgebra of a Hom-algebra (A, μ, α) is a Hom-submodule U of A , such that $\mu(U, U) \subset U$.

An ideal of a Hom-algebra (A, μ, α) is a Hom-submodule I of A , such that $\mu(I, A) \subset I$ and $\mu(A, I) \subset A$.

Given a Hom-algebra A and an ideal I , one can define on the quotient A/I a Hom-algebra. In the case of Hom-associative and Hom-Lie algebras the quotient is again of this type.

There is a forgetful functor $\mathcal{O} : \mathbf{HomMag} \rightarrow \mathbf{HomMod}$, which has a left adjoint $\mathbb{F}_{HM ag} : \mathbf{HomMod} \rightarrow \mathbf{HomMag}$ and for a Hom-module V we call $\mathbb{F}_{HM ag}(V)$ the free Hom-magmatic algebra.

The free Hom-magmatic algebra for a Hom-module (V, α) can be explicitly constructed by using planar binary trees. We denote the set of planar binary trees with n inputs by n -pbTree. Given two such trees φ, ψ we define a new tree $\varphi \vee \psi$, which

we call the grafting of φ and ψ , by 

We define

$$\mathbb{F}_{HM ag}(V) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{n\text{-pbTree}} \bigotimes_{i=1}^n V. \quad (1)$$

Given a planar binary n -tree φ and $x_1, \dots, x_n \in V$, we define an element in $\mathbb{F}_{HM ag}(V)$ by $(x_1, \dots, x_n)_\varphi$. This is the tree φ where the outputs are decorated from left to right by x_1 to x_n . The product of two such elements is given by $(x_1, \dots, x_n)_\varphi (y_1, \dots, y_n)_\psi = (x_1, \dots, x_n, y_1, \dots, y_n)_{\varphi \vee \psi}$.

The structure map is given by $\alpha((x_1, \dots, x_n)_\varphi) = (\alpha(x_1), \dots, \alpha(x_n))_\varphi$.

There is an operad, which describes multiplicative Hom-magmatic algebras, we denote it by \mathbf{HMag} . It can be defined by $\mathbb{F}_{Op}(V) / \langle R \rangle$, where V is spanned by μ, α , where μ is a binary operation and α takes one argument, and $R = \alpha\mu - \mu(\alpha, \alpha)$. It can be described by planar binary trees, where each output carries an additional weight (or height), similar to the free magmatic algebra described in [LGMT18].

In Appendix C.2 we defined operads where each output of each operations carries a height. In the case of Hom-algebras we assign to α and μ the height 1. The defining relations of \mathbf{HMag} respect this height, so it can be extended to \mathbf{HMag} .

There is a forgetful functor $\mathcal{O} : \mathbf{HomMag} \rightarrow \mathbf{Vec}$, it has a right adjoint $\mathbb{F}_{HM od}$, which assign to any vector space the free multiplicative Hom-magmatic algebra. It can be constructed as the composition $\mathbb{F}_{HM od} \mathbb{F}_{HM ag}$.

We also define regular Hom-magmatic algebras as Hom-magmatic algebras with invertible structure map. Again there is an operad describing these and the free regular Hom-magmatic algebra can be constructed as $\mathbb{F}_{HM ag} \mathbb{F}_{rHM od}$.

5. Definitions

We define derivations and α -derivation for Hom-magmatic algebras. Derivations for other types of (Hom-)algebras can be defined in the same way.

Definition 5.1.5 Let (A, μ, α) be a Hom-magmatic algebra, and $\varphi : A \rightarrow A$ a linear map, then

- we call φ a derivation if $\varphi(ab) = \varphi(a)b + a\varphi(b)$,
- we call φ an α -derivation if $\varphi(ab) = \varphi(a)\alpha(b) + \alpha(a)\varphi(b)$,
- we call φ a conjugate α -derivation if α is invertible and $\varphi(xy) = \alpha(x)\alpha^{-1}\varphi\alpha(y) + \alpha^{-1}\varphi\alpha(x)\alpha(y)$.

5.2 HOM-ASSOCIATIVE ALGEBRAS

Definition 5.2.1 (Hom-(associative) algebra) A Hom-associative algebra (A, μ, α) , is a Hom-module (A, α) , with a linear map $A \otimes A \rightarrow A$, called multiplication, such that

$$\mu \circ (\mu \otimes \alpha) = \mu \circ (\alpha \otimes \mu). \quad (2)$$

It is called multiplicative if α is an algebra morphism or equivalently μ is a Hom-morphism, i.e.

$$\mu(\alpha \otimes \alpha) = \alpha\mu. \quad (3)$$

A unit (sometimes called weak or Hom-unit) is an element $\mathbb{1} \in A$ such that $\mu(\mathbb{1} \otimes \text{id}) = \alpha = \mu(\text{id} \otimes \mathbb{1})$ and $\alpha(\mathbb{1}) = \mathbb{1}$. A Hom-associative algebra with a unit we call a unital Hom-associative algebra.

Normally we write xy or $x \cdot y$ for $\mu(x \otimes y)$, so the Hom-associativity becomes $\alpha(x)(yz) = (xy)\alpha(z)$ for all $x, y, z \in A$, and the condition for the unit becomes $\mathbb{1} \cdot x = \alpha(x) = x \cdot \mathbb{1}$.

We call a Hom-associative algebra (A, μ, α) regular if α is invertible. We call it simple if it has no proper ideal and $\alpha \neq 0$.

A morphism of Hom-associative algebras (A, μ, α) and (B, μ, β) is a linear map $\varphi : A \rightarrow B$, such that $\varphi\alpha = \beta\varphi$ and $\mu(\alpha \otimes \alpha) = \beta\mu$. For unital Hom-associative algebras we also require $\varphi(\mathbb{1}_A) = \mathbb{1}_B$.

Given a multiplicative Hom-associative algebra A one can always adjoin a Hom-unit. For this one defines $A^+ = A \oplus \mathbb{K}$, with multiplication $(a, \lambda)(b, \lambda') = (ab + \lambda\alpha(b) + \alpha(a)\lambda', \lambda\lambda')$ for $a, b \in A, \lambda, \lambda' \in \mathbb{K}$. Then it is clear that A^+ is a Hom-associative algebra with unit $(0, 1)$ and $a \mapsto (a, 0)$ is an injective morphism of Hom-algebras.

Also every unital Hom-associative algebra A is multiplicative. For $a, b \in A$, we have $\alpha(ab) = \mathbb{1}(ab) = \alpha(\mathbb{1})(ab) = (\mathbb{1}a)\alpha(b) = \alpha(a)\alpha(b)$.

Remark 5.2.2 One cannot adjoin a true unit to a Hom-associative algebra, since the new algebra would no longer be Hom-associative. Also if a Hom-associative algebra

A has a true unit 1 , which satisfies $\alpha(1) = 1$, one has $ab = \alpha(1)(ab) = (1a)\alpha(b) = a\alpha(b)$ for all $a, b \in A$ and further for $a = 1$ one gets $b = \alpha(b)$. So α is the identity.

The tensor product of two Hom-algebras is again a Hom-algebra.

It is problematic to speak of inverses in the case of Hom-associative objects, since for example we do not even have the usual definition of unit. Anyhow, we call $x \in A$ the inverse of $y \in A$ if $x \cdot y = \mathbb{1} = y \cdot x$. Note that if the multiplication comes from a Yau twist, which is defined later, the inverses stay the same.

We call an element $y \in A$ a weak inverse of $x \in A$ if there exists an $n \in \mathbb{N}_0$, such that $\alpha^n(yx) = \mathbb{1} = \alpha^n(xy)$. The set of weak invertible elements is closed under multiplication. The set of invertible elements however is not.

There is an operad describing Hom-associative algebras, which we denote by HAss . Again we can assign to the generating operations α and μ the height 1. This induces a height on HAss since all relations are homogeneous with height 2. There is an obvious morphism of operads $\text{HAss} \rightarrow \text{Ass}$, which is given by $\mu \mapsto \mu, \alpha \mapsto \text{id}$. On algebras it is given by the fact that any associative algebra (A, μ) can be regarded as a Hom-associative algebra (A, μ, id) .

By abstract nonsense one can prove that the functor $\text{AssAlg} \rightarrow \text{HomAlg}$ has a left adjoint. In fact we have

Proposition 5.2.3 *Let (A, μ, α) be a multiplicative Hom-associative algebra then the eigenspace of α to the eigenvalue 1 is an associative algebra, which we denote by A_1 . Further if α is diagonalizable, given an associative algebra B , there is a natural isomorphism*

$$\text{Hom}(A_1, B) \cong \text{Hom}(A, B). \quad (4)$$

Proof. Since $\alpha(xy) = \alpha(x)\alpha(y)$ for $x, y \in A$, it is clear that if x, y are eigenvectors for α with eigenvalues λ_x, λ_y resp. so is xy with eigenvalue $\lambda_x \lambda_y$. So $\text{Eig}(\alpha, 1)$ is a subalgebra of A . Since α restricted to it is the identity it is associative.

Given $\varphi \in \text{Hom}(A, B)$, it must satisfy $\varphi\alpha(x) = \varphi(x)$, so $\varphi(\alpha(x) - x) = 0$. Since α can be assumed to be diagonal, we can assume that x is an eigenvector and we get $x \in \text{Eig}(\alpha, 1)$ or $\varphi(x) = 0$. So φ is determined on A_1 . On the other hand every algebra morphism $A_1 \rightarrow B$ can be extended by zero to a map $A \rightarrow B$. This is a Hom-algebra morphism. \square

In the general case one can construct the left adjoint functor F as follows. Let (A, μ, α) be a Hom-algebra and $T(A)$ the free algebra on A . We consider in $T(A)$ the ideal I generated by $xy - \alpha\mu(a, b)$ and $\alpha(x) - x$ for $x, y \in A$. Then we define $FA := T(A)/I$. It is easy to see that in the case that α is diagonal this is isomorphic to A_1 by choosing a basis of eigenvectors. If $\mathbb{K} = \mathbb{C}$ and A is finite dimensional one can consider the Jordan normalform of α to calculate FA . If there is a nontrivial

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Jordan block to 1, FA is not given by A_1 but by the highest generalized eigenvector for each block to 1.

Proposition 5.2.4 (Yau twist) *Let (A, μ, α) be a not necessarily Hom-associative algebra and γ a morphism of the product, then $(A, \gamma \circ \mu, \gamma \circ \alpha)$ is again a Hom-associative algebra, which we denote by A_γ . If A is multiplicative and γ commutes with α then A_γ is again multiplicative.*

Given two Hom-associative algebras A and B , endomorphism $\gamma_A : A \rightarrow A$ and $\gamma_B : B \rightarrow B$ and a morphism $\varphi : A \rightarrow B$ which satisfies $\varphi\gamma_A = \gamma_B\varphi$, there is an induced morphism $\varphi : A_{\gamma_A} \rightarrow B_{\gamma_B}$.

We define the operad of associative algebras with an endomorphism, which we denote by $EAss$. An algebra over this operad is an associative algebra (A, μ) with an endomorphism $\gamma : A \rightarrow A$ of it. The free $EAss$ -algebra for a vector space V is given by the free algebra over $\mathbb{F}_{HMod}(V) = V[\gamma]$, i.e. the tensor algebra $T(V[\gamma])$.

Similar one can define the operad for other types of algebras with an endomorphism.

In the case A is an associative algebra the Yau construction is induced by a morphism $Y : HAss \rightarrow EAss$ between operads. On generators it is given by $\mu \mapsto \gamma\mu$ and $\alpha \mapsto \gamma$. This morphism is not injective, since e.g. $\mu(\alpha \otimes \mu(\mu \otimes \mu))$ and $\mu(\mu(\text{id} \otimes \alpha) \otimes \mu(\mu \otimes \alpha))$ are both mapped to $\mu_5(\alpha^2 \otimes \alpha^3 \otimes \alpha^3 \otimes \alpha^3 \otimes \alpha^4)$. Here μ_5 is the composition of 5 elements.

By abstract nonsense there is again a left adjoint functor to the induced functor on the category of algebras. In the general case this can be described by the free $EAss$ -algebra modulo some relations. In the regular case the situation is simpler and we have:

Proposition 5.2.5 *Let (A, μ, α) be a regular Hom-associative algebra and (B, μ, γ) an associative algebra with an endomorphism then*

$$\text{Hom}_{EAss}(A_{\alpha^{-1}}, B) \cong \text{Hom}(A, B_\gamma). \quad (5)$$

Proof. Given $\varphi \in \text{Hom}(A, B_\gamma)$, we have $\varphi\alpha = \gamma\varphi$ and $\varphi(ab) = \gamma(\varphi(a)\varphi(b))$ for $a, b \in A$. This implies $\varphi(\alpha^{-1}(\alpha(a)\alpha(b))) = \varphi(ab) = \gamma(\varphi(a)\varphi(b)) = \varphi(\alpha(a))\varphi(\alpha(b))$. This proves $\varphi \in \text{Hom}_{EAss}(A_{\alpha^{-1}}, B)$ since α is surjective. \square

The operad of regular Hom-associative algebras is isomorphic to the operad of associative algebras with an invertible endomorphism. The isomorphism is given by Y .

We define the operad $EHAss$ for a Hom-algebra with an endomorphism. It is generated by μ in arity two and α and γ in arity one, with the obvious relations. Then the Yau construction corresponds to a morphism of operads $Y : EHAss \rightarrow HAss$ given on generators by $\mu \mapsto \gamma\mu$ and $\alpha \mapsto \gamma\alpha$.

Definition 5.2.6 A Hom-algebra A , which is of the form \tilde{A}_γ for an associative algebra \tilde{A} and an endomorphism γ of it, is called of associative-type.

Proposition 5.2.7 Let (A, μ, α) be a Hom-associative algebra, such that α is invertible then $A_{\alpha^{-1}}$ is an ordinary associative algebra. So A is of associative type.

Proof. Follows directly from Proposition 5.2.4. □

We recall the definitions for the dual concept of Hom-coassociative coalgebras.

Definition 5.2.8 (Hom-(coassociative) coalgebra) A Hom-(coassociative) coalgebra (C, Δ, β) is a Hom-module (C, β) with a comultiplication $\Delta : C \rightarrow C \otimes C$, which satisfies

$$(\Delta \otimes \beta)\Delta = (\beta \otimes \Delta)\Delta. \quad (6)$$

It is called comultiplicative if $\Delta(\beta \otimes \beta) = \beta\Delta$. A counit is a map $\varepsilon : C \rightarrow \mathbb{K}$, such that $(\text{id} \otimes \varepsilon)\Delta = \beta = (\varepsilon \otimes \text{id})\Delta$ and $\varepsilon\beta = \varepsilon$.

A Hom-coalgebra with a counit we call a counital Hom-coalgebra.

We will in the following make use of the Sweedler notation. This means we write $\Delta(x) = x^{(1)} \otimes x^{(2)}$, where on the right hand side there is an implicit sum.

A morphism $\varphi : C \rightarrow D$ between Hom-coalgebras is defined in the obvious way. For counital Hom-coalgebras with also require $\varepsilon_B \varphi = \varepsilon_A$.

Given a Hom-coalgebra (C, Δ, β) , its linear dual with the duals maps of the co-product and structure map (C^*, Δ^*, β^*) is a Hom-associative algebra. Here $\Delta^*(\varphi \otimes \psi)(a) = \varphi(a^{(1)})\psi(a^{(2)})$ and $\beta^*(\varphi)(a) = \varphi(\beta(a))$ for $\varphi, \psi \in C^*$ and $a \in C$.

Given a Hom-coalgebra (C, Δ, β) and a Hom-algebra (A, μ, α) , we define the convolution product on $\text{Hom}(C, A)$ by

$$\varphi * \psi = \mu(\varphi \otimes \psi)\Delta \text{ for } \varphi, \psi \in \text{Hom}(C, A). \quad (7)$$

Proposition 5.2.9 $(\text{Hom}(C, A), *, \gamma : \varphi \mapsto \alpha \circ \varphi \circ \beta)$ is a Hom-algebra. Which is multiplicative if A and C are multiplicative and comultiplicative resp. Further it is commutative is A is commutative and C cocommutative.

Proof. This is a simply calculation. □

5.2.1 Free Hom-associative algebra

Let (V, α) be a Hom-module then there exists a free Hom-associative algebra on V . There is a forgetful functor $\mathcal{O} : \mathbf{HomAlg} \rightarrow \mathbf{HomMod}$, which given a Hom-associative algebra forgets the multiplication. It has a left adjoint $\mathbb{F}_{\text{HAss}} : \mathbf{HomMod} \rightarrow$

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HomAlg and we call $\mathbb{F}_{HAss}(V, \alpha)$ the free Hom-algebra on V . So for all Hom-algebras A there is a natural bijection

$$\text{Hom}_H(V, \mathcal{O}A) \cong \text{Hom}_{HAlg}(\mathbb{F}_{HAss}(V), A). \quad (8)$$

The free Hom-algebra can be constructed, out of the free Hom-magmatic algebra, by taking the quotient with respect to an ideal. Let I be the ideal generated by $\alpha(x)(yz) - (xy)\alpha(z)$. Then we have $\alpha(I) = I$ and $\mathbb{F}_{HAss} = \mathbb{F}_{HMAg}/I$.

Given a vector space V , one can also construct the free Hom-associative algebra by $\mathbb{F}_{HAss}\mathbb{F}_{HMod}(V)$. It is also possible to define the free regular Hom-associative algebra by $\mathbb{F}_{HAss}\mathbb{F}_{rHMod}(V)$.

In general it is difficult to describe the free Hom-associative algebra or the operad HAss but we have:

Proposition 5.2.10 *Given any two n -ary operations in HAss with the same height greater or equal than $n - 1$ they are equal.*

Proof. This follows from [LGMT18, Prop. 4.21]. □

This implies that in any Hom-associative algebra the corresponding operations are equal.

We define $\mu_\alpha^2 = \mu$ and further $\mu_\alpha^{n+1} = \mu(\mu_\alpha^n \otimes \alpha^{n-1})$. The corresponding tree is of level $n - 1$. Using Proposition 5.2.10 we get that $\mu_\alpha^{n+m} = \mu(\alpha^{m-1} \mu_\alpha^n \otimes \alpha^{n-1} \mu_\alpha^m)$.

Let $\beta : V \rightarrow V$ be a linear map which commutes with α , then one can define a comultiplication Δ on $\mathbb{F}_{HAss}(V)$ by setting $\Delta(x) = 1 \otimes \beta(x) + \beta(x) \otimes 1$ for $x \in V$. This extends to an algebra morphism. The map β can also be extended to a Hom-algebra morphism. In Section 5.3 we will define Hom-bialgebras. With that we get:

Proposition 5.2.11 *The free Hom-algebra, with the comultiplication given above, is a (α, β) -Hom-bialgebra.*

Proof. The compatibility follows from the construction so we only need to prove the coassociativity. For $x \in V$ we have

$$\begin{aligned} (\Delta \otimes \beta)\Delta(x) &= (\Delta \otimes \beta)(\beta(x) \otimes 1 + 1 \otimes \beta(x)) \\ &= \beta^2(x) \otimes 1 \otimes 1 + 1 \otimes \beta^2(x) \otimes 1 + 1 \otimes 1 \otimes \beta^2(x) \end{aligned}$$

and similarly on the other hand

$$(\beta \otimes \Delta)\Delta(x) = \beta^2(x) \otimes 1 \otimes 1 + 1 \otimes \beta^2(x) \otimes 1 + 1 \otimes 1 \otimes \beta^2(x).$$

So we have the coassociativity on generators and by freeness everywhere. □

5.2.2 Modules and Bimodules

Definition 5.2.12 (Hom-algebra module) Let (A, μ, α) be a Hom-associative algebra and (M, β) a Hom-module, with a map $\rho : A \otimes M \rightarrow M, (a, m) \mapsto a \cdot m$, then V is called an (left) A -module if

$$(ab) \cdot \beta(m) = \alpha(a)(b \cdot m). \quad (9)$$

We also require it to be multiplicative, i.e.

$$\beta(a \cdot m) = \alpha(a) \cdot \beta(m). \quad (10)$$

Similarly one can define a right A -module.

If (M, ρ, β) is an A -module and γ is a Hom-algebra endomorphism of A then $(M, \rho(\gamma \otimes \text{id}), \beta)$ is again an A -module. We will mostly use this in the case $\gamma = \alpha^n$ for $n \in \mathbb{N}$.

The product (or equally the sum) of two A -modules M, N is again an A -module, by $a \cdot (m, n) = (a \cdot m, a \cdot n)$ for $m \in M$ and $n \in N$. The same is true for products and sums with more factors.

Definition 5.2.13 (Hom-algebra bimodule) Let A be a Hom-algebra, then a A -bimodule is a Hom-module (M, β) , with two maps $\rho : A \otimes M \rightarrow M, a \otimes m \mapsto a \cdot m$ and $\lambda : M \otimes A \rightarrow A, m \otimes a \mapsto m \cdot a$, such that ρ is a left and λ a right module structure and

$$\alpha(a) \cdot (m \cdot b) = (a \cdot m) \cdot \alpha(b). \quad (11)$$

It is called multiplicative if ρ and λ are.

If A is commutative, we call an A -bimodule symmetric if $a \cdot m = m \cdot a$ for all $a \in A$ and $m \in M$.

Obviously A is an A -bimodule, where the left and right action is given by the multiplication.

If V is an A -bimodule then $A \oplus V$ is a Hom-algebra. The multiplication for $a, b \in A$ and $v, w \in V$ is given by $(a, v)(b, w) = (ab, a \cdot w + b \cdot v)$. On the other hand if $(A, \alpha) \oplus (V, \beta)$ is a Hom-associative algebra, such that the multiplication on V is trivial, i.e. $vw = 0$ for all $v, w \in V$, V is an A -bimodule.

Given a Hom-coassociative coalgebra one can regard its comodules, dual to the case of Hom-associative algebras and modules.

Definition 5.2.14 (Hom-coalgebra comodule) Let (C, Δ, β) be a Hom-coalgebra and (M, γ) a Hom-module, with a map $\rho : M \rightarrow A \otimes M$, then M is called a (left) C -comodule if

$$(\beta \otimes \rho)\rho = (\Delta \otimes \beta)\rho. \quad (12)$$

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We also assume that it is comultiplicative, i.e.

$$\rho\gamma = (\beta \otimes \gamma)\rho. \quad (13)$$

Similarly one can define a right C -comodule.

Definition 5.2.15 (Hom-coalgebra bicomodule) Let (C, Δ, β) be a Hom-coalgebra, then a C -bicomodule is a Hom-module (M, γ) , with two maps $\rho : M \rightarrow A \otimes M$ and $\lambda : M \otimes A \rightarrow M \otimes A, a \otimes m \mapsto m \cdot a$, such that ρ is a left and λ a right comodule structure and

$$(\beta \otimes \rho)\lambda = (\lambda \otimes \beta)\rho. \quad (14)$$

5.3 HOM-BIALGEBRAS

In this section we will only consider multiplicative Hom-(co)algebras and only give the definition for multiplicative Hom-bialgebras and simply call them Hom-bialgebras.

Definition 5.3.1 (Hom-(associative) bialgebra) A Hom-(associative) bialgebra is tuple $(A, \mu, \Delta, \mathbb{1}, \varepsilon, \alpha, \beta)$ such that $(A, \mu, \mathbb{1}, \alpha)$ is a unital Hom-associative algebra, $(A, \Delta, \varepsilon, \beta)$ is a counital Hom-coalgebra and they are compatible in the sense that Δ is a Hom-algebra morphism and μ is a Hom-coalgebra morphism. Also the unit is assumed to be a Hom-coalgebra morphisms and the counit a Hom-algebra morphisms. We also require that α and β commute. Written as equations this means

$$\begin{aligned} \alpha\beta &= \beta\alpha, & \Delta(xy) &= \Delta(x)\Delta(y), \\ \Delta\alpha &= (\alpha \otimes \alpha)\Delta, & \beta\mu &= \mu(\beta \otimes \beta), \\ \Delta(\mathbb{1}) &= \mathbb{1} \otimes \mathbb{1}, & \varepsilon(xy) &= \varepsilon(x)\varepsilon(y) \\ \varepsilon\alpha &= \varepsilon & \beta(\mathbb{1}) &= \mathbb{1}, \varepsilon(\mathbb{1}) = 1. \end{aligned}$$

Definition 5.3.2 An antipode is a convolution inverse of the identity. A weak antipode is a weak convolution inverse of the antipode.

Definition 5.3.3 A Hom-Hopf algebra is a Hom-bialgebra with an antipode.

One can also consider the case, where $\alpha = \beta$ we will call such a Hom-bialgebra a α -Hom-bialgebra. Similar we call a Hom-bialgebra with two different structure maps an (α, β) -Hom-bialgebra.

One can also consider non-unital bialgebra, where we do not assume the existence of a unit or counit.

A morphism of a Hom-bialgebra is a unital morphism of the Hom-algebra and Hom-coalgebra structure.

Similar to the non-Hom-case there is a properad, which describes non-unital Hom-bialgebras. So it can be described as the free properad generated by $\Delta, \mu, \alpha, \beta$ modulo an ideal, which corresponds to the relations these maps have to satisfy.

We recall the following construction method for Hom-bialgebras:

Proposition 5.3.4 (Yau twist) *Let $(A, \mu, \Delta, \mathbb{1}, \varepsilon, \alpha, \beta)$ be a Hom-algebra and γ a morphism of it then $(A, \gamma\mu, \Delta, \mathbb{1}, \varepsilon, \gamma\alpha, \beta)$, $(A, \mu, \Delta\gamma, \mathbb{1}, \varepsilon, \alpha, \gamma\beta)$ and $(A, \gamma\mu, \Delta\gamma, \mathbb{1}, \varepsilon, \gamma\alpha, \gamma\beta)$ are again Hom-bialgebras.*

An element of a bialgebra A is called γ -primitive if $\Delta(x) = \gamma(x) \otimes \mathbb{1} + \mathbb{1} \otimes \gamma(x)$. Here γ could be an arbitrary linear map on A , but only the case $\gamma = \text{id}$, α or β are of interest. We denote the set of γ -primitive elements by $\gamma\text{-Prim}(A)$, and write simply $\text{Prim}(A)$ for $\text{id-Prim}(A)$.

Similar to [LGMT18, section 4.3] one gets the following result, which shows that in general there are more primitive elements than in the non-Hom-case.

Proposition 5.3.5 *Let A be a bialgebra and γ a morphisms of the product, i.e. $\gamma\mu = \mu(\gamma \otimes \gamma)$, then $\gamma\text{-Prim}(A)$ is a Hom-Lie algebra with respect to the commutator in A and α .*

Proof. Since A is a Hom-Lie algebra with respect the the commutator and α , it is enough to show that for $x, y \in \gamma\text{-Prim}(A)$ the commutator $[x, y]$ is again in $\gamma\text{-Prim}(A)$. So we compute

$$\begin{aligned}
 \Delta([x, y]) &= \Delta(xy) - \Delta(yx) \\
 &= (\mathbb{1} \otimes \gamma(x) + \gamma(x) \otimes \mathbb{1})(\mathbb{1} \otimes \gamma(y) + \gamma(y) \otimes \mathbb{1}) \\
 &\quad - (\mathbb{1} \otimes \gamma(y) + \gamma(y) \otimes \mathbb{1})(\mathbb{1} \otimes \gamma(x) + \gamma(x) \otimes \mathbb{1}) \\
 &= \mathbb{1} \otimes \gamma(x)\gamma(y) + \alpha(\gamma(x)) \otimes \alpha(\gamma(y)) + \alpha(\gamma(y)) \otimes \alpha(\gamma(x)) + \gamma(x)\gamma(y) \otimes \mathbb{1} \\
 &\quad - \gamma(y)\gamma(x) \otimes \mathbb{1} - \alpha(\gamma(x)) \otimes \alpha(\gamma(y)) - \alpha(\gamma(y)) \otimes \alpha(\gamma(x)) - \mathbb{1} \otimes \gamma(y)\gamma(x) \\
 &= [\gamma(x), \gamma(y)] \otimes \mathbb{1} + \mathbb{1} \otimes [\gamma(x), \gamma(y)] \\
 &= \gamma([x, y]) \otimes \mathbb{1} + \mathbb{1} \otimes \gamma([x, y]).
 \end{aligned}$$

□

Proposition 5.3.6 *Let A be a bialgebra and $x, y \in \gamma\text{-Prim}(A)$, such that $\alpha(\gamma(x)) = \alpha(\gamma(y)) = 0$, then $xy \in \gamma\text{-Prim}$. Particularly, if α and γ commute and $\alpha(x) = 0$, then the subalgebra generated by x lies in $\gamma\text{-Prim}$.*

Proof. We have

$$\begin{aligned}
 \Delta(xy) &= \mathbb{1} \otimes \gamma(x)\gamma(y) + \alpha(\gamma(x)) \otimes \alpha(\gamma(y)) + \alpha(\gamma(y)) \otimes \alpha(\gamma(x)) + \gamma(x)\gamma(y) \otimes \mathbb{1} \\
 &= \mathbb{1} \otimes \gamma(x)\gamma(y) + \gamma(x)\gamma(y) \otimes \mathbb{1}
 \end{aligned}$$

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The second part follows easily by induction. \square

We note that given a Hom-bialgebra A and morphism $\gamma : A \rightarrow A$, we have $\text{Prim } A \subset \gamma\text{-Prim}(A_\gamma)$. Here A_γ denotes A with the the coproduct twisted by γ . In general the inclusion is strict consider for example the case $\gamma = 1\varepsilon$.

Modules

Let A be a Hom-bialgebra then the tensor product of two A -modules M and N is again an A -module, where the action is given by $(\rho_M \otimes \rho_N)\tau_{23}(\Delta \otimes \text{id} \otimes \text{id})$ or

$$a \cdot (m \otimes n) = a^{(1)} \cdot m \otimes a^{(2)} \cdot n \text{ for } a \in A, m, n \in M. \quad (15)$$

Note that given three or more A -modules the module structure on the tensor product depends in general on the bracketing of the modules, i.e. $(M_1 \otimes M_2) \otimes M_3$ is not isomorphic to $M_1 \otimes (M_2 \otimes M_3)$, since the comultiplication is not coassociative. So we define the action on tensor products with more factors by including α and β in the definition.

We define $\Delta_\beta^2 = \Delta$ and

$$\Delta_\beta^{n+1} = (\Delta \otimes \beta^{\otimes(n-1)})\Delta_\beta^n. \quad (16)$$

This satisfies

$$\Delta_\beta^{n+m} = (\Delta_\beta^n \beta^{m-1} \otimes \Delta_\beta^m \beta^{n-1})\Delta. \quad (17)$$

Let (M_i, ρ_i) be A modules for $i = 1, \dots, n$ then the action on $M_1 \otimes \dots \otimes M_n$ is defined by

$$(\rho_1 \otimes \dots \otimes \rho_n)\tau_{(2,n)}(\Delta_\beta^n \otimes \text{id}^n). \quad (18)$$

Here $\tau_{(2,n)}$ denotes the permutation $(1, \dots, 2n) \mapsto (1, n+1, 2, n+2, \dots, n, 2n)$.

We also have

$$x \cdot (v_1 \otimes \dots \otimes v_{m+n}) = \beta^{m-1}(x^{(1)}) \cdot (v_1 \otimes \dots \otimes v_n) \otimes \beta^{n-1}(x^{(2)}) \cdot (v_{n+1} \otimes \dots \otimes v_{n+m}). \quad (19)$$

If all the M_i are bimodules the tensor product is again a bimodule. Similarly the tensor product of comodules is again in a comodule.

The category of modules over a Hom-bialgebra is in general not a monoidal category, since one needs to include the structure map in the definition of the tensor products of several factors to make it independent of the bracketing. However one can define Hom-tensor categories as a generalization of monoidal categories and it turns out this category is a Hom-tensor category, see [PSS17].

5.4 HOM-LIE ALGEBRAS

Definition 5.4.1 A Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is a Hom-module (\mathfrak{g}, α) with a skew-symmetric linear map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ such that the Hom-Jacobi identity is satisfied, i.e.

$$[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)] = 0 \text{ for all } x, y, z \in \mathfrak{g}. \quad (20)$$

It is called multiplicative if $[\alpha(x), \alpha(y)] = \alpha([x, y])$.

We will sometimes write $(\mathfrak{g}, \nu, \alpha)$ instead of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$, with $\nu : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, $\nu(x \otimes y) = [x, y]$ for $x, y \in \mathfrak{g}$.

There is an operad describing Hom-Lie algebras, which we denote by HLie . We denote the generating operations by ν and α .

Proposition 5.4.2 ([MS08]) *Let (A, μ, α) be a Hom-associative algebra then the commutator $[x, y] = xy - yx$ defines a Hom-Lie structure on A , which is multiplicative if A was so. We denote it by A_L .*

This corresponds to a morphism $\text{HLie} \rightarrow \text{HAss}$ of operads, which on generators is given by $\nu \mapsto \mu - (12)\mu$.

Similar to the Hom-associative case one can construct a free Hom-Lie algebra $\mathbb{F}_{\text{HLie}}(V)$ for a Hom-module (V, α) . It can be constructed as the Hom-Lie subalgebra of the free Hom-associative algebra $\mathbb{F}_{\text{HAss}}(V)$ generated by V under the commutator. It is clear that this is a Hom-Lie algebra. So one has to prove the universal property, i.e. there is a natural isomorphism

$$\text{Hom}(\mathbb{F}_{\text{HLie}}(V), \mathfrak{g}) \cong \text{Hom}_H(V, \mathfrak{g}) \quad (21)$$

for all Hom-Lie algebras \mathfrak{g} .

Similar to the case of associative algebras one can define a Yau twist.

Proposition 5.4.3 *Let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra and $\gamma : \mathfrak{g} \rightarrow \mathfrak{g}$ an endomorphism of it, then $(\mathfrak{g}, \gamma\nu, \gamma\alpha)$ is a Hom-Lie algebra.*

All the remarks concerning the operadic description of the Yau twist can be transferred from the Hom-associative to the Hom-Lie setting.

Definition 5.4.4 (Hom-Lie module, Representation of Hom-Lie algebra) Let \mathfrak{g} be a Hom-Lie algebra. A Hom-Lie module over \mathfrak{g} or a representation of \mathfrak{g} is a Hom-module (V, β) , with an action $\rho : \mathfrak{g} \otimes V \rightarrow V$, $g \otimes v \mapsto x \cdot v$, such that

$$[x, y] \cdot \beta(v) = \alpha(x) \cdot (y \cdot v) - \alpha(y) \cdot (x \cdot v). \quad (22)$$

It is multiplicative if

$$\beta(x \cdot v) = \alpha(x) \cdot \beta(v). \quad (23)$$

5. Definitions

Given a Hom-Lie algebra \mathfrak{g} we can define the adjoint representation of \mathfrak{g} on itself, by $x \cdot y = [x, y]$.

Given n \mathfrak{g} -modules $(M_1, \alpha_1), \dots, (M_n, \alpha_n)$, we can define a \mathfrak{g} -module on the tensor product $M_1 \otimes \dots \otimes M_n$ by

$$x \cdot (y_1 \otimes \dots \otimes y_n) = \sum_{i=1}^k \alpha_1(y_1) \otimes \dots \otimes [x, y_i] \otimes \dots \otimes \alpha_n(y_n) \quad (24)$$

for $y_i \in M_i$.

Given a Hom-Lie algebra \mathfrak{g} we can define the coadjoint action of \mathfrak{g} on \mathfrak{g}^* by

$$(\text{ad}_x \varphi)(y) = \varphi([x, y]). \quad (25)$$

We define the dual notion of a Hom-Lie algebra this is a Hom-Lie coalgebra. We denote by σ_c the cyclic permutation map $x \otimes y \otimes z \mapsto z \otimes x \otimes y$.

Definition 5.4.5 A Hom-Lie coalgebra $(\mathfrak{g}, \delta, \alpha)$ is a Hom-module (\mathfrak{g}, α) with a co-bracket $\delta : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, which satisfies

$$\delta(\delta \otimes \beta)(\text{id} + \sigma_c + \sigma_c^2) = 0. \quad (26)$$

Universal enveloping algebra

Let \mathfrak{g} be a Hom-Lie algebra then we define its universal enveloping algebra as follows [Yau08]:

Let $\mathbb{F}_{\text{HAss}}(\mathfrak{g})$ be the free Hom-associative algebra on \mathfrak{g} and I be the ideal generated by elements of the form $xy - yx - [x, y]$ for $x, y \in \mathfrak{g}$. Then I is closed under α and $U(\mathfrak{g}) := \mathbb{F}_{\text{HAss}}(\mathfrak{g}) / I$ is again a Hom-associative algebra, which we call the universal enveloping algebra of \mathfrak{g} .

There is an adjunction

$$\text{Hom}_{\text{HLie}}(\mathfrak{g}, A_L) \cong \text{Hom}_{\text{HAss}}(U(\mathfrak{g}), A). \quad (27)$$

Proposition 5.4.6 *If the Hom-Lie algebra $(\mathfrak{g}, \nu, \alpha)$ is regular, then the universal enveloping algebra is of associative type, and we have $U(\mathfrak{g}) = U(\mathfrak{g}_{\alpha^{-1}})_{\alpha}$.*

Proof. There is a morphism $\mathfrak{g} \rightarrow U(\mathfrak{g}_{\alpha^{-1}})_{\alpha}$, which can be extended by the universal property to a morphism $U(\mathfrak{g}) \rightarrow U(\mathfrak{g}_{\alpha^{-1}})_{\alpha}$. On the other hand there is an morphism $\mathfrak{g}_{\alpha^{-1}} \rightarrow U(\mathfrak{g})_{\alpha^{-1}}$ this can be extended to a morphism $\varphi : U(\mathfrak{g}_{\alpha^{-1}}) \rightarrow U(\mathfrak{g})_{\alpha^{-1}}$. Since φ is compatible with α this induces a morphism $U(\mathfrak{g}_{\alpha^{-1}})_{\alpha} \rightarrow U(\mathfrak{g})$. It is easy to see that this two morphism are inverse to each other. \square

Proposition 5.4.7 *Given two Hom-Lie algebras \mathfrak{g} and \mathfrak{h} , we have $U(\mathfrak{g} \oplus \mathfrak{h}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{h})$.*

Proof. We can define a morphisms $\mathfrak{g} \oplus \mathfrak{h} \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{h})$ by $(g, h) \mapsto g \otimes \mathbb{1} + \mathbb{1} \otimes h$. This is clearly a morphism of Hom-Lie algebras, which by the universal property of the universal enveloping algebra can be extended to $U(\mathfrak{g} \oplus \mathfrak{h})$. \square

The comultiplication on $\mathbb{F}_{HAss}(\mathfrak{g})$ induces a comultiplication on $U(\mathfrak{g})$, so $U(\mathfrak{g})$ is a Hom-bialgebra.

The primitive elements form a Hom-Lie algebra, see Proposition 5.3.6, but in general it is not equal to \mathfrak{g} , but only contains \mathfrak{g} as a Hom-Lie subalgebra.

However if α and β are invertible the set of β -primitive elements is precisely \mathfrak{g} .

Different from the ordinary case a \mathfrak{g} -module is in general not a $U(\mathfrak{g})$ -modules. However, it is clear that a $U(\mathfrak{g})$ -bimodule is also a \mathfrak{g} -module.

5.5 HOM-LIE BIALGEBRAS

Definition 5.5.1 (Hom-Lie bialgebra) A Hom-Lie bialgebra is a tuple $(\mathfrak{g}, \nu, \delta, \alpha, \beta)$, such that $(\mathfrak{g}, \nu, \alpha)$ is a Hom-Lie algebra, $(\mathfrak{g}, \delta, \beta)$ is a Hom-Lie coalgebra and they are compatible in the sense that

$$\begin{aligned} \delta([x, y]) &= \alpha(x^{(1)}) \otimes [x^{(2)}, \beta(y)] + [x^{(1)}, \beta(y)] \otimes \alpha(x^{(1)}) \\ &\quad + [\beta(x), y^{(1)}] \otimes \alpha(y^{(1)}) + \alpha(y^{(1)}) \otimes [\beta(x), y^{(2)}]. \end{aligned} \quad (28)$$

The condition (28) is often written as

$$\delta([x, y]) = \text{ad}_{\beta(x)} \delta(y) - \text{ad}_{\beta(y)} \delta(x), \quad (29)$$

where the adjoint representation of \mathfrak{g} on $\mathfrak{g}^{\otimes k}$ is defined by

$$\text{ad}_x(y_1 \otimes \cdots \otimes y_k) = \sum_{i=1}^k \alpha(y_1) \otimes \cdots \otimes [x, y_i] \otimes \cdots \otimes \alpha(y_k). \quad (30)$$

Often only the case $\alpha = \beta$ or $\alpha = \beta^{-1}$ is considered.

There is a properad which described Hom-Lie bialgebras, similar to the properad describing Lie bialgebras. It is generated by $\nu, \delta, \alpha, \beta$.

We note that there is a different properad which describes α -Hom-Lie algebras, where we only have one structure map.

Proposition 5.5.2 (Yau twist) *Let \mathfrak{g} be a Hom-Lie bialgebra and $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Hom-Lie bialgebra morphism then $(\mathfrak{g}, \varphi \circ \nu, \delta, \varphi \circ \alpha, \beta)$, $(\mathfrak{g}, \nu, \varphi \circ \delta, \alpha, \varphi \circ \beta)$ and $(\mathfrak{g}, \varphi \circ \nu, \varphi \circ \delta, \varphi \circ \alpha, \varphi \circ \beta)$ are again Hom-Lie bialgebras.*

Proof. This is a simple calculation. \square

5.6 HOM-MANIN TRIPLES

In this section we give a new definition for a Hom-Manin triple generalizing the known Manin-triples for Lie bialgebras. We will only consider the case where the two structure maps are the same.

We first recall the approach due to Sheng [SB14]: Let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra then we call an inner product $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ invariant if

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle \quad (31)$$

and

$$\langle \alpha(x), y \rangle = \langle x, \alpha(y) \rangle. \quad (32)$$

This approach needs admissible Hom-Lie algebras, i.e. the following condition must be satisfied

$$[\alpha^2(x) - x, \alpha(y)] = 0 \text{ for all } x, y \in \mathfrak{g}. \quad (33)$$

Then given a Hom-Lie bialgebra one can construct a Manin-triple and vice versa.

We want to consider a different approach here. We call an inner product $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ Hom-invariant if it satisfies

$$\langle [x, y], \alpha(z) \rangle = \langle \alpha(x), [y, z] \rangle \quad (34)$$

and

$$\langle \alpha(x), y \rangle = \langle x, \alpha(y) \rangle. \quad (35)$$

The advantage of our approach is that we do not need to consider admissible Lie algebras, but can consider arbitrary ones. The disadvantage is that we do not obtain an equivalence between Hom-Lie bialgebras and Hom-Manin triples, but have to consider weak Manin triples and weak Hom-Lie bialgebras, which we will define below.

Definition 5.6.1 A quadratic Hom-Lie algebra is a Hom-Lie algebra with a symmetric nondegenerate Hom-invariant product.

Definition 5.6.2 A Hom-Manin triple is a triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, such that \mathfrak{g} is a quadratic Hom-Lie algebra, \mathfrak{g}_\pm are isotropic Hom-Lie subalgebras and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Hom-modules.

We will here only consider finite dimensional ones, so Manin triple here always means finite dimensional Manin triple.

If the Hom-Jacobi identity is replaced by the weaker one

$$\sum_{\text{zykl.}} \alpha^2(\langle [x, y], \alpha(z) \rangle) = 0, \quad (36)$$

we call it a weak Manin-triple. Note that if α is invertible a weak Hom-Manin triple is the same as a Hom-Manin triple.

A weak Hom-Lie bialgebra is defined as a Hom-Lie algebra but the compatibility is defined as

$$\alpha^2(\delta([x, y])) = \alpha^2(\text{ad}_{\beta(x)}\delta(y) - \text{ad}_{\beta(y)}\delta(x)). \quad (37)$$

Again if α is invertible this is the same as a Hom-Lie bialgebra.

Given a weak Manin triple, the twisted algebra \mathfrak{g}_α is a Manin triple, and given a weak Hom-Lie bialgebra \mathfrak{g} , \mathfrak{g}_α is a Hom-Lie bialgebra.

Given a Manin triple the inner product induces an isomorphism $\mathfrak{g}_- \cong \mathfrak{g}_+^*$.

Proposition 5.6.3 *Let \mathfrak{g} be a Hom-Lie bialgebra then $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$, with inner product*

$$\langle (x, \varphi), (y, \psi) \rangle = \varphi(y) + \psi(x) \quad (38)$$

for $x, y \in \mathfrak{g}, \varphi, \psi \in \mathfrak{g}^*$ and bracket

$$[(x, \varphi), (y, \psi)] = ([x, y] + \text{ad}_{\alpha(\varphi)}y - \text{ad}_{\alpha(\psi)}x, [\varphi, \psi] + \text{ad}_{\alpha(x)}\psi - \text{ad}_{\alpha(y)}\varphi), \quad (39)$$

is a weak Manin triple.

On the other hand given a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, \mathfrak{g}_+ is a weak Hom-Lie algebra. The coproduct is given by the dual of the bracket restricted to $\mathfrak{g}_- \cong \mathfrak{g}_+^*$.

Proof. The inner product is clearly nondegenerate and symmetric. Also clearly $\langle \alpha(x), y \rangle = \langle x, \alpha(y) \rangle$ holds. For the invariance we have

$$\langle \alpha(x), [y, \psi] \rangle = \langle \alpha(x), \text{ad}_{\alpha(y)}\psi \rangle = \langle [\alpha(y), \alpha(x)], \psi \rangle = \langle [x, y], \alpha(\psi) \rangle.$$

We only need to check the weak-Jacobi identity in the case where one element is in \mathfrak{g}^* . The other cases are clear or dual. For this we have

$$\begin{aligned} \langle \alpha^2([x, [y, \varphi]]), y \rangle &= \langle [x, [y, \varphi]], \alpha^2(z) \rangle \\ &= -\langle \alpha([y, \varphi]), \alpha([x, z]) \rangle \\ &= \langle \alpha^2(\varphi), [\alpha(y), [x, z]] \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \alpha^2([\alpha(\varphi), [x, y]]), z \rangle &= \langle [\alpha(\varphi), [x, y]], \alpha^2(z) \rangle \\ &= \langle \alpha^2(\varphi), [[x, y], \alpha(z)] \rangle. \end{aligned}$$

This shows $\langle [\alpha(\varphi), [x, y]] + [\alpha(y), [\varphi, x]] + [\alpha(x), [y, \varphi]], z \rangle = 0$. On the other hand we have

$$\begin{aligned} \langle [[x, y], \alpha(\varphi)], \alpha^2(\psi) \rangle &= \langle \alpha[x, y], \alpha[\varphi, \psi] \rangle \\ &= \langle \delta(\alpha^2[x, y]), \varphi \otimes \psi \rangle \end{aligned}$$

and

$$\begin{aligned}\langle [\alpha(x), [y, \varphi]], \alpha^2(\psi) \rangle &= \langle \alpha(\text{ad}_{\alpha(y)} \varphi), \text{ad}_{\alpha^2(\psi)} \alpha(x) \rangle + \langle \alpha(\text{ad}_{\alpha(\varphi)} y), \text{ad}_{\alpha^2(x)\alpha(\psi)} \rangle \\ &= \langle [\alpha^2(\psi), \text{ad}_{\alpha^2(y)} \alpha(\varphi)], \alpha(x) \rangle + \langle \alpha(y), [\alpha^2(\varphi), \text{ad}_{\alpha^2(x)} \alpha\psi] \rangle \\ &= \langle (\alpha \otimes \text{ad}_{\alpha(y)}) \delta \alpha^2(x), \varphi \otimes \psi \rangle + \langle \alpha \otimes \text{ad}_{\alpha(x)} \delta \alpha^2(y), \varphi \otimes \psi \rangle.\end{aligned}$$

This shows

$$\langle [\alpha(\varphi), [x, y]] + [\alpha(y), [\varphi, x]] + [\alpha(x), [y, \varphi]], \psi \rangle = 0. \quad (40)$$

Now let \mathfrak{g} be a Hom-Manin triple, we need to show that \mathfrak{g}_+ is a Hom-Lie bialgebra. It is clear that \mathfrak{g}_+ is a Hom-Lie algebra. Further since \mathfrak{g}_- is a Hom-Lie algebra \mathfrak{g}_+ is also a Hom-Lie coalgebra. It remains to show to compatibility. We first note that

$$\begin{aligned}\langle \alpha^2(y), [[x, \varphi]_+, \alpha(\psi)] \rangle &= -\langle \alpha([x, \varphi]_+), \alpha([y, \psi]) \rangle \\ &= -\langle \alpha([x, \varphi]), \alpha([y, \psi]_-) \rangle \\ &= -\langle \alpha^2(x), [\alpha(\varphi), [y, \psi]_-] \rangle.\end{aligned}$$

Here $[\cdot, \cdot]_-$ denotes the projection to \mathfrak{g}_- and similar for \mathfrak{g}_+ . With this we get

$$\begin{aligned}&\langle \alpha^2(\delta([x, y]) - (\text{ad}_{\alpha(x)} \otimes \alpha) \delta(y) - \alpha \otimes (\text{ad}_{\alpha(x)}) \delta(y) \\ &+ (\text{ad}_{\alpha(y)} \otimes \alpha) \delta(x) + \alpha \otimes (\text{ad}_{\alpha(y)}) \delta(x)), \varphi \otimes \psi \rangle \\ &= \langle \alpha[x, y], \alpha([\varphi, \psi]) \rangle + \langle \alpha^2(y), [[x, \varphi]_-, \alpha(\psi)] \rangle + \langle \alpha^2(y), [\alpha(\varphi), [x, \psi]_-] \rangle \\ &\quad - \langle \alpha^2(x), [[y, \varphi]_-, \alpha(\psi)] \rangle - \langle \alpha^2(x), [\alpha(\varphi), [y, \psi]_-] \rangle \\ &= \langle \alpha^2(y), -[\alpha(x), [\varphi, \psi]] + [[x, \varphi], \alpha(\psi)] + [\alpha(\varphi), [x, \psi]] \rangle = 0.\end{aligned}$$

Where the last equality follows from the Hom-Jacobi identity in \mathfrak{g} . \square

5.7 HOM-POISSON ALGEBRAS

Definition 5.7.1 A Hom-Poisson algebra is a tuple $(A, \mu, \{\cdot, \cdot\}, \alpha)$, such that (A, μ, α) is a Hom-commutative algebra and $(A, \{\cdot, \cdot\}, \alpha)$ is a Hom-Lie algebra, which are compatible in the sense that they satisfy the Hom-Leibniz identity

$$\{\alpha(a), bc\} = \alpha(b)\{a, c\} + \{a, b\}\alpha(c). \quad (41)$$

It is called multiplicative if μ and the Hom-Poisson bracket are multiplicative.

One can also consider the case, where μ is not commutative, in this case, we call A a noncommutative (nc.) Hom-Poisson algebra. In this case also the other Leibniz identity $\{ab, \alpha(c)\} = \alpha(a)\{b, c\} + \{a, c\}\alpha(b)$ has to be satisfied. A Hom-algebra, with the

Lie bracket given by the commutator is a nc. Hom-Poisson algebra. (And this is the only case we need).

One can give a slight generalization of this, where one considers the structure maps for the commutative and Lie algebra to be different.

Definition 5.7.2 A Hom-Poisson algebra is a tuple $(A, \mu, \alpha, \{\cdot, \cdot\}, \beta)$, such that (A, μ, α) is a Hom-commutative algebra and $(A, \{\cdot, \cdot\}, \beta)$ is a Hom-Lie algebra, which are compatible in the sense that they satisfy the Hom-Leibniz identity

$$\{\alpha(a), bc\} = \beta(b)\{a, c\} + \{a, b\}\beta(c). \quad (42)$$

A morphism of a Hom-Poisson algebra is a morphism of the commutative Hom- and the Hom-Poisson algebra.

Proposition 5.7.3 (Yau twist) *Let $(A, \mu, \alpha, \{\cdot, \cdot\}, \beta)$ be a Hom-Poisson algebra and γ a morphism of it. Then $(A, \gamma\mu, \gamma\alpha, \{\cdot, \cdot\}, \beta)$ and $(A, \mu, \alpha, \gamma\{\cdot, \cdot\}, \gamma\beta)$ are again Hom-Poisson algebras. So the two products can be twisted separately.*

Let $(A, \mu, \{\cdot, \cdot\}, \alpha)$ be a Hom-Poisson algebra and γ a morphism of it. Then $(A, \gamma\mu, \gamma\{\cdot, \cdot\}, \gamma\alpha)$ is again a Hom-Poisson algebra.

Definition 5.7.4 (Hom-Poisson module) Let P be a Hom-Poisson algebra. Then a Hom-Poisson module is a Hom-module (M, α_M) with a module structure $P \otimes M \rightarrow M, a \otimes m \mapsto a \cdot m$ and an map $P \otimes M \rightarrow M$ written as $a \otimes m \mapsto \{a, m\}$, such that

$$\{\alpha(a), \{b, m\}\} = \{\{a, b\}, \alpha_M(m)\} + \{\alpha(b), \{a, m\}\} \quad (43)$$

$$\{ab, \alpha_M(m)\} = \alpha(a) \cdot \{b, m\} + \{a, m\} \cdot \alpha(b) \quad (44)$$

$$\{\alpha(a), b \cdot m\} = \alpha(b) \cdot \{a, m\} + \{a, b\} \cdot \alpha_M(m) \quad (45)$$

for $a, b \in P$ and $m \in M$. In the multiplicative case we also require

$$\alpha_M\{a, m\} = \{\alpha(a), \alpha_M(m)\}. \quad (46)$$

There is also the dual notion of a Hom-Poisson coalgebra.

6. COHOMOLOGY

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Now we came to one of the main parts of this thesis. In this chapter we first recall the definitions for the different cohomologies for multiplicative Hom-algebras and Hom-bialgebras and then extend them to α -type cohomologies. The advantage of these α -type cohomologies is that they can be used to study deformations of Hom-algebras, where the multiplication and the structure map are deformed. This is one reason why we call them α -type cohomologies. In the α -type cohomology the cochains are given by pairs of multilinear maps. This chapter has several sections. We first give some remarks on the cohomology of endomorphism, since we will need it later in the study of α -type cohomologies of Hom-algebras of associative

6. Cohomology

or Lie-type. Then we recall the known Hochschild cohomology for Hom-associative algebras and the Gerstenhaber bracket, which can be used to define it. In Section 6.3 we define the α -type Hochschild cohomology and study it in more detail. In the following we do the same for Hom-bialgebras, Hom-Lie algebras and bialgebras. In Section 6.12 we define a cohomology for a morphism of Hom-associative algebras.

6.1 ALGEBRA ENDOMORPHISMS

We want to give a general definition of a cohomology for an algebra and an endomorphism of it. This is well known for the different types of algebras and arbitrary morphisms. For the case of associative algebras see [GS83, GS85] and for the case of Lie algebras [NR67]. More recently an L_∞ -structure for associative algebras, Lie and Lie-bialgebras was found in [FMY09, FZ15].

So let A be an algebra (one could consider algebras over a Koszul operad, but we will only consider the case of associative and Lie algebras here). We denote by $C^\bullet(A)$ the cohomology complex of it. Then we define a new complex for the cohomology of an algebra endomorphism $\gamma : A \rightarrow A$ by

$$C^n(\gamma) = C_\mu^n(\gamma) \oplus C_\gamma^n(A) = C^n(A) \oplus C^{n-1}(A, \tilde{A}) \quad (1)$$

for $n \geq 2$, $C^1(A, \gamma) := C(A)$ and $C^n(A, \gamma) = 0$ for $n \leq 0$, where \tilde{A} is the algebra A regarded as an A -module with the action twisted by γ , i.e. the action is given by $a \cdot b = \gamma(a)b$ for $a \in A, b \in \tilde{A}$.

Note that again we have used $\{0\}$ instead of $\text{Hom}(\mathbb{K}, \cdot)$ in the low degrees since this simplifies making the connection to the α -type cohomology.

We denote the differential of $C^\bullet(A)$ by ∂_A and define $\partial_\gamma \varphi = \gamma\varphi - \varphi\gamma^{\otimes n}$ for $\varphi \in C^n(A)$. With this we further define

$$\partial(\varphi, \psi) = (\partial_A \varphi, \partial_\gamma \varphi - \partial_A \psi). \quad (2)$$

Proposition 6.1.1 *The map ∂ defined above is a differential. The complex $C^\bullet(\gamma)$ can be seen as a bicomplex, with only two rows.*

Proof. We only have to prove $\partial_A \partial_\gamma = \partial_\gamma \partial_A$. This can be done using the explicit differential for Lie and associative algebras and is a short and straight forward calculation. We only do it for the case of associative algebras. Here we have

$$\begin{aligned} (\partial_A \partial_\gamma \varphi)(x_1, \dots, x_{n+1}) &= \gamma(x_1) \gamma \varphi(x_1, \dots, x_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i \gamma \varphi(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) \\ &+ (-1)^{n+1} \gamma \varphi(x_1, \dots, x_n) \gamma(x_{n+1}) \\ &- \gamma(x_1) \varphi(\gamma(x_1), \dots, \gamma(x_{n+1})) \\ &- \sum_{i=1}^n (-1)^i \varphi(\gamma(x_1), \dots, \gamma(x_i x_{i+1}), \dots, \gamma(x_{n+1})) \\ &- (-1)^{n+1} \varphi(\gamma(x_1), \dots, \gamma(x_n)) \gamma(x_{n+1}) \end{aligned}$$

$$\begin{aligned}
 (\partial_\gamma \partial_A \varphi) &= \gamma(x_1 \varphi(x_1, \dots, x_{n+1})) \\
 &+ \sum_{i=1}^n (-1)^i \gamma \varphi(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) \\
 &+ (-1)^{n+1} \gamma(\varphi(x_1, \dots, x_n) x_{n+1}) \\
 &- \gamma(x_1) \varphi(\gamma(x_1), \dots, \gamma(x_{n+1})) \\
 &- \sum_{i=1}^n (-1)^i \varphi(\gamma(x_1), \dots, \gamma(x_i x_{i+1}), \dots, \gamma(x_{n+1})) \\
 &- (-1)^{n+1} \varphi(\gamma(x_1), \dots, \gamma(x_n)) \gamma(x_{n+1}).
 \end{aligned}$$

It is obvious that the two sides agree since γ is an algebra morphism. \square

We assign a height to the elements in the cohomology. We set $\text{hgt}(\varphi) = 0$ for $\varphi \in C_\mu(\gamma)$ and $\text{hgt}(\psi) = 1$ for $\psi \in C_\gamma(\gamma)$. We also note that with $\text{hgt}(\gamma) = 1$ and $\text{hgt}(\mu) = 0$ the different parts of the differential are homogeneous, namely ∂_γ has height 1 and ∂_A has height 0. This is compatible with the heights we assigned to the different parts of the complex.

We denote the associated cohomology by $H^\bullet(A, \gamma)$ or just $H^\bullet(\gamma)$.

This cohomology can be computed using spectral sequences. But since the bi-complex only has two rows not much of the theory is needed. For more on spectral sequences see e.g. [Wei94, Section 5]. We denote by $H(C(\gamma), \partial_A)$ the cohomology with respect to ∂_A . On this there is a differential induced by ∂_γ . With this we have

Proposition 6.1.2 *For the cohomology of the total complex we have*

$$H(C^\bullet(\gamma), \partial) = H(H(C(\gamma), \partial_A), \partial_\gamma).$$

Proof. This follows directly considering the spectral sequence associated to the bi-complex $C(\gamma)$ with the vertical filtration, since then the differential on the second sheet already vanishes, since the complex only has two rows. \square

If one knows the cohomology of the algebra A well, $H(H(C(\gamma), \partial_A), \partial_\gamma)$ can be computed quite easily. Also note that there is a chain map $(C_\mu^\bullet(\gamma), \partial_A) \rightarrow (C_\gamma^{\bullet+1}(\gamma), \partial_A)$ given by $\varphi \mapsto \gamma\varphi$, which is even an isomorphism if γ is invertible.

On the other hand the other iterated cohomology $H(H(C(\gamma), \partial_\gamma), \partial_A)$ is in general not isomorphic to the cohomology $H(\gamma)$, since there is still a non-trivial differential on it. Only the cohomology with respect this differential gives the total cohomology. However it turns out that this differential vanishes in nice situations. If this is the case every cocycle is cohomologous to either one if the form $(\varphi, 0)$ or one of the form $(0, \psi)$.

For the case of Hom-bialgebras, we also need the case of two commuting endomorphisms. So let A be an algebra and γ_1, γ_2 be two commuting algebra endomorphisms, then we define

$$C^n(\gamma_1, \gamma_2) = C_\mu^n \oplus C_{\gamma_1}^n \oplus C_{\gamma_2}^n \oplus C_{com}^n = C^n(A) \oplus C^{n-1}(A) \oplus C^{n-1}(A) \oplus C^{n-2}(A). \quad (3)$$

The left action on A in $C_{\gamma_i}^n$ is given by $a \cdot b = \gamma_i(a)b$ for $i = 1, 2$ and similar for the right action. On A in C_{com}^n it is given by $a \cdot b = \gamma_1\gamma_2(a)b$.

The differential for $(\varphi, \psi, \chi, \xi) \in C^n(\gamma_1, \gamma_2)$ is given by

$$\partial\varphi = (\partial_A\varphi, \partial_{\gamma_1}\varphi, \partial_{\gamma_2}\varphi, 0) \quad (4)$$

$$\partial\psi = (0, -\partial_A\psi, 0, \partial_{\gamma_2}\psi) \quad (5)$$

$$\partial\chi = (0, 0, -\partial_A\chi, -\partial_{\gamma_1}\psi) \quad (6)$$

$$\partial\xi = (0, 0, 0, \partial_A\xi). \quad (7)$$

This can also be seen as a bicomplex with three rows, the first being C_μ^n , the second $C_{\gamma_1}^n \oplus C_{\gamma_2}^n$ and the third C_{com}^n . It can also be seen as a tricomplex, with degree $C_\mu^n = (n, 0, 0)$, $C_{\gamma_1}^n = (n-1, 0, 1)$, $C_{\gamma_2}^n = (n-1, 0, 1)$ and $C_{com}^n = (n-2, 1, 1)$. Similar to Proposition 6.1.1 it is easy to see that $\partial \circ \partial = 0$ so $C^\bullet(\gamma_1, \gamma_2)$ is in a fact a cochain complex and one can define the corresponding cohomology.

Similar to the above one can construct a cohomology for a coalgebra and an endomorphism of it.

Next we give a similar description for the cohomology of a bialgebra morphism. Here we will focus on the case of Lie-bialgebras, but it works similar for associative bialgebras.

Given a Lie-bialgebra \mathfrak{g} and an endomorphism $\gamma : \mathfrak{g} \rightarrow \mathfrak{g}$, we can define a cohomology for γ . For the case of arbitrary Lie-algebra morphisms this can be found in [FZ15]. We modify the definition slightly to better agree with the definition of the α -type cohomology. The complex is given by

$$C^n(\mathfrak{g}, \gamma) = C_\mu^n(\mathfrak{g}) \oplus C_\gamma^n = C^n(\mathfrak{g}) \oplus C^{n-1}(\mathfrak{g}) \quad (8)$$

for $n \in \mathbb{N}$. Here $C^n(\mathfrak{g})$ denotes the Chevalley-Eilenberg complex of \mathfrak{g} , modified such that it begins in degree 1. We note that $C^\bullet(\mathfrak{g})$ is a bicomplex, so the same is true for $C_\mu^n(\mathfrak{g})$ and C_γ^n . Note however that the degree of $C_\gamma^{i,j}$ is $i + j + 1$.

The differential is given by

$$\partial\varphi = (\partial_{CE}\varphi, \partial_\gamma\varphi, \partial_{CE}^c\varphi) \in C_\mu^{i+1,j} \oplus C_\alpha^{i,j} \oplus C_\mu^{i,j+1} \quad (9)$$

$$\partial\psi = (-\partial_{CE}\psi, -\partial_{CE}^c\psi) \in C_\alpha^{i+1,j} \oplus C_\alpha^{i,j+1} \quad (10)$$

for $\varphi \in C_\mu^{i,j}(\gamma)$ and $\psi \in C_\gamma^{i,j}(\gamma)$. Here ∂_{CE} is the ordinary Chevalley-Eilenberg differential and ∂_{CE}^c its dual. The action on ψ is given by $x \cdot \psi = \gamma(x)\psi$ and similar for the coaction.

It is also possible to define a slightly different complex, where the morphism is only regarded as a morphism of the Lie algebra or the Lie-coalgebra.

At last we consider the situation of two commuting morphism, where one is regarded as an algebra and one as a coaction morphism. This is interesting since in the case Hom-associative bialgebras the complex is the same as the one for the α -type cohomology and they are related by the Yau twist.

The complex is a bicomplex and given by

$$\begin{aligned} C^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) &= C_\mu^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \oplus C_{\gamma_1}^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \oplus C_{\gamma_2}^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \oplus C_{com}^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2) \\ &= C^{i,j}(\mathfrak{g}) \oplus C^{i-1,j}(\mathfrak{g}) \oplus C^{i,j-1}(\mathfrak{g}) \oplus C^{i-1,j-1}(\mathfrak{g}). \end{aligned}$$

The algebra differential, i.e. the horizontal differential, for $(\varphi, \psi, \chi, \xi) \in C^{i,j}(\mathfrak{g}, \gamma_1, \gamma_2)$ is given by

$$\partial(\varphi, \psi, \chi, \xi) = (\partial_{CE}\varphi, \partial_{\gamma_1}\varphi - \partial_{CE}\psi, \partial_{\gamma_2}\varphi - \partial_{CE}\chi, -\partial_{\gamma_2}\psi - \partial_{\gamma_1}\chi + \partial_{CE}\xi).$$

Dually one can define a coalgebra differential. In fact this is a tetra-complex, where in two directions there are only two elements. The differential in the different directions is given by the algebra differential, its dual, ∂_{γ_1} and ∂_{γ_2} . Since all differentials in the different directions anti-commute it is clear that ∂ defined above is a differential. So we can define the corresponding cohomology which we denote by $H^\bullet(\gamma_1, \gamma_2)$.

6.2 HOM-ASSOCIATIVE ALGEBRAS

We recall the definition of the cohomology of Hom-associative algebras as given in [AEM11, MS08].

We define the complex of cochains for the Hochschild cohomology of a Hom-associative algebra (A, μ, α) with values in an A -module (M, β) by the set of Hom-module morphisms

$$HC^n(A, M) = \text{Hom}((A^{\otimes n}, \alpha^{\otimes n}), (M, \beta)) \quad (11)$$

for $n \geq 1$ and $HC^n(A, M) = 0$ for $n < 1$. We call this the Hochschild complex of A with values in M . Note that for $\alpha = \text{id}$ this agrees with the classical Hochschild cohomology, starting in degree 1, so the name and notation are always unambiguous. We let the complex begin in degree 1, because the differential in degree zero does not exist since it would involve α^{-1} .

Note that since we are considering multiplicative Hom-algebras the multiplication is in this complex. In the non multiplicative case this would not be true.

Next we define the codifferential $\delta : \text{HC}^\bullet(A, M) \rightarrow \text{HC}^{\bullet+1}(A, M)$ by

$$\begin{aligned} (\delta\varphi)(a_1, \dots, a_{n+1}) &= \alpha^{n-1}(a_1) \cdot \varphi(a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i \varphi(\alpha(a_1), \dots, a_i a_{i+1}, \dots, \alpha(a_{n+1})) \\ &+ (-1)^{n+1} \varphi(a_1, \dots, a_n) \cdot \alpha^{n-1}(a_{n+1}). \end{aligned} \quad (12)$$

In low degree the differential is given explicitly by

$$\begin{aligned} (\delta\varphi)(a, b) &= a\varphi(b) - \varphi(ab) + \varphi(a)b, \\ (\delta\varphi)(a, b, c) &= \alpha(a)\varphi(b, c) - \varphi(\alpha(a), bc) + \varphi(ab, \alpha(c)) - \varphi(a, b)\alpha(c). \end{aligned}$$

So the first cohomology is given precisely by the derivations of A , which commute with α .

The map δ squares to zero, so we can define a cohomology, which we call Hochschild cohomology and denote by $\text{HH}^\bullet(A, M)$.

We can associate a height to the different maps in the cohomology by setting $\text{hgt } \varphi = n-1$ for $\varphi \in \text{HC}^n(A)$. This is consistent with $\text{hgt } \mu = 1$, since μ can be regarded as an element of $\text{HC}^2(A)$. The structure map α cannot be seen as part of the complex. Also the differential respects this height.

Gerstenhaber algebra

The cohomology in this case can be described by using a modified version of the classical Gerstenhaber bracket [AEM11]. For this we need a shifted version of the Hochschild complex. First we note that the Hochschild complex, without differential, can be defined for a Hom-module instead of a Hom-associative algebra. So given a Hom-module (A, α) we define $\text{deg } \varphi = n-1$ for $\varphi \in \text{HC}^n(A)$, and call this the deg-degree of phi. The other non-shifted degree we call tensor degree.

For $\varphi \in \text{HC}^{k+1}(A)$ and $\psi \in \text{HC}^{l+1}(A)$ we define

$$(\varphi \circ_i \psi)(a_0, \dots, a_{k+l}) = \varphi(\alpha^l(a_0), \dots, \alpha^l(a_{i-1}), \psi(a_i, \dots, a_{i+l}), \alpha^l(a_{i+l+1}), \alpha^l(a_{k+l})) \quad (13)$$

and with this

$$\varphi \circ \psi = \sum_i (-1)^{il} \varphi \circ_i \psi \quad (14)$$

and finally the Gerstenhaber bracket by

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{kl} \psi \circ \varphi. \quad (15)$$

6. Cohomology

Lemma 6.2.1 *The product \circ defined above is graded preLie, with respect to the degree, this means it satisfies*

$$(\varphi \circ \psi) \circ \chi - \varphi \circ (\psi \circ \chi) = (-1)^{\deg(\psi)\deg(\chi)}((\varphi \circ \chi) \circ \psi - \varphi \circ (\psi \circ \chi)).$$

Proof. This is a straight forward calculation. \square

This proves the following

Proposition 6.2.2 *The Gerstenhaber bracket $[\cdot, \cdot]$ defined above is a graded Lie-bracket.*

Note that the Gerstenhaber product is defined such the height of $\varphi \circ \psi$ is $k+l$ as necessary for a map in $\mathrm{HC}^{k+l+1}(A)$.

For this Gerstenhaber bracket a Maurer-Cartan element μ , i.e. an element which satisfies $[\mu, \mu] = 0$, gives a Hom-associative algebra (A, μ, α) . Note that α is determined by the complex and not part of the Maurer-Cartan element.

With this the differential in the previous section, can be defined by $\delta\varphi = -[\varphi, \mu]$.

Given a Hom-associative algebra, there is also a cup product $\cup : \mathrm{HC}^j(A) \otimes \mathrm{HC}^k(A) \rightarrow \mathrm{HC}^{j+k}(A)$ defined by

$$(\psi \cup \varphi)(x_1, \dots, x_{j+k}) = \alpha^{k-1}(\psi(x_1, \dots, x_j))\alpha^{j-1}(\varphi(x_{j+1}, \dots, x_{j+k})) = (\mu \circ_0 \psi) \circ_j \varphi. \quad (16)$$

Again the cup product is defined such that it respects the height of the different maps.

Proposition 6.2.3 *The cup product is associative and symmetric up to a coboundary, i.e. for $\varphi \in \mathrm{HC}^k(A), \psi \in \mathrm{HC}^l(A)$ we have*

$$\psi \cup \varphi - (-1)^{kl} \varphi \cup \psi = \delta\varphi \cup \psi - (-1)^{k-1} \delta(\varphi \cup \psi) + (-1)^{l-1} \varphi \cup \delta\psi. \quad (17)$$

Proof. In this proof and the following ones we use the notation $x_{1,n} := x_1 \otimes \dots \otimes x_n$. We compute for $\varphi \in C^{k-1}(A), \psi \in C^{l-1}(A), \chi \in C^{m-1}(A)$

$$((\varphi \cup \psi) \cup \chi)(x_{1,k+l+m}) = \left(\alpha^{l-1+m-1} \varphi(x_{1,k}) \alpha^{k-1+m-1} \psi(x_{k+1,k+l}) \right) \alpha^{l+k-1} \chi(x_{k+l+1,k+l+m}) \quad (18)$$

and on the other side

$$(\varphi \cup (\psi \cup \chi))(x_{1,k+l+m}) = \alpha^{l+m-1} \varphi(x_{1,k}) \left(\alpha^{k-1+m-1} \psi(x_{k+1,k+l}) \alpha^{l-1+k-1} \chi(x_{k+l+1,k+l+m}) \right) \quad (19)$$

So the cup product is associative since the product on A is Hom-associative.

Equation (17) follows similar to the calculation [Ger63] since \circ is also graded preLie. \square

Proposition 6.2.4 *The Hochschild differential is a graded derivation of the cup product, i.e. $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$. So it induces a commutative associative product in cohomology.*

Proof. We calculate

$$\begin{aligned} (\delta(\varphi \cup \psi))(x_1, \dots, x_{k+l+1}) &= \alpha^{k+l-1}(x_1)(\alpha^{l-1}\varphi(x_2, \dots)\alpha^{k-1}\psi(x_{k+2}, \dots)) \\ &\quad + \sum_{i=1}^k (-1)^i \alpha^{l-1}\varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots)\alpha^{k-1}\psi(\alpha(x_{k+2}), \dots) \\ &\quad + \sum_{i=k+1}^{k+l} (-1)^i \alpha^{l-1}\varphi(\alpha(x_1), \dots)\alpha^{k-1}\psi(\alpha(x_{k+1}), \dots, x_i x_{i+1}, \dots) \\ &\quad + (-1)^{k+l+1}(\alpha^{l-1}\varphi(x_1, \dots)\alpha^{k-1}\psi(x_{k+1}, \dots))\alpha^{k+l-1}(x_{k+l+1}) \end{aligned}$$

and for the right hand side

$$\begin{aligned} (\delta\varphi \cup \psi)(x_1, \dots, x_{k+l+1}) &= \alpha^{l-1}(\alpha^{k-1}(x_1)\varphi(x_2, \dots))\alpha^k\psi(x_{k+2}, \dots) \\ &\quad + \sum_{i=1}^k (-1)^i \alpha^{l-1}\varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots)\alpha^k\psi(x_{k+2}, \dots) \\ &\quad + (-1)^{k+1}\alpha^{l-1}(\varphi(x_1, \dots)\alpha^{k-1}(x_{k+1}))\alpha^k\psi(x_{k+2}, \dots), \end{aligned}$$

$$\begin{aligned} (\varphi \cup \delta\psi)(x_1, \dots, x_{k+l+1}) &= \alpha^l\varphi(x_1, \dots)\alpha^{k-1}(\alpha^{l-1}(x_{k+1})\psi(x_{k+2}, \dots)) \\ &= \sum_{i=k+1}^{k+l} (-1)^{i-k}\alpha^l\varphi(x_1, \dots)\alpha^{k-1}(\varphi(\alpha(x_{k+1}), \dots, x_i x_{i+1}, \dots)) \\ &= (-1)^{l+1}\alpha^l\varphi(x_1, \dots)\alpha^{k-1}(\varphi(x_{k+1}, \dots)\alpha^{l-1}(x_{k+l+1})). \end{aligned}$$

Using the fact that α commutes with everything and the Hom-associativity, this gives the desired equality. The fact that \cup induces a product in cohomology is standard. It is commutative because of Proposition 6.2.3. \square

Remark 6.2.5 Using the same method as in [Ger63] it is possible to prove that the cup-product satisfies a graded Leibniz identity with respect to the Gerstenhaber bracket in cohomology. So in cohomology one has in fact a Gerstenhaber algebra. It is also easy to see that one can define a braces algebra on $\text{HC}(A)$, for this it also follows that in cohomology there is a Gerstenhaber algebra [DW17]. The braces are defined as

$$\{\varphi | \psi_1, \dots, \psi_n\}(a_1, \dots, a_N) = \pm \varphi(\alpha^{l-n}(a_1), \dots, \alpha^{l-n-l_1+1}\psi_1(a, \dots), a, \dots, \alpha^{l-n-l_2+1}\psi_2(a, \dots)) \quad (20)$$

with $\psi_i \in \mathrm{HC}^{l_i}(A)$ for $i = 1, \dots, n$ and $\varphi \in \mathrm{HC}^k(A)$ $l = \sum_{i=1}^n l_i, N = l - n + k$. This is homogeneous of height $l - n + k - 1 = N - 1$ as expect for an element in $\mathrm{HC}^N(A)$. Since every map is compatible with α this satisfies the same relation as in the ordinary case.

It is also possible to define a cup product of the form $\varphi \cup \psi = \mu(\varphi \otimes \psi)$, which is Hom-associative. However the differential is not a derivation for it, so it not clear whether this induces a product in cohomology.

6.3 α -TYPE COHOMOLOGY FOR HOM-ASSOCIATIVE ALGEBRAS

In this section we define a cohomology for Hom-associative algebras, which takes also into account the structure map α . This section closely follows [HM18], where this cohomology was introduced. We will see that it is a generalization of the cohomology which is usually considered, see Section 6.2. The complex is different from the one used for the Hochschild cohomology, in the sense that the cochains are given by pairs. In the case of associative algebras, i.e. $\alpha = \mathrm{id}$, it is completely determined by the Hochschild cohomology, but in the general case it contains more information. It extends the cohomology given in [MS10]. With this cohomology it is also possible to consider deformations of Hom-associative algebras, where the multiplication and the structure map are deformed. We will study this in Section 8.1.

We start by giving the complex for the cohomology of a Hom-associative algebra A with values in itself. The cochains are given by

$$\tilde{\mathrm{HC}}^n(A) = \tilde{\mathrm{HC}}_\mu^n(A) \oplus \tilde{\mathrm{HC}}_\alpha^n(A) = \mathrm{Hom}(A^{\otimes n}, A) \oplus \mathrm{Hom}(A^{\otimes n-1}, A) \text{ for } n \geq 2, \quad (21)$$

$\tilde{\mathrm{HC}}^1(A) = \tilde{\mathrm{HC}}_\mu^1(A) \oplus \tilde{\mathrm{HC}}_\alpha^1(A) = \mathrm{Hom}(A, A) \oplus \{0\}$ and $\tilde{\mathrm{HC}}^n(A) = \{0\}$ for $n \leq 0$. In general we will denote an n -cochain by the pair (φ, ψ) , where $\varphi \in \tilde{\mathrm{HC}}_\mu^n(A)$ and $\psi \in \tilde{\mathrm{HC}}_\alpha^n(A)$. We have $\alpha \in \tilde{\mathrm{HC}}_\alpha^2(A)$ and $\mu \in \tilde{\mathrm{HC}}_\mu^2(A)$, which motivates the names for the two summands.

Remark 6.3.1 Note that $\tilde{\mathrm{HC}}^0(A)$ is the zero space and not as one might expect $\mathrm{Hom}(\mathbb{K}, A) \cong A$. This is so, because otherwise the differential would involve α^{-1} , what we do not want, since we consider α to be non-invertible. The same is true for $\tilde{\mathrm{HC}}_\alpha^1$. If α is invertible one can add these components and the corresponding differentials. In this case also Theorem 6.3.5 simplifies. But we will not consider this further.

We define four maps with domain and range given in the following diagram:

$$\begin{array}{ccc}
 \tilde{\mathrm{HC}}_{\mu}^n & \xrightarrow{\partial_{\mu\mu}} & \tilde{\mathrm{HC}}_{\mu}^{n+1} \\
 & \searrow \partial_{\mu\alpha} & \nearrow \partial_{\alpha\mu} \\
 \oplus & & \oplus \\
 \tilde{\mathrm{HC}}_{\alpha}^n & \xrightarrow{\partial_{\alpha\alpha}} & \tilde{\mathrm{HC}}_{\alpha}^{n+1}
 \end{array}$$

First the classical Hochschild differential for Hom-algebras $\partial_{\mu\mu} : \tilde{\mathrm{HC}}_{\mu}^n \rightarrow \tilde{\mathrm{HC}}_{\mu}^{n+1}$

$$\begin{aligned}
 \partial_{\mu\mu}\varphi(x_1, \dots, x_{n+1}) &= \alpha^{n-1}(x_1)\varphi(x_2, \dots, x_{n+1}) \\
 &+ \sum_{i=1}^n (-1)^i \varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1})) \\
 &+ (-1)^{n+1} \varphi(x_1, \dots, x_{n-1}) \alpha^{n-1}(x_{n+1}).
 \end{aligned} \tag{22}$$

The map $\partial_{\alpha\alpha} : \tilde{\mathrm{HC}}_{\alpha}^{n-1} \rightarrow \tilde{\mathrm{HC}}_{\alpha}^n$ is also the classical differential for Hom-associative algebras, but the bimodules structure is modified by α , this means

$$\begin{aligned}
 \partial_{\alpha\alpha}\psi(x_1, \dots, x_n) &= \alpha^{n-1}(x_1)\psi(x_2, \dots, x_n) \\
 &+ \sum_{i=1}^{n-1} (-1)^i \psi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n)) \\
 &+ (-1)^n \psi(x_1, \dots, x_{n-1}) \alpha^{n-1}(x_n),
 \end{aligned} \tag{23}$$

the map $\partial_{\mu\alpha} : \tilde{\mathrm{HC}}_{\mu}^{n-1} \rightarrow \tilde{\mathrm{HC}}_{\alpha}^n$ is the commutator of α and φ defined by

$$\partial_{\mu\alpha}\varphi(x_1, \dots, x_n) = \alpha(\varphi(x_1, \dots, x_n)) - \varphi(\alpha(x_1), \dots, \alpha(x_n)) \tag{24}$$

and finally $\partial_{\alpha\mu} : \tilde{\mathrm{HC}}_{\alpha}^{n-1} \rightarrow \tilde{\mathrm{HC}}_{\mu}^n$ by

$$\partial_{\alpha\mu}\psi(x_1, \dots, x_{n+1}) = \alpha^{n-2}(x_1 x_2) \psi(x_3, \dots, x_{n+1}) - \psi(x_1, \dots, x_{n-1}) \alpha^{n-2}(x_n x_{n+1}). \tag{25}$$

With this we set

$$\begin{aligned}
 \partial(\varphi + \psi) &= (\partial_{\mu\mu} + \partial_{\mu\alpha})\varphi - (\partial_{\alpha\mu} + \partial_{\alpha\alpha})\psi \\
 &= (\partial_{\mu\mu}\varphi - \partial_{\alpha\mu}\psi, \partial_{\mu\alpha}\varphi - \partial_{\alpha\alpha}\psi).
 \end{aligned} \tag{26}$$

Note that we are considering a difference in this formula. We have chosen this convention, because this way the single differentials look more natural.

In total we get the following:

6. Cohomology

Theorem 6.3.2 *The complex $\tilde{H}C^\bullet(A)$ is a chain complex with differential $\partial(\varphi, \psi) = (\partial_{\mu\mu} + \partial_{\mu\alpha})\varphi - (\partial_{\alpha\mu} + \partial_{\alpha\alpha})\psi$ defined as above.*

Proof. We only need to show $\partial^2 = 0$. This is a lengthy but straight forward calculation. We will give it here to some extent.

$$\begin{aligned} \partial_{\mu\mu}\partial_{\mu\mu}\varphi(x_1, \dots, x_{n+2}) &= \alpha^n(x_1)(\partial_{\mu\mu}\varphi)(x_2, \dots, x_{n+2}) \\ &+ \sum_{i=1}^{n+1} (-1)^i (\partial_{\mu\mu}\varphi)(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+2})) \\ &+ (-1)^n (\partial_{\mu\mu}\varphi)(x_1, \dots, x_{n+1}) \alpha^n(x_{n+2}) \\ &= \alpha^n(x_1)(\alpha^{n-1}(x_2)\varphi(x_3, \dots, x_{n+2})) \\ &- \sum_{i=2}^{n+1} (-1)^i \alpha^n(x_1)\varphi(\alpha(x_2), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+2})) \end{aligned} \quad (27)$$

$$\begin{aligned} &- (-1)^n \alpha^n(x_1)(\varphi(x_2, \dots, x_{n+1}) \alpha^{n-1}(x_{n+2})) \\ &+ \alpha^{n-1}(x_1 x_2)\varphi(\alpha(x_3), \dots, \alpha(x_{n+2})) \end{aligned} \quad (28)$$

$$+ \sum_{i=2}^{n+1} (-1)^i \alpha^n(x_1)\varphi(\alpha(x_2), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+2})) \quad (29)$$

$$+ \sum_{i=1}^{n+1} \sum_{j=1}^{i-2} (-1)^{i+j} \varphi(\alpha^2(x_1), \dots, \alpha(x_j x_{j+1}), \dots, \alpha(x_i x_{i+1}), \dots, \alpha^2(x_{n+2})) \quad (30)$$

$$- \sum_{i=2}^{n+1} \varphi(\alpha^2(x_1), \dots, \alpha(x_{i-1})(x_i x_{i+1}), \dots, \alpha^2(x_{n+2})) \quad (31)$$

$$+ \sum_{i=1}^n \varphi(\alpha^2(x_1), \dots, (x_{i-1} x_i) \alpha(x_{i+1}), \dots, \alpha^2(x_{n+2})) \quad (32)$$

$$+ \sum_{i=1}^{n+1} \sum_{j=i+2}^{n+1} (-1)^{i+j-1} \varphi(\alpha^2(x_1), \dots, \alpha(x_i x_{i+1}), \dots, \alpha(x_j x_{j+1}), \dots, \alpha^2(x_{n+2})) \quad (33)$$

$$+ \sum_{i=1}^n (-1)^{i+n+1} \varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1})) \alpha^n(x_{n+2}) \quad (34)$$

$$\begin{aligned} &+ \varphi(\alpha(x_1), \dots, \alpha(x_n)) \alpha^{n-1}(x_{n+1} x_{n+2}) \\ &+ (-1)^n (\alpha^n(x_1)\varphi(x_2, \dots, x_{n+1})) \alpha^n(x_{n+2}) \end{aligned} \quad (35)$$

$$+ \sum_{i=1}^n (-1)^{i+n} \varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1})) \alpha^n(x_{n+2}) \quad (36)$$

$$-(\varphi(x_1, \dots, x_n) \alpha^{n-1}(x_{n+1})) \alpha^n(x_{n+2}).$$

The terms (31) cancels with (32), (27) with (29), (34) with (36) and (28) with (35). Exchanging the indices in (33) one sees that it cancels with (30). The remaining four terms can be easily arranged to

$$\partial_{\alpha\mu} \partial_{\mu\alpha} \varphi(x_1, \dots, x_{n+2}) = \alpha^{n-1}(x_1 x_2) (\partial_{\mu\alpha} \varphi)(x_3, \dots, x_{n+2}) \quad (37)$$

$$\begin{aligned} & - (\partial_{\mu\alpha} \varphi)(x_1, \dots, x_n) \alpha^{n-1}(x_{n+1} x_{n+2}) \\ & = \alpha^{n-1}(x_1 x_2) \alpha(\varphi(x_3, \dots, x_{n+2})) \end{aligned} \quad (38)$$

$$- \alpha^{n-1}(x_1 x_2) \varphi(\alpha(x_3), \dots, \alpha(x_{n+2})) \quad (39)$$

$$- \alpha(\varphi(x_1, \dots, x_n)) \alpha^{n-1}(x_{n+1} x_{n+2}) \quad (40)$$

$$+ \varphi(\alpha(x_1), \dots, \alpha(x_n)) \alpha^{n-1}(x_{n+1} x_{n+2}). \quad (41)$$

Next we check $\partial_{\mu\mu} \partial_{\alpha\mu} = \partial_{\alpha\mu} \partial_{\alpha\alpha}$ explicitly. On one side we get

$$\begin{aligned} \partial_{\mu\mu} \partial_{\alpha\mu} \psi(x_1, \dots, x_{n+2}) &= \alpha^n(x_1) (\partial_{\alpha\mu} \psi)(x_2, \dots, x_{n+2}) \\ &+ \sum_{i=1}^{n+1} (-1)^i (\partial_{\alpha\mu} \psi)(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+2})) \\ &+ (-1)^{n+2} (\partial_{\alpha\mu} \psi)(x_1, \dots, x_{n+1}) \alpha(x_{n+2}) \\ &= \alpha^n(x_1) (\alpha^{n-2}(x_2 x_3) \psi(x_4, \dots, x_{n+2})) \end{aligned} \quad (42)$$

$$- \alpha^n(x_1) (\psi(x_2, \dots, x_n) \alpha^{n-2}(x_{n+1} x_{n+2})) \quad (43)$$

$$- (\alpha^{n-2}(x_1 x_2) \alpha^{n-1}(x_3)) \psi(\alpha(x_4), \dots, \alpha(x_{n+2})) \quad (44)$$

$$+ (\alpha^{n-1}(x_1) \alpha^{n-2}(x_2 x_3)) \psi(\alpha(x_4), \dots, \alpha(x_{n+2})) \quad (45)$$

$$+ \sum_{i=3}^{n+1} (-1)^i \alpha^{n-1}(x_1 x_2) \psi(\alpha(x_3), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+2})) \quad (46)$$

$$- \sum_{i=1}^{n-1} (-1)^i \psi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n)) \alpha^{n-1}(x_{n+1} x_{n+2}) \quad (47)$$

$$- (-1)^n \psi(\alpha(x_1), \dots, \alpha(x_n)) (\alpha^{n-2}(x_n x_{n+1}) \alpha^{n-1}(x_{n+2})) \quad (48)$$

$$+ (-1)^n \psi(\alpha(x_1), \dots, \alpha(x_n)) (\alpha^{n-1}(x_n) \alpha^{n-2}(x_{n+1} x_{n+2})) \quad (49)$$

$$+ (-1)^{n+2} (\alpha^{n-2}(x_1 x_2) \psi(x_3, \dots, x_{n+1})) \alpha^n(x_{n+2}) \quad (50)$$

$$- (-1)^{n+2} (\psi(x_1, \dots, x_{n-1}) \alpha(x_n x_{n+1})) \alpha^n(x_{n+2}). \quad (51)$$

6. Cohomology

On the other side one has

$$\begin{aligned} \partial_{\alpha\mu}\partial_{\alpha\alpha}\psi(x_1, \dots, x_{n+2}) &= \alpha^{n-1}(x_1x_2)(\partial_{\alpha\alpha}\psi)(x_3, \dots, x_{n+2}) \\ &\quad - (\partial_{\alpha\alpha}\psi)(x_1, \dots, x_n)\alpha^{n-1}(x_{n+1}x_{n+2}) \\ &= \alpha^{n-1}(x_1x_2)(\alpha^{n-1}(x_3)\psi(x_4, \dots, x_{n+2})) \end{aligned} \quad (52)$$

$$+ \sum_{i=3}^{n+1} (-1)^i \alpha^{n-1}(x_1x_2)\psi(\alpha(x_3, \dots, x_i x_{i+1}, \dots, \alpha(x_{n+2}))) \quad (53)$$

$$+ (-1)^{n+2} \alpha^{n-1}(x_1x_2)(\psi(x_3, \dots, x_{n+1})\alpha^{n-1}(x_{n+2})) \quad (54)$$

$$- (\alpha^{n-1}(x_1)\psi(x_2, \dots, x_n))\alpha(x_{n+1}x_{n+2}) \quad (55)$$

$$- \sum_{i=1}^{n-1} \psi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n))\alpha^{n-1}(x_{n+1}x_{n+2}) \quad (56)$$

$$- (-1)^n (\psi(x_1, \dots, x_n)\alpha^{n-1}(x_n))\alpha^{n-1}(x_{n+1}x_{n+2}). \quad (57)$$

Now (44) cancels with (45) and (48) with (49). Using the Hom-associativity it is also easy to see that (46) equals (53), (47) equals (56), (51) equals (57), (50) equals (54), (42) equals (52) and (43) equals (55).

Next we verify $\partial_{\mu\alpha}\partial_{\mu\mu}\varphi = \partial_{\alpha\alpha}\partial_{\mu\alpha}\varphi$. For this we compute

$$\begin{aligned} \partial_{\alpha\alpha}\partial_{\mu\alpha}\varphi(x_1, \dots, x_{n+1}) &= \alpha^n(x_1)(\partial_{\mu\alpha}\varphi)(x_2, \dots, x_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i (\partial_{\mu\alpha}\varphi)(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n)) + (-1)^{n+1} (\partial_{\mu\alpha}\varphi)(x_1, \dots, x_n)\alpha^n(x_{n+1}) \\ &= \alpha^n(x_1)\alpha(\varphi(x_2, \dots, x_{n+1})) - \alpha^n(x_1)\varphi(\alpha(x_2), \dots, \alpha(x_{n+1})) \\ &\quad + \sum_{i=1}^n (-1)^i \alpha(\varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1}))) \\ &\quad - \sum_{i=1}^n (-1)^i \varphi(\alpha^2(x_1), \dots, \alpha(x_i x_{i+1}), \dots, \alpha^2(x_{n+1})) \\ &\quad + (-1)^{i+1} \alpha(\varphi(x_1, \dots, x_n))\alpha^n(x_{n+1}) - (-1)^{i+1} \varphi(\alpha(x_1), \dots, \alpha(x_n))\alpha^n(x_{n+1}) \end{aligned}$$

and

$$\begin{aligned} \partial_{\mu\alpha}\partial_{\mu\mu}\varphi(x_1, \dots, x_{n+1}) &= \alpha((\partial_{\mu\mu}\varphi)(x_1, \dots, x_{n+1})) - (\partial_{\mu\mu}\varphi)(\alpha(x_1), \dots, \alpha(x_{n+1})) \\ &= \alpha(\alpha^{n-1}(x_1)\varphi(x_2, \dots, x_{n+1})) + \sum_{i=1}^n (-1)^i \alpha(\varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1}))) \\ &\quad + (-1)^{i+1} \alpha(\varphi(x_1, \dots, x_n))\alpha^{n-1}(x_{n+1}) - \alpha^n(x_1)\varphi(\alpha(x_2), \dots, \alpha(x_{n+1})) \end{aligned}$$

$$\begin{aligned} & - \sum_{i=1}^n (-1)^i \varphi(\alpha^2(x_1), \dots, \alpha(x_i) \alpha(x_{i+1}), \dots, \alpha^2(x_{n+1})) \\ & - (-1)^{i+1} \varphi(\alpha(x_1), \dots, \alpha(x_n)) \alpha^n(x_{n+1}). \end{aligned}$$

Using the multiplicativity it is easily seen that the two sides agree. This completes the proof. \square

Definition 6.3.3 We denote by $\tilde{\text{HB}}^\bullet(A) = \partial \tilde{\text{HC}}^{\bullet-1}(A)$ the coboundaries and by $\tilde{\text{HZ}}(A) = \{(\varphi, \psi) \in \tilde{\text{HC}}(A) | \partial(\varphi, \psi) = 0\}$ the cocycles. The cohomology of $(\tilde{\text{HC}}(A), \partial) = \tilde{\text{HC}}(A) / \tilde{\text{HB}}(A)$ we call α -type Hochschild cohomology of A with value in itself and denote it by $\tilde{\text{HH}}^\bullet(A)$.

Similarly one can define the cohomology $\tilde{\text{HH}}^\bullet(A, M)$ with values in an A -bimodule (M, α_M) . We denote the left and right action by \cdot . In this case we have:

$$\tilde{\text{HC}}^n(A, M) = \tilde{\text{HC}}_\mu^n(A, M) \oplus \tilde{\text{HC}}_\alpha^n(A, M) = \text{Hom}(A^{\otimes n}, M) \oplus \text{Hom}(A^{\otimes n-1}, M) \quad (58)$$

for $n \geq 2$, $\tilde{\text{HC}}^1(A, M) = \text{Hom}(A, M)$, $\tilde{\text{HC}}^n(A, M) = 0$ for $n \leq 0$ and differentials

$$\begin{aligned} \partial_{\mu\mu} \varphi(x_1, \dots, x_{n+1}) &= \alpha^{n-1}(x_1) \cdot \varphi(x_2, \dots, x_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i \varphi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1})), \\ &+ (-1)^{n+1} \varphi(x_1, \dots, x_{n-1}) \cdot \alpha^{n-1}(x_{n+1}) \end{aligned} \quad (59)$$

$$\begin{aligned} \partial_{\alpha\alpha} \psi(x_1, \dots, x_n) &= \alpha^{n-1}(x_1) \cdot \psi(x_2, \dots, x_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i \psi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n)) \\ &+ (-1)^n \psi(x_1, \dots, x_{n-1}) \cdot \alpha^{n-1}(x_n), \end{aligned} \quad (60)$$

$$\partial_{\mu\alpha} \varphi(x_1, \dots, x_n) = \alpha_M(\varphi(x_1, \dots, x_n)) - \varphi(\alpha(x_1), \dots, \alpha(x_n)), \quad (61)$$

$$\partial_{\alpha\alpha} \psi(x_1, \dots, x_{n+1}) = \alpha^{n-2}(x_1 x_2) \cdot \psi(x_3, \dots, x_{n+1}) - \psi(x_1, \dots, x_{n-1}) \cdot \alpha^{n-2}(x_n x_{n+1}). \quad (62)$$

With this we set

$$\partial(\varphi + \psi) = (\partial_{\mu\mu} + \partial_{\mu\alpha})\varphi - (\partial_{\alpha\mu} + \partial_{\alpha\alpha})\psi \quad (63)$$

$$= (\partial_{\mu\mu}\varphi - \partial_{\alpha\mu}\psi, \partial_{\mu\alpha}\varphi - \partial_{\alpha\alpha}\psi). \quad (64)$$

The proof for $\partial^2 = 0$ is complete analog to the proof of Theorem 6.3.2.

We define the height of an element $(\varphi, \psi) \in \tilde{\text{HC}}^n(A)$ as $\text{hgt}(\varphi) = n - 1$ and $\text{hgt} \psi = n - 1$. This is compatible with $(\mu, \alpha) \in \tilde{\text{HC}}^2(A)$ and also the differential.

6.3.1 Relation to other Hochschild cohomologies

First we establish the connection to the cohomology given in Section 6.2. For this we consider only elements where the summand in $\tilde{\text{HC}}_\alpha$ is zero, this means pairs of the form $(\varphi, 0)$. The condition $\partial(\varphi, 0) = 0$, corresponds to $\partial_{\mu\mu}\varphi = 0$ and $\partial_{\mu\alpha}\varphi = 0$, since the other two parts vanish. Since $\partial_{\mu\alpha}\varphi \in \tilde{\text{HC}}(A)_\alpha$ and we want this part to be zero, we consider only the subcomplex where $\partial_{\mu\alpha}$ vanishes, this is $\text{HC}_\alpha^n(A) = \{\varphi \in \tilde{\text{HC}}_\mu^n(A) \mid \alpha\varphi = \varphi\alpha^{\otimes n}\}$. The remaining map $\partial_{\mu\mu}$ is a differential on this. In fact we have

Proposition 6.3.4 *The cohomology of $\text{HC}_\alpha^\bullet(A)$ with differential $\partial_{\mu\mu}$ is the one given in Section 6.2.*

Proof. This follows directly since the complexes and the differentials agree by definition. \square

As stated above in the associative case the cohomology is completely determined by the Hochschild cohomology. In fact we have the following

Theorem 6.3.5 *Let A be an associative algebra, then $\tilde{\text{HH}}^k(A) \cong \text{HH}^k(A) \oplus \text{HH}^{k-1}(A)$ for $k \geq 1$ with $\text{HH}^1(A)$ set to $\text{Der}(A)$ and $\text{HH}^0(A)$ to $\{0\}$.*

Proof. Since $\alpha = \text{id}$, we have $\partial_{\mu\alpha} = 0$, $\partial_{\alpha\alpha}^{k+1} = \partial_{\mu\mu}^k$ and $\partial_{\mu\mu}$ agrees with the Hochschild differential. We prove the statement by induction over k . For $k = 1$ the statement is clear. For $(\varphi, \psi) \in \tilde{\text{HC}}(A)$ to be closed, we must have $\partial_{\alpha\alpha}\psi = 0$, so it is a cocycle in the ordinary Hochschild cohomology $\text{HC}^{k-1}(A)$ and so cohomologous to an element in $\text{HH}(A)$. W.l.o.g. we can assume ψ is this element. Then $\tilde{\varphi} \in \tilde{\text{HC}}_\mu^k(A)$ defined by $\tilde{\varphi}(x_1, \dots, x_n) = x_1\psi(x_2, \dots, x_n) + (-1)^n\psi(x_1, \dots, x_{n-1})x_n$ satisfies $\partial_{\mu\mu}\tilde{\varphi} = \partial_{\alpha\mu}\psi$, so $(\tilde{\varphi}, \psi)$ is closed. Further $\partial_{\mu\mu}(\tilde{\varphi} - \varphi) = 0$, so it is cohomologous to an element in $\text{HH}^k(A)$. \square

The previous theorem is also true for $\text{HH}^\bullet(A, M)$ for an A -bimodule M .

We consider the case where α is invertible, in this case the construction in the previous proof can be generalized. For a linear map $\psi : A^{\otimes n-1} \rightarrow A$ we define $\varphi_\psi(x_1, \dots, x_n) = \alpha^{n-2}(x_1)\alpha^{-1}(\psi(x_2, \dots, x_n)) + (-1)^n\alpha^{-1}\psi(x_1, \dots, x_{n-1})\alpha^{n-2}(x_n)$.

Proposition 6.3.6 *Let $\psi : A^{\otimes n-1} \rightarrow A$ be a linear map which satisfies $\partial_{\alpha\alpha}\psi = \partial_{\mu\alpha}\varphi_\psi$. Then (φ_ψ, ψ) is an n -cocycle of A .*

Proof. We consider $\xi(x_1, \dots, x_n) = \alpha^{n-2}(x_1)\alpha^{-1}(\psi(x_2, \dots, x_n))$ and calculate $\partial_{\mu\mu}\xi$.

$$\begin{aligned}
 (\partial_{\mu\mu}\xi)(x_1, \dots, x_{n+1}) &= \alpha^{n-1}(x_1)(\alpha^{n-2}(x_2)\alpha^{-1}\psi(x_3, \dots, x_{n+1})) \\
 &\quad + \alpha^{n-2}(x_1x_2)\alpha^{-1}\psi(\alpha(x_3), \dots, \alpha(x_{n+1})) \\
 &\quad + \sum_{i=2}^n (-1)^i \alpha^{n-1}(x_1)\alpha^{-1}\psi(\alpha(x_2), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1})) \\
 &\quad + (-1)^{n+1} (\alpha^{n-2}\alpha^{-1}\psi(x_2, \dots, x_n)) \alpha^{n-1}(x_{n+1}) \\
 &= (\alpha^{n-2}(x_1)\alpha^{n-2})\psi(x_3, \dots, x_{n+1}) - \alpha^{n-1}(x_1) (\alpha^{n-2}(x_2)\alpha^{-2}\psi(\alpha(x_3), \dots, \alpha(x_{n+1}))) \\
 &\quad + \alpha^{n-1}(x_1)\alpha^{-1}(\alpha^{n-1}(x_2)\psi(x_3, \dots, x_{n+1})) - \alpha^{n-1}(x_1)\alpha^{-1}(\alpha^{n-1}(x_2)\psi(x_3, \dots, x_{n+1})) \\
 &\quad + \sum_{i=2}^n (-1)^i \alpha^{n-1}(x_1)\alpha^{-1}\psi(\alpha(x_2), \dots, x_i x_{i+1}, \dots, \alpha(x_{n+1})) \\
 &\quad + \alpha^{n-1}(x_1)\alpha^{-1} (\psi(x_1, \dots, x_n)\alpha^{n-1}(x_n)) \\
 &= \alpha^{n-2}(x_1x_2)\psi(x_3, \dots, x_{n+1}) \\
 &\quad + \alpha^{n-1}(x_1)\alpha^{-1} \left((\partial_{\mu\alpha}\xi)(x_2, \dots, x_{n+1}) - (\partial_{\alpha\alpha}\psi)(x_2, \dots, x_{n+1}) \right)
 \end{aligned}$$

Using this it is easy to see that

$$\begin{aligned}
 \partial_{\mu\mu}\varphi\psi(x_1, \dots, x_{n+1}) &= \partial_{\alpha\mu}\psi(x_1, \dots, x_{n+1}) + \alpha^{n-1}(x_1)\alpha^{-1} \left((\partial_{\mu\alpha}\varphi\psi)(x_2, \dots, x_{n+1}) \right. \\
 &\quad \left. - (\partial_{\alpha\alpha}\psi)(x_2, \dots, x_{n+1}) \right).
 \end{aligned}$$

Since we assumed $\partial_{\alpha\alpha}\psi = \partial_{\mu\alpha}\varphi\psi$ we get the result. \square

Note that the condition in the previous theorem can be written explicitly as

$$\begin{aligned}
 \alpha^{n-1}(x_1)\alpha^{-1}\psi(\alpha(x_2), \dots, \alpha(x_n)) + \sum_{i=1}^{n-1} (-1)^i \psi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n)) \\
 + (-1)^n \alpha^{-1}\psi(\alpha(x_1), \dots, \alpha(x_{n-1}))\alpha^{n-1}(x_n) = 0
 \end{aligned} \tag{65}$$

If ψ commutes with α this reduces to $\partial_{\alpha\alpha}\psi = 0$. For $n = 2$ the condition means that ψ is a conjugate α -derivation.

Note that in the associative case, i.e. $\alpha = \text{id}$, every cocycle is cohomologous to a sum of two cocycles, where one has the form $(\varphi, 0)$ and the other one the form $(\varphi\psi, \varphi)$. It would be interesting to know whether this is always the case if α is invertible, but see also Section 6.3.2.

6.3.2 α -type cohomology under Yau twist

We study the relation between two Hom-associative algebra related by a Yau twist. For this we use the cohomology for an algebra endomorphism defined in Section 6.1.

Let A be a Hom-associative algebra and γ a Hom-algebra morphism of A . We consider the Hom-associative algebra A_γ obtained by Yau twist.

Proposition 6.3.7 *If (φ, ψ) is an n -cocycle of A , which commutes with γ , i.e. $\gamma\varphi = \varphi\gamma^{\otimes n}$ and $\gamma\psi = \psi\gamma^{\otimes n-1}$, then $(\tilde{\varphi}, \tilde{\psi}) = (\gamma^{n-1}\varphi, \gamma^{n-1}\psi)$ is an n -cocycle of A_γ .*

Proof. We need to show $\tilde{\partial}(\tilde{\varphi}, \tilde{\psi}) = \gamma^n(\partial(\varphi, \psi))$, where ∂ denotes the differential in A and $\tilde{\partial}$ the one in A_γ . This is an easy calculation, we show it for one part of the differential the others work analogously.

$$\begin{aligned} (\partial_{\alpha\alpha}\tilde{\psi})(x_1, \dots, x_n) &= \tilde{\alpha}^{n-1}(x_1)\tilde{\psi}(x_2, \dots, x_n) + \sum_i (-1)^i \tilde{\psi}(\tilde{\alpha}(x_1)\dots, x_i\tilde{x}_{i+1}, \dots, \tilde{\alpha}(x_n)) \\ &\quad + \tilde{\psi}(x_1, \dots, x_{n-1})\tilde{\alpha}^{n-1}(x_n) \\ &= \gamma(\gamma^{n-1}(\alpha^{n-1}(x_1))\gamma^{n-1}(\psi(x_2, \dots, x_n))) \\ &\quad + \sum_i (-1)^i \gamma^{n-1}(\psi(\gamma(\alpha(x_1)), \dots, \gamma(x_i x_{i+1}), \dots, \gamma(\alpha(x_n)))) \\ &\quad + \gamma(\gamma^{n-1}(\psi(x_1, \dots, x_{n-1}))\gamma^{n-1}(\alpha^{n-1}(x_n))) \\ &= \gamma^n((\partial_{\alpha\alpha}\psi)(x_1, \dots, c_n)). \end{aligned}$$

□

Next we study the α -type cohomology of Hom-associative algebras of associative type. This includes in particular the case where α is invertible.

Now let A be an associative algebra, γ an endomorphism of it and A_γ the Yau twist of A by γ . Then we can define a map $\Phi : C(A, \gamma) \rightarrow \tilde{\text{HC}}(A_\gamma)$ for $(\varphi, \psi) \in C^n(A, \gamma)$ by

$$(\varphi, \psi) \mapsto (\gamma^{n-1}\varphi + \gamma^{n-2}\psi \circ \mu, \gamma^{n-2}\psi), \quad (66)$$

where $\psi \circ \mu = \sum_{i=0}^{n-2} (-1)^i \psi \circ_i \mu$ with $\varphi \circ_i \mu := \varphi(\text{id}^{\otimes i} \otimes \mu \otimes \text{id}^{\otimes n-i-2})$ is the Gerstenhaber product in $\text{HC}^\bullet(A)$.

Theorem 6.3.8 *The map Φ is a chain map, so it induces a map in cohomology. If γ is invertible it is an isomorphism and especially the corresponding cohomologies are also isomorphic.*

Proof. The fact that it is a chain map is a straightforward calculation.

$$\Phi \partial(\varphi, 0) = \Phi(\partial_\mu \varphi, \partial_\gamma \varphi) \quad (67)$$

$$= (\gamma^n \partial_\mu \varphi + \gamma^{n-1}(\partial_\gamma \varphi) \circ \mu, \gamma^{n-1} \partial_\gamma \varphi) \quad (68)$$

$$\partial \Phi(\varphi, 0) = \partial(\gamma^{n-1}, 0) \quad (69)$$

$$= (\partial_{\mu\mu} \gamma^{n-1} \varphi, \partial_{\mu\alpha} \gamma^{n-1} \varphi) \quad (70)$$

The second components agree obviously and for the first we get

$$\begin{aligned} & (\gamma^n \partial_\mu \varphi)(x_1, \dots, x_{n+1}) + \gamma^{n-1}(\partial_\gamma \varphi) \circ \mu(x_1, \dots, x_{n+1}) = \gamma^n(x_1 \cdot \varphi(x_2, \dots, x_n)) \\ & - \sum_{i=1}^n (-1)^{i+1} \varphi(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) + (-1)^{n+1} \varphi(x_1, \dots, x_n) \cdot x_{n+1} \\ & + \sum_{i=1}^n (-1)^{i+1} \gamma^n \varphi(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) \\ & - \sum_{i=1}^n (-1)^{i+1} \gamma^{n-1} \varphi(\gamma(x_1), \dots, \gamma(x_i x_{i+1}), \dots, \gamma(x_{n+1})) \\ & = \partial_{\mu\mu} \gamma^{n-1} \varphi(x_1, \dots, x_{n+1}). \end{aligned}$$

$$\Phi \partial(0, \psi) = (\gamma^{n-1}(\partial_\mu \psi) \circ \mu, \gamma^{n-1} \partial_\mu \psi)$$

$$\partial \Phi(0, \psi) = (\partial_{\mu\mu}(\gamma^{n-2} \psi \circ \mu) - \partial_{\alpha\mu}(\gamma^{n-2} \psi), \partial_{\mu\alpha}(\gamma^{n-2} \psi \circ \mu) - \partial_{\alpha\alpha}(\gamma^{n-2} \psi))$$

The second components agree using a similar calculation as above and for the first one uses

$$\begin{aligned} \partial_{\mu\mu}(\gamma^{n-2} \psi \circ \mu)(x_1, \dots, x_{n+1}) &= (-1)^i \gamma^{n-1}(x_1 \cdot \psi(\gamma(x_2), \dots, \gamma(x_i x_{i+1}, \dots, x_{n+1}))) \\ & \quad (-1)^{i+n} \gamma^{n-1}(\psi(\gamma(x_1), \dots, \gamma(x_i x_{i+1}, \dots, x_n)) \cdot x_{n+1}) \\ (\partial_\mu \psi) \circ \mu(x_1, \dots, x_n) &= (x_1 x_2) \psi(x_3, \dots, x_{n+1}) + (-1)^i x_1 \cdot \psi(x_2, \dots, x_i x_{i+1}, \dots, x_{n+1}) \\ & \quad - \psi(x_1, \dots, x_{n-1}) \cdot (x_n x_{n+1}) + (-1)^{i+n} \psi(x_2, \dots, x_i x_{i+1}, \dots, x_n) \cdot x_{n+1}. \end{aligned}$$

If γ is invertible, the inverse of Φ is given by

$$(\varphi, \psi) \mapsto (\gamma^{-n+1} \varphi - \gamma^{-n+1} \psi \circ \mu, \gamma^{-(n-2)} \psi), \quad (71)$$

which is easy to check. \square

Using this proposition and the remarks on how to compute $H(\gamma)$ in Section 6.1 it is possible to compute the α -type Hochschild cohomology of A_γ , if one knows the Hochschild cohomology of A . Especially we have

Corollary 6.3.9 *If (A, μ, α) is a Hom-associative algebra, such that α is invertible, then its α -type cohomology $\tilde{H}H(A)$ is isomorphic to $H(A_{\alpha^{-1}}, \alpha)$, where $A_{\alpha^{-1}}$ is the associative algebra obtained by Yau twist of A with α^{-1} .*

This implies that if $HC(\gamma)$ splits as a direct sum like the underlying complex, this means every cocycle is cohomologous to one of the form $(\varphi, 0)$ or $(0, \psi)$, also $\tilde{H}C(A_\gamma)$ splits in a similar way. This is interesting, when considering the deformation of commutative Hom-associative algebras.

6.4 EXAMPLES

In the case where α is invertible, the cohomology can be computed using Theorem 6.3.8. Let V be a vector space with an endomorphism α . Then we consider the symmetric algebra $S(V)$ over V and $\tilde{\alpha}$ the extension of α to $S(V)$. With this we can define $S(V, \alpha)$ as the Yau twist of $S(V)$ by $\tilde{\alpha}$. If α is invertible this is isomorphic to the free commutative Hom-associative algebra over (V, α) .

Using the Hochschild-Kostant-Rosenberg Theorem it is not difficult to see that $H^k(C(\tilde{\alpha}), \partial_\mu) = \Lambda^k \text{Der}(S(V)) \oplus \tilde{\alpha} \Lambda^{k-1} \text{Der}(S(V))$. Because the HKR-Theorem directly gives $H^k(C_\mu(\tilde{\alpha}), \partial_\mu) = HH^k(A, A) = \Lambda^k \text{Der}(S(V))$ and $\varphi \mapsto \alpha\varphi$ is a cochain isomorphism from $HC(A, A) \rightarrow HC(\tilde{A}, \tilde{A}) = H^k(C_\gamma(\tilde{\alpha}), \partial_\mu)$. The differential ∂_γ is well defined on this complex. So we get $H(\gamma) = \Lambda^k \text{Der}_{\tilde{\alpha}}(S(V)) \oplus \tilde{\alpha} \Lambda^{k-1} \text{Der}(S(V)) / \text{im } \partial_\gamma$, where $\text{Der}_\gamma(A) = \{\varphi \in \text{Der}(A) | \varphi\gamma = \gamma\varphi\}$. If V is finite dimensional and α and consequently $\tilde{\alpha}$ are diagonal the second summand simplifies to $\tilde{\alpha} \Lambda^{k-1} \text{Der}_{\tilde{\alpha}}(S(V))$.

Next we consider the case $\alpha = 0$. In this case every bilinear map defines a Hom-associative algebra. It turns out that all differentials except low degrees are zero. In degree one we have $\partial_{\mu\mu}\varphi(x, y) = x\varphi(y) - \varphi(xy) + \varphi(x)y$ and in degree two $\partial_{\alpha\alpha}\psi(x, y) = \psi(xy)$ and $\partial_{\alpha\mu}\psi(x, y, z) = (xy)\psi(z) - \psi(x)(yz)$. So the cohomology in degree 1 consists of the derivations as always and starting from degree 4 it is the whole complex.

The other extreme case is when the multiplication is zero. In this case all differentials except of $\partial_{\mu\alpha}$ are zero. So the cohomology can be easily computed to be $\tilde{H}H^n = \text{Hom}_\alpha(A^{\otimes n}, A) \oplus \text{Hom}(A^{\otimes n-1}, A) / \text{im } \partial_{\mu\alpha}$, where

$$\text{Hom}_\alpha(A^{\otimes n}, A) = \{\varphi \in \text{Hom}(A^{\otimes n}, A) | \partial_{\mu\alpha}\varphi = 0\}$$

are the linear maps commuting with α . If α is diagonalizable, so is $\partial_{\mu\alpha}$ and we have $\text{Hom}(A^{\otimes n-1}, A) / \text{im } \partial_{\mu\alpha} \cong \text{Hom}_\alpha(A^{\otimes n-1}, A)$.

Next we give two examples, where we compute the cohomology – or at least its dimension – in low degrees explicitly using computer algebra systems. The first is a Hom-associative algebra, which is not of associative type, the second one a truncated polynomial algebra.

Hom-associative algebra not of associative type We consider the Hom-associative algebra, spanned by e_1, e_2 with structure map $\alpha(e_1) = e_1 + e_2, \alpha(e_2) = 0$ and multiplication $e_1 e_1 = e_1, e_i e_j = e_2$ for all other i, j . Then using the computer algebra system Maxima, we computed the lower cohomologies. It turns out that $\tilde{H}H^1 = \tilde{H}H^2 = 0, \dim(\tilde{H}H^3) = 2$ and $\dim(\tilde{H}H^4) = 10$. The third cohomology is spanned by (φ_1, ψ_1) and $(\varphi_2, 0)$ with

$$\begin{aligned} \psi_1(e_1, e_2) &= e_2, & \psi_1(e_2, e_1) &= e_2, & \psi_1(e_2, e_2) &= e_2, \\ \varphi_1(e_1, e_1, e_1) &= e_2, & \varphi_1(e_1, e_1, e_2) &= 2e_2 - e_1, & \varphi_1(e_2, e_1, e_1) &= e_1, & \varphi_1(e_2, e_1, e_2) &= e_2 \\ \text{and } \varphi_2(e_1, e_1, e_1) &= -e_2, & \varphi_2(e_1, e_2, e_1) &= -e_2, & \varphi_2(e_2, e_1, e_2) &= e_2, & \varphi_2(e_2, e_2, e_2) &= e_2. \end{aligned}$$

Truncated polynomial algebra We consider a vector space V spanned by x, y and an endomorphism α of it. Then we can form the polynomial algebra $S(V)$ of it. We consider $A = S(V)/S^3(V)$. This is spanned by $\{1, x, y, x^2, xy, y^2\}$. The map α can be extended to this as an algebra morphism, which we denote by $\tilde{\alpha}$, so we can consider the Yau twist A_α . The cohomology of course depends on α . So we consider the different possibilities. We start with the case $\alpha = \text{id}$. In this case $\dim(\tilde{H}H^1) = 10, \dim(\tilde{H}H^2) = 25$ and $\dim(\tilde{H}H^3) = 41$.

In the case $\alpha = \lambda \text{id}, \lambda \neq 1$, we have $\dim(\tilde{H}H^1) = 4$, and the derivations are determined by

$$\varphi(x) = \lambda_1 x + \lambda_2 y, \quad \varphi(y) = \lambda_3 x + \lambda_4 y.$$

For the second cohomology we have $\dim(\tilde{H}H^2) = 7$. Four generators are of the form (φ_ψ, ψ) with $\psi = \alpha\varphi$ for $\varphi \in \tilde{H}H^1$. The rest is given by

$$\varphi(x, y) = \lambda_3 y^2 - \lambda_1 xy, \quad \varphi(y, x) = \lambda_2 x^2 + \lambda_1 xy.$$

For the third cohomology we get $\dim(\tilde{H}H^3) = 3$, so everything comes from $\tilde{H}H^2$ as described before.

In the case $\alpha(x) = \lambda(x), \alpha(y) = x + \lambda y$, this is, α is not diagonalizable, we have $\dim(\tilde{H}H^1) = 2$, and it is given by

$$\varphi(x) = \lambda_1 x, \quad \varphi(y) = \lambda_1 y + \lambda_2 x.$$

For the second cohomology we have $\dim(\tilde{H}H^2) = 3$, with two generators coming from derivations and

$$\varphi(x, y) = x^2 = -\varphi(y, x).$$

For the third cohomology we get $\dim(\tilde{H}H^3) = 1$.

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The next case is a $\alpha(x) = \lambda x, \alpha(y) = \mu y$, with $\lambda^i \neq \mu^k$ for all $i, k \in \mathbb{N}$. In this case $\dim(\tilde{H}H^1) = 2$, namely

$$\varphi(x) = \lambda_1 x, \quad \varphi(y) = \lambda_2 y.$$

For the second cohomology we get $\dim(\tilde{H}H^2) = 3$, with two generators coming from derivations and

$$\varphi(x, y) = \lambda_1 xy.$$

For the third cohomology we get $\dim(\tilde{H}H^3) = 1$.

As a last case we consider an example of a diagonal α , but such that the eigenvalues are algebraically dependent, so we consider $\alpha(x) = 2x, \alpha(y) = 4y$. In this case $\dim(\tilde{H}H^1) = 3$, namely the derivations determined by

$$\varphi(x) = \lambda_1 x, \quad \varphi(y) = \lambda_2 y + \lambda_3 x^2.$$

For the second cohomology we get $\dim(\tilde{H}H^2) = 6$, with three generators coming from derivations, one is the same as in the previous case and

$$\begin{aligned} \varphi(x, y) &= \lambda_1 xy, \\ \varphi(x, xy) &= \lambda_2 y^2 = \varphi(xy, x), \\ \varphi(y, x^2) &= \lambda_2 y^2 = \varphi(x^2, y), \\ \varphi(x, x^2) &= \varphi(x^2, x) = \lambda_3 xy. \end{aligned}$$

6.4.1 Cohomology of free Hom-algebras

In this section we compute the α -type cohomology of some special free Hom-associative algebras. This is given a Hom-module (V, α) , we compute the cohomology of its free Hom-associative algebra $\tilde{H}C(\mathbb{F}_{HAss}(V))$.

We start with the case that α is invertible. In this case $A = \mathbb{F}(V)$ is of associative type and using Corollary 6.3.9 it is possible to compute $\tilde{H}C(A)$ by computing $H(A_{\alpha^{-1}}, \alpha)$.

Lemma 6.4.1 *The associative algebra $A_{\alpha^{-1}}$ is isomorphic to the free algebra $T(V)$.*

Proof. On generators the isomorphism can be defined by the identity. Since $T(V)$ is a free algebra it can be extended to a homomorphism $A \rightarrow A_{\alpha^{-1}}$. It is explicitly given by $(x_1 \dots x_n) \mapsto (\dots (\alpha^{-n+1}(x_1 \alpha^{-n+1}(x_2)) \alpha^{-n+2}(x_3)) \dots) x_n$, where the product on the right hand side is the product in A . \square

So it is enough to compute $H(T(V), \alpha)$.

Proposition 6.4.2 *Given a vector space V and a linear map $\gamma : V \rightarrow V$ we have*

$$H^1(T(V), \gamma) = \text{Hom}_\alpha(V, T(V)), \quad H^2(T(V), \gamma) = \text{Hom}(V, T(V)) / \partial_\gamma \text{Hom}(V, T(V))$$

and all other parts of the cohomology are zero.

Proof. It is well known that for an arbitrary $T(V)$ -module M the Hochschild cohomology vanish for degree greater or equal to 2. Since we begin the complex in degree one, we have $H^1(C_\mu(T(V), \gamma), \partial_\mu) = \text{Hom}(V, T(V)) = \text{Der}(T(V))$ and similar $H^1(C_\gamma(T(V), \gamma), \partial_\mu) = \text{Hom}(V, T(V)) = \gamma - \text{Der}(T(V))$. So the first sheet of the spectral sequence associated to $C(T(V), \gamma)$ consists only of these two terms, and the second sheets gives the claimed result. \square

We consider the case where (V, α) is a free Hom-module, then we have

Lemma 6.4.3 *Let V be a vector space and $(V[\gamma], \gamma)$ the free Hom-module on it, then $\partial_\gamma : \text{Hom}(V[\gamma], T(V)) \rightarrow \text{Hom}(V[\gamma], T(V)), \varphi \mapsto \gamma\varphi - \varphi\gamma$ is surjective.*

Proof. Let $\varphi \in \text{Hom}(V, T(V))$ we construct a $\psi \in \text{Hom}(V, T(V))$ such that $\partial_\gamma \psi = \varphi$, by induction on the degree in γ . We set $\psi(x) = 0$ for all $x \in V$, and $\psi(\gamma^{n+1}(x)) = \gamma\psi(\gamma^n(x)) - \varphi(\gamma^n(x))$. Then a simply calculation shows that indeed $\partial_\gamma \psi = \varphi$. \square

This proves $H^\bullet(T(V[\gamma]), \gamma) = \text{Der}(T(V[\gamma]))$.

Similar one can show that also $H^\bullet(T(V[\gamma, \gamma^{-1}]), \gamma) = \text{Der}(T(V[\gamma, \gamma^{-1}]))$. Using the Yau twist this shows that given a vector space, the free Hom-algebra with invertible structure map has as cohomology only the derivations.

6.5 HOM-COASSOCIATIVE COALGEBRAS

In this section we mention very briefly, how the Hochschild cohomology can be dualized to a cohomology for Hom-coassociative coalgebras.

So let (C, Δ, β) be a Hom-coassociative coalgebra and (M, β_M) a cobimodule of it, then we define its cohomology complex by

$$C^n(C, M) = \text{Hom}_H((M, \beta_M), (C^{\otimes n}, \beta)). \quad (72)$$

The differential on it for $\varphi \in C(C, M)$ is given by

$$\partial\varphi = (\text{id} \otimes \varphi)\lambda + \sum_{i=1}^{n-1} (-1)^i (\text{id}^{i-1} \otimes \Delta \otimes \text{id}^{n-i})\varphi + (-1)^{n+1} (\varphi \otimes \text{id})\rho, \quad (73)$$

where λ and ρ denote the left and right coaction resp.

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One can also define an α -type cohomology for coassociative coalgebras. For a coalgebra C and a C -cobimodule M the complex is given by

$$C^n(A) = C(A)_\Delta^n \oplus C_\beta^n(A) = \text{Hom}(M, C^{\otimes n}) \oplus \text{Hom}(M, C^{\otimes n-1}) \quad (74)$$

for $n \in \mathbb{N}$ with $\text{Hom}(M, C)$ replace with the zero space. The differential consists of four maps dual to the ones in Section 6.3.

$$\partial_{\Delta\Delta}\varphi = (\beta^{n-1} \otimes \varphi)\lambda - \sum_{i=0}^{n-1} (-1)^i (\beta^{\otimes i} \otimes \Delta \otimes \beta^{\otimes n-i-1})\varphi - (-1)^n (\varphi \otimes \beta^{n-1})\rho \quad (75)$$

$$\partial_{\Delta\beta}\varphi = \beta\psi - \psi\beta^{\otimes n} \quad (76)$$

$$\partial_{\beta\Delta}\psi = (\Delta\beta^{n-2} \otimes \varphi)\lambda - (\varphi \otimes \Delta\beta^{n-2})\rho \quad (77)$$

$$\partial_{\Delta\Delta}\psi = (\beta^{n-1} \otimes \psi)\lambda - \sum_{i=0}^{n-1} (-1)^i (\beta^{\otimes i} \otimes \Delta \otimes \beta^{\otimes n-i-1})\psi - (-1)^n (\varphi \otimes \beta^{n-1})\rho \quad (78)$$

We also want to note that the theorems from Section 6.3 also hold for coalgebras in a similar way.

6.6 HOM-BIALGEBRAS

Given a Hom-bialgebra we recall the cohomology for it given in [DM17], which we will call the Gerstenhaber-Schack cohomology. We stick to the case where $\alpha = \beta$ here since only this case is considered in [DM17].

In the following let A be a Hom-bialgebra then $T^\bullet A$ is an A -bimodule, and dually it is an A -cobimodule. We set $C_{GS}^{\bullet\bullet}(A) = \text{Hom}_H(T^\bullet A, T^\bullet A)$. Then it is a bicomplex with differentials δ_Δ and δ_μ .

As usual the total complex is given by

$$C_{GS}^k = \bigoplus_{k=i+j} C_{GS}^{i,j} = \bigoplus_{k=i+j} \text{Hom}_H(T^i A, T^j A)$$

with differential $\delta = \delta_\mu + \delta_\Delta$. Here δ_μ denotes the Hochschild differential defined in Section 6.2 and δ_Δ denotes the differential defined in Section 6.5. The complex $C^{\bullet,k}$ can be seen as the Hochschild complex of the algebra (A, μ, α) with values in $A^{\otimes k}$ and similarly $C^{k,\bullet}$ can be seen as the cohomology complex for the coalgebra (A, Δ, β) .

The Hom-algebra A acts on itself from the left and right by multiplication, with this we have on $A^{\otimes n}$ the action defined in Section 5.3. It is given by $x \cdot (a_1 \otimes \dots \otimes a_n) = \Delta_\beta^n(x)(a_1 \otimes \dots \otimes a_n) = x^{(1)}a_1 \otimes x^{(2)}a_2 \otimes \dots \otimes x^{(n)}a_n$. With $\Delta_\beta^n(x) = x^{(1)} \otimes \dots \otimes x^{(n)}$. Dually we have a left coaction ρ on $A^{\otimes n}$ given by $\rho(a_1 \otimes \dots \otimes a_n) = \mu_\alpha^n(a_1^{(1)}, \dots, a_n^{(1)}) \otimes a_1^{(2)} \otimes \dots \otimes a_n^{(2)}$. Similarly the right action and coaction can be defined.

6.7 α -TYPE COHOMOLOGY FOR HOM-BIALGEBRAS

In this section we extend the α -type Hochschild cohomology defined for Hom-associative algebras to Hom-bialgebras.

Let A be a Hom-bialgebra. As a shorthand notation we set $H^{i,j} = \text{Hom}(A^{\otimes i}, A^{\otimes j})$ for $i, j \in \mathbb{N}$. For i or $j \leq 0$ we set $H^{i,j} = 0$. (This is necessary since the differentials we are going to define, would involve α^{-1} or β^{-1} otherwise.) We define a bicomplex $C_{GS}^{\bullet, \bullet}(A)$ by

$$\begin{aligned} C_{GS}^{n,m}(A) &= C_{GS}^{n,m}_{\mu\Delta}(A) \oplus C_{GS}^{n,m}_{\alpha\Delta}(A) \oplus C_{GS}^{n,m}_{\mu\beta}(A) \oplus C_{GS}^{n,m}_{\alpha\beta}(A) \\ &= H^{n,m} \oplus H^{n-1,m} \oplus H^{n,m-1} \oplus H^{n-1,m-1}. \end{aligned} \quad (79)$$

For every summand there is algebra and a coalgebra differential.

We denote an element in $C_{GS}^{n,m}$ by $(\varphi, \psi, \chi, \xi)$. The algebra differential $\partial_\mu : C_{GS}^{n,m} \rightarrow C_{GS}^{n+1,m}$ is given by

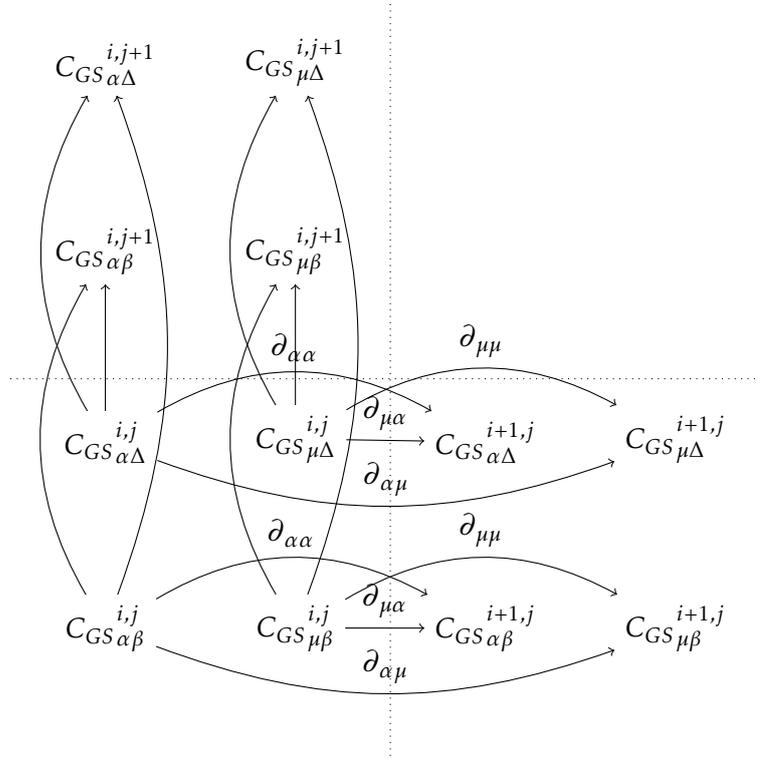
$$\partial_\mu(\varphi, \psi, \chi, \xi) = (\partial_{\mu\mu}\varphi - \partial_{\mu\alpha}\psi, \partial_{\mu\alpha}\varphi - \partial_{\alpha\alpha}\psi, \partial_{\mu\mu}\chi - \partial_{\alpha\mu}\xi, \partial_{\mu\alpha}\chi - \partial_{\alpha\alpha}\xi), \quad (80)$$

where the differentials are the α -type Hochschild differentials defined before, where the left action on $A^{\otimes m}$ in $C_{GS}^{n,m}_{\alpha\Delta}$ and $C_{GS}^{n,m}_{\mu\Delta}$ is the usual one and the left-action on $A^{\otimes m-1}$ in $C_{GS}^{n,m}_{\alpha\beta}$ and $C_{GS}^{n,m}_{\mu\beta}$ is given by $\beta(x) \cdot y$, where \cdot denotes the usual left-action on $A^{\otimes m-1}$, and similarly for the right-action.

Dually the coaction on $A^{\otimes m-1}$ in $C_{GS}^{n,m}_{\alpha\beta}$ and $C_{GS}^{n,m}_{\alpha\Delta}$ is given by $\alpha(x_{(-1)}) \cdot x_{(0)}$, where $x \mapsto x_{(-1)} \otimes x_{(0)}$ denotes the usual left coaction on $A^{\otimes m-1}$.

Since there are a lot of different maps involved we give a diagram depicting all maps, which start in $C_{GS}^{i,j}$:

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Proposition 6.7.1 *The differentials defined above make C_{GS} to a bicomplex.*

Proof. From the known differential of Hom-bialgebras [DM17] it follows that $\partial_{\mu\mu}\partial_{\Delta\Delta} = \partial_{\Delta\Delta}\partial_{\mu\mu}$. And essentially doing the same calculation, one gets the same for all parts $C_{GSij}^{n,m} \rightarrow C_{GSij}^{n+1,m+1}$ for $i \in \{\mu, \alpha\}$ and $j \in \{\Delta, \beta\}$.

Next we consider the part $C_{GS\mu\Delta}^{n,m} \rightarrow C_{GS\alpha\beta}^{n,m}$, for this we compute

$$\begin{aligned} \partial_{\Delta\beta}\partial_{\mu\alpha}\varphi(x_1, \dots, x_n) &= \beta^{\otimes m}\alpha^{\otimes m}\varphi(x_1, \dots, x_n) - \beta^{\otimes m}\varphi(\alpha(x_1), \dots, \alpha(x_n)) \\ &\quad - \alpha^{\otimes m}\varphi(\beta(x_1), \dots, \beta(x_n)) + \varphi(\beta(\alpha(x_1)), \dots, \beta(\alpha(x_n))) \\ &= \partial_{\mu\alpha}\partial_{\Delta\beta}\varphi(x_1, \dots, x_n) \end{aligned}$$

Which holds, since α and β commute.

$$C_{GS\mu\Delta}^{n,m} \rightarrow C_{GS\alpha\beta}^{n,m}$$

$$\begin{aligned} \partial_{\Delta\Delta}\partial_{\mu\alpha}\varphi(x_1, \dots, x_n) &= \alpha\mu_\alpha^n(x_1^{(1)}, \dots, x_n^{(1)}) \otimes (\partial_{\mu\alpha}\varphi)(x_1^{(2)}, \dots, x_n^{(2)}) \\ &\quad + \sum_{i=1}^m (-1)^i \Delta_i(\partial_{\mu\alpha}\varphi)(x_1, \dots, x_n) + (-1)^{m+1}(\partial_{\mu\alpha}\varphi)(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \alpha\mu_\alpha^n(x_1^{(2)}, \dots, x_n^{(2)}) \\ &= \alpha\mu_\alpha^n(x_1^{(1)}, \dots, x_n^{(1)}) \otimes (\alpha^{\otimes m}\varphi)(x_1^{(2)}, \dots, x_n^{(2)}) - \alpha\mu_\alpha^n(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \varphi(\alpha(x_1^{(2)}), \dots, \alpha(x_n^{(2)})) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^m (-1)^i \Delta_i (\alpha^{\otimes m} \varphi)(x_1, \dots, x_n) - \sum_{i=1}^m (-1)^i \Delta_i \varphi(\alpha(x_1), \dots, \alpha(x_n)) \\
 & + (-1)^{m+1} (\alpha^{\otimes m} \varphi)(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \alpha \mu_\alpha^n(x_1^{(2)}, \dots, x_n^{(2)}) \\
 & - (-1)^{m+1} \varphi(\alpha(x_1^{(1)}), \dots, \alpha(x_n^{(1)})) \otimes \alpha \mu_\alpha^n(x_1^{(2)}, \dots, x_n^{(2)}) \\
 \\
 & \partial_{\mu\alpha} \partial_{\Delta\Delta} \varphi(x_1, \dots, x_n) = \alpha^{\otimes m+1} (\partial_{\Delta\Delta} \varphi)(x_1, \dots, x_n) - (\partial_{\Delta\Delta} \varphi)(x_1, \dots, x_n) \\
 & = \alpha^{\otimes m+1} (\mu_\alpha^n(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \varphi(x_1^{(2)}, \dots, x_n^{(2)})) \\
 & + \sum_{i=1}^m (-1)^i \alpha^{\otimes m+1} \Delta_i \varphi(x_1, \dots, x_n) \\
 & + (-1)^{m+1} \alpha^{m+1} (\varphi(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \mu_\alpha^n(x_1^{(2)}, \dots, x_n^{(2)})) \\
 & - \mu_\alpha^n(\alpha(x_1^{(1)}), \dots, \alpha(x_n^{(1)})) \otimes \varphi(\alpha(x_1^{(2)}), \dots, \alpha(x_n^{(2)})) \\
 & - \sum_{i=1}^m (-1)^i \Delta_i \varphi(\alpha(x_1), \dots, \alpha(x_n)) \\
 & - (-1)^{m+1} \varphi(\alpha(x_1^{(1)}), \dots, \alpha(x_n^{(1)})) \otimes \mu_\alpha^n(\alpha(x_1^{(2)}), \dots, \alpha(x_n^{(2)}))
 \end{aligned}$$

Using that α is a morphism of Δ , it is easy to see that the two sides agree.

$$C_{GS\alpha\beta}^{n,m} \rightarrow C_{GS\mu\Delta}^{n,m}$$

$$\begin{aligned}
 (\partial_{\beta\Delta} \partial_{\mu\mu} \xi)(x_1, \dots, x_{n+1}) & = \Delta \beta^{m-2} \mu_\alpha^n(w_1^{(1)}, \dots, x_n^{(1)}) \otimes (\partial_{\alpha\mu} \xi)(x_1^{(2)}, \dots, x_1^{(2)}) \\
 & - (\partial_{\alpha\mu} \xi)(x_1^{(1)}, \dots, x_1^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^n(w_1^{(2)}, \dots, x_n^{(2)}) \\
 & = \Delta \beta^{m-2} \mu_\alpha^n(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \beta \alpha^{n-2} (x_1^{(2)} x_2^{(2)}) \cdot \xi(x_3^{(2)}, \dots, x_{n+1}^{(2)}) \\
 & - (-1)^n \Delta \beta^{m-2} \mu_\alpha^n(w_1^{(1)}, \dots, x_n^{(1)}) \otimes \xi(x_1^{(2)}, \dots, x_{n-1}^{(2)}) \cdot \beta \alpha^{n-2} (x_n^{(2)} x_{n+1}^{(2)}) \\
 & - \beta \alpha^{n-2} (x_1^{(1)} x_2^{(1)}) \cdot \xi(x_3^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^n(x_1^{(2)}, \dots, x_n^{(2)}) \\
 & + (-1)^n \xi(x_1^{(1)}, \dots, x_{n-1}^{(1)}) \cdot \beta \alpha^{n-2} (x_n^{(1)} x_{n+1}^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^n(x_1^{(2)}, \dots, x_n^{(2)})
 \end{aligned}$$

On the other side we get

$$\begin{aligned}
 (\partial_{\alpha\mu} \partial_{\beta\Delta} \xi)(x_1, \dots, x_{n+1}) & = \alpha^{n-2} (x_1 x_2) \cdot (\partial_{\beta\Delta} \xi)(x_3, \dots, x_{n+1}) \\
 & - (\partial_{\beta\Delta} \xi)(x_1, \dots, x_{n-1}) \cdot \alpha^{n-2} (x_n x_{n+1}) \\
 & = \alpha^{n-2} (x_1 x_2) \cdot \left(\Delta \alpha \beta^{m-2} \mu_\alpha^{n-i} (x_3^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \xi(x_3^{(2)}, \dots, x_{n+1}^{(2)}) \right) \\
 & - \alpha^{n-2} (x_1 x_2) \cdot \left(\xi(x_3^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \Delta \alpha \beta^{m-2} \mu_\alpha^{n-i} (x_3^{(2)}, \dots, x_{n+1}^{(2)}) \right) \\
 & - \left(\Delta \alpha \beta^{m-2} \mu_\alpha^{n-i} (x_1^{(1)}, \dots, x_{n-1}^{(1)}) \otimes \xi(x_1^{(2)}, \dots, x_{n-1}^{(2)}) \right) \cdot \alpha^{n-2} (x_n x_{n+1}) \\
 & + \left(\xi(x_1^{(1)}, \dots, x_{n-1}^{(1)}) \otimes \Delta \alpha \beta^{m-2} \mu_\alpha^{n-i} (x_1^{(2)}, \dots, x_{n-1}^{(2)}) \right) \cdot \alpha^{n-2} (x_n x_{n+1})
 \end{aligned}$$

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Using (19) and (17) we get

$$\begin{aligned}
& \alpha^{n-2}(x_1 x_2) \cdot \left(\Delta \alpha \beta^{m-2} \mu_\alpha^{n-i}(x_3^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \xi(x_3^{(2)}, \dots, x_{n+1}^{(2)}) \right) \\
&= \left(\beta^{m-2} \alpha^{n-2}(x_1^{(1)} x_2^{(1)}) \cdot \Delta \alpha \beta^{m-2} \mu_\alpha^{n-i}(x_3^{(1)}, \dots, x_{n+1}^{(1)}) \right) \otimes \left(\beta \alpha^{n-2}(x_1^{(2)} x_2^{(2)}) \cdot \xi(x_3^{(2)}, \dots, x_{n+1}^{(2)}) \right) \\
&= \left(\Delta \alpha \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(1)}, \dots, x_{n+1}^{(1)}) \right) \otimes \left(\beta \alpha^{n-2}(x_1^{(2)} x_2^{(2)}) \cdot \xi(x_3^{(2)}, \dots, x_{n+1}^{(2)}) \right)
\end{aligned}$$

Similarly the other three terms can be manipulated, which shows that the two sides agree.

$$C_{GS} \alpha_\Delta^{n,m} \rightarrow C_{GS} \mu_\Delta^{n,m}$$

$$\begin{aligned}
\partial_{\mu\mu} \partial_{\beta\Delta} \chi(x_1, \dots, x_{n+1}) &= \alpha^{n-1}(x_1) \cdot (\partial_{\beta\Delta} \chi)(x_2, \dots, x_{n+1}) \\
&+ \sum_{i=1}^n (-1)^i \partial_{\beta\Delta} \chi(\alpha(x_1), \dots, x_i x_{i+1}, \dots, \alpha(x_n)) + (-1)^{n+1} (\partial_{\beta\Delta} \chi)(x_1, \dots, x_n) \cdot \alpha^{n-1}(x_{n+1}) \\
&= \alpha^{n-1}(x_1) \cdot \left(\Delta \beta^{m-2} \mu_\alpha^n(x_2^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \chi(x_2^{(2)}, \dots, x_{n+1}^{(2)}) \right) \\
&- \alpha^{n-1}(x_1) \cdot \left(\chi(x_2^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^n(x_2^{(2)}, \dots, x_{n+1}^{(2)}) \right) \\
&+ \sum_{i=1}^n \beta^{m-2} \mu_\alpha^n(\alpha(x_1)^{(1)}, \dots, (x_i x_{i+1})^{(1)}, \dots, \alpha(x_{n+1})^{(1)}) \otimes \chi(\alpha(x_1)^{(2)}, \dots, (x_i x_{i+1})^{(2)}, \dots, \alpha(x_{n+1})^{(2)}) \\
&- \sum_{i=1}^n \chi(\alpha(x_1)^{(1)}, \dots, (x_i x_{i+1})^{(1)}, \dots, \alpha(x_{n+1})^{(1)}) \otimes \beta^{m-2} \mu_\alpha^n(\alpha(x_1)^{(2)}, \dots, (x_i x_{i+1})^{(2)}, \dots, \alpha(x_{n+1})^{(2)}) \\
&+ (-1)^{n+1} \left(\Delta \beta^{m-2} \mu_\alpha^n(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \chi(x_1^{(2)}, \dots, x_n^{(2)}) \right) \cdot \alpha^{n-1}(x_{n+1}) \\
&- (-1)^{n+1} \left(\chi(x_1^{(1)}, \dots, x_n^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^n(x_1^{(2)}, \dots, x_n^{(2)}) \right) \cdot \alpha^{n-1}(x_{n+1})
\end{aligned}$$

We note that $\mu_\alpha^n(\alpha(x_1), \dots, (x_i x_{i+1}), \dots, \alpha(x_{n+1})) = \mu_\alpha^{n+1}(x_1, \dots, x_{n+1})$.

$$\begin{aligned}
(\partial_{\beta\Delta} \partial_{\mu\mu} \chi)(x_1, \dots, x_{n+1}) &= \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(1)}, \dots, x_{n+1}^{(1)}) \otimes (\partial_{\beta\Delta} \chi)(x_1^{(2)}, \dots, x_{n+1}^{(2)}) \\
&- (\partial_{\beta\Delta} \chi)(x_1^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(2)}, \dots, x_{n+1}^{(2)}) \\
&= \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \beta \alpha^{n-1}(x_1^{(2)}) \cdot \chi(x_2^{(2)}, \dots, x_{n+1}^{(2)}) \\
&+ \sum_{i=1}^n (-1)^i \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \chi(\alpha(x_1^{(2)}), \dots, x_i^{(2)} x_{i+1}^{(2)}, \dots, \alpha(x_{n+1}^{(2)})) \\
&+ (-1)^{n+1} \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \beta \chi(x_1^{(2)}, \dots, x_n^{(2)}) \cdot \alpha^{n-1}(x_{n+1}^{(2)}) \\
&- \beta \alpha^{n-1}(x_1^{(1)}) \cdot \chi(x_2^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(2)}, \dots, x_{n+1}^{(2)}) \\
&- \sum_{i=1}^n (-1)^i \chi(\alpha(x_1^{(1)}), \dots, x_i^{(2)} x_{i+1}^{(1)}, \dots, \alpha(x_{n+1}^{(1)})) \otimes \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(2)}, \dots, x_{n+1}^{(2)})
\end{aligned}$$

$$-(-1)^{n+1} \beta \chi(x_1^{(1)}, \dots, x_n^{(1)}) \cdot \alpha^{n-1}(x_{n+1}^{(1)}) \otimes \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(2)}, \dots, x_{n+1}^{(2)})$$

Again using (19) and (17) we get

$$\begin{aligned} & \alpha^{n-1}(x_1) \cdot \left(\Delta \beta^{m-2} \mu_\alpha^n(x_2^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \chi(x_2^{(2)}, \dots, x_{n+1}^{(2)}) \right) = \\ & \Delta \beta^{m-2} \mu_\alpha^{n+1}(x_1^{(1)}, \dots, x_{n+1}^{(1)}) \otimes \beta \alpha^{n-1}(x_1^{(2)}) \cdot \chi(x_2^{(2)}, \dots, x_{n+1}^{(2)}). \end{aligned}$$

With this and similar equalities it is easy to see that the two sides agrees. \square

This describes the deformation of a general Hom-bialgebra. More on deformation will be given in Section 8.4.

If one wants to consider only the case where $\alpha = \beta$, i.e. both structure maps are the same, and one wants the deformation to respect this or only consider algebras where this is the case, one has to identify $C_{\alpha\Delta}^{i+1,j}$ with $C_{\mu\beta}^{i,j+1}$ and set $C_{\alpha\beta}^{i,j}$ to $\{0\}$. Note that this does not respect the bicomplex structure, so the resulting complex is no longer a bicomplex. For example we have $C_{GS}^2 = H^{2,1} \oplus H^{1,1} \oplus H^{1,2}$. Since this is often considered we give some more details.

Proposition 6.7.2 *Let $(A, \mu, \alpha, \Delta, \alpha)$ be a Hom-bialgebra. Then the subspace of C_{GS} given in degree n by elements of the form $(\varphi_{i,j}, \psi_{i,j}, \chi_{i,j}, \xi_{i,j})_{i+j=n}$, with $\varphi_{i,j} \in C_{GS\mu\Delta}^{i,j}$, $\psi_{i,j} \in C_{GS\alpha\Delta}^{i,j}$, $\chi_{i,j} \in C_{GS\alpha\Delta}^{i,j}$ and $\xi_{i,j} \in C_{GS\alpha\beta}^{i,j}$ for all i, j , which satisfy $\psi_{i,j} = \chi_{i-1,j+1}$ and $\xi_{i,j} = 0$, is a subcomplex, i.e. it is preserved by the differential.*

Proof. We need to verify that $\partial(\varphi_{i,j}, \psi_{i,j}, \chi_{i,j}, \xi_{i,j})_{i+j=n}$ again satisfies the two constraints. For the first we have $\partial_{\mu\mu}\chi_{i,j+1} = \partial_{\alpha\alpha}\psi_{i+1,j}$ and $\partial_{\mu\alpha}\varphi_{i,j} = \partial_{\Delta\beta}\varphi_{i,j}$. For the second we have $\partial_{\mu\alpha}\chi_{i-1,j} = \partial_{\Delta\beta}\psi_{i,j-1}$ since $\alpha = \beta$. \square

So the complex can be given by

$$C_{GS}^i = \bigoplus_{j=1}^{i-1} C_{GS\mu}^{j,i-j} \oplus \bigoplus_{j=1}^{i-2} C_{GS\alpha}^{j,i-j-1} \quad (81)$$

$$= \bigoplus_{j=1}^{i-1} H^{j,i-j} \oplus \bigoplus_{j=1}^{i-2} H^{j,i-j-1}. \quad (82)$$

Note that the degree of $C_{GS\mu}^{i,j}$ is $i + j$, while the degree of $C_{GS\alpha}^{i,j}$ is $i + j + 1$. The

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differential consists of maps

$$\begin{aligned}
 \partial_{\mu\mu} : C_{GS\mu}^{i,j} &\rightarrow C_{GS\mu}^{i+1,j} : \\
 (\partial_{\mu\mu}\varphi)(x_1, \dots, x_{i+1}) &= \alpha^{i-1}(x_1) \cdot \varphi(x_2, \dots, x_{i+1}) \\
 &\quad + \sum_{k=1}^i (-1)^k \varphi(\alpha(x_1), \dots, x_k x_{k+1}, \dots, \alpha(x_{i+1})) + (-1)^{i+1} \varphi(x_1, \dots, x_i) \cdot \alpha^{i-1}(x_{i+1}), \\
 \partial_{\mu\alpha} : C_{GS\mu}^{i,j} &\rightarrow C_{GS\alpha}^{i,j} : \\
 (\partial_{\mu\alpha}\varphi)(x_1, \dots, x_i) &= \alpha^{\otimes j} \varphi(x_1, \dots, x_i) - \varphi(\alpha(x_1), \dots, \alpha(x_i)), \\
 \partial_{\alpha\alpha} : C_{GS\alpha}^{i,j} &\rightarrow C_{GS\alpha}^{i+1,j} : \\
 (\partial_{\alpha\alpha}\psi)(x_1, \dots, x_{i+1}) &= \alpha^i(x_1) \cdot \psi(x_2, \dots, x_{i+1}) \\
 &\quad + \sum_{k=1}^i (-1)^k \psi(\alpha(x_1), \dots, x_k x_{k+1}, \dots, \alpha(x_{i+1})) + (-1)^{i+1} \psi(x_1, \dots, x_i) \cdot \alpha^i(x_{i+1}), \\
 \partial_{\alpha\mu} : C_{GS\alpha}^{i,j} &\rightarrow C_{GS\mu}^{i+2,j} : \\
 (\partial_{\alpha\mu}\psi)(x_1, \dots, x_{i+2}) &= \alpha^i(x_1 x_2) \cdot \psi(x_3, \dots, x_{i+2}) - \psi(x_1, \dots, x_i) \cdot \alpha^i(x_{i+1} x_{i+2}),
 \end{aligned}$$

Dually we define the differentials $\partial_{\Delta\Delta} : C_{GS\mu}^{i,j} \rightarrow C_{GS\mu}^{i,j+1}$, $\partial_{\beta\beta} : C_{GS\alpha}^{i,j} \rightarrow C_{GS\alpha}^{i,j+1}$ and $\partial_{\beta\Delta} : C_{GS\alpha}^{i,j} \rightarrow C_{GS\mu}^{i,j+2}$.

The different parts of the differential can be seen in the following diagram:

$$\begin{array}{ccccc}
 & & C_{GS\mu}^{i,j+2} & & \\
 & & \nearrow & & \\
 & & C_{GS\alpha}^{i,j+1} & & \\
 & & \nearrow & & \\
 & & C_{GS\mu}^{i,j+1} & & \\
 & & \nearrow & & \\
 & & C_{GS\alpha}^{i,j} & \longrightarrow & C_{GS\alpha}^{i+1,j} \\
 & \nearrow & \searrow & & \\
 C_{GS\mu}^{i,j} & \longrightarrow & C_{GS\mu}^{i+1,j} & & C_{GS\mu}^{i+2,j}
 \end{array}$$

6.7.1 Relation to other cohomologies

We relate this to the cohomology given in Section 6.6.

We first consider the situation, of α -Hom-bialgebras. We want to find a subcomplex of the form such that all elements in $C_{GS\alpha}(A)$ are zero. For the differential to respect this we must have $\partial_\alpha\varphi = 0$ for $\varphi \in C_{GS\mu}$. This describes precisely the complex of Section 6.6. It is also clear that the remaining differential is the one described there.

In the more general situation of (α, β) -Hom-algebras one can search for different types of subcomplexes. The most simple one is to search for a subcomplex in $C_{GS\mu\Delta}$. In this case $\partial_{\mu\alpha}$ and $\partial_{\Delta\beta}$ must vanishes, and we get the bicomplex

$$\text{Hom}_{\alpha,\beta}(A^{\otimes\bullet}, A^{\otimes\bullet}) = \{\varphi \in \text{Hom}(A^{\otimes\bullet}, A^{\otimes\bullet}) \mid \alpha\varphi = \varphi\alpha, \beta\varphi = \varphi\beta\}.$$

It is clear that $\partial_{\mu\mu}$ and $\partial_{\Delta\Delta}$ restrict to this complex. In the case $\alpha = \beta$ one gets the same as before.

6.7.2 Yau twist

There are different cases to consider: The product and the coproduct are twisted at the same time with the same map. Or they are twisted with different maps. This includes as a special case the situation where only the product or coproduct are twisted.

We first consider the case that for an α -Hom-bialgebra the product and coproduct are twisted with the same map.

In this case we define a map Φ from $C(A, \gamma)$ to $\tilde{\text{HC}}(A)$. For $\varphi \in C_\mu^{i,j}(A, \gamma), \psi \in C_\alpha^{i,j}(A, \gamma)$ it is given by

$$\Phi(\varphi) = \gamma^{i-1}\varphi\gamma^{j-1} \in \tilde{\text{HC}}_\mu^{i,j}(A) \quad (83)$$

$$\begin{aligned} \Phi(\psi) &= (\gamma^{i-1}\psi \circ \mu\gamma^{j-1}, \gamma^{i-1}\psi\gamma^{j-1}, \gamma^{i-1}\Delta \circ \psi\gamma^{j-1}) \\ &\in \tilde{\text{HC}}_\mu^{i+1,j} \oplus \tilde{\text{HC}}_\alpha^{i,j} \oplus \tilde{\text{HC}}_\mu^{i,j+1} \end{aligned} \quad (84)$$

Proposition 6.7.3 *The map $\Phi : C(A, \gamma) \rightarrow \tilde{\text{HC}}(A)$ defined above is a chain map.*

Proof. Let $\varphi \in C_\mu(A, \gamma)$. We need to verify $\Phi(\partial^E\varphi) = \partial^H\Phi(\varphi)$, where ∂^E is the differential of $C(A, \gamma)$ and ∂^H the one of $\tilde{\text{HC}}(A_\gamma)$. This is simply calculation using Theorem 6.3.8. Now we consider $\psi \in C_\gamma^{i,j}(A, \gamma)$. Then we have

$$\begin{aligned} \partial\Phi(\psi) &= \partial(-\gamma^{i-1}(\psi \circ \mu)\gamma^{j-1}, \gamma^{i-1}\psi\gamma^{j-1}, -\gamma^{i-1}(\Delta \circ \psi)\gamma^{j-1}) \\ &= (-\partial_{\mu\mu}(\gamma^{i-1}(\psi \circ \mu)\gamma^{j-1}) + \partial_{\alpha\mu}(\gamma^{i-1}\psi\gamma^{j-1}), \\ &\quad -\partial_{\mu\alpha}(\gamma^{i-1}(\psi \circ \mu)\gamma^{j-1}) - \partial_{\alpha\alpha}(\gamma^{i-1}\psi\gamma^{j-1}), \\ &\quad -\partial_{\mu\mu}(\gamma^{i-1}(\Delta \circ \psi)\gamma^{j-1}) - \partial_{\mu\mu}^c(\gamma^{i-1}(\psi \circ \mu)\gamma^{j-1}), \text{dual}) \end{aligned}$$

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and

$$\begin{aligned}\Phi(\partial\psi) &= \Phi(\partial_\mu\psi, \partial_\mu^c\psi) \\ &= (\gamma^i(\partial_\mu) \circ \mu\gamma^{j-1}, \gamma^i(\partial_\mu\psi)\gamma^{j-1}, \gamma^i\delta \circ (\partial_\mu\psi)\gamma^{j-1} + \gamma^{i-1}(\partial_\mu^c\psi) \circ \mu\gamma^j, dual)\end{aligned}$$

Here *dual* denotes terms which are dual to the ones already given so we omit them. Now using calculations similar to Theorem 6.3.8 one can show that the first two components agree. The third one are equal since

$$\Delta_k\gamma^i(\gamma(x_1)\cdot\psi(\gamma^{j-1}(x_2), \dots, \gamma^{j-1}(x_{i+1}))) = \gamma^{i-1}(x_1)\cdot\gamma\gamma^{i-1}(\Delta_k\psi)(\gamma^{j-1}(x_2), \dots, \gamma^{j-1}(x_{i+1})),$$

where $\Delta_k = \text{id}^{\otimes k} \otimes \Delta \otimes \text{id}^{\otimes j-k-1}$. \square

Now we come to the case where product and product are twisted by different maps. We recall what we mean with Yau twist in this case. Let A be a bialgebra and γ_1, γ_2 be two commuting morphisms of it then $(A, \gamma_1\mu, \Delta\gamma_2, \gamma_1, \gamma_2)$ is a Hom-bialgebra, which we denote by A_{γ_1, γ_2} .

Then we define a map $C(A, \gamma_1, \gamma_2) \rightarrow \tilde{\text{HC}}(A_{\gamma_1, \gamma_2})$ for $(\varphi, \psi, \chi, \xi) \in C^{i,j}(A, \gamma_1, \gamma_2)$ by

$$\begin{aligned}\Phi(\varphi, \psi, \chi, \xi) &= (\gamma_1^{i-1}\varphi\gamma_2^{j-1} + \gamma_1^{i-2}(\psi \circ \mu)\gamma_2^{j-1} + \gamma_1^{i-1}(\Delta \circ \chi)\gamma_2^{j-2} + \gamma_1^{i-2}(\Delta \circ \xi \circ \mu)\gamma_2^{j-2}, \\ &\quad \gamma_1^{i-2}\psi\gamma_2^{j-1} + \gamma_1^{i-2}(\Delta \circ \xi)\gamma_2^{j-2}, \\ &\quad \gamma_1^{i-1}\chi\gamma_2^{j-2} + \gamma_1^{i-2}(\xi \circ \mu)\gamma_2^{j-2}, \\ &\quad \gamma_1^{i-2}\xi\gamma_2^{j-2}).\end{aligned}\tag{85}$$

Proposition 6.7.4 *The map Φ defined above is a chain map. If γ_1 and γ_2 are invertible it is an isomorphism.*

Proof. For $\varphi \in C_{\Delta\mu}(A, \gamma)$ doing calculations similar to Theorem 6.3.8, one gets $\Phi(\partial\varphi) = \partial\Phi(\varphi)$. Similar for the other components of the complex. Note that since the definition of the cohomology is self dual one does only need to check certain equations.

It is also clear that Φ is invertible if γ_1 and γ_2 are so. \square

With this one can compute the cohomology for regular Hom-bialgebras if one can compute the cohomology for the corresponding commuting endomorphism. As a special case one can calculate the α -type GS cohomology of an ordinary bialgebra considered as Hom-bialgebra and one gets:

Proposition 6.7.5 *Let A be a bialgebra then*

$$\tilde{\text{HC}}^{i,j}(A) = C_{GS}^{i,j}(A) \oplus C_{GS}^{i-1,j}(A) \oplus C_{GS}^{i,j-1}(A) \oplus C_{GS}^{i-1,j-1}(A).\tag{86}$$

Proof. It is clear that $H(A, \text{id}, \text{id})$ is given by the above. So the result follows from the previous proposition. \square

6.8 MULT. HOM-LIE AND LIE-BIALGEBRA

In this section we recall the definition of the Chevalley-Eilenberg cohomology for Hom-Lie algebras given in [AEM11]. We also define a generalization of the big bracket to the Hom case, which allows us to define a cohomology for Hom-Lie bialgebras with values in itself [HM].

We recall the cohomology complex for a Hom-Lie algebra $(\mathfrak{g}, \nu, \alpha)$ and a \mathfrak{g} -module (M, ρ, β) . For this we consider the Grassman algebra $\Lambda \mathfrak{g}$ over \mathfrak{g} , which is a Hopf-algebra. See Section 2.2.2 for a definition. The structure map α can be extended to Hopf-algebra morphism on $\Lambda \mathfrak{g}$, which we also denote by α .

With this we define the Chevalley-Eilenberg complex by $C_{CE}^n(\mathfrak{g}, M) = \text{Hom}_H(\Lambda^n(\mathfrak{g}, \alpha), (M, \beta))$. This is the set of alternating multilinear maps, which are compatible with the structure maps. Further we define a map $\delta : C_{CE}^n(\mathfrak{g}) \rightarrow C_{CE}^{n+1}(\mathfrak{g})$ by

$$\begin{aligned} (\delta\varphi)(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \alpha^{n-1}(x_i) \cdot \varphi(x_1, \dots, \hat{x}_i, \dots) \\ &\quad - \sum_{i < j} (-1)^{j-i+1} \varphi(\alpha(x_1), \dots, [x_i, x_j], \dots, \hat{x}_j, \dots). \end{aligned} \tag{87}$$

One can show that $\delta \circ \delta = 0$ so one can define the associated cohomology. We will call this the Chevalley-Eilenberg cohomology of \mathfrak{g} with values in M and denote it by $H_{CE}(\mathfrak{g}, M)$. In the case $\alpha = \text{id}$ this definition agrees with the definition of the Chevalley-Eilenberg cohomology for Lie algebras so no ambiguities arises. Again the complex starts in degree 1 and not 0 as it is normally the case.

6.8.1 Big bracket

To define the cohomology for a Hom-Lie bialgebra with values in itself we first define a Hom-version of the big bracket. The original version comes from [LR90]. For the Hom-case and $\beta = \alpha^{-1}$ this was already done in [CS16].

Let V be a vector space and $\alpha, \beta : V \rightarrow V$ be two commuting structure maps. One can think of (A, α, β) as a Hom-Lie bialgebra with zero bracket and cobracket. We will call the triple (V, α, β) a biHom-module.

Let $\varphi \in \text{Hom}(\Lambda V, \Lambda V)$. We define the α - and β -height of φ , as integers $\text{hgt}_\alpha \varphi$ and $\text{hgt}_\beta \varphi$ resp. We also write $\text{hgt} \varphi = (\text{hgt}_\alpha \varphi, \text{hgt}_\beta \varphi)$. The height of α is $(1, 0)$ and the height of β is $(0, 1)$. This explains the name. Further if V is a Lie-bialgebra we set $\text{hgt} \nu = (1, 0)$ and $\text{hgt} \delta = (0, 1)$. And more general $\text{hgt} \varphi = (i - 1, j - 1)$ for $\varphi \in \text{Hom}(\Lambda V, \Lambda V)$ with the usual action and coaction, so for example in $C_{CE}(V)^{i,j}$. If the action is twisted by α^k this is $a \cdot x = [\alpha(a), x]$ and the coaction by β^l then $\text{hgt} \varphi = (i + k - 1, j + l - 1)$. So e.g. $\text{hgt} \varphi = (i, 0)$ for $\varphi \in \tilde{C}_{CE \alpha}^i(V)$. On the other hand if

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we say φ has a certain height, we also assume the actions to be twisted accordingly. This will not be needed in this section but in Sections 6.9 and 6.11

We set the height of the product and coproduct on ΛV to zero. By pr_V or simply pr we denote the projection from $\Lambda V \rightarrow V$. Note that $\text{pr} \mu = \mu(\text{pr} \otimes \text{id} + \text{id} \otimes \text{pr})$.

Now we can define the big bracket. For this we consider $\text{Hom}(\Lambda V, \Lambda V)$ to be graded, such that $\varphi \in \text{Hom}(\Lambda^i V, \Lambda^j V)$ has degree $\deg(\varphi) = i + j$.

Definition 6.8.1 Let $\varphi, \psi \in \text{Hom}(\Lambda V, \Lambda V)$ be maps with height then we define a product by

$$\psi \circ \varphi = \mu(\psi \otimes \alpha^\psi)(\mu \otimes \text{id})(\text{id} \otimes \text{pr} \otimes \text{id})(\text{id} \otimes \Delta)(\alpha^\varphi \otimes \varphi)\Delta \quad (88)$$

and with that

$$\{\varphi, \psi\}_{\text{BB}} = \varphi \circ \psi - (-1)^{\deg(\varphi)\deg(\psi)} \psi \circ \varphi. \quad (89)$$

Here we used the shorthand $\alpha^\varphi = \alpha^{\text{hgt}_\alpha \varphi} \beta^{\text{hgt}_\beta \varphi}$, which has the same height as φ . We call $\{\cdot, \cdot\}_{\text{BB}}$ big bracket.

For $\varphi \in \text{Hom}(\Lambda^k V, V)$ and $\psi \in \text{Hom}(\Lambda^l V, V)$ the product \circ can be written as

$$(\varphi \circ \psi)(x_1, \dots, x_{k+l-1}) = \frac{1}{k!(l-1)!} \sum_{\sigma \in S_{n+k}} \text{sign}(\sigma) \varphi(\psi(x_{\sigma(1)}, \dots, x_{\sigma(k)}), \alpha^{\text{hgt}_\alpha \varphi}(x_{\sigma(k+1)}), \dots, \alpha^{\text{hgt}_\alpha \varphi}(x_{\sigma(k+l-1)})). \quad (90)$$

One can use the big bracket to define the Chevalley-Eilenberg differential:

Proposition 6.8.2 Let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra. For $\varphi \in C_{\text{CE}}(V, V) = \text{Hom}_H(\Lambda V, V)$ we have $\partial \varphi = (-1)^{k-1} \{\nu, \varphi\}_{\text{BB}}$.

Proof. We have that

$$\nu \circ \varphi = (-1)^{k-1} [\alpha^k(x^{(1)}), \varphi(x^{(2)})], \varphi \circ \nu = (-1)^{k-1} \varphi(\alpha(x^{(1)}) \wedge \nu(x^{(2)})). \quad (91)$$

Where we used the Sweedler notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$ for the coproduct in ΛV . This shows that $\partial_{\mu\mu} \varphi = (-1)^{k-1} \{\nu, \varphi\}_{\text{BB}}$. \square

We consider the complex

$$B^{i,j} = \text{Hom}_{\alpha,\beta}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) = \{\varphi \in \text{Hom}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) \mid \alpha^{\otimes j} \varphi = \varphi \alpha^{\otimes i}, \beta^{\otimes j} \varphi = \varphi \beta^{\otimes i}\}$$

for $i, j \geq 1$ and $\{0\}$ otherwise.

Theorem 6.8.3 The bracket defined in Eq. (89) defines a graded Lie algebra structure on $B^{\bullet\bullet}$.

Proof. We compute $\varphi \circ (\psi \circ \chi)$, for this we introduce

$$Q = (\mu \otimes \text{id})(\text{id} \otimes \text{pr} \otimes \text{id})(\text{id} \otimes \Delta),$$

and get

$$\begin{aligned} \varphi \circ (\psi \circ \chi) &= \mu(\varphi \otimes \alpha^\varphi)(\text{id} \otimes \mu)Q(\text{id} \otimes \psi \otimes \alpha^\psi)(\text{id} \otimes Q)(\alpha^{\varphi\psi} \otimes \alpha^\chi \otimes \chi)(\text{id} \otimes \Delta)\Delta \\ &= \mu(\varphi \otimes \alpha^\varphi)(\text{id} \otimes \mu)(Q \otimes \text{id})(\alpha^\psi \otimes \psi \otimes \alpha^\psi)(\text{id} \otimes Q)(\Delta \otimes \text{id})(\alpha^\chi \otimes \chi)\Delta \quad (92) \\ &\quad + \mu^3(\alpha^\varphi \otimes \varphi \otimes \alpha^\varphi)(\text{id} \otimes Q)(\tau \otimes \text{id})(\alpha^\psi \otimes \psi \otimes \alpha^\psi)(\alpha^\chi \otimes \alpha^\chi \otimes \chi)\Delta^3. \end{aligned}$$

Here we used the abbreviations $\mu^3 = \mu(\text{id} \otimes \mu)$ and $\Delta^3 = (\Delta \otimes \text{id})\Delta$. Using that φ commutes with α and β , the last term can be rearranged to

$$\mu^3(\alpha^\varphi \otimes \alpha^\psi \otimes \text{id})(\psi \otimes \varphi \otimes \text{id})(\text{id} \otimes Q)(\tau \otimes \text{id})(\text{id} \otimes Q)(\alpha^\chi \otimes \alpha^\chi \otimes \chi)\Delta^3. \quad (93)$$

This can be seen to be symmetric in φ and ψ . The term $(\varphi \circ \psi) \circ \chi$ can be computed similarly. One gets the term (92) plus a term symmetric in ψ and χ . So in total we get $\sum_{\text{perm. } \varphi, \psi, \chi} (\varphi \circ (\psi \circ \chi) - (\varphi \circ \psi) \circ \chi) = 0$. This is equivalent to the fact that $\{\cdot, \cdot\}_{\text{BB}}$ satisfies the Jacobi identity. \square

Proposition 6.8.4 *Let \mathfrak{g} be a vector space with two commuting structure maps $\alpha, \beta : \mathfrak{g} \rightarrow \mathfrak{g}$ then*

- *If $v \in B^{2,1}$ satisfies $\{v, v\}_{\text{BB}} = 0$ then $(\mathfrak{g}, v, \alpha)$ is a Hom-Lie algebra.*
- *If $\delta \in B^{1,2}$ satisfies $\{\delta, \delta\}_{\text{BB}} = 0$ then $(\mathfrak{g}, \delta, \beta)$ is a Hom-Lie coalgebra.*
- *If a pair (v, δ) satisfies $\{v + \delta, v + \delta\}_{\text{BB}} = 0$ then $(\mathfrak{g}, v, \delta, \alpha, \beta)$ is a Hom-Lie bialgebra. These are the Maurer-Cartan elements of $\{\cdot, \cdot\}_{\text{BB}}$.*

Proof. This follows easily using the computation done in the proof of Proposition 6.9.4. \square

Moreover the big bracket can be restricted to the complex $\text{Hom}_\alpha(\Lambda\mathfrak{g}, \mathfrak{g})$, which is the complex of the Chevalley-Eilenberg cohomology for Hom-Lie algebras. On this complex it agrees with the generalization of the Nijenhuis-Richardson bracket given in [AAM15].

6.8.2 Hom-Lie bialgebras

Given a Hom-Lie bialgebra \mathfrak{g} , we define the total complex $B^i(\mathfrak{g}) = \bigoplus_{j=1}^i B^{j, i-j}(\mathfrak{g})$.

Proposition 6.8.5 *Let \mathfrak{g} be a Hom-Lie bialgebra. The map $\partial : B^\bullet \rightarrow B^\bullet$ defined by $\partial\varphi = \{v + \delta, \varphi\}_{\text{BB}}$ is a differential, i.e. $\partial \circ \partial = 0$.*

Proof. This is standard and follows from the Jacobi-identity of the big bracket. \square

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With this one can define a cohomology for Hom-Lie bialgebras, which we will denote by $H_{CE}(\mathfrak{g})$ and call Chevalley-Eilenberg cohomology of \mathfrak{g} .

Remark 6.8.6 Note that in the Hom-context it is not straightforward to define an analog to quasi-Lie bialgebras. Since extending the bracket to including maps in $\text{Hom}(\mathbb{K}, \Lambda \mathfrak{g})$ would involve α^{-1} . So it is possible if α is invertible. It is also possible if one considers the case $\alpha = \beta$ since in this case α^{-1} cancels with β .

Remark 6.8.7 In the case that $\alpha = \beta = \text{id}$ the cohomology we defined here is precisely the normal cohomology for Lie bialgebras, since in this case the complex is just $\text{Hom}(\Lambda^\bullet \mathfrak{g}, \Lambda^\bullet \mathfrak{g})$ and the big bracket we defined here is the ordinary one.

6.9 α -TYPE CHEVALLEY-EILENBERG COHOMOLOGY FOR HOM-LIE ALGEBRAS

Similarly to Section 6.3, we define an α -type cohomology for Hom-Lie algebras. Let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra and (M, β) a \mathfrak{g} -module. We denote by $\Lambda^k \mathfrak{g}$ the k -th exterior power of \mathfrak{g} . Then the complex for the cohomology of \mathfrak{g} with values in M is given by

$$\tilde{C}_{CE}^n(\mathfrak{g}, M) = \tilde{C}_{CE \mu}^n(\mathfrak{g}, M) \oplus \tilde{C}_{CE \alpha}^n(\mathfrak{g}, M) = \text{Hom}(\Lambda^n \mathfrak{g}, M) \oplus \text{Hom}(\Lambda^{n-1} \mathfrak{g}, M). \quad (94)$$

Here $\text{Hom}(\Lambda^0 \mathfrak{g}, M)$ is set to $\{0\}$, instead of \mathbb{K} as usual, since otherwise α^{-1} would be needed in the definition of the differential. We write (φ, ψ) or $\varphi + \psi$ with $\varphi \in \tilde{C}_{CE \mu}^n(\mathfrak{g}, M)$ and $\psi \in \tilde{C}_{CE \alpha}^n(\mathfrak{g}, M)$ for an element in $\tilde{C}_{CE}^n(\mathfrak{g}, M)$. We define four linear maps, with domain and range given in the following diagram:

$$\begin{array}{ccc} \tilde{C}_{CE \mu}^n & \xrightarrow{\partial_{\mu\mu}} & \tilde{C}_{CE \mu}^{n+1} \\ & \searrow \partial_{\mu\alpha} & \uparrow \\ \oplus & & \oplus \\ \tilde{C}_{CE \alpha}^n & \xrightarrow{\partial_{\alpha\alpha}} & \tilde{C}_{CE \alpha}^{n+1} \end{array}$$

$$\begin{aligned} (\partial_{\mu\mu}\varphi)(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \alpha^{n-1}(x_i) \cdot \varphi(x_1, \dots, x_{n+1}) \\ &\quad - \sum_{i < j} (-1)^{i+j-1} \varphi([x_i, x_j], \alpha(x_1), \dots, \hat{x}_i, \hat{x}_j, \dots, \alpha(x_n)) \end{aligned} \quad (95)$$

$$(\partial_{\alpha\alpha}\psi)(x_1, \dots, x_n) = \sum_{i=1}^n (-1)^{i+1} \alpha^{n-1}(x_i) \cdot \psi(x_1, \dots, x_n) \quad (96)$$

$$- \sum_{i < j} (-1)^{i+j-1} \psi([x_i, x_j], \alpha(x_1), \dots, \hat{x}_i, \hat{x}_j, \dots, \alpha(x_n))$$

$$(\partial_{\mu\alpha}\varphi)(x_1, \dots, x_n) = \beta(\varphi(x_1, \dots, x_n)) - \varphi(\alpha(x_1), \dots, \alpha(x_n)) \quad (97)$$

$$(\partial_{\alpha\mu}\psi)(x_1, \dots, x_{n+1}) = \sum_{i \leq j} (-1)^{i+j-1} [\alpha^{n-2}(x_i), \alpha^{n-2}(x_j)] \cdot \psi(x_1, \dots, \hat{x}_i, \hat{x}_j, x_{n+1}). \quad (98)$$

The sign given by $(-1)^{\cdot}$ is always determined by the permutation of the x_i .

Theorem 6.9.1 *The map $\partial : \tilde{C}_{CE}^n(\mathfrak{g}, M) \rightarrow \tilde{C}_{CE}^{n+1}(\mathfrak{g}, M)$ defined by $\partial(\varphi, \psi) = (\partial_{\mu\mu}\varphi - \partial_{\alpha\mu}\psi, \partial_{\mu\alpha}\varphi - \partial_{\alpha\alpha}\psi)$ is a differential, i.e. $\partial \circ \partial = 0$.*

Proof. This is a straightforward calculation. One has to take care of the signs, but as stated above most of the sign come from permutations of the x_i . We will omit these in the following, and simply write \pm , since this simplifies the formulas and the correct sign is easy to obtain. Further x_1, \dots stands for the remaining x_i , such that each x_i appears once in each expression.

$$\partial_{\mu\mu}(\partial_{\mu\mu}\varphi)(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \sum_{j=1, i \neq j}^{n-1} \alpha^n(x_i) \pm \cdot (\alpha^{n-1}(x_j) \cdot \varphi(x_1, \dots)) \quad (99)$$

$$- \sum_i \sum_{j < k, j, k \neq i} \pm \alpha^n(x_i) \cdot \varphi([x_j, x_k], \alpha(x_1), \dots) \quad (100)$$

$$- \sum_{i < j} \sum_{k \neq i, j} \pm \alpha^n(x_k) \cdot \varphi([x_i, x_j], \alpha(x_1), \dots) \quad (101)$$

$$- \sum_{i < j} \pm \alpha^{n-1}([x_i, x_j]) \cdot \varphi(\alpha(x_1), \dots) \quad (102)$$

$$+ \sum_{i < j} \sum_{k < l, k, l \neq i, j} \pm \varphi(\alpha[x_k, x_l], \alpha([x_i, x_j]), \alpha^2(x_1), \dots) \quad (103)$$

$$+ \sum_{i < j} \sum_{k \neq i, j} \pm \varphi([x_i, x_j], \alpha(x_k), \alpha^2(x_1), \dots) \quad (104)$$

$$= \sum_{i < j} \sum_{i < j} \pm \alpha^{n-1}([x_i, x_j]) \cdot \varphi(\alpha(x_1), \dots) \quad (105)$$

$$- \sum_{i < j} \alpha^{n-1}([x_i, x_j]) \cdot \beta\varphi(x_1, \dots) \quad (106)$$

The terms (100) and (101) cancel each other and (103) cancel itself due to antisymmetry, and (104) due to the Hom-Jacobi identity. Equation (99) gives the term (106)

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by the Hom-Jacobi identity and (102) is equal to (105). It is easy to see that this is the same as $\partial_{\alpha\mu}\partial_{\mu\alpha}\varphi$.

One can compute $\partial_{\alpha\alpha}(\partial_{\alpha\alpha}\psi)$ similar to $\partial_{\mu\mu}(\partial_{\mu\mu}\varphi)$ and gets

$$\sum_{i<j} \sum_{i<j} \alpha^n([x_i, x_j]) \cdot \varphi(\alpha(x_1), \dots) - \sum_{i<j} \alpha^n([x_i, x_j]) \cdot \beta\varphi(x_1, \dots).$$

This is the same as $\partial_{\mu\alpha}\partial_{\alpha\mu}\psi$. It remains to show $\partial_{\mu\alpha}\partial_{\mu\mu} = \partial_{\alpha\alpha}\partial_{\mu\alpha}$ and $\partial_{\alpha\mu}\partial_{\alpha\alpha} = \partial_{\mu\mu}\partial_{\mu\alpha}$. We compute

$$\begin{aligned} \partial_{\mu\alpha}\partial_{\mu\mu}\varphi(x_1, \dots, x_{n+1}) &= \sum_{i=1}^n (-1)^{i-1} \alpha^n(x_i) \cdot \beta\varphi(x_1, \dots, x_{n+1}) \\ &\quad - \sum_{i=1}^n (-1)^{i-1} \alpha^n(x_i) \cdot \varphi(\alpha(x_1), \dots, \hat{x}_i \alpha(x_n)) \\ &\quad - \sum_{i<j} (-1)^{i+j-1} \varphi(\alpha([x_i, x_j]), \alpha^2(x_1), \dots) \\ &\quad + \sum_{i<j} (-1)^{i+j-1} \beta\varphi([x_i, x_j], \alpha(x_1), \dots) \\ &= \partial_{\alpha\alpha}\partial_{\mu\alpha}\varphi(x_1, \dots, x_{n+1}) \end{aligned}$$

and

$$\begin{aligned} \partial_{\alpha\mu}\partial_{\alpha\alpha}\psi(x_1, \dots, x_n) &= \sum_{i<j} \sum_{k \neq i, j} \pm \alpha^{n-1}([x_i, x_j]) \cdot \alpha^{n-1}([x_i, x_j]) \cdot \psi(x_1, \dots) \\ &\quad - \sum_{i<j} \sum_{k<l, k, l \neq i, j} \pm \alpha^{n-1}([x_i, x_j]) \cdot \psi([x_k, x_l], \alpha(x_1), \dots). \end{aligned}$$

The first term is zero, which can be seen using the module structure and the Hom-Jacobi identity. It is also easy to see that $\partial_{\mu\mu}\partial_{\alpha\mu}\psi$ gives the same. \square

So we can define

Definition 6.9.2 Let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra and (M, β) be a \mathfrak{g} -module then we call $\tilde{C}_{\text{CE}}^\bullet(\mathfrak{g}, M)$ with differential ∂ the α -type Chevalley-Eilenberg complex and the corresponding cohomology α -type Chevalley-Eilenberg cohomology of \mathfrak{g} with values in M and denote it by $H_{\text{CE}}^\bullet(\mathfrak{g}, M)$.

One can of course specialize this, to define the α -type cohomology with values in itself, where the action is given by the adjoint action. So let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra then we define

$$\tilde{C}_{\text{CE}}^n(\mathfrak{g}) = \text{Hom}(\Lambda^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\Lambda^{n-1} \mathfrak{g}, \mathfrak{g}) \quad (107)$$

and maps

$$\begin{aligned} \partial_{\mu\mu}\varphi(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} [\alpha^{n-1}(x_i), \varphi(x_1, \dots, x_{n+1})] \\ &\quad - \sum_{i<j} (-1)^{i+j-1} \varphi([x_i, x_j], \alpha(x_1), \dots, \hat{x}_i, \hat{x}_j, \dots, \alpha(x_n)) \end{aligned} \quad (108)$$

$$\begin{aligned} \partial_{\alpha\alpha}\psi(x_1, \dots, x_n) &= \sum_{i=1}^n (-1)^{i+1} [\alpha^{n-1}(x_i), \psi(x_1, \dots, x_n)] \\ &\quad - \sum_{i<j} (-1)^{i+j-1} \psi([x_i, x_j], \alpha(x_1), \dots, \hat{x}_i, \hat{x}_j, \dots, \alpha(x_n)) \end{aligned} \quad (109)$$

$$\partial_{\mu\alpha}\varphi(x_1, \dots, x_n) = \alpha(\varphi(x_1, \dots, x_n)) - \varphi(\alpha(x_1), \dots, \alpha(x_n)) \quad (110)$$

$$\partial_{\alpha\mu}\psi(x_1, \dots, x_{n+1}) = \sum_{i \leq j} (-1)^{i+j-1} [[\alpha^{n-2}(x_i), \alpha^{n-2}(x_j)], \psi(x_1, \dots, \hat{x}_i, \hat{x}_j, x_{n+1})]. \quad (111)$$

The differential ∂ is defined as before.

This complex is called α -type Chevalley-Eilenberg complex of \mathfrak{g} with values in itself and its cohomology is denoted by $\tilde{H}_{\text{CE}}(\mathfrak{g})$.

Remark 6.9.3 Note that since φ is completely antisymmetric we can write

$$\begin{aligned} \partial_{\alpha\mu}\psi(x_1, \dots, x_{n+1}) &= \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \frac{1}{2 \cdot (n-1)!} \alpha^{n-2}([x_{\sigma(1)}, x_{\sigma(2)}]) \psi(x_{\sigma(3)}, \dots, x_{\sigma(n+1)}) \\ \partial_{\mu\mu}\varphi(x_1, \dots, x_{n+1}) &= \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \frac{1}{n!} (\alpha^{n-1}(x_{\sigma(1)}) \cdot \varphi(x_{\sigma(2)}, \dots, x_{\sigma(n+1)})) \\ &\quad - \frac{1}{2} \varphi([x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}, \dots, x_{\sigma(n+1)}) \\ \partial_{\alpha\alpha}\psi(x_1, \dots, x_n) &= \sum_{\sigma \in S_n} \text{sign}(\sigma) \frac{1}{(n-1)!} (\alpha^{n-1}(x_{\sigma(1)}) \cdot \psi(x_{\sigma(2)}, \dots, x_{\sigma(n)})) \\ &\quad - \frac{1}{2} \psi([x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}, \dots, x_{\sigma(n)}). \end{aligned}$$

Here S_n denotes the symmetric group and sign the signature of a permutation.

Parts of the differential and relations between them can also be written with help of the big bracket defined earlier.

Proposition 6.9.4 For $(\varphi, \psi) \in \tilde{C}_{\text{CE}}(V, V) = \text{Hom}(\Lambda V, V)$ and $\chi_1, \chi_2 \in \text{Hom}(\Lambda V, \Lambda V)$ with arbitrary heights we have:

$$- \partial_{\mu\mu}\varphi = (-1)^{k-1} \{v, \varphi\}_{\text{BB}}.$$

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- $\partial_{\alpha\alpha}\psi = (-1)^{k-1}\{\nu, \psi\}_{\text{BB}}$.
- $\partial_{\mu\alpha}(\chi_1 \circ \chi_2) = (\partial_{\mu\alpha}\chi_1 \circ \chi_2) + (\chi_1 \circ \partial_{\mu\alpha}\chi_2)$

Proof. We have that

$$\nu \circ \varphi = (-1)^{k-1}[\alpha^k(x^{(1)}), \varphi(x^{(2)})], \varphi \circ \nu = (-1)^{k-1}\varphi(\alpha(x^{(1)}) \wedge \nu(x^{(2)})). \quad (112)$$

Where we used the Sweedler notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$ for the coproduct in ΛV . This shows that $\partial_{\mu\mu}\varphi = (-1)^{k-1}\{\nu, \varphi\}_{\text{BB}}$.

The second statement follows from

$$\begin{aligned} \partial_{\mu\alpha}\chi_1 \circ \chi_2 + \chi_1 \partial_{\mu\alpha}\chi_2 &= \alpha'\chi_1 \circ \chi_2 - \chi_1\alpha' \circ \chi_2 + \chi_1 \circ \alpha'\chi_2 - \chi_1\chi_2\alpha' \\ &= \alpha'\chi_1 \circ \chi_2 - \chi_1\chi_2\alpha' = \partial_{\mu\alpha}(\chi_1 \circ \chi_2) \end{aligned}$$

Using $\chi_1 \circ \alpha'\chi_2 = \mu(\chi_1 \otimes \alpha_1^{\chi})Q(\alpha^{\chi_2}\alpha' \otimes \alpha'\chi_2) = \mu(\chi_1\alpha' \otimes \alpha_1^{\chi}\alpha')Q(\alpha^{\chi_2} \otimes \chi_2) = \chi_1\alpha' \circ \chi_2$. Also remember that $\alpha'\chi_1$ and $\partial_{\mu\alpha}\chi_1$ have the height $\text{hgt } \chi_1 + \text{hgt } \alpha$. \square

Let A be a Hom-associative algebra and M an A -module. We consider the α -type Hochschild cohomology defined in Section 6.3. Then we can define a map $\Phi : \text{HC}(A, M) \rightarrow \widetilde{\text{C}}_{\text{CE}}(A_L, M_L)$ by

$$\Phi(\varphi)(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \text{sign}(\sigma)\varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \quad (113)$$

Here S_n denotes the symmetric group and sign the signature of a permutation.

Theorem 6.9.5 *The map Φ is a surjective chain map. So it induces a map in cohomology.*

Proof. It is clear that it is surjective. We have to check $\Phi\partial^H = \partial\Phi$, where ∂^H denotes the α -type Hochschild differential. Clearly $\Phi\partial_{\mu\alpha} = \partial_{\mu\alpha}\Phi$. Further, we have

$$\Phi\partial_{\mu\mu}^H\varphi(x_1, \dots, x_{n+1}) = \sum_{\sigma \in S_n} \text{sign}(\sigma)\left(\alpha^{n-1}(x_{\sigma(1)})\varphi(x_{\sigma(2)}, \dots, x_{\sigma(n+1)})\right) \quad (114)$$

$$+ (-1)^n\varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)})\alpha^{n-1}(x_{\sigma(n+1)}) \quad (115)$$

$$+ \sum_i (-1)^i\varphi(\alpha(x_{\sigma(1)}), \dots, x_{\sigma(i)}x_{\sigma(i+1)}, \dots, \alpha(x_{\sigma(n+1)})) \quad (116)$$

$$= \sum_{i=1}^n \sum_{\sigma \in S_n} \text{sign}(\sigma)\left(\alpha^{n-1}(x_i) \cdot_L \varphi(x_{\sigma(1)}, \dots, \hat{x}_i, \dots, x_{\sigma(n)})\right) \quad (117)$$

$$+ \sum_i \frac{1}{2}\varphi(\alpha(x_{\sigma(1)}), \dots, [x_{\sigma(i)}, x_{\sigma(i+1)}], \dots, \alpha(x_{\sigma(n+1)})), \quad (118)$$

where \tilde{S}_n is the symmetric group on $\{1, \dots, \hat{i}, \dots, n+1\}$. Using Remark 6.9.3 it is to see that this equals $\partial_{\mu\mu}\Phi$. The proof for the other parts of the differential works similarly. \square

We also want to relate the α -type cohomology defined here to the Hom-Lie cohomology given in Section 3.2.2. For this we construct a subcomplex spanned by cocycles of the form $(\varphi, 0)$. To get that $\partial(\varphi, 0)$ is again of this form we need $\partial_{\mu\alpha}\varphi = 0$. So we define the complex $C_{CE}^n(A) = \{\varphi \in \text{Hom}(\Lambda^n \mathfrak{g}, \mathfrak{g}) \mid \alpha\varphi = \varphi\alpha^{\otimes n}\}$. The map $\partial_{\mu\mu}$ defines a differential on this complex. This is precisely the complex given in Section 3.2.2.

6.9.1 Cohomology for Hom-Lie algebras of Lie type

In this section we use the cohomology for a Lie algebra endomorphism defined in Section 6.1 to study the α -type Chevalley-Eilenberg cohomology of a Lie algebra of Lie type.

We regard the Hom-Lie algebra \mathfrak{g}_γ obtained by Yau twist of \mathfrak{g} by γ . We define a linear map $\Phi : C(\mathfrak{g}, \gamma) \rightarrow \tilde{C}_{CE}(\mathfrak{g}_\gamma)$ for $\varphi \in C_\mu^n(\gamma)$ and $\psi \in C_\gamma^n(\gamma)$ by

$$(\varphi, \psi) \mapsto (\gamma^{n-1}\varphi + \gamma^{n-2}\psi \circ \nu, \gamma^{n-2}\psi), \quad (119)$$

where

$$\begin{aligned} (\varphi_1 \circ \varphi_2)(x_1, \dots, x_{k+l-1}) &= \varphi_1(\varphi_2 \wedge \text{id}^{\wedge k-1})(x_1, \dots, x_{k+l-1}) \\ &= \sum_{\sigma \in \text{Sh}(l, k-1)} \text{sign}(\sigma) \varphi_1(\varphi_2(x_{\sigma(1)}, \dots, x_{\sigma(l)}, x_{\sigma(l+1)}, \dots, x_{\sigma(k+l-1)})) \end{aligned}$$

for $\varphi_1 \in \text{Hom}(\Lambda^k \mathfrak{g}, \mathfrak{g})$ and $\varphi_2 \in \text{Hom}(\Lambda^l \mathfrak{g}, \mathfrak{g})$. Then we get the following

Theorem 6.9.6 *Let \mathfrak{g} be a Lie algebra and γ an endomorphism of it, then Φ defined as above is chain map, which is an isomorphism if γ is invertible. So in particular it induces a homomorphism in cohomology.*

Proof. One has to show $\partial_L \Phi(\varphi, \psi) = \Phi \partial_L(\varphi, \psi)$ for $(\varphi, \psi) \in C(\mathfrak{g}, \gamma)$, where ∂_E denotes

the differential in $C(\mathfrak{g}, \gamma)$ and ∂_L the one in $\tilde{C}_{CE}(\mathfrak{g}_\gamma)$. We have

$$\begin{aligned}
 \Phi(\partial_E \varphi)(x_1, \dots, x_{n+1}) &= \Phi\left(\sum_i \pm x_i \cdot \varphi(x_1, \dots) - \sum_{i < j} \pm \varphi([x_i, x_j], x_1, \dots), \right. \\
 &\quad \left. \gamma \varphi(x_1, \dots) - \varphi(\gamma(x_1), \dots)\right) \\
 &= \left(\sum_i \gamma^n(x_i \cdot \varphi(x_1, \dots)) - \sum_{i < j} \pm \gamma^n(\varphi([x_i, x_j], x_1, \dots))\right. \\
 &\quad \left. + \sum_{i < j} \gamma^n \varphi([x_i, x_j], x_1, \dots) - \gamma^{n-1} \varphi(\gamma([x_i, x_j], \gamma(x_1), \dots), \right. \\
 &\quad \left. \gamma^n \varphi(x_1, \dots) - \gamma^{n-1} \varphi(\gamma(x_1), \dots)\right) \\
 &= \left(\sum_i \gamma^n(x_i \cdot \varphi(x_1, \dots)) - \sum_{i < j} \gamma^{n-1} \varphi(\gamma([x_i, x_j], \gamma(x_1), \dots), \right. \\
 &\quad \left. \gamma^n \varphi(x_1, \dots) - \gamma^{n-1} \varphi(\gamma(x_1), \dots)\right) \\
 &= \partial_L(\gamma^{n-1} \varphi(x_1, \dots), 0) = \partial_L \Phi(\varphi)(x_1, \dots).
 \end{aligned}$$

Similarly one can check that $\partial_L \Phi(0, \psi) = \Phi \partial_E(0, \psi)$. \square

Using this it is easy to compute the α -type cohomology of Hom-Lie algebras which are in fact ordinary Lie algebras. We have the following

Theorem 6.9.7 *Let $(\mathfrak{g}, \nu, \text{id})$ be a Lie-algebra considered as a Hom-Lie algebra. Then*

$$\tilde{H}_{CE}^n(\mathfrak{g}) = H_{CE}^n(\mathfrak{g}) \oplus H_{CE}^{n-1}(\mathfrak{g}). \quad (120)$$

Here one has to replace $H_{CE}^1(\mathfrak{g})$ with $\text{Der}(\mathfrak{g})$ and $H_{CE}^0(\mathfrak{g})$ with $\{0\}$.

Proof. This follows directly from Theorem 6.9.6, since $\partial_\gamma = 0$ and the other two parts of the differential are precisely the ordinary Chevalley-Eilenberg differential on C_{CE}^n and C_{CE}^{n-1} . One has to replace H_{CE}^1 with Der and $H_{CE}^0(\mathfrak{g})$ with $\{0\}$ since our complex starts with \tilde{C}_{CE}^1 . \square

6.9.2 Whitehead Theorem

In this section we need the field \mathbb{K} to be of characteristic 0. If \mathfrak{g} is a simple Lie algebra, the well known Whitehead Lemma states that $H_{CE}(\mathfrak{g})$ is trivial. So the α -type Chevalley-Eilenberg cohomology consists of the derivations. But this is easy to compute since all derivations are inner. Also all non-trivial endomorphism are invertible. So we can compute the cohomology of Yau-twist of simple Lie algebras by Theorem 6.9.6. On the other hand all (finite dimensional) simple Hom-Lie algebras are of Lie type.

Proposition 6.9.8 *Let \mathfrak{g} be a (finite dimensional) semi-simple Lie algebra and γ an automorphism of it. Then $H^1(\gamma) = \text{Der}_\gamma(\mathfrak{g})$, $H^2(\gamma) = \gamma \text{Der}(\mathfrak{g}) / \partial_\gamma \text{Der}(\mathfrak{g})$ and $H^k(\gamma) = 0$ for $k \geq 2$. Here $\text{Der}_\gamma(\mathfrak{g}) = \{\varphi \in \text{Der}(\mathfrak{g}) \mid \gamma\varphi = \varphi\gamma\}$.*

Proof. By the usual Whitehead Lemma we get that $H_{\text{CE}}(\mathfrak{g}, M) = 0$ for a simple \mathfrak{g} -module M . So in particular $H_{\text{CE}}(\mathfrak{g}, \mathfrak{g}) = 0$ and $H_{\text{CE}}(\mathfrak{g}, \tilde{\mathfrak{g}}) = 0$, where $\tilde{\mathfrak{g}}$ denotes \mathfrak{g} with the action given by $x \cdot y = [\gamma(x), y]$ for $x \in \mathfrak{g}$ and $y \in \tilde{\mathfrak{g}}$. So if we consider the spectral sequence associated to the vertical filtration of the bicomplex $C(\gamma)$, this is the cohomology with respect to ∂_{CE} , we get

$$H^1(C_\mu^\bullet(\mathfrak{g})) = \text{Der}(\mathfrak{g}) = \text{InnDer}(\mathfrak{g}), H^1(C_\gamma^\bullet(\mathfrak{g})) = \text{Der}(\tilde{\mathfrak{g}}) = \text{InnDer}(\tilde{\mathfrak{g}}). \quad (121)$$

We do not get zero here, since we started the complex for $H(\gamma)$ by $\text{Hom}(\mathfrak{g}, \mathfrak{g})$ and dropped $\text{Hom}(\mathbb{K}, \mathfrak{g})$. So the second page of the spectral sequence gives the claimed result. We have $\gamma\text{-Der}(\mathfrak{g}) = \gamma(\text{Der}(\mathfrak{g}))$, so every γ -derivation is of the form $\gamma\varphi$ for a derivation φ . \square

If γ is diagonalizable, we further have $H^2(\gamma) = \gamma \text{Der}(\mathfrak{g}) / \partial_\gamma \text{Der}(\mathfrak{g}) = \gamma \text{Der}_\gamma(\mathfrak{g})$.

Corollary 6.9.9 (Whitehead theorem) *Let $(\mathfrak{g}, \nu, \alpha)$ be a finite dimensional simple Hom-Lie algebra, then $\tilde{H}_{\text{CE}}^1(\mathfrak{g}) = \text{Der}_\alpha(\mathfrak{g})$, $\tilde{H}_{\text{CE}}^2(\mathfrak{g}) = \alpha \text{Der}(\mathfrak{g}) / \partial_\alpha \text{Der}(\mathfrak{g})$ and $\tilde{H}_{\text{CE}}^k(\mathfrak{g}) = 0$ for $k \geq 2$.*

Proof. For a simple Hom-Lie algebra we have that α is invertible and in [CH16] it is proven that $\mathfrak{g}_{\alpha^{-1}}$ is a semi-simple Lie algebra. So it is enough to compute $H(\mathfrak{g}_{\alpha^{-1}}, \alpha)$. This can be done by the previous proposition and gives the result. \square

6.10 EXAMPLES

Next we compute the α -type Chevalley-Eilenberg cohomology explicitly for some low dimensional Hom-Lie algebras by using computer software.

Hom-Lie algebra not of Lie-type First we consider a Hom-Lie algebra, which is not of Lie-type. Let \mathfrak{g} be a vector space with basis given by (x, y, z) . Then we define α by $\alpha(x) = x, \alpha(y) = y, \alpha(z) = 0$ and a bracket by

$$[x, y] = x, [y, z] = z \text{ and } [x, z] = z. \quad (122)$$

It is easy to verify that this is in fact a Hom-Lie algebra and it is not of Lie-type since $z \in \text{im } \nu$ but $z \notin \text{im}(\alpha)$. The following table gives the dimensions of the cohomology and of the coboundaries and cocycles:

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| i | $\dim \tilde{C}_{CE}^i$ | $\dim \operatorname{im} \partial^i$ | $\dim \ker \partial^i$ | $\dim \tilde{H}_{CE}^i$ |
|-----|-------------------------|-------------------------------------|------------------------|-------------------------|
| 1 | 9 | 8 | 1 | 1 |
| 2 | 18 | 8 | 10 | 2 |
| 3 | 12 | 2 | 10 | 2 |
| 4 | 3 | 0 | 3 | 1 |

The cohomology space can be spanned by the following maps, where $\varphi \in \tilde{C}_{CE\mu}^n(\mathfrak{g}) = \operatorname{Hom}(\Lambda^n, \mathfrak{g})$ and $\psi \in \tilde{C}_{CE\alpha}^n(\mathfrak{g}) = \operatorname{Hom}(\Lambda^{n-1}, \mathfrak{g})$:

$$\tilde{H}_{CE}^1 : \varphi(z) = \lambda z$$

$$\tilde{H}_{CE}^2 : \varphi \equiv 0, \psi(z) = \lambda_1 z, g(x) = (\lambda_1 + \lambda_2)x, \psi(y) = \lambda_2 y$$

$$\tilde{H}_{CE}^3 : \varphi \equiv 0, \psi(x, z) = \lambda_1(z)x, \psi(y, z) = \lambda_2 z$$

$$\tilde{H}_{CE}^4 : \varphi \equiv 0, \psi(x, y, z) = \lambda z$$

Yau twist We consider a vector space \mathfrak{g} spanned by x, y, z and a Lie bracket on it defined by

$$[x, y] = x, [x, z] = x, [y, z] = y - z \quad (123)$$

| i | $\dim \tilde{C}_{CE}^i$ | $\dim \operatorname{im} \partial^i$ | $\dim \ker \partial^i$ | $\dim \tilde{H}_{CE}^i$ |
|-----|-------------------------|-------------------------------------|------------------------|-------------------------|
| 1 | 9 | 3 | 6 | 6 |
| 2 | 18 | 6 | 12 | 9 |
| 3 | 12 | 3 | 9 | 3 |
| 4 | 3 | 0 | 3 | 0 |

It is enough to give its ordinary Chevalley-Eilenberg cohomology to know the α -type Chevalley-Eilenberg cohomology.

$$H_{CE}^1 : \varphi(x) = \lambda_1 x - \lambda_2 y + \lambda_2 z$$

$$\varphi(y) = \lambda_3 x - \lambda_4 y + \lambda_4 z$$

$$\varphi(z) = \lambda_5 x - \lambda_6 y + \lambda_6 z$$

$$H_{CE}^2 : \varphi(x, y) = \lambda_1 z, \varphi(x, z) = -\lambda_1 y$$

$$\varphi(y, z) = \lambda_2 x + \lambda_3 z - \lambda_3 y$$

It is easy to see that $\gamma(x) = 0, \gamma(y) = y, \gamma(z) = z$ defines a morphism of \mathfrak{g} . So the Yau twist \mathfrak{g}_γ is a Hom-Lie algebra, with structure map γ and the only non-vanishing bracket is $[y, z] = y - z$.

| i | $\dim \tilde{C}_{CE}^i$ | $\dim \operatorname{im} \partial^i$ | $\dim \ker \partial^i$ | $\dim \tilde{H}_{CE}^i$ |
|-----|-------------------------|-------------------------------------|------------------------|-------------------------|
| 1 | 9 | 6 | 3 | 3 |
| 2 | 18 | 7 | 11 | 5 |
| 3 | 12 | 2 | 10 | 3 |
| 4 | 3 | 0 | 3 | 1 |

The derivations and α derivation can be obtained from the derivation of \mathfrak{g} , which are compatible with γ . They are both given by:

$$\varphi(x) = \lambda_1 x, \varphi(y) = \lambda_2 y - \lambda_2 z, \varphi(z) = \lambda_3 y - \lambda_3 z \quad (124)$$

The remaining cohomology is spanned by

$$\tilde{H}_{CE}^1 : \varphi(x, y) = \lambda_1 x, \varphi(x, z) = \lambda_2 x, \quad (125)$$

$$\tilde{H}_{CE}^2 : \varphi(x, y, z) = \lambda_1 x, \psi(x, y) = \lambda_2 x, \psi(x, z) = \lambda_3 x, \quad (126)$$

$$\tilde{H}_{CE}^3 : \psi(x, y, z) = \lambda_1 x \quad (127)$$

This cannot be obtained using Theorem 6.9.6, since $x \notin \operatorname{im} \gamma$.

6.11 α -TYPE CHEVALLEY-EILENBERG COHOMOLOGY FOR HOM-LIE BIALGEBRAS

In this section we define an α -type cohomology for Hom-Lie bialgebras. For simplicity we only consider the case where $\alpha = \beta$. In this case the cohomology complex is not a bicomplex as in Section 6.8.

Instead we have for a Hom-Lie bialgebra $(\mathfrak{g}, \nu, \delta, \alpha)$

$$\tilde{C}_{CE}^k(\mathfrak{g}) = \tilde{C}_{CE \mu}^k(\mathfrak{g}) \oplus \tilde{C}_{CE \alpha}^k(\mathfrak{g}) = \bigoplus_{l=1}^k \tilde{C}_{CE \mu}^{l, k-l+1}(\mathfrak{g}) \oplus \bigoplus_{l=1}^{k-1} \tilde{C}_{CE \alpha}^{l, k-l}(\mathfrak{g}) \quad (128)$$

$$= \bigoplus_{l=1}^k \operatorname{Hom}(\Lambda^l \mathfrak{g}, \Lambda^{k-l+1} \mathfrak{g}) \oplus \bigoplus_{l=1}^{k-1} \operatorname{Hom}(\Lambda^l \mathfrak{g}, \Lambda^{k-l} \mathfrak{g}) \quad (129)$$

Similarly to the previous section it is convenient to define the height of an element in the cohomology. Since we here do not distinguish between α and β , the height is a natural number and not a pair. With this we define a big bracket as before. The height of an element in $\tilde{C}_{CE \alpha}^{ij}$ is $i + j - 1$ and of one in $\tilde{C}_{CE \mu}^{ij}$ is $i + j - 2$.

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For $\varphi_{ij} \in \text{Hom}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) \subset \tilde{C}_{CE}^{i+j-1}{}_{\mu}$ and $\psi_{ij} \in \text{Hom}(\Lambda^i \mathfrak{g}, \Lambda^j \mathfrak{g}) \subset \tilde{C}_{CE\alpha}^{i+j}$ we set

$$\partial\varphi = (\partial_{\mu\mu}\varphi, \partial_{\mu\alpha}\varphi, \partial_{\mu\mu}^c\varphi) \in \tilde{C}_{CE\mu}^{i+1,j} \oplus \tilde{C}_{CE\alpha}^{i,j} \oplus \tilde{C}_{CE\mu}^{i,j+1}, \quad (130)$$

$$\begin{aligned} \partial\psi &= (\partial_{\alpha\mu}\psi, -\partial_{\alpha\alpha}\psi, \partial_b\psi, -\partial_{\alpha\alpha}^c\psi, \partial_{\alpha\mu}^c\psi) \\ &\in \tilde{C}_{CE\mu}^{i+2,j} \oplus \tilde{C}_{CE\alpha}^{i+1,j} \oplus \tilde{C}_{CE\mu}^{i+1,j+1} \oplus \tilde{C}_{CE\alpha}^{i,j+1} \oplus \tilde{C}_{CE\mu}^{i,j+2}. \end{aligned} \quad (131)$$

Here $\partial_{\alpha\alpha} = \{\nu, \cdot\}_{\text{BB}}$, $\partial_{\alpha\alpha}^c = \{\delta, \cdot\}_{\text{BB}}$, $\partial_{\mu\alpha}\varphi = \alpha\varphi - \varphi\alpha$ and $\partial_{\mu\alpha}\varphi = \mu(\alpha^{\text{hgt}\varphi-1}\nu \wedge \varphi)\Delta$. So there are the same as in the definition of the α -type cohomology for Lie-algebras and their duals. The map ∂_b is defined by

$$\begin{aligned} \partial_b\psi &= \partial_{b_1} + \partial_{b_2} \\ &= \mu^3(\text{id} \otimes \nu \otimes \text{id})(\text{pr} \otimes \mu \otimes \text{id})(\Delta \otimes \Delta)(\delta\alpha^{\text{hgt}\psi-1} \otimes \psi)\Delta \\ &\quad + \mu(\alpha^{\text{hgt}\psi-1}\nu \otimes \psi)(\mu \otimes \mu)(\text{pr} \otimes \Delta \otimes \text{id})(\text{id} \otimes \delta \otimes \text{id})\Delta^3. \end{aligned} \quad (132)$$

This means for $\psi \in \tilde{C}_{CE\alpha}^{k,l}$

$$\partial_{b_1}\psi(x_1, \dots, x_{k+1}) = \sum_{i=1}^{k+1} (-1)^i \delta(\alpha^{k+l} x_i) \cdot \psi(x_1, \dots) \quad (133)$$

$$= \sum_{i=1}^{k+1} (-1)^i \alpha^{k+l}(x^{[1]}) \wedge (\alpha^{k+l}(x^{[1]})) \cdot \psi(x_1, \dots) \quad (134)$$

The maps can be arranged in the following diagram

$$\begin{array}{ccccc} & & C_{CE\mu}^{i+2,j} & & \\ & & \uparrow & & \\ & & C_{CE\mu}^{i,j+1} & \nearrow & C_{CE\alpha}^{i,j+1} \\ & & \uparrow & & \uparrow \\ C_{CE\mu}^{i,j+1} & \longrightarrow & C_{CE\mu}^{i+1,j+1} & & \\ & & \uparrow & & \uparrow \\ & & C_{CE\alpha}^{i,j} & \longrightarrow & C_{CE\alpha}^{i+1,j} \\ & & \uparrow & & \uparrow \\ & & C_{CE\mu}^{i+1,j} & \longrightarrow & C_{CE\mu}^{i,j+2} \end{array}$$

Theorem 6.11.1 *The map ∂ defined above is a differential for the complex $\tilde{C}_{CE}(\mathfrak{g})$.*

Proof. This is a very lengthy calculation, which we do not want to give in full detail here. There are several different equations one has to check, namely

$$\partial_{\mu\mu}\partial_{\mu\mu} = \partial_{\alpha\mu}\partial_{\mu\alpha} \quad (135)$$

$$\partial_{\mu\alpha}\partial_{\mu\mu} = \partial_{\alpha\alpha}\partial_{\mu\alpha} \quad (136)$$

$$\partial_{\mu\mu}\partial_{\mu\mu}^c = \partial_{\mu\mu}^c\partial_{\mu\mu} - \partial_b\partial_{\mu\alpha} \quad (137)$$

$$\partial_{\mu\mu}\partial_{\alpha\mu} = \partial_{\alpha\mu}\partial_{\alpha\alpha} \quad (138)$$

$$\partial_{\mu\alpha}\partial_{\alpha\mu} = \partial_{\alpha\alpha}\partial_{\alpha\alpha} \quad (139)$$

$$\partial_{\mu\mu}^c\partial_{\alpha\mu} = \partial_{\alpha\mu}\partial_{\alpha\alpha}^c - \partial_b\partial_{\alpha\alpha} + \partial_{\mu\mu}\partial_b \quad (140)$$

$$\partial_{\alpha\alpha}^c\partial_{\mu\mu} = \partial_{\alpha\alpha}\partial_{\alpha\alpha}^c - \partial_{\mu\alpha}\partial_b \quad (141)$$

and their duals. All of them besides Eqs. (137) and (140) and Eq. (141) follow from the identity proven for the α -type Chevalley-Eilenberg cohomology. Equation (137) and Eq. (141) are similar so we only give the prove for Eq. (137). For $\varphi \in \widetilde{C}_{CE\mu}^{p,q}(\mathfrak{g})$ and $n = p + q - 1$ we get

$$(\partial_{\mu\mu}\partial_{\mu\mu}^c\varphi)(x_1, \dots, x_{p+1}) = \alpha^{n+1}(x_1) \cdot (x_j^{[1]} \wedge \varphi(x_i^{[2]}, \alpha(x_1), \dots)) \quad (142)$$

$$- \alpha^{n+1}(x_j) \cdot (\delta \wedge \alpha^{\wedge q-1})\varphi(x_1, \dots) \quad (143)$$

$$- \alpha^n([x_i, x_j]^{[1]}) \wedge \varphi([x_i, x_j]^{[2]}, \alpha^2(x_1), \dots) \quad (144)$$

$$- \alpha^{n+1}(x_1)^{[1]} \wedge \varphi(\alpha(x_i^{[2]}), [x_j, x_k], \alpha^2(x_1), \dots) \quad (145)$$

$$- (\delta \wedge \alpha^{\wedge q-1})\varphi([x_i, x_j], \alpha(x_1), \dots) \quad (146)$$

$$(\partial_{\mu\mu}^c\partial_{\mu\mu}\varphi)(x_1, \dots, x_{p+1}) = \alpha^{n+1}(x_i^{[1]}) \wedge (\alpha^n(x_j) \cdot \varphi(x_i^{[2]}, \alpha(x_1), \dots)) \quad (147)$$

$$+ \alpha^{n+1}(x_i^{[1]}) \wedge (\alpha^n(x_j^{[2]}) \cdot \varphi(\alpha(x_1), \dots)) \quad (148)$$

$$- \alpha^{n+1}(x_i^{[1]}) \wedge \varphi([x_i^{[2]}, x_j], \alpha^2(x_1), \dots) \quad (149)$$

$$- \alpha^n(x_i^{[1]}) \cdot \varphi(\alpha(x_i^{[2]}), \alpha([x_j, x_k]), \alpha^2(x_1), \dots) \quad (150)$$

$$- (\delta \wedge \alpha^{\wedge q-1})(\alpha^n(x_j) \cdot \varphi(x_2, \dots)) \quad (151)$$

$$+ (\delta \wedge \alpha^{\wedge q-1})(\varphi([x_i, x_j], \alpha(x_1), \dots)) \quad (152)$$

The terms Eqs. (145), (146), (150) and (152) cancel each other. Further (148), (143) and (151) give $\partial_{b_1}\partial_{\mu\alpha}\varphi$ and (142), (147), (144), and (149) give $\partial_{b_2}\partial_{\mu\alpha}\varphi$.

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Now we prove Eq. (140). For $\psi \in \tilde{\mathcal{C}}_{\text{CE}}(\mathfrak{g})^{p,q}$ and $n = i + j$ we get

$$(\partial_b \partial_{\alpha\alpha} \psi)(x_1, \dots, x_{p+2}) = \alpha^n(x_i^{[1]}) \wedge \alpha^n(x_i^{[2]}) \cdot (\alpha^{n-1}(x_j) \cdot \psi(x_1, \text{dots})) \quad (153)$$

$$- \alpha^n(x_i^{[1]}) \wedge \alpha^n(x_i^{[2]}) \cdot \psi([x_j, x_k], \alpha(x_1), \dots) \quad (154)$$

$$+ \alpha^{n-1}([x_i^{[1]}, \alpha x_j]) \wedge \alpha^n(x_k) \cdot \psi(x_i^{[1]}, \alpha(x_1), \dots) \quad (155)$$

$$+ \alpha^{n-1}([x_i^{[1]}, \alpha x_j]) \wedge \alpha^n(x_i^{[1]}) \cdot \psi(\alpha(x_1), \dots) \quad (156)$$

$$- \alpha^{n-1}([x_i^{[1]}, \alpha x_j]) \wedge \psi(\alpha([x_k, x_l]), \alpha^2(x_1), \dots) \quad (157)$$

$$- \alpha^{n-1}([x_i^{[1]}, \alpha x_j]) \wedge \psi([x_i^{[2]}, \alpha(x_k)], \alpha^2(x_1), \dots) \quad (158)$$

$$(\partial_{\mu\mu} \partial_b \psi)(x_1, \dots, x_{p+2}) = \alpha^n(x_i) \cdot (\alpha^{n-1}(x_j^{[1]}) \wedge \alpha^{n-2}(x_i^{[2]}) \cdot \psi(x_1, \dots)) \quad (159)$$

$$+ \alpha^n(x_i) \cdot (\alpha^{n-2}([x_j^{[1]}, \alpha(x_k)]) \wedge \psi(x_j^{[2]}, \alpha(x_1), \dots)) \quad (160)$$

$$- \alpha^{n-1}([x_i, x_j]^{[1]}) \wedge \alpha^{n-2}([x_i, x_j]^{[2]}) \cdot \psi(\alpha(x_1), \dots) \quad (161)$$

$$- \alpha^n(x_k^{[1]}) \wedge \alpha^{n-1}(x_k^{[2]}) \cdot \psi([x_i, x_j], \alpha(x_1), \dots) \quad (162)$$

$$- \alpha^{n-2}([\alpha(x_k^{[1]}), \alpha([x_i, x_j])]) \wedge \psi(\alpha(x_k^{[2]}), \alpha^2(x_1), \dots) \quad (163)$$

$$- \alpha^{n-2}([x_i, x_j]^{[1]}, \alpha(x_k)) \wedge \psi([x_i, x_j]^{[2]}, \alpha^2(x_1), \dots) \quad (164)$$

$$- \alpha^{n-1}([x_k^{[1]}, \alpha(x_l)]) \wedge \psi(\alpha(x_k^{[2]}), \alpha([x_i, x_j]), \alpha^2(x_1), \dots) \quad (165)$$

$$(\partial_{\alpha\mu} \partial_{\alpha\alpha}^c \psi)(x_1, \dots, x_{p+2}) = \alpha^{n-1}([x_i, x_j]) \cdot (\alpha^{n-1}(x_k^{[1]}) \wedge \psi(x_k^{[2]}, \alpha(x_1), \dots)) \quad (166)$$

$$- \alpha^{n-1}([x_i, x_j]) \cdot (\delta \wedge \alpha^{\wedge q-1}) \psi(x_1, \dots) \quad (167)$$

$$(\partial_{\mu\mu}^c \partial_{\alpha\mu} \psi)(x_1, \dots, x_{p+2}) = \alpha^n(x_i^{[1]}) \wedge \alpha^{n-1}([x_j, x_k]) \cdot \psi(x_i^{[2]}, \alpha(x_1), \dots) \quad (168)$$

$$+ \alpha^n(x_i^{[1]}) \wedge \alpha^{n-2}([x_i^{[2]}, \alpha(x_j)]) \cdot \psi(\alpha(x_1), \dots) \quad (169)$$

$$- (\delta \wedge \alpha^{\wedge q-1})(\alpha^{n-2}([x_j, x_k]) \cdot \psi(x_1, \dots)) \quad (170)$$

The terms Eqs. (154), (157), (162) and (165) cancel directly. Doing a bit of calculation using the Hom-Jacobi identity and the compatibility it is possible to see that Eqs. (153), (159), (167) and (170) also sum up to zero. The same is true for Eqs. (158), (163) and (164). Using the Hom-Jacobi identity one also gets that Eqs. (155), (160), (167) and (168) sum up to zero. Finally the remaining term cancel because of the compatibility condition. \square

As in the case of bialgebras the Chevalley-Eilenberg cohomology defined in Section 6.8 can be seen as a subcomplex of the α -type cohomology.

Again we consider the case that the Hom-Lie bialgebra can be obtained as a Yau-twist of an Lie bialgebra. In this case we consider the cohomology of an Lie bialgebra morphism.

Proposition 6.11.2 *Let \mathfrak{g} be a Lie bialgebra and γ an endomorphism of it. Then $\Phi : C(\gamma) \rightarrow \tilde{C}_{CE}(\mathfrak{g}, \gamma)$ defined by*

$$\Phi(\varphi) = (\gamma^{i-1}\varphi\gamma^{j-1}) \quad (171)$$

$$\Phi(\psi) = (\gamma^{i-1}(\psi \circ \nu)\gamma^{j-1}, \gamma^{i-1}\psi\gamma^{j-1}, \gamma^{i-1}(\delta \circ \psi)\gamma^{j-1}) \quad (172)$$

is a morphism of complexes. If γ is invertible, so is Φ .

Proof. We compute for $\varphi \in C_{\mu}^{i,j}(\gamma)$

$$\begin{aligned} \Phi(\partial\varphi) &= \Phi(\partial_{\mu}\varphi, \partial_{\gamma}\varphi, \partial_{\mu}^c\varphi) \\ &= (\gamma^i\partial_{\mu}\varphi\gamma^{j-1}, -\gamma^{i-1}(\partial_{\gamma}\varphi) \circ \nu\gamma^{j-1}, \gamma^{i-1}\partial_{\gamma}\varphi\gamma^{j-1}, dual) \end{aligned}$$

$$\begin{aligned} \Phi(\partial\varphi) &= \partial(\gamma^{i-1}\varphi\gamma^{j-1}) \\ &= (\partial_{\mu\mu}(\gamma^{i-1}\varphi\gamma^{j-1}), 0, \partial_{\gamma}(\gamma^{i-1}\varphi\gamma^{j-1}), dual). \end{aligned}$$

Now in Theorem 6.9.6 we proved that the first component agrees and it is clear that the second agrees, too.

Next we compute for $\psi \in C_{\gamma}^{i,j}(\gamma)$

$$\begin{aligned} \Phi(\partial\psi) &= \Phi(\partial_{\mu}\psi, 0, \partial_{\mu}^c\psi) \\ &= (\gamma^i(\partial\psi) \circ \nu\gamma^{j-1}, \gamma^i\partial_{\mu}\psi\gamma^{j-1}, -\gamma^{i-1}\delta \circ (\partial_{\mu}\psi)\gamma^{j-1} - \gamma^{i-1}(\partial_{\mu}^c\psi) \circ \nu\gamma^j, dual) \end{aligned}$$

$$\begin{aligned} \partial\Phi(\psi) &= (-\gamma^{i-1}(\psi \circ \nu)\gamma^{j-1}, \gamma^{i-1}\psi\gamma^{j-1}, -\gamma^{i-1}\delta \circ \psi\gamma^{j-1}) \\ &= (\partial_{\mu\mu}\gamma^{i-1}(\psi \circ \nu)\gamma^{j-1} - \partial_{\alpha\mu}(\gamma^{i-1}\psi\gamma^{j-1}), \\ &\quad -\partial_{\mu\alpha}(\gamma^{i-1}(\psi \circ \nu)\gamma^{j-1}) - \partial_{\alpha\alpha}(\gamma^{i-1}\psi\gamma^{j-1}), \\ &\quad \partial_b\gamma^{i-1}\psi\gamma^{j-1} - \partial_{\mu\mu}^c\gamma^{i-1}(\psi \circ \nu)\gamma^{j-1} - \partial_{\mu\mu}(\gamma^{i-1}\delta \circ \psi\gamma^{j-1}), dual) \end{aligned}$$

The first and second components agree similar to Theorem 6.9.6. For the third we first calculate

$$\partial_{b1}\gamma^{i-1}\psi\gamma^{j-1} = \gamma^i\delta \circ (\nu \circ \psi)\gamma^{j-1} - \gamma^i\nu \circ (\delta \circ \psi)\gamma^{j-1}$$

using the compatibility condition in \mathfrak{g} . Then writing the differential with the \circ product gives the result. \square

With this one can compute the α -type Chevalley-Eilenberg cohomology of an Lie bialgebra considered as a Hom-Lie bialgebra and gets:

Proposition 6.11.3 *Let $(\mathfrak{g}, \nu, \delta, \text{id})$ be a Hom-Lie bialgebra then $\tilde{C}_{\text{CE}}^\bullet(\mathfrak{g}) = C_{\text{CE}}^\bullet(\mathfrak{g}) \oplus C_{\text{CE}}^{\bullet-1}(\mathfrak{g})$.*

Proof. This follows from the previous proposition. \square

We do not have a cohomology for two different structure maps. However we do note that if they are invertible one gets by Yau twist a Lie bialgebra with two commuting morphism for which we have given a cohomology in Section 6.1.

6.12 HOMOMORPHISMS OF HOM-ALGEBRAS

It is possible the generalize the cohomology for an algebra morphism to the case of Hom-algebras and define an α -type cohomology for an algebra morphism. We will only deal with the case of Hom-associative algebras here. The case of Hom-Lie algebras is similar.

So let (A, μ, α) and (B, μ, β) be two Hom-associative algebras. Then the complex for a morphism $\gamma : A \rightarrow B$ between the two Hom-algebras is given by

$$\tilde{\text{HC}}(\gamma) = \tilde{\text{HC}}(A) \oplus \tilde{\text{HC}}(B) \oplus \tilde{\text{HC}}(A, B), \quad (173)$$

where B is considered as an A -bimodule by γ . This means the left and right action is given by $a \cdot b = \gamma(a)b$ and $b \cdot a = b\gamma(a)$ resp. for $a \in A$ and $b \in B$.

We will write $\chi_A = (\varphi_A, \psi_A)$ for an element in $\tilde{\text{HC}}(A) \subset \tilde{\text{HC}}(\gamma)$ and similar for $\chi_B \in \tilde{\text{HC}}(B)$ and $\chi_\gamma \in \tilde{\text{HC}}(A, B)$.

Proposition 6.12.1 *Let $(\chi_A, \chi_\gamma, \chi_B) \in \tilde{\text{HC}}^\bullet(\gamma)$, then the map $\partial : \tilde{\text{HC}}^\bullet(\gamma) \rightarrow \tilde{\text{HC}}^{\bullet+1}(\gamma)$ defined by*

$$\partial(\chi_A, \chi_\gamma, \chi_B) = (\partial_H \chi_A, \gamma \chi_A - \chi_B \gamma - \partial_H \chi_\gamma, \partial_H \chi_B), \quad (174)$$

where ∂_H is the α -type Hochschild differential defined in Section 6.3, is a codifferential.

Proof. We simply have to compute $\partial \circ \partial$ and get

$$\begin{aligned} \partial \circ \partial(\chi_A, \chi_\gamma, \chi_B) &= (\partial_H \chi_A, \gamma \chi_A - \chi_B \gamma - \partial_H \chi_\gamma, \partial_H \chi_B) \\ &= (\partial_H^2 \chi_A, \gamma \partial_H \chi_A - \partial_H(\gamma \chi_A) + \partial_H(\chi_B \gamma) - (\partial_H \chi_B) \gamma) + \partial_H^2 \chi_\gamma, \partial_H^2 \chi_B). \end{aligned}$$

We know that $\partial_H^2 = 0$ and we have $\gamma \partial_H \chi_A = \partial_H(\gamma \chi_A)$ since the action on $\tilde{\text{HC}}(A, B)$ is induced by γ . \square

Note that in the previous theorem χ_A, χ_γ and χ_B are again pairs, so in fact a cocycle in this cohomology consists of a sextuple of linear maps.

One can search for a subcomplex such that the $\chi_A = (\varphi_A, 0)$, $\chi_\gamma = (\varphi_\gamma, 0)$ and $\chi_B = (\varphi_B, 0)$. This means that $\partial_{\mu\alpha}\varphi = 0$ must hold so $\varphi_A \in \text{Hom}_\alpha(A, A)$, $\varphi_\gamma \in \text{Hom}_\alpha(A, B)$ and $\varphi_B \in \text{Hom}_\alpha(B, B)$. Then it is easy to see that $\text{Hom}_\alpha(A, A) \oplus \text{Hom}_\alpha(A, B) \oplus \text{Hom}_\alpha(A, B)$ is a subcomplex of $\tilde{\text{HC}}(\gamma)$. And we get that this subcomplex with the restricted differential gives the one considered in [AFM17].

Similarly to Section 6.1 this can be modified to give a cohomology for an endomorphism. So if $A = B$ we can consider the subcomplex of $C(\gamma)$ of elements of the form $(\chi_A, \chi_\gamma, \chi_A)$. Clearly the differential restrict to this. This subcomplex is isomorphic to the complex $C^\bullet(\gamma) = \tilde{\text{HC}}^\bullet(A) \oplus \tilde{\text{HC}}^{\bullet-1}(A, \tilde{A})$, where the action on \tilde{A} is twisted by γ . The differential can be given for $(\chi_A, \chi_\gamma) \in C^n(\gamma)$ as

$$\partial(\chi_A, \chi_\gamma) = (\partial_H \chi_A, \partial_\gamma \chi_A - \partial_H \chi_\gamma), \quad (175)$$

where $\partial_\gamma \chi = \gamma \chi - \chi \gamma^{\otimes n}$ as before.

We next study how this cohomology of an endomorphism can be used to study the cohomology of a Hom-associative algebra obtained by Yau twist.

Let A be a Hom-algebra and γ an endomorphism then there is a morphism from $C(\gamma) \rightarrow \tilde{\text{HC}}(A_\gamma)$, where A_γ is the Yau twist of A by γ . The morphism for $(\varphi, \psi) \in \tilde{\text{HC}}(A)$ and $(\chi, \xi) \in \tilde{\text{HC}}(A, \tilde{A})$ is given by

$$(\varphi, \psi, \chi, \xi) \mapsto (\gamma^{n-1} \varphi + \gamma^{n-2} \chi \circ \mu, \gamma^{n-2} \chi \alpha^{\otimes n-1} + \gamma^{n-1} \psi + \gamma^{n-2} \xi \circ \mu). \quad (176)$$

Here $\varphi \circ \mu := \sum_{i=1}^n (-1)^{i+1} \varphi(\alpha^{\otimes i-1} \otimes \mu \otimes \alpha^{\otimes n-i-1})$ for an arbitrary map $\varphi \in \text{Hom}(A^{\otimes n}, A)$.

Proposition 6.12.2 *The map $\Phi : C(\gamma) \rightarrow \tilde{\text{HC}}(A_\gamma)$ is a chain map.*

Proof. We have to prove $\Phi \partial_E = \partial_H \Phi$, where ∂_E is the codifferential of $C(\gamma)$. We denote the parts of the differential with respect to A_γ by ∂^γ . First we note that

$$(\gamma^n \partial_{\mu\mu} \varphi - \gamma^{n-1} (\partial_\gamma \varphi) \circ \mu) = \partial_{\mu\mu}^\gamma (\gamma^{n-1} \varphi). \quad (177)$$

For $\varphi \in \tilde{\text{HC}}_\mu^n(A) \subset C^n(\gamma)$ we have

$$\begin{aligned} \Phi \partial(\varphi, 0, 0, 0) &= (\gamma^n \partial_{\mu\mu} \varphi + \gamma^{n-1} (\partial_\gamma \varphi) \circ \mu, \gamma^{n-1} (\partial_\gamma \varphi) \alpha + \gamma^n (\partial_{\mu\alpha} \varphi)) \\ &= (\partial_{\mu\mu}^\gamma (\gamma^{n-1} \varphi), \gamma^n \varphi \alpha + \gamma^{n-1} \varphi \gamma \alpha + \gamma^n \alpha \varphi + \gamma^n \varphi \alpha) \\ &= (\partial_{\mu\mu}^\gamma (\gamma^{n-1} \varphi), \partial_{\mu\alpha} (\gamma^{n-1} \varphi)) = \partial_H \Phi(\varphi, 0, 0, 0) \end{aligned}$$

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and for $\xi \in \tilde{\text{HC}}_\alpha^{n-1}(A, \tilde{A}) \subset C^n(\gamma)$ we have

$$\begin{aligned}\Phi\partial(\xi) &= \Phi(0, 0, \partial_{\mu\alpha}\xi, \partial_{\alpha\alpha}\xi) \\ &= (\gamma^{n-1}(\partial_{\mu\alpha}\xi) \circ \mu, \gamma^{n-1}(\partial_{\alpha\alpha}\xi) \circ \mu + \gamma^{n-1}(\partial_{\mu\alpha}\xi)\alpha) \\ \partial\Phi(\xi) &= \partial(0, \gamma^{n-2}\xi \circ \mu) \\ &= (\partial_{\mu\alpha}(\gamma^{n-2}\xi \circ \mu), \partial_{\mu\mu}(\gamma^{n-2}\xi \circ \mu))\end{aligned}$$

For the first summand we get

$$\begin{aligned}\gamma^{n-1}(\partial_{\mu\alpha}\xi) \circ \mu(x_1, \dots, x_{n+1}) &= \gamma^{n-1}(\alpha^{n-2}((x_1x_2)\alpha(x_3) - \alpha(x_1)(x_2x_3))\xi(\alpha(x_4), \dots)) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \gamma^{n-1}(\alpha^{n-3}\gamma(x_1x_2)\xi(\alpha(x_3), \dots, x_i x_{i+1}, \dots)) \\ &= \sum_{i=1}^{n-1} (-1)^{i-2} (\gamma\alpha)^{n-2} (\gamma(x_1x_2) \cdot_\gamma \gamma^{n-2}\xi(\alpha(x_1), \dots, x_i x_{i+1}, \dots)) \\ &= \partial_{\mu\alpha}(\gamma^{n-2}\xi \circ \mu)(x_1, \dots, x_{n+1})\end{aligned}$$

The second summand is similar to Eq. (177). Similarly the equality can be checked on the other two summands. \square

So every cocycle in $\tilde{\text{HC}}(\gamma)$ is mapped by Φ to a cocycle in $\tilde{\text{HC}}(A_\gamma)$ this can be seen as a generalization of Proposition 6.3.7.

Remark 6.12.3 Everything done in this section can be done in a similar way for morphism of Hom-Lie algebras.

7. L_∞ STRUCTURES

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IN this chapter we give L_∞ structures which can be used to describe Hom-Lie and Hom-associative algebras resp. We only give their low order terms, which are computed using computer software. We also give some conjectures and how a full L_∞ structure could look like. It is clear that these should generalize the well known Gerstenhaber bracket and Nijenhuis bracket.

7.1 HOM-ASSOCIATIVE ALGEBRAS

It would be nice to have an L_∞ -structure on the complex $\tilde{\text{HC}}(A)$, for a vector space A , such that the Maurer-Cartan elements are precisely the Hom-associative algebras on A , and the differential is given as usual in this context. It is clear that on elements of the form $(\varphi, 0)$ this should reduce to the Gerstenhaber structure given in [AEM11]. It is also clear that since the equations for the Hom-associativity and multiplicativity are not binary that it cannot be an ordinary Lie algebra but must be a true L_∞ -structure. Unfortunately we do not know such a structure, but one can try to construct it degree by degree. Using a computer we were able to do this for the low degrees, up to degree 5. Since the terms become quite long we only give the terms needed to do deformation theory here and the others in the appendix. But we first state the following conjecture:

Conjecture 7.1.1 *Given a vector space V , there is an L_∞ -algebra structure on the complex $\tilde{\text{HC}}(A)$, such that the Maurer-Cartan elements are precisely multiplicative Hom-associative algebras on A and the differential defined in Section 6.2 is induced from it.*

We give the brackets with values in degree up to 2, where the degree here is shifted such that $\deg \tilde{\text{HC}}^n = n - 2$, so the multiplication and structure map are of degree zero. With $\varphi_i \in \tilde{\text{HC}}_\mu^i$, $\psi_i \in \tilde{\text{HC}}_\alpha^i$, $\alpha_i \in \tilde{\text{HC}}_\alpha^2$ and $\mu_i \in \tilde{\text{HC}}_\mu^2$ we have:

$$\begin{aligned}
 & \text{deg 1 :} \\
 & [\mu_1, \mu_2, \alpha]_\mu = \mu_1(\mu_2 \otimes \alpha) - \mu_1(\alpha \otimes \mu_2) + \mu_2(\mu_1 \otimes \alpha) - \mu_2(\alpha \otimes \mu_1) \\
 & [\mu, \alpha]_\alpha = \alpha \mu \\
 & [\mu, \alpha_1, \alpha_2]_\alpha = -\mu(\alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \alpha_1) \quad \text{deg 2 :} \\
 & [\varphi_3, \mu, \alpha_1, \alpha_2]_\mu = \varphi_3(\mu \otimes \alpha_1 \otimes \alpha_2) - \varphi_3(\alpha_1 \otimes \mu \otimes \alpha_2) + \varphi_3(\alpha_1 \otimes \alpha_2 \otimes \mu) \\
 & \quad + \varphi_3(\mu \otimes \alpha_2 \otimes \alpha_1) - \varphi_3(\alpha_2 \otimes \mu \otimes \alpha_1) + \varphi_3(\alpha_2 \otimes \alpha_1 \otimes \mu) \\
 & \quad - \mu(\alpha_1 \alpha_2 \otimes \varphi_3) - \mu(\varphi_3 \otimes \alpha_1 \alpha_2) - \mu(\alpha_2 \alpha_1 \otimes \varphi_3) - \mu(\varphi_3 \otimes \alpha_2 \alpha_1) \\
 & [\psi_2, \mu_1, \mu_2, \alpha_1, \alpha_2] = \sum_{\sigma, \tau \in \mathcal{S}_2} \mu_{\sigma 1}(\psi_2 \otimes \mu_{\sigma 2}(\alpha_{\tau 1} \otimes \alpha_{\tau 2})) - \mu_{\sigma 1}(\mu_{\sigma 2}(\alpha_{\tau 1} \otimes \alpha_{\tau 2}) \otimes \psi_2) \\
 & [\varphi_2, \alpha]_\alpha = \alpha \varphi_3 \\
 & [\varphi_2, \alpha_1, \alpha_2, \alpha_3]_\alpha = - \sum_{\sigma \in \mathcal{S}_3} \varphi_3(\alpha_{\sigma(1)} \otimes \alpha_{\sigma(2)} \otimes \alpha_{\sigma(3)}) \\
 & [\psi_2, \mu, \alpha]_\alpha = \psi_2(\alpha \otimes \mu) - \psi_2(\mu \otimes \alpha) \\
 & [\psi_2, \mu, \alpha_1, \alpha_2]_\alpha = \mu(\alpha_1 \alpha_2 \otimes \psi_2) + \mu(\alpha_2 \alpha_1 \otimes \psi_2) - \mu(\psi_2 \otimes \alpha_1 \alpha_2) - \mu(\psi_2 \otimes \alpha_2 \alpha_1)
 \end{aligned}$$

It is well known that an L_∞ -algebra on a graded vector space V can be given by a coderivation \bar{l} on the graded symmetric algebra $S(V[1])$, which squares to zero.

The derivation \bar{l} is completely determined by its corestriction to V , which we will denote by $l : S(V[1]) \rightarrow V[1]$. In our case we have $V = \tilde{\text{HC}}(A)$, and l is given by the brackets defined above.

To proof that this is in fact an L_∞ -structure, it is easiest to consider a graph complex. This is based on the approach for example in [Mar10] for defining L_∞ structure governing deformations if one knows a model for the corresponding operad. For details see there. Another way to see this is using the fact that a codifferential on the cofree conilpotent cooperad gives an L_∞ structures as we want it here, see [LV12, Section 10.5]. To use this one has to use the weight graded dual of the free operad we give here, i.e. graded by the numbers of generated. We only give the general ideas here since, we only need it as a tool to motivate the brackets defined above and to show that they form in fact an L_∞ structure, which one could also do by hand.

The graph complex consists of planar rooted trees formed by vertices \blacktriangleleft , \blacktriangleright , \circ , \circ . This means it correspond the the free operad generated by these operations. We consider the graphs to be graded, such that $\text{deg}(\blacktriangleleft_k) = k - 2$ and $\text{deg}(\circ_k) = k - 1$. This means in particular that \blacktriangleleft and \circ , which correspond to α and μ , are of degree 0. We also associate a height to every graph. It is given by $\text{hgt}(\blacktriangleleft_k) = k - 1$ and $\text{hgt}(\circ_k) = k$.

The differential is given by $\partial \blacktriangleleft = \partial \circ = 0$,

$$\begin{aligned} \partial \blacktriangleleft &= \blacktriangleleft \circ - \circ \blacktriangleleft, & \partial \circ &= \blacktriangleleft - \circ \circ, \\ \partial \blacktriangleleft_k &= \blacktriangleleft \circ_k - \circ_k \blacktriangleleft + \blacktriangleleft \circ \circ - \circ \circ \blacktriangleleft + \blacktriangleleft \circ \circ \circ - \circ \circ \circ \blacktriangleleft - \blacktriangleleft \circ \circ \circ + \circ \circ \circ \blacktriangleleft - \blacktriangleleft \circ \circ \circ \circ + \circ \circ \circ \circ \blacktriangleleft, \\ \partial \circ_k &= \circ \circ_k - \circ \circ \circ - \circ \circ \circ + \circ \circ \circ + \circ \circ \circ - \circ \circ \circ \circ. \end{aligned}$$

It is an easy calculation to show that ∂ squares to zero. Also note that differential respects the height in the sense that for a graph Γ with have $\text{hgt}(\partial\Gamma) = \text{hgt}\Gamma$.

In fact using computer software we found an extensions of the differential above. Where the operad is spanned by vertices with up to 6 inputs in can be found in Appendix D.1.

There is a pairing between a graph and an element $\varphi_1 \cdots \varphi_k \cdot \psi_1 \cdots \psi_l$ of the graded symmetric algebra $S(\tilde{\text{HC}})$ with values in $\tilde{\text{HC}}(A)$. It is given on the generators by $\langle \blacktriangleleft_k, \varphi \rangle = \varphi$ if $\varphi \in \tilde{\text{HC}}_\mu^k$, $\langle \circ_k, \psi \rangle = \psi$ if $\psi \in \tilde{\text{HC}}_\alpha^{k-1}$. For general graphs it is given as

the sum over all permutations of possible assignments. Note that since the objects are graded this includes signs. This results in a tree with each element decorated by an element of $\check{H}\mathbb{C}$ this can be made to an element in $\check{H}\mathbb{C}$ again by composition as the tree describes.

The L_∞ -structure is given by $[\psi_1, \dots, \varphi_1, \dots]_\mu = \langle \partial \langle \text{graph with } k \text{ inputs and } 1 \text{ output} \rangle, \varphi_1 \cdot \varphi_k \cdots \psi_1 \cdots \psi_l \rangle$ and $[\psi_1, \dots, \varphi_1, \dots]_\alpha = \langle \partial \langle \text{graph with } k \text{ inputs and } 1 \text{ output} \rangle, \varphi_1 \cdot \varphi_k \cdots \psi_1 \cdots \psi_l \rangle$, for the restriction of the bracket to $\check{H}\mathbb{C}(A)_\mu^k$ and $\check{H}\mathbb{C}(A)_\alpha^{k+1}$ respectively.

The bracket can be extended to a coderivation \bar{l} of $S(\check{H}\mathbb{C})$, and it is an L_∞ -structure iff $l^2 = 0$. The fact that $\bar{l}^2 = 0$ follows directly from $\partial^2 = 0$.

Proposition 7.1.2 *A Maurer-Cartan element on this L_∞ -algebra is a Hom-associative algebra, and for a Hom-associative algebra structure (μ, α) the differential in degree two and three on $\check{H}\mathbb{C}$ are given by $\partial(\varphi, \psi) = l(e^{(\mu, \alpha)}(\varphi, \psi))$*

Proof. The bracket defined by graphs are the same as the brackets given above. It is clear by looking at the defining equations that a Maurer-Cartan element for them is a Hom-associative algebra and the differential in degree two and three are given by these brackets as $\partial(\varphi, \psi) = ([\mu, \dots, \alpha, \dots, \varphi]_\mu + [\mu, \dots, \alpha, \dots, \psi]_\mu, [\mu, \dots, \alpha, \dots, \varphi]_\alpha + [\mu, \dots, \alpha, \dots, \psi]_\alpha)$, $\mu, \dots, \alpha, \dots$ stands for zero or more insertions of α and μ . This can be written as $\partial(\varphi, \psi) = l(e^{(\mu, \alpha)} \cdot (\varphi, \psi))$. \square

7.2 HOM-LIE ALGEBRAS

As in the Hom-associative case, we suspect that there is an L_∞ -structure on the complex $\widetilde{C}_{CE}(\mathfrak{g})$, where \mathfrak{g} is a vector space considered as Hom-Lie algebra with the zero Hom-Lie structure, such that Maurer-Cartan elements are Hom-Lie algebra structures on \mathfrak{g} and the cohomology, one can construct from this, is the one we defined in Section 6.9. In fact we suspect that the L_∞ -structure describing Hom-Lie algebras can be derived from the one describing Hom-associative algebras by total antisymmetrization.

However using a graph complex, which corresponds to a free operad, we were able to calculate the low degrees.

We consider the free symmetric operad, spanned by totally antisymmetric operations . We consider them to be graded with $\deg(\langle \text{graph with } k \text{ inputs and } 1 \text{ output} \rangle) = k - 1$ and $\deg(\langle \text{graph with } k \text{ inputs and } 1 \text{ output} \rangle) = k - 2$. The height is given by $\text{hgt}(\langle \text{graph with } k \text{ inputs and } 1 \text{ output} \rangle) = k - 1$ and $\text{hgt}(\langle \text{graph with } k \text{ inputs and } 1 \text{ output} \rangle) = k$ as in the associative case.

On this free operad we define a differential on generators by $\partial \wedge = \partial \circ = 0$,

$$\begin{aligned} \partial \wedge &= \wedge \circ - \wedge \circ, & \partial \circ &= \circ \wedge - \circ \wedge, \\ \partial \circ &= \circ \wedge \circ - \circ \wedge \circ + \circ \wedge \circ, \\ \partial \circ &= \circ \wedge \circ - \circ \wedge \circ - \circ \wedge \circ - \circ \wedge \circ. \end{aligned}$$

Here one has to take the total antisymmetrization of each term, but only take terms, which are not equal using the antisymmetry of the operations.

Note that the differential preserves the height.

Using the same technique as in the previous section, which is based on [Mar06, FMY09], this gives an L_∞ -structure on \tilde{C}_{CE} by replacing the nodes with maps in the complex.

We give the corresponding brackets for $\varphi_i \in \tilde{C}_{CE}^i_\mu, \psi_i \in \tilde{C}_{CE}^i_\alpha, \alpha_i \in \tilde{C}_{CE}^2_\alpha$ and $\nu_i \in \tilde{C}_{CE}^2_\nu$

$$\begin{aligned} \text{deg 1 :} \\ [\nu_1, \nu_2, \alpha]_\nu &= \nu_1(\nu_2 \wedge \alpha) + \nu_2(\nu_1 \wedge \alpha) \\ [\nu, \alpha]_\alpha &= \alpha \nu \\ [\nu, \alpha_1, \alpha_2]_\alpha &= -\nu(\alpha_1 \wedge \alpha_2) \\ \text{deg 2 :} \\ [\varphi_3, \nu, \alpha_1, \alpha_2]_\nu &= \varphi_3(\nu \wedge \alpha_1 \wedge \alpha_2) - \nu(\varphi_3 \wedge \alpha_1 \alpha_2) - \nu(\varphi_3 \wedge \alpha_2 \alpha_1) \\ [\psi_3, \nu_1, \nu_2, \alpha_1, \alpha_2]_\nu &= \sum_{\sigma \in S_2} \nu_{\sigma(1)}(\psi_3 \wedge \nu_{\sigma(2)}(\alpha_1 \wedge \alpha_2)) \\ [\varphi_3, \alpha]_\alpha &= \alpha \varphi_3 \\ [\varphi_3, \alpha_1, \alpha_2, \alpha_3]_\alpha &= -\varphi_3(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \\ [\psi_3, \nu, \alpha]_\alpha &= \psi_3(\alpha \wedge \nu) \\ [\psi_3, \nu, \alpha_1, \alpha_2]_\alpha &= -\nu(\psi_3 \wedge \alpha_1 \alpha_2) - \nu(\psi_3 \wedge \alpha_2 \alpha_1) \end{aligned}$$

Theorem 7.2.1 *The Maurer-Cartan elements of this L_∞ -structure are Hom-Lie algebras, and the differential on \tilde{C}_{CE}^2 and \tilde{C}_{CE}^3 comes from it.*

Proof. It is clear, by regarding the brackets in degree 0, that a Maurer-Cartan element is a Hom-Lie algebra. The differentials of $(\varphi, \psi) \in \tilde{C}_{CE}(\mathfrak{g})$ can be computed by $\partial \varphi = [\alpha, \dots, \nu, \dots, \varphi]$ and similar for ψ . Here α, \dots stands for the insertion of zero or

more α and the same for ν . Inspection of the brackets shows that this agrees with the definition given before. \square

We want to end with the following conjecture:

Conjecture 7.2.2 *The free operad above can be extended to a minimal model (or resolution) for HLie. It is generated by totally antisymmetric graphs of the form $\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$ for $k \in \mathbb{N}$. The L_∞ structure it induces on $\widetilde{\mathcal{C}}_{\text{CE}}$ can be used to define the cohomology given in Section 6.9. Furthermore the map $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \mapsto \text{id}$ and $\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \mapsto 0$ for $k \leq 2$ defines a morphism from this minimal model to the standard minimal model of Lie given by the Koszul resolution.*

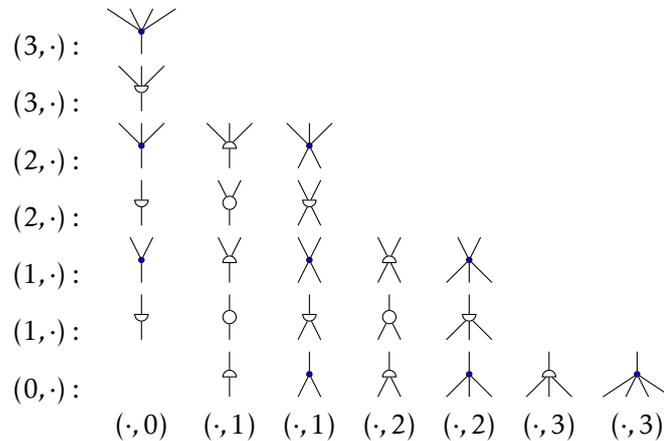
If this conjecture is true it also implies that there is a morphism from this minimal model to a minimal of ELie extending the map $Y : \text{HLie} \rightarrow \text{ELie}$.

7.3 HOM-LIE BIALGEBRAS

We give this complex for the general case, i.e. α and β can be different, even though for the α -type cohomology we only considered the case $\alpha = \beta$.

Here we need a free properad to describe the L_∞ structures.

These free properad can be considered as a bicomplex and is spanned by the following graphs, which are totally antisymmetric:



Here $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$ corresponds to α , $\begin{array}{c} \text{---} \\ \text{---} \end{array}$ to β , $\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$ to δ and $\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$ to ν . One can also associate a height to this vertices, in this case it is a pair of integers, similar to Section 6.8. It is given by the bidegree.

Note that the degree has to be shifted from the total degree of the bicomplex, since we work in the setting where the Maurer-Cartan elements are considered of degree zero.

The differential on the complex can be found in Appendix D.4.

Note that the first line is precisely the complex considered for Hom-Lie algebras.

With this one can construct an L_∞ structures on $\tilde{C}_{CE}(\mathfrak{g}) = \text{Hom}(\Lambda^\bullet \mathfrak{g}, \Lambda^\bullet \mathfrak{g})$ for a vector space \mathfrak{g} considered as trivial Hom-Lie bialgebra, similar to the previous sections.

Proposition 7.3.1 *Let \mathfrak{g} be a vector space then a Maurer-Cartan element for the L_∞ algebra defined above is a Hom-Lie bialgebra.*

Proof. This is clear by construction. □

One can also define the first degrees of a cohomology for Hom-Lie bialgebra with this. But for the moment we do not know how to generalize it to the full complex. However, we conjecture that the free properad given here can be extended to a minimal model for HBiLie.

7.4 HOM-ASSOCIATIVE BIALGEBRAS

Here the free properad is spanned by the same graphs as in the previous section but they have no symmetry. They also have the same height and degree as there. The differential is given in Appendix D.3.

Again one can define a pairing between this free properad and the complex defined in Section 6.7, which induces the structure of an L_∞ algebra on C_{GS} , at least in low degrees. We do not want to go into more detail here.

Proposition 7.4.1 *Let A be a vector space then a Maurer-Cartan element for the L_∞ structures defined above is precisely a Hom-bialgebra. Further the α -type GS cohomology in degrees 2 and 3 is induced from it.*

We want to state a conjecture that this free properad we have given can be extended to a minimal model for HBiAlg. Furthermore the cohomology one can derive from it is given by the one described in Section 6.7.

8. DEFORMATIONS

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WE give some basics for the formal deformation of different type of (Hom-)algebras. The study of formal deformations of especially associative algebras goes back to Gerstenhaber [Ger64].

So as before let \mathbb{K} be a field of characteristic zero. Then we denote by $\mathbb{K}[[t]]$ the ring of formal power series over \mathbb{K} . This is an element is given as a series $x = \sum_{i=0}^{\infty} x_i t^i$. More on formal power series and formal deformations can e.g. be found in [Wal07, Kapitel 6], but we want to state some basic facts. We define the t -adic topology on $\mathbb{K}[[t]]$ such that $t^n \mathbb{K}[[t]]$ for $n \in \mathbb{N}$ is a system of neighborhoods of zero. We denote by $\text{ord}(x) := \min\{i | x_i \neq 0\}$. With this we define a metric on $\mathbb{K}[[t]]$ by $d(x, y) = 2^{-\text{ord}(x-y)}$, where we set $2^{-\infty} = 0$, so that $d(x, x) = 0$.

Given a vector space V we denote by $V[[t]]$ the set of series of the form $v = \sum_{i=0}^{\infty} v_i t^i$. For finite dimensional V we have $V[[t]] = V \otimes \mathbb{K}[[t]]$ and for infinite dimensional V we have $V[[t]] = V \hat{\otimes} \mathbb{K}[[t]]$, where $\hat{\otimes}$ denotes the completion of the tensor product in the t -adic topology. It is clear that $V[[t]]$ is a $\mathbb{K}[[t]]$ module. We have $(A \otimes A)[[t]] = A[[t]] \hat{\otimes}_{\mathbb{K}[[t]]} A[[t]]$.

We want to consider the morphisms between two vector spaces of formal power series. So let V, W be two vector spaces, then we have $\text{Hom}(V, W)[[t]] = \text{Hom}_{\mathbb{K}[[t]]}(V[[t]], W[[t]])$. So every $\mathbb{K}[[t]]$ -linear map $\varphi : V \rightarrow W$ can be written as $\varphi : \sum_{i=0}^{\infty} \varphi_i t^i$, where each φ_i is a linear map $V \rightarrow W$, which is extended to a $\mathbb{K}[[t]]$ -linear map $V[[t]] \rightarrow W[[t]]$. We will use this in the following quite often and not always state this explicitly.

8.1 HOM-ASSOCIATIVE ALGEBRAS

In this section we consider deformations of Hom-associative algebras and how they can be described using the cohomology defined in the Section 6.2. Up to now mostly the case where only the multiplication is changed is discussed. But here we consider the more general case where the multiplication as well as the structure map is deformed. We will show that the order by order construction of such deformations gives rise to equations in the α -type cohomology described in Section 6.3. We also give a generalization of the well known fact that a deformation of a commutative Hom-algebra gives rise to a Hom-Poisson algebra.

First we recall the basic definitions of formal deformations using formal power series as described by Gerstenhaber [Ger64].

Definition 8.1.1 Let (A, μ, α) be a Hom-associative algebra over \mathbb{K} , then a deformation of A is a Hom-associative algebra $(A[[t]], \star, \alpha_\star)$ over $\mathbb{K}[[t]]$, such that $ab = a \star b \pmod t$ for all $a, b \in A$ and $\alpha_\star = \alpha \pmod t$.

In the following we will always write $a \star b = \mu_\star(a, b) = \sum_{i=0}^{\infty} \mu_i(a, b) t^i$ and $\alpha_\star = \sum_{i=0}^{\infty} \alpha_i t^i$. So for a deformation we have $\alpha_0 = \alpha$ and $\mu_0(a, b) = ab$.

Definition 8.1.2 Two deformation (\star, α_\star) and (\star', α'_\star) of a Hom-associative algebra A are called equivalent if there exists a linear isomorphism T of $A[[t]]$ of the form $T = \text{id} + \sum_{i=0}^{\infty} T_i t^i$, such that $T(a \star' b) = T(a) \star T(b)$ and $T(\alpha'_\star(a)) = \alpha(T(a))$.

Given a deformation of A and a linear map T as above one can also define an equivalent deformation by $a \star' b = T^{-1}(T(a) \star T(b))$ and $\alpha'_\star(a) = T^{-1}(\alpha(T(a)))$. It is clear that equivalence is an equivalence relation on the set of deformations of a Hom-algebra A .

Let $(A[[t]], \star, \alpha_\star)$ be a deformation of the Hom-associative algebra (A, μ, α) , where $a \star b = \mu_\star(a, b) = \sum_{i=0}^{\infty} \mu_i(a, b) t^i$ and $\alpha_\star = \sum_{i=0}^{\infty} \alpha_i t^i$.

The Hom-associativity condition $\mu_\star(\mu_\star(a, b), \alpha_\star(c)) - \mu_\star(\alpha_\star(a), \mu_\star(b, c))$ can be written as

$$\sum_{i,j,k=0}^{\infty} t^{i+j+k} (\mu_i(\alpha_j(a), \mu_k(b, c)) - \mu_i(\mu_k(a, b), \alpha_j(c))) = 0 \quad (1)$$

and is equivalent to an infinite system of equations, called deformation equation with respect to Hom-associativity, such that the n -th equation is of the form

$$\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \mu_i(\alpha_j(a), \mu_k(b, c)) - \mu_i(\mu_k(a, b), \alpha_j(c)) = 0.$$

Notice that the 0-th equation expresses the Hom-associativity of A .

Now, rearranging the terms and using coboundary operators from Section 6.3, one may write the previous equation as

$$(\partial_{\mu\mu}\mu_n + \partial_{\alpha\mu}\alpha_n)(a, b, c) = \sum_{\substack{i,j,k=0,\dots,n-1 \\ i+j+k=n}} \mu_i(\alpha_j(a), \mu_k(b, c)) - \mu_i(\mu_k(a, b), \alpha_j(c)), \quad (2)$$

where $\partial_{\mu\mu}\mu_n(a, b, c) = \alpha_0(a)\mu_n(b, c) - \mu_n(ab, \alpha_0(c)) + \mu_n(\alpha_0(a), bc) - \mu_n(a, b)\alpha_0(c)$ and $\partial_{\alpha\mu}\alpha_n(a, b, c) = \alpha_n(a)(bc) - (ab)\alpha_n(c)$.

Similarly the multiplicativity condition $\alpha_\star(\mu_\star(a, b)) = \mu_\star(\alpha_\star(a), \alpha_\star(b))$ can be written as

$$\sum_{i,j=0}^{\infty} t^{i+j} \alpha_i(\mu_j(a, b)) - \sum_{i,j,k=0}^{\infty} t^{i+j+k} \mu_i(\alpha_j(a), \alpha_k(b)) = 0. \quad (3)$$

This again is equivalent to an infinite system of equations, called deformation equation with respect to multiplicativity, with the n -th equation given by

$$\sum_{\substack{i,j=0,\dots,n \\ i+j=n}} \alpha_i(\mu_j(a, b)) - \sum_{\substack{i,j,k=0,\dots,n \\ i+j+k=n}} \mu_i(\alpha_j(a), \alpha_k(b)) = 0, \quad (4)$$

which can be rearranged to

$$(\partial_{\alpha\alpha}\alpha_n - \partial_{\alpha\mu}\mu_n)(a, b) = \sum_{\substack{i,j=0,\dots,n-1 \\ i+j=n}} \alpha_i(\mu_j(a, b)) - \sum_{\substack{i,j,k=0,\dots,n-1 \\ i+j+k=n}} \mu_i(\alpha_j(a), \alpha_k(b)), \quad (5)$$

where $\partial_{\alpha\alpha}\alpha_n(a, b) = \alpha_0(a)\alpha(b) - \alpha_n(ab) - \alpha_n(a)\alpha_0(b)$ and $\partial_{\alpha\mu}\mu_n(a, b) = \alpha_0(\mu_n(a, b)) - \mu_n(\alpha_0(a), \alpha_0(b))$.

We denote by R_n^1 the right hand side of equation (2) and R_n^2 the right hand side of equation (5). The pair (R_n^1, R_n^2) is called the n -th obstruction.

Since the deformation is governed by an L_∞ -algebra, we have the usual statement relating deformations and cohomology.

Theorem 8.1.3 *Let (A, μ, α) be a Hom-associative algebra and $(A[[t]], \star, \alpha_\star)$ be a deformation of A . Then we have*

1. *The first order term of the deformation is a 2-cocycle, i.e. we have $\partial(\mu_1, \alpha_1) = 0$, and its cohomology class is invariant under equivalence.*
2. *The n -th deformation equations with respect to Hom-associativity and multiplicativity respectively, are equivalent to $\partial(\mu_n, \alpha_n) = (R_n^1, R_n^2)$. Moreover, (R_n^1, R_n^2) is a 3-cocycle, i.e. $\partial(R_n^1, R_n^2) = 0$.*

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Proof. The equation $\partial(\mu_n, \alpha_n) = (R_n^1, R_n^2)$ is equivalent to Eqs. (2) and (5). For $n = 1$ Eq. (2) gives $(\partial_{\mu\mu}\mu_n + \partial_{\alpha\mu}\alpha_n)(a, b, c) = 0$ and Eq. (5) gives $(\partial_{\alpha\alpha}\alpha_n - \partial_{\alpha\mu}\mu_n)(a, b) = 0$, since obviously the right hand side vanishes. This means we have $\partial(\mu_1, \alpha_1) = 0$.

For an equivalent deformation (\star', α'_\star) , we get from $S(a \star b) = S(a) \star S(b)$ in first order that $\mu'_1(a, b) + S_1(ab) = S_1(a)b + aS_1(b) + \mu_1(a, b)$. Further from $S(\alpha'_\star(a)) = \alpha_\star(S)(a)$, we get $S_1(\alpha_0(a)) + \alpha'_1(a) = \alpha_1(a) + \alpha_0 S_1(a)$. Rearranging gives $(\mu'_1 - \mu_1, \alpha'_1 - \alpha_1) = \partial(S, 0)$. So the cohomology class of (μ_1, α_1) and (μ'_1, α'_1) are the same as claimed.

Using the L_∞ -structure l defined in Chapter 7, we have $\partial(\mu_n, \alpha_n) - R_n = l(e^{(\mu_\star, \alpha_\star)})$ at order n in t , since both are equivalent to the fact that $(\mu_\star, \alpha_\star)$ is a Hom-associative algebra. Also $l(e^{(\mu_\star, \alpha_\star)})$ vanishes up to order $n - 1$ in t . In the following we write ν_\star for $(\mu_\star, \alpha_\star)$ for shortness. Since l is an L_∞ -structure, it satisfies $l(e^{\nu_\star} l(e^{\nu_\star})) = 0$ and since ν_\star is a Maurer-Cartan element up to order $n - 1$ it satisfies $l(e^{\nu_\star}) = 0 \pmod{t^n}$. So the n -th order of $(e^{\nu_\star} l(e^{\nu_\star}))$ is given by $l(e^{\nu_0}(\partial\nu_n - R_n)) = \partial\partial\nu_n - \partial R_n = \partial R_n$ and has to vanish as claimed. \square

Corollary 8.1.4 *If $\tilde{H}H^3(A) = \{0\}$ every deformation up to order n can be extended to a full deformation. So especially for every 2-cocycle there exists a deformation, with it as first order term.*

Proof. We use the notation of the previous theorem. If $\mu_{(n)} = \sum_{i=0}^n \mu_i t^i, \alpha_{(n)} = \sum_{i=0}^n \alpha_i t^i$ is a deformation up to order n , the deformation equation $\partial(\mu_{n+1}, \alpha_{n+1}) = R_{n+1}$ can be solved, since $\tilde{H}H^3(A) = \{0\}$ and $(\mu_{(n)} + \mu_{n+1} t^{n+1}, \alpha_{(n)} + \alpha_{n+1} t^{n+1})$ is a deformation up to order $n + 1$. Continuing this for each order, gives a full deformation. If (μ_1, α_1) is a 2-cocycle $(\mu_0 + \mu_1 t, \alpha_0 + \alpha_1 t)$ is a deformation up to order 1, so be the previous it can be extended to a deformation of A . \square

Proposition 8.1.5 *If two deformations $(\mu_\star, \alpha_\star)$ and $(\mu'_\star, \alpha'_\star)$ have the same terms up to order $n - 1$, then we have $\partial(\mu_n - \mu'_n, \alpha_n - \alpha'_n) = 0$ and there exists an equivalence up to order n if there exists a linear map $S_n : A \rightarrow A$ such that $\partial(S_n, 0) = (\mu_n - \mu'_n, \alpha_n - \alpha'_n)$.*

Proof. Let $S = \text{id} + S_n t^n$, we have

$$S(a \star' b) = \sum_{i=0}^n \mu'_i(a, b) t^i + S_n(ab) t^n \pmod{t^{n+1}} \text{ and}$$

$$S(a) \star S(b) = \sum_{i=0}^n \mu'_i(a, b) t^i + (S_n(a)b + aS_n(b)) t^n \pmod{t^{n+1}}.$$

So S is an equivalence up to order n if

$$S_n(ab) + \mu'_n(ab) = S_n(a)b + aS_n(b) + \mu_n(ab),$$

which can be written as $\partial_{\mu\mu}S_n = \mu'_n - \mu_n$. Similarly from $\alpha S - S\alpha'$ in order n we get

$$\alpha_n + \alpha_0 S_n - S_n - \alpha'_n = \partial_{\mu\alpha}S_n + \alpha_n - \alpha'_n = 0.$$

So $S = 1 + S_n t^n$ is an equivalence up to order n .

Using the L_∞ -structure, we calculate with $\nu_\star = (\mu_\star, \alpha_\star)$

$$\begin{aligned} l((e^{-\nu_\star} - 1)(e^{\nu_\star - \nu'_\star} - 1)) &= l(e^{-\nu_\star} e^{\nu_\star - \nu'_\star}) - l(e^{-\nu_\star}) + l(e^{\nu_\star - \nu'_\star}) + l(1) \\ &= l(e^{\nu_\star}) - l(e^{\nu_\star - \nu'_\star}) = -l(e^{\nu_\star - \nu'_\star}). \end{aligned}$$

Since ν and ν' equal up to order $n-1$ the last term vanishes up to order $2(n-1)$, so especially in order n . The first term in order n equals $l(e^{\nu_0}(\nu_n - \nu'_n)) = \partial(\nu_n - \mu'_n)$. \square

Corollary 8.1.6 *Every deformation is equivalent to one of the form $\mu_\star = \mu_0 + \sum_{i=k}^\infty \mu_i t^i$ and $\alpha_\star = \sum_{i=k}^\infty \alpha_i t^i$, such that (μ_k, α_k) is a 2-cocycle which is not a coboundary.*

If $\tilde{\text{HH}}^2(A) = \{0\}$ every deformation is equivalent to the undeformed algebra.

Proof. If $\mu_\star = \mu_0 + \sum_{i=k}^\infty \mu_i t^i$ and $\alpha_\star = \sum_{i=k}^\infty \alpha_i t^i$ is a deformation of A , then (μ_k, α_k) is a 2-cocycle. If it is a 2-coboundary, by the previous proposition, there exists an equivalent deformation of the form $(\mu_\star = \mu_0 + \sum_{i=k+1}^\infty \mu_i t^i, \alpha_\star = \sum_{i=k+1}^\infty \alpha_i t^i)$, where the first non-trivial term is in order $k+1$. Repeating this if necessary, we arrive at a deformation such that the first nontrivial term is not a 2-coboundary. If $\tilde{\text{HH}}^2(A) = \{0\}$ every 2-cocycle is coboundary and the second statement is clear. \square

We will now consider the deformation of commutative algebras.

We define the star commutator $[\cdot, \cdot]_\star : A[[t]] \otimes A[[t]] \rightarrow A[[t]]$ by

$$[a, b]_\star = a \star b - b \star a. \quad (6)$$

It is easy to see that it satisfies the Hom-Jacobi identity and the Hom-Leibniz identity, so it defines a nc. Hom-Poisson algebra. The following propositions shows that if A is commutative its first order gives rise to a Hom-Poisson algebra on A .

Proposition 8.1.7 *Let A be a commutative Hom-associative algebra and A_\star be a deformation of it. Then $\{a, b\} = \frac{1}{2t}(a \star b - b \star a) \bmod t$ defines a Hom-Poisson algebra on A*

Proof. The bracket is obviously skewsymmetric. We have $[a, b]_\star = ab - ba + t(\mu_1(a, b) - \mu_1(b, a)) + \mathcal{O}(t^2) = 2t\{a, b\} + \mathcal{O}(t^2)$. In second order we have

$$[[a, b]_\star, \alpha_\star(c)]_\star = 4\{a, b, \alpha(c)\}.$$

Taking the cycling sum over this gives the Hom-Jacobi identity. For the Hom-Leibniz identity we consider the Hom-Leibniz identity for the star commutator in first order this is

$$[ab, \alpha_\star(c)] = 2\{ab, \alpha(c)\} \quad (7)$$

$$\alpha_\star(a)[b, c] + [a, c]\alpha_\star(b) = 2\alpha(a)\{b, c\} + 2\{a, c\}\alpha(b), \quad (8)$$

which gives what we want. Next we want to show that the Poisson bracket is multiplicative with respect to α_0 . This follows from

$$\begin{aligned} \alpha_\star([a, b]) &= t\alpha_0(2\{a, b\}) + t^2\alpha_1(2\{a, b\}) + \alpha_2(\mu_2(a, b) - \mu_2(b, a)) + \mathcal{O}(t^3) \\ [\alpha_\star(a), \alpha_\star(b)] &= t2\{\alpha_0(a), \alpha_0(b)\} + t^2(2\{\alpha_1(a), \alpha_0(b)\} + 2\{\alpha_0(a), \alpha_1(b)\}) \\ &\quad + \mu_2(\alpha_0(a), \alpha_0(b)) - \mu_2(\alpha_0(b), \alpha_0(a)) + \mathcal{O}(t^3) \end{aligned} \quad (9)$$

if one compares the first order terms. □

Dually for the deformation of a cocommutative Hom-coassociative algebra, $\delta(x) = \frac{1}{2t}(\Delta(x) - \Delta^{\text{opp}}(x)) \pmod t$ defines a Hom-Poisson coalgebra.

As in the associative case the Poisson bracket is invariant under equivalence and also here we have

Proposition 8.1.8 *Given two equivalent deformation of a Hom-algebra A the Poisson bracket $\{a, b\}$ is the same.*

Proof. If \star and \star' are equivalent there exists an $T = \text{id} + \sum_{i=1}^{\infty} T_i t^i : A[[t]] \rightarrow A[[t]]$, such that $a \star' b = T^{-1}(T(a) \star T(b))$. Using $T^{-1} = \text{id} - T_1 t + \mathcal{O}(t^2)$, this gives

$$ab + \mu'_1(a, b)t = ab + t(\mu_1(a, b) + T_1(a)b + aT_1(b) - T_1(ab)). \quad (10)$$

From which follows $\{a, b\}' = \frac{1}{2}(\mu'_1(a, b) - \mu'_1(b, a)) = \frac{1}{2}(\mu(a, b) - \mu(b, a)) = \{a, b\}$, as claimed. □

However different then in the associative case the Poisson bracket is not cohomologous to (μ_1, α_1) . Instead we have that $(\{\cdot, \cdot\}, 0)$ is a cocycle. One can construct another cocycle, which is given by $(\frac{1}{2}(\mu_1(a, b) + \mu_1(b, a)), \alpha_1)$. This however is not invariant under equivalence. We suspect however that if α is invertible, one can construct a conjugate α_0 -derivation out of it, which is invariant, as in the case where $\alpha_0 = \text{id}$, see Proposition 8.1.9.

Next we consider the case, when A is indeed associative, but is deformed into a Hom-associative algebra.

Proposition 8.1.9 *Let (A, μ, id) be a commutative Hom-associative algebra and (A, \star, α) a deformation of it, then α_1 is a derivation of the associated Poisson algebra and invariant under equivalence.*

Proof. Using the proof of Proposition 8.1.7, especially the second order in Eq. (9), we get that it is a derivation of the Poisson bracket, since the terms involving μ_2 cancel. Since for two equivalent transformations α_1 differs by the commutator $[S_1, \alpha_0]$, the second statement is clear. \square

So in this case we are closer to the associative case, since we get from the first order terms a canonical invariant algebraic structure, which at least in simple cases should be enough to determine all deformation up to equivalence.

In the associative case especially the deformation of the symmetric algebra $S(V)$ for a vector space V is well studied and understood. In this case there is the well known theorem that any Poisson bracket admits a deformation, which was first proofed by Kontsevich in [Kon03]. We can extend this to Hom-context in the following way:

Theorem 8.1.10 *Let $S(V)$ be the symmetric algebra over a vector space V , considered as a Hom-associative algebra, and $\{\cdot, \cdot\}$ a Poisson bracket on $S(V)$ and α_1 a derivation of it, then there exists a deformation of it.*

Proof. Using the theorem of Kontsevich there exist a deformation of the Poisson bracket, which we denote by $\tilde{\star}$. Since α_1 is a derivation and Poisson derivation there exists an derivation $D = \alpha_1 + \sum_i D_i t^i$ of \star as shown e.g. in [Sha17]. With this e^{tD} is an automorphism of the form $\varphi = \text{id} + t\alpha_1 + o(t^2)$. Now the Yau twist $\star = \tilde{\star}_\varphi$ has the desired properties. \square

If α is invertible we have that $S(V, \alpha)$ is isomorphic to $S(V)_\alpha$. Also the first order term gives a cohomology class in $C(S(V), \alpha)$. So it is given by a pair $(\{\cdot, \cdot\}, \psi)$, where $\{\cdot, \cdot\}$ is a Poisson-bracket which is compatible with α and ψ is an α -derivation.

8.1.1 Deformations and Yau twist

We look at the relation between deformations and the Yau twist. One question is for example, if a deformation is of associative type, was the undeformed algebra also of associative type and vice versa.

In the best case given an associative algebra A and a morphism $\gamma : A \rightarrow A$, one gets a commutative diagram of the form

$$\begin{array}{ccc}
 (A, \mu), \gamma & \xrightarrow{\gamma} & (A, \gamma, \mu) \\
 \downarrow \mu_1, \gamma_1 & & \downarrow \\
 (A, \mu_\star), \gamma_\star & \xrightarrow{\gamma_\star} & (A, \gamma_\star \mu_\star, \gamma_\star)
 \end{array}$$

Here the horizontal arrows indicate Yau twists and the vertical ones a deformation.

So let (A, μ) be an associative algebra and γ a morphism of it. Then if one can deform this one of course gets a deformed Hom-associative algebra. We note that the first order term of A_{γ_\star} is given by $\Phi(\mu_1, \gamma_1) = \gamma_0 \mu + \gamma_0 \mu_1, \gamma_1$.

If (A, α, μ, α) is a Hom-associative algebra of associative type it is easy to see that a deformation is in general not of associative type. To see this consider $\alpha = 0$. Then any multiplication, which is zero in zeroth order, and $\alpha_\star = \alpha = 0$ is a deformation of it. If however A is regular then also $A_\star[[t]]$ is regular and hence of associative type. So in this case one always gets a commuting square as above.

Further if (A, μ, α) is a Hom-associative algebra and $(A, \mu_\star, \alpha_\star)$ a deformation of it, which is of associative type, then also A is of associative type.

Now we briefly consider the case of twisting a Hom-algebra. So let (A, μ, α) be a Hom-associative algebra and γ a morphism of it. Than again it is clear if this can be deformed one again gets a commutative diagram similar to the one above. Not further that the first order term of A_γ is given by $\Phi(\mu_1, \alpha_1, \gamma_1, 0) = (\gamma_1 \mu_0 + \gamma_0 \mu_1, \gamma_0 \alpha_1 + \gamma_1 \alpha_0, \gamma_1, 0)$.

We also want to study how the Poisson bracket and α_1 change under Yau twists.

Proposition 8.1.11 *Let A be a commutative Hom-algebra, $(A[[t]], \star)$ a deformation of it and $\varphi = \sum_{i=0}^{\infty} t^i \varphi_i$ a morphism of \star , then the algebra $(A, \star_\varphi, \varphi \alpha)$ obtained by Yau-twist is a deformation of $(A, \varphi_0 \mu, \varphi_0 \alpha_0)$ with Poisson bracket $\varphi_0 \{, \cdot\}$.*

Proof. Using $a \star_\varphi b = \varphi(a \star b) = \varphi_0(ab) + t(\varphi_0 \mu_1(a, b) + \varphi_1(ab)) + \mathcal{O}(t^2)$ and $\varphi \alpha = \varphi_0 \alpha_0 + t(\varphi_1 \alpha_0 + \varphi_0 \alpha_1) + \mathcal{O}(t)$ this is easy to see. We also note that $(\alpha_\varphi)_1 = \varphi_1 \alpha_0 + \varphi_0 \alpha_1$. \square

In the case $\varphi_0 = \text{id}$ in the above proposition we get a deformation of the same algebra, which also has the same Poisson bracket, but in general with different α_1 so the deformations are not equivalent in general.

8.1.2 Examples

We consider some examples. We begin with an arbitrary commutative associative algebra A . Let Y, X_i for $i = 1, \dots, n$ be commuting derivations. They are for example easy to find in the case $A = S(V)$ for a vector space V . Then we define a new product on $A[[t]]$ by $\mu_\star = \mu e^{t a^{ij} X_i \otimes X_j}$, where a^{ij} is a constant matrix. This is sometimes called Gerstenhaber's formula and it is well known that this is a deformation of A with

Poisson bracket $\frac{1}{2}ta^{ij}(X_i \otimes X_j - X_j \otimes X_i)$. Further it is easy to see that e^Y is a morphism of it. So one can perform a Yau twist and gets that $(A, \gamma_\star \mu_\star, \gamma_\star)$ is a deformation of (A, μ, id) .

One can similarly consider the case, where A is a commutative Hom-associative algebra and X_i for $i = 1 \dots n$ are commuting derivations, which also commute with α . Then one can again define

$$\mu_\star = \mu e^{ta^{ij}X_i \otimes X_j}. \quad (11)$$

We first compute

$$(X_i \otimes X_j)(\mu \otimes \alpha) = (\mu \otimes \alpha)(X_i \otimes \text{id} \otimes X_j + \text{id} \otimes X_i \otimes X_j).$$

With this one gets

$$\begin{aligned} \mu_\star(\mu_\star \otimes \alpha) &= \mu e^{ta^{ij}X_i \otimes X_j} (\mu e^{ta^{ij}X_i \otimes X_j} \otimes \alpha) \\ &= \mu(\mu \otimes \alpha) e^{ta^{ij}X_i \otimes X_j} e^{ta^{ij}(X_i \otimes \text{id} \otimes X_j + \text{id} \otimes X_i \otimes X_j)} \\ &= \mu(\mu \otimes \alpha) e^{ta^{ij}(X_i \otimes X_j + X_i \otimes \text{id} \otimes X_j + \text{id} \otimes X_i \otimes X_j)} \end{aligned}$$

and doing a similar calculation

$$\mu_\star(\alpha \otimes \mu_\star) = \mu(\alpha \otimes \mu) e^{ta^{ij}(X_i \otimes X_j + X_i \otimes \text{id} \otimes X_j + \text{id} \otimes X_i \otimes X_j)}.$$

So the product μ_\star is Hom-associative since μ is so. Since the derivations X_i commute with α it is clear that α is also a morphism of μ_\star .

It this case the Hom-Poisson bracket is given by $\{a, b\} = \pi^{ij}X_i(a)X_j(b)$ for $a, b \in A$, and it is easy to see that this is in fact a Hom-Poisson bracket.

8.2 HOM-ALGEBRA BIMODULES

In this section we give some remarks on the deformation of Hom-algebra modules and bimodules.

Definition 8.2.1 (Module deformation) Let A be a Hom-associative algebra and (M, ρ, β) be an A right module. Further let $(A[[t]], \star, \alpha_\star)$ be a deformation of A . Then an A_\star module $(M[[t]], \lambda_\star, \beta_\star)$ is called a Hom-algebra module deformation of M if λ is given by $a \bullet m = a \cdot m + \sum_{i=1}^{\infty} \lambda_i(a, m)t^i$ and $\beta_\star = \beta + \sum_{i=1}^{\infty} \beta^i t^i$, i.e. in zero order they agree with M .

Similarly one can define a right module deformation.

Definition 8.2.2 Let A be a Hom-algebra and M be a Hom-bimodule. Then a deformation of M is a left and right module deformation, such that they form a A -bimodule.

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Similar to the associative case considered in [Hur18] the deformations of a symmetric Hom-bimodule gives rise to a Hom-Poisson module.

Theorem 8.2.3 *Let M be a symmetric A -Hom-bimodule and $(M[[t]], \bullet)$ a deformation of it. Then the bracket $\{a, m\} := \frac{1}{t}(a \bullet m - m \bullet a) \pmod{t}$ together with the undeformed bimodule defines a Hom-Poisson module.*

Proof. We define $[a, m] := a \bullet m - m \bullet a$, then $[a, m] = \{a, m\} + \mathcal{O}(t^2)$. It is easy to check that $[\alpha(a), [b, m]] = [[a, b]_{\star}, \beta_{\star}(m)] + [\alpha_{\star}(a), [b, m]]$. Evaluating this is second order gives $\{\alpha(a), [b, m]\} = \{\{a, b\}, \alpha_M(m)\} + \{\alpha(b), \{a, m\}\}$, since $[\cdot, \cdot]$ is zero in zeroth order. We also have $[\alpha(a), b \cdot m] = [a, m] \bullet \beta_{\star}(m) + \alpha_{\star}(a) \bullet [b, m]$, which in first order gives $\{\alpha(a), b \cdot m\} = \alpha(b) \cdot \{a, m\} + \{a, b\} \cdot \alpha_M(m)$. Similarly $\{\alpha(a), b \cdot m\} = \alpha(b) \cdot \{a, m\} + \{a, b\} \cdot \alpha_M(m)$ can be checked. \square

We remark that other than in the case of module deformation of an associative algebra, this deformation cannot be described by the cohomology $\mathrm{HH}(A, \mathrm{Hom}(M, M))$, since $\mathrm{Hom}(M, M)$ is in general not a Hom-module, since one cannot define a structure map. Also the deformation of bimodules cannot be reduced to the module case since for Hom-algebras A, B in general a A - B -bimodule is not a module over $A \otimes B^{\mathrm{opp}}$.

However it is possible to consider to consider the algebra $A \oplus M$. And restrict the complex to maps of the form $\mathrm{Hom}(A^{\otimes n} \otimes M, M)$. It is easy to see that the α -type differential restricts to this complex and one gets an associated cohomology. In the ordinary case this gives the same as $\mathrm{HC}(A, \mathrm{Hom}(M, M))$. For bimodules one can proceed similarly in this case one restricts to the complex $\mathrm{Hom}(A^{\otimes i} \otimes M \otimes A^{\otimes n-i}, M)$.

8.3 HOM-LIE ALGEBRAS

In this section we briefly discuss how the α -type Chevalley-Eilenberg cohomology can be used to study deformations of Hom-Lie algebras, where the bracket and the structure map α are deformed.

Definition 8.3.1 Let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra. A deformation of it is a Hom-Lie algebra $(\mathfrak{g}[[t]], \nu_t, \alpha_t)$ over $\mathbb{K}[[t]]$, such that $\alpha_t = \alpha + \sum_{i=1}^{\infty} \alpha_i t^i$ and $[x, y]_t = [x, y] + \sum_{i=1}^{\infty} t^i [x, y]_i$. Here we denote $\nu_t(x \otimes y) = [x, y]_t$.

Definition 8.3.2 Two deformations ν_t, α_t and ν'_t, α'_t are called equivalent if there exists a formal power series $S = \mathrm{id} + \sum_{i=1}^{\infty} S_i t^i$ such that $S([x, y]'_t) = [S(x), S(y)]_t$ and $S\alpha'_t = \alpha_t S$.

It is clear that this equivalence is an equivalence relation on the set of deformations of a Hom-Lie algebra.

Let \mathfrak{g} be a Lie algebra regard as Hom-Lie algebra, then for a deformation ν_t, α_t of it, α_1 is a derivation. This follows from the multiplicativity and $\alpha_t([x, y]_t) = [x, y]_0 + t(\alpha_1[x, y] + [x, y]_1)$ and $[\alpha_t(x), \alpha_t(y)] = [x, y]_0 + t([\alpha_1 x, y] + [x, \alpha_1 y])$.

The condition that $(\mathfrak{g}, \nu_t, \alpha_t)$ is a deformation of $(\mathfrak{g}, \nu, \alpha)$ is equivalent to the following two conditions:

First it has to satisfy the Hom-Jacobi identity, which gives

$$\sum_{i,j,k \geq 0} \sum_{cycl.} \nu_i(\alpha_j(x), \nu_k(y, z)) t^{i+j+k} = 0. \quad (12)$$

Here and in the following equations $\sum_{cycl.}$ denotes the cyclic sum over the elements $x, y, z \in \mathfrak{g}$. At order n in t this gives

$$\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \sum_{cycl.} \nu_i(\alpha_j(x), \nu_k(y, z)) = 0, \quad (13)$$

This is called the n -th deformation equation with respect to the Hom-Jacobi identity. It can be rearranged to

$$(\partial_{\mu\mu} \nu_n + \partial_{\alpha\mu} \alpha_n)(x, y, z) = \sum_{\substack{0 \leq i,j,k \leq n-1 \\ i+j+k=n}} \sum_{cycl.} \nu_i(\alpha_j(x), \nu_k(y, z)) = 0, \quad (14)$$

where $\partial_{\mu\mu} \nu_n(x, y, z) = -\sum_{cycl.} \nu_n([x, y], \alpha(z)) + \sum_{cycl.} [\alpha(x), \nu_n(y, z)]$ and $\partial_{\alpha\mu} \alpha_n(x, y, z) = \sum_{cycl.} [\alpha_n(x), [y, z]]$ are the parts of the differential defined in Section 6.9. We denote the right hand side by R_n^1 .

Second it has to satisfy the multiplicativity, this is

$$\sum_{i,j \geq 0} \alpha_i(\nu_j(x, y)) t^{i+j} - \sum_{i,j,k \geq 0} \nu_i(\alpha_j(x), \alpha_k(y)) t^{i+j+k} = 0. \quad (15)$$

At order n this gives

$$\sum_{i=0}^n \alpha_i(\nu_{n-i}(x, y)) - \sum_{\substack{0 \leq i,j,k \leq n \\ i+j+k=n}} \nu_i(\alpha_j(x), \alpha_k(y)) = 0. \quad (16)$$

We call this the n -th deformation equation with respect to the multiplicativity. Again this can be rewritten as

$$(\partial_{\alpha\alpha} \alpha_n + \partial_{\mu\alpha} \nu_n)(x, y) = -\sum_{i=0}^{n-1} \alpha_i(\nu_{n-i}(x, y)) + \sum_{\substack{0 \leq i,j,k \leq n-1 \\ i+j+k=n}} \nu_i(\alpha_j(x), \alpha_k(y)), \quad (17)$$

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where $\partial_{\mu\alpha} \nu_n = \alpha \nu_n - \nu_n(\alpha \otimes \alpha)$ and $\partial_{\alpha\alpha} \alpha_n(x, y) = [\alpha(x), \alpha_n(y)] - [\alpha(y), \alpha_n(x)] - \alpha_n([x, y])$. We denote the right hand side by R_n^2 .

Since the deformation is governed by an L_∞ -algebra, we have the usual statement relating deformations and cohomology. We will omit the proof here since this follows from the deformation equations given above and are almost identical to the Hom-associative case given in the previous section.

Theorem 8.3.3 *Let $(\mathfrak{g}, \nu, \alpha)$ be a Hom-Lie algebra and $(\mathfrak{g}[[t]], \nu_t, \alpha_t)$ a deformation of it. Then we have*

1. *The first order term is a 2-cocycle, i.e. we have $\partial(\nu_1, \alpha_1) = 0$, whose cohomology class is invariant under equivalence.*
2. *The n -th deformation equations with respect to the Hom-Jacobi identity and multiplicativity respectively, are equivalent to $\partial(\nu_n, \alpha_n) = (R_n^1, R_n^2)$. Furthermore, (R_n^1, R_n^2) is a 3-cocycle, i.e. $\partial(R_n^1, R_n^2) = 0$.*

This means that the construction of a deformation order by order gives equations in $\tilde{H}_{\text{CE}}^3(\mathfrak{g})$, we also get:

Corollary 8.3.4 *If $\tilde{H}_{\text{CE}}^3(\mathfrak{g}) = 0$ any finite deformation up to order n can be extended to a full deformation. So especially every 2-cocycle can be extended to a deformation.*

Proposition 8.3.5 *If two deformations (ν_t, α_t) and (ν'_t, α'_t) agree up to order $n-1$, i.e. $\alpha_i = \alpha'_i, \nu_i = \nu'_i$ for $i = 1, \dots, n-1$, then we have $\partial(\mu_n - \nu'_n, \alpha_n - \alpha'_n) = 0$ and there exists an equivalence up to order n if there exists an $S_n : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\partial(S_n, 0) = (\mu_n - \nu'_n, \alpha_n - \alpha'_n)$.*

So the construction of an equivalence is a problem in $\tilde{H}_{\text{CE}}^2(\mathfrak{g})$, and we get

Corollary 8.3.6 *If $\tilde{H}_{\text{CE}}^2(\mathfrak{g}) = 0$ then all deformations are trivial, i.e. are equivalent to the undeformed algebra.*

8.4 HOM-BIALGEBRAS

We briefly also discuss the deformation of Hom-bialgebras. We will only consider the case of (α, β) -Hom-bialgebras here.

Definition 8.4.1 A deformation of a Hom-bialgebra $(\mathfrak{g}, \mu, \Delta, \alpha, \beta)$ over \mathbb{K} is a Hom bialgebra $(\mathfrak{g}[[t]], \mu_t, \Delta_t, \alpha_t, \beta_t)$ over $\mathbb{K}[[t]]$ such that $\mu_0 = \mu, \delta_0 = \Delta$ and $\alpha_0 = \alpha$. Two deformations $(\mathfrak{g}[[t]], \mu_t, \Delta_t, \alpha_t, \beta_t)$ and $(\mathfrak{g}[[t]], \mu'_t, \Delta'_t, \alpha'_t, \beta'_t)$ are called equivalent if there exists a formal series $S : \mathfrak{g}[[t]] \rightarrow \mathfrak{g}[[t]]$ of the form $S = \text{id} + \sum_{i=1}^{\infty} S_i$, such that $S(\mu'_t(a, b)) = \mu_t(S(a), S(b))$, $(S \otimes S)\Delta'_t(a) = \Delta_t(S(a))$, $S(\alpha'_t) = \alpha_t(S)$ and $S(\beta'_t) = \beta_t(S)$.

The deformation equations with respect to algebra structure are the same as in Section 8.1. Dually the deformation of the coalgebra structure can be described. It remains to consider the compatibility conditions the compatibility of the structure maps with the coproduct and product are essentially the same as the multiplicativity of the (co)algebra structure, so it remains the condition relating the multiplication and comultiplication.

$$\sum_{i,j=0}^{\infty} t^{i+j} \Delta_i(\mu_j(a,b)) = \sum_{i,j,k,l} t^{i+j+k+l} \mu_i(a_k^{(1)}, b_l^{(1)}) \otimes \mu_j(a_k^{(2)}, b_l^{(2)})$$

Here we used $\Delta_i(a) = a_i^{(1)} \otimes a^{(2)i}$. The n -th order term of this equation can be written as

$$\partial_{\mu\mu} \Delta_n(a,b) + \partial_{\Delta\Delta} \Delta_n(a,b) = \sum_{i=1}^{n-1} \Delta_i(\mu_{n-i}(a,b)) - \sum_{\substack{i,j,k,l=0,\dots,n-1 \\ i+j+k+l=n}} \mu_i(a_k^{(1)}, b_l^{(1)}) \otimes \mu_j(a_k^{(2)}, b_l^{(2)}). \quad (18)$$

We recall that $\partial_{\mu\mu} \Delta_n(a,b) = \Delta_0(a)\Delta_n(b) - \Delta_n(ab) + \Delta_n(a)\Delta_0(b)$ and $\partial_{\Delta\Delta} \mu_n(a,b) = a^{(1)}b^{(1)} \otimes \mu_n(a^{(2)}, b^{(2)}) - \Delta(\mu_n(a,b)) + \mu_n(a^{(1)}, b^{(1)}) \otimes a^{(2)}b^{(2)}$.

Using this and the L_∞ structure defined in Chapter 7, one gets the theorems linking the GS cohomology and deformations.

We want to consider the special case of deforming the symmetric algebra over a vector space since in the ordinary case the first order term gives rise to a Lie bialgebra structure.

In the Hom-case however this does not work since the algebra even as Hom-modules are in general not isomorphic. For this consider the case $(V, \alpha = 0)$, where V is a vector space with dimension greater than 1. Then one can define the free commutative Hom-algebra $S(V, 0)$, which is given by the free Hom-associative algebra on V , which in this case is the free magmatic algebra, modulo the ideal generated by $xy - yx$ for $x, y \in \mathbb{F}_{\text{HAss}}(V, 0)$. One can also consider V to be a Hom-Lie algebra with zero bracket. Then however $U(V)$ is not commutative, since for example $(xy)z - z(xy) \neq 0$. Now one could consider the case whether the universal enveloping algebra of a Hom-Lie algebra $(\mathfrak{g}, \nu, \alpha)$ can be considered as a deformation of $U(V, 0, \alpha)$. Here of course one has to place the formal parameter t in front of ν to get a formal deformation.

In the case that α and β are invertible things are easier. Here one has a commutative diagram relating Yau-twist and deformation. We consider the general case. Let A, μ, Δ be a bialgebra with two commuting endomorphisms γ and γ' . Then the diagram

$$\begin{array}{ccc}
 (A, \mu, \Delta), \gamma, \gamma' & \xrightarrow{\gamma, \gamma'} & (A, \gamma\mu, \gamma, \Delta\gamma', \gamma') \\
 \mu_1, \gamma_1, \Delta_1, \gamma_1' \downarrow & & \downarrow \\
 (A, \mu_t, \Delta_t), \gamma_{1t} & \xrightarrow{\gamma_t, \gamma_t'} & (A, \gamma_t\mu_t, \gamma_t, \Delta\gamma_t', \gamma_t')
 \end{array}$$

Here $(\mu_1, \gamma_1, \Delta_1, \gamma_1')$ is on cocycle in the complex $C(\gamma)$ for an bialgebra morphism defined in Section 6.1. The 1-order term of the deformation of the Yau-twist is given by $\Phi(\mu_1, \Delta_1, \gamma_1, \gamma_1') = (\gamma_1\mu_0 + \gamma_0\mu_1, \gamma_0\alpha_1 + \gamma_1\alpha_0, \gamma_1\Delta_0 + \gamma_0\Delta_1, \beta_0\gamma_1' + \beta_1\gamma_0')$. If for a Hom-bialgebra both structure maps are invertible all deformations can be obtained this way.

8.5 HOM-LIE BIALGEBRAS

Here we consider the situation where $\alpha = \beta$ since we defined the cohomology only for this case. We do not have an L_∞ structure, which can be used to describe this cohomology. So the situation is not as good as in the previous sections. However we defined the differential in such a way that the deformation equation can be written with the differential.

Definition 8.5.1 A deformation of a Hom-Lie bialgebra $(\mathfrak{g}, \nu, \delta, \alpha)$ over \mathbb{K} is a Hom-Lie bialgebra $(\mathfrak{g}[[t]], \nu_t, \delta_t, \alpha_t)$ over $\mathbb{K}[[t]]$ such that $\nu_0 = \nu, \delta_0 = \delta$ and $\alpha_0 = \alpha$. Two deformations $(\mathfrak{g}[[t]], \nu_t, \delta_t, \alpha_t)$ and $(\mathfrak{g}[[t]], \nu_t', \delta_t', \alpha_t')$ are called equivalent if there exists a formal series $S : \mathfrak{g}[[t]] \rightarrow \mathfrak{g}[[t]]$ of the form $S = \text{id} + \sum_{i=1}^{\infty} S_i$, such that $S([a, b]_t) = [S(a), S(b)]_t$, $(S \otimes S)\delta_t'(a) = \delta_t(S(a))$ and $S(\alpha_t') = \alpha_t(S)$.

The deformation equation with respect to the Hom-Lie algebra structure is the same as in Section 8.3, also the compatibility of the structure maps with the multiplication is similar to there. The corresponding equations for the cobracket are dual to these. So we will consider here only the compatibility condition. Different from the Hom-bialgebra case it involves the structure maps. We have

$$\begin{aligned}
 \sum_{i,j=0}^{\infty} t^{i+j} \delta_i([a, b]_j) &= \sum_{i,j,k,l=0}^{\infty} t^{i+j+k+l} \left([\alpha_i(a), b_j^{[1]}]_k \otimes \alpha_l(b_j^{[2]}) + \alpha_l(b_j^{[1]}) \otimes [\alpha_i(a), b_j^{[2]}]_k \right. \\
 &\quad \left. - [\alpha_i(b), a_j^{[1]}]_k \otimes \alpha_l(a_j^{[2]}) + \alpha_l(b_j^{[1]}) \otimes [\alpha_i(b), a_j^{[2]}]_k \right)
 \end{aligned}$$

$$\begin{aligned}
 \partial_{\mu\mu}^c \nu_n(a, b) + \partial_{\mu\mu} \delta(a, b) + \partial_b \alpha(a, b) &= \sum_{i=1}^{n-1} \delta_i([a, b]_{n-i}) \\
 &- \sum_{\substack{0 \leq i, j, k, l < n \\ i+j+k+l=n}} [\alpha_i(a), b_j^{[1]}]_k \otimes \alpha_l(b_j^{[2]}) + \alpha_l(b_j^{[1]}) \otimes [\alpha_i(a), b_j^{[2]}]_k \\
 &- [\alpha_i(b), a_j^{[1]}]_k \otimes \alpha_l(a_j^{[2]}) + \alpha_l(b_j^{[1]}) \otimes [\alpha_i(b), a_j^{[2]}]_k
 \end{aligned}$$

Here $\partial_{\mu\mu}^c \nu_n(a, b) = a^{[1]} \wedge \nu_n(a^{[2]}, \alpha(b)) + b^{[1]} \wedge \nu_n(b^{[2]}, \alpha(a)) - \delta_0 \nu_n(a, b)$, $\partial_{\mu\mu} \delta_n(a, b) = a \cdot \delta_n(b) - b \cdot \delta_n(a)$ and $\text{del}_b(\alpha) = a \cdot \delta_0(\alpha_1(b)) - b \cdot \delta_0(\alpha_1(a)) + \alpha_0(a^{[1]}) \wedge \alpha_1([a^{[2]}, b]) - \alpha_0(b^{[1]}) \wedge \alpha_1([b^{[2]}, a])$.

Proposition 8.5.2 *The first order term is a 2-cocycle it is invariant under equivalence.*

Proof. This follows directly from the previous equations and the fact that a Lie-bialgebra deformation is clearly a Lie-algebra and coalgebra deformation. \square

8.6 QUANTIZATION OF HOM-LIE BIALGEBRAS

In Chapter 4 we studied the quantization of color Lie bialgebras, we here study the quantization of Hom-Lie bialgebras. We will focus on the regular case since in this case many results from the non-Hom-case can be transferred. The general case is more difficult since here for example the structure of the universal enveloping algebra is more complicated and less well understood.

Notation: For an algebra over $\mathbb{K}[[t]]$ we denote $\Delta = \sum_i \Delta_i$ and $a \cdot b = \sum_i a \cdot_i b$ and simple ab for $a \cdot_0 b$.

Let H be a Hom-Hopf algebra over $\mathbb{K}[[t]]$ such that $H \bmod t$ is isomorphic to the universal enveloping algebra of a Hom-Lie algebra. In this case we call H a HomQUEA.

Lemma 8.6.1 *Let H be the (α, β) -universal enveloping algebra of a Hom-Lie algebra, then $[\Delta(x), y \otimes z] = \text{ad}_{\beta(x)}(y \otimes z)$ for $x \in \mathfrak{g}$ and $y, z \in H$.*

Proof. This is a simple computation:

$$\begin{aligned}
 [\Delta(x), y \otimes z] &= [1 \otimes \beta(x) + \beta(x) + 1, y \otimes z] \\
 &= \alpha(y) \otimes [\beta(x), z] + [\beta(x), y] \otimes \alpha(z) \\
 &= \alpha(y) \otimes \text{ad}_{\beta(x)} z + \text{ad}_{\beta(x)}(y) \otimes \alpha(z)
 \end{aligned}$$

\square

8. Deformations

Proposition 8.6.2 *Let H be a HomQUEA, then if $\delta(x) = \Delta_1(x) - \Delta_1^{\text{opp}}(x) \pmod t$ defines a map from $\mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, δ defines a Hom-Lie cobracket on \mathfrak{g} , such that \mathfrak{g} becomes a Hom-Lie bialgebra. If α and β are invertible then the condition is automatic.*

Proof. The map δ is clearly skew-symmetric. We want to consider if δ maps into $\text{Prim}(H)$ for this we compute for $x \in \mathfrak{g}$

$$(\beta \otimes \Delta_0)\delta(x) = (\delta \otimes \beta)\Delta_0(x) + \tau_{12}(\beta \otimes \delta)\Delta_0(x) = \delta(\beta(x)) \otimes \mathbb{1} + \tau_{12}\delta(\beta(x)).$$

Here we used that δ is a Hom-Poisson cobracket and x is β -primitive. From this and the symmetry of δ it follows that if β is invertible then δ maps into β -Prim. If α is also invertible β -Prim = \mathfrak{g} and we get what we wanted.

The map δ defines a Hom-Poisson coalgebra. This follows from Proposition 8.1.7.

To show that it gives a Hom-Lie bialgebra we compute

$$\Delta([x, y]) - \Delta^{\text{opp}}([x, y]) = \Delta_0([x, y]_1) + \Delta(xy - yx) - \Delta_0([x, y]_1) - \Delta_1^{\text{opp}}(xy - yx) = \delta([x, y])$$

$$\begin{aligned} \Delta(x) \cdot \Delta(y) - \Delta^{\text{opp}}(x) \cdot \Delta^{\text{opp}}(y) &= \\ &= \Delta_0(x)\Delta_1(y) + \Delta_1(x)\Delta_0(y) - \Delta_0(x)\Delta_1^{\text{opp}}(y) + \Delta_1^{\text{opp}}(x)\Delta_0(y) \\ &= \Delta_0(x)\delta(y) + \delta(x)\Delta_0(y) \end{aligned}$$

Comparing the two sides and using Lemma 8.6.1 we get the result. \square

Theorem 8.6.3 *Let \mathfrak{g} be a Hom-Lie bialgebra with invertible structure map then there exists a quantization of \mathfrak{g} .*

Proof. This follows from the Etingof-Kazhdan quantization and its functoriality. By Proposition 5.2.7 \mathfrak{g} Yau-twisted by α^{-1} is on ordinary Lie bialgebra, which we denote by \mathfrak{g}_0 . Also the corresponding universal enveloping algebras are $\tilde{\alpha}$ related, where $\tilde{\alpha}$ is the extension of α to $U(\mathfrak{g})$. There exists a quantization of \mathfrak{g}_0 , which we denote by $U_q(\mathfrak{g}_0)$. Then $\tilde{\alpha}$ is still a morphism of $U_q(\mathfrak{g}_0)$ and the Yau twist is a quantization of \mathfrak{g} . \square

The case when α is not invertible is more difficult since in this case, the universal enveloping algebra even as vector space is not isomorphic to some universal enveloping algebra of a Lie-algebra even if the considered Hom-Lie algebra is of Lie-type.

A. BASIC CATEGORY THEORY

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WE first recall some basic definitions from ordinary category theory. We will limit ourselves to only giving definitions and some theorems but no proofs, for more details see a good book on category theory, e.g. [ML98].

A.1 BASISCS

Definition A.1.1 A category \mathcal{C} consists of a class of objects $\text{ob } \mathcal{C}$ and for two objects $A, B \in \text{ob } \mathcal{C}$ a set of morphisms denoted by $\text{Hom}_{\mathcal{C}}(A, B)$. Given two morphisms $f \in \text{Hom}(A, B), g \in \text{Hom}(B, C)$, there exists a morphism $gf \in \text{Hom}(A, C)$, such that the composition is associative. For all $A \in \mathcal{C}$ there exists an identity morphism $\text{id}_A \in \text{Hom}(A, A)$, which is a neutral element with respect to the composition.

We will normally only write $A \in \mathcal{C}$ instead of $A \in \text{ob } \mathcal{C}$, and Hom instead of $\text{Hom}_{\mathcal{C}}$ if the used category is clear. The composition gives a map $\text{Hom}(B, C) \times \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$. Note that $\text{ob } \mathcal{C}$ is in general not a set but a proper class. However we assume that $\text{Hom}(A, B)$ is a set for all $A, B \in \mathcal{C}$. There are some set theoretic problems surrounding this, since it is not clear what axioms a class should satisfy. So some authors use different approaches for examples using so called Grothendieck universes. We here try to avoid all these problems.

We want to give some examples of categories, also to fix the notation.

Example A.1.2 Given a set S , we can define the discrete category $\mathbb{F}_{\text{Cat}}(S)$ on S . It is given by $\text{ob } \mathbb{F}_{\text{Cat}}(S) = S$ and $\text{Hom}(x, x) := *$ and $\text{Hom}(x, y) = \emptyset$ if $x \neq y$. Here $*$ denotes a singleton.

There is the category **Set** of sets, where the objects are sets and the morphisms are maps between sets. This is somehow the most basic category.

Given a field \mathbb{K} there is the category $\mathbb{K}\text{-Vec}$ of \mathbb{K} vector spaces, where the objects are \mathbb{K} -vector spaces and the morphisms are linear maps. Similarly given a ring R one can define the category $R\text{-Mod}$ of R -modules.

Given some type of algebras or group one can also define the category of these with morphisms the maps which respect the structure.

We call a category small if the objects form a set. So we can form the category of small categories, where the objects are small categories and the morphisms are functors between them.

There is no category of all categories since the “set” of objects would be too big.

Given two categories \mathcal{C} and \mathcal{D} we can form their product $\mathcal{C} \times \mathcal{D}$, with objects all pairs (C, D) with $C \in \mathcal{C}$ and $D \in \mathcal{D}$ and morphisms $\text{Hom}((C, D), (C', D')) = \text{Hom}(C, C') \times \text{Hom}(D, D')$.

Given a category \mathcal{C} one can define its opposite category \mathcal{C}^{opp} , with the same objects but reversed morphisms, i.e. $\text{Hom}_{\mathcal{C}^{\text{opp}}}(A, B) := \text{Hom}_{\mathcal{C}}(B, A)$ for all $A, B \in \mathcal{C}$.

Definition A.1.3 (Mono, Epi, ...) A morphism $f : A \rightarrow B$ is called a monomorphism or monic if for all $C \in \mathcal{C}$ and $g, g' \in \text{Hom}(C, A)$ from $fg = fg'$ follows $g = g'$. It is called an epimorphism or epic if $gf = g'f$ implies $g = g'$ for all $g, g' \in \text{Hom}(B, C)$. It is called an isomorphism if there exists an inverse $g : B \rightarrow A$, such that $fg =$

id_B and $gf = \text{id}_A$. If f, g satisfy $fg = \text{id}$ then f is an epimorphism and called a split epimorphism or a retraction of g and g is a monomorphism and called a split monomorphism or a section of f .

Definition A.1.4 (Functor) A functor F between two categories \mathcal{C} and \mathcal{D} consists of a map $F : \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$, and a map $F : \text{Hom}(A, B) \rightarrow \text{Hom}(FA, FB)$ for all $A, B \in \mathcal{C}$, such that $F(gf) = F(g)F(f)$ for all $f \in \text{Hom}(A, B), g \in \text{Hom}(B, C)$.

We denote the set of all functors between two categories \mathcal{C} and \mathcal{D} by $\mathcal{D}^{\mathcal{C}}$ or $\text{Hom}(\mathcal{C}, \mathcal{D})$.

Functors can be composed. So given functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, one can define a functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ by $A \mapsto G(F(A))$ for $A \in \mathcal{C}$ on objects, and in a similar way on morphisms. Note that the composition of functors is associative.

Definition A.1.5 A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called faithful, if for all $C, D \in \mathcal{C}$ the map $F : \text{Hom}(C, D) \rightarrow \text{Hom}(FC, FD)$ is injective, it is called full if this map is surjective. It is called fully faithful if it is full and faithful. It is called conservative if for all $f : C \rightarrow D$ the morphism f is an isomorphism whenever $F(f)$ is an isomorphism.

Note that if f is an isomorphism then $F(f)$ is always an isomorphism.

The most important example of a functor is maybe the Hom functor. It is a functor $\mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathbf{Set}$ given on objects by $(A, B) \mapsto \text{Hom}(A, B)$ and on morphisms by $\text{Hom}(f, f') : g \mapsto f'gf$ for $f \in \text{Hom}(A', A), f' \in \text{Hom}(B, B')$ and $g \in \text{Hom}(A, B)$. Also for all $A \in \mathcal{C}$ there is a functor $\text{Hom}(A, \cdot) : \mathcal{C} \rightarrow \mathbf{Set} : B \mapsto \text{Hom}(A, B)$ and one $\text{Hom}(\cdot, B) : \mathcal{C}^{\text{opp}} \rightarrow \mathbf{Set} : B \mapsto \text{Hom}(B, A)$.

There is the identity functor $\text{id}_{\mathcal{C}}$ on \mathcal{C} , which simply is the identity on objects and on morphisms.

Given two categories \mathcal{C} and \mathcal{D} and an object $A \in \mathcal{D}$ we define the constant functor $\mathcal{C} \rightarrow \mathcal{D}$ with value A . Each object in \mathcal{C} is mapped to the A and all morphisms are mapped to the identity id_A .

On the product of two categories one can define the flip functor $\tau_{\mathcal{C} \times \mathcal{D}} : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D} \times \mathcal{C}$, which simply interchanges the components in the pairs of objects and morphisms.

A functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ is called representable if there exists an element $X \in \mathcal{C}$, such that $\text{Hom}(X, C) \cong FC$ for all $C \in \mathcal{C}$ natural in C , this means there exists a natural isomorphism $\text{Hom}(X, \cdot) \rightarrow F$.

Definition A.1.6 (Natural transformation) A natural transformation $\eta : \mathcal{C} \begin{matrix} \xrightarrow{G} \\ \xrightarrow{F} \end{matrix} \mathcal{D}$ between two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ consists of a family $\{\eta_A\}_{A \in \mathcal{C}}$ of maps $\eta_A : \text{Hom}(FA, GA)$, such that the following diagram commutes for all $A, B \in \mathcal{C}$ and $f \in \text{Hom}(A, B)$:

$$\begin{array}{ccc}
 FA & \xrightarrow{Ff} & FB \\
 \eta_A \downarrow & & \downarrow \eta_B \\
 GA & \xrightarrow{Gf} & GB
 \end{array}$$

A natural transformation η is called a natural isomorphism if all η_A are isomorphism.

Given two functors F, G we denote the set of natural transformation between them by $\text{Nat}(F, G)$. For a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ we denote the natural transformation $F \rightarrow F$, which has as components the identity by id_F .

Given two constant functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ with values $A, B \in \mathcal{D}$ resp., we have $\text{Nat}(F, G) = \text{Hom}(A, B)$.

They are two different ways to compose natural transformations. Given natural transformation $\eta : \mathcal{C} \xrightarrow[F]{G} \mathcal{D}$ and $\zeta : \mathcal{D} \xrightarrow[F']{G'} \mathcal{E}$ one can define the horizontal composition $\eta\zeta : \mathcal{C} \xrightarrow[FF']{GG'} \mathcal{E}$ by $\zeta\eta_A : G'(\eta_A)\zeta_{FA}$.

Given natural transformations η as before and $\zeta : \mathcal{C} \xrightarrow[G]{H} \mathcal{D}$ their vertical composition $\eta \cdot \zeta : \mathcal{C} \xrightarrow[F]{H} \mathcal{D}$ is defined by $(\eta\zeta)_A := \zeta_A\eta_A$.

Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\varphi : G \rightarrow H : \mathcal{D} \rightarrow \mathcal{E}$, we can define φF by $(\varphi F)_A = \varphi_{FA}F$.

These two composition satisfy some compatibility conditions. These conditions are used to define bicategories, but we will not consider this here.

Definition A.1.7 (Adjoint functors) Two functors $F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$ are called adjoint if for all $A \in \mathcal{C}, B \in \mathcal{D}$ there exists an isomorphism

$$\text{Hom}(FA, B) \rightarrow \text{Hom}(A, GB) \tag{1}$$

natural in A and B .

This is equivalent to the existence of two natural transformation $\eta : \text{id}_{\mathcal{C}} \rightarrow GF$ and $\epsilon : FG \rightarrow \text{id}_{\mathcal{D}}$, which satisfy $G\epsilon \cdot \eta G = \text{id}_G$ and $\eta F \cdot F\epsilon = \text{id}_F$

An adjunction, such that $FG \cong \text{id}_{\mathcal{D}}$ and $GF \cong \text{id}_{\mathcal{C}}$, is called an equivalence.

Definition A.1.8 (Cone, Limit) Let J be a small and often finite category and $F \in \mathcal{C}^J$ a functor. A cone φ to F is a natural transformation from a constant functor φ_0 to F . It is called universal if for all other cones ψ there exists a $\psi' \in \text{Nat}(\psi_0, \varphi_0)$, such that $\psi = \varphi\psi'$. A cone consists of an objects $c := \varphi_0(i) \in \mathcal{C}$ for $i \in J$ and a collection of morphism $\varphi_i \in \text{Hom}(c, F(i))$ for all $i \in J$. Given a universal cone the element $\varphi(i)$

is called the limit of F . A limit is unique up to unique isomorphisms, this means given two limits there exists a unique isomorphism between them.

We define the diagonal functor $\Delta : \mathcal{C} \rightarrow \mathcal{C}^J$, which assigns to every objects $i \in J$ the constant functor. A limit can also be defined as a right adjoint to the diagonal functor.

Consider the category (2) with two elements and only identity morphisms then a limit $(2) \rightarrow \mathcal{C}$ is called a product. Similarly one can define products involving more objects.

The limit where J is the empty category is called a terminal object. From every object there is a exactly one morphism to the terminal object.

Example A.1.9 The product of two sets is there cartesian product, similar for more sets. The terminal objects is the singleton. The product in **Vec** is the direct product of vector spaces and the terminal object is the zero space.

An equalizers can be defined as a limit where $J = \cdot \rightrightarrows \cdot$.

A pullback is a limit where $J = \cdot \rightarrow \cdot \leftarrow \cdot$. This is usually denoted by a pullout square

$$\begin{array}{ccc} A \times_C B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array} .$$

Dually to limit one can define colimit.

Example A.1.10 The coproduct in **Set** is the disjoint union and the coproduct in **Vec** is the direct sum. Note that in **Vec** the product and coproduct agree for finitely many objects. In the category **Grp** of groups the coproduct is given by the free product.

Definition A.1.11 (complete category) A category is called complete if all small limits exist, and it is called cocomplete if all small colimits exists.

For example the categories **Set** and **Vec** are complete and cocomplete.

We call a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ continuous if it preserves all limits, which exist in \mathcal{C} . This means given the limit $\lim G$ of a functor $G : J \rightarrow \mathcal{C}$ in \mathcal{C} , we have that $\lim FG = F \lim G$. Dually we call a functor cocontinuous if it preserves all colimits.

Proposition A.1.12 Any right adjoint functor is continuous and any left adjoint is cocontinuous.

A. Basic Category Theory

Proposition A.1.13 (Yoneda lemma) *Given a functor $F : \mathcal{C} \rightarrow \mathbf{Set}$, and $X \in \mathcal{C}$, there exists an isomorphism*

$$F(X) \cong \text{Nat}(\text{Hom}(X, \cdot), T). \quad (2)$$

Proof. Given a natural transformation η we can define an element in FX by $\eta_X(\text{id}_X)$. On the other hand given a element $x \in FX$ we can define a natural transformation by $\eta_C(f) = Ff(x)$ for $f \in \text{Hom}(X, C)$ for all $C \in \mathcal{C}$. \square

We give the definition of end and coends, since these are needed in enriched category theory. For this we first have to define dinatural transformations.

A dinatural transformations α between two functors $F, G : \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{D}$, written as $\alpha : F \dashrightarrow G$, consists of a family $\{\alpha_A\}_{A \in \mathcal{C}}$ of morphism $\alpha_A : F(A, A) \rightarrow G(A, A)$, such that for all $A, B \in \mathcal{C}$ and $f : A \rightarrow B$ the following diagram commutes

$$\begin{array}{ccccc} F(B, A) & \xrightarrow{\text{Hom}(f, \text{id})} & F(A, A) & \xrightarrow{\alpha_A} & G(A, A) \\ \text{Hom}(\text{id}, f) \downarrow & & & & \downarrow \text{Hom}(\text{id}, f) \\ F(B, B) & \xrightarrow{\alpha_B} & G(B, B) & \xrightarrow{\text{Hom}(f, \text{id})} & G(A, B) \end{array}$$

Definition A.1.14 An end of a functor $S : \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow X$ is a universal dinatural transformation from a constant e to S , that is an end is a pair (e, ω) , where e is an object and $\omega : e \dashrightarrow S$ a dinatural transformation (or wedge), such that for every wedge $\beta : x \dashrightarrow S$ there is an unique morphisms $h : x \rightarrow e$ such that $\beta_a = \omega_a h$ for all $a \in \mathcal{C}$.

$$\begin{array}{ccccc} x & \xrightarrow{\beta_b} & S(b, b) & \xrightarrow{S(1, f)} & S(b, c) \\ & \searrow \beta_c & & \nearrow S(f, 1) & \\ & & & & \\ e & \xrightarrow{\omega_c} & S(c, c) & & \end{array}$$

$\downarrow h$ (vertical arrow from x to e)

We write

$$e = \int_c S(c, c) \quad (3)$$

for the end of S .

If the functor F is trivial in the first argument, an end of F is the same as an ordinary limit.

Dually one can define cowedges and coends.

Proposition A.1.15 *If X is small-complete and C is small, every functor $S : C^{\text{opp}} \times C \rightarrow X$ has an end in X .*

Actually we need the coend of a functor $S : C^{\text{opp}} \times C \rightarrow X$, which is defined dual to the end. This means a coend is a pair $(d, \zeta : S \twoheadrightarrow d)$, where d is an object in X and ζ is a dinatural transformation. For the object d we write

$$d = \int^c S(c, c) \tag{4}$$

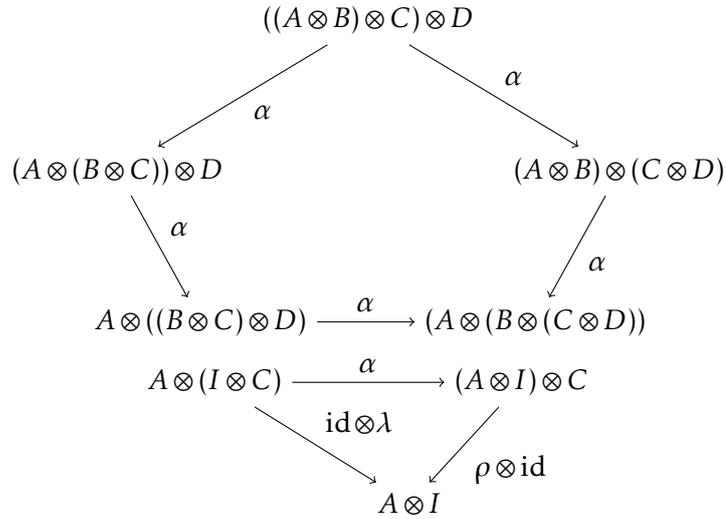
The properties of coends are dual to those of ends, so a coend always exists if C is small and X small cocomplete.

Proposition A.1.16 *Given a natural transformation γ between functors $S, S' : C^{\text{opp}} \times C \rightarrow X$, with ends (e, ω) and (e', ω') resp., there is a unique morphism $g = \int_c \gamma_{c,c}$ in X such that the following diagram commutes for every $c \in C$*

$$\begin{array}{ccc} \int_c S(c, c) & \xrightarrow{\omega_e} & S'(c, c) \\ \downarrow g & & \downarrow \gamma_{c,c'} \\ \int_c S(c, c) & \xrightarrow{\omega'_e} & S'(c, c) \end{array}$$

A.2 MONOIDAL CATEGORIES

Definition A.2.1 (Monoidal category) A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category equipped with a bifunctor, its multiplication or tensor product, $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. This multiplication is associative up to a natural isomorphism $\alpha : \otimes(\otimes \times \text{id}) \rightarrow \otimes(\text{id} \times \otimes)$, called associator, and has a unit object $I \in \mathcal{C}$, which is a left and right unit up to natural isomorphisms $\lambda : I \otimes \text{id} \rightarrow \text{id}$ resp. $\rho : \text{id} \otimes I \rightarrow \text{id}$, called unitors. To be precise this means that the following diagrams commute:

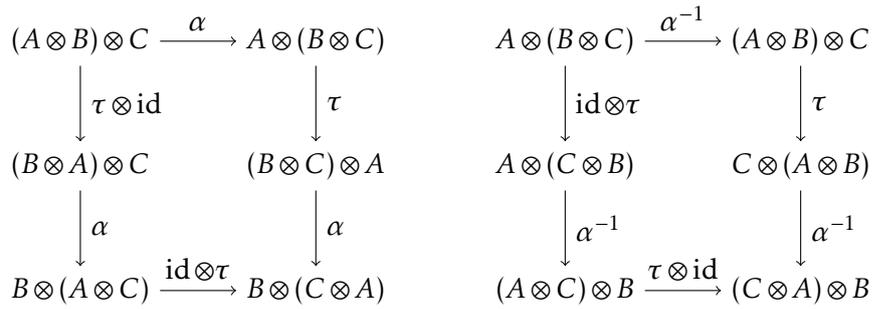
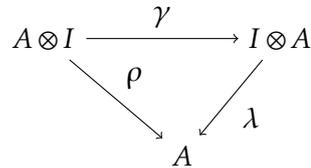


and $\lambda_I = \rho_I : I \otimes I \rightarrow I$.

A braiding for a monoidal category is a natural transformation

$$\gamma_{AB} : A \otimes B \rightarrow B \otimes A, \tag{5}$$

such that the following diagrams commute



The later two are called hexagon identities.

A braiding is called a symmetry if additionally

$$\gamma_{BA} \gamma_{AB} = \text{id}_{A \otimes B}. \tag{6}$$

In this case one hexagon identity implies the other.

A monoidal category \mathcal{C} is called closed if the functor $\cdot \otimes A$ has a right adjoint for each $A \in \mathcal{C}$, which we denote by $\text{HOM}(A, \cdot) : \mathcal{C} \rightarrow \mathcal{C}$ and call the internal homs, this means for all $A, B, C \in \mathcal{C}$ there is a natural isomorphism

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{HOM}(B, C)), \quad (7)$$

and also the unit and counit of the adjunction, i.e. natural isomorphisms

$$\begin{aligned} \eta : A &\rightarrow [B, A \otimes B] \\ \text{ev}_A : [A, B] \otimes A &\rightarrow B. \end{aligned}$$

Example A.2.2 The category **Set** is closed monoidal with the monoidal product given by the cartesian product and $\text{HOM}(A, B) = \text{Hom}(A, B)$. The category **Vec** is closed monoidal with the monoidal product given by the tensor product and the internal homs are the regular ones but regarded as a vector space.

Note that these examples are quite particular since the set $\text{Hom}(A, B)$ can be considered as an object in \mathcal{C} and as such is the internal homs, if this not the case it is not clear what the relation between the internal homs and the ordinary homs is. However we can define a functor $\mathcal{V} : \mathcal{C} \rightarrow \mathbf{Set}$ by $\mathcal{V} = \text{Hom}(I, -)$. Then we get a natural isomorphism

$$\text{Hom}(A, B) = \mathcal{V}[A, B]. \quad (8)$$

This is given by $\mathcal{V}\text{HOM}(A, B) = \text{Hom}(I, \text{HOM}(A, B)) \rightarrow \text{Hom}(I \otimes A, B) \rightarrow \text{Hom}(A, B)$.

For every closed monoidal category one has

$$X \cong \text{HOM}(I, X) \quad (9)$$

for all $X \in \mathcal{V}$, since $\text{Hom}(A, A) \cong \text{Hom}(A \otimes I, A) \cong \text{Hom}(A, \text{HOM}(I, A))$ and id_A under this isomorphism gives the desired isomorphism.

Definition A.2.3 A functor F between two monoidal categories \mathcal{C} and \mathcal{D} is called a monoidal functor if in addition there is given a morphism $F_0 \in \text{Hom}(I_{\mathcal{D}}, F(I_{\mathcal{C}}))$ and a natural transformation $F_2 : \otimes_{\mathcal{D}}(F \times F) \rightarrow F \otimes_{\mathcal{C}}$ such that the following diagrams commute

$$\begin{array}{ccccc} (FA \otimes FB) \otimes FC & \xrightarrow{F_2 \otimes \text{id}} & F(A \otimes B) \otimes FC & \xrightarrow{F(\alpha)} & F((A \otimes B) \otimes C) \\ \downarrow \alpha & & & & \downarrow F(\alpha) \\ FA \otimes (FB \otimes FC) & \xrightarrow{\text{id} \otimes F_2} & F(A) \otimes F(B \otimes C) & \xrightarrow{F_2} & F(A \otimes (B \otimes C)) \end{array}$$

$$\begin{array}{ccc}
 FA \otimes I_{\mathcal{D}} & \xrightarrow{\rho} & F(A) \\
 \downarrow \text{id} \otimes F_0 & & \uparrow F(\rho) \\
 FA \otimes F(I_{\mathcal{C}}) & \xrightarrow{F_2} & F(A \otimes I_{\mathcal{C}})
 \end{array}$$

and a similar one for λ .

If \mathcal{C} and \mathcal{D} are also braided F is called a braided monoidal functor if in addition $F_2\gamma = F(\gamma)F_2$.

A monoidal functor is called strong monoidal if F_0 is an isomorphism and F_2 a natural isomorphism. What we call monoidal functor is also called lax monoidal functor. There is also the concept of a colax monoidal functor, where $F_0 \in \text{Hom}(F(I_{\mathcal{C}}), I_{\mathcal{D}})$ and $F_2 : F \otimes_{\mathcal{C}} \rightarrow \otimes_{\mathcal{D}}(F \times F)$ and the corresponding arrows in the diagrams above reversed.

The composition of two braided monoidal functors F, G is again braided monoidal, with $(GF)_0 = G(F_0)G_0$ and $(GF)_2 = G(F_2)G_2(F \times F)$.

B. ENRICHED CATEGORY THEORY

CONTENTS

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WE study enriched categories in this chapter. This means categories, where the morphisms are no longer a set, but given by an object in a given closed monoidal category \mathcal{V} . This is why we call enriched categories also \mathcal{V} -categories. We will follow [Kel82] here.

B.1 \mathcal{V} -CATEGORIES, FUNCTORS AND NATURAL TRANSFORMATIONS

From now on let \mathcal{V} be a fixed closed symmetric monoidal category. For some constructions one does not need that \mathcal{V} is symmetric but we will assume this anyway. We will also assume that \mathcal{V} is complete and cocomplete since this simplifies things and this is the case for all examples we are interested in. In this section we define the “category” of \mathcal{V} -categories.

Definition B.1.1 (\mathcal{V} -category) A \mathcal{V} -category \mathcal{C} consists of

- a set (or more generally a class) of objects $\text{ob } \mathcal{C}$,
- for all objects A, B a \mathcal{V} -object $\mathcal{C}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$ of morphisms,
- for all objects A, B, C a composition morphisms in \mathcal{V} : $c_{ABC} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$,
- for every object an identity $\iota_A : I \rightarrow \mathcal{C}(A, A)$.

This must satisfy the associativity

$$\begin{array}{ccc}
 \mathcal{C}(C, D) \otimes (\mathcal{C}(B, C) \otimes \mathcal{C}(A, B)) & \xrightarrow{\quad} & (\mathcal{C}(C, D) \otimes \mathcal{C}(B, C)) \otimes \mathcal{C}(A, B) \\
 \downarrow \text{id} \otimes c_{ABC} & & \uparrow c_{BCD} \otimes \text{id} \\
 \mathcal{C}(C, D) \otimes \mathcal{C}(A, C) & \xrightarrow{c_{ACD}} & \mathcal{C}(A, D) \xleftarrow{c_{ABD}} \mathcal{C}(C, B) \otimes \mathcal{C}(A, B)
 \end{array}$$

and the identity

$$\begin{array}{ccc}
 \mathcal{C}(A, B) & \xrightarrow{\quad} & \mathcal{C}(A, B) \otimes I \\
 \swarrow c_{AAB} & & \downarrow \text{id} \otimes \iota_A \\
 & & \mathcal{C}(A, B) \otimes \mathcal{C}(A, AC)
 \end{array}$$

For example a **Set** enriched category is precisely an ordinary category, since we always assume that all Hom objects are sets.

There exists a tensor product of two \mathcal{V} -categories. For two \mathcal{V} -categories \mathcal{C} and \mathcal{D} it has as objects pairs (C, D) with $C \in \mathcal{C}$ and $D \in \mathcal{D}$ and the morphisms are given by $\text{Hom}((C, D), (C', D')) = \text{Hom}(C, C') \otimes \text{Hom}(D, D')$. Also we define the opposite category by \mathcal{C}^{opp} .

Definition B.1.2 Let \mathcal{A}, \mathcal{B} be two \mathcal{V} -categories then a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is given by

- a map from $\text{ob } A$ to $\text{ob } B$
 - for all objects $A, B \in \mathcal{A}$ a \mathcal{V} -morphisms $F_{A,B} : \mathcal{A}(A, B) \rightarrow \mathcal{B}(F(A), F(B))$
- This must satisfy the functoriality

$$\begin{array}{ccc}
 \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) & \xrightarrow{F_{BC} \otimes F_{AB}} & \mathcal{B}(F(B), F(C)) \otimes \mathcal{B}(F(A), F(B)) \\
 \downarrow c_{ABC} & & \downarrow c_{F(A)F(B)} \\
 \mathcal{A}(A, C) & \xrightarrow{F_{AC}} & \mathcal{A}(F(A), F(B))
 \end{array}$$

and respect the identity

$$\begin{array}{ccc}
 I & \xrightarrow{\iota} & \mathcal{A}(A, A) \\
 & \searrow \iota & \downarrow F \\
 & & \mathcal{B}(FA, FB)
 \end{array}$$

F is fully faithful if all F_{AB} are isomorphisms.

Definition B.1.3 (natural transformation) Let \mathcal{A}, \mathcal{B} be two \mathcal{V} -categories and $F, G : \mathcal{A} \rightarrow \mathcal{B}$ be two functors between them, then a \mathcal{V} -natural transformation α consists of a morphisms $\alpha_A : I \rightarrow \mathcal{B}(F(A), G(A))$ for each object $A \in \mathcal{A}_0$, which must satisfy the naturality condition

$$\begin{array}{ccccc}
 \mathcal{A}(A, B) & \xrightarrow{F_{AB}} & \mathcal{B}(F(A), F(B)) & \xrightarrow{\alpha_B \otimes \text{id} \circ \lambda} & \mathcal{B}(F(B), G(B)) \otimes \mathcal{B}(F(A), F(B)) \\
 \downarrow G_{AB} & & & & \downarrow c \\
 \mathcal{B}(G(A), G(B)) & \xrightarrow{\text{id} \otimes \alpha_A \circ \rho} & \mathcal{B}(G(A), G(B)) \otimes \mathcal{B}(F(A), G(A)) & \xrightarrow{c} & \mathcal{B}(F(A), G(B))
 \end{array}$$

The vertical composite of two natural transformation $\alpha : T \rightarrow S, \beta : S \rightarrow R$ can be defined by

$$(\beta \cdot \alpha)_A : I \cong I \otimes I \xrightarrow{\beta_A \otimes \alpha_A} \mathcal{B}(SA, RA) \otimes \mathcal{B}(TA, SA) \xrightarrow{M} \mathcal{B}(TA, RA) \quad (1)$$

B. Enriched Category Theory

The horizontal composition $\alpha\beta$ of two \mathcal{V} -natural transformations $\alpha : F \rightarrow F'$ and $\beta : G \rightarrow G'$ is defined by

$$I \xrightarrow{\lambda\beta_A} I \otimes \text{Hom}(GA, G'A) \xrightarrow{\alpha_{G'A} \otimes F} \text{Hom}(FG'A, F'G'A) \otimes \text{Hom}(FGA, FG'A) \xrightarrow{\zeta} \text{Hom}(FGA, F'G'A)$$

The concept of natural transformations can be extended to (extraordinary) natural transformations. Let \mathcal{C} and \mathcal{D} be \mathcal{V} -categories, $B \in \mathcal{D}$ and $T : \mathcal{C}^{\text{opp}} \otimes \mathcal{C} \rightarrow \mathcal{D}$. We call an family $\{\alpha_A\}_{A \in \mathcal{C}}$ of maps $\alpha_A : K \rightarrow T(A, A)$ an (extraordinary) \mathcal{V} -natural transformation if for all $A, B \in \mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} \text{Hom}(A, B) & \xrightarrow{T(A, \cdot)} & \text{Hom}(T(A, A), T(A, B)) \\ \downarrow T(\cdot, B) & & \downarrow \text{Hom}(\alpha_A, \text{id}) \\ \text{Hom}(T(B, B), T(A, B)) & \xrightarrow{\text{Hom}(\alpha_B, \text{id})} & \text{Hom}(K, T(A, B)) \end{array}$$

Similarly one can define \mathcal{V} natural transformations $T \rightarrow K$.

Using these and combining it with natural transformations, one can define natural transformations between functors $F : \mathcal{C} \otimes \mathcal{D}^{\text{opp}} \otimes \mathcal{D} \rightarrow \mathcal{F}$ and $G : \mathcal{C} \otimes \mathcal{E}^{\text{opp}} \otimes \mathcal{E} \rightarrow \mathcal{F}$ for \mathcal{V} -categories $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$.

Proposition B.1.4 *The category \mathcal{V} itself is a \mathcal{V} -category.*

Proof. For $\text{ob } \mathcal{V}$, we take the objects of \mathcal{V} . We set $\mathcal{V}(A, B) = \text{HOM}(A, B)$. The composition is given by the adjunction $\text{HOM}(B, C) \otimes \text{HOM}(A, B) \otimes A \xrightarrow{\text{id} \otimes \text{ev}_A} \text{HOM}(B, C) \otimes B \xrightarrow{\text{ev}_B} C$. The identity is given by the identity on A under the isomorphism

$$\text{Hom}(A, A) \cong \text{Hom}(I \otimes A, A) \cong \text{Hom}(I, \text{HOM}(A, A))$$

□

Since for a closed monoidal category there is the functor $\mathcal{U} : \mathcal{V} \rightarrow \mathbf{Set}$, we can for any \mathcal{V} -category \mathcal{C} define the underlying ordinary category \mathcal{C}_0 . The objects are the same and for the morphism we set $\text{Hom}_{\mathcal{C}_0}(A, B) = \mathcal{U}\mathcal{C}(A, B)$.

We will be in particular be interested in the case $\mathcal{V} = \mathbf{Vec}$. In this case a \mathbf{Vec} -category is close to an ordinary category.

A \mathcal{V} -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between two \mathcal{V} -categories \mathcal{C} and \mathcal{D} is called left adjoint the the functor $G : \mathcal{D} \rightarrow \mathcal{C}$ if there exists a \mathcal{V} -natural transformation

$$\text{Hom}(FD, C) \cong \text{Hom}(D, GC). \quad (2)$$

By the Yoneda lemma this is equivalent to two natural transformations $\eta : \text{id} \rightarrow FG$, called unit, and $\epsilon : GF \rightarrow \text{id}$, called counit, which satisfy $(G\epsilon) \cdot (\eta G) = \text{id}_G$ and $(\epsilon F) \cdot (F\eta) = \text{id}_F$.

The functor \mathcal{U} carries such a \mathcal{V} -adjunction to an ordinary adjunction between F_0 and G_0 .

When the functor $\mathcal{U} : \mathcal{V} \rightarrow \mathbf{Set}$ is conservative and the functor G_0 has a left adjoint then G has also a left adjoint.

In \mathcal{V} we have that the HOM functor and the tensor product are adjoint as \mathcal{V} -functors. This means there exists a \mathcal{V} -natural isomorphism

$$\text{HOM}(A \otimes B, C) \cong \text{HOM}(A, \text{HOM}(B, C)). \quad (3)$$

Given an ordinary category \mathcal{C} we can define its associated \mathcal{V} -category, which we denote by $\mathcal{C}_{\mathcal{V}}$. As objects we take those of \mathcal{C} , and for the morphisms we set $\text{Hom}_{\mathcal{C}_{\mathcal{V}}}(A, B) = \coprod_{f \in \text{Hom}_{\mathcal{C}}(A, B)} I$, where I is the unit object in \mathcal{V} . By the universal property of the coproduct the composition in \mathcal{C} induces a composition in $\mathcal{C}_{\mathcal{V}}$.

Given a set S one can define the discrete \mathcal{V} -category as the \mathcal{V} -category associated to the discrete category on S , this is $\text{Hom}(x, y) = *$ for $w \neq y \in S$, where $*$ is the terminal object in \mathcal{V} and $\text{Hom}(x, x) = I$ for $x \in S$, where I is the unit object in \mathcal{V} .

B.2 ENRICHED FUNCTOR CATEGORIES

We want to extend the notion of ends and coends to the enriched setting. For now we only consider the case of functors with values in \mathcal{V} . Let \mathcal{C} be a \mathcal{V} -category and $F : \mathcal{C}^{\text{opp}} \otimes \mathcal{C} \rightarrow \mathcal{V}$ a \mathcal{V} -functor.

Definition B.2.1 An end X in \mathcal{V} of F is a universal \mathcal{V} -natural transformation $\lambda : X \rightarrow T(A, A)$, such that for every \mathcal{V} -natural $\alpha_A : Y \rightarrow F(A, A)$ there exists a unique morphism $f : Y \rightarrow X$ such that $\alpha_A = \lambda_A f$. We denote X by $\int_{C \in \mathcal{C}} F(C, C)$.

The end of F is clearly unique up to a unique isomorphism if it exists. Since we assume \mathcal{V} to be complete it always exists if \mathcal{C} is small.

The functor category between two \mathcal{V} -categories \mathcal{A}, \mathcal{B} is defined by

$$[\mathcal{A}, \mathcal{B}](F, G) = \text{Hom}(F, G) = \int_a \text{Hom}(F(a), G(a)) \quad (4)$$

for two functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$.

This does not need to exist in general for all F, G but it does if \mathcal{V} is small and (co)complete.

Proposition B.2.2 *The functor category between two \mathcal{V} -categories is again a \mathcal{V} -category.*

Proposition B.2.3 (Yoneda lemma) *Let \mathcal{C} be a \mathcal{V} -category and $F : \mathcal{C} \rightarrow \mathcal{V}$ be a \mathcal{V} -functor then there is an isomorphism*

$$FK \cong [\mathcal{C}, \mathcal{V}](\text{Hom}(K, \cdot), F), \quad (5)$$

which is \mathcal{V} -natural in K and F .

B.3 CATEGORIES OVER A BASE

Given a symmetric monoidal category \mathcal{V} , we define a category over \mathcal{V} . This can for example be found in [Fre09]. This is a category \mathcal{C} equipped with an external tensor product, this is a functor $\otimes : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$, which satisfies

$$I \otimes C \cong C \text{ for all } C \in \mathcal{C} \quad (6)$$

and the associativity

$$(X \otimes Y) \otimes C \cong X \otimes (Y \otimes C). \quad (7)$$

This isomorphisms must also satisfy some pentagon identity.

If \mathcal{C} is a monoidal category we also require that the internal and external tensor product are compatible and satisfy

$$X \otimes (C \otimes D) \cong (X \otimes C) \otimes D \cong C \otimes (X \otimes D) \quad (8)$$

for all $X \in \mathcal{V}, C, D \in \mathcal{C}$. Again this isomorphisms must satisfy some coherence.

Any cocomplete category can be regarded as a category over **Set** be defining the internal tensor product by

$$S \otimes C := \coprod_{s \in S} C. \quad (9)$$

If for all $C \in \mathcal{C}$ the functor $\cdot \otimes C : \mathcal{V} \rightarrow \mathcal{C}$ admits a right adjoint $\text{Hom}^{\mathcal{V}}(C, \cdot) : \mathcal{C} \rightarrow \mathcal{V}$ this means we have

$$\text{Hom}(X \otimes C, D) \cong \text{Hom}_{\mathcal{V}}(X, \text{HOM}(C, D)), \quad (10)$$

then \mathcal{C} is a \mathcal{V} enriched category with morphism object $\text{Hom}^{\mathcal{V}}(C, D)$.

On the other hand if we have a \mathcal{V} -category \mathcal{C} and the \mathcal{V} -functor $\text{HOM}(C, \cdot)$ admits a left adjoint $\cdot \otimes C$ for all $C \in \mathcal{C}$, then \otimes makes $\mathcal{V}\mathcal{C}$ into a category over \mathcal{V} .

B.4 INDEXED LIMITS

To define the notion of limits in enriched categories is a bit more complicated, we will not need this in too much details, but we want to give the basic definitions.

Let J be a \mathcal{V} -category, often small, than we call a functor $J \rightarrow \mathcal{V}$ an indexing type. We will call a functor $J \rightarrow \mathcal{C}$ a diagram of type F . For all $C \in \mathcal{C}$ we have a functor $\text{Hom}(C, G \cdot) : J \rightarrow \mathcal{V}$ and we assume that $[J, \mathcal{V}](F, \text{Hom}(C, G \cdot))$ exists.

Definition B.4.1 If there exists an object $\{F, G\}$ such that

$$\text{Hom}(C, \{F, G\}) \cong [J, \mathcal{V}](F, \text{Hom}(C, G)). \quad (11)$$

Then we call $\{F, G\}$ the limit of G indexed by F .

In the case $\mathcal{V} = \mathbf{Set}$ there is a simple relation between indexed limits and ordinary limits. Let $*$ be a one element set and Δ^* the constant functor $J \rightarrow \mathbf{Set}$ with values $*$. Then $\lim G = \{\Delta^*, G\}$. In this construction one can of course replace \mathbf{Set} by an arbitrary \mathcal{V} and $*$ by the unit in \mathcal{V} . On the other hand any indexed functor in the case $\mathcal{V} = \mathbf{Set}$ is equal to an ordinary functor. This is however not true for arbitrary \mathcal{V} .

We give a Fubini like theorem relating repeated limits to double limits.

Theorem B.4.2 Let $F : J \rightarrow \mathcal{V}$ be a indexing type and $G : J \rightarrow [\mathcal{C}, \mathcal{D}]$ a functor. We define a functor $P : J \otimes \mathcal{C} \rightarrow \mathcal{D}$ by $(i, C) \mapsto G(i)(C)$ for $i \in J, C \in \mathcal{C}$. Assume that the limit $\{F, G\}$ exists pointwise and let $H : \mathcal{C} \rightarrow \mathcal{V}$ be an indexing type then there exists a canonical isomorphism

$$\{H, \{F, G\}\} \cong \{H \otimes F, P\} \quad (12)$$

and each side exists if the other does.

We consider the case where the domain of the limit J is the unit \mathcal{V} -category $I_{\mathcal{V}}$. Then $G : I_{\mathcal{V}} \rightarrow \mathcal{V}$ is given by an object $X \in \mathcal{V}$ and $F : I_{\mathcal{V}} \rightarrow \mathcal{C}$ by an object $C \in \mathcal{C}$. In this case we call the limit $\{F, G\}$ the cotensor product of X and B and denote it by $X \pitchfork C$. It is defined by a \mathcal{V} -natural isomorphism

$$\text{Hom}(D, X \pitchfork C) \cong \text{HOM}(X, \text{Hom}(B, C)) \quad (13)$$

If the cotensor product exists for all $X \in \mathcal{V}$ and $C \in \mathcal{C}$, we say that \mathcal{C} is cotensored.

We have that \mathcal{V} itself is cotensored, with $X \pitchfork Y = \text{HOM}(X, Y)$.

Dually we can define the (external) tensor product $X \otimes C$ for $X \in \mathcal{V}$ and $C \in \mathcal{C}$ by

$$\text{Hom}(X \otimes C, D) \cong \text{HOM}(X, \text{Hom}(C, B)). \quad (14)$$

If all tensor products exist we call \mathcal{C} tensored. For example \mathcal{V} is tensored with the tensor product given by the monoidal product.

Using indexed limits it is also possible to define ends and coends in \mathcal{V} -categories. We take $\text{Hom} : \mathcal{C}^{\text{opp}} \otimes \mathcal{C} \rightarrow \mathcal{V}$ as indexing type. Then a diagram of this type is a functor $G : \mathcal{C}^{\text{opp}} \otimes \mathcal{C} \rightarrow \mathcal{C}$. We denote $\{\text{Hom}_{\mathcal{C}}, G\}$ by $\int_{C \in \mathcal{C}} G(C, C)$ if it exists and call it the end of G .

By definition the end is given by a natural isomorphism

$$\text{Hom}(D, \int_A G(C, C)) \cong \int_C \text{Hom}(D, G(C, C)), \quad (15)$$

where the end on the right is an end in \mathcal{V} and has been defined before.

Using Theorem B.4.2 we get

$$\int_C \int_D G(C, C, D, D) \cong \int_{(C, D)} G(C, C, D, D), \quad (16)$$

what looks like the classical Fubini theorem for integrals.

We note the following three propositions, which can be found in [Kel82], since we will need them in the main part.

Proposition B.4.3 *A natural transformation $\alpha : F \rightarrow G$ gives a natural transformation*

$$\int^a \alpha : \int^a F(a) \rightarrow \int^a G(a). \quad (17)$$

Proposition B.4.4 *Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and an object $a \in \mathcal{C}$ we have*

$$F(a) = \int^{b \in \mathcal{C}} F(b) \otimes \text{Hom}_{\mathcal{C}}(b, a) \quad (18)$$

Since the end is a limit and the Hom-functor is (co)continuous we also get:

Proposition B.4.5 *Given a functor $S : \mathcal{C} \otimes \mathcal{C}^{\text{opp}} \rightarrow \mathcal{D}$ and an object $b \in \mathcal{D}$ we have*

$$\begin{aligned} \text{Hom}\left(\int^a S(a, a), b\right) &= \int_a \text{Hom}(S(a, a), b) \\ \text{Hom}\left(b, \int_a S(a, a)\right) &= \int_a \text{Hom}(b, S(a, a)) \end{aligned}$$

B.5 ENRICHED MONOIDAL CATEGORIES

In this section we define enriched monoidal categories.

A \mathcal{V} -monoidal category \mathcal{C} consists of a \mathcal{V} -category \mathcal{C} , a \mathcal{V} -functor $\boxtimes : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$, a unit object $I_{\mathcal{C}} \in \mathcal{C}$ and \mathcal{V} -natural transformations α, λ, ρ . They have to satisfy the same diagrams as in the ordinary case. One has to take additional care if the category \mathcal{V} is not strict monoidal, since then also the associator of \mathcal{V} appears in the diagrams.

An closed monoidal category \mathcal{V} is also a \mathcal{V} -monoidal category.

Let \mathbb{K} be a field and $\mathcal{V} = \mathbb{K}\text{-Vec}$. Further let R be an algebra over \mathbb{K} then the category of R -modules is a \mathcal{V} -category. If R is a bialgebra $\mathbf{R}\text{-Mod}$ is a monoidal category and even a \mathcal{V} -monoidal category.

C. OPERADS AND PROPERADS

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WE recall some basic definitions about operads and properads and how they can be described by trees and graphs resp. For a more complete introduction see e.g. [LV12, Val07, MSS02, Mar08] We also give the definition of an operad with heights. Finally we give some remarks on operadic Kan-extension, which appear to be quite useful, but -to our knowledge- are not widely considered.

C.1 OPERADS AND PROPS

We will follow [Val07] for this short introduction.

Definition C.1.1 (*S*-bimodule) An *S*-bimodule \mathcal{P} is a collection $\mathcal{P}(m, n)$ of $S_m - S_n$ -bimodules. It is called reductive if $\mathcal{P}(m, n) = 0$ if $m = 0$ or $n = 0$.

When we consider dg modules we assume the differential to be of degree -1.

Given an augmentation $\epsilon : \mathcal{P} \rightarrow I$. We denote by $\bar{\mathcal{P}} := \ker(\epsilon)$ the augmentation ideal and we have $\mathcal{P} = \bar{\mathcal{P}} \oplus I$. Here I is defined by $I(1, 1) = \mathbb{K}$ and the zero space else.

We denote by $\text{Hom}(\mathcal{P}, \mathcal{Q})$ the set of linear maps and by $\text{Hom}^S(\mathcal{P}, \mathcal{Q})$ the set of invariant linear maps.

We denote the category of *S*-bimodules by $S - \mathbf{BiMod}$.

The composition is modeled on directed graphs. A direct graph is graphs with a global flow. This means every node has inputs and outputs and the outputs can only be connected to inputs. It is also not allowed to have loops. Also we assume the inputs and outputs of each vertex to be numbered as well as the global in- and outputs.

A two level graph is a graph such that it can be decomposed in two levels, such that in each level the graphs are not connected and the levels respect to flow. We write \mathcal{G}^2 for the set of two level graphs and for a given graph Γ we write \mathcal{N}_Γ^i for the set of vertices in level i .

Given two *S*-bimodules \mathcal{P}, \mathcal{Q} we define their composition product as follows

$$\mathcal{P} \boxtimes \mathcal{Q} = \bigoplus_{\Gamma \in \mathcal{G}^2} \bigotimes_{v \in \mathcal{N}_1} \mathcal{P}(|\text{In}(v)|, |\text{Out}(v)|) \otimes \bigotimes_{v \in \mathcal{N}_2} \mathcal{Q}(|\text{In}(v)|, |\text{Out}(v)|) / \sim. \quad (1)$$

Here \sim is the equivalence relation given by renumbering the in- and outputs of each vertex and applying the corresponding permutation to the element of the *S*-bimodule associated to it.

It can be shown that this product is associative.

We can restrict the direct sum in Eq. (1). We denoted this product by \boxtimes_c . Is is again associative.

We define a unit *S*-bimodule I by $I(1, 1) = \mathbb{K}$ and $I(m, n) = 0$ otherwise.

Proposition C.1.2 $(S - \mathbf{BiMod}, \boxtimes_c, I)$ is a monoidal category.

One can also define a concatenation product \otimes in this case one simply puts two graphs next to each other.

Definition C.1.3 A monoid in the category $(S - \mathbf{BiMod}, \boxtimes_c, I)$ is called a properad.

This means a properad an *S*-bimodule \mathcal{P} , with a multiplication $\mu : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ and a unit $\eta : I \rightarrow \mathcal{P}$, such that the following diagrams commute

$$\begin{array}{ccc}
 I \circ \mathcal{P} & \xrightarrow{\eta \circ \mathcal{P}} & \mathcal{P} \circ \mathcal{P} & \xleftarrow{\mathcal{P} \circ \eta} & \mathcal{P} \circ I \\
 & \searrow = & \downarrow \mu & \swarrow = & \\
 & & \mathcal{P} & &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 (\mathcal{P} \circ \mathcal{P}) \circ \mathcal{P} & \xrightarrow{\text{id} \circ \gamma} & \mathcal{P} \circ (\mathcal{P} \circ \mathcal{P}) & \xrightarrow{\gamma} & \mathcal{P} \circ \mathcal{P} \\
 \downarrow \gamma \circ \text{id} & & & & \downarrow \gamma \\
 \mathcal{P} \circ \mathcal{P} & \xrightarrow{\gamma} & & & \mathcal{P}
 \end{array}$$

One can consider $S - \mathbf{Mod}$ as a subcategory of $S - \mathbf{BiMod}$, where $\mathcal{P}(m, n) = \{0\}$ for $m \geq 2$. In this case the composition product can be modeled on rooted trees.

An monoid in $S - \mathbf{Mod}, \boxtimes, I$ is called an operad. So in particular every operad is a properad.

Some well know examples are the operad of associative algebras Ass or Lie-algebra Lie . We will also often use the operad EAss describing an algebra and an endomorphisms. It can be defined as the free operad generated by $\mu \in \text{EAss}(2)$ and $\gamma \in \text{EAss}(1)$ corresponding to the multiplication and endomorphisms resp. modulo the ideal generated by $\mu(\gamma, \gamma) - \gamma(\mu)$ and $\mu(\text{id}, \mu) - \mu(\mu, \text{id})$.

For operads one can define the partial composition. For $v_1 \in \mathcal{P}(m)$ and $v_2 \in \mathcal{P}(n)$ we define $v_1 \circ_i v_2 \in \mathcal{P}(m+n-1)$ by the composition corresponding to connecting the output of v_2 with the i -th input of v_1 .

One can define morphism of properads and operads in the obvious way.

To every vector space one can associate its endomorphisms properad $\text{End}(V)$ it is given by $\text{End}(V)(m, n) = \text{Hom}(V^{\otimes m}, V^{\otimes n})$ with the S_n action given by permutation of the tensor factors and the composition product by the composition and tensor product of linear maps. Similarly one can define the endomorphisms operad, which we also call $\text{End}(V)$.

An algebra over an properad \mathcal{P} is given by an properad morphism $\mathcal{P} \rightarrow \text{End}(V)$ and similar for operads. We denote the category of \mathcal{P} algebras by $\mathcal{P} - \mathbf{Alg}$.

Given a morphism of properads $\mathcal{P} \rightarrow \mathcal{Q}$ there exists a functor $\mathcal{Q} - \mathbf{Alg} \rightarrow \mathcal{P} - \mathbf{Alg}$.

The free properad over an S -module \mathcal{P} can be constructed by using directed graphs and associate to every node an element of \mathcal{P} with the corresponding inputs and outputs. As space it is given by

$$\mathbb{F}(\mathcal{P}) = \bigoplus_{\Gamma \in \mathcal{G}} \bigotimes_{v \in \Gamma} \mathcal{P}(|\text{Out}(v)|, |\text{In}(v)|). \quad (2)$$

The composition is given by the composition of the corresponding graphs.

Similarly the free operad can be constructed by using rooted trees.

We note that for a operad \mathcal{P} one can define the free algebra on a vector space V by $\mathcal{P}(V) = \bigoplus_{n \in \mathbb{N}} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$.

C.2 HEIGHT

We define properads and operads with height. For this we first consider direct graphs. Let Γ be a direct graph, such that additionally the every vertices one associates a height, i.e. a natural number or more general an element in a commutative monoid. Then given a path from an input i to an output o and can calculate the height of this path by adding up the height of the vertices it passes. If there only is one path from i to o or there are several path and the all give the same result we will denote this by $\text{hgt}(i, \Gamma, o)$. If the height for all paths connecting in and outputs is the same we call this number $\text{hgt}\Gamma$ and call Γ homogeneous.

This is particularly useful in the setting of rooted trees since there all inputs are connected to the root by a unique path.

Definition C.2.1 An operad with heights is an operad \mathcal{P} such that any output of any element $v \in \mathcal{P}$ has a height $\text{hgt}(i, v)$ and the composition respects this height, so $\text{hgt}(i + j - 1, v \circ_i v') = \text{hgt}(i, v) + \text{hgt}(j, v')$.

For example given the free operad on a S -module \mathcal{P} , where each element of \mathcal{P} has a height, is an operad with heights. The height is simply given by the height of the associate graph. If an operad can be obtained as the quotient of $\mathbb{F}(\mathcal{P})$ by an ideal generated by homogeneous elements it is also an operad with heights.

An example is the operad describing on associative algebra and a morphism of it. Here one can associate to μ the height zero and to γ the height 1.

Similar to the above, given a set of morphism one can also associate heights to them and then any composition of them also has an associated height this will be used in the case of Hom-algebras. For example given an operad \mathcal{P} with height and an algebra A over \mathcal{P} , then one can associate a height to the corresponding operations in A .

C.3 OPERADIC KAN EXTENSIONS

In this section, we briefly discuss operadic Kan-extension. We mention the following theorem [Hor17, Prop. 1.15]:

Theorem C.3.1 Let \mathcal{P} and \mathcal{Q} be two operads and $\Phi : \mathcal{P} \rightarrow \mathcal{Q}$ a morphism between them, then there exists a left adjoint functor $\mathcal{P} - \mathbf{Alg} \rightarrow \mathcal{Q} - \mathbf{Alg}$ to the functor $\Phi^* : \mathcal{Q} - \mathbf{Alg} \rightarrow \mathcal{P} - \mathbf{Alg}$, i.e. there is a natural isomorphism

$$\text{Hom}_{\mathcal{P}}(P, \Phi^* Q) \cong \text{Hom}_{\mathcal{Q}}(\Phi_*(P), Q) \quad (3)$$

for $P \in \mathcal{P} - \mathbf{Alg}, Q \in \mathcal{Q} - \mathbf{Alg}$.

This left adjoint for an \mathcal{P} -algebra P can be calculated as the free \mathcal{Q} algebra, modulo the ideal generated by $\nu - \Phi(\nu)$ for an n -ary operation $\nu \in \mathcal{P}$. Where ν is considered as a map $P \otimes P \rightarrow P \rightarrow \mathbb{F}_{\mathcal{Q}}(P)$ and $\Phi(\nu)$ as a map $P \otimes P \rightarrow \mathbb{F}_{\mathcal{Q}}(P)$.

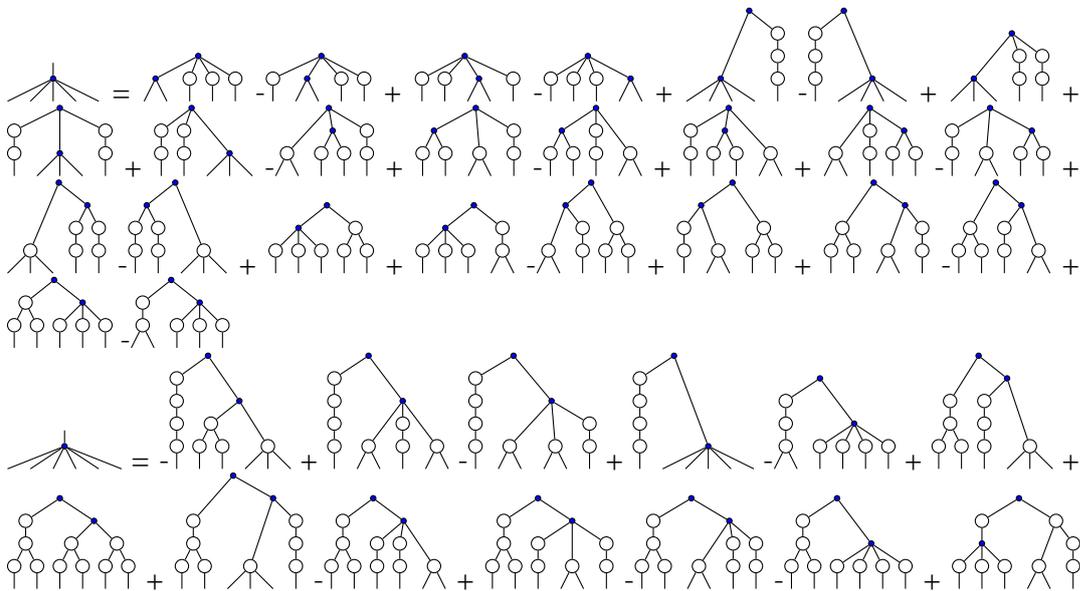
As an example the universal enveloping algebra of a Lie algebra can be constructed this way by considering the operad morphism $\text{Lie} \rightarrow \text{Ass}$.

D. DIFFERENTIALS FOR GRAPH COMPLEXES

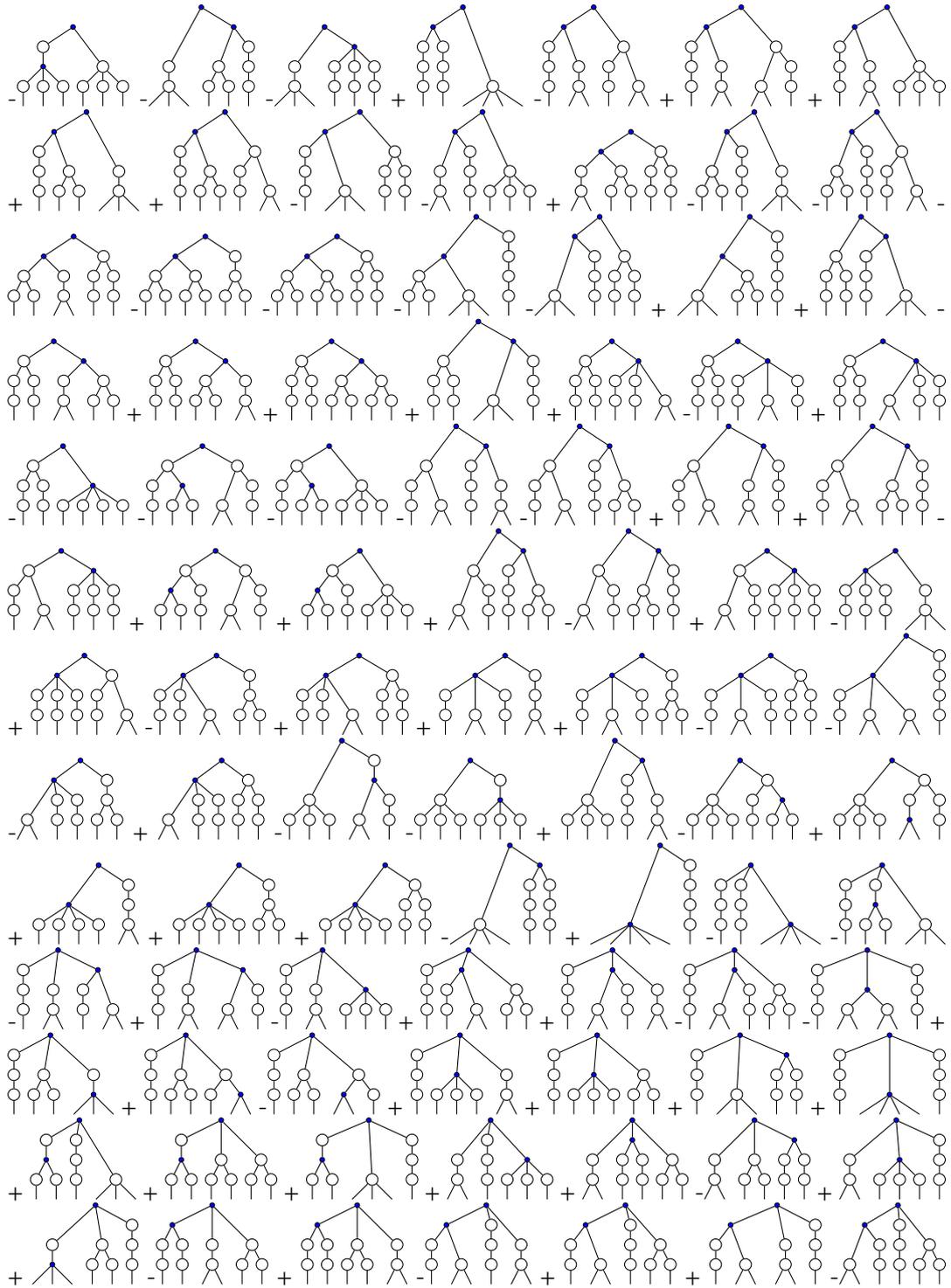
Here we show the differentials for the operads and properads for the different types of Hom-algebras and bialgebras we considered. These are computer generated. We note that for Hom-Lie algebras we give all terms obtained by total antisymmetrization and the small numbers below the leafs give the permutation. In the case of Hom-Lie bialgebras, one has to take to the total antisymmetrization of the given terms but only keep the terms which are different.

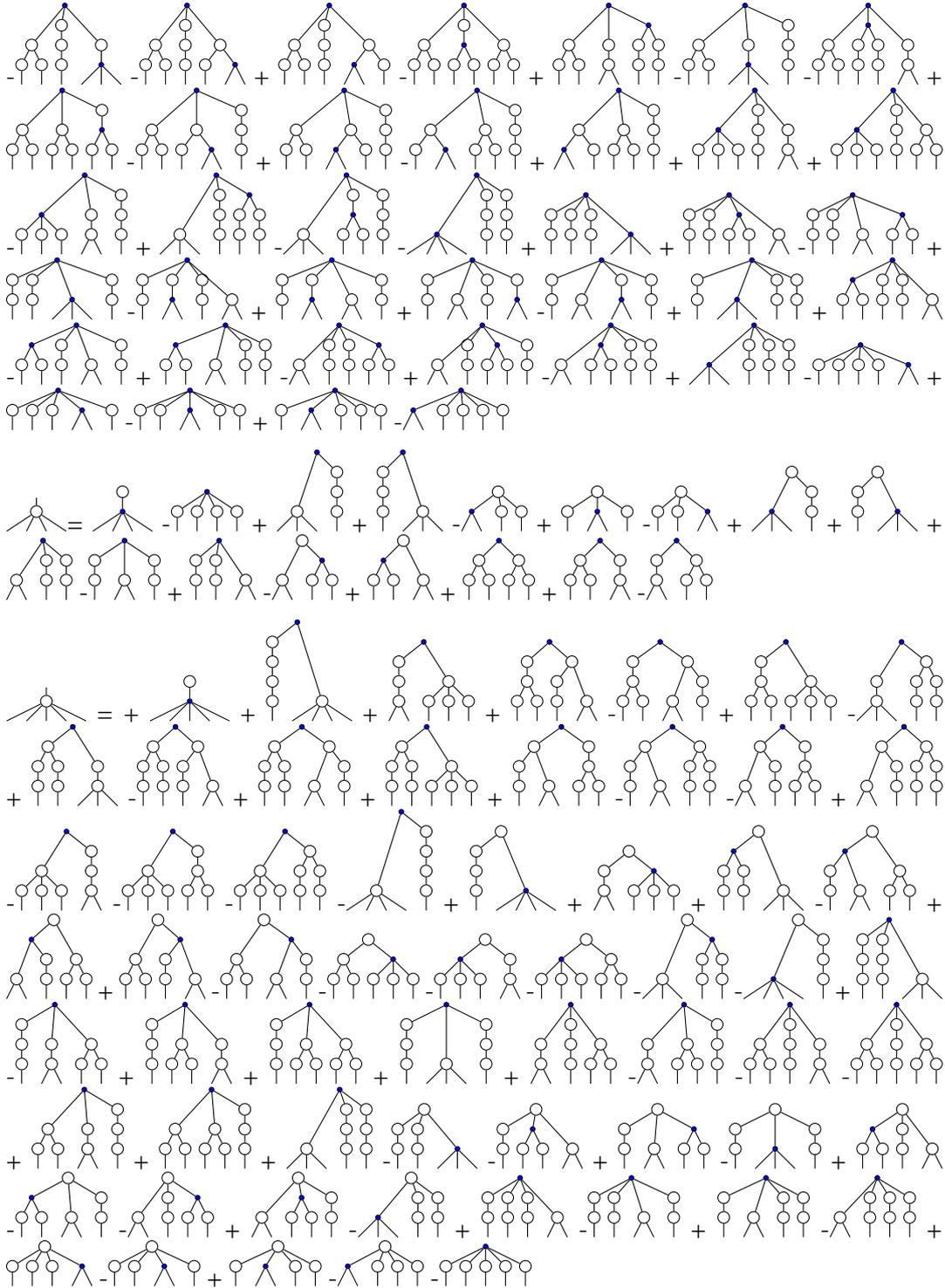
Since the vertices are graded the order of them is important, since reordering changes the sign. In the case of operads the root is the first vertex. Then the vertices connected to the root of numbered from left to right. This is the first row. Then the vertices in the second row, i.e. the ones connected to the first row, are numbered from left to right and so on. In the case of properads the ordering is given in a similar way. The graph is split into levels beginning with the inputs and each node is put as low as possible. Then in each levels the nodes are numbered from left to right.

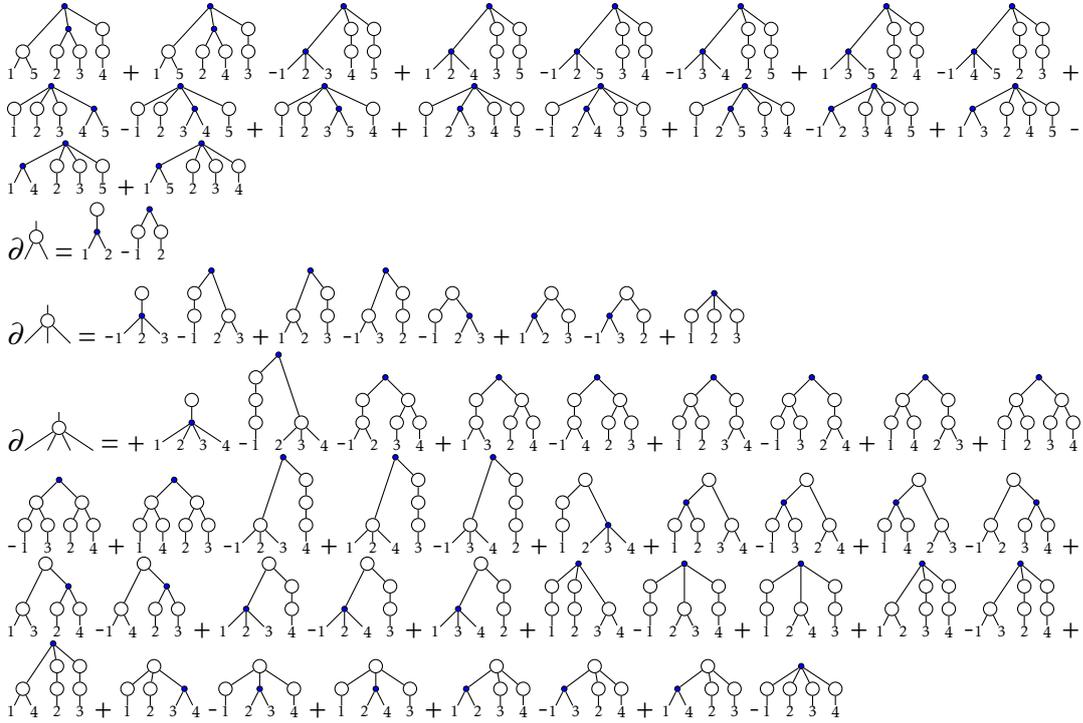
D.1 HOM-ASSOCIATIVE ALGEBRAS



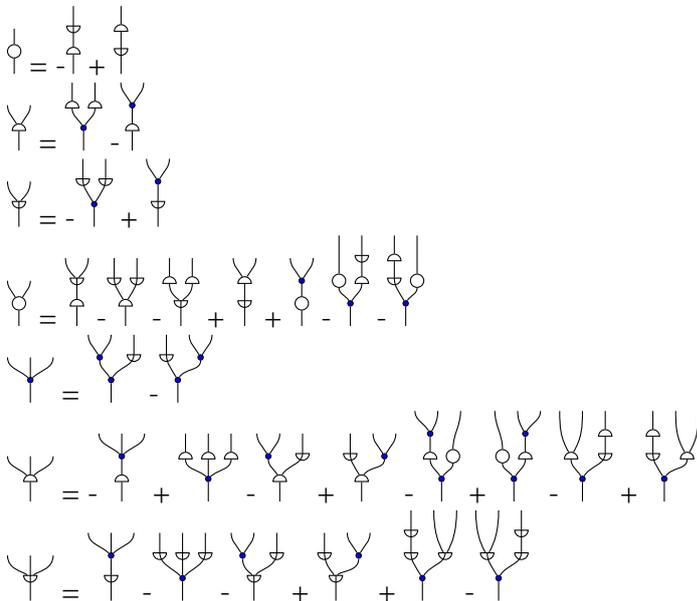
D. Differentials for graph complexes



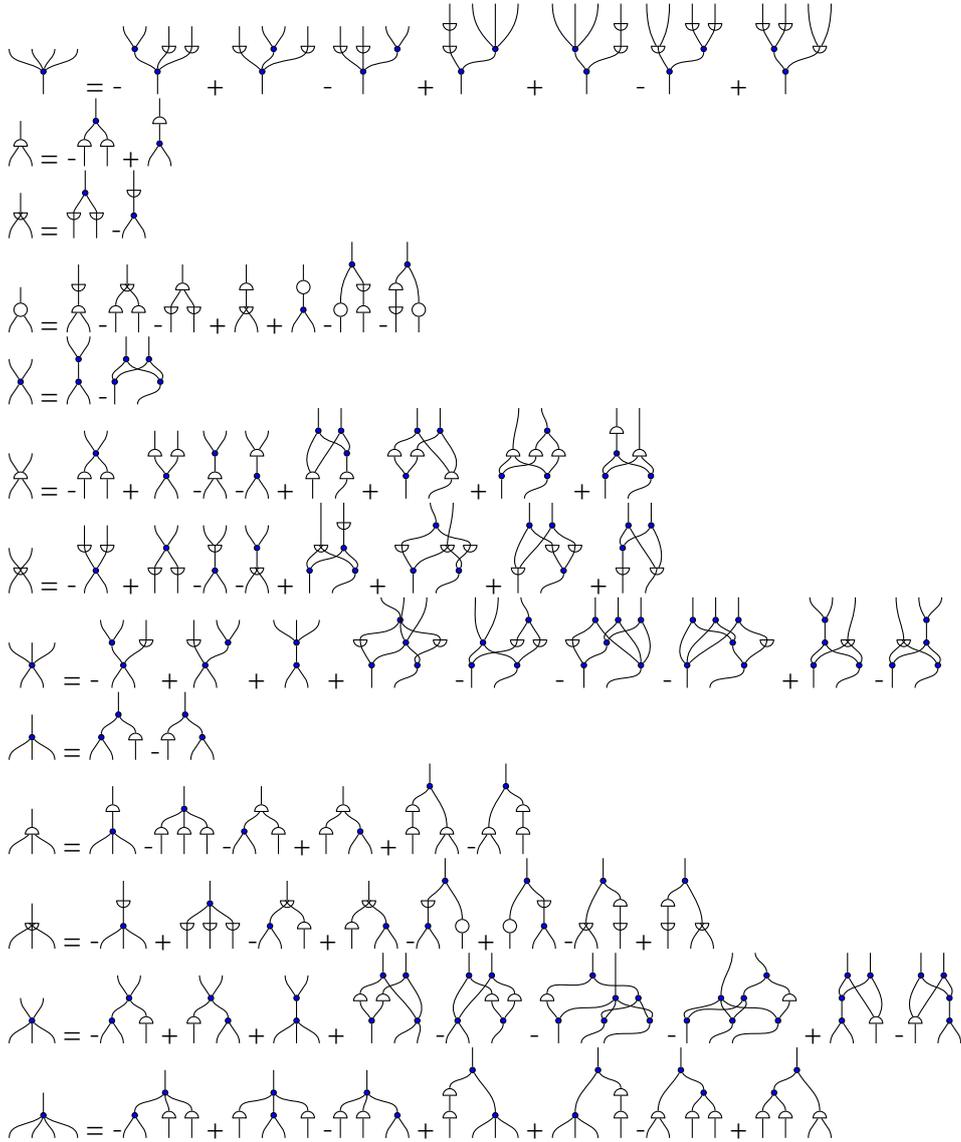




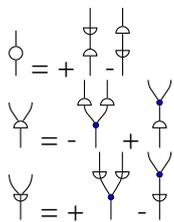
D.3 HOM-ASSOCIATIVE BIALGEBRAS



D. Differentials for graph complexes



D.4 HOM-LIE BIALGEBRAS



$$\begin{aligned}
 & \text{Diagram} = -2 \text{Diagram} + 2 \text{Diagram} + 2 \text{Diagram} - 2 \text{Diagram} - 2 \text{Diagram} + \text{Diagram} + \text{Diagram} \\
 & \text{Diagram} = - \text{Diagram} \\
 & \text{Diagram} = - \text{Diagram} + \text{Diagram} - \text{Diagram} - \text{Diagram} - \text{Diagram} \\
 & \text{Diagram} = - \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} \\
 & \text{Diagram} = -6 \text{Diagram} - 2 \text{Diagram} + 3 \text{Diagram} - \text{Diagram} - 6 \text{Diagram} - \text{Diagram} + 3 \text{Diagram} - 2 \text{Diagram} - 6 \text{Diagram} + 6 \text{Diagram} - 3 \text{Diagram} \\
 & -2 \text{Diagram} - 6 \text{Diagram} - 6 \text{Diagram} + 6 \text{Diagram} + 6 \text{Diagram} + 3 \text{Diagram} - 3 \text{Diagram} + 6 \text{Diagram} - 6 \text{Diagram} + 6 \text{Diagram} \\
 & \text{Diagram} = - \text{Diagram} + \text{Diagram} - \text{Diagram} \\
 & \text{Diagram} = -4 \text{Diagram} + 4 \text{Diagram} + 4 \text{Diagram} - 2 \text{Diagram} + 2 \text{Diagram} - 2 \text{Diagram} - 2 \text{Diagram} + 4 \text{Diagram} \\
 & -2 \text{Diagram} - 2 \text{Diagram} - \text{Diagram} - \text{Diagram} + 4 \text{Diagram} - 2 \text{Diagram} + 4 \text{Diagram} + 4 \text{Diagram} + 2 \text{Diagram} \\
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 \end{aligned}$$

D. Differentials for graph complexes

$$\begin{aligned}
 & -\text{graph} + 2\text{graph} - 2\text{graph} - 2\text{graph} - 2\text{graph} + 2\text{graph} - 2\text{graph} - 2\text{graph} - 2\text{graph} + 2\text{graph} - 2\text{graph} \\
 & + \text{graph} - 2\text{graph} - 2\text{graph} - 2\text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} - 2\text{graph} + 2\text{graph} - 2\text{graph} + \text{graph} + 2\text{graph} \\
 & + \text{graph} - 2\text{graph} - 2\text{graph} + \text{graph} + 2\text{graph} + 2\text{graph}
 \end{aligned}$$

$$\text{graph} = -\text{graph} - \text{graph} + \text{graph} - \text{graph} + \text{graph} + \text{graph} + \text{graph} + \text{graph} + \text{graph}$$

$$\text{graph} = -2\text{graph} + 2\text{graph} + 2\text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph}$$

$$-2\text{graph} + 2\text{graph} - 2\text{graph} - 2\text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} - 2\text{graph} - 2\text{graph} - 2\text{graph}$$

$$+ 2\text{graph} - 2\text{graph} - 2\text{graph} - 2\text{graph} + 2\text{graph} - 2\text{graph} + 2\text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph}$$

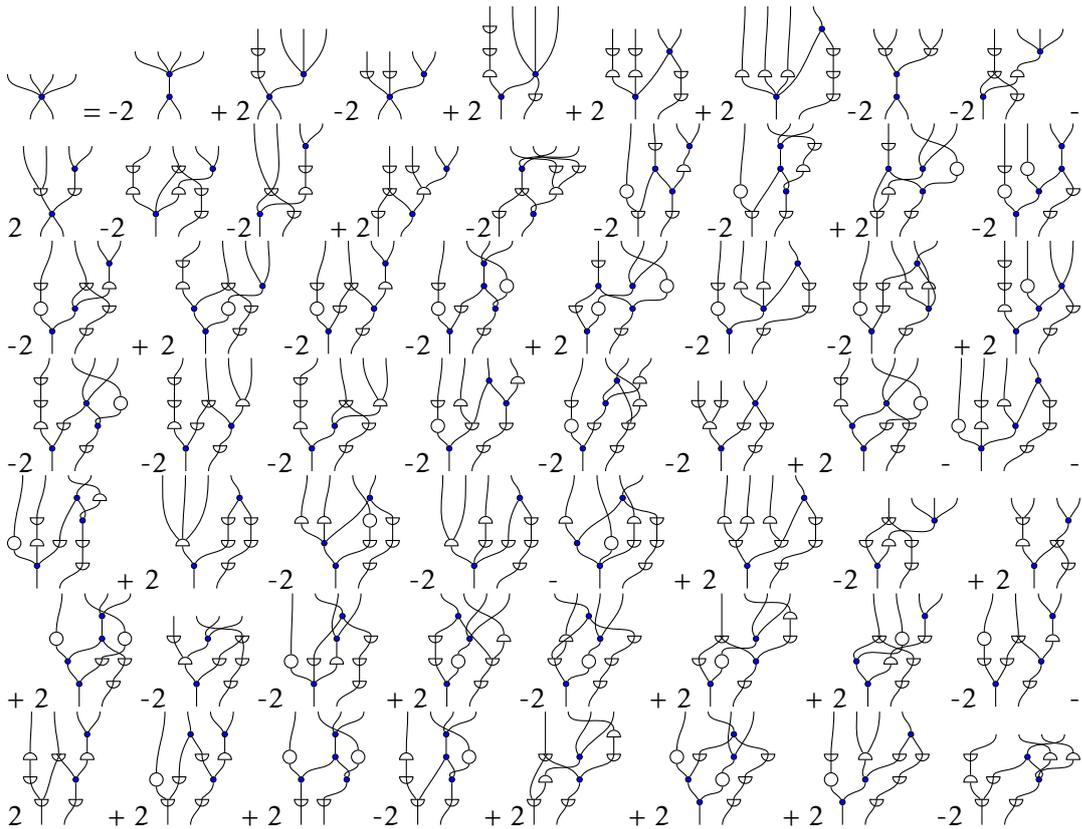
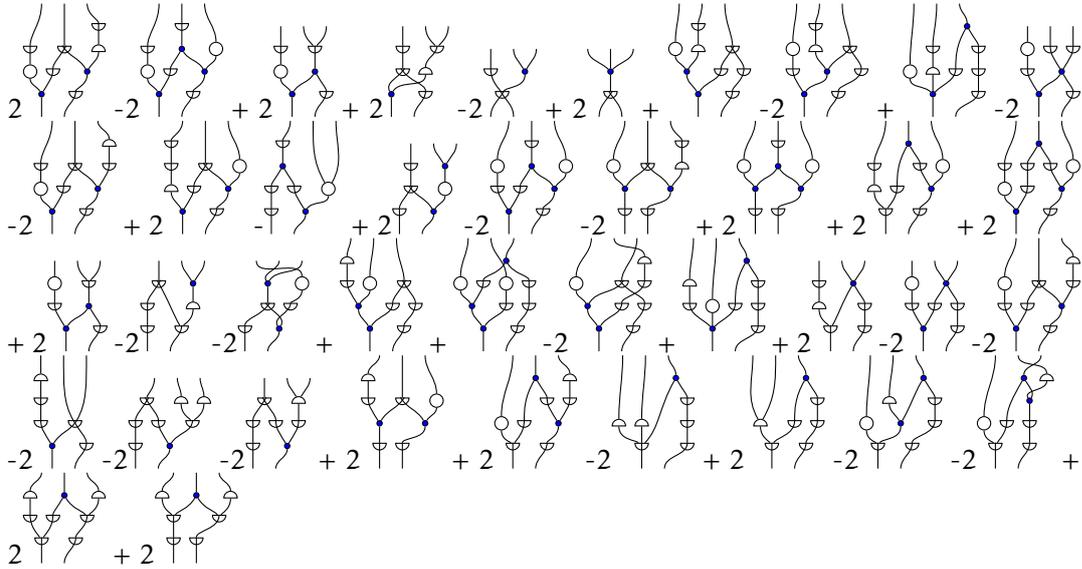
$$+ 2\text{graph} + 2\text{graph} - \text{graph} + 2\text{graph} + \text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} + \text{graph} -$$

$$2\text{graph} - 2\text{graph} - 2\text{graph} + 2\text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph} - 2\text{graph} - 2\text{graph}$$

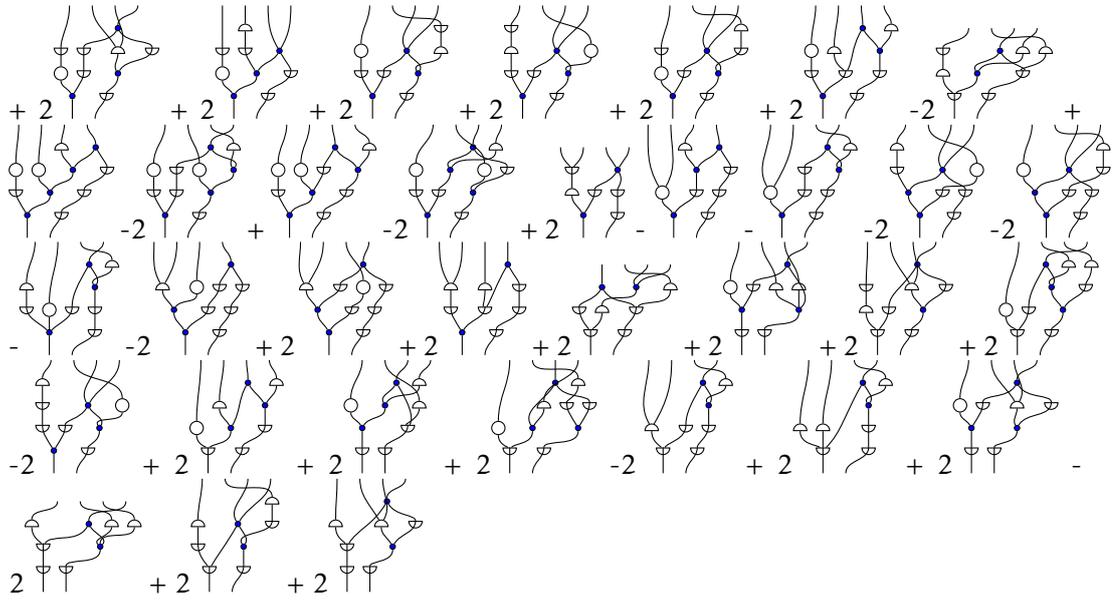
$$+ 2\text{graph} - 2\text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph} - 2\text{graph} - 2\text{graph}$$

$$-2\text{graph} + 2\text{graph} + \text{graph} + 2\text{graph}$$

$$\text{graph} = -2\text{graph} + 2\text{graph} + 2\text{graph} - 2\text{graph} + 2\text{graph} + 2\text{graph} + 2\text{graph} - 2\text{graph} + 2\text{graph} +$$



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